Bernard Dacorogna

Introduction to The Calculus of Variations

Imperial College Press

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ICP
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Published by

Imperial College Press
57 Shelton Street
Covent Garden
London WC2H 9HE

Distributed by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Originally published in French under the title: ‹‹Introduction au calcul des variations››

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ISBN 1-86094-499-X
ISBN 1-86094-508-2 (pbk)

Printed in Singapore.

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Preface to the English Edition

The present monograph is a translation of *Introduction au calcul des variations* that was published by Presses Polytechniques et Universitaires Romandes. In fact it is more than a translation, it can be considered as a new edition. Indeed, I have substantially modified many proofs and exercises, with their corrections, adding also several new ones. In doing so I have benefited from many comments of students and colleagues who used the French version in their courses on the calculus of variations.

After several years of experience, I think that the present book can adequately serve as a concise and broad introduction to the calculus of variations. It can be used at undergraduate as well as graduate level. Of course at a more advanced level it has to be complemented by more specialized materials and I have indicated, in every chapter, appropriate books for further readings. The numerous exercises, integrally corrected in Chapter 7, will also be important to help understand the subject better.

I would like to thank all students and colleagues for their comments on the French version, in particular O. Besson and M. M. Marques who commented in writing. Ms. M. F. DeCarmine helped me by efficiently typing the manuscript. Finally my thanks go to C. Hebeisen for the drawing of the figures.

Preface to the French Edition

The present book is a result of a graduate course that I gave at the Ecole Polytechnique Fédérale of Lausanne during the winter semester of 1990–1991.

The calculus of variations is one of the classical subjects in mathematics. Several outstanding mathematicians have contributed, over several centuries, to its development. It is still a very alive and evolving subject. Besides its mathematical importance and its links with other branches of mathematics, such as geometry or differential equations, it is widely used in physics, engineering, economics and biology. I have decided, in order to remain as unified and concise as possible, not to speak of any applications other than mathematical ones. Every interested reader, whether physicist, engineer or biologist, will easily see where, in his own subject, the results of the present monograph are used. This fact is clearly asserted by the numerous engineers and physicists that followed the course that resulted in the present book.

Let us now examine the content of the monograph. It should first be emphasized that it is not a reference book. Every individual chapter can be, on its own, the subject of a book, For example, I have written one that, essentially, covers the subject of Chapter 3. Furthermore several aspects of the calculus of variations are not discussed here. One of the aims is to serve as a guide in the extensive existing literature. However, the main purpose is to help the non specialist, whether mathematician, physicist, engineer, student or researcher, to discover the most important problems, results and techniques of the subject. Despite the aim of addressing the non specialists, I have tried not to sacrifice the mathematical rigor. Most of the theorems are either fully proved or proved under stronger, but significant, assumptions than stated.

The different chapters may be read more or less independently. In Chapter 1, I have recalled some standard results on spaces of functions (continuous, L^p or Sobolev spaces) and on convex analysis. The reader, familiar or not with these subjects, can, at first reading, omit this chapter and refer to it when needed in the next ones. It is much used in Chapters 3 and 4 but less in the others. All of them, besides numerous examples, contain exercises that are fully corrected in Chapter 7.

Finally I would like to thank the students and assistants that followed my course; their interest has been a strong motivation for writing these notes. I would like to thank J. Sesiano for several discussions concerning the history of the calculus of variations, F. Weissbaum for the figures contained in the book and S. D. Chatterji who accepted my manuscript in his collection at Presses Polytechniques et Universitaires Romandes (PPUR). My thanks also go to the staff of PPUR for their excellent job.

Chapter 0

Introduction

0.1 Brief historical comments

The calculus of variations is one of the classical branches of mathematics. It was Euler who, looking at the work of Lagrange, gave the present name, not really self explanatory, to this field of mathematics.

In fact the subject is much older. It starts with one of the oldest problems in mathematics: the isoperimetric inequality. A variant of this inequality is known as the Dido problem (Dido was a semi historical Phoenician princess and later a Carthaginian queen). Several more or less rigorous proofs were known since the times of Zenodorus around 200 BC, who proved the inequality for polygons. There are also significant contributions by Archimedes and Pappus. Important attempts for proving the inequality are due to Euler, Galileo, Legendre, L'Huilier, Riccati, Simpson or Steiner. The first proof that agrees with modern standards is due to Weierstrass and it has been extended or proved with different tools by Blaschke, Bonnesen, Carathéodory, Edler, Frobenius, Hurwitz, Lebesgue, Liebmann, Minkowski, H.A. Schwarz, Sturm, and Tonelli among others. We refer to Porter [86] for an interesting article on the history of the inequality.

Other important problems of the calculus of variations were considered in the seventeenth century in Europe, such as the work of Fermat on geometrical optics (1662), the problem of Newton (1685) for the study of bodies moving in fluids (see also Huygens in 1691 on the same problem) or the problem of the brachistochrone formulated by Galileo in 1638. This last problem had a very strong influence on the development of the calculus of variations. It was resolved by John Bernoulli in 1696 and almost immediately after also by James, his brother, Leibniz and Newton. A decisive step was achieved with the work of

Euler and Lagrange who found a systematic way of dealing with problems in this field by introducing what is now known as the Euler-Lagrange equation. This work was then extended in many ways by Bliss, Bolza, Carathéodory, Clebsch, Hahn, Hamilton, Hilbert, Kneser, Jacobi, Legendre, Mayer, Weierstrass, just to quote a few. For an interesting historical book on the one dimensional problems of the calculus of variations, see Goldstine [52].

In the nineteenth century and in parallel to some of the work that was mentioned above, probably, the most celebrated problem of the calculus of variations emerged, namely the study of the Dirichlet integral; a problem of multiple integrals. The importance of this problem was motivated by its relationship with the Laplace equation. Many important contributions were made by Dirichlet, Gauss, Thompson and Riemann among others. It was Hilbert who, at the turn of the twentieth century, solved the problem and was immediately after imitated by Lebesgue and then Tonelli. Their methods for solving the problem were, essentially, what are now known as the direct methods of the calculus of variations. We should also emphasize that the problem was very important in the development of analysis in general and more notably functional analysis, measure theory, distribution theory, Sobolev spaces or partial differential equations. This influence is studied in the book by Monna [73].

The problem of minimal surfaces has also had, almost at the same time as the previous one, a strong influence on the calculus of variations. The problem was formulated by Lagrange in 1762. Many attempts to solve the problem were made by Ampère, Beltrami, Bernstein, Bonnet, Catalan, Darboux, Enneper, Haar, Korn, Legendre, Lie, Meusnier, Monge, Müntz, Riemann, H.A. Schwarz, Serret, Weierstrass, Weingarten and others. Douglas and Rado in 1930 gave, simultaneously and independently, the first complete proof. One of the first two Fields medals was awarded to Douglas in 1936 for having solved the problem. Immediately after the results of Douglas and Rado, many generalizations and improvements were made by Courant, Leray, Mac Shane, Morrey, Morse, Tonelli and many others since then. We refer for historical notes to Dierkes-Hildebrandt-Küster-Wohlrab [39] and Nitsche [78].

In 1900 at the International Congress of Mathematicians in Paris, Hilbert formulated 23 problems that he considered to be important for the development of mathematics in the twentieth century. Three of them (the 19th, 20th and 23rd) were devoted to the calculus of variations. These "predictions" of Hilbert have been amply justified all along the twentieth century and the field is at the turn of the twenty first one as active as in the previous century.

Finally we should mention that we will not speak of many important topics of the calculus of variations such as Morse or Liusternik-Schnirelman theories. The interested reader is referred to Ekeland [40], Mawhin-Willem [72], Struwe [92] or Zeidler [99].

