SCHAUM'S OUTLINE SERIES

THEORY AND PROBLEMS OF

TRANSMISSION LINES

ROBERT A. CHIPMAN

INCLUDING 165 SOLVED PROBLEMS

SCHAUM'S OUTLINE SERIES IN ENGINEERING

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TRANSMISSION LINES

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Chapter 1

Introduction

1.1. Electrical transmission systems.

The electrical transmission of signals and power is perhaps the most vital single contribution of engineering technology to modern civilization.

Among its visible manifestations, the most impressive are the high voltage transmission lines on tall steel towers that cross the countryside in all directions. Carrying thousands of megawatts of power, these lines link remote generating stations to urban load centers, or unite into cooperative complexes the power production facilities of large geographical areas.

Equally obvious and important, if neither impressive nor attractive, are the millions of miles of pole-lines that parallel city streets, rural highways and railways everywhere. The conductors of these lines may be delivering kilowatts or megawatts of power to domestic and industrial users, or may be transmitting telephone, teletype and data signals, at milliwatt power levels and kilohertz frequencies, over distances usually not exceeding a few dozen miles.

Every radio and television receiver is a terminal of another kind of electrical transmission system, in which the signal power at megahertz frequencies propagates freely from a transmitting antenna through the earth's atmosphere, guided only by the conductivity of the earth's surface, or by that of the ionospheric layers of the upper atmosphere. An essential but unobtrusive component of most television receiver installations is an electrical transmission line that conveys signal power at the picowatt or nanowatt level from the antenna to the receiver terminals.

At microwave frequencies of several gigahertz, wavelengths are small and antennas in the form of arrays, horns or paraboloidal "dishes" can have apertures many wavelengths wide. This makes possible the confinement of microwave signals or power into directed beams with small divergence angles. Towers supporting such antennas can be seen at 20 to 50 mile intervals in most regions of the United States. The electrical transmission systems of which they are a part carry all types of communication signals, including television video, telephone, data and control, at power levels of a few watts, over distances up to a few thousand miles. Similar antennas with even greater directivity are employed in microwave radar stations, in the uhf scatter-propagation circuits of the Arctic, and for communication, control and telemetering in all satellite projects, whether terrestrial, or exploratory in outer space.

Unknown to most laymen, a substantial amount of the electrical transmission of signals and power occurs on buried transmission circuits. The greatest mileage of these is in the form of twisted pairs of small wires, paper or plastic insulated, which are packed by the hundreds into cables in underground conduits. Carrying telephone, teletype and data signals within the exchange areas of cities, some of these lines operate at voice frequencies, others at carrier frequencies as high as 100 kilohertz and still others are multiplexed with pulse code modulation utilizing frequencies up to 1 or 2 megahertz. Also in city areas, partly buried and partly airborne are the vhf coaxial or shielded pair transmission lines handling the signals of multichannel cable television services.

Between cities, in the United States and in many parts of Europe, buried coaxial transmission lines operating at frequencies up to several megahertz carry all types of communication signals from teletype to television over distances of hundreds or thousands of miles, providing better security against enemy action, natural catastrophes and various kinds of signal interference than is offered by microwave link circuits.

The reliable worldwide intercontinental communication provided via terrestrial satellites is backed up by thousands of miles of less romantic but equally sophisticated carrier frequency circuits on submarine coaxial cable, with built-in amplifiers every few miles, that cross most of the oceans of the world.

Finally, in addition to all the relatively long transmission systems that have been enumerated, the totality of electrical transmission engineering includes the endless variety of shorter transmission line segments that perform many different functions within the terminal units of the systems. Ranging in length from a few millimeters in microwave circuits to inches or feet or hundreds of feet in devices at lower frequencies, and serving not only as transmission paths but in such other applications as resonant elements, filters and wave-shaping networks, many of these transmission line circuit components present more challenging design problems than all the miles of a long uniform line.

From the above summary, it can be seen that electrical transmission systems fall structurally into two distinct groups, according to whether the signals or power are guided along a path by a material "line", or propagate in the earth's atmosphere or space.

The attention of this book is confined solely to systems in the first of these two groups, and more specifically to systems consisting of two uniform parallel conductors, as defined in Chapter 2. The methods developed are applicable equally to two-wire low frequency electric power transmission lines, to telephone and data transmission lines of all types at all frequencies, and to all of the vhf, uhf and microwave two-conductor lines used in contemporary electrical engineering, at frequencies up to many gigahertz.

1.2. Analytical methods.

Electrical engineers are fully aware that, in principle, a complete analysis of any electrical problem involving time-varying signals can be made only through the use of electromagnetic theory as expressed in Maxwell's equations, with explicit recognition that the electric and magnetic fields throughout the region of the problem are the primary physical variables. In studying the modes of propagation of microwaves in hollow metal pipes, the radiation properties of antennas, or the interaction of electromagnetic waves with plasmas, for example, electrical engineering students always make direct use of electromagnetic theory.

However, it is not possible to solve the electromagnetic integral or differential equations either conveniently or rigorously in regions containing or bounded by geometrically complicated metal or dielectric structures, and the basic analytical discipline of electrical engineering curricula has from the beginning been that of lumped element circuit analysis, not electromagnetic theory. This employs the idealized concepts of two-terminal resistances, inductances and capacitances to represent the localized functions of energy dissipation, magnetic-field energy storage, and electric-field energy storage, respectively. Voltages and currents, which are related by integral or differential expressions to electric and magnetic fields, are the primary electrical variables.

The method is an adequate substitute for electromagnetic theory when the occurrences of the three functions mentioned can be separately identified, and when the dimensions of a circuit are sufficiently small that no appreciable change will occur in the voltage or current at any point during the time electromagnetic waves would require to propagate through the entire circuit. The size criterion is obviously a function of frequency. At the power line frequency of 60 hertz, the methods of lumped element circuit analysis are applicable with high accuracy to circuits several miles long, while at microwave gigahertz frequencies the same methods may be useless for analyzing a circuit less than an inch across.

In addition to the techniques of electromagnetic theory and of lumped element circuit analysis, electrical engineers make use of a third analytical procedure for electrical problems, which combines features that are separately characteristic of each of the other two methods. It extends the application of the concepts of lumped element circuit analysis to circuits which can be indefinitely long in one dimension, but which must be restricted and uniform in the other dimensions throughout their length. The analysis discloses propagating waves of the voltage and current variables, analogous to the waves of electric and magnetic fields that are solutions of Maxwell's equations. The method is known as distributed circuit analysis.

The principal subject matter of this book is the application of distributed circuit analysis to uniform two-conductor transmission lines. Frequent use is necessarily made of lumped-element circuit concepts and methods, particularly when dealing with situations at transmission line terminals, and electromagnetic theory is required to develop expressions relating the distributed circuit coefficients of a uniform line to its materials, geometry and dimensions.

By judicious conversion of symbols, much of the theory presented in this book can also be applied directly to the analysis of any other physical forms of uniform one-dimensional transmission systems. Examples are the propagation of plane transverse electromagnetic waves in homogeneous media, and mechanical wave transmission systems including physical and architectural acoustics, underwater sound, and vibrations of strings, wires and solid rods.

1.3. The evolution of electrical transmission systems.

A brief review of the historical development of electrical transmission engineering can help to explain several features of the present scene.

The subject had an improbable beginning in 1729 when Stephen Gray, a 63 year old pensioner in a charitable institution for elderly men, discovered that the electrostatic phenomenon of attraction of small bits of matter could occur at one end of a damp string several hundred feet long when an electrostatically charged body (a rubbed glass tube) was touched to the other end. He concluded that "electric effluvia" were transmitted along the line.

Sixty years earlier, councillor Otto von Guericke of Magdeburg, Germany, (famous for the evacuated Magdeburg hemispheres that teams of horses could not pull apart) had noted that short threads connected at one end to his primitive electrostatic machine became charged throughout their length, but no one deduced the concept of electrical transmission from this observation until after Gray's time.

Gray established that electrostatic transmission occurred along his moist packthread lines if they were supported by dry silk threads, but not if they were supported by fine brass wires. This first distinction between electric conductors and electric insulators was further developed in the succeeding five years by French botanist Charles DuFay, who also reported the existence of two different kinds of electricity, eventually labeled positive and negative by Benjamin Franklin in 1747.

Only 24 years elapsed after Gray's experiments before the inevitable armchair inventor proposed in the "Scots Magazine" of Edinburgh for February 17, 1753, that an electrical communication system for use over considerable distances might be constructed by employing a transmission line of 26 parallel wires (each wire identified at each end for one letter of the alphabet) supported at 60-foot intervals by insulators of glass or "jewellers' cement". A sequence of letters was to be transmitted, using Gray's technique, by touching a charged object to each of the appropriate wires in turn. At the receiving end, bits of paper or straw would be seen jumping successively to the ends of the corresponding wires. Between 1770 and 1830 several electrostatic telegraph systems were constructed in various parts of the world, over distances up to a few miles. None of these achieved any practical success, but the earlier ones may be said to have firmly established the transmission line concept.

Volta's discovery of the chemical pile in 1800 and Oersted's discovery of the magnetic effect of a current in 1820 resulted in the experimental magnetic telegraphs of Gauss, Henry and others in the early 1830's and these were followed by the first commercial electromagnetic telegraphs of Wheatstone and Cook in England in 1839, and of Morse in America in 1844. In both cases, transmission lines of buried insulated wires were tested first, but with poor results, and open wire lines on poles or trees were quickly adopted. By 1850 there were thousands of miles of fairly crude telegraph transmission lines in operation in the United States and Europe. Their design and construction was purely empirical. No test instruments existed, and even Ohm's law was unknown to most of the "electricians" of the time.

As telegraph land-lines were extended, the need arose for underwater cables across rivers, lakes, and larger expanses of water. In spite of the inadequacies of the insulating material available at the time, a successful 40-mile submarine cable was laid across the English Channel in 1851, and within two or three years submarine cables up to 300 miles in length had been laid in various parts of Europe. The operation of these long underwater telegraph circuits soon revealed a new transmission phenomenon, that of signal distortion. The received signals, recorded as pen traces on paper tapes, lost the squareness familiar on land-line circuits and became blurred attenuated waverings of a jittery baseline.

Promoters were naturally stimulated by the prospect that a transatlantic submarine cable might provide the first instantaneous communication link between Europe and America, but they hesitated to risk the large amount of capital required without some reasonable assurance that useful signals could traverse an underwater cable many times longer than the longest then in use. For advice they turned to William Thomson, professor of natural philosophy at the University of Edinburgh. This may be the first instance of the employment of a professional consultant on a major commercial venture in electrical engineering. Thomson (later Sir William and ultimately Lord Kelvin) carried out in 1855 the first distributed circuit analysis of a uniform transmission line. He represented the cable by series resistance and shunt capacitance uniformly distributed along its length. He fully appreciated that a more complete investigation of the cable's signal transmission properties might require attributing distributed series inductance and distributed shunt conductance (leakage) to the cable. By trial calculations, however, he found that at telegraph-signal frequencies the effects of the inductance would be negligible, and from measurements on cable samples he satisfied himself that leakage conductance could be kept low enough to be unimportant. Thomson's published paper of 1855 makes profitable reading for electrical engineers even today.

Thomson provided additional assistance of a more practical nature to the cable promoters by inventing more sensitive receiving galvanometers than any in existence, and by directing the manufacture of purer copper, with conductivity several times greater than that of the commercial metal then available. After various mechanical difficulties were surmounted, a cable designed to Thomson's specifications was successfully laid across the Atlantic in 1858, and carried messages for a few weeks before the insulation failed. Further financing of the cable project was delayed by the Civil War in the United States, but in the decade after 1866 numerous cables were laid, some of which were still in operation as telegraph circuits until fairly recently.

The technological problems encountered in the design and operation of long submarine cables were so much more sophisticated than those associated with land-line telegraphs that a new class of technical personnel gradually developed, consisting of men with a substantial knowledge of electrical principles and techniques. The world's first professional association of electrical engineers was the Society of Telegraph Engineers, founded in England in 1871, and most of its charter members were submarine telegraph engineers. The Society added "and Electricians" to its title in 1880, and became the present day Institution of Electrical Engineers in 1889, five years after the formation of the American Institute of Electrical Engineers in the United States.

The invention of the telephone in 1876 immediately made evident further complications in the use of transmission lines for electrical communication. The frequencies required for voice reproduction were hundreds of times higher than those used in telegraphy. Attempts at inter-city telephony over the telegraph lines of the time, which generally consisted of single iron wires with the ground as a return circuit, were frustrated by the low level and garbled unintelligibility of the received signals. In the ensuing 40 years, improvements in long distance voice frequency telephony developed slowly but steadily through a combination of empirical discoveries and theoretical studies.

The man chiefly responsible for a new and more complete mathematical analysis of signal propagation on transmission lines was Oliver Heaviside, one of the most unusual, and at the same time most productive engineer-mathematicians of all time. A nephew of Charles Wheatstone, the prominent electrical scientist and telegraph inventor, and brother of a well-known telegraph engineer, Oliver Heaviside worked for a few years in the British telegraph industry, then "retired" (according to a major encyclopedia) in 1874 at age 24, to spend the next fifty years of his life in almost total seclusion. During that half century his publications on transmission lines, electric circuit theory, vector analysis, operational calculus, electromagnetic field theory, and numerous other topics, did more to define the concepts and establish the theoretical methods of modern electrical engineering than has the work of any other one individual. Transmission line theory as developed in several of the chapters of this book is entirely the work of Oliver Heaviside, and was first published by him during the 1880's. Modern presentations of Maxwellian electromagnetic theory are also essentially in the form created by Heaviside.

By the end of the 19th century, experience and analysis had indicated that long voice frequency telephone circuits, still the most challenging transmission line problem, worked best when constructed of two large low-resistance copper wires, mounted as widely spaced, well insulated open-wire pole lines. The use of ground-return was abandoned. From his equations Heaviside had noted that on most practical lines, voice signals should travel with reduced loss and with greater fidelity if the distributed inductance of the line could be increased without adversely changing the other distributed circuit coefficients. In the United States, Michael Pupin of Columbia University and George Campbell of the Bell Telephone Laboratories conceived about 1900 that a practical alternative to the difficult process of increasing the uniformly distributed inductance of a line might be the insertion of low resistance lumped inductance coils at intervals of a mile or so along the line. The technique, known as "loading", is discussed briefly in Section 5.8 on page 60. It proved extremely successful, and loading coils were connected into tens of thousands of miles of open wire and cable telephone circuits in a period of about thirty years. Loading permitted the economy, in long telephone lines, of using smaller gauge copper wires than would otherwise have been needed to give the same electrical efficiency and quality of transmission.

The carbon microphone of telephony is an electromechanical amplifier, whose electrical power output can be a thousand times greater than its mechanical voice-power input. From about 1890 on, much effort was expended in attempts to develop this property into an amplifier unit that could be inserted into long telephone lines to offset the effects of resistance and leakage. The results never became sufficiently satisfactory to be used on a commercial scale. The peak potential of passive voice-frequency telephone circuits is well exemplified by the historic occurrence in 1911 of a brief telephone conversation between New York City and Denver, Colorado. The circuit consisted of two large-conductor pole-mounted transmission lines between the two cities, connected in parallel. No amplifiers were involved, and the low-resistance, high-inductance lines could not benefit from any practical form of loading. The circuit could handle only a single voice frequency signal, and its operation proved that telephony over a distance of two thousand miles was an economic impossibility for the technology of the time. In a country more than three thousand miles across, this was an unpleasant conclusion.

At this critical juncture in the progress of electrical communication, Lee De Forest in 1912 offered the telephone industry his primitive erratic triode amplifier, incorporating the thermionic "audion" he had invented in 1907. Two years of intensive research in the industry's laboratories improved the device to the point of making transcontinental telephony a realized achievement in 1915. With vacuum tube amplifiers the losses of small conductor lines could be offset inexpensively by stable amplifier gain, and distortion could be reduced to any desired value by networks "equalizing" the characteristics of a line over any range of signal frequencies.

In possession of a practical solution to the problems of distance and signal quality, telephone engineers looked for new methods of cost reduction. It was obvious that the greatest rewards lay in the possibility of using the expensive transmission lines for several voice frequency signals simultaneously. Pursuit of this goal, in many different ways, became a major activity of the telephone industry for the next fifty years and still receives considerable attention.

Military interest in electronic tubes and their associated circuitry during World War I greatly accelerated the development of amplifiers, oscillators, filters and other devices, and helped make feasible by 1919 the first installations of long-distance carrier-frequency telephone systems, in which several voice frequency channels of bandwidth about 4 kilohertz are translated to different higher frequency intervals for transmission. Early carrier systems (the technique is now known as "frequency division multiplexing") handled three telephone channels in each direction on a single two-wire cable pair or open-wire transmission line, using frequencies up to about 30 kilohertz.

It is perhaps the greatest irony of electrical engineering history that the loading technique, which was the salvation of the long distance telephone industry in the first quarter of the century, had converted every loaded transmission line into a low-pass filter, incapable of transmitting any frequencies above 3 or 4 kilohertz, and hence useless for carrier frequency systems. From 1925 to 1940, most of the millions of loading coils previously installed were removed.

While the wire-line communication industry was evolving from the very limited capabilities of passive voice-frequency transmission line circuits to the virtually unlimited potentialities, for continental purposes at least, of multi-channel carrier-frequency circuits using vacuum tube amplifiers, the technology of communication by freely propagating electromagnetic waves was developing simultaneously. Marconi's experiments of 1895 to 1902 showed that local and intercontinental telegraphy could be accomplished without wires, using hertzian waves. Wireless telegraphy quickly became a glamorous and highly publicized activity, but in spite of dire forebodings it offered no significant competitive threat to any of the established telephone and telegraph services using land lines or submarine cables until after the development, during World War I, of medium power thermionic vacuum tubes as transmitters and various sensitive vacuum tube circuits as receivers. (It did create still another distinctive body of technical personnel, however, and the Institute of Radio Engineers, founded in 1912, resulted from the merger of two small societies of wireless telegraph engineers. In 1963 the I.R.E. merged with the A.I.E.E. to form the Institute of Electrical and Electronics Engineers.)

Wartime progress led to the founding in the early 1920's of the radio broadcasting industry, using frequencies around one megahertz, whose technological principles have undergone only minor changes since that time. Subsequent exploration of higher frequencies discovered the extraordinary world-wide propagation characteristics of "short waves" between 3 and 30 megahertz, and radio telephone circuits at these frequencies, adopted by the telephone industry before 1930 for the first intercontinental telephone service, remained the only commercial solution to that problem for the next thirty years.

During this decade the rapidly increasing proportion of electrical engineers employed in carrier frequency telephony and high frequency radio found it necessary to deal with transmission circuits many wavelengths long, whose analysis required the use of distributed circuit methods. The study of Heaviside's transmission line theory began to appear in a few electrical engineering curricula, generally at the graduate level, using textbooks by Steinmetz, Dwight, Fleming, Pierce, Johnson and Kennelly, among others.

The 1930's witnessed the extension of the technology of electronic devices and circuits from frequencies in the tens of megahertz to frequencies of several hundred megahertz, with such applications as FM broadcasting and mobile telephone service, and the beginnings of television and radar. The theory of uniform transmission lines was incorporated into several undergraduate curricula, often as an optional subject. Popular textbooks dealing with the subject included those by Terman, Everitt and Guillemin.

Finally, the concentrated attention given to the whole field of uhf and microwave engineering during World War II, with the subsequent development of commercial television, microwave communication links, radio and radar astronomy, and the innumerable applications of these frequencies in space exploration and continuing military uses, made it obvious that transmission line theory must become a basic topic in all electrical engineering curricula.

The realization of this result was marked by the publication in the period 1949-1954 of a large number of widely used textbooks. Among the authors were Skilling, Johnson, Kimbark, Cramer, Ryder, Jackson and Karakash. These were the successors to a few textbooks that appeared during the war period, of which the most influential were probably the two by Sarbacher and Edson and by Ramo and Whinnery. Since 1955 new textbooks on transmission line theory have appeared less frequently.

The "high frequency frontier" of electrical engineering has in the last two decades been pushed to frequencies beyond one hundred gigahertz, using solid state and free electron devices, and with quantum-electronic lasers able to generate power at frequencies all the way from hundreds of gigahertz through hundreds of terahertz in the infra-red, visible and ultra-violet regions of the spectrum, the very concept of a frontier has lost most of its meaning.

At frequencies for which the free space wavelength is less than a few millimeters, two-conductor electrical transmission systems operating in the mode implicit in distributed circuit theory are little used. Other analytical formulations, such as those of electromagnetic theory, or of geometrical or physical optics, become appropriate. To the extent that the interests of electrical communication engineers are scattered over a far wider region of the electromagnetic spectrum than was the case a few years ago, the relative importance of transmission line theory in their total study program has begun to diminish. Two-conductor lines, however, will always remain a basic transmission technique, and for their analysis Heaviside's theory will always be as fundamental as Maxwell's equations are to electromagnetics, since the theory is in fact a direct application of the primary electromagnetic relations to the principal mode of propagation on such lines.

1.4. References.

The following is a list of a few historic references and some of the more recent textbooks in the field of distributed circuit transmission line theory.

An exposure to the pithy personality of Oliver Heaviside, through the introductory Chapter 1 of his *Electromagnetic Theory,* is an interesting experience.

King's *Transmission Line Theory* contains a complete bibliography of transmission line textbooks published before 1955.

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Chapter 2

Postulates, Symbols and Notation

2.1. Postulates of distributed circuit analysis.

The distributed circuit analysis of uniform transmission lines, begun by William Thomson (Lord Kelvin) in 1855 and completed by Oliver Heaviside about 1885, is derived by applying the basic laws of electric circuit analysis to systems described by the following postulates.

Postulate 1. The uniform system or line consists of two straight parallel conductors.

The adjective "uniform" means that the materials, dimensions and cross-sectional geometry of the line and its surrounding medium remain constant throughout the length of the line. Typically, a signal source is connected at one end of the system and a terminal load is connected at the other end, as shown in Fig. 2-1.

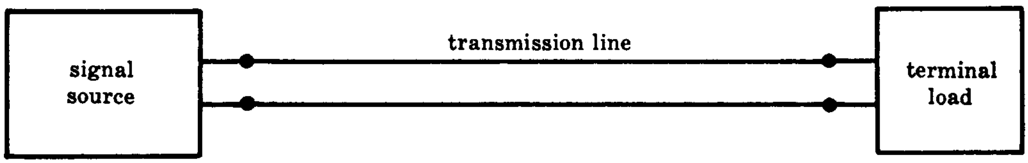


Fig. 2-1. Basic transmission line circuit.

This postulate does not require that the two conductors be of the same material or have the same cross-sectional shape. The analysis is therefore valid for a conductor of any material and cross section enclosing another conductor of any material and cross section, for a wire parallel to any conducting plane or strip, and for many other useful constructions in addition to the simple example of two parallel wires of circular cross section and of the same diameter and material.

The analysis is applicable to systems with more than two parallel conductors, provided these are interconnected in such a way as to present only two terminals at the points of connection of source and load. Systems may also involve shielding conductors that are not connected to the line at any point.

Fig. 2-2 shows the cross-sectional configurations of the conductors for several uniform two-conductor transmission lines used in engineering practice.

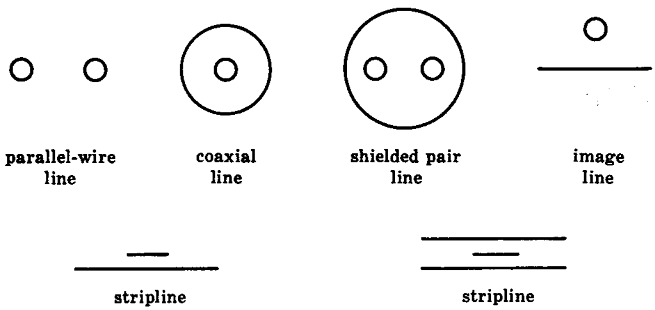


Fig. 2-2. Conductor cross sections for several practical transmission lines.

In general, twists or bends in a transmission line violate the "uniformity" postulate and create effects not explainable by the distributed circuit theory. These effects will be negligible if the rate of the twist or bend does not exceed about one degree in a length of line comparable with the separation of the line conductors.

The uniformity postulate is also violated by any discontinuities in a line, such as the termination points of an otherwise uniform system, or the point of connection between two uniform lines that differ physically in some respect. In the vicinity of discontinuities like these, phenomena will occur which are not in accord with distributed circuit theory. The anomalous behavior is usually confined to distances not greater than a few times the separation of the line conductors, on either side of the discontinuity.

Postulate 2. The currents in the line conductors flow only in the direction of the length of the line.

This is a basic premise of elementary electric circuit analysis, and it may seem unnecessary that such a requirement should have to be stated in making a distributed circuit analysis of transmission lines. It is a fact, however, that under certain conditions signals can propagate on any uniform transmission line with either the whole of the line current or a component of it flowing *around* the conductors rather than along them. These cases are known as "waveguide" modes of propagation. They are discussed further in Section 2.2.

Postulate 2 means that distributed circuit transmission line theory does not recognize the existence of waveguide modes.

Postulate 3. At the intersection of any transverse plane with the conductors of a transmission line, the instantaneous total currents in the two conductors are equal in magnitude and flow in opposite directions.

In elementary network theory the postulate equivalent to this for a simple loop circuit such as Fig. 2-1 would be that the current is the same at *all* points of the circuit at a given instant. Postulate 3 allows the instantaneous currents to be different at different cross sections of the line at the same instant. Clearly this is not possible without violating Kirchhoff's current law unless currents can flow transversely between the two conductors at any region of the line's length. Provision for such transverse currents is made in postulate 5.

Postulate 4. At the intersection of any transverse plane with the line conductors there is a unique value of potential difference between the conductors at any instant, which is equal to the line integral of the electric field along all paths in the transverse plane, between any point on the periphery of one of the conductors and any point on the periphery of the other.

Like postulate 3, this postulate has the consequence of ruling out waveguide modes, for which the line integral of the electric field is in general not independent of the path.

Postulate 5. The electrical behavior of the line is completely described by four distributed electric circuit coefficients, whose values per unit length of line are constant everywhere on the line. These electric circuit coefficients are resistance and inductance uniformly distributed as series circuit elements along the length of the line, together with capacitance and leakage conductance uniformly distributed as shunt circuit elements along the length of the line.

It is an essential part of this postulate that the values of these distributed circuit coefficients at a given frequency are determined only by the materials and dimensions of the line conductors and the surrounding medium. They do not vary with time, nor with line voltage or current. The line is thus a linear passive network.

Postulate 5 has a direct relation to postulates 3 and 4. The line currents of postulate 3 are accompanied by a magnetic field. The distributed line inductance is a measure of the energy stored in this magnetic field for unit length of line, per unit current. There is power loss as the line currents flow in the conductors. The distributed line resistance is a measure of the power loss in unit length of line per unit current.

Similarly the line potential difference of postulate 4 is associated with an electric field. The distributed line capacitance is a measure of the energy stored in this electric field per unit length of line, for unit potential difference. There is power loss in the medium between the conductors because of the potential difference. The distributed line conductance is a measure of the power loss in unit length of line, for unit potential difference.

The existence of distributed *shunt* circuit coefficients accounts for the possibility discussed in connection with postulate 3, that the conductor currents can be different at different cross sections of the line. Conduction currents or displacement currents will flow transversely between the conductors, as a function of the potential difference between them or its time rate of change, respectively. The line currents at two separated cross sections of the line will differ by the amount of the transverse current in the intervening length of line.

2.2. Waveguide modes and electromagnetic theory.

In Chapter 1 it was noted that a complete analysis of the transmission properties of any transmission line system can be made by starting with Maxwell's equations, and seeking a solution subject to the boundary conditions imposed by the line conductors. Such an analysis reveals all the "waveguide" modes mentioned under postulates 2 and 4. These fall into two categories known as TE (*transverse electric*) and TM (*transverse magnetic*) waves, distinguished respectively by field distributions with components of magnetic or electric field parallel to the length of the line. For any transmission line structure there is an infinite number of these modes, each with its own specific patterns of electric and magnetic fields.

Any TE or TM mode can be propagated on a particular transmission line only at frequencies above some minimum cutoff frequency, which is calculable for each separate mode from the dimensions and materials of the transmission line. For lines whose conductor separations do not exceed a few inches, these cutoff frequencies range from thousands to tens of thousands of megahertz. Hence in most practical uses of transmission lines, at frequencies from d-c to uhf there is no possibility of TE or TM modes being propagated. When transmission lines are used at microwave or millimeter-wave frequencies, care must sometimes be taken to avoid the occurrence of such modes, since their presence will invariably result in excessive line losses and other undesirable consequences.

No useful applications have yet been made of the TE and TM modes that can propagate on two-conductor transmission lines at extremely high frequencies. Within single conductors in the form of hollow metal pipes, however, the TE and TM modes are the basis of the invaluable microwave technique of waveguide transmission.

Although the TE and TM modes cannot propagate in any transmission system at frequencies below a cutoff frequency which is in the microwave frequency range for typical line constructions, the field patterns of one or more of these modes are invariably generated by irregularities and discontinuities in a system. When the frequency is below the cutoff frequency for the modes excited, the field patterns are unable to propagate as waves. They do, however, diffuse or penetrate a short distance from their point of origin, a distance not greater than a few times the conductor separation of the transmission system. They are responsible for the anomalous behavior of transmission lines, mentioned in Section 2.1, postulate 1, that occurs in the vicinity of discontinuities and is not explainable by distributed circuit theory.

2.3. The TEM mode.

The analysis of any uniform two-conductor transmission line by the methods of electromagnetic theory reveals one other unique mode in addition to the TE and TM infinite sequences of modes. This single mode differs from the others in that its electric and magnetic fields are everywhere transverse to the direction of the conductors' length, and there is no cutoff frequency other than zero. Designated TEM (*transverse electromagnetic*), this mode has the field representation that corresponds to the voltages and currents of the distributed circuit theory of transmission lines.

The electric and magnetic field patterns for the TEM mode in a circular coaxial line are shown in Fig. 2-3. They are the only possible distribution of electric and magnetic fields that can simultaneously satisfy the postulates listed in Section 2.1 and the basic laws of electromagnetism, for this geometry of conductors.

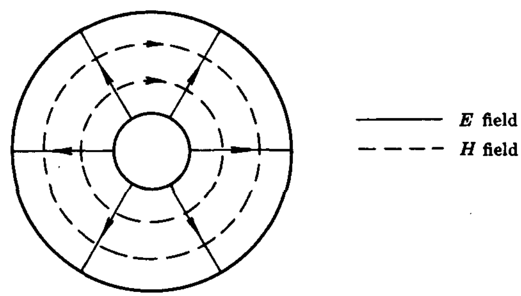


Fig. 2-3. Electromagnetic field pattern for the TEM (transverse electromagnetic) mode in a coaxial transmission line.

2.4. Distributed circuit analysis and electromagnetic theory.

The analysis of transmission lines by distributed circuit methods is not independent of the analysis by field methods, since the calculation of the circuit coefficients used in the former can be made only from a knowledge of the electric and magnetic fields associated with the line voltages and currents.

The practical advantage of the distributed circuit method of analysis is that it uses the circuit analysis language of voltages, currents, impedances, etc. It is customary in engineering practice, even at frequencies up through the microwave region, to designate the sources and loads used with transmission systems by their equivalent circuits, rather than by a statement of the spatial conditions they impose on electric and magnetic fields.

The combination of an analysis of transmission line behavior in circuit terms with the equivalent circuit specification of sources and loads, permits studying the overall system of source, line and load with the help of all the powerful techniques that have been developed in electric network theory.

A complete analysis by electromagnetic theory, on the other hand, of the detailed distributions of the electric and magnetic fields associated with a source-line-load system would raise insuperable mathematical difficulties for any structure with other than exceedingly simple and continuous geometry throughout.

2.5. Coordinates, coefficients and variables.

Before the postulates of equivalent circuit transmission line analysis stated in Section 2.1 can be embodied in equations, it is necessary to select symbols for coordinates, variables and physical coefficients, and to establish a few sign conventions. There are "standard" symbols recommended by major electrical professional societies for a few of the quantities to be labelled, but for most of the decisions involved there are no generally accepted guides. The result is that current textbooks and journal articles on transmission line topics contain a regrettable variety of notation, to which the reader must adjust in each context.

A brief discussion of the reasons for the choices used in this book, together with comments on the possible merits of alternatives, is offered in the next few sections for the benefit of readers who like to compare several accounts of the same topic from different sources.

2.6. Choice of coordinate notation.

So far as transmission is concerned, the transmission line problem is one-dimensional, with a single coordinate axis parallel to the length of the line. In this book the symbol z is selected for this coordinate.

It is tempting to choose x for this coordinate, as in the one-dimensional problems of elementary mathematics. However, the analysis of many other transmission systems of interest to engineers, such as waveguides, vibrating strings, transversely vibrating rods, electromagnetic wave beams, etc. requires the use of coordinates in the transverse plane as well as in the direction of propagation. The use of x and y for these transverse coordinates has been adopted quite unanimously, and it is therefore appropriate to select z as the universal symbol for the longitudinal coordinate of all transmission systems. Supporting this choice is the applicability of cylindrical coordinates, with their longitudinal z-coordinate, to the analysis of the many forms of transmission line that have circular elements or circular symmetry.

2.7. Choice of origin for longitudinal coordinate.

An unexpected development that arises in setting up the distributed circuit theory of transmission lines is that, for certain important purposes, more significance attaches to the distance of a point on the line from the terminal load than to its distance from the signal source. This has led some writers to base their entire analysis on a longitudinal coordinate whose origin is at the load. Some take the additional step of locating this load-origin at the left end of the line, with the signal source at the right.

On the whole it seems more appropriate that a signal moving from a source to a load should be moving in the direction of an increasing coordinate, and since it is universal practice mathematically to have the coordinates of a one-dimensional problem increase from left to right, the analysis in this book uses the convention that the signal source is at the left-hand end of the line, the terminal load is at the right-hand end, and the longitudinal coordinate z has its origin at the signal source.

When occasion demands, the distance of a point on the line from the terminal load is indicated by a coordinate d, with origin at the load and increasing from right to left.

The symbol l is used at all times for the total length of the line.

Fig. 2-4 shows a complete transmission line system, with all the symbols for longitudinal coordinates and distances according to these chosen conventions.

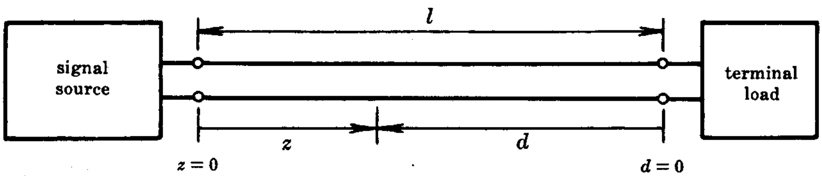


Fig. 2-4. Coordinates on a transmission line circuit.

2.8. Symbols for current and voltage.

The dependent variables in the distributed circuit analysis of transmission lines are current and voltage, which are functions of time at any point on the line, and functions of position on the line at any instant. The functional expressions describing these relations are determined by the signal source, the terminal load, and the distributed circuit coefficients and total length of the line.

In elementary circuit analysis it is generally accepted notation to use lower case letters as symbols for the instantaneous values of time-varying dependent variables. Capital letters are used for d-c quantities, and for the complex number or phasor values of a-c quantities which have constant amplitude harmonic variation with time.

In the analyses in this book, d-c currents and voltages seldom occur. Capital letter symbols for current and voltage therefore represent complex number phasor values. Their magnitudes are rms magnitudes, not peak values. Unless specifically designated as quantities at the signal source or terminal load ends of the line, they are functions of position along the line. Lower case symbols for current or voltage represent instantaneous values, which must also be understood to be functions of the coordinate z.

As a symbol for current the letter i, I (from the French word *intensité*) replaced all competitors long ago.

For voltage or potential difference the letter e, E (presumably for *electromotive force*) has had majority approval for several decades.

As long as physicists paid little attention to electric circuit theory, and engineers even less to electromagnetic field theory, only minor confusion resulted from using the same symbol E for the two important concepts of potential difference and electric field strength. This segregation no longer exists and the confusion is now serious. It is sometimes avoided by using special symbols, such as script letters, but these are awkward for note-takers and typists. The situation can be satisfactorily resolved by using the symbol v, V for voltage. In technical writing V is widely used for the concept of volume, and v for the concept of velocity. Neither of these occurs frequently enough in the same context with the concept of voltage to constitute a hazard.

In summary, the notation for dependent variables in the transmission line theory of this book is as follows:

i or $i(z,t)$ = instantaneous current at a specific point on a transmission line, i.e. the current at time t at coordinate z;

I or $I(z)$ = complex rms value of a constant amplitude harmonically varying current at coordinate z;

$|I|$ or $|I(z)|$ = rms magnitude of I or $I(z)$.

The symbols v, $v(z,t)$, V, $V(z)$, $|V|$ and $|V(z)|$ have corresponding meanings for voltage or potential difference.

A sign convention relating current directions and voltage polarities must be adopted, to avoid ambiguities when a circuit analysis is made of a transmission line section. The convention used is standard in elementary circuit theory, for two-terminal networks.

Thus at a coordinate z on a transmission line as shown in Fig. 2-5(a) below, an instantaneous voltage $v(z,t)$ in the time domain may be represented by an arrow drawn from one line conductor to the other in the transverse plane at z. The head of the arrow has positive polarity, and the voltage $v(z,t)$ is positive when the arrow is directed from the lower conductor to the upper, as in the figure.

Similarly, the line currents at coordinate z are indicated by *two* arrowheads, one in each line conductor, the two pointing in opposite directions according to postulate 3 of Section 2.1. The sign of the current is positive when the current in the *upper* line conductor flows in the direction of *increasing* z, as in the figure.

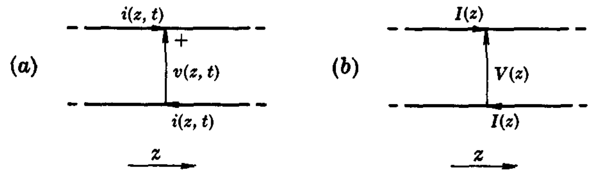


Fig. 2-5. Sign conventions for current and voltage in transmission line analysis: (*a*) in the time domain; (*b*) for phasors.

It can be noted that if the portion of the transmission line circuit to the left of the coordinate z is removed, the sign conventions adopted have the usual simple implication for the two-terminal passive network to the right of the coordinate z, that a positive applied voltage causes a positive current to flow.

Fig. 2-5(*b*) illustrates the corresponding conventions for phasor notation. Here the implication is that if $V(z)$ in the direction shown is chosen as a real reference phasor, the direction shown for $I(z)$ represents a phasor current with a non-negative real part.

Example 2.1.

What meaning can be given to the symbols $i(t)$, $I(t)$, $|v|$, $V(z,t)$ in view of the definitions given above?

Three of these four symbols have no meaning within the definitions stated.

Since the symbol i is for a current which is a function of both z and t, to write it as $i(t)$, a function of t alone would require additional explanation.

The symbol I is for a current whose manner of time-variation is implicitly understood. To designate it as $I(t)$, explicitly a function of time, is either unnecessary or implies some additional form of time variation (such as modulation) superimposed on the harmonic variation. An auxiliary definition would have to be supplied to give meaning to this symbol. The same reasoning applies to the term $V(z,t)$.

Since the symbol v stands for an instantaneous value of voltage at a particular point on the line, its meaning is a scalar quantity, which could be either positive or negative. $|v|$ is the magnitude of v, and is a positive quantity by definition.

2.9. Symbols for distributed circuit coefficients.

These are the symbols about which there is the greatest measure of agreement in transmission line writings. The definitions are:

R = total series resistance of the transmission line per unit length, including both line conductors, or both combinations of conductors making up the two sides of the line. In mks units, R is in ohms/meter. (If the interconductor space of a transmission line is filled with a material that is *magnetically* lossy, i.e. which converts electromagnetic field energy into heat in proportion to the square of the magnetic field B in the medium, these losses will be represented by a contribution to R, in the equivalent circuit.)

L = total series inductance of the transmission line per unit length, including inductance due to magnetic flux both internal and external to the line conductors. In mks units L is in henries/meter.

G = shunt conductance of the transmission line per unit length. This is the circuit representation of losses that are proportional to the square of the voltage between the conductors or the square of the electric field in the medium. Usually G represents internal molecular lossiness of dielectric insulating materials, rather than an actual charge flow leakage current. In mks units G is in mhos/meter.

C = shunt capacity of the transmission line per unit length. In mks units C is in farads/meter.

It must be noted carefully that the symbols R, L, G and C as defined here have *different meanings and dimensions* from those with which the reader has become familiar in the study of lumped element networks. In that context the symbols mean the resistance, inductance, etc. of a two-terminal element postulated to have negligible dimensions. Here they mean resistance, inductance, etc. *per unit length* of a four-terminal or two-port circuit with non-zero length and with uniformly distributed circuit coefficients. Using mks units, R in the former case would be expressed in ohms, while in the transmission line case it is expressed in ohms/meter, and a similar distinction holds for the other three symbols.

To avoid this ambiguity, writers have occasionally used modified symbols for the distributed circuit quantities, such as R', r, $\bar{R}$, R/l, $\mathcal{R}$, etc. Most, however, use the notational procedure followed in this book, that R, L, G and C when *not subscripted* are the distributed circuit coefficients. When subscripted they represent lumped circuit elements in two terminal networks, the subscript indicating the function of the network, as described in the next section.

2.10. Symbols for terminal quantities and elements.

Notation is needed to distinguish voltages, currents and connected impedances at the signal source and terminal load ends of a transmission line from the same quantities at arbitrary points along the length of the line. This book follows common practice in using subscripts for the purpose. Transmission line writings show much variety in the choice of these subscripts. For the signal source end of the line, subscripts s or S (for source or sending-end), g or G (for generator) and o or O (for origin) are found, while for the terminal load end of the line r or R (for receiver), l or L (for load) and t or T (for termination) are all in common use.

Because R, L and G occur prominently in the analysis as symbols for the distributed circuit coefficients, it seems unwise to use them as subscripts with a totally different reference. The symbols S and T seem to offer minimum opportunity for confusion, and are adopted in this book as the subscripts referring respectively to the signal source end of the line and the terminal load end.

Example 2.2.

What are the meanings of i_T, V_S, I_G, I_L, V_t, $|I_T|$, and L_T, in the notation defined above?

i_T is the instantaneous current (at time t) at the terminal load end of the line.

V_S is the complex rms value or phasor rms value of the time-harmonic voltage of a generator connected at the input or signal source end of the line. It is *not* the symbol for the phasor voltage at the input *terminals* of the line, which would be $V(z=0)$. If the connected generator had zero internal impedance, or if the line current were zero at $z=0$, V_S and $V(z=0)$ would have the same value, but as symbols they represent different quantities.

I_G, I_L and V_t have no meaning within the definitions stated. $|I_T|$ is the rms magnitude of the phasor current at the terminal load end of the line. It is identical with $|I(z=l)|$ where $I(z=l)$ is the phasor line current at $z=l$, since by definition the total length of the line is l.

L_T represents a lumped inductance connected at the terminal load end of the line. It could also be the inductance component of the simple series circuit equivalent to any more complicated linear network connected as terminal load.

2.11. Notation for impedance and admittance.

Three occurrences of impedance quantities (or admittance quantities) need to be distinguished in transmission line theory. These are:

1. Impedances or admittances connected at the line terminals. A terminal load impedance has the symbol Z_T in this text, with components R_T and jX_T. From previous definitions it is clear that $Z_T = V_T/I_T$ always, this being a complex number equation involving the ratio of two phasors.

Similarly, the internal impedance of a generator connected at the signal source end of the line has the symbol $Z_S\,(=R_S+jX_S)$. In general Z_S is *not* equal to V_S/I_S, since the latter ratio depends on the total transmission line circuit, while Z_S can have any arbitrary value, entirely independent of the transmission line.

2. A unique quantity appearing in transmission line equations, that has the dimensions of impedance and is determined solely by the distributed circuit coefficients of the line and the signal frequency. This is now universally called the "characteristic impedance" of a transmission line and given the symbol Z_0 (with components R_0 and X_0). In the past it has often been called the "surge impedance" or the "iterative impedance" of a line. Symbols Z_c, Z_s and Z_i have occasionally been used for it.

3. The impedance given by the ratio of the phasor line voltage to the phasor line current at any cross section of the line. The notation $Z(z)$ or $Z(d)$ is appropriate for this (with corresponding symbols for real and imaginary components) according to whether the particular point on the line is located by its coordinate relative to the signal source end or the terminal load end of the line as origin. The impedance at a specific numerical coordinate requires expanded notation such as $Z\,(z=3)$ for a point 3 meters from the source end of the line.

This text does not make explicit use of the concepts of distributed line impedance $Z = R + j\omega L$ or distributed line admittance $Y = G + j\omega C$ found in many treatments of transmission line theory.

The corresponding admittance definition and symbol can be substituted for any of the above occurrences of impedance language. Thus:

$$Y_T = 1/Z_T = G_T + jB_T, \qquad Y_0 = 1/Z_0 = G_0 + jB_0, \qquad Y(z) = 1/Z(z) = G(z) + jB(z)$$

Either impedance or admittance may of course also be expressed in polar form: $Z(z) = |Z(z)|\,e^{j\theta(z)}$.

Example 2.3. What are the meanings of $G\,(z=0)$, $B\,(z=l)$, $X\,(d=5)$, $Y(3)$, $Z(0)$?

$G\,(z=0)$ means the conductance component (i.e. the real part) of that admittance which is the ratio of the phasor current at the signal source end of the line to the phasor voltage at the same position. This is clearly entitled to the notation G_{inp}, i.e. the conductance component of the input admittance Y_{inp} of the line. (Note that the symbol G here has no connection with the symbol G used for one of the distributed circuit coefficients of the line. The duplication is unfortunate but is well entrenched in contemporary notation.)

$B\,(z=l)$ means the susceptance component (i.e. the imaginary part) of that admittance which is the ratio of the phasor current to the phasor voltage at the terminal load end of the line. Since this ratio is necessarily identically equal to the terminal load admittance *connected* to the line, it follows that $B\,(z=l) = B_T$.

$X\,(d=5)$ means the reactive component (i.e. the imaginary part) of that impedance which is the ratio of the phasor voltage to the phasor current at a point on the line 5 m from the terminal load end.

$Y(3)$ and $Z(0)$ have no meaning since they are ambiguous as to whether the coordinate referred to is z or d.

2.12. Notation for transient response.

In introductory textbooks on the theory of lumped element circuits it is common practice to use a generalized notation in which voltages, currents, impedances and admittances are shown as functions of the complex frequency variable s. Circuit response for steady state harmonic signals of angular frequency ω is then obtained by substituting $j\omega$ for s, while time-domain response is obtained by Laplace transform methods.

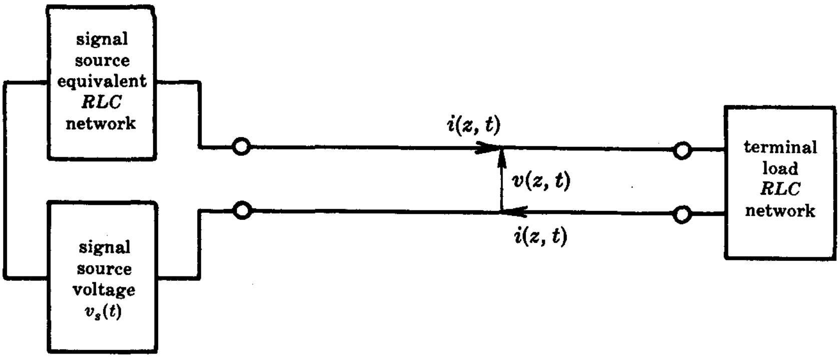
Since the theory of distributed circuits is no more than the extension of lumped element circuit theory by the addition of a space variable, it might seem reasonable to take the same generalized approach to transmission line theory by writing all equations in terms of the complex frequency variable s. That no textbooks have so far been written in this form is evidence that elementary methods have not yet proved profitable for studying the general transient response of transmission line circuits. It is therefore realistic to develop transmission line theory in terms of the steady state angular frequency variable ω.

Chapter 3

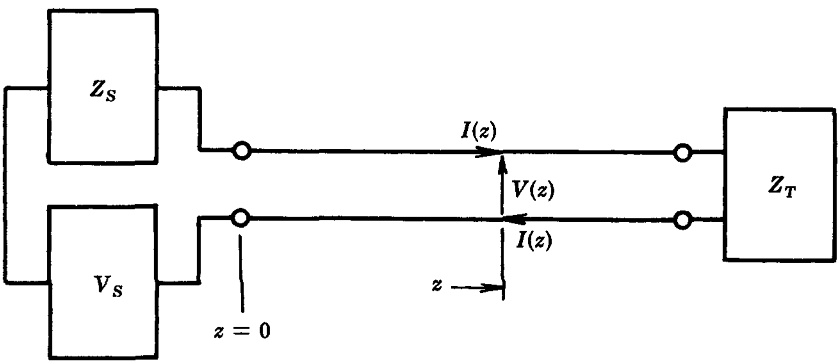
The Differential Equations of the Uniform Transmission Line

3.1. Time domain and frequency domain.

Two illustrations of complete transmission line circuits are shown in Fig. 3-1, incorporating the notational definitions of Chapter 2. In Fig. 3-1(*a*) the source signal voltage is any general function of time, and the electric circuit features of the source and terminal load are described by networks of resistance, inductance and capacitance. In Fig. 3-1(*b*) the source signal voltage is a constant amplitude harmonic function of time at some angular frequency ω rad/sec. The circuit representations of the source and terminal load are replaced by the respective impedance values Z_S and Z_T at that frequency.



(*a*)



(*b*)

Fig. 3-1. Complete transmission line circuits, including signal sources and terminal loads: (*a*) in the time domain; (*b*) in the frequency domain.

Fig. 3-1(*b*) is obviously the form in which the circuit would be drawn when it is desired to determine a steady state a-c solution at a specified frequency. In a typical situation values might be given for the angular frequency ω, the rms phasor source voltage V_S, the impedances Z_S and Z_T at frequency ω, and the distributed circuit coefficients R, L, G and C for the transmission line at that frequency. The desired information would be the rms phasor values of the voltages and currents at the input and terminal load ends of the line, and the variation of the phasor voltage and current along the length of the line. The analysis developed in the next few chapters permits simple and precise determination of all of these quantities.

A complication arises from the fact that in practical cases the actual signals handled by a transmission line circuit are never confined to a single frequency, but cover a finite bandwidth. Common examples are the audio frequency range of voice signals, the video frequency range of television picture signals, and the harmonic frequencies in the Fourier spectrum of switching transients or lightning induced surges on 60-hertz power lines.

Because the behavior of real transmission lines always varies finitely with frequency, a signal of non-zero bandwidth is always perceptibly distorted in transmission. The nature of the distortion for such a multi-frequency signal is that the phase and amplitude relations among the signal's component frequencies are not the same at the output of the transmission line as they were for the original signal at the input.

Pursuing the discussion of Section 2.12, page 17, the distortion suffered by any general signal pattern on a transmission line might in principle be analyzed by solving the circuit equations of the line in the time domain, i.e. as differential equations involving time and space derivatives of instantaneous currents and voltages, or by solving the equations in the complex frequency domain. The main obstacle to the practicability of these approaches is the physical fact that the transmission line distributed circuit coefficients R, L, G and C are not independent of frequency and are not simple functions of frequency. This raises formidable mathematical difficulties.

The alternative is to solve the transmission line circuit for phase and amplitude changes between input and output at several different frequencies in the bandwidth of the signal. The distortion of the time pattern of a complex signal can then be obtained by a synthesis procedure, if desired. Except for signals involving pulses or other sharp discontinuities, however, transmission line distortion of finite-bandwidth signals is generally interpreted directly from data on phase and amplitude variations with frequency over the signal bandwidth, rather than from comparison of time patterns.

Differential equations are derived in this chapter for both the time domain circuit of Fig. 3-1(*a*) and the frequency domain circuit of Fig. 3-1(*b*), assuming the distributed circuit coefficients constant in each case.

3.2. Equations in the time domain.

The differential equations for a uniform transmission line are found by focusing attention on an infinitesimal section of line of length Δz, located at coordinate z on the line, remote from the line's terminations. This line section has total series resistance $R\,\Delta z$, series inductance $L\,\Delta z$, shunt capacitance $C\,\Delta z$ and shunt conductance $G\,\Delta z$, from the postulates of Chapter 2. Its equivalent circuit as a two-port network can be drawn in a number of different ways, incorporating these circuit elements. One such circuit is the *L*-section of Fig. 3-2 below.

Inspection of this circuit shows that the output voltage of the section differs from the input voltage because of the series voltages across the resistance and inductance elements, while the output current differs from the input current because of the shunt currents

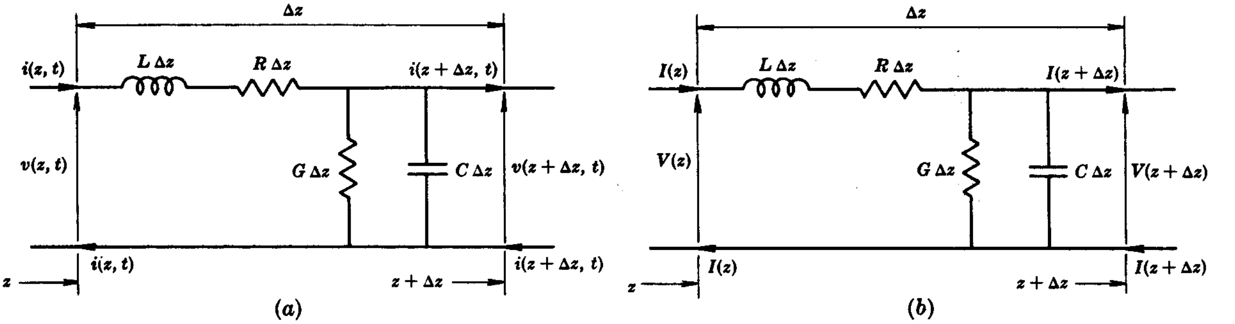


Fig. 3-2. Equivalent circuits of an infinitesimal portion of a uniform transmission line: (*a*) in the time domain; (*b*) in the frequency domain.

through the conductance and capacitance elements. Using the instantaneous quantities of Fig. 3-2(*a*), these relations are expressed by the following equations, derived by application of Kirchhoff's laws to the circuit:

$$v(z+\Delta z, t) - v(z,t) \equiv \Delta v(z,t) = -R\,\Delta z\, i(z,t) - L\,\Delta z\,\frac{\partial i(z,t)}{\partial t} \tag{3.1}$$

$$i(z+\Delta z, t) - i(z,t) \equiv \Delta i(z,t) = -G\,\Delta z\, v(z,t) - C\,\Delta z\,\frac{\partial v(z,t)}{\partial t} \tag{3.2}$$

Dividing by Δz and then letting Δz approach zero leads to the partial differential equations

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t} \tag{3.3}$$

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t} \tag{3.4}$$

The objection can be raised, legitimately, that in equation (*3.2*) it is not $v(z,t)$ and its derivative that produce the current $\Delta i(z,t)$, but the voltage $v(z+\Delta z, t)$ and its derivative. However, if the voltage $v(z+\Delta z, t)$ is expanded in a Taylor's series about z, the terms of (*3.2*) are obtained, together with additional terms involving $(\Delta z)^2$, $(\Delta z)^3$, etc. These are higher order small quantities which can be dropped on proceeding to the limit. There is no way of arranging the equivalent circuit elements of the length Δz of line that avoids this step.

The physical facts involved in equations (*3.3*) and (*3.4*) are very simply stated and could have been written directly from the postulates of Chapter 2 without the help of Fig. 3-2(*a*) or equations (*3.1*) and (*3.2*). Equation (*3.3*), for example, states in the language of differential calculus that the rate of change of voltage with distance along the line at any point of the line is the sum of two longitudinally distributed voltages. One of these, caused by the line current flowing through the distributed series resistance of the line, is proportional to the instantaneous value of the line current at the point. The other, caused by the time varying line current flowing through the distributed inductance of the line, is proportional to the instantaneous value of the time rate of change of the line current at the point. The respective constants of proportionality for the two are the line's series resistance per unit length and series inductance per unit length.

From the postulates of the analysis these are the only possible circuit phenomena that could cause the line voltage to vary with position on the line. The signs of the various terms are dictated by the original choice of sign conventions for voltage and current on the line.

Equation (*3.4*) can be interpreted in a similar fashion.

Any intuitive impression that the signs in (*3.3*) and (*3.4*) must result in the line voltage and current always decreasing with increasing distance along the line, is without foundation. The instantaneous values of the line voltage and its time derivative, at a specific point on the line at a specific instant, may have either sign and the signs of the two are independent of each other. The same is true for the current and its time derivative. The actual instantaneous values of all of these four quantities at any point will be functions of the total line circuit of Fig. 3-1(*a*).

Equations (*3.3*) and (*3.4*) are two simultaneous first order linear partial differential equations, with constant coefficients, in the dependent variables v and i and the independent variables z and t. They are of a sufficiently complicated nature that no conclusions can be drawn from them by simple inspection.

3.3. Solving the equations in the time domain.

A complete solution of equations (*3.3*) and (*3.4*) would find expressions for v and i as functions of z and t, subject to boundary conditions determined by the nature of the devices connected at the two ends of the line, i.e. the source signal generator (with its equivalent circuit) at $z = 0$, and the terminal load circuit at $z = l$.

The usual first step to take in attempting to solve such simultaneous equations is to eliminate one of the variables. This is achieved by taking the partial derivative with respect to z of all the terms in equation (*3.3*), thus:

$$\frac{\partial^2 v(z,t)}{\partial z^2} = -R\frac{\partial i(z,t)}{\partial z} - L\frac{\partial}{\partial z}\left(\frac{\partial i(z,t)}{\partial t}\right) \tag{3.5}$$

The order of differentiation in the right-hand term can be changed, giving

$$\frac{\partial^2 v(z,t)}{\partial z^2} = -R\frac{\partial i(z,t)}{\partial z} - L\frac{\partial}{\partial t}\left(\frac{\partial i(z,t)}{\partial z}\right) \tag{3.6}$$

When the right-hand side of (*3.4*) is substituted for $\partial i(z,t)/\partial z$ in (*3.6*), the result is

$$\frac{\partial^2 v(z,t)}{\partial z^2} = LC\frac{\partial^2 v(z,t)}{\partial t^2} + (LG+RC)\frac{\partial v(z,t)}{\partial t} + RG\,v(z,t) \tag{3.7}$$

If the alternative procedure of taking the partial derivative with respect to z of all terms in (*3.4*) is followed, substituting in the result an expression for $\partial v(z,t)/\partial z$ obtained from (*3.3*), a differential equation in $i(z,t)$ is obtained which is similar to (*3.7*), i.e.,

$$\frac{\partial^2 i(z,t)}{\partial z^2} = LC\frac{\partial^2 i(z,t)}{\partial t^2} + (LG+RC)\frac{\partial i(z,t)}{\partial t} + RG\,i(z,t) \tag{3.8}$$

The fact that $i(z,t)$ and $v(z,t)$ obey the same differential equation does not mean that they are identical functions of z and t in a practical problem, since in general the boundary conditions are not the same for the two variables.

The derivation of equations (*3.7*) and (*3.8*) has involved no special assumptions or approximations beyond the postulates of Chapter 2. They are therefore complete descriptions of the possible interrelations of the voltage and current and their derivatives on a transmission line, under conditions for which the postulates are valid. It has already been noted that one of the postulates of the analysis is violated for practical lines by the fact that the distributed circuit coefficients R, L, G and C are always to some extent variable with frequency, and hence are functions of the time derivatives of voltage and current. This is particularly true of R and G. For any practical problem, the accuracy of solutions of (*3.7*) and (*3.8*) with R, L, G and C assumed constant will depend in a complicated manner on the range of values of the instantaneous current and voltage and their derivatives, and on the range of variability of the distributed circuit coefficients for the signals being transmitted. No simple criteria can be established.

When R, L, G and C are assumed to be constants for all values of the current and voltage and their derivatives, equations (*3.7*) and (*3.8*) are second order linear partial differential equations in the time coordinate and one space coordinate. They are similar to several "standard" partial differential equations of mathematical physics, for which numerous solutions can be found in reference books. Unfortunately these two equations are just a little more complicated than any of the standard equations, and it is not possible to write any simple complete general solution for them, in the form of i or v as a function of z and t.

If the generality of the transmission line specifications is reduced by postulating that one or two of R, L, G and C are sufficiently small to be set equal to zero, one or more of the terms on the right of equations (*3.7*) and (*3.8*) will disappear. Most of the simpler equations that result fall into the "standard" categories mentioned, and useful solutions to them are easily found for meaningful sets of boundary conditions. Some of these "reduced" equations represent specific transmission line applications closely enough for practical purposes. The case $L = G = 0$, for example, is an adequate description of a single conductor submarine telegraph cable as used at low frequencies for so-called d-c telegraph transmission. The distributed circuit coefficients R and C for such a cable are very precisely constant for all aspects of the signals involved. The case $R = G = 0$ describes a "lossless" line. While no ordinary line is completely lossless, the resulting simple equation gives useful information about the transmission properties of short lengths of large conductor (hence low loss) high frequency transmission lines, such as might be used to carry television signal power from a transmitter to an antenna.

3.4. Equations in the frequency domain.

To develop differential equations describing transmission line behavior in the frequency domain, the simplest procedure is to return to the infinitesimal L-section of Fig. 3-2(*b*). Two phasor equations can be written:

$$V(z+\Delta z) - V(z) = \Delta V(z) = -R\Delta z\, I(z) - j\omega L\,\Delta z\, I(z) \tag{3.9}$$

$$I(z+\Delta z) - I(z) = \Delta I(z) = -G\,\Delta z\, V(z) - j\omega C\,\Delta z\, V(z) \tag{3.10}$$

Each term of these equations is a complex number. Each is implicitly a harmonic function of time at angular frequency ω radians per second. The zero phase angle reference for the complex numbers when expressed in polar form is arbitrary. Convenient choices for this reference may be the source voltage phasor, or the voltage phasor at the input end or the terminal load end of the line.

As an equation in instantaneous voltages (the form to which Kirchhoff's law applies), equation (*3.9*) would take the following form, using the customary conventions of phasor circuit analysis:

$$\text{Re}\,\{V(z)\,e^{j\omega t}\} = -R\,\Delta z\,\text{Re}\,\{I(z)\,e^{j\omega t}\} - L\,\Delta z\,\text{Re}\,\{j\omega\, I(z)\,e^{j\omega t}\} \tag{3.9a}$$

where Re { } means the real part of the complex number in the braces. $R\,\Delta z$ and $L\,\Delta z$ are real numbers by definition.

A similar equation can be written for instantaneous currents from equation (*3.10*).

Dividing equations (*3.9*) and (*3.10*) by Δz and letting Δz approach zero, leads to the differential equations

$$dV(z)/dz = -(R + j\omega L)\, I(z) \tag{3.11}$$

$$dI(z)/dz = -(G + j\omega C)\, V(z) \tag{3.12}$$

These are not written as partial differential equations, since V and I are here explicitly functions of only the single variable z. It is not even necessary to be perpetually reminded that V and I are functions of z, and the equations can be further simplified to

$$dV/dz = -(R + j\omega L)I \tag{3.13}$$

$$dI/dz = -(G + j\omega C)V \tag{3.14}$$

As was noted for equations (*3.3*) and (*3.4*), the physical facts embodied in these equations are easily comprehended, and the equations could be written down directly from the postulates of Chapter 2. Equation (*3.13*) states that the rate of change of phasor voltage with distance along the line, at a specific point of the line, is equal to the series impedance of the line per unit length multiplied by the phasor current at the point. Equation (*3.14*) states that the rate of change of phasor current with distance along the line, at a specific point, is equal to the shunt admittance per unit length of the line multiplied by the phasor voltage at the point.

Since these are complex number equations, each carries information about both magnitude and phase angle variations. No simple conclusions can be drawn from them by inspection. As was noted for the corresponding equations (*3.3*) and (*3.4*) in the time domain, it is incorrect to deduce from the negative signs in equations (*3.13*) and (*3.14*) that the voltage and current diminish steadily with distance along the line.

Solving these simultaneous linear first order ordinary differential equations with constant coefficients for separate equations in V and I, produces two second order equations:

$$d^2V/dz^2 - (R + j\omega L)(G + j\omega C)V = 0 \tag{3.15}$$

$$d^2I/dz^2 - (R + j\omega L)(G + j\omega C)I = 0 \tag{3.16}$$

These equations are of much more elementary form than equations (*3.7*) and (*3.8*) and their solutions in terms of V and I as phasor functions of z can be written directly in simple expressions, as is done in the next chapter. This greater simplicity is the basis for the comments in Section 3.3 that it is realistic to study transmission line theory with primary attention to solutions which are functions of the real angular frequency variable ω, while lumped constant networks can be usefully studied through more general solutions of the equations in the time domain or in the complex frequency domain.

Most of the remainder of this book consists of investigation of the solutions of equations (*3.15*), (*3.16*), (*3.7*) and (*3.8*) for various line constructions (expressed by values of the distributed circuit coefficients R, L, G and C which are either constants or expressible as functions of the frequency ω), and various terminal connections of sources and loads (which are the boundary values for the solutions).

All of the possible voltage and current relations that can occur on transmission lines defined by the postulates of Chapter 2 must be solutions of these equations.

Solved Problems

3.1. The distributed circuit coefficients of a 19 gauge cable pair transmission line at $\omega = 10^4$ rad/sec are: $R = 0.053$ ohms/m, $L = 0.62$ microhenries/m, $G = 950$ micromicromhos/m, $C = 39.5$ micromicrofarads/m. At a coordinate z on the line the instantaneous current is given by $i(t) = 75 \cos 10^4 t$ milliamperes. (*a*) Find an expression for the voltage gradient along the line at the point z, in volts/m. (*b*) What is the maximum possible value of the voltage gradient?

(*a*) The voltage gradient is given in the time domain by equation (*3.3*) as

$$\begin{aligned} \partial v/\partial z &= -Ri - L\,\partial i/\partial t = -0.053(0.075 \cos 10^4 t) + (0.62 \times 10^{-6})(10^4 \times 0.075 \sin 10^4 t) \\ &= -3.98 \times 10^{-3} \cos 10^4 t + 0.46 \times 10^{-3} \sin 10^4 t = 4.01 \cos(10^4 t - 3.03) \text{ millivolts/m} \end{aligned}$$

(*b*) The maximum voltage gradient is 4.01 millivolts/m, when $\cos(10^4 t - 3.03) = 1$.

3.2. For the transmission line of Problem 3.1, the phasor voltage at a point on the line has an rms magnitude of 16.5 volts, the signal frequency being 1100 hertz. Assume the values given for R, L, G and C to be valid at this slightly different frequency. (*a*) Find an expression for the phasor current gradient along the line at the same point. (*b*) What is the rms phasor magnitude of the transverse current between the two conductors along 10 cm of line length at the point, and what is the phase angle of this current relative to the line voltage at the point? (*c*) What is the maximum instantaneous current gradient along the line at the point?

(*a*) The current gradient along the line is given in the frequency domain by equation (*3.14*) as

$$\begin{aligned} dI/dz &= -(G + j\omega C)V \\ &= -[950 \times 10^{-12} + j(2\pi \times 1100)(39.5 \times 10^{-12})](16.5 + j0) \\ &= -(0.016 + j4.51) \times 10^{-6} \text{ amperes/m} \end{aligned}$$

(*b*) Equation (*3.10*) indicates that the change of the longitudinal current along a short section of line is the negative of the transverse current for the same section. Hence the transverse current for 10 cm length of line is $0.1(0.016 + j4.51) \times 10^{-6}$ amperes, expressed as a phasor quantity, the reference phasor of zero phase angle being the line voltage at the point. The transverse current leads the line voltage by $\tan^{-1} 4.51/0.016 = 89.7°$ and is almost a purely capacitive current.

(*c*) Since the magnitudes of all the phasor quantities used in (*a*) and (*b*) are rms values, the maximum *instantaneous current gradient along the line is* $\sqrt{2}\,|0.016 + j4.51| \times 10^{-6} = 6.38$ *micro*-amperes/m.

Supplementary Problems

3.3. (*a*) Redraw the circuit of Fig. 3-2 as a symmetrical T-network, with the series resistance and series inductance of the section divided into two equal portions.

(*b*) Redraw the circuit of Fig. 3-2 as a symmetrical π-network, with the shunt conductance and shunt capacitance of the section divided into two equal portions.

(*c*) Redraw the circuit of Fig. 3-2 as an inverted L-network, with the shunt conductance and shunt capacitance at the left.

3.4. For each of the circuits of Problem 3.3, establish appropriate time-domain notation for v and i at each node of the network. For each circuit write the *exact* equations corresponding to equations (*3.1*) and (*3.2*). By making a Taylor's series expansion of either v or i about the coordinate z, show that the exact equations reduce in every case to (*3.1*) and (*3.2*) on proceeding to the limit by letting Δz approach zero.

3.5. Rewrite equation (*3.7*) for a lossless transmission line having $R = G = 0$.

(*a*) Show that for the resulting equation

$$v(z,t) = f_1(t - \sqrt{LC}\,z) + f_2(t + \sqrt{LC}\,z)$$

is a solution, where f_1 and f_2 are any continuous functions having first and second derivatives.

(*b*) What dimensions are indicated for the quantity $\sqrt{LC}$? *Ans.* (*b*) Reciprocal velocity.

(The discussion presented in deriving equation (*4.6*), page 27, shows that a term of the form $f_1(t - \sqrt{LC}\,z)$ describes a traveling pattern, moving in the direction of increasing z with a velocity given by the ratio (coefficient of t)/(coefficient of z) and subject to no distortion as it travels. The time shape of the pattern can be found by plotting f_1 as a function of time for some constant value of z, such as $z = 0$. A term $f_2(t + \sqrt{LC}\,z)$ similarly describes a pattern moving in the direction of decreasing z.)

3.6. Show that for the very special case of a transmission line whose distributed circuit coefficients are related by $R/L = G/C$ a solution to equation (*3.8*) is

$$i(z,t) \quad = \quad e^{-R\sqrt{C/L}\,z} f_1(t - \sqrt{LC}\,z) \; + \; e^{R\sqrt{C/L}\,z} f_2(t + \sqrt{LC}\,z)$$

A line with these properties is known as a "Heaviside distortionless line" and is mentioned further in Chapter 5, page 49. The time pattern of any signal traveling in one direction on such a line is the same at every cross section, except for a scale factor. The amplitude diminishes exponentially as the pattern travels.

3.7. For the transmission line of Problem 3.1, the voltage at a coordinate z_1 is found to be given by $v(t) = 30 \sin(10^2 t + \pi/6)$ volts, as a function of time. Derive an expression for the current gradient along the line at the point z_1 in amperes/m, assuming the values given for R, L, G and C to be valid at this much lower frequency. *Ans.* $0.121 \times 10^{-6} \cos(10^2 t - 2.86)$ amperes/m.

3.8. (*a*) For the lossless transmission line of Problem 3.5, show that if the voltage on the line is given by $v(z,t) = v_1 f_1(t - \sqrt{LC}\,z)$ where v_1 is a constant and f_1 is any continuous function with first and second derivatives, then the current is given by $i(z,t) = v(z,t)/\sqrt{L/C}$.

(*b*) For the Heaviside distortionless line of Problem 3.6, show that if the voltage on the line is given by $v(z,t) = v_1 e^{-R\sqrt{C/L}\,z} f_1(t - \sqrt{LC}\,z)$, where f_1 meets the conditions stated in (*a*), then the current on the line is $i(z,t) = v(z,t)/\sqrt{L/C}$.

(The current-voltage relation that holds in these special cases (*a*) and (*b*) is shown in Chapter 5 to be an adequate approximation for all "low-loss" transmission lines. It does not, however, apply to typical lines used at low signal frequencies such as voice frequencies.)

3.9. For all ordinary transmission lines used at frequencies up to hundreds of megahertz, the ratio R/L is much larger than the ratio G/C. However, for lines with solid dielectric insulation used at microwave frequencies, the relation for the Heaviside distortionless line of Problem 3.6 might be attained, with distributed circuit coefficients having the values: $R = 0.15$ ohms/m, $L = 0.375$ microhenries/m, $G = 30$ micromhos/m, $C = 75$ micromicrofarads/m. If the time pattern of input voltage to this line is $v(t) = v_1 f_1(t)$, determine (*a*) the time pattern of the voltage at a point 800 m along the line from the input terminals and (*b*) the time pattern of the current at the input terminals of the line. It is assumed that the signals on the line are traveling away from the input terminals.

Ans. (*a*) The time pattern of voltage at the distant point is the same as at the input terminals, but is delayed by 4.2 microseconds and reduced in magnitude by a factor 0.18. (*b*) The time pattern of current at the input terminals is the same as the time pattern of voltage, with the ordinate scale changed from volts to amperes in the proportion 0.014 amperes/volt.

3.10. Show that if $L = G = 0$, equations (*3.7*) and (*3.8*) become identical with the one-dimensional "heat flow" or "diffusion" equation of classical mathematical physics, when appropriate changes are made in the symbols.

(Thus if the voltage at the input end of such a transmission line is increased from zero to a constant value at $t = 0$ (the beginning of a dot or dash of telegraph code), the subsequent variation of voltage with time at the distant terminal load end of the line will have the same time pattern as that of the temperature rise at one end of a thermally insulated rod of homogeneous material after the temperature at the other end has been raised at $t = 0$ by some constant amount above an initial thermal equilibrium value. This is the problem solved by William Thomson (Lord Kelvin) in 1853-55 when he was investigating the practicability of a transatlantic submarine telegraph cable.)

Chapter 4

Traveling Harmonic Waves

4.1. Solutions of the differential equations.

Equations (*3.15*) and (*3.16*), page 23, are the differential equations governing the voltage and current distributions along a transmission line, when the voltage and current have time-harmonic variation at angular frequency ω, and the values used for the distributed circuit coefficients R, L, G and C are the values appropriate to that frequency.

The solutions of these relatively simple equations are

$$V(z) = V_1 e^{-\gamma z} + V_2 e^{+\gamma z} \tag{4.1}$$

$$I(z) = I_1 e^{-\gamma z} + I_2 e^{+\gamma z} \tag{4.2}$$

Here $V(z)$ and $I(z)$ are respectively the phasor voltage and phasor current at any coordinate z on the line. V_1, V_2 and I_1, I_2 are also phasors (i.e. complex number quantities) and are the sets of two arbitrary coefficients that occur in the solution of second order ordinary differential equations. γ is defined by

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$

from which

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{4.3}$$

From equation (*4.3*) the value of γ is expressed by a complex number.

Equations (*4.1*) and (*4.2*) have been derived from (*3.15*) and (*3.16*) by straightforward mathematical processes. Their physical implications can be appreciated most directly by separating the magnitude and phase aspects of the terms $e^{-\gamma z}$ and $e^{+\gamma z}$, and by temporarily reintroducing the harmonic time variation term $e^{j\omega t}$. Defining the real and imaginary parts of γ by the relation

$$\gamma = \alpha + j\beta \tag{4.4}$$

(*4.1*) can be written as an equation in instantaneous voltages, following the rules used in presenting (*3.9a*):

$$v(z,t) = |\hat{V}_1|\, e^{-\alpha z}\,\mathrm{Re}\,\{e^{j(\omega t - \beta z + \xi_1)}\} + |\hat{V}_2|\, e^{+\alpha z}\,\mathrm{Re}\,\{e^{j(\omega t + \beta z + \xi_2)}\} \tag{4.1a}$$

where $|\hat{V}_1|$ and $|\hat{V}_2|$ are the *peak* amplitudes of the arbitrary phasor coefficients V_1 and V_2 (the phasor symbols V_1 and V_2 as used in this book stand for *rms* amplitudes), $e^{-\alpha z}$ and $e^{+\alpha z}$ are real numbers by definition, and ξ_1 and ξ_2 are the respective phase angles of the arbitrary phasors V_1 and V_2. The terms $\mathrm{Re}\{e^{j(\omega t - \beta z + \xi_1)}\}$ and $\mathrm{Re}\{e^{j(\omega t + \beta z + \xi_2)}\}$ are numbers varying harmonically in time, and in distance along the line, with peak value unity and with values at $t = z = 0$ determined by ξ_1 and ξ_2.

Equation (*4.2*) can be rewritten similarly as an equation in instantaneous currents.

4.2. The meaning of the solutions.

The physical meaning of equations (*4.1*) and (*4.2*) is found by focusing attention on the harmonic variation terms. Thus the first term on the right of (*4.1a*) describes an instantaneous voltage which is a function of both z and t. Its greatest instantaneous value is

$|\hat{V}_1|$. At the coordinate $z = 0$ the maximum instantaneous value occurs at the times t which satisfy $(\omega t + \xi_1) = n\pi$, where $n = 0, 1, 2, \ldots$. At every coordinate z on the line the voltage varies harmonically with time, with constant amplitude $|\hat{V}_1| e^{-\alpha z}$. At any instant the pattern of voltage as a function of position along the line is a harmonic pattern in the coordinate z, with amplitude diminishing exponentially as z increases. The interval between zero-value points of the pattern (determined by the wavelength, see Section 4.8) has the same value everywhere on the line.

At a selected position z_1 on the line, at a chosen instant t_1, the voltage represented by the first term on the right of equation (*4.1a*) is determined by the value of the phase factor $(\omega t_1 - \beta z_1 + \xi_1)$ and the value of the exponential amplitude factor $e^{-\alpha z_1}$. At a slightly later instant of time $t_1 + \Delta t$, the value of the term $(\omega t - \beta z_1 + \xi_1)$ will have changed *at the point* z_1, but the original value will be found at a slightly different location on the line $z_1 + \Delta z$ such that

$$\omega(t_1 + \Delta t) - \beta(z_1 + \Delta z) + \xi_1 \;=\; \omega t_1 - \beta z_1 + \xi_1$$

Hence

$$\omega\,\Delta t - \beta\,\Delta z \;=\; 0 \tag{4.5}$$

The interpretation of this result is that a point of specific phase value on the harmonic voltage pattern moves to greater values of z as time increases, according to the relation $\Delta z/\Delta t = \omega/\beta$. Proceeding to the limit,

$$\lim_{\Delta t = 0} \Delta z/\Delta t \;=\; dz/dt \;=\; v_p \;=\; \omega/\beta \tag{4.6}$$

The derivative dz/dt is a velocity. It is represented by the symbol v_p for *phase velocity* because it is the velocity with which a point of *constant phase value* travels along the transmission line. (An additional velocity concept, that of *group velocity,* is needed when dealing with situations where the phase velocity on a transmission system is not the same for all the Fourier component frequencies of the signal. For most transmission lines used at high frequencies this complication does not arise, and the phase velocity and group velocity have the same value.)

Applying the same reasoning to the phase factor $(\omega t + \beta z + \xi_2)$ in (*4.1a*), it is found that the change of sign of the βz term relative to the ωt term alters the meaning to a traveling voltage pattern moving in the direction of *decreasing* z, with the same magnitude of phase velocity as for the first wave. Thus the two terms on the right of (*4.1a*) describe harmonic voltage patterns traveling in the only two possible directions on the transmission line, and the sign of the index of the real exponential term in each case indicates that the amplitude of the wave pattern diminishes *as it travels.*

Returning to equation (*4.1*) it can now be stated that a term of the form $V_1 e^{-\gamma z}$ or $V_1 e^{-(\alpha + j\beta)z}$ represents a harmonic voltage pattern or wave, of phasor value V_1 at $z = 0$, traveling in the direction of increasing z with phase velocity $v_p = \omega/\beta$, diminishing exponentially in amplitude as it travels, according to the term $e^{-\alpha z}$. Similarly a term of the form $V_2 e^{+\gamma z}$ or $V_2 e^{+(\alpha + j\beta)z}$ represents a harmonic voltage wave, of phasor value V_2 at $z = 0$, traveling in the direction of decreasing z with phase velocity of magnitude $|v_p| = \omega/\beta$, diminishing exponentially in amplitude as it travels, according to the term $e^{-\alpha(-z)}$. It is important to note that V_1 is the phasor value of the first wave as it *leaves* the point $z = 0$, while V_2 is the phasor value of the second wave as it *arrives* at $z = 0$. Clearly $V_1 + V_2 = V_{\text{inp}}$, the phasor voltage at the input terminals of the line.

4.3. Current waves.

All references to voltage patterns or voltage waves in Sections 4.1 and 4.2 apply identically to current patterns and current waves, because of the fact that equation (*4.2*) is of exactly the same form as (*4.1*).

It was noted earlier in a similar context that this does not mean that the current pattern on a transmission line is of necessity a replica of the voltage pattern. It does mean that harmonic current waves can travel in either direction on the line, that they will always have phase velocity of magnitude $|v_p| = \omega/\beta$, and that they will diminish exponentially in amplitude by a factor $e^{-\alpha|z|}$ as they travel over a length $|z|$ of the line. The current wave traveling in the direction of increasing z will have phasor value I_1 at $z = 0$, and the current wave traveling in the direction of decreasing z will have phasor value I_2 at $z = 0$. It will be seen in subsequent chapters, however, that the phase relation of I_2 to I_1 is never the same as that of V_2 to V_1.

4.4. Reflected waves.

It is reasonable to ask at this point how, for the circuit of Fig. 3-1(*b*), page 18, on which the above analysis is based, there can be voltage and current waves traveling in both directions on the transmission line when there is only a single signal source. The answer lies in the phenomenon of reflection, which is very familiar in the case of light waves, sound waves, and water waves. Whenever traveling waves of any of these kinds meet an obstacle, i.e. encounter a discontinuous change from the medium in which they have been traveling, they are partially or totally reflected.

In the transmission line case, if a switch is closed at time $t = 0$ to connect the source V_S to the circuit of Fig. 3-1(*b*), voltage and current waves will start traveling along the line in the direction of increasing z. If when they reach the end of the line at $z = l$, the terminal load impedance Z_T connected there requires different magnitude and phase relations between voltage and current than the relations that exist for the arriving waves, then reflected voltage and current waves will come into existence at the termination. The phasor values of the reflected waves will be such that when they are combined with the phasor values of the arriving waves, the boundary conditions at the termination imposed by the connected impedance Z_T will be satisfied.

The reflected voltage and current waves will travel back along the line to the point $z = 0$, and in general will be partially re-reflected there, depending on the boundary conditions established by the source impedance Z_S. The detailed analysis of the resulting infinite series of multiple reflections is given in Chapter 8.

4.5. The line with no reflected waves.

For the remainder of this chapter attention is directed to equations (*4.1*) and (*4.2*) with $V_2 = I_2 = 0$, i.e. transmission lines on which waves are traveling only in the direction away from the source. This case is often referred to as the "infinite line", since if a hypothetical line were infinitely long, no reflected wave would return in any finite time after connecting the signal source, or alternatively, for any finite value of the factor α the magnitude of voltage and current waves reflected back over an infinite length of line would necessarily always be zero. However, the concept of an infinitely long line is an unattractive one, and it is sufficient to say that the case to be studied first is that of a line on which there are no reflected waves. It will be seen later that this situation can be achieved for a line of finite length by suitable choice of the terminal load impedance.

In the absence of reflected waves, equations (*4.1*) and (*4.2*) become

$$V = V_1 e^{-\alpha z} e^{-j\beta z} \tag{4.7}$$

$$I = I_1 e^{-\alpha z} e^{-j\beta z} \tag{4.8}$$

4.6. The attenuation factor α.

When a physical quantity diminishes steadily as a function of some increasing independent variable, it is common usage to say that the quantity is "attenuated". Thus the loudness or intensity of a sound wave from a localized source is *attenuated* as the spherical wave pattern travels away from the source, or the concentration of a solution is *attenuated* when additional solvent is added. It is therefore appropriate to say that the voltage and current waves on a transmission line are attenuated with distance according to the term $e^{-\alpha z}$, and to refer to the quantity αz as the measure of the attenuation produced by length z of line. The quantity α is then called the "attenuation factor" of the line. In textbooks α is very often referred to as the "attenuation constant" of a line, but since it varies markedly with frequency for typical lines, the implication of the word "constant" is not satisfactory. α is also commonly called the "attenuation coefficient" of a line.

Since the index of an exponential number must be dimensionless, the quantity αz is a dimensionless real number, α and z being real by definition. It has proved to be convenient, nevertheless, to establish a named unit for a specific increment of this dimensionless index of an exponential term.

A current or voltage wave is said to experience an attenuation of N *nepers* when its *magnitude* changes by a factor e^{-N} as the wave travels between two points on a transmission line. Correspondingly, a length of transmission line is said to have an attenuation of N nepers when a current or voltage wave changes in magnitude by a factor e^{-N} on traveling over the length of line. The word neper comes from the Latin form of Napier, the name of the 16th century Scottish mathematician who invented logarithms.

Example 4.1.

By what factor is the magnitude of a phasor voltage wave reduced if the wave experiences an attenuation of 1 neper?

The voltage magnitude is reduced by the factor e^{-1}, or 0.368.

Example 4.2.

If a current wave suffers a reduction in magnitude by a factor of 10 on traveling over a particular length of transmission line, what is the attenuation of the transmission line section in nepers?

The attenuation of the line section is N nepers, where $e^{-N} = 0.100$.

Then $-N = \log_e 0.100 = -2.303$, and $N = 2.303$ nepers.

The relation between the neper and the decibel is explained in Section 4.11, where certain limitations in the applicability of the above definition of attenuation are also discussed.

From the definition of the neper, and the fact that αz is dimensionless, the unit for the attenuation factor α must be one neper per unit length, or one neper/meter in the mks system. This is consistent with the dimensions of α (i.e. reciprocal length) dictated by the defining equations (*4.3*) and (*4.4*).

Example 4.3.

Electric power at a certain frequency is transmitted from a source to a load by a 500 m length of uniform transmission line, with no reflection of voltage or current waves at the load. The input voltage to the line is 250 volts rms, and the voltage at the load is 220 volts rms. What is the total attenuation of the line, and what is the attenuation factor of the line, at the power frequency?

The total attenuation of the line is N nepers, where $e^{-N} = 220/250 = 0.880$. Then $-N = \log_e 0.880 = -0.128$, and $N = 0.128$ nepers.

The attenuation factor α is the attenuation per unit length. Hence

$$\alpha = N/500 = 0.128/500 = 2.56 \times 10^{-4} \text{ nepers/m}$$

Example 4.4.

A telephone line has an attenuation of 1.50×10^{-4} nepers/m at a frequency of 1000 Hz. The line is 50 km long. If the input current to the line is 650 mA rms at 1000 Hz, what is the line current at the load, assuming no reflected current wave is present on the line?

The attenuation of the 50 km length of line is $\alpha l = (50 \times 10^3)(1.50 \times 10^{-4}) = 7.50$ nepers. If I_T is the line current at the load in mA rms, then $e^{-7.50} = I_T/650$, and $I_T = 650e^{-7.50} = 0.360$ mA rms.

Example 4.5.

If the phasor voltage is $V(z_1)$ at a point distant z_1 from the input terminals of a transmission line, and is $V(z_2)$ at a point distant z_2 from the input terminals, and $z_2 > z_1$, find an expression for the total attenuation in nepers of the length of line between z_1 and z_2, assuming no reflected waves on the line. What is the attenuation factor α for the line?

The total attenuation of the line length $(z_2 - z_1)$ is N nepers, where $e^{-N} = |V(z_2)/V(z_1)|$. Thus $N = -\log_e |V(z_2)/V(z_1)|$.

For $z_2 > z_1$ the ratio of the voltage magnitudes is less than unity and its logarithm is negative, making N a positive number. Since $N = \alpha l$, where l is the length of line whose attenuation is N nepers, $\alpha = N/l = N/(z_2 - z_1)$ nepers per unit length of line. The unit of length in the expression for α will be the unit of length in which z_1 and z_2 are measured.

4.7. The phase factor β.

The term $e^{-j\beta z}$ that appears in equations (*4.7*) and (*4.8*) is a complex number of magnitude unity and phase angle $-\beta z$ radians. Hence this term does not affect the magnitude of the phasors V and I as a function of z, but only the phase angle. The term states that when there are harmonic voltage and current waves traveling on a transmission line in the direction of increasing z, the phase angles of the phasor voltage and current decrease uniformly and at the same rate, with increasing z along the line. The rate of decrease of phase angle with distance is evidently β radians per unit length of line, since the product of β and a line length gives a dimensionless angle in radians.

β is called the "phase factor" of the line. It has also been called the "phase propagation constant", or the "phase propagation coefficient". It is measured in units of radians per unit length, or radians/meter in the mks system. This is consistent with the dimension of reciprocal length required by the defining equations (*4.3*) and (*4.4*).

Equations (*4.7*) and (*4.8*) state that the phase of the voltage and current phasors at coordinate z on the line differ from the corresponding phases at the input terminals of the line ($z = 0$) by an angle $-\beta z$ radians. The time taken for a point of constant phase on a harmonic signal pattern to travel a distance z is by definition z/v_p, where v_p is the phase velocity for harmonic waves at the signal frequency ω rad/sec. The phase of the signal supplied by the source to the input terminals of the line increases at the rate ω rad/sec, and therefore increases by $\omega z/v_p$ radians in the time a reference point of *constant* phase on the signal pattern takes to travel the distance z. This is clearly the same amount by which the phase angle of the voltage or current at the point z lags the respective phase angle at $z = 0$.

*Hence $-\beta z = -\omega z/v_p$, or $v_p = \omega/\beta$, a result established by somewhat different reason-*ing in Section 4.2.

4.8. The wavelength of waves on the line.

In a harmonic space pattern (for example, a graph of a sine wave), the distance over which the phase changes by 2π rad is called a "wavelength" of the pattern. Giving this distance the generally accepted symbol λ leads to the relation $\beta\lambda = 2\pi$, hence

$$\beta = 2\pi/\lambda \quad \text{or} \quad \lambda = 2\pi/\beta \tag{4.9}$$

Calculation of the quantity β for a transmission line, using equations (*4.3*) and (*4.4*), is therefore the basis for determining analytically both the phase velocity and the wavelength for harmonic voltage and current signals on the line at any angular frequency ω rad/sec. For harmonic waves of any kind it is always true that phase velocity is equal to the product of frequency and wavelength. Combining (*4.6*) and (*4.9*) verifies this result for harmonic waves on transmission lines.

Example 4.6.

On a radio frequency transmission line the velocity of signals at a frequency of 125 MHz is 2.10×10^8 m/sec. What is the wavelength λ of the signals on the line, and what is the value of the phase factor β at that frequency?

From the relation $\lambda = v_p/f$, where f is the frequency in hertz, λ is in some length unit, and v_p is in the same units of length per second,

$$\lambda = (2.10 \times 10^8 \text{ m/sec})/(125 \times 10^6 \text{ Hz}) = 1.68 \text{ m} \quad \text{and hence} \quad \beta = 2\pi/\lambda = 3.74 \text{ rad/m}$$

Example 4.7.

The phase factor of a transmission line is calculated to be 0.123 rad/m at a frequency of 4.50 MHz. The line is 500 m long. Find the wavelength and phase velocity of the waves on the line, the phase difference between the phasor voltages at the two ends of the line, and the time required for a reference point on the signal pattern to travel the full length of the line.

Wavelength $\lambda = 2\pi/\beta = 51.1$ m.

Phase velocity $v_p = \omega/\beta = 2\pi \times 4.50 \times 10^6/0.123 = 2.30 \times 10^8$ m/sec.

Phase difference over 500 m of line $= \beta l = 61.5$ rad.

Travel time for a point of constant phase $= l/v_p = 2.17\ \mu$ sec.

4.9. Some implications of α and β.

Equations (*4.3*) and (*4.4*) show that α and β have the same physical dimensions of reciprocal length. The addition of the words nepers and radians respectively to the names of their units does not change this fact. As has been seen, however, α and β represent totally different aspects of the propagation of harmonic waves on transmission lines, because mathematically α occurs as a factor in the real index of an exponential term, while β occurs as a factor in the imaginary index of an exponential term.

When the electrical properties of a transmission line are given in the form of values of the distributed circuit coefficients R, L, G and C at signal angular frequency ω rad/sec, α and β must be calculated from (*4.3*) and (*4.4*), i.e. from

$$\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{4.10}$$

This is a somewhat complicated complex number relation involving the numerical values of five different quantities, and for arbitrary values of these quantities no simple statements can be made about the dependence of α and β on R, L, G and C, or about the variation of α and β with frequency.

Any *a priori* opinion that voltage and current waves *always* travel on transmission lines with the "velocity of light" (i.e. the velocity of plane electromagnetic waves in unbounded space or 3.00×10^8 m/sec) is obviously invalid, since the velocity must be calculated from equations (*4.6*) and (*4.10*), and the latter does not give a constant value for ω/β that is independent of the actual values of the individual distributed circuit coefficients.

On the other hand, the values of R, L, G and C for any specific transmission line are by no means entirely independent of one another, and it does turn out that for a transmission line whose interconductor medium is predominantly air or "space", if R and G are small enough or if the frequency is high enough the velocity of voltage or current waves on the line is extremely close to the "velocity of light".

These conditions do not hold for typical telephone transmission lines at voice frequencies, and the velocity of waves on such lines can be much less than fifty percent of the velocity of light and highly variable with frequency.

4.10. The characteristic impedance Z_0.

Equations (*4.7*) and (*4.8*) show that the ratio of the magnitudes of V and I, and the relative phases of V and I are the same at all points on a uniform transmission line on which there are no reflected waves, since the terms $e^{-\alpha z}$ and $e^{-j\beta z}$ are identical in the two equations.

To find an expression from which to calculate this ratio of magnitudes and this relative phase, it is necessary to return to equation (*3.13*):

$$dV/dz \;=\; -(R+j\omega L)I \tag{3.13}$$

Differentiating equation (*4.7*) with respect to z,

$$dV/dz \;=\; -(\alpha+j\beta)\,V_1\,e^{-\alpha z}\,e^{-j\beta z}$$

Substituting into (*3.13*) this value for dV/dz, and the value for I from (*4.8*),

$$-(\alpha+j\beta)\,V_1\,e^{-\alpha z}\,e^{-j\beta z} \;=\; -(R+j\omega L)\,I_1\,e^{-\alpha z}\,e^{-j\beta z}$$

or

$$\frac{V_1}{I_1} \;=\; \frac{R+j\omega L}{\alpha+j\beta}$$

Substituting for $\alpha + j\beta$ from (*4.10*) into this result, and making use also of (*4.7*) and (*4.8*) for V_1 and I_1,

$$\frac{V}{I} \;=\; \frac{R+j\omega L}{\sqrt{(R+j\omega L)(G+j\omega C)}} \;=\; \sqrt{\frac{R+j\omega L}{G+j\omega C}} \tag{4.11}$$

Thus the ratio of the phasor voltage to the phasor current on a uniform transmission line on which there are no reflected waves is the same at all points of the line, and is a complex number quantity determined entirely by the distributed circuit coefficients of the line and the signal frequency.

Since physically this quantity on the right of (*4.11*) has the dimensions of impedance, and since it is "characteristic" of the line itself and of nothing else except the frequency, it is appropriately named the "characteristic impedance" of the line. It is assigned the symbol $Z_0 = R_0 + jX_0$, and as a function of the line's distributed circuit coefficients is given by

$$Z_0 \;=\; R_0 + jX_0 \;=\; \sqrt{\frac{R+j\omega L}{G+j\omega C}} \tag{4.12}$$

In many contexts it is more convenient to use the reciprocal quantity, the "characteristic admittance" of a transmission line, defined by

$$Y_0 \;=\; G_0 + jB_0 \;=\; \frac{1}{Z_0} \;=\; \sqrt{\frac{G+j\omega C}{R+j\omega L}} \tag{4.13}$$

The unit for Z_0 is ohms and for Y_0 is mhos, when R, L, G and C are in the electrical units of the mks system.

Although the characteristic impedance of a transmission line is a very important and realistic physical quantity that directly governs the phasor relations between harmonic voltages and currents on a line, it is nevertheless a somewhat intangible entity. It does not "exist" on a line in any simple and obvious sense. It cannot be measured directly with an impedance-measuring bridge by making a single impedance measurement on an arbitrary finite length of line. It can be calculated from the distributed circuit coefficients of the line at any frequency using equation (*4.12*). For certain idealized conditions it can be determined directly from the dimensions and materials of the line, using formulas developed in Chapter 6. Experimentally it can be determined for a given sample of transmission line by making at least two impedance measurements of the input impedance of suitably chosen lengths of the sample with suitably chosen terminal load impedances. (See Section 7.5.)

Commercially available transmission lines are commonly labeled as having certain definite values of characteristic impedance such as 50 ohms, 300 ohms, etc., with the implication that the value is not only independent of frequency, but is purely resistive. This is obviously not in agreement with the nature of equation (*4.12*), which for fixed values of R, L, G and C might be expected to give a wide range of magnitudes and phase angles for Z_0 as ω is varied from zero to very high frequencies. It will be seen in Chapter 5 that for most practical transmission lines the actual values of R, L, G and C are such that at frequencies above a few tens or hundreds of kilohertz, the characteristic impedance does attain an approximately constant value, whose phase angle does not exceed a few degrees. For these same lines at lower frequencies, however, the magnitude of Z_0 may rise by a factor of 10 or more over the asymptotic high frequency value, and its phase angle may become as large as 45°.

Fig. 4-1 shows a transmission line circuit where the terminal load has been designated simply as a "nonreflecting termination". From what has been said earlier in this section,

$$V_1/I_1 = V/I = Z_0 \tag{4.14}$$

where V_1 and I_1 are the phasor voltage and current at the input terminals of the line and V and I are the phasor voltage and current at any coordinate z on the line, including the terminal load end of the line where $z = l$.

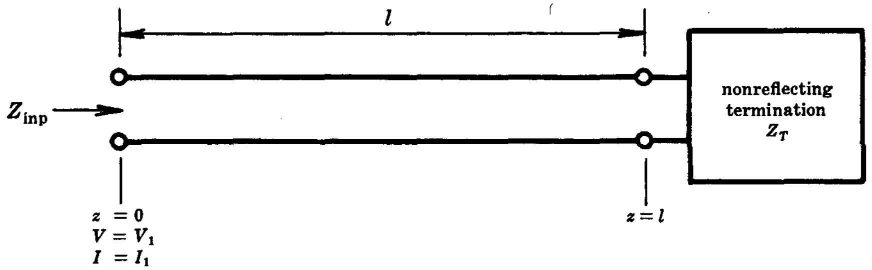


Fig. 4-1. A transmission line circuit of length l with nonreflecting termination, illustrating notation at the input terminals. The line has characteristic impedance Z_0.

The concept of the "input impedance" of the line can only mean the ratio of the phasor voltage to the phasor current at the input terminals. Hence for the circuit of Fig. 4-1,

$$Z_{\text{inp}} = V_1/I_1 = Z_0$$

and since the same phasor ratio must hold at $z = l$, it follows that $Z_T = Z_0$ also, since the phasor voltage $V(z=l)$ is identically the phasor voltage across Z_T at $z = l$, and the phasor current $I(z=l)$ is identically the phasor current through Z_T at $z = l$.

These results apply only for the conditions postulated in writing equations (*4.7*) and (*4.8*), i.e. that there are no reflected waves on the line. Two significant conclusions can be drawn:

(1) The only value of impedance that can be connected as a terminal load on a transmission line and constitute a nonreflecting termination is an impedance equal to the characteristic impedance of the line.

(2) The input impedance of any length of a uniform transmission line terminated in its characteristic impedance (i.e. nonreflectively) is equal to the characteristic impedance of the line.

From equation (*4.14*) the second conclusion can be generalized to the statement that the impedance at *any* point on a transmission line terminated nonreflectively is equal to the characteristic impedance of the line. The meaning of "the impedance at any coordinate z of the line" must be correctly understood, however. It is *not* the impedance that would be measured by an impedance bridge connected to that point of the line (with appropriate short-circuiting of the voltage source or open-circuiting of the current source connected to the line's input terminals), because the phasor voltage V applied by the generator in the impedance bridge to the line at the cross section z would create not only the phasor line current I on the load side of that cross section as shown in Fig. 4-2, but would also create a current I' in the portion of the line on the signal source side of the cross section. The two currents would be entirely independent of one another. Evidently the physical meaning of the impedance $Z(z)$ at a coordinate z of a transmission line, defined as the ratio of the phasor voltage V at the coordinate z to the phasor current I at coordinate z, and following the sign conventions in Fig. 3-1(*b*), must be the *input impedance of the total line circuit on the terminal load side of the cross section at coordinate z*. It could be measured directly by an impedance bridge connected to the line at that point only by cutting the line at that cross section and removing the portion on the signal source side.

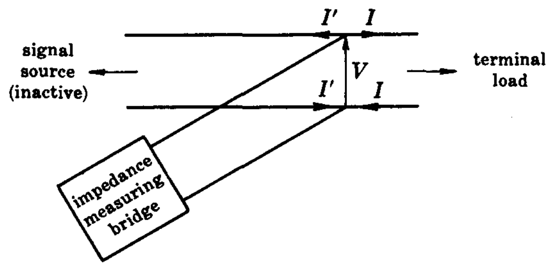


Fig. 4-2. An impedance measuring bridge connected at a point on a transmission line causes currents to flow in both directions from the point, and does not measure the quantity defined as the impedance of the line at that cross section.

Example 4.8.

A parallel wire transmission line used in a carrier telephone system has a characteristic impedance of $700 - j150$ ohms at a frequency of 8.00 kHz. The terminal load impedance connected to the line is equal to the characteristic impedance. If a signal voltage of 10.0 volts rms at 8.00 kHz is connected to the input terminals of the line, what is the phasor input current and what is the real power supplied by the signal source to the line?

From the conditions stated, the input impedance of the line is equal to its characteristic impedance. The current and power calculations are then simply those for an a-c circuit consisting of a voltage of 10.0 volts rms connected across an impedance of $700 - j150$ ohms. Taking the voltage as a reference phasor $10 + j0$ volts,

$$I_{\text{inp}} \;=\; (10.0 + j0)/(700 - j150) \;=\; 13.7 + j2.93 \text{ mA}$$

The real power input to the line is given most directly by

$$|I_{\text{inp}}|^2 \cdot R_{\text{inp}} \;=\; (14.0 \times 10^{-3})^2 (700) \;=\; 0.137 \text{ watts}$$

This power is dissipated partly in the attenuation of the line, and partly in the load. [From equation (*4.12*) the fact that Z_0 is complex indicates that R and G are not both equal to zero, and from equation (*4.10*) this means that the line has a finite attenuation factor α.]

Example 4.9.

A coaxial transmission line used to transmit substantial amounts of power at a frequency of 100 MHz has the following distributed circuit coefficients at that frequency: $R = 0.098$ ohms/m; $L = 0.32 \times 10^{-6}$ henries/m; $G = 1.50 \times 10^{-6}$ mhos/m; $C = 34.5 \times 10^{-12}$ farads/m. Find the characteristic impedance of the line at the frequency of operation.

From equation (*4.12*),

$$Z_0 \;=\; R_0 + jX_0 \;=\; \sqrt{\frac{0.098 \;+\; j(2\pi \times 10^8)(0.32 \times 10^{-6})}{1.50 \times 10^{-6} \;+\; j(2\pi \times 10^8)(34.5 \times 10^{-12})}}$$

$$= \sqrt{\frac{0.098 + j201}{(1.50 + j21{,}700) \times 10^{-6}}} \;=\; 10^3 \sqrt{\frac{201\underline{/(\pi/2) \;-\; 4.87 \times 10^{-4} \text{ rad}}}{21{,}700\underline{/(\pi/2) \;-\; 6.91 \times 10^{-5} \text{ rad}}}}$$

$$= 96.2\underline{/-2.09 \times 10^{-4} \text{ rad}} \;=\; 96.2 - j0.02 \text{ ohms}$$

The nearly real value for Z_0 obtained in Example 4.9 is typical of low-loss lines at very high frequencies. Inspection of the arithmetic shows that the inclusion of R and G in the calculations has not affected the value of R_0, but has been responsible for the appearance of the reactance component X_0, which is far too small to be of any practical consequence. Criteria for deciding when R and G can be ignored in calculating Z_0 from equation (*4.12*) are developed in Chapter 5.

In converting a complex number to polar form, the usual procedure is to express the phase angle as the angle whose tangent is the ratio of the imaginary part of the complex number to the real part. For the complex numbers encountered in Example 4.9, this would require stating the phase angles to the nearest 0.001°, to retain the significant figure accuracy of the data, i.e. $0.098 + j201 = 201\underline{/89.972°}$, and similarly for $(1.50 + j21{,}700) \times 10^{-6}$. To complete the problem, the angle for this latter number must then be subtracted from the former angle, the result divided by two, and the sine and cosine of the resultant angle (in this case a very small fraction of a degree) used to determine the real and imaginary components of the final complex number answer.

Trigonometric tables for making calculations when angles are expressed to the nearest 0.001° are awkward to use and not readily available. For most transmission line problems the solutions can be obtained more quickly and easily by small angle approximation methods. The details of the solution given for Example 4.9 illustrate the process of stating the phase angles of $R + j\omega L$ and $G + j\omega C$ in the form of their deviation in radians from $\pi/2$. Thus the actual phase angle of the complex number $0.098 + j201$ is $\tan^{-1}(201/0.098) = \tan^{-1} 2050$, an angle that cannot be read meaningfully from most standard sets of tables. The deviation of the phase angle of $0.098 + j201$ from $\pi/2$ rad, however, is $\tan^{-1}(0.098/201)$, and the radian measure of this angle can be written directly without tables as $0.098/201 = 4.87 \times 10^{-4}$ rad.

If the deviation of the phase angle of a complex number from $\pi/2$ does not exceed 0.1 rad, the sine of the angle can be taken as unity and the cosine as the value of the deviation in radians, with accuracy better than $\frac{1}{2}$%. Conversely, for the final calculation of the components of Z_0 from a magnitude and a very small phase angle, the cosine of the small phase angle is unity and the sine is equal to the value of the angle in radians.