0.2 Model problem and some examples

We now describe in more detail the problems that we will consider. The model case takes the following form

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx : u \in X \right\} = m\,.$$

This means that we want to minimize the integral, $I(u)$, among all functions $u \in X$ (and we call m the minimal value that can take such an integral), where

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded open set, a point in Ω will be denoted by $x = (x_1, ..., x_n)$;

- $u : \Omega \to \mathbb{R}^N$, $N \geq 1$, $u = \left(u^1, ..., u^N\right)$, and hence

$$\nabla u = \left(\frac{\partial u^j}{\partial x_i} \right)_{1 \leq i \leq n}^{1 \leq j \leq N} \in \mathbb{R}^{N \times n}\,;$$

- $f : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}$, $f = f(x, u, \xi)$, is continuous;

- X is the space of admissible functions (for example, $u \in C^1\left(\overline{\Omega}\right)$ with $u = u_0$ on $\partial\Omega$).

We will be concerned with finding a *minimizer* $\overline{u} \in X$ of (P), meaning that

$$I(\overline{u}) \leq I(u)\,,\ \forall u \in X\,.$$

Many problems coming from analysis, geometry or applied mathematics (in physics, economics or biology) can be formulated as above. Many other problems, even though not entering in this framework, can be solved by the very same techniques.

We now give several classical examples.

Example: Fermat principle. We want to find the trajectory that should follow a light ray in a medium with non constant refraction index. We can formulate the problem in the above formalism. We have $n = N = 1$,

$$f(x, u, \xi) = g(x, u) \sqrt{1 + \xi^2}$$

and

$$(P) \quad \inf \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx : u(a) = \alpha,\ u(b) = \beta \right\} = m\,.$$

Example: Newton problem. We seek for a surface of revolution moving in a fluid with least resistance. The problem can be mathematically formulated as follows. Let $n = N = 1$,

$$f(x, u, \xi) = f(u, \xi) = 2\pi u \frac{\xi^3}{1 + \xi^2}$$

and

$$(P) \quad \inf \left\{ I(u) = \int_a^b f(u(x), u'(x))\, dx : u(a) = \alpha,\ u(b) = \beta \right\} = m\,.$$

We will not treat this problem in the present book and we refer to Buttazzo-Kawohl [18] for a review.

Example: Brachistochrone. The aim is to find the shortest path between two points that follows a point mass moving under the influence of gravity. We place the initial point at the origin and the end one at $(b, -\beta)$, with $b, \beta > 0$. We let the gravity act downwards along the y-axis and we represent any point along the path by $(x, -u(x))$, $0 \le x \le b$.

In terms of our notation we have that $n = N = 1$ and the function, under consideration, is $f(x, u, \xi) = f(u, \xi) = \sqrt{1 + \xi^2}/\sqrt{2gu}$ and

$$(P) \quad \inf \left\{ I(u) = \int_0^b f(u(x), u'(x))\, dx \ : u \in X \right\} = m$$

where $X = \left\{ u \in C^1([0, b]) : u(0) = 0,\ u(b) = \beta \text{ and } u(x) > 0,\ \forall x \in (0, b] \right\}$. The shortest path turns out to be a *cycloid*.

Example: Minimal surface of revolution. We have to determine among all surfaces of revolution of the form

$$v(x, y) = (x, u(x) \cos y, u(x) \sin y)$$

with fixed end points $u(a) = \alpha,\ u(b) = \beta$ one with minimal area. We still have $n = N = 1$,

$$f(x, u, \xi) = f(u, \xi) = 2\pi u \sqrt{1 + \xi^2}$$

and

$$(P) \quad \inf \left\{ I(u) = \int_a^b f(u(x), u'(x))\, dx : u(a) = \alpha,\ u(b) = \beta,\ u > 0 \right\} = m\,.$$

Solutions of this problem, when they exist, are *catenoids*. More precisely the minimizer is given, $\lambda > 0$ and μ denoting some constants, by

$$u(x) = \lambda \cosh \frac{x+\mu}{\lambda}.$$

Example: Mechanical system. Consider a mechanical system with M particles whose respective masses are m_i and positions at time t are $u_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$, $1 \le i \le M$. Let

$$T(u') = \frac{1}{2} \sum_{i=1}^{M} m_i |u_i'|^2 = \frac{1}{2} \sum_{i=1}^{M} m_i \left(x_i'^2 + y_i'^2 + z_i'^2 \right)$$

be the kinetic energy and denote the potential energy with $U = U(t,u)$. Finally let

$$f(t,u,\xi) = T(\xi) - U(t,u)$$

be the Lagrangian. In our formalism we have $n = 1$ and $N = 3M$.

Example: Dirichlet integral. This is the most celebrated problem of the calculus of variations. We have here $n > 1$, $N = 1$ and

$$(P) \quad \inf \left\{ I(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx : u = u_0 \text{ on } \partial\Omega \right\}.$$

As for every variational problem we associate a differential equation which is nothing other than *Laplace equation*, namely $\Delta u = 0$.

Example: Minimal surfaces. This problem is almost as famous as the preceding one. The question is to find among all surfaces $\Sigma \subset \mathbb{R}^3$ (or more generally in $\mathbb{R}^{n+1}$, $n \ge 2$) with prescribed boundary, $\partial\Sigma = \Gamma$, where Γ is a closed curve, one that is of minimal area. A variant of this problem is known as *Plateau problem*. One can realize experimentally such surfaces by dipping a wire into a soapy water; the surface obtained when pulling the wire out from the water is then a minimal surface.

The precise formulation of the problem depends on the kind of surfaces that we are considering. We have seen above how to write the problem for minimal surfaces of revolution. We now formulate the problem for more general surfaces.

Case 1: Nonparametric surfaces. We consider (hyper) surfaces of the form

$$\Sigma = \left\{ v(x) = (x, u(x)) \in \mathbb{R}^{n+1} : x \in \overline{\Omega} \right\}$$

with $u : \overline{\Omega} \to \mathbb{R}$ and where $\Omega \subset \mathbb{R}^n$ is a bounded domain. These surfaces are therefore graphs of functions. The fact that $\partial\Sigma$ is a preassigned curve Γ, reads now as $u = u_0$ on $\partial\Omega$, where u_0 is a given function. The area of such a surface is given by

$$\text{Area}(\Sigma) = I(u) = \int_\Omega f(\nabla u(x))\, dx$$

where, for $\xi \in \mathbb{R}^n$, we have set

$$f(\xi) = \sqrt{1 + |\xi|^2}\,.$$

The problem is then written in the usual form

$$(P) \quad \inf\left\{ I(u) = \int_\Omega f(\nabla u(x))\, dx : u = u_0 \text{ on } \partial\Omega \right\}.$$

Associated with (P) we have the so called *minimal surface equation*

$$(E) \quad Mu \equiv \left(1 + |\nabla u|^2\right)\Delta u - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

which is the equation that any minimizer u of (P) should satisfy. In geometrical terms this equation just expresses the fact that the corresponding surface Σ has its *mean curvature* that vanishes everywhere.

Case 2: Parametric surfaces. Nonparametric surfaces are clearly too restrictive from the geometrical point of view and one is lead to consider *parametric surfaces*. These are sets $\Sigma \subset \mathbb{R}^{n+1}$ so that there exist a domain $\Omega \subset \mathbb{R}^n$ and a map $v : \overline{\Omega} \to \mathbb{R}^{n+1}$ such that

$$\Sigma = v\left(\overline{\Omega}\right) = \left\{ v(x) : x \in \overline{\Omega} \right\}.$$

For example, when $n = 2$ and $v = v(x, y) \in \mathbb{R}^3$, if we denote by $v_x \times v_y$ the normal to the surface (where $a \times b$ stands for the vectorial product of $a, b \in \mathbb{R}^3$ and $v_x = \partial v / \partial x$, $v_y = \partial v / \partial y$) we find that the area is given by

$$\text{Area}(\Sigma) = J(v) = \iint_\Omega |v_x \times v_y|\, dxdy\,.$$

In terms of the notations introduced at the beginning of the present section we have $n = 2$ and $N = 3$.