4.11. Nepers and decibels.

Electrical engineers make much use of the language of decibels, which originated in the telephone industry. Its basic justification is that the response of human senses to stimuli such as sound and light is fairly closely proportional to the logarithm of the power level of the stimulus, when other factors such as frequency are held constant. Hence a quantity proportional to the logarithm of the power level of such a signal is an approximate measure of its physiological effect. In telephony, where physiological effect is the delivered commodity, these logarithmic measures are highly appropriate.

Even apart from the physiological justification, however, logarithmic measures simplify many calculations and designations that are commonly needed in communication systems. Between the signal source and the terminal load in a typical extended communication circuit, the signal will pass through many structurally distinct units, including lengths of different types of uniform line, amplifiers, attenuating networks, filters, etc. Each of these will modify the signal power level by some factor greater or less than unity. The ratio of the power delivered to the terminal load to the power supplied by the source is the product of all these factors. If the effect of each of these circuit units is designated by the logarithm of the factor by which it modifies the signal power, instead of by the factor itself, then the logarithm of the effect of all the units combined is the sum of the logarithms of the effects for each of the separate units. Addition is thus substituted for multiplication in calculating the cumulative effect of cascaded elements in a transmission system.

Decibels by definition are units for the logarithmic measure of the ratio of two *power levels,* using logarithms to the base 10. Any two power levels P_1 and P_2 differ by D decibels according to

$$D \text{ (db)} \;=\; 10 \log_{10}(P_1/P_2) \tag{4.15}$$

If the logarithm is positive, P_1 is said to be D db *above* P_2; if negative, P_1 is D db *below* P_2. An absolute meaning can be given to the decibel value of a power level by making P_2 a *standard reference value. If P_2 is 1 watt, the decibel value of a power level P_1* calculated by equation (*4.15*) is designated $\pm D$ dbw, i.e. D decibels above or below a power level of 1 watt. One milliwatt is also commonly used as a reference value, with the corresponding designation $\pm D$ dbm.

For an arbitrary two terminal pair network as shown in Fig. 4-3, the input impedance and the terminal load impedance will in general have different values. If these are respectively $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$, the real power input P_1 to the network and the real power P_2 reaching the load will be given by

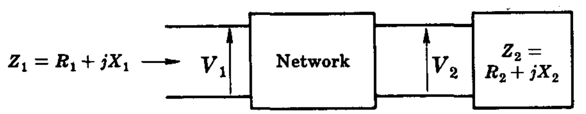


Fig. 4-3. *A two-port linear passive network terminated in an arbitrary impedance.*

$$P_1 \;=\; R_1|V_1/Z_1|^2 \qquad \text{and} \qquad P_2 \;=\; R_2|V_2/Z_2|^2$$

Applying equation (*4.15*) to these power values, P_1 is D decibels above P_2, where

$$D \text{ (db)} \;=\; 10 \log_{10} \frac{|V_1/Z_1|^2}{|V_2/Z_2|^2}(R_1/R_2)$$

In the special case of $Z_1 = Z_2$ (hence $R_1 = R_2$), this equation simplifies to

$$D \text{ (db)} \;=\; 20 \log_{10}|V_1/V_2| \tag{4.16}$$

Equation (*4.16*) appears to be an equation expressing a decibel relation between two voltages, but it has been derived from a decibel relation between power levels, and is a formally correct use of decibel language only when the two power levels have the same ratio to the squares of the respective voltage magnitudes. This requires that the impedances across which the voltages exist have admittance values with equal conductance components, a condition which is obviously satisfied if $Z_1 = Z_2$.

It is fairly general practice among communication engineers to apply (*4.16*) to two signal voltages in a circuit that are across impedances not meeting the stated requirement. Thus a voltage amplifier with a ratio of 1000 between output voltage magnitude and input voltage magnitude will often be described as having a gain of 60 decibels, even in a case where the input impedance and the load impedance have arbitrarily independent values.

This is improper use of decibel notation, and will lead to erroneous results if equation (*4.15*) is used to determine a power ratio from the decibel figure.

Nepers by definition are a logarithmic measure of the ratio of two *voltage magnitudes* or two *current magnitudes*, the logarithm being to the base e. The equations from which the definition is formulated are (*4.7*) and (*4.8*), which relate specifically to uniform transmission systems on which waves are traveling in only one direction.

It has been shown in Section 4.10 that for the case of waves traveling in only one direction on a uniform transmission line, the impedance at every point is equal to the characteristic impedance of the line, $Z_0 = R_0 + jX_0$. If then the rms phasor voltage is $V(z_1)$ at coordinate z_1 on the line and $V(z_2)$ at coordinate z_2, the power levels at the two points will be $P(z_1) = |V(z_1)/Z_0|^2 R_0$ and $P(z_2) = |V(z_2)/Z_0|^2 R_0$. Since the impedance is the same in the two locations, the decibel value of the ratio of the two power levels will be, from equation (*4.16*), $D\text{ (db)} = 20 \log_{10} |V(z_1)/V(z_2)|$. From the defining equation (*4.7*) and from Example 4.5, the neper value of the ratio of the voltages is $N = \log_e |V(z_1)/V(z_2)|$. It follows that

$$D\text{ (db)} = 20 \log_{10} e^N = 8.686\, N \tag{4.17}$$

The attenuation factor α of a uniform transmission line appears in its defining equations in the natural units of nepers per unit length. It can also be expressed in decibels per unit length, using the conversion factor from equation (*4.17*) that 1 neper = 8.686 decibels, without violating the definitions on which decibel notation is based. However, it must be emphasized again that the attenuation between two points on a transmission line, whether given in nepers or decibels, can be used to relate the voltages at the two points by equation (*4.16*), or the power levels at the two points by equation (*4.15*), only if the impedance is the same at the two locations. It is shown in Chapter 7 that when there are reflected waves on a transmission line the impedance is a fairly complicated function of position, involving the distributed circuit coefficients of the line, the frequency of the source, and the value of the terminal load impedance. Under such conditions equations (*4.15*) and (*4.16*) are inapplicable, and the attenuation between two points on a transmission line is not the only consideration determining the ratio of the power levels or of the phasor voltage magnitudes at the points.

Several specialized uses of decibel notation, for transmission lines on which there are reflected waves, are discussed in Chapters 8 and 9.

4.12. Phasor diagrams for V and I.

In the steady state analysis of lumped constant networks, phasor diagrams make use of directed line segments to describe the phase and amplitude relations of the various harmonic voltages and currents in a network.

When the technique is applied to transmission lines an additional variable appears — the position coordinate along the line. Fig. 4-4 is a representative phasor diagram for the voltage and current at the input terminals of a transmission line on which there are no reflected waves, using the phasor voltage V_1 as the zero degree reference phasor. (Since C/G exceeds L/R for all conceivable practical transmission lines under reasonable conditions, the phase angle of Z_0 is invariably negative and I_1 leads V_1 when there are no

Fig. 4-4. Phasor diagram of the time-harmonic voltage and current at the input terminals (i.e. $z = 0$) of a transmission line terminated in its characteristic impedance. The phasors are related by $V_1/I_1 = Z_0$.

reflected waves.) At a coordinate z_1 along the line, equations (*4.7*) and (*4.8*) show that the voltage and current phasors are smaller in magnitude by the factor $e^{-\alpha z_1}$ and retarded in phase by angle βz_1 relative to the phasors at the input terminals where $z = 0$. The corresponding phasor diagram is Fig. 4-5, for βz_1 approximately 0.5 rad, and αz_1 about 0.2 nepers, the reference for magnitude and phase being the same as in Fig. 4-4. Similarly, at a coordinate $z_2 = 2z_1$, the phasor diagram with the same reference is that of Fig. 4-6.

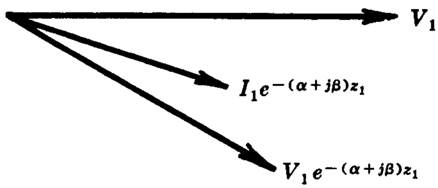


Fig. 4-5. Phasor diagram of the time-harmonic voltage and current at the coordinate z_1 of the same transmission line referred to in Fig. 4-4. The line has attenuation factor α and phase factor β.

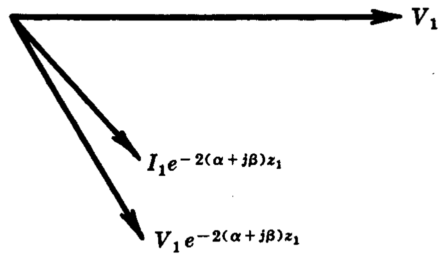


Fig. 4-6. Phasor diagram of the time-harmonic voltage and current at the coordinate $z_2 = 2z_1$ of the same transmission line referred to in Fig. 4-4 and 4-5.

An attempt to show the continuous variation with coordinate z of the phasor relations on a transmission line, by combining diagrams like Fig. 4-4, 4-5 and 4-6 for small intervals of z on a single chart, leads to undue confusion. Taking the voltage phasors separately, the desired result is achieved in useful form by drawing the envelope of the tips of the phasors. This gives a logarithmic spiral such as Fig. 4-7, whose angular coordinate is $-\beta z$, linearly proportional to distance along the transmission line. The envelope for the current phasors is a similar spiral with a different starting reference and a different radial scale.

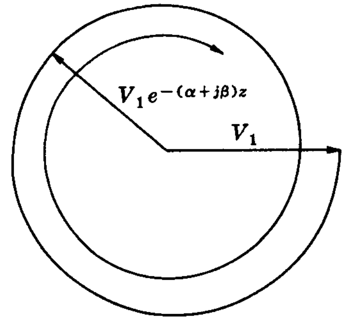


Fig. 4-7. The tip of the voltage phasor $V_1 e^{-(\alpha+j\beta)z}$ at any coordinate z on a transmission line of attenuation factor α and phase factor β and terminated nonreflectively, traces part of a logarithmic spiral as z increases from $z = 0$ to $z = l$.

Example 4.10.

If the diagrams of Fig. 4-4 and 4-5 are combined on a single chart, what is the meaning of the phasor joining the tips of the two voltage phasors, and of the phasor joining the tips of the two current phasors?

The phasor joining the tips of the two voltage phasors is the *longitudinal* phasor voltage along the line between coordinates $z = 0$ and $z = z_1$. The phasor joining the tips of the two current phasors is the phasor current flowing transversely between the line conductors (i.e. through the shunt conductance and susceptance of the line) in the line section between $z = 0$ and $z = z_1$.

Phasor diagrams like Fig. 4-4, 4-5, 4-6 and 4-7 convey exactly the same information as equations (*4.7*) and (*4.8*), information which is easily comprehended from the equations themselves. For the more general case of transmission lines on which there are reflected waves, the use of phasor diagrams provides unique assistance in visualizing voltage and current relations along a line, the pattern of which is not directly obvious from the corresponding equations. Such diagrams are discussed in Section 8.8.

Solved Problems

4.1. A uniform transmission line is 15,000 m long. Its characteristic impedance is $200 + j0$ ohms, and the signal velocity on the line is 65% of the "velocity of light" in free space. The terminal load impedance connected to the line is *not* equal to the line's characteristic impedance. At the input end of the line a time-harmonic signal source of rms amplitude 10 volts at a frequency of 50 MHz, and internal impedance $100 + j0$ ohms, is connected in series with a switch and the line's input terminals. The switch is initially open.

(*a*) When the switch is closed, what is the initial phasor voltage at the input terminals of the line, and what is the initial phasor input current to the line?

(*b*) For how long do the input voltage and current remain at the initial phasor values?

Equation (*4.14*) states that at any point on a transmission line where there are waves traveling in only one direction, the ratio of the phasor voltage to the phasor current must be equal to the characteristic impedance of the line.

At the instant the switch is closed, harmonic voltage and current waves will start to travel along the transmission line toward the terminal load end, with velocity $0.65(3.00 \times 10^8) = 1.95 \times 10^8$ m/sec. Signals will travel the full length of the line in a time $15{,}000/(1.95 \times 10^8) = 77$ microseconds, after the closing of the switch. Since the terminal load impedance is *not* equal to the characteristic impedance of the line, voltage and current waves will be reflected from the terminal load, and will reach the input terminals of the line 154 microseconds after the switch was initially closed. During these 154 microseconds it will remain true *at the input terminals of the line* that there are waves traveling in only one direction, and the relation $V/I = Z_0$ will be valid there for that length of time.

(*a*) If V_i and I_i are the initial phasor voltage and current at the input terminals of the line just after closing the switch, then $V_i/I_i = 200$. On applying Kirchhoff's voltage law to the circuit consisting of the source, the internal impedance of the source, and the input terminals of the line, $10 - 100I_i - V_i = 0$, where the sign conventions of Chapter 3 have been observed. Simultaneous solution of these two equations gives $V_i = 6.67$ volts rms, and $I_i = 0.033$ amperes rms. It is obvious that this process is equivalent to assuming that the *initial* input impedance of the line (before any reflected waves return to the input terminals) is equal to its characteristic impedance.

(*b*) The input voltage and current retain their initial phasor values for 154 microseconds. At any time later than 154 microseconds after the closing of the switch there will be waves traveling in both directions at all points on the line. The input impedance of the line will no longer be equal to its characteristic impedance, and the values of input voltage and current cannot be determined by the methods of Chapter 4. Chapters 7, 8 and 9 deal with procedures for solving transmission line circuits in these more general circumstances.

Although the source frequency of 50 MHz was not used in the solution, it is nevertheless of some significance in the problem. If it had been 1 kHz, for example, the source voltage and current would not in 154 microseconds be able to vary through a full cycle of the signal. There would then be no literal meaning to a phasor value. Since the period of the 50 MHz source frequency is only 0.02 microseconds, the steady state time-harmonic variation of the voltage and current at the input terminals is established, after the closing of the switch, in a time very short compared with the time required for the signal to travel to the terminal load end of the line and back.

(The term "signal velocity", not previously defined, has been used in the statement of the problem. This is justified by demonstrations given in Chapters 5 and 6 that for all reasonable and practical transmission line designs the phase velocity becomes quite independent of frequency at frequencies above 10^4 or 10^5 Hz. Under these conditions all high-frequency signals, regardless of their frequency spectrum or bandwidth, travel with the same velocity which can appropriately be called the signal velocity. Under the same conditions the characteristic impedance of the line is also independent of frequency and purely resistive. For typical transmission lines these simplified relationships do not hold for signals at voice frequencies.)

4.2. A signal source at a frequency of 2.5 MHz is connected to the input terminals of a low-loss transmission line which is 125 m long and is terminated in its characteristic impedance. The wavelength of the signals on the line is measured (by methods described in Chapter 8) and found to be 92 m.

(*a*) Determine the time delay between the instant of connecting the source to the line and the arrival of a signal at the terminal load.

(*b*) What is the phase difference between the voltages at the two ends of the line in the steady state?

(*a*) The signal velocity can be found from either of the equivalent relations $v_p = f\lambda$ or $v_p = \omega/\beta$, where $\beta = 2\pi/\lambda$ and λ is the wavelength on the line. Using the first expression, $v_p = 92(2.5 \times 10^6) = 2.3 \times 10^8$ m/sec. Then the required time delay is $t = l/v_p = 125/(2.3 \times 10^8) = 0.54$ microseconds.

(*b*) The phase of the voltage at the output terminals relative to that at the input terminals is given by the term $e^{-j\beta z}$ of equation (*4.7*), with $z = l$. This term states that the phase of the voltage at the terminal load will be less than (i.e. will lag) the phase of the voltage at the input terminals by the amount βl rad. Here $\beta = 2\pi/\lambda = 2\pi/92 = 0.068$ rad/m, and $\beta l = 8.5$ rad.

It may also be observed that the phase changes in time at the rate 2π rad per period of the signal. Hence the phase lag can be found from the time delay of the line as

$$2\pi(0.54 \times 10^{-6})/(0.4 \times 10^{-6}) \;=\; 8.5 \text{ rad}$$

4.3. On an air dielectric transmission line for which the attenuation per wavelength is negligible, the signal velocity is 100% (this is standard commercial notation, meaning 100% of the "free space" velocity of light, or 3.00×10^8 m/sec). A harmonic signal voltage $v(t) = 50 \cos(10^7 t + \pi/6)$ is connected to the input terminals of the line at $t = 0$. Draw a graph of each of the following:

(*a*) The voltage at the input terminals of the line as a function of time from $t = 0$ to $t = 1.0$ microseconds.

(*b*) The voltage as a function of position on the transmission line at time $t = 0.2$ microseconds.

(*c*) The voltage as a function of position on the line at time $t = 1.0$ microseconds.

(*a*) The voltage at the input terminals of the line is simply $v(t)$. This is +43.3 volts at $t = 0$; it passes through zero at $10^7 t + \pi/6 = \pi/2$, or $t = 0.105$ microseconds; subsequent zero crossings are at intervals of $10^7 t = \pi$, or $t = 0.314$ microseconds. The graph is shown in Fig. 4-8.

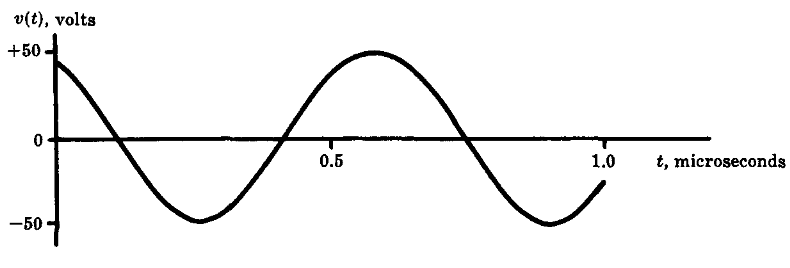


Fig. 4-8

(*b*) For a wave traveling in the direction of increasing z on a transmission line, the voltage at any coordinate z on the line at any time t is given by a function of the form $f_1(t - z/v_p)$, according to Problem 3.5, page 24, where f_1 is any function. Since in the present problem $v_p = 3.00 \times 10^8$ m/sec and the voltage at $z = 0$ is $v(t) = 50 \cos(10^7 t + \pi/6)$, it follows that the voltage as a function of z and t is $v(z, t) = 50 \cos\{10^7[t - z/(3.00 \times 10^8)] + \pi/6\}$, with the additional stipulation that there is no voltage on the line at any time t for values of z greater than $z = v_p t$, this being the distance the signal advances on the line from the input terminals in time t after being connected to the terminals.

At $t = 0.2$ microseconds,

$$v(z) = 50 \cos [10^7(0.2 \times 10^6 - 3.33 \times 10^{-9} z) + \pi/6] = 50 \cos (2.00 + \pi/6 - 0.033z)$$

for $0 < z < v_p t$, where $v_p t = (3.00 \times 10^8)(0.2 \times 10^{-6}) = 60$ m. The resulting graph of $v(z)$ is shown in Fig. 4-9.

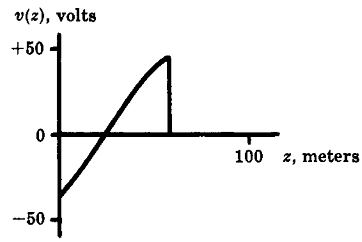


Fig. 4-9

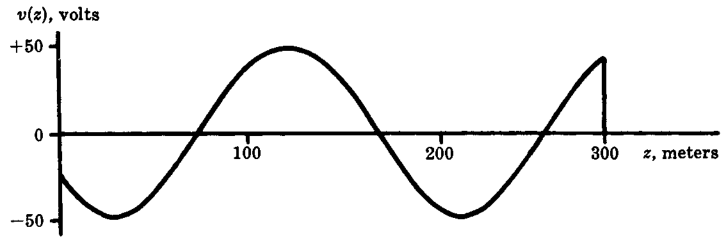


Fig. 4-10

(c) The reasoning is exactly the same as in (b), but the pattern extends for a greater distance $z = v_p t = 300$ m along the line, as shown in Fig. 4-10.

(It should be noted that because the terms containing z and t in the expression for $v(z, t)$ have opposite signs, a graph of $v(z, t)$ as a function of z at fixed t will be reversed along the coordinate scale, or backward, relative to a graph of the same $v(z, t)$ as a function of t at fixed z. For the harmonic voltage function of Figs. 4-8, 4-9 and 4-10, this reversal appears only at the advancing front of the wave. If the voltage function had been an unsymmetrical one such as a sawtooth pattern, the reversal would be clearly exhibited over the whole of the pattern.)

4.4. At time $t = 0$ a switch is closed to connect a voltage source

$$v(t) = 10 \sin (2\pi 10^7 t - \pi/3)$$

with negligible internal impedance, to the input terminals of a section of lossless transmission line 36 m long. The line is terminated in its characteristic impedance of $200 + j0$ ohms. The signal velocity on the line is 80% (see Problem 4.3). (a) Graph the instantaneous voltage as a function of position along the line 0.15 microseconds after the switch is closed. (b) Graph the instantaneous current as a function of position along the line at the same instant used in (a). (c) What is the next earliest time at which the voltage pattern on the line will be the same as in part (a)?

(a) The signal velocity on the line is $v_p = 80\%$ of $3.00 \times 10^8 = 2.40 \times 10^8$ m/sec. In 0.15 microseconds after the switch is closed the front of the wave will advance a distance $z = v_p t = (2.40 \times 10^8)(0.15 \times 10^{-6}) = 36$ m, i.e. exactly to the terminal load end of the line. The wavelength of the voltage pattern on the line is $\lambda = v_p/f = 2.40 \times 10^8/10^7 = 24$ m. Hence the line is 1.50 wavelengths long. As a function of z and t the voltage on the line is

$$v(z, t) = 10 \sin [2\pi 10^7(t - z/(2.40 \times 10^8)) - \pi/3] = 10 \sin (2\pi 10^7 t - 0.262z - \pi/3)$$

As a function of z at $t = 0.15$ microseconds, this is $v(z) = 10 \sin (-0.262z + 8.37)$, which is valid from $z = 0$ to $z = v_p t = 36$ m. Fig. 4-11 is a graph of this voltage.

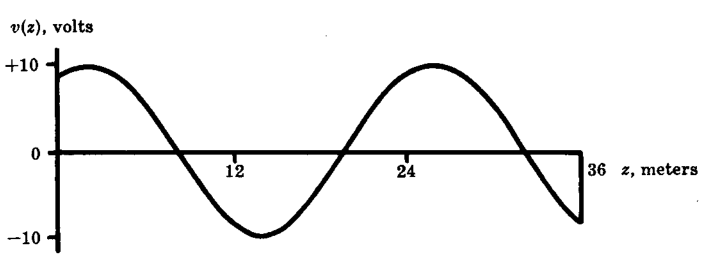


Fig. 4-11

(b) Since the transmission line is terminated nonreflectively, $i(z,t) = v(z,t)/Z_0$ everywhere and always; and since Z_0 is real, the relation between the current pattern and the voltage pattern is that they are identical in shape. The current pattern at the instant stated is therefore the pattern of Fig. 4-11 with a current ordinate scale of $10/200 = 0.050$ amperes per division substituted for the voltage scale of 10 volts per division.

(c) The voltage and current patterns on the line will again be the same as in Fig. 4-11 after the entire pattern has traveled an additional distance of one wavelength along the line. The time required for this is $t = v_p/\lambda = 1/f = 1/10^7 = 0.1$ microseconds. This time is the period of the source frequency.

4.5. A coaxial transmission line for carrier telephony is 900 miles long, from New York City to Chicago. The distributed inductance of the line is 0.425 millihenries/mile, and its distributed capacitance is 0.0715 microfarads/mile. The distributed resistance R and the distributed conductance G are small enough to have no effect on the values of the phase factor β and the characteristic impedance Z_0 at the frequencies of operation. (R and G do cause attenuation, however, which is offset by amplifiers at intervals along the line.) There are no reflected waves on the line. (a) How long is a telephone signal delayed in transmission from New York City to Chicago? (b) Is the delay time of part (a) sufficient to be annoying to persons conducting a conversation via the line? (c) When the rms signal voltage at a particular point on the line is 5.0 volts, what is the rms signal current at the point and what is the signal power passing the point?

(a) The condition that R and G do not affect β or Z_0 must mean that equation (4.10) can be written as $\beta = \omega\sqrt{LC}$, and equation (4.12) can be written as $Z_0 = \sqrt{L/C}$. (These simplifications do not imply that $\alpha = 0$ and $X_0 = 0$, but only that $\alpha \ll \beta$ and $X_0 \ll R_0$, conditions that are typical for most transmission lines at frequencies above a few kilohertz, as shown in Chapter 5.) Then

$$v_p = \omega/\beta = 1/\sqrt{LC} = 1/\sqrt{(4.25 \times 10^{-4})(7.15 \times 10^{-8})} = 181{,}500 \text{ miles/sec}$$

The time delay of a signal over the length of the line is then

$$t = l/v_p = 900/181{,}500 = 4.96 \text{ milliseconds}$$

(b) The annoyance of transmission delay on a telephone circuit is a matter for experimental investigation, and is very much dependent on the loudness and quality of the signal. Evidence shows that for signals of reasonable strength and clarity, delays as great as 25 to 50 milliseconds cause little inconvenience. A channel using re-transmission from a satellite in a fixed position over the earth's surface involves a delay of about 250 milliseconds, which is long enough to create considerable confusion in telephone communication.

(c) The characteristic impedance of the line is

$$Z_0 = \sqrt{L/C} = \sqrt{(4.25 \times 10^{-4})/(7.15 \times 10^{-8})} = 77.1 \text{ ohms}$$

The rms signal current will therefore be $I = V/Z_0 = 5.0/77.1 = 0.065$ amperes, and the signal power will be $P = VI = V^2/Z_0 = I^2 Z_0 = 32.5$ milliwatts.

4.6. Two parallel wire transmission lines operated at radio frequencies have nominal characteristic impedances of 300 ohms and 200 ohms respectively. Each is terminated in its characteristic impedance. At their input terminals the two lines are connected in parallel, and their inputs serve as the terminal load of a third parallel wire transmission line which supplies power to the other two. (a) What must be the characteristic impedance of the third transmission line if there are to be no reflected waves on it? (b) How does power flowing on the third line divide between the other two lines at the junction?

(a) The first two lines are terminated nonreflectively, so their input impedances are respectively equal to their characteristic impedances of 300 ohms and 200 ohms. These impedances connected in parallel constitute the terminal load impedance of the third line, which is therefore given by $Z_T = 1/(1/200 + 1/300) = 120$ ohms. If the third line is to be terminated nonreflectively, this must be the value of its characteristic impedance.

(b) At the junction of the three lines the signal voltage is common to all of them. Taking it to be a time-harmonic voltage of rms value V, the power on the third line is $V^2/120$ watts, and on the other two lines $V^2/200$ and $V^2/300$ watts respectively. Hence 60% of the power on the third line flows onto the line of characteristic impedance 200 ohms, and 40% onto the line of characteristic impedance 300 ohms.

4.7. A transmission line 400 ft long has an attenuation of 4.00 db/(100 ft) at its operating frequency of 5.00×10^6 Hz. The phase velocity at that frequency is 170,000 miles/sec. The line is terminated nonreflectively. For steady state conditions, make a graph of the pattern of instantaneous voltage as a function of position along the line at an instant when the voltage at the input terminals is passing through zero and rising.

From equation (4.1a), if there are no reflected waves on the line $V_2 = 0$, and

$$v(z,t) = |\hat{V}_1|\, e^{-\alpha z} \cos(\omega t - \beta z + \xi_1)$$

where ξ_1 is the phase angle of V_1.

A time t at which the input voltage of the line (i.e. at $z = 0$) is passing through zero and rising is identified by $(\omega t + \xi_1) = 3\pi/2$.

Thus an expression for the instantaneous voltage as a function of position on the line that satisfies all the stated conditions is

$$v(z) = |\hat{V}_1|\, e^{-\alpha z} \cos\left(\frac{3\pi}{2} - \beta z\right) = -|\hat{V}_1|\, e^{-\alpha z} \sin \beta z$$

where $|\hat{V}_1|$ is an arbitrary amplitude factor.

The attenuation factor α of the line is $4.00/(100 \times 8.686) = 4.61 \times 10^{-3}$ nepers/ft. The wavelength on the line is $\lambda = v_p/f = (170{,}000 \times 5280)/(5 \times 10^6) = 178$ ft and $\beta = 2\pi/\lambda = 0.0353$ rad/ft. Then $v(z) = e^{-4.61 \times 10^{-3} z} \sin 0.0353z$ if $|\hat{V}_1|$ is taken as unity.

The graph of this voltage pattern is a distorted sine wave lying between symmetrical positive and negative exponentially decaying envelopes, with the axis crossings at the equally spaced intervals of a true sine wave. The ordinates of the envelopes are ± 1.00 at $z = 0$, $\pm e^{-0.461} = \pm 0.632$ at $z = 100$ ft, $\pm(0.632)^2 = \pm 0.399$ at $z = 200$ ft, $\pm(0.632)^3 = \pm 0.251$ at $z = 300$ ft, and $\pm(0.632)^4 = 0.159$ at $z = 400$ ft. The pattern crosses the axis at intervals of one-half wavelength, or 89 ft. Hence the zero crossings occur at $z = 0, 89, 178, 267$ and 356 ft. From these data the graph is easily drawn. Note that the voltage initially increases negatively as z increases from zero.

4.8. If for the transmission line of Problem 4.7 the input signal has a phasor amplitude of 50 volts at the frequency of 5.00×10^6 Hz, make a graph of the phasor amplitude of the voltage as a function of position along the line.

From equation (4.7), the phasor amplitude of the voltage as a function of position on the line is given by $|\hat{V}| = |\hat{V}_1|\, |e^{-\alpha z}|\, |e^{-j\beta z}|$. But $|e^{-\alpha z}| = e^{-\alpha z}$, $|e^{-j\beta z}| = 1$, and from the data of the problem $|\hat{V}_1| = 50$ volts. The equation for the required graph is then simply $|\hat{V}| = 50e^{-\alpha z}$. This is the upper envelope of the graph of Problem 4.7, with a change of voltage scale.

4.9. For any lossless transmission line ($R = G = 0$) terminated in its characteristic impedance, and carrying voltage and current waves of any arbitrary pattern, show that the total energy stored in the distributed inductance of the line is equal to the total energy stored in the distributed capacitance of the line.

For the theorem to be true in the general form stated, it must be true for each infinitesimal length dz of the line. The energy stored in an inductance L_0 carrying an instantaneous current $i(t)$ is $W_L = \frac{1}{2}L_0[i(t)]^2$. Hence the energy stored in the inductance of a length dz of the transmission line at any coordinate z at any time t is $dW_L = \frac{1}{2}L\, dz\, [i(z,t)]^2$. The energy stored in the capacitance of the same line element at the same instant is $dW_C = \frac{1}{2}C\, dz\, [v(z,t)]^2$. The line is terminated in its characteristic impedance, and from equation (4.12) this is $Z_0 = \sqrt{L/C}$ when $R = G = 0$. Then $i(z,t) = v(z,t)/\sqrt{L/C}$, and substituting this into the expression for dW_L shows that $dW_L = dW_C$, which proves the theorem.

4.10. A uniform transmission line 50 m long and terminated in its characteristic impedance delivers 1250 watts of radio frequency power to its terminal load. The input power to the line is 1600 watts. Determine (*a*) the attenuation factor of the line and (*b*) the efficiency of the line as a transmission system. (*c*) What is the ratio of the peak phasor voltage magnitude at the midpoint of the line to the peak phasor voltage magnitude at the input terminals?

(a) The total attenuation of the line in decibels is $10 \log_{10}(1600/1250) = 1.07$. The attenuation factor is then $\alpha = 1.07/(8.686 \times 50) = 2.47 \times 10^{-3}$ nepers/m. (If the power ratio is taken as 1250/1600, the decibel value is found to be -1.07. This is equivalent to saying that the transmission line as a system element has a gain of -1.07 db. Attenuation is positive when gain is negative, and vice versa. A transmission system with active elements, such as amplifiers, that give it a net gain, is commonly said to have negative attenuation.)

(b) The efficiency of the line as a transmission system is defined in the usual manner appropriate to any passive two-port device, as efficiency = (output power)/(input power) × 100%. In this case it is $(1250/1600) \times 100 = 78.3\%$.

(c) The phasor voltage at any coordinate z is given in terms of the phasor voltage at the input terminals of the line by equation (*4.7*). The magnitudes of the corresponding voltages are related by $|\hat{V}| = |\hat{V}_1| e^{-\alpha z}$. Then $|\hat{V}(z=25)| = |\hat{V}_1| e^{-2.47 \times 10^{-3} \times 25} = 0.938\,|\hat{V}_1|$.

4.11. A signal of 2.50 volts rms is connected to the input terminals of a uniform transmission line whose attenuation at the signal frequency is 0.0010 nepers/m. The line runs for 4000 m and is terminated by the input terminals of an amplifier with a gain of 50 db, the input and output impedances of the amplifier being equal to the characteristic impedance of the line. The output of the amplifier is connected to the input terminals of another 4000 m length of the same transmission line, which is followed by a second identical amplifier and a third 4000 m length of the same transmission line. The final transmission line section is terminated in its characteristic impedance. What is the signal voltage at the final terminal load impedance?

Attenuation is loss or negative gain. When expressed in nepers or decibels, attenuations and amplifier gains are algebraically additive. The attenuation of each transmission line section in decibels is $0.0010 \times 4000 \times 8.686 = 34.7$ db. The total *attenuation* of the system is then $3 \times 34.7 - 2 \times 50 = 4.1$ db. From equation (*4.16*), the rms output voltage V_{out} for the system is given by

$$4.1 = 20 \log_{10} |2.50/V_{out}| \quad \text{or} \quad V_{out} = 1.56 \text{ volts rms}$$

4.12. The specifications for a form of "Twin-Lead" plastic insulated parallel wire transmission line used as lead-in wire from television receiving antennas are: characteristic impedance = 300 ohms, signal velocity = 82%, attenuation = 2.8 db/(100 ft) at 144 MHz. Find approximate values for the distributed circuit coefficients L and C for the cable.

At the television frequency of 144 MHz, the attenuation factor of the line is $\alpha = 0.028/8.686 = 3.22 \times 10^{-3}$ nepers/ft. Since the signal velocity is $82\% = 0.82 \times 3.00 \times 10^8 \times 3.28 = 8.07 \times 10^8$ ft/sec, the wavelength λ on the line is $\lambda = v_p/f = (8.07 \times 10^8)/(144 \times 10^6) = 5.60$ ft, and $\beta = 2\pi/\lambda = 1.12$ rad/ft. Clearly $\alpha \ll \beta$. This justifies the approximation from equation (*4.10*) that $\beta = \omega\sqrt{LC}$ (so that $v_p = 1/\sqrt{LC}$) and the approximation from equation (*4.12*) that $Z_0 = \sqrt{L/C}$. Using these approximations, $L = Z_0/v_p = 300/(8.07 \times 10^8) = 0.373$ microhenries/ft, and

$$C = 1/v_p Z_0 = 1/(8.07 \times 10^8 \times 300) = 4.1 \text{ picofarads/ft}$$

Supplementary Problems

4.13. A switch is closed at $t = 0$ to connect a signal voltage $v(t) = 5.0 \sin(2\pi 10^4 t - \pi)$ to the input terminals of a 30 mile length of uniform transmission line whose attenuation is 0.010 nepers/mile, and on which the signal velocity is 165,000 miles/sec. The line is terminated in its characteristic impedance. On a common time scale extending from $t = 0$ to $t = 0.3$ milliseconds, graph the voltage as a function of time at (a) the input terminals of the line, (b) the midpoint of the line, (c) the terminal load end of the line.

Ans. (a) $v(t) = -5.0 \sin 2\pi 10^4 t$, from $t = 0$ to $t = 0.3$ milliseconds.
(b) $v(t) = -4.3 \sin 2\pi 10^4 (t - 0.091 \times 10^{-3})$, from $t = 0.091$ milliseconds to $t = 0.3$ milliseconds.
(c) $v(t) = -3.7 \sin 2\pi 10^4 (t - 0.182 \times 10^{-3})$, from $t = 0.182$ milliseconds to $t = 0.3$ milliseconds.

4.14. The signal voltage connected to the input terminals of a lossless transmission line 300 meters long is given by: $v(t) = 0,\ t < 0$; $v(t) = 10(1 - 10^7 t),\ 0 < t < 0.1$ microseconds; $v(t) = 0,\ t > 0.1$ microseconds. The signal velocity on the line is 80% of the free space velocity of electromagnetic waves. Graph the voltage pattern on the line at (a) $t = 0.05$ microseconds, (b) $t = 0.5$ microseconds.

Ans. (a) $v(z) = 5$ at $z = 0$; $v(z) = 5 + 5z/12,\ 0 < z < 12$ m; $v(z) = 0,\ z > 12$ m.
(b) $v(z) = 0,\ z < 96$ m; $v(z) = 10(z/24 - 4),\ 96 < z < 120$ m; $v(z) = 0,\ z > 120$ m.

4.15. A transmission line of characteristic impedance $80 + j0$ ohms and negligible losses branches at its load terminals into three secondary transmission lines which all have the same characteristic impedance Z_0 and are all terminated nonreflectively. Their input terminals are connected in parallel as the terminal load of the first transmission line. (a) If the first transmission line is terminated nonreflectively, what is the value of Z_0? (b) If the phasor voltage magnitude is 1.00 volts rms on the first transmission line, what is the power delivered to the terminal load impedances of each of the secondary transmission lines?

Ans. (a) $Z_0 = 240$ ohms. (b) Power to each terminal load = 4.2 milliwatts.

4.16. From consideration of equation (4.12) show that the phase angle of the characteristic impedance Z_0 of a uniform transmission line cannot lie outside the range $-45°$ to $+45°$ for a line constructed of ordinary passive materials (i.e. R, L, G and C all positive and real).

4.17. Show that for a uniform transmission line of low total attenuation $(\alpha l \ll 1)$ and terminated in its characteristic impedance, the efficiency of power transmission is given by $(1 - 2\alpha l) \times 100\%$.

4.18. A transmission line has an attenuation factor 0.050 nepers/mile and is terminated in its characteristic impedance. For what length of line will the efficiency of power transmission by the line be 50%? What is the attenuation in decibels of such a length of line? *Ans.* 6.93 miles, 3.0 db.

4.19. For standard RG/58-U plastic dielectric flexible coaxial transmission line, two of the specifications are that the signal velocity is 66% and that the attenuation at a frequency of 50 MHz is 2.7 db/100 ft. If it were desired to use a length of this cable to delay a signal by 1.00 microseconds between two points of a circuit operating at 50 MHz, what length of cable would be required and by what factor would the signal voltage be reduced while undergoing the delay?

Ans. 650 ft. Voltage would be reduced to 0.133 of initial value, assuming the line terminated in its characteristic impedance. (In radar circuits and many other applications it is often desired to delay a signal by times of the order of a few microseconds. The numerical answers to this problem explain why cables are awkward and inefficient for the purpose. Mechanical delay lines, involving the much lower velocity of sound waves in solids, are commonly used. These require transducers at each end to convert the electrical signal into mechanical form and back.)

4.20. If the characteristic impedance of a transmission line at a frequency of 5.0 kHz is $350 - j125$ ohms, what is its characteristic admittance at the same frequency? What parallel combination of resistance and capacitance connected as terminal load on the line would provide a nonreflective termination? *Ans.* $0.00253 + j0.000906$ mhos. 395 ohms in parallel with 0.0288 microfarads.

4.21. Referring to equation (4.12), what ratio relations must exist among the four distributed circuit coefficients R, L, G and C of a uniform transmission line at any frequency ω rad/sec to make the characteristic impedance Z_0 real?

Ans. Z_0 as given by equation (4.12) will be real if the phase angle of the numerator term on the right is equal to the phase angle of the denominator. This will be true if $L/R = C/G$, for any value of ω.

4.22. If the relation $R/L = G/C$ exists among the distributed circuit coefficients of a uniform transmission line, show that $\alpha = R\sqrt{C/L} = \sqrt{RG}$, $v_p = 1/\sqrt{LC} = \sqrt{G/R}/C$, and $Z_0 = \sqrt{L/C} = \sqrt{R/G}$, all of which are independent of frequency.

Chapter 5

Propagation Characteristics and Distributed Circuit Coefficients

5.1. The nature of transmission line problems.

Transmission line engineering consists of designing transmission lines to meet desired operating specifications. The statement of these specifications will usually include values of the characteristic impedance Z_0, the attenuation factor α, and the phase factor β (or phase velocity v_p) that must be provided at one or more signal frequencies. Requirements in addition to these wave propagation characteristics may include ratings for voltage breakdown, temperature rise at full load, or various aspects of mechanical behavior. Detailed discussion of these non-propagation features of transmission lines is outside the scope of this book.

The final solution of a practical transmission line problem will be in the form of a set of prescribed data for the dimensions, cross-sectional configuration and materials of the line conductors, and of the dielectric between them or surrounding them. It might therefore be expected that appropriate design formulas should directly relate the physical attributes of a line to its wave propagation behavior. Such expressions can indeed be constructed, but their complexity makes their solution mathematically awkward, and they provide poor insight into the consequences of varying any of the individual design parameters.

The experience of several decades seems to suggest that the best foundation for the practice of transmission line *design* is a thorough familiarity with the results of transmission line *analysis*, for a wide variety of line constructions. Transmission line analysis, in turn, appears to be best comprehended when the distributed circuit coefficients R, L, G and C are used as an intermediate step between the physical data of dimensions and materials, and the ultimate signal propagation characteristics.

This chapter deals with practical algebraic and numerical processes for evaluating α, β and Z_0 for a transmission line at angular frequency ω from the values of R, L, G and C at that frequency. The reverse problem, finding R, L, G and C that will give desired values of α, β and Z_0, is also discussed. In Chapter 6 the functional relations between R, L, G and C and the physical structure of a line are developed. When the results of this chapter are combined with those of Chapter 6, the attenuation factor, phase factor and characteristic impedance of any uniform line can be determined from its geometry and materials, or conversely, a line can be designed to provide desired values of those operating parameters.

5.2. Polar number solutions.

In Chapter 4 it has been shown that the propagation of voltage and current waves along a uniform transmission line at angular frequency ω is completely described by the attenuation factor α, the phase factor β, and the characteristic impedance $Z_0 = R_0 + jX_0$, when there are no reflected waves present. The quantities α, β and Z_0 are expressed in terms of the distributed circuit coefficients of the line at angular frequency ω, by the equations

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{5.1}$$

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{5.2}$$

It is obvious that for any set of values of R, L, G, C and ω these equations can be solved in a straightforward manner by the standard methods of complex number algebra. To perform the square root operation, the product and quotient of the complex numbers involving the distributed circuit coefficients must be brought to polar form.

The arithmetical operations to be performed are:

$$\alpha = |\gamma| \cos\psi, \qquad \beta = |\gamma| \sin\psi$$

where $$|\gamma| = \sqrt{(R^2+\omega^2L^2)^{1/2}(G^2+\omega^2C^2)^{1/2}}, \qquad \psi = \frac{1}{2}\left(\tan^{-1}\frac{\omega L}{R} + \tan^{-1}\frac{\omega C}{G}\right)$$

and $$R_0 = |Z_0| \cos\theta_0, \qquad X_0 = |Z_0| \sin\theta_0$$

where $$|Z_0| = \sqrt{\frac{(R^2+\omega^2L^2)^{1/2}}{(G^2+\omega^2C^2)^{1/2}}}, \qquad \theta_0 = \frac{1}{2}\left(\tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega C}{G}\right)$$

Example 5.1.

The distributed circuit coefficients of a parallel wire pole-mounted telephone line with copper conductors 0.128 in. in diameter spaced 12 in. between centers are

$$R = 6.74 \text{ ohms/mile} \qquad G = 0.29 \text{ micromhos/mile}$$
$$L = 0.00352 \text{ henries/mile} \qquad C = 0.0087 \text{ microfarads/mile}$$

at a frequency of 1000 Hz. Find the attenuation, phase velocity and characteristic impedance of the line at that frequency.

$$|\gamma| = \sqrt{\{6.74^2 + (2\pi\times1000\times0.00352)^2\}^{1/2} \times \{(0.29\times10^{-6})^2 + (2\pi\times1000\times0.0087\times10^{-6})^2\}^{1/2}}$$
$$= 0.0356/\text{mile}$$

$$\psi = \frac{1}{2}\left(\tan^{-1}\frac{22.1}{6.74} + \tan^{-1}\frac{54.7\times10^{-6}}{0.29\times10^{-6}}\right) = \frac{1}{2}(73.0° + 89.7°) = 81.4°$$

Then
$$\alpha = 0.0356 \cos 81.4° = 0.00534 \text{ nepers/mile} = 0.00534 \times 8.686 = 0.0434 \text{ db/mile}$$
$$\beta = 0.0356 \sin 81.4° = 0.0352 \text{ rad/mile}$$
$$v_p = \omega/\beta = 6283/0.0352 = 178{,}500 \text{ miles/sec}$$

and
$$|Z_0| = \sqrt{(6.74^2 + 22.1^2)^{1/2} \times 10^6/(0.29^2 + 54.7^2)^{1/2}} = 650 \text{ ohms}$$
$$\theta_0 = \tfrac{1}{2}(73.0° - 89.7°) = -8.4°$$
$$R_0 = 650 \cos(-8.4°) = 643 \text{ ohms}$$
$$X_0 = 650 \sin(-8.4°) = -95 \text{ ohms}$$

In this example the phase angle of $G + j\omega C$ differs from 90° by only 0.3° of angle, while the difference from 90° of the phase angle of $R + j\omega L$ is about fifty times as great. This is a very typical illustration of the relative values that $R, \omega L, G$ and ωC are likely to have for virtually all types of transmission lines over frequencies from kilohertz to gigahertz.

Setting $G = 0$ in the above example, instead of 0.29 micromhos/mile, would have no effect on the calculated values of $\beta, v_p, |Z_0|$ or R_0, and would change α and X_0 by only 2%.

The fact that the shunt conductance coefficient G of a transmission line is often variable with time or not amenable to simple theoretical treatment (e.g. the effect of rain or ice on a pole-mounted open wire line, or the local variations in paper-insulated twisted pair cable) is therefore generally of little importance.

The dominant role of the phase angle of $R + j\omega L$ (i.e. its deviation from 90°) in determining the phase angles ψ and θ_0 means that the attenuation of most practical transmission lines is due largely to the distributed resistance R, and the distributed conductance G usually makes little or no contribution to it. Exceptions are most likely to occur at very low frequencies of a few hertz or extremely high frequencies of several gigahertz.

Example 5.2.

Find the attenuation and phase velocity at a frequency of 100 MHz for the transmission line whose distributed circuit coefficients at that frequency are given in Example 4.9, page 35.

The magnitudes and phase angles of $R + j\omega L$ and $G + j\omega C$ were determined in Example 4.9, with the phase angles expressed in terms of their small deviations from $\pi/2$ rad. Using those values,

$$\alpha + j\beta = \sqrt{201/\underline{(\pi/2 - 4.88\times10^{-4})} \times 0.0217/\underline{(\pi/2 - 6.91\times10^{-5})}}$$
$$= \sqrt{4.36/\underline{(\pi - 5.57\times10^{-4})}} = 2.09/\underline{(\pi/2 - 2.78\times10^{-4})}$$

Again, trigonometric tables are not needed, since the sine of this angle is 1.000 and the cosine is equal to the radian deviation of the angle from $\pi/2$.

Then $\alpha + j\beta = 5.81 \times 10^{-4} + j2.09 \text{ m}^{-1}$. Hence

$$\alpha = 5.81 \times 10^{-4} \text{ nepers/m} = 5.05 \times 10^{-3} \text{ db/m}$$

$$v_p = \omega/\beta = \frac{2\pi \times 10^8 \text{ rad/sec}}{2.09 \text{ rad/m}} = 3.00 \times 10^8 \text{ m/sec}$$

In contrast with the earlier calculation for the characteristic impedance of this same line, where the values of R and G were responsible only for the appearance of a small and unimportant reactance component X_0 in the result, it is evident in this analysis that R and G determine the value of the highly important attenuation factor α, which would be zero if R and G were both zero.

Thus in solving equation (*5.1*) by polar number arithmetic, it is much more important to retain full significant figure precision in expressing the phase angles of $R + j\omega L$ and $G + j\omega C$ than in solving equation (*5.2*). The imaginary part of Z_0 is of no practical concern if it is small compared to the real part, but the real part of γ is of prime interest no matter how small it may be relative to the imaginary part.

Although it is always possible to solve (*5.1*) and (*5.2*) by the use of polar numbers, with or without the help of small-angle approximations when applicable, the calculations are in many cases made more quickly and conveniently using derivations that avoid complex numbers. For cases in which the small-angle approximations *are* valid, Section 5.3 shows that very simple approximate real number expressions can be found for each of α, β, R_0 and X_0. Section 5.4 develops perfectly general real number solutions of (*5.1*) and (*5.2*) which are most effective when the phase angles of $R + j\omega L$ and $G + j\omega C$ are *not* close to 90°. The two techniques are complementary in that the method of Section 5.3 can be used only at frequencies *above* some minimum value for any particular transmission line, while the method of Section 5.4 lacks precision unless the frequency is *below* a value that is specific for each line. The latter method is particularly advantageous for making a series of calculations of α, β, R_0 and X_0 at small frequency intervals in the range of low frequencies used for transmission of teletype signals, control data, and voice frequency telephony.

5.3. The "high-frequency" solutions.

The conditions that produce phase angles near 90° for the terms $R + j\omega L$ and $G + j\omega C$ are $\omega L \gg R$ and $\omega C \gg G$, conditions that can always be met at some value of frequency for any transmission line provided R and G do not themselves increase with frequency as rapidly as the first power of the frequency. (It is shown in Chapter 6 that R is independent of frequency at low frequencies and increases as the square root of the frequency at high frequencies, while G, although usually increasing linearly with frequency, is too small at low frequencies to affect the propagation characteristics of any line, and at high frequencies approaches asymptotically a constant fraction of ωC which is very small compared to unity.)

Equations (*5.1*) and (*5.2*) can be rewritten in the form

$$\alpha + j\beta = j\omega\sqrt{LC}\,(1 + R/j\omega L)^{1/2}\,(1 + G/j\omega C)^{1/2} \tag{5.3}$$

$$Z_0 = R_0 + jX_0 = \sqrt{L/C}\,(1 + R/j\omega L)^{1/2}\,(1 + G/j\omega C)^{1/2} \tag{5.4}$$

When the inequalities $\omega L/R \gg 1$ and $\omega C/G \gg 1$ are satisfied, the imaginary terms of each of the expressions in parentheses on the right are very small compared to unity. Expanding these expressions by the binomial theorem, performing the indicated multiplications, and equating real and imaginary terms respectively on the two sides of each equation, independent explicit relations are obtained for α, β, R_0 and X_0 as real-number functions of

R, L, G, C and ω. If terms in the expansion as far as $(R/j\omega L)^2$ and $(G/j\omega C)^2$ are retained, the results are

$$\alpha = (\tfrac{1}{2}R\sqrt{C/L} + \tfrac{1}{2}G\sqrt{L/C})\{1 - \tfrac{1}{2}(R/2\omega L - G/2\omega C)^2\} \tag{5.5}$$

$$\beta = \omega\sqrt{LC}\,\{1 + \tfrac{1}{2}(R/2\omega L - G/2\omega C)^2\} \tag{5.6}$$

$$R_0 = \sqrt{L/C}\,\{1 + \tfrac{1}{2}(R/2\omega L - G/2\omega C)(R/2\omega L + 3G/2\omega C)\} \tag{5.7}$$

$$X_0 = -\sqrt{L/C}\,(R/2\omega L - G/2\omega C)\,\{1 - \tfrac{1}{2}[(R/2\omega L)^2 + 2(R/2\omega L)(G/2\omega C) + 5(G/2\omega C)^2]\} \tag{5.8}$$

One of the most obvious features of these four equations is that each contains the term $(R/2\omega L - G/2\omega C)$. It is apparent that if a transmission line's distributed circuit coefficients could be adjusted to make this factor equal to zero, then α, $v_p = \omega/\beta$, and R_0 would be independent of frequency (to the extent that R, L, G and C are themselves independent of frequency), and X_0 would be zero, making Z_0 real and independent of frequency. The relation among the coefficients required to achieve this result is $R/L = G/C$. It was first noticed by Oliver Heaviside in the 1880's, and a line whose distributed circuit coefficients obey these proportions is known as a "Heaviside distortionless line", since it transmits signals of any time pattern without change of wave shape.

Although the propagation characteristics of the Heaviside distortionless line are in themselves highly desirable, the case is of little or no practical importance. A line designed to meet this specification turns out to be uneconomical in its use of materials, and electrically inefficient. Furthermore, the unavoidable frequency dependence of some of the distributed circuit coefficients means that the Heaviside relation cannot be maintained with high precision over a large signal bandwidth.

The more important conclusion that can be drawn from equations (*5.3*) to (*5.6*) is that if $\omega L/R$ and $\omega C/G$ are *sufficiently large compared to unity*, then *all* the terms involving these ratios can be dropped, and α, ω/β, R_0 and X_0 are again independent of frequency (X_0 being zero), still subject to R, L, G and C themselves being independent of frequency.

The resulting expressions, known as the "high frequency approximations" are

$$\alpha_{\text{hf}} = \tfrac{1}{2}R/Z_0 + \tfrac{1}{2}GZ_0 \tag{5.9}$$

$$v_{p\,\text{hf}} = \omega/\beta_{\text{hf}} = 1/\sqrt{LC} \tag{5.10}$$

$$Z_{0\,\text{hf}} = R_{0\,\text{hf}} = \sqrt{L/C} \tag{5.11}$$

$$X_{0\,\text{hf}} = 0 \tag{5.12}$$

where the result of (*5.11*) has been incorporated in (*5.9*).

The minimum frequency that will qualify as a "high" frequency and justify the use of these simplified equations, depends on the actual values of R, L, G and C for a particular line, and on the accuracy desired in the result. For some open wire transmission lines the approximate formulas are adequately accurate at all frequencies above a few kilohertz. Coaxial lines have lower ratios of L/R and C/G in general, and the approximate formulas may be useful for them only when the frequency is above a few hundred kilohertz. Before using (*5.9*) to (*5.12*) to evaluate α, v_p and Z_0 from the distributed circuit coefficients at frequency ω, the values of the ratios $\omega L/R$ and $\omega C/G$ must always be checked to make sure the approximate formulas will have the needed accuracy.

Inspection of equations (*5.5*) to (*5.8*) indicates that any fixed set of values for the ratios $\omega L/R$ and $\omega C/G$ will not provide the same accuracy in all four of the approximate high frequency formulas. For example, the approximate calculations for both α and β (or v_p) from equations (*5.9*) and (*5.10*) will differ by less than 1% from the values given by equation (*5.1*) if $\omega L/R$ and $\omega C/G$ are both greater than 5. For these same conditions the error in the approximation for R_0 could amount to as much as $1\tfrac{1}{2}$%, and X_0 instead of being zero could be as great as 10% of R_0.

Conservatively, it can be said that the four approximate formulas are all sufficiently accurate for practical purposes if both $\omega L/R$ and $\omega C/G$ are greater than 10, but the calculations may be good enough for many applications if these ratios exceed 4 or 5. Almost invariably $\omega C/G$ is much larger than $\omega L/R$, so that it is usually sufficient to check the value of the latter. If the two ratios are nearly equal, the accuracy of the approximate formulas is considerably increased, because of the term $(R/2\omega L - G/2\omega C)$. The estimates listed above are for the least favorable case, where one of the ratios has the minimum value stated and the other is much larger.

It is shown in Chapter 6 that for most transmission lines of engineering interest, the distributed circuit coefficients L and C may be very precisely independent of frequency over the entire range of frequencies from a few tens or hundreds of kilohertz up to microwave frequencies of several gigahertz. It is then true that for all operating frequencies above the minimum frequency at which the approximate transmission line formulas are valid, the characteristic impedance Z_0 $(= R_0 = \sqrt{L/C})$ and the phase velocity v_p $(= \omega/\beta = 1/\sqrt{LC})$ are quite literally "constants" of the line, for any particular line.

The distributed circuit coefficients R and G, on the other hand, increase steadily and considerably with frequency. The attenuation factor α is a linear function of each and hence cannot be considered a "constant" of a line over any appreciable range of frequencies. It must be evaluated at suitable intervals of frequency, from values of R and G appropriate to each frequency.

Although the reactance component X_0 of the characteristic impedance is also to a first approximation a linear function of R or G (in equation *(5.8)*), it is in addition inversely proportional to frequency. At frequencies for which the approximate high frequency formulas give adequate accuracy for the calculations of α, v_p and R_0, the value of X_0 will correspond to a phase angle for Z_0 not exceeding a few degrees. This angle will diminish as the frequency increases. Consequently no attention need be paid to the behavior of X_0 (or the phase angle of Z_0) as the frequency increases beyond the minimum value for which X_0 is acceptably small.

Example 5.3.

The distributed circuit coefficients for a certain coaxial transmission line at its operating frequency of 2.00 megahertz are $R = 24.5$ ohms/mile, $L = 296$ microhenries/mile, $G = 14.2$ micromhos/mile and $C = 0.111$ microfarads/mile. (*a*) Are the approximate high frequency formulas accurate enough for calculating α, v_p and Z_0 at the stated frequency? (*b*) What are the values of $\alpha, \beta, \lambda, v_p, R_0$ and X_0 at that frequency?

(*a*) At 2.00 megahertz,

$$\omega L/R = (2\pi \times 2.00 \times 10^6)(296 \times 10^{-6})/24.5 = 152$$

$$\omega C/G = (2\pi \times 2.00 \times 10^6)(0.111 \times 10^{-6})/(14.2 \times 10^{-6}) = 98{,}200$$

Hence the approximate formulas will be extremely accurate.

(*b*) It is advisable to calculate Z_0 first, since its value is needed in calculating α. From equations *(5.11)* and *(5.12)*,

$$Z_{0\text{hf}} = \sqrt{L/C} + j0 = \sqrt{(296 \times 10^{-6})/(0.111 \times 10^{-6})} + j0 = 51.6 + j0 \text{ ohms}$$

From equation *(5.9)*,

$$\alpha_{\text{hf}} = \tfrac{1}{2}R/Z_0 + \tfrac{1}{2}GZ_0 = 24.5/(2 \times 51.6) + \tfrac{1}{2}(14.2 \times 10^{-6} \times 51.6)$$

$$= 0.237 + 0.00037 = 0.237 \text{ nepers/mile}$$

At this moderately low frequency the losses caused by G are typically very small compared with the losses caused by R. However, when G represents lossiness in dielectric insulation or supports, as is usually the case, it increases with frequency more rapidly than R does, and for this same transmission line at gigahertz frequencies, G might contribute at least a few percent of the total attenuation.

From equation *(5.10)*,

$$\beta_{\text{hf}} = \omega\sqrt{LC} = 2\pi \times 2.00 \times 10^6 \sqrt{(296 \times 10^{-6})(0.111 \times 10^{-6})} = 72.0 \text{ rad/mile}$$

Then $\lambda = 2\pi/\beta = 2\pi/72.0 = 0.0873$ miles, and

$$v_p = \omega/\beta = (2\pi \times 2.00 \times 10^6)/72.0 = 174{,}500 \text{ miles/sec}$$

The fact that this phase velocity is somewhat below the free space velocity of electromagnetic waves (186,300 miles/sec) must be attributed to the presence of insulating material, whether continuous or periodic, used to support the center conductor of the line.

That the imaginary part of Z_0 is indeed negligible can be checked to a good approximation by using the first terms in equation (*5.8*),

$$\begin{aligned} X_0 &= -\sqrt{L/C}\,(R/2\omega L - G/2\omega C) \\ &= -51.6[1/(2 \times 152) - 1/(2 \times 98{,}300)] \quad \text{from } (a) \text{ above,} \\ &= -0.17 \text{ ohms} \end{aligned}$$

5.4. Solutions in the transition ranges of frequency.

Each of the examples presented so far in this chapter has provided transmission line data in the form of values for the distributed circuit coefficients R, L, G and C *at a single specified frequency,* from which the line's propagation factors α, β, v_p and Z_0 were calculated *at that frequency.* In such problems the ratios $\omega L/R$ and $\omega C/G$ can be evaluated from the coefficients *at the specified frequency* to determine whether or not the simplified high frequency approximate formulas may be used for the calculations. However, it would in general not be legitimate to use the same data in attempting to find the lowest frequency at which, for example, $\omega L/R \geqq 10$, since the values of L and R might not be the same at that lowest frequency as at the specified frequency.

Because of this fact that R, L, G and C each varies with frequency in individual ways and for different reasons, it is never possible to state a single set of values of these quantities for any transmission line, from which to determine the behavior of the line over a range of frequencies from a few hertz to several gigahertz. In spite of this limitation two important statements can be made that provide a basis for further classification of forms of solution of equations (*5.1*) and (*5.2*), page 46. The justification for these statements is developed in Chapter 6.

(1) If a frequency is found for which $\omega L/R \gg 1$ and $\omega C/G \gg 1$, so that the high frequency approximate solutions can be used at that frequency, then the inequalities continue to hold for all higher frequencies. This is equivalent to saying that R and G never increase more rapidly than the first power of the frequency, which is true for all materials used in practical transmission lines.

(2) For any transmission line there is a range of low frequencies starting at zero frequency (d-c) and extending typically to a frequency in the region of several kilohertz or several tens of kilohertz, throughout which the distributed circuit coefficients R, L and C have constant values independent of frequency, and the distributed conductance G is either constant or small enough to have no effect on the propagation behavior of the line. (In the latter case G is considered to be constant and equal to zero.)

The range of low frequencies over which the distributed circuit coefficients of a line remain effectively constant usually includes the frequencies regularly used for transmission of voice, data and telegraph signals. The variations of the propagation factors of a line with frequency in this range are therefore of interest.

Inspection of equations (*5.1*) and (*5.2*) shows that even when R, L, G and C do not themselves vary with frequency, the magnitudes and phase angles of γ and Z_0 can be expected to experience large fluctuations as the frequency rises from zero. The value of Z_0, for example, will change from $\sqrt{R/G} + j0$ at very low frequencies to $\sqrt{L/C} + j0$ at very high frequencies, and these values may differ by a factor of 10 or 100 or more. The changes with frequency will occur most rapidly in the frequency ranges where the transitions occur from

$\omega C/G \ll 1$ to $\omega C/G \gg 1$ and from $\omega L/R \ll 1$ to $\omega L/R \gg 1$, since in these frequency ranges the phase angles of the terms $G + j\omega C$ and $R + j\omega L$ change from approximately zero to approximately 90°, and the magnitudes change from the magnitudes of the real parts to the magnitudes of the imaginary parts. The two transition ranges of frequency are determined by distinctly different aspects of a transmission line. They can be thought of as extending approximately one decade above and one decade below the frequencies at which $\omega C = G$ and $\omega L = R$, respectively. For most lines the former frequency is of the order of one hertz or less, while the latter is larger by a factor of several hundred or several thousand. Hence the transition frequencies for the term $G + j\omega C$ generally lie in a region of no practical interest, while those for the term $R + j\omega L$ may be of considerable importance.

Example 5.4.

The distributed circuit coefficients R, L and C of a 19 gauge paper-insulated twisted-pair transmission line in a telephone cable are constant over the frequency range from d-c to 30 kilohertz, having the values $R = 86$ ohms/mile, $L = 1.00$ millihenries/mile, $C = 0.062$ microfarads/mile. The distributed conductance G has a d-c value of 0.010 micromhos/mile, a value at 1 kilohertz of approximately 1.0 micromhos/mile, and is directly proportional to frequency for frequencies above about 20 hertz. (*a*) At what frequency is $\omega L = R$? (*b*) At what frequency is $\omega C = G$? (*c*) At the frequency for which $\omega L = R$, what are the values of α, β, v_p, R_0 and X_0? (*d*) At the frequency for which $\omega C = G$, what are the values of the same five quantities? (*e*) What estimate can be made of the lowest frequency above which the high frequency approximate solutions for α, β, v_p and Z_0 will be reasonably accurate? (*f*) What values are given for α, β, v_p and Z_0 by the high frequency approximate formulas at the frequency found in (*e*), if R, L and C are assumed to have their constant low frequency values at that frequency, and G is assumed determined by the law of variation stated for it?

(*a*) The frequency f at which $\omega L = R$ is given by $2\pi f(1.00 \times 10^{-3}) = 86$ from which $f = 13.7$ kilohertz.

(*b*) At 1 kilohertz, $\omega C = 2\pi \times 1000(0.062 \times 10^{-6}) = 390$ micromhos/mile, and $\omega C/G = 390$. Over the frequency range in which G is directly proportional to frequency, the ratio $\omega C/G$ will be constant at this value of 390. Hence the condition $\omega C = G$ can be reached only at a frequency so low that G has attained its constant d-c value. This will occur at a frequency f given by $2\pi f(0.062 \times 10^{-6}) = 0.010 \times 10^{-6}$ or $f = 0.026$ hertz.

(*c*)
$$\alpha + j\beta = \sqrt{(86 + j86)(13.7 \times 10^{-6} + j0.0053)} = (0.31 + j0.74) \text{ miles}^{-1}$$
$$v_p = \omega/\beta = (2\pi \times 13{,}700)/0.74 = 116{,}000 \text{ miles/sec}$$
$$R_0 + jX_0 = \sqrt{\frac{86 + j86}{13.7 \times 10^{-6} + j0.0053}} = 140 - j58 \text{ ohms}$$

(*d*)
$$\alpha + j\beta = \sqrt{(86 + j0.00016)(0.010 + j0.010) \times 10^{-6}} = (0.00102 + j0.00042) \text{ miles}^{-1}$$
$$v_p = \omega/\beta = (2\pi \times 0.026)/0.00042 = 390 \text{ miles/sec}$$
$$R_0 + jX_0 = \sqrt{\frac{86 + j0.00016}{(0.010 + j0.010) \times 10^{-6}}} = 72{,}000 - j30{,}000 \text{ ohms}$$

(*e*) The high frequency approximate formulas will be usefully accurate at frequencies for which $\omega L/R$ and $\omega C/G$ exceed about 5 or 10. It has been seen that $\omega C/G = 390$ for all frequencies above a few hertz. On the other hand, $\omega L/R = 1$ at 13.7 kilohertz and is directly proportional to frequency in the frequency range where L and R are constant. The minimum frequency at which the approximate high frequency formulas should be used is therefore governed entirely by the variation with frequency of the term $\omega L/R$, and a value of about 100 kilohertz would appear to be a reasonable estimate. More exact information in Chapter 6 about the variation of R and L with frequency for a transmission line using 19 gauge copper wires shows that at a frequency of 100 kilohertz, R will have risen about 35% above its constant low frequency value, and L will have fallen by a few percent. These trends continue at higher frequencies. The estimated minimum frequency for using the high frequency approximate formulas might therefore be increased to about 200 kilohertz.

(*f*) $Z_{0\,\text{hf}} = \sqrt{L/C} + j0 = 127 + j0$ ohms. Using the value of G at 200 kilohertz,
$$\alpha_{\text{hf}} = \tfrac{1}{2}R/Z_0 + \tfrac{1}{2}GZ_0 = 0.34 + 0.01 = 0.35 \text{ nepers/mile}$$
$$v_{p\,\text{hf}} = 1/\sqrt{LC} = 127{,}000 \text{ miles/sec}$$

The simplicity of the calculations in part (*f*) is alluring, but for the data of this particular problem none of the propagation factors determined from the high frequency approximate formulas can be considered to be of useful accuracy. This is because the lowest frequency at which the high frequency approximate formulas should be used (200 kilohertz) is much higher than the highest frequency (30 kilohertz) at which the distributed circuit coefficients R, L and C remain constant at their low frequency values. The disparity between these two frequencies is not a universal feature of transmission line data, and there are practical transmission lines for which the gap between the two disappears.

For the cable pair of the above example, it would of course always be legitimate to use the simplified high frequency approximate formulas to determine α, v_p and Z_0 at any frequency above about 200 kilohertz, provided the values of R, L, G and C employed were the values *at the frequency of the calculation.* It is true that at all frequencies above 200 kilohertz, v_p and Z_0 for this line are no longer affected by the values of R and G and are functions only of the distributed circuit coefficients L and C. However, it must not be concluded that at frequencies above 200 kilohertz the values of v_p and Z_0 become constant and independent of frequency. It is shown in Chapter 6 that for this 19 gauge line, the distributed inductance L reaches a constant high frequency value only at frequencies above about 100 megahertz. Both v_p and Z_0 for this particular transmission line will change by several percent in the frequency range from 200 kilohertz to 100 megahertz, even though the high frequency approximate formulas are valid throughout this range.

The results of parts (*c*) and (*d*) in Example 5.4 provide a typical illustration of the enormous difference between the values of all the propagation factors of a transmission line at the low transition frequency of the term $G + j\omega C$ and the values at the much higher transition frequency of the term $R + j\omega L$. Example 5.5 gives further detail on the variation of the propagation factors of the same transmission line at frequencies between the two transition frequencies. The large variations with frequency of α, v_p, R_0 and X_0 in the range of frequencies used for transmission of voice, teletype and other low frequency signals, means that such signals would suffer serious amplitude distortion and phase or delay distortion when transmitted over considerable lengths of any simple uniform transmission line. In practice these distortions are counteracted by the periodic insertion in the transmission line of lumped-element equalizing networks whose transfer functions are complementary to those of the line sections between networks.

In addition to the polar number and high frequency approximation techniques for solving equations (*5.1*) and (*5.2*), a third form of solution is available which is particularly useful when it is desired to trace the variations with frequency of the attenuation factor, phase velocity and characteristic impedance through the range of low frequencies in which the distributed circuit coefficients R, L and C are effectively constant, and the distributed conductance G is small enough to be neglected. The method is well suited to computer programming.

Equation (*5.1*) is a complex number equation. It can therefore be written as two independent equations, one relating the real parts on the two sides, the other relating the imaginary parts. These two independent equations can then be solved simultaneously to obtain explicit expressions for α and β as functions of R, L, G, C and ω. The same procedure applied to (*5.2*) yields explicit expressions for R_0 and X_0 as functions of the same quantities. It turns out that the same terms occur in all four of the resulting expressions, with permutations of signs.

Squaring both sides of (*5.1*),

$$\alpha^2 - \beta^2 + 2j\alpha\beta = RG - \omega^2 LC + j(\omega CR + \omega LG)$$

From the real terms,
$$\alpha^2 - \beta^2 = RG - \omega^2 LC \qquad (5.13)$$

From the imaginary terms,
$$2\alpha\beta = \omega CR + \omega LG \qquad (5.14)$$

Eliminating first β, then α, from (*5.13*) and (*5.14*),

$$\alpha = \{\tfrac{1}{2}[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} - \omega^2LC + RG]\}^{1/2} \tag{5.15}$$

$$\beta = \{\tfrac{1}{2}[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} + \omega^2LC - RG]\}^{1/2} \tag{5.16}$$

Similar operations on (*5.2*) result in

$$R_0 = \frac{1}{\sqrt{G^2+\omega^2C^2}}\{\tfrac{1}{2}[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} + \omega^2LC + RG]\}^{1/2} \tag{5.17}$$

$$X_0 = \frac{\pm 1}{\sqrt{G^2+\omega^2C^2}}\{\tfrac{1}{2}[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} - \omega^2LC - RG]\}^{1/2} \tag{5.18}$$

X_0 can have either sign, determined by the relative values of R, L, G, C and ω, but in the development of equation (*5.18*) information about the sign of X_0 has been lost in the various squaring and square root operations. It must be determined by an additional direct reference to equation (*5.2*), which shows that X_0 is positive if the phase angle of the numerator is greater than the phase angle of the denominator, and conversely. Thus if $\omega L/R > \omega C/G$, X_0 is positive. For most practical transmission lines $\omega C/G > \omega L/R$, and X_0 is negative.

Equations (*5.15*) to (*5.18*) have been derived from (*5.1*) and (*5.2*) without making any approximations. They involve only elementary operations in real number arithmetic. It might seem logical therefore that they would constitute the preferred method of calculating α, β, R_0 and X_0 for *any* transmission line at *any* frequency, from the distributed circuit coefficients R, L, G and C at that frequency. There are two principal reasons why this is not so.

In the first place, for calculations at a single frequency equations (*5.15*) to (*5.18*) lack convenience. They are difficult to remember, and the arithmetic involved provides numerous opportunities for trivial errors. The fundamental equations (*5.1*) and (*5.2*), on the other hand, are more easily remembered because they are so basic, and the expressions on the right contain their own instructions as to the polar number operations to be performed. Calculations using (*5.1*) and (*5.2*) never lose sight of the magnitudes and phase angles of the terms $R + j\omega L$ and $G + j\omega C$, and it is a simple matter to check the order of magnitude of the final results from the original data.

Secondly, and of greater importance, at frequencies for which $\omega L/R \gg 1$ and $\omega C/G \gg 1$, equation (*5.15*) gives an indeterminate answer for α. This is because the term under the square root sign and the term ω^2LC become very nearly equal. The significant figures for their difference disappear, although their difference is in general greater than the residual term RG. The same indeterminacy occurs in (*5.18*) under the same conditions, but this is of no consequence since it is merely saying that X_0 is small.

It is clear therefore that equations (*5.15*) to (*5.18*) are not the universal equations they might appear to be, but are in fact exactly complementary to the high frequency approximate equations (*5.9*) to (*5.12*). The latter provide the simplest solutions to (*5.1*) and (*5.2*) when the ratios $\omega L/R$ and $\omega C/G$ are greater than 5 or 10, while the former provide workable solutions when these ratios are less than 5 or 10. The polar number technique, employing small angle approximations when necessary, can be used for any values of the ratios.

Example 5.5.

For the 19 gauge paper-insulated twisted-pair telephone cable transmission line whose distributed circuit coefficients are given in Example 5.4, find the values of α, β, v_p, R_0 and X_0 at frequencies of 100, 300, 1000, 3000, 10,000 and 30,000 hertz.

Table 5.1 below shows how the various separate terms of equations (*5.15*) to (*5.18*) are conveniently evaluated in sequence at the stated frequencies. α, β, v_p, R_0 and X_0 are then easily found from the appropriate combinations of these terms. It will be noted that if $G = 0$ were used everywhere, instead of the values for G given by the original data of the problem, there would be no perceptible change in the calculated results. This is likely to be true for most transmission lines at low frequencies.

Table 5.1

Frequency hertz	R^2 (ohms/mile)^2	G^2 (mhos/mile)^2	ω^2L^2 (ohms/mile)^2	ω^2C^2 (mhos/mile)^2
				D*
100	7400	1.00×10^{-14}	0.395	1.52×10^{-9}
300	7400	9.0×10^{-14}	3.55	1.37×10^{-8}
1000	7400	1.00×10^{-12}	39.5	1.52×10^{-7}
3000	7400	9.0×10^{-12}	355	1.37×10^{-6}
10,000	7400	1.00×10^{-10}	3950	1.52×10^{-5}
30,000	7400	9.0×10^{-10}	35,500	1.37×10^{-4}

Frequency hertz	$R^2+\omega^2L^2$	ω^2LC	RG	$\sqrt{DE}$
	E	A	B	F
100	7400	2.45×10^{-5}	8.6×10^{-6}	3.35×10^{-3}
300	7400	2.20×10^{-4}	2.6×10^{-5}	1.01×10^{-2}
1000	7440	2.45×10^{-3}	8.6×10^{-5}	3.36×10^{-2}
3000	7760	2.20×10^{-2}	2.6×10^{-4}	1.03×10^{-1}
10,000	11,350	2.45×10^{-1}	8.6×10^{-4}	4.15×10^{-1}
30,000	42,900	2.20	2.6×10^{-3}	2.42

Frequency hertz	$\alpha=\sqrt{\frac{1}{2}(F-A+B)}$ nepers/mile	$\beta=\sqrt{\frac{1}{2}(F+A-B)}$ rad/mile	$v_p=\omega/\beta$ miles/sec	$R_0=\sqrt{\frac{1}{2}(F+A+B)/D}$ ohms	$X_0=\sqrt{\frac{1}{2}(F-A-B)/D}$ ohms
100	0.040	0.041	15,300	1060	−1030
300	0.071	0.072	26,200	616	−598
1000	0.125	0.134	46,900	345	−319
3000	0.20	0.25	75,400	214	−172
10,000	0.29	0.58	108,000	147	−75
30,000	0.33	1.52	124,000	130	−28

*G is so small that $G^2+\omega^2C^2=\omega^2C^2$.

Although the previously mentioned indeterminacy in the calculations of α and X_0 is not numerically demonstrated at the frequencies covered by the table, it is evident that the percentage difference between the terms F and A is diminishing rapidly with increasing frequency, while at the same time the ratio of either of these terms to the term B is increasing.

Example 5.6.

At a frequency of 1000 hertz the distributed circuit coefficients of a transmission line are $R=10.1$ ohms/mile, $L=0.0040$ henries/mile, $G=0.30$ micromhos/mile and $C=0.0080$ microfarads/mile. Should values for α, β, v_p, R_0 and X_0 at that frequency be calculated from equations *(5.1)* and *(5.2)* using polar numbers, from the high frequency approximate equations *(5.9)* to *(5.12)*, or from the transition frequency equations *(5.15)* to *(5.18)*?

At the frequency of 1000 hertz, $\omega L/R=2.5$ and $\omega C/G=168$. The low value of the former means that the high frequency approximate formulas would not give usefully accurate results. If the calculations are to be made at only the one frequency, they will probably be made most quickly and with fewest opportunities for error by the polar number method using equations *(5.1)* and *(5.2)*.

The calculations of Table 5.1 were based on the assumption that for the 19 gauge copper conductors of the transmission line involved, the distributed circuit coefficients R, L and C were independent of frequency over the frequency range of the table, and the distributed conductance G was too small to affect the results at any frequency. The criteria developed in Chapter 6 show that the constancy of C is reliably established by the properties of insulating media, and that for the relatively small wires of this cable pair, the phenomenon of "skin effect" just begins to affect the values of R and L at the top frequencies of the table. Up to 10 kilohertz the deviation of R and L from the constant low frequency values is less than $\frac{1}{2}\%$. At 30 kilohertz, L diminishes by about 1% and R increases by 4%.

For a transmission line with larger conductors the distributed circuit coefficients R and L will vary much more over the same frequency range. Some relevant data and results are shown in Table 5.2 for an open wire transmission line consisting of 165 mil diameter copper-steel wires spaced 12 in. between centers, the ratio of copper sheath cross-sectional area to steel core cross-sectional area producing a resultant d-c conductivity 40% of the value for solid copper conductors of the same outer diameter. The values given for the distributed circuit coefficients were determined by experimental measurements at each frequency.

Table 5.2

Frequency kilohertz	R ohms/mile	L mH/mile	G*	C*	α nepers/mile	v_p miles/sec	R_0 ohms	X_0 ohms
0.3	9.9	3.38			0.0071	149,000	731	−386
1.0	10.3	3.32			0.0085	174,000	617	−140
3.0	10.8	3.29			0.0092	178,500	597	−51
10.0	11.4	3.25			0.0098	182,000	591	−16
30.0	12.6	3.24			0.0108	182,300	590	−5
140.0	24.7	3.23			0.0211	182,500	589	−2

*The value of C is constant over the frequency range at 0.0093 microfarads/mile. For an open wire line the value of G is largely governed by surface leakage at the supporting insulators. It shows little variation with frequency, but is very much dependent on ambient atmospheric conditions. Measurements on this line indicate a nominal average value for G of 0.50 micromhos/mile at all frequencies, too small to affect any of the results.

It is strikingly apparent that over the frequency range 300 hertz to 30 kilohertz, the percentage variations in the propagation factors α, v_p, R_0 and X_0 are very much smaller for the transmission line of Table 5.2 than for the line of Table 5.1. This is due to the fact that the upper transition frequency defined by $\omega L/R = 1$ occurs at about 500 hertz for the high-inductance low-resistance line of Table 5.2, and at 13.7 kilohertz for the relatively low-inductance high-resistance line of Table 5.1. For the line of Table 5.2, in fact, the high frequency approximate formulas are quite accurate above a frequency of about 5 kilohertz, and since C is constant and L is very nearly constant for frequencies above that value, both v_p and R_0 show little change at higher frequencies, and X_0 is small enough to be negligible.

For the composite conductors of the line of Table 5.2, the distributed resistance R does not vary according to any simple law in the frequency range covered by the table. At frequencies above about 50 kilohertz both R and α should increase directly as the square root of the frequency.

5.5. Summary concerning the solutions of equations (*5.1*) and (*5.2*).

The criteria for choosing among the various solution procedures for equations (*5.1*) and (*5.2*) that have been discussed in Sections 5.2, 5.3 and 5.4, and the principal reliable generalities about the variations of R, L, G and C with frequency that affect the solutions of those equations, can be summarized as follows:

(1) The distributed capacitance C of a transmission line is the least variable with frequency of all the distributed circuit coefficients. If most of the medium surrounding the conductors of a line is air, the value of C may remain constant within 1% from zero frequency to gigahertz frequencies. If there is a continuous solid dielectric associated with the conductors, as in a plastic filled coaxial line, the value of C may in some cases change by as much as a few percent over several decades of frequency.

(2) For any transmission line there is a range of low frequencies, typically extending from d-c to several kilohertz, in which the distributed circuit coefficients R and L do not vary with frequency.

(3) It is shown in Chapter 6 that when the frequency increases above the range defined by (2), there is a frequency interval of three or four decades in which the variation of R and L with frequency obeys a very complicated law. For frequencies above this interval, R increases directly as the square root of the frequency, and L becomes independent of frequency.

(4) The distributed conductance G is directly proportional to the frequency, within a few percent or less over many decades of frequency, for all transmission lines in which G is due to molecular lossiness of dielectric material surrounding or supporting the conductors. For such lines G is usually too small to affect the propagation factors of the line at frequencies below the megahertz or gigahertz region.

If a line is exposed to outdoor weather, or to any other form of contaminating environment, the distributed conductance G is likely to be highly variable with time and unpredictable in value. Its variations must be measured experimentally before its effect on line behavior can be understood.

(5) At frequencies for which $\omega L/R \gg 1$ and $\omega C/G \gg 1$ for any line, the simplified high frequency approximate formulas (*5.9*) to (*5.12*) should always be used in calculating α, β, v_p, and Z_0 ($\doteq R_0 + j0$). Under these conditions v_p and Z_0 are independent of R and G, and Z_0 has a sufficiently small phase angle to be considered real.

(6) The lowest frequency defined in (5), at which the high frequency approximate formulas may be used, is not the same as the lowest frequency defined in (3), at which L becomes independent of frequency. At frequencies above *both* of these lowest frequencies, v_p and Z_0 ($\doteq R_0 + j0$) become independent of frequency, and the term $\frac{1}{2}R/Z_0$ in equation (*5.9*) for α becomes proportional to the square root of the frequency. According to (4), the term $\frac{1}{2}GZ_0$ for any unexposed transmission line at high frequencies is directly proportional to frequency. At low megahertz frequencies the latter term is usually very small compared to $R/2Z_0$, but since it increases more rapidly with frequency, the two terms may become of comparable magnitude at high megahertz or gigahertz frequencies.

(7) At the low frequencies of (2), the calculations of α, β, v_p, R_0 and X_0 may be made by either the polar number method using equations (*5.1*) and (*5.2*) directly, or by the real number equations (*5.15*) to (*5.18*). For a calculation at a single frequency the polar number procedure is likely to be preferred. For calculations at several frequencies in the low frequency range for the same line, (*5.15*) to (*5.18*) may be more efficient, and can be programmed for computer use.

(8) At frequencies between the highest frequency defined in (2) and the lowest frequency defined in (5) no helpful simplifications or quantitative generalizations are available. Calculations may be made by either the polar number method or the real number equations

(*5.15*) to (*5.18*). The values of R, L and G may all be undergoing considerable variation with frequency, so that separate values for them must be used at each separate frequency. Often G will be small enough to be neglected.

(9) If the conductors of a line are iron, or contain ferromagnetic material of any kind, or if the interconductor space of a line contains any nonlinear materials such as ferrites or ferroelectrics, the variations with frequency of the line's distributed circuit coefficients will not agree with the generalizations that have been listed, and the coefficients may also vary with signal amplitude. Direct measurement of α, v_p and Z_0 may be easier than any attempt at analysis through R, L, G and C.

5.6. Solutions of the inverse form $R, L, G, C = f(\alpha, \beta, R_0, X_0, \omega)$.

Further manipulations of (*5.1*) and (*5.2*) produce several additional relations between the distributed circuit coefficients of a transmission line and its propagation factors, some of which have computational usefulness in certain situations.

Multiplication of corresponding sides of (*5.1*) and (*5.2*) gives

$$(\alpha + j\beta)(R_0 + jX_0) = R + j\omega L \tag{5.19}$$

From the real terms of (*5.19*),

$$R = \alpha R_0 - \beta X_0 \tag{5.20}$$

From the imaginary terms of (*5.19*),

$$L = \frac{\alpha X_0 + \beta R_0}{\omega} \tag{5.21}$$

Dividing corresponding sides of (*5.1*) by those of (*5.2*) gives

$$\frac{\alpha + j\beta}{R_0 + jX_0} = G + j\omega C \tag{5.22}$$

From the real terms of (*5.22*),

$$G = \frac{\alpha R_0 + \beta X_0}{R_0^2 + X_0^2} \tag{5.23}$$

From the imaginary terms of (*5.22*),

$$C = \frac{-\alpha X_0 + \beta R_0}{\omega(R_0^2 + X_0^2)} \tag{5.24}$$

Equations (*5.20*), (*5.21*), (*5.23*) and (*5.24*) *appear to be* general design equations for determining the distributed circuit coefficients that a transmission line would have to possess to give a set of desired operating characteristics specified by values of α, β, R_0 and X_0 at frequency ω. The situation is, however, not as simple and straightforward as that statement suggests.

It has been seen in the preceding sections of this chapter that at frequencies of a few megahertz, the characteristic impedance of a typical transmission line can be very nearly a pure resistance. It has also been pointed out that for the dielectric materials normally used in high frequency transmission lines the value of G is too small to have any significant effect on the propagation factors of a line at low megahertz frequencies. In proposing to design a line for use at such frequencies, it might therefore seem entirely reasonable to adopt the specifications $X_0 = 0$ and $G = 0$, along with specific values for α, β and R_0.

Substituting these postulates into (*5.23*) gives the result $\alpha = 0$, which is incorrect because the distributed line resistance R has not been required to be zero. Substituting these postulates into (*5.20*) gives a value for R which is only half as great as the value given by the high frequency approximate equation (*5.9*) for the same case.

The explanation of these contradictions lies in the fact that it is not mathematically possible as a consequence of (*5.1*) and (*5.2*) to have X_0 identically zero when $G = 0$ and α is finite. X_0 can be exactly zero for a line with finite attenuation only if the relation $R/L = G/C$ of the Heaviside "distortionless line" holds, as can be seen from (*5.8*), but this means that G cannot be zero. It is possible for X_0 to be *small enough to be unimportant* (relative to R_0) with $G = 0$, but even such a small value of X_0 can be significant in equations (*5.20*) and (*5.23*), since for the high frequency line under discussion β is always much larger than α.

The same difficulty does not arise in using equations (*5.21*) and (*5.24*), since the term αX_0 is small compared with βR_0 for the reasons already given, and can be dropped. Then using $v_p = \omega/\beta$, (*5.21*) becomes

$$L = Z_0/v_p \tag{5.25}$$

and (*5.24*) becomes (assuming $X_0 \ll R_0$)

$$C = 1/(Z_0 v_p) \tag{5.26}$$

Equations (*5.25*) and (*5.26*) could have been derived directly from (*5.10*) and (*5.11*).

Further insight into the intriguing but somewhat academic problem associated with equations (*5.20*) and (*5.23*) can be gained through another derivation. Let $R_0^2 + X_0^2 = |Z_0|^2$. Then (*5.23*) becomes

$$G\,|Z_0|^2 = \alpha R_0 + \beta X_0 \tag{5.27}$$

Eliminating βX_0 between (*5.20*) and (*5.25*),

$$2\alpha R_0 = G\,|Z_0|^2 + R \tag{5.28}$$

Eliminating R_0 between (*5.20*) and (*5.25*),

$$2\beta X_0 = G\,|Z_0|^2 - R \tag{5.29}$$

Dividing (*5.29*) by (*5.28*),

$$\frac{X_0}{R_0} = \frac{\alpha}{\beta}\left(\frac{G\,|Z_0|^2 - R}{G\,|Z_0|^2 + R}\right) \tag{5.30}$$

From (*5.28*),

$$\alpha = \frac{R}{2R_0} + \frac{G\,|Z_0|^2}{2R_0} \tag{5.31}$$

Equation (*5.31*) involves no approximations, and applies to all transmission lines at all frequencies. It states that the relative contributions of a line's distributed resistance R and distributed conductance G to its attenuation factor α are proportional to R and $G|Z_0|^2$ respectively. When $|Z_0| \doteq R_0$, equation (*5.31*) becomes the high frequency approximate equation (*5.9*).

If the design postulates $G = 0$ and $X_0 = 0$ for a high frequency line are substituted into (*5.31*), there results

$$R = 2\alpha R_0 \tag{5.32}$$

which could also have been obtained from (*5.9*).

Equations (*5.25*), (*5.26*), and (*5.32*) are simple and useful design equations for determining the values of the distributed circuit coefficients R, L and C that a transmission line must have to attain specified values of α, v_p and Z_0, provided the inequalities $\omega L/R \gg 1$ and $\omega C/G \gg 1$ hold. The second of these is ensured by the postulate $G = 0$. There are no correspondingly useful and simple design formulas for the general case at lower frequencies.

Equation (*5.30*) shows in clear and explicit form that the phase angle of the characteristic impedance of a transmission line, given by $\theta_0 = \tan^{-1}(X_0/R_0)$ can vary between $+\alpha/\beta$ (when $R = 0$ and the line losses are all due to G) and $-\alpha/\beta$ (when $G = 0$ and the line losses are all due to R). Reference to (*5.31*) shows that the phase angle of a line's characteristic impedance will always be negative if the line's distributed resistance contributes more than its distributed conductance to the total attenuation.

Physically the ratio α/β represents $1/2\pi$ of the attenuation of one wavelength of line in nepers. For practical lines at radio frequencies this is always a number much smaller than unity, but the results of Table 5.1 show that at low frequencies α/β can be very close to unity for a line with small conductors of relatively high resistance. Equation (*5.1*) shows that $|\alpha/\beta| \leqq 1$ always.

5.7. Concluding remarks on design of high frequency lines.

It has been seen that the specification $X_0 = 0$ can be identically met only by making the line losses due to G equal to the line losses due to R. But the insulating material in a line that gives rise to G plays only a mechanical support role. The conductors are the essential electrical elements. If the conductors of a line are designed to have the maximum value of R that will limit the attenuation to a specified value, from equation (*5.32*), it is always possible to provide the mechanical support economically with material for which the resulting value of G will contribute much less to the attenuation α than R does, at least at frequencies up to the high megahertz or gigahertz region. To achieve equality of losses from R and G while retaining the same total attenuation, the designer would have to make the conductors larger (to reduce R) and the insulating material more lossy. Most of the cost and weight of a line is for the conductors, and this procedure would be economically indefensible. The designer therefore chooses a line whose losses are due predominantly to conductor resistance, and for which the phase angle of the characteristic impedance is consequently not identically zero. At frequencies from a few tens of kilohertz to several gigahertz the phase angle is always small enough to have no adverse effects.

5.8. Inductive loading.

Inductive loading, as mentioned in Chapter 1, is the technique of inserting identical lumped magnetic-core inductance coils in series with the conductors of a transmission line at equal intervals along the line.

The advantages of inductive loading of a transmission line can be appreciated in terms of the solutions developed in this chapter in the light of two fundamental facts. First, at all frequencies for which there are at least a few loading coils per wavelength on a transmission line, the resistance and inductance of the lumped coils have exactly the same effect on the line's transmission properties as if they were uniformly distributed along the length of the line. Secondly, it is possible in a lumped inductance coil with a magnetic core to achieve a much higher ratio of inductance to resistance than exists for any ordinary transmission lines. A loaded line therefore has a considerably higher ratio of L/R and hence of $\omega L/R$ at all frequencies.

An obvious consequence of a higher value of $\omega L/R$ for a line is that the high frequency approximate equations become applicable down to lower frequencies. Practical loading pushes the lowest frequency for which $\omega L/R \gg 1$ and $\omega C/G \gg 1$ well down into the voice frequency range for all telephone lines, and even below the lowest voice frequency, for low resistance lines. The improvement resulting from this is not just the trivial one of simplified calculations, but lies in the fact that since R, L and C are all nearly constant for telephone lines over the voice frequency range, and G is too small to have significant effects, the values of α, v_p and Z_0 given by equations (*5.9*) to (*5.11*) approach the very desirable condition of being independent of frequency on a loaded line, while they might vary by a factor of 2 or 3 or more over the same frequency range of 300 to 3500 hertz, on the same line without loading.

When the high frequency approximate equations are valid, it follows from (*5.7*) that loading increases the characteristic impedance of a line. For a given level of transmitted power this means the voltage is increased and the current decreased relative to their respective values on the same line without loading. The power losses in the distributed resistance R are therefore reduced, and those in the distributed conductance G are increased, as can also be seen directly from (*5.15*). The losses in G are still only a very small fraction of the total losses, so the net result is a substantial decrease in the total attenuation factor of the line, particularly at the higher voice frequencies.

Lumped inductance-coil loading of transmission lines has two serious disadvantages. The first follows from equation (*5.10*), which indicates that the phase velocity on a line is reduced when the effective distributed inductance is increased, for a fixed value of distributed capacitance. Two-way telephone conversation on trans-continental voice-frequency telephone circuits two or three thousand miles long would be adversely affected by the extra signal delay introduced by inductive loading.

The second disadvantage of lumped inductance-coil loading is a consequence of the fact that a line loaded in this way is no longer a uniform distributed system. Instead it must be regarded as a sequence of finite line sections, each consisting of a loading coil and the section of line between two consecutive coils. Analysis of this composite system reveals that it has the properties of a low-pass filter, characterized by a "cut-off" frequency that is inversely proportional to the square root of the product of the total series inductance per line section and the total shunt capacitance per line section. If the circuit components themselves are independent of frequency and signal strength, the transmission properties of the loaded line regarded as a sequence of filter sections are practically independent of frequency from zero frequency to within a few percent of the cut-off frequency, and there is no transmission at frequencies above the cut-off frequency.

For any specified amount of loading (i.e. of added coil-inductance *per unit length of line*), the cut-off frequency can be increased without limit by using more coils at smaller separations, with less inductance per coil. However, it turns out to be prohibitively expensive to achieve useful amounts of loading with cut-off frequencies higher than a few kilohertz, and there has been very little use of loading on commercial transmission line circuits operating at frequencies above the voice-frequency range.

Coaxial lines have been continuously loaded by winding magnetic tape around the center conductor. This avoids the cut-off frequency limitation, but the non-linearity of magnetic materials introduces cross-talk when such a circuit is multiplexed.

Although loading is nowadays not able to make any useful contribution to commercial telephone practice, there are specialized applications such as small-wire light weight telephone lines for emergency military field use where it may still occasionally play a helpful role.

Solved Problems

5.1. For the transmission line of Example 5.1, page 47, find α, β, R_0 and X_0 at 1000 hertz using the real number equations (*5.15*) to (*5.18*), pages 53 and 54.

From (*5.15*),

$$\alpha = \{\tfrac{1}{2}[\{6.74^2 + (2\pi \times 1000 \times 0.00352)^2\}^{1/2} \times \{(0.29 \times 10^{-6})^2 + (2\pi \times 1000 \times 0.0087 \times 10^{-6})^2\}^{1/2}$$
$$- (2\pi \times 1000)^2 \times 0.00352 \times (0.0087 \times 10^{-6}) + 6.74 \times (0.29 \times 10^{-6})]\}^{1/2}$$
$$= \{\tfrac{1}{2}[0.00127 - 0.00121 + 0.0000019]\}^{1/2} = (0.00003)^{1/2} \doteq 0.005 \text{ nepers/mile}$$

It will be noted that although the significant figures of the original data imply a precision of about 1% or better, the meaningful precision in this calculated value for α is only about 10%. A check shows that $\omega L/R \doteq 3$ and $\omega C/G \doteq 200$. Under these conditions equations (*5.15*) and (*5.18*) begin to show the indeterminacy in the calculations of α and X_0 which becomes total indeterminacy when both ratios are greater than about 10. The indeterminacy does not occur for β (hence v_p) and R_0.

Within the estimated indeterminacy the above value for α agrees with the more precise value found in Example 5.1.

Substituting the values calculated above for the appropriate terms into (*5.16*),

$$\beta = \{\tfrac{1}{2}[0.00127 + 0.00121 - 0.0000019]\}^{1/2} = (0.00124)^{1/2} = 0.0352 \text{ rad/miles}$$

which agrees with the result determined by the polar number method.

With better than 0.1% precision, for the data of this problem $\sqrt{G^2 + \omega^2C^2} = \omega C = 54.6 \times 10^{-6}$, and $1/\sqrt{G^2 + \omega^2C^2} = 1830$. Then from (*5.17*),

$$R_0 = 1830\{\tfrac{1}{2}[0.00127 + 0.00121 + 0.0000019]\}^{1/2} = 644 \text{ ohms}$$

which also agrees with the earlier result.

From (*5.18*),

$$X_0 = -1830\{\tfrac{1}{2}[0.00127 - 0.00121 - 0.0000019]\}^{1/2} \doteq -1830 \times 0.005 \doteq -92 \text{ ohms}$$

The deviation of this value from the correct value is much smaller than the inherent indeterminacy in the calculation. The negative sign has been determined independently, from the fact that $\omega C/G > \omega L/R$.

5.2. For the transmission line of Example 5.1, page 47, find α, v_p and Z_0 using the high frequency approximate formulas (*5.9*) to (*5.12*). Compare the deviations of the results from those obtained in Example 5.1 with the errors to be expected according to equations (*5.5*), (*5.6*) and (*5.7*).

From (*5.11*), $Z_{0\,hf} = R_0 = \sqrt{L/C} = \sqrt{0.00352/(0.0087 \times 10^{-6})} = 636$ ohms. Since $R/2\omega L = 0.153$ and $G/2\omega C = 0.00266$, R_0 according to equation (*5.7*) should be greater than $\sqrt{L/C}$ by a factor

$$1 + \tfrac{1}{2}(0.153 - 0.00266)(0.153 + 0.00898) \doteq 1.011$$

The correct value of R_0 should then be $636 \times 1.011 = 643$ ohms, as found in Example 5.1.

From (*5.9*), using $Z_{0\,hf}$ from (*5.11*),

$$\alpha_{hf} = 6.74/1272 + (0.29 \times 10^{-6}) \times 318 = 0.00530 + 0.000092 = 0.00539 \text{ nepers/mile}$$

According to equation (*5.5*) the correct value of α should differ from this high frequency approximate value by the factor $1 - \tfrac{1}{2}(0.153 - 0.00266)^2 = 0.989$. Hence the correct value should be $0.00539 \times 0.989 = 0.00534$ nepers/mile, confirming the result obtained by polar numbers.

From (*5.10*), $v_{p\,hf} = 1/\sqrt{LC} = 180{,}700$ miles/sec. This corresponds to the high frequency approximation $\beta_{hf} = \omega\sqrt{LC} = 0.0348$ rad/mile. According to equation (*5.6*) the correct value of β should be greater than this by the factor $1 + \tfrac{1}{8}(0.153 - 0.00266)^2 = 1.011$, or $\beta = 0.0348 \times 1.011 = 0.0352$ rad/mile, as found previously, and $v_p = 178{,}500$ miles/sec.

Finally, from (*5.8*), X_0 should be $-636(0.150)(0.988) = -94$ ohms, as previously obtained.

Although in this problem the lower of the two ratios $\omega L/R$ and $\omega C/G$ is only 3.3, the error in the values of α, v_p and R_0 incurred by using the high frequency approximate equations is just slightly over 1%. The true phase angle for Z_0, however, is about $-8°$, instead of the value zero given by the high frequency approximation.

5.3. For the transmission line of Example 5.3, page 50, at all frequencies above 1 megahertz, the distributed resistance R is directly proportional to the square root of the frequency and the distributed conductance G is directly proportional to the frequency. The distributed inductance L and distributed capacitance C are constant over the same frequency range. Determine the values of the attenuation factor α, the phase velocity v_p and the characteristic impedance Z_0 at frequencies of 1, 5, 10, 50 and 100 megahertz.

From the values calculated in Example 5.3 for $\omega L/R$ and $\omega C/G$ at 2 megahertz, it is readily seen that these ratios will always be greater than 100 for all of the frequencies listed. Hence the high frequency approximate equations (5.9) to (5.12) will be very accurate.

Since (5.10), (5.11) and (5.12) do not involve R or G, the characteristic impedance Z_0 will be constant at the 2 megahertz value of $51.6 + j0$ ohms, and the phase velocity v_p will be constant at the 2 megahertz value of 174,500 miles/sec, for all frequencies from 1 megahertz to 100 megahertz (and higher).

At 2 megahertz the attenuation factor α has a component 0.237 nepers/mile caused by distributed resistance R, and a component 0.00037 nepers/mile caused by distributed conductance G. At any frequency f megahertz, the former component will change to $0.237\sqrt{f/2}$ and the latter component will change to $0.00037(f/2)$, from equation (5.9) and the stated laws of variation of R and G with frequency. Then

$$\begin{aligned}
&\text{At 1 megahertz,} && \alpha = 0.237\sqrt{1/2} + 0.00037(1/2) = 0.168 \text{ nepers/mile}\\
&\text{At 5 megahertz,} && \alpha = 0.237\sqrt{5/2} + 0.00037(5/2) = 0.375 \text{ nepers/mile}\\
&\text{At 10 megahertz,} && \alpha = 0.237\sqrt{10/2} + 0.00037(10/2) = 0.530 \text{ nepers/mile}\\
&\text{At 50 megahertz,} && \alpha = 0.237\sqrt{50/2} + 0.00037(50/2) = 1.19 \text{ nepers/mile}\\
&\text{At 100 megahertz,} && \alpha = 0.237\sqrt{100/2} + 0.00037(100/2) = 1.69 \text{ nepers/mile}
\end{aligned}$$

At 1 megahertz the contribution of G to the attenuation factor is not perceptible, but at 100 megahertz it is slightly greater than 1%.

5.4. A high frequency transmission line operating at a frequency of 10 megahertz has an attenuation factor $\alpha = 0.0022$ db/m. Its phase velocity is 85%. What is the phase angle of its characteristic impedance if the losses due to distributed conductance G are negligible?

The phase angle of the characteristic impedance of a transmission line in terms of propagation factors and distributed resistance and conductance is obtainable from equation (5.30). If $G = 0$, $\theta_0 = \tan^{-1}(X_0/R_0) = \tan^{-1}(-\alpha/\beta)$. Here α must be in nepers/unit length and β in rad/unit length, the length unit being the same for both.

From the data of the problem, $\alpha = 0.0022/8.686 = 0.000253$ nepers/m, and $\beta = \omega/v_p = (2\pi \times 10^7)/(0.85 \times 3.00 \times 10^8) = 0.246$ rad/m. Then $\theta_0 = \tan^{-1}(-0.000253/0.246) = -0.00103 \text{ rad} \doteq 0.06°$. This very small phase angle is typical for low loss lines at high frequencies.

5.5. At a frequency of 3000 hertz, measurements (using methods described in Chapter 7) on a transmission line whose dielectric is mainly air show the characteristic impedance Z_0 to be $560 - j115$ ohms, and the phase velocity to be 105,000 miles/sec. Determine the attenuation factor of the line and the values of the distributed circuit coefficients R, L and C at the frequency of the measurements.

The specification that the dielectric of the line is mainly air implies that the distributed conductance G is to be taken as zero. From equation (5.30) with $G = 0$, $\alpha/\beta = -X_0/R_0$, where $\beta = \omega/v_p = (2\pi \times 3000)/105{,}000 = 0.180$ rad/mile. Then $\alpha = -0.180(-115/560) = 0.0370$ nepers/mile.

From equation (5.20),

$$R = \alpha R_0 - \beta X_0 = 0.0370 \times 560 - 0.180 \times (-115) = 20.7 + 20.7 = 41.4 \text{ ohms/mile}$$

The equality of the two terms in equation (5.20) is an identity when $G = 0$.

From equation (5.21),

$$L = (\alpha X_0 + \beta R_0)/\omega = [0.0370 \times (-115) + 0.180 \times 560]/(2\pi \times 3000) = 0.00512 \text{ henries/mile}$$

From equation (5.24),

$$\begin{aligned}
C &= (-\alpha X_0 + \beta R_0)/[\omega(R_0^2 + X_0^2)] = (0.0370 \times 115 + 0.180 \times 560)/[(2\pi \times 3000)(560^2 + 115^2)]\\
&= 0.0170 \text{ microfarads/mile}
\end{aligned}$$

5.6. A type of commercial coaxial transmission line widely used for transmission of a few kilowatts of power in the frequency range 1 to 1000 megahertz is known as 50 ohm 7/8″ standard rigid line. The designation means that the outer diameter of the outer conductor of the line is 7/8″, that the characteristic impedance over the frequency range is precisely $50 + j0$ ohms, and that the conductors are smooth nonflexible metal tubes. Additional specifications given by the manufacturer for this type of line are: velocity v_p = 99.8% over the frequency range; attenuation factor α = 0.0425 db/(100 ft) at 1 megahertz, 0.135 db/(100 ft) at 10 megahertz, 0.440 db/(100 ft) at 100 megahertz, and 1.49 db/(100 ft) at 1000 megahertz. The line conductors are of copper, and the center conductor of the line is supported by regularly spaced thin discs of low loss teflon. Determine the distributed circuit coefficients R, L, G and C of the line at frequencies of 1, 10, 100 and 1000 megahertz.