Example: Isoperimetric inequality. Let $A \subset \mathbb{R}^2$ be a bounded open set whose boundary, ∂A, is a sufficiently regular simple closed curve. Denote by $L(\partial A)$ the length of the boundary and by $M(A)$ the measure (the area) of A. The isoperimetric inequality states that

$$[L(\partial A)]^2 - 4\pi M(A) \geq 0\,.$$

Furthermore, equality holds if and only if A is a disk (i.e., ∂A is a circle).

We can rewrite it into our formalism (here $n = 1$ and $N = 2$) by parametrizing the curve

$$\partial A = \{u(x) = (u_1(x), u_2(x)) : x \in [a, b]\}$$

and setting

$$\begin{aligned} L(\partial A) &= L(u) = \int_a^b \sqrt{u_1'^2 + u_2'^2}\, dx \\ M(A) &= M(u) = \frac{1}{2} \int_a^b (u_1 u_2' - u_2 u_1')\, dx = \int_a^b u_1 u_2'\, dx\,. \end{aligned}$$

The problem is then to show that

$$(P) \quad \inf\{L(u) : M(u) = 1;\ u(a) = u(b)\} = 2\sqrt{\pi}\,.$$

The problem can then be generalized to open sets $A \subset \mathbb{R}^n$ with sufficiently regular boundary, ∂A, and it reads as

$$[L(\partial A)]^n - n^n \omega_n [M(A)]^{n-1} \geq 0$$

where ω_n is the measure of the unit ball of $\mathbb{R}^n$, $M(A)$ stands for the measure of A and $L(\partial A)$ for the $(n-1)$ measure of ∂A. Moreover, if A is sufficiently regular (for example, convex), there is equality if and only if A is a ball.

0.3 Presentation of the content of the monograph

To deal with problems of the type considered in the previous section, there are, roughly speaking, two ways of proceeding: the classical and the direct methods. Before describing a little more precisely these two methods, it might be enlightening to first discuss minimization problems in $\mathbb{R}^N$.

Let $X \subset \mathbb{R}^N$, $F : X \to \mathbb{R}$ and

$$(P) \quad \inf\{F(x) : x \in X\}\,.$$

The first method consists, if F is continuously differentiable, in finding solutions $\overline{x} \in X$ of

$$F'(x) = 0,\ x \in X\,.$$

Then, by analyzing the behavior of the higher derivatives of F, we determine if $\overline{x}$ is a minimum (global or local), a maximum (global or local) or just a stationary point.

The second method consists in considering a minimizing sequence $\{x_\nu\} \subset X$ so that

$$F(x_\nu) \to \inf\{F(x) : x \in X\}.$$

We then, with appropriate hypotheses on F, prove that the sequence is compact in X, meaning that

$$x_\nu \to \overline{x} \in X, \text{ as } \nu \to \infty.$$

Finally if F is lower semicontinuous, meaning that

$$\liminf_{\nu \to \infty} F(x_\nu) \geq F(\overline{x})$$

we have indeed shown that $\overline{x}$ is a minimizer of (P).

We can proceed in a similar manner for problems of the calculus of variations. The first and second methods are then called, respectively, classical and direct methods. However, the problem is now considerably harder because we are working in infinite dimensional spaces.

Let us recall the problem under consideration

$$(P) \quad \inf\left\{I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx : u \in X\right\} = m$$

where

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded open set, points in Ω are denoted by $x = (x_1, ..., x_n)$;
- $u : \Omega \to \mathbb{R}^N$, $N \geq 1$, $u = (u^1, ..., u^N)$ and $\nabla u = \left(\frac{\partial u^j}{\partial x_i}\right)_{1 \leq i \leq n}^{1 \leq j \leq N} \in \mathbb{R}^{N \times n}$;
- $f : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}$, $f = f(x, u, \xi)$, is continuous;
- X is a space of admissible functions which satisfy $u = u_0$ on $\partial\Omega$, where u_0 is a given function.

Here, contrary to the case of $\mathbb{R}^N$, we encounter a preliminary problem, namely: what is the best choice for the space X of admissible functions. A natural one seems to be $X = C^1(\overline{\Omega})$. There are several reasons, which will be clearer during the course of the book, that indicate that this is not the best choice. A better one is the *Sobolev space* $W^{1,p}(\Omega)$, $p \geq 1$. We will say that $u \in W^{1,p}(\Omega)$, if u is (weakly) differentiable and if

$$\|u\|_{W^{1,p}} = \left[\int_\Omega \left(|u(x)|^p + |\nabla u(x)|^p\right) dx\right]^{\frac{1}{p}} < \infty$$

The most important properties of these spaces will be recalled in Chapter 1.

In Chapter 2, we will briefly discuss the *classical methods* introduced by Euler, Hamilton, Hilbert, Jacobi, Lagrange, Legendre, Weierstrass and others. The most important tool is the *Euler-Lagrange equation*, the equivalent of $F'(x) = 0$ in the finite dimensional case, that should satisfy any $\overline{u} \in C^2\left(\overline{\Omega}\right)$ minimizer of (P), namely (we write here the equation in the case $N = 1$)

$$(E) \quad \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[f_{\xi_i}(x, \overline{u}, \nabla\overline{u})\right] = f_u(x, \overline{u}, \nabla\overline{u}), \ \forall x \in \overline{\Omega}$$

where $f_{\xi_i} = \partial f/\partial\xi_i$ and $f_u = \partial f/\partial u$.

In the case of the Dirichlet integral

$$(P) \quad \inf\left\{I(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, dx : u = u_0 \text{ on } \partial\Omega\right\}$$

the Euler-Lagrange equation reduces to *Laplace equation*, namely $\Delta\overline{u} = 0$.

We immediately note that, in general, finding a C^2 solution of (E) is a difficult task, unless, perhaps, $n = 1$ or the equation (E) is linear. The next step is to know if a solution $\overline{u}$ of (E), called sometimes a stationary point of I, is, in fact, a minimizer of (P). If $(u, \xi) \to f(x, u, \xi)$ is convex for every $x \in \Omega$ then $\overline{u}$ is indeed a minimum of (P); in the above examples this happens for the Dirichlet integral or the problem of minimal surfaces in nonparametric form. If, however, $(u, \xi) \to f(x, u, \xi)$ is not convex, several criteria, specially in the case $n = 1$, can be used to determine the nature of the stationary point. Such criteria are for example, Legendre, Weierstrass, Weierstrass-Erdmann, Jacobi conditions or the fields theories.

In Chapters 3 and 4 we will present the *direct methods* introduced by Hilbert, Lebesgue and Tonelli. The idea is to break the problem into two pieces: *existence* of minimizers in Sobolev spaces and then *regularity* of the solution. We will start by establishing, in Chapter 3, the existence of minimizers of (P) in Sobolev spaces $W^{1,p}(\Omega)$. In Chapter 4 we will see that, sometimes, minimizers of (P) are more regular than in a Sobolev space they are in C^1 or even in C^∞, if the data Ω, f and u_0 are sufficiently regular.