The fact that v_p and Z_0 are independent of frequency over the stated frequency range means that the high frequency approximate equations (*5.9*) to (*5.12*) are valid with high accuracy in that range. Then from equation (*5.25*), $L = Z_0/v_p = 50/(0.998 \times 3.00 \times 10^8) = 0.167$ microhenries/m; and from equation (*5.26*), $C = 1/(Z_0 v_p) = 1/(50 \times 2.99 \times 10^8) = 66.8$ micromicrofarads/m.

It has been noted above, and is proved in Chapter 6, that for a transmission line whose conductors are of nonmagnetic material, the distributed resistance R increases directly as the square root of the frequency above some minimum frequency (below which the rate of increase with frequency is less rapid), and that for a line protected against contaminating environments (such as a coaxial line with a solid metal outer conductor) the distributed conductance G resulting from dielectric losses in the insulating supports is directly proportional to frequency.

From equation (*5.9*) it follows conversely that if the attenuation factor α of a line increases directly as the square root of the frequency over an appreciable range of frequencies, it must be caused entirely by distributed resistance R in that range; while if α increases more rapidly than the square root of the frequency over a frequency interval, it contains a component caused by distributed conductance G in that interval.

Inspection of the attenuation factor data for the 7/8″ standard rigid coaxial line shows that between 1 and 10 megahertz α increases almost precisely as the square root of the frequency. It is therefore a reasonable assumption that at 1 megahertz the contribution to the attenuation factor from the distributed conductance G is negligible. Then from equation (*5.32*) or (*5.9*), $R = 2\alpha R_0 = 2 \times 0.0425 \times 50/(8.686 \times 30.48) = 0.0161$ ohms/m at 1 megahertz. Increasing as the square root of the frequency, the values of R at the other frequencies will be 0.0509 ohms/m at 10 megahertz, 0.161 ohms/m at 100 megahertz, and 0.509 ohms/m at 1000 megahertz.

If the attenuation factor at 1 megahertz is due entirely to the distributed resistance R, then the contribution of R to the attenuation factor at 1000 megahertz would be $0.0425\sqrt{1000} = 1.34$ db/(100 ft). The fact that the actual attenuation factor at that frequency is 1.49 db/(100 ft) indicates that there is a contribution of 0.15 db/(100 ft) from the distributed conductance G. Using part of equation (*5.9*), G at 1000 megahertz can be calculated from $0.15/(8.686 \times 30.48) = G \times 50/2$; hence $G = 23$ micromhos/m. Writing $\alpha = \alpha_R + \alpha_G$ where $\alpha_R = \frac{1}{2}R/Z_0$ and $\alpha_G = \frac{1}{2}GZ_0$, the results for all four frequencies can be tabulated as follows.

Frequency megahertz	R ohms/m	G mhos/m	α_R db/(100 ft)	α_G db/(100 ft)	α db/(100 ft)
1	*0.0161*	23×10^{-9}	*0.0425*	*0.00015*	*0.0426*
10	0.0509	23×10^{-8}	0.134	0.0015	0.135
100	0.161	23×10^{-7}	0.425	0.015	0.440
1000	0.509	23×10^{-6}	1.34	0.15	1.49

The initial hypothesis that the attenuation at 1 megahertz is due entirely to distributed resistance R is seen to have been in error by less than $\frac{1}{2}$%.

(A separate proof, based on other relations derived in Chapter 6, shows that the frequency of 1 megahertz is in fact *high* enough for the resistance of the copper conductors of the 7/8″ standard rigid line to be increasing directly as the square root of the frequency. For the same conductors at a frequency of 0.1 megahertz, the rate of increase of resistance with frequency would be much less.)

5.7. For the transmission line of Problem 5.6 find at each of the frequencies 1, 10, 100 and 1000 megahertz, the length of line that will have a transmission efficiency of (*a*) 99%, (*b*) 90%, (*c*) 50%, (*d*) 10%, (*e*) 1%. The transmission line is assumed to be terminated always in its characteristic impedance. Plot the results on log-log coordinates of length and frequency, and draw contours of constant percent transmission efficiency.

Transmission efficiency means (output power)/(input power). When a transmission line of length l has an attenuation factor α, the ratio of output voltage to input voltage or output current to input current when the line is terminated nonreflectively is $e^{-\alpha l}$. Hence the ratio of output power to input power is $e^{-2\alpha l}$. As a sample of the 20 calculations to be made, the length l of 7/8" standard rigid copper coaxial line having a transmission efficiency of 90% at a frequency of 1000 megahertz will be given by $e^{-2 \times 1.491l/(8.686 \times 30.48)} = 0.90$, from which $l = 9.36$ m. Toward the other extreme, a line length l having 10% transmission efficiency at a frequency of 1 megahertz will be given by $e^{-2 \times 0.0425l/(8.686 \times 30.48)} = 0.10$, and $l = 7170$ m. (Note: 100 ft = 30.48 m).

An alternative form of solution is to use the attenuation factor in db/(100 ft) directly, and solve for a length l' in hundreds of feet. For the second case calculated the answer would be obtained from $0.0425l' = -10 \log_{10} 0.10$ and $l' = 235$. Then the length is $235 \times 100 = 23{,}500$ ft $= 7170$ m.

5.8. For the transmission line of Examples 4.9, page 35, and 5.2, page 47, check that the high frequency approximate equations are valid at the frequency of operation, and use the equations to find α, v_p and Z_0 at that frequency.

For the line in question, $R = 0.098$ ohms/m, $L = 0.32$ microhenries/m, $G = 1.5$ micromhos/m and $C = 34.5$ micromicrofarads/m, all at a frequency of 100 megahertz. Then $\omega L/R = 2060$ and $\omega C/G = 14{,}500$ at that frequency, and the high frequency approximate equations should give results as accurate as those obtainable by any other method. From equations (*5.11*), (*5.9*) and (*5.10*) respectively,

$$Z_0 = R_0 + j0 = \sqrt{L/C} = 96.3 + j0 \text{ ohms}$$

$$\alpha = \tfrac{1}{2}R/Z_0 + \tfrac{1}{2}GZ_0 = 5.09 \times 10^{-4} + 0.72 \times 10^{-4} = 5.81 \times 10^{-4} \text{ nepers/m}$$

$$v_p = 1/\sqrt{LC} = 3.01 \times 10^8 \text{ m/sec}$$

The trivial deviations from the results obtained with polar numbers are due to rounding off of significant figures. At this fairly high frequency the distributed conductance G of the line has contributed almost 15% of the total attenuation factor.

5.9. A standard RG-11/U flexible coaxial transmission line for use at frequencies up to a few hundred megahertz, has the following specifications according to a handbook: $Z_0 = 75$ ohms (nominally real); velocity = 66%; distributed capacity $C = 20.5$ micromicrofarads/ft; attenuation factor in decibels/(100 ft) = 0.27 at 3.5 megahertz, 0.41 at 7 megahertz, 0.61 at 14 megahertz, 0.92 at 28 megahertz, 1.3 at 50 megahertz, and 2.4 at 144 megahertz.

(*a*) Are the data for Z_0, v_p and C consistent?
(*b*) What is the value of the distributed inductance L?
(*c*) Does the distributed conductance G contribute to the attenuation at all frequencies?
(*d*) Determine the distributed resistance R and the distributed conductance G over the frequency range.

(*a*) It can be taken for granted that the high frequency approximation equations are valid for any practical transmission line at frequencies of 3.5 megahertz and higher. When this is the case equation (*5.26*) indicates a relation between Z_0 (real), v_p and C. The stated data to be consistent must agree with this equation. It is most convenient to use v_p in m/sec. Then $C = 1/(Z_0 v_p) = 1/(75 \times 0.66 \times 3.00 \times 10^8) = 67.3$ micromicrofarads/m = 20.5 micromicrofarads/ft. The data for Z_0, v_p and C is therefore consistent within better than $\tfrac{1}{2}$%.

(*b*) From (*5.25*), $L = Z_0/v_p = 0.378$ microhenries/m = 0.115 microhenries/ft.

(c) If the attenuation factor were due entirely to distributed resistance R, it would increase directly as the square root of the frequency over the frequency range, assuming the lowest frequency 3.5 megahertz is above the range of low frequencies in which R has a complicated law of variation with frequency, as discussed in Chapter 6. Starting from 0.27 db/(100 ft) at 3.5 megahertz, it should then be 0.38 db/(100 ft) at 7 megahertz, 0.54 db/(100 ft) at 14 megahertz, 0.76 db/(100 ft) at 28 megahertz, etc. Clearly the attenuation factor is increasing considerably more rapidly than the square root of the frequency, even at 3.5 megahertz, and at all the listed frequencies α must contain a component caused by distributed conductance G.

A suggested approach for determining R and G as a function of frequency is to designate symbols for the values of R and G at any one frequency, and then to write equation (*5.9*) at that frequency and one other frequency, incorporating the postulate that R should increase directly as the square root of the frequency and G should increase directly as the frequency in the range containing the two frequencies. The two equations can then be solved simultaneously for the values of R and G at the chosen frequency.

The postulated laws of variation are likely to be most accurate at the highest frequencies, and familiarity with the manner of variation of resistance with frequency for fairly small wires such as the center conductor of this coaxial line suggests that R may not be increasing as rapidly as the square root of the frequency at 3.5 megahertz (compare Table 6.2, page 80). Then let R_{14} be the distributed resistance of the line in ohms/ft at 14 megahertz, and G_{14} be the distributed conductance of the line in mhos/ft at that frequency. Equation (*5.9*) at 14 megahertz gives

$$0.61/(8.686 \times 100) \;=\; R_{14}/(2 \times 75) + G_{14} \times 75/2$$

and at 144 megahertz (*5.9*) becomes

$$2.4/(8.686 \times 100) \;=\; (R_{14} \times \sqrt{144/14})/(2 \times 75) + (G_{14} \times 144/14) \times 75/2$$

The unit of length in all terms is the foot. Solving these equations simultaneously, $R_{14} =$ 0.095 ohms/ft and $G_{14} = 1.91$ micromhos/ft. The values of R and G at other frequencies are then easily found from the assumed laws of variation. When the resulting values of R and G are used to compute the attenuation factor at other frequencies, deviations of a few percent from the handbook values are found in some cases, the discrepancy being greatest at 3.5 megahertz.

5.10. A widely used type of loading for 19 gauge cable pair telephone transmission lines consisted of coils having 44 millihenries inductance and about 3 ohms resistance inserted in the line at intervals of 1.15 miles. The distributed circuit coefficients of the loaded line were then $R = 89$ ohms/mile, $L = 0.039$ henries/mile, $C = 0.062$ microfarads/mile, and G as a function of frequency had the same values stated in Example 5.4, page 52. Determine the attenuation factor, phase velocity and characteristic impedance of the loaded 19 gauge cable pair at frequencies of 300, 1000 and 3000 hertz, and compare the results with those in Table 5.1, page 55, for the same line without loading, at the same frequencies.

The values of $\omega L/R$ are not quite high enough at frequencies of 300 and 1000 hertz for the high frequency approximate equations (*5.9*) to (*5.12*) to be adequately accurate, but the values of the ratio are too high at those frequencies for the real number equations (*5.15*) to (*5.18*) to be entirely free of indeterminacy in the calculations of α and X_0 (see Problem 5.1). Hence the polar number method is advisable at all three frequencies. The results are:

Frequency hertz	α nepers/mile	v_p miles/sec	Z_0 ohms
300	0.050	17,900	$901 - j423$
1000	0.056	20,000	$806 - j141$
3000	0.057	20,300	$796 - j47$

Comparison with the results of Table 5.1 at these frequencies shows that:

(1) The attenuation factor is reduced more than 71% at 3000 hertz, 55% at 1000 hertz, and 39% at 300 hertz.

(2) Over the frequency range 300 to 3000 hertz, the attenuation factor varies by 65% of its 3000 hertz value, for the unloaded line, but by only 12% for the loaded line. The percent equalization improvement is approximately the same for the phase velocity and the real part of the characteristic impedance.

(3) The phase angle of the characteristic impedance is greatly reduced at all frequencies, becoming less than 10° at 1000 and 3000 hertz.

(4) Although the phase velocity shows much less variation over the voice frequency range for the loaded line than for the same line without loading, it is also reduced in value at all frequencies. This is an unavoidable disadvantage of inductive loading that would be a serious hazard on lines more than 1000 to 2000 miles long, because of the resulting transmission delay time.

Supplementary Problems

5.11. An open wire transmission line with copper conductors 0.165″ in diameter spaced 8″ between centers has the following distributed circuit coefficients at a frequency of 1000 hertz: $R = 2.55$ ohms/km, $L = 1.94$ millihenries/km, $C = 0.0062$ microfarads/km and $G = 0.07$ micromhos/km.

(*a*) Find the attenuation, phase velocity and characteristic impedance of the line at the stated frequency, using polar numbers.

(*b*) Assuming R, L, G and C independent of frequency, what is the lowest frequency at which the high frequency approximate formulas would be usefully accurate for this line?

(*c*) If α, v_p and Z_0 are calculated from the high frequency approximate formulas, what is the error in each compared with the accurate values obtained by the polar number method?

(*d*) How useful are the real number equations (*5.15*) and (*5.18*) for calculating α and X_0 for this line at the frequency of 1000 hertz?

Ans. (*a*) $R + j\omega L = 12.46\underline{/78.2^\circ}$ ohms/km; $G + j\omega C = (39.0 \times 10^{-6})\underline{/89.9^\circ}$ mhos/km; $\alpha = 0.00229$ nepers/km; $\beta = 0.0219$ rad/km; $v_p = 287{,}400$ km/sec $= 96\%$; $Z_0 = 565\underline{/-5.9^\circ} = 562 - j58$ ohms.

(*b*) Approximately 1000 hertz.

(*c*) $Z_0 = \sqrt{L/C} = 559$ ohms; $v_p = 1/\sqrt{LC} = 288{,}300$ km/sec $= 96\%$; $\alpha = \frac{1}{2}R/Z_0 + \frac{1}{2}GZ_0 = 0.00230$ nepers/km. These values differ from those in part (*a*) by about $\frac{1}{2}\%$ in each case, in agreement with the indications of equations (*5.5*)-(*5.7*).

(*d*) The calculations for R_0 and X_0 have an arithmetical precision of about 5%, compared with better than 1% for the calculations of part (*a*).

5.12. Show that if the calculations of Table 5.1, page 55, are extended to a frequency of 500 kilohertz, using the low frequency values for R, L and C, and the formula value for G, the justified significant figures for the terms F and A become identical, and the value of B is too small to be meaningfully combined with $(F - A)$ in the evaluation of α or X_0. (This result would still occur if the true values of R, L, G and C at 500 kilohertz were used.)

5.13. Derive equations (*5.5*), (*5.6*), (*5.7*) and (*5.8*) from (*5.1*) and (*5.2*).

5.14. On log-log coordinates of $\omega L/R$ and $\omega C/G$ covering the range 1 to 1000 for each, plot contours of the percent error to be expected in using the high frequency approximate equations (*5.9*)-(*5.11*) according to equations (*5.5*)-(*5.7*), and contours of the phase angle of Z_0 according to equation (*5.8*). Use separate sheets of graph paper for each set of contours. (For $\omega L/R$ and $\omega C/G$ less than 2 or 3, the contours will be inaccurate because of the higher order terms that were dropped when writing equations (*5.5*)-(*5.8*).)

5.15. Write a computer program for equations (*5.15*)-(*5.18*) and use it to check the results of Table 5.1, page 55. Insert additional frequencies of 20, 30, 50, 200, 500, 2000, 5000 and 20,000 hertz.

5.16. From the results of Problem 5.14 or the data of Table 5.1, page 55, plot α, v_p, R_0, X_0 and $|Z_0|$ as functions of frequency on semi-log graph paper, with frequency on the logarithmic scale. What evidence is there, from straight line portions of the resulting curves, that any of the plotted quantities varies as the logarithm of the frequency over appreciable ranges of frequency?

5.17. Repeat Problem 5.14, plotting the data on log-log graph paper. What evidence is there, from straight line portions of the resulting curves, that any of the plotted quantities varies as some power of the frequency over appreciable ranges of frequency? If there are such indications, what power of the frequency is involved in each case?

5.18. Repeat Problem 5.14, plotting the data on semi-log graph paper, with frequency on the linear scale. What evidence is there, from straight line portions of the resulting curves, that any of the plotted quantities varies exponentially with frequency over appreciable ranges of frequency?

5.19. Using the data of Example 5.6, page 55, find α, β, v_p, R_0 and X_0 for the transmission line at a frequency of 1000 hertz, using (a) the polar number method with equations (*5.1*) and (*5.2*), and (b) the real number equations (*5.15*) to (*5.18*). Compare the times required to make each of the calculations. What would be the effect of letting $G = 0$?

5.20. Show that if a transmission line has a component α_G of its attenuation factor caused by losses in the distributed conductance G, and a component α_R caused by losses in the distributed resistance R, then the phase angle θ_0 of the characteristic impedance is given by $\theta_0 = \tan^{-1}(\alpha_G - \alpha_R)/\beta$.

5.21. Derive equations (*5.20*) and (*5.21*) from equation (*5.19*), page 58.

5.22. Derive equations (*5.23*) and (*5.24*) from equation (*5.22*), page 58.

5.23. Derive the relation $\beta = \dfrac{\omega L}{2R_0} + \dfrac{\omega C\,|Z_0|^2}{2R_0}$, analogous to equation (*5.31*). From this write a universal expression for v_p in terms of L, C, R_0 and $|Z_0|$, valid for all transmission lines at all frequencies. Show that if $|Z_0| = R_0$, the expression reduces to equation (*5.10*).

5.24. Show that if $G = 0$, equations (*5.20*) and (*5.30*) lead to equation (*5.32*) for all values of X_0. (Eliminate X_0 from the first two equations.)

5.25. A standard 9″ rigid copper coaxial transmission line, used to transmit power levels of 1 megawatt at frequencies up to 10 megahertz, and $\frac{1}{4}$ megawatt at 200 megahertz, has a characteristic impedance of $50 + j0$ ohms and a velocity of 99.8% at frequencies above 1 megahertz. The attenuation factor of the line varies directly as the square root of the frequency over the frequency range 1 to 100 megahertz, being 0.0038 db/(100 ft) at 1 megahertz and 0.038 db/(100 ft) at 100 megahertz. Find the values of R, L and C for the line over the frequency range 1 to 100 megahertz, and determine a formula for the maximum value of G over the range, on the assumption that G is directly proportional to frequency.

Ans. $L = 0.167$ microhenries/m; $C = 66.8$ micromicrofarads/m; these are the same as for the standard 7/8″ line of Problem 5.6, page 64, since Z_0 and v_p have the same values for the two lines. $R = 0.00144\sqrt{\text{frequency in megahertz}}$ ohms/m. $G < 6 \times 10^{-9}$(frequency in megahertz) mhos/m, to contribute less than 1% to the attenuation factor at any frequency between 1 and 100 megahertz.

5.26. Derive equations (*5.15*)-(*5.18*), page 53 and 54, from equations (*5.1*) and (*5.2*), page 46.

5.27. Show that if G is negligible, the condition $\omega L/R \gg 1$ is equivalent to the condition $\omega/2\alpha v_p \gg 1$, and that the high frequency approximate equations are therefore valid for a transmission line if a harmonic wave on the line experiences an attenuation of not more than 2 or 3 db in one wavelength on the line.

5.28. There is no apparent limit to the number of relations that can be discovered among transmission line factors and coefficients and characteristics. The following are all exact relations when $G = 0$. Proof of some of them involves a little ingenuity. Any of the equations from (*5.15*) to (*5.24*) and from (*5.27*) to (*5.31*) may be used in the proofs, since all of these are also exact equations, derived without approximation from equations (*5.1*) and (*5.2*). Show that, if $G = 0$:

(a) $\alpha = \beta \tan \theta_0$, where $Z_0 = |Z_0|\underline{/\theta_0}$.

(b) $X_0 = R_0\sqrt{1 - LCv_p^2}$, where $v_p = \omega/\beta$.

(c) $\alpha = \beta\sqrt{1 - (v_p/v_0)^2}$, where $v_0 = 1/\sqrt{LC}$, = phase velocity on the same line if lossless, i.e. if L and C retain the same values but $R = G = 0$.

(*d*) $\alpha = \beta\sqrt{(W_C - W_L)/(W_C + W_L)}$, where W_C is the average energy stored per unit length in the distributed capacitance of the line, and W_L is the average energy stored per unit length in the distributed inductance of the line, when the line is terminated in its characteristic impedance. *Hint*: Start by writing the ratio of (*5.15*) to (*5.16*). Note that $W_C = \frac{1}{2}C\,|V|^2$ and $W_L = \frac{1}{2}L\,|I|^2$ where V and I are the rms phasor voltage and phasor current on the line respectively, and $V/I = Z_0$ for the conditions stated. For purposes of proof the line can be assumed of infinitesimal length, although the relation holds for any length of line, regardless of its total attenuation.)

(*e*) $Z_0 = 1/(Cv_p) - jRv_p/2\omega$.

5.29. It is undoubtedly true that the distributed inductance of a transmission line, of either the parallel wire or coaxial configuration, could be increased by winding either or both of the line conductors in the form of solenoidal coils, while retaining the same outer dimensions of the conductors.

For any of the lines whose distributed circuit coefficients are listed in the examples or problems of this chapter, show that replacing any of the conductors by solenoids of the same outer diameter, regardless of the wire size used in the solenoid, will not result in reducing the attenuation factor of the line.

5.30. Serious consideration is given to the idea of operating long transmission lines at liquid helium temperatures. The conductor resistance for superconducting metals then becomes extremely small even at radio frequencies, and the attenuation is greatly reduced, being determined by the distributed conductance G.

If the distributed resistance R of a transmission line could be made effectively zero, would capacitive loading in the form of lumped capacitors connected across the line at regular intervals promise the same improvements that inductive loading is able to realize for lines whose losses are due mainly to distributed resistance?

It can be seen from the second term on the right of equation (*5.9*), page 49, that if R were negligible for a transmission line, a reduced attenuation factor would result from capacitive loading provided the ratio of the total distributed capacitance to the square of the total distributed conductance for the loaded line was greater than for the original line. If the dielectric surrounding the line conductors was largely solid material, improvement could undoubtedly be obtained by adding loading capacitors with air dielectric, but if the dielectric of the line itself was mainly air, as in the case of a coaxial line with the center conductor supported by periodic small pins or thin discs of dielectric material, it might be difficult or impossible to construct loading capacitors with a sufficiently higher ratio of capacitance to conductance.

Chapter 6

Distributed Circuit Coefficients and Physical Design

6.1. Introduction.

Chapters 2 through 5 have developed transmission line analysis as an investigation of the propagation of voltage and current waves on uniform transmission lines defined by uniformly distributed electric circuit coefficients. Chapter 7 and subsequent chapters continue the same analysis. The present chapter, dealing with the derivation of expressions relating the distributed circuit coefficients of a uniform line to its dimensions and materials, involves topics and methods that are not directly part of this mainstream of transmission line theory.

Books on introductory circuit analysis, using the circuit element concepts of lumped resistance, inductance and capacitance, almost invariably omit any reference to the physical nature or construction of the units embodying these circuit properties. It is assumed that information is available in other sources on how to calculate the equivalent circuit of a specific real object, or how to design an assemblage of metals and dielectrics and ferromagnetic substances that will provide a circuit element meeting a desired specification. Since resistance, inductance and capacitance are in effect shorthand notations for relations between currents, charges and electromagnetic fields in bounded physical structures, the creation of formulas for the circuit representation of such structures is undertaken, in various degrees, by textbooks on electricity and magnetism or electromagnetic theory.

It is equally true that many books on electromagnetic theory develop equations for some or all of the distributed circuit coefficients for at least the simpler configurations of uniform transmission lines, and this could be used as a justification for omitting all such information from a transmission line textbook. There is, however, a fundamental difference of intent between the study of elementary circuit analysis and the study of transmission line engineering. The former seeks to convey a working knowledge of a few abstract relations between currents, voltages and circuit elements, with no thought of the specific situations in which these may occur. The latter, on the other hand, has an inherent concern to maintain a contact with physical reality, and to illustrate its theoretical analyses in terms of actual lines used for the transmission of signals and power. To achieve this purpose, it is essential that a transmission line textbook present a full discussion of the ways in which the distributed circuit coefficients of a line are dependent on its geometry and materials. Most of the standard textbooks on the subject have accepted this obligation.

The solution of boundary value problems in electromagnetic theory has been the business of mathematical physicists for a century. So far as the calculation of resistance, inductance and capacitance for either lumped or distributed devices is concerned, there now exists a voluminous amount of data on a wide variety of structures, but the mathematical form of the results is reasonably elementary only for constant unidirectional currents or voltages, and for structures with the simplest of geometries, such as concentric spheres, concentric circular cylinders, or infinite parallel planes. When the frequency is other than zero, even for these idealized geometries, calculations of resistance and reactance in some frequency ranges require uncommon mathematical functions for their expression, with the consequence that approximate formulas and graphical representations are widely used. Slightly less simple symmetries, such as parallel or concentric square or rectangular conductors, pose very difficult mathematical problems. Computer methods are needed to obtain adequate solutions, and the results are given in tables or charts.

It is an exceedingly fortunate fact that the transmission line constructions which are found to be experimentally optimal, in the sense of making most effective use of materials by providing minimum attenuation or maximum power handling capacity at the lowest cost and in the least space, involve only the simplest possible geometries. These are the lines whose cross sections are illustrated in Fig. 2-2, page 9. No important advantages, on either technical or economic grounds, have ever been claimed for lines with more unusual cross sections.

6.2. Distributed resistance and internal inductance of solid circular conductors.

By far the most widely used transmission line conductors are solid homogeneous wires of circular cross section. They are used as the center conductors of coaxial lines, the conductors of parallel wire or shielded pair or multi-conductor lines, and as the single conductor of image lines. Next in importance are tubular conductors of circular periphery, which are used in all of the above applications, and also as the outer conductor of coaxial lines and as the shield of shielded pair lines. The analysis that follows shows that an exact solution in functional form can be found for the distributed resistance and distributed internal inductance of homogeneous isotropic circular conductors, both solid and tubular, for all frequencies at which such conductors are used in transmission lines. Such exact solutions are not possible for the "stripline" constructions shown in Fig. 2-2, nor for any other practical transmission line cross section involving finite widths of plane surfaces.

If a wire of circular cross section has radius a meters, and is made of homogeneous isotropic material of conductivity σ mhos/m, its resistance per unit length at zero frequency (i.e. its d-c distributed resistance) is given by

$$R_{\text{d-c}} = \frac{1}{\sigma\pi a^2}\text{ ohms/m} \tag{6.1}$$

For the same isolated wire, considered indefinitely long, the inductance per unit length derived as the total magnetic flux linking unit length of the wire when the wire carries unit current is found to be infinite. It does not follow from this that the distributed circuit coefficient L for a transmission line is infinite. The conductors of a transmission line are not infinitely long isolated wires, and one of them always prescribes a finite limit for the integral determining the magnetic flux linking the other.

The distributed inductance of any conductor, whether isolated or not, consists of two parts. One part is caused by flux-current linkages inside the conductor itself, the other by linkages of the total conductor current with flux external to the conductor. These will be designated as L_i and L_x, respectively, in units of henries/m. For a solid circular conductor of radius a carrying current I of zero frequency, an expression for the internal inductance $L_{i\,\text{d-c}}$ is found as follows, referring to Fig. 6-1 below.

The current in an infinitesimal tube of radius r_1 and thickness dr_1 in the conductor's cross section is $(I/\pi a^2)(2\pi r_1\,dr_1)$. Referring to the definition that inductance is the flux linking a "circuit" per unit current in the circuit, this tube of radius r_1 and thickness dr_1 evidently constitutes a fraction $(2\pi r_1\,dr_1)/(\pi a^2)$ of the conductor as a "circuit".

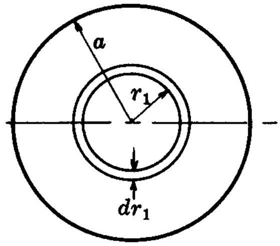


Fig. 6-1. Fractional "circuit" within a solid circular conductor carrying a d-c current.

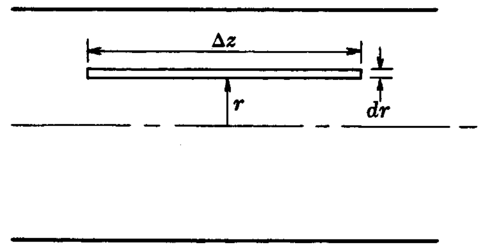


Fig. 6-2. Longitudinal cross section of solid circular conductor in diametral plane.

This fractional circuit is linked by all the magnetic flux inside the conductor between the radii r_1 and a (the flux lines being circles concentric with the conductor). At any radius r in this interval the magnetic flux density $B(r)$ is given by

$$B(r) \quad = \quad \frac{\mu I}{2\pi r}\left(\frac{\pi r^2}{\pi a^2}\right) \quad \text{teslas} \tag{6.2}$$

where μ is the mks permeability of the conductor material. Thus the contribution to the distributed internal inductance $L_{i\,\text{d-c}}$ of the fractional circuit consisting of the tube of thickness dr_1 at radius r_1 is

$$dL_{i\,\text{d-c}} \quad = \quad \frac{1}{I}\,\frac{2\pi r_1\,dr_1}{\pi a^2}\int_{r_1}^{a} dr \int_0^1 dz\,\frac{\mu I}{2\pi r}\,\frac{r^2}{a^2} \quad \text{henries/m} \tag{6.3}$$

where z is the coordinate in the direction of the length of the line, and r_1 is a constant during the integration with respect to r. The result is

$$dL_{i\,\text{d-c}} \quad = \quad \mu r_1\,dr_1\,(a^2 - r_1^2) \text{ henries/m} \tag{6.4}$$

The total internal inductance at zero frequency is obtained by integrating this with respect to r_1 from 0 to a, with the result

$$L_{i\,\text{d-c}} \quad = \quad \frac{\mu}{8\pi}\text{ henries/m} \tag{6.5}$$

Thus the internal inductance of a solid circular conductor, when the current is uniformly distributed over the conductor's cross section, is independent of the radius of the conductor. Calculations later in this chapter show that the internal inductance of the conductors of a transmission line may constitute 10% or even more of the line's total distributed inductance at low frequencies. At much higher frequencies the distributed internal inductance of a circular conductor becomes very small compared to its d-c value, and the distributed reactance of its internal inductance asymptotically approaches being identically equal to the frequency-dependent high frequency distributed resistance of the conductor.

Inspection of equation (*6.4*) indicates that for tubes of constant annular cross-sectional area, i.e. $2\pi r_1\,dr_1 = \text{constant}$, the distributed internal inductance is greatest for small values of r_1 and approaches zero as r_1 approaches a. This means that at a given frequency the distributed internal reactance of a small circular area at the center of a circular conductor is much greater than the distributed reactance of the same area of conductor near the periphery. If an a-c voltage exists between the ends of a section of the conductor, less current will flow in the high reactance region at the center of the conductor than in an equal area of cross section at a greater radius. The current distribution will then no longer be one of constant density as was the case at zero frequency. The effect becomes more pronounced the higher the frequency, until at sufficiently high frequencies the current flows only in a very thin *skin* at the conductor's surface. Known as *skin effect,* this phenomenon causes the resistance of conductors of any shape or material to increase markedly and continuously with frequency for frequencies above some minimum value that depends on the conductor's size, permeability and conductivity, while at the same time the internal inductance decreases continuously.

A quantitative analysis for skin effect in a homogeneous isotropic circular conductor is obtained by applying Faraday's law to a rectangular path in a radial plane of the conductor as illustrated in Fig. 6-2 above. The radius of the conductor is a. The rectangle has length Δz in the coordinate direction z parallel to the length of the conductor, and infinitesimal width dr in the radial direction. It is located at distance r from the center of the conductor.

A postulate of this analysis is that an external source is causing current to flow in the conductor in the z direction, and that the resulting current density J_z at any point in the conductor's cross section is in general a function of r, but for symmetry reasons is not a function of angular position around the center of the conductor. The purpose of the analysis is to find the manner in which $J_z(r)$ varies with r, and from this result to find the effective resistance and internal inductance of the conductor per unit length, as a function of frequency and conductor material.

At any radius r in the conductor there will be an electric field $E_z(r)$ associated with the total current density $J_z(r)$ according to the time-harmonic electromagnetic relation

$$J_z = \sigma E_z + j\omega\epsilon E_z \tag{6.6}$$

where σ is the conductivity of the conductor and ϵ its permittivity. For the metals used in transmission line conductors, information about the value of the permittivity ϵ is nebulous, but there is no reason to believe that it differs appreciably from the value for free space, $\epsilon_0 = 8.85 \times 10^{-12}$ farads/m. Since the conductivity of the metals is 10^7 mhos/m or higher, it is readily seen that at all conceivable transmission line frequencies equation (*6.6*) becomes

$$J_z(r) = \sigma E_z(r) \tag{6.7}$$

which means that $J_z(r)$ is entirely conduction current density.

Voltage will be induced in the rectangle of Fig. 6-2 because of the time-changing magnetic flux through it. This flux is produced by all the conductor current inside radius r. Faraday's law $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S}$ becomes

$$\left\{E_z(r) + \frac{\partial E_z(r)}{\partial r}\right\}\Delta z - E_z(r)\,\Delta z = \frac{j\omega\, dr\,\Delta z}{2\pi r}\int_0^r \mu\, J_z(r')\, 2\pi r'\, dr' \tag{6.8}$$

where r' is a dummy radial variable of integration, not to be confused with the coordinate r giving the location of the rectangle.

Substituting $E_z(r) = J_z(r)/\sigma$, multiplying both sides by $r/(dr\,\Delta z)$, differentiating both sides with respect to r, and finally dividing all terms by r,

$$\frac{\partial^2 J_z(r)}{\partial r^2} + \frac{1}{r}\frac{\partial J_z(r)}{\partial r} - j\omega\mu\sigma\, J_z(r) = 0 \tag{6.9}$$

The differentiation on the right merely removes the integration and substitutes the upper limit for the variable.

Equation (*6.9*) is a modified form of Bessel's equation of order zero. The order is zero because the term $-\nu^2 J_z(r)/r^2$ in a Bessel equation of order ν has coefficient zero. The equation is a "modified" Bessel equation because the coefficient of the term in $J_z(r)$ is a negative imaginary number rather than a positive real number. Formally, the solution of (*6.9*) can be written

$$J_z(r) \quad = \quad A_1\, J_0(\sqrt{-j\omega\mu\sigma}\, r) \;+\; A_2\, Y_0(\sqrt{-j\omega\mu\sigma}\, r) \tag{6.10}$$

where the symbols J_0 and Y_0 stand for Bessel functions of the first and second kinds respectively, of order zero. (Other symbols are frequently used for Y_0.) For real variables these functions are evaluated from infinite series and are readily available in mathematical tables. Such tables are not applicable, however, when the coefficient of r in the variable is the square root of a negative imaginary number, since in this case the series expansion will contain both real and imaginary terms. Equation (*6.9*) is of sufficient importance that separate names have been given to the functions which are respectively the real and imaginary parts of the Bessel functions of the first and second kinds of order zero, for the specific form of complex variable occurring in that equation. One of several equivalent sets of definitions for these special functions is

$$\text{ber}\,(x) \;=\; \text{real part of } J_0(\sqrt{-j}\,x) \tag{6.11}$$

$$\text{bei}\,(x) \;=\; \text{imaginary part of } J_0(\sqrt{-j}\,x) \tag{6.12}$$

$$\text{ker}\,(x) \;=\; \text{real part of } Y_0(\sqrt{-j}\,x) \tag{6.13}$$

$$\text{kei}\,(x) \;=\; \text{imaginary part of } Y_0(\sqrt{-j}\,x) \tag{6.14}$$

where x is real.

For reasons explained later, it is desirable to write the variable in (*6.10*) in the form $\sqrt{-j}\,\sqrt{2}\,r/\delta$, where

$$\delta \;=\; \sqrt{\frac{2}{\omega\mu\sigma}} \tag{6.15}$$

Using (*6.11*) to (*6.15*), the general solution to (*6.10*) is

$$J_z(r) \quad = \quad A_1\left(\text{ber}\,\frac{\sqrt{2}\,r}{\delta} + j\,\text{bei}\,\frac{\sqrt{2}\,r}{\delta}\right) \;+\; A_2\left(\text{ker}\,\frac{\sqrt{2}\,r}{\delta} + j\,\text{kei}\,\frac{\sqrt{2}\,r}{\delta}\right) \tag{6.16}$$

Because ker (x) is infinite at $x = 0$, A_2 must equal zero when equation (*6.16*) is applied to a solid circular conductor, since the location $r = 0$ is a line within the conductor. For a tubular conductor the location $r = 0$ is not within the conductor's cross section and A_2 is in general not zero. For the solid circular conductor, then

$$J_z(r) \quad = \quad A_1\left(\text{ber}\,\frac{\sqrt{2}\,r}{\delta} + j\,\text{bei}\,\frac{\sqrt{2}\,r}{\delta}\right) \tag{6.17}$$

Although the primary purpose of this analysis is to find expressions for the distributed resistance and internal inductance of a homogeneous solid circular conductor, as a function of the conductor material and the frequency, it is of incidental interest to note the distribution of current density over the conductor's cross section, which is given directly by equation (*6.17*).

Convenient tables of ber (x) and bei (x) are available in H. B. Dwight's *Tables of Integrals and Other Mathematical Data*. Fig. 6-3 shows graphs of the magnitude and phase of $J_z(r)$ plotted from equation (*6.17*) using these tables, with $A_1 = 1$. Since $\text{ber}\,(0) = 1$ and $\text{bei}\,(0) = 0$, these graphs all show the magnitude and phase angle of the current density at any value of r/δ relative to the magnitude and zero reference phase angle respectively at the center of the conductor.

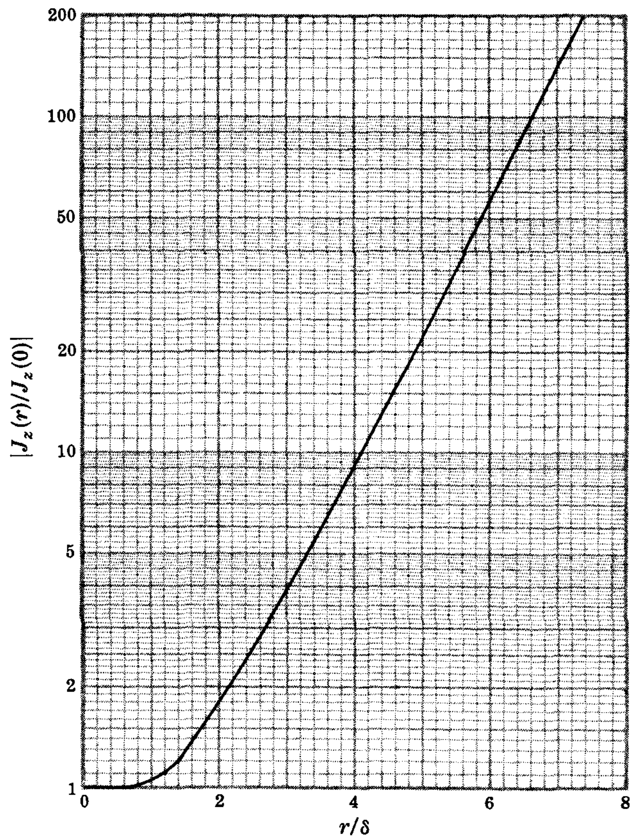


Fig. 6-3(*a*). Ratio of the magnitude of the current density $|J_z(r)|$ at any radius r, inside a solid circular conductor, to the magnitude of the current density $|J_z(0)|$ at the center of the conductor, as a function of r in skin depths.

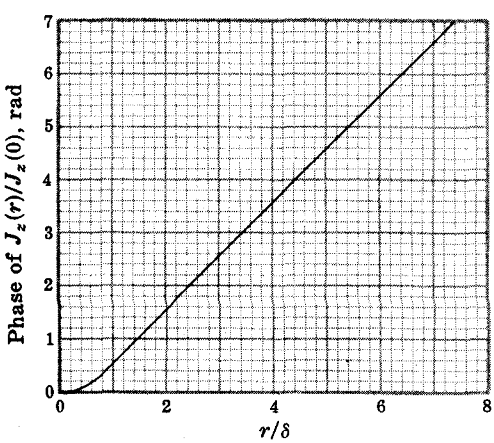


Fig. 6-3(*b*). Phase of the current density $J_z(r)$ at any radius r, inside a solid circular conductor, relative to the phase of the current density $J_z(0)$ at the center of the conductor, as a function of r in skin depths.

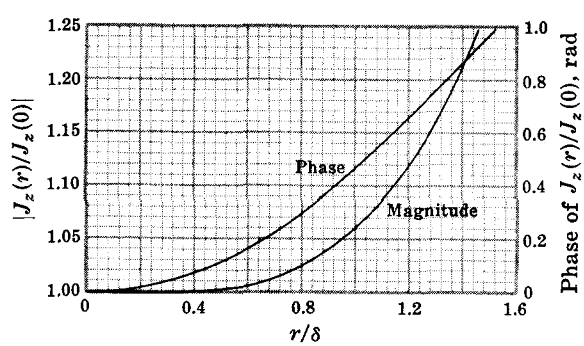


Fig. 6-3(*c*). An expanded linear presentation of the portions of Fig. 6-3(*a*) and (*b*) at low values of r/δ.

According to electromagnetic theory the currents and fields inside a solid circular conductor are to be regarded as having penetrated into the conductor from the interconductor fields of the transmission line at the conductor's surface. It is therefore quantitatively more significant to consider the current density at any radius r relative to the current density at the periphery of the conductor, in magnitude and phase. The current density at the surface is

$$J_z(a) \quad = \quad A_1\left(\text{ber}\frac{\sqrt{2}\,a}{\delta} + j\,\text{bei}\frac{\sqrt{2}\,a}{\delta}\right) \qquad (6.18)$$

where a is the radius of the conductor. Substituting for A_1 from (*6.18*) into (*6.17*),

$$\frac{J_z(r)}{J_z(a)} \quad = \quad \frac{\text{ber}\,\sqrt{2}\,r/\delta + j\,\text{bei}\,\sqrt{2}\,r/\delta}{\text{ber}\,\sqrt{2}\,a/\delta + j\,\text{bei}\,\sqrt{2}\,a/\delta} \qquad (6.19)$$

which is the desired relation. The same result can be obtained graphically, for a specific conductor at a specific frequency, by evaluating a/δ for the conductor and marking a vertical line at that value of r/δ on one of the graphs of Fig. 6-3. If then all the current density magnitude values for smaller values of r/δ are divided by the current density magnitudes at $r/\delta = a/\delta$, and the phase angle at $r/\delta = a/\delta$ is subtracted from the phase angle at all smaller values of r/δ, the graph of the resulting magnitudes and phase angles versus $(r/\delta)/(a/\delta)$ or r/a will show the magnitude and phase angle of the current density at any radius r, relative to the magnitude and zero reference phase angle of the current density at the conductor's surface. Fig. 6-4 is such a graph for $a/\delta = 4$.

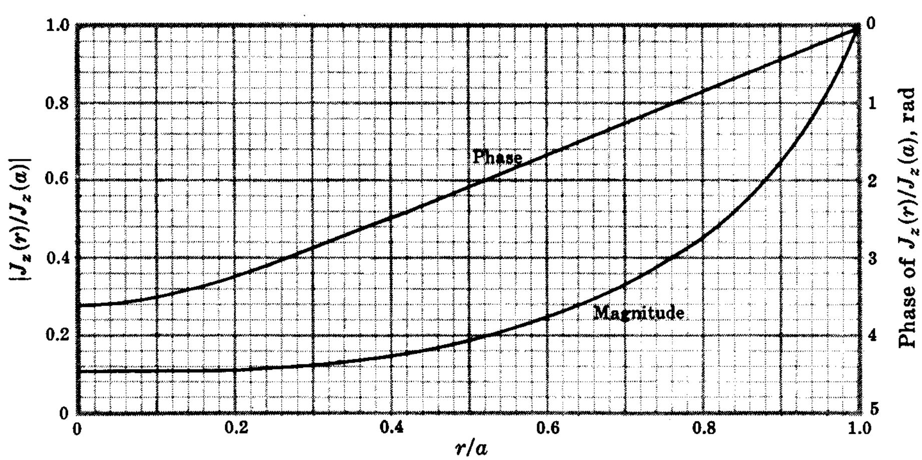


Fig. 6-4. Phase and magnitude relations for the current density at radius r in a solid circular conductor of radius a relative to the current density at the surface, at a frequency for which $a/\delta = 4$.

It is evident from Fig. 6-3 that for values of a/δ less than 0.5, the current distribution in a solid circular conductor is not perceptibly different from that for d-c, when $a/\delta = 0$. At $a/\delta = 1$, however, the change is quite noticeable, and when $a/\delta = 5$ there is a very marked concentration of current close to the conductor's surface. As a numerical example, a/δ has the value 0.5 for a copper conductor 2 millimeters in diameter at a frequency of 1090 hertz, the value 1 at 4360 hertz, and the value 5 at 109,000 hertz. The suggestion from Fig. 6-3 that at high enough values of a/δ a solid circular conductor might be replaced by a thin-walled circular tube with negligible change in current distribution, and consequently in a-c resistance, is perfectly correct. Surprisingly, it turns out that for a wall thickness of about 1.6δ the distributed resistance of a tubular conductor at high values of a/δ is actually a few percent less than the distributed resistance of a solid conductor of the same metal and outer diameter at the same frequency.

The distributed resistance R and distributed internal inductance L_i of a solid circular conductor at angular frequency ω rad/sec can be combined in the concept of the distributed internal impedance Z_i of the conductors,

$$Z_i = R_i + j\omega L_i \quad \text{ohms/m} \tag{6.20}$$

where it is not necessary to use a symbol R_i since the distributed internal resistance of the conductor is its total distributed resistance. In terms of electrical variables, the distributed internal impedance of the conductor is the ratio of the longitudinal potential difference over unit length of the conductor at the conductor's surface to the total current in the conductor. The longitudinal potential difference per unit length is identically the longitudinal electric field at the conductor's surface, which from equation (6.7) is $E_z(a) = J_z(a)/\sigma$. Then if the total current in the conductor is I_z,

$$Z_i = R + j\omega L_i = \frac{J_z(a)}{\sigma I_z} \tag{6.21}$$

To relate the total current I_z to quantities that have already appeared in the analysis, it is necessary to refer to another electromagnetic relation, the Maxwell equation for time-harmonic fields: curl $\mathbf{E} = -j\omega\mu\mathbf{H}$. From the postulated symmetry of the problem, the only field components present are E_z and H_ϕ, and these quantities are functions only of the coordinate r. The Maxwell equation in cylindrical coordinates then reduces to the single relationship

$$\frac{\partial E_z}{\partial r} = j\omega\mu H_\phi(r) \tag{6.22}$$

From Ampere's law (i.e. the integral form of the second Maxwell curl equation), the total current in the conductor will be equal to the line integral of H_ϕ around the conductor's periphery,

$$I_z = 2\pi a H_\phi(a) \tag{6.23}$$

Combining equations (6.7), (6.17) and (6.22),

$$H_\phi(r) = \frac{1}{j\omega\mu\sigma}\frac{\partial J_z(r)}{\partial r} = \frac{\sqrt{2}/\delta}{j\omega\mu\sigma} A_1\left(\text{ber}'\frac{\sqrt{2}\,r}{\delta} + j\,\text{bei}'\frac{\sqrt{2}\,r}{\delta}\right) \tag{6.24}$$

where $\text{ber}'(x) = d\,\text{ber}(x)/dx$ and $\text{bei}'(x) = d\,\text{bei}(x)/dx$.

Putting $r = a$ in equation (6.24) and combining with (6.23),

$$I_z = \frac{2\pi a A_1 \sqrt{2}/\delta}{j\omega\mu\sigma}\left(\text{ber}'\frac{\sqrt{2}\,a}{\delta} + j\,\text{bei}'\frac{\sqrt{2}\,a}{\delta}\right) \tag{6.25}$$

Substituting for A_1 from (6.18),

$$I_z = \frac{2\pi a \sqrt{2}/\delta}{j\omega\mu\sigma} J_z(a)\left(\frac{\text{ber}'\sqrt{2}\,a/\delta + j\,\text{bei}'\sqrt{2}\,a/\delta}{\text{ber}\sqrt{2}\,a/\delta + j\,\text{bei}\sqrt{2}\,a/\delta}\right) \tag{6.26}$$

Finally, substituting this value of I_z in (6.21) gives

$$R + j\omega L_i = \frac{jR_s}{\sqrt{2}\,\pi a}\left(\frac{\text{ber}\sqrt{2}\,a/\delta + j\,\text{bei}\sqrt{2}\,a/\delta}{\text{ber}'\sqrt{2}\,a/\delta + j\,\text{bei}'\sqrt{2}\,a/\delta}\right) \tag{6.27}$$

Here the symbol R_s has been adopted for $\dfrac{1}{\sigma\delta} = \sqrt{\dfrac{\omega\mu}{2\sigma}}$.

R_s is a surface resistivity in ohms per square, sometimes called the skin resistivity or high frequency surface resistivity, of the material defined by μ and σ at angular frequency ω. Physically it is the d-c resistance between opposite edges of a square sheet of the metal having thickness equal to the skin depth δ, since it is experimentally confirmed for the usual non-ferromagnetic conductor materials that the d-c value of σ continues to hold at all frequencies used on transmission lines.

Separate expressions for R and ωL_i are easily obtained from equation (*6.27*), and with the help of (*6.1*) and (*6.5*), equations giving the ratios of the distributed resistance and distributed internal inductance at any frequency to the same quantities at zero frequency can be formulated. (See Problem 6.36.)

For practical purposes, calculations of data from (*6.27*) or from the equations derived in Problem 6.36 can be divided into five sections, according to the value of a/δ:

(1) When a/δ is less than about 0.5, the distributed a-c resistance of a conductor increases over its d-c value given in (*6.1*) by less than $\frac{1}{2}\%$, and the distributed internal inductance decreases by less than $\frac{1}{4}\%$ from the d-c value given in (*6.5*).

(2) For all values of a/δ less than about 1.5, better than $\frac{1}{2}\%$ accuracy is obtained from the approximate formulas

$$R/R_{\text{d-c}} = 1 + (a/\delta)^4/48 \tag{6.28}$$

$$L_i/L_{i\,\text{d-c}} = 1 - (a/\delta)^4/96 \tag{6.29}$$

(3) When a/δ is greater than about 100, $\frac{1}{2}\%$ accuracy is given by the very simple formulas

$$R = R_s/(2\pi a) \tag{6.30}$$

where $R_s = 1/(\sigma\delta) = \sqrt{\omega\mu/(2\sigma)}$ as defined in equation (*6.27*), and

$$\omega L_i = R \quad \text{or} \quad L_i = R/\omega \tag{6.31}$$

Equation (*6.30*) states that for very high values of a/δ, which occur at frequencies of tens or hundreds of megahertz for typical conductor diameters, the distributed a-c resistance of a solid circular conductor is equal to the d-c resistance per unit length of a plane strip of the conductor material having thickness δ and a width equal to the periphery of the circular conductor. This is an expression of the skin effect theorem derived in Section 6.3. Alternative interpretations of (*6.30*) are that the distributed a-c resistance of a solid circular conductor at sufficiently high frequencies is equal to the d-c resistance per unit length of a surface skin of the conductor of thickness δ, or is equal to the resistance per unit length of a strip of indefinitely thin sheet resistance material having surface resistivity R_s ohms/square, wrapped around a nonconducting cylinder of radius equal to that of the conductor.

Equation (*6.31*) states that under the same conditions, the distributed internal reactance of the solid circular conductor is equal to its distributed a-c resistance.

It is obvious that (*6.30*) and (*6.31*) also apply to tubular circular conductors whose wall thickness is great enough to contain essentially the whole of the current distribution. For high values of a/δ a wall thickness greater than about 3δ ensures $\frac{1}{2}\%$ accuracy in using the equations. (See Section 6.3 and Fig. 6-7.)

(4) Equation (*6.30*) has better than $\frac{1}{2}\%$ accuracy for all values of a/δ greater than 4 if the effective circumference of the peripheral skin of the conductor is calculated more appropriately from a radius $(a - \frac{1}{2}\delta)$ instead of a. Then

$$R = \frac{1}{2\pi\sigma(a - \frac{1}{2}\delta)\delta} = \frac{R_s}{2\pi(a - \frac{1}{2}\delta)} \tag{6.32}$$

and

$$\frac{R}{R_{\text{d-c}}} = \frac{\frac{1}{2}a^2}{(a - \frac{1}{2}\delta)\delta} = \frac{(a/\delta)^2}{2(a/\delta) - 1} \tag{6.33}$$

For the same range of a/δ the inductance ratio $L_i/L_{i\,\text{d-c}}$ is given with equal accuracy by an empirical formula

$$\frac{L_i}{L_{i\,\text{d-c}}} = \frac{1}{R/R_{\text{d-c}}} + \frac{1}{(R/R_{\text{d-c}})^3} \tag{6.34}$$

(5) For values of a/δ between 1.5 and 4, there is no standard alternative to using data calculated directly from equation (*6.27*). Tables and graphs of such data are available in many sources. Table 6.1 shows the variation of $R/R_{\text{d-c}}$ and $L_i/L_{i\,\text{d-c}}$ for small intervals of a/δ in the range 0 to 4. Linear interpolation in the intervals is accurate enough for engineering purposes.

Table 6.1

a/δ	$R/R_{\text{d-c}}$	$L_i/L_{i\,\text{d-c}}$	a/δ	$R/R_{\text{d-c}}$	$L_i/L_{i\,\text{d-c}}$
0	1.000	1.000	2.3	1.404	0.805
0.5	1.001	1.000	2.4	1.454	0.783
0.7	1.005	0.998	2.5	1.505	0.760
0.8	1.009	0.996	2.6	1.557	0.737
0.9	1.014	0.993	2.7	1.610	0.715
1.0	1.021	0.989	2.8	1.663	0.693
1.1	1.030	0.984	2.9	1.716	0.672
1.2	1.042	0.978	3.0	1.769	0.652
1.3	1.057	0.971	3.1	1.821	0.632
1.4	1.075	0.962	3.2	1.873	0.613
1.5	1.097	0.951	3.3	1.924	0.595
1.6	1.122	0.938	3.4	1.974	0.578
1.7	1.152	0.924	3.5	2.024	0.562
1.8	1.187	0.908	3.6	2.074	0.547
1.9	1.225	0.890	3.7	2.124	0.533
2.0	1.266	0.870	3.8	2.174	0.520
2.1	1.309	0.849	3.9	2.224	0.507
2.2	1.355	0.827	4.0	2.274	0.495

Example 6.1.

Determine the distributed resistance and distributed internal reactance in ohms/m of a 19 gauge copper wire at frequencies of 0, 60, 10^3, 10^4, 10^5, 10^6, 10^8 and 10^{10} hertz.

From wire tables the radius of a 19 gauge wire is 0.4558×10^{-3} m, and $R_{\text{d-c}}$ is given as 26.42 ohms/km at 20°C. These figures are consistent with the copper having a conductivity of 5.80×10^7 mhos/m, which is officially defined as "100% conductivity" for copper at 20°C, and is the value invariably used for "room temperature" resistance calculations on copper conductors, unless some other specific value is stated.

For this value of conductivity the skin depth in copper as given by equation (*6.15*) is $\delta = 0.0661/\sqrt{f}$ m, where f is the frequency in hertz. Thus for the 19 gauge wire of this problem, the ratio a/δ at the frequencies listed has the sequence of values 0, 0.0533, 0.218, 0.689, 2.18, 6.89, 68.9 and 689. The first three values fall in the first category of calculations listed above. These are followed by one in the second category, one in the fifth, two in the fourth, and the final one in the third category.

The reference values for R and L_i are $R_{\text{d-c}} = 0.0264$ ohms/m and $L_{i\,\text{d-c}} = \mu_0/8\pi = 5.00 \times 10^{-8}$ henries/m. These are then also the values of R and L_i for the 19 gauge copper wire at frequencies of 0, 60 and 10^3 hertz, where $a/\delta < 0.5$.

At 10 kilohertz, with $a/\delta = 0.689$, equations (*6.28*) and (*6.29*) should be used. The results are $R/R_{\text{d-c}} = 1.005$ and $L_i/L_{i\,\text{d-c}} = 0.998$, which could also have been taken from Table 6.1 at $a/\delta = 0.7$.

At 100 kilohertz, Table 6.1 is used, to find $R/R_{\text{d-c}} = 1.346$ and $L_i/L_{i\,\text{d-c}} = 0.831$.

For frequencies of 1 megahertz and higher, (*6.33*) and (*6.34*) are used. These automatically reduce to the simpler forms of (*6.30*) and (*6.31*) when a/δ is large enough.

A tabulation of all the results for 19 gauge copper wire at frequencies from 0 to 10^{10} hertz is given in Table 6.2 below.

Table 6.2

Frequency hertz	a/δ	$R/R_{d\text{-}c}$	R ohms/m	$L_i/L_{i\,d\text{-}c}$	L_i henries/m	ωL_i ohms/m	$\omega L_i/R$
0	0	1.000	0.0264	1.000	5.00×10^{-8}	0	0
60	0.0533	1.000	0.0264	1.000	5.00×10^{-8}	1.88×10^{-5}	7.1×10^{-4}
10^3	0.218	1.000	0.0264	1.000	5.00×10^{-8}	3.14×10^{-4}	0.0119
10^4	0.689	1.005	0.0265	0.998	4.99×10^{-8}	3.13×10^{-3}	0.118
10^5	2.18	1.346	0.0355	0.831	4.16×10^{-8}	0.0261	0.736
10^6	6.89	3.71	0.0980	0.289	1.45×10^{-8}	0.0909	0.930
10^8	68.9	34.7	0.914	0.0288	1.44×10^{-7}	0.905	0.991
10^{10}	689	344	9.09	0.00291	1.45×10^{-10}	9.09	1.000

It is quite clear from Tables 6.1 and 6.2 that a/δ increasing above unity marks the beginning of rapid increases with frequency for the ratios $R/R_{d\text{-}c}$ and $L_i/L_{i\,d\text{-}c}$. Also, from Table 6.2 it can be seen that when a/δ approaches 100, the distributed resistance R begins to increase quite precisely in proportion to the square root of the frequency, while the distributed internal inductance L_i begins to vary inversely as the square root of the frequency.

The ratio a/δ varies directly with conductor radius a, and with the square root of the frequency, for solid circular conductors of the same material. Thus the same values of $R/R_{d\text{-}c}$ and $L_i/L_{i\,d\text{-}c}$ that apply to a conductor of radius a at frequency f, will hold for a conductor of the same material with radius $10a$ at a frequency $f/100$, or with radius $a/10$ at a frequency $100f$. Pursuing these figures in both directions, a 40 gauge copper wire shows no perceptible change in resistance from its d-c value for frequencies up to 1 megahertz, while a solid copper conductor 2″ in diameter shows about 10% increase of resistance from skin effect at a frequency of 60 hertz.

The ratio a/δ varies directly as the square root of the permeability and the square root of the conductivity of the conductor material. Hence conductors of any non-magnetic material except silver will have smaller values of a/δ than copper wires of the same diameter at the same frequency, and the changes in their values of distributed resistance R and distributed internal inductance L_i from the d-c values will be less than for the copper conductors. Wires of iron, nickel, or other ferromagnetic material may have values of a/δ, and hence of $R/R_{d\text{-}c}$, either larger or smaller than for copper wires of the same diameter at the same frequency, depending on whether or not their relative permeability at the frequency exceeds the ratio of the conductivity of copper to the conductivity of the ferromagnetic material. Iron wires may have quite large values of relative permeability at frequencies up to the low megahertz range, in which case the ratio of the distributed resistance of iron wires to the distributed resistance of copper wires of the same diameter may become much higher than would be determined by the ratio of their conductivities alone.

Table 6.3 below lists the conductivities at 20°C, and the temperature coefficient of the conductivity at that temperature, for some of the metals most commonly used as transmission line conductors, or as resistor materials or plating materials in high frequency applications.

From Example 6.1 it has been seen that the skin depth δ for 100% conductivity copper at frequency f hertz and 20°C is given by $\delta = 0.0661/\sqrt{f}$ m. Fig. 6-3 shows that for a/δ greater than about 5 or 10, most of the a-c current in a solid circular conductor flows in a peripheral skin of the conductor about two or three skin depths in thickness. At a frequency of 1 megahertz in copper this thickness is approximately 0.1 millimeters, and it is smaller

Table 6.3. Conductivities and temperature coefficients of common metals at 20°C.

Metal	Conductivity mhos/m	Temperature Coefficient /°C (all negative)
Aluminum	3.54×10^7	0.0039
Brass (somewhat variable)	1.4×10^7	0.002
Copper (annealed)	5.80×10^7	0.00393
Copper (hard drawn)	5.65×10^7	0.00382
Constantan	2.04×10^6	0.000008
Gold (pure)	4.10×10^7	0.0034
Iron* (pure)	1.00×10^7	0.0050
Lead	4.54×10^6	0.0039
Mercury	1.04×10^6	0.00089
Nickel*	1.28×10^7	0.0006
Silver	6.15×10^7	0.0038
Tin	8.67×10^6	0.0042
Zinc	1.76×10^7	0.0037

*The permeability of iron and nickel is very much dependent on processing techniques, and must be determined experimentally for any specific conductors.

at higher frequencies. This suggests the possibility of using plated conductors in high frequency applications, having a thin skin of more expensive high conductivity metal on a core of inexpensive material whose conductivity does not affect the situation. The technique is extensively used. For many years it was thought that silver was necessarily the best plating material, since silver has the highest conductivity of all metals. However, careful measurements have shown that the corrosion products on a silver surface in ordinary atmospheres have intermediate conductivity, while those on a copper surface have very low conductivity. The result is that high frequency currents in a copper conductor flow almost entirely in the copper, below the surface corrosion layers, and the conductor's effective conductivity is that of the copper. For a silver conductor, on the other hand, an appreciable fraction of the current flows in the corrosion material of intermediate conductivity (the corrosion products are generally oxides and sulfides) and the effective conductivity of the conductor as a whole may be substantially less than that of silver. If a silver surface is protected against corrosion, including oxidation, by an extremely thin layer of plated or evaporated gold or by a low-loss dielectric coating, a silver plated conductor will have the lowest possible distributed resistance.

It is theoretically true, when $a/\delta \gg 1$, that a plating thickness of 1.2δ or greater, on any base material whether conducting or not, will ensure a distributed conductor resistance equal to or less than that of a solid circular conductor of the plating material, if there is adequate protection from corrosion and from surface roughness effects.

The problem of surface roughness exists whether a conductor is solid or plated, and is an obvious consequence of the small values of δ at high frequencies. In the commercial fabrication of circular metal rods or wire, microscopic surface imperfections appear, in the form of pits, cracks, grooves, fissures, etc. At frequencies in the hundreds or thousands of megahertz, the skin depth in the material may be no greater than the dimensions of these imperfections, with the result that the current flow path along the conductor's surface is not a straight line equal to the conductor's length, but is a meandering path that may be much longer than the conductor itself. The high frequency resistance will increase by a corresponding factor over the theoretical value. At microwave frequencies the realization of distributed conductor resistances close to theoretical values for a given conductor material may require special surface polishing techniques in addition to protection against corrosion.

6.3. Distributed resistance and internal inductance of thick plane conductors.

Plane conductors of finite width occur in several of the transmission lines illustrated in Fig. 2-2, page 9, and it would be useful to be able to calculate the distributed resistance and internal inductance of those conductors. However, a reasonably simple analysis of the high frequency current distribution and distributed internal impedance of plane conductors is possible only if the conductors are postulated to extend indefinitely in both directions in the conductor plane. The general case of the distributed internal impedance of plane parallel conductors of finite width, or such special cases as the distributed internal impedance of finite width plane conductors within a rectangular, elliptical or circular shield or outer conductor, can be solved only by approximation and computer methods, and there are no universally accepted presentations of results for these cases, in the form of equations, tables, or graphs. In practice, experimental measurements are often likely to provide information on the distributed resistance and inductance of uniform transmission lines involving plane conductors as quickly and accurately as attempts to solve the analytical problem.

The analysis of the idealized unbounded-plane case is worth inspection in spite of these limitations, because for high values of a/δ it offers an easier solution to the practical problem of the thin-walled circular tube than can be obtained from the equations of Section 6.2, and because it provides a basis for approximate calculations in various other cases.

The longitudinal currents postulated in the distributed circuit analysis of transmission lines will flow only if there is a longitudinal electric field at the conductor's surface. An electric field component normal to the surface will not generate such currents, nor will a component parallel to the surface but transverse to the direction of propagation. In an investigation of the power losses associated with high frequency current flow in a plane metal surface, it is therefore most appropriate to consider a constant amplitude plane transverse electromagnetic harmonic wave incident normally on such a metal surface and partly reflected from it. The field components of the incident and reflected waves will combine at the surface to produce a total tangential electric field component which is continuous across the boundary, and an accompanying tangential magnetic field component, perpendicular to the electric field component and also continuous across the boundary. Let these field components, as phasors, have directions and rms magnitudes given by E_{z0} and H_{x0} at the metal surface, and by $E_z(y)$ and $H_x(y)$ at any distance y measured into the metal normally from the surface.

The unbounded plane conductor can be thought of as one of two such conductors constituting a parallel plane transmission line, with voltage and current waves propagating in the longitudinal z direction on the line. The normal to the conductor surfaces is in the y direction, and the conductor surfaces extend indefinitely in the transverse x direction. The distributed resistance to be determined is the resistance of the conductors per unit length in the z direction, per unit width in the x direction. Initially the conductors are assumed to be indefinitely thick in the y direction. The distance between the conductors is not relevant to the problem, and attention is directed to only one of the conductors, whose surface is in the coordinate plane at $y = 0$, the direction of increasing y being into the metal. The coordinate relations are shown in Fig. 6-5, below.

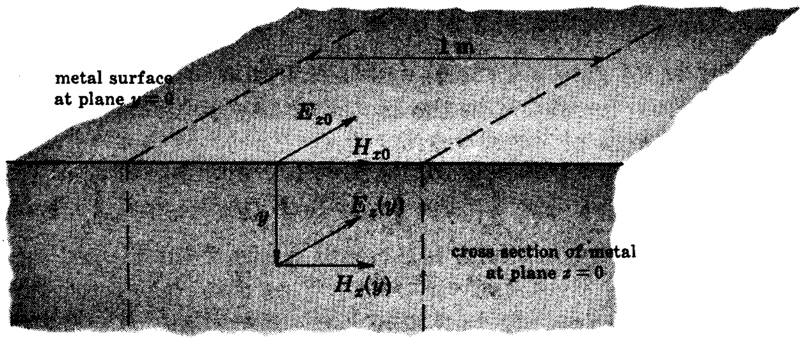


Fig. 6-5. Coordinate relations for investigating a-c current flow in a thick metal sheet.

Maxwell's equations for these time-harmonic fields are

$$\text{curl}\,\mathbf{E} = -j\omega\mu\mathbf{H} \tag{6.35}$$

$$\text{curl}\,\mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E} \tag{6.36}$$

It has been seen in connection with equation (*6.6*) that the displacement current density $\omega\epsilon E$ in (*6.36*) is negligible compared to the conduction current density σE for all ordinary metals at all frequencies up to 10^{12} hertz or higher. (This would not be true if the conductors were made, for example, of silicon.)

Dropping the term $j\omega\epsilon$, the wave equation is obtained from (*6.35*) and (*6.36*) by taking the curl of either and substituting into it from the other. This leads to

$$\nabla^2\mathbf{E} = j\omega\mu\sigma\mathbf{E} \tag{6.37}$$

and the same equation in **H**, where it is to be understood that (*6.37*) is actually three equations, one for each separate rectangular coordinate component of **E**. Since it has been postulated that the only component of **E** is E_z, equation (*6.37*) becomes

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = j\omega\mu\sigma E_z \tag{6.38}$$

Since it has also been postulated that the wave planes of the plane electromagnetic wave being considered extend indefinitely in the x and z directions, the derivatives $\partial/\partial x$ and $\partial/\partial z$ must both be zero. Finally, then, (*6.38*) takes the elementary form

$$\frac{\partial^2 E_z}{\partial y^2} = j\omega\mu\sigma E_z \tag{6.39}$$

whose solution is

$$E_z(y) = E_{z0}\, e^{\pm\gamma y} \tag{6.40}$$

where $\gamma = \sqrt{j\omega\mu\sigma} = (1+j)/\delta$ and $\delta = \sqrt{2/\omega\mu\sigma}$ is the same skin depth defined by equation (*6.15*). It follows that

$$E_z(y) = E_{z0}\, e^{-y/\delta}\, e^{-jy/\delta} \tag{6.41}$$

for a wave traveling into the metal in the positive y direction.

The meaning of equation (*6.41*) is that the electric field of the wave suffers an attenuation of 1 neper and a phase delay of 1 radian in traveling distance δ through the metal. (The propagation of voltage and current waves on a transmission line would be analogous if the line had the properties $R \ll \omega L$ and $G \gg \omega C$. The corresponding value of δ would be $\sqrt{2/(\omega LG)}$.)

At any point in the metal the current density produced by the electric field of equation (*6.41*) flows in the z direction and is given by

$$J_z(y) = \sigma E_z(y) \qquad (6.42)$$

The power loss per unit volume in the metal at any point is $|J_z(y)|^2/\sigma = \sigma|E_z(y)|^2$, and the power loss per unit area of metal surface is

$$\begin{aligned} P_L &= \int_0^1 dx \int_0^\infty dy \int_0^1 dz \,\{\sigma|E_z(y)|^2\} = \int_0^\infty \sigma|E_z(y)|^2\,dy \\ &= \int_0^\infty \sigma|E_{z0}|^2 e^{-2y/\delta}\,dy = \sigma|E_{z0}|^2\,\delta/2 \end{aligned} \qquad (6.43)$$

where $|E_z(y)|$ is obtained from equation (*6.41*).

From (*6.41*), the field E_z and consequently the current density J_z fall to negligible fractions of the surface values at a depth of less than 10δ, which at high frequencies can be a very small distance. The limit of infinity on the integrals with respect to y in (*6.43*) could be replaced by this distance.

A more tangible electrical variable in this skin effect situation than the tangential electric field at the surface of the metal is the total current in the conductor per unit width of surface. This is contained within a thickness of less than 10δ at the surface of the metal. It must be identified notationally as a *surface-current density* (mks units, amperes/meter), physically distinct from the current density J_z at a point (mks units, amperes/square meter). If J_{sz} is this *surface* current density,

$$J_{sz} = \int_0^1 dx \int_0^\infty dy\, J_z(y) = \int_0^\infty \sigma E_{z0}\, e^{-(1+j)y/\delta}\,dy = \sigma E_{z0}\,\delta/(1+j) \qquad (6.44)$$

The phase relation here is meaningful, establishing that the total surface-current density J_{sz} lags the surface electric field E_{z0} by 45°. (Note, however, that the *current density at a point at the surface* J_{z0} must be in phase with E_{z0} according to equation (*6.42*). The distinction between the total surface-current density and the current density at a point at the surface is a vital one.)

In the case of the solid circular conductor, a distributed internal conductor impedance was defined in equation (*6.21*) by the ratio of the tangential electric field at the surface to the total current in the conductor. The corresponding concept for plane conductors is a longitudinally distributed internal impedance for unit width of conductor. Designating this as $Z_s = R_s + jX_s$, equation (*6.44*) shows that

$$Z_s = R_s + jX_s = (1+j)/\sigma\delta = R_s(1+j) \qquad (6.45)$$

Thus the real and imaginary parts of this distributed internal impedance, for unit width of a plane conductor having thickness not less than about 10 skin depths, are both equal to the quantity R_s previously defined in (*6.27*), commonly known as the surface resistivity of the material and numerically equal to the d-c resistance between opposite edges of any square sheet of the material of thickness δ. The units of R_s and hence of X_s and Z_s are ohms, or ohms/square.

Substituting for $|E_{z0}|$ from (*6.44*) into (*6.43*),

$$P_L = |J_{sz}|^2/\sigma\delta = |J_{sz}|^2 R_s \qquad (6.46)$$

According to (*6.46*) the power loss per unit area associated with the a-c surface-current density J_{sz} in amperes/(meter width of surface), for which the point current density J_z diminishes exponentially with depth into the metal, is the same as the power loss per unit area that would occur if a current of the same total rms value (whether a-c or d-c) flowed with constant point current density in a skin of the conductor of thickness δ. This result is known as the *skin effect theorem.* Its application to solid circular conductors when a/δ is of the order of 100 or greater has already been seen in equation (*6.30*).

The agreement of equations (*6.30*) and (*6.31*) with (*6.45*) for very large values of a/δ is a recognition of the fact that when a circular conductor carries an a-c current, if the radius of curvature of the conductor's surface is very large compared to the skin depth, then the surface can be regarded as a plane, and the skin effect calculations for plane surfaces are applicable.

The use of the name "skin depth" and a superficial misinterpretation of the skin effect theorem sometimes lead to the erroneous impression that high frequency currents are physically totally contained within a skin of thickness δ, at the surface of a plane conductor or of a curved conductor whose radius of curvature is very large compared with δ. The actual distribution of high frequency current in such cases, as given by (*6.41*) and (*6.42*) is of course that the magnitude of the current density at a point diminishes exponentially with distance into the metal, while the phase retards linearly with distance. At any point within the metal, the amplitude reduction of the current density in nepers and the phase lag in radians, relative to the surface values, are numerically equal.

Simple calculations show that the current density falls to somewhat less than 1% of its surface magnitude at a depth of 5δ, and consequently to less than 0.01% of its surface magnitude at a depth of 10δ. Thus conductors carrying time-harmonic currents of any frequency need never be more than 5δ to 10δ thick for that frequency. Additional metal would serve no electrical purpose. For reasons to be explained, a plane conductor or a circular metal tube conductor of fixed outside radius $a \gg \delta$ actually have lower a-c resistance when the metal thickness is about 1.6δ than for either smaller or larger thicknesses, and their distributed resistance per unit width of surface remains within $\frac{1}{2}$% of R_s for all thicknesses greater than 3δ.

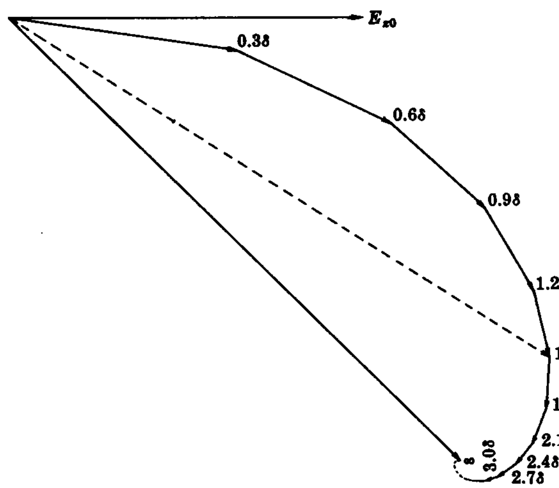


Fig. 6-6. Summation of surface-current density phasors in consecutive layers of a thick metal sheet carrying a-c current. The layers are 0.3 skin depths thick in the direction normal to the metal's plane surface, and 1 m wide parallel to the surface. The current flow is parallel to the surface. The largest surface-current density phasor at the top of the diagram is for the layer at the surface of the metal. The reference phasor is E_{z0}, the tangential electric field at the surface of the metal. The dashed line suggests that if the current beyond 1.5 skin depths into the metal could be eliminated, the total surface-current density would be the same as in an indefinitely thick sheet, but the losses would be reduced.

Closer inspection of equations (*6.41*) and (*6.42*) reveals an interesting phenomenon associated with a-c current flow in plane conductors. If the plane conductor is imagined to consist of a large number of uniform plane layers of equal thickness, the thickness Δy of each layer being conveniently about 0.3δ, then the longitudinal phasor current ΔJ_{sz} in unit width of any layer (this is an increment of the total surface-current density J_{sz}) would be given by $\Delta J_{sz}(y) = \Delta J_{sz}(0)\, e^{-(1+j)y/\delta}$ for a layer at depth y below the surface, where $\Delta J_{sz}(0)$ is the phasor current in unit width of the layer at the surface. Relative to the phasor current in the surface layer, the phasor currents in layers farther into the metal are smaller in magnitude and retarded in phase. Fig. 6-6 is a phasor diagram showing the cumulative addition of the phasor currents in consecutive layers, for layers totalling 3δ in thickness.

The value of y for each layer has been measured to the center of the layer. The value of $|\Delta J_{sz}(0)|$ in the first layer is an arbitrary scale factor. Since the center of the first layer is at $y = 0.15\delta$, its phase angle is -0.15 rad relative to the zero reference phasor given by the tangential electric field at the surface.

Fig. 6-6 could also be obtained as the envelope of the successive phasors given by the integral used in obtaining equation (*6.44*), with the upper limit varied through a sequence of values ranging from 0 to 3δ in suitable steps of 0.3δ.

The procedure of equation (*6.43*) for obtaining the power loss per unit area in an indefinitely thick plane conductor for a-c currents is equivalent to evaluating $|\Delta J_{sz}(y)|^2/(\sigma\,\Delta y)$ over an infinite set of layers, each having thickness Δy, on letting Δy become infinitesimally small. The result would be only slightly different for $\Delta y = 0.3$ skin depths. It is obvious from the convoluted form of the cumulative-phasor curve in Fig. 6-6 that the summation or integral of (*6.43*) must be quite a bit greater, for the same total conductor current magnitude, than if the incremental current phasors in successive layers all had the same phase.

These facts suggest qualitatively that some reduction of distributed surface resistance for plane conductors (and circular tube conductors with large a/δ) might be achieved by making the metal thin enough to eliminate the "backward current" portion of the phasor diagram of Fig. 6-6, without reducing the total current magnitude. This would make the length of the cumulative-phasor curve more nearly equal to the chord representing the total conductor current phasor.

The dashed total current phasor in Fig. 6-6 illustrates this hypothesis for a conductor 1.5 skin depths thick. The total current phasor in this case has almost exactly the same magnitude as for the indefinitely thick metal, but the curve constructed by adding the current phasors in successive layers is much shorter than before. The implication is that for the same total current the losses should be less in metal of thickness 1.5δ than for greater thicknesses. Although the model from which this conclusion has been drawn is somewhat oversimplified, because the postulated electromagnetic waves in the metal are actually reflected from the second surface of the thin sheet, and the combination of reflected waves with original waves gives a current phasor pattern differing appreciably from that for 1.5δ in Fig. 6-6, the reduction of a-c resistance in thin metal surfaces is nevertheless real.

Quantitative proof is given in Problem 7.10, page 147, where the analysis is made of a transmission line analog. A graph of the variation with thickness of the distributed resistance of a plane conductor, relative to the distributed resistance of an indefinitely thick conductor, is shown in Fig. 6-7 below. It appears that the resistance can be reduced by a maximum of about 8% for thicknesses of 1.5δ to 1.6δ, and that 5% reduction is obtained from about 1.3δ to 2.0δ. Since the abscissa of the curve is metal thickness in skin depths, it can be converted to a linear scale of thickness in meters at constant frequency, or to a scale proportional to the square root of the frequency for a given metal of constant thickness.

The quantitative conclusions drawn from Fig. 6-7 are directly applicable to tubular circular conductors for which the outer radius is very large compared to the skin depth.

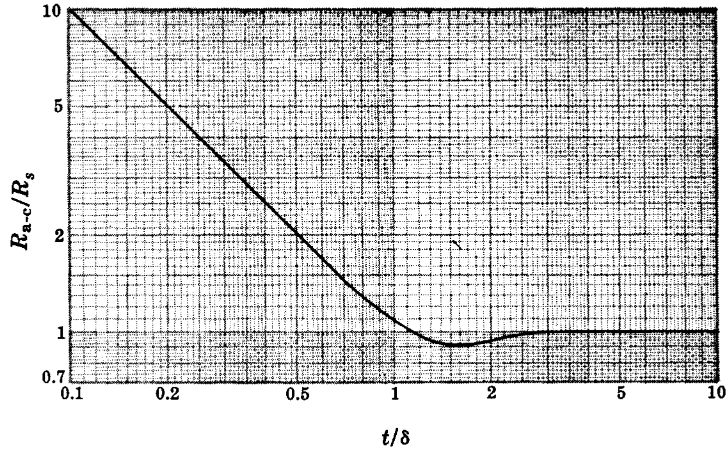


Fig. 6-7. Ratio of the distributed a-c resistance per unit width for an unbounded plane conductor of thickness t/δ skin depths to the distributed a-c resistance per unit width of an indefinitely thick conductor of the same metal. For t/δ less than about 0.5, the distributed a-c resistance is equal to the distributed d-c resistance. For t/δ greater than about 3, the distributed a-c resistance is equal to R_s.

6.4. Distributed resistance of tubular circular conductors.

Circular metal tubes are the optimum outer conductors for a coaxial transmission line, and they may be used for the center conductor of such a line also, or for the two identical conductors of a parallel wire line, in cases where solid conductors would be excessively heavy or would involve unduly inefficient use of metal.

Unfortunately, rigorous analysis of the distributed resistance and distributed internal inductance of circular tubular conductors, for all ranges of the ratio of wall thickness to outside diameter and of the ratio of outside diameter to skin depth, is a considerably more complicated and tedious procedure than the corresponding analysis for solid circular conductors given in Section 6.2. Equations (*6.9*) and (*6.10*) are still the basic equations, but boundary conditions must be met at two boundaries instead of one, and the coefficient A_2 in (*6.10*) is not zero.

The results of a complete analysis of the distributed internal *inductance* of tubular circular conductors do not appear to be available in convenient form. Discussion of the relative magnitudes of the distributed *internal* inductance and the distributed *external* inductance for various transmission line designs is continued in Section 6.8, which also includes a review of the procedures for estimating the value of the former.

The distributed a-c *resistance* of an isolated circular tubular conductor for a wide range of the variables was computed several decades ago by H. B. Dwight, with results shown graphically in Fig. 6-8 below. The top linear horizontal scale in $\sqrt{f/R_{d\text{-}c}}$, where f is the frequency in hertz and $R_{d\text{-}c}$ the distributed d-c resistance in ohms/meter length, applies to both solid and tubular circular conductors made of nonmagnetic material. For solid circular nonmagnetic conductors $\sqrt{f/R_{d\text{-}c}}$ is easily shown to be directly proportional to the variable a/δ already used in discussing the distributed a-c resistance of such conductors. The constant of proportionality is such that $\sqrt{f/R_{d\text{-}c}} = 892$ corresponds to $a/\delta = 1$. In calculating this, it is found to be given by $1/\sqrt{\mu_0}$ where μ_0 ($= 4\pi \times 10^{-7}$ henries/m) is the mks value of the permeability of nonmagnetic materials. Since $t/a = 1$ corresponds to a solid conductor, the curve in Fig. 6-8 for $t/a = 1$ is a plot of the data of Table 6.1.

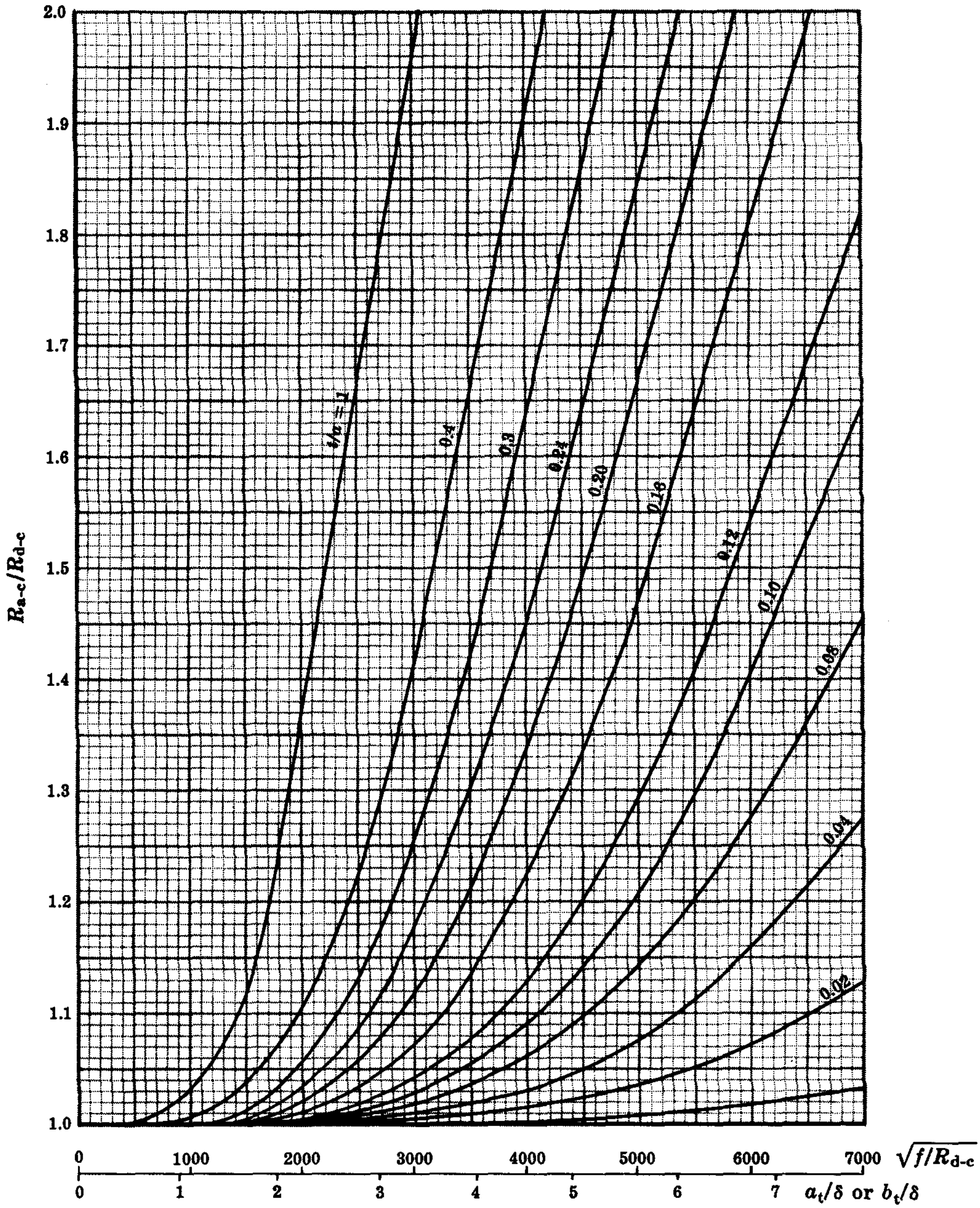


Fig. 6-8. Ratio of the distributed a-c resistance to the distributed d-c resistance of a circular tubular conductor of outside radius a, inside radius b, and wall thickness t as a function of dimension parameters in skin depths, for several values of the ratio of wall thickness to outside radius. The variables a_t and b_t are given by equations (*6.47*) and (*6.48*) respectively. The horizontal scale $\sqrt{f/R_{d\text{-}c}}$, for which f is in hertz and $R_{d\text{-}c}$ is in ohms/m, applies only to nonmagnetic conductor materials. The scales a_t and b_t apply to both magnetic and nonmagnetic materials. The ratio of the two scales at any point is 892. (After H. B. Dwight.)

For tubular conductors the scale conversion in Fig. 6-8 is a less direct one. The quantity corresponding to a/δ for the solid circular conductor is $\sqrt{2at-t^2}/\delta$ where a is the *outside* radius of the conductor and t its wall thickness. It reduces to a/δ when $t = a$. To permit further reference, this quantity is given the symbol a_t/δ. Then

$$a_t/\delta = \sqrt{2at-t^2}/\delta \tag{6.47}$$

When reference to the inside radius b of a tubular conductor is more convenient, the variable b_t/δ is defined by

$$b_t/\delta = \sqrt{2bt+t^2}/\delta \tag{6.48}$$

For a given tube a_t and b_t have identical values. The bottom horizontal scale of Fig. 6-8 is the same for a/δ, a_t/δ and b_t/δ.

The coordinate quantity $\sqrt{f/R_{\text{d-c}}}$ in Fig. 6-8 is unaffected by whether the conductor is magnetic or not. The question therefore arises as to how the graph should be used for conductors made of magnetic metals. The fact that the curve in Fig. 6-8 for solid conductors ($t/a = 1$) agrees with the $R/R_{\text{d-c}}$ figures from Table 6.1 where the permeability of the metal is accounted for in the skin depth δ, indicates that the abscissa scale of a_t/δ and b_t/δ (or a/δ for solid conductors) is applicable to either magnetic or nonmagnetic conductors. For conductors made of magnetic materials the scale $\sqrt{f/R_{\text{d-c}}}$ must not be used. It is clear from Dwight's writings that Fig. 6-8 was in fact constructed from expressions that derive from equations (*6.10*) or (*6.16*), in which the variable has the form $\sqrt{2}\,r/\delta$.

Since there are no electromagnetic fields in the interior space of the tubular conductor for which Fig. 6-8 was computed, and since the external fields and those within the metal must be the same at all angular positions around such a conductor, the only transmission line conductors to which Fig. 6-8 is directly applicable are the cases of a circular metal tube used as either the center conductor of a coaxial line or as a conductor of a parallel wire line whose two conductors are widely separated. However, if the ratio of a tube's wall thickness t to its external radius a does not exceed about 20%, the distributed resistance of a circular tube used as the *outer* conductor of a coaxial line can also be taken from Fig. 6-8 with adequate accuracy.

It is useful to classify the procedures for calculating the distributed a-c resistance of circular tubular conductors into the same five categories that were established in Section 6.2 for solid circular conductors.

(1) At the lowest frequencies, the quantitative relation for solid circular conductors stated in Section 6.2, that $R_{\text{a-c}}/R_{\text{d-c}} < 1.005$ for a/δ from 0 to 0.5, applies directly to tubular conductors if the modified variable $(\sqrt{t/a})(a/\delta)$ is substituted for a/δ. This statement is derived empirically from Fig. 6-8 and is a consequence of the shapes and interrelations of the curves of that figure. It has no connection with the variables a_t/δ or b_t/δ derived from the horizontal scale conversions.

(2) Using the modified variable $(\sqrt{t/a})(a/\delta)$ in place of a/δ, equation (*6.28*) gives the ratio of the distributed a-c resistance to the distributed d-c resistance for circular tubular conductors within $\frac{1}{2}$% for values of the modified variable up to 1.5. Equation (*6.28*) then covers all of the portion of Fig. 6-8 for which $R_{\text{a-c}}/R_{\text{d-c}}$ lies between 1.0 and 1.1.

(3) For values of the *unmodified* variable a/δ greater than 100, the distributed a-c resistance of circular tubular conductors is calculated from the same plane surface approximation used for solid circular conductors. Equation (*6.30*) will give results accurate to $\frac{1}{2}$%, when a is the *radius of the surface carrying the surface current,* provided the tube's wall is "thick" (i.e. t/δ greater than about 3), a condition that will almost invariably be satisfied for such large values of a/δ. For tubular conductors used in transmission lines the current carrying surface is the outside surface except when the tube is the

outer conductor of a coaxial line, in which case it is the inside surface. In the unlikely event that $t/\delta < 3$ when $a/\delta > 100$, the distributed a-c resistance value given by (*6.30*) for the appropriate surface must be multiplied by a correction factor from Fig. 6-7 for conductors of "finite" thickness.

(4) For values of the unmodified variable a/δ down to 4, equations (*6.32*) and (*6.33*) derived for solid circular conductors also apply to tubular circular conductors, subject to the two conditions stated in (3), and with the further modification that if the tube is the *outer* conductor of a coaxial line, the sign in the denominator of each equation must be changed to positive.

As in (3), if $t/\delta < 3$, the multiplying factor from Fig. 6-7 must be used. With a/δ as low as 4, if t/a is very small then t/δ may be considerably less than unity, in which case the distributed a-c resistance of the conductor becomes equal to its d-c resistance, as can in fact be seen directly in Fig. 6-8. What has happened in this case is that conditions have fallen into a category already covered in (1) or (2) for the modified variable $(\sqrt{t/a})(a/\delta)$.

(5) For combinations of a/δ and t/a or t/δ not covered by any of the four categories above, Fig. 6-8 is used directly, with the horizontal scale in a_t or b_t, after one of those quantities has been calculated from (*6.47*) or (*6.48*).

A final precaution must be repeated, that for the unusual combination of a/δ fairly small and t/a fairly large (i.e. a very thick walled tube at a low to intermediate frequency) the value given by Fig. 6-8 for $R_{a\text{-}c}/R_{d\text{-}c}$ may be as much as several percent in error for a tubular conductor used as the *outer* conductor of a coaxial line. The inaccuracy is partially discounted if the distributed resistance of the outer conductor of the coaxial line is much less than that of the inner conductor, as is often true.

Example 6.2.

A coaxial transmission line has a solid copper center conductor 0.100″ in diameter. The outer conductor is a tube of outside diameter 0.375″ and wall thickness 0.0100″. The line is used at carrier frequencies between 60 kilohertz and 10 megahertz. Determine the distributed resistance of each of the conductors at the highest and lowest frequencies used on the line.

Using a for the outer radius of the inner conductor and b for the inner radius of the outer conductor, the quantities a/δ, b/δ, t/δ, a_t/δ, and t/b must be evaluated first, to identify the category of each of the four distributed resistance calculations.

The skin depth δ is given by equation (*6.15*) which for 100% conductivity copper ($\sigma = 5.80 \times 10^7$ mhos/m) becomes $\delta = 0.0661/\sqrt{f}$ m when f is in hertz. From this, $\delta = 2.70 \times 10^{-4}$ m at 60 kilohertz, and $\delta = 2.09 \times 10^{-5}$ m at 10 megahertz. The required values of the five ratios at the two frequencies are:

	60 kilohertz	10 megahertz
For the inner conductor, a/δ ($a = 1.27 \times 10^{-3}$ m)	4.71	61
For the outer conductor, b/δ ($b = 4.51 \times 10^{-3}$ m)	16.7	216
t/δ ($t = 2.54 \times 10^{-4}$ m)	0.94	12.2
a_t/δ (equation (*6.47*))	5.7	73.5
t/b	0.056	0.056

Two of the distributed resistance calculations fall clearly into category (4), one clearly into (3), and one marginally ($a/\delta = 61$) into (3). For both calculations in category (4) the values of a/δ or a_t/δ are low enough for the results to be checked by the procedure of category (5), i.e. by Fig. 6-8. For one of the calculations in category (4) the value of t/δ is small enough to require a correction for insufficient wall thickness. Evaluating the modified variable $(\sqrt{t/a})(a/\delta)$ does not change the classification of any of the calculations.

The distributed resistance calculations are as follows.

Inner conductor at 60 kilohertz, category (4):

$$R_{\text{a-c}} = \frac{1}{\sigma 2\pi(a - \frac{1}{2}\delta)\delta} = \frac{1}{2\pi(5.80 \times 10^7)(1.27 \times 10^{-3} - 1.35 \times 10^{-4})(2.70 \times 10^{-4})} = 0.00894 \text{ ohms/m}$$

In this calculation the length units for a, b and δ must be the same as for σ, in meters.

This result can be checked on Fig. 6-8 by a linear extrapolation of the curve for $t/a = 1$ to the ordinate $a/\delta = 4.71$. The result is $R_{\text{a-c}}/R_{\text{d-c}} = 2.63$ and since $R_{\text{d-c}}$ from equation (*6.1*) is found to be 0.00340 ohms/m, $R_{\text{a-c}}$ is given as 0.00894 ohms/m.

Inner conductor at 10 megahertz, category (4):

$$R_{\text{a-c}} = \frac{1}{2\pi(5.80 \times 10^7)(1.27 \times 10^{-3} - 1.04 \times 10^{-5})(2.09 \times 10^{-5})} = 0.104 \text{ ohms/m}$$

If this case is calculated as category (5) the result is 0.103 ohms/m, the difference being very small for such a high value of a/δ.

Outer conductor at 60 kilohertz, category (4):

$$R_{\text{a-c}} = \frac{1}{2\pi(5.80 \times 10^7)(4.51 \times 10^{-3} + 1.35 \times 10^{-4})(2.70 \times 10^{-4})} = 0.00219 \text{ ohms/m}$$

The change of sign in the denominator term, relative to equation (*6.32*) as used in the two previous calculations, results from this being the *outer* conductor of a coaxial line. The current carrying surface is the inside surface of the tube.

Since t/δ is only 0.94 for this tube at this frequency, there is a correction factor from Fig. 6-7 of 1.14 and the distributed resistance of the outer tube at 60 kilohertz is $0.00219 \times 1.14 = 0.00250$ ohms/m.

The result can be checked directly from Fig. 6-8. For $a_t/\delta = 5.7$, $R_{\text{a-c}}/R_{\text{d-c}}$ is found to be between 1.07 and 1.08 for a tube whose ratio of wall thickness to radius is 0.056. From a modification of equation (*6.1*), $R_{\text{d-c}} = \dfrac{1}{\sigma 2\pi(b + \frac{1}{2}t)t} = 0.00233$ ohms/m, and hence $R_{\text{a-c}} \doteq 0.00233 \times 1.07 \doteq 0.00250$ ohms/m, within about $\frac{1}{2}$%.

Outer conductor at 10 megahertz, category (5):

$$R_{\text{a-c}} = \frac{1}{\sigma 2\pi b\delta} = \frac{1}{2\pi(5.80 \times 10^7)(4.51 \times 10^{-3})(2.09 \times 10^{-5})} = 0.0292 \text{ ohms/m}$$

6.5. Distributed circuit coefficients of coaxial lines.

(a) Distributed resistance.

Methods for calculating distributed resistances have been fully covered in Sections 6.2, 6.3 and 6.4, including Example 6.2. For coaxial lines used at frequencies of tens to hundreds of megahertz and higher, a specific expression obtained from equation (*6.30*) is worth noting:

$$R_{\text{hf}}(\text{coax}) = \frac{R_s}{2\pi b}\left(1 + \frac{b}{a}\right) \qquad (6.49)$$

where $R_{\text{hf}}(\text{coax})$ is the *total* distributed resistance of the coaxial line at very high frequencies. Here a is the outside radius of the inner conductor, b the inside radius of the outer conductor, and $R_s = 1/(\sigma\delta) = \sqrt{\omega\mu/2\sigma}$ is the surface resistivity of the conductor material defined in connection with equation (*6.27*) and assumed the same for both conductors. This expression requires that the wall thickness of both conductors be greater than 3δ, and will give better than $\frac{1}{2}$% accuracy when $a/\delta > 100$, which ensures that $b/\delta \gg 100$. If the two conductors are of different materials, equation (*6.30*) must be applied separately to each. For values of $a/\delta < 100$ the distributed resistance must be calculated by one of the other methods described in the preceding sections.

(b) Distributed capacitance.

Derivation of an expression for the distributed capacitance of a coaxial line is the simplest of all the distributed circuit coefficient derivations, and no case has fewer complicating factors in its practical calculations.

Fig. 6-9 shows the central cross section, in a plane containing the axis, for a portion of coaxial line included between two transverse planes separated by distance Δl. The facing surfaces of the two conductors have radii a and b respectively, as prescribed in (a) above. It is postulated that there is free charge $+\Delta Q$ coulombs uniformly distributed over the length Δl of the outside surface of the inner conductor of the line and free charge $-\Delta Q$ coulombs similarly uniformly distributed over the length Δl of the inside surface of the outer conductor. It is also postulated that the electric charge distribution on the two conductors is continuous on either side of the length Δl so that the electric flux lines throughout Δl are everywhere radial. From symmetry the electric flux density at any point in the interconductor space over the length Δl is independent of angular position around the axis of the conductors.

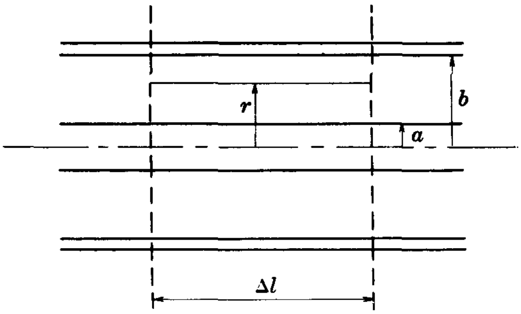


Fig. 6-9. Cross section of a coaxial transmission line in a longitudinal diametral plane with notation for determining distributed capacitance.

Crossing the length Δl of any cylinder of radius r concentric with the line's conductors, where $a < r < b$, the total electric flux is ΔQ coulombs, and the radial electric flux density $D_r(r)$ at any point on the cylinder is

$$D_r(r) \;=\; \frac{\Delta Q}{\Delta l}\frac{1}{2\pi r} \;=\; \frac{\rho_l}{2\pi r} \quad \text{coulombs/m}^2 \tag{6.50}$$

where ρ_l is the uniformly distributed longitudinal charge density in coulombs/m over Δl.

If the space between the conductors is filled with a homogeneous lossless isotropic medium of (real) permittivity ϵ' farads/m, the electric field $E_r(r)$ at radius r is

$$E_r(r) \;=\; \frac{D_r(r)}{\epsilon'} \;=\; \frac{\rho_l}{2\pi\epsilon' r} \quad \text{volts/m} \tag{6.51}$$

Postulate 4 of transmission line analysis, as given in Chapter 2, states that the potential difference between the two conductors of a line at any cross section has a unique value given by the line integral of the electric field along any path between the conductor surfaces, which are equipotentials. Since in the present derivation Δl can always be taken small enough to ensure that there is negligible longitudinal potential difference along the length Δl of the conductors, it follows that the potential difference between the two conductors at any cross section within Δl is given by $V_a - V_b = \int_a^b -E_r(r)\,dr$, or

$$V_b - V_a \;=\; \int_a^b \frac{\rho_l\,dr}{2\pi\epsilon' r} \;=\; \frac{\rho_l}{2\pi\epsilon'}\log_e\frac{b}{a} \quad \text{volts} \tag{6.52}$$

By definition the capacitance between any two conductors or conductor elements is the ratio of the magnitude of either of the equal and opposite charges on them to the potential difference associated with the charges. Then if ΔC is the capacitance between the conductors of a coaxial line for length Δl, $\Delta C = \Delta Q/(V_b - V_a)$, and the *distributed* capacitance of the line is $C = \Delta C/\Delta l = (\Delta Q/\Delta l)/(V_b - V_a)$. Using equation (6.52) and the relation $\Delta Q/\Delta l = \rho_l$,

$$C = \frac{2\pi\epsilon'}{\log_e b/a} \text{ farads/m} \qquad (6.53)$$

If k'_e is the real dielectric constant of the lossless material filling the interconductor space, $k'_e = \epsilon'/\epsilon_0$, where $\epsilon_0 = 8.85 \times 10^{-12}$ farads/m is the permittivity of free space. Equation (6.53) can be written

$$C = \frac{55.6\, k'_e}{\log_e b/a} \text{ micromicrofarads/m} \qquad (6.54)$$

For most practical coaxial transmission lines the ratio b/a lies within a fairly narrow range from about 2 to less than 10, and the logarithm is then confined to an even smaller range from about 0.7 to 2. Except in unusual cases k'_e is the low dielectric constant of a low-loss low-density material, or is a small average value for a line partly filled with dielectric beads or discs. The distributed capacitance of a coaxial transmission line therefore usually lies within the range of about 25 to 200 micromicrofarads/m, and values between 50 and 100 micromicrofarads/m are most common.

Distributed capacitance calculations differ from those for distributed resistance and distributed inductance in that the effects of fields and currents within the metal of the conductors are completely negligible, for all conceivable physical conditions. Equation (6.54) is therefore a highly accurate and complete expression for the case to which it applies, of smooth uniform conductors with isotropic homogeneous dielectric filling the interconductor space. The complications introduced by a multi-strand center conductor or a braided outer conductor are not amenable to simple analytical treatment, and the distributed capacitance of such lines is usually determined by experimental measurement.

The dielectric constant k'_e of the insulating materials most widely used in coaxial transmission lines, such as teflon, polyethylene and polystyrene plastics, and steatite ceramics, varies by less than ½% over the frequency range from 60 hertz to 10 gigahertz or more, so that the distributed capacitance is generally a true "constant" of the line. Other less common insulating materials, including bakelite, glasses, acrylic plastics, rubbers, etc., show several percent variation of k'_e over the same frequency range.

(c) Distributed conductance.

In very rare instances the interconductor space of a coaxial transmission line is filled with material which conducts electricity by the actual flow of charged carriers, either electrons or ions. Damp earth, electrolytic solutions, and plasters or ceramics containing dispersed carbon would be examples of such materials, whose current carrying properties would be described by a true value of conductivity having the same significance as the conductivity of a metal. The distributed conductance of a transmission line filled with such a conducting dielectric is found, by appropriate application of equation (6.1), as the conductance between the inner and outer cylindrical surfaces for the piece of material filling unit length of the line (see Problem 6.31). Depending on the material, such a charge-flow conductance value may remain constant from zero frequency up to a few hertz, or to a few kilohertz, or even into the gigahertz frequency range.

Apart from these exceptional cases, which would exist only for special purposes and not as lines intended for efficient transmission of signals or power, the zero frequency distributed conductance of coaxial transmission lines insulated with plastics or ceramics is normally so small as to be difficult to measure, and the distributed conductance at any operating frequency is not caused by the flow of free charges but is a measure of internal dielectric losses in the insulating material resulting from repeated reversals of the dielectric polarization by the a-c electric field. Since the loss is thus on a per cycle basis, it tends to be directly proportional to frequency over wide ranges of frequency. It is then desirable to have an expression for the distributed conductance of a transmission line that contains the frequency explicitly, the remaining terms in the expression being more or less independent of frequency. This goal is achieved by the standard convention of designating the permittivity or dielectric constant of a lossy (but nonconducting for d-c) dielectric by a complex number. Thus for such a material

$$\epsilon \equiv \epsilon' - j\epsilon'', \qquad k_e \equiv k_e' - jk_e'' = \epsilon'/\epsilon_0 - j\epsilon''/\epsilon_0 \tag{6.55}$$

The distributed admittance of the distributed capacitance of a uniform transmission line is given by $G + jB$, where G is one of the line's four distributed circuit coefficients and $B = \omega C$ is the distributed susceptance of the line's distributed capacitance C. If in equation (*6.53*) for the distributed capacitance of a coaxial line the permittivity ϵ' or the dielectric constant k_e' is replaced by a complex number form from (*6.55*), then the quantity calculated as the distributed susceptance B will have both real and imaginary parts. The negative imaginary part of ϵ' contributes a positive real part to the distributed admittance, which is the distributed conductance of the line. Thus $G + jB = j\omega C = j\omega 2\pi(\epsilon' - j\epsilon'')/(\log_e b/a) = \omega\epsilon''2\pi/(\log_e b/a) + j\omega\epsilon'2\pi/(\log_e b/a)$. From the real parts of this relation,

$$G = \omega\frac{\epsilon''}{\epsilon'}\frac{2\pi\epsilon'}{\log_e b/a} = \omega C \tan\delta \tag{6.56}$$

where $C = 2\pi\epsilon'/(\log_e b/a)$ as in equation (*6.53*), and $\tan\delta = \epsilon''/\epsilon'$.

The quantity $\tan\delta$ used to designate the lossiness of a dielectric in a-c electric fields is called the loss factor or the tangent of the loss angle for the material. *It cannot be too strongly emphasized that this use of the Greek letter* δ *is in no way connected with the previous use of the letter* δ *for the skin depth in a conductor carrying a-c currents.* It is an unfortunate duplication that is firmly entrenched as the established notation in each case.

Equation (*6.56*) shows that the distributed conductance of a coaxial transmission line is directly proportional to frequency, if the loss factor $\tan\delta$ and the line's distributed capacitance C are independent of frequency. The constancy of C has been discussed in (*b*) above. For the most commonly used low loss insulating materials mentioned there, the loss factor $\tan\delta$ lies between 10^{-3} and 10^{-5}. Tabulated values often indicate only that the loss factor of polyethylene, for example, is less than 0.0002 for frequencies from hertz to gigahertz. This usually means that the value was below the sensitivity of the measuring equipment. Except at gigahertz frequencies, the value of G resulting from such low values of $\tan\delta$ is likely to be too small to affect the attenuation factor α as determined by the methods of Chapter 5. There is evidence that the precise value of $\tan\delta$ may vary considerably among different samples of polymer plastic dielectrics, depending on impurities and thermal history. Values of $\tan\delta$ listed in standard handbooks for low loss materials should usually be considered as approximate typical values only.

Example 6.3.

If the interconductor space of the coaxial transmission line of Example 6.2 is filled with teflon dielectric having constant $k_e' = 2.10$ and constant $\tan\delta = 0.00015$ over the frequency range 10 megahertz to 10 gigahertz, determine the distributed capacitance C and the distributed conductance G of the line at frequencies of 10^7, 10^8, 10^9 and 10^{10} hertz.

The distributed capacitance C has the same value at all the frequencies listed, given by equation (*6.53*) as

$$C = 2\pi(2.10 \times 8.85 \times 10^{-12})/(\log_e 0.1775/0.050) = 92.2 \text{ micromicrofarads/m}$$

The distributed conductance G is then directly proportional to frequency, according to (*6.56*), and at 10^7 hertz has the value

$$G = 2\pi \times 10^7(92.2 \times 10^{-12}) \times 0.00015 = 0.87 \text{ micromhos/m}$$

The values of G at the other frequencies are respectively 8.7, 87 and 870 micromhos/m.

(d) Distributed inductance.

Of the three distributed circuit coefficients so far considered for a coaxial line, the distributed resistance R is a quantity entirely "internal" to the conductors, being determined completely by the conductor materials and dimensions and the frequency. It is in no way dependent on the properties of the uniform medium filling the interconductor space. The distributed capacitance C and the distributed conductance G, on the other hand, are "external" quantities, being totally independent of the material of the conductors or of the transverse extension of the conductors on either side of the interconductor space. They are functions only of the nature and dimensions of the material filling the interconductor space, and of the frequency.

The fourth distributed circuit coefficient, distributed inductance, is the only one of the four such coefficients for a transmission line that in some cases has to be determined as the sum of both "external" and "internal" components. It is true that at frequencies above some minimum frequency that is typically in the megahertz region the distributed reactance of the distributed internal inductance of circular or plane conductors becomes equal to the distributed resistance of the conductors, as shown by equations (*6.31*) and (*6.45*), and since $\omega L/R \gg 1$ always under such conditions, it follows that the distributed external inductance must be very much greater than the distributed internal inductance for coaxial lines at such frequencies. However, at low frequencies the distributed internal inductance of a solid circular conductor as given by equation (*6.5*) can be a substantial fraction of the total distributed inductance, for a coaxial line.

The expression for the distributed external inductance L_x to be derived for a coaxial line is independent of frequency if the magnetic properties of the material in the interconductor space are not functions of frequency. It is the total distributed inductance of the coaxial line at frequencies high enough to make $a/\delta \gg 100$. At lower frequencies it must be added to a distributed internal inductance term L_i determined by methods reviewed in Section 6.8.

Referring to Fig. 6-10, a uniform coaxial transmission line has circular conductors, the outside radius of the inner conductor being a and the inside radius of the outer conductor being b, as in Fig. 6-9. Distributed external inductance is a measure of the linkage of magnetic flux in the interconductor space with the center conductor of the line as a distributed "circuit".

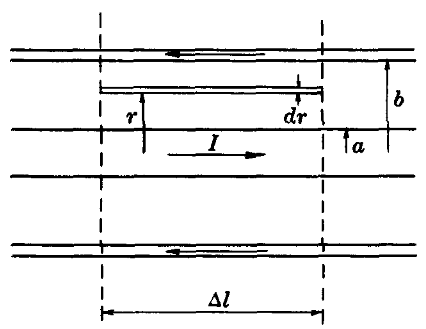


Fig. 6-10. Cross section of a coaxial transmission line in a longitudinal diametral plane with notation for determining distributed external inductance.

Considering a section of the transmission line of length Δl between transverse planes, a total current I flows longitudinally in one direction in the inner conductor, and an equal current flows in the opposite direction in the outer conductor. From the symmetry of the line the currents are uniformly distributed with respect to angular position around the periphery of the conductors, and the magnetic flux lines produced in the interconductor space by the current in the center conductor are circles concentric with the conductor. Δl is assumed short enough that no quantities in the problem vary in the longitudinal direction.

The only component of magnetic flux density is B_ϕ and its value $B_\phi(r)$ at any point in the interconductor space distant r from the central axis is

$$B_\phi(r) = \mu_m' I/2\pi r \text{ teslas} \tag{6.57}$$

where μ_m' is the (real) permeability of *the medium in the interconductor space.*

The magnetic flux passing through the small rectangle of length Δl and radial width dr at coordinate dr in the radial plane is $d\psi = B_\phi(r)\,\Delta l\, dr$. Then the total flux in the interconductor space linking length Δl of the center conductor is

$$\Delta\psi = \Delta l \int_a^b (\mu_m' I)/(2\pi r)\, dr = (1/2\pi)\,\Delta l\, \mu_m' I \log_e b/a$$

The distributed external inductance of the line, L_x, defined as the amount of external flux-circuit linkage per unit length per unit current, is then

$$L_x = \frac{1}{I}\frac{1}{\Delta l}\Delta\psi = \frac{\mu_m'}{2\pi}\log_e\frac{b}{a} \text{ henries/m} \tag{6.58}$$

Except for peculiar situations, such as filling the interconductor space of a coaxial line with a ferrite material, the permeability μ_m' always has the value for free space, $\mu_0 = 4\pi \times 10^{-7}$ henries/m, which is an exactly defined value in the mks system of units.

The comments in (*c*) above about the limited variation of b/a and of $\log_e(b/a)$ for practical coaxial lines are directly applicable here again, and mean that the distributed external inductance L_x of coaxial lines usually lies between about 0.1 and 10 microhenries/m. (In contrast to the small range of values of L_x and C for practical coaxial lines, the distributed resistance R varies by a factor of 10^5 among common types of lines.)

Example 6.4.

For the transmission line of Examples 6.2 and 6.3 determine the distributed external inductance. Then at a frequency of 10 megahertz find the characteristic impedance, phase velocity and attenuation factor of the line.

The distributed external inductance L_x at all frequencies is given by equation (*6.58*) with $\mu_m' = \mu_0$ for all nonmagnetic media in the interconductor space. Thus $L_x = 2 \times 10^{-7} \log_e 0.1775/0.050 = 0.253$ microhenries/m.

From Example 6.2, the values of a/δ and b/δ for this coaxial line at 10 megahertz are 61 and 216 respectively, which according to Table 6.2 ensures that the distributed internal inductance of the center conductor of the line is less than about 3% of its zero frequency value of 0.05 microhenries/m. It is therefore somewhat under 1% of the distributed external inductance and can be neglected. At the value $b/\delta = 216$ for the tubular outer conductor, equation (*6.31*) applies and $L_i = R/\omega = 0.00046$ microhenries/m, where R has the value 0.0292 ohms/m given in Example 6.2. This also is negligible in comparison with L_x. Hence for this coaxial line at a frequency of 10 megahertz, $L = L_x + L_i = L_x = 0.253$ microhenries/m. A check shows that at 10 megahertz $\omega L/R \doteq 120$, where R is the total distributed resistance of the line, and $\omega C/G = 1/(\tan\delta) \doteq 6000$. Hence the high frequency approximate equations of Chapter 5 are to be used in determining Z_0, v_p and α. The results are

$$Z_0 = \sqrt{L/C} = \sqrt{(2.53\times 10^{-7})/(92.2\times 10^{-12})} = 52.4 \text{ ohms}$$

$$v_p = 1/\sqrt{LC} = 2.07\times 10^8 \text{ m/sec}$$

$$\alpha = R/2Z_0 + GZ_0/2 = 0.00127 + 0.000023 = 0.00129 \text{ nepers/m}$$

It is to be noted that dielectric losses contribute about 2% of the attenuation factor at this frequency.

(e) Summary of high frequency relations for the coaxial line.

If the conditions $a/\delta > 100$, $b/\delta > 100$, $\omega L/R > 10$ and $\omega C/G > 10$ *all* hold for a coaxial line, which is likely to be the case for most lines at frequencies above some value between 1 and 100 megahertz depending on the line dimensions and materials, the following simplified expressions are easily derived

$$Z_0 = \sqrt{\frac{\mu_m'/2\pi}{2\pi\epsilon'}}\log_e\frac{b}{a} = \frac{1}{2\pi\sqrt{k_e'}}\sqrt{\frac{\mu_0}{\epsilon_0}}\log_e\frac{b}{a} = \frac{60}{\sqrt{k_e'}}\log_e\frac{b}{a} = \frac{138}{\sqrt{k_e'}}\log_{10}\frac{b}{a} \text{ ohms} \tag{6.59}$$

$$v_p = \frac{1}{\sqrt{\mu'_m \epsilon'}} = \frac{1}{\sqrt{k'_e}} \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{3.00 \times 10^8}{\sqrt{k'_e}} \text{ m/sec} \qquad (6.60)$$

Equations (*6.59*) and (*6.60*) both make the assumption, usually valid, that the material in the interconductor space of the coaxial line has the magnetic properties of free space. If this is not the case, Z_0 must be multiplied by, and v_p must be divided by, the square root of the relative permeability of the medium.

6.6. Distributed circuit coefficients of transmission lines with parallel circular conductors.

Measured by the number of miles in practical operation, parallel wire transmission lines must constitute a large majority of transmission line installations. Telephone pole lines and cable pairs and television antenna lead-in lines are among the most common examples.

The derivation of expressions for the distributed circuit coefficients of parallel wire lines follows the same basic procedures used in the derivations for coaxial lines, but the departure from cylindrical symmetry adds a complication. The effects of this distortion of symmetry are known collectively as "proximity effect". In the analyses for distributed capacitance, conductance and external inductance of parallel wire lines, proximity effect can be allowed for without too much difficulty, but the complete analyses of distributed resistance and internal inductance for solid and circular tubular conductors are mathematically too cumbersome to be presented here. Tabulated data and approximate expressions are given in (*a*) below and in Section 6.8.

(*a*) Distributed resistance.

For parallel wire transmission lines with identical solid circular conductors, the amount by which proximity effect increases the resistance depends on the material and radius of the conductors, and the frequency, combined in the variable a/δ as previously, and on the "proximity" of the conductors, expressed by the ratio of the separation s of their centers to the diameter $2a$ of each of them. The most complete analysis available is that of A. H. M. Arnold.

In a d-c circuit $(a/\delta = 0)$ there is no proximity effect even when the conductors are virtually in contact at their adjacent surfaces $(s/2a \doteq 1)$. At any finite value of the conductor separation there is a minimum value of a/δ at which proximity effect increases the distributed resistance perceptibly. With the conductors almost touching, for example, the distributed resistance will increase about $\frac{1}{2}\%$ from proximity effect when a/δ is approximately 0.5; but if the conductor axes are 8 diameters apart, the same increase in distributed resistance will not occur until a/δ is increased to about 2. As $s/2a$ increases from 8 to just over 10, the minimum value of a/δ at which proximity effect increases the distributed resistance by $\frac{1}{2}\%$ rises from 2 to infinity; and for axial separations greater than 10 or 12 conductor diameters, proximity effect is negligible at all frequencies. This statement applies to all the distributed circuit coefficients, and to lines with either solid or tubular circular conductors. In some cases the effects become negligible at smaller separations.

At high frequencies for which $a/\delta > 100$, proximity effect increases the distributed resistance of solid circular conductors by a factor P_{hf} given by the simple formula

$$P_{hf} = \frac{1}{\sqrt{1 - 1/(s/2a)^2}} \qquad (6.61)$$

While the implication of infinite distributed resistance for conductors approaching contact is not to be taken literally, predicted values greater than 2 for P_{hf} when the adjacent surfaces of the line's conductors are only 1% of a diameter apart $(s/2a = 1.01)$ have been confirmed experimentally.

Equation (*6.61*) applies also to circular tubular conductors for the stated conditions of $a/\delta > 100$, if the tube wall is thick, i.e. $t/\delta > 3$.

Example 6.5.

If the axes of the copper conductors in the 19 gauge cable pair transmission line of Example 5.4, page 52, are separated on the average by 2.0 conductor diameters, determine the lowest frequency at which proximity effect will increase the distributed resistance of the line by about $\frac{1}{2}$%, using data for isolated 19 gauge conductors from Table 6.2, page 80. At a frequency of 3×10^8 hertz, find the distributed resistance of the line.

Interpolating among the approximate figures in the second paragraph of (*a*) above, it appears that for $s/2a = 2.0$ there should be a $\frac{1}{2}$% increase in distributed resistance for $a/\delta = 0.7$ approximately. From Table 6.2 this ratio occurs for 19 gauge copper wires at a frequency close to 10,000 hertz.

At a frequency of 3×10^8 hertz for the same wires, $a/\delta \doteq 100$. The distributed a-c resistance at that frequency from equation (*6.30*) is 1.57 ohms/m, for each conductor of the transmission line. The high frequency proximity effect factor from equation (*6.61*) is $P_{\text{hf}} = 1/\sqrt{1-(1/2)^2} = 1.15$. Hence the distributed resistance of the parallel wire line at 3×10^8 hertz is $1.57 \times 2 \times 1.15 = 3.61$ ohms/m. A small additional factor caused by the increase in effective length due to twist of the wire pair has been neglected.

After equation (*6.61*), the next simplest approximate equation for the proximity effect factor in parallel wire transmission lines having solid circular conductors applies to lines for which $s/2a > 2$, for all values of a/δ. Designating the factor as P_s under these conditions,

$$P_s \quad = \quad \frac{1}{\sqrt{1 - f_1(\sqrt{2}\,a/\delta)/(s/2a)^2}} \qquad (6.62)$$

where $f_1(\sqrt{2}\,a/\delta)$ is tabulated in Table 6.4 as given by Arnold. The actual function f_1 derives from a Bessel equation similar to equation (*6.27*).

Table 6.4

$\sqrt{2}\,a_t/\delta$	f_1	f_4	$\sqrt{2}\,a_t/\delta$	f_1	f_4	$\sqrt{2}\,a_t/\delta$	f_1	f_4
0.2	0.000	0.000	2.3	0.436	0.201	4.8	0.731	0.240
0.3	0.000	0.000	2.4	0.470	0.205	5.0	0.739	0.236
0.4	0.001	0.001	2.5	0.502	0.208	5.5	0.760	0.223
0.5	0.002	0.002	2.6	0.530	0.210	6.0	0.778	0.209
0.6	0.004	0.004	2.7	0.556	0.213	6.5	0.795	0.196
0.7	0.007	0.007	2.8	0.578	0.215	7.0	0.809	0.185
0.8	0.013	0.012	2.9	0.598	0.218	7.5	0.821	0.174
0.9	0.020	0.019	3.0	0.614	0.221	8.0	0.832	0.165
1.0	0.030	0.029	3.1	0.629	0.224	9	0.849	0.148
1.1	0.044	0.040	3.2	0.641	0.227	10	0.864	0.135
1.2	0.061	0.054	3.3	0.652	0.230	11	0.876	0.123
1.3	0.081	0.070	3.4	0.661	0.233	12	0.886	0.113
1.4	0.106	0.088	3.5	0.668	0.235	14	0.902	0.098
1.5	0.135	0.106	3.6	0.675	0.238	16	0.914	0.086
1.6	0.167	0.123	3.7	0.681	0.240	18	0.923	0.077
1.7	0.203	0.140	3.8	0.687	0.242	20	0.931	0.069
1.8	0.240	0.156	3.9	0.692	0.244	25	0.944	0.056
1.9	0.280	0.169	4.0	0.696	0.245	30	0.953	0.047
2.0	0.320	0.180	4.2	0.705	0.246	35	0.960	0.040
2.1	0.360	0.189	4.4	0.714	0.245	40	0.965	0.035
2.2	0.399	0.196	4.6	0.722	0.243	50	0.972	0.028
						Above 50	$1 - \delta/a_t$	δ/a_t

Finally, an equation stated by Arnold to give the value of the proximity effect factor for all circular solid and tubular conductors at all separations and all frequencies, with an accuracy of better than $\frac{1}{2}$% in almost all cases, is

$$P = \frac{1}{\sqrt{1 - A_1/(s/2a)^2 + [A_2/(s/2a)^4]/[1 - A_3/(s/2a)^2]}} \tag{6.63}$$

where

$$A_1 = f_1(\sqrt{2}\,a_t/\delta) + \{1 - (a_t/a)^2 - (a_t/a)[1 - (a_t/a)]\,f_7(\sqrt{2}\,a_t/\delta)\}\,f_4(\sqrt{2}\,a_t/\delta)$$

$$A_2 = f_2(\sqrt{2}\,a_t/\delta) + [1 - (a_t/a)^2]\,f_5(\sqrt{2}\,a_t/\delta)$$

$$A_3 = f_3(\sqrt{2}\,a_t/\delta) + [1 - (a_t/a)^2]\,f_6(\sqrt{2}\,a_t/\delta)$$

In these equations, $a_t = \sqrt{2at - t^2}$ for a circular tubular conductor of outside radius a and wall thickness t, as in equation (*6.47*). For a solid circular conductor $t = a$ and $a_t = a$.

The functions f_1, f_2, f_3, f_4, f_5 and f_6 are tabulated in Tables 6.4, 6.5 and 6.6 for all values of the variable $\sqrt{2}\,a_t/\delta$ (which is $\sqrt{2}\,a/\delta$ for solid conductors). The function f_7 is an empirical one given by $f_7 = (\sqrt{2}\,a_t/\delta)^3/[400 + (\sqrt{2}\,a_t/\delta)^3]$.

Table 6.5

$\sqrt{2}\,a_t/\delta$	f_2	f_5	$\sqrt{2}\,a_t/\delta$	f_2	f_5
0.5	0.000	0.000	3.6	0.053	−0.026
1.0	−0.001	−0.002	3.8	0.046	−0.021
1.2	−0.001	−0.003	4.0	0.039	−0.017
1.4	−0.000	−0.005	4.2	0.033	−0.014
1.6	+0.003	−0.006	4.4	0.027	−0.011
1.8	0.011	−0.007	4.6	0.023	−0.009
2.0	0.022	−0.010	4.8	0.020	−0.008
2.2	0.037	−0.015	5.0	0.018	−0.007
2.4	0.051	−0.022	5.5	0.014	−0.006
2.6	0.062	−0.028	6.0	0.012	−0.006
2.8	0.068	−0.033	7	0.009	−0.006
3.0	0.069	−0.034	8	0.007	−0.005
3.2	0.066	−0.033	10	0.005	−0.004
3.4	0.060	−0.030	Above 10	$\frac{1}{4}(\delta/a_t)^2$	$2(\delta/a_t)^4 - \frac{1}{4}(\delta/a_t)^2$

Table 6.6

$\sqrt{2}\,a_t/\delta$	f_3	f_6	$\sqrt{2}\,a_t/\delta$	f_3	f_6
0.0	0.09	0.03	4.0	0.41	0.33
0.2	0.09	0.03	4.2	0.46	0.30
0.4	0.09	0.03	4.4	0.51	0.27
0.6	0.08	0.02	4.6	0.56	0.24
0.8	0.08	0.02	4.8	0.60	0.21
1.0	0.06	0.00	5.0	0.64	0.19
1.2	0.02	*0.00 (−0.05)	5.2	0.67	0.17
1.4	*0.00 (−1.2)	*0.00 (0.91)	5.4	0.69	0.16
1.6	*0.02 (0.17)	*0.00 (−3.2)	5.6	0.70	0.15
1.8	*0.05 (0.11)	*0.00 (−0.09)	5.8	0.71	0.15
2.0	*0.07 (0.09)	0.44	6	0.72	0.14
2.2	0.08	0.44	7	0.76	0.13
2.4	0.08	0.44	8	0.79	0.11
2.6	0.10	0.44	9	0.83	0.08
2.8	0.12	0.44	10	0.85	0.07
3.0	0.15	0.44	12	0.87	0.05
3.2	0.19	0.43	14	0.89	0.03
3.4	0.24	0.42	16	0.90	0.02
3.6	0.29	0.39	18	0.93	0.00
3.8	0.35	0.36	20	0.93	*0.00
			Above 20	$1 - \delta/a_t$	*0.00

Values marked with an asterisk are artificial values which should be used in equations (*6.62*) and (*6.63*). When $\sqrt{2}\,a_t/\delta > 20$, the true value of f_6 is $\delta/a_t - 5/64$.

Arnold notes that the peculiar behavior of equation (*6.63*) when $A_3/(s/2a)^2 = 1$ has no physical meaning and the arithmetical anomaly should be avoided by using the starred figures in Table 6.6 instead of the proper values of the functions f_3 and f_6 in a certain range of $\sqrt{2}\,a/\delta$. This is a very minor complication that will seldom be encountered.

(b) Distributed capacitance.

The derivation of an expression for the distributed capacitance of a parallel wire transmission line, including proximity effect, is a somewhat indirect procedure. Consider two indefinitely long parallel lines lying in the xz coordinate plane at locations $x = +d$ and $x = -d$, as shown in cross section in Fig. 6-11. On each line, electric charge is uniformly distributed longitudinally, the linear charge density being $+\rho_l$ coulombs/m for the line at $x = +d$, and $-\rho_l$ coulombs/m for the line at $x = -d$.

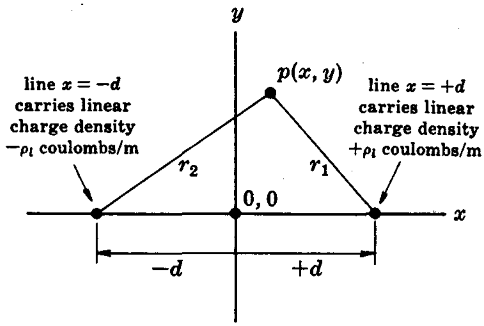


Fig. 6-11. Coordinates in a plane transverse to two infinitely long parallel lines of charge, having equal and opposite linear densities of charge.

The electric potential difference between any two points in the field of an indefinitely long uniformly distributed line of charge is a function of the linear charge density on the line, the permittivity of the surrounding medium, and the radial distances of the two points from the line. At any point $p(x, y)$ in an xy plane transverse to the charge lines of Fig. 6-11, the potential relative to a zero reference potential on the axis $x = y = 0$ is $V_p^+ = +(\rho_l/2\pi\epsilon')(\log_e d/r_1)$ from the field of the positively charged line, and $V_p^- = -(\rho_l/2\pi\epsilon')(\log_e d/r_2)$ from the field of the negatively charged line, r_1 and r_2 being respectively the distances from the point p to the positively and negatively charged lines. The total potential difference between the point p and the axis $x = y = 0$ is then

$$V_p \quad = \quad \frac{\rho_l}{2\pi\epsilon'} \log_e \frac{r_2}{r_1} \quad \text{volts} \tag{6.64}$$

An equipotential line in the transverse plane will be described by the relation

$$r_2/r_1 \quad = \quad \text{constant } K \quad = \quad e^{2\pi\epsilon' V_p/\rho_l} \tag{6.65}$$

From the coordinates of the point p, $r_1 = \sqrt{(d-x)^2 + y^2}$ and $r_2 = \sqrt{(d+x)^2 + y^2}$. Combining these expressions through $r_2/r_1 = K$, the equation of an equipotential line becomes

$$x^2 - 2xd\left(\frac{1+K^2}{1-K^2}\right) + y^2 \quad = \quad -d^2 \tag{6.66}$$

where the actual potential at any specific equipotential line (relative to zero potential at $x = y = 0$) is determined by the value of K, using (*6.65*).

Adding $d^2\left(\frac{1+K^2}{1-K^2}\right)^2$ to both sides of (*6.66*) completes a square with the first two terms, resulting in a more comprehensible equation for an equipotential line

$$\left[x - d\left(\frac{1+K^2}{1-K^2}\right)\right]^2 + y^2 \quad = \quad \left(\frac{2Kd}{1-K^2}\right)^2 \tag{6.67}$$

Equation (6.67) is the equation of a family of circles, with K as a parameter and d as a scale factor. For any potential (i.e. value of K) an equipotential line is a circle of radius $2Kd/(1-K^2)$ whose center has x coordinate $d(1+K^2)/(1-K^2)$ and y coordinate zero. Fig. 6-12 shows a few equipotential circles as given by equation (6.67) for parallel line charges.

The derivation of an expression for the capacitance of a parallel wire transmission line from equation (6.67) is accomplished by making use of the physical fact that at any cross section the circumferences of a transmission line's conductors are equipotential lines in the electric field. Hence the field pattern of any specific parallel wire transmission line will be found by fitting the cross section pattern of the outside surface of the two conductors to a pair of equipotential circles in Fig. 6-12, making whatever scale changes are required.

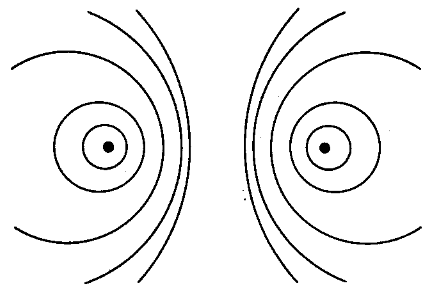


Fig. 6-12. Equipotential lines in a plane transverse to two infinitely long parallel lines of charge, having equal and opposite linear densities of charge.

For any specified potential difference between the conductors (balanced relative to the center point between them) the equivalent linear charge density ρ_l can be determined from the equations, and the distributed capacitance is the ratio of the distributed charge to the potential difference.

It is important to note that for conductors of finite radius, the separation $2d$ of the equivalent lines of charge producing the field is not the same as the separation s of the axes of the conductors. The difference between these two quantities represents the embodiment of proximity effect in the calculation.

If a parallel wire transmission line has circular conductors of radius a, the axes of the two conductors being separated by distance s, then from equation (6.67)

$$a = 2Kd/(1-K^2) \tag{6.68}$$

$$s/2 = d(1+K^2)/(1-K^2) \tag{6.69}$$

Eliminating d from these equations and solving for $s/2a$,

$$s/2a = \tfrac{1}{2}(K+1/K) \tag{6.70}$$

From (6.65), $K = e^{2\pi\epsilon' V_p/\rho_l}$. If there is a potential difference V between the conductors of a parallel wire line, balanced with respect to the central axis, then one conductor is at potential $+V/2$ and the other at potential $-V/2$. In (6.65) this corresponds to $V_p = \pm V/2$. For $V_p = +V/2$, $K = e^{+\pi\epsilon' V/\rho_l}$, and substituting this into (6.70),

$$s/2a = \tfrac{1}{2}(e^{+\pi\epsilon' V/\rho_l} + e^{-\pi\epsilon' V/\rho_l}) = \cosh \pi\epsilon' V/\rho_l \tag{6.71}$$

Since ρ_l/V is the distributed capacitance C of the line, being the ratio of the magnitude of one of the equal and opposite linear charge densities on the conductors to the potential difference between them, it follows from (6.71) that

$$C = \frac{\pi\epsilon'}{\cosh^{-1} s/2a} \text{ farads/m} \tag{6.72}$$

or

$$C = \frac{27.8\, k_e'}{\cosh^{-1} s/2a} \text{ micromicrofarads/m}$$

It is an identity that $\cosh^{-1} x = \log_e (x + \sqrt{x^2 - 1})$. For $x \gg 1$ this becomes $\cosh^{-1} x \doteq \log_e 2x$. The approximation involves an error of less than ½% for $x > 5$. Applying this result to (*6.72*),

$$C \doteq \frac{\pi\epsilon'}{\log_e s/a} \text{ farads/m} \quad \text{if } s/a > 10 \tag{6.73}$$

Equation (*6.73*) is the expression for the distributed capacity of a parallel wire transmission line that would be arrived at by elementary methods ignoring proximity effect, for $s/a \gg 1$. Referring back to comments in (*a*) above, it appears that in the range of $s/2a$ from 5 to 10, proximity effect modifies the distributed resistance of a parallel wire transmission line appreciably, but has no effect on the distributed capacitance.

Comparison of numerical values for the distributed capacitance of parallel wire transmission lines from (*6.73*) with values for the distributed capacitance of coaxial lines from (*6.53*) shows that for lines of average design the former are usually less than 50% of the latter.

(c) Distributed conductance.

The distributed conductance of any transmission line depends on the loss factor or conductivity of the medium surrounding the conductors, and on the geometry of the line, in exactly the same manner that the distributed capacitance depends on the permittivity of the medium and the line's geometry. This means that equation (*6.56*), presented in the discussion of coaxial lines, is in fact directly applicable to uniform transmission lines having conductors of any shape or arrangement, provided all of the electric field pattern of the line lies within the medium described by the dielectric constant k_e' and loss factor $\tan \delta$. The bounded geometry of a coaxial line, or of any other line with a self-shielding configuration of conductors, assures that this requirement is satisfied if the medium fills the interconductor space inside the outer conductor. For a coaxial line whose interconductor space is only partly filled with insulating beads or discs, separate determinations of G and C can be made for the air-dielectric and material-dielectric fractions of the line, and from these reliable average values for the line as a whole can be calculated.

For a parallel wire transmission line, on the other hand, the requirement that the electric field surrounding the conductors be entirely contained within a surrounding dielectric medium would literally demand that the medium be of indefinitely great extent, and even a more practical goal such as 99% containment would necessitate that the medium around and between the conductors have a thickness considerably greater than the distance between the conductor axes.

The presence of solid dielectric material in the interconductor electric field of any transmission line increases the distributed circuit coefficients C and G, relative to air dielectric, without affecting R or L. Both of the changes increase the attenuation factor α. This can be seen directly, for high frequencies, from equations (*5.9*) and (*5.11*); and a consideration of phase angle relations in equation (*5.1*) shows that it is true at all useful frequencies. For the efficient transmission of signals and power, there is therefore always a premium on minimizing the amount of solid dielectric material in the interconductor electric fields of a transmission line.

The many commercial types of small diameter coaxial transmission lines whose interconductor space is filled with plastic dielectric have been designed primarily to achieve a high degree of flexibility, combined with adequate electrical uniformity and mechanical stability. This objective is attained at an increase in cost and an increase in attenuation factor over a line with identical conductors but with an interconductor space only partially filled with dielectric.

To minimize the cost and the undesirable effects of dielectric material, practical parallel wire transmission lines use the smallest possible amount of insulating material for periodic supports or spacers, or in the form of a thin web maintaining the spacing between the conductors of a flexible line. Unfortunately for analytical purposes, expressions for the distributed capacitance and conductance of such lines cannot be derived in any generalized form.

Frequently an estimate of relative volumes of air and solid dielectric can suggest that the solid insulating material in the structure of a particular line occupies too little space to produce a significant value of distributed conductance or to have any measurable effect on the distributed capacitance. For lines incorporating a substantial amount of solid dielectric, such as the standard "Twinline" used for connecting television receivers to antennas, the values of C and G must be determined by direct measurement at any operating frequency.

Example 6.6.

The conductors of the 19 gauge cable pair transmission line of Example 6.5 are insulated by a machine which embeds them in liquid paper pulp. With the separation between conductor axes maintained at 2.0 conductor diameters, the pulp is dried and forms a solid dielectric that extends far enough in all directions around the conductors to protect the line electrically and mechanically from dozens of similar parallel wire transmission lines in the same cable. The average values of distributed capacitance and distributed conductance for the 19 gauge cable pair line are respectively $C = 0.062$ microfarads/mile and $G = 1.0$ micromhos/mile at a frequency of 1 kilohertz. Assuming the electric field of the line is completely contained within the homogeneous paper insulation, and that the value of distributed conductance is entirely due to dielectric loss rather than charge-flow conductivity, determine the average value of the dielectric constant k_e' and the loss factor $\tan\delta$ for the insulating material.

It is convenient to do the calculation in metric units, using (*6.72*). The distributed capacitance of the line is $(0.062 \times 10^{-6})/1609 = 38.5$ micromicrofarads/m. From the dimension data, $s/2a = 2.0$ and $\cosh^{-1} s/2a = 1.32$. Then from the second equation of (*6.72*), $27.8\,k_e'/1.32 = 38.5$, or $k_e' = 1.83$, a reasonable value for the dielectric constant of a rather porous material.

In using (*6.56*) to determine $\tan\delta$, any length units may be used for C and G provided they are the same for both. Using the values per mile, and a frequency of 1 kilohertz,

$$1.0 \times 10^{-6} = (2\pi \times 1000)(0.062 \times 10^{-6})(\tan\delta) \quad \text{or} \quad \tan\delta = 0.0026$$

This is typical of the values of $\tan\delta$ for many organic materials, such as paper, wood, and bakelite and acetate plastics, but is higher by a factor of 10 or more than the values for the best low loss plastics.

(d) Distributed inductance.

All the introductory remarks in Section 6.4(*d*) about the relative magnitudes of the distributed internal inductance and the distributed external inductance of coaxial lines also apply to parallel wire lines. Although the distributed external inductance of parallel wire lines tends to be considerably higher than that of coaxial lines, for the same geometrical reasons that make the distributed capacitance tend to be lower, the distributed internal inductance also tends to be somewhat higher because the parallel wire line has two identical conductors, usually solid, while one of the coaxial line's conductors is a thin walled tube. The distributed internal inductance of the two cases at various frequencies is discussed in Section 6.8. The net result is that the ratio of the distributed internal inductance to the distributed external inductance is typically only a little smaller for parallel wire transmission lines than for coaxial lines.

The derivation of an expression for the distributed external inductance L_x of a parallel wire transmission line, taking proximity effect into account, follows a pattern identical to that of the derivation of equation (*6.72*) for the distributed capacitance. Magnetic quantities take the place of electric quantities, and the conductor circumferences are identified as lines of constant magnetic vector potential. The result is

$$L_x = (\mu_m'/\pi)(\cosh^{-1} s/2a) \text{ henries/m} \quad \text{or} \quad L_x = 0.4 \cosh^{-1} s/2a \text{ microhenries/m} \qquad (6.74)$$

where μ_m' is the real part of the mks permeability of the medium surrounding the conductors. For all insulating materials used in practical transmission lines, $\mu_m' = \mu_0 = 4\pi \times 10^{-7}$ henries/m.

Example 6.7.

To complete the calculations on the 19 gauge cable pair transmission line of Examples 6.5 and 6.6, determine its distributed external inductance and total distributed inductance at a frequency of 1 kilohertz, and compare the results with the stated distributed inductance at that frequency (Example 5.4, page 52) of 1.00 millihenries/mile.

For $s/2a = 2.0$, $\cosh^{-1}(s/2a) = 1.32$, as in Example 6.5. From equation (*6.74*) the distributed external inductance of the line is then $0.4 \times 1.32 = 0.53$ microhenries/m $= 0.53 \times 1609 = 0.85$ millihenries/mile. At the low frequency of 1 kilohertz, Table 6.2 shows that the distributed internal inductance of a 19 gauge isolated wire is the same as its d-c value of 0.050 microhenries/m = 0.080 millihenries/mile.

At the frequency of 1 kilohertz, a/δ from Table 6.2 is 0.22, and from Fig. 6-8 or the discussion in Section 6.6(*a*) the distributed resistance of the conductors is not increased by proximity effect under these conditions. It can therefore be assumed that the distributed internal inductance is also not affected.

The total distributed internal inductance for the two wires of the line is then 0.160 millihenries/mile, and the total distributed inductance of the line is $0.85 + 0.16 = 1.01$ millihenries/mile. The difference between this calculated value and the stated average value of 1.00 millihenries/mile is covered by the slightly nominal nature of the latter figure.

(e) Summary of high frequency relations for the parallel wire transmission line.

If the conditions $a/\delta > 100$, $\omega L/R > 10$ and $\omega C/G > 10$ *all* hold for a parallel wire transmission line, which is usually the case at frequencies above some value between 1 and 100 megahertz depending on the line dimensions and materials, the following simplified expressions can be obtained from equations (*6.72*) and (*6.74*),

$$Z_0 = \sqrt{\frac{\mu_m'/\pi}{\pi\epsilon'}}\cosh^{-1}(s/2a) = \frac{1}{\pi\sqrt{k_e'}}\sqrt{\frac{\mu_0}{\epsilon_0}}\cosh^{-1}s/2a = \frac{120}{\sqrt{k_e'}}\cosh^{-1}s/2a \tag{6.75}$$

$$= \frac{120}{\sqrt{k_e'}}\log_e\left(\frac{s}{2a}+\sqrt{\left(\frac{s}{2a}\right)^2-1}\right) = \frac{276}{\sqrt{k_e'}}\log_{10}\left(\frac{s}{2a}+\sqrt{\left(\frac{s}{2a}\right)^2-1}\right) \text{ ohms}$$

and if $s/2a > 10$, then within $\frac{1}{2}\%$,

$$Z_0 \doteq \frac{120}{\sqrt{k_e'}}\log_e s/a \doteq \frac{276}{\sqrt{k_e'}}\log_{10} s/a \quad \text{ohms} \tag{6.76}$$

For all values of $s/2a$,

$$v_p = \frac{1}{\sqrt{\mu_m'\epsilon'}} = \frac{1}{\sqrt{k_e'}}\frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{3.00 \times 10^8}{\sqrt{k_e'}} \quad \text{m/sec} \tag{6.77}$$

Equations (*6.60*) and (*6.77*), for the high frequency phase velocity on coaxial lines and parallel wire lines respectively, are identical. They carry no information about the geometry of the lines or the metal of their conductors. The numerical result is the value of the phase velocity for plane transverse electromagnetic waves in an unbounded medium having the properties of the insulating material filling or surrounding the line conductors. This follows from the fact, discussed in Chapter 2, Section 2.3, that the electric power traveling along a transmission line is a plane transverse electromagnetic (TEM) wave guided by the conductors. At high enough frequencies the propagation of the waves in the interconductor medium is not affected by the bounding conductors. At lower frequencies equation (*5.6*), page 49, shows that either distributed resistance or distributed conductance separately will reduce the phase velocity below the value for an unbounded medium, but if both are present there is some mutual cancellation of effects according to the term $\{(R/2\omega L) - (G/2\omega C)\}^2$.

As was the case for equations (*6.59*) and (*6.60*), the interconductor medium is assumed in (*6.75*), (*6.76*) and (*6.77*) to have the magnetic properties of free space. Otherwise Z_0 must be multiplied by and v_p divided by the square root of the relative permeability of the medium. If the relative permeability is complex and has a sufficiently large phase angle, an additional contribution to the distributed resistance of the line will be created. This situation is likely to occur only in a research context. Its analysis is easily developed by analogy with the method used in Section 6.5(*c*) to handle complex dielectric permittivity. (See Problem 6.32.)

Example 6.8.

Three sizes of circular copper tubing, having outside diameters respectively of 1.00″, 1.50″ and 4.00″ and all with wall thickness 0.100″, are available to make a transmission line to be used at 100 megahertz. The maximum transverse dimensions of the line must not exceed 4.00″. Assuming that dielectric supports will have negligible effect on either the distributed capacitance or the losses of any proposed line, what arrangement of the conductors, as either a coaxial line, or a parallel wire line with identical conductors, will result in the lowest attenuation factor? (Make a guess before proceeding.)

There are five possible designs of transmission line that use the available conductors and meet the conditions stated. Two are coaxial with the 4″ tube as outer conductor, one is coaxial with the 1.5″ tube as outer conductor, and two are parallel wire lines with the 1″ and 1.5″ tubes respectively as conductors. The coaxial line with the 1.5″ tube as outer conductor will obviously have higher attenuation than the coaxial lines with a 4″ tube as outer conductor, since its distributed resistance is higher and its characteristic impedance is lower than for either of them. No calculations need therefore be made for it. There is no self-evident basis for rejecting any of the other four possibilities. (Some design criteria for "optimum" transmission lines for various purposes and subject to various specifications are developed in Section 6.9.)

Since δ for copper at 100 megahertz is 6.61×10^{-6} m, it is obvious that $a/\delta \gg 100$ for all of the conductors, and the resistance calculations can be made by the simplest high frequency formulas (*6.30*) or (*6.49*), with the simplest correction for proximity effect, equation (*6.61*), used where necessary. The attenuation calculations for each of the four lines are then as follows:

(1) *Coaxial line with* 4″ *outer conductor and* 1″ *inner conductor.*

From (*6.59*), characteristic impedance $= 138 \log_{10} 1.90/0.50 = 80.1$ ohms.

From (*6.49*), distributed resistance $= \dfrac{0.00261}{2\pi(1.90 \times 0.0254)}\left(1 + \dfrac{1.90}{0.50}\right) = 0.0413$ ohms/m. (Here use has been made of $R_s = 1/\sigma\delta = 2.61 \times 10^{-7}\sqrt{f}$ ohms/square for copper when f is in hertz, and of the conversion factor $1'' = 0.0254$ m.)

From (*5.9*), attenuation factor $= 0.0413/(2 \times 80.1) = 2.57 \times 10^{-4}$ nepers/m.

(2) *Coaxial line with* 4″ *outer conductor and* 1.5″ *inner conductor.*

From (*6.59*), characteristic impedance $= 138 \log_{10} 1.90/0.75 = 57.9$ ohms.

From (*6.49*), distributed resistance $= \dfrac{0.00261}{2\pi(1.90 \times 0.0254)}\left(1 + \dfrac{1.90}{0.75}\right) = 0.0304$ ohms/m.

From (*5.9*), attenuation factor $= 0.0304/(2 \times 57.9) = 2.62 \times 10^{-4}$ nepers/m.

(3) *Parallel wire line with* 1″ *conductors.*

Minimum attenuation for this line will occur with the conductors as far apart as possible, since this will maximize Z_0 in equation (*5.9*) and will at the same time minimize R by minimizing proximity effect. Therefore $s = 4.00'' - 2a = 3.00''$.

From (*6.75*), characteristic impedance $= 276 \log_{10}\left(3.00/1.00 + \sqrt{(3.00/1.00)^2 - 1}\right) = 211.6$ ohms.

From (*6.30*), distributed resistance of conductors if isolated $= 2\,\dfrac{0.00261}{2\pi(0.050 \times 0.0254)} = 0.0656$ ohms/m.

From (*6.61*), proximity effect factor $= 1/\sqrt{1 - 1/(3.00/1.00)^2} = 1.060$.

Distributed resistance of conductors including proximity effect = 0.0695 ohms/m.

From (*5.9*), attenuation factor $= 0.0695/(2 \times 211.6) = 1.64 \times 10^{-4}$ nepers/m.

(4) *Parallel wire line with* 1.5″ *conductors,* $s = 2.50''$.

From (*6.75*), characteristic impedance $= 276 \log_{10}\left(2.50/1.50 + \sqrt{(2.50/1.50)^2 - 1}\right) = 131.9$ ohms.

From (*6.30*), distributed resistance of conductors if isolated $= 2\,\dfrac{0.00261}{2\pi(0.75 \times 0.0254)} = 0.0437$ ohms/m.

From (*6.61*), proximity effect factor $= 1/\sqrt{1 - 1/(2.50/1.50)^2} = 1.118$.

Distributed resistance of conductors including proximity effect $= 0.0489$ ohms/m.

From (*5.9*), attenuation factor $= 0.0489/(2 \times 131.9) = 1.86 \times 10^{-4}$ nepers/m.

Worth noting from these results are the fact that the two parallel wire lines have substantially lower attenuation factors than either of the coaxial lines, in spite of having higher distributed resistance values, and the fact that for each type of line the 1″ conductor gives lower attenuation than the 1.5″ conductor. Both facts are of course due to the relative values of the characteristic impedances involved. It does not follow that conductors of 0.75″ or 0.50″ outside diameter will give still lower attenuation in either type of line. For each case, subject to the fixed maximum lateral dimension, there is an optimum value of conductor diameter for the parallel wire line or inner conductor diameter for the coaxial line that provides minimum attenuation. With smaller conductors the distributed resistance rises more rapidly than the characteristic impedance, and the attenuation factor increases. (See Section 6.9 and Problem 6.15.)

6.7. Distributed circuit coefficients of transmission lines with parallel plane conductors.

Elementary methods can derive exact expressions for the distributed circuit coefficients of transmission lines with parallel plane conductors only in the idealized case of conductors which are portions of infinite parallel planes. The infinite plane specification eliminates the effects of the curved electric and magnetic field lines that occur at the edges of finite plane conductors, effects which are not easy to analyze mathematically. The equations for the idealized case apply with useful accuracy to lines with conductors of finite width if the conductor width is sufficiently large compared with the separation of the conductors.

Parallel plane transmission lines with conductors many times wider than their separation have no particular electrical virtues, their attenuation being greater than for parallel wire or coaxial transmission lines of comparable maximum transverse dimensions, or containing comparable amounts of metal, but profitable use can sometimes be made of their space-saving geometry and the fact that they have a higher degree of self shielding than lines with parallel circular conductors. The stripline constructions of Fig. 2-2, page 9, are commercial types of line that take advantage of these properties.

Even though the expressions for the distributed circuit coefficients of the idealized parallel plane transmission line are seldom accurately applicable to practical situations, they are worth noting because their simple form is so easily remembered, and they can often serve as a basis for a useful estimate of coefficient values for a line having some other design.

(a) Distributed resistance.

The distributed resistance per unit width of single infinite plane conductors, in which the currents are excited by fields from one side only, has been fully dealt with in Section 6.3. For a two-conductor line with plane parallel conductors of width w and thickness t, assuming the idealized fields of infinite planes, the distributed resistance at any value of the ratio t/δ will be $2/w$ times the values given in Section 6.3 for conductors of finite thickness. At low frequencies the result is equal to the distributed d-c resistance of the conductors, as usual. At frequencies sufficiently high that the conductors are at least three skin depths thick, the result is

$$R = 2R_s/w \text{ ohms/m} \tag{6.78}$$

For intermediate frequencies the distributed resistance is the value given by equation (*6.78*) multiplied by a factor from Fig. 6-7, page 87.

(b) Distributed capacitance.

If two infinite parallel plane conductors carry equal and opposite uniform surface charge densities of magnitude ρ_s coulombs/m^2, the electric flux field is entirely confined to the space between them, is normal to the conductor surfaces, and has constant density $D = \rho_s$ coulombs/m^2. The electric field E, also constant everywhere and normal to the surfaces, is given by $E = D/\epsilon' = \rho_s/\epsilon'$ volts/m, where ϵ' is the real part of the permittivity of the medium between the conductors. The potential difference V between the conductors is then $V = Ed = \rho_s d/\epsilon'$ volts. The capacitance per unit area, being the ratio of the magnitude of one of the surface charges per unit area to the potential difference between the conductors, is ϵ'/d farads/m^2. The distributed capacitance of the parallel plane conductor is therefore given approximately, when $w/d \gg 1$, by

$$C = \epsilon' w/d \text{ farads/m} = 8.85\, k_e' w/d \text{ micromicrofarads/m} \qquad (6.79)$$

(c) Distributed conductance.

The distributed conductance of all transmission lines is given by equation (*6.56*), in terms of the distributed capacitance of the line and the loss factor of the interconductor medium, assuming that the distributed conductance is due entirely to molecular dielectric loss mechanisms. For the idealized parallel plane transmission line neglecting edge effects this takes the specific form

$$G = \omega\, 8.85\, k_e' w/d \tan\delta \text{ micromicromhos/m} \qquad (6.80)$$

For this expression to be applicable, the interconductor medium whose loss factor is $\tan\delta$ must contain all of the electric field surrounding the line conductors. The condition is fulfilled for parallel plane conductors if the medium fills the space between them.

(d) Distributed inductance.

Practical applications of parallel plane transmission lines are based primarily on the geometrical fact that the total volume occupied by the line can be made small and one dimension can be made very small, with no reduction in the surface area of the conductors or increase in the high frequency distributed resistance R. At the same time the lines retain a high degree of self-shielding.

If a transmission line has to be designed so that its cross section is contained within a rectangular area of which one edge is much longer than the other, a line with parallel plane conductors will have a lower attenuation factor and much better power handling capacity than either a coaxial line or a parallel wire line with circular conductors when the aperture ratio of the rectangle exceeds some minimum value, and will have much smaller external fields than the parallel wire line.

It is shown in Section 6.8 that the distributed internal inductance of tubular and plane conductors is proportional to the thickness t at low frequencies for which $\delta > t$, and to the skin depth δ at high frequencies for which $t > \delta$. The expression derived in this section for the distributed external inductance of a line with parallel plane conductors shows that it is proportional to the separation d between the facing surfaces of the conductors. Since d might be comparable to or even smaller than t for a parallel plane line, it is possible at low frequencies for the distributed internal inductance of such a line to constitute the greater part of the total distributed inductance, a result not possible for any reasonable design of coaxial or parallel wire line with circular conductors.

In calculating the distributed inductance at low and intermediate frequencies for lines with parallel plane conductors, therefore, particular attention must be paid to the relative importance of the distributed internal inductance. If $d/\delta > 100$ the distributed internal inductance is always negligible, for all values of conductor thickness t.

A single isolated infinite plane conductor parallel to the xz plane and carrying an instantaneous surface current in the z direction of density J_{sz} amperes/(meter width of plane in the x direction) has a constant magnetic field of value $H_x = J_{sz}/2$ throughout the whole of space on one side of the plane, and a constant field of equal magnitude and opposite sign on the other side of the plane. When two such conductors are parallel to one another and carry currents of equal density in opposite directions, the fields between the conductors are additive and those outside the conductors cancel. Thus the magnetic flux density between the conductors of a parallel plane transmission line, neglecting edge effects caused by finite conductor width, is constant at the value $B_x = \pm \mu'_m J_{sz}$ teslas, where μ'_m is the real part of the mks permeability of the medium between the conductors. The sign is determined by the sign of the z directed current in one of the conductors. The magnitude of the total magnetic flux $|\psi|$ between the conductors per unit length of conductor in the z direction is $|\psi| = |B_x| d = \mu'_m d J_{sz}$. The conclusion that all of this flux links all of the current in the conductors as a "circuit" requires the hypothesis that the flux lines after extending indefinitely laterally in the space between the conductors are self-closing by returning in the unbounded space outside the conductors, where their vanishingly small density constitutes zero field.

If the actual conductors in the parallel plane transmission line have finite width w, but the currents and fields are those of infinite planes, the current magnitude in each line conductor will be $I = J_{sz}w$ amperes. The distributed external inductance of the line, defined as the total flux external to the conductors linking the circuit per unit current, becomes

$$L_x = \frac{|\psi|}{I} = \mu'_m \frac{d}{w} \text{ henries/m} \qquad (6.81)$$

For a nonmagnetic medium in the interconductor space, $\mu'_m = \mu_0 = 4\pi \times 10^{-7}$ henries/m and $L_x = 1.256d/w$ microhenries/m.

(e) Summary of high frequency relations for the parallel plane transmission line.

Subject to the conditions that $d/\delta > 100$, $w/d \gg 1$, $\omega L/R > 10$ and $\omega C/G > 10$, the following simplified expressions can be obtained from equations (*6.79*) and (*6.81*):

$$Z_0 = \sqrt{\frac{\mu'_m d/w}{\epsilon' w/d}} = \frac{1}{\sqrt{k'_e}}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{d}{w} = \frac{377}{\sqrt{k'_e}}\frac{d}{w} \text{ ohms} \qquad (6.82)$$

$$v_p = \frac{1}{\sqrt{(\mu'_m d/w)(\epsilon' w/d)}} = \frac{1}{\sqrt{k'_e}}\frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{3.00 \times 10^8}{\sqrt{k'_e}} \text{ m/sec} \qquad (6.83)$$

In these equations it is again assumed that the interconductor medium is nonmagnetic.

Example 6.9.

A parallel plane transmission line used at 10 megahertz has copper conductors 1.00″ wide and 0.050″ thick, spaced 0.100″. The interconductor space is filled with material of dielectric constant 2.25 and loss factor 0.00025. Neglecting edge effects, determine the characteristic impedance, the attenuation factor and the phase velocity at the frequency of operation. Compare the results with the values for a coaxial line whose total conductor periphery is equal to the combined width of the two plane conductors (i.e. $2\pi a + 2\pi b = 2w$) and for which the ratio $b/a = 3.5$, the interconductor space being filled with the same medium. (The reason for choosing $b/a = 3.5$ is given in Section 6.9.)

The skin depth in copper at 10 megahertz is 2.09×10^{-5} m so that $d/\delta > 100$ and $t/\delta > 100$. A check shows that $\omega L/R > 10$ and $\omega C/G > 10$ are both satisfied.

The characteristic impedance of the parallel conductor line, from the high frequency equation (*6.82*) is 25.1 ohms.

The distributed resistance from (*6.78*) is $R = 2 \times 0.000825/0.0254 = 0.0650$ ohms/m.

The distributed capacitance from (*6.79*) is 199 micromicrofarads/m, and the distributed conductance from (*6.80*) is 3.13×10^{-6} mhos/m.

The attenuation factor is therefore

$$\alpha = 0.0650/(2 \times 25.1) + \tfrac{1}{2}(3.13 \times 10^{-6}) \times 25.1 = 0.00129 + 0.000039 = 1.33 \times 10^{-3} \text{ nepers/m}$$

The phase velocity from *(6.83)* is

$$v_p = 3.00 \times 10^8/1.5 = 2.00 \times 10^8 \text{ m/sec}$$

For the coaxial line meeting the stated conditions, the relations $a + b = w/\pi$ and $b/a = 3.5$ give $a = 0.0707'' = 1.79 \times 10^{-3}$ m, and $b = 3.5a = 0.247'' = 6.28 \times 10^{-3}$ m.

The characteristic impedance from equation *(6.59)* is 50.1 ohms.

The distributed capacitance from *(6.54)* is 99.7 micromicrofarads/m, and the distributed conductance from *(6.56)* is then 1.56×10^{-6} mhos/m.

Finally the attenuation factor of the coaxial line is

$$\alpha = 0.094/(2 \times 50.1) + \tfrac{1}{2}(1.56 \times 10^{-6}) \times 50.1 = 0.000937 + 0.000039 = 9.8 \times 10^{-4} \text{ nepers/m}$$

The coaxial line in the above example has an attenuation factor about 25% smaller than that of the parallel plane line, with the same area of metal sheet of the same thickness in unit length of each. The calculation for the parallel plane line neglected edge effects, and allowance for these would increase the advantage of the coaxial line by a few percent. The equality of the contribution to the attenuation factor from the distributed conductance in the two cases is an identity (see Problem 6.29).

If the same amount of metal sheet were made into two circular conductors for a parallel wire transmission line with the maximum transverse dimension equal to that of the parallel plane transmission line, the conductors being fully surrounded by the same medium, the attenuation factor would be much less than for either the coaxial or the parallel plane lines, but the external fields of the parallel wires line would be appreciable at greater distances from the conductors than for the parallel plane line.

Reducing the separation of the parallel plane conductors by 50% would reduce the characteristic impedance, as given by the idealized equations, by the same amount. The distributed resistance would not change, and the contribution of the distributed resistance to the attenuation factor would therefore be doubled. The contribution from the distributed conductance would remain constant, so that the total attenuation factor would increase by somewhat less than 100%.

If the separation between the parallel plane conductors is steadily increased, the entire field pattern begins to change drastically, becoming similar to that of a parallel wire transmission line for values of d/w much greater than unity. The attenuation factor decreases sharply, but the idealized theory ceases to be usefully accurate when d/w exceeds 0.1.

6.8. Distributed internal inductance for plane and tubular circular conductors of finite thickness.

In Sections 6.2 and 6.3, full derivations were presented of expressions for the distributed resistance of solid circular conductors, and plane conductors of unlimited area and thickness, respectively, as a function of frequency. An inherent feature of those derivations was that the resulting expressions, equations *(6.27)* and *(6.45)*, also gave values for the distributed internal inductance of the two cases.

The much more elaborate mathematical investigations that resulted in Fig. 6-8, page 88, for the distributed resistance of circular tubular conductors at low and intermediate values of the parameter a_t/δ, and in equation *(6.63)* for the influence of proximity effect on the distributed resistance of solid and tubular circular conductors under all conditions, would also have provided information about distributed internal inductance. Unfortunately, the main practical application which the authors of those two major studies had in mind was power transmission at frequencies of 50 and 60 hertz, where ωL_i is generally too small to merit attention. Their published works contain the very complicated equations from which L_i can be determined, with or without proximity effect, but the extensive numerical computations required were made only for the real parts of the expressions, yielding $R_{a\text{-}c}/R_{d\text{-}c}$.

It is the purpose of this section to discuss the conditions under which the distributed internal inductance of a transmission line's conductors is a significant part of its total distributed inductance, and to review the methods and data available for calculating distributed internal inductance for the types of transmission lines dealt with in Sections 6.5, 6.6 and 6.7. For some situations exact values are readily determined, for others adequate approximations can be made, and for still others a judicious guess may be the best available solution.

A simple and useful criterion for estimating whether or not the distributed internal inductance can be ignored in any particular transmission line application, can be derived from the flux-circuit linkage definition of inductance, combined with the fundamental electromagnetic fact that across any boundary between two materials the tangential component of magnetic flux density B is continuous.

The tangential B-field at the surface of a conductor being continuous means that its value just inside the metal is equal to its value just outside the metal in the adjacent interconductor space. Inside the metal the B-field diminishes to zero, either exponentially (as in a plane conductor several skin depths thick), or linearly (as in an isolated solid circular conductor at zero frequency) or according to some other law. The diminishing flux densities farther from the surface into the metal also link a decreasing fraction of the conductor current as a "circuit". If the tangential B-field at the surface of a conductor in a transmission line is designated B_s, and most of the flux of this field is contained within a metal thickness l_i from the surface, then as a rough average estimate, about half the flux within the metal will link about half the circuit, per unit length, and a resulting approximate expression for the distributed internal inductance is $\frac{1}{4}B_s l_i/i_C$, where i_C is the instantaneous total current in the conductor, causing the field B_s at the surface. (It is an encouraging coincidence that this procedure happens to produce exactly the correct answer for the distributed internal inductance of a solid circular conductor at zero frequency. For that case $B_s = \mu i_C/(2\pi a)$, $l_i = a$, and the resulting distributed internal inductance is given as $\mu_0/8\pi$ henries/m, the value found in equation (*6.5*) by formal analysis.)

In the interconductor space, the value of B does not fall to zero, and all of its flux links the whole circuit. If the distance between facing conductor surfaces of a transmission line is l_x, the amount of flux-circuit linkage in the interconductor space will be fairly represented in most cases by the product $B_s l_x$. It may be greater or less than this, depending on the cross section of the line, but usually by only a small factor less than 2. The distributed external inductance of the line will therefore be given approximately by $B_s l_x/i_C$.

The ratio of distributed external inductance to distributed internal inductance is finally $2l_x/l_i$, where an additional factor of 2 has been introduced because a line has two conductors.

The distance l_x is an obvious quantity, independent of frequency, for any transmission line. The distance l_i, on the other hand, varies widely with frequency. At frequencies low enough for the current density to be nearly constant over a conductor's cross section, l_i is the radius of a solid circular conductor, or the thickness of tubular or plane conductors. At frequencies high enough to make a/δ or t/δ greater than 3 or 4, l_i can be equated with sufficient accuracy to the skin depth δ. In the small range of intermediate frequencies between these two regions, l_i can be taken conservatively as whichever value is greater.

Example 6.10.

For the 19 gauge cable pair line of Examples 6.5 to 6.7, the ratio $s/2a = 2.0$, and $a = 4.56 \times 10^{-4}$ m. Estimate what percent of the total distributed inductance of the line is distributed internal inductance, at frequencies of 10^4, 10^6 and 10^8 hertz. Make the same estimate at the same frequencies for a $3\frac{1}{8}''$ standard rigid transmission line whose inner conductor has an O.D. of 3.34×10^{-2} m with wall thickness 2.14×10^{-3} m, and whose outer conductor has I.D. of 7.80×10^{-2} m and wall thickness 2.49×10^{-3} m.

At each separate frequency the skin depth in the copper conductor metal is the same for the two lines, with values 6.61×10^{-4} m at 10^4 hertz, 6.61×10^{-5} m at 10^6 hertz, and 6.61×10^{-6} m at 10^8 hertz.

For the 19 gauge cable pair the distance between facing conductor surfaces is $2a = 9.12 \times 10^{-4}$ m. At 10^4 hertz, l_i must be taken as a since $a/\delta < 1$. Therefore $2l_x/l_i \doteq 4$ and the distributed internal inductance must not be disregarded. At 10^6 hertz, $a/\delta \doteq 10$ and l_i can be taken as δ. Then $2l_x/l_i \doteq 30$. In this case the distributed internal inductance is estimated as being a few percent of the total, and if not considered negligible would at least not need to be calculated very accurately.

At 10^8 hertz the distributed internal inductance will clearly be negligible.

The distance between facing surfaces of the $3\frac{1}{8}''$ standard rigid coaxial line is 2.23×10^{-2} m. At 10^4 hertz the wall thicknesses are somewhat over 3 skin depths and $l_i = \delta$ can be used, giving $2l_x/l_i \doteq 35$. The situation is the same as for the 19 gauge cable pair at 10^6 hertz. At 10^6 hertz and 10^8 hertz distributed internal inductance is negligible for this large diameter coaxial line.

Distributed resistance values for all the types of conductors considered earlier in this chapter are always given, in the low frequency range from zero to some upper limit, by a factor multiplying the distributed d-c resistance for the entire cross section of the conductor (Table 6.1 and Fig. 6-8, for example). At frequencies diminishing from very high values down to some lower limit, calculations make use of the skin effect theorem and distributed resistance values are given by a factor multiplying the distributed d-c resistance of a peripheral skin of the conductor of thickness δ, this skin depth δ being itself a function of frequency (Fig. 6-7, for example). Use of the theorem requires that metal thicknesses and conductor radii be not less than 3 or 4 skin depths. At frequencies between the upper and lower limiting frequencies for these two categories of calculations, tables, graphs, or elaborate formulas are required.

The same classifications apply to calculations of distributed internal inductance of circular conductors. Expressions for the d-c distributed internal inductance are therefore needed for each type of conductor investigated. The only such expression developed so far is equation (*6.5*) for the solid circular conductor.

The d-c distributed internal inductance of a circular tubular conductor depends somewhat, in the case of coaxial lines, on whether the tube is the inner or outer conductor of the line. The difference in the two values, for the same tube, is large in the case of thick walled tubes ($t/a = 0.3$, for example), but only a few percent for thin walled tubes ($t/a = 0.05$) and disappears for extremely thin walled tubes ($t/a = 0.005$). The value for a tube used as the inner conductor of a coaxial line also applies to the same tube used in a parallel wire line or an image line, with correction by a proximity effect factor if necessary.

Expressions for the d-c distributed internal inductance of a circular tubular conductor are developed through the same mathematical procedures employed to obtain equation (*6.5*) for a solid circular conductor. The result for an isolated tube, which applies to the inner conductor of a coaxial line and the other equivalent cases, is

$$L_{i\,\text{d-c}}\ (\text{tube}) \;=\; \frac{\mu}{8\pi}\cdot\frac{1 - 4(a_i/a)^2 + 3(a_i/a)^4 + 4(a_i/a)^4\log_e(a/a_i)}{[1-(a_i/a)^2]^2} \quad \text{henries/m} \qquad (6.84)$$

where a is the external radius of the tube and a_i its internal radius. Equation (*6.84*) reduces to (*6.5*) when $a_i = 0$.

Making the substitution $t = a - a_i$, and expanding the logarithm as a power series in (t/a), the first few terms of (*6.84*) are

$$L_{i\,\text{d-c}}\ (\text{tube}) \;\doteq\; \frac{\mu}{8\pi}\left[\frac{4}{3}\left(\frac{t}{a}\right) - \frac{2}{15}\left(\frac{t}{a}\right)^3 - \frac{1}{10}\left(\frac{t}{a}\right)^4\right] \quad \text{henries/m} \qquad (6.85)$$

The accuracy of this expression is better than ½% for $t/a < 0.5$, and for small values of t/a it is much more convenient to use than equation (*6.84*).

When a circular tubular conductor is the outside conductor of a coaxial line, the magnetic flux density in the metal of the outer conductor at any point is caused by the whole of the current in the inner conductor and a portion of the current in the outer conductor. The flux in any increment of radius in the outer conductor also links all of the current in the inner conductor and a portion of the current in the outer conductor. The total situation can be expressed by an integral similar to that used in obtaining (*6.84*), with the result

$$L_{i\,\text{d-c}}\ (\text{coax. outer}) \;=\; \frac{\mu}{8\pi}\cdot\frac{4\log_e(a/a_i) - 3 + 4(a_i/a)^2 - (a_i/a)^4}{[1-(a_i/a)^2]} \quad \text{henries/m} \qquad (6.86)$$

which expanded in powers of t/a becomes

$$L_{i\,\text{d-c}}\ (\text{coax. outer}) \;\doteq\; \frac{\mu}{8\pi}\left[\frac{4}{3}\left(\frac{t}{a}\right) + \frac{4}{3}\left(\frac{t}{a}\right)^2 + \frac{6}{5}\left(\frac{t}{a}\right)^3 + \frac{31}{30}\left(\frac{t}{a}\right)^4\right] \quad \text{henries/m} \qquad (6.87)$$

Equation (*6.87*) converges less rapidly than (*6.85*) but is accurate to better than ½% for t/a as large as 0.25.

Equations (*6.84*) and (*6.86*) are plotted in Fig. 6-13 for a wider range of wall thicknesses than would ever be encountered in practical transmission lines. Inspection of the numerical values shows that for the same outside diameter a, the ratio of the d-c distributed internal inductance of a circular tube to that of a solid circular conductor is roughly equal to the ratio of their metal cross sections.

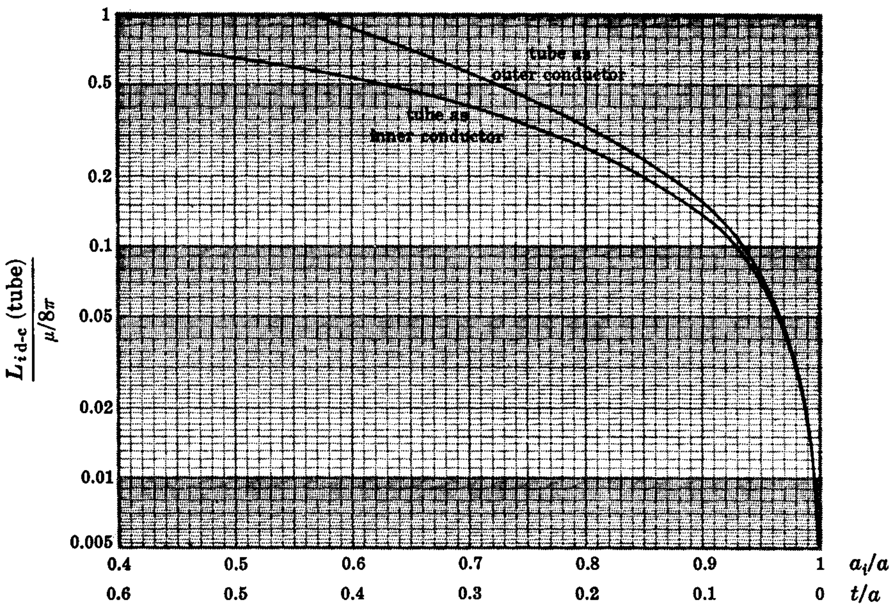


Fig. 6-13. Ratio of the d-c distributed internal inductance of a circular metal tubular conductor of inside radius a_i, outside radius a, and wall thickness t, to the distributed d-c internal inductance of a solid circular conductor of the same metal and same outside radius.

An expression for the d-c distributed internal inductance per unit width of infinite plane conductors of finite thickness is obtained by extending the analysis of Section 6.7(*d*). It derives from the fact that if the longitudinal d-c current in the plane conductors has density

J_{sz} as a surface current, and flows in opposite directions in the two conductors, the tangential B field at the facing conductor surfaces is μJ_{sz}. Inside the conductors the field diminishes linearly from this value to zero at the outside surfaces. The resulting value for the distributed internal inductance per unit width of plane is

$$L_{i\,\text{d-c}} \text{ (plane, per unit width)} \quad = \quad \tfrac{1}{3}\mu t \quad \text{henries/m} \tag{6.88}$$

where t is the thickness of each conductor. For a parallel plane transmission line with identical conductors of width w, but with t/w small enough that the field pattern in the interconductor space is essentially that for infinite planes, equation (*6.88*) gives

$$L_{i\,\text{d-c}} \text{ (plane transmission line)} \quad = \quad \tfrac{2}{3}\mu t/w \quad \text{henries/m} \tag{6.89}$$

Applied to a tubular conductor of radius a, with $t \ll a$, the conductor would approximate a plane of width $2\pi a$, with the result $L_{i\,\text{d-c}}\text{ (tube)} \doteq (\mu/8\pi)(4/3)(t/a)$ henries/m, in agreement with the first terms in equations (*6.85*) and (*6.87*).

In Section 6.3 a transmission line analog was used (Problem 7.10) to investigate the ratio $R_{\text{a-c}}/R_s$ for unit width of plane conductors of thickness t, over the critical range of t/δ from about 0.5 to 3, the results being presented in Fig. 6-7. The same analog provides information about the ratio $\omega L_i/R_s$ for the plane conductors, as shown in Fig. 6-14. It is known from equation (*6.45*) that for large enough values of t/δ, $R_{\text{a-c}} = \omega L_i = R_s$ for unit width of infinite plane conductors, and it is known that for low enough frequencies ($t/\delta < 0.5$), $\omega L_i = \omega L_{i\,\text{d-c}}$. Fig. 6-14 covers the transition between these two limits. For small values of t/a it applies also to tubular conductors.

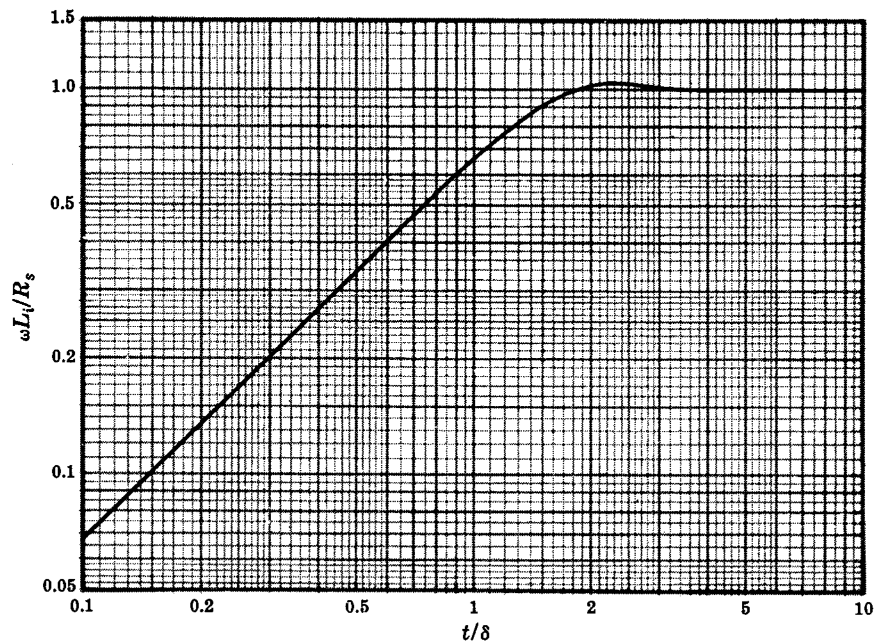


Fig. 6-14. Ratio of the distributed internal inductance per unit width of an unbounded plane conductor of thickness t/δ skin depths to the limiting high frequency value R_s/ω for a conductor of indefinite thickness. For values of t/δ less than about 0.5, the value of L_i is equal to the distributed d-c internal inductance of the metal sheet.

For $t/\delta < 0.5$ in Fig. 6-14 inspection shows that $\omega L_i/R_s = \tfrac{2}{3}t/\delta$ with high accuracy. From this $L_i = 2R_s t/(3\omega\delta) = 2t/(3\omega\sigma\delta^2) = 2\omega\mu\sigma t/6\omega\sigma = \mu t/3$ henries/m for unit width of the plane. From equation (*6.88*) this is equal to $L_{i\,\text{d-c}}$.

Analogous to equation (*6.29*) for solid circular conductors, a power series expression can be used to express the variation of $L_i/L_{i\,\text{d-c}}$ for t/δ as high as 1.5. This is

$$\frac{L_i}{L_{i\,\text{d-c}}}\ (\text{plane}) \quad = \quad 1 \ - \ 0.020(t/\delta)^4 \qquad (6.90)$$

This equation is also adequately accurate for tubular conductors having the reasonable values of t/a likely to be encountered in practical transmission line conductors.

A similar expression can be found for $R/R_{\text{d-c}}$ for plane and tubular conductors,

$$\frac{R}{R_{\text{d-c}}}\ (\text{plane}) \quad = \quad 1 \ + \ 0.075(t/\delta)^4 \qquad (6.91)$$

The question to which no universal answer seems to have been published is that of the influence of proximity effect on distributed internal inductance. Considering the evidence from equations (*6.28*) and (*6.29*) for solid circular conductors and from equations (*6.90*) and (*6.91*) for plane and tubular conductors, it can be concluded that proximity effect like skin effect will reduce the distributed internal inductance of a conductor and that the percent reduction will be considerably less in any situation than the percent by which proximity effect increases the distributed resistance of the same conductor.

Fortunately, the combination of circumstances that would require accurate information about the proximity effect factor for distributed internal inductance occurs rather rarely in transmission line practice. The most unfavorable situation would be a parallel wire line with solid circular conductors, the facing surfaces of the conductors being separated by only a few percent of a conductor radius, operating at a frequency to make a/δ have a value near 2. These conditions make the distributed internal inductance comparable in magnitude to the distributed external inductance, with a proximity effect factor that might be as small as 0.8 or 0.85. There is no recognized basis for making an accurate analysis of the total distributed inductance of a line for such a case.

Appropriate changes in any of the factors specified can improve the situation. Increasing the conductor separation, changing to tubular conductors, or increasing the frequency *will all reduce the relative value of the distributed internal inductance, and the first change* will reduce the proximity effect factor. When the facing conductor surfaces are at least a conductor diameter apart ($s/2a = 2$), the distributed internal inductance will be less than 20% of the total distributed inductance, and the proximity effect factor will be not less than 0.87 according to equation (*6.62*) and Table 6.4. Proximity effect can then not modify the total distributed inductance value by more than about 2%, and the factor need be known only very roughly.

All of the above facts combine to justify the general conclusion that the influence of proximity effect on the total distributed inductance of a parallel wire transmission line will always be less than, and except in inconceivably extreme situations much less than, its influence on the line's distributed resistance. It is therefore suggested as a working *hypothesis that unless the proximity effect factor for distributed resistance is found from* equations (*6.61*), (*6.62*), or (*6.63*) to be at least 1.05, it can be assumed that proximity effect will not change the value of the line's total distributed inductance. If the proximity effect factor for distributed resistance lies between 1.05 and 1.25 it may be a fair guess that the distributed *internal* inductance value should be divided by the square root of that factor. For higher values of the distributed resistance factor no suggestions are offered for accurate determination of the total distributed inductance, and experimental measurements may be the best procedure.

Example 6.11.

Complete the analysis of the 19 gauge cable pair transmission line of Examples 6.5, 6.6 and 6.7, by finding the distributed internal inductance at frequencies of 10^2, 10^4, 10^6 and 10^8 hertz.

The information about the line needed for purposes of the calculation is $s/2a = 2.0$, $a = 4.56 \times 10^{-4}$ m, and from Table 6.2, page 80, $a/\delta = 0.0689, 0.689, 6.89$ and 68.9 at the four frequencies listed.

At 10^2 hertz, with $a/\delta = 0.0689$, $L_i = L_{i\,\text{d-c}} = \mu_0/8\pi$ henries/m for an isolated 19 gauge conductor, from Tables 6.1 or 6.2 and equation (*6.5*). From equation (*6.62*) and Table 6.4, page 98, the proximity effect factor for resistance for solid conductors at $a/\delta = 0.0689$ is 1.000. Hence the total distributed internal inductance of the line is (for two conductors) 0.100 microhenries/m. The distributed external inductance of this line was determined in Example 6.6 as 0.53 microhenries/m. The distributed internal inductance is therefore about 16% of the total.

At 10^4 hertz, with $a/\delta = 0.689$, the situation has changed by only about 0.3% from the value of distributed internal inductance at 10^2 hertz.

At 10^6 hertz, with $a/\delta = 6.89$, Table 6.2 shows that the distributed internal inductance for the isolated conductors has dropped to 29% of its d-c value. Equation (*6.62*) shows a proximity effect factor of 1.12 for resistance for these solid conductors, close to the limiting high frequency value from equation (*6.61*) of 1.15. The procedure suggested above is that the distributed internal inductance in this case be estimated as being reduced by a factor $1/\sqrt{1.12} \doteq 0.95$. The total distributed internal inductance of the line is then $2(\mu_0/8\pi) \times 0.29 \times 0.95 = 0.0274$ microhenries/m, which should be accurate within better than 2%. Since this is only about 5% of the total distributed inductance of the line, it need not be known with better than 10% accuracy, and the inclusion of the proximity effect factor has no effect on the final total value. This illustrates the typical phenomenon that, as proximity effect becomes greater, the fraction of the total distributed inductance affected by it becomes less. Only for extremely closely spaced conductors will a critical calculation problem arise, in a small range of frequencies.

At the frequency of 10^8 hertz, $L_i/L_{i\,\text{d-c}}$ is about 3%, and this percentage of $2\mu_0/8\pi$ has dropped to the negligible amount of about $\frac{1}{2}$% of the total distributed inductance of the line.

6.9. Optimum geometries for coaxial lines.

In designing a coaxial transmission line, the diameter of the outer conductor is often determined by considerations of space or cost. The size of the center conductor does not change the space occupied by the line and its cost is a minor part of the total. It can then be adjusted to achieve various desirable electrical properties. It can be seen, for example, from equation (*6.49*) for the high frequency distributed resistance of a coaxial line, and from equation (*6.59*) for its high frequency characteristic impedance, that the high frequency attenuation factor of such a line as given by equation (*5.9*), page 49, becomes indefinitely large when a is approximately equal to b ($Z_0 \doteq 0$) and when a becomes vanishingly small (R increases to large values more rapidly than the logarithmic term in Z_0.) There is therefore an optimum intermediate value of a, when b is fixed, for which the line has a minimum high frequency attenuation factor. (Note, however, that if a is fixed and b allowed to vary, the attenuation factor will diminish continuously for indefinite increase of b, and there is no optimum value for b other than infinity.)

Making the indicated substitutions for R from (*6.49*) and for Z_0 from (*6.59*) into (*5.9*), the high frequency attenuation factor of a coaxial line having $G = 0$ is

$$\alpha_{\text{hf}}\,(\text{coax.}) \;=\; (R_s/2\pi b)[1 + (b/a)]/[120 \log_e (b/a)] \;\text{ nepers/m} \qquad (6.92)$$

Differentiating with respect to b/a and equating to zero leads to $\log_e(b/a) = 1 + (a/b)$. This is a transcendental equation which must be solved graphically or from tables. The result is $b/a = 3.592$, and the corresponding characteristic impedance for an air dielectric line from equation (*6.59*) is 76.64 ohms. The minimum in the attenuation factor as a function of b/a is a broad one, showing less than $\frac{1}{2}$% variation from $b/a = 3.2$ to $b/a = 4.1$, and only 5% increase at $b/a = 2.6$ and $b/a = 5.2$.

If a transmission line is to handle a maximum amount of power for a fixed value of b, the design must be optimized to avoid breakdown rather than to reduce the attenuation factor. The principal types of failure are dielectric breakdown due to excessive electric field in the interconductor space, and thermal breakdown due to excessive temperature rise of the center conductor.

From the geometry of a coaxial line, maximum electric field will always occur at the surface of the inner conductor. Substituting into equation (*6.51*) an expression for the distributed longitudinal charge density ρ_l obtained from (*6.52*), the maximum electric field at $r = a$ is found in terms of the potential difference $V_b - V_a$ between the conductors to be $E_{max} = (V_b - V_a)/[a \log_e (b/a)]$. If the voltage on the line varies harmonically with time and has rms value V volts, the peak value of $V_b - V_a$ will be $\sqrt{2}\,V$, and the power transmitted by the line is V^2/Z_0 since Z_0 is real at high frequencies. The final expression for the maximum electric field in terms of the power level and the line dimensions is

$$E_{max} = \sqrt{120P}/[a\sqrt{\log_e (b/a)}\,] \text{ volts/m}$$

where P is the power level in watts. The differentiation is much easier if the expression is inverted. Taking $d(1/E_{max})/da = 0$ gives directly $\log_e (b/a) = \frac{1}{2}$, $b/a = 1.649$, and $Z_0 = 30$ ohms.

Optimum design for protection against thermal breakdown through overheating of the center conductor requires assumptions about the processes of heat transfer between the conductors and from the outer conductor to its surroundings, both of which are very sensitive to the condition of the conductor surfaces. On the simple but rather inaccurate assumption that the outer conductor temperature remains close to the line's ambient temperature, regardless of the temperature of the inner conductor, it is easily shown that minimum power dissipation occurs in the inner conductor, for constant transmitted power and constant dimension b, with $b/a = 1$ and $Z_0 = 60$ ohms if the dielectric is air.

If a section of coaxial line is used as a capacitor, or as a delay line or wave-shaping network, the desired objective may be that it should withstand the highest possible value of applied voltage, for a fixed value of outside diameter. From the equation for E_{max} in terms of $V_b - V_a$ given above, this requires maximizing $a \log_e (b/a)$ with b constant. The result is $b/a = 1$ and $Z_0 = 60$ ohms.

Large rigid conductor coaxial lines for high power use are generally designed with $Z_0 = 50$ ohms, which appears to be a compromise value for optimization against breakdown. The attenuation factor is 10% higher than for a line of the same outside diameter having $Z_0 = 76.6$ ohms. Low power dielectric filled flexible coaxial lines are available in several values of Z_0, the most widely used having characteristic impedances near 50 ohms or near 75 ohms. The latter are effectively optimum design for minimum attenuation factor at constant outside diameter. The former are close to optimum design for minimum center conductor heating, for constant diameter and temperature of the outside conductor. When the interconductor space of a coaxial line is filled with solid dielectric, the assumption becomes reasonably correct that the temperature of the outer conductor is not affected by the heating of the center conductor. Dielectric, whether lossy or lossless, does not affect the optimum design for minimum attenuation factor.

Solved Problems

6.1. An indefinitely long straight solid circular conductor has radius a, carries a d-c current of I amperes, and is made of metal whose mks permeability is μ henries/m. The surrounding medium has zero conductivity and permeability μ_m henries/m. Describe the variation of the magnetic flux density B in the conductor and the surrounding medium, as a function of the radial distance r from the central axis of the conductor.

The H field within and surrounding the wire is determined by the current distribution alone and is independent of the magnetic properties of the materials. From the symmetry of the problem the H field and the B field have only H_ϕ and B_ϕ components in the cylindrical coordinate system whose z axis coincides with the axis of the conductor. Applying Ampere's law to any transverse circular path of radius r inside the conductor and concentric with it gives $H_\phi = Ir/(2\pi a^2)$ amperes/m, since the current enclosed by the path is the fraction r^2/a^2 of the total conductor current I. Outside the conductor all paths enclose the total current I, and $H_\phi = I/(2\pi r)$ amperes/m. The B field is everywhere equal to the H field multiplied by the mks permeability. Thus for $r < a$, $B_\phi = \mu I r/2\pi a^2$ teslas (or webers/m²) and increases linearly with r from the center of the conductor to its periphery. For $r > a$, $B_\phi = \mu_m I/2\pi r$. There is a discontinuity in B_ϕ at $r = a$ unless $\mu_m = \mu$. In the medium outside the conductor, B_ϕ falls off inversely as the distance r from the conductor's axis. For a copper conductor in air or in a plastic dielectric, $\mu = \mu_m = \mu_0$, the mks permeability of free space; but for an iron conductor in air or for a copper conductor embedded in ferrite, the two permeabilities will have different values and B_ϕ will either decrease or increase discontinuously at the conductor surface.

6.2. (*a*) A current of 5 amperes d-c flows in a 16 gauge copper wire having 100% conductivity. Determine the current density as a function of position inside the wire.
(*b*) At a frequency which makes $\sqrt{2}\,a/\delta = 10$, a current of rms value 5 amperes flows in the same wire. What is the ratio of the rms magnitude of the current density at the surface of the wire to the value found in (*a*)?
(*c*) What is the frequency in (*b*)?

(*a*) The radius of 16 gauge wire is 6.455×10^{-4} m. The d-c current density is constant over the cross section of the wire at the value $J_{z\,\text{d-c}} = I/(\pi a^2) = 3.82 \times 10^6$ amperes/m².

(*b*) The surface current density in the a-c case is given in terms of the total conductor current by equation (*6.26*), page 77, where $J_z(a)$ will be an rms current density if I_z is an rms current. From tables, ber (10) = 138.84, bei (10) = 56.37, ber′ (10) = 51.20 and bei′ (10) = 135.31. Substituting $I_z = \pi a^2 J_{z\,\text{d-c}}$, and noting that $2/(\omega\mu\sigma) = \delta^2$, equation (*6.26*) gives

$$|J_z(a)/J_{z\,\text{d-c}}| = \tfrac{1}{2}(\sqrt{2}\,a/\delta)\sqrt{\{[\text{ber}(10)]^2 + [\text{bei}(10)]^2\}/\{[\text{ber}'(10)]^2 + [\text{bei}'(10)]^2\}} = 5.19$$

(*c*) When $\sqrt{2}\,a/\delta = 10$, $\delta = 9.13 \times 10^{-5}$ m for 16 gauge wire. From $\delta = 0.0661/\sqrt{f}$ for 100% conductivity copper when f is in hertz, the frequency is 523 kilohertz.

6.3. If a solid circular conductor of radius a carries an a-c current of sufficiently high frequency, the reactance of the distributed internal inductance ωL_i is equal to the distributed internal resistance R according to equation (*6.31*), page 78. This requires $a/\delta = 100$. Show that under these conditions $L_i/L_{i\,\text{d-c}} = 1/(R/R_{\text{d-c}})$, a relation that is confirmed in Table 6.2, page 80, for large values of a/δ.

It is known from equation (*6.1*), page 71, that $R_{\text{d-c}} = 1/(\sigma\pi a^2)$ ohms/m, and from (*6.5*) that $L_{i\,\text{d-c}} = \mu 8\pi$ henries/m. Since under the conditions stated $L_i = R/\omega$, the problem is to demonstrate the identity $(R/\omega)/(\mu/8\pi) = (1/\sigma\pi a^2)/R$, or $R^2 = (\omega\mu)/(8\pi^2\sigma a^2)$. Using $\omega\mu\sigma/2 = 1/\delta^2$, this becomes $R^2 = 1/(4\pi^2 a^2 \sigma\delta)$, which is in agreement with equation (*6.30*), using $R_s = 1/(\sigma\delta)$. Hence the identity is established.

6.4. An iron wire of diameter 0.128″ used in a telephone circuit has relative permeability 150 at frequency 1000 hertz. Determine the distributed resistance and distributed internal inductance of the wire at that frequency at 20°C.

The radius a of the wire in metric units is 1.63×10^{-3} m. The conductivity of iron at 20°C from Table 6.3 is 1.00×10^7 mhos/m. The skin depth δ in iron at 1000 hertz must be calculated directly from equation (*6.15*), page 74, and is 4.12×10^{-4} m. The value of a/δ is then 3.96. From Table 6.1, $R/R_{\text{d-c}}$ for this value of a/δ is about 2.25, and $L_i/L_{i\,\text{d-c}}$ is about 0.499. The value of $R_{\text{d-c}}$ for the iron wire from equation (*6.1*) is 0.0120 ohms/m. Hence the distributed resistance R for the wire at 1000 hertz is 2.70 ohms/m. Since $L_{i\,\text{d-c}}$ from (*6.5*) is 7.50 microhenries/m, the distributed internal inductance of the wire at 1000 hertz is 3.74 microhenries/m.

6.5. For an a-c surface current produced in a plane conductor of indefinite thickness by a uniform tangential a-c electric field, determine the percent of the total power loss in the conductor, per unit area of surface, that occurs in distances from the surface of 0.5, 1, 1.5, 1.6, 2, 3, 4 and 5 skin depths.

The result for any distance is obtained from equation (*6.43*) on changing the upper limit of the integral from infinity to the desired distance y in skin depths. The percent is then found to be given by $100(1 - e^{-2y/\delta})$. Numerical values are

Distance from surface in skin depths	0.5	1	1.5	1.6	2	3	4	5
Percent of total power loss, from surface to that distance	63.2	86.5	95.0	95.9	98.2	99.75	99.97	99.995

Although the magnitude of the current density diminishes as $e^{-y/\delta}$, so that at a distance of 5δ into the metal from the surface the current density is still nearly 1% of the surface value, the losses vary as the *square* of the current density at any distance and it can be seen that more than 99% of the losses occur in about 2.5 skin depths.

It is mentioned in Section 6.3, referring to the results of a transmission line analog in Problem 7.10, that if an unbounded plane conductor of thickness about 1.6δ is substituted for an unbounded plane conductor of indefinite thickness (i.e. any thickness greater than a few δ) the power loss per unit area of surface in the total metal thickness is reduced about 8%. Fig. 6-6, page 85, shows that the magnitude of the total surface current density for a given tangential electric field is the same, within about 1%, in a surface layer of thickness 1.6δ on a thick plane conductor as in an indefinitely thick surface layer (compare also the results of Problem 6.6 below). The data table of this problem, however, suggests that "removing" the current pattern beyond a thickness of 1.6δ would reduce the losses by only 5%. The extra 3% is a consequence of the reflection of the electromagnetic waves from the second surface of the metal, mentioned in Section 6.3. The resulting modified current distribution in the metal is slightly more uniform and hence produces less loss for the same total current than the exponentially decaying current distribution from which Fig. 6-6 was constructed. An alternative explanation is that the presence of the reflected wave modifies the distributed internal admittance of the conductor, making the real part a little smaller than the value given by the reciprocal of equation (*6.45*).

6.6. A tangential electric field of rms phasor value E_{z0} exists over the surface of a plane conductor. The plane surface of the conductor extends indefinitely in the x and z directions and the conductor is indefinitely thick in the y direction perpendicular to the surface. Using E_{z0} as a real reference phasor, determine an equation for the total phasor surface-current density per unit width of surface in the x direction, contained between the surface plane $y = 0$, and a parallel plane at distance y into the metal from the surface. Make calculations from the equation, a graph of which would give the curve of Fig. 6-6, page 85, for the case $\Delta y = 0$.

The answer is obtained in the same manner as equation (*6.44*), page 84, but with the upper limit of the integral changed to y instead of infinity. Thus

$$J_{sz}\,(0 \text{ to } y) \quad = \quad \int_0^1 dx \int_0^y dy\,[\sigma E_{z0}(1 + j0)\,e^{-(1+j)y/\delta}]$$

Carrying out the integration, and using the relation $R_s = 1/\sigma\delta$, this becomes

$$J_{sz}\,(0 \text{ to } y) \quad = \quad \{E_{z0}/[R_s(1+j)]\}(1 - e^{-(1+j)y/\delta})$$

Rationalizing the first term and using $e^{-jx} = \cos x - j\sin x$,

$$J_{sz}\,(0 \text{ to } y) \quad = \quad (E_{z0}/2R_s)\{1 - e^{-y/\delta}(\cos y/\delta - \sin y/\delta) - j[1 - e^{-y/\delta}(\cos y/\delta + \sin y/\delta)]\}$$

Several points on the required curve are shown in the following table.

y/δ	$J_{sz}\,(0 \text{ to } y)/(E_{z0}/2R_s)$	Magnitude	Phase Angle
0.3	$0.512 - j0.075$	0.518	−8.3°
0.6	$0.856 - j0.250$	0.891	−16.3°
0.9	$1.064 - j0.428$	1.147	−21.9°
1.2	$1.171 - j0.611$	1.323	−27.6°
1.5	$1.206 - j0.762$	1.430	−32.3°
1.8	$1.198 - j0.877$	1.486	−36.2°
2.1	$1.167 - j0.957$	1.510	−39.4°
2.4	$1.128 - j1.006$	1.511	−41.8°
2.7	$1.088 - j1.032$	1.502	−43.5°
3.0	$1.057 - j1.042$	1.486	−44.6°
4	$0.999 - j1.025$	1.433	−45.7°
5	$0.992 - j1.004$	1.412	−45.3°
10	$1.000 - j1.000$	1.414	−45°

The result is constant at the last value for all thicknesses greater than 10δ, and is in agreement with the magnitude and phase relations given by equation (*6.45*), page 84. When the above points are plotted on the graph of Fig. 6-6, the locations of the first ten points agree very precisely with the locations of the points determined by summing surface-current incremental phasors for layers 0.3δ thick.

6.7. For the same plane conductor and tangential phasor value of electric field described in Problem 6.6, determine the length of the arc of the curve of Fig. 6-6 from the origin to each of the points tabulated in Problem 6.6, relative to the length of the chord from the origin to each point, and relative to the length of the chord from the origin to the point at infinite y.

The magnitude values in Problem 6.6 are the chord lengths from the origin to any point on the curve of Fig. 6-6, expressed in units of $E_{z0}/2R_s$, the chord length to the point for infinite y being 1.414 in these units. The length of the arc of the curve from the origin to any coordinate y in the conductor is given by the same integral used in Problem 6.6, with the phase information removed. The result has no particular physical meaning, but the ratio of the arc length from the origin to the chord length over the same portion of the curve gives some impression of the relative inefficiency of a "thick" conductor compared to a conductor of optimum thickness. In the same units used in Problem 6.6, the arc length from the origin to coordinate y is

$$\int_0^1 dx \int_0^y dy\, |\sigma E_{z0}(1+j0)|\, |e^{-(1+j)y/\delta}| \quad = \quad (E_{z0}/R_s)(1-e^{-y/\delta})$$

The results are tabulated below, with the chord lengths repeated from Problem 6.6 for comparison.

y/δ	(arc length)/(length of chord for infinite y)	(chord length)/(length of chord for infinite y)	(arc length)/(chord length)
0.3	0.366	0.366	1.00
0.6	0.637	0.630	1.01
0.9	0.838	0.812	1.03
1.2	0.987	0.938	1.05
1.5	1.097	1.012	1.08
1.8	1.179	1.050	1.12
2.1	1.240	1.068	1.16
2.4	1.285	1.068	1.20
2.7	1.318	1.062	1.24
3	1.343	1.050	1.28
4	1.388	1.014	1.37
5	1.405	0.998	1.41
10	1.414	1.000	1.414

6.8. The copper inner conductor of a coaxial transmission line is a circular tube of outside diameter 0.250″ and wall thickness 0.015″. Determine its distributed resistance at frequencies of 10, 10^3, 10^5, 10^7 and 10^9 hertz. Compare the range of frequencies over which its distributed resistance remains within ½% of its d-c distributed resistance with the corresponding frequency range for a solid circular copper conductor of the same outside diameter.

The calculation requires significant quantities which have the following values at the different frequencies:

	10 hertz	10^3 hertz	10^5 hertz	10^7 hertz	10^9 hertz
δ, m	2.09×10^{-2}	2.09×10^{-3}	2.09×10^{-4}	2.09×10^{-5}	2.05×10^{-6}
a/δ	0.152	1.52	15.2	152	1520
$a_t = 1.51\times10^{-3}$ m					
a_t/δ	0.0723	0.723	7.23	72.3	723
t/δ	0.0182	0.182	1.82	18.2	182
R_s, ohms	8.25×10^{-7}	8.25×10^{-6}	8.25×10^{-5}	8.25×10^{-4}	8.25×10^{-3}
$R_{d\text{-}c} = 2.42\times10^{-3}$ ohms/m					

At 10 hertz, with $a/\delta = 0.152$, $R_{a\text{-}c} = R_{d\text{-}c} = 2.42 \times 10^{-3}$ ohms/m.

At 10^3 hertz, with $a/\delta = 1.52$, $R_{a\text{-}c}/R_{d\text{-}c}$ would be almost 1.10 for a solid conductor, but on consulting Fig. 6-8 for $a_t/\delta = 0.723$ for a tubular conductor with $t/a = 0.12$, it is clear that $R_{a\text{-}c}/R_{d\text{-}c}$ is less than 1.005, and the distributed resistance is again 2.42×10^{-3} ohms/m.

At 10^5 hertz, with $a/\delta = 15.2$, $R_{a\text{-}c}/R_{d\text{-}c}$ for a solid conductor can be found quickly from equation (*6.33*), page 78, as 7.86. But $R_{d\text{-}c}$ for the tube is greater than that of a solid conductor of the same outside diameter by a factor of 4.43. Hence if (*6.33*) were directly applicable to the tube, the result would be $R_{a\text{-}c}/R_{d\text{-}c} = 7.86/4.43 = 1.77$. However, the tube wall at this frequency is only 1.84 skin depths thick, and there is a correction factor from Fig. 6-7 to be applied. The distributed a-c resistance for a tube of wall thickness 1.84 skin depths is in fact less than that for an indefinitely thick tube, to which equations (*6.32*) and (*6.33*) would apply, by a factor of about 0.93, indicating a corrected ratio for $R_{a\text{-}c}/R_{d\text{-}c}$ of $1.77 \times 0.93 = 1.65$. The graphical result from Fig. 6-8 for $a_t/\delta = 7.23$ and $t/\delta = 0.12$ is 1.66. Hence the distributed resistance of the conductor is 4.02×10^{-3} ohms/m.

At 10^7 hertz, with $a/\delta = 152$ and $t/\delta = 18.4$, the distributed resistance is given by equation (*6.30*) and is exactly the same as for a solid conductor of the same outside diameter. The result is 4.14×10^{-2} ohms/m, and $R_{a\text{-}c}/R_{d\text{-}c} = 17.1$.

At 10^9 hertz, equation (*6.30*) again applies, giving a distributed resistance value higher by precisely a factor of 10. The result is 0.414 ohms/m, and $R_{a\text{-}c}/R_{d\text{-}c} = 171$.

The highest frequency at which $R/R_{d\text{-}c}$ remains less than 1.005 for the solid conductor is taken as the frequency for which $a/\delta \doteq 0.5$. This is found to be about 104 hertz. For the tubular conductor it is not possible to calculate the corresponding limiting frequency directly and it must be found empirically from Fig. 6-8. With $t/a = 0.12$, it appears from Fig. 6-8 that $R_{a\text{-}c}/R_{d\text{-}c} < 1.005$ when $a_t/\delta < 2$, approximately. This occurs at a frequency of about 7700 hertz, with $\delta = 7.55 \times 10^{-4}$ m and $t/\delta = 0.50$. Although Fig. 6-7 cannot be considered applicable with high precision to this situation with a/δ as low as 4.4, it does show that for plane conductors the distributed a-c resistance is quite precisely equal to the d-c resistance when t/δ is as low as 0.5. Hence the figure of 7700 hertz should be reasonably accurate, indicating that the frequency range of constant distributed resistance for the tube is about 70 times as great as the range for a solid conductor of the same outside diameter.

6.9. A pair of circular coaxial metal tubes is used as an electrostatic capacitor. The outside diameter of the inner conductor is 2 cm and the inside diameter of the outer conductor is 15 cm. The tubes are 3.5 m long. The center conductor is supported by thin transverse dielectric discs, which occupy 5% of the interconductor volume and are made of material having dielectric constant 3.2.

(*a*) What is the total capacitance of the capacitor, neglecting anomalous "edge effects" at the ends?

(*b*) If breakdown occurs in the line when the electric field in the air dielectric portions exceeds 1.5×10^6 volts/m, what is the maximum voltage that can be applied to the capacitor?

(*c*) Find the breakdown voltage of the capacitor if the outside diameter of the inner conductor is increased to 4 cm without changing the outer conductor.

(*d*) Find the breakdown voltage of the capacitor if the outside diameter of the inner conductor is increased to 8 cm without changing the outer conductor.

(*a*) The capacitance must be calculated as the sum of two separate components, one for the air dielectric portion of the line, and the other for the portion filled with solid dielectric. Using equation (*6.54*), page 93, the *distributed* capacitance of the air dielectric portion is $55.6/(\log_e 7.5) =$ 27.5 micromicrofarads/m. The air dielectric portion is 95% of the total length. Hence its capacitance is $3.5 \times 0.95 \times 27.5 = 91.5$ micromicrofarads. For the solid dielectric portion the capacitance is similarly $3.5 \times 0.05 \times 55.6 \times 3.2/(\log_e 7.5) = 15.5$ micromicrofarads. The total capacity of the line is then $91.5 + 15.5 = 107$ micromicrofarads.

(*b*) For any specific line geometry, equation (*6.52*) shows that the linearly distributed charges on coaxial conductors are directly proportional to the applied voltage. Equation (*6.51*) states that the electric field in the interconductor space is directly proportional to the linearly distributed charges, and is a maximum at the smallest value of r in the interconductor space, i.e. at the outer surface of the inner conductor.

When the breakdown field of 1.5×10^6 volts exists at the surface of the inner conductor in the air dielectric portions of the line, the magnitude of the distributed charge on each conductor is

$$2\pi(8.85 \times 10^{-12})(1 \times 10^{-2})(1.5 \times 10^6) = 8.33 \times 10^{-7} \text{ coulombs/m}$$

From equation (*6.52*) the voltage between the conductors will then be

$$8.33 \times 10^{-7} \log_e 7.5/(2\pi \times 8.85 \times 10^{-12}) = 30{,}300 \text{ volts}$$

(*c*) When the outside radius of the center conductor is 2 cm, the breakdown voltage is

$$(55.6 \times 10^{-12})(2 \times 10^{-2})(1.5 \times 10^6) \log_e 3.75/(55.6 \times 10^{-12}) = 39{,}700 \text{ volts}$$

(*d*) When the outside radius of the center conductor is 4 cm the breakdown voltage is

$$(4 \times 10^{-2})(1.5 \times 10^6) \log_e 1.875 = 37{,}600 \text{ volts}$$

The results of parts (*b*), (*c*) and (*d*) illustrate a phenomenon investigated analytically in Section 6.9, that for a fixed size of the outer conductor there is an optimum size of the inner conductor that results in minimum electric field in the interconductor space for a given applied voltage.

6.10. A coaxial transmission line has circular tubular copper conductors with walls 0.050″ thick, the inside diameter of the outer conductor being 1.25″, and the outside diameter of the inner conductor 0.375″. The center conductor is supported by a continuously spiraled dielectric webbing which fills 20% of the interconductor space. At a frequency of 200 megahertz the phase velocity on the line is measured to be 93%, and the attenuation factor to be 0.635 db/(100 ft). Determine the equivalent average dielectric constant and loss factor of the material in the interconductor space.

The frequency is high enough to ensure that $a/\delta \gg 100$ and $t/\delta \gg 1$ for both conductors. A rough check shows also that $\omega L/R \gg 1$ and $\omega C/G \gg 1$. It follows that the attenuation factor of the line is given by equation (*5.9*), page 49, using equation (*6.59*) for the characteristic impedance Z_0, (*6.49*) for the distributed resistance R, and (*6.56*) for the distributed conductance G. Using (*6.60*) for the phase velocity, the real part of the average dielectric constant of the interconductor medium is found from

$$k_e' \text{ (average)} = [(3.00 \times 10^8)/(0.93 \times 3.00 \times 10^8)]^2 = 1.16$$

The portion α_R of the attenuation factor caused by conductor resistance is

$$[R_s(1 + b/a)/(2\pi b)]/[60 \log_e (b/a)/\sqrt{k_e'}] = 2.26 \times 10^{-3} \text{ nepers/m} = 0.598 \text{ db/(100 ft)}$$

The portion of the attenuation factor due to dielectric loss is then $\alpha_G = 0.635 - 0.598 = 0.037$ db/(100 ft) $= 1.40 \times 10^{-4}$ nepers/m. From (*5.9*) the distributed conductance that would produce this attenuation is $G = 2\alpha_G/Z_0 = 2.80 \times 10^{-4}/67.3 = 4.17 \times 10^{-6}$ mhos/m. Using equation (*6.56*),

$$\tan \delta = G/\omega C = 4.17 \times 10^{-6}/[(2\pi \times 200 \times 10^6)(2\pi \times 1.16 \times 8.85 \times 10^{-12})/1.206] = 7.2 \times 10^{-5}$$

6.11. Show that the characteristic impedance Z_0 of a coaxial transmission line as given by equation (*6.59*), page 96, is equal to the d-c resistance between two concentric circle metallic electrodes on a surface resistance sheet (such as a thin carbon film deposited on a plane of nonconducting material), the d-c surface resistivity of the sheet being $377/\sqrt{k_e'} \doteq 120\pi/\sqrt{k_e'}$ ohms/square, the outside radius of the inner circular electrode being a, and the inside radius of the outer circular electrode being b.