We now briefly describe the ideas behind the proof of existence of minimizers in Sobolev spaces. As for the finite dimensional case we start by considering a minimizing sequence $\{u_\nu\} \subset W^{1,p}(\Omega)$, which means that

$$I(u_\nu) \to \inf\left\{I(u) : u = u_0 \text{ on } \partial\Omega \text{ and } u \in W^{1,p}(\Omega)\right\} = m, \text{ as } \nu \to \infty.$$

The first step consists in showing that the sequence is compact, i.e., that the sequence converges to an element $\overline{u} \in W^{1,p}(\Omega)$. This, of course, depends on the topology that we have on $W^{1,p}$. The natural one is the one induced by the norm, that we call *strong convergence* and that we denote by

$$u_\nu \to \overline{u} \text{ in } W^{1,p}.$$

However, it is, in general, not an easy matter to show that the sequence converges in such a strong topology. It is often better to weaken the notion of convergence and to consider the so called *weak convergence*, denoted by $\rightharpoonup$. To obtain that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p}, \text{ as } \nu \to \infty$$

is much easier and it is enough, for example if $p > 1$, to show (up to the extraction of a subsequence) that

$$\|u_\nu\|_{W^{1,p}} \leq \gamma$$

where γ is a constant independent of ν. Such an estimate follows, for instance, if we impose a *coercivity* assumption on the function f of the type

$$\lim_{|\xi| \to \infty} \frac{f(x, u, \xi)}{|\xi|} = +\infty, \ \forall (x, u) \in \overline{\Omega} \times \mathbb{R}\,.$$

We observe that the Dirichlet integral, with $f(x, u, \xi) = |\xi|^2 / 2$, satisfies this hypothesis but not the minimal surface in nonparametric form, where $f(x, u, \xi) = \sqrt{1 + |\xi|^2}$.

The second step consists in showing that the functional I is lower semicontinuous with respect to weak convergence, namely

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \ \Rightarrow \ \liminf_{\nu \to \infty} I(u_\nu) \geq I(\overline{u})\ .$$

We will see that this conclusion is true if

$$\xi \to f(x, u, \xi) \text{ is convex}, \forall (x, u) \in \overline{\Omega} \times \mathbb{R}\,.$$

Since $\{u_\nu\}$ was a minimizing sequence we deduce that $\overline{u}$ is indeed a minimizer of (P).

In Chapter 5 we will consider the problem of minimal surfaces. The methods of Chapter 3 cannot be directly applied. In fact the step of compactness of the minimizing sequences is much harder to obtain, for reasons that we will detail in Chapter 5. There are, moreover, difficulties related to the geometrical nature of the problem; for instance, the type of surfaces that we consider, or the notion of area. We will present a method due to Douglas and refined by Courant and Tonelli to deal with this problem. However the techniques are, in essence, direct methods similar to those of Chapter 3.

In Chapter 6 we will discuss the isoperimetric inequality in $\mathbb{R}^n$. Depending on the dimension the way of solving the problem is very different. When $n = 2$, we will present a proof which is essentially the one of Hurwitz and is in the spirit of the techniques developed in Chapter 2. In higher dimensions the proof is more geometrical; it will use as a main tool the *Brunn-Minkowski theorem*.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we will introduce several notions that will be used throughout the book. Most of them are concerned with different spaces of functions. We recommend for the first reading to omit this chapter and to refer to it only when needed in the next chapters.

In Section 1.2, we just fix the notations concerning spaces of k-times, $k \geq 0$ an integer, continuously differentiable functions, $C^k(\Omega)$. We next introduce the spaces of Hölder continuous functions, $C^{k,\alpha}(\Omega)$, where $k \geq 0$ is an integer and $0 < \alpha \leq 1$.

In Section 1.3 we consider the Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$. We will assume that the reader is familiar with Lebesgue integration and we will not recall theorems such as, Fatou lemma, Lebesgue dominated convergence theorem or Fubini theorem. We will however state, mostly without proofs, some other important facts such as, Hölder inequality, Riesz theorem and some density results. We will also discuss the notion of weak convergence in L^p and the Riemann-Lebesgue theorem. We will conclude with the fundamental lemma of the calculus of variations that will be used throughout the book, in particular for deriving the Euler-Lagrange equations. There are many excellent books on this subject and we refer, for example to Adams [1], Brézis [14], De Barra [37].

In Section 1.4 we define the Sobolev spaces $W^{k,p}(\Omega)$, where $1 \leq p \leq \infty$ and $k \geq 1$ is an integer. We will recall several important results concerning these spaces, notably the Sobolev imbedding theorem and Rellich-Kondrachov theorem. We will, in some instances, give some proofs for the one dimensional case in order to help the reader to get more familiar with these spaces. We recommend the books of Brézis [14] and Evans [43] for a very clear introduction to the subject. The monograph of Gilbarg-Trudinger [49] can also be of great help. The book of Adams [1] is surely one of the most complete in this field, but its reading is harder than the three others.

Finally in Section 1.5 we will gather some important properties of convex functions such as, Jensen inequality, the Legendre transform and Carathéodory theorem. The book of Rockafellar [87] is classical in this field. One can also consult Hörmander [60] or Webster [96], see also [31].

1.2 Continuous and Hölder continuous functions

Definition 1.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set and define*

(i) $C^0(\Omega) = C(\Omega)$ is the set of continuous functions $u : \Omega \to \mathbb{R}$. *Similarly we let $C^0(\Omega;\mathbb{R}^N) = C(\Omega;\mathbb{R}^N)$ be the* set of continuous maps $u : \Omega \to \mathbb{R}^N$.

(ii) $C^0(\overline{\Omega}) = C(\overline{\Omega})$ is the set of continuous functions $u : \Omega \to \mathbb{R}$, which can be continuously extended to $\overline{\Omega}$. When we are dealing with maps, $u : \Omega \to \mathbb{R}^N$, we will write, similarly as above, $C^0(\overline{\Omega};\mathbb{R}^N) = C(\overline{\Omega};\mathbb{R}^N)$.

(iii) The support *of a function $u : \Omega \to \mathbb{R}$ is defined as*

$$\operatorname{supp} u = \overline{\{x \in \Omega : u(x) \neq 0\}}\,.$$

(iv) $C_0(\Omega) = \{u \in C(\Omega) : \operatorname{supp} u \subset \Omega \text{ is compact}\}$.

(v) We define the norm over $C(\overline{\Omega})$, by

$$\|u\|_{C^0} = \sup_{x \in \overline{\Omega}} |u(x)| \ .$$

Remark 1.2 *$C(\overline{\Omega})$ equipped with the norm $\|\cdot\|_{C^0}$ is a Banach space.*

Theorem 1.3 ***(Ascoli-Arzela Theorem)*** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $K \subset C(\overline{\Omega})$ be bounded and such that the following property of equicontinuity holds: for every $\epsilon > 0$ there exists $\delta > 0$ so that*

$$|x - y| < \delta \Rightarrow |u(x) - u(y)| < \varepsilon,\ \forall x, y \in \overline{\Omega} \text{ and } \forall u \in K\,,$$

then $\overline{K}$ is compact.

We will also use the following notations.

(i) If $u:\mathbb{R}^n\to\mathbb{R}$, $u=u(x_1,...,x_n)$, we will denote partial derivatives by either of the following ways

$$\begin{aligned} D_j u &= u_{x_j}=\frac{\partial u}{\partial x_j} \\ \nabla u &= \operatorname{grad} u=\left(\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_n}\right)=\left(u_{x_1},...,u_{x_n}\right). \end{aligned}$$

(ii) We now introduce the notations for the higher derivatives. Let $k\geq 1$ be an integer; an element of

$$\mathcal{A}_k=\left\{a=(a_1,...,a_n),\ a_j\geq 0 \text{ an integer and } \sum_{j=1}^n a_j=k\right\},$$

will be called a multi-index of order k. We will also write, sometimes, for such elements

$$|a|=\sum_{j=1}^n a_j=k\,.$$

Let $a\in\mathcal{A}_k$, we will write

$$D^a u=D_1^{a_1}...D_n^{a_n}u=\frac{\partial^{|a|}u}{\partial x_1^{a_1}...\partial x_n^{a_n}}\,.$$

We will also let $\nabla^k u=(D^a u)_{a\in A_k}$. In other words, $\nabla^k u$ contains all the partial derivatives of order k of the function u (for example $\nabla^0 u=u$, $\nabla^1 u=\nabla u$).

Definition 1.4 *Let $\Omega\subset\mathbb{R}^n$ be an open set and $k\geq 0$ be an integer.*

(i) The set of functions $u:\Omega\to\mathbb{R}$ which have all partial derivatives, $D^a u$, $a\in\mathcal{A}_m$, $0\leq m\leq k$, continuous will be denoted by $C^k(\Omega)$.

(ii) $C^k\left(\overline{\Omega}\right)$ is the set of $C^k(\Omega)$ functions whose derivatives up to the order k can be extended continuously to $\overline{\Omega}$. It is equipped with the following norm

$$\|u\|_{C^k}=\max_{0\leq|a|\leq k}\sup_{x\in\overline{\Omega}}|D^a u(x)|\,.$$

(iii) $C_0^k(\Omega)=C^k(\Omega)\cap C_0(\Omega)$.

(iv) $C^\infty(\Omega)=\bigcap_{k=0}^{\infty}C^k(\Omega)$, $C^\infty\left(\overline{\Omega}\right)=\bigcap_{k=0}^{\infty}C^k\left(\overline{\Omega}\right)$.

(v) $C_0^\infty(\Omega) = \mathcal{D}(\Omega) = C^\infty(\Omega) \cap C_0(\Omega)$.

(vi) When dealing with maps $u : \Omega \to \mathbb{R}^N$, *we will write, for example,* $C^k\left(\Omega; \mathbb{R}^N\right)$, *and similarly for the other cases.*

Remark 1.5 $C^k\left(\overline{\Omega}\right)$ *with its norm* $\|\cdot\|_{C^k}$ *is a Banach space.*

We will also need to define the set of piecewise continuous functions.

Definition 1.6 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set.*

(i) Define $C_{piec}^0\left(\overline{\Omega}\right) = C_{piec}\left(\overline{\Omega}\right)$ *to be the* set of piecewise continuous functions $u : \overline{\Omega} \to \mathbb{R}$. *This means that there exists a finite (or more generally a countable) partition of* Ω *into open sets* $\Omega_i \subset \Omega$, $i = 1, ..., I$, *so that*

$$\overline{\Omega} = \bigcup_{i=1}^{I} \overline{\Omega}_i, \ \Omega_i \cap \Omega_j = \emptyset, \ \text{if } i \neq j, \ 1 \leq i, j \leq I$$

and $u|_{\overline{\Omega}_i}$ *is continuous.*

(ii) Similarly $C_{piec}^k\left(\overline{\Omega}\right)$, $k \geq 1$, *is the set of functions* $u \in C^{k-1}\left(\overline{\Omega}\right)$, *whose partial derivatives of order* k *are in* $C_{piec}^0\left(\overline{\Omega}\right)$.

We now turn to the notion of Hölder continuous functions.

Definition 1.7 *Let* $D \subset \mathbb{R}^n$, $u : D \to \mathbb{R}$ *and* $0 < \alpha \leq 1$. *We let*

$$[u]_{C^{0,\alpha}(D)} = \sup_{\substack{x,y \in D \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\}.$$

Let $\Omega \subset \mathbb{R}^n$ *be open,* $k \geq 0$ *be an integer. We define the different spaces of* Hölder continuous functions *in the following way.*

(i) $C^{0,\alpha}(\Omega)$ *is the set of* $u \in C(\Omega)$ *so that*

$$[u]_{C^{0,\alpha}(K)} = \sup_{\substack{x,y \in K \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\} < \infty$$

for every compact set $K \subset \Omega$.

(ii) $C^{0,\alpha}\left(\overline{\Omega}\right)$ *is the set of functions* $u \in C\left(\overline{\Omega}\right)$ *so that*

$$[u]_{C^{0,\alpha}\left(\overline{\Omega}\right)} < \infty.$$

It is equipped with the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\alpha}(\overline{\Omega})} \ .$$

If there is no ambiguity we drop the dependence on the set $\overline{\Omega}$ *and write simply*

$$\|u\|_{C^{0,\alpha}} = \|u\|_{C^0} + [u]_{C^{0,\alpha}} \ .$$

(iii) $C^{k,\alpha}(\Omega)$ *is the set of* $u \in C^k(\Omega)$ *so that*

$$[D^a u]_{C^{0,\alpha}(K)} < \infty$$

for every compact set $K \subset \Omega$ *and every* $a \in \mathcal{A}_k$.

(iv) $C^{k,\alpha}\left(\overline{\Omega}\right)$ *is the set of functions* $u \in C^k\left(\overline{\Omega}\right)$ *so that*

$$[D^a u]_{C^{0,\alpha}(\overline{\Omega})} < \infty$$

for every multi-index $a \in \mathcal{A}_k$. *It is equipped with the following norm*

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \max_{a \in \mathcal{A}_k} [D^a u]_{C^{0,\alpha}} \ .$$

Remark 1.8 *(i)* $C^{k,\alpha}\left(\overline{\Omega}\right)$ *with its norm* $\|\cdot\|_{C^{k,\alpha}}$ *is a Banach space.*

(ii) By abuse of notations we write $C^k(\Omega) = C^{k,0}(\Omega)$; *or in other words, the set of continuous functions is identified with the set of Hölder continuous functions with exponent* 0.

(iii) Similarly when $\alpha = 1$, *we see that* $C^{0,1}\left(\overline{\Omega}\right)$ *is in fact the set of* Lipschitz continuous functions, *namely the set of functions* u *such that there exists a constant* $\gamma > 0$ *so that*

$$|u(x) - u(y)| \leq \gamma |x - y| \,, \ \forall x, y \in \overline{\Omega}.$$

The best such constant is $\gamma = [u]_{C^{0,1}}$.

Example 1.9 *Let* $\Omega = (0,1)$ *and* $u_\alpha(x) = x^\alpha$ *with* $\alpha \in [0,1]$. *It is easy to see that* $u_\alpha \in C^{0,\alpha}([0,1])$. *Moreover, if* $0 < \alpha \leq 1$, *then*

$$[u_\alpha]_{C^{0,\alpha}} = \sup_{\substack{x \neq y \\ x,y \in [0,1]}} \left\{ \frac{|x^\alpha - y^\alpha|}{|x - y|^\alpha} \right\} = 1 \,.$$

Proposition 1.10 *Let* $\Omega \subset \mathbb{R}^n$ *be open and* $0 \leq \alpha \leq 1$. *The following properties then hold.*

(i) If $u, v \in C^{0,\alpha}\left(\overline{\Omega}\right)$ *then* $uv \in C^{0,\alpha}\left(\overline{\Omega}\right)$.

(ii) If $0 \leq \alpha \leq \beta \leq 1$ and $k \geq 0$ is an integer, then

$$C^k\left(\overline{\Omega}\right) \supset C^{k,\alpha}\left(\overline{\Omega}\right) \supset C^{k,\beta}\left(\overline{\Omega}\right) \supset C^{k,1}\left(\overline{\Omega}\right).$$

(iii) If, in addition, Ω is bounded and convex, then

$$C^{k,1}\left(\overline{\Omega}\right) \supset C^{k+1}\left(\overline{\Omega}\right).$$

1.2.1 Exercises

Exercise 1.2.1 *Show Proposition 1.10.*

1.3 L^p spaces

Definition 1.11 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to $L^p(\Omega)$ if*

$$\|u\|_{L^p} = \begin{cases} \left(\displaystyle\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \\ \inf\{\alpha : |u(x)| \leq \alpha \text{ a.e. in } \Omega\} & \text{if } p = \infty \end{cases}$$

is finite. As above if $u : \Omega \to \mathbb{R}^N$, $u = \left(u^1, ..., u^N\right)$, is such that $u^i \in L^p(\Omega)$, for every $i = 1, ..., N$, we write $u \in L^p\left(\Omega; \mathbb{R}^N\right)$.