For a circular ring of radius r and radial width dr on the resistance sheet, the d-c resistance between the inner circumference and the outer circumference is $dR = \rho_s(dr)/2\pi r$ ohms, where ρ_s is the d-c surface resistivity of the sheet in ohms/square. The resistances of such rings filling the area between the contact electrodes at $r = a$ and $r = b$ are in series between the electrodes. Hence the total resistance between the electrodes is

$$R = \int_a^b dR = (\rho_s/2\pi) \log_e (b/a) \quad \text{ohms}$$

and if $R_s = 120\pi/\sqrt{k_e'}$ ohms/square, $R = (60/\sqrt{k_e'}) \log_e (b/a) = Z_0$ ohms.

This relation derives from the fact that the wave impedance or intrinsic impedance for plane transverse electromagnetic waves in an unbounded medium of dielectric constant k_e' and low loss factor is

$$Z_{\text{TEM}} = \sqrt{\mu/\epsilon} = (1/\sqrt{k_e'})\sqrt{\mu_0/\epsilon_0} = 377/\sqrt{k_e'} \text{ ohms}$$

6.12. Show that the thickness 1.5δ to 1.6δ of plane metal sheet conductor shown in Fig. 6-7, page 87, to have minimum distributed high frequency resistance is very close to $\frac{1}{4}$ wavelength thick for the waves propagating in the metal.

In the analysis of Section 6.3, all field quantities for the plane waves propagating in the metal vary in amplitude by a term $e^{-y/\delta}$ and in phase by a term $e^{-jy/\delta}$, when the wave is traveling in the direction of increasing y. According to the latter term, the phase will change by 2π radians in a distance Δy given by $\Delta y/\delta = 2\pi$. But in a harmonic space pattern, the distance in which the phase changes by 2π radians is defined as the wavelength λ of the pattern. Hence in the metal, $\lambda = 2\pi\delta \doteq 6.28\delta$, and $\frac{1}{4}$ wavelength would be 1.57δ, very close to the sheet thickness found to have minimum distributed a-c resistance.

6.13. At a high frequency ω rad/sec a coaxial transmission line has distributed external inductance L_x henries/m, distributed resistance R ohms/m, and negligible distributed internal inductance L_i when the interconductor space is air filled. Show that if the interconductor space could be filled with a magnetic medium having relative permeability k_m' and magnetic loss factor tan δ_m such that $\sqrt{k_m'} - 1 > \omega L_x(\tan\delta_m)/R$, the attenuation factor of the line will be reduced if the medium has a dielectric constant of unity and no dielectric losses.

Since the attenuation factor of an air dielectric transmission line at high frequencies is given by $\alpha_R = R/(2\sqrt{L_x/C})$ nepers/m when there are no dielectric losses, the requirement on the medium is that the square root of the factor by which the distributed external inductance is increased on adding the medium must exceed the factor by which the distributed resistance is increased from magnetic losses. (See Problem 6.32.) The total distributed external inductance in the presence of the medium is $k_m'L_x$, and the total distributed resistance is $(R + \omega L_x \tan\delta_m)$. The properties of the medium must then satisfy $\sqrt{k_m'L_x/L_x} > (R + \omega L_x \tan\delta_m)/R$ which converts to the expression stated in the problem.

At low frequencies a useful amount of the desired result can be achieved by winding thin magnetic metal tape (such as Permalloy) around the center conductor of a coaxial line. At higher frequencies ferrite materials have high values of k_m' and low values of tan δ_m but they also exhibit dielectric losses, which according to equation (*5.9*) are accentuated by the value of k_m'. Their use as an interconductor medium in coaxial lines can result in reduced attenuation for certain designs of line at low and intermediate frequencies.

6.14. A coaxial transmission line is to be designed to transmit 50 kilowatts of power at a frequency of 100 megahertz over a distance of 100 ft with at least 90% efficiency. What is the minimum diameter of a coaxial line with copper conductors and air dielectric that will meet the efficiency specification? Will the minimum diameter line handle the stated amount of power?

The attenuation of the line in decibels is $10\log_{10}(50{,}000/45{,}000) = 0.457$ db $= 0.0526$ nepers. Since the line is 30.48 m long, the attenuation factor is $0.0526/30.48 = 0.00172$ nepers/m. For fixed outside diameter, a coaxial line of minimum attenuation has $b/a \doteq 3.6$ (see Section 6.9) and for an air dielectric line this corresponds to $Z_0 = 76.6$ ohms. Then $\alpha = [4.6R_s/(2\pi b)]/153.2$, and since $R_s = 2.61 \times 10^{-3}$ ohms for copper at 10^8 hertz, b to give the desired value of attenuation is found to be 0.725 cm.

The peak voltage on the line, assuming it to be terminated in its characteristic impedance will be $\sqrt{2}\sqrt{50{,}000 \times 76.6} = 2760$ volts. The interconductor distance being about 0.5 cm, the maximum electric field is of the order of 500,000 volts/m (a more precise value could be determined from equations (*6.51*) and (*6.52*)). This is a fairly high value of electric field, but would be tolerable in a well-maintained line. From the thermal point of view the line must dissipate 5 kilowatts in 100 feet, or 50 watts per foot of length. There is no simple basis for demonstrating that this would result in the temperature of the center conductor rising to more than 300°F, which may be considered excessive. For steady power transmission of 50 kilowatts at 100 megahertz, manufacturers recommend a copper transmission line with outside diameter about 3″.

6.15. Assuming that a parallel wire transmission line with circular tubular or solid conductors is operated at high enough frequencies that its characteristic impedance is given by equation (*6.75*), the resistance of its conductors if isolated is given by equation (*6.30*), and the proximity effect factor for resistance is given by (*6.61*), show that if the radius of the conductors is varied while the separation of their axes is held constant, there is an intermediate value of the conductor radius at which the line has a minimum attenuation factor. Find an approximate value for the ratio $s/2a$ (s being constant) that gives the minimum value for the attenuation factor.

Combining the conditions stated, the attenuation factor of the line is

$$\alpha \;=\; \frac{2R_s}{\pi s}\,\frac{s/2a}{\sqrt{1-(2a/s)^2}}\,\frac{1}{120\cosh^{-1} s/2a}\ \text{nepers/m}$$

Letting $s/2a = x$, the problem stated in its simplest terms is to find the value of x that minimizes $x/[\sqrt{1-(1/x)^2}\,(\cosh^{-1} x)]$. Although this is not an impossible analytical task, an adequate solution is found much more quickly by making calculations for several values of x. The attenuation factor is found to be constant within about $\frac{1}{2}$% for values of $s/2a$ from 2.1 to almost 2.5, with a minimum close to 2.32.

6.16. Determine the attenuation factor at a frequency of 1 kilohertz for a parallel conductor transmission line whose copper conductors are tubes of outside diameter $\frac{3}{8}''$ and wall thickness $\frac{1}{16}''$, the separation between the adjacent surfaces of the conductors being $\frac{1}{8}''$. The line is assumed to have air dielectric.

Since $a/\delta \doteq 2.3$, equation (*6.61*) cannot be used for calculating the proximity effect factor for resistance, and since $s/2a = 1.333$, equation (*6.62*) is also not considered to be accurate. The full calculation of equation (*6.63*) must therefore be used.

The various quantities required in the calculation are $a_t = 3.55 \times 10^{-3}$, $a_t/a = 0.746$, $a_t/\delta = 1.70$, and $s/2a = 1.333$.

The next stage of the calculations gives $A_1 = 0.56$, $A_2 = 0.041$, $A_3 = 0.28$.

The relative effect of the various terms is then finally indicated by $P = 1/\sqrt{1 - 0.314 + 0.016} = 1.19$. It is evident that the error incurred in P by dropping A_2 and A_3 from the calculations would be about 1%. Equation (*6.62*) gives $P = 1.17$, a deviation of 2% from the value given by the more complete formula.

For $a_t/\delta = 1.70$ and $t/a = 0.333$, Fig. 6-8 gives $R_{\text{a-c}}/R_{\text{d-c}} \doteq 1.02$. $R_{\text{d-c}}$ is found by the usual formulas to be 8.70×10^{-4} ohms/m for the two conductors. Hence the value of the distributed line resistance $R = 1.19(1.02)(8.70 \times 10^{-4}) = 1.06 \times 10^{-3}$ ohms/m.

In Section 6.9 a rough criterion $L_x/L_i = 2l_x l_i$ is developed for the ratio of a line's distributed external inductance L_x to its distributed internal inductance L_i. In this expression l_x is the interconductor distance in which magnetic flux contributes to L_x, and l_i is the estimated thickness of the region inside the conductors in which magnetic flux contributes to L_i. In the present problem $t/\delta \doteq 0.8$, so l_i must be taken as equal to t. Since the facing surfaces of the conductors are separated by $2t$, $2l_x/l_i \doteq 4$, and L_i is a substantial part of the line's total distributed inductance L. The value of L_x from equation (*6.74*) is 0.318 microhenries/m. The value of $L_{i\,\text{d-c}}$ for the two tubular conductors from equation (*6.84*) is 0.044 microhenries/m. Equation (*6.90*) and the accompanying discussion in Section 6.8 suggest that this should be reduced about 1% because t/δ, being approximately 0.8, is a little higher than the maximum value at which L_i for tubular and plane sheet conductors can be assumed to remain within $\frac{1}{2}$% of $L_{i\,\text{d-c}}$. Section 6.8 emphasizes that there is no very accurate information available about the influence of proximity effect on distributed internal inductance, but that it is plausible to reduce L_i by the square root of the proximity effect factor for resistance, found above to be 1.19. The net result of all these corrections is to give a value 0.040 microhenries/m for L_i, which adds to L_x to give $L = 0.358$ microhenries/m, with a probable error not exceeding about 1%.

The distributed capacitance of the line from equation (*6.72*) is 35.0 micromicrofarads/m, and since it is obvious that $\omega L/R \gg 1$ and $\omega C/G \gg 1$, the characteristic impedance of the line is $\sqrt{L/C} = 101$ ohms, more than 5% higher than the value given by equation (*6.75*), which neglects distributed internal inductance.

Finally, the attenuation factor is $\alpha = R/2Z_0 = 5.25 \times 10^{-6}$ nepers/m.

Supplementary Problems

6.17. The distributed internal inductance and the distributed resistance of a solid circular conductor change by less than $\frac{1}{2}\%$ from the d-c values at all low frequencies for which $a/\delta < 0.5$, where a is the radius of the conductor, and δ the skin depth given by equation (*6.15*), page 74. Show that over this same range of frequencies $\omega L_i/R = \frac{1}{4}(a/\delta)^2$, within $\frac{1}{2}\%$.

6.18. (*a*) For isolated wires of each of the following AWG sizes (same as B & S sizes), determine the highest frequency at which the distributed resistance will remain within $\frac{1}{2}\%$ of the d-c value: 000, 0, 2, 4, 8, 12, 16, 20, 24, 28, 32, 36, and 40.

(*b*) Find an empirical equation from this data, relating $f_{1/2}(X)$ to $f_{1/2}(16)$, where $f_{1/2}(16)$ is the frequency determined in part (*a*) for 16 gauge wire, and $f_{1/2}(X)$ is the frequency for wire of any gauge X.

Ans. (*a*) 79.3, 126, 200, 318, 804, 2030, 5140, 13,000, 32,850, 83,100, 210,000, 530,000 and 1,337,000 hertz.

(*b*) The equation is $f_{1/2}(X) = 5140(1.261)^{(X-16)}$ hertz. For use in the equation the wire size 000 must be called size -2. The nature of the equation can be found from plotting $f_{1/2}(X)$ against X on various types of graph paper. The plot on semilog paper with X on the linear scale is a straight line.

6.19. From some source (e.g. Dwight's *Tables of Integrals and Other Mathematical Data*) find power series suitable for expressing ber x, bei x, ber$'$ x, and bei$'$ x, fcr small values of x. Then derive equations (*6.28*) and (*6.29*) from (*6.27*).

6.20. Show that for a solid circular conductor having $a/\delta \gg 100$, equation (*6.33*) is consistent with (*6.30*), and that under these conditions $R/R_{\text{d-c}} = \frac{1}{2}a/\delta$ and $L_i/L_{i\,\text{d-c}} = 1/(\frac{1}{2}a/\delta)$.

6.21. A solid circular aluminum conductor has diameter 1.50″. Determine the ratio of the current density at the center of the conductor to the current density at the surface, at frequencies of 60, 200 and 1000 hertz. Assume a temperature of 20°C.

Ans. 66% at 60 hertz, 22% at 200 hertz, 0.60% at 1000 hertz.

6.22. Extend Table 6.1, page 79, by calculating $R_{\text{a-c}}/R_{\text{d-c}}$ for a solid circular wire at the following additional values of a/δ: 5, 10, 20, 40, 60, 80, 100. *Ans.* 2.77, 5.26, 10.25, 20.25, 30.25, 40.25, 50.25.

6.23. From the preceding problem it appears that for $a/\delta > 20$, $R_{\text{a-c}}/R_{\text{d-c}} = \frac{1}{2}(a/\delta) + \frac{1}{4}$. Show that this is consistent with equation (*6.33*) when the division of the latter is carried out to two terms.

6.24. Determine the distributed internal impedance of a copper sheet at frequencies of 60, 10^3, 10^6 and 10^9 hertz, assuming copper of 100% conductivity at 20°C. Assume the metal to be many skin depths thick at each frequency.

Ans. $(2.02 + j2.02) \times 10^{-6}$ ohms/square at 60 hertz; $(8.23 + j8.23) \times 10^{-6}$ ohms/square at 10^3 hertz; $(2.61 + j2.61) \times 10^{-4}$ ohms/square at 10^6 hertz; $(8.23 + j8.23) \times 10^{-3}$ ohms/square at 10^9 hertz.

6.25. At a frequency of 10^6 hertz determine the distributed internal impedance of plane sheets of aluminum and lead, and of a plane sheet of iron having a relative permeability of 200 at that frequency. The temperature is 20°C in each case.

Ans. $(3.33 + j3.33) \times 10^{-4}$ ohms/square for aluminum; $(9.33 + j9.33) \times 10^{-4}$ ohms/square for lead; $(8.91 + j8.91) \times 10^{-3}$ ohms/square for iron.

6.26. What wall thickness should a circular tubular copper conductor of outside radius 0.100″ have if its distributed resistance is to remain within $\frac{1}{2}\%$ of the d-c value for all frequencies up to 10^8 hertz?

Ans. a/δ being very large, a tubular conductor with $t/\delta = 0.5$ or $t = 3.3 \times 10^{-6}$ m will achieve the desired result.

6.27. Show that for a circular metal tube of outside radius a, inside radius b, and made of *nonmagnetic* material, $\sqrt{f/R_{\text{d-c}}} = 892a_t/\delta = 892b_t/\delta$ where f is in hertz, $R_{\text{d-c}}$ is the distributed d-c resistance of the tube in ohms/m, and a_t and b_t are defined by equations (*6.47*) and (*6.48*) respectively.

6.28. A 12 gauge copper conductor carries a current of 5 amperes at a frequency of 10^7 hertz. What is the power loss per meter length of conductor, and how does it compare with the result for a d-c current of the same magnitude flowing in the same conductor?

Ans. At 10^7 hertz the power loss is 3.2 watts/m, about 25 times as great as the d-c power loss of 0.130 watts/m.

6.29. Show that if a medium of dielectric constant k_e' and loss factor $\tan\delta$ fills the interconductor space of any transmission line, the contribution of the distributed conductance to the attenuation factor at high frequencies is $\alpha_G = (\omega \tan\delta)/2v_p$ nepers/m.

6.30. If a coaxial transmission line with circular conductors has an outer conductor whose inside radius is 3.50″, what must be the radial distance between the facing conductor surfaces of the line to give the line a distributed capacitance of 1000 micromicrofarads/m, the interconductor medium being air?
Ans. 0.189 in.

6.31. Show that if the medium filling the interconductor space of a coaxial transmission line has a conductivity σ_m mhos/m, the distributed conductance of the line is given by $G = 2\pi\sigma_m/(\log_e b/a)$ mhos/m.

6.32. Show that if the interconductor space of any transmission line is filled with material having a complex permeability $\mu_m' - j\mu_m''$ in mks units, or relative permeability $(\mu_m' - \mu_m'')/\mu_0 = k_m' - jk_m''$, and $k_m''/km' = \tan\delta_m$, then the distributed external inductance of the line is increased by the factor k_m' over its value with a nonmagnetic medium in the space, and a contribution $R_m = \omega L_x \tan\delta_m$ ohms/m must be added to the usual distributed resistance of the conductors, where L_x is the distributed external inductance with the magnetic medium present.

6.33. For a coaxial line operated at 10^8 hertz, the outside diameter of the inner conductor is 0.500″ and the inside diameter of the outer conductor is 1.75″. Both conductors are several skin depths thick.

(*a*) What is the distributed resistance of the line if both conductors are copper?

(*b*) By what percent will the distributed resistance of the line be increased if the outer conductor is changed to aluminum, the center conductor being copper?

(*c*) By what percent will the distributed resistance of the line be increased if the inner conductor of the line is changed to aluminum, the outer conductor being copper?

(*d*) By what percent will the distributed resistance be increased if the outer conductor is changed to pure iron with a relative permeability of 50 at 10^8 hertz, the inner conductor being copper?

Ans. (*a*) 0.0842 ohms/m; (*b*) 4%; (*c*) 18%; (*d*) 113%. The result shows that changing the outer conductor of a coaxial transmission line from copper to lighter and less expensive aluminum has little effect on the distributed resistance and attenuation factor of the line. The magnetic permeability of iron makes it unsuitable for the same purpose.

6.34. Show that for copper $R_s = 2.61 \times 10^{-7}\sqrt{f}$ ohms/square, where f is in hertz.

6.35. Show that if the interconductor space of a coaxial transmission line is filled with a material of dielectric constant k_e' and loss factor $\tan\delta$, the ratio b/a of the radii of the facing conductor surfaces that will give minimum high frequency attenuation for a fixed size of outer conductor is 3.592, for all values of k_e' or $\tan\delta$.

6.36. From equation (*6.27*), page 77, determine explicit expressions for $R_{\text{a-c}}/R_{\text{d-c}}$ and $\omega L_i/R_{\text{d-c}}$ for a solid circular conductor of radius a defined by mks permeability μ and conductivity σ at angular frequency ω.

Ans.
$$\frac{R_{\text{a-c}}}{R_{\text{d-c}}} = \frac{x}{2}\cdot\frac{\operatorname{ber} x \operatorname{bei}' x - \operatorname{bei} x \operatorname{ber}' x}{\operatorname{ber}'^2 x + \operatorname{bei}'^2 x} \qquad \text{where } x = \sqrt{2}\,a/\delta$$
$$\frac{\omega L_i}{R_{\text{d-c}}} = \frac{x}{2}\cdot\frac{\operatorname{ber} x \operatorname{ber}' x + \operatorname{bei} x \operatorname{bei}' x}{\operatorname{ber}'^2 x + \operatorname{bei}'^2 x}$$

Chapter 7

Impedance Relations

7.1. Reflection coefficient for voltage waves.

In Chapter 4 it was shown that when a uniform transmission line is terminated in an impedance equal to its characteristic impedance, there are no reflected waves on the line, and the impedance at any point of the line (including the input terminals) is also equal to the line's characteristic impedance. "The impedance at any point of the line" was found to mean the input impedance of the line section on the load side of the point, when the portion of the line on the generator side of the point is removed.

When a uniform transmission line is *not* terminated in its characteristic impedance, but is terminated in some arbitrary impedance $Z_T \neq Z_0$, there are *always* reflected waves on the line, and the impedance at *every* point of the line differs from the characteristic impedance Z_0.

Referring to the general transmission line circuit of Fig. 7-1, the expression for the phasor voltage at any coordinate z of the line is, from Section 4.1,

$$V = V_1 e^{-\gamma z} + V_2 e^{+\gamma z} \tag{7.1}$$

where V_1 and V_2 are phasor coefficients whose values are determined by the voltage and internal impedance of the signal source connected at $z = 0$, the attenuation and phase factors of the line, the line length l, and the terminal load impedance Z_T connected at $z = l$.

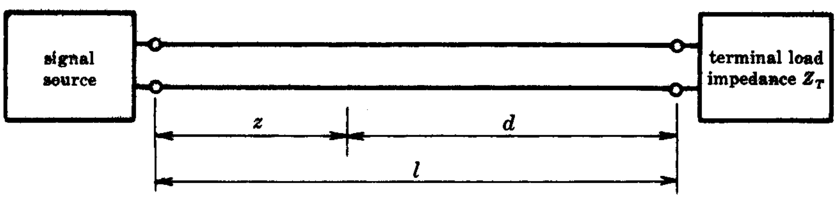


Fig. 7-1. General transmission line circuit.

Applying equation (*3.13*), page 23, to (*7.1*),

$$dV/dz = -(R + j\omega L)I = -\gamma V_1 e^{-\gamma z} + \gamma V_2 e^{+\gamma z}$$

from which

$$I = \frac{\gamma}{R + j\omega L}(V_1 e^{-\gamma z} - V_2 e^{+\gamma z}) \tag{7.2}$$

From equation (*4.12*), page 32,

$$\frac{\gamma}{R + j\omega L} = \sqrt{\frac{G + j\omega C}{R + j\omega L}} = \frac{1}{Z_0}$$

Thus

$$I = \frac{1}{Z_0}(V_1 e^{-\gamma z} - V_2 e^{+\gamma z}) \tag{7.3}$$

A comparison of equation (*7.3*) with (*4.2*) shows that when the phasor current on the line is described in the general form

$$I = I_1 e^{-\gamma z} + I_2 e^{+\gamma z} \tag{7.4}$$

the relation between the phasor current coefficients I_1 and I_2 (for the harmonic current waves traveling respectively in the direction of increasing z and the direction of decreasing z) and the corresponding phasor voltage coefficients V_1 and V_2 is

$$I_1 = V_1/Z_0 \qquad I_2 = -V_2/Z_0 \tag{7.5}$$

The difference of sign is a fundamental distinction between the two pairs of waves (each pair consisting of a voltage wave and a current wave) traveling in the two directions on a transmission line. A direct consequence is that if the phasor values of the two voltage waves at a specific point on a transmission line differ in phase by ψ radians, the phasor values of the two current waves at the same point will differ in phase by $\psi + \pi$ radians. From (*7.3*) the magnitudes of the current phasors are directly proportional to the magnitudes of the corresponding voltage phasors. The manner in which these magnitude and phase relations affect the "standing wave" patterns of voltage and current along a transmission line is discussed in Chapter 8.

The impedance at any point on a transmission line is given by the ratio of the phasor voltage (equation (*7.1*)) to the phasor current (equation (*7.3*)) at the point. At the terminal load end of the line this ratio is constrained to be equal to the connected terminal impedance. Thus

$$\frac{V\,(z=l)}{I\,(z=l)} = Z_T = Z_0\left[\frac{V_1 e^{-\gamma l} + V_2 e^{+\gamma l}}{V_1 e^{-\gamma l} - V_2 e^{+\gamma l}}\right] \tag{7.6}$$

where $V_1 e^{-\gamma l}$ is the phasor value at $z = l$ of a harmonic voltage wave traveling in the direction of increasing z, and $V_2 e^{+\gamma l}$ is the phasor value at $z = l$ of a harmonic voltage wave traveling in the direction of decreasing z. In the general transmission line circuit of Fig. 7-1 to which this analysis applies, the only connected signal source initiates a harmonic voltage wave traveling in the direction of increasing z. It must be concluded that the wave traveling in the direction of decreasing z comes into existence through a physical process of reflection, the reflection occurring at the terminal load end of the line and being a function of the connected impedance Z_T.

In any discussion of reflected wave phenomena, whether the waves be sound waves, water waves, light waves, etc., the concept of a "reflection coefficient" is introduced. The natural definition of such a concept is

$$\text{reflection coefficient} = \frac{\text{value of reflected wave at point of reflection}}{\text{value of incident wave at point of reflection}}$$

The word "value" here is deliberately unspecific, since it may have different meanings in different situations.

For a linear system, if the incident waves are harmonic in time, the reflected waves will also be harmonic in time and of the same frequency. For harmonic voltage waves on a transmission line the appropriate definition of a reflection coefficient becomes

$$\begin{matrix}\text{reflection coefficient}\\ \text{for harmonic voltage waves}\end{matrix} = \frac{\begin{matrix}\text{phasor value of reflected voltage wave}\\ \text{at point of reflection}\end{matrix}}{\begin{matrix}\text{phasor value of incident voltage wave}\\ \text{at point of reflection}\end{matrix}}$$

The ratio of two phasor quantities is a complex number. For the terminal load end of the line the reflection coefficient for phasor voltage waves is designated as ρ_T, where

$$\rho_T = |\rho_T|\, e^{j\phi_T} \tag{7.7}$$

From (*7.6*),

$$\rho_T = V_2 e^{+\gamma l}/V_1 e^{-\gamma l} = (V_2/V_1)e^{2\gamma l} \tag{7.8}$$

The magnitude of this complex number reflection coefficient is the ratio of the magnitude of the reflected wave to the magnitude of the incident wave at the point of reflection. The phase angle of the reflection coefficient establishes the phase relation between the reflected and incident waves at the point of reflection, often referred to as "the phase change on reflection".

Dividing all four terms on the right of (*7.6*) by $V_1 e^{-\gamma l}$, and making use of (*7.8*), gives a simple relation between the reflection coefficient, the terminal load impedance Z_T and the characteristic impedance Z_0 of the line,

$$Z_T/Z_0 = (1+\rho_T)/(1-\rho_T) \tag{7.9}$$

The dimensionless ratio Z_T/Z_0 *is called the normalized value of the impedance* Z_T. Any actual complex terminal load impedance in ohms will have *different normalized values* when it is connected to transmission lines with different values of characteristic impedance Z_0.

In Chapter 8 there is described a basic measurement technique, used with transmission lines at very high frequencies, that easily and directly yields values of the magnitude $|\rho_T|$ and the phase angle ϕ_T of the complex reflection coefficient ρ_T. If the reflection coefficient is written as $\rho_T = |\rho_T|\cos\phi_T + j\,|\rho_T|\sin\phi_T$, and the normalized terminal load impedance is written as $Z_T/Z_0 = R_T/Z_0 + jX_T/Z_0$, the components of the normalized terminal load impedance can be found from the reflection coefficient data by using the following expressions derived from (*7.9*):

$$\frac{R_T}{Z_0} = \frac{1 - |\rho_T|^2}{1 + |\rho_T|^2 - 2\,|\rho_T|\cos\phi_T} \tag{7.9a}$$

$$\frac{X_T}{Z_0} = \frac{2\,|\rho_T|\sin\phi_T}{1 + |\rho_T|^2 - 2\,|\rho_T|\cos\phi_T} \tag{7.9b}$$

Solving (*7.9*) for ρ_T,

$$\rho_T = (Z_T - Z_0)/(Z_T + Z_0) = (Z_T/Z_0 - 1)/(Z_T/Z_0 + 1) \tag{7.10}$$

Equation (*7.10*) states that the reflection coefficient at the terminal load end of a transmission line is, as might be expected, a function solely of the terminal load impedance connected to the line and of the characteristic impedance of the line. More succinctly, it is a function solely of the normalized value of the terminal load impedance.

The following table shows the value of the reflection coefficient as a complex number, and its magnitude and phase angle, for several easily calculated cases of representative terminal load impedances.

Table 7.1

Z_T/Z_0	ρ_T	$\lvert\rho_T\rvert$	ϕ_T	Nature of the termination
$1 + j0$	$0 + j0$	0	indeterminate	Equal to characteristic impedance.
$0 + j0$	$-1 + j0$	1	π	Short circuit.
infinite	$1 + j0$	1	0	Open circuit.
$0 + j1$	$0 + j1$	1	$\pi/2$	If Z_0 is real, Z_T is a pure inductive reactance, equal in magnitude to Z_0.
$0 - j1$	$0 - j1$	1	$-\pi/2$	If Z_0 is real, Z_T is a pure capacitive reactance, equal in magnitude to Z_0.
$2 + j0$	$\frac{1}{3} + j0$	$\frac{1}{3}$	0	If Z_0 is real, Z_T is a pure resistance, equal to $2Z_0$.
$\frac{1}{2} + j0$	$-\frac{1}{3} + j0$	$\frac{1}{3}$	π	If Z_0 is real, Z_T is a pure resistance, equal to $\frac{1}{2}Z_0$.
$n + j0$ $(n > 1)$	$\frac{n-1}{n+1} + j0$	$\frac{n-1}{n+1}$	0	If Z_0 is real, Z_T is a pure resistance, greater than Z_0.
$n + j0$ $(n < 1)$	$-\left(\frac{n-1}{n+1}\right) + j0$	$\frac{n-1}{n+1}$	π	If Z_0 is real, Z_T is a pure resistance, less than Z_0.

It will be noted that except for the special cases of short circuit, open circuit, and nonreflective terminations, the normalized terminal load impedances expressed by simple real or imaginary numbers correspond to physically simple terminations of pure resistance or pure reactance only when Z_0 is essentially a pure resistance, i.e. when the "high frequency" approximations of Chapter 5 are valid. Equations (*7.9*) and (*7.10*) correctly relate impedances and reflection coefficients, however, whether Z_0 is real or complex. Section 7.6 discusses some unusual situations that arise when Z_0 is complex. In the remainder of this section, and in Sections 7.3, 7.4 and 7.5, all stated relations between the voltage reflection coefficient ρ_T and the normalized terminal load impedance Z_T/Z_0 are true when Z_0 is real but many of them are not true if Z_0 is complex.

Some of the reflection coefficient values in Table 7.1 can be understood directly from physical reasoning. A terminal load impedance that contains no resistive component, for example, cannot absorb power from an incident wave and must be totally reflecting. The magnitude of the reflection coefficient is therefore unity for all purely reactive terminations, as well as for open circuit and short circuit terminations. Purely resistive terminations of normalized value other than unity dissipate a finite fraction of the power of the incident wave and reflect the balance. The magnitude of the voltage reflection coefficient is then necessarily less than unity.

At a short circuit termination the voltage must always be identically zero, a result that can only be achieved by having the reflected voltage wave always instantaneously equal in magnitude and opposite in sign to the incident voltage wave. For harmonic waves this is equivalent to saying that the voltage reflection coefficient must have a magnitude of unity and a phase angle of π radians. The result shown in Table 7.1 for an open circuit termination, that the reflection coefficient has magnitude unity and phase angle zero, shows that the instantaneous voltage at such a termination has the maximum possible value and is always equal to twice the instantaneous voltage of the incident wave. Normalized terminal load impedances with a finite imaginary component always produce reflection coefficients with phase angles other than 0 or π, the phase angles lying between 0 and $+\pi$ for inductively reactive terminations, and between 0 and $-\pi$ for capacitively reactive terminations, when Z_0 is real.

Example 7.1.

A transmission line with characteristic impedance $50 + j0$ ohms is terminated in a load impedance $25 - j75$ ohms. What is the reflection coefficient for voltage waves at the terminal load end of the line?

By equation (*7.10*),

$$\rho_T = (Z_T - Z_0)/(Z_T + Z_0) = \{(25 - j75) - (50 + j0)\}/\{(25 - j75) + (50 + j0)\}$$
$$= 0.333 - j0.667 = 0.745\underline{/63.4^\circ}$$

Example 7.2.

From the data of Example 7.1, find the normalized value of the terminal load impedance, and use it to determine the reflection coefficient.

The normalized value of Z_T is $Z_T/Z_0 = (25 - j75)/(50 + j0) = 0.50 - j1.50$. Then from equation (*7.10*),

$$\rho_T = (Z_T/Z_0 - 1)/(Z_T/Z_0 + 1) = (0.50 - j1.50 - 1)/(0.50 - j1.50 + 1) = 0.333 - j0.667, \text{ as before}$$

Example 7.3.

The reflection coefficient produced by the connected terminal load impedance on a low loss transmission line is $-0.65 + j0.25$. The characteristic impedance of the line is $52 + j0$ ohms. Find the numerical value of the terminal load impedance.

The magnitude $|\rho_T|$ of the reflection coefficient is 0.696. By equations (*7.9a*) and (*7.9b*),

$$\frac{R_T}{Z_0} = \frac{1 - |\rho_T|^2}{1 + |\rho_T|^2 - 2\,|\rho_T| \cos\phi_T} = \frac{1 - |0.696|^2}{1 + |0.696|^2 - 2(-0.65)} = 0.183$$

$$\frac{X_T}{Z_0} = \frac{2\,|\rho_T| \sin\phi_T}{1 + |\rho_T|^2 - 2\,|\rho_T| \cos\phi_T} = \frac{2(0.25)}{1 + |0.696|^2 - 2(-0.65)} = 0.177$$

Then $$R_T + jX_T = (0.183 + j0.177)(52 + j0) = 9.52 + j9.20 \text{ ohms}$$

The transmission line circle diagram introduced in Chapter 9 provides an easily visualized perspective of the relation between normalized terminal impedances and the reflection coefficients they produce, for all values of impedance. The results of the table above, and of Examples 7.1, 7.2, and 7.3 should be confirmed later using the chart.

7.2. Input impedance of a transmission line.

The impedance at any point on a general transmission line terminated in an arbitrary impedance Z_T is, from equations (*7.1*) and (*7.3*),

$$Z(z) \quad = \quad \frac{V(z)}{I(z)} \quad = \quad Z_0\left[\frac{V_1 e^{-\gamma z} + V_2 e^{+\gamma z}}{V_1 e^{-\gamma z} - V_2 e^{+\gamma z}}\right] \tag{7.11}$$

Since $Z(z)$ differs from Z_0 only because Z_T differs from Z_0, $Z(z)$ must be expressible in terms of Z_T and Z_0, together with the coordinate z, and the length l and propagation factors α and β for the line.

Dividing all the terms on the right of (*7.11*) by V_1 and substituting $\rho_T e^{-2\gamma l} = V_2/V_1$ from (*7.8*),

$$\frac{Z(z)}{Z_0} \quad = \quad \left[\frac{e^{-\gamma z} + \rho_T e^{-2\gamma l} e^{+\gamma z}}{e^{-\gamma z} - \rho_T e^{-2\gamma l} e^{+\gamma z}}\right] \tag{7.12}$$

Multiplying all terms on the right of (*7.12*) by $e^{\gamma l}$,

$$\frac{Z(z)}{Z_0} \quad = \quad \frac{e^{+\gamma(l-z)} + \rho_T e^{-\gamma(l-z)}}{e^{+\gamma(l-z)} - \rho_T e^{-\gamma(l-z)}} \tag{7.13}$$

which shows that the normalized value of $Z(z)$ at any coordinate z on a line is most simply expressed as a function of the distance $l - z$ measured from the terminal load end of the line. The coordinate of any point on a line measured from the terminal load end has already been designated by the symbol d in Chapter 2. Thus

$$\frac{Z(d)}{Z_0} \quad = \quad \frac{e^{+\gamma d} + \rho_T e^{-\gamma d}}{e^{+\gamma d} - \rho_T e^{-\gamma d}} \tag{7.14}$$

Substituting for ρ_T from equation (*7.10*),

$$\frac{Z(d)}{Z_0} \quad = \quad \frac{e^{\gamma d}(Z_T/Z_0 + 1) + e^{-\gamma d}(Z_T/Z_0 - 1)}{e^{\gamma d}(Z_T/Z_0 + 1) - e^{-\gamma d}(Z_T/Z_0 - 1)} \tag{7.15}$$

or

$$\frac{Z(d)}{Z_0} \quad = \quad \frac{(Z_T/Z_0)(e^{+\gamma d} + e^{-\gamma d}) + (e^{+\gamma d} - e^{-\gamma d})}{(e^{+\gamma d} + e^{-\gamma d}) + (Z_T/Z_0)(e^{+\gamma d} - e^{-\gamma d})} \tag{7.16}$$

Equations (*7.14*), (*7.15*) and (*7.16*) are all expressions from which it is possible to calculate by complex number arithmetic the impedance at any point on a transmission line, the point being distant d from the terminal load end of the line, the line being terminated in impedance Z_T, and the properties of the line itself being described by the complex propagation factor γ $(= \alpha + j\beta)$ and the characteristic impedance Z_0.

Equation (*7.16*) can be stated more concisely using hyperbolic functions, in three different forms:

$$\frac{Z(d)}{Z_0} \quad = \quad \frac{Z_T \cosh(\alpha + j\beta)d + Z_0 \sinh(\alpha + j\beta)d}{Z_0 \cosh(\alpha + j\beta)d + Z_T \sinh(\alpha + j\beta)d} \tag{7.17}$$

$$\frac{Z(d)}{Z_0} \quad = \quad \frac{(Z_T/Z_0) + \tanh(\alpha + j\beta)d}{1 + (Z_T/Z_0)\tanh(\alpha + j\beta)d} \tag{7.18}$$

$$\frac{Z(d)}{Z_0} \quad = \quad \tanh\left[(\alpha + j\beta)d + \tanh^{-1}(Z_T/Z_0)\right] \tag{7.19}$$

With sufficient accuracy for most engineering purposes, the normalized impedance at a coordinate d on a transmission line, $Z(d)/Z_0$, is determined very quickly and easily by making use of the transmission line circle diagram described in Chapter 9. However, if greater numerical precision is desired, the calculation is probably made most efficiently by using equation (*7.14*). Equation (*7.18*) is a particularly useful form for demonstrating the relation between $Z(d)/Z_0$ and Z_T/Z_0 in several special cases.

As noted at the beginning of this chapter, "the impedance at a specific point on a line" means the input impedance of the portion of the line on the terminal load side of the point. It is convenient to rewrite (*7.18*) to make it an explicit expression for the normalized input impedance Z_{inp}/Z_0 of a length l of transmission line:

$$\frac{Z_{\text{inp}}}{Z_0} = \frac{Z_T/Z_0 + \tanh(\alpha + j\beta)l}{1 + (Z_T/Z_0)\tanh(\alpha + j\beta)l} \qquad (7.20)$$

It is easily seen that in the simplest special case of $Z_T = Z_0$, this somewhat formidable equation reduces to $Z_{\text{inp}}/Z_0 = 1$, consistent with all previous results for a transmission line terminated nonreflectively.

7.3. "Stub" lines with open circuit and short circuit terminations.

Postulating $\alpha = 0$ and $Z_T = 0$ in (*7.20*), the equation will give the normalized input impedance of a length l of transmission line, terminated in a short circuit and having "negligible" attenuation. The result is, making use of the relation $\tanh j\beta l = j\tan\beta l$,

$$Z_{\text{inp}}/Z_0 = j\tan\beta l \qquad (7.21)$$

If the same transmission line section has an open circuit termination $(Z_T = \infty)$, the result is

$$Z_{\text{inp}}/Z_0 = -j\cot\beta l \qquad (7.22)$$

Since for low-loss transmission lines Z_0 is very accurately a pure resistance, these equations show that the input impedances of low-loss line sections with either short circuit or open circuit terminations are pure reactances. From the relation $\beta = 2\pi/\lambda$ (equation (*4.9*), page 30), it follows that over the range of lengths $l = 0$ to $l = \lambda/2$, where λ is the wavelength on the transmission line, the input reactances of these line sections with either termination span the range from $-\infty$ to $+\infty$, thus providing the equivalent of all possible values of inductance and capacitance.

At frequencies above a few hundred megahertz, a wavelength becomes a physical length small enough to be incorporated in laboratory or industrial apparatus. The attenuation of such line sections is easily made very small (i.e. $\alpha \ll \beta$). Under these conditions short lengths of low-loss line with either open circuit or short circuit terminations can play the role that lumped inductances and capacitances play in low-frequency circuits. Such line sections are known as "stub" lines, and are important components of very-high-frequency circuitry. Some of their applications are discussed in Chapters 8 and 9.

Example 7.4.

A section of low-loss transmission line is 0.40 wavelengths long and is terminated in a short circuit. Its characteristic impedance is $73 + j0$ ohms. The frequency of operation is 200 megahertz. Determine the input reactance of the line section, and the value of inductance or capacitance to which it is equivalent.

By equation (*7.21*),

$$Z_{\text{inp}} = jZ_0\tan 2\pi l/\lambda = j(73 + j0)\tan(2.51\text{ rad}) = -j\,53.4\text{ ohms}$$

This is a capacitive reactance, and the equivalent value of capacitance C_{inp} is obtained from $53.4 = 1/(\omega C)$. Thus

$$C_{\text{inp}} = 1/(53.4\omega) = 1/(53.4 \times 2\pi \times 200 \times 10^6) = 14.9\text{ micromicrofarads}$$

Example 7.5.

A low-loss transmission line has distributed capacitance of 52.5×10^{-12} farads/m, distributed inductance of 2.43×10^{-7} henries/m, and negligible distributed resistance and conductance. It is operated at a frequency of 500 megahertz. What is the shortest length of line, with open circuit termination, that will have an input susceptance of +0.025 mhos?

The calculations can be made entirely in admittance notation by writing, from (7.22), $Y_{\text{inp}}/Y_0 = 1/(Z_{\text{inp}}/Z_0) = j \tan \beta l$ for a line with open circuit termination and negligible losses. From the distributed circuit coefficients L and C of the line, $Y_0 = 1/Z_0 = \sqrt{C/L} = 0.0147 + j0$ mhos, and $\beta = \omega\sqrt{LC} = 11.2$ rad/m. Then the required length l of transmission line is given by $(0 + j0.025)/(0.0147 + j0) = j \tan 11.2l$, and $l = 0.093\text{ m} + n\lambda/2$, where $\lambda = 2\pi/\beta = 0.56$ m, and n is any integer.

7.4. Half-wavelength and quarter-wavelength transformers.

If $\alpha = 0$ and $\beta l = n\pi$ in equation (7.20), n being any integer, the result is

$$Z_{\text{inp}}/Z_0 = Z_T/Z_0 \tag{7.23}$$

since $\tan n\pi = 0$.

This relation states that the input impedance of any low-loss section of transmission line an integral number of half wavelengths long is identically equal to the terminal load impedance connected to the section. Such a line section is therefore a "one-to-one" impedance transformer, differing from a lumped element low frequency transformer because of the requirement of specific physical length. It is generally referred to as a "half-wavelength transformer" and can serve a useful purpose in high frequency transmission line circuitry as a device to present at some convenient location an impedance existing at some other location that may be inaccessible or awkwardly located.

The same function could obviously be performed by a transmission line of characteristic impedance equal to the circuit impedance in question, but it has been seen in Chapter 6 that the range of characteristic impedance magnitudes that transmission lines can be designed to have is in fact very limited, and the phase angle of a characteristic impedance can differ appreciably from 0° only for lines whose attenuation per wavelength exceeds several decibels.

Example 7.6.

A high frequency generator has been designed to have an output impedance of $25 - j50$ ohms, so that it will be conjugately matched to a load of $25 + j50$ ohms, to which it is to supply power. The operating frequency is 150 megahertz. The load terminals are separated by about 8 ft from the generator terminals. What length and what characteristic impedance must a section of air dielectric low-loss transmission line have, to connect the load to the generator and present at the generator terminals an input impedance equal to the load impedance?

The required transmission line must be a "half-wavelength transformer" (i.e. a line section some integral number of half-wavelengths long) since it is not possible for a transmission line to have a characteristic impedance of $25 + j50$ ohms.

The distance of "about 8 ft" is approximately 2.4 m. From Chapter 6, the phase velocity of voltage waves on an "air dielectric" transmission line is the velocity of light in free space, or 3.00×10^8 m/sec. The wavelength on such a line at the operating frequency of 150 megahertz is then

$$\lambda = v_p/f = (3.00 \times 10^8)/(150 \times 10^6) = 2.00 \text{ m}$$

A suitable length of transmission line to connect the load to the generator would therefore be $1\frac{1}{2}$ wavelengths or 3.00 m, = 9.84 ft. A section one wavelength long would be too short.

Although the line could in principle have any value of characteristic impedance, it is indicated in Chapter 8 that the peak values of voltage and current along such a line (which may lead to breakdown if the power level is high) will be lowest if the purely resistive characteristic impedance of this low loss line is equal to the *magnitude* of the complex terminal load impedance, in this case 56 ohms.

An important consideration in any engineering installation of a half-wavelength transformer of this sort would be the behavior of the system as a whole over the finite bandwidth of frequencies that all practical transmissions must utilize. At frequencies different from the exact designated operating frequency of the problem, the transformer section of line will not be precisely an integral number of half-wavelengths long, and its input impedance will not be identically equal to its terminal impedance.

The general problem of connecting a source to a load by a transmission line, subject to various stated specifications about matching impedances, minimizing peak voltages or currents, etc., can have many different solutions and the "half-wavelength transformer" is a relatively minor one. Another solution is discussed in the next paragraph, and others are dealt with in Problem 7.7 and in Chapters 8 and 9.

If $\alpha = 0$ and $\beta l = n\pi/2$ in (*7.20*), where n is any integer, the $\tan \beta l$ term becomes indefinitely large and any finite value of Z_T/Z_0 is negligible by comparison. The result is

$$Z_{\text{inp}}/Z_0 = 1/(Z_T/Z_0) \tag{7.24}$$

This relation states that the normalized input impedance of any low-loss section of transmission line an odd number of quarter-wavelengths long is the reciprocal of the normalized terminal load impedance connected to the section. Such a line section is referred to as a "quarter-wavelength transformer" and is used in high-frequency transmission line circuitry as a device for connecting a high-impedance load to a low-impedance source or a low-impedance load to a high-impedance source. The two impedances involved may not have arbitrary "high" and "low" values, but must meet two specific requirements.

From (*7.24*), $Z_{\text{inp}} \times Z_T = Z_0^2$. Since Z_0 is purely resistive for the low-loss high-frequency transmission line section being used, a first requirement on Z_{inp} and Z_T is that they must have phase angles of equal magnitude and opposite sign. Secondly, their geometric mean must be a value which is physically achievable as a characteristic impedance for the type of transmission line being used. In practice, the quarter-wavelength transformer is generally used to connect resistive loads to sources with resistive output impedances.

Example 7.7.

An antenna array used at a frequency of 40 megahertz has an input impedance of $36 + j0$ ohms at that frequency. The generator supplying power to the antenna has an output impedance of $500 + j0$ ohms and is located about 100 feet from the antenna terminals. A parallel wire transmission line with characteristic impedance $500 + j0$ ohms runs from the generator to the vicinity of the antenna terminals. Design a quarter-wavelength transformer, using parallel wire transmission line on which the phase velocity is 97% of the free space velocity of light, to connect the antenna to the main transmission line and provide an impedance match.

The characteristic impedance of the transmission line section used for the quarter-wavelength transformer must be $Z_0 = \sqrt{Z_{\text{inp}} Z_T} = \sqrt{(36 + j0)(500 + j0)} = 134 + j0$ ohms, a value that can be obtained for a parallel wire line by spacing the conductors just a few diameters apart.

The free space wavelength at the operating frequency of 40 megahertz is 7.50 m. On a line for which the phase velocity is 97% of the free space value, the wavelength at the same frequency is 7.27 m. The quarter-wavelength transformer must therefore be 1.82 m. long. The overall final result would have the appearance shown in Fig. 7-2.

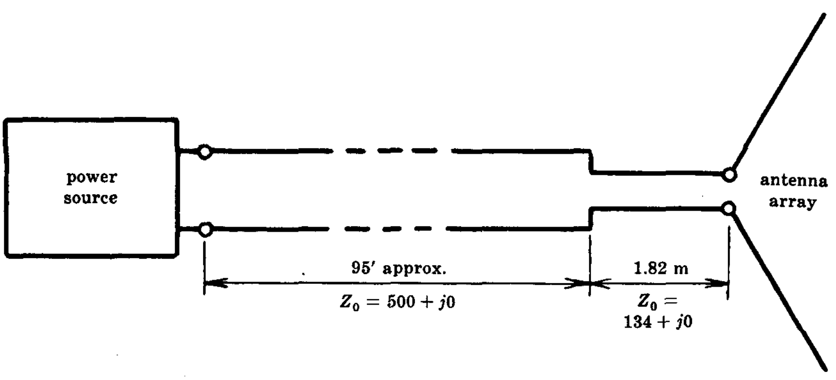


Fig. 7-2. A high impedance source connected to a low impedance load by a quarter wavelength impedance matching transformer.

Since the "quarter-wavelength transformer" principle operates for low-loss line sections that are *any odd number* of quarter-wavelengths long, another possible solution of Example 7.7 would be to use a $4\frac{1}{4}$ wavelength section of the line whose characteristic impedance is 134 ohms. The length of this section would be about 101.5 ft. In practice, however, this solution would be a poor one, since actual signals always occupy a finite range of frequencies. Over any small frequency interval the range of variation of the term βl in (*7.20*) for a $4\frac{1}{4}$ wavelength line section would be 17 times as great as for a $\frac{1}{4}$ wavelength line section. In the solution illustrated by Fig. 7-2, the 95 ft of line with characteristic impedance 500 ohms would have little effect on the frequency sensitivity of the system, because this characteristic impedance is equal to the impedance of the source to which the line is connected. These statements can easily be checked quantitatively by using the transmission-line circle diagram discussed in Chapter 9.

It has been pointed out in Section 2.1 that the uniformity postulate underlying the transmission line analysis of this book is violated in the vicinity of terminations and other discontinuities on transmission lines. In the use of quarter-wavelength or half-wavelength transformers, such discontinuities occur at each end, either where the line section is connected to a source or load, or where it is connected to a transmission line section of different characteristic impedance. The main practical consequence of this departure from idealized conditions is that the optimum length of these transformers in specific applications is likely to differ slightly from the value calculated using the equations of uniform lines. The difference is generally much less than one transverse dimension of the line, and the experimentally optimum length is best found by starting from the calculated value and making small adjustments.

7.5. Determination of transmission line characteristics from impedance measurements.

When an arbitrary length of any general transmission line is terminated in an open circuit or a short circuit, its input impedance is determined completely by the propagation factors α and β, the characteristic impedance Z_0 and the line length l. From (*7.20*), if $Z_T = 0$ (but $\alpha \neq 0$) the input impedance Z_{sc} of a line of length l with short circuit termination is

$$Z_{sc} = Z_0 \tanh(\alpha + j\beta)l \tag{7.25}$$

The input impedance Z_{oc} of the same line with open circuit termination is

$$Z_{oc} = Z_0 \coth(\alpha + j\beta)l \tag{7.26}$$

If Z_{sc} and Z_{oc} are measured at the same frequency, for a line section of length l, then Z_0, α, β, and l will have the same values in both of the equations (*7.25*) and (*7.26*). Multiplying together the corresponding sides of these equations gives

$$Z_0 = \sqrt{Z_{sc} Z_{oc}} \tag{7.27}$$

This is a valuable and universally valid equation by which the characteristic impedance of any type of uniform transmission line can be obtained from two impedance measurements made on a sample length of the line, using two readily available terminal load impedances.

Two precautions must be observed in making the measurements needed for this calculation. First, the impedance-measuring device must be capable of measuring "balanced" impedances if the line conductors are symmetrical (e.g. a parallel-wire line or a shielded pair), or of measuring "unbalanced" impedances if the line has one of its two conductors acting as a shield or "ground" (e.g., a coaxial line or a stripline). Second, the length of line l cannot be completely arbitrary but must be chosen so that both Z_{sc} and Z_{oc} have values appropriate to the impedance-measuring device. It is obvious, for example, that for an extremely short section of any line Z_{sc} might be too small and Z_{oc} too large to be accurately measurable by any available bridges. Use of the transmission line charts discussed in

Chapter 9 will show that line-section lengths close to any odd number of eighths of a wavelength are particularly appropriate. For such line lengths Z_{sc}, Z_{oc} and Z_0 will all have similar magnitudes. If the wavelength is only approximately known, measurements can be made at several values of l until this condition is found.

The attenuation factor α and the phase propagation factor β can also be calculated from the measured impedances Z_{sc} and Z_{oc}. Dividing corresponding sides of (*7.25*) by those of (*7.26*),

$$\sqrt{Z_{sc}/Z_{oc}} = \tanh(\alpha + j\beta)l$$

Expanding this hyperbolic tangent in exponential form, using $\gamma = \alpha + j\beta$,

$$\sqrt{Z_{sc}/Z_{oc}} = (1 - e^{-2\gamma l})/(1 + e^{-2\gamma l})$$

which gives

$$e^{2\gamma l} = \frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}$$

Taking logarithms of both sides,

$$(\alpha + j\beta)l = \tfrac{1}{2}\log_e \frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}$$

The logarithm of a complex number expressed in polar form $Ae^{j\phi}$ is defined by

$$\log_e Ae^{j\phi} = \log_e A + j(\phi + 2n\pi)$$

The attenuation factor α is therefore given by

$$\alpha = \frac{1}{2l}\log_e\left|\frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}\right| \quad \text{nepers/m} \tag{7.28}$$

when l is in meters. The phase propagation factor β is given by

$$\beta = \frac{1}{2l}\left\{\left(\text{phase angle of } \frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}\right) + 2n\pi\right\} \quad \text{radians/m} \tag{7.29}$$

This method does not determine a unique value for β, but a series of values differing consecutively by π/l rad/m. In a practical case it may sometimes be difficult to decide which value in the series is the correct answer.

Example 7.8.

At a frequency of 20.0 megahertz the input impedance of a section of flexible coaxial transmission line 32.0 m long is measured, first with the line terminated in a short circuit and then with the line terminated in an open circuit. The respective values obtained are $Z_{sc} = 17.0 + j19.4$ ohms and $Z_{oc} = 115 - j138$ ohms. Find the attenuation factor, the phase propagation factor, and the characteristic impedance of the line.

The impedances are needed in polar form for all of the calculations. $Z_{sc} = 25.7\underline{/48.8°}$ and $Z_{oc} = 179\underline{/-50.2°}$ ohms. The characteristic impedance from (*7.27*) is then $Z_0 = \sqrt{Z_{sc}Z_{oc}} = 68\underline{/-0.7°}$ ohms.

For determining α and β the quantity $\sqrt{Z_{sc}/Z_{oc}} = 0.378\underline{/49.5°} = 0.245 + j0.288$ is required. Using equation (*7.28*),

$$\alpha = \frac{1}{2(32.0)}\log_e\left|\frac{1.245 + j0.288}{0.755 - j0.288}\right| = 0.0072 \quad \text{nepers/m}$$

The phase angle of the term $(1.245 + j0.288)/(0.755 - j0.288)$ is found to be 33.9°. From equation (*7.29*), $\beta = (0.59 + 2n\pi)/(2 \times 32.0)$ rad/m, but there is no basis for choosing the value of n.

At the frequency of the measurements, the free space wavelength is 15.0 m and the corresponding value of β would be found from $\beta = 2\pi/\lambda = 0.419$. Since the line contains plastic dielectric it is expected that the wavelength on the line may be as much as 30% shorter than the free space value, but the figure is not known accurately. Hence β might conceivably lie between about 0.50 and 0.65. The above equation gives $\beta = 0.40$ for $n = 4$, $\beta = 0.50$ for $n = 5$, $\beta = 0.60$ for $n = 6$, and $\beta = 0.70$ for $n = 7$.

On the evidence available, there is no conclusive basis for choosing between the two intermediate values. The data indicates that the line section is between two and three wavelengths long. By making the same impedance measurements on a shorter length of line, lower values of n will occur in the equation for β and there will be less doubt about which value should be chosen.

For a section of the same line 1.50 m long, the impedance values measured were $Z_{sc} = 0 + j88$ ohms and $Z_{oc} = 0 - j52$ ohms, the resistive component in each case being less than 1 ohm. From these values the characteristic impedance is calculated as $68\underline{/0^\circ}$ ohms.

Because the quantity $\sqrt{Z_{sc}Z_{oc}} = 0 + j1.30$ is purely imaginary, the attenuation factor α is indicated as having value zero. The phase angle of the term $(1 + j1.30)/(1 - j1.30)$ is 105° or 1.83 rad. From equation *(7.29)*, $\beta = (1.83 + 2n\pi)/(2 \times 1.50) = 0.61$ rad/m for $n = 0$, or 2.70 rad/m for $n = 1$. It is clear now that $n = 6$ gave the correct value in the previous data and that the measured value of β is 0.60 (or 0.61) rad/m. It is also evident that measurements on this short length of line cannot be used to obtain a value for the attenuation factor α.

Consideration of the equations shows that this method of determining α and β from measurements of Z_{sc} and Z_{oc} will give the best results for α when the line section has a total attenuation of about 3 db, and will give the least ambiguous results for β when the line is about one-eighth wavelength long. Except at frequencies in the kilohertz range, a single piece of any practical transmission line will not satisfy both of these conditions, so that measurements of α and β should usually be made on two different line sections, one much longer than the other. Measurements on either section will give satisfactory data for the determination of Z_0.

7.6. Complex characteristic impedance.

The characteristic impedance of a transmission line was originally defined in terms of the line's distributed circuit constants, and given by equation *(4.12)* as

$$Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)}$$

Since R, L, G, C and ω are all positive real numbers for a passive transmission line, it follows from this expression that the phase angle of Z_0 must lie between −45° and +45° or, if $Z_0 = R_0 + jX_0$, the ratio X_0/R_0 must lie between −1 and +1. The extreme values occur when *either* $R \gg \omega L$ and $G \ll \omega C$, *or* $R \ll \omega L$ and $G \gg \omega C$. For either of these sets of conditions the defining equation *(4.10)*, $\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$, shows that $\alpha = \beta$.

Noting that $\beta = 2\pi/\lambda$, the relation $\alpha = \beta$ has the physical meaning that the attenuation of the line is 2π *nepers per wavelength* or 54.6 decibels per wavelength. Transmission lines useful at high frequencies have attenuations per wavelength smaller than this value by several orders of magnitude, but Table 5.1, page 55, shows that a standard type of telephone cable-pair can have α very nearly equal to β (and $|X_0|$ nearly equal to R_0) at frequencies below about 1 kilohertz.

Neither a large value of attenuation factor α in nepers per meter, nor a large total attenuation αl in nepers, is a sufficient condition to ensure that the characteristic impedance of a transmission line will have a substantial phase angle. The attenuation over *one wavelength* of line, $\alpha\lambda$ nepers, must be large, and it must be caused predominantly by *one* of R *or* G and not by a combination of the two. It has already been noted that if the losses are due equally to R and G, Z_0 is real, no matter how high the losses are.

When the characteristic impedance of a transmission line has an appreciable phase angle, some peculiar results arise, which need further discussion. If, for example, $Z_0 = R_0 + jX_0$ and the line has a terminal load impedance $Z_T = 0 - jX_0$, the reflection coefficient determined from equation *(7.10)* is

$$\rho_T = (-jX_0 - R_0 - jX_0)/(-jX_0 + R_0 + jX_0) = -1 - j2X_0/R_0$$

and the magnitude of this is greater than unity. The question arises as to whether the transmission line equations predict a reflected wave at the termination having a higher power level than the wave incident on the termination, in violation of the principle of conservation of energy.

To determine the conditions leading to the maximum possible magnitude of the reflection coefficient, and the value of this maximum, equation (*7.10*) is written in the form

$$\rho_T = \frac{Z_T/Z_0 - 1}{Z_T/Z_0 + 1} = \frac{|Z_T/Z_0|\, e^{j\theta} - 1}{|Z_T/Z_0|\, e^{j\theta} + 1} = \frac{|Z_T/Z_0| \cos\theta - 1 + j\,|Z_T/Z_0| \sin\theta}{|Z_T/Z_0| \cos\theta + 1 + j\,|Z_T/Z_0| \sin\theta}$$

Taking the magnitude of numerator and denominator,

$$|\rho_T|^2 = \frac{|Z_T/Z_0|^2 + 1 - 2\,|Z_T/Z_0| \cos\theta}{|Z_T/Z_0|^2 + 1 + 2\,|Z_T/Z_0| \cos\theta} \tag{7.30}$$

For any value of $|Z_T/Z_0|$, the magnitude of the normalized terminal load impedance, this expression shows that the magnitude of the reflection coefficient will be greatest for the largest negative value of $\cos\theta$, where θ is the phase angle of the normalized terminal load impedance. It has been seen that the phase angle of Z_0 lies between the limits $\pm 45°$. There are no limitations on the value of Z_T except that it must be a passive impedance. Hence its phase angle can lie anywhere in the range $\pm 90°$. The phase angle of Z_T/Z_0 can therefore have any value between $\pm 135°$ and the angles in this range whose cosines have the greatest negative values are the extreme angles $+135°$ and $-135°$, for which the cosine is $-1/\sqrt{2}$.

Inserting this value for $\cos\theta$ into equation (*7.30*), the value of $|Z_T/Z_0|$ that will give maximum $|\rho_T|$ is found by differentiating the right-hand side and equating to zero. The result is $|Z_T/Z_0| = 1$. When this is substituted into (*7.30*), the maximum possible value of $|\rho_T|$ is found to be $1 + \sqrt{2}$, or about 2.41.

The basic equation for steady-state power calculations in linear electric circuits with single-frequency voltages and currents is $P = VI^*$, where V is the rms phasor voltage between two points of the circuit and I^* is the complex conjugate of the rms phasor current at those points, expressed with reference to a suitable convention for polarity. P is the complex power in the circuit, its real part representing real power and its imaginary part representing an oscillating flow of stored energy.

The power passing any point z on a transmission line will be given by $V(z)\,I(z)^*$ where $V(z)$ and $I(z)$ are the usual rms phasor values of the voltage and current, respectively, at the coordinate z. If the reflected wave at any point of a transmission line had a real power level higher than that of the incident wave at the same point, the net power would be in the direction of the reflected wave, and the real part of $V(z)\,I(z)^*$ at that point would be negative.

To establish that the principle of conservation of energy is not violated on a transmission line even when the magnitude of the reflection coefficient at a point on the line exceeds unity, it is sufficient to show that the real part of $V(z)\,I(z)^*$ can never be negative for any possible values of Z_0 and Z_T.

Appropriate general expressions for the phasor voltage and current at any point of a transmission line on which there are reflected waves are equations (*7.1*) and (*7.3*). These can be rewritten

$$V(z) = V_1 e^{-\gamma z}\{1 + (V_2/V_1)e^{2\gamma z}\} \tag{7.31}$$

$$I(z) = (V_1/Z_0)e^{-\gamma z}\{1 - (V_2/V_1)e^{2\gamma z}\} \tag{7.32}$$

In these equations V_1, V_2, Z_0 and $e^{\pm\gamma z}$ are all complex numbers. Converting the term $(V_2/V_1)e^{2\gamma z}$ to the form $(V_2 e^{\gamma z})/(V_1 e^{-\gamma z})$, it is seen to be the ratio, at coordinate z, of the phasor voltage of the reflected wave to the phasor voltage of the incident wave. It can therefore be symbolized by an equivalent reflection coefficient $\rho(z)$ at that point. It is in fact the reflection coefficient that would be created at coordinate z if the line were terminated there by impedance $Z(z)$, the input impedance of the line section on the terminal load side of that point. It follows that

$$\rho(z) = (Z(z) - Z_0)/(Z(z) + Z_0) \tag{7.33}$$

The power $P = V(z)\,I(z)^*$ at coordinate z of the transmission line is then

$$\begin{aligned} P &= V_1 e^{-\alpha z} e^{-j\beta z}\{1 + \rho(z)\}\,(V_1^*/Z_0^*)\,e^{-\alpha z} e^{j\beta z}\{1 - [\rho(z)]^*\} \\ &= |V_1/Z_0|^2 Z_0 e^{-2\alpha z}\{1 - |\rho(z)|^2 + \rho(z) - [\rho(z)]^*\} \end{aligned}$$

It is obvious that $\rho(z) - [\rho(z)]^* = j2\,\{\text{imaginary part of } \rho(z)\}$, which will be written $j2\,\mathrm{Im}\,\rho(z)$. Using $Z_0 = R_0 + jX_0$ and $P = P_r + jP_i$, separate expressions for the real and imaginary parts of P are obtained:

$$P_r = \left|\frac{V_1}{Z_0}\right|^2 R_0 e^{-2\alpha z}\left\{1 - |\rho(z)|^2 - 2\,\frac{X_0}{R_0}\,\mathrm{Im}\,\rho(z)\right\} \tag{7.34}$$

$$P_i = \left|\frac{V_1}{Z_0}\right|^2 X_0 e^{-2\alpha z}\left\{1 - |\rho(z)|^2 + 2\,\frac{R_0}{X_0}\,\mathrm{Im}\,\rho(z)\right\} \tag{7.35}$$

The condition that P_r should never become negative is that

$$|\rho(z)|^2 + 2(X_0/R_0)\,\mathrm{Im}\,\rho(z) \leqq 1$$

Expanding $\rho(z)$ from (*7.33*) with $Z_0 = R_0 + jX_0$ and $Z(z) = R(z) + jX(z)$, it is easily found that this reduces to the condition $|X_0/R_0| \leqq 1$, which has already been seen to be true.

The conclusion is somewhat surprising, though inescapable, that a transmission line can be terminated with a reflection coefficient whose magnitude is as great as 2.41 without there being any implication that the power level of the reflected wave is greater than that of the incident wave. Such a reflection coefficient can exist only on a line whose attenuation per wavelength is high, so that even if the reflected wave is in some sense large at the point of reflection, it remains so for only a small fraction of a wavelength along the line away from that point. These large reflection coefficients are an example of the phenomenon of "resonant rise of voltage" in series resonant circuits. The resonant circuit in this case consists of the Thévenin equivalent circuit of the transmission line and its source, relative to the load terminals, combined with the load impedance. The large reflection coefficients are obtained only when the reactance of the terminal load impedance is of opposite sign to the reactance component of the characteristic impedance.

Consideration of equation (*7.34*) reveals that the coefficient on the right, multiplied by the first term (unity) in the braces, represents the real power that would be calculated for the incident wave alone (i.e. the wave moving in the direction of increasing z) at coordinate z, while the coefficient multiplied by the second term represents the real power that would be calculated for the reflected wave alone, at the same point. The third term on the right represents interaction between the two waves.

The fact that the third term is zero if $X_0 = 0$ indicates that on transmission lines whose characteristic impedance is real (whether the line is lossy or not), the power at any point of the line is correctly given by the difference between the incident power and the reflected power at the point. When the characteristic impedance of the line is complex, however, (which *requires* that the line be lossy), this is not the case and calculation of the power at any point requires the full detail of equation (*7.34*). A line with losses can have a real characteristic impedance only if the distributed-circuit coefficients obey the Heaviside relation $R/L = G/C$.

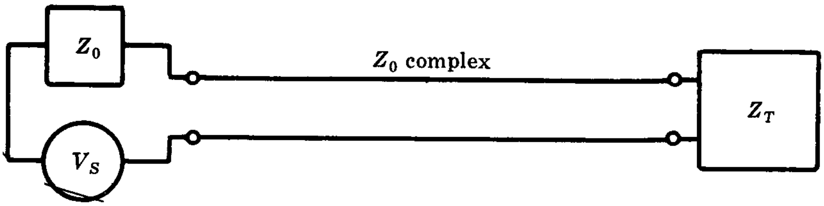


Fig. 7-3. Transmission line circuit having source impedance equal to the complex characteristic impedance of the transmission line.

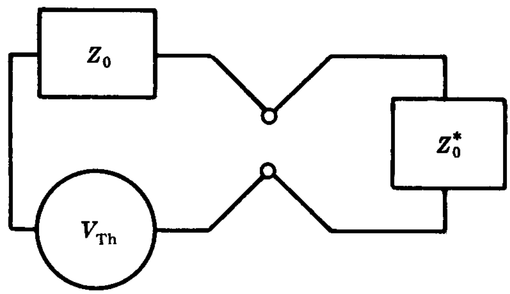


Fig. 7-4. Thévenin equivalent circuit of a length of transmission line having complex characteristic impedance, the source impedance being equal to the line's characteristic impedance, and the terminal load impedance being the conjugate of that impedance. The terminals shown are the load terminals of the transmission line.

A second interesting problem that arises in connection with transmission lines having complex characteristic impedance is illustrated in Fig. 7-3. A source whose internal impedance is equal to the transmission line's characteristic impedance Z_0, is connected to a lossy line for which Z_0 is complex. If the terminal load impedance Z_T connected to the line is made equal to Z_0, there will be no reflected waves on the line and all the real power incident on Z_T will be absorbed there. If, however, the transmission line circuit of Fig. 7-3 is redrawn as an equivalent lumped element circuit in Fig. 7-4, replacing the transmission line and its source by an equivalent Thévenin source, the internal impedance of this source will be Z_0. By the maximum power transfer theorem, this Thévenin source will deliver maximum power to a load impedance Z_0^*, the complex conjugate of the line's characteristic impedance, and not to a load impedance Z_0. Applying this result back to the transmission line circuit of Fig. 7-3, the conclusion must be drawn that more power will be delivered to a terminal load impedance Z_0^* that produces a reflected wave on the line than to a terminal load impedance Z_0 that produces no reflected wave.

The proof that this is indeed the case is straightforward, but is too long to be presented here. The explanation, briefly, lies in the fact that with the Z_0^* terminal load impedance the reflected and incident voltage and current waves combine in such a manner that the line losses, in the two or three eighths of a wavelength of the line adjacent to the termination, are reduced, by exactly the amount of the extra power that reaches the load. Since the extra power is appreciable only when Z_0^* differs significantly from Z_0, and since this occurs only if the line's attenuation is at least a few decibels per wavelength, the reflected wave has a negligible effect on the line losses beyond a small fraction of a wavelength from the termination.

It follows from remarks at the beginning of this section, that the distinction between terminating a transmission line in its characteristic impedance and terminating the line in the conjugate of its characteristic impedance, is likely to have practical consequences only at frequencies below the kilohertz region. The technique of reducing the attenuation factor of a voice frequency telephone line by "loading" the line with inductance coils, as discussed in Section 5.8, is an application of the principle that there is greater power transfer to the conjugate impedance termination.

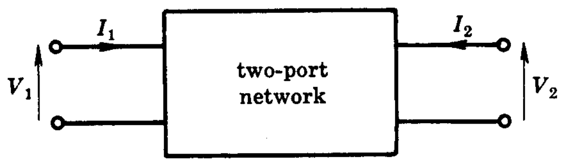
7.7. Transmission line sections as two-port networks.

One of the basic categories of networks studied in network theory is that of two-port networks, also known as two-terminal-pair networks, four-terminal networks, four-poles and quadripoles. A "port" of a network is defined as any pair of terminals at which the instantaneous current into one of the terminals is equal to the instantaneous current out of the other terminal. It is obvious that any section of uniform transmission line is a two-port network, and more specifically that all uniform transmission line sections are reciprocal symmetrical two-port networks. Unless they involve ferroelectric or ferromagnetic materials they are also linear networks.

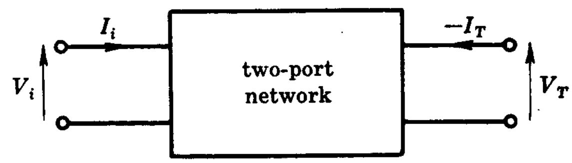
The variables in the study of linear passive reciprocal two-port networks are the time-dependent voltages and currents at the two ports, four quantities in all. The object of two-port network theory is to develop formulas expressing useful relationships among various choices of pairs of these four quantities, in terms of the nature of the network. The nature of the network is defined by the results of specified measurements made at the network terminals. If the detailed structure of the network is known the results of these measurements can be calculated from elementary circuit theory. For transmission line two-ports the results are calculated from transmission line theory.

Among the four quantities comprising the input and output currents and voltages of a two-port network, it is possible to choose two pairs in six different ways. For each set of pairs two simultaneous equations can be written expressing each of the quantities of one pair as functions of the two quantities in the other pair. Of the six possible sets of equations, four are distinct and are assigned specific names. The other two involve only reversing the network or interchanging the variables.

Two-port network theory is developed with reference to the circuit of Fig. 7-5(*a*). The notation normally used in the theory identifies the voltage and current at the left port of the network by the symbols V_1 and I_1 respectively, and the voltage and current at the port on the right by the symbols V_2 and I_2 respectively. Since all of these symbols have been extensively used with quite different meanings in this book, it is necessary to adopt some other notation for the two-port analysis of transmission lines. The choice of V_i and I_i for quantities on the left as "input" quantities, and V_T and I_T for quantities on the right as terminal load or "output" quantities is consistent with notation employed earlier in this and in previous chapters. The conventional choice of sign in two-port network analysis for the current on the right is opposite to that used for a terminal load current in transmission line theory. Fig. 7-5(*b*) shows the two-port of Fig. 7-5(*a*) with transmission line notation for the variables.



(*a*) Notation in network theory.



(*b*) Notation in transmission line theory.

Fig. 7-5. Notation for two-port network analysis.

For lumped element two-port networks it is customary to perform the analysis in the complex frequency domain, the time-variable currents and voltages being identified by their Laplace transforms. It has already been pointed out in Chapter 2 that for transmission line circuits this more generalized analysis has no particular advantages. The analysis of transmission line two-port networks is therefore carried out in the radian frequency or ω domain, with the currents and voltages expressed as phasors.

Referring to Fig. 7-5(a), one set of simultaneous equations relating the four current and voltage phasors can be written formally as

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned} \tag{7.36}$$

In the notation of Fig. 7-5(b) these become

$$\begin{aligned} V_i &= z_{ii}I_i + z_{iT}(-I_T) \\ V_T &= z_{Ti}I_i + z_{TT}(-I_T) \end{aligned} \tag{7.37}$$

The coefficients on the right have the dimensions of impedance, since they transform phasor currents to phasor voltages. Furthermore, it is obvious that $z_{ii} = V_i/I_i$ when $I_T = 0$, i.e. when the terminals on the right of the network are open circuited. Similar statements can be made for all four of the impedance coefficients, with the result that they are collectively known as the "open circuit impedance parameters" of the network. They are commonly written as a 2×2 matrix. (The use of the lower case letter z for these coefficients is the standard convention.)

Example 7.9.

Determine the open circuit impedance matrix of a length l of uniform transmission line, having characteristic impedance Z_0, attenuation factor α and phase factor β.

The input impedance from either side of the transmission line section, when the terminals on the other side are open circuited, is given by equation (*7.20*) as $Z_{\text{inp}} = Z_0 \coth(\alpha + j\beta)l$. Hence this is the value of z_{ii} and z_{TT}. The coefficient z_{Ti} is the value of the ratio V_T/I_i when $I_T = 0$. This cannot be written by inspection, but must be determined from (*7.1*) and (*7.3*) with the help of (*7.8*).

When the transmission line circuit of Fig. 7-1 has an open circuit at the right (Z_T infinite), and an input current I_i at the left, (*7.1*) and (*7.3*) lead to

$$V_T = V(z) \text{ with } z = l, \quad \text{or} \quad V_T = V_1 e^{-\gamma l} + V_2 e^{\gamma l} \text{ where } \gamma = \alpha + j\beta$$

$$I_i = I(z) \text{ with } z = 0, \quad \text{or} \quad I_i = V_1/Z_0 - V_2/Z_0$$

Then

$$z_{Ti} = (V_T/I_i)_{I_T=0} = Z_0 \frac{V_1\{e^{-\gamma l} + (V_2/V_1)e^{\gamma l}\}}{V_1(1 - V_2/V_1)}$$

Noting from equation (*7.8*) that $V_2/V_1 = \rho_T e^{-2\gamma l}$, and that for open circuit termination $\rho_T = 1 + j0$, it is easily found that $z_{Ti} = Z_0/[\sinh(\alpha + j\beta)l]$. A similar derivation for $z_{iT} = V_i/(-I_T)$ with $I_i = 0$ gives the same result, as is to be expected from the symmetry of the transmission line network. The open circuit impedance matrix of the section of uniform transmission line is therefore

$$\begin{bmatrix} Z_0 \coth(\alpha + j\beta)l & Z_0 \operatorname{cosech}(\alpha + j\beta)l \\ Z_0 \operatorname{cosech}(\alpha + j\beta)l & Z_0 \coth(\alpha + j\beta)l \end{bmatrix}$$

The open circuit impedance matrix has application in solving problems involving two or more uniform transmission line sections connected in series at both their input and output terminals, or problems in which the input and output ports of a single section of uniform transmission line are connected in series with the input and output ports respectively of any two port network.

If the ports of a section of uniform transmission line are connected in parallel at each end with the corresponding ports of other uniform transmission line sections, or any other two-port networks, the useful matrix representation of the transmission line section is known as the short circuit admittance matrix, derived from the equations

$$\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned} \tag{7.38}$$

referred to Fig. 7-5(a), which in the notation of Fig. 7-5(b) become

$$\begin{aligned} I_i &= y_{ii}V_i + y_{iT}V_T \\ (-I_T) &= y_{Ti}V_i + y_{TT}V_T \end{aligned} \tag{7.39}$$

In these equations the coefficients on the right are admittances, and it is easily seen that each is the ratio of a phasor current to a phasor voltage when one or the other port of the network is short circuited. They are known collectively as the "short-circuit admittance parameters" of the network, and their matrix is the short circuit admittance matrix. The terms of this matrix for a section of uniform transmission line are given in Problem 7.35.

The interconnection of two two-port networks in series at the input ports and in parallel at the output ports is an important combination in feedback control systems. Analysis of the case is carried out using the equations

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned} \qquad (7.40)$$

for Fig. 7-5(*a*), or

$$\begin{aligned} V_i &= h_{ii}I_i + h_{iT}V_T \\ (-I_T) &= h_{Ti}I_i + h_{TT}V_T \end{aligned} \qquad (7.41)$$

for Fig. 7-5(*b*).

The coefficients on the right are known as "hybrid parameters", since one is an impedance, one an admittance, one a voltage ratio and one a current ratio. All are different from the coefficients in (*7.37*) and (*7.39*). Their matrix is the hybrid matrix of the network. Interchanging I and V produces the inverse hybrid matrix, useful in analyzing the case of two-port networks connected in parallel at their input ports and in series at their output ports. The symbols g_{ii}, etc., are used for the terms in the matrix when referred to Fig. 7-5(*b*). For the hybrid matrix and inverse hybrid matrix of a section of uniform transmission line, see Problem 7.36.

In practical transmission line systems incorporating sections of uniform transmission line as component two-ports, cascade connection of the transmission line sections with other two-ports such as attenuators, equalizers, amplifiers, etc., occurs far more frequently than the series connection, parallel connection, or series-parallel connection to which the matrices of (*7.37*), (*7.39*) and (*7.41*) are relevant. In the cascade connection the output port of one two-port is connected to the input port of another, and the appropriate equations relate the pair of input quantities of a two-port to the pair of output quantities. Referred to Fig. 7-5(*a*) the "transmission equations" are

$$\begin{aligned} V_1 &= AV_2 - BI_2 \\ I_1 &= CV_2 - DI_2 \end{aligned} \qquad (7.42)$$

The sign convention for the output current is changed, since the direction of the output current of one two-port must be identical with that of the input current of the succeeding two-port in cascade connection. In transmission line notation these equations become

$$\begin{aligned} V_i &= AV_T + BI_T \\ I_i &= CV_T + DI_T \end{aligned} \qquad (7.43)$$

Like the hybrid parameters, the "transmission parameters" A, B, C and D all have different physical connotations. Their matrix is the transmission matrix or chain matrix of the network. Interchanging the subscripts i and T produces the inverse transmission matrix. For the terms of the transmission matrix and inverse transmission matrix of a uniform transmission line see Problem 7.37.

The various sets of equations (*7.36*) through (*7.43*), together with the two inverse sets mentioned, express all possible relations between pairs of phasor input and output voltages and currents for two-port networks, with no reference to any sources or terminal loads connected to the networks. In all of these equations two of the signal variables must be known before the others can be calculated. Connecting a terminal load impedance Z_T to the output port of a two-port network establishes a relation between two of the variables, since it must be true that $V_T/I_T = Z_T$. It is then possible to eliminate either I_T or V_T from at least one of the equations in each of the above pairs, and in every case the two equations that result can be solved to give a relation between some pair of the signal variables. Using the equations (*7.39*) for example, if $V_T/I_T = Z_T$, the second equation gives $I_T = z_{Ti}I_i/(z_{TT} + Z_T)$ and on substituting this into the first equation, $V_i/I_i = z_{ii} + z_{iT}z_{Ti}/(z_{TT} + Z_T)$. For a transmission line section $z_{ii} = z_{TT} = Z_0 \coth \gamma l$ and $z_{Ti} = z_{iT} = Z_0 \operatorname{cosech} \gamma l$; hence

$$V_i/I_i \quad = \quad Z_{\text{inp}} \quad = \quad Z_0\left\{\coth \gamma l - \frac{\operatorname{cosech}^2 \gamma l}{\coth \gamma l + (Z_T/Z_0)}\right\}$$

Use of the identity $\coth^2 x - \operatorname{cosech}^2 x = 1$ establishes that this result is identical with that given by equation (*7.20*).

If I_i is eliminated from equations (*7.39*) instead of I_T, the result for a transmission line section is

$$V_i/I_T \quad = \quad Z_0\{\coth^2 \gamma l + (Z_T/Z_0) \coth \gamma l - \operatorname{cosech}^2 \gamma l\}/(\operatorname{cosech} \gamma l)$$

which reduces to

$$V_i/I_T \quad = \quad Z_0 \sinh \gamma l + Z_T \cosh \gamma l \tag{7.44}$$

This is a useful "transfer impedance" formula for evaluating terminal load current in terms of input voltage, or vice versa, for the transmission line circuit of Fig. 7-1, page 126. The input voltage is the value at the actual input terminals of the line. A similar expression for the transfer admittance I_i/V_T for the circuit of Fig. 7-1 can be determined from (*7.39*) or directly from (*7.1*) and (*7.3*) in the form

$$I_i/V_T \quad = \quad (Z_T \sinh \gamma l + Z_0 \cosh \gamma l)/Z_0^2 \tag{7.45}$$

Example 7.10.

A transmission line 1250 ft long has a characteristic impedance of $50 + j0$ ohms, an attenuation factor 1.50 db/(100 ft) at the operating frequency of 2.00 megahertz, and the phase velocity on the line at that frequency is 70%. The line is terminated in an impedance $100 - j200$ ohms. If the voltage at the input terminals of the line has the rms reference phasor value $10 + j0$ volts, determine the phasor current in the terminal load impedance.

Using equation (*7.44*), the calculations can be made either by expressing the hyperbolic functions in exponential form, or by using the identities

$$\sinh(x + jy) \;=\; \sinh x \cos y + j \cosh x \sin y, \qquad \cosh(x + jy) \;=\; \cosh x \cos y + j \sinh x \sin y$$

The total attenuation of the line is $\alpha l = 1.50 \times 12.5/8.686 = 2.16$ nepers. The total phase shift is

$$\beta l \;=\; \omega l/v_p \;=\; (2\pi \times 2.00 \times 10^6) \times (1250 \times 0.305)/(0.70 \times 3.00 \times 10^8) \;=\; 22.798 \;=\; 7\pi + 0.807 \text{ rad}$$

Then $\gamma l = \alpha l + j\beta l = 2.16 + j22.798$.

For use in the identities, $\sinh \alpha l = 4.278$, $\cosh \alpha l = 4.393$, $\sin \beta l = -0.722$, $\cos \beta l = -0.692$. Hence $\sinh \gamma l = -2.960 - j3.172$ and $\cosh \gamma l = -3.040 - j3.089$. Finally

$$I_T \;=\; \frac{10 + j0}{50(-2.960 - j3.172) + (100 - j200)(-3.040 - j3.089)}$$

$$= \; (-9.19 - j1.21) \times 10^{-3} \text{ rms amperes} \;=\; (9.27 \times 10^{-3})\underline{/3.272 \text{ rad}} \text{ rms amperes}$$

Solved Problems

7.1. Show that the reflection coefficient for harmonic current waves produced by a terminal load impedance Z_T connected to a transmission line of characteristic impedance Z_0 is the negative of the reflection coefficient for harmonic voltage waves on the line produced by the same terminal load impedance.

Referring to Fig. 7-1, page 126, the definition of the reflection coefficient for harmonic current waves at the terminal load end of the line must be {phasor value at $z = l$ of harmonic current wave traveling in direction of decreasing z (i.e. reflected wave at point of reflection)}/{phasor value at $z = l$ of harmonic current wave traveling in direction of increasing z (i.e. incident wave at point of reflection)}. From equations (*7.4*) and (*7.5*) this ratio is $(-V_2 e^{\gamma l}/Z_0)/(V_1 e^{-\gamma l}/Z_0) = -V_2 e^{2\gamma l}/V_1$. This is the negative of ρ_T, the complex number reflection coefficient for harmonic voltage waves at $z = l$ as given by (*7.8*).

It follows from (*7.10*) that the reflection coefficient for harmonic current waves is given in terms of the terminal load impedance Z_T and the line's characteristic impedance Z_0 by $(Z_0 - Z_T)/(Z_0 + Z_T)$. To avoid undue confusion in this text, no separate symbol is assigned to the reflection coefficient for harmonic current waves. All references to reflection coefficients are to reflection coefficients for harmonic voltage waves unless there is a specific statement to the contrary; and the reflection coefficient for harmonic current waves, when used, is simply taken as $-\rho_T$.

7.2. Find expressions corresponding to equations (*7.9*), (*7.9a*), (*7.9b*) and (*7.10*), page 128, for a transmission line of characteristic admittance Y_0 $(= 1/Z_0)$ terminated in a terminal load admittance $Y_T = G_T + jB_T$ $(= 1/Z_T)$.

It is evident that $Y_T/Y_0 = 1/(Z_T/Z_0)$. Equation (*7.9*) then becomes simply $Y_T/Y_0 = (1 - \rho_T)/(1 + \rho_T)$. Equation (*7.9a*) cannot be rewritten directly, since it is not true that $G_T = 1/R_T$, and a similar statement applies to (*7.9b*). The easiest procedure is to return to the expression $Y_T/Y_0 = (1 - \rho_T)/(1 + \rho_T)$, letting $\rho_T = |\rho_T|\, e^{j\phi_T} = |\rho_T|\,(\cos\phi_T + j\sin\phi_T)$. On rationalizing the result, the equations in real and imaginary terms respectively are the equations equivalent in admittance notation to (*7.9a*) and (*7.9b*) in impedance notation. Thus

$$\frac{G_T}{Y_0} = \frac{1 - |\rho_T|^2}{1 + |\rho_T|^2 + 2\,|\rho_T|\cos\phi_T} \qquad \text{and} \qquad \frac{B_T}{Y_0} = \frac{-2\,|\rho_T|\sin\phi_T}{1 + |\rho_T|^2 + 2\,|\rho_T|\cos\phi_T}$$

The actual terms are the same in admittance and impedance notation, but two of the signs are different.

Equation (*7.10*) in admittance notation becomes

$$\rho_T = \frac{1/(Y_T/Y_0) - 1}{1/(Y_T/Y_0) + 1} = \frac{1 - Y_T/Y_0}{1 + Y_T/Y_0}$$

7.3. The reflection coefficient at the terminal load end of a transmission line is measured (by techniques described in Chapter 8) as $-0.45 - j0.15$. The transmission line has a characteristic impedance of $75 + j0$ ohms. What is the value of the terminal load admittance connected to the line?

The problem can be solved using admittance notation throughout, or in impedance notation with ultimate conversion of the final answer. In admittance notation the equations to be used are those developed in Problem 7.2. The characteristic admittance Y_0 of the transmission line is $1/(75 + j0) = 0.01333 + j0$ mhos. The square of the magnitude of the reflection coefficient $|\rho_T|^2 = (-0.45)^2 + (-0.15)^2 = 0.225$, and the components of the reflection coefficient are $|\rho_T|\cos\phi_T = -0.45$ and $|\rho_T|\sin\phi_T = -0.15$. Then

$$\frac{G_T}{Y_0} = \frac{1 - 0.225}{1 + 0.225 - 2(0.45)} = 2.38 \qquad \text{and} \qquad \frac{B_T}{Y_0} = \frac{-2(-0.15)}{0.335} = 0.90$$

Finally, the value of the terminal load admittance Y_T in mhos is

$$(2.38 + j0.90)(0.01333) = 0.0317 + j0.0120 \text{ mhos}$$

7.4. A transmission line with a characteristic impedance of $50 + j0$ ohms is operated at a frequency of 15.0 megahertz. The line is terminated in a resistance of 80 ohms connected in parallel with a lossless capacitance of 450 micromicrofarads. Determine the value of the complex reflection coefficient for harmonic voltage waves at the terminal load end of the line, in component form and in polar form.

The characteristic admittance Y_0 of the line is $0.0200 + j0$ mhos. The terminal load admittance consists of 0.0125 mhos conductance for the resistor, and $+\omega C = +0.0424$ mhos susceptance for the capacitor. Thus $Y_T = 0.0125 + j0.0424$ mhos and $Y_T/Y_0 = 0.625 + j2.12$. Using the expression for the reflection coefficient in admittance notation developed in Problem 7.2,

$$\rho_T = \frac{1 - Y_T/Y_0}{1 + Y_T/Y_0} = \frac{1 - 0.625 - j2.12}{1 + 0.625 + j2.12} = -0.544 - j0.594 = 0.807\underline{/227.6^\circ}$$

7.5. A section of transmission line connecting two units of a high frequency system is 1.380 wavelengths long at its frequency of operation, and has a total attenuation of 0.85 decibels. The characteristic impedance of the line is $60 + j0$ ohms, and it is terminated in an impedance of $40 + j30$ ohms. Determine the input impedance of the line.

The result could be obtained from any of the seven equations (*7.14*) to (*7.20*). To illustrate the quantity of numerical work involved, the calculations will be presented for (*7.14*), for a modified form of (*7.14*), and for (*7.20*).

Equation (*7.14*) requires numerical values for ρ_T, $e^{\gamma d}$ and $e^{-\gamma d}$. Since the normalized value of the terminal load impedance is $Z_T/Z_0 = 0.667 + j0.500$, the reflection coefficient from equation (*7.10*) is

$$\rho_T = (Z_T/Z_0 - 1)/(Z_T/Z_0 + 1) = -0.101 + j0.330 = 0.345\underline{/107.3^\circ}$$

The term $e^{\gamma d}$ must be expanded as $e^{\alpha d}\,e^{j\beta d} = e^{\alpha d}(\cos\beta d + j\sin\beta d)$, and similarly for the term $e^{-\gamma d}$. From the data of the problem, $\alpha d = 0.85$ decibels $= 0.85/8.686 = 0.0979$ nepers, and $\beta d = 2\pi d/\lambda = 2\pi(1.38) = 8.6705$ rad. Then $e^{\alpha d} = 1.103$, $e^{-\alpha d} = 0.907$, $e^{j\beta d} = -0.7286 + j0.6850$, and $e^{-j\beta d} = -0.7286 - j0.6850$. The total complex number calculation to be performed is therefore

$$\frac{Z(d)}{Z_0} = \frac{1.103(-0.7286 + j0.6850) + 0.907(-0.101 + j0.330)(-0.7286 - j0.6850)}{1.103(-0.7286 + j0.6850) - 0.907(-0.101 + j0.330)(-0.7286 - j0.6850)}$$

where the terms in the denominator are the same as those in the numerator, but one sign is changed. The result is $Z(d)/Z_0 = 0.563 - j0.081$, and $Z(d) = 33.8 - j4.9$ ohms. The time consuming and error inviting parts of the calculation are the evaluation of $e^{j\beta d}$, and of the products of complex numbers, including the rationalization process.

A small but helpful reduction of the arithmetical work involved is achieved by dividing all terms in equation (*7.14*) by $e^{\gamma d}$, giving $Z(d)/Z_0 = (1 + \rho_T e^{-2\gamma d})/(1 - \rho_T e^{-2\gamma d})$. For this calculation $e^{-2\alpha d} = 0.822$ and $e^{-j2\beta d} = 0.0622 + j0.9981$. The total complex number expression to be evaluated is then

$$\frac{Z(d)}{Z_0} = \frac{1 + 0.822(-0.101 + j0.330)(0.0622 + j0.9981)}{1 - 0.822(-0.101 + j0.330)(0.0622 + j0.9981)} = 0.563 - j0.081$$

If the calculation is performed using equation (*7.20*), it is necessary to evaluate $\tanh(\alpha + j\beta)l = \tanh(0.0979 + j8.6705)$. Using

$$\tanh(x + jy) = (\sinh 2x + j\sin 2y)/(\cosh 2x + \cos 2y)$$

this is found to be $0.1774 - j0.9230$. The complex number expression to be evaluated is then

$$\frac{Z(d)}{Z_0} = \frac{(0.667 + j0.500) + (0.1774 - j0.9230)}{1 + (0.667 + j0.500)(0.1774 - j0.9230)} = 0.563 - j0.081$$

The final arithmetic is simplest in this last calculation, but is partly offset by the work of evaluating the complex hyperbolic tangent.

7.6. Show that equation (*7.20*) can be written in admittance notation by simply substituting Y_0 for Z_0, Y_T for Z_T, and Y_{inp} for Z_{inp}.

Inverting both sides of the equation, Y_{inp}/Y_0 appears on the left. Substituting $1/(Y_T/Y_0)$ for Z_T/Z_0 on the right, the equation becomes

$$\frac{Y_{\text{inp}}}{Y_0} = \frac{1 + [1/(Y_T/Y_0)] \tanh \gamma l}{1/(Y_T/Y_0) + \tanh \gamma l}$$

and multiplying all terms on the right by Y_T/Y_0 produces the required result.

7.7. When a lossless high frequency transmission line of attenuation factor α, phase factor β and real characteristic admittance Y_0 is terminated in an arbitrary load admittance Y_T, there are two locations on the line in each half wavelength at which the normalized admittance $Y(d)/Y_0$ $(= [G(d) + jB(d)]/Y_0)$ has a normalized conductance component $G(d)/Y_0$ of unit value. If a susceptance $-B(d)$ is connected in shunt with the transmission line at any such point, the total normalized admittance at the point will be $1 + j0$, and there will be no reflected waves on the generator side of the point. The susceptance $-B(d)$ can be supplied by connecting a stub line (Section 7.3) in shunt with the transmission line. The process is commonly known as "single stub matching", since the combination of the terminal load admittance, a short intervening portion of the transmission line, and the stub line of suitable length, have produced a non-reflecting or "matched" termination effective at the location of the stub. It is also said that the terminal load admittance has been matched to the line by the single stub.

Derive an analytical expression for the location of the points on a general lossless transmission line circuit at which the normalized admittance has the form $Y(d)/Y_0 = 1 + jB(d)/Y_0$, and an expression for the value of $B(d)/Y_0$ at the points.

The normalized admittance at any point in a lossless transmission line circuit is given by equation (*7.18*) and Problem 7.6 as

$$\frac{Y(d)}{Y_0} = \frac{Y_T/Y_0 + j \tan \beta d}{1 + j(Y_T/Y_0) \tan \beta d}$$

where d is the coordinate of a point on the line measured as a positive quantity from the terminal load end of the line, and Y_T/Y_0 is the normalized value of the connected terminal load admittance. To simplify the notation somewhat, let $Y_T/Y_0 = Y_{Tn} = G_{Tn} + jB_{Tn}$. Then

$$Y(d)/Y_0 = \{G_{Tn} + j(B_{Tn} + \tan \beta d)\}/\{1 - B_{Tn} \tan \beta d + jG_{Tn} \tan \beta d\}$$

and the real part of this is

$$G(d)/Y_0 = \frac{G_{Tn}(1 + \tan^2 \beta d)}{(1 - B_{Tn} \tan \beta d)^2 + (G_{Tn} \tan \beta d)^2}$$

This normalized conductance will equal unity at values of d such that

$$(G_{Tn}^2 + B_{Tn}^2 - G_{Tn})(\tan^2 \beta d) - 2B_{Tn} \tan \beta d + (1 - G_{Tn}) = 0$$

or

$$d = (1/\beta) \tan^{-1} \left(\frac{B_{Tn} \pm \sqrt{G_{Tn}(1 + |Y_{Tn}|^2 - 2G_{Tn})}}{|Y_{Tn}|^2 - G_{Tn}} \right) \pm n\lambda/2$$

To find the normalized susceptance at the points where the normalized conductance is unity, the least complicated procedure is to return to the original equations and write

$$Y(d)/Y_0 = 1 + jB(d)/Y_0 = (Y_T/Y_0 + j \tan \beta d)/[1 + j(Y_T/Y_0) \tan \beta d]$$

Cross multiplying, writing separate equations for the real and imaginary terms, and eliminating $\tan \beta d$ from the two equations leads to

$$B(d)/Y_0 = \pm \sqrt{(1 + |Y_{Tn}|^2 - 2G_{Tn})/G_{Tn}}$$

Calculations for this matching problem from reflection coefficient and standing wave pattern data are discussed in Problem 8.3, page 178. The calculations can also be made graphically using the transmission line circle diagram (Smith chart), as described in Chapter 9. In the absence of a Smith chart or standing wave data, the analytical expressions of this problem may be used; they require only simple real number operations.

7.8. Show that if the attenuation is negligible in the general transmission line circuit of Fig. 7-1, page 126, the magnitude of the phasor voltage on the line has minimum values at locations given by $d_{V(\min)}/\lambda = \frac{1}{4}(1+\phi_T/\pi) + \frac{1}{2}n$, where ϕ_T is the phase angle of the complex reflection coefficient ρ_T and n is any integer.

The phasor voltage at any point on the line is given by equation (*7.1*) as $V(z) = V_1 e^{-\gamma z} + V_2 e^{\gamma z}$, where V_1 and V_2 are phasor voltages determined by the connected source and load, and $\gamma = \alpha + j\beta$. Taking out V_1 as a factor, and substituting for V_2/V_1 from (*7.8*),

$$V(z) = V_1(e^{-\gamma z} + \rho_T\, e^{-2\gamma l}\, e^{\gamma z})$$

Changing the coordinate to $d = l - z$,

$$V(d) = V_1\, e^{-\gamma(l-d)}(1 + \rho_T\, e^{-2\gamma d})$$

With negligible attenuation, $\gamma = j\beta$. The magnitude of the phasor voltage at a location d is then

$$|V(d)| = |V_1\, e^{-j\beta(l-d)}|\,|1 + |\rho_T|\, e^{j(\phi_T - 2\beta d)}|$$

The magnitude of $V_1 e^{-j\beta(l-d)}$ is $|V_1|$ and is independent of d. The magnitude of $1 + |\rho_T|\, e^{j(\phi_T - 2\beta d)}$ has a minimum value of $1 - |\rho_T|$ when $(\phi_T - 2\beta d) = \pi \pm 2n\pi$ where n is any integer; hence this equation determines the values of d at which $|V(d)|$ is a minimum. Introducing $\beta = 2\pi/\lambda$ and solving for the locations of the minima, defined as $d_{V(\min)}$, the result is $d_{V(\min)}/\lambda = \frac{1}{4}(1+\phi_T/\pi) + \frac{1}{2}n$. This shows that the locations of the voltage minima along the line are determined solely by the phase angle of the reflection coefficient produced by the terminal load, and that successive voltage minima are separated by one half wavelength. A more complete discussion of voltage magnitude patterns on transmission lines is given in Chapter 8.

7.9. A transmission line 80 m long and operating at a frequency of 10.0 megahertz has an attenuation factor of 1.50×10^{-3} nepers/m and a phase velocity of 2.75×10^8 m/sec. The characteristic impedance is $50 + j0$ ohms. The input impedance of the line is measured to be $31.2 - j10.0$ ohms. What is the value of the terminal load impedance in ohms?

An explicit expression for Z_T/Z_0 in terms of the data of the problem can be obtained from equation (*7.20*), page 131, as

$$\frac{Z_T}{Z_0} = \frac{Z_{\text{inp}}/Z_0 - \tanh(\alpha + j\beta)l}{1 - (Z_{\text{inp}}/Z_0)\tanh(\alpha + j\beta)l}$$

which is symmetrical with (*7.20*) except for the signs. From the data of the problem, $\alpha l = 80(1.50 \times 10^{-3}) = 0.120$ nepers, and $\beta l = \omega l/v_p = (2\pi \times 10^7)(80)/(2.75 \times 10^8) = 18.278$ rad. The quantity $\tanh(0.120 + j18.278)$ can be evaluated using the expansion

$$\tanh(x \pm jy) = (\sinh 2x \pm j \sin 2y)/(\cosh 2x + \cos 2y)$$

The terms needed are $\sinh 0.24 = 0.2423$, $\cosh 0.24 = 1.0289$, $\sin 36.556 = -0.9100$, $\cos 36.556 = 0.4147$. The result is $\tanh(0.12 + j18.278) = 0.1678 - j0.6303$. The normalized input impedance $Z_{\text{inp}}/Z_0 = 0.624 - j0.200$. The terminal load impedance is finally calculated from

$$\frac{Z_T}{Z_0} = \frac{(0.624 - j0.200) - (0.1678 - j0.6303)}{1 - (0.624 - j0.200)(0.1678 - j0.6303)} = 0.531 + j0.199$$

and $Z_T = (0.531 + j0.199)(50 + j0) = 26.6 + j9.9$ ohms.

7.10. In Section 6.3 it was shown that the propagation in the direction of increasing z of the electric and magnetic fields of a plane electromagnetic wave of angular frequency ω rad/sec into a metal of conductivity σ mhos/m and permeability μ henries/m is described by the term $e^{-\gamma z} = e^{-\alpha z}\, e^{-j\beta z}$, with $\alpha = \beta = 1/\delta = \sqrt{\omega\mu\sigma/2}\ \text{m}^{-1}$. Analytically this situation is exactly analogous to the propagation of voltage and current waves of angular frequency ω rad/sec on a transmission line having finite values for the distributed circuit coefficients L and G, the values of the distributed circuit coefficients R and C being zero. The analog of a finite thickness t for the metal is a finite length l for the transmission line. The analog of a value of surface-current density J_{sz} (or tangential component of magnetic field H_t) at the "input" surface of the metal is an input current I_{inp} for the transmission line. The analog of the air beyond the metal, at $z > t$, is an infinite terminal load impedance connected to the transmission line at $z = l$.

Show that a uniform transmission line having distributed circuit coefficients L henries/m and G mhos/m, and terminated in an open circuit, has a minimum input resistance component at a frequency for which the line is between 1 and 2 rad in length, and determine the variation of the input resistance as a function of frequency in the vicinity of the minimum value.

From equation (*5.1*), page 46, the propagation factors for the analog transmission line are given by

$$\alpha + j\beta \;=\; \sqrt{j\omega LG} \;=\; (1+j)\sqrt{\omega LG/2} \qquad \text{and} \qquad \alpha \;=\; \beta \;=\; \sqrt{\omega LG/2}\ \text{m}^{-1}$$

Similarly, $Z_0 = R_0 + jX_0 = \sqrt{j\omega L/G} = (1+j)\sqrt{\omega L/2G}$ ohms. It will be noted that the distributed inductance L of the transmission line has the same units as the permeability μ of the metal, and that the distributed conductance G of the transmission line has the same units as the conductivity σ of the metal. The characteristic impedance Z_0 of the transmission line is in every way analogous to the surface impedance of the metal $Z_s = R_s + jX_s = (1+j)R_s$, as described in equations (*6.27*) and (*6.45*). The general concept Z_{TEM} of the "wave impedance" of any medium at any angular frequency ω is given by the relation $Z_{\text{TEM}} = \sqrt{j\omega\mu/(\sigma + j\omega\epsilon)}$ in terms of the conductivity, permeability and permittivity of the medium. For free space (or air) the value is enormously greater than for a metal, at all frequencies, and this is the justification for considering the analog transmission line to be terminated in an open circuit.

The normalized input impedance of any transmission line terminated in an open circuit is given by equation (*7.20*) as

$$\frac{Z_{\text{inp}}}{Z_0} \;=\; \coth \gamma l \;=\; \frac{1 + e^{-2\gamma l}}{1 - e^{-2\gamma l}}$$

Using expressions stated above for γ and Z_0, the real part of this input impedance, normalized with respect to the real part for a line of infinite length (i.e. R_s), is given by

$$\frac{R_{\text{inp}}}{\sqrt{\omega L/2G}} \;=\; \frac{1 \;+\; e^{-2l\sqrt{\omega LG/2} \,-\, j2l\sqrt{\omega LG/2}}}{1 \;-\; e^{-2l\sqrt{\omega LG/2} \,-\, j2l\sqrt{\omega LG/2}}}$$

Simplifying the notation by letting $A = \sqrt{\omega LG/2}\, l$, and rationalizing, the result is

$$\frac{R_{\text{inp}}}{\sqrt{\omega L/2G}} \;=\; \frac{1 \;-\; e^{-4A} \;+\; 2e^{-2A}\sin 2A}{1 \;+\; e^{-4A} \;-\; 2e^{-2A}\cos 2A}$$

Interpreting the analog again, this expression gives the ratio at any angular frequency ω of the distributed resistance per unit width of a metal sheet of thickness $A = t/\delta$ skin depths to the distributed resistance per unit width of an indefinitely thick sheet of the same metal. Clearly, when A is large enough the ratio is unity.

The imaginary part of Z_{inp}/Z_0 is also of interest, since its analog is the distributed internal reactance per unit width of the metal sheet. If this is normalized relative to the value for an indefinitely thick sheet (which is the same as the distributed resistance per unit width for an indefinitely thick sheet) the result is

$$\frac{X_{\text{inp}}}{\sqrt{\omega L/2G}} \;=\; \frac{1 \;-\; e^{-4A} \;-\; 2e^{-2A}\sin 2A}{1 \;+\; e^{-4A} \;-\; 2e^{-2A}\cos 2A}$$

Values of the two ratios are given in the following table for a range of the variable A (whose analog is metal sheet thickness in skin depths) from 0.05 to 4.

$A = \sqrt{\omega LG/2}\, l$ (analog t/δ)	$R_{inp}/\sqrt{\omega L/2G}$ (analog R(plane)$/R_s$)	$X_{inp}/\sqrt{\omega L/2G}$ (analog ωL_i(plane)$/R_s$)
0.05	20.00	0.0333
0.10	10.00	0.0667
0.20	5.000	0.1333
0.40	2.506	0.2665
0.60	1.686	0.3987
0.80	1.295	0.5279
1.00	1.0856	0.6504
1.20	0.9758	0.7611
1.30	0.9455	0.8103
1.40	0.9273	0.8545
1.50	0.9187	0.8932
1.60	0.9174	0.9262
1.70	0.9214	0.9534
1.80	0.9289	0.9699
1.90	0.9384	0.9913
2.00	0.9489	0.9993
2.20	0.9690	1.0154
2.50	0.9840	1.0082
3.00	1.0034	1.0062
4.00	1.0006	0.9992

7.11. A transmission line 250 ft long and having $Z_0 = 51.5 + j0$ ohms is terminated in an impedance of $150 - j20$ ohms. The line has an attenuation factor of 1.45 db/(100 ft), and the wavelength on the line is 63 ft. The harmonic input voltage to the line has an rms value of 30.0 volts. Relative to the input voltage as a zero-angle reference phasor, determine the rms phasor values at the line's input terminals of the harmonic voltage wave traveling in the direction from source to load, and of the harmonic voltage wave traveling in the direction from load to source.

The phasor voltages to be evaluated are V_1 and V_2 in equation (*7.1*), page 126. From that equation $V_1 + V_2 = 30.0 + j0$, the rms phasor input voltage to the line. Eliminating ρ_T between equations (*7.8*) and (*7.10*) supplies a second relation between V_1, V_2, and other data of the problem, in the form

$$V_2/V_1 \;=\; e^{-2\alpha l}\, e^{-j2\beta l}\,(Z_T/Z_0 - 1)/(Z_T/Z_0 + 1)$$

The normalized value Z_T/Z_0 of the terminal load impedance is $2.913 - j0.3883$. The total attenuation αl of the line is $0.0145(250) = 3.625$ db $= 0.4173$ nepers, and $e^{-2\alpha l} = e^{-0.8346} = 0.4340$. The length of the line in wavelengths $l/\lambda = 250/63 = 3.9683$, and $\beta l = 2\pi l/\lambda = 24.9336$ rad. Then $e^{-j2\beta l} = \cos 2\beta l - j \sin 2\beta l = 0.9216 + j0.3880$. The result of substituting these various quantities is $V_2/V_1 = 0.2060 + j0.0631$. Eliminating V_2 between this equation and $V_1 + V_2 = 30.0 + j0$ gives $V_1 = 30/(1 + 0.2060 + j0.0631)$ and $V_1 = 24.808 - j1.298$ volts rms. Finally, $V_2 = 5.192 + j1.298$ volts rms.

7.12. Determine an equivalent lumped element series *R-L-C* circuit that will have the same impedance at a frequency ω rad/sec as the input impedance of a section of low loss transmission line with short circuit termination, length less than 1% of a wavelength, and total attenuation less than 0.5 db, the attenuation being caused entirely by distributed resistance R (i.e. $G = 0$).

The transmission line section described is actually too short in wavelengths to require analysis as a distributed circuit. It is in effect a rectangular loop of wire being used as an inductor at low frequencies. Its total series resistance is obviously Rl ohms and its total series inductance (ignoring the inductance of the short-circuiting conductor) is Ll henries, where l is the length of the line and R and L are respectively the distributed resistance and distributed inductance of the line.

The required equivalent lumped circuit is thus a resistance of Rl ohms in series with an inductance of Ll henries. Because of the short circuit termination and the short line length in wavelengths, the shunt reactance of the distributed capacitance of the line section is too great to affect this result.

The attempt to derive this equivalent circuit from equation (*7.20*), is of interest because it encounters an unexpected hazard that must sometimes be allowed for in using that equation.

For $Z_T = 0$, equation (*7.20*) becomes, without approximation, $Z_{inp}/Z_0 = \tanh(\alpha + j\beta)l$. Any expansion of $\tanh(x+jy)$ shows that if $x < 0.1$ and $y < 0.1$, then $\tanh(x+jy) \doteq x + jy$, within $\frac{1}{2}$% for each term. It has been specified for the line in this problem that $\alpha l < 0.5$ db or $\alpha l < 0.06$ nepers, and $l/\lambda < 0.01$ or $\beta l < 0.06$ rad; hence $Z_{inp}/Z_0 \doteq \alpha l + j\beta l$. The most inviting procedure for introducing the line's distributed circuit coefficients into the right hand side of this equation is to use the approximate high frequency equations (*5.11*), (*5.10*) and (*5.9*) for a low-loss line having $G = 0$. The equations give respectively $Z_0 = \sqrt{L/C}$, $\alpha = R/2Z_0$ and $\beta = \omega\sqrt{LC}$. Substituting all of these leads to $Z_{inp} = Rl/2 + j\omega Ll$ ohms, corresponding to an equivalent lumped constant circuit of a resistance $Rl/2$ ohms in series with an inductance Ll henries.

The discrepancy between this series resistance component $Rl/2$ and the intuitively obvious value Rl can be accounted for, in the case of a section of high frequency transmission line, by the inadequacy of the approximation $Z_0 = \sqrt{L/C}$. For a line whose losses are due entirely to distributed resistance R, equation (*5.30*) gives $Z_0 = \sqrt{L/C}\,\{1 - j(\alpha/\beta)\}$, which is approximate in magnitude but exact in phase angle. Substituting this instead of $Z_0 = \sqrt{L/C}$ into the approximate expression for Z_{inp}/Z_0 leads to

$$Z_{inp} \doteq Rl + j\omega Ll\{1 - (\alpha/\beta)^2\}$$

Since for low-loss lines at "high" frequencies $\alpha/\beta \ll 1$, this gives the correct result for such lines.

However, the relation $Z_{inp} = Rl + j\omega Ll$ should apply to all lines having $\alpha l \ll 1$ and $\beta l \ll 1$, not just to lines operating under conditions meeting "high frequency" specifications. That it does in fact apply to all lines can be seen by substituting the exact expressions

$$Z_0 = \sqrt{(R+j\omega L)/(G+j\omega C)} \quad \text{and} \quad \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$

into $Z_{inp}/Z_0 = \alpha l + j\beta l$. Thus the term $(\alpha/\beta)^2$ obtained on using the high frequency approximate equations is a consequence of the approximate nature of the expressions used for α and β and the magnitude of Z_0.

It is worth noting that if the transmission line of this problem had been specified as a Heaviside distortionless line with $R/L = G/C$, then Z_0 would be real and the losses would be equally divided between the distributed resistance R and the distributed conductance G. From equations (*5.5*) to (*5.8*) it would follow that $Z_0 = \sqrt{L/C} + j0$, $\alpha = R/Z_0$ and $\beta = \omega\sqrt{LC}$, all of these being exact expressions. These substituted into $Z_{inp}/Z_0 \doteq \alpha l + j\beta l$ give directly the correct result $Z_{inp} = Rl + j\omega Ll$.

For the other extreme of the distribution of line losses, a line with finite G but $R = 0$, the analysis using high frequency values for α and β and the complex value for Z_0 gives to a first approximation $Z_{inp} = 0 + j\omega Ll$, the expected result.

7.13. Repeat the analysis of Problem 7.12, for a transmission line section having $\alpha l < 0.1$ and $\beta l < 0.1$, with *open* circuit termination.

It is intuitively evident that the reactive element in a lumped circuit equivalent to an electrically short ($l/\lambda \ll 1$) section of low loss transmission line with open circuit termination must be simply Cl, the total capacitance between the line conductors of length l. It is not directly obvious what the resistive element should be, although it will not be vanishingly small, since the capacitive current between the line conductors at any location must pass through the resistance of the line conductors between the input terminals and that location.

The transmission line section is a distributed capacitor with terminals at one end. The conditions $\alpha l < 0.1$ and $\beta l < 0.1$ have the consequence that the voltage between the line conductors is essentially constant at all points along the section. Let the rms phasor value of this voltage be V volts and the operating frequency ω rad/sec. Then the total transverse displacement current between the line conductors is ωClV; and if R_{inp} is the effective resistance component of the input impedance of the line section, the total power loss in the capacitor is $(\omega ClV)^2 R_{inp}$.

Considering a cross section of the line at coordinate z measured from the input terminals, the current in the line conductors at the point is the displacement current to the length $l - z$ of line beyond z, of value $\omega C(l-z)V$. The power loss in length dz of line at z is therefore $\{\omega C(l-z)V\}^2 R\,dz$, and the total power loss in the line section is the integral of this from $z = 0$ to $z = l$. The result is $(\omega ClV)^2 Rl/3$; hence $R_{inp} = Rl/3$.

The derivation of this result from equation (7.20) proves to be quite devious. For open circuit termination (Z_T = infinite), equation (7.20) becomes $Z_{inp}/Z_0 = \coth(\alpha + j\beta)l$, with no approximations. Since $\coth(\alpha + j\beta)l = 1/[\tanh(\alpha + j\beta)l]$, the results of the approximations used in Problem 7.12 are easily tested. Using $\tanh(\alpha + j\beta)l \doteq (\alpha + j\beta)l$, $Z_{inp} \doteq Z_0/\{(\alpha + j\beta)l\}$. Then the exact expressions $Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)}$ and $\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$, with $G = 0$, lead to the result $Z_{inp} = 0 - j(1/\omega C)$. The reactance term is correct, but the resistance term is not acceptable since the distributed resistance R of the line is not zero. If the high frequency approximations $\alpha = R/2Z_0$ and $\beta = \omega\sqrt{LC}$ are combined with the expression $Z_0 = \sqrt{L/C}\{1 - j(\alpha/\beta)\}$, as in Problem 7.12, the result obtained is $Z_{inp} = 0 - j(1/\omega Cl)\{1 + (\alpha/\beta)^2\}$, and again the resistance term is incorrect.

To obtain a meaningful solution it is necessary to return to the original equation $Z_{inp}/Z_0 = \coth(\alpha + j\beta)l$ and retain an additional term in the power series expansion for a hyperbolic function of a small complex number. A convenient identity is

$$\coth(x + jy) = (\sinh 2x - j \sin 2y)/(\cosh 2x - \cos 2y)$$

With $x = \alpha l$ and $y = \beta l$, the power series for the respective functions to two terms are: $\sinh 2\alpha l = 2\alpha l + (2\alpha l)^3/3!$; $\sin 2\beta l = 2\beta l - (2\beta l)^3/3!$; $\cosh 2\alpha l = 1 + (2\alpha l)^2/2!$; $\cos 2\beta l = 1 - (2\beta l)^2/2!$. Making these substitutions,

$$Z_{inp}/Z_0 = \frac{\alpha l + \frac{2}{3}(\alpha l)^3 - j\{\beta l - \frac{2}{3}(\beta l)^3\}}{(\alpha l)^2 + (\beta l)^2}$$

Introducing $Z_0 = \sqrt{L/C}\{1 - j(\alpha/\beta)\}$, and evaluating the resistive component only, $R_{inp} = \sqrt{L/C}(2\alpha l/3)$. Finally, substituting $\alpha = R/2Z_0$ gives the desired result $R_{inp} = Rl/3$.

This problem and Problem 7.12 illustrate that combinations of approximations can create errors that are not obvious consequences of any of the approximations separately.

7.14. Show that for high frequency transmission line sections with short circuit termination and low total attenuation ($\alpha l < 0.05$ nepers) the input impedance is real and approximately equal to $Z_0/\alpha l$ for line lengths close to an odd number of quarter wavelengths, and the input impedance is real and approximately equal to $Z_0 \alpha l$ for line lengths close to an integral number of half wavelengths.

For the line sections mentioned the smallest value of βl is $\pi/2$, for the one quarter wavelength line. Hence α/β is less than about 0.03 for all of the sections, and the phase angle of the characteristic impedance does not exceed 2°. The calculation will be carried out for the largest negative phase angle, which occurs when the attenuation of a line is due entirely to distributed resistance R. The characteristic impedance of the line is then, to a good approximation, given by $Z_0 = \sqrt{L/C}\{1 - j(\alpha/\beta)\}$. The input impedance of the line with short circuit termination is $Z_{inp} = Z_0 \tanh(\alpha + j\beta)l$. The complex hyperbolic tangent can be expanded as

$$(\sinh 2\alpha l + j \sin 2\beta l)/(\cosh 2\alpha l + \cos 2\beta l)$$

Combining this with the expression for Z_0 gives

$$Z_{inp} = \sqrt{L/C}\,\frac{\{\sinh 2\alpha l + (\alpha/\beta)\sin 2\beta l + j[\sin 2\beta l - (\alpha/\beta)\sinh 2\alpha l]\}}{\cosh 2\alpha l + \cos 2\beta l}$$

For the values of αl specified in the problem, good approximations are $\sinh 2\alpha l = 2\alpha l$ and $\cosh 2\alpha l = 1 + (2\alpha l)^2/2!$. The input impedance of the line is then real for values of l that make $\sin 2\beta l = 2\alpha l(\alpha/\beta)$, or $\cos 2\beta l = \pm\sqrt{1 - 4(\alpha l)^2(\alpha/\beta)^2} = \pm\{1 - 2(\alpha l)^2(\alpha/\beta)^2\}$ since $\alpha l(\alpha/\beta) \ll 1$. The positive sign corresponds to line lengths that are very close to an integral number of half wavelengths, and the negative sign to line lengths that are close to an odd number of quarter wavelengths. Substituting into the expression for Z_{inp} gives

$$Z_{inp} = \sqrt{L/C}\,\frac{2\alpha l\{1 + (\alpha/\beta)^2\}}{1 + 2(\alpha l)^2 \pm \{1 - 2(\alpha l)^2(\alpha/\beta)^2\}}$$

For the minus sign in the denominator the final result is

$$Z_{inp} = \sqrt{L/C}/\alpha l \doteq Z_0/\alpha l$$

For the plus sign in the denominator the final result is

$$Z_{inp} = \sqrt{L/C}\,\alpha l\,\frac{1 + (\alpha/\beta)^2}{1 + (\alpha l)^2\{1 - (\alpha/\beta)^2\}} \doteq Z_0 \alpha l$$

It is evident that, in contrast with the case of Problem 7.12, the small phase angle of the characteristic impedance has negligible effect on these results, and it is immaterial how the attenuation is divided between distributed resistance R and distributed conductance G.

If the transmission line sections of this problem had open circuit termination instead of short circuit termination, the input impedance would be $Z_0\alpha l$ for line sections an odd number of quarter wavelengths long and $Z_0/\alpha l$ for line sections an integral number of half wavelengths long.

7.15. A transmission line circuit consists of a source having a resistive output impedance Z_S, and a terminal load having a resistive impedance Z_T, the impedance match between the two being achieved by a quarter wavelength transformer section of lossless transmission line, of purely resistive characteristic impedance $Z_0' = \sqrt{Z_S Z_T}$. The signals on the circuit occupy a frequency bandwidth from $f_0 - \Delta f$ hertz to $f_0 + \Delta f$ hertz, with $\Delta f/f_0 \ll 1$. The transmission line section between source and load is one quarter wavelength long at the frequency f_0.

Derive an expression for the power transfer from source to load at the frequencies $f_0 \pm \Delta f$, relative to the maximum power transfer at frequency f_0. Assume the impedances Z_S and Z_T are not functions of frequency.

The power transfer will be less than maximum at a frequency slightly different from f_0 because the transmission line section will not be exactly one quarter wavelength long, and its input impedance will not be exactly equal to the source impedance.

If the phase velocity on the line section is v_p, assumed independent of frequency, then $\beta = \omega/v_p$ and $\beta_0 = 2\pi f_0/v_p$ at the center frequency of operation. The line length l is determined by $\beta_0 l = \pi/2$, or $l = v_p/4f_0$. At frequency $f_0 + \Delta f$, $\beta = 2\pi(f_0 + \Delta f)/v_p$ and the input impedance of the lossless line section of characteristic impedance Z_0' and length l terminated in Z_T is, from equation (*7.20*),

$$\frac{Z_{\text{inp}}}{Z_0'} = \frac{Z_T/Z_0' + j\tan\{(\pi/2)(1+\Delta f/f_0)\}}{1 + j(Z_T/Z_0')\tan\{(\pi/2)(1+\Delta f/f_0)\}}$$

Using the identity $\tan(\pi/2 + x) = -\cot x$,

$$\tan\{(\pi/2)(1+\Delta f/f_0)\} = -\cot\{\pi\,\Delta f/(2f_0)\} = -2f_0/(\pi\,\Delta f) \quad \text{if } \Delta f/f_0 \ll 1$$

Then

$$\frac{Z_{\text{inp}}}{Z_0'} = \frac{Z_T/Z_0' - j\,2f_0/(\pi\,\Delta f)}{1 - j(Z_T/Z_0')(2f_0)/(\pi\,\Delta f)}$$

$$= \frac{(Z_T/Z_0')\{1 + (2f_0/\pi\,\Delta f)^2\} - j\,(2f_0/\pi\,\Delta f)\{1 - (Z_T/Z_0')^2\}}{1 + (Z_T/Z_0')^2(2f_0/\pi\,\Delta f)^2}$$

Using $Z_0' = \sqrt{Z_S Z_T}$ and assuming $f_0/\Delta f \gg 1$ and $\sqrt{Z_T/Z_S}\,(f_0/\Delta f) \gg 1$, this can be simplified to

$$Z_{\text{inp}} = Z_S + jZ_0'(\pi/2)(\Delta f/f_0)(1 - Z_S/Z_T)$$

which recognizes that for the trivial case of $Z_S = Z_T = Z_0'$, the input impedance of the transmission line section has this same value, at all frequencies.

Letting $X(\Delta f) = Z_0'(\pi/2)(\Delta f/f_0)(1 - Z_S/Z_T)$, the power transfer from a source of rms harmonic voltage V and resistive output impedance Z_S to the input of the transmission line section will have the maximum possible value of $V^2/4Z_S$ at frequency f_0 when $X(\Delta f) = 0$, and will be given by $|V/\{2Z_S + jX(\Delta f)\}|^2 Z_S$ at frequency $f_0 + \Delta f$. The ratio of the power transfer in the latter case to the maximum value is then $1/\{1 + [X(\Delta f)/2Z_S]^2\}$.

The approximations that have been made are sufficiently accurate for most purposes if the impedance transformation ratio of the transmission line section does not exceed 5 to 10, and if the signal bandwidth is not more than a few percent. Within these same limitations, $X(\Delta f)$ will never be a large fraction of Z_S, and the power transfer will remain near maximum over the signal bandwidth. When the limitations are exceeded the properties of the transformer must be calculated from the initial equation for Z_{inp}/Z_0'.

Supplementary Problems

7.16. Derive equations (*7.9a*) and (*7.9b*) from equation (*7.9*), page 128.

7.17. Show that if there is a reflection coefficient for harmonic voltages waves $\rho_T = |\rho_T|\, e^{j\phi_T}$ at the terminal load end of a transmission line, the normalized magnitude of the terminal load impedance is

$$|Z_T/Z_0| \;=\; \sqrt{1 + 2\,|\rho_T|\,\cos\phi_T + |\rho_T|^2}\Big/\sqrt{1 - 2\,|\rho_T|\,\cos\phi_T + |\rho_T|^2}$$

and the phase angle θ of the normalized terminal load impedance is

$$\theta \;=\; \tan^{-1}\{(2\,|\rho_T|\,\sin\phi_T)/(1 - |\rho_T|^2)\}$$

7.18. If the reflection coefficient for harmonic voltage waves at the terminal load end of a transmission line is $1/n + j0$, show that the line is terminated in an impedance $Z_T = nZ_0 + j0$, or an admittance $Y_T = Y_0/n + j0$, where Z_0 is the characteristic impedance of the line and $Y_0 = 1/Z_0$.

7.19. Show that all terminal load impedances or admittances of normalized magnitude unity connected to any transmission line produce reflection coefficients whose phase angles are $\pm 90°$, the angle being $+90°$ when the connected terminal load is inductive, and $-90°$ when the connected terminal load is capacitive.

7.20. Determine the value of a terminal load impedance of normalized form $A + jA$, where A is real, which when connected to a transmission line will produce a reflection coefficient of magnitude 0.50.
Ans. $1.275 + j1.275$ or $0.393 + j0.393$

7.21. What identity of complex hyperbolic trigonometry is used to obtain equation (*7.19*) from equation (*7.18*), page 130? *Ans.* $\tanh(A + B) = (\tanh A + \tanh B)/(1 + \tanh A\,\tanh B)$

7.22. A particular transmission line when terminated in a normalized impedance $1.25 - j0.42$ has a normalized input impedance of $0.84 + j0.32$. If the same transmission line were terminated in a normalized admittance $1.25 - j0.42$, what would be the value of its normalized input admittance?
Ans. $0.84 + j0.32$ (see Problem 7.6)

7.23. From equations (*7.21*) and (*7.22*), page 131, write expressions for the normalized input admittance of lossless stub lines having length l, phase factor β at their frequency of operation, and either short circuit or open circuit terminations. *Ans.* $Y_{\text{inp}}/Y_0 \;=\; 0 - j\cot\beta l$; $Y_{\text{inp}}/Y_0 \;=\; 0 + j\tan\beta l$

7.24. Using equation (*7.22*), show that if a transmission line has negligible attenuation, is terminated in an open circuit and has a length l less than 1% of a wavelength, its input impedance is equal to the impedance of a capacitance of value Cl farads, where C is the distributed capacitance of the line in farads/unit length.

7.25. A lossless transmission line of characteristic impedance $50 + j0$ ohms is terminated in an admittance $0.0080 - j0.0120$ mhos.

(*a*) Determine the locations of the points on the line, relative to the terminal load end, at which the normalized admittance has the form $1 + jB(d)/Y_0$.

(*b*) Determine the values of $B(d)/Y_0$ at the locations found in (*a*).

(*c*) Find the lengths of stub line with short circuit termination that must be connected in shunt with the line at the locations found in (*a*) to achieve single stub matching.

Ans. (*a*) $0.424 + n/2$ wavelengths, and $0.266 + n/2$ wavelengths. (*b*) $B(d)/Y_0 = -1.34$ at the former points and $+1.34$ at the latter. (*c*) $0.398 + n/2$ wavelengths at the former points and $0.102 + n/2$ wavelengths at the latter.

7.26. A signal source operating at 50 megahertz has an output impedance of $20 + j0$ ohms. It is to supply power through a coaxial line to a load having impedance $150 + j40$ ohms. Design a quarter wavelength transformer to match the source to the load.

Ans. The phase angles of the two impedances are not equal, so a quarter wavelength transformer cannot provide perfect impedance matching. The load impedance can be made real by adding a series capacitor of reactance -40 ohms. The matching transformer would then have a characteristic impedance $Z_0' = \sqrt{20 \times 150} = 54.8$ ohms. The free space wavelength at 50 megahertz is 6.00 m. If the transformer has predominantly air dielectric, its length would be 1.50 m.

7.27. The input impedance of a section of transmission line with short circuit termination is measured to be $33.5 - j34.0$ ohms. The line section has a total attenuation of 3.75 db and is 15.38 wavelengths long at its frequency of operation. What is the characteristic impedance of the line?

Ans. $50 + j0$ ohms

7.28. Show that the normalized input *impedance* of any section of transmission line terminated in a short circuit is equal to the normalized input *admittance* of the same line section terminated in an open circuit.

7.29. A coaxial transmission line 12 ft long is constructed of two copper tubes. The outside diameters of the tubes are 0.250″ and 0.500″ respectively, and the wall thickness of each is 0.030″. The interconductor space is air filled. If there are "input" terminals at one end of the line and the other end of the line is open circuited, determine an *R-L-C* lumped element series circuit whose impedance would be the same as the input impedance of the transmission line section at (*a*) 60 hertz, (*b*) 1.50×10^8 hertz.

Ans. (*a*) 0.0069 ohms in series with 361 micromicrofarads. (At this frequency $\alpha l \ll 1$, $\beta l \ll 1$, $R_{a\text{-}c} = R_{d\text{-}c}$, and the results of Problem 7.13 apply.)

(*b*) Approximately 0.4 ohms in series with an inductance of 0.020 microhenries. (At this frequency $a/\delta \gg 1$, $t/\delta \gg 1$, $l/\lambda > 1$, and high frequency transmission line calculations must be used.)

7.30. An air dielectric lossless high frequency coaxial transmission line is terminated in its characteristic impedance. The wavelength on the line is λ meters. Show that if a length of the line $n\lambda/\sqrt{k_e'}$ meters (where n is any integer) is filled with a lossless insulating material of dielectric constant k_e', the input impedance of the line will continue to be equal to its characteristic impedance. (This is an application of the half wavelength transformer principle.)

7.31. A lossless high frequency transmission line of length l with open circuit termination is measured to have at a certain frequency an input capacitance three times as great as its low frequency input capacitance Cl (where C is the distributed capacitance of the line). Determine the length of the line in wavelengths.

Ans. There is an infinite sequence of solutions, the lowest value being 0.21 wavelengths. The problem reduces to finding βl such that $(\tan \beta l)/\beta l = 3$. Tables of $(\tan x)/x$ are available. The various solutions in the infinite sequence are not related by any exact periodicity or integer ratio relationships.

7.32. A 300 foot long section of flexible coaxial cable with plastic dielectric is measured to have an input impedance of $96.8 + j0$ ohms at a frequency of 17.4 megahertz, a frequency at which the cable is exactly 8.00 wavelengths long. The line is terminated in an open circuit. The line's characteristic impedance is $50 + j0$ ohms. What is the attenuation factor of the line? *Ans.* 1.66 db/(100 ft)

7.33. A lossless transmission line 1.25 wavelengths long having a characteristic impedance of $50 + j0$ ohms is terminated by the input terminals of a second lossless transmission line 1.25 wavelengths long having a characteristic impedance of $75 + j0$ ohms. The second line is terminated in a pure resistance of 100 ohms. Determine the input impedance of the first line.

Ans. 44.5 ohms (Each line section is a quarter wavelength transformer.)

7.34. If the 19 gauge cable pair transmission line of Table 5.1, page 55, were terminated in an impedance of $100 + j300$ ohms at a frequency of 1000 hertz, what would be the magnitude and phase angle of the reflection coefficient for voltage waves at the termination? *Ans.* $1.49\underline{/113.8^\circ}$

7.35. Show that the short circuit admittance matrix for a section of uniform transmission line is

$$\begin{bmatrix} \dfrac{\coth \gamma l}{Z_0} & \dfrac{-\operatorname{cosech} \gamma l}{Z_0} \\[2ex] \dfrac{-\operatorname{cosech} \gamma l}{Z_0} & \dfrac{\coth \gamma l}{Z_0} \end{bmatrix}$$

7.36. Show that the hybrid matrix for a section of uniform transmission line is

$$\begin{bmatrix} Z_0 \tanh \gamma l & \operatorname{sech} \gamma l \\ -\operatorname{sech} \gamma l & (\tanh \gamma l)/Z_0 \end{bmatrix}$$

and the inverse hybrid matrix is $\begin{bmatrix} (\tanh \gamma l)/Z_0 & -\operatorname{sech} \gamma l \\ \operatorname{sech} \gamma l & Z_0 \tanh \gamma l \end{bmatrix}$

7.37. Show that the transmission matrix of a section of uniform transmission line is

$$\begin{bmatrix} \cosh \gamma l & Z_0 \sinh \gamma l \\ (\sinh \gamma l)/Z_0 & \cosh \gamma l \end{bmatrix}$$

and the inverse transmission matrix is the same.

7.38. Show that for the transmission line circuit of Fig. 7-1, page 126,

$$V_T/V_i = \frac{1}{\cosh \gamma l + (Z_0/Z_T) \sinh \gamma l}, \quad I_T/I_i = \frac{1}{\cosh \gamma l + (Z_T/Z_0) \sinh \gamma l}$$

where V_i and I_i are respectively the phasor voltage and current at the input terminals of the line, and V_T and I_T are respectively the phasor voltage and current at the load terminals of the line.

Chapter 8

Standing Wave Patterns

8.1. The phenomenon of interference.

It has been established in earlier chapters that all possible single frequency time harmonic voltage distributions on a uniform transmission line are described by the equation

$$V(z) \quad = \quad V_1 e^{-\gamma z} + V_2 e^{+\gamma z} \tag{8.1}$$

where V_1 and V_2 are arbitrary voltage phasors to be determined by boundary conditions at the ends of the line, and $\gamma = \alpha + j\beta$. The time harmonic variation of the voltage is represented by an implicit multiplying factor $e^{j\omega t}$, where $\omega/2\pi$ is the signal frequency in hertz. The phase velocity of the voltage waves is $v_p = \omega/\beta$, their wavelength is $\lambda = 2\pi/\beta$, and they attenuate at the rate of α nepers per unit length of line. The corresponding equation for single frequency time harmonic current distributions is

$$I(z) \quad = \quad \frac{1}{Z_0}(V_1 e^{-\gamma z} - V_2 e^{+\gamma z}) \tag{8.2}$$

where Z_0 is the characteristic impedance of the transmission line. The current waves have the same frequency, phase velocity, wavelength and attenuation as the voltage waves.

Equations (*8.1*) and (*8.2*) have specific reference to the transmission line circuit of Fig. 8-1. The first term on the right of each of the equations describes a wave traveling from the source toward the load, and the second term describes a wave traveling from the load toward the source. The former may be called the "incident" wave, incident on the terminal load, and the latter the "reflected" wave, produced by reflection from the terminal load.

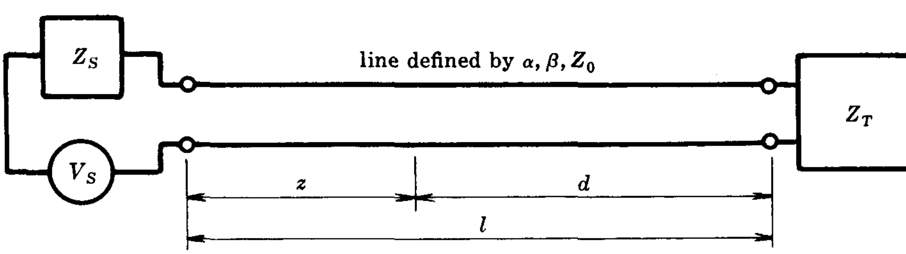


Fig. 8-1. Basic transmission line circuit. When waves are reflected from the impedance Z_T, a standing wave pattern is produced on the line.

(It is worth noting that (*8.1*) and (*8.2*) are also applicable to Fig. 8-2 below, a distinctly different transmission line circuit. Here a single source supplies signals to both ends of a transmission line section, through networks that terminate the section in its characteristic impedance at each end. There are waves of identical frequency traveling in both directions on the line, but their amplitudes and phases are independently variable, and neither can be called "incident" or "reflected" waves. The arrangement can be used to measure the phase shift produced by a transmission line component inserted at one end of the line.)

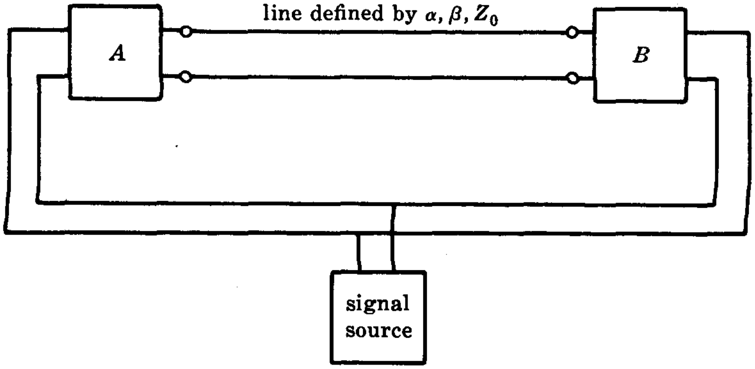


Fig. 8-2. A transmission line circuit on which standing waves will occur even when networks A and B terminate the line nonreflectively at each end. The networks may have different values of attenuation and phase shift.

In Chapter 7 it was shown that reflected waves occur on a transmission line whenever the terminal load impedance is not equal to the characteristic impedance Z_0 of the line. The complex number reflection coefficient ρ_T for harmonic voltage waves, produced at the end of the line $z = l$ by a connected terminal load impedance Z_T, was defined as the phasor value of the reflected voltage wave to the phasor value of the incident voltage wave at the load terminals. It was found to be given by

$$\rho_T = \frac{V_2 e^{+\gamma l}}{V_1 e^{-\gamma l}} = \frac{Z_T/Z_0 - 1}{Z_T/Z_0 + 1} \tag{8.3}$$

The reflection coefficient for harmonic current waves was found to be $-\rho_T$.

Equations (*8.1*) and (*8.2*) can be rewritten to contain the reflection coefficient explicitly:

$$V(z) = V_1\left(e^{-\gamma z} + \frac{V_2}{V_1}e^{+\gamma z}\right) = V_1(e^{-\gamma z} + \rho_T e^{-2\gamma l} e^{+\gamma z})$$

$$= V_1 e^{-\gamma l}(e^{\gamma(l-z)} + \rho_T e^{-\gamma(l-z)})$$

or

$$V(d) = V_1 e^{-\gamma l}(e^{\gamma d} + \rho_T e^{-\gamma d}) \tag{8.4}$$

and

$$I(d) = \frac{V_1 e^{-\gamma l}}{Z_0}(e^{\gamma d} - \rho_T e^{-\gamma d}) \tag{8.5}$$

Whenever two waves of identical frequency travel in opposite directions on a transmission system, whether the system be electrical or mechanical, solid, liquid or gaseous, the fundamental phenomenon of interference or "standing waves" occurs. The magnitude of each phasor wave variable, instead of diminishing steadily and exponentially from the source to the terminal load end of the system, as is true when only one wave is present, exhibits periodic maxima and minima along the system, at intervals determined by the wavelength of the individual waves. The effect appears in its most striking form when the two oppositely directed waves have equal amplitude and the transmission system has zero attenuation. For harmonic voltage waves on a lossless transmission line this case is most simply represented by letting $\alpha = 0$ (hence $\gamma = j\beta$), and $\rho_T = 1 + j0$. Then the voltage magnitude as a function of position along the line is

$$|V(d)| = A\,|\cos\beta d| \tag{8.6}$$

and the current magnitude as a function of position is

$$|I(d)| = (A/Z_0)\,|\sin\beta d| \tag{8.7}$$

where $A = |2V_1 e^{-\gamma l}|$ is a scale factor determined by the total circuit of Fig. 8-1.

Fig. 8-3 is a graph of equation (*8.6*). The ordinate of the graph at any position d on the line is proportional to the rms voltage that would actually be measured between the line conductors at that cross-section by an a-c voltmeter or similar indicating instrument. From the graph and the equation it is seen that there is a sequence of locations d_n along the line, given by

$$\beta d_n = \pi/2 + n\pi \quad (n = 0, 1, 2, \ldots)$$

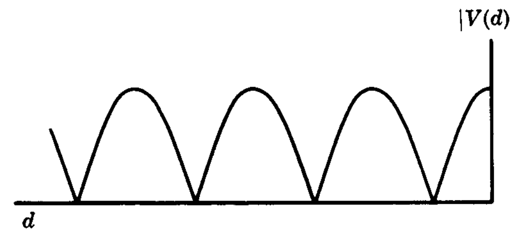


Fig. 8-3. A standing wave pattern on a lossless transmission line when the incident and reflected waves have equal amplitudes.

at which the voltage is zero at all times. The separation of a consecutive pair of these nodes is

$$\beta d_n - \beta d_{n-1} = (\pi/2 + n\pi) - \{\pi/2 + (n-1)\pi\} = \pi$$

Using $\beta = 2\pi/\lambda$,

$$d_n - d_{n-1} = \lambda/2 \qquad (8.8)$$

Midway between the points of zero voltage in Fig. 8-3 are points where the phasor voltage magnitude is a maximum and is twice the value for each of the individual traveling waves. The two waves are said to combine with "constructive interference" at the points of maximum voltage, and with "destructive interference" at the points of zero voltage. The instantaneous voltage at any point of the pattern oscillates harmonically with time at the signal frequency, but the amplitude of the oscillation ranges from zero at the points $d_1, d_2, \ldots$ to a maximum at the points midway between.

Comparison of equations (*8.7*) and (*8.6*) shows that for the particular case of a lossless transmission line terminated in a voltage reflection coefficient $\rho_T = 1 + j0$, the standing wave pattern for the current waves is similar in shape to that for the voltage waves but is displaced one quarter wavelength along the line. The general analysis of standing wave patterns in Section 8.4 establishes that current maxima are always coincident with voltage minima (and vice versa) on transmission lines having low attenuation per wavelength, for all values of the reflection coefficient at the terminal load.

The standing wave patterns of voltage magnitude and current magnitude on a lossless transmission line when there are waves traveling in only one direction on the line are found from equations (*8.4*) and (*8.5*) by substituting $\alpha = 0$ and $\rho_T = 0$, with the results

$$|V(d)| = A\,|e^{j\beta d}| = A \qquad (8.9)$$

$$|I(d)| = A/Z_0 \qquad (8.10)$$

where A is the scale factor $|V_1 e^{-\gamma l}|$ as before, and Z_0 is necessarily real for a lossless line. Fig. 8-4 is a graph of these equations. Again, the instantaneous voltage at any point on the line oscillates in value harmonically with time, but in this case the amplitude of the oscillation is everywhere constant and equal to the amplitude of the traveling voltage wave. A similar statement applies to the pattern of current magnitude. With waves traveling in only one direction on the line there can be no interference, and no maxima or minima in the patterns.

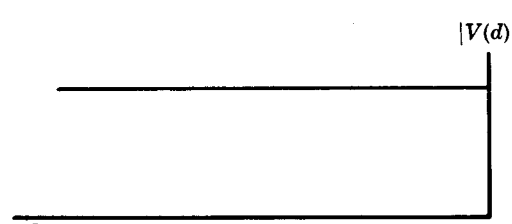


Fig. 8-4. The standing wave pattern on a lossless transmission line when the line is terminated non-reflectively.

Like the a-c meter readings of $V(d)$ or $I(d)$ whose graphs they represent, Fig. 8-3 and 8-4 give no information about the relative phases of voltage or current at each point. Phase information is retained if equation (*8.6*) is written $V(d) = A\cos\beta d$. This expression indicates that in the voltage standing wave pattern produced on a lossless transmission line by a voltage reflection coefficient ρ_T at the terminal load end, the phase is constant over any half wavelength of the pattern between successive points of zero voltage magnitude, but changes by π radians (i.e. changes sign) in the adjacent half wavelengths. Similarly, writing (*8.9*) as $V(d) = Ae^{j\beta d}$ shows that the phase of the voltage in the pattern of Fig. 8-4 changes at a constant rate of β radians per unit length along the entire length of the line.

A direct illustration of how interference produces standing waves is presented in Fig. 8-5. Fig. 8-5(*a*) shows two segments of sine wave patterns, identical in amplitude and wavelength. These represent the localized instantaneous values, along a portion of their common line of travel, of two identical harmonic waves of some physical variable, traveling in opposite directions. At points *a*, *b* and *c* the algebraic sum of the ordinates of the two curves is zero, and at points *a′*, *b′* and *c′* the algebraic sum has maximum magnitude.

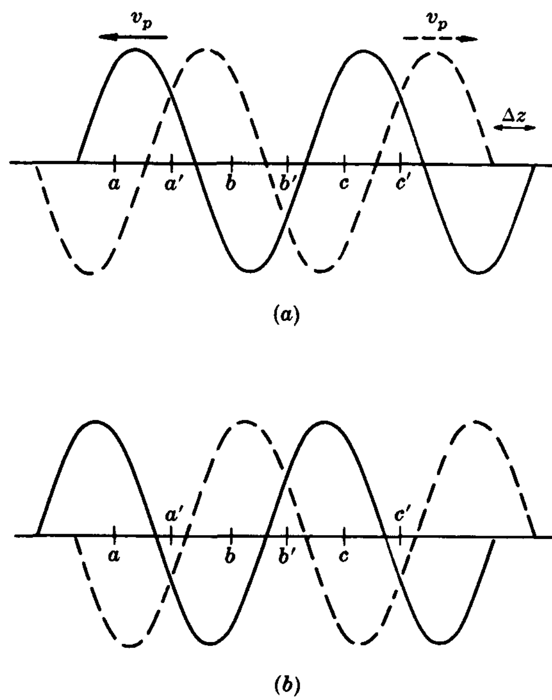


Fig. 8-5. An illustration of the production of standing waves by interference. The two waves of equal wavelength and amplitude are traveling in opposite directions with equal velocities. At the instant shown in (*a*), the waves interfere destructively at points *a*, *b* and *c*. In (*b*) each wave has traveled a distance Δz from its position in (*a*). Destructive interference still occurs at locations *a*, *b* and *c*. In each case constructive interference occurs at *a′*, *b′* and *c′*.

Fig. 8-5(*b*) shows the same two patterns after a short time interval during which the two waves have traveled with equal speeds in opposite directions through distance Δz. The algebraic sum of the ordinates is still zero at points *a*, *b* and *c*, and has maximum magnitude at points *a′*, *b′* and *c′*.

Repeating this observation for several values of Δz establishes that the instantaneous value of the wave variable is always zero at points *a*, *b* and *c*, and oscillates harmonically in time with maximum amplitude at points *a′*, *b′* and *c′*. The total result, as a magnitude pattern, resembles Fig. 8-3.

Fig. 8-3 and 8-4 illustrate two extremes of the possible voltage standing wave patterns on a lossless transmission line. The latter occurs only for $\rho_T = 0$, which requires $Z_T = Z_0$. Fig. 8-3 was obtained from the specific case of $\rho_T = 1 + j0$ (Z_T infinite) but similar patterns consisting of a succession of positive half sine-wave segments are produced by all reflection coefficients of magnitude unity, which result when Z_T is a short circuit, an open circuit, or any value of pure reactance. The locations along the line of the zeros or nodes of the patterns differ for different values of purely reactive Z_T.

Any general value of terminal load impedance $Z_T = R_T + jX_T$, with both R_T and X_T finite, when connected to a uniform lossless transmission line will create a standing wave pattern of voltage (or current) in which the minima are of nonzero amplitude, and the ordinate of the pattern at the load terminals is neither a maximum nor a minimum of the pattern. A typical case is suggested by Fig. 8-6.

The nature of the patterns of $|V(d)|$ and $|I(d)|$ on a uniform transmission line in the completely general case of an arbitrary terminal load impedance Z_T connected to a line of arbitrary attenuation in nepers per wavelength is discussed in Section 8.4.

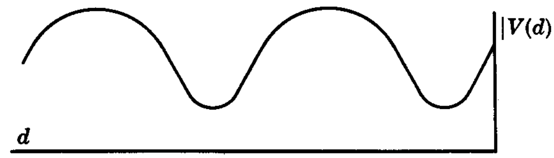


Fig. 8-6. Standing wave pattern on a lossless line, caused by a terminal load impedance that produces a reflection coefficient of magnitude about 0.4.

8.2. The practical importance of standing wave observations.

The observation and measurement of voltage or current standing wave patterns on high frequency transmission systems has become an experimental technique of prime importance for two reasons:

(1) An analytical expression can be derived (Section 8.4) relating the quantitative aspects of a standing wave pattern on a uniform transmission line to the normalized value of the terminal load impedance Z_T/Z_0 connected to the line and the propagation factor $\gamma = \alpha + j\beta$ of the line. Connecting known values of Z_T/Z_0 (open circuit, short circuit, etc.) to the line, the attenuation factor α and phase velocity $v_p = \omega/\beta$ can be found from standing wave pattern measurements. When α and β are known, the normalized value of any unknown terminal load impedance Z_T/Z_0 connected to the line can be calculated from the details of the standing wave pattern it produces. Standing wave observations then provide a simple and precise impedance measuring procedure in a range of high frequencies where impedance bridges and other techniques lack both simplicity and precision.

(2) When the intended function of a transmission line circuit in the form of Fig. 8-1 is to convey power or signals efficiently from a source to a load, the existence of voltage and current standing waves on the line can impair the performance of the circuit in several different ways. Standing wave observations provide convenient and direct data from which to calculate or estimate the magnitude of these various effects. Standing wave measurements are so quickly and easily carried out that they can also be used to monitor a circuit's performance while adjustments are being made to achieve optimum conditions.

The occurrence of standing waves on a transmission line circuit of the form of Fig. 8-1 is synonymous with the presence of reflected waves on the line, the reflected waves being caused by a terminal load impedance that is not equal to the characteristic impedance of the line. On practical high frequency transmission lines, for which the attenuation per wavelength is low and the phase angle of the characteristic impedance is small, standing waves can be responsible for any of the following adverse effects:

(*a*) At the maxima in the voltage standing wave pattern the voltage between the line conductors exceeds the value required for delivering the same amount of power to a nonreflecting load impedance $Z_T = Z_0$. The power capacity of the line, if limited by voltage breakdown or by local heating of the dielectric in the interconductor space, is therefore reduced.

(*b*) At the maxima in the current standing wave pattern, located between the voltage maxima, the current in the line conductors exceeds the value required for delivering the same amount of power to a nonreflecting load impedance. The power capacity of the line, if limited by local heating of the line conductors, is therefore reduced. (For practical high frequency transmission lines the peak pulse power rating is usually determined by voltage breakdown and the continuous power rating by conductor heating.)

(*c*) In the presence of standing waves the losses per wavelength in both the distributed resistance R and the distributed conductance G of the line are greater than they would be if the same amount of power were being delivered to a nonreflecting load.

(*d*) Since the presence of standing waves on a line means that Z_T is not equal to Z_0, it follows from equation (*7.20*), page 131, that the input impedance of a transmission line in the circuit of Fig. 8-1 will vary with frequency (i.e. with line length in wavelengths). The efficiency of power transfer from the source to the input terminals of the line will therefore vary over the operating bandwidth of the system. The effect may be increased if the terminal load impedance Z_T has a reactive component which is itself a function of frequency.

(*e*) If the source impedance $Z_S = Z_0$ in the circuit of Fig. 8-1, then maximum power will be delivered to a terminal load impedance $Z_T = Z_0$, and the presence of standing waves on the line is positive evidence of less than optimum transmission efficiency. However, when the source impedance is not equal to the characteristic impedance of the line, this conclusion does not apply. The general case is discussed more fully in Chapter 9.

8.3. Instrumentation for standing wave measurements.

The most common commercial form of device for making standing wave measurements on transmission lines consists of a section of air-dielectric coaxial line whose conductors are rigid metal tubes two or three feet long, the center conductor being supported by dielectric inserts at the ends of the section. A narrow longitudinal slot is cut in the outer conductor along most of its length. An external carriage is arranged to move mechanically along the length of the section, carrying a small probe-conductor which penetrates slightly through the slot into the interconductor space of the line.

When this "slotted line section" is connected between a source and a terminal load in the manner indicated by Fig. 8-7, the probe receives a small signal-frequency excitation proportional to the magnitude of the voltage or electric field between the line conductors at the probe's location. The received signal is detected and amplified, and its magnitude displayed or recorded as a function of the probe position along the slot. If the detector is linear, the resulting graph is a standing wave pattern of relative r.f. voltage magnitude on the slotted section of the line, of the type shown in Fig. 8-6. Very often the simple crystal or bolometer detectors used with slotted line sections have square-law response at the low signal levels received by the probe and the ordinates of the observed standing wave pattern are proportional to the square of the r.f. voltage magnitude at each point of the line.

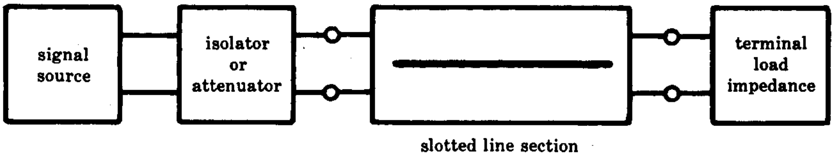


Fig. 8-7. Use of a slotted line section to measure an impedance connected directly to its output terminals. The attenuator or isolator prevents changes in the measuring circuit from affecting the frequency or power output of the signal source.

The radii of the conductors of the slotted line section, dictated mainly by mechanical considerations, are large enough to ensure that the attenuation of the section has no detectable effect on the observed standing wave patterns, in most cases.

For the circuit of Fig. 8-7, the details of the standing wave pattern observed in the slotted section, whether the detector is linear or square law, provide data from which the value of the terminal load impedance Z_T, normalized relative to the characteristic impedance of the slotted section, can be calculated. The geometry of the slotted section is sufficiently precise that its high frequency characteristic impedance can be accurately determined from equation (*6.59*), page 96, using $k_e' = 1$. (The effect of the slot is negligible.) Thus the calculated normalized value of Z_T can be converted to a value in ohms.

When a slotted line section is inserted into a general transmission line circuit like that of Fig. 8-1, either between the source and the input terminals of the transmission line as in Fig. 8-8, or at any intermediate point of the line, it is important that there be no abrupt change in transmission line geometry between the slotted section and the portion of the main line on the terminal load side. Otherwise the uniformity postulate of Chapter 2 will be violated, and phenomena will occur which are not covered by transmission line theory. The reliability of calculations made from standing wave patterns in the slotted line section may then be seriously affected.

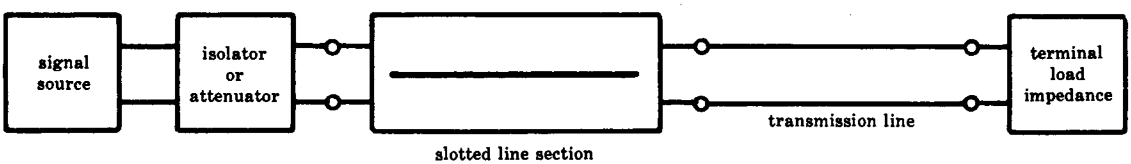


Fig. 8-8. Use of a slotted line section to measure the input impedance of a section of transmission line having terminal load impedance Z_T.

Standard practice is to use a slotted line section whose characteristic impedance is the same as that of the transmission line to which it is connected and, if necessary, to make the transition from the conductor radii of the slotted section to those of the main transmission line by means of a coaxial line section with tapered conductors having a length of at least a few quarter wavelengths. Under these conditions the presence of a standing wave on the slotted section indicates that there are reflected waves and standing waves on the main line. Conversely, adjustment of the terminal load impedance to eliminate the standing wave in the slotted section will mean that the main transmission line is terminated nonreflectively.

The only quantity that can be calculated from a standing wave pattern observed on a slotted line section inserted in a transmission line circuit in the manner described by Fig. 8-8 is the input impedance of the portion of the transmission line on the terminal load side of the slotted section, normalized with respect to the characteristic impedance of the slotted section. To obtain information about the attenuation or phase velocity of the main transmission line, or about the normalized or actual value of the terminal load impedance Z_T, standing wave patterns must be observed using two or three different lengths of the main line, with open circuit and short circuit termination in addition to the Z_T termination.

Although in principle the slotted line technique could be used to observe standing waves on a transmission line at any frequency, the method becomes impractical at frequencies below about one hundred megahertz. Inspection of Fig. 8-6 shows that the full description of a voltage standing wave pattern on a transmission line having negligible attenuation per wavelength requires observation over at least one quarter wavelength of line, including a voltage minimum and an adjacent voltage maximum. For random positioning of the pattern in the slotted section this would require that the slot be not less than one half wavelength long. Mechanical considerations make the construction of a reasonably precise slotted line section difficult for lengths greater than three or four feet. This establishes a lower frequency limit in the hundred megahertz region. The upper frequency limit of a slotted line unit may be determined by any one of several factors, such as connector design, probe circuitry, or propagation of the TE and TM modes discussed in Chapter 3. Typical commercial slotted line sections are useful in the frequency range from a few hundred to a few thousand megahertz. Special units can be obtained for higher and lower frequencies, as can standing wave detectors for transmission lines with noncoaxial configurations, such as parallel wires, striplines, and others illustrated in Fig. 2-2.

At frequencies for which the wavelength is between several feet and several hundred feet, it is often practicable on open types of transmission line to obtain the data for a standing wave pattern by direct measurement of the voltage at intervals along the line itself, using a portable meter. At frequencies below the hundred megahertz region, however, the information available from standing wave patterns is usually obtained more conveniently by other methods.

8.4. The analysis of standing wave patterns.

Since the pattern of the variations of voltage and current magnitude along the line is not directly comprehensible in the general case from the complex number equations (*8.1*) and (*8.2*), the analytical problem is to find expressions for $|V(z)|$ and $|I(z)|$, in terms of the components in the transmission line circuit of Fig. 8-1, whose graphical representation as a standing wave pattern is easily visualized, and which directly relate measurable features of the pattern to desired circuit information.

The analysis will be carried out in full for the standing wave pattern of voltage magnitude on a line. A minor modification makes the same analysis applicable to the standing wave pattern of current magnitude. The latter is little used in practice, because its observation requires a coupling *loop* in the slot of the slotted line section, which is mechanically less convenient than a coupling *probe*.

Equation (*8.4*) is a general expression for the harmonic voltage $V(d)$ at any coordinate d on a uniform transmission line. The line has length l, characteristic impedance Z_0, propagation factor $\gamma = \alpha + j\beta$, and is terminated in an impedance Z_T that produces a voltage reflection coefficient ρ_T at the terminal load end of the line in accordance with (*8.3*). A voltage standing wave pattern, being a graph of $|V(d)|$ as a function of d, has a shape which is determined entirely by the terms γd and ρ_T, i.e. by the normalized value of the terminal load impedance, and by the total attenuation and total phase shift of the length of line between the terminal load impedance and the point of observation of the pattern. The shape of the pattern is not affected by the strength or internal impedance of the signal source connected to the line, nor by the length or properties of the line between the source and the point of observation. As noted previously, the term $|V_1 e^{-\gamma l}|$ is a scale factor for the pattern but does not influence its shape. The scale factor is a function of all the components of the total transmission line circuit.

To obtain from equation (*8.4*) a graphable expression for $|V(d)|$, the voltage reflection coefficient ρ_T must be expressed as an exponential number,

$$\rho_T = |\rho_T|\, e^{j\phi_T} = e^{-2(p+jq)} \tag{8.11}$$

from which

$$p = \log_e \frac{1}{\sqrt{|\rho_T|}} \quad \text{and} \quad q = -\tfrac{1}{2}\phi_T \tag{8.12}$$

Substituting from (*8.12*) into (*8.4*) and taking out a term $e^{-(p+jq)} = \sqrt{\rho_T}$,

$$V(d) = V_1 e^{-\gamma l} \sqrt{\rho_T}\,(e^{+(p+jq)}\, e^{+\gamma d} + e^{-(p+jq)}\, e^{-\gamma d}) \tag{8.13}$$

Finally, using $\gamma = \alpha + j\beta$ and regrouping exponents,

$$\begin{aligned} V(d) &= V_1 e^{-\gamma l}\sqrt{\rho_T}\{e^{+(\alpha d+p)+j(\beta d+q)} + e^{-(\alpha d+p)-j(\beta d+q)}\} \\ &= 2V_1 e^{-\gamma l}\sqrt{\rho_T}\cosh\{(\alpha d+p)+j(\beta d+q)\} \end{aligned} \tag{8.14}$$

The complex hyperbolic cosine can be expanded using the identity

$$\cosh(x+jy) = \cosh x\cos y + j\sinh x\sin y$$

from which there follows the magnitude relation

$$\begin{aligned} |\cosh(x+jy)| &= \{\cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y\}^{1/2} \\ &= \{(\sinh^2 x + 1)\cos^2 y + \sinh^2 x\sin^2 y\}^{1/2} \\ &= \{\sinh^2 x + \cos^2 y\}^{1/2} \end{aligned}$$

The phase angle of $\cosh(x+jy)$ is $\tan^{-1}\left(\dfrac{\sinh x\sin y}{\cosh x\cos y}\right) = \tan^{-1}(\tanh x\tan y)$.

Applying these results to equation (*8.14*), the magnitude of the phasor voltage on a transmission line as a function of distance from the terminal load end of the line is

$$|V(d)| = |2V_1 e^{-\gamma l}\sqrt{\rho_T}|\{\sinh^2(\alpha d+p) + \cos^2(\beta d+q)\}^{1/2} \tag{8.15}$$

The variation of the phase angle ξ of the phasor voltage as a function of the coordinate d is given by

$$\xi(d) = \tan^{-1}\{\tanh(\alpha d+p)\tan(\beta d+q)\} \tag{8.16}$$

Equation (*8.15*) is a graphable expression for the voltage standing wave pattern on a transmission line for the perfectly general case of the transmission line circuit of Fig. 8-1.

Phase variation of the voltage along a transmission line when there are standing waves present can be evaluated and plotted from (*8.16*), although the shape of the graph is not easily visualized directly from this equation. The usual processes for making transmission line circuit calculations from standing wave measurements make no reference to the relative voltage phase $\xi(d)$, but the phase information is important in many applications, such as the feeding of the component antennas of an array from a common transmission line on which there are standing waves.

It is worth noting that no approximations have been used in deriving (*8.15*) and (*8.16*) from the general equation (*8.1*), and these equations are therefore valid for all values of α, β and ρ_T.

In drawing a graph of a voltage standing wave pattern from (*8.15*), no significance attaches to the value or the functional form of the scale factor term $|2V_1 e^{-\gamma l}\sqrt{\rho_T}|$. It can therefore be replaced by any arbitrary real value of phasor voltage. Notation is simplified by letting this value be unity. Then

$$|V(d)| = \{\sinh^2(\alpha d+p) + \cos^2(\beta d+q)\}^{1/2} \tag{8.17}$$

8.5. Standing wave patterns on lossless lines.

When $\alpha = 0$, equation (*8.17*) becomes

$$|V(d)| = \{\sinh^2 p + \cos^2(\beta d+q)\}^{1/2} \tag{8.18}$$

Since $\sinh^2 p$ is a constant for fixed values of Z_T and Z_0, the standing wave pattern for $|V(d)|^2$ is particularly easy to draw, being simply a cosine squared pattern with all ordinates offset a constant amount from the zero axis. The term $\cos^2(\beta d+q)$ oscillates in magnitude between zero and unity, as a function of d. The offset term $\sinh^2 p$ is a function of $|\rho_T|$, according to equation (*8.12*). The voltage maxima, which occur when $\cos^2(\beta d+q) = 1$, are all of the same magnitude, as are the voltage minima, which occur when $\cos^2(\beta d+q) = 0$.

Using the symbol $d_{V(\min)}$ for the d-coordinates of the voltage minima, and noting from (*8.12*) that $q = -\frac{1}{2}\phi_T$, the relation $\cos^2(\beta d_{V(\min)} + q) = 0$ leads to

$$d_{V(\min)}/\lambda = \tfrac{1}{4}(1 + \phi_T/\pi) \pm \tfrac{1}{2}n \qquad n = 0, 1, 2, \ldots \tag{8.19}$$

where λ is the wavelength on the line.

The true voltage standing wave pattern, a graph of $|V(d)|$ as a function of d, can be obtained only by plotting the square root of the ordinates of the pattern of $|V(d)|^2$ at a sufficient number of values of d. No simple functional expression for the curve is available. The general nature of the result, already illustrated in Fig. 8-6, is that the minima become sharper and the maxima become broader than in the curve for $|V(d)|^2$. This is the reason for establishing the reference to minimum voltage points of the pattern in (*8.19*) rather than maximum voltage points. If a voltage standing wave pattern is observed with a square-law detector, the actual standing wave pattern obtained will be that of $|V(d)|^2$ and there is no difference in sharpness between the maxima and minima.

The specification of three quantities is sufficient to ensure that the shape of a voltage standing wave pattern on a lossless transmission line can be drawn in full using equation (*8.17*). These are:

(*a*) the ratio of the maximum voltage magnitude in the pattern to the minimum voltage magnitude;
(*b*) the location of any voltage minimum relative to the origin of coordinates $d = 0$;
(*c*) the distance between successive voltage minima.

Conversely, these are three quantities which can be directly measured from the data of any voltage standing wave pattern.

Normally the frequency of the source will be a known quantity. From (*8.19*) the distance required in item (*c*) is one half wavelength of the waves in the air dielectric slotted line section at the frequency of the source, which is given by $\lambda/2 = c/2f$, where f is the frequency of the source in hertz, $c = 3.00 \times 10^8$ m/sec and λ is in meters. If the frequency of the source is not known, it can be determined from this same equation, using the measured distance between successive voltage minima.

Item (*a*) suggests the desirability of defining a quantity known as the "voltage standing wave ratio" of a voltage standing wave pattern. This is designated by the notation VSWR in most transmission line writing, and is defined by

$$\text{VSWR} = \frac{|V(d)|_{\max}}{|V(d)|_{\min}} \tag{8.20}$$

where $|V(d)|_{\max}$ is the voltage magnitude at a maximum in the voltage standing wave pattern, and $|V(d)|_{\min}$ is the voltage magnitude at a minimum.

Referring to (*8.14*) it is apparent that

$$\text{VSWR} = \left\{\frac{\sinh^2 p + 1}{\sinh^2 p}\right\}^{1/2} = \left\{\frac{\cosh^2 p}{\sinh^2 p}\right\}^{1/2} = \coth p = \frac{1 + e^{-2p}}{1 - e^{-2p}}$$

or

$$\text{VSWR} = \frac{1 + |\rho_T|}{1 - |\rho_T|} \tag{8.21}$$

using equation (*8.12*). Since $|\rho_T|$ is the ratio of the magnitude of the reflected wave to the magnitude of the incident wave on the lossless line, this result shows that the maximum voltage magnitude in the standing wave patterns is the sum of the magnitudes of the incident and reflected waves, while the minimum voltage magnitude in the standing wave pattern is the difference of the same two magnitudes.

Equations (*8.19*) and (*8.21*) present the important and valuable facts that the VSWR of a voltage standing wave pattern on a lossless transmission line is a function only of the *magnitude* of the voltage reflection coefficient at the reference location $d = 0$, while the locations of the voltage minima in the pattern, expressed in wavelengths, are functions only of the phase angle of the same reflection coefficient.

Example 8.1.

An air dielectric transmission line with low attenuation per wavelength at its operating frequency of 400 megahertz has a characteristic impedance of $50 + j0$ ohms and is terminated in an impedance of $20 - j80$ ohms. Describe the voltage standing wave pattern on the line.

The voltage reflection coefficient produced by the terminal load impedance is

$$\rho_T = \frac{Z_T/Z_0 - 1}{Z_T/Z_0 + 1} = \frac{0.4 - j1.6 - 1}{0.4 - j1.6 + 1} = 0.80\underline{/5.20 \text{ rad}}$$

Then $\text{VSWR} = \dfrac{1 + |\rho_T|}{1 - |\rho_T|} = \dfrac{1.80}{0.20} = 9.0$, and

$$d_{V(\min)}/\lambda = \tfrac{1}{4}(1 + \phi_T/\pi) \pm \tfrac{1}{2}n = \tfrac{1}{4}(1 + 5.20/\pi) \pm \tfrac{1}{2}n = 0.664 - 0.50 = 0.164$$

The wavelength on the air dielectric line is $\lambda = c/f = (3.00 \times 10^8)/(400 \times 10^6) = 0.75$ m, so that $d_{V(\min)} = 0.164 \times 0.75 = 0.123$ m. The pattern can be sketched from the values of VSWR and $d_{V(\min)}/\lambda$.

Equations (*8.19*) and (*8.21*) can be solved for $|\rho_T|$ and ϕ_T:

$$|\rho_T| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \tag{8.22}$$

$$\phi_T = 4\pi d_{V(\min)}/\lambda - \pi \pm 2n\pi \tag{8.23}$$

Using equations (*8.22*) and (*8.23*) together with (*7.9a*) and (*7.9b*), page 129, the components of the normalized impedance terminating a lossless transmission line can be calculated from the numerical data of the voltage standing wave pattern.

Example 8.2.

On a lossless transmission line having a characteristic impedance of $73 + j0$ ohms, there is a VSWR of 3.50. The distance between successive minima of the pattern is 45.5 cm, and there is a voltage minimum 36.8 cm from the load terminals. Determine the value of the terminal load impedance connected to the line.

Since the distance between successive minima of any standing wave pattern on a lossless line is one half wavelength, $\lambda = 2 \times 45.5 = 91.0$ cm and $d_{V(\min)}/\lambda = 36.8/91.0 = 0.404$. From (*8.22*), $|\rho_T| = (3.50 - 1)/(3.50 + 1) = 0.556$, and from (*8.23*), $\phi_T = 4\pi \times 0.404 - \pi \pm 2n\pi = 1.940$ rad. For use in equations (*7.9a*) and (*7.9b*), $\cos\phi_T = -0.360$ and $\sin\phi_T = 0.933$. Then

$$R_T/Z_0 = \frac{1 - |\rho_T|^2}{1 + |\rho_T|^2 - 2|\rho_T|\cos\phi_T} = \frac{1 - 0.308}{1 + 0.308 - 2(0.556 \times 0.360)} = 0.405$$

and $R_T = 0.405 \times 73 = 29.6$ ohms. Similarly,

$$X_T/Z_0 = \frac{2|\rho_T|\sin\phi_T}{1 + |\rho_T|^2 - 2|\rho_T|\cos\phi_T} = \frac{2(0.556 \times 0.933)}{1.708} = 0.608$$

and $X_T = 0.608 \times 73 = 44.4$ ohms.

Table 8.1 below summarizes the details of the voltage standing wave patterns that are produced on lossless transmission lines by several easily designated values of normalized terminal load impedance.

The table shows that a purely resistive terminal load, of normalized value $n + j0$, produces a VSWR of n with a voltage *maximum* at the load if $n > 1$, and produces a VSWR of $1/n$ with a voltage *minimum* at the load if $n < 1$.

The table suggests, and this is demonstrated graphically for the general case in Chapter 9, that $d_{V(\min)}/\lambda$ lies between 0 and $\frac{1}{4}$ for all capacitively reactive terminations, and between $\frac{1}{4}$ and $\frac{1}{2}$ for all inductively reactive terminations.

The table confirms a statement made previously, that open circuit, short circuit, and all purely reactive terminations produce patterns for which the VSWR is infinite, i.e. patterns for which the minima are zero. It has been seen that such patterns consist of consecutive positive half sine-wave segments.

Table 8.1

Z_T/Z_0	ρ_T equation (*8.3*)	$\|\rho_T\|$	ϕ_T	VSWR equation (*8.21*)	$d_{V(\min)}/\lambda$ equation (*8.19*)
0	$-1 + j0$	1	π	infinite	0, 1/2
infinite	$1 + j0$	1	0	infinite	1/4
$1 + j0$	0	0	none	1	none
$0 + j1$	$0 + j1$	1	$\pi/2$	infinite	3/8
$0 - j1$	$0 - j1$	1	$-\pi/2$	infinite	1/8
$n + j0 \ (n > 1)$	$\dfrac{n-1}{n+1} + j0$	$\dfrac{n-1}{n+1}$	0	n	1/4
$n + j0 \ (n < 1)$	$\dfrac{n-1}{n+1} + j0$	$\dfrac{1-n}{1+n}$	π	$1/n$	0, 1/2

8.6. Standing wave patterns on transmission lines with attenuation.

In a voltage standing wave pattern on a lossless transmission line, the variations of the voltage magnitude are confined between constant upper and lower limits along the line, shown by (*8.18*) to be proportional respectively to $\cosh p \ (= \frac{1}{2}[1/\sqrt{|\rho_T|} + \sqrt{|\rho_T|}\,]$ from equation (*8.12*)), and $\sinh p \ (= \frac{1}{2}[1/\sqrt{|\rho_T|} - \sqrt{|\rho_T|}\,])$. The ratio of the two limits is everywhere $\coth p = \dfrac{1 + |\rho_T|}{1 - |\rho_T|} = \text{VSWR}$, a constant.

If a line has attenuation factor α, it is seen from (*8.17*) that the corresponding upper and lower limits or envelopes of the standing wave pattern are proportional to $\cosh(\alpha d + p)$ and $\sinh(\alpha d + p)$ respectively. The ratio of the limits at any value of d is

$$\coth(\alpha d + p) \quad = \quad \frac{1 + e^{-2(\alpha d + p)}}{1 - e^{-2(\alpha d + p)}} \quad = \quad \frac{1 + |\rho_T|\, e^{-2\alpha d}}{1 - |\rho_T|\, e^{-2\alpha d}}$$

Since all maxima and minima in any standing wave pattern occur at different values of d, it appears from this equation that in a pattern on a line with attenuation all the maxima and minima will each have different magnitude. This suggests that the VSWR concept and its uses might not be applicable. Quantitatively it is found that for the important case of lines with low attenuation per wavelength $(\alpha/\beta \ll 1)$, but any value of total attenuation αl, terminated in impedances that produce reflection coefficient magnitudes not too close to unity ($|\rho_T| < 0.5$ is a conservative criterion), the ratio of a maximum voltage magnitude in the standing wave pattern to an *adjacent* minimum voltage magnitude one quarter wavelength away gives a VSWR value from which the line impedance $Z(d)$ at any point *in that quarter wavelength of the line* can be calculated with useful accuracy, using the value of $d_{V(\min)}/\lambda$ appropriate to the point d in the interval at which $Z(d)$ is evaluated.

Fig. 8-9 is a voltage standing wave pattern calculated from equation (*8.17*) for a transmission line with an attenuation of approximately one neper/wavelength $(\alpha/\beta = 1/6$

exactly), a case which is far from qualifying as "low attenuation per wavelength". Reference to Table 5.1, page 55, shows that this value of α/β could occur for a 19 gauge cable pair at a frequency somewhat higher than 30 kilohertz, and that the three half wavelengths of Fig. 8-9 would cover about 5 miles of the cable pair. On the other hand, for the standard 7/8″ rigid 50 ohm copper coaxial line of Problem 5.6, page 64, at a frequency of 100 megahertz, the attenuation factor $\alpha = 1.60 \times 10^{-3}$ nepers/m, while the phase factor $\beta = \omega/v_p = 2.09$ rad/m and $\alpha/\beta = 7.7 \times 10^{-4}$. Thus a standing wave pattern like Fig. 8-9 will normally occur only under conditions where it would not be usual practice to make standing wave measurements, and a comparable pattern could occur on a high frequency slotted line section only if the line conductors were made of some extraordinary material such as intrinsic germanium, sintered ferrite, or carbon dispersed in a nonconducting binder. For an ordinary metal conductor line with low dielectric losses the attenuation factor α increases as the square root of the frequency at high frequencies, as shown in Chapter 6, and it is evident that $\beta = \omega/v_p$ increases as the first power of the frequency, since v_p is quite accurately constant for such a line. Thus the quantity

$$\alpha/\beta \;=\; (1/2\pi)(\text{line attenuation in nepers/wavelength})$$

usually varies approximately inversely as the square root of the frequency in the frequency range from a few megahertz to several gigahertz, for transmission lines of reasonable construction.

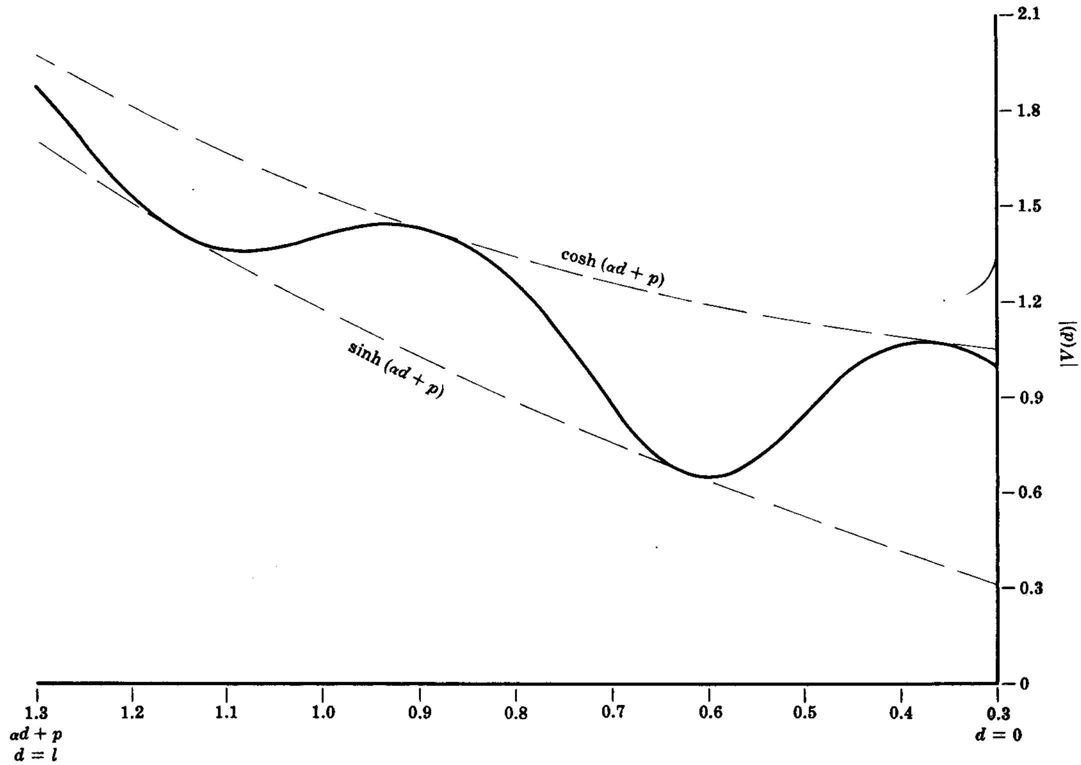


Fig. 8-9. Voltage standing wave pattern on a transmission line having a substantial attenuation per wavelength. The magnitude of the reflection coefficient produced by the terminal load impedance is about 0.5, and the phase angle of the reflection coefficient is about $\pi/4$ radians.

Although Fig. 8-9 is therefore not a realistic illustration of a possible voltage standing wave pattern on a high frequency transmission line, its exaggerations help to emphasize the important features of all standing wave patterns on lines having attenuation. The pattern has four parameters which, graphically, are entirely independent. These are:

(a) The total attenuation αl covered by the range of the pattern. Since the median ordinate of the two envelopes is $\frac{1}{2}[\cosh(\alpha d + p) + \sinh(\alpha d + p)] = (\frac{1}{2}e^p)e^{\alpha d}$, and $e^p = 1/\sqrt{|\rho_T|}$ is a constant, the ratio of the median ordinate at $d = l$ to the median ordinate at $d = 0$ is $e^{\alpha l}$, and the total attenuation can be calculated from appropriate measurements on the envelopes of any pattern.

(b) The total length of the pattern in wavelengths, $\beta l/2\pi$. One half wavelength on the line is the distance between consecutive points of contact of the pattern with *one* of the envelopes.

(c) The value of p, which determines the ordinates of the envelope curves $\sinh(\alpha d + p)$ and $\cosh(\alpha d + p)$ at $d = 0$.

(d) The value of q, which determines the phase at $d = 0$ of the oscillating portion of the standing wave pattern deriving from the term $\cos^2(\beta d + q)$ in equation (*8.17*).

Because of the independence of these four features, it is not possible to draw a "universal" standing wave pattern, applicable to all transmission line circuits through simple scale changes. If, however, the functions $\sinh x$ and $\cosh x$ are plotted against x, on a linear scale of x increasing from 0 at the right to about 3 or 4 at the left, the following procedure shows that all standing wave patterns of practical interest that might occur on any uniform transmission line in the circuit of Fig. 8-1 lie between the two curves as envelopes, and extend over some interval of the x scale determined by the quantitative details of the circuit.

The right hand limit x_1 of the interval on the x scale occupied by the standing wave pattern occurring on a specific transmission line circuit is given by $x_1 = p = \log_e(1/\sqrt{|\rho_T|})$. This point then corresponds to $d = 0$ on the transmission line, where the connected terminal load impedance Z_T produces the reflection coefficient ρ_T. The left hand limit x_2 of the interval occupied on the x scale is given by $x_2 = \alpha l + p$, where αl is the total line attenuation in nepers. This point then corresponds to $d = l$ on the transmission line, where the source is connected. Thus the interval x_1 to x_2 on the x scale is converted to represent the length $d = 0$ to $d = l$ on the transmission line, by the use of a scale factor and a shift of the origin.

At $d = 0$, the ratio of the magnitude of the reflected voltage wave to the magnitude of the incident voltage wave is $|\rho_T|$, by definition. At any other value of d, on a line with attenuation, the incident wave is larger in magnitude by the factor $e^{\alpha d}$, and the reflected wave is smaller by the factor $e^{-\alpha d}$. Hence at a general point the ratio of the magnitude of the reflected voltage wave to that of the incident voltage wave is $|\rho_T|\,e^{-2\alpha d} = e^{-2(\alpha d + p)}$. When $(\alpha d + p) = 3$, $e^{-2(\alpha d + p)} = 0.0025$. The reflected wave magnitude is then only $\frac{1}{4}\%$ of the incident wave magnitude, the fluctuations in the standing wave pattern are about $\frac{1}{2}\%$ of its average amplitude, and the undesirable consequences of standing waves listed in Section 8.2 are present to only a trivial extent. This is a major reason why the x scale in the procedure described above need not be extended beyond $x = 3$ or 4. A second reason is that most commercial slotted line sections will not measure VSWR values less than about 1.01.

Having established that any specific standing wave pattern will lie between the curves for $\sinh x$ and $\cosh x$, in an interval of x from x_1 to x_2, and having converted the interval between x_1 and x_2 to correspond to the d coordinates of the transmission line, the final step is to fit into this bounded space the fluctuating component of the pattern. This will resemble a damped sinusoidal oscillation curve, diminishing in amplitude from right to left, making contact with the $\sinh x$ lower envelope at values of d given by equation (*8.19*) and with the $\cosh x$ upper envelope at values of d midway between the lower contact points. It can be seen from Fig. 8-9 that when the attenuation per wavelength is large, the locations of the points of contact with the lower envelope, given correctly by equation (*8.19*), are not in fact points of minimum voltage magnitude in the pattern, and the symbol $d_{V(\min)}$ is inappropriate. For practical high frequency lines the error introduced by this discrepancy is not important.

For an idealized lossless transmission line having $\alpha = 0$ identically, the interval x_1 to x_2 in the procedure described becomes a point of magnitude zero. The step of converting it to a d coordinate scale introduces the indeterminacy 0/0. The implication is simply the fact presented in Section 8.4 and Fig. 8-6, that the upper and lower envelopes of the standing wave patterns on such a transmission line are straight lines parallel to the d axis.

For the same transmission line circuit whose voltage standing wave pattern is given by Fig. 8-9, the current standing wave pattern as determined in Problem 8.1, page 178, is shown superimposed on the voltage pattern in Fig. 8-10. The points of contact of the current pattern with either envelope lie midway between the points of contact of the voltage pattern. If the attenuation does not exceed a few tenths of a decibel per wavelength, the maxima and minima of the current pattern will occur at the same values of d as the minima and maxima respectively of the voltage pattern, within the precision of experimental measurements. However, when the attenuation per wavelength is as large as in Fig. 8-9 and 8-10, there is an obvious difference in the locations of corresponding maxima and minima of the two patterns.

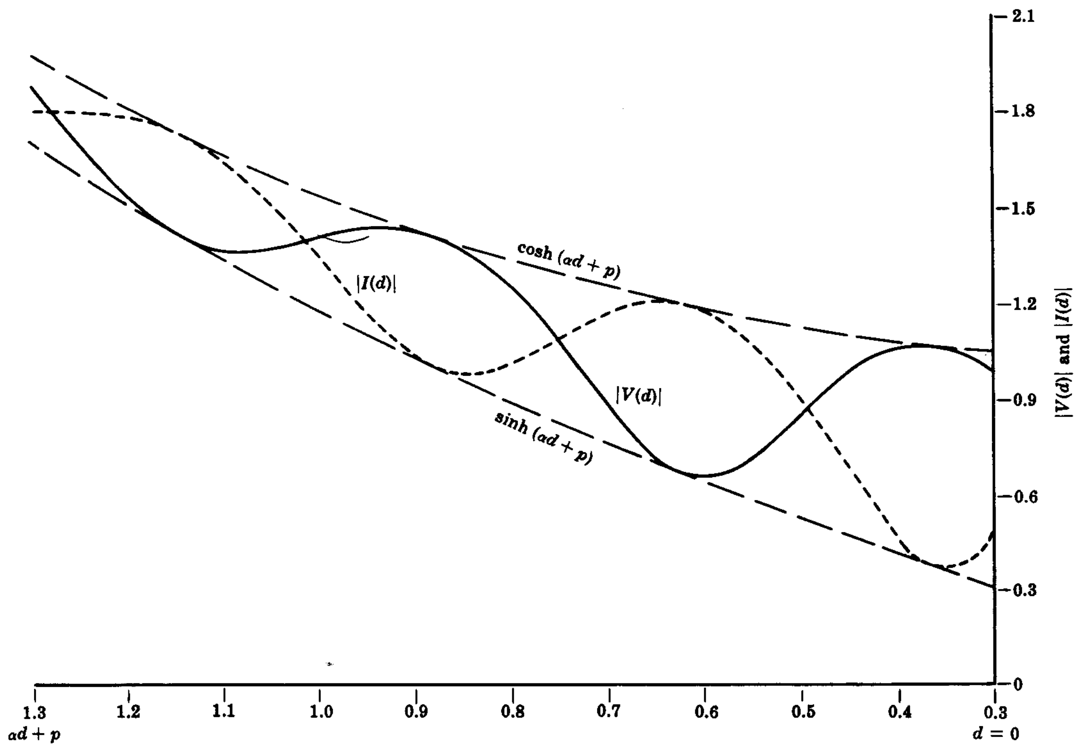


Fig. 8-10. Same as Fig. 8-9 with the addition of the current standing wave pattern.

Fig. 8-11 below is a graph of $|Z(d)| = |V(d)|/|I(d)|$ obtained from Fig. 8-10. It shows that the fluctuations of the normalized impedance magnitude $|Z(d)/Z_0|$ along the line, corresponding to the voltage and current standing wave patterns of Fig. 8-10, are confined between the envelopes $\coth(\alpha d + p)$ and $\tanh(\alpha d + p)$. As noted previously, $Z(d)$ is the input impedance of the portion of the transmission line circuit on the terminal load side of the coordinate d. When $(\alpha d + p) = 3$, the value of $|Z(d)/Z_0|$ fluctuates only $\frac{1}{2}\%$ on either side of unity. This means that the input impedance of any combination of transmission line section and terminal load impedance for which $(\alpha d + p) \geqq 3$, deviates less than $\frac{1}{2}\%$ from the characteristic impedance of the line. If $(\alpha d + p) \geqq 4$, the deviation is less than 0.1%.

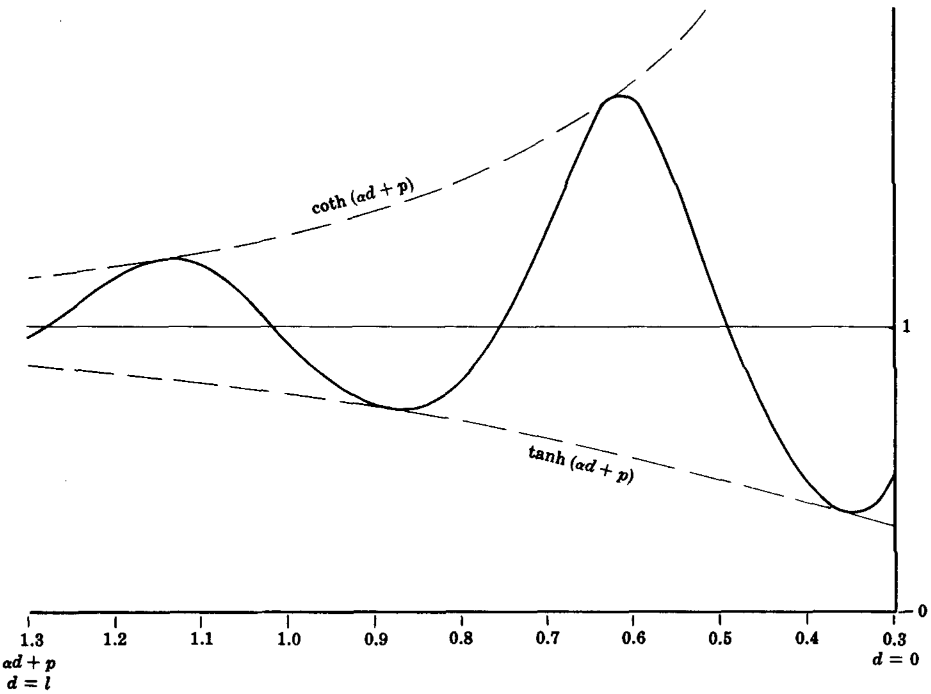


Fig. 8-11. The normalized impedance magnitude as a function of position along the line for the standing wave pattern of Fig. 8-10.

The relative phase of the voltage in the standing wave pattern of Fig. 8-9, calculated from (*8.16*), is shown in Fig. 8-12. In the absence of reflected waves the relative phase would increase linearly at the rate of 2π rad/wavelength from $d = 0$ to $d = l$.

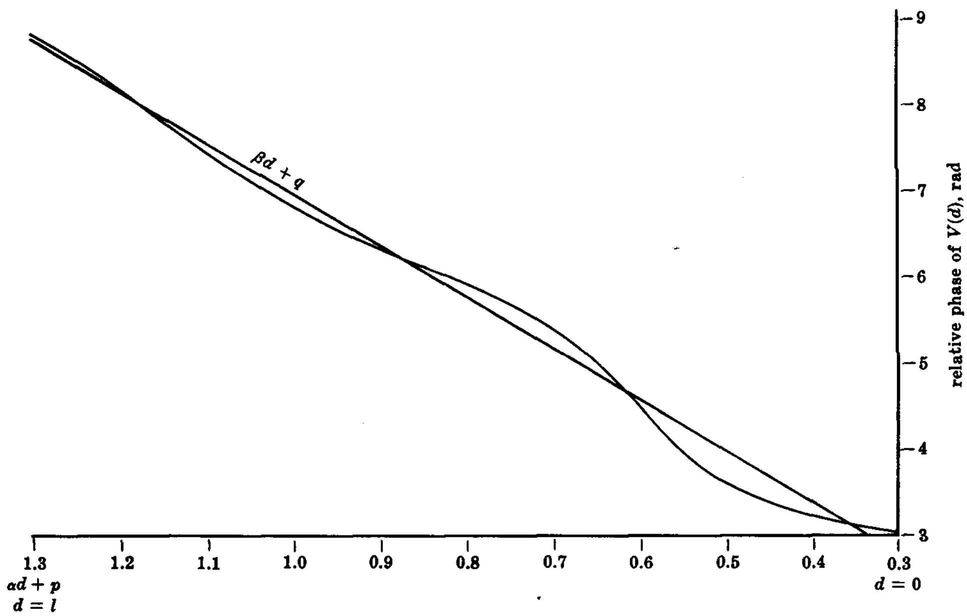


Fig. 8-12. The relative phase of the voltage as a function of position along the line for the standing wave pattern of Fig. 8-9.

8.7. The problem of measuring high VSWR values.

On transmission line circuits intended for the efficient transfer of power from a source to a load, the occurrence of VSWR values as high as 5 or 10 would be virtually inconceivable, but there are transmission line measurement techniques at high frequencies that involve observing VSWR values of 100 or higher.

Accuracy in experimental VSWR values requires that both the minimum and the maximum voltage magnitudes in a standing wave pattern be observed with the same adequate precision. This in turn requires that the minimum voltage magnitude must be far above the threshold sensitivity or noise level of the voltage-indicating circuitry. Unless there is provision for insertion of a calibrated high frequency attenuator between the pickup probe and the detector of a slotted line section, which is seldom the case in standard commercial units, the ratio of two output-meter readings gives a VSWR value accurately only if the assumed response law of the detector is valid over the range of signal levels in the pattern. For the sensitive crystal and bolometer detectors generally used with slotted line sections, the ideal small-signal square law response is unlikely to hold at the voltage maxima of a pattern with high VSWR.

A second hazard in the measurement of high VSWR values arises from the fact that the voltage magnitudes at the minima are very small. The attainment of sufficient sensitivity to measure such low signal levels accurately may require increasing the penetration of the pickup probe into the slot of the slotted section. This introduces a localized shunt admittance across the line at the location of the probe. At a voltage minimum the line admittance $Y(d)$ is high and the effect of a small probe admittance is likely to be negligible, but at a voltage maximum the line admittance is very low and the probe admittance may alter the circuit sufficiently to change the voltage from its correct value.

Both of the difficulties described are avoided if high VSWR values are measured by an indirect method often referred to as the "double minimum" technique. It is exactly analogous to the familiar procedure of measuring the Q of a resonant circuit by observing the width of a resonance curve between the points on either side of resonance at which the observed power level differs by 3 db from the power level at resonance.

A high value of VSWR can exist on the transmission line circuit of Fig. 8-1 only if the magnitude of the reflection coefficient produced by the terminal load impedance is very near unity, and only along that portion of the line, adjacent to the load terminals, for which the total attenuation αd does not exceed a very small fraction of one decibel. This is equivalent to saying that a high value of VSWR is synonymous with a value of $(\alpha d + p)$ very small compared to unity. For such values of $(\alpha d + p)$, the approximations $\cosh(\alpha d + p) = 1$ and $\sinh(\alpha d + p) = (\alpha d + p)$ are accurate to better than $\frac{1}{2}\%$ for $(\alpha d + p) \leq 0.1$, which covers all VSWR values greater than 10. The "double minimum" method usually has no particular advantages for VSWR values below 10.

Introducing these approximations into equation (*8.17*), the maxima of the voltage standing wave pattern occurring on that portion of a transmission line circuit for which $(\alpha d + p) \leq 1$ will all have the same relative magnitude of unity at locations where $\cos^2(\beta d + q) = 1$, and the voltage minima on the same scale will have magnitudes of $(\alpha d + p)$ at the values of d for which $\cos^2(\beta d + q) = 0$. There is therefore no unique value of VSWR on the line, but the relation

$$\text{VSWR} = 1/(\alpha d_{V(\min)} + p) \tag{8.24}$$

gives the VSWR accurately at each voltage minimum in the pattern.

On each side of any voltage minimum in the pattern, the voltage magnitude will increase rapidly as the term $\cos^2(\beta d + q)$ increases from zero. At $d = d_{V(\min)} \pm \frac{1}{2}\Delta d$ such that $|\cos[\beta(d_{V(\min)} \pm \frac{1}{2}\Delta d) + q]| = (\alpha d + p)$, the voltage magnitude will have increased by a factor $\sqrt{2}$ above the minimum value. For the case $(\frac{1}{2}\beta\,\Delta d) \ll 1$, a Taylor series expansion produces the result $\frac{1}{2}\beta\,\Delta d = (\alpha d + p)$ on dropping second order small quantities, and from equation (*8.24*) it follows that

$$\text{VSWR} = 2/(\beta\,\Delta d) \qquad (8.25)$$

In equation (*8.25*), Δd is evidently the width of the minimum in the standing wave pattern between points where the voltage magnitude is greater than the adjacent minimum voltage by the factor $\sqrt{2}$. The name "double minimum" method arises from the fact that with the square law detectors usually used in standing wave observations, the output meter reading at these same points is double the reading at the voltage minimum.

The advantages of this method over direct measurement of the maximum and minimum voltages in a standing wave pattern are that the ideal law of the detector must be maintained over a voltage range of only 1.4 : 1, instead of VSWR : 1, and all the observations are made in a region of the line where $Y(d)$ is large, so that any effects of the shunt admittance of the pickup probe are minimized.

In deriving equation (*8.25*) a term $\alpha\,\Delta d$, representing the attenuation of the line across the length increment Δd, was ignored. This is always justified, partly because $\alpha/\beta \ll 1$ for the conditions postulated, and partly because the effects of line attenuation on the respective locations of the observed points on the two sides of any voltage minimum are of opposite sign, and cancel one another to a first approximation.

Example 8.3.

A slotted section of coaxial transmission line with copper conductors has a characteristic impedance of 50 ohms at high frequencies. The inner diameter of the outer conductor is 0.96 cm. The section is terminated in a short circuit consisting of a plane transverse copper disc making firm contact with the peripheries of the facing surfaces of the two conductors. Assuming no dielectric in the line, (*a*) locate the voltage minima in the slotted section at a frequency of 800 megahertz, (*b*) find the widths (at $\sqrt{2}$ times minimum amplitude in each case) of the three voltage minima nearest the short circuit plane, and (*c*) find the VSWR values at the locations of the minima.

The resistance between concentric circular contacts of facing radii a and b $(b > a)$ on a resistive sheet of surface resistivity R_s ohms/square is $R_T = (R_s/2\pi)\log_e(b/a)$ ohms. (See Problem 6.11, page 121.) For a copper plane under skin effect conditions, $R_s = 2.61 \times 10^{-7}\sqrt{f}$ ohms/square, where f is in hertz. The characteristic impedance of an air-dielectric coaxial transmission line is $Z_0 = 60\log_e(b/a)$. Substituting $Z_0 = 50$ gives $\log_e(b/a) = 0.833$. Hence the resistive component of the terminal load impedance presented by the short circuiting plane to the coaxial transmission line at the frequency of 800 megahertz is

$$R_T = (2.61 \times 10^{-7}\sqrt{8 \times 10^8}) \times 0.833/(2\pi) = 9.8 \times 10^{-4} \text{ ohms}$$

From equation (*6.45*), page 85, the inductive reactance component X_T of the terminal load impedance has the same value.

To determine the required widths of the standing wave pattern at the voltage minima, it is necessary to evaluate the attenuation factor α and the quantity $p = \log_e(1/\sqrt{|\rho_T|})$. The attenuation factor of the line is given by equations (*5.9*) and (*6.49*) as

$$\alpha = \frac{(R_s/2\pi b)(1 + b/a)}{2Z_0} = \frac{[(7.38 \times 10^{-3})/(2\pi \times 0.0096)] \times 3.30}{100} = 4.03 \times 10^{-3} \text{ nepers/m}$$

The reflection coefficient magnitude is

$$|\rho_T| = \left|\frac{Z_T/Z_0 - 1}{Z_T/Z_0 + 1}\right| = \left|\frac{(9.8 \times 10^{-4})(1 + j)/50 - 1}{(9.8 \times 10^{-4})(1 + j)/50 + 1}\right| = 1 - 3.92 \times 10^{-5}$$

and $p = \log_e(1 + 1.96 \times 10^{-5}) = 1.96 \times 10^{-5}$, with high accuracy. The imaginary part of Z_T has no measurable effect on $|\rho_T|$ or on the departure of the phase angle of ρ_T from π radians.

With $\phi_T = \pi$, the minima in the standing wave pattern are located at precise half wavelength intervals from the plane of the short circuit, according to equation (*8.19*). The values of d at the first three minima are therefore respectively 18.75, 37.50 and 56.25 cm on the air dielectric line at a frequency of 800 megahertz. The corresponding values of $(\alpha d + p)$ are 7.75×10^{-4}, 1.53×10^{-3} and 2.29×10^{-3} at the three minima, and from (*8.24*) the VSWR values obtained as the ratio of any of the equal voltage magnitudes at the maxima along the slotted section to the voltage magnitudes at the minima are about 1290, 650 and 436, the highest value occurring nearest to the load terminals. Using (*8.25*) and noting that $\beta = 2\pi/0.375 = 16.7$ rad/m, the widths of the three minima in the standing wave pattern, measured in the manner defined are 9.3×10^{-5} m, 1.84×10^{-4} m, and 2.74×10^{-4} m. With an ordinary micrometer these distances can be measured to an accuracy of a few percent.

The results of the example show that the impedance of a physically excellent short circuit connected to a typical slotted line section is at or beyond the limit of sensitivity of standing wave measurement methods, and that the measurement of any terminal load impedance producing a VSWR as high as only 10 or 20 must be corrected for the attenuation of the slotted line section between the observed voltage minimum and the point of connection of the impedance, if accuracy better than 5% to 10% is expected. With higher values of VSWR the correction factors can become very large.

The procedure illustrated by the example can obviously be reversed, to provide a method for determining the attenuation factor of a transmission line by measurement of the widths of two or more successive minima in a voltage standing wave pattern produced on a line by a short circuit termination.

8.8. Multiple reflections.

Steady state harmonic conditions on a transmission line circuit have been assumed in all discussions of wave propagation, impedance relations and standing wave patterns, up to this point. In a lumped constant circuit the transient time interval during which steady state conditions are established after switching a time-harmonic source into the circuit is determined by a time constant which is a function of the value of the circuit components. In a transmission line circuit there is in general a similar switching transient time interval after the source is connected to the input terminals, but in addition there is another transient interval, usually far longer, in which waves are partially reflected back and forth along the length of the line while equilibrium is being established. An analysis of this multiple reflection process produces some useful relations.

If in the circuit of Fig. 8-1, page 156, the source is connected to the transmission line by closing a series switch at a certain instant, a harmonic voltage wave will start to travel along the line. (The accompanying current wave is ignored in this analysis, as is any switching transient.) The phasor value of this initial voltage wave at the input terminals of the line is $V_S Z_0/(Z_S + Z_0)$, since the input impedance of the line is equal to its characteristic impedance when there are waves traveling in only one direction on the line at the input terminals. At coordinate z this initial harmonic voltage wave will have the phasor value $\dfrac{V_S Z_0}{Z_S + Z_0} e^{-\gamma z}$ at any time after the wave reaches that location. On reaching the terminal load end of the line the voltage wave will be reflected by the complex reflection coefficient $\rho_T = (Z_T/Z_0 - 1)/(Z_T/Z_0 + 1)$, and the phasor value of this first reflected wave when it returns to the coordinate z will be

$$\frac{V_S Z_0}{Z_S + Z_0} e^{-\gamma l} \rho_T e^{-\gamma(l-z)}$$

At the input terminals of the line this first reflected wave will experience a reflection coefficient $\rho_S = (Z_S/Z_0 - 1)/(Z_S/Z_0 + 1)$, and its phasor value when it returns to coordinate z will be

$$\frac{V_S Z_0}{Z_S + Z_0} e^{-\gamma l} \rho_T e^{-\gamma l} \rho_S e^{-\gamma z}$$

By the superposition theorem the total phasor voltage at the coordinate z after an indefinite time, i.e. the steady state value, is the sum of an infinite series of such contributions. Hence

$$V(z) = \frac{V_S Z_0}{Z_S + Z_0}(e^{-\gamma z} + \rho_T e^{-2\gamma l} e^{\gamma z})[1 + \rho_T\rho_S e^{-2\gamma l} + (\rho_T\rho_S e^{-2\gamma l})^2 + \cdots]$$

$$= \frac{V_S Z_0}{Z_S + Z_0} \cdot \frac{e^{-\gamma z} + \rho_T e^{-2\gamma l} e^{\gamma z}}{1 - \rho_T\rho_S e^{-2\gamma l}} \qquad (8.26)$$

and

$$V(d) = \frac{V_S Z_0 e^{-\gamma l}}{Z_S + Z_0} \cdot \frac{e^{\gamma d} + \rho_T e^{-\gamma d}}{1 - \rho_T\rho_S e^{-2\gamma l}} \qquad (8.27)$$

Comparison of (*8.27*) with (*8.4*), page 157, shows that the undetermined coefficient V_1 in the latter is given by

$$V_1 = \frac{V_S Z_0}{(Z_S + Z_0)(1 - \rho_T\rho_S e^{-2\gamma l})} \qquad (8.28)$$

in terms of the various elements of the circuit of Fig. 8-1.

Equation (*8.27*) demonstrates explicitly that the shape of a standing wave pattern representing $|V(d)|$ as a function of d on a transmission line is in no way affected by the quantities V_S, Z_S and ρ_S at the source. The effect of circuit parameters on the ordinate scale of such a pattern can be calculated from (*8.28*) and (*8.15*).

A hint at the possibility of a resonance phenomenon in transmission line circuits is also provided by (*8.27*), since the magnitude and phase of ρ_T, ρ_S and $e^{-2\gamma l}$ in the denominator term are all independently adjustable, and their product can be made very nearly equal to $1 + j0$ in a practical circuit. The investigation of transmission line resonant circuits by means of (*8.27*) is discussed in Chapter 10.

8.9. Standing wave patterns from phasor diagrams.

Phasor diagrams provide a simple graphical perspective of the formation of standing wave patterns, and the ordinates for constructing such patterns can be taken directly from the diagrams.

For a lossless line, (*8.4*) becomes

$$V(d) = V_1 e^{-j\beta l}(e^{j\beta d} + \rho_T e^{-j\beta d})$$

In this equation $V_1 e^{-j\beta l}$ is a constant for any transmission line circuit and can be given the arbitrary phasor value $1 + j0$. Then at the load terminals of the transmission line where $d = 0$, the phasor value of the incident voltage wave represented by the term $e^{j\beta d}$ is also $1 + j0$, and the phasor value of the reflected wave at the same location is ρ_T, a function only of Z_T/Z_0.

Fig. 8-13(*a*) below shows these two phasors, and their sum which on the same scale is the phasor value of $V(d)$ at $d = 0$. At any other value of d, the phase of the voltage wave traveling in the direction of decreasing d (i.e. the incident wave before it reaches the load terminals) is greater by the amount βd radians, while the phase of the reflected wave traveling in the direction of increasing d has decreased by the same amount. On a lossless line there has been no change in the magnitudes of the phasors. The resultant phasor diagram, for a value of βd of about 1 rad, is shown in Fig. 8-13(*b*) below, obtained from Fig. 8-13(*a*) by rotating the two phasors through equal angles in opposite directions.

For a standing wave pattern, only the magnitude of the sum of the two phasors is of interest. This quantity is more easily visualized if the incident wave phasor is kept constant at the value $1 + j0$ for all values of d, and the reflected wave phasor is rotated clockwise at the rate of $2\beta d$ rad/m. It is then obvious from Fig. 8-13(*c*) that the maximum ordinate

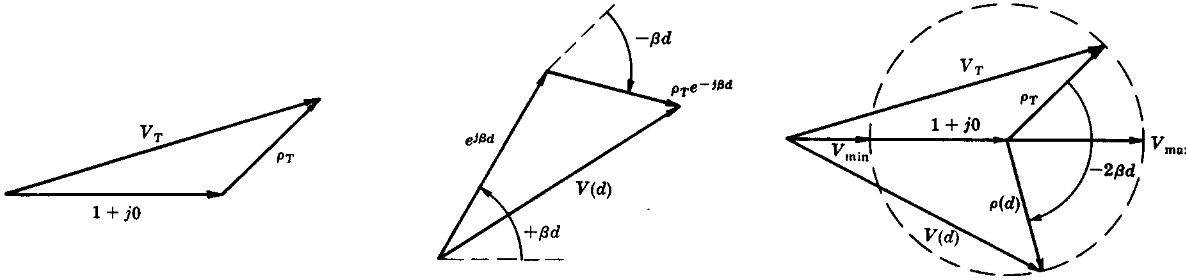


Fig. 8-13. Phasor diagrams for the voltage of the incident wave, the voltage of the reflected wave, and the total voltage, at two points on a lossless transmission line.
(a) At the load terminals. The incident wave voltage phasor has the reference value $1 + j0$. The reflection coefficient is $0.625\underline{/\pi/4 \text{ rad}}$.
(b) At a coordinate d such that $\beta d = \pi/3$ rad.
(c) Phasor diagrams (a) and (b) combined to have a common incident voltage phasor. The circle is then the locus of the tip of all total-voltage phasors on the line. The diagram gives correct relative magnitude information for all $V(d)$ phasors, but to obtain true relative phases each $V(d)$ must be multiplied by $e^{j\beta d}$.

in a standing wave pattern on a lossless line is proportional to $1 + |\rho_T|$ and the minimum ordinate is proportional on the same scale to $1 - |\rho_T|$, in agreement with (*8.21*). It is also directly apparent that the lowest value of d at which a voltage minimum will occur will be given by $2\beta d_{V(\min)} = \phi_T + \pi$, in agreement with (*8.19*). Finally, if $|\rho_T| = 1$, the voltage magnitude is seen to vary much more rapidly with d at a voltage minimum than at a voltage maximum, but if $|\rho_T| \ll 1$, the small fluctuations in the pattern are approximately sinusoidal.

When the transmission line has a finite attenuation factor α, the magnitude of the phasor representing the incident wave increases with d by a term $e^{\alpha d}$, while that representing the reflected wave decreases by a term $e^{-\alpha d}$. The phase changes are the same as in the lossless case. The phasor diagram for any value of d is easily drawn, but no simple diagram displays the phenomena for all values of d.

8.10. The generalized reflection coefficient.

All previous references to voltage reflection coefficients have been to the voltage reflection coefficient ρ_T produced at the terminal load end of a transmission line by a connected impedance Z_T.

There has been considerable use, however, of the concept $Z(z)$ or $Z(d)$, the impedance at a general coordinate z or d on a line, being the ratio of the phasor voltage to the phasor current on the line at that coordinate. It has been demonstrated that the meaning of $Z(z)$ or $Z(d)$ is the input impedance of that portion of a total transmission line circuit on the terminal load side of the coordinate z or d, when separated from the portion of the circuit on the signal source side.