Remark 1.12 *The abbreviation "a.e." means that a property holds almost everywhere. For example, the function*

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of rational numbers, is such that $\chi_{\mathbb{Q}} = 0$ a.e.

In the next theorem we summarize the most important properties of L^p spaces that we will need. We however will not recall Fatou lemma, the dominated convergence theorem and other basic theorems of Lebesgue integral.

Theorem 1.13 *Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p \leq \infty$.*

(i) $\|\cdot\|_{L^p}$ is a norm and $L^p(\Omega)$, equipped with this norm, is a Banach space. The space $L^2(\Omega)$ is a Hilbert space with scalar product given by

$$\langle u; v \rangle = \int_\Omega u(x) \, v(x) \, dx.$$

(ii) ***Hölder inequality*** *asserts that if* $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$ *where* $1/p + 1/p' = 1$ *(i.e.,* $p' = p/(p-1)$*) and* $1 \leq p \leq \infty$ *then* $uv \in L^1(\Omega)$ *and moreover*

$$\|uv\|_{L^1} \leq \|u\|_{L^p} \|v\|_{L^{p'}} .$$

In the case $p = 2$ *and hence* $p' = 2$*, Hölder inequality is nothing else than Cauchy-Schwarz inequality*

$$\|uv\|_{L^1} \leq \|u\|_{L^2} \|v\|_{L^2} , \text{ i.e. } \int_\Omega |uv| \leq \left(\int_\Omega u^2\right)^{\frac{1}{2}} \left(\int_\Omega v^2\right)^{\frac{1}{2}} .$$

(iii) ***Minkowski inequality*** *asserts that*

$$\|u+v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p} .$$

(iv) ***Riesz Theorem****: the dual space of* L^p*, denoted by* $(L^p)'$*, can be identified with* $L^{p'}(\Omega)$ *where* $1/p + 1/p' = 1$ *provided* $1 \leq p < \infty$*. The result is false if* $p = \infty$ *(and hence* $p' = 1$*). The theorem has to be understood as follows: if* $\varphi \in (L^p)'$ *with* $1 \leq p < \infty$ *then there exists a unique* $u \in L^{p'}$ *so that*

$$\langle \varphi; f \rangle = \varphi(f) = \int_\Omega u(x) f(x)\, dx, \ \forall f \in L^p(\Omega)$$

and moreover

$$\|u\|_{L^{p'}} = \|\varphi\|_{(L^p)'} .$$

(v) L^p *is separable if* $1 \leq p < \infty$ *and reflexive (which means that the bidual of* L^p*,* $(L^p)''$*, can be identified with* L^p*) if* $1 < p < \infty$*.*

(vi) Let $1 \leq p < \infty$*. The piecewise constant functions (also called step functions if* $\Omega \subset \mathbb{R}$*), or the* $C_0^\infty(\Omega)$ *functions (i.e., those functions that are* $C^\infty(\Omega)$ *and have compact support) are dense in* L^p*. More precisely if* $u \in L^p(\Omega)$ *then there exist* $u_\nu \in C_0^\infty(\Omega)$ *(or* u_ν *piecewise constants) so that*

$$\lim_{\nu \to \infty} \|u_\nu - u\|_{L^p} = 0 .$$

The result is false if $p = \infty$*.*

Remark 1.14 *We will always make the identification* $(L^p)' = L^{p'}$*. Summarizing the results on duality we have*

$$(L^p)' = L^{p'} \text{ if } 1 < p < \infty,$$

$$(L^2)' = L^2, \ (L^1)' = L^\infty, \ L^1 \subsetneq (L^\infty)' .$$

We now turn our attention to the notions of convergence in L^p spaces. The natural notion, called *strong convergence*, is the one induced by the $\|\cdot\|_{L^p}$ norm. We will often need a weaker notion of convergence known as *weak convergence*. We now define these notions.

Definition 1.15 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$.*

(i) A sequence u_ν is said to (strongly) converge *to u if u_ν, $u \in L^p$ and if*

$$\lim_{\nu \to \infty} \|u_\nu - u\|_{L^p} = 0\,.$$

We will denote this convergence by: $u_\nu \to u$ in L^p.

(ii) If $1 \leq p < \infty$, we say that the sequence u_ν weakly converges *to u if u_ν, $u \in L^p$ and if*

$$\lim_{\nu \to \infty} \int_\Omega [u_\nu(x) - u(x)]\, \varphi(x)\, dx = 0, \ \forall \varphi \in L^{p'}(\Omega)\ .$$

This convergence will be denoted by: $u_\nu \rightharpoonup u$ in L^p.

(iii) If $p = \infty$, the sequence u_ν is said to weak $*$ converge *to u if u_ν, $u \in L^\infty$ and if*

$$\lim_{\nu \to \infty} \int_\Omega [u_\nu(x) - u(x)]\, \varphi(x)\, dx = 0, \ \forall \varphi \in L^1(\Omega)$$

and will be denoted by: $u_\nu \overset{*}{\rightharpoonup} u$ in L^∞.

Remark 1.16 *(i) We speak of weak $*$ convergence in L^∞ instead of weak convergence, because as seen above the dual of L^∞ is strictly larger than L^1. Formally, however, weak convergence in L^p and weak $*$ convergence in L^∞ take the same form.*

(ii) The limit (weak or strong) is unique.

(iii) It is obvious that

$$u_\nu \to u \text{ in } L^p \Rightarrow \begin{cases} u_\nu \rightharpoonup u \text{ in } L^p & \text{if } 1 \leq p < \infty \\ \\ u_\nu \overset{*}{\rightharpoonup} u \text{ in } L^\infty & \text{if } p = \infty\,. \end{cases}$$

Example 1.17 *Let $\Omega = (0,1)$, $\alpha > 0$ and*

$$u_\nu(x) = \begin{cases} \nu^\alpha & \text{if } x \in (0, 1/\nu) \\ \\ 0 & \text{if } x \in (1/\nu, 1)\ . \end{cases}$$

If $1 < p < \infty$, we find

$$u_\nu \to 0 \text{ in } L^p \iff 0 \le \alpha < \frac{1}{p}$$
$$u_\nu \rightharpoonup 0 \text{ in } L^p \iff 0 \le \alpha \le \frac{1}{p}$$

(cf. Exercise 1.3.2).

Example 1.18 *Let $\Omega = (0, 2\pi)$ and $u_\nu(x) = \sin \nu x$, then*

$$\sin \nu x \nrightarrow 0 \text{ in } L^p,\ \forall\, 1 \le p \le \infty$$
$$\sin \nu x \rightharpoonup 0 \text{ in } L^p,\ \forall\, 1 \le p < \infty$$

and

$$\sin \nu x \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty .$$

These facts will be consequences of Riemann-Lebesgue Theorem (cf. Theorem 1.22).

Example 1.19 *Let $\Omega = (0, 1)$, $\alpha, \beta \in \mathbb{R}$*

$$u(x) = \begin{cases} \alpha & \text{if } x \in (0, 1/2) \\ \\ \beta & \text{if } x \in (1/2, 1) . \end{cases}$$

Extend u by periodicity from $(0, 1)$ to $\mathbb{R}$ and define

$$u_\nu(x) = u(\nu x) .$$

Note that u_ν takes only the values α and β and the sets where it takes such values are, both, sets of measure $1/2$. It is clear that $\{u_\nu\}$ cannot be compact in any L^p spaces; however from Riemann-Lebesgue Theorem (cf. Theorem 1.22), we will find

$$u_\nu \rightharpoonup \frac{\alpha + \beta}{2} \text{ in } L^p,\ \forall\, 1 \le p < \infty \text{ and } u_\nu \overset{*}{\rightharpoonup} \frac{\alpha + \beta}{2} \text{ in } L^\infty.$$

Theorem 1.20 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The following properties then hold.*

(i) If $u_\nu \overset{*}{\rightharpoonup} u$ in L^∞, then $u_\nu \rightharpoonup u$ in L^p, $\forall\, 1 \le p < \infty$.