It is meaningful to consider that at a general coordinate z or d a transmission line is in effect "terminated" by the impedance $Z(z)$ or $Z(d)$ at that point, and that there is a voltage reflection coefficient $\rho(z)$ or $\rho(d)$ at the point which is determined by the normalized value of $Z(z)$ or $Z(d)$ in the same manner that ρ_T is determined by Z_T/Z_0. Using the abbreviated notation ρ for this reflection coefficient at any general point on a transmission line, and the abbreviated notation Z for the impedance at the point, we have

$$\rho \quad = \quad \frac{Z/Z_0 - 1}{Z/Z_0 + 1} \quad = \quad |\rho|\, e^{j\phi} \tag{8.29}$$

It must also be true that the standing wave pattern along the portion of a transmission line circuit on the signal source side of any general point on the line is determined by the reflection coefficient and impedance at that point, through exactly the same relations that apply at the load terminals of the line. Thus

$$\text{VSWR} \quad = \quad \frac{1+|\rho|}{1-|\rho|} \tag{8.30}$$

at any point, or conversely

$$|\rho| \quad = \quad \frac{\text{VSWR}-1}{\text{VSWR}+1} \tag{8.31}$$

Since on a lossless line the VSWR value is the same everywhere, (*8.31*) indicates that the reflection coefficient magnitude is the same for all points on such a line for any one value of terminal load impedance, a conclusion that is easily understood physically.

Similarly,

$$d_{V(\min)}/\lambda \quad = \quad \tfrac{1}{4}(1+\phi/\pi) + \tfrac{1}{2}n \tag{8.32}$$

where the meaning of $d_{V(\min)}$ is now the distance of any general point on a transmission *line from a voltage minimum in the standing wave pattern.* In this case negative values of $d_{V(\min)}/\lambda$ result if the distance is measured to a voltage minimum on the terminal load side of the point.

The converse equation to (*8.32*) is

$$\phi \quad = \quad 4\pi(d_{V(\min)}/\lambda) - \pi + 2n\pi \tag{8.33}$$

Using (*8.31*) and (*8.33*), with the help of (*7.9a*) and (*7.9b*), page 128, or their admittance equivalents in Problem 7.2, page 144, the impedance or admittance at *any* point on a transmission line whose attenuation per wavelength is small can be found directly from the VSWR on the line, when the point's location is measured in wavelengths from a voltage minimum of the pattern. No reference need be made to either the value or the location of the terminal load impedance connected to the line.

By physical definition, the voltage reflection coefficient ρ at any general point on a transmission line is given by

$$\rho \quad = \quad \frac{\text{phasor value, at the point, of the voltage wave traveling toward the terminal load}}{\text{phasor value, at the point, of the voltage wave traveling toward the signal source}}$$

It follows from (*8.1*) and (*8.3*) that

$$\rho \quad = \quad \frac{V_2e^{\gamma z}}{V_1e^{-\gamma z}} \quad = \quad \frac{V_2}{V_1}e^{-2\gamma z} \quad = \quad \frac{V_2}{V_1}e^{2\gamma l}\,e^{-2\gamma d} \quad = \quad \rho_T e^{-2\gamma d} \quad = \quad |\rho_T|\,e^{-2\alpha d}\,\underline{/\phi_T - 2\beta d} \tag{8.34}$$

Thus the magnitude of the voltage reflection coefficient at any general coordinate d on a transmission line circuit is less than the value at the load terminals of the circuit by the factor $e^{-2\alpha d}$, involving twice the attenuation of the length of line between the two locations, and the phase angle of the reflection coefficient at the coordinate d is less than that at the load terminals by the amount $2\beta d$, which is twice the phase change that a single traveling wave would experience between the points.

In evaluating the impedance at a coordinate d on a transmission line in terms of the terminal load impedance and the propagation factors of the line, it may be easier to evaluate ρ_T from (*8.3*), ρ from (*8.34*), and the normalized components of the impedance at d from (*7.9a*) and (*7.9b*) using ρ in place of ρ_T, than to attempt to calculate the impedance at d more directly from (*7.20*), page 131.

Solved Problems

8.1. Derive an expression of the form of equation (*8.15*) from which the standing wave pattern of current on the transmission line circuit of Fig. 8-1 can be plotted.

Starting with (*8.5*) and using (*8.11*) and (*8.12*), an equation for $I(d)$ is obtained which differs from (*8.13*) for $V(d)$ only in the appearance of a multiplying factor $1/Z_0$ and a negative sign for the term $e^{-(p+jq)}e^{-\gamma d}$. The result is that (*8.14*) becomes

$$I(d) \quad = \quad \frac{2V_1 e^{-\gamma l}\sqrt{\rho_T}}{Z_0}\ \sinh\{(\alpha d + p) + j(\beta d + q)\}$$

The identity used is $\sinh(x + jy) = \sinh x \cos y + j \cosh x \sin y$, from which it is easily established that $|\sinh(x + jy)| = (\sinh^2 x + \sin^2 y)^{1/2}$. Finally,

$$|I(d)| \quad = \quad \left|\frac{2V_1 e^{-\gamma l}\sqrt{\rho_T}}{Z_0}\right| \{\sinh^2(\alpha d + p) + \sin^2(\beta d + q)\}^{1/2}$$

and the phase angle $\eta(d)$ of the current as a function of position is

$$\eta(d) \quad = \quad \tan^{-1}\{\coth(\alpha d + p)\tan(\beta d + q)\}$$

The ratio of the coefficient of $|V(d)|$ in (*8.14*) to the coefficient of $|I(d)|$ in the equation above is obviously $|Z_0|$, so that in Fig. 8-10 the ratio of the scales for $|V(d)|$ and $|I(d)|$ is $|Z_0|$.

If a line has negligible losses, it follows from the above expression for $|I(d)|$ that

$$|I(d)|_{max}/|I(d)|_{min} \quad = \quad \coth p \quad = \quad (1 + |\rho_T|)/(1 - |\rho_T|) \quad = \quad \text{VSWR}$$

Hence the "current standing wave ratio" and the "voltage standing wave ratio" have identical values on any low loss transmission line circuit and the symbol VSWR serves for both. (Some authors have used SWR as a common symbol.)

8.2. Show that if a low loss transmission line is to be designed to supply high frequency power to a load impedance Z_T, the lowest VSWR will occur on the transmission line if the characteristic impedance of the line has the value $Z_0 = |Z_T| + j0$.

Since the VSWR on a transmission line is a function of $|\rho_T|$ only, it is required to show that for arbitrary fixed Z_T and variable real Z_0, the quantity $|\rho_T| = \left|\dfrac{Z_T/Z_0 - 1}{Z_T/Z_0 + 1}\right|$ is minimized by $Z_0 = |Z_T| + j0$. The procedure is straightforward. Thus

$$|\rho_T| \quad = \quad \left|\frac{R_T/Z_0 + jX_T/Z_0 - 1}{R_T/Z_0 + jX_T/Z_0 + 1}\right| \quad = \quad \left\{\frac{R_T^2 - 2R_TZ_0 + Z_0^2 + X_T^2}{R_T^2 + 2R_TZ_0 + Z_0^2 + X_T^2}\right\}^{1/2}$$

The value of $\text{VSWR} = (1 + |\rho_T|)/(1 - |\rho_T|)$ is minimized by a minimum value of $|\rho_T|$ and hence also by a minimum value of $|\rho_T|^2$.

Equating to zero the derivative with respect to Z_0 of the square of the above expression for $|\rho_T|$ leads directly to the result $Z_0^2 = R_T^2 + X_T^2$, which proves the theorem.

Other practical solutions to this situation would make use of single stub matching or quarter wavelength transformers. The choice among them would depend on the performance feature to be optimized.

8.3. In Problem 7.7, page 146, design formulas were developed for the technique of "single stub matching". The technique is based on the fact that when there are reflected waves on a transmission line, there are two locations in every half wavelength of the line at which the real part of the normalized admittance is equal to unity. If the susceptance at either of these locations is cancelled by the equal and opposite susceptance of an appropriate stub line connected in parallel with the main line, the normalized admittance at the point becomes $1 + j0$. Reflected waves are eliminated on the portion of the main transmission line between the signal source and the point of connection of the matching stub. In Problem 7.7 the locations and lengths of the matching stubs were determined in terms of the normalized value of the terminal load admittance, the locations being referred to the load terminals of the line. There was no mention of a standing wave pattern on the line.

Show that if the voltage standing wave pattern on a transmission line is observed, the solution of the single stub matching procedure can be expressed completely in terms of the VSWR on the line, the locations of the two possible matching stubs per half wavelength being referred to a voltage minimum in the pattern.

The normalized conductance at any point on a transmission line is given in terms of the voltage reflection coefficient at the point by an expression from Problem 7.2, page 144, using notation appropriate to a general point on the line:

$$G/Y_0 = (1-|\rho|^2)/(1+|\rho|^2+2|\rho|\cos\phi)$$

Equating this to unity gives $\cos\phi = -|\rho|$, or $\phi = \cos^{-1}(-|\rho|)$. Then using (*8.31*), ϕ can be expressed as a function of VSWR.

The locations of the points at which matching stubs can be connected to the line are found, relative to any voltage minimum in the standing wave pattern, by substituting for ϕ into (*8.32*). Thus $d_{V(\min)}/\lambda = \frac{1}{4}\left\{1+\frac{\cos^{-1}(-|\rho|)}{\pi}\right\} \pm \frac{1}{2}n$. The two solutions of $\cos^{-1}(-|\rho|)$ can be expressed as $-\pi \pm |\cos^{-1}|\rho||$ where $|\cos^{-1}|\rho||$ is an angle less than $\pi/2$ radians. It follows that the points for connecting matching stubs are given by $d_{V(\min)}/\lambda = \pm|\cos^{-1}|\rho||/4\pi$. They are equidistant on either side of a voltage minimum and cannot be more than one eighth wavelength from a voltage minimum. Using (*8.31*) their location can be expressed as a function of VSWR.

To find the lengths of the required matching stubs, use is made of the expression in Problem 7.2 for the normalized susceptance at any point on a transmission line where the reflection coefficient is ρ:

$$B/Y_0 = -2|\rho|(\sin\phi)/(1+|\rho|^2+2|\rho|\cos\phi)$$

At the points where the stub line is to be connected, $\cos\phi = -|\rho|$ and hence $\sin\phi = \pm\sqrt{1-|\rho|^2}$. Substituting these expressions, $B/Y_0 = \pm 2|\rho|/\sqrt{1-|\rho|^2}$. Since the normalized input susceptance of a lossless stub line of length l with the customary short circuit termination is, as given by (*7.21*), $-\cot\beta l$, the required stub lengths are $l = (1/\beta)\cot^{-1}(\pm 2|\rho|/\sqrt{1-|\rho|^2})$. On substituting for $|\rho|$ from (*8.31*), this becomes $l = (1/\beta)\cot^{-1}\{\pm(\text{VSWR}-1)/\sqrt{\text{VSWR}}\}$.

Inspection shows that the stub line less than one quarter wavelength long is to be used at the location on the terminal load side of the voltage minimum. The sum of the stub lengths for the two different solutions is one half wavelength.

8.4. A transmission line 30 ft long operating at a frequency of 100 megahertz has a characteristic impedance of $60+j0$ ohms, an attenuation factor of 8.0 db/(100 ft) and a phase velocity of 75%. The line is to provide a signal voltage of 40 volts rms across a terminal load resistance of 1000 ohms. (*a*) What is the required rms voltage at the input terminals of the line? (*b*) What are the peak values of voltage and current along the line and where do they occur?

(*a*) From equations (*8.15*) or (*8.17*), page 164, the ratio of the voltage magnitude at $d=0$ on a transmission line circuit to the voltage magnitude at $d=l$ is

$$|V_T/V_{\text{inp}}| = \left\{\frac{\sinh^2 p + \cos^2 q}{\sinh^2(\alpha l+p)+\cos^2(\beta l+q)}\right\}^{1/2}$$

Quantities required in making the calculation are:

$$\alpha = 8.0/(8.686\times 30.48) = 3.02\times 10^{-2}\text{ nepers/m}$$

$$\beta = \omega/v_p = (2\pi\times 10^8)/(0.75\times 3.00\times 10^8) = 2.79\text{ rad/m}$$

$$\rho_T = (Z_T/Z_0-1)/(Z_T/Z_0+1) = (1000/60-1)/(1000/60+1) = 0.887\underline{/0}$$

$$p = \log_e(1/\sqrt{|\rho_T|}) = 0.0598; \quad q = -\tfrac{1}{2}\phi_T = 0.$$

Since $l = 9.84$ m, $\alpha l + p = 0.357$ and $\beta l + q = 27.45$. Also, $\sinh p = 0.0598$; $\sinh(\alpha l+p) = 0.365$; $\cos q = 1$; $\cos(\beta l+q) = -0.682$. Then

$$|V_T/V_{\text{inp}}| = \sqrt{\frac{0.0036+1}{0.133+0.465}} = 1.297$$

and the rms value of the input voltage to the line must be $40/1.297 = 30.8$ volts.

(b) It is clear from the preceding calculation that the scale factor $|2V_1 e^{-\gamma l}\sqrt{\rho_T}|$ has the value $40/\sqrt{1.0036} = 39.9$ volts rms. The attenuation per wavelength is $2\pi\alpha/\beta = 2\pi \times 3.02 \times 10^{-2}/2.79 = 0.068$ nepers, which is small enough to justify the approximation that the voltage maxima occur where $\cos^2(\beta d + q) = 1$, whence $d_{V(\max)}/\lambda = \phi_T/(4\pi) + n/2 = n/2$. The wavelength on the line is $v_p/f = 2.25 \times 10^8/10^8 = 2.25$ m. Hence the voltage maxima are located at $0, 1.12, 2.25, \ldots$ meters from the load terminals. Because the line has a finite attenuation factor, the highest voltage on the line will occur at the value of $d_{V(\max)}$ nearest the input terminals, i.e. at $d = 9.00$ m.

At this location $(\alpha d + p) = 0.332$, $\sinh(\alpha d + p) = 0.338$, and $\cos(\beta d + q) = 1$ by definition. Then

$$|V_{\max}| \;=\; 39.9\sqrt{0.332^2 + 1} \;=\; 42.1 \text{ volts rms or } 59.6 \text{ volts peak}$$

A current maximum will occur one quarter wavelength closer to the input terminals than the above voltage maximum, at $d = 9.56$ m. The scale factor for the current standing wave pattern is $39.9/Z_0 = 39.9/60 = 0.665$ rms amperes. The maximum current at any point on the line is then

$$|I_{\max}| \;=\; 0.665\sqrt{0.355^2 + 1} \;=\; 0.705 \text{ amperes rms or } 1.00 \text{ amperes peak}$$

8.5. On a lossless transmission line two voltage waves traveling in opposite directions are represented by the equations

$$V_a(z,t) \;=\; \hat{V}_a \cos(\omega t - \beta z) \quad \text{and} \quad V_b(z,t) \;=\; \hat{V}_b \cos(\omega t + \beta z)$$

respectively, where $\hat{V}_a$ and $\hat{V}_b$ are real. Derive an expression for the magnitude of the peak phasor voltage as a function of position along the line, and for the value of the VSWR on the line.

The instantaneous voltage at any point on the line is $V_a(z,t) + V_b(z,t)$. Using

$$V_a(z,t) \;=\; \hat{V}_a(\cos\omega t \cos\beta z \;+\; \sin\omega t \sin\beta z)$$

and

$$V_b(z,t) \;=\; \hat{V}_b(\cos\omega t \cos\beta z \;-\; \sin\omega t \sin\beta z)$$

the instantaneous voltage at any point is

$$\{(\hat{V}_a + \hat{V}_b)\cos\beta z\}\cos\omega t \;+\; \{(\hat{V}_a - \hat{V}_b)\sin\beta z\}\sin\omega t$$

An inversion of the same identity has the form

$$A\cos\omega t \;+\; B\sin\omega t \;=\; \sqrt{A^2 + B^2}\cos(\omega t - \psi), \quad \text{where } \psi = \tan^{-1}(B/A)$$

Using this identity, the instantaneous voltage at any point is given by

$$(\hat{V}_a^2 + \hat{V}_b^2 + 2\,\hat{V}_a\,\hat{V}_b \cos 2\beta z)^{1/2}\cos(\omega t - \psi)$$

The peak phasor magnitude of this voltage is the coefficient of the term $\cos(\omega t + \psi)$, and the functional form of ψ is of no concern. The maximum phasor voltage occurs at the points on the line where $\cos 2\beta z = +1$, and the minimum phasor voltage where $\cos 2\beta z = -1$. It follows that $\text{VSWR} = (\hat{V}_a + \hat{V}_b)/(\hat{V}_a - \hat{V}_b)$. This result assumes $\hat{V}_a > \hat{V}_b$, which would be true on a low-loss transmission line using the coordinate system of Fig. 8-1, with $Z_T \neq Z_0$.

8.6. A standard 7/8″ rigid copper coaxial transmission line of characteristic impedance $50 + j0$ ohms is to deliver 4.0 kilowatts of power at a frequency of 10 megahertz over a distance of several hundred feet to a load. The rms value of current in the line conductors must not exceed 15 amperes at any point. What is the highest value of VSWR that can be tolerated on the line?

On a low loss line such as this the characteristic impedance has a very small phase angle, and the power can be calculated independently for the waves traveling in the two directions on the line. (See Section 7.6, page 136, specifically equations (*7.34*) and (*7.35*) with $X_0 = 0$.)

If I_a is the rms value of the harmonic current wave traveling toward the load and I_b the rms value of the harmonic current wave traveling toward the signal source, it follows from Section 7.6 that the power delivered to the load is $I_a^2 Z_0 - I_b^2 Z_0 = Z_0(I_a + I_b)(I_a - I_b) = 4000$ watts. From Problem 8.5, the rms current value at the maxima in the current standing wave pattern is $I_a + I_b$, which must not exceed 15 amperes. With $I_a + I_b = 15$, the above expression for the power gives $I_a - I_b = 4000/(50 \times 15) = 5.33$ rms amperes. From Problems 8.1 and 8.5, the VSWR is then given under these limiting conditions by $\text{VSWR} = (I_a + I_b)/(I_a - I_b) = 2.81$.

The impedance of the load receiving the 4.0 kilowatts of power would therefore have to be such as to produce a VSWR of less than 2.81 on the line of characteristic impedance $50 + j0$ ohms.

8.7. It is desired to find the attenuation factor and phase velocity for a flexible high frequency coaxial cable of characteristic impedance $50 + j0$ ohms at a frequency of 500 megahertz. When a 50 foot length of the cable with short circuit termination is connected as the transmission line element in the circuit of Fig. 8-8, page 162, the standing wave pattern observed in the low loss slotted line section of characteristic impedance $50 + j0$ ohms has a VSWR of 3.25 with a voltage minimum 28.5 cm from the input terminals of the cable being tested. When 18.50 cm is cut off the length of the cable, the VSWR value changes only slightly, but the voltage minima in the slotted section are found at 54.9 and 24.9 cm from the input terminals of the cable. When a 25 foot length of the same cable is terminated in a short circuit and connected to the slotted section, the observed VSWR is 6.35. Calculate the attenuation factor and phase velocity for the cable.

From equation (*8.34*), the ratio of the reflection coefficient magnitudes at two locations d and d' on a transmission line having attenuation factor α is $|\rho(d)|/|\rho(d')| = e^{-2\alpha(d-d')}$. From the observed VSWR values, $|\rho(d)| = (\text{VSWR} - 1)/(\text{VSWR} + 1) = 2.25/4.25 = 0.530$, and $|\rho(d')| = 5.35/7.35 = 0.728$. Using $d - d' = 25 \text{ ft} = 7.62$ m,

$$\alpha = \frac{\log_e (0.728/0.530)}{2 \times 7.62} = 2.08 \times 10^{-2} \text{ nepers/m} = 5.51 \text{ db/(100 ft)}$$

The nature of the terminal load impedance is not a factor in this measurement or calculation, but it must be the same for each line length and should produce a reflection coefficient of high magnitude.

From equation (*8.34*) the difference in the phase angles of the reflection coefficients at two locations d and d'' on a transmission line is $\phi(d) - \phi(d'') = -2\beta(d - d'')$. From the observed values of $d_{V(\min)}$, and since the wavelength on the slotted section is $3.00 \times 10^8/(5.00 \times 10^8) = 0.60$ m,

$$\phi(d) = 4\pi d_{V(\min)}/\lambda - \pi = 4\pi(0.285/0.60) - \pi = 2.83 \text{ rad}$$

and $\phi(d'') = 4\pi(0.249/0.60) - \pi = 2.08$ rad. Used directly, this data would give a negative value for β and for v_p. The confusion arises from the fact that in calculating $\phi(d)$ and $\phi(d'')$, both angles are subject to an additional term $\pm 2n\pi$ where n may be zero or any integer. The choice of $n = 0$ in each calculation has given an incorrect answer. It is therefore necessary to test other values of n and to decide from the results whether a unique acceptable answer is indicated. If two or more answers are about equally reasonable on physical grounds, additional measurements must be taken on other lengths of line until the uncertainty is resolved.

In the present case the simplest way to obtain a positive value for β is to increase $\phi(d'')$ by 2π. (This is equivalent to choosing $d_{V(\min)} = 0.549$ from the original data.) Then $\phi(d'') = 2.08 + 2\pi = 8.36$ rad, $\beta = (2.83 - 8.36)/(-2 \times 0.185) = 14.9$, and $v_p = 2\pi \times 5.00 \times 10^8/14.9 = 2.11 \times 10^8$ m/sec $= 70\%$. This is an acceptable result, and a test for other values of n shows that none gives a velocity greater than about one half of this. The usual plastic dielectrics used in flexible cables have dielectric constants much too small to produce such low velocities.

8.8. A low loss transmission line with phase velocity of 82% and characteristic impedance of $75 + j0$ ohms at the operating frequency of 50.0 megahertz is terminated in an impedance of $100 - j25$ ohms. What phase difference exists between the phasor voltage at the load terminals and the phasor voltage at a point 3.40 m from the load terminals?

The answer will be obtained from equation (*8.16*), page 164. From the given data,

$$\rho_T = (Z_T/Z_0 - 1)/(Z_T/Z_0 + 1) = (1.333 - j0.333 - 1)/(1.333 - j0.333 + 1) = 0.160 - j0.120$$
$$= 0.20\underline{/-0.644 \text{ rad}}$$

Then $p = \log_e (1/\sqrt{|\rho_T|}) = 0.805$ and $\tanh p = 0.667$; $q = 0.322$. Since

$$\beta = \omega/v_p = 2\pi \times 50 \times 10^6/(0.82 \times 3.00 \times 10^8) = 1.275 \text{ rad/m}$$

$\beta d + q = 1.275 \times 3.40 + 0.322 = 4.657$ rad at the point in question, and $\tan(\beta d + q) = +18.0$. The statement of the problem implies $\alpha = 0$. The phase of the voltage at the point is therefore $\xi(3.40) = \tan^{-1}(0.667 \times 18.0) = \tan^{-1} 12.0 = 1.49 \pm n\pi$ rad. The phase must also be evaluated at $d = 0$, where $\tan(\beta d + q) = \tan q = 0.334$. Then $\xi(0) = \tan^{-1}(0.667 \times 0.334) = \tan^{-1} 0.223 = 0.219 \pm n\pi$ rad. The phase difference between the voltages at the two points is $1.49 - 0.22 \pm n\pi = 1.27 \pm n\pi$ rad. If there were no standing waves on the line, the phase difference between the voltages at the same two points would be $\beta d = 1.275 \times 3.40 = 4.33$ rad. Since the VSWR on the line is not large, the actual phase difference should be not far from this value, and is evidently $1.27 + \pi = 4.41$ rad. With high VSWR values the phase discrepancy between the two cases can exceed one radian.

Supplementary Problems

8.9. An air dielectric transmission line operating at a frequency of 200 megahertz has a characteristic impedance of $73 + j0$ ohms. It is terminated in a 30 ohm resistor shunted by a capacitance of 15 picofarads. What is the VSWR on the line, and how far is the nearest voltage minimum in the standing wave pattern from the load terminals? The line losses are assumed negligible.

Ans. VSWR = 3.35; $d_{V(\min)}/\lambda = 0.030$, or $d_{V(\min)} = 0.045$ m

8.10. Solve Problem 8.9 if (*a*) the capacitance is disconnected, (*b*) the resistor is disconnected.

Ans. (*a*) VSWR = 2.43; $d_{V(\min)} = 0$. (*b*) VSWR = infinite; $d_{V(\min)} = 0.149$ m

8.11. Show by scaling values from Fig. 8-9, page 168, that if the total length of the diagram represents a transmission line 30 m long:

(*a*) the attenuation factor of the line is approximately 0.033 nepers/m = 0.29 db/m = 8.9 db/(100 ft);

(*b*) the wavelength on the line is approximately 31 m;

(*c*) the magnitude of the reflection coefficient at $d = 0$ is approximately 0.55;

(*d*) the phase angle of the reflection coefficient at $d = 0$ is approximately 43°;

(*e*) the normalized value of the terminal load impedance is approximately $1.4 + j1.5$.

(More precise data, from which the curve was calculated, are: total attenuation αd for the length of the diagram = 1.000 nepers; ratio $\beta/\alpha = 6.00$; $p = 0.300$; $q = 1.200 - \pi/2 = -0.371$).

8.12. Show by consideration of multiple reflection of current waves on a transmission line that the expression for $I(d)$ corresponding to equation (*8.27*) for $V(d)$ is

$$I(d) = \frac{V_S e^{-\gamma l}}{Z_S + Z_0} \cdot \frac{e^{\gamma d} - \rho_T e^{-\gamma d}}{1 - \rho_T \rho_S e^{-2\gamma l}}$$

and that the ratio of $V(d)$ from (*8.27*) to this expression for $I(d)$ gives equation (*7.20*), page 131, for $Z(d)$. (*Note.* The reflection coefficient for current waves is always the negative of the reflection coefficient for voltage waves.)

8.13. In equation (*8.16*), what is the location of the reference points of zero phase angle for the phasor voltage in any standing wave pattern if the line has (*a*) negligible attenuation per wavelength, (*b*) considerable attenuation per wavelength?

Ans. (*a*) The voltage maxima in the pattern. (*b*) The points of contact of the pattern with its upper envelope.

8.14. Show from equations (*8.3*) and (*8.19*) that $d_{V(\min)}/\lambda$ lies between 0 and 0.25 for all impedances whose normalized imaginary component is negative, and lies between 0.25 and 0.5 for all impedances whose normalized imaginary component is positive.

8.15. If a transmission line has an attenuation of a few decibels per wavelength, successive maxima and successive minima in a standing wave pattern on the line differ so much in magnitude that there is no literal meaning to the concept of VSWR.

Show that if a graph is made of the standing wave pattern of voltage magnitude produced on such a line by some terminal load impedance, and the envelopes of the pattern are drawn, it is meaningful to define the VSWR of the pattern at any coordinate d as the ratio of the ordinate of the upper envelope to that of the lower envelope at that location, and that such a VSWR value can be used in equation (*8.31*) to give the magnitude of the reflection coefficient at that point. Show also that $d_{V(\min)}/\lambda$ for the point at coordinate d can be usefully defined as the distance in wavelengths from that location to the nearest point of contact of the voltage standing wave pattern with its lower envelope, in the direction of the signal source, and that this value of $d_{V(\min)}/\lambda$ can be used in (*8.33*) to obtain the phase angle of the reflection coefficient at d. From the reflection coefficient the normalized impedance at d can be calculated using (*7.9a*) and (*7.9b*).

8.16. There is a VSWR of 2.55 on a low-loss transmission line. Where can a stub line be connected in shunt with the transmission line to remove the standing waves from the line on the signal source side of the stub, and what must be the length of the stub?

Ans. Use the results of Problem 8.3. The stub line may be connected at a distance of 0.089 wavelengths on either side of any voltage minimum in the standing wave pattern. If the stub is connected on the terminal load side of a voltage minimum, it should have a length of 0.127 wavelengths (plus any number of half wavelengths). If the stub is connected on the signal source side of a voltage minimum, it should have a length of 0.373 wavelengths (plus any number of half wavelengths).

8.17. There is a VSWR of 1.75 on a low loss transmission line whose characteristic impedance is $50 + j0$ ohms. What is the value of the impedance at the voltage maxima and at the voltage minima in the standing wave pattern on the line? *Ans.* $87.5 + j0$ ohms at the maxima; 28.5 ohms at the minima

8.18. Show that on a low loss transmission line the voltage magnitude at any coordinate d can be expressed in terms of the voltage magnitude at a minimum of the pattern by

$$|V(d)| \;=\; |V(d)|_{\min}\left\{1 + \frac{4\,|\rho|}{(1+|\rho|)^2}\cos^2(\beta d + q)\right\}^{1/2}$$

where $|\rho|$ is the constant magnitude of the constant voltage reflection coefficient at all points of the low-loss line, and $-2q$ is the phase angle of the reflection coefficient at the origin of the d coordinates, which may be chosen anywhere on the line.

Then show that if the origin of the d coordinates is chosen at a voltage minimum (which makes $q = \pi/2$), the equation can be written in terms of the VSWR on the line as

$$|V(d)| \;=\; |V(d)|_{\min}\,\{1 + (\text{VSWR}^2 - 1)\sin^2\beta d\}^{1/2}$$

Finally, show that if Δd is the distance between two points on either side of any voltage minimum, at which $|V(d)| = \sqrt{2}\,|V(d)|_{\min}$, then

$$\sin^2(\tfrac{1}{2}\beta\,\Delta d) \;=\; 1/(\text{VSWR}^2 - 1) \quad\text{and}\quad \text{VSWR} \;=\; \frac{\sqrt{1 + \sin^2(\frac{1}{2}\beta\,\Delta d)}}{\sin(\frac{1}{2}\beta\,\Delta d)}$$

and for $\text{VSWR} \gg 1$ (corresponding to $\frac{1}{2}\beta\,\Delta d \ll 1$) this result is in agreement with equation (*8.25*).

8.19. Show that the lowest VSWR value that can be measured by the method of Problem 8.18 is 1.414, and that in measuring such a VSWR, the distance $\Delta d = \lambda/2$.

8.20. Show that the normalized impedance at any point on a transmission line is given by $Z/Z_0 = (1+\rho)/(1-\rho)$ where ρ is the reflection coefficient at the point. Then show that the phase angle γ of the normalized impedance at any point is

$$\gamma \;=\; \tan^{-1}\frac{2\,|\rho|\sin(-2\beta d)}{1 - |\rho|^2}$$

when d is measured from a voltage maximum in the standing wave pattern if the line has low losses, or from a point of contact of the pattern with its upper envelope otherwise.

Chapter 9

Graphical Aids to Transmission Line Calculations

9.1. Transmission line charts.

The preceding chapters have developed several equations for calculating voltages, currents, impedances, reflection coefficients and standing wave data on transmission lines. The variables in these equations have generally been complex numbers, and there have been frequent occurrences of exponential numbers with complex exponents and of hyperbolic functions of complex arguments.

Arithmetical evaluation of complex exponential and hyperbolic functions and of the functions inverse to these are time-consuming, and the assistance available from mathematical tables is less effective than for the corresponding functions of real variables. This is undoubtedly the reason for the long history of the use of graphical aids in transmission line calculations.

The graphs have taken many different forms. A *Chart Atlas of Complex Hyperbolic Functions,* published by A. E. Kennelly of Harvard University in 1914 and widely used for several decades, presented loci of the real and imaginary parts of the complex hyperbolic tangent and other functions over the complex variable or neper-radian plane. The charts were intended particularly for the solution of impedance problems by the use of equations (*7.18*), (*7.19*) and (*7.20*). Good significant-figure precision over a large portion of the neper-radian plane was achieved by the large graph size of about twenty inches square, and by presenting separate graphs of various portions of the plane on different scales. For calculations on systems such as cable pairs and open-wire lines at voice frequencies and low carrier frequencies, the charts are still useful; but for high frequency systems with low loss per wavelength they are more cumbersome than other graphical forms now available.

Since the 1940's there has developed a quite general agreement that one particular form of transmission line chart, commonly known as the "Smith" chart, is more versatile and more generally satisfactory than any of the others for solving the most commonly encountered problems, particularly on high-frequency systems. It is named for P. H. Smith of the Bell Telephone Laboratories, who in 1939 published one of the first descriptions of the uses of the chart.

The Smith chart is plotted on the voltage reflection coefficient plane or ρ-plane, i.e. on linear polar coordinates of $\rho = |\rho| e^{j\phi}$ where ρ is a general voltage reflection coefficient at any point of a transmission line. Naturally the chart can also be considered as plotted on rectangular coordinates of the real and imaginary components of ρ.

A third type of chart that was much used in the past and may still be encountered occasionally is plotted on the normalized impedance plane, i.e. on rectangular coordinates of general normalized impedance components R/Z_0 and X/Z_0. To distinguish it from the Smith chart in references, it is often euphemistically designated as the "Jones" chart. The label "rectangular impedance" chart is also applied to it.

One of the several important advantages of the Smith chart is that, within a circular contour enclosing a finite area of the voltage reflection coefficient plane, it presents complete information relating all possible values of normalized impedances, reflection coefficients and standing wave pattern data for all transmission line circuits involving only passive connected impedances. For the simple case when the characteristic impedance of a transmission line is real, as is effectively true for most transmission lines used at high frequencies, equation *(7.10)*, page 128, shows that the magnitude of the voltage reflection coefficient at the terminal load must lie between zero and unity for all passive values of terminal-load impedance Z_T, and equation *(8.29)*, page 176, shows that this is then also true of the general voltage reflection coefficient ρ at any point of any line, subject only to the limitation of low attenuation per wavelength. The Smith chart for this case is completely contained within a *unit* circle of the ρ-plane centered at the origin. This is the situation for which the Smith chart as normally printed is primarily intended, i.e. passive transmission line circuits at high frequencies, having low attenuation per wavelength to ensure that the characteristic impedance is very nearly real and that the relation between standing wave data and reflection coefficient values is simple, but without restriction as to the total attenuation of the circuit.

When the characteristic impedance of a transmission line is complex, as it is for telephone lines at frequencies near 1 kilohertz, it has been seen in Section 7.6 that the reflection coefficient may have a magnitude as great as $1+\sqrt{2}$, or 2.414. From equation *(7.10)* it is easily found that a reflection coefficient of magnitude greater than unity can be produced only by *impedances whose normalized value has a negative real part.* Such normalized impedances can occur for certain ranges of Z_T when Z_0 is complex, or can occur when Z_0 is real if Z_T has a negative resistance component, i.e. is an active network or device.

Extending the Smith chart to a radius of 2.414 in the reflection coefficient plane allows it to handle all possible problems of transmission lines with complex characteristic impedance and a partial range of situations involving active network elements connected to transmission lines.

From the defining equation $\rho = (Z/Z_0 - 1)/(Z/Z_0 + 1)$ it is easily seen that changing Z/Z_0 to $-Z/Z_0$ results in a reflection coefficient ρ' given by $\rho' = -1/\rho$. This suggests that a separate complete Smith chart for normalized impedances with negative real parts will be identical with the standard chart for normalized impedances with positive real parts if the plane on which the standard chart is plotted is recalibrated as the $-1/\rho$ plane or negative reciprocal reflection coefficient plane, by substituting $1/|\rho|$ for each $|\rho|$ value of radial coordinate, and $\pi - \phi$ for each ϕ value of angular coordinate.

The two charts taken together will handle transmission line problems with all possible values of characteristic impedance and all possible values of connected impedances with both positive and negative resistance components.

9.2. Equations for constructing the Smith chart.

In the form in which it is now always printed, the Smith chart displays orthogonal curvilinear coordinates of normalized impedance components on the voltage reflection coefficient plane. It is thus derived from the relation

$$\rho = \frac{Z/Z_0 - 1}{Z/Z_0 + 1} \tag{9.1}$$

where ρ is the complex voltage reflection coefficient at a point on a transmission line, and Z/Z_0 is the normalized value of the impedance at that point, understood to be the input impedance of the total transmission line circuit on the terminal-load side of the point. In terms of the traveling voltage waves on the line, ρ is the ratio at any point on the line of the phasor value of the reflected voltage wave, traveling toward the signal source, to the phasor value of the incident voltage wave, traveling toward the terminal load.

The algebra of deriving the equations for the Smith chart is simplified by assigning complex number notation to ρ and to Z/Z_0. Let $\rho = u + jv$, and let

$$Z/Z_0\ (= R/Z_0 + jX/Z_0) \;=\; z_n \;=\; r_n + jx_n$$

the subscript n and the lower case letters implying that the impedance and its components are in normalized form. Equation *(9.1)* becomes

$$u + jv \;=\; \frac{r_n + jx_n - 1}{r_n + jx_n + 1} \tag{9.2}$$

Cross multiplying and grouping real and imaginary terms yields two equations,

$$r_n(u-1) - x_n v \;=\; -(u+1) \tag{9.3}$$

$$r_n v + x_n(u-1) \;=\; -v \tag{9.4}$$

Eliminating x_n and regrouping terms in order of powers of u and powers of v, gives

$$u^2(r_n+1) - 2ur_n + v^2(r_n+1) \;=\; 1 - r_n \tag{9.5}$$

Dividing all terms by (r_n+1) and completing the square of the resulting two terms containing u^2 and u, the result is

$$\left\{u - \frac{r_n}{r_n+1}\right\}^2 + v^2 \;=\; \frac{1}{(r_n+1)^2} \tag{9.6}$$

On rectangular coordinates of u and v this is the equation of a circle whose center for any value of r_n is located at $u = r_n/(r_n+1)$, $v = 0$, and whose radius is $1/(r_n+1)$.

The following table gives the coordinates of the center, and the radius, for the circles which are the loci of several constant values of r_n distributed over the zero-to-infinity range of that variable. From the definition of u and v, these circles are on the reflection coefficient plane or ρ-plane.

Table 9.1

$r_n = R/Z_0$	Coordinates of center of circle		Radius of circle
	u	v	
0	0	0	1
1/7	1/8	0	7/8
1/3	1/4	0	3/4
1	1/2	0	1/2
3	3/4	0	1/4
7	7/8	0	1/8
15	15/16	0	1/16

All of these circles except the last one are plotted in Fig. 9-1 below, the first being the bounding circle of the standard Smith chart, $|\rho| = 1$. The values listed have been chosen to illustrate two features of the construction of the chart which are useful memory aids for visualizing the relation of the r_n circles to one another. The unique positions of the normalized resistance circles for $r_n = 0$ and $r_n = 1$ are easily remembered. The table shows that the circles for normalized resistance values $0, 1, 3, 7, 15, \ldots, (2^k - 1)$ constitute a series in which the radius of any circle is half the radius of the previous circle. All the circles pass through the point $1, 0$.

The second feature to be noticed is that for any value of r_n, the intersections with the central horizontal axis of the chart of the circles for r_n and $r_n' = 1/r_n$ occur at points symmetrical with respect to the center of the chart.

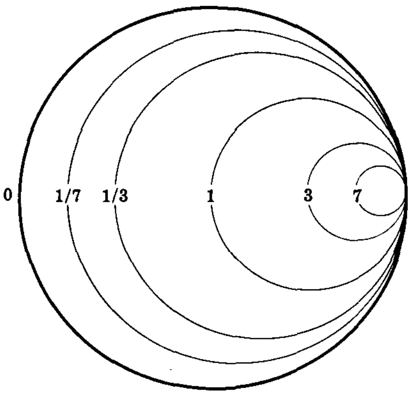


Fig. 9-1. Coordinate circles for constant normalized resistance on the Smith chart. The radii of the particular circles shown are related by simple fractions.

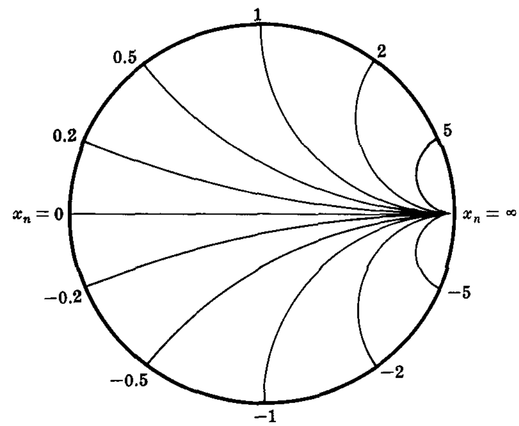


Fig. 9-2. Coordinate circles for constant normalized reactance on the Smith chart.

If the procedure in deriving (*9.6*) from (*9.3*) and (*9.4*) is repeated eliminating r_n instead of x_n, an equation is found for the locus of any constant value of x_n on the u, v coordinates. The result is

$$(u-1)^2 + (v - 1/x_n)^2 \;=\; (1/x_n)^2 \tag{9.7}$$

from which the following table is constructed.

Table 9.2

$x_n = X/Z_0$	Coordinates of center of circle		Radius of circle
	u	v	
0	1	infinite	infinite
0.2	1	5	5
−0.2	1	−5	5
±0.5	1	±2	2
±1	1	±1	1
±2	1	±0.5	0.5
±5	1	±0.2	0.2

These circles are plotted in Fig. 9-2, within the bounding circle $|\rho| = 1$.

The most obvious symmetry exhibited here is the mirror-image symmetry about the horizontal central axis of the chart, for the circles corresponding to values of x_n of equal magnitude but opposite sign. Inspection reveals another symmetrical aspect, similar to one of the symmetries of the r_n circles. The point of intersection of any x_n circle with the bounding circle of the chart is diametrically opposite to the point of intersection of the circle for $x'_n = -1/x_n$ with the bounding circle. These two symmetries combine to give the result that the point of intersection with the bounding circle for any x_n circle is the mirror image, with respect to the central vertical axis of the chart, of the point of intersection of the x'_n circle, when $x'_n = 1/x_n$.

The construction of the detailed Smith chart in its standard published form shown in Fig. 9-3 is an extension of the procedures that have been described. The meanings of the

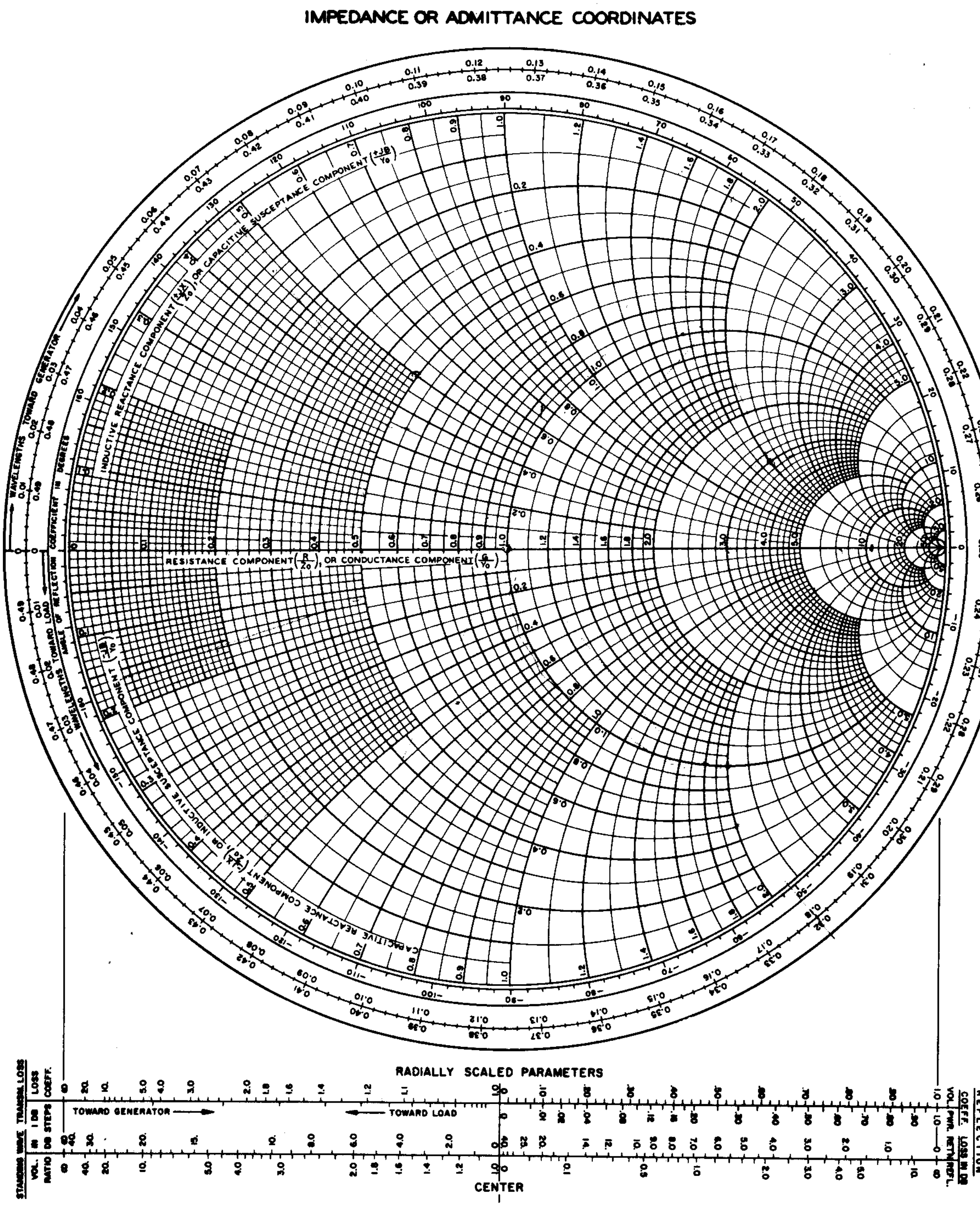


Fig. 9-3. A standard commercially available form of Smith chart graph paper. Copyrighted 1949 by Kay Electric Company, Pine Brook, N. J., and reprinted with their permission.

scales around the periphery of the chart and of the set of auxiliary scales accompanying the chart at the bottom of the figure are discussed in succeeding sections of this chapter.

9.3. Reflection coefficient and normalized impedance.

From the description of the Smith chart given in Section 9.2, it can be used to solve graphically problems that would be solved analytically by using equation (*9.1*).

Example 9.1.

A transmission line with characteristic impedance $Z_0 = 50 + j0$ ohms is terminated in an impedance $25 - j100$ ohms. Determine the reflection coefficient at the terminal load end of the line.

The normalized terminal load impedance is $z_n = r_n + jx_n = (25 - j100)/(50 + j0) = 0.50 - j2.00$. Fig. 9-4 shows where this value of normalized impedance is located on the Smith chart. The angular scale immediately outside the periphery of the Smith chart of Fig. 9-3 is a scale of reflection coefficient phase angle, which shows a phase angle of 309° to the radial line through the normalized impedance $0.50 - j2.00$. Reflection coefficient magnitude is a linear radial scale reading from zero at the center of the chart to unity at the periphery. The value for the point plotted on Fig. 9-4 can therefore be found as the ratio of two lengths, (radius to the point)/(outer radius of the chart). In the standard chart of Fig. 9-3, the upper right hand scale of the radial scales at the bottom of the chart is provided for making this measurement, and when laid along the radial line through the normalized impedance $0.50 - j2.00$, with the zero of the scale at the center of the chart, it shows the magnitude of the reflection coefficient produced by this value of normalized impedance to be 0.82. Thus the reflection coefficient produced by a normalized impedance $0.50 - j2.00$ is $\rho = 0.82\underline{/309°} = 0.64 - j0.52$, as shown in Fig. 9-5.

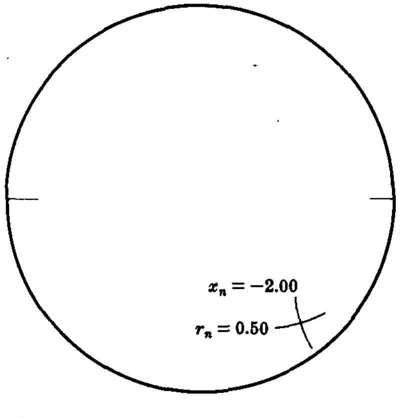


Fig. 9-4. Location on the Smith chart of the normalized impedance $z_n = 0.50 - j2.00$.

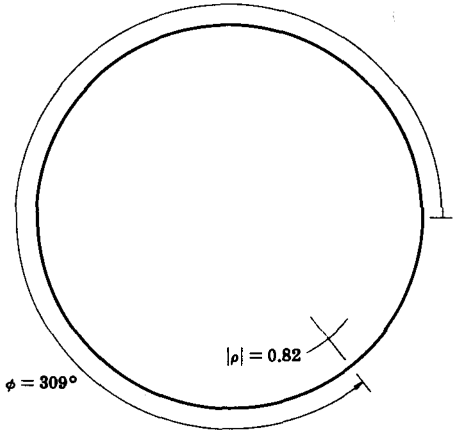


Fig. 9-5. Reflection coefficient coordinates of the point having normalized impedance coordinates $z_n = 0.50 - j2.00$.

Example 9.2.

At a point on a transmission line the reflection coefficient is measured as having a magnitude of 0.64. (A device called a reflectometer can make such a measurement.) If the impedance at that point of the line is a pure resistance, and the characteristic impedance of the line is real, what is the normalized value of the impedance at the point?

Fig. 9-6 shows the locus of all reflection coefficients of magnitude 0.64, and the locus of all normalized impedances which are purely resistive when normalized relative to a real characteristic impedance. There are two answers to the problem, one at each of the intersections of the two loci. The answers are $r_n + jx_n = 4.55 + j0$ or $0.22 + j0$.

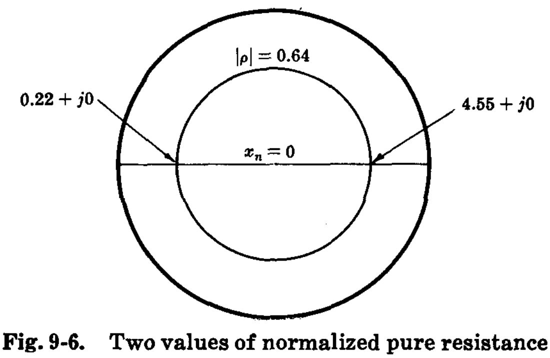


Fig. 9-6. Two values of normalized pure resistance that produce a reflection coefficient of 0.64.

9.4. Coordinates for standing-wave data.

In Chapter 8 two simple equations were derived which respectively related voltage standing-wave ratio to reflection coefficient magnitude, and the locations of voltage minima to reflection coefficient phase angle. These were

$$\text{VSWR} = \frac{1+|\rho|}{1-|\rho|} \tag{9.8}$$

$$d_{V(\min)}/\lambda = \tfrac{1}{4}(1+\phi/\pi) \pm \tfrac{1}{2}n \tag{9.9}$$

Here $\rho = |\rho|e^{j\phi}$ is the reflection coefficient at any point on a transmission line, VSWR is the voltage standing-wave ratio it produces, and there is a voltage minimum in the standing-wave pattern at a distance $d_{V(\min)}/\lambda$ in wavelengths from the point, in the direction of the signal-source. The relations are valid only on lines which have low attenuation per wavelength, since it is only for this case that the concept of voltage standing-wave ratio has a useful empirical meaning. For such lines the characteristic impedance is real.

From equations (*9.8*) and (*9.9*) it is a simple matter to place coordinates of VSWR and $d_{V(\min)}/\lambda$ on the reflection coefficient plane on which the Smith chart is drawn. Table 9.3 gives data for a few such coordinates.

Table 9.3

Reflection coefficient magnitude $\lvert\rho\rvert$	Voltage standing-wave ratio VSWR	Reflection coefficient phase angle ϕ	Distance of voltage minimum in wavelengths from point of reflection $d_{V(\min)}/\lambda$
0	1	0	0.25
0.2	1.5	$\pi/4$	0.3125
0.5	3	$\pi/2$	0.375
0.75	7	π	0.50 or 0
0.875	15	$3\pi/2$	0.125
0.9375	31	2π	0.25

The VSWR coordinates listed are plotted in Fig. 9-7 and the $d_{V(\min)}/\lambda$ coordinates in Fig. 9-8.

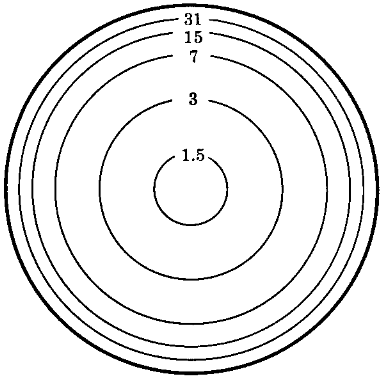


Fig. 9-7. Circle loci of constant VSWR on the reflection coefficient plane. The four intermediate circles have radii proportional to $1-(\frac{1}{2})^n$, with $n = 1, 2, 3,$ and 4.

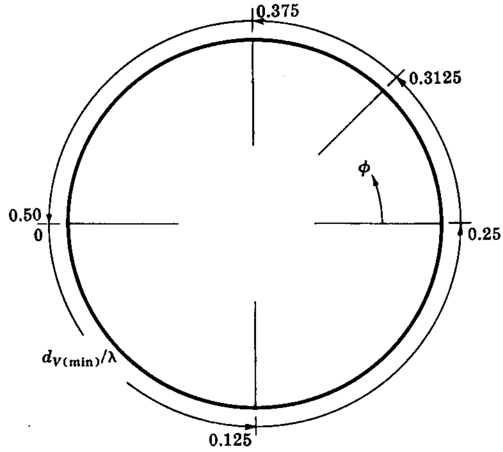


Fig. 9-8. Radial line loci of constant $d_{V(\min)}/\lambda$ on the reflection coefficient plane.

Comparison of the data and form of Fig. 9-7 with the data and form of Fig. 9-1 shows that each VSWR circle is tangent to the r_n circle of the same numerical value, the point of tangency being on the $\phi = 0$ radius. This is in agreement with a result shown in Chapter 8, that if a transmission line is terminated in a normalized impedance of value $r_n + j0$, with $r_n > 1$, the VSWR produced is numerically equal to r_n.

Printed forms of the Smith chart displaying both the $(r_n; x_n)$ and the (VSWR, $d_{V(\min)}/\lambda$) pairs of coordinates are occasionally encountered, but the resulting density of lines in some parts of the chart is confusing. Since VSWR is a function of $|\rho|$ only, it can be determined for any point on the chart by a radial scale derived from the radial scale for reflection coefficient magnitude. Of the radial scales at the bottom of Fig. 9-3, the lowest one at the left is a scale of VSWR determined from equation (*9.8*) in terms of the linear radial scale of reflection coefficient magnitude. Similarly, since $d_{V(\min)}/\lambda$ is a function of ϕ only, it can be measured for any point on the chart by a linear angular scale derived from the linear angular scale for ϕ, using equation (*9.9*). Such a scale is printed on the chart of Fig. 9-3 immediately outside the angular scale of reflection coefficient phase angle. (The inscription "WAVELENGTHS TOWARD LOAD" refers to a different use of the chart, discussed in Section 9.6.)

Example 9.3.

At the terminal-load end of a low-loss transmission line there is a reflection coefficient of $-0.30 + j0.55$. What is the voltage standing-wave ratio on the line, and where are the minima in the voltage standing-wave pattern located relative to the terminal-load end of the line?

To enter the chart, the reflection coefficient must be expressed in polar form as $\rho = 0.63\underline{/118.6^\circ}$. Fig. 9-9 shows this point on a Smith chart and indicates how the resulting VSWR value can be found, either by reference to a radial VSWR scale, or by making use of the fact mentioned above that a normalized impedance of value $r_n + j0$ produces a VSWR equal to r_n, when $r_n > 1$. The answer is VSWR = 4.4. Fig. 9-10 shows that the value of $d_{V(\min)}/\lambda$ for the same point is 0.415, meaning that minima in the voltage standing-wave pattern are located at 0.415, 0.915, 1.415, etc., wavelengths from the terminal-load end of the line.

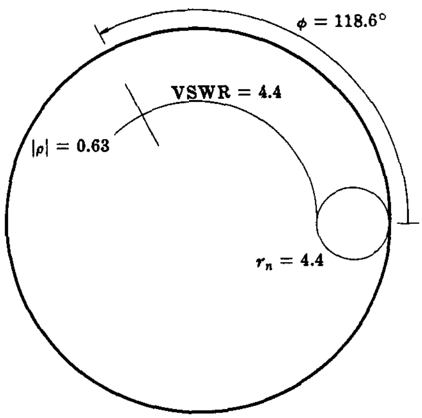


Fig. 9-9. The numerical value for a VSWR coordinate circle is equal to the numerical value of the normalized resistance coordinate circle ($r_n > 1$) to which it is tangent.

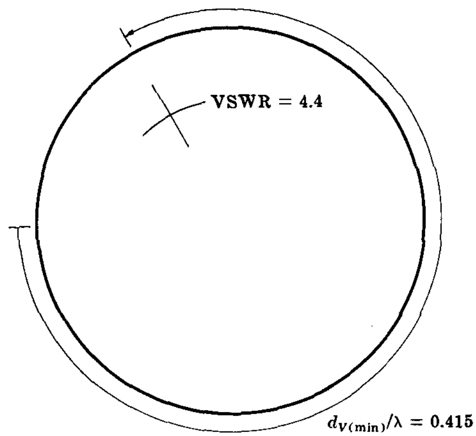


Fig. 9-10. VSWR and $d_{V(\min)}/\lambda$ coordinates of a point on the Smith chart.

Two more of the radial scales at the bottom of Fig. 9-3 can now be explained in terms of the scales that have been described above. On a line whose characteristic impedance is real, it was shown in Section 7.6 that the power in a reflected harmonic wave is proportional to the square of the phasor voltage magnitude of the wave. The magnitude of the reflection coefficient for power under these conditions is then the square of the magnitude of the voltage reflection coefficient. The second scale from the top in the right-hand group of radial scales in Fig. 9-3 is a radial scale of power reflection coefficient magnitude, derived as the square of the voltage reflection coefficient magnitude scale immediately above it.

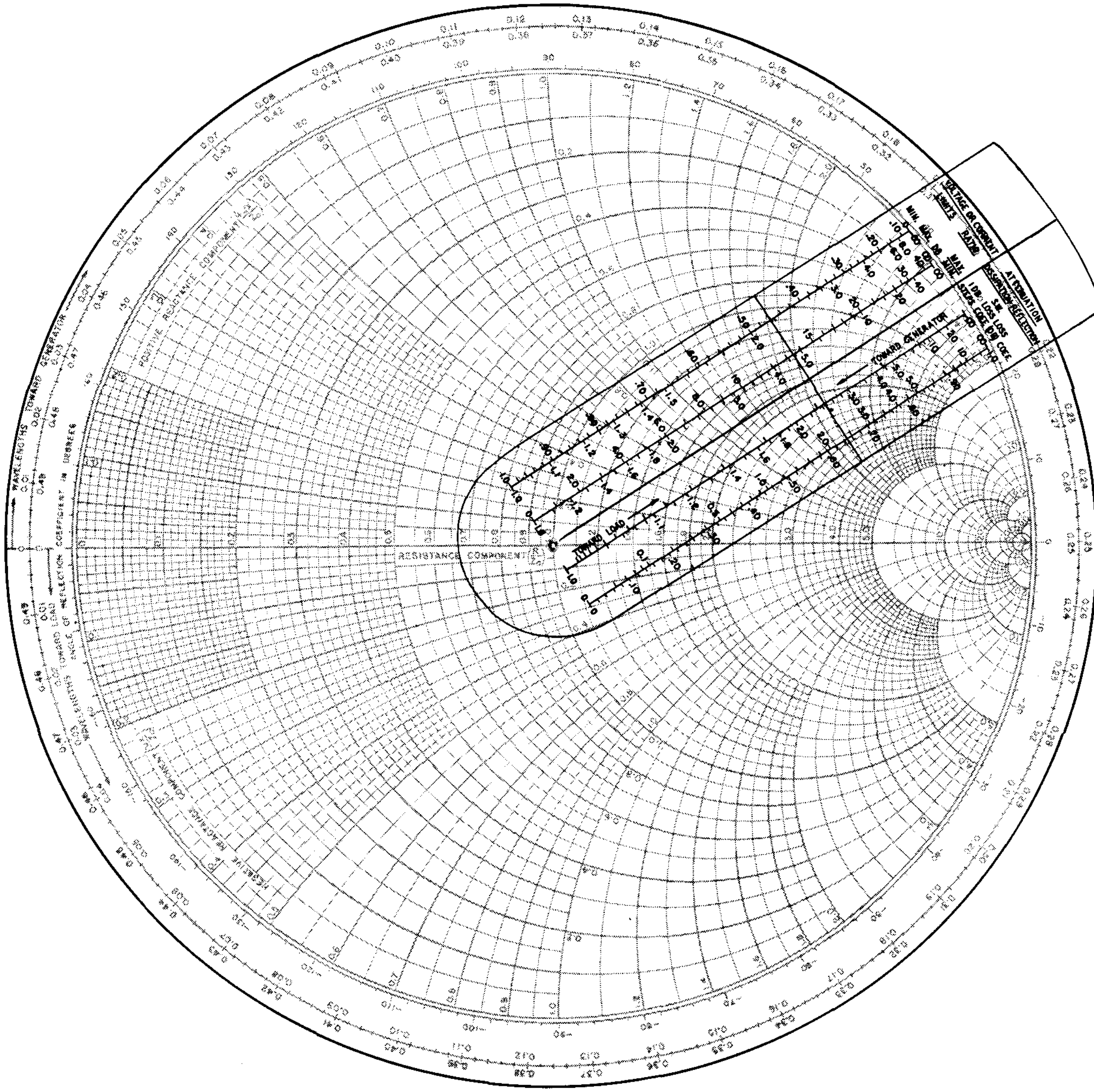


Fig. 9-11. A Smith chart "slide rule". The transparent radial strip, with one end pivoted at the center of the Smith chart, has eight radial scales similar to those of Fig. 9-3 printed on it. A transparent cursor slides along the strip and is marked with a single line transverse to the center line of the strip. When the strip's center line and the cursor's transverse line cross at a point on the chart, the cursor's transverse line shows eight types of information about the point, on the strip's radial scales. With permission of The Emeloid Co., Hillside, N. J.

Many communication engineers become so attached to decibel notation that they prefer to express all possible quantities pertaining to voltage, current or power in decibels relative to some reference. Since VSWR is by definition a ratio of voltages, it lends itself directly to a decibel formula like equation (*4.15*), page 36, with the result

$$\text{VSWR in db} \;=\; 20\log_{10}\left|\frac{V_{\max}}{V_{\min}}\right| \;=\; 20\log_{10}\text{VSWR} \qquad (9.10)$$

Expressing a VSWR in decibels has no particular practical value, such as facilitating other calculations, but has become a widely accepted conventional notation. Of the radial scales in Fig. 9-3, the second from the bottom on the left gives the decibel values for the VSWR values on the scale immediately below it, calculated from equation (*9.10*).

For the data of Example 9.3, the power reflection coefficient is 0.40, and the VSWR is 13.0 db.

In a commercially available form of Smith chart "slide rule", eight radial scales similar to those of Fig. 9-3 are printed on a strip of transparent plastic, which is mounted at its center point to rotate about the center point of a Smith chart printed on opaque plastic. With the addition of a central diametral line on the transparent strip, and a transverse line on a transparent plastic cursor that moves along the strip, the radial distance from the center of the chart to any point on the chart can be referred to any of the eight radial scales, in a manner indicated by Fig. 9-11 above.

Example 9.4.

The voltage standing wave pattern observed on an air-dielectric coaxial line slotted section shows a VSWR of 2.50, and there is a voltage minimum in the standing wave pattern 8.75 cm from the terminal load end of the section. The characteristic impedance of the section at the operating frequency of 800 megahertz is $50 + j0$ ohms. What is the value of the terminal load impedance?

The adjective "air-dielectric" establishes that the phase velocity on the slotted section is the free space value of 3.00×10^8 m/sec. The wavelength on the line is therefore

$$\lambda \;=\; (3.00\times10^8)/(800\times10^6) \;=\; 0.375\text{ m}$$

Hence $\quad d_{V(\min)}/\lambda \;=\; 0.0875/0.375 \;=\; 0.233$

The point on the chart with $\text{VSWR} = 2.50$ and this value of $d_{V(\min)}/\lambda$ is located as shown in Fig. 9-12. The normalized impedance coordinates of this point are found to be $r_n + jx_n = 2.35 - j0.50$. The actual value of the terminal load impedance in ohms is therefore $(2.35 - j0.50)(50 + j0) \;=\; 117 - j25$ ohms.

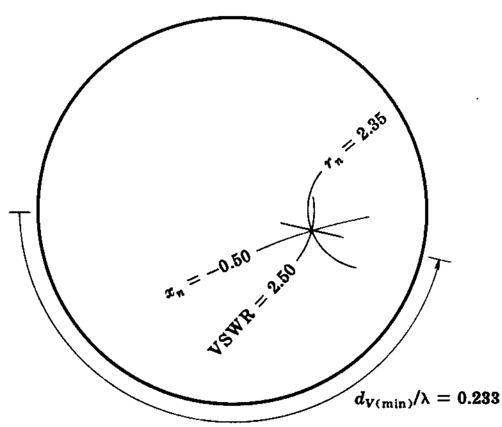


Fig. 9-12. Determination of the normalized terminal load impedance on a low-loss transmission line from standing wave pattern data.

9.5. Coordinates of magnitude and phase angle of normalized impedance.

It is sometimes convenient to work with impedances expressed in polar form rather than in complex-number component form. It would then be helpful to be able to perform Smith-chart computations without having to bring all impedances to the complex-number form required by the chart of Fig. 9-3. The derivation of the equations for the coordinates $(|z_n|, \theta)$ on the reflection coefficient plane, where $z_n = R/Z_0 + jX/Z_0 = |z_n|\underline{/\theta}$, is assigned as Problem 9.21. The graphical nature of the result is shown in Fig. 9-13 below. The Smith chart in this form is sometimes called a "Carter" chart, because it was first published by P. S. Carter of R.C.A. in 1939. In addition to being useful for handling data in the coordinates indicated, this chart illustrates another and very striking form of symmetry of the

Smith chart. The vertical central diameter of the chart is the coordinate $|z_n| = 1$, i.e. the locus of all impedances of normalized magnitude unity. Two impedances of equal phase angle but with reciprocally related normalized magnitudes lie at mirror-image points relative to this central vertical diameter.

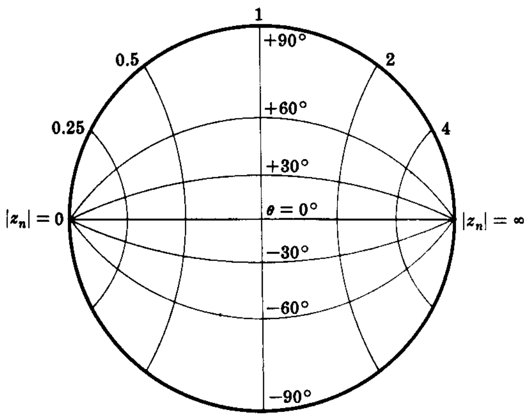


Fig. 9-13. The Carter chart. Coordinates of normalized impedance magnitude and phase angle on the reflection coefficient plane.

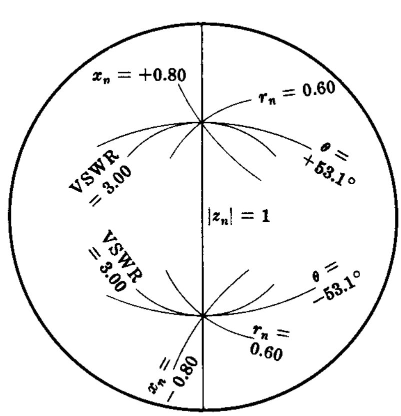


Fig. 9-14. Determining an impedance of normalized magnitude unity that will produce a VSWR of 3.00.

Example 9.5.

What impedance of normalized magnitude unity connected as the terminal-load impedance on a transmission line will produce a VSWR of 3.00?

Fig. 9-14 shows the straight-line locus of all impedances of normalized magnitude unity, and the circle locus of all points for which the VSWR is 3.00. These loci intersect in two points whose (r_n, x_n) or $(|z_n|, \theta)$ coordinates are the answer to the question. The results are: $|z_n| = |Z_T/Z_0| = 1$, $\theta = \pm 53.1°$; or $r_n = R_T/Z_0 = 0.60$, $x_n = X_T/Z_0 = \pm 0.80$.

9.6. Impedance transformations on the Smith chart.

It was shown in Section 8.10 that if $\rho = |\rho|\, e^{j\phi}$ is the voltage reflection coefficient at any coordinate d on a uniform transmission line, then the reflection coefficient $\rho_1 = |\rho_1|\, e^{j\phi_1}$ at any other coordinate d_1 is given by

$$\rho_1 = \rho\, e^{-2\alpha(d_1 - d)}\, e^{-2j\beta(d_1 - d)} \tag{9.11}$$

It follows that

$$|\rho_1| = |\rho|\, e^{-2\alpha(d_1 - d)} \tag{9.12}$$

$$\phi_1 = \phi - 2\beta(d_1 - d) \tag{9.13}$$

At points on the signal-source side of d (i.e. $d_1 > d$), the reflection coefficient magnitude and phase angle will both be less than at d. At points on the terminal load side of d $(d_1 < d)$, the reflection coefficient magnitude and phase angle will both be greater than at d.

Applied to the Smith chart, these results provide a very simple and direct graphical procedure for finding the normalized impedance at any point on a transmission line in terms of the normalized impedance at any other point on the line and the values of total attenuation αl and phase shift βl for the length of line l between the two points.

In Fig. 9-15, r_n and x_n are the normalized components of the impedance at some coordinate d on a transmission line having a finite attenuation factor. It is desired to know the normalized impedance components r_n' and x_n' at a point d_1 on the line, d_1 being on the signal source side of d. From equation (*9.13*) the reflection coefficient at coordinate d_1 will have a phase angle smaller by $2\beta(d_1 - d) = (4\pi/\lambda)(d_1 - d)$ than the phase angle of the reflection coefficient at coordinate d. The normalized impedance at the point d_1 will therefore be on a radius of the chart that lies clockwise (decreasing ϕ) from the angular location of the normalized impedance at d, by an angle in radians which is 4π times the length of transmission line in wavelengths between the two points. The angular scale on the chart to be used for determining the impedance transformation between two locations on a transmission line should therefore be a linear scale at the rate of 2π radians (one full rotation) around the chart for every half wavelength. The outermost angular scale on the Smith chart of Fig. 9-3 is such a scale, and the inscription WAVELENGTHS TOWARD GENERATOR indicates that transformation from a coordinate d to a coordinate d_1 is clockwise on this scale if d_1 is closer to the signal source or generator than d is. Paired with the above scale and just inside it, is an identical scale increasing in the counterclockwise direction and labeled WAVELENGTHS TOWARD LOAD for transformations from d to d_1 when d_1 is on the terminal load side of d. For these transformation problems the fact that the origin for both angular coordinate scales is on the left hand horizontal axis has no significance. The Smith chart "slide rule" mentioned in Section 9.4 permits rotation of the angular coordinate scales relative to the main body of the chart, so that the origin of these scales can be set at any desired point.

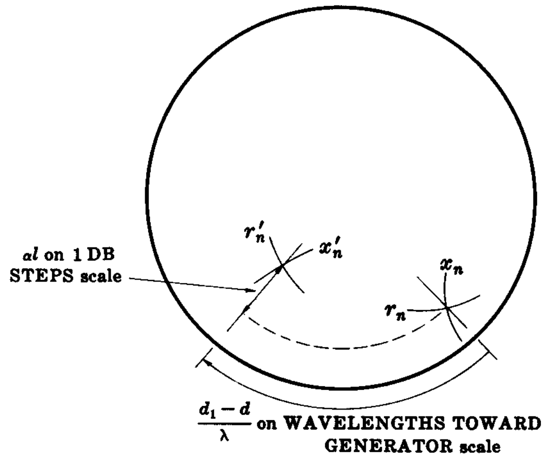


Fig. 9-15. Use of Smith chart for determining normalized impedance transformations between two locations on a lossy transmission line.

The magnitude of the reflection coefficient at the coordinate d_1 will differ from the value at d by the factor $e^{-2\alpha(d_1-d)}$. The normalized impedance at d_1 will therefore be closer to the center of the chart than the normalized impedance at d, by a factor dependent on the total attenuation of the line between the two locations. This attenuation is conveniently expressed in decibels. Since $1 \text{ neper} = 8.686$ decibels, the reflection coefficient magnitude at $d_1 > d$ will be reduced by a factor $e^{-2/8.686} = 0.794$ for every decibel attenuation of the line between coordinates d and d_1. The procedure adopted on the Smith chart of Fig. 9-3 for handling this factor graphically is the provision of a radial scale marked at 1 decibel intervals, but without numbers, starting at the periphery of the chart where $|\rho| = 1$. On the radial scales of Fig. 9-3 this is the second scale from the top at the left. Comparison with the top right-hand linear scale of reflection coefficient magnitude shows that the first 1-db step occurs at $|\rho| = 0.794$, the second at $|\rho| = 0.794^2 = 0.631$, etc. The intervals of this scale as printed on the standard chart are too large, especially near the periphery, to permit taking full advantage of the inherent precision of the chart. In interpolating between the marked points, some effort must be made to allow for the nonlinearity of the scale.

Undoubtedly a large majority of the impedance-transformation calculations most commonly made using the Smith chart are for situations where the total attenuation of the line length over which the transformation occurs is quite negligible. The calculations in these cases involve only angular motion around the chart, the final point being at the same $|\rho|$ or VSWR coordinate as the original. Examples are the input susceptances of stub lines

(Section 7.3), the transformations of quarter-wavelength transformers (Section 7.4), the input impedances of short feed lines terminated in transmitting antennas, and in general the transformations occurring whenever a short section of transmission line acts as a connector between two components of a high frequency circuit.

Example 9.6.

An air-dielectric slotted section is connected to an air-dielectric transmission line of the same characteristic impedance $50 + j0$ ohms by a reflectionless connector. The transmission line is 3.75 m long and is terminated in an antenna. On the slotted section the voltage standing wave pattern is observed to have a VSWR of 2.25, and there are successive voltage minima at 0.180 and 0.630 m from the connector. The total attenuation of the line and slotted section is negligible. What is the impedance of the antenna at the frequency of the measurements?

As in Example 9.4, the use of the adjective "air-dielectric" is a way of indicating that the phase velocity of the voltage waves on both the slotted section and the transmission line is the free space value for TEM electromagnetic waves, 3.00×10^8 m/sec.

The separation of 0.450 m between consecutive voltage minima shows that the wavelength on the slotted section is 0.900 m. The value is then the same on the transmission line, and the line length in wavelengths is $3.75/0.900 = 4.17$ wavelengths.

For calculating the normalized impedance at the connector, the values VSWR = 2.25 and $d_{V(\min)}/\lambda = 0.180/0.900 = 0.200$ are used. The result is $r_n + jx_n = 1.62 - j0.86$, using the method of Example 9.4. The location of the point on the Smith chart is shown in Fig. 9-16. This value of $r_n + jx_n$ is the normalized input impedance of the transmission line, which is 4.17 wavelengths long. The normalized terminal load impedance connected to the line is therefore found by moving 0.17 WAVELENGTHS TOWARD LOAD, i.e. counterclockwise, along the constant VSWR circle through the normalized input impedance point. The 4.00 wavelengths of the transmission line length have no effect on the result, since they merely represent eight complete rotations around the chart back to the starting point. The normalized impedance of the antenna as terminal load impedance, is found to be $0.77 + j0.70$; and the impedance is $(0.77 + j0.70)(50 + j0) = 37.5 + j35$ ohms. The frequency of measurement is $3.00 \times 10^8/0.900 = 333$ megahertz.

Since the slotted section and the transmission line in this problem have the same characteristic impedance, the same phase velocity, and the same attenuation (zero), it was not in fact necessary to calculate the impedance at the connector as an intermediate step. If the slotted section and transmission line are regarded as a single continuous length of uniform system, the terminal load antenna impedance can be evaluated directly from the fact that it produces a VSWR of 2.25 and a $d_{V(\min)}/\lambda$ of $0.20 + 4.17 = 4.37$.

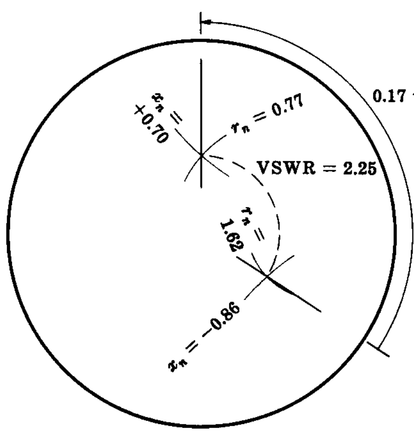


Fig. 9-16. Normalized impedance transformation toward the load on a lossless transmission line.

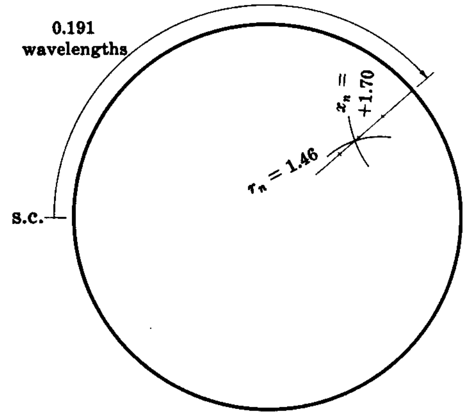


Fig. 9-17. Determining the attenuation factor and phase velocity of a transmission line from the normalized input impedance of a section of the line with short circuit termination.

Example 9.7.

A section of flexible plastic-dielectric coaxial high-frequency transmission line is 24.25 m long. At a frequency of 50.0 megahertz, its characteristic impedance is $72 + j0$ ohms, and the input impedance of the section is measured to be $105 + j122$ ohms at that frequency when the terminal-load end of the section is short-circuited. What is the attenuation factor of the line? What can be determined about the phase velocity on the line from the measured impedance?

The normalized value of the measured input impedance is $1.46 + j1.70$. The location of this point on the Smith chart is shown in Fig. 9-17 above. Since the terminal load impedance is a short circuit, it is located at the zero impedance point of the chart, at the left end of the central horizontal axis. Scaling the radial distance from the periphery of the chart to the normalized input impedance point along the scale of 1 db steps, second from the top at the left in Fig. 9-3, it is found that the 24.25 m length of line must have an attenuation of about 2.25 decibels. Hence the attenuation factor of the line is $2.25/24.25 = 0.093$ db/m = 0.0107 nepers/m.

The angular distance measured clockwise from the radius through the short circuit point to the radius through the input impedance point is 0.193 wavelengths. However, it is impossible that 24.25 m of transmission line should be only 0.193 wavelengths long at a frequency of 50.0 megahertz, since this would indicate a wavelength of 127 m and a phase velocity of 6.35×10^9 m/sec, more than twenty times the free space velocity of plane electromagnetic waves. The length in wavelengths must therefore be some other term in the series 0.193, 0.693, 1.193, 1.693, etc., all of which would involve identical motions on the chart.

The free space wavelength for TEM waves at a frequency of 50.0 megahertz is 6.00 m. The plastic dielectrics most commonly used in flexible coaxial lines have a dielectric constant of about 2.25, and the wave velocity on transmission lines filled with these materials will be less than the free space value by a factor of about $1/\sqrt{2.25} = 0.67$. The wavelength on the line may therefore be about 4.0 m, and the line length in wavelengths would in that case be about 6.0. However, this approximate reasoning cannot justify a decisive choice among the possible values 5.193, 5.693, 6.193, 6.693 and 7.193 offered by the measurements. If the input impedance is measured for a shorter length of the same line, with short circuit termination, another sequence of values of the line length in wavelengths and hence of β and v_p will be found. Generally only one pair of terms will coincide closely in the two series of β values or v_p values, and this will be the correct value for the line.

Example 9.8.

What must be the length of a stub line with open circuit termination, in order that the stub shall have a normalized input reactance of +0.75?

Entering the Smith chart at the open circuit or infinite normalized impedance point as in Fig. 9-18, on the outer boundary of the chart at the right hand end of the central horizontal axis, the normalized impedance of any length of stub line (the name implies negligible losses) will also be found on the outer boundary of the chart, after clockwise rotation (WAVELENGTHS TOWARD GENERATOR, the generator being implicitly at the end opposite the terminal load open circuit impedance) through an angle corresponding to the length of the line in wavelengths.

Fig. 9-18 shows that the rotation required to reach the normalized reactance value of +0.75 is 0.352 wavelengths, which is an answer to the question. Other answers are 0.852, 1.352, 1.852, etc., wavelengths.

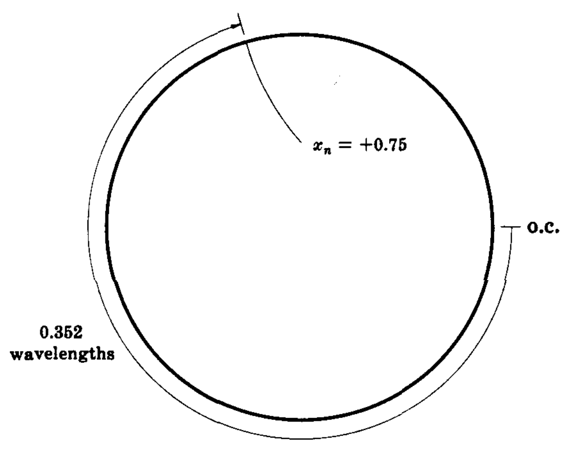


Fig. 9-18. The normalized input susceptance of a section of lossless line with open circuit termination.

9.7. Normalized admittance coordinates.

A quarter wavelength section of lossless transmission line has the property of being an inverter of normalized impedance values (see Section 7.4). The normalized value of the input impedance of such a section is the reciprocal of the normalized value of its terminal load impedance.

If any point on the Smith chart is taken as the normalized terminal load impedance of a lossless transmission line section one quarter wavelength long, the normalized input impedance of the section according to the method of Section 9.6 will be found by rotation through one quarter wavelength (i.e. halfway around the chart) with no change of radial distance from the center of the chart. The new point will thus be diametrically opposite the original point. The quarter wavelength transformer property states that the numerical value of this normalized input impedance is the reciprocal of the normalized value of the initially chosen terminal load impedance which, by definition, is the normalized admittance value of that impedance.

It follows from this example that rotation of the (r_n, x_n) coordinates for the chart as a whole through 180° on the reflection coefficient plane around the center of the chart will substitute normalized admittance coordinates (g_n, b_n) at every point for the normalized impedance coordinates (r_n, x_n). The nature of the (g_n, b_n) coordinates is shown in Fig. 9-19.

The equation $(g_n + jb_n) = 1/(r_n + jx_n) = r_n/(r_n^2 + x_n^2) - jx_n/(r_n^2 + x_n^2)$ shows that the sign of b_n is always opposite to that of x_n. The upper half of the Smith chart contains all normalized impedances for which x_n is positive, or all normalized admittances for which b_n is negative. In the lower half of the chart the signs are reversed.

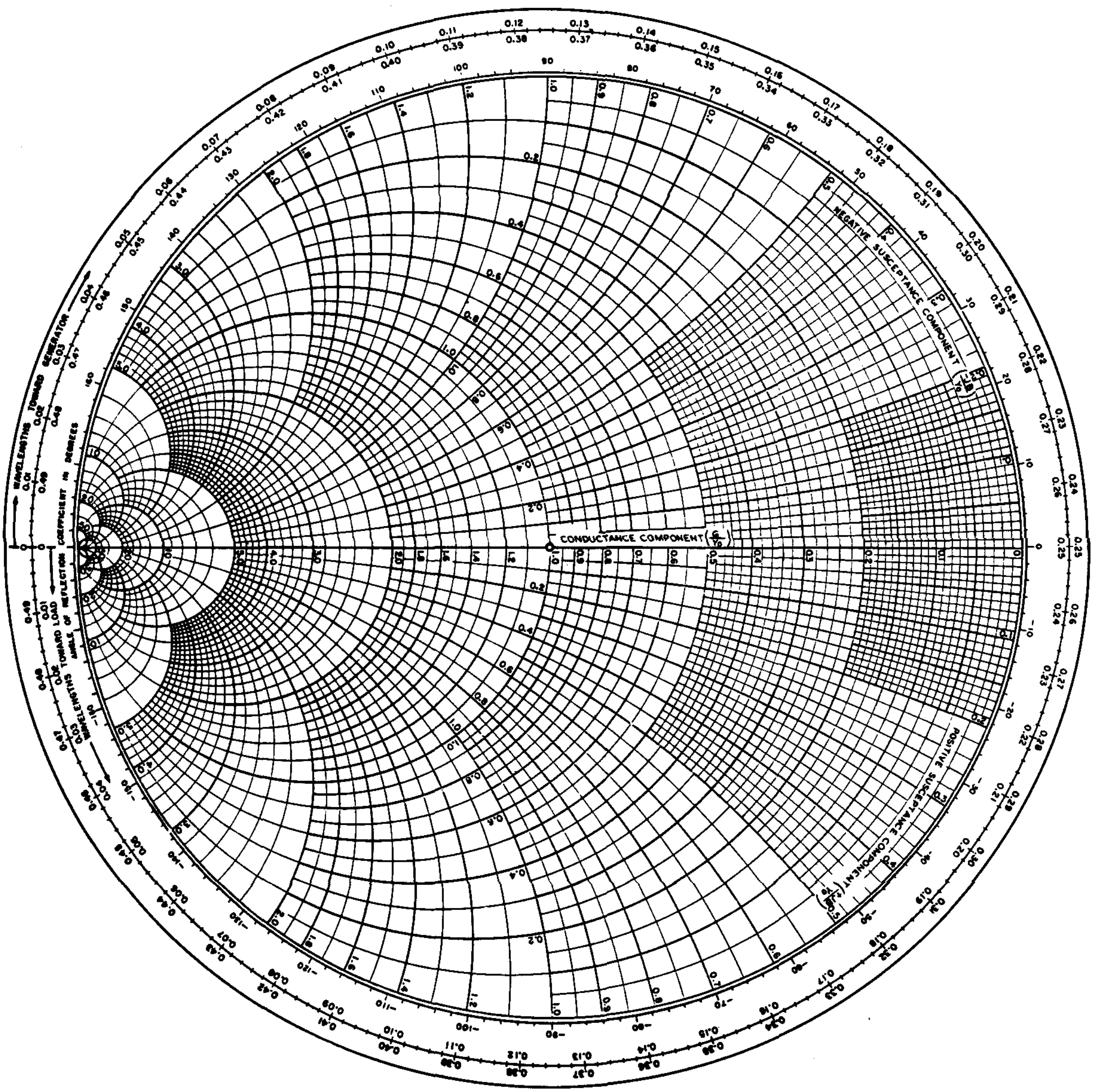
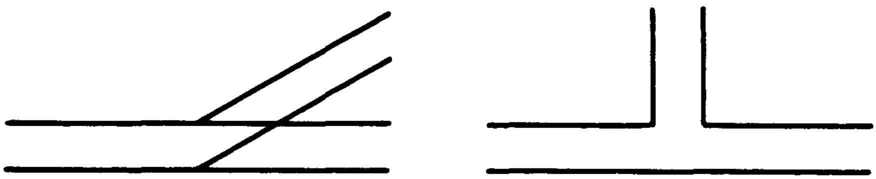


Fig. 9-19. The Smith chart with normalized conductance and susceptance coordinates on the reflection coefficient plane.

In the general study of circuit analysis and its application to such practical situations as filters, matching networks, and electronic amplifiers, experience indicates that the language of admittance, appropriate to parallel-connected circuit elements, has an importance fully comparable to the language of impedance, appropriate to series-connected circuit elements. In the case of transmission lines, when different transmission line sections are connected together in a multi-branch system, they are almost invariably connected in parallel at the junctions (Fig. 9-20(*a*)) rather than in series (Fig. 9-20(*b*)), for reasons that may be partly electrical and partly mechanical. It is not in fact possible to connect a coaxial stub-line in series with another coaxial line without destroying the shielding property of the outside conductors, but a parallel-connected junction of the two raises no difficulties.



(*a*) (*b*)

Fig. 9-20. A branch transmission line connected (*a*) in parallel and (*b*) in series with a main transmission line.

The analyses of Chapter 7 and the discussions of the Smith chart in the preceding sections have used impedance terminology much more freely than that of admittance, but this was solely in the interest of better uniformity and continuity in the presentations, and must not be taken to imply that the impedance language has priority of any kind. An engineer using the Smith chart for calculations on transmission line circuits should be equally well prepared to use the chart in either normalized impedance or normalized admittance coordinates.

Unfortunately, textbooks and other writings about the Smith chart have used two distinctly different conventions as to the manipulative procedures to be used in switching between normalized impedance and normalized admittance coordinates. The distinction between the two conventions is a simple one on the surface, but it can be a source of considerable confusion and error.

In the one convention, used in this book and already described above, the normalized impedance and normalized admittance coordinate grids illustrated by Fig. 9-3 and Fig. 9-19 respectively, are used as separate plots or overlays on the reflection coefficient plane. The absolute orientation of the coordinates of this plane is kept constant, with reflection coefficient phase angle increasing counterclockwise from zero at the right. Since any physical structure connected to a transmission line as a terminal load impedance is uniquely identified by the reflection coefficient it produces, this procedure leaves unchanged the geometric location on the chart of all *physical* connotations, such as the point representing a short-circuit, the point representing an open-circuit, the point representing any given assembly of resistance-inductance-capacitance components, and the zero-reference angle for $d_{V(\min)}/\lambda$ coordinates. This stability of all the physical correspondences on the chart is achieved at the expense of having to rotate the (r_n, x_n) coordinate grid to obtain the (g_n, b_n) coordinate grid. The design of the Smith-chart "slide rule" mentioned above facilitates the use of this convention by permitting rotation of the coordinates within the unit circle relative to the peripheral angular coordinates. In using this convention, the appearance of the unit normalized-real-part circle on the right of the symbolic Smith chart of Fig. 9-21(*a*) below signifies that normalized impedance coordinates are being used, while if the unit normalized-real-part circle is on the left as in Fig. 9-21(*b*) below, normalized admittance coordinates are being used.

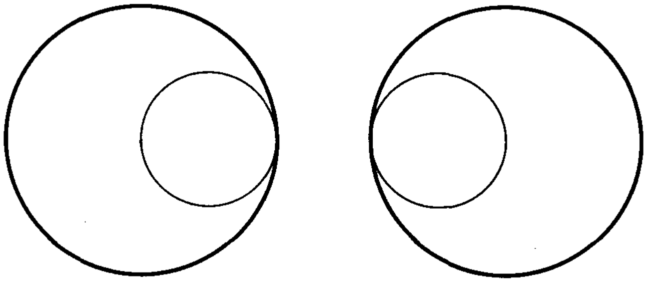


Fig. 9-21. Symbolic representation of the relation between Smith chart orientation and the use of normalized impedance or normalized admittance coordinates. With the unit-normalized-real-part circle on the right of the chart as in (*a*), the chart is oriented for normalized impedance coordinates, and the small circle represents unit normalized resistance. When the small circle is on the left, as in (*b*), the chart is oriented for normalized admittance coordinates, and the small circle represents unit normalized conductance.

In the alternative convention, the (r_n, x_n) *coordinates as printed in Fig.* 9-3 *are declared* to be either normalized impedance or normalized admittance coordinates as needed, being kept in the orientation shown in both cases. All the physical features of the chart are then rotated through 180° when changing from the (r_n, x_n) coordinates to the (g_n, b_n) coordinates. The short-circuit point, for example, will be at the left for normalized impedance coordinates and at the right for normalized admittance coordinates.

Printed Smith chart graph sheets generally have a statement on them reading "Impedance or Admittance Coordinates", which can be seen at the top of Fig. 9-3. They also have, as mentioned in Section 9.3, a fixed peripheral angular scale identified as "Angle of Reflection Coefficient in Degrees", reading counterclockwise from zero at the right. The discussion of Section 9.2 has shown that (r_n, x_n) coordinates appear in the form of Fig. 9-3 for such a reflection-coefficient phase angle scale. Hence the label "Impedance or Admittance Coordinates" is valid only when accompanied by the additional instruction that if the chart is to be used in admittance coordinates, *either* the (r_n, x_n) coordinates must be rotated 180° on the reflection coefficient plane to become (g_n, b_n) coordinates, *or* the reflection coefficient phase angle scale (i.e. the entire reflection coefficient plane) and all of its physical concomitants must be rotated 180° to allow the (r_n, x_n) coordinates to become the (g_n, b_n) coordinates without change of position. This book chooses the former of these alternatives.

Example 9.9.

A VSWR of 3.25 is observed on a slotted-line section, with a voltage minimum 0.205 wavelengths from the terminal load end of the section. What is the value of the normalized admittance at the terminal load end?

Following the convention of Fig. 9-21(*b*), the point corresponding to VSWR = 3.25 and $d_{V(\min)}/\lambda =$ 0.205 is located on the chart oriented for (g_n, b_n) coordinates as in Fig. 9-22 below. The normalized admittance value is found to be $0.33 + j0.26$. Note that the origin of the $d_{V(\min)}/\lambda$ coordinates is still the left hand radius of the chart.

Example 9.10.

What length of lossless transmission line with short circuit termination and characteristic impedance of $75 + j0$ ohms will have a capacitive input susceptance of 0.0250 mhos?

The characteristic admittance of the transmission line is $Y_0 = 1/Z_0 = 1/(75 + j0) = 0.0133 + j0$ mhos. The normalized input susceptance required is therefore $0.0250/0.0133 = 1.87$. The normalized input admittance $0 + j1.87$ is located on (g_n, b_n) coordinates as shown in Fig. 9-23 below. The required line length in wavelengths is found as the angular distance in WAVELENGTHS TOWARD LOAD over which this normalized admittance will transform to the infinite admittance of a short circuit. (Alternatively, the required length is the angular distance in WAVELENGTHS TOWARD GENERATOR over which a short

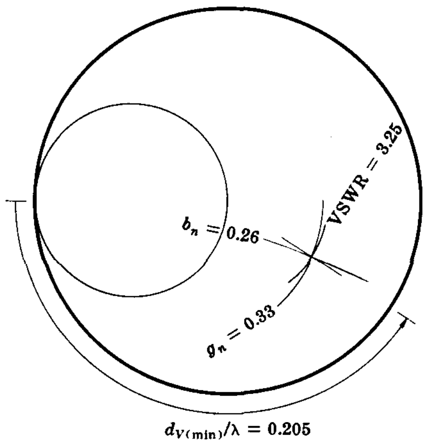


Fig. 9-22. Determination of normalized terminal load admittance from standing wave data on a lossless line.

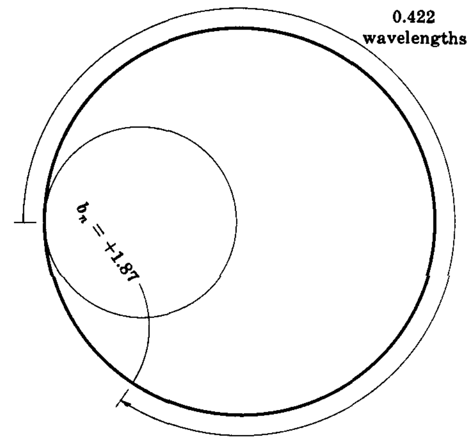


Fig. 9-23. Normalized input susceptance of a lossless line section with short circuit termination.

circuit will transform to the desired value of normalized admittance.) The result is found to be 0.422 wavelengths. Answers of 0.922, 1.422, etc., wavelengths are equally correct.

Just as the Carter chart of Section 9.5 and Fig. 9-13 presented coordinates of magnitude and phase angle of normalized impedance on the reflection coefficient plane, so the same coordinates rotated 180° on the plane become coordinates of magnitude and phase angle of normalized admittance, as indicated in Fig. 9-24. Fig. 9-24 is obtained from Fig. 9-13 by changing the sign of the phase angle coordinates and changing the value on each normalized magnitude coordinate to its reciprocal.

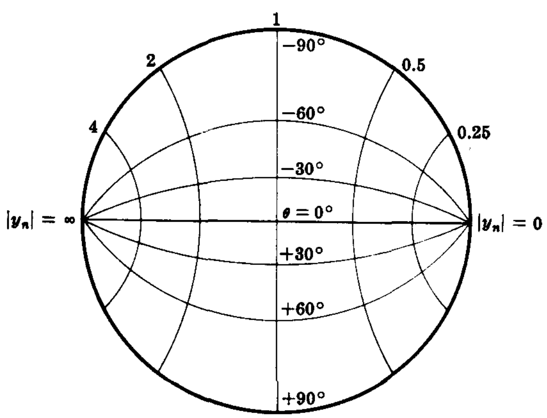


Fig. 9-24. The Carter chart with coordinates of magnitude and phase angle of normalized admittance.

9.8. Inversion of complex numbers.

It is worth noting that the Smith chart can be used as a graphical device for finding the reciprocal of any complex number, even if the calculation has no reference to transmission lines.

Example 9.11.

Find the reciprocal of $244 - j38$.

It is obviously not satisfactory to take these numbers directly into the Smith chart as (r_n, x_n) coordinates, since the result would be indistinguishable from the infinity point of the chart. Inspection of Fig. 9-13 shows that the magnitude and phase angle scales for normalized impedance or admittance are most expanded in the vicinity of the central vertical line of the chart, i.e. where the normalized magnitude is of the order of unity.

If the complex number $244 - j38$ is "normalized" relative to some simple real number such as 200 or 300, approximately equal to the magnitude of $244 - j38$, the arithmetic is easy and the result will be located on the (r_n, x_n) or (g_n, b_n) coordinates of the chart near the unit normalized magnitude locus. The "normalized" reciprocal is found at the diametrically opposite point on the chart, and the final result obtained by denormalizing this relative to the original normalizing reference.

Normalizing $244 - j38$ relative to 200 gives $1.22 - j0.19$. The location of this point on (r_n, x_n) coordinates is shown in Fig. 9-25. Its reciprocal, found as the coordinates of the diametrically opposite point, has the value $0.80 + j0.124$. Multiplying by 1/200 denormalizes this, giving the result

$$(4.00 + j0.62) \times 10^{-3}$$

which is the answer found for the reciprocal of $244 - j38$. The probable error in each component in such a calculation should not exceed about 1% of the larger component. It will be seen that the "normalizing" and "denormalizing" processes in this operation are not reciprocally related as they were in the transformation calculations of Section 9.6, but are identical.

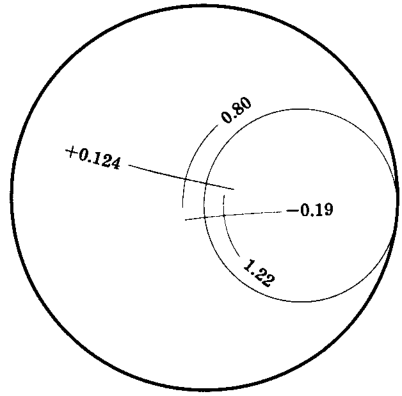


Fig. 9-25. Use of the Smith chart to find the reciprocal of a complex number.

9.9. Other mathematical uses of the Smith chart.

From derivations given previously in this chapter and in Chapter 7, fairly simple graphical procedures make it possible to use the Smith chart to evaluate complex hyperbolic tangents and cotangents, i.e. $\tanh(x + jy)$ and $\coth(x + jy)$ and, as special cases of these, circular tangents and cotangents $\tan x$ and $\cot x$. It can also be used to evaluate complex exponential numbers, $e^{\pm(x+jy)}$, and by extension $\sinh(x + jy)$ and $\cosh(x + jy)$.

The demonstrations of the uses of the chart for some of these purposes are assigned as problems below.

9.10. Return loss, reflection loss, and transmission loss.

Three of the radial scales shown on the Smith chart of Fig. 9-3 remain to be explained. These are the two at the bottom right, and the one at the top left, designated respectively as "return loss in db", "reflection loss in db", and "transmission loss coefficient". As the names suggest, they are various ways of expressing power relations on transmission lines in the presence of both reflected waves and line attenuation. All three scales are valid only for lines whose characteristic impedance is real. This requires that the attenuation *per wavelength* on the lines be much less than one neper, but places no limitation on the allowed total attenuation of a line. (If a line's attenuation factor is caused partly by distributed resistance R and partly by distributed conductance G, the specification on attenuation per wavelength applies to the *difference* between the two contributions. For Heaviside's "distortionless" line the difference is zero, and the characteristic impedance is always real, for all values of attenuation per wavelength.)

The concept of "return loss" is a simple and straightforward one. At any specific point on a uniform transmission line the power carried by the reflected wave (traveling from the terminal load toward the signal source) will be less than the power carried by the incident wave (traveling from the signal source toward the terminal load) when the magnitude of the reflection coefficient at the terminal load end of the line is less than unity, or when there is attenuation between the specific point and the terminal load.

The return loss in decibels at a point on a transmission circuit is defined as the total loss or attenuation in decibels which the incident wave power at the point would have to experience to be reduced to the reflected wave power at the point. From this definition, and from relations derived in Section 8.10,

$$\text{return loss} \;=\; 10 \log_{10} |\rho|^2 \quad \text{decibels} \qquad (9.14)$$

where ρ is the voltage reflection coefficient at the point in question.

From equations (*4.16*), page 36, and (*9.12*), page 194, if the voltage reflection coefficient at the terminal load end of a line is ρ_T, and the line has an attenuation factor α nepers/m, the return loss at a point distant d meters from the terminal load end of the line will be given by

$$\text{return loss} \;=\; 10\log_{10}|\rho_T|^2 + 8.686\times 2\alpha d \quad \text{decibels} \qquad (9.15)$$

The value of the return loss at a point on a system is useful as an indication of the extent to which reflected waves may be degrading the operation of the system at that point. Such degradation might result, for example, from the reflected waves being disturbing echo signals, or from their affecting the frequency or power output of a signal source by causing the input impedance of the connected transmission line circuit to be different from the characteristic impedance of the line. The design specifications for some forms of transmission line communication circuits may state a minimum value of return loss that must be maintained over some portion of the system.

It will be noted that the return loss scale (second from the bottom at the right in Fig. 9-3) is identical in structure to the scale of 1 db steps of transmission loss (second from the top at the left) discussed in Section 9.6. It differs in having a numerical scale reading from zero at the periphery of the chart to infinity at the center, and because of the "return" factor, the scale increases by 2 decibels for each 1 db step of the transmission loss scale.

The concept of "reflection loss" (presented by the bottom scale on the right in Fig. 9-3) embodies the proposition that power reflected by a terminal load impedance Z_T not equal to Z_0 is lost power, relative to the power that would be delivered to a nonreflecting terminal load. The concept is directly applicable only to transmission line circuits of the form shown in Fig. 9-26, in which the source impedance is equal to the characteristic impedance of the line. The line may have any value of total attenuation provided the attenuation per wavelength is small enough to ensure that the characteristic impedance has a negligible phase angle.

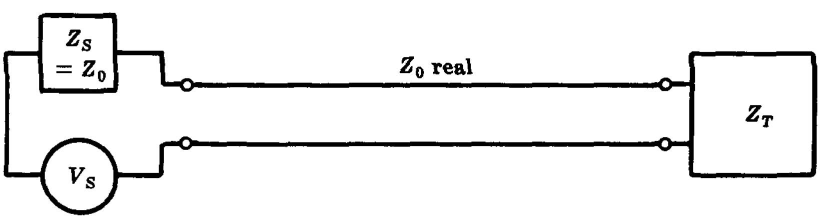


Fig. 9-26. A transmission line circuit to which the Smith chart's radial "Reflection Loss" scale is applicable.

For this circuit the power delivered to a nonreflecting terminal load impedance $Z_T = Z_0$ is $\frac{1}{4}(|V_s|^2/Z_0)e^{-2\alpha l}$ watts, since the input impedance of the line is also Z_0. Applying the multiple-reflection analysis of Section 8.8, page 174, to the circuit, this is also the power of the initial wave that travels the length of the line and is incident on the terminal load, after the source is connected to the line. If the terminal load impedance Z_T is not equal to Z_0, the power in the first wave reflected by the termination will be $\frac{1}{4}(|V_s|^2/Z_0)e^{-2\alpha l}|\rho_T|^2$, where $\rho_T = (Z_T - Z_0)(Z_T + Z_0)$. At the signal source end of the line the reflection coefficient $\rho_S = 0$ (because $Z_S = Z_0$), so that none of the power reflected by the terminal load impedance is re-reflected on returning to the input end of the line. The above expressions for power therefore give, respectively, the *total* power in the incident wave and the *total* power in the reflected wave, at the terminal load impedance where reflection occurs. The power delivered to the load is therefore given by $\frac{1}{4}(|V_s|^2/Z_0)e^{-2\alpha l}(1 - |\rho_T|^2)$. Reflection loss is defined from the ratio of this power to the power in the incident wave, and is invariably expressed in decibels. It is often referred to in circuit language as "mismatch loss". The definition is

$$\text{reflection loss} \quad = \quad -10 \log_{10}(1-|\rho_T|^2) \quad \text{decibels} \tag{9.16}$$

The negative sign in equation (*9.16*) is required by the convention that reflection loss is always stated as a positive number of decibels.

If the transmission line of Fig. 9-26 is lossless, conservation of energy demands that the presence of a singly reflected wave must cause the source to deliver less power to the input terminals of the line by exactly the amount of power in the reflected wave. Since the line length is arbitrary, the input impedance of the line may lie anywhere on the circle of constant $|\rho_T|$ on the Smith chart, where ρ_T is the reflection coefficient produced by the terminal load. Viewing the situation as a circuit problem, it is not at all obvious that the required reduction in input power will occur identically for all of these possible values of input impedance.

The following analysis confirms that the indicated reduction in power does occur, and shows in addition that if the line has finite total attenuation (but a real characteristic impedance) the reduction in power delivered by a nonreflecting source to a line, when the terminal load impedance is changed from nonreflecting to reflecting, is equal to the power in the reflected wave at the point of reflection less the power lost by the reflected wave in the attenuation of the line.

Applying equations (*7.1*), (*7.3*) and (*7.8*) to the circuit of Fig. 9-26,

$$V(z=0) \equiv V_{\text{inp}} = V_1 + V_2 = V_1(1+\rho_T e^{-2\gamma l}) \tag{9.17}$$

$$I(z=0) \equiv I_{\text{inp}} = (V_1/Z_0)(1-\rho_T e^{-2\gamma l}) \tag{9.18}$$

But $V_{\text{inp}} = V_S - I_{\text{inp}}Z_S = V_S - I_{\text{inp}}Z_0$. Therefore $V_1(1+\rho_T e^{-2\gamma l}) = V_S - V_1(1-\rho_T e^{-2\gamma l})$, and $V_1 = \frac{1}{2}V_S$, a result which could have been written directly by using the multiple reflection analysis of Section 8.8. It is important to note that this statement is not equivalent to $V_{\text{inp}} = \frac{1}{2}V_S$, which would be true only if $Z_T = Z_0$ and $\rho_T = 0$.