(ii) If $u_\nu \to u$ in L^p, then $\|u_\nu\|_{L^p} \to \|u\|_{L^p}$, $1 \le p \le \infty$.

(iii) If $1 \leq p < \infty$ *and if* $u_\nu \rightharpoonup u$ *in* L^p*, then there exists a constant* $\gamma > 0$ *so that* $\|u_\nu\|_{L^p} \leq \gamma$*, moreover* $\|u\|_{L^p} \leq \liminf_{\nu \to \infty} \|u_\nu\|_{L^p}$*. The result remains valid if* $p = \infty$ *and if* $u_\nu \overset{*}{\rightharpoonup} u$ *in* L^∞*.*

(iv) If $1 < p < \infty$ *and if there exists a constant* $\gamma > 0$ *so that* $\|u_\nu\|_{L^p} \leq \gamma$*, then there exist a subsequence* $\{u_{\nu_i}\}$ *and* $u \in L^p$ *so that* $u_{\nu_i} \rightharpoonup u$ *in* L^p*. The result remains valid if* $p = \infty$ *and we then have* $u_{\nu_i} \overset{*}{\rightharpoonup} u$ *in* L^∞*.*

(v) Let $1 \leq p \leq \infty$ *and* $u_\nu \to u$ *in* L^p*, then there exist a subsequence* $\{u_{\nu_i}\}$ *and* $h \in L^p$ *such that* $u_{\nu_i} \to u$ *a.e. and* $|u_{\nu_i}| \leq h$ *a.e.*

Remark 1.21 *(i) Comparing (ii) and (iii) of the theorem, we see that the weak convergence ensures the lower semicontinuity of the norm, while strong convergence guarantees its continuity.*

(ii) The most interesting part of the theorem is (iv). We know that in $\mathbb{R}^n$*, Bolzano-Weierstrass Theorem ascertains that from any bounded sequence we can extract a convergent subsequence. This is false in* L^p *spaces (and more generally in infinite dimensional spaces); but it is true if we replace strong convergence by weak convergence.*

(iii) The result (iv) is, however, false if $p = 1$*; this is a consequence of the fact that* L^1 *is not a reflexive space. To deduce, up to the extraction of a subsequence, weak convergence, it is not sufficient to have* $\|u_\nu\|_{L^1} \leq \gamma$*, we need a condition known as "equiintegrability" (cf. the bibliography). This fact is the reason that explains the difficulty of the minimal surface problem that we will discuss in Chapter 5.*

We now turn to Riemann-Lebesgue theorem that allows to easily construct weakly convergent sequences that do not converge strongly. This theorem is particularly useful when dealing with *Fourier series* (there $u(x) = \sin x$ or $\cos x$).

Theorem 1.22 ***(Riemann-Lebesgue Theorem)****. Let* $1 \leq p \leq \infty$*,* $\Omega = \prod_{i=1}^n (a_i, b_i)$ *and* $u \in L^p(\Omega)$*. Let* u *be extended by periodicity from* Ω *to* $\mathbb{R}^n$ *and define*

$$u_\nu(x) = u(\nu x) \quad \textit{and} \quad \overline{u} = \frac{1}{\operatorname{meas}\Omega} \int_\Omega u(x)\, dx$$

then $u_\nu \rightharpoonup \overline{u}$ *in* L^p *if* $1 \leq p < \infty$ *and, if* $p = \infty$*,* $u_\nu \overset{*}{\rightharpoonup} \overline{u}$ *in* L^∞*.*

Proof. To make the argument simpler we will assume in the proof that $\Omega = (0, 1)$ and $1 < p \leq \infty$. For the proof of the general case ($\Omega \subset \mathbb{R}^n$ or $p = 1$)

see, for example, Theorem 2.1.5 in [31]. We will also assume, without loss of generality, that

$$\overline{u} = \int_0^1 u(x)\, dx = 0\,.$$

Step 1. Observe that if $1 \leq p < \infty$, then

$$\begin{aligned} \|u_\nu\|_{L^p}^p &= \int_0^1 |u_\nu(x)|^p\, dx = \int_0^1 |u(\nu x)|^p\, dx \\ &= \frac{1}{\nu}\int_0^\nu |u(y)|^p\, dy = \int_0^1 |u(y)|^p\, dy\,. \end{aligned}$$

The last identity being a consequence of the 1-periodicity of u. We therefore find that

$$\|u_\nu\|_{L^p} = \|u\|_{L^p}\,. \tag{1.1}$$

The result is trivially true if $p = \infty$.

Step 2. (For a slightly different proof of this step see Exercise 1.3.5). We therefore have that $u_\nu \in L^p$ and, since $\overline{u} = 0$, we have to show that

$$\lim_{\nu \to \infty} \int_0^1 u_\nu(x)\, \varphi(x)\, dx = 0,\ \forall \varphi \in L^{p'}(0,1)\,. \tag{1.2}$$

Let $\epsilon > 0$ be arbitrary. Since $\varphi \in L^{p'}(0,1)$ and $1 < p \leq \infty$, which implies $1 \leq p' < \infty$ (i.e., $p' \neq \infty$), we have from Theorem 1.13 that there exists h a step function so that

$$\|\varphi - h\|_{L^{p'}} \leq \epsilon\,. \tag{1.3}$$

Since h is a step function, we can find $a_0 = 0 < a_1 < ... < a_I = 1$ and $\alpha_i \in \mathbb{R}$ such that

$$h(x) = \alpha_i \text{ if } x \in (a_{i-1}; a_i)\,,\ 1 \leq i \leq I\,.$$

We now compute

$$\int_0^1 u_\nu(x)\, \varphi(x)\, dx = \int_0^1 u_\nu(x)\, [\varphi(x) - h(x)]\, dx + \int_0^1 u_\nu(x)\, h(x)\, dx$$

and get that

$$\left| \int_0^1 u_\nu(x)\, \varphi(x)\, dx \right| \leq \int_0^1 |u_\nu(x)|\, |\varphi(x) - h(x)|\, dx + \left| \int_0^1 u_\nu(x)\, h(x)\, dx \right|.$$

Using Hölder inequality, (1.1) and (1.3) for the first term in the right hand side of the inequality, we obtain

$$\begin{aligned} \left|\int_0^1 u_\nu(x)\,\varphi(x)\;dx\right| &\leq \|u_\nu\|_{L^p}\,\|\varphi-h\|_{L^{p'}}+\sum_{i=1}^I|\alpha_i|\left|\int_{a_{i-1}}^{a_i}u_\nu(x)\;dx\right| \\ &\leq \epsilon\,\|u\|_{L^p}+\sum_{i=1}^I|\alpha_i|\left|\int_{a_{i-1}}^{a_i}u_\nu(x)\;dx\right|. \end{aligned} \tag{1.4}$$

To conclude we still have to evaluate

$$\begin{aligned} \int_{a_{i-1}}^{a_i}u_\nu(x)\;dx &= \int_{a_{i-1}}^{a_i}u(\nu x)\;dx=\frac{1}{\nu}\int_{\nu a_{i-1}}^{\nu a_i}u(y)\;dy \\ &= \frac{1}{\nu}\left\{\int_{\nu a_{i-1}}^{[\nu a_{i-1}]+1}u\,dy+\int_{[\nu a_{i-1}]+1}^{[\nu a_i]}u\,dy+\int_{[\nu a_i]}^{\nu a_i}u\,dy\right\} \end{aligned}$$

where $[a]$ stands for the integer part of $a\geq 0$. We now use the periodicity of u in the second term, this is legal since $[\nu a_i]-([\nu a_{i-1}]+1)$ is an integer, we therefore find that

$$\left|\int_{a_{i-1}}^{a_i}u_\nu(x)\;dx\right|\leq\frac{2}{\nu}\int_0^1|u|\;dy+\frac{[\nu a_i]-[\nu a_{i-1}-1]}{\nu}\left|\int_0^1u\,dy\right|.$$