Equations (*9.17*) and (*9.18*) can then be rewritten,

$$V_{\text{inp}} = \tfrac{1}{2}V_S(1+\rho_T e^{-2\gamma l}) \tag{9.19}$$

$$I_{\text{inp}} = \tfrac{1}{2}(V_S/Z_0)(1-\rho_T e^{-2\gamma l}) \tag{9.20}$$

From equations (*9.19*) and (*9.20*),

$$\frac{Z_{\text{inp}}}{Z_0} = \frac{R_{\text{inp}}}{Z_0} + j\frac{X_{\text{inp}}}{Z_0} = \frac{1}{Z_0}\cdot\frac{V_{\text{inp}}}{I_{\text{inp}}} = \frac{1+\rho_T e^{-2\gamma l}}{1-\rho_T e^{-2\gamma l}} \tag{9.21}$$

which agrees with equation (*7.14*), page 130.

If $Z_T = Z_0$, $\rho_T = 0$. Then the power input to the line is $P_0 = \frac{1}{4}|V_S|^2/Z_0$, and the power reaching the terminal load is $P_0 e^{-2\alpha l}$. If $Z_T \neq Z_0$, the power in the initial incident wave reaching the terminal load (which is the *only* incident wave) is still $P_0 e^{-2\alpha l}$. The power in the reflected wave *at the point of reflection* is $P_0 e^{-2\alpha l}|\rho_T|^2$, and the power received by the load is $P_0 e^{-2\alpha l}(1-|\rho_T|^2)$.

When the reflected wave returns to the input end of the line its power level is reduced by a further factor $e^{-2\alpha l}$, to become $P_0 e^{-4\alpha l}|\rho_T|^2$.

It is desired to prove, using circuit analysis concepts, that for all values of Z_T and γl the reduction in the power delivered by the source to the line when the terminal load impedance is changed from $Z_T = Z_0$ to $Z_T \neq Z_0$ is identically equal to the amount of reflected wave power that returns to the input end of the line.

If P_0' is the input power to the line when $Z_T \neq Z_0$, the problem is therefore to show that

$$P_0 - P_0' = P_0 e^{-4\alpha l}|\rho_T|^2$$

or

$$P_0' = P_0(1 - |\rho_T|^2 e^{-4\alpha l}) = \tfrac{1}{4}(|V_S|^2/Z_0)(1 - |\rho_T|^2 e^{-4\alpha l})$$

Since $P_0' = \left|\dfrac{V_S}{Z_0 + Z_{\text{inp}}}\right|^2 \cdot R_{\text{inp}} = \dfrac{V_S^2}{Z_0}\cdot\dfrac{1}{|1 + Z_{\text{inp}}/Z_0|^2}\cdot\dfrac{R_{\text{inp}}}{Z_0}$, the requirement is to prove that

$$\frac{1}{|1 + Z_{\text{inp}}/Z_0|^2}\cdot\frac{R_{\text{inp}}}{Z_0} = \tfrac{1}{4}(1 - |\rho_T|^2 e^{-4\alpha l}) \tag{9.22}$$

To simplify the arithmetical work, let $\rho_T e^{-2\gamma l} = C + jD$. Then $C^2 + D^2 = |\rho_T|^2 e^{-4\alpha l}$.

From equation (*9.21*),

$$\frac{Z_{\text{inp}}}{Z_0} = \frac{1 + (C + jD)}{1 - (C + jD)} = \frac{1 - (C^2 + D^2) + 2jD}{1 + (C^2 + D^2) - 2C}$$

so that

$$\frac{R_{\text{inp}}}{Z_0} = \frac{1 - (C^2 + D^2)}{1 + (C^2 + D^2) - 2C} \tag{9.23}$$

Also,

$$\frac{1}{|1 + Z_{\text{inp}}/Z_0|^2} = \frac{1}{\left|\dfrac{2(1 + C + jD)}{1 + (C^2 + D^2) - 2C}\right|^2} = \frac{1 + (C^2 + D^2) - 2C}{4} \tag{9.24}$$

From equations (*9.23*) and (*9.24*),

$$\frac{1}{|1 + Z_{\text{inp}}/Z_0|^2}\cdot\frac{R_{\text{inp}}}{Z_0} = \tfrac{1}{4}\{1 - (C^2 + D^2)\} = \tfrac{1}{4}(1 - |\rho_T|^2 e^{-4\alpha l})$$

which proves the theorem.

Although the power delivered by the source to the line is thus shown to be reduced by the amount of the reflected power returning to the input terminals, in agreement with the conclusion obtained by applying the principle of energy conservation to the multiply reflected wave model of the system, the implication of the latter reasoning that the reflected wave power is entirely absorbed in the source impedance without affecting the total output of the signal source generator, is incorrect. This is easily seen by considering the simple case of changing Z_T from Z_0 to $2Z_0$ on a lossless line whose length is an integral number of half-wavelengths.

The multiply reflected wave model deals directly only with current or voltage waves, not with power. The phasor voltage at the input terminals of a line is the sum of the phasor voltages of the incident and reflected voltage waves at that point, and the relative phase of these two voltages can have any value whatever, depending on the terminal load reflection coefficient ρ_T and the electrical length of the line βl in radians. For the circuit of Fig. 9-26, when the power input to the line is reduced by the amount of any reflected wave power reaching the input terminals, this will in general result from a change in *both* the output power of the signal source, and the amount of power dissipated in the source impedance $Z_S = Z_0$.

The eighth and final radial scale in Fig. 9-3 to be discussed is the "transmission loss coefficient" scale at the top left. This scale purports to give the numerical ratio (*not* in decibels) of a line's attenuation losses in the presence of reflected waves to the attenuation losses in the absence of reflected waves, for the same power delivered to the terminal load in each case. In fact, the scale is applicable with useful accuracy only to certain relatively unimportant situations.

The analytical basis of the scale is as follows:

For the circuit of Fig. 9-26, with Z_0 real, the power input to the transmission line when $Z_T = Z_0$ is $P_0 = \frac{1}{4}|V_S|^2/Z_0$, as seen previously. The power reaching the load is $P_0 e^{-2\alpha l}$, and the power dissipated by attenuation is the difference between these two figures, i.e. $P_0(1 - e^{-2\alpha l})$. When $Z_T \neq Z_0$, but produces a reflection coefficient ρ_T at the terminal load, it has been shown above that the power input to the line falls to $P_0(1 - |\rho_T|^2 e^{-4\alpha l})$, while the power delivered to the load becomes $P_0 e^{-2\alpha l}(1 - |\rho_T|^2)$. The difference is $P_0(1 - e^{-2\alpha l})(1 + |\rho_T|^2 e^{-2\alpha l})$, which is the power dissipated in the line. The ratio of this to the power dissipated in the first case is $(1 + |\rho_T|^2 e^{-2\alpha l})$. Since this quantity must always be greater than unity, there is *always* an absolute increase of attenuation losses for the transmission line circuit of Fig. 9-26, when the terminal load impedance Z_T is changed from Z_0 (real) to some other value, with no alteration in the signal source. (This argument requires that Z_0 be *exactly* real, which is the unusual case of the Heaviside line whose losses are equally divided between distributed resistance and distributed conductance.)

To make the power delivered to the load have the same value in both of the above cases, all quantities in the second calculation must be multiplied by $1/(1 - |\rho_T|^2)$. It then follows from the definition that

$$\text{transmission loss coefficient} = \frac{1 + |\rho_T|^2 e^{-2\alpha l}}{1 - |\rho_T|^2} \qquad (9.25)$$

The transmission loss coefficient scale in Fig. 9-3 is calculated from this equation, with $e^{-2\alpha l} = 1$. Apparently, therefore, the scale is directly applicable only to lossless lines, for which the transmission loss coefficient is meaningless, or to impractical lines whose characteristic impedance is identically real. Actually, for practical high frequency lines whose characteristic impedance is nearly but not exactly real, the scale gives useful approximate values of the factor by which line losses are increased in the presence of reflected waves in two cases. The first is that of lines many wavelengths long, with terminal reflection coefficient ρ_T and low total attenuation $(\alpha l \ll 1)$. A second somewhat more general application is to any half-wavelength segment of a line with fairly low losses per wavelength $(\alpha/\beta \ll 1)$, ρ_T being replaced in this case by the reflection coefficient ρ at the center of the particular half-wavelength segment.

It will be noted that if the total line attenuation exceeds about 20 db, the transmission loss coefficient becomes $1/(1 - |\rho_T|^2)$, independent of the line attenuation. There is no radial scale in Fig. 9-3 for this result.

In using the concepts of reflection loss and transmission loss coefficient to make calculations on transmission line circuits having the form of Fig. 9-26, with finite line attenuation, care must be taken not to count any of the loss components twice. When the terminal load impedance Z_T is changed from Z_0 to some other value, without changing V_S, it has been seen that several changes in the power relations in the circuit occur simultaneously: (*a*) a reflected wave is created, which constitutes a nondissipative reduction in the power delivered to the terminal load; (*b*) the reflected wave suffers dissipative power loss in the attenuation of the line; (*c*) the input power to the line is reduced; (*d*) the total attenuation losses in the line are increased.

The analysis has shown that these different aspects of the situation are not independent. The power reduction (*a*) is equal to the sum of the power loss (*b*) and the power reduction (*c*). The extra dissipative power loss (*d*) is identical to the power loss (*b*).

Example 9.12.

In a transmission line circuit having the form of Fig. 9-26, the magnitude of the source voltage is 10.0 volts r.m.s. The source impedance is equal to the characteristic impedance $Z_0 = 50 + j0$ ohms. The terminal load impedance Z_T is $150 + j0$ ohms. The line has a total attenuation of 6.00 db and is 100 wavelengths long. Find the reflection loss, the power reaching the load, the total power losses in the line, and the power supplied by the source to the line.

The terminal load reflection coefficient is $0.5 + j0$, so that $|\rho_T| = 0.5$. From the radial scale of Fig. 9-3, the reflection loss is found to be 1.25 db. For the same circuit with $Z_T = Z_0 = 50 + j0$ ohms, the input power to the line is easily calculated as 0.500 watts, the power delivered to the load is 0.125 watts, and the power dissipated in the line attenuation is the difference, or 0.375 watts. From the definition of reflection loss, the power reaching the load when $Z_T = 150 + j0$ ohms is P_T, given by $10 \log_{10}(0.125/P_T) = 1.25$. The result is $P_T = 0.0937$ watts, a reduction of 0.031 watts from the case $Z_T = Z_0$. Referring to the derivation of equation (*9.25*), the factor by which reflected waves increase line losses when V_S remains constant is given by the numerator of the equation, which has the value 1.062 for the above data. Hence the line losses with $Z_T = 150 + j0$ ohms are $0.375 \times 1.062 = 0.398$ watts, an increase of 0.023 watts relative to the case $Z_T = Z_0$. The reduction in the power delivered by the source to the line is finally $0.031 - 0.023 = 0.008$ watts which can also be calculated from $0.500|\rho_T|^2 e^{-4\alpha l}$.

Solved Problems

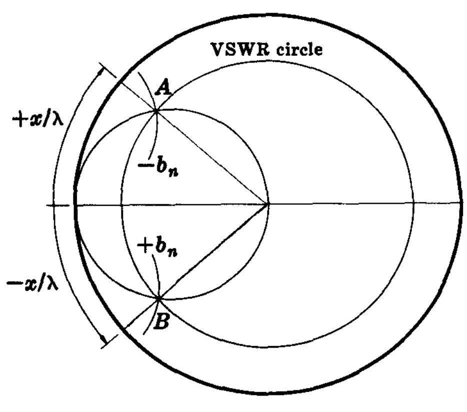
9.1. Use the Smith chart to solve the "single stub matching" problem previously considered in Problems 7.7, page 146, and 8.3, page 178.

The problem has two parts:

(*a*) To show that on a transmission line having negligible attenuation per wavelength there are two locations in every half wavelength at which the real part of the normalized admittance is unity, to find the locations of these points, and to determine the normalized susceptance on the line at each of the locations.

(*b*) To find the length of lossless stub line with open circuit or short circuit termination whose normalized input susceptance will be equal and opposite to the values found in (*a*).

On the transmission line of part (*a*) there will be some value of VSWR created by the connected terminal load. If the circle for this VSWR value is drawn on a Smith chart oriented for admittance coordinates (Fig. 9-27(*a*)), it intersects the unit normalized conductance circle in two points *A* and *B*, which are symmetrically above and below the left hand horizontal radius of the chart. Since this radius represents the location of all voltage minima on the line (it has the coordinate $d_{V(\min)} = 0$), the points at which the normalized conductance on the line is unity occur in pairs at locations equidistant on either side of any voltage minimum. The distance of each such point from a voltage



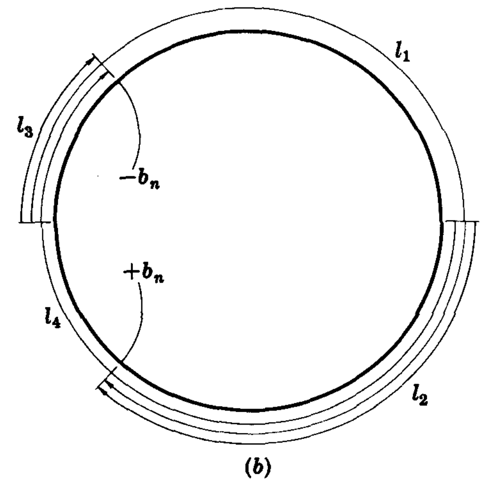


Fig. 9-27. Use of the Smith chart to solve the single stub matching procedure.

(*a*) The stub may be placed at $+x/\lambda$ (WAVELENGTHS TOWARD GENERATOR) or $-x/\lambda$ (WAVELENGTHS TOWARD LOAD) from any voltage minimum in the standing wave pattern. The normalized input susceptance of a stub placed at $+x/\lambda$ must be $+b_n$, and the normalized input susceptance of a stub placed at $-x/\lambda$ must be $-b_n$.

(*b*) Determination of the matching stub lengths.
l_1 is for a stub with short circuit termination to be placed at $+x/\lambda$;
l_2 is for a stub with open circuit termination to be placed at $+x/\lambda$;
l_3 is for a stub with short circuit termination to be placed at $-x/\lambda$;
l_4 is for a stub with open circuit termination to be placed at $-x/\lambda$.

minimum can be read directly from the chart, as $\pm x/\lambda$ in Fig. 9-27(*a*). The normalized susceptance coordinates of the two points are of equal magnitude but opposite sign, and are found as $\mp b_n$ in Fig. 9-27(*a*). To find the lengths of stub lines required to achieve the matching operation, confusion is reduced if reference is made to a second Smith chart, Fig. 9-27(*b*), on which the normalized admittances $y_n = 0 \pm jb_n$ have been located. By the processes described in Section 9.6, four possible lengths of stub line with open circuit or short circuit termination are found to be solutions to the problem, as indicated in Fig. 9-27(*b*). l_1 is the length of a stub line with short circuit termination that could be connected at point A; l_2 is the length of a stub line with open circuit termination that could be connected at point A; l_3 is the length of a stub line with short circuit termination that could be connected at point B, and l_4 is the length of a stub line with open circuit termination that could be connected at point B.

9.2. Because of the poor detail in the scale of "1 db steps" accompanying the commercially printed Smith chart, graphical calculations on lossy transmission lines cannot be made with as good accuracy as calculations for lossless lines. Create an appropriate table that can be used in place of the scale.

From equation (*9.11*), if a transmission line section of length l and attenuation factor α is terminated in a short circuit at $d = 0$, the magnitude of the reflection coefficient at the input terminals will be $|\rho| = e^{-2\alpha l}$, since the magnitude of the reflection coefficient produced by a short circuit is unity.

If the input terminals of the transmission line section were connected to a slotted line section of the same characteristic impedance and negligible losses, the VSWR on the slotted section would be $(1+|\rho|)/(1-|\rho|) = (1+e^{-2\alpha l})/(1-e^{-2\alpha l}) = \coth \alpha l$. If the transmission line section itself has low attenuation *per wavelength*, the VSWR near its input terminals will have this same value.

Table 9.4 shows these values of $|\rho|$ and VSWR as functions of αl in decibels. Since $|\rho|$ is the linear radial coordinate of the Smith chart, the values of αl are exactly the values that would be read from the "1 db steps" scale, reading radially inward from the periphery of the chart, at any value of $|\rho|$ on the chart. The table can therefore be used to solve all problems dependent on equation (*9.12*), such as Example 9.7.

Table 9.4

$\|\rho\|$	VSWR	αl db	$\|\rho\|$	VSWR	αl db	$\|\rho\|$	VSWR	αl db	$\|\rho\|$	VSWR	αl db
1.00	inf.	0	.74	6.70	1.31	.48	2.85	3.19	.22	1.57	6.59
.99	199	.043	.73	6.42	1.37	.47	2.78	3.28	.21	1.53	6.79
.98	99	.087	.72	6.15	1.43	.46	2.71	3.37	.20	1.50	7.00
.97	66	.132	.71	5.90	1.49	.45	2.64	3.47	.19	1.47	7.22
.96	49	.177	.70	5.67	1.55	.44	2.57	3.57	.18	1.44	7.46
.95	39	.222	.69	5.45	1.61	.43	2.51	3.67	.17	1.41	7.70
.94	32	.268	.68	5.25	1.67	.42	2.45	3.77	.16	1.38	7.96
.93	27.6	.314	.67	5.06	1.73	.41	2.39	3.87	.15	1.35	8.24
.92	24.0	.360	.66	4.88	1.80	.40	2.33	3.98	.14	1.32	8.54
.91	21.2	.407	.65	4.72	1.87	.39	2.28	4.09	.13	1.29	8.86
.90	19.0	.455	.64	4.56	1.94	.38	2.23	4.20	.12	1.26	9.21
.89	17.2	.51	.63	4.41	2.01	.37	2.18	4.31	.11	1.24	9.59
.88	15.7	.56	.62	4.27	2.08	.36	2.13	4.43	.10	1.22	10.0
.87	14.4	.61	.61	4.13	2.15	.35	2.08	4.55	.09	1.20	10.4
.86	13.3	.66	.60	4.00	2.22	.34	2.03	4.68	.08	1.17	10.9
.85	12.3	.71	.59	3.87	2.29	.33	1.99	4.81	.07	1.15	11.5
.84	11.5	.76	.58	3.76	2.36	.32	1.95	4.94	.06	1.13	12.2
.83	10.8	.81	.57	3.65	2.44	.31	1.91	5.08	.05	1.10	13.0
.82	10.1	.86	.56	3.55	2.52	.30	1.87	5.22	.04	1.08	14.0
.81	9.53	.91	.55	3.45	2.60	.29	1.83	5.37	.03	1.06	15.2
.80	9.00	.96	.54	3.35	2.68	.28	1.79	5.52	.02	1.04	17.0
.79	8.52	1.01	.53	3.26	2.76	.27	1.75	5.68	.01	1.02	20.0
.78	8.10	1.07	.52	3.17	2.84	.26	1.71	5.85	.005	1.01	23.0
.77	7.70	1.13	.51	3.08	2.92	.25	1.67	6.02	.0025	1.005	26.0
.76	7.33	1.19	.50	3.00	3.01	.24	1.63	6.20	.0010	1.002	30.0
.75	7.00	1.25	.49	2.92	3.10	.23	1.60	6.39	.0005	1.001	33.0

9.3. A transmission line is 2.00 wavelengths long at its frequency of operation. It is terminated in a normalized impedance $0.25 - j1.80$. What is the normalized input impedance of the section if its total attenuation is (*a*) zero, (*b*) 1.0 db, (*c*) 3.0 db, (*d*) 10.0 db?

The normalized terminal load impedance is located in the fourth quadrant of the Smith chart, fairly close to the perimeter, at a VSWR of approximately 20.

To find the normalized input impedance of the section, the first motion required is 2.00 WAVELENGTHS TOWARD GENERATOR on a circle of constant VSWR. This results in returning identically to the original point. The second motion required is to move radially inward by the indicated amount of attenuation on the "1 db steps" radial scale of the Smith chart. This second motion can be performed more accurately using Table 9.4, transferring between chart and table via either reflection coefficient magnitude or VSWR.

The resulting normalized input impedances for the transmission line section are (*a*) $0.25 - j1.80$, (*b*) $0.68 - j1.62$, (*c*) $1.11 - j1.06$, (*d*) $1.08 - j0.17$. A progression toward the value $1 + j0$ is evident, and that would be the graphically determined answer if the total attenuation of the section exceeded 25 or 30 db.

9.4. There is a VSWR of 1.25 on a low loss transmission line. Determine the maximum phase angle possible for the impedance at any point on the line.

Inspection of the Carter chart of Fig. 9-13 and 9-24 shows that the maximum phase angle of the normalized impedance or admittance that can exist on a low loss transmission line for any specific value of VSWR occurs at points where the normalized impedance magnitude is unity. At such points the reflection coefficient phase angle is ±90°. With a VSWR of 1.25, the normalized impedance at such points is found from the Smith chart to be $0.98 \pm j0.21$, and the phase angle is $\tan^{-1}(\pm 0.21/0.98) = \pm 12.0°$.

9.5. A resistive load of $800 + j0$ ohms is matched to a parallel wire transmission line for which $Z_0 = 200 + j0$ ohms by a quarter wavelength transformer consisting of a quarter wavelength section of lossless parallel wire transmission line having a characteristic impedance of $400 + j0$ ohms. At the design frequency the VSWR on the 200 ohm line is less than 1.01. Use the Smith chart to determine the percent by which the frequency can be varied above and below the design value without causing this VSWR to increase above 1.30.

The normalized value of the terminal load impedance relative to the characteristic impedance of the transformer section of line is $2.00 + j0$, which is assumed not to vary with frequency. This is

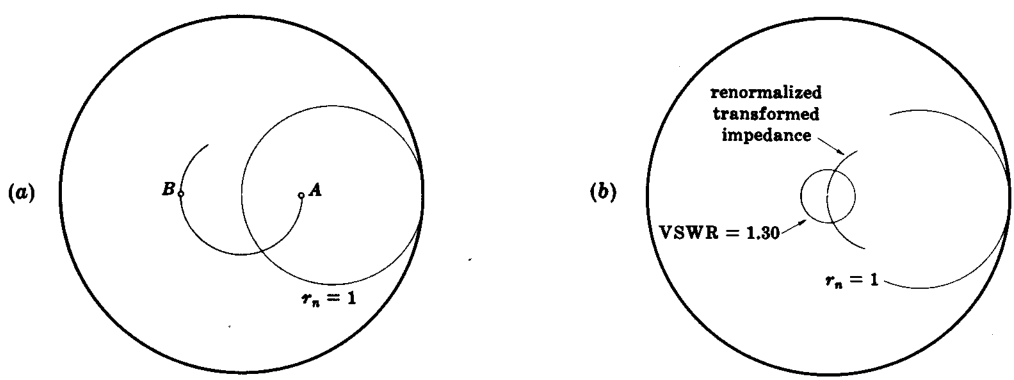


Fig. 9-28. Determining the bandwidth of a quarter wavelength transformer matching a constant resistive load to a low loss transmission line, for a specified maximum value of VSWR.

(*a*) The locus of the impedance at the transformer input terminals, normalized relative to the characteristic impedance of the transformer section.

(*b*) The transformer input impedance renormalized relative to the characteristic impedance of the main transmission line, shown with the VSWR specification.

point A on the Smith chart of Fig. 9-28(*a*). Transformation of this normalized impedance over the quarter wavelength of the transformer at the design frequency produces the point B at $0.50 + j0$, which is the input impedance of the transformer section *normalized relative to the characteristic impedance of the transformer section.* At frequencies above and below the design frequency the input impedance of the transformer section normalized in this way will lie respectively above or below the point B, on the constant VSWR circle through A and B, part of which is shown in the figure. When any normalized impedance on this circle is *denormalized* with respect to the characteristic impedance of the transformer section (by multiplying by $400 + j0$ ohms) and then re-normalized with respect to the main transmission line (by dividing by $200 + j0$ ohms), the resulting normalized value when plotted on a separate Smith chart determines the VSWR on the main transmission line at the frequency corresponding to the original point. The locus of such denormalized re-normalized values for a range of frequencies above and below the design frequency is shown on the Smith chart of Fig. 9-28(*b*). It passes through the center point of the chart at the design frequency, in agreement with the VSWR on the 200 ohm line being less than 1.01 at that frequency.

Also shown on Fig. 9-28(*b*) is the circle for constant VSWR = 1.30. This intersects the plotted locus line at normalized impedance values of $1.02 \pm j0.27$. These values transform back to the chart of Fig. 9-28(*a*) on multiplying by $(200 + j0)/(400 + j0)$, and become the points $0.51 \pm j0.135$ on the VSWR = 2.00 circle for the transformer section of line. Drawing radial lines through these points in Fig. 9-28(*a*) (not shown) indicates that the normalized load impedance $2.00 + j0$ transforms to these values at the input of the transformer section for transformer section lengths of 0.278 and 0.222 wavelengths respectively. The transformer section which is 0.250 wavelengths long at the design frequency has these lengths at 111% and 89% respectively of the design frequency. The total circuit therefore has a bandwidth of ±11%, usually given as 22%, within which the VSWR on the main transmission line does not exceed 1.30. For a lower maximum VSWR specification the bandwidth would be smaller.

9.6. Evaluate $\tanh(0.82 - j2.14)$ using the Smith chart.

From equation (*7.20*) the normalized input impedance of a transmission line section of length l with short circuit termination is $Z_{\text{inp}}/Z_0 = \tanh(\alpha l + j\beta l)$. The graphical procedure for finding the normalized input impedance of such a line section is to start from the short circuit point $Z/Z_0 = 0$, move clockwise on the periphery of the chart $\beta l/2\pi$ wavelengths on the scale WAVELENGTHS TOWARD GENERATOR, and then move radially inward from the periphery through $8.686\alpha l$ decibels on the "1 db steps" scale.

For this problem $\alpha l = 0.82$ nepers, $\beta l = 2.14$ rad, and the answer is Z_{inp}/Z_0.

From the data, $\beta l/2\pi = 0.340$ wavelengths and $\alpha l = 0.82 \times 8.686 = 7.12$ db. Performing the corresponding motions determines a point in the fourth quadrant of the Smith chart, for which $Z_{\text{inp}}/Z_0 = 1.10 - j0.40$. Hence $\tanh(0.82 - j2.14) = 1.10 - j0.40$ with an accuracy of about 0.01 in each component.

Inspection of the process shows that the real part of the hyperbolic angle fixes limits on the range of both components of the hyperbolic tangent. When the real part of the angle has unit value, the real part of the hyperbolic tangent is confined between about 0.75 and 1.3, and the imaginary part cannot lie outside the range ±0.35. When the real part of the angle exceeds about 5, the hyperbolic tangent is $1.00 + j0$, with an accuracy better than 1%, for all values of the imaginary part of the angle.

9.7. Evaluate $\cot 133.2°$ using the Smith chart.

The basis for the calculation is the equation for the normalized input impedance of a length l of lossless transmission line terminated in an open circuit, $Z_{\text{inp}}/Z_0 = 0 - j\cot\beta l$. The angle in degrees must be converted to radians, then divided by 2π to become a line length in wavelengths. Thus $l/\lambda = 133.2(\pi/180)/2\pi = 0.370$.

Starting from the open circuit point on the Smith chart (Z/Z_0 = infinite) and moving clockwise 0.370 wavelengths on the periphery (lossless line) finds a normalized input impedance $0 + j0.935$ in the second quadrant. Hence $\cot 133.2° = -0.935$, with an accuracy of about $\frac{1}{2}$%.

9.8. A "triple stub tuner" is a standard device used in many forms of high frequency transmission systems to perform the function of matching a load to the system. It is an alternative to the single stub matching process, and differs in employing three variable-length stubs at fixed locations in the system instead of a single stub variable in both length and location. In systems such as coaxial lines it has obvious mechanical advantages.

Fig. 9-29 represents a triple stub tuner diagrammatically, showing three stubs of variable lengths l_1, l_2 and l_3, with equal spacings x between adjacent stubs. The length of the stub lines is generally adjustable by means of movable short circuiting conductors.

Show that if the distance x between center lines of adjacent stubs is $3\lambda/8$, and the stub lengths l_1, l_2 and l_3 are variable over at least one half-wavelength, a triple stub tuner can match any impedance or admittance, connected at one end of it, to the transmission system of which the tuner's longitudinal section is a part, at the other end. Assume that all parts of the system, including the stubs, have the same characteristic impedance.

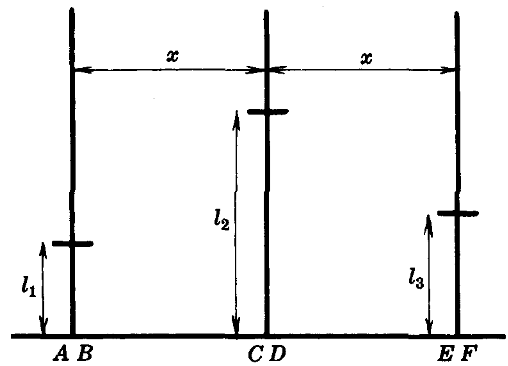


Fig. 9-29. Schematic diagram of a triple stub tuner.

In Fig. 9-29, reading from the signal source side of the triple stub tuner to the terminal load side, A, B and C, D and E, F are three pairs of points such that the two points of each pair are located at infinitesimal distances on either side of the points of connection of the three variable-length stub lines. This means that there is no measurable length of transmission line between the points A and B, but that adjustment of the length l_1 of the left hand stub line can cause the normalized susceptance at point A to differ from that at point B by any amount of either sign. Similar statements apply to the other two pairs of points. Point C is 3/8 wavelengths from point B, and point E is 3/8 wavelengths from point D. The locations of the signal source and terminal load are immaterial.

Since the stub lines are invariably connected in parallel with the main transmission line, as is also the case in single stub matching, reference is made to the Smith chart in admittance coordinates.

If the triple stub tuner correctly performs its function of matching a terminal load admittance, connected at the right hand end, to the transmission line at the left hand end, the normalized admittance at the point A must have the unique value $1 + j0$. At the point B any normalized admittance of the form $1 \pm jb_n$, where b_n is any magnitude of normalized susceptance, can be brought to the value $1 + j0$ at the point A by adjustment of the stub length l_1. The unit conductance circle on the Smith chart of Fig. 9-30(*a*) is therefore *the locus of all matchable admittances at the location B.*

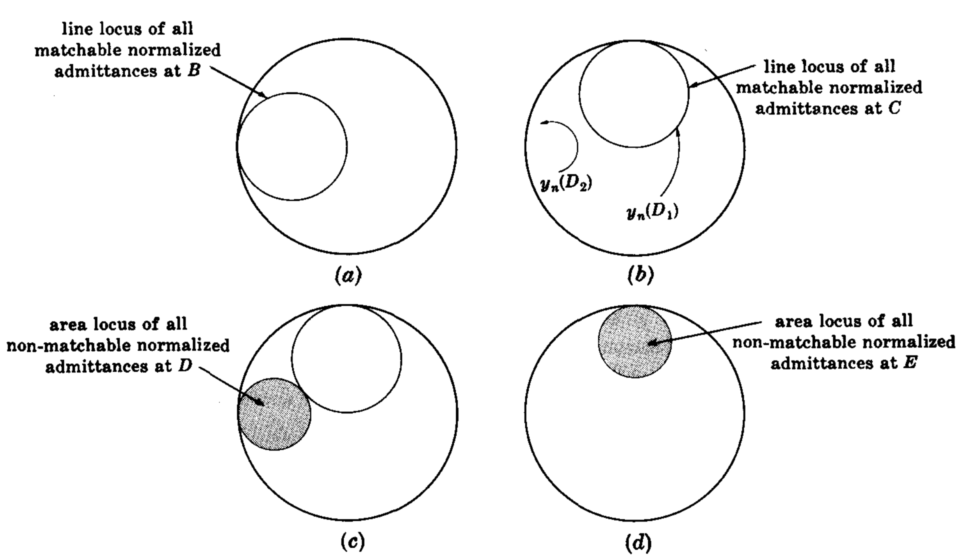


Fig. 9-30. Loci of matchable and non-matchable normalized admittances at various locations in a triple stub tuner.

Any normalized admittance at location B transforms to a value at location C that is found by moving 3/8 wavelengths counterclockwise (toward load) along a circle of constant VSWR. The entire circle locus of matchable normalized admittances at B therefore transforms to another circle 3/8 wavelengths counterclockwise, as shown in Fig. 9-30(*b*), which is *the locus of all matchable normalized admittances at C.*

The graphical process of adding normalized susceptance to the normalized admittance value at any point on the Smith chart consists of moving along a circle of constant normalized conductance. Thus if the normalized admittance at point D had the value $y_n(D_1)$ in Fig. 9-30(*b*), the adjustment of stub length l_2 could change it at point C to a value on the locus of matchable normalized admittances at C. If the normalized admittance at D had the value $y_n(D_2)$ in the figure, on the other hand, no value of l_2 could achieve this result. Hence the *area* within the normalized conductance circle tangent to the circle which is the locus of all matchable normalized admittances at the point C is *the locus of all non-matchable normalized admittances at the point D,* as indicated in Fig. 9-30(*c*). By the same transformation process used between points B and C, the shaded circle shown in Fig. 9-30(*d*) is *the locus of all non-matchable normalized admittances at the point E.*

The conclusion of the proof of universality for this particular triple stub tuner is that if the normalized admittance presented at location F by the connected terminal load lies *outside* the shaded circle of Fig. 9-30(*d*), then stub length l_3 could be adjusted for zero normalized input susceptance and the matching process could be completed by adjustment of the stub lengths l_1 and l_2. If, however, the normalized admittance presented at the point F lies *inside* the shaded circle, stub length l_3 can *always* be adjusted to produce at point E a normalized admittance lying *outside* the shaded circle, and the matching procedure can be completed by adjustment of stub lengths l_1 and l_2 as before. All normalized admittances at point F can therefore be matched to the transmission line at point A.

It will be noted that for any particular value of the normalized admittance at point F there is no unique solution to the problem. Once stub length l_3 has been set at an appropriate value, however, only two specific settings are possible for stub length l_2, and for each of these a unique setting of l_1 is required.

Triple stub tuners are invariably adjusted by trial and error, not by calculation, and the purpose of this problem is to demonstrate that matching can always be achieved if a tuner is used whose design meets the conditions specified.

A review of the operations undertaken in the various steps of Fig. 9-30 will show clearly that if the separation of adjacent stubs in a triple stub tuner were any integral number of quarter wavelengths, the matching procedure would not work, since the two outer stubs would then be effectively in parallel. Spacings near 3/8 wavelength are often a good compromise between mechanical convenience and electrical performance. In principle, spacings of any odd multiple of 1/8 wavelength are satisfactory and variations of a few percent from these values are not serious. Large values of the stub separation in wavelengths have the disadvantage of a reduced operating bandwidth.

Supplementary Problems

9.9. Determine the range of reflection coefficient phase angles ϕ that can be produced by any normalized impedance of the form $z_n = 5.0 \pm jx_n$, where $0 < x_n < \infty$. *Ans.* $-11.5° < \phi < +11.5°$.

9.10. What value of normalized admittance of the form $A - jA$ will produce a reflection coefficient of phase angle 45°? *Ans.* $y_n \doteq 0.365 - j0.365$.

9.11. What value of normalized admittance of the form $A + jA$ will produce a reflection coefficient of phase angle 45°?

Ans. This is not possible. All admittances $A + jA$ are in the lower half of the Smith chart, while all reflection coefficients of phase angle 45° are in the upper half.

9.12. A low-loss transmission line of characteristic impedance $50 + j0$ ohms is terminated in a resistance of 150 ohms in series with a capacitive reactance of 30 ohms.

(a) What is the VSWR on the line?

(b) If the resistance component can be varied from 20 ohms to 500 ohms without changing the reactance component, what is the lowest possible VSWR and what value of resistance will produce it? *Ans.* (a) About 3.18. (b) About 1.77 with a resistance of 58.5 ohms.

9.13. Derive equation (*9.7*) from equation (*9.2*), page 186.

9.14. Use the Smith chart to show that the normalized input impedance of any transmission line section with open circuit termination is numerically equal to the normalized input admittance of the same transmission line section with short circuit termination, and that the normalized input admittance of any transmission line section with open circuit termination is numerically equal to the normalized input impedance of the same transmission line section with short circuit termination.

9.15. Use the Smith chart to verify the results of Examples 8.1 and 8.2, page 166.

9.16. The normalized admittances $0.5 + j\sqrt{3.75}$, $1.0 + j\sqrt{3.0}$, and $1.5 + j\sqrt{1.75}$ all have the same normalized magnitude of 2.0. Which produces the lowest VSWR when connected as terminal load admittance on a transmission line?

Ans. The last of the three. The Carter chart shows directly that for all terminal load impedances or admittances of any given normalized magnitude value, the lowest VSWR will be produced by the one with the smallest phase angle.

9.17. A transmission line is terminated in a normalized conductance of 3.75 in parallel with a variable capacitor whose normalized susceptance at the operating frequency is variable from 0.20 to 5.0. What range of values of $d_{V(\min)}/\lambda$ results when the capacitive susceptance is varied over its full range?

Ans. 0.0025 at $b_n = 0.20$, to a maximum of 0.0215 at $b_n = 3.7$, diminishing to 0.0205 at $b_n = 5.0$.

9.18. Use the Smith chart to show that if a lossless transmission line is terminated in a load impedance that produces a reflection coefficient of magnitude 0.44, the maximum impedance at any point on the line has a normalized value of $2.57 + j0$, and that the maximum normalized reactance component of the impedance at any point on the line is ± 1.08. Show that on the same line the maximum normalized admittance at any point is $2.57 + j0$ and the maximum normalized susceptance is ± 1.08.

9.19. Use the Smith chart to verify the answers of Problems 7.3, 7.4, 7.5 and 7.9.

9.20. Use the Carter chart to demonstrate that the maximum and minimum admittance and impedance values on any transmission line of low attenuation per wavelength are always purely resistive, regardless of the terminal load connected to the line, and that they occur at minima or maxima of the voltage or current standing wave patterns.

9.21. Starting from equation (*9.1*) and using the notation of equation (*9.2*) with $z_n = r_n + jx_n = |z_n|\underline{/\theta}$, show that the locus on the reflection coefficient (u, v) plane of all points having any constant value of $|z_n|$ is a circle with center at $u = (|z_n|^2 + 1)/(|z_n|^2 - 1)$, $v = 0$, and radius $2|z_n|/|(|z_n|^2 - 1)|$, and that the locus on the same plane of all points having any constant value of θ is a circle with center at $u = 0, v = -\cot\theta$, and radius $\operatorname{cosec}\theta$. These are the data for drawing the Carter chart of Fig. 9-13.

9.22. A low-loss transmission line has a characteristic impedance of $50 + j0$ ohms. When three different terminal load impedances are separately connected to the line, the resulting standing wave patterns on the line are described respectively by the following data:

(a) VSWR = 2.35, $d_{V(\min)}/\lambda = 0.395$; (c) VSWR = 2.35, $d_{V(\min)}/\lambda = 0.062$

(b) VSWR = 2.35, $d_{V(\min)}/\lambda = 0.271$;

What will happen to the VSWR value in each case when a 50 ohm resistor is connected in parallel with the terminal load impedance?

Ans. (a) Remains unchanged at 2.35; (b) reduced to 1.44; (c) increased to about 2.9.

9.23. There is a VSWR of 3.0 on a lossless transmission line. (*a*) Where, relative to any voltage minimum on the line, might stub lines be placed to remove the standing waves on the signal source side of the stub? (*b*) What lengths of stub lines with either open circuit or short circuit termination are required to perform the matching operation, if the characteristic impedance of the stub lines is the same as that of the transmission line itself?

Ans. (*a*) In the notation of Fig. 9-27(*a*), the locations A and B at which matching stubs might be placed are respectively 0.083 wavelengths on the generator and load sides of any voltage minimum on the line. (*b*) The normalized susceptance at point A is -1.15. A stub with short circuit termination connected at A should be 0.386 wavelengths long to have a normalized susceptance of $+1.15$ to cancel this value. With open circuit termination the stub required at A should be 0.136 wavelengths long. At point B, required stub lengths are 0.114 wavelengths with short circuit termination and 0.364 wavelengths with open circuit termination.

9.24. The Smith chart lends itself to direct construction of phasor diagrams, from which graphical evaluation can be made of phase and amplitude relations between harmonic voltages and currents at any point on a transmission line.

In Fig. 9-31, if the left hand horizontal radius of the chart (the directed line segment A) is designated as a reference phasor $1 + j0$, representing in normalized form the phasor value $V_1 e^{-\gamma z}$ at any coordinate z on a transmission line of the harmonic voltage wave traveling in the direction of increasing z, then the directed line segment B, joining the center of the chart to the point on the chart identified as associated with the coordinate z, represents the phasor value $V_2 e^{+\gamma z}$ at z of the harmonic voltage wave traveling in the direction of decreasing z. This follows from the fact that $A = 1\underline{/0}$ and $B = |\rho|\underline{/\phi}$ in the reflection coefficient plane.

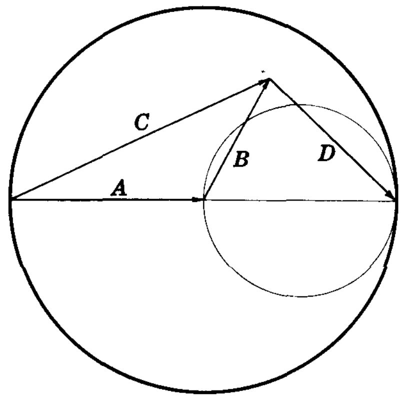


Fig. 9-31. A phasor diagram constructed on the Smith chart.

The directed line segment C is then the total phasor voltage $V(z)$ at z, normalized relative to $V_1 e^{-\gamma z}$.

Show that the directed line segment D is the phasor value of the current $I(z)$ at coordinate z, normalized relative to $(V_1/Z_0)e^{-\gamma z}$, and that the acute angle between the extended line segments C and D is the phase angle of the normalized impedance Z/Z_0 at coordinate z, being positive in the upper half of the chart and negative in the lower half. This makes it possible to measure the phase angle of the normalized impedance at any point on the Smith chart with a protractor.

Show that $C/D = (1+\rho)/(1-\rho)$ and note the result of substituting for ρ from equation (*9.1*), page 185.

The phasor diagrams of Sections 8.9 and 4.12 can be drawn on the Smith chart in the manner of Fig. 9-31, after suitable normalization.

Chapter 10

Resonant Transmission Line Circuits

10.1. The nature of resonance.

Any passive lumped element linear n-port electric circuit that contains at least one inductor and one capacitor can exhibit the important phenomenon of resonance, if the resistors in the circuit do not introduce excessive dissipation. Experimentally, the occurrence of resonance in a circuit is investigated by observing the variation with frequency of some impedance, admittance, or hybrid parameter of the circuit, such as those described for the special case of two-port networks in Section 7.7, page 140. When the magnitude of the measured parameter passes through a maximum or a minimum at some frequency, and its phase angle changes sign at or very close to the same frequency, resonance is the most likely explanation. The diagnosis is confirmed if the circuit shows a decaying oscillatory response at the same or nearly the same frequency when excited by a discontinuous signal such as a voltage step.

The principal practical applications of resonant circuits exploit the frequency sensitivity property, in filter networks, or the oscillatory response property, in harmonic signal generators.

A lumped element circuit containing a single inductor and a single capacitor has just one resonant frequency, for measurements made at any prescribed pair of terminals. For a transmission line circuit consisting of a length of low-loss line with low-loss terminations, on the other hand, the uniform distribution of inductance and capacitance along the line produces the distinctly different result that the circuit has an infinite series of resonant frequencies, which under certain conditions may be quite precisely integral multiples of a lowest or fundamental frequency.

In the frequency range from a few tens of megahertz to several gigahertz, resonant transmission line sections are widely used in amplifiers, oscillators, filters, etc., because their quantitative resonant properties, even for simple and inexpensive types of transmission line, can be far superior to those of lumped element circuits in the same frequency range. At frequencies above a few gigahertz the same functions are performed more efficiently by cavity resonators.

10.2. The basic lumped element series resonant circuit.

The resonant properties of transmission line circuits are best appreciated through their analogies with the familiar resonant properties of lumped element resistance-inductance-capacitance circuits. A brief review of the latter will therefore be given to provide the notation and context in which to understand the former. For reasons stated in earlier chapters the analysis is presented in the radian frequency domain rather than the complex frequency domain.

The simplest analytical expression for resonant behavior in a lumped element circuit is obtained from the circuit model of Fig. 10-1, consisting of an ideal resistor, an ideal inductor and an ideal capacitor in series. The elements are ideal in that each embodies only a single circuit property, and in the assumption that the value of each is independent of frequency or signal strength.

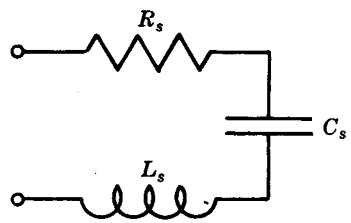


Fig. 10-1. Lumped element series resonant circuit consisting of an ideal resistor R_s, an ideal inductor L_s and an ideal capacitor C_s in series.

The input impedance of the circuit of Fig. 10-1 is

$$Z_{\text{inp}} = R_s + j[\omega L_s - 1/(\omega C_s)] \qquad (10.1)$$

The use of a subscript on R, L and C serves as a reminder that the quantities are lumped, and are not the distributed circuit coefficients of transmission line theory. The specific subscript s refers to the elements being in series.

The radian resonant frequency ω_r of any resonant circuit is *defined* to be the radian frequency at which the input impedance (or other observed variable) is real. Hence for the circuit of Fig. 10-1 $\omega_r L_s = 1/(\omega_r C_s)$ and

$$\omega_r = 1/\sqrt{L_s C_s} \qquad (10.2)$$

The resonant impedance Z_r of the circuit, *defined* as the impedance at the resonant frequency, is

$$Z_r = R_s + j0 \qquad (10.3)$$

and in this particular case the resonant impedance is identical with the minimum value of the input impedance magnitude $|Z_{\text{inp}}|$.

Equation (*10.1*) can now be written

$$Z_{\text{inp}} = Z_r[1 + j(\omega_r L_s/R_s)(\omega/\omega_r - \omega_r/\omega)] \qquad (10.4)$$

The analysis of a practical high frequency resonant circuit is of interest only in a very narrow range of frequencies centering around ω_r. It is therefore appropriate to express the angular frequency variable ω as $\omega = \omega_r + \Delta\omega$, i.e. in terms of its deviation $\Delta\omega$ from the resonant angular frequency. The coefficient $\omega_r L_s/R_s$ plays such a fundamental role in the analysis of resonant circuits that it has become universal practice to designate it by the symbol Q_r or Q. It is sometimes called the "quality factor" of the circuit, but more often is referred to simply as the Q-value or the resonant Q-value for the circuit. Introducing Q_r and $\Delta\omega$ into equation (*10.4*) and expanding the final term by the binomial theorem gives

$$Z_{\text{inp}} = Z_r\{1 + j2Q_r(\Delta\omega/\omega_r)[1 - \tfrac{1}{2}(\Delta\omega/\omega_r) + \tfrac{1}{2}(\Delta\omega/\omega_r)^2 - \tfrac{1}{2}(\Delta\omega/\omega_r)^3 + \cdots]\} \qquad (10.5)$$

For frequency deviations from resonance not exceeding one percent, the reactive component, the phase angle and the magnitude of Z_{inp} can all be obtained with better than $\tfrac{1}{2}\%$ accuracy from the approximate equation

$$Z_{\text{inp}} \doteq Z_r[1 + j2Q_r(\Delta\omega/\omega_r)] \qquad (10.6)$$

The Q value of a resonant circuit at its resonant frequency is a measure, exact or approximate, of all the important aspects of the circuit's resonant behavior.

If a constant harmonic voltage at the resonant angular frequency ω_r, having a reference phasor value $V_{\text{inp}} + j0$, is connected to the input terminals of the circuit of Fig. 10-1, it is evident from equation (*10.1*) that the resulting phasor current will be $V_{\text{inp}}/R_s + j0$. This current flows through the inductor L_s and the capacitor C_s, whose impedances at the resonant frequency are $j\omega_r L_s$ and $-j/(\omega_r C_s)$ respectively. The phasor voltages across the two elements are therefore respectively $j\omega_r L_s V_{\text{inp}}/R_s = jQ_r V_{\text{inp}}$, and $-jV_{\text{inp}}/(\omega_r C_s R_s) = -jQ_r V_{\text{inp}}$, making use of equation (*10.2*). Thus the magnitude of the phasor voltage across each of the reactive components of the series circuit at the resonant frequency is exactly Q_r times as great as the magnitude of the phasor input voltage to the circuit. Since $Q_r \gg 1$ for circuits of practical interest, there is a "resonant rise of voltage" in the circuit.

If the harmonic voltage of constant magnitude V_{inp} connected to the terminals of the circuit of Fig. 10-1 is varied in frequency, there will be a radian frequency ω_1 above resonance at which $Z_{\text{inp}} = Z_r(1+j1)$ and a radian frequency ω_2 below resonance at which $Z_{\text{inp}} = Z_r(1-j1)$. Simple calculation shows that at each of these frequencies the input power to the circuit will be one half of the input power at the resonant frequency. They are therefore generally known as the "half power frequencies" or the "3 db frequencies" of the circuit. The difference between the two frequencies is a measure of the "sharpness" of the resonance behavior of the circuit, and when expressed in hertz is commonly called the "bandwidth" or the "3 db bandwidth" of the circuit.

From equation (*10.4*) it is easily found that the difference between the 3 db angular frequencies ω_1 and ω_2 is related to the resonant Q value and the resonant angular frequency ω_r by the simple and exact expression

$$\omega_1 - \omega_2 = \omega_r/Q_r \tag{10.7}$$

The frequency deviations of ω_1 and ω_2 from ω_r are not equal in magnitude, and are not given by any comparably simple expressions. (See Problem 10.7.)

It follows from equation (*10.7*) that the 3 db bandwidth of the circuit in hertz, designated Δf_3, is given by

$$\Delta f_3 = f_r/Q_r \tag{10.8}$$

where $f_r = \omega_r/2\pi$ is the resonant frequency of the circuit in hertz.

With an rms phasor input voltage $V_{\text{inp}} + j0$ at the resonant frequency ω_r applied to the input terminals of the circuit of Fig. 10-1, the rms phasor input current is $I_{\text{inp}} = V_{\text{inp}}/R_s + j0$. The maximum instantaneous energy stored in the magnetic field of the inductor L_s then occurs at the peak value of the harmonic current and is given by $\hat{W}_{L_s} = \frac{1}{2}L_s(\hat{I}_{\text{inp}})^2 = \frac{1}{2}L_s(\sqrt{2}\,V_{\text{inp}}/R_s)^2$. The average power dissipation in the circuit is $P_{R_s} = (I_{\text{inp}})^2 R_s = (V_{\text{inp}})^2/R_s$. From the defining relation $Q_r = \omega_r L_s/R_s$ it follows directly that $\hat{W}_{L_s}/P_{R_s} = Q_r/\omega_r$.

This is a particular instance of a general expression for the resonant Q value of any circuit or system,

$$Q_r = 2\pi \frac{\text{maximum instantaneous energy stored during cycle}}{\text{total energy dissipated during cycle}} \tag{10.9}$$

where it is understood that the excitation is harmonic and at the resonant frequency. Since (*10.9*) makes no reference to specific elements, it can be regarded as a very basic physical definition of resonant Q value, applicable to any type of system.

If in the circuit of Fig. 10-1 the capacitor is initially charged to voltage V_0 from a d-c source and the input terminals of the circuit are then short circuited, the resulting well-known "natural" response of the circuit is in the form of a damped oscillatory current

$$i(t) = (V_0/\omega_n L_s)\, e^{-(\frac{1}{2}R_s/L_s)t} \sin \omega_n t \tag{10.10}$$

where

$$\omega_n = \sqrt{1/(L_s C_s) - (\tfrac{1}{2}R_s/L_s)^2} \tag{10.11}$$

is the natural angular frequency of oscillation. Using (*10.2*) and the definition of Q_r, (*10.10*) can be written

$$i(t) = I_0 e^{-(\omega_r/2Q_r)t} \sin\left[\omega_r\sqrt{1 - 1/(2Q_r)^2}\right] \tag{10.12}$$

Equation (*10.12*) shows that the resonant Q-value of the basic series resonant circuit determines the fractional deviation of the circuit's self-oscillation frequency from its defined resonant frequency, and in combination with the angular resonant frequency ω_r determines the damping factor of the natural oscillations.

A physical implication of equation (*10.12*) is that, after excitation to the same initial signal level at the same resonant frequency, a high Q circuit will "ring" longer than a low Q circuit. This governs the design of microwave "echo" cavities.

10.3. The basic lumped element parallel resonant circuit.

Of more practical importance than the series resonant circuit of Fig. 10-1 is the parallel resonant circuit whose input impedance magnitude passes through a maximum at or near the resonant frequency at which it is real. The circuit of Fig. 10-2 is the simplest representation of a parallel resonant circuit, and is the dual of the circuit of Fig. 10-1. It is evident that for this circuit the resonant frequency ω_r at which the input impedance is real is identical with the frequency at which the input impedance magnitude is a maximum, and that the reactance of the inductor and capacitor are also equal in magnitude at this same frequency. Thus

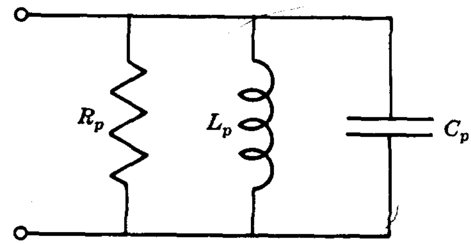


Fig. 10-2. Lumped element parallel resonant circuit consisting of an ideal resistor R_p, an ideal inductor L_p and an ideal capacitor C_p in parallel.

$$\omega_r = 1/\sqrt{L_p C_p} \tag{10.13}$$

and

$$Z_r = R_p + j0 \tag{10.14}$$

In the admittance notation appropriate to this circuit, the methods used in Section 10.2 produce the result

$$Y_{\text{inp}} = Y_r[1 + jQ_r(\omega/\omega_r - \omega_r/\omega)] \tag{10.15}$$

where $Y_r = 1/Z_r = 1/R_p + j0$, and

$$Q_r = R_p/(\omega_r L_p) \tag{10.16}$$

Equation (*10.15*) is similar in form to (*10.4*). Correspondingly similar forms of equations (*10.5*) and (*10.6*) can be written directly, and equations (*10.7*), (*10.8*), (*10.9*) and (*10.12*) are then found to be applicable to the parallel resonant circuit without change.

Practical lumped element circuits deviate from the ideal representations of Fig. 10-1 and 10-2 in several ways. Inductors have internal distributed capacitance between windings. The plates and leads of capacitors have distributed inductance. All conductors are subject to skin effect. Losses in capacitor dielectrics and inductor magnetic cores vary with frequency. There may be radiation losses or various forms of coupling to the surroundings. The diverse physical forms of lumped elements prohibit any general analysis of these phenomena. Except in extreme cases, however, the fractional errors they cause in calculations using equations (*10.1*) to (*10.16*) are only of the order $1/Q_r^2$, provided the values of R_s, L_s and C_s, or R_p, L_p and C_p used in the equations are the *effective values of these quantities at the resonant frequency*. Subject to this stipulation, the errors are negligibly small in circuits of high Q-value, and the equations establish meaningful concepts, relations and notation for use in the analysis of resonant transmission line sections.

10.4. The nature of resonance in transmission line circuits.

Before proceeding to a formal analysis of transmission line resonant circuits it is instructive to consider a few simple situations and suggestions.

Example 10.1.

On a Smith chart show that if a length l_1 of low-loss high-frequency transmission line with short circuit termination has an inductive input reactance, the reactance increases with increasing frequency. Show that if a length l_2 of the same line with short circuit termination has a capacitive input reactance, the reactance decreases with increasing frequency. What is the total length $l_r = l_1 + l_2$ of two such low-loss high-frequency transmission line sections with short circuit termination, whose input reactances (or susceptances) are equal in magnitude but opposite in sign?

Referring to the Smith chart of Fig. 10-3, any point A on or near the periphery of the upper half of the chart represents the normalized input impedance of a length l_1 of transmission line with short circuit termination and low total attenuation, the normalized impedance having a small real part and a positive imaginary part. Since the characteristic impedance of a "low-loss high-frequency" line is to be assumed real, the input reactance of the line will be inductive if $0 < l_1/\lambda < 0.25$.

The angular location of the point A, being proportional to l_1/λ, varies with the angular frequency ω according to $l_1\omega/(2\pi v_p)$. Since v_p is virtually independent of frequency for the type of line specified, the point A for a fixed line length l_1 will move clockwise as the frequency rises, indicating that the input reactance of the line section increases with increasing frequency. The line section is therefore analogous to a lumped inductor, but with the difference that the section's input reactance is not in general directly proportional to frequency.

Applying the above reasoning to a point B, on or near the periphery of the lower half of the chart, shows that the input impedance of a length l_2 of low-loss high-frequency transmission line with short circuit termination has a capacitive reactance if $0.25 < l_2/\lambda < 0.50$, and that the capacitive reactance decreases with increasing frequency. The line section is therefore analogous to a lumped capacitor, but with the difference that the section's input reactance is not in general reciprocally proportional to the frequency.

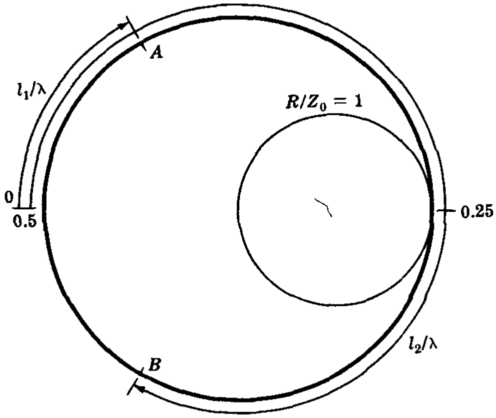


Fig. 10-3. Point A on the Smith chart marks the normalized input impedance of a low-loss transmission line section with short circuit termination whose length in wavelengths l_1/λ is less than 0.25. Point B marks the normalized input impedance of a similar line section whose length in wavelengths is between 0.25 and 0.50.

It is clear that if the reactance magnitudes are equal at points A and B, the points are symmetrical relative to the central horizontal axis of the chart, and $l_1 + l_2 = \lambda/2$. Hence the transmission line circuit of Fig. 10-4 has at least one of the attributes of a series resonant circuit at the terminals X-X, that the input reactance is zero at the frequency for which the circuit length is one half wavelength. Furthermore, as the frequency is increased or decreased from this value, the input reactance becomes inductive or capacitive, respectively, in analogy with the behavior of the circuit of Fig. 10-1.

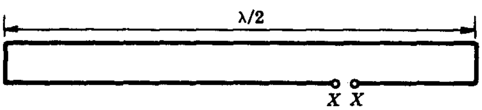


Fig. 10-4. Around the frequency for which it is one half wavelength long, the normalized input impedance at terminals X-X of a low-loss transmission line circuit with short circuit terminations at each end varies with frequency in the same manner as the input impedance of a lumped element series resonant circuit in the vicinity of resonance.

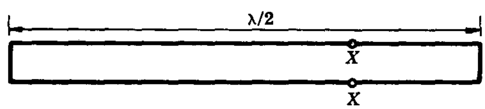


Fig. 10-5. Around the frequency for which it is one half wavelength long, the normalized input impedance at terminals X-X of a low-loss transmission line circuit with short circuit terminations at each end varies with frequency in the same manner as the input impedance of a lumped element parallel resonant circuit in the vicinity of resonance.

Similarly, the transmission line circuit of Fig. 10-5 above has zero input susceptance at the terminals *X-X*, at the frequency for which the circuit length is one half wavelength, and as the frequency is increased or decreased from this value, the input susceptance becomes capacitive or inductive, respectively, in analogy with the behavior of the circuit of Fig. 10-2.

It is easily seen that the above statements continue to apply if the circuits of Fig. 10-4 and 10-5 are any integral number of half wavelengths in length, and that the terminals *X-X* in either case may be located anywhere along the length of the line.

Example 10.2.

A low-loss high-frequency transmission line has a distributed inductance of L henries/m and a distributed capacitance of C farads/m. For a length l m of the line the total inductance L_t is therefore given by $L_t = Ll$, and the total capacitance C_t is given by $C_t = Cl$. At the frequency at which a lumped element circuit having inductance L_t and capacitance C_t would be resonant, what is the length of the transmission line section in wavelengths?

Equation (*10.2*) or (*10.13*) gives the required angular frequency as $\omega = 1/\sqrt{C_t L_t}$. The phase velocity on the low-loss high-frequency line is $v_p = 1/\sqrt{LC}$. It follows directly that $\omega = v_p/l$ and $l/\lambda = 1/(2\pi) = 0.159$. Since this result is independent of the nature of the terminations connected to the line section, and since it disagrees with the result of Example 10.1, it must be concluded that the resonant frequencies of transmission line circuits are not related in any significant way to the total inductance and total capacitance of the circuits.

Example 10.3.

Using a Carter chart, show that as the frequency increases continuously from zero, the magnitude of the normalized input impedance of any low-loss transmission line section with short circuit termination also increases continuously from zero, until it reaches a maximum value at a frequency f_1 for which the line length is very close to one quarter wavelength, that it then diminishes to a minimum value at a frequency $f_2 \doteq 2f_1$, rises to another maximum value at $f_3 \doteq 3f_1$, and continues to oscillate in this manner indefinitely with increasing frequency. Show also that the impedance magnitudes at the maxima and minima are respectively very large and very small compared to the characteristic impedance of the line.

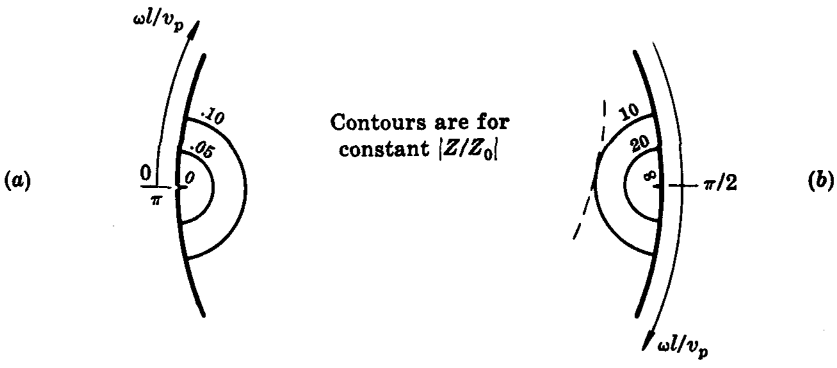


Fig. 10-6. (*a*) A portion of a Carter chart showing that the magnitude of the normalized input impedance of a section of low-loss transmission line with short circuit termination increases as the frequency increases from zero.

(*b*) When the frequency reaches the value at which the line length is one quarter wavelength, the magnitude of the normalized input impedance passes through an indefinitely large maximum if the line has negligible losses (periphery of the chart). The dashed line shows that if the transmission line has appreciable attenuation that increases with frequency, the maximum of the normalized input impedance magnitude will occur at a frequency for which the line length is slightly less than one quarter wavelength.

Fig. 10-6 shows enlargements of portions of the Carter chart (Fig. 9-13, page 194) at the two ends of the horizontal axis. Fig. 10-6(*a*) contains the short circuit terminal impedance of normalized magnitude zero. As the frequency increases from zero, the length in wavelengths of a transmission line section also increases, and the point on the chart representing the normalized input impedance of a section with negligible losses moves clockwise along the periphery of the chart from the short circuit point. From the coordinates of the chart it is obvious that the normalized input impedance magnitude increases with frequency.

As the frequency continues to increase, the point representing the normalized input impedance enters the portion of the chart's periphery shown in Fig. 10-6(*b*), reaching infinite magnitude at a frequency for which the line length is exactly one quarter wavelength. It then decreases to zero at precisely twice this frequency, and the sequence continues indefinitely.

If the line losses are small but finite, the point representing the normalized input impedance of the section at any frequency is displaced radially inward from the periphery of the chart by an amount that depends on the total attenuation of the section at that frequency, as described in Section 9.6. For a total attenuation independent of frequency, the point would move on a circular locus just inside the bounding circle of the chart. For a total attenuation directly proportional to frequency, the point would move along a true logarithmic spiral. Because of skin effect, the attenuation of air dielectric transmission lines at high frequencies increases approximately with the square root of the frequency. The normalized input impedance point on the Carter chart (or Smith chart) then moves on a spiral path with increasing frequency, but the spiral is not an elementary one.

A portion of an exaggerated spiral is shown in Fig. 10-6(*b*). It demonstrates that if the attenuation of a line section increases in any manner with increasing frequency, the frequencies for maximum normalized input impedance magnitude will be slightly less than they would be for a lossless line section of the same length and phase velocity. This is one of many second order effects in resonant transmission line circuits analogous to those listed for lumped element circuits in Section 10.3.

10.5. Resonant transmission line sections with short circuit termination.

In the design of transmission line resonant circuits, the goal, almost invariably, is to maximize the resonant Q-value. From the energy definition of Q in equation (*10.9*), the terminations forming parts of such circuits should then be chosen to have the smallest possible losses. In principle, terminations of zero impedance, infinite impedance, or any purely reactive impedance satisfy this requirement by having zero losses, and open circuit, short circuit, purely inductive, or purely capacitive terminations should all be satisfactory. As a practical matter, however, terminal inductors or capacitors always have higher ratios of resistance to inductance, or of conductance to capacitance, than the line itself, and they reduce the resonant Q-value of any line section to which they are connected. Under most conditions open circuit terminations are electrically adequate, but they provide no mechanical support between the line conductors. Except in special cases, resonant transmission line circuits usually have a short circuit termination at one end, to combine low terminal loss with conductor support, and to achieve the additional advantages, for coaxial lines, of complete electrical shielding and precise termination location (by avoiding the fringing fields that occur at any form of open circuit design).

The input impedance of any length l of uniform transmission line with short circuit termination, the line having characteristic impedance Z_0 ohms, attenuation factor α nepers/m and phase factor β rad/m, at angular frequency ω rad/sec, is given by equation (*7.25*), page 134, as

$$Z_{\text{inp}} = Z_0 \tanh(\alpha + j\beta)l \qquad (10.17)$$

Using a standard identity this expands to

$$Z_{\text{inp}} = Z_0 \frac{\sinh 2\alpha l + j \sin 2\beta l}{\cosh 2\alpha l + \cos 2\beta l} \qquad (10.18)$$

Adopting the simplifying assumption that Z_0 is real, the input impedance of the line section will be real at all frequencies for which $\sin 2\beta l = 0$, or $2\beta l = n\pi$, where n is any integer. Since $\beta = 2\pi/\lambda$, this relation is equivalent to $l/\lambda = n/4$, i.e. the frequencies at which the input impedance is real are those for which the line length is an integral number of quarter wavelengths. Substituting $\beta = \omega/v_p$, the corresponding frequencies are given by $\omega_r = \pi n v_p/(2l)$. If the phase velocity v_p is independent of frequency, the indicated frequencies are integrally related.

To establish that each of the frequencies at which the input impedance of the transmission line circuit is real is a "resonant" frequency in the sense of Sections 10.2 or 10.3, it is necessary to show from (*10.18*) that the variation of Z_{inp} with frequency in the vicinity of each of these frequencies is similar to the variation with frequency of the input impedance of a lumped element resonant circuit near resonance.

Two additional simplifying assumptions must be made. The first is that over a narrow range of frequency centered at each of the resonant frequencies given by $\omega_r = \pi n v_p/(2l)$, there is negligible variation of the total line attenuation αl. The second is that the actual value of the total attenuation αl is small enough to permit use of the approximations $\sinh 2\alpha l \doteq 2\alpha l$ and $\cosh 2\alpha l \doteq 1 + 2(\alpha l)^2$. (For reasonable accuracy this requires $\alpha l < 0.05$ nepers.)

With these assumptions it is easily found that $|Z_{\text{inp}}|$ has a maximum value $Z_0/(\alpha l)$ when $\sin 2\beta l = 0$ and $\cos 2\beta l = -1$ (i.e. when n is odd in the above relations), and has a minimum value $Z_0 \alpha l$ when $\sin 2\beta l = 0$ and $\cos 2\beta l = +1$ (i.e. when n is even).

At a frequency differing by a small fraction $\Delta\omega/\omega_r \ll 1$ from the resonant frequency ω_r at any impedance *minimum*,

$$\sin 2\beta l = \sin\{2(\omega_r + \Delta\omega)l/v_p\} = \sin(2\Delta\omega l/v_p)$$
$$= \sin\{2\beta_r l(\Delta\omega/\omega_r)\} \doteq 2\beta_r l(\Delta\omega/\omega_r)$$

and

$$\cos 2\beta l = +\sqrt{1 - \sin^2 2\beta l} \doteq +1, \quad \text{where } \beta_r = \omega_r/v_p$$

Defining $Z_r = Z_0 \alpha_r l$, where α_r is the line's attenuation factor at the frequency ω_r, equation (*10.18*) can now be written

$$Z_{\text{inp}} \doteq Z_r\left\{1 + j\frac{\beta_r}{\alpha_r}\left(\frac{\Delta\omega}{\omega_r}\right)\right\} \tag{10.19}$$

A term $(\alpha l)^2$ has been dropped, relative to unity, in the denominator.

Since this equation is identical in form with (*10.6*), it follows that in the vicinity of every frequency ω_r at which $|Z_{\text{inp}}|$ is a minimum, the input impedance of a low-loss transmission line section with short circuit termination displays the resonance behavior of a lumped element series resonant circuit near resonance.

The indicated resonant Q-value of the transmission line resonant circuit is

$$Q_r = \beta_r/2\alpha_r \tag{10.20}$$

To demonstrate that resonance behavior also occurs near the frequencies ω_r at which $|Z_{\text{inp}}|$ is a *maximum*, the simplest procedure is to rewrite (*10.17*) in admittance form,

$$Y_{\text{inp}} = Y_0 \coth(\alpha + j\beta)l \tag{10.21}$$

which can be expanded as

$$Y_{\text{inp}} = Y_0 \frac{\sinh 2\alpha l - j\sin 2\beta l}{\cosh 2\alpha l - j\sin 2\beta l} \tag{10.22}$$

Making the same assumptions and approximations as before, the fact that n is now odd in $2\beta l = n\pi$ results in two sign changes, $\sin 2\beta l \doteq -2\beta_r l(\Delta\omega/\omega_r)$ and $\cos 2\beta l \doteq -1$. Defining $Y_r = Y_0 \alpha_r l$, equation (*10.22*) becomes

$$Y_{\text{inp}} \doteq Y_r\left\{1 + j\frac{\beta_r}{\alpha_r}\left(\frac{\Delta\omega}{\omega_r}\right)\right\} \tag{10.23}$$

which is functionally identical to (*10.6*) and (*10.19*). The indicated resonant Q-value is the value given in (*10.20*).

In summary, the input impedance of a section of low-loss transmission line with short circuit termination varies analogously to the input impedance of a series lumped element resonant circuit around resonance, in the vicinity of all frequencies for which the line length is an integral number of half wavelengths, and varies analogously to the input impedance of a parallel lumped element resonant circuit around resonance, in the vicinity of all frequencies for which the line length is an odd number of quarter wavelengths. The resonant Q-value at any resonant frequency is easily determined from the phase factor and the attenuation factor of the line and is independent of the line length.

Example 10.4.

The specifications of standard rigid 7/8″ copper coaxial line are given in Problem 5.6, page 64, for frequencies of 1, 10, 100 and 1000 megahertz. At each of these frequencies determine the length of a quarter wavelength resonant line section with short circuit termination, and calculate the resonant Q-value and resonant input impedance in each case.

The phase velocity is given as 99.8% at all four frequencies. Hence $v_p = 0.998 \times 3.00 \times 10^8 = 2.99 \times 10^8$ m/sec. The values of $\beta_r = \omega_r/v_p$ and $l = \lambda/4 = 2\pi v_p/(4\omega_r)$ at the four frequencies are

f_r megahertz	ω_r rad/sec	β_r rad/m	$\lambda/4$ m
1	6.28×10^6	0.0210	74.9
10	6.28×10^7	0.210	7.49
100	6.28×10^8	2.10	0.749
1000	6.28×10^9	21.0	0.0749

The attenuation factors given for the line at the four frequencies in db/100 ft are respectively 0.0425, 0.135, 0.440 and 1.49. The characteristic impedance is 50.0 ohms. The resonant Q-values determined from $Q_r = \beta_r/(2\alpha_r)$, and the resonant input impedances determined from $Z_r = Z_0/(\alpha_r l)$ are

f_r megahertz	α_r nepers/m	$\alpha_r l$ nepers	Q_r	Z_r ohms	$\omega_r L/R$
1	1.61×10^{-4}	1.20×10^{-2}	65.4	4,170	65.2
10	5.10×10^{-4}	3.82×10^{-3}	206	13,100	206
100	1.67×10^{-3}	1.24×10^{-3}	632	40,200	652
1000	5.63×10^{-3}	4.22×10^{-4}	1866	119,000	2062

The values of $\omega_r L/R$ are added for comparison, since it is an identity (see Problem 10.10) that $Q_r = \omega_r L/R$, where L and R are respectively the distributed inductance and resistance of the line, when the losses are due entirely to distributed resistance. In Problem 5.6, page 64, the distributed inductance of the line was calculated to be 0.167 microhenries/m, and the distributed resistances at the four frequencies had the respective values 0.0161, 0.0509, 0.161 and 0.509 ohms/m.

From the above results it is seen that at frequencies of 1 megahertz and 10 megahertz, and almost up to 100 megahertz, the values of Q_r and Z_r available from quarter wavelength sections of even this heavy, bulky, and expensive low-loss transmission line are not as high as can be obtained from simple compact lumped element resonant circuits consisting of a wound coil and a parallel plate condenser. At the two lower frequencies the circuit lengths are totally impractical.

At 100 megahertz the circuit length is about 30 in., but the Q-value and resonant impedance are considerably higher than lumped element circuits could provide. Critical applications might justify use of the line at this frequency.

At 1000 megahertz, the Q-value and resonant impedance are enormous, by the standards of lumped element circuits. The circuit length, however, is about 3 in., only three times the line's diameter. Depending on the total structure to which the unit is connected, this could result in the stray fields at the open input end substantially modifying the resonance performance.

As a general conclusion, it appears that resonant quarter wavelength sections of standard rigid 7/8″ copper coaxial line with short circuit termination would be very superior resonant circuits over the frequency range from about 100 megahertz to 500 megahertz or somewhat higher, and their use might be indicated in applications involving high power levels or requiring high selectivity.

10.6. The validity of the approximations.

All of the approximations made in deriving the resonant transmission line circuit relations of Section 10.5 improve in accuracy for circuits of higher resonant Q-value. The assumption that the line's characteristic impedance is real, for example, is an approximation to the fact first stated in equation (*5.30*), page 59, that the phase angle of the characteristic impedance of a lossy line lies between the values $\tan^{-1}(+\alpha/\beta)$ and $\tan^{-1}(-\alpha/\beta)$. The former value applies if the losses are all due to distributed conductance, and the latter if the losses are all due to distributed resistance. For high frequency lines with a substantial amount of dielectric in the interconductor space, the phase angle will lie somewhere between the two extremes. It follows from equation (*10.20*) that if a resonant transmission line circuit is calculated or measured to have a resonant Q-value of Q_r, the phase angle of the characteristic impedance of the line cannot exceed approximately $1/(2Q_r)$ rad and might be much smaller.

The accuracy of the approximations for $\sinh \alpha l$ and $\cosh \alpha l$ depends on the total line attenuation αl being small. But $\alpha l/(\beta l) \doteq 1/(2Q_r)$ and βl for resonant circuits with short circuit or open circuit terminations is a small multiple of $\pi/2$. Specifically, for a quarter-wavelength circuit $\beta l = \pi/2$ and $\alpha l \doteq 1/Q_r$.

The accuracy of the approximation for $\sin[2(\omega_r + \Delta\omega)l/v_p]$ is good if it is not necessary to vary $\Delta\omega$ more than a few percent on either side of ω_r. The half-power or 3 db points of a resonance curve will be covered by such variations if the value of Q_r for the circuit exceeds the reciprocal of twice the maximum fractional frequency deviation for which the approximation is acceptable.

The fact that the attenuation factor of high frequency transmission lines varies finitely with frequency across the frequency range of a resonance curve introduces only a second order correction in deriving (*10.20*) from (*10.19*) or (*10.23*) because the effects of this variation on the deviation of the two half power frequencies from the resonant frequency are of opposite sign and cancel to a first approximation. The error introduced is of the order $1/Q_r^2$.

For many reasons, including limits to tolerable physical length, practical resonant transmission line circuits very seldom have resonant Q-values less than 100, and values of several hundred are more typical. The approximations made in Section 10.5 are therefore almost invariably highly accurate.

When resonant circuits are constructed from parallel wire transmission line, allowance may have to be made for an additional phenomenon. It has been established both theoretically and experimentally that open circuit or short circuit terminations of parallel wire lines radiate as small dipole elements, with radiation resistance given by

$$R_{\text{rad}} \doteq 60\pi^2(s/\lambda)^2 \text{ ohms} \tag{10.24}$$

where s is the separation between conductor centers. Even with s/λ as small as 0.01 this resistance is likely to be considerably larger than the resistance of a typical short circuit termination, and it may have a marked effect on the Q-value of the resulting circuit. The radiation loss can be eliminated by terminating a parallel wire line with a short circuit in the form of a plane transverse metal sheet about 1.3 wavelengths in diameter.

In contrast to the situation for lumped circuit elements, transmission line resonant circuits designed from the formulas of this chapter and Chapter 6 can be expected to have experimental characteristics agreeing very closely with the design specifications. Particularly in the case of fully shielded coaxial line circuits or shielded pair circuits with short circuit terminations at each end in the manner of Fig. 10-5, there are no significant intangible factors not covered by the theory.

10.7. Resonance curve methods for impedance measurement.

In Chapter 8 it has been seen that the normalized value of any impedance connected as the terminal load of a transmission line can be determined from measurements of the voltage standing wave pattern it produces on the line. When the unknown impedance produces a reflection coefficient of magnitude fairly close to unity, the VSWR on the line is high. Section 8.7, page 172, describes a technique by which such high VSWR values can be determined in terms of the separation between two locations on the line, one on each side of a voltage minimum, at which the voltage magnitude is $\sqrt{2}$ times the value at the minimum. This procedure is obviously analogous to that of finding the "3 db" bandwidth of a resonant circuit, but the circuit on which the measurement is made is not a resonant circuit, and the independent variable is neither source frequency nor a circuit reactance. Because this method observes signals at a voltage minimum, it requires a sensitive detector, and attention to the reduction of noise and interference.

An alternative method of measuring transmission line terminal impedances that produce reflection coefficients of large magnitude takes advantage of the phenomenon of resonance and observes the width of a resonance curve maximum of current or voltage, instead of a standing wave minimum. The practical instrumentation of the method can have any of several configurations, one of which is shown in Fig. 10-7. Here Z_T is the unknown terminal load impedance to be measured, which produces a voltage reflection coefficient ρ_T of magnitude close to unity. At the other end of the line the signal source produces a reflection coefficient ρ_S, whose magnitude should be comparable to or larger than that of ρ_T for best accuracy in the final results. This can be achieved by using a source having low internal impedance. The detector shown is a voltage probe, such as that used in a slotted line section. Its location on the line can be varied, relative to the location of the impedance Z_T. The resonance curve of output at the detector is obtained by varying the line length l through a resonant value, usually by sliding contacts at or near the source end of the line.

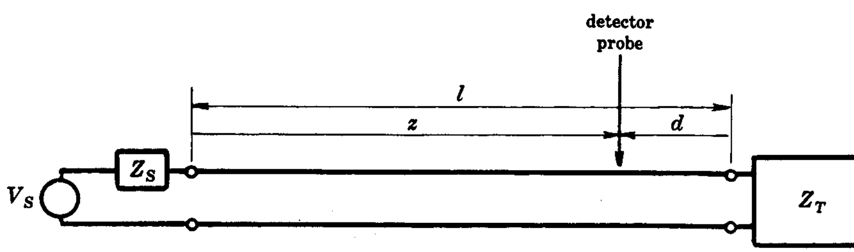


Fig. 10-7. Circuit for measuring an unknown impedance Z_T by resonance curve observations. With distance d adjusted for maximum detector output, the circuit length l is varied through a resonant value.

The voltage at a coordinate z on the transmission line circuit of Fig. 10-7 is given by equation (*8.26*), page 175, where V_S is the rms source voltage, Z_S the source impedance, Z_0 the line's characteristic impedance, $\gamma = \alpha + j\beta$ the propagation factor of the line, and the other terms are as defined above. Since both z and l are independent variables,

$$V(z,l) \;=\; \frac{V_S Z_0}{Z_S + Z_0}\,\frac{e^{-\gamma z} + \rho_T e^{-2\gamma l} e^{\gamma z}}{1 - \rho_T \rho_S e^{-2\gamma l}} \qquad (10.25)$$

Taking out a factor $e^{-\gamma l}$ in the numerator, and noting that $l - z = d$,

$$V(d,l) \;=\; \frac{V_S Z_0}{Z_S + Z_0}\,\frac{e^{-\gamma l}\,(e^{\gamma d} + \rho_T e^{-\gamma d})}{1 - \rho_S \rho_T e^{-2\gamma l}} \qquad (10.26)$$

The experimental procedure, after the circuit has been assembled with the source, detector and unknown Z_T connected, is to vary l and d until a detector indication is found. Then d is varied at fixed l until the detector output is a maximum. Since only the exponential terms in (*10.26*) are functions of d or l, it is easily seen that the value of d that maximizes $|V(d,l)|$ is the same for all values of l. The distance between the detector probe and the terminal impedance Z_T is therefore fixed at the experimentally determined optimum value, throughout a measurement. The adjustment of this value is not critical, since it is at a maximum in a standing wave pattern.

As a function of the circuit length l, the voltage magnitude $|V_d(l)|$ at the fixed coordinate d is now given by

$$|V_d(l)| \quad = \quad \frac{|V'|\,|e^{-\gamma l}|}{|1 - \rho_S\rho_T e^{-2\gamma l}|} \tag{10.27}$$

where $|V'|$ is a phasor voltage magnitude that is not a function of l. Let $|V'| = 1$.

Using the mathematical procedure of Section 8.4, page 163, let

$$\rho_S\rho_T \quad = \quad |\rho_S\rho_T|\,e^{j\psi} \quad = \quad e^{-2(u+jv)} \tag{10.28}$$

Then
$$|V_d(l)| \quad = \quad \frac{1/(2\sqrt{|\rho_S\rho_T|})}{|\sinh[(\alpha l + u) + j(\beta l + v)]|} \quad = \quad \frac{1/(2\sqrt{|\rho_S\rho_T|})}{[\sinh^2(\alpha l + u) + \sin^2(\beta l + v)]^{1/2}} \tag{10.29}$$

Although this resonance curve method of impedance measurement is usable for values of $|\rho_S\rho_T|$ from unity down to about 0.18, its advantages outweigh its additional mechanical complexities in comparison with the slotted line standing wave method only for fairly large values of $|\rho_S\rho_T|$, say 0.8 or greater. For such cases, $u = \log_e(1/\sqrt{|\rho_S\rho_T|}) < 0.1$, and since α is usually of the order 10^{-3} nepers/m for suitable transmission lines at frequencies appropriate to the method, $\sinh^2(\alpha l + u)$ changes only infinitesimally across the width of a resonance curve. The maxima of $|V_d(l)|$ therefore occur quite precisely at the resonant line lengths l_r for which $\sin^2(\beta l_r + v) = 0$ or $\beta l_r + v = n\pi$, where n is zero or any integer. From (*10.28*) the phase angle ψ of $|\rho_S\rho_T|$ is determined from any resonant line length l_r by

$$\psi \quad = \quad -2v \quad = \quad 4\pi(l_r/\lambda - n/2) \tag{10.30}$$

The resonant value of $|V_d(l)|$ is $|V_{dr}|$, given by

$$|V_{dr}| \quad = \quad \frac{1/(2\sqrt{|\rho_S\rho_T|})}{\sinh(\alpha l_r + u)} \tag{10.31}$$

which has the same voltage scale factor as (*10.29*).

When $(\alpha l_r + u) < 0.1$, the usual approximation can be made that $\sinh(\alpha l_r + u) \doteq (\alpha l_r + u)$. For a small line length change Δl from a resonant line length l_r, $\sin[\beta(l_r + \Delta l) + v] \doteq \beta\,\Delta l$. Substituting these approximations and the expression for $|V_{dr}|$ from (*10.31*) into (*10.29*),

$$|V_d(l)| \quad = \quad \frac{|V_{dr}|}{\left[1 + \left(\dfrac{\beta\,\Delta l}{\alpha l_r + u}\right)^2\right]^{1/2}} \tag{10.32}$$

Comparison with equation (*10.23*) shows that $|V_d(l)|$ varies with l as a true resonance curve, with a maximum at resonance. The value of $|V_d(l)|$ drops to $|V_{dr}|/\sqrt{2}$ for line length changes $\Delta l'$ on either side of any resonant length l_r, given by $\Delta l'/\lambda = (\alpha l_r + u)/(2\pi)$. If W/λ is the width of the resonance curve in wavelengths between these "3 db" points,

$$W/\lambda \quad = \quad 2\Delta l'/\lambda \quad = \quad (\alpha l_r + u)/\pi \tag{10.33}$$

Finally, from (*10.30*) and (*10.33*),

$$|\rho_S \rho_T| = e^{-2(\pi W/\lambda - \alpha l_r)} \tag{10.34}$$

If $2(\pi W/\lambda - \alpha l_r)$ is less than about 0.01, which can easily be the case for low-loss terminations and low line attenuation, equation (*10.34*) can be simplified to

$$|\rho_S \rho_T| = 1 - 2(\pi W/\lambda - \alpha l_r) \tag{10.35}$$

With the values of α and λ known, the complex number value of $\rho_S \rho_T$ can therefore be determined from measured values of W and l_r, using equations (*10.30*) and either (*10.34*) or (*10.35*).

Provided the total circuit length l is variable over a sufficient range, both α and λ can be measured directly and accurately with the circuit itself, by observing resonance curves at two consecutive values of l_r without changing the line's terminations. If the resonant lengths are l_r and l_r', with $l_r' > l_r$, the wavelength is obtained from $l_r' - l_r = \lambda/2$. If the widths of the corresponding resonance curves are respectively W and W', it follows from equation (*10.33*) that

$$\alpha = 2\pi(W' - W)/\lambda^2 \tag{10.36}$$

since u has the same value in both cases.

The information desired from this resonance curve procedure is the complex number value of ρ_T, from which the normalized components of the unknown terminal impedance Z_T can be calculated using equations (*7.9a*) and (*7.9b*), page 128. It is therefore necessary to determine ρ_S. This is done by connecting in place of Z_T a short circuit for which $\rho_T = -1 + j0 = 1\underline{/\pi}$. If the resulting resonance curve measurements are W'' and l_r'', the final determination of $\rho_T = |\rho_T| e^{j\phi_T}$ will be made from

$$\phi_T = \pi + 4\pi(l_r - l_r'')/\lambda \tag{10.37}$$

and

$$|\rho_T| = e^{-2\{\pi(W - W'')/\lambda - \alpha(l_r - l_r'')\}} \tag{10.38}$$

where W and l_r are the measurements with Z_T connected. Equation (*10.38*) can be written as

$$|\rho_T| \doteq 1 - 2\{\pi(W - W'')/\lambda - \alpha(l_r - l_r'')\} \tag{10.39}$$

when the exponent is less than 0.01.

When this resonance curve method of impedance measurement is extended to values of $|\rho_S \rho_T|$ between 0.18 and 0.80, the width in wavelengths of a resonance curve at the "3 db" points can increase to a maximum value of $\frac{1}{2}$. Many of the approximations made after equation (*10.29*) are then unsatisfactory. The best procedure is to construct graphs relating W/λ to $|\rho_S \rho_T|$ for various values of the parameter αl_r.

As alternatives to the detector arrangement in the circuit of Fig. 10-7, the resonance curve procedure can be used with a low impedance coupling loop detector in series at either the source end of the line or the Z_T end of the line, or the source and Z_T can be at the same end of the line with a low impedance coupling loop detector connected as the termination at the other end.

Solved Problems

10.1. For a resonant section of low-loss transmission line terminated in a short circuit, show that the definition of resonant Q-value given by equation (*10.9*) is equivalent to the value given by equation (*10.20*).

A quarter wavelength section will be assumed. The result will then apply to each quarter wavelength section of any longer circuit.

For any element of length Δd at coordinate d of the line, the instantaneous energy stored in the distributed capacitance of the element is $W_C = \frac{1}{2}C\,\Delta d\, v(d,t)^2$, and the instantaneous energy stored in the distributed inductance of the element is $W_L = \frac{1}{2}L\,\Delta d\, i(d,t)^2$. The simultaneous power loss is $P_1 = R\,\Delta d\, i(d,t)^2 + G\,\Delta d\, v(d,t)^2$, where R, L, G and C are the usual distributed circuit coefficients of the line at the operating frequency, and $v(d,t)$ and $i(d,t)$ are respectively the instantaneous voltage and current values at coordinate d on the line at some instant t.

Since the short circuit termination on the line produces a voltage reflection coefficient $\rho_T = -1 + j0$, and the line losses are very small, it follows from the derivations in Chapter 8 that the standing wave pattern of voltage magnitude on the line is very accurately one quarter of a sine wave, increasing from zero at the short circuit to a maximum at the input terminals, and the standing wave pattern of current magnitude is a mirror image of the voltage pattern, rising from zero at the input terminals to a maximum at the short circuit.

Thus in phasor magnitude notation, $|V(d)| = |V_{\text{inp}}| \sin\beta d$ and $|I(d)| = |I_T| \cos\beta d$ where the coordinate d increases from the short circuit toward the input terminals. Comparing the scale factor for $|V(d)|$ in equation (*8.15*), page 164, with the scale factor for $|I(d)|$ in the corresponding equation in Problem 8.1, page 178, it is evident that $|I_T| = |V_{\text{inp}}|/Z_0$, where Z_0 is the line's characteristic impedance.

In Chapter 8, attention was focused on the standing wave patterns of voltage and current magnitude along a line as a function of the distance coordinate d, and the time variation of the voltage and current were not discussed, except in Problem 8.5, page 180. In that problem it was shown that if two equal amplitude harmonic voltage waves of angular frequency ω and phase factor β travel in opposite directions on a transmission line having negligible losses, the instantaneous voltage at any time t at any coordinate z can be represented for the two waves by the expressions $V_1\cos(\omega t - \beta z)$ and $V_1\cos(\omega t + \beta z)$ respectively. By a trigonometric identity, the sum of these two expressions, which is the total voltage on the line as a function of z and t, becomes $2V_1\cos\omega t\cos\beta z$. Referring to equation (*7.5*), page 127, the corresponding current waves are represented by $(V_1/Z_0)\cos(\omega t - \beta z)$ and $(-V_1/Z_0)\cos(\omega t + \beta z)$ respectively, and their sum is $(2V_1/Z_0)\sin\omega t\sin\omega z$.

The functional difference of the expressions for current and voltage in the coordinate z is equivalent to the difference noted above for the standing wave patterns of $|V(d)|$ and $|I(d)|$ on a low-loss line terminated in a reflection coefficient of magnitude unity. The implications of the functional difference of the time-varying terms, however, has not been explored in previous chapters. For purposes of the present problem it has the important significance that at an instant when the voltage at every point on the line is a maximum, the current is everywhere zero, and vice versa. This is analogous to the fact that in the lumped element circuits of Fig. 10-1 and 10-2 the voltage across the capacitor is a maximum at an instant when the current in the inductor is zero, and vice versa.

In calculating the *peak* energy stored in a transmission line resonant circuit, therefore, as required in equation (*10.9*), it is sufficient to calculate *either* the total energy stored in the line section's distributed capacitance at an instant when the voltage on the line is everywhere a maximum *or* the total energy stored in the line section's distributed inductance at an instant when the current on the line is everywhere a maximum. The losses calculated for a full cycle, however, which are also required in the equation, must include *both* the losses produced by line current in the line's distributed resistance, and the losses produced by line voltage in the line's distributed conductance.

Evaluating the energy stored, and the line losses, along a quarter wavelength section of line involves only the integrals

$$(1/\beta)\int_0^{\pi/2} \sin^2\beta d\; d(\beta d) \;=\; \lambda/8 \qquad\text{and}\qquad (1/\beta)\int_0^{\pi/2} \cos^2\beta d\; d(\beta d) \;=\; \lambda/8$$

Combining all of the above, and taking V_{inp} to be an rms phasor quantity, equation (*10.9*) becomes

$$Q_r \;=\; 2\pi\,\frac{\frac{1}{2}C(\sqrt{2}\,|V_{\text{inp}}|)^2(\lambda/8)}{[R|V_{\text{inp}}/Z_0|^2(\lambda/8) + G|V_{\text{inp}}|^2(\lambda/8)](1/f_r)} \;=\; 2\pi f_r C/(R/Z_0^2 + G)$$

where f_r is the resonant frequency in hertz. Using $Z_0 = \sqrt{L/C}$ and multiplying all terms by Z_0,

$$Q_r = \frac{\omega_r\sqrt{LC}}{2(\frac{1}{2}R\sqrt{C/L} + \frac{1}{2}G\sqrt{L/C})} = \frac{\beta_r}{2\alpha_r}$$

from equations (*5.5*) and (*5.6*), subject to the "high frequency" approximations.

It is obvious that the result will be the same for every separate quarter-wavelength of the standing wave pattern on a line of negligible losses, terminated in either an open circuit or a short circuit.

10.2. A parallel resonant circuit is to be created at the input terminals of an amplifier, resonant at the amplifier's operating frequency of 250 megahertz, consisting of the amplifier's input capacitance of 7.5 micromicrofarads connected in parallel with the input terminals of a section of low-loss air-dielectric transmission line having short circuit termination. What length of transmission line section is required, if the line's characteristic impedance is $80 + j0$ ohms?

For parallel resonance, the input susceptance of the transmission line must be equal in magnitude and opposite in sign to the input susceptance of the amplifier. Then, from equation (*7.21*), $-Y_0 \cot(\omega_r l/v_p) = -\omega_r C_{amp}$, where ω_r is the resonant angular frequency, l the line length, C_{amp} the input capacitance of the amplifier and $Y_0 = 1/Z_0$ is the characteristic admittance of the line. Solving,

$$\begin{aligned} l &= (v_p/\omega_r)\cot^{-1}(\omega_r C_{amp} Z_0) \\ &= (3.00\times 10^8)/(2\pi\times 2.50\times 10^8)\cot^{-1}[(2\pi\times 2.5\times 10^8)\times 7.5\times 10^{-12}\times 80] = 0.156 \text{ m} \end{aligned}$$

This solution gives the shortest line length that will produce the desired resonance. Any integral number of half wavelengths (0.60 m) can be added to this value.

10.3. A section of low-loss transmission line of length l/λ wavelengths has characteristic impedance Z_0 and is terminated in a short circuit. There is a capacitance C connected across the input terminals. What are the resonant frequencies of the combination?

The equation for stating the problem is that used in Problem 10.2. Thus

$$Y_0 \cot(\omega_r l/v_p) = \omega_r C$$

This is a transcendental equation in ω_r. It can be solved graphically or by testing a series of numerical values. Also, tables of the function $(\cot x)/x$ are available, which can be used if the above equation is brought to the form

$$[\cot(\omega_r l/v_p)]/(\omega_r l/v_p) = v_p C/(lY_0)$$

Plotted against ω_r, the cotangent function for any value of l/v_p has an infinite number of parts, each extending in ordinate from minus infinity to plus infinity. The straight line represented by the function $\omega_r C Z_0$ on the same coordinates passes through the origin and has a finite slope. It therefore intersects each of the cotangent curves at an ω_r coordinate which is a resonant frequency of the circuit. It is evident that the resulting frequencies are not harmonically related.

10.4. For a resonant transmission line section one quarter wavelength long with short circuit termination, determine the impedance that would be measured between the line conductors at a cross section distant d from the short circuit, at the resonant frequency. The line has attenuation factor α_r, phase factor β_r and characteristic impedance Z_0 at that frequency.

The input admittance Y_d at the location d is the sum of the input admittances of the two line sections on either side. One section has length d and is terminated in a short circuit. The other section has length $(\lambda/4) - d$ and is terminated in an open circuit. Hence

$$Y_d = Y_0[\coth(\alpha_r + j\beta_r)d + \tanh(\alpha_r + j\beta_r)(\lambda/4 - d)]$$

Let $(\lambda/4) - d = z$. Expanding the hyperbolic cotangent and tangent by equations (*10.22*) and (*10.18*) respectively, noting that $\sin 2\beta_r z = \sin 2\beta_r d$ and $\cos 2\beta_r z = -\cos 2\beta_r d$, and using the approximations $\sinh 2\alpha_r d \doteq 2\alpha_r d$, $\sinh 2\alpha_r z \doteq 2\alpha_r z$, $\cosh 2\alpha_r d \doteq \cosh 2\alpha_r z \doteq 1$, both denominators become $1 - \cos 2\beta_r d = 2\sin^2 \beta_r d$. Then

$$Y_d = Y_0\left(\frac{2\alpha_r d - j\sin 2\beta_r d}{2\sin^2\beta_r d} + \frac{2\alpha_r z + j\sin 2\beta_r d}{2\sin^2\beta_r d}\right) = \frac{\alpha_r(d+z)}{\sin^2\beta_r d} = \frac{\alpha_r\lambda/4}{\sin^2\beta_r d}$$

Y_d is real at every cross section because the circuit is resonant. The input impedance at the cross section d is $Z_d = 1/Y_d = Z_0/(\alpha_r\lambda/4)\sin^2\beta_r d = Z_r\sin^2\beta_r d$ where Z_r is the resonant impedance at the input terminals of the resonant quarter wavelength section.

Since the energy storage and power loss relations are always those of the quarter wavelength resonant circuit, the resonant Q-value governing the impedance variations at any location d must have the constant value $\beta_r/2\alpha_r$. Thus the selectivity properties of the circuit are available at any level of resonant impedance less than Z_r by suitable choice of the connection location d. Although the resonant impedance at the center of the quarter wavelength section is one half the resonant impedance at its input terminals, this simple proportion does not hold at other locations.

10.5. A quarter wavelength resonant section of low-loss air-dielectric high-frequency coaxial transmission line with short circuit termination has a resonant input impedance $Z_r = Z_0/\alpha l$ according to Section 10.5, where l is the line length, Z_0 the characteristic impedance (assumed real) and α the attenuation factor at the resonant frequency. What ratio of the radii of the facing conductor surfaces b/a will result in the highest value of Z_r, if the value of the inner radius of the outer conductor b is fixed?

Since there are no dielectric losses, the attenuation factor is given by $\alpha = R/(2Z_0)$ where R is the total conductor resistance per unit length. Thus $Z_r = 2Z_0^2/(Rl)$. In terms of conductor radii, using equations (*6.49*), page 91 and (*6.59*), page 96,

$$Z_r = \frac{7200[\log_e(b/a)]^2}{(R_s/2\pi b)(1+b/a)}$$

The value of b/a (at constant b) that maximizes Z_r is found in the usual way.

$$\frac{dZ_r}{d(b/a)} = \frac{2(a/b)\log_e(b/a)}{1+b/a} - \frac{[\log_e(b/a)]^2}{(1+b/a)^2} = 0$$

after deleting the coefficients. This reduces to $2/(b/a) + 2 = \log_e(b/a)$, a transcendental equation to be solved graphically or by trial. The approximate result is $b/a = 9.1$. The resulting coaxial line has the very high characteristic impedance of 132 ohms, and is far from optimum by any other criterion. The resonant Q-value of the circuit constructed from this line would be about 20% less than for a line with the same outer conductor and a ratio $b/a = 3.60$ for minimum attenuation.

10.6. With a circuit of the form of Fig. 10-7, consisting of a length of air dielectric transmission line operating at a frequency of 400 megahertz, a resonance curve of width 3.37 mm at the half power level is observed with a resonant line length of 0.703 m. With the same termination a resonance curve of width 4.36 mm is observed when the resonant line length is 1.078 m. Determine the attenuation factor of the line, and the reflection coefficient at the source end if the termination is assumed to be a perfect short circuit.

At the stated frequency the wavelength on the air dielectric line is 0.750 m, and the two resonant line lengths differ by one half wavelength. Equation (*10.36*) therefore applies directly, and

$$\alpha = 2\pi(4.36\times10^{-3} - 3.37\times10^{-3})/0.750^2 = 1.11\times10^{-2}\ \text{nepers/m}$$

Using this value in equation (*10.35*) with $|\rho_T| = 1$, the value of $|\rho_S|$ is found directly from either set of data. Thus

$$|\rho_S| = 1 - 2[\pi(3.37\times10^{-3})/0.75 - 1.11\times10^{-2}\times0.703] = 0.9874$$

Supplementary Problems

10.7. From equation (*10.4*), show that for the circuit of Fig. 10-1 the fractional frequency deviation $(\omega_1 - \omega_r)/\omega_r$ above resonance at which the circuit's input impedance is $Z_r(1 + j1)$, is given by

$$(\omega_1 - \omega_r)/\omega_r \;=\; 1/(2Q_r) + 1/(8Q_r^2) - 1/(128Q_r^4) + \cdots$$

and that the equivalent statement for the corresponding frequency ω_2 below resonance is

$$(\omega_r - \omega_2)/\omega_r \;=\; 1/(2Q_r) - 1/(8Q_r)^2 + 1/(128Q_r^4) + \cdots$$

Thus at the level of the half power points, the resonance curve for this ideal circuit is "off center" by a fraction of approximately $1/(8Q_r)$ of the width of the resonance curve at that level.

10.8. Show that for the circuit of Fig. 10-2, the phasor magnitude of the current in each of the reactive elements of the circuit is Q_r times as great as the phasor magnitude of the current supplied to the circuit by the source, at the resonant frequency. This is the phenomenon of "resonant rise of current".

10.9. If equation (*10.4*) is equated to $Z_{\text{inp}} = R_{\text{inp}} + jX_{\text{inp}}$, show that the graphs of $|X_{\text{inp}}|$ and $|Z_{\text{inp}}|$ plotted against the frequency ω are symmetrical about the ordinate $\omega = \omega_r$, *provided the frequency scale is logarithmic* but not otherwise. If the frequency coordinates are normalized relative to ω_r and the reactance and impedance magnitude coordinates are normalized relative to Z_r, such graphs become universal resonance curves for series resonant circuits. Equation (*10.15*) shows that the same graphs are also universal resonance curves for parallel resonant circuits if $|B_{\text{inp}}|$ is substituted for $|X_{\text{inp}}|$ and $|Y_{\text{inp}}|$ for $|Z_{\text{inp}}|$.

10.10. Show that the approximate expression $Q_r \doteq \beta_r/2\alpha_r$ for the resonant Q-value of any resonant section of low-loss transmission line terminated in an open circuit or a short circuit is equivalent to the expression $Q_r = \omega_r L/R$ *if the line losses are caused entirely by* R (i.e. $G = 0$), to the expression $Q_r = \omega_r C/G$ *if the line losses are caused entirely by* G (i.e. $R = 0$), and to the expressions $Q_r = x\omega_r L/R = (1-x)\omega_r C/G$ if a fraction x of the losses is due to R and a fraction $(1-x)$ is due to G. In these relations R, L, G and C are the line's distributed circuit coefficients.

10.11. Standard RG-8/U flexible coaxial cable has a characteristic impedance of 52 ohms, a phase velocity of 66%, and an attenuation factor of 2.05 db/(100 ft) at a frequency of 100 megahertz. What Q-value will a resonant section of the line with open circuit or short circuit termination have at that frequency and what are the resonant input impedances of a quarter wavelength section and a three-quarter wavelength section with short circuit termination?

Ans. $Q_r = 205$ for all the circuits mentioned. $Z_r = 13{,}600$ ohms for a one quarter wavelength circuit and 4,530 ohms for a three-quarter wavelength circuit.

10.12. Applying the result of Problem 10.3 to the circuit of Problem 10.2, consisting of a 0.156 m length of low-loss air-dielectric transmission line ($Z_0 = 80 + j0$ ohms) with short circuit termination and a capacitance of 7.5 micromicrofarads across its input terminals, determine the next two frequencies above 250 megahertz at which parallel resonance will occur at the input terminals.

Ans. Approximately 1210 and 2180 megahertz.

10.13. Show by the methods of Section 7.7 or otherwise that if a voltage of rms phasor magnitude $|V_i|$ is applied to the input terminals of any quarter wavelength section of transmission line with open circuit termination, the rms phasor magnitude of the voltage V_T at the open circuit termination is given by $|V_T| = |V_i|/(\sinh \alpha l) \doteq |V_i|/(\alpha l)$ if the total attenuation of the line is small. This is a form of "resonant rise of voltage". Show that $|V_T|/|V_i| = 1.27\,Q_r$, analogous to the result obtained for a lumped element circuit.

The device can be used as a transformer to develop high voltages, but is subject to the complication that the input impedance has the low value $Z_0\alpha l$ for low-loss lines. Hence a large source current is required to produce a large output voltage.

This phenomenon must be protected against in very long high voltage commercial power lines at 60 hertz. If for any reason the terminal load becomes disconnected from the line, while the generator remains connected, the voltage at the end of the line remote from the generator can rise to values far in excess of the operating voltage of the line.

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