Since $\overline{u}=\int_0^1 u=0$, we have, using the above inequality, and returning to (1.4)

$$\left|\int_0^1u_\nu\varphi\,dx\right|\leq\epsilon\,\|u\|_{L^p}+\frac{2}{\nu}\,\|u\|_{L^1}\sum_{i=1}^I|\alpha_i|\;.$$

Let $\nu\to\infty$, we hence obtain

$$0\leq\limsup_{\nu\to\infty}\left|\int_0^1u_\nu\varphi\,dx\right|\leq\epsilon\,\|u\|_{L^p}\;.$$

Since ϵ is arbitrary, we immediately have (1.2) and thus the result. ■

We conclude the present Section with a result that will be used on several occasions when deriving the Euler-Lagrange equation associated to the problems of the calculus of variations. We start with a definition.

Definition 1.23 *Let $\Omega\subset\mathbb{R}^n$ be an open set and $1\leq p\leq\infty$. We say that $u\in L^p_{loc}(\Omega)$ if $u\in L^p(\Omega')$ for every open set Ω' compactly contained in Ω (i.e. $\overline{\Omega'}\subset\Omega$ and $\overline{\Omega'}$ is compact).*

Theorem 1.24 ***(Fundamental lemma of the calculus of variations).*** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u(x)\,\psi(x)\,dx = 0,\ \forall \psi \in C_0^\infty(\Omega) \tag{1.5}$$

then $u = 0$, almost everywhere in Ω.

Proof. We will show the theorem under the stronger hypothesis that $u \in L^2(\Omega)$ and not only $u \in L^1_{\text{loc}}(\Omega)$ (recall that $L^2(\Omega) \subset L^1_{\text{loc}}(\Omega)$); for a proof in the general framework see, for example, Corollary 3.26 in Adams [1] or Lemma IV.2 in Brézis [14] . Let $\varepsilon > 0$. Since $u \in L^2(\Omega)$, invoking Theorem 1.13, we can find $\psi \in C_0^\infty(\Omega)$ so that

$$\|u - \psi\|_{L^2} \le \varepsilon\,.$$

Using (1.5) we deduce that

$$\|u\|_{L^2}^2 = \int_\Omega u^2\,dx = \int_\Omega u\,(u - \psi)\,dx\,.$$

Combining the above identity and Hölder inequality, we find

$$\|u\|_{L^2}^2 \le \|u\|_{L^2}\,\|u - \psi\|_{L^2} \le \varepsilon\,\|u\|_{L^2}\ .$$

Since $\varepsilon > 0$ is arbitrary we deduce that $\|u\|_{L^2} = 0$ and hence the claim. ■

We next have as a consequence the following result (for a proof see Exercise 1.3.6)

Corollary 1.25 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u(x)\,\psi(x)\,dx = 0,\ \forall \psi \in C_0^\infty(\Omega)\ \text{ with } \int_\Omega \psi(x)\,dx = 0$$

then $u =$constant, almost everywhere in Ω.

1.3.1 Exercises

Exercise 1.3.1 *(i) Prove Hölder and Minkowski inequalities.*

(ii) Show that if $p, q \ge 1$ with $pq/(p+q) \ge 1$, $u \in L^p$ and $v \in L^q$, then

$$uv \in L^{pq/p+q} \quad \text{and} \quad \|uv\|_{L^{pq/p+q}} \le \|u\|_{L^p}\,\|v\|_{L^q}\ .$$

(iii) Deduce that if Ω is bounded, then

$$L^\infty(\Omega) \subset L^p(\Omega) \subset L^q(\Omega) \subset L^1(\Omega)\,, \quad 1 \le q \le p \le \infty\,.$$

Show, by exhibiting an example, that (iii) is false if Ω is unbounded.

Exercise 1.3.2 *Establish the results in Example 1.17.*

Exercise 1.3.3 *(i) Prove that if $1 \leq p < \infty$, then*

$$\left.\begin{array}{l} u_\nu \rightharpoonup u \text{ in } L^p \\ \\ v_\nu \to v \text{ in } L^{p'} \end{array}\right\} \Rightarrow u_\nu v_\nu \rightharpoonup uv \text{ in } L^1.$$

Find an example showing that the result is false if we replace $v_\nu \to v$ in $L^{p'}$ by $v_\nu \rightharpoonup v$ in $L^{p'}$.

(ii) Show that

$$\left.\begin{array}{l} u_\nu \rightharpoonup u \text{ in } L^2 \\ \\ u_\nu^2 \rightharpoonup u^2 \text{ in } L^1 \end{array}\right\} \Rightarrow u_\nu \to u \text{ in } L^2.$$

Exercise 1.3.4 *(**Mollifiers**). Let $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geq 0$, $\varphi(x) = 0$ if $|x| > 1$ and $\int_{-\infty}^{+\infty} \varphi(x)\, dx = 1$, for example*

$$\varphi(x) = \begin{cases} c \exp\left\{\dfrac{1}{x^2 - 1}\right\} & \text{if } |x| < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

and c is chosen so that $\int_{-\infty}^{+\infty} \varphi\, dx = 1$. Define

$$\begin{aligned} \varphi_\nu(x) &= \nu \varphi(\nu x) \\ u_\nu(x) &= (\varphi_\nu * u)(x) = \int_{-\infty}^{+\infty} \varphi_\nu(x - y)\, u(y)\, dy\,. \end{aligned}$$

(i) Show that if $1 \leq p \leq \infty$ then

$$\|u_\nu\|_{L^p} \leq \|u\|_{L^p}\ .$$

(ii) Prove that if $u \in L^p(\mathbb{R})$, then $u_\nu \in C^\infty(\mathbb{R})$.

(iii) Establish that if $u \in C(\mathbb{R})$, then

$$u_\nu \to u \text{ uniformly on every compact set of } \mathbb{R}.$$

(iv) Show that if $u \in L^p(\mathbb{R})$ and if $1 \leq p < \infty$, then

$$u_\nu \to u \text{ in } L^p(\mathbb{R})\,.$$

Exercise 1.3.5 *In Step 2 of Theorem 1.22 use approximation by smooth functions instead of by step functions.*

Exercise 1.3.6 *(i) Show Corollary 1.25.*

(ii) Prove that if $u \in L^1_{loc}(a,b)$ *is such that*

$$\int_a^b u(x)\,\varphi'(x)\,dx = 0,\ \forall \varphi \in C_0^\infty(a,b)$$

then $u = \text{constant}$*, almost everywhere in* (a,b)*.*

Exercise 1.3.7 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $u \in L^1(\Omega)$*. Show that for every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that for any measurable set* $E \subset \Omega$

$$\operatorname{meas} E \leq \delta \ \Rightarrow\ \int_E |u(x)|\,dx \leq \epsilon\,.$$

1.4 Sobolev spaces

Before giving the definition of Sobolev spaces, we need to weaken the notion of derivative. In doing so we want to keep the right to integrate by parts; this is one of the reasons of the following definition.

Definition 1.26 *Let* $\Omega \subset \mathbb{R}^n$ *be open and* $u \in L^1_{loc}(\Omega)$*. We say that* $v \in L^1_{loc}(\Omega)$ *is the* weak partial derivative *of* u *with respect to* x_i *if*

$$\int_\Omega v(x)\,\varphi(x)\,dx = -\int_\Omega u(x)\,\frac{\partial \varphi}{\partial x_i}(x)\,dx,\ \forall \varphi \in C_0^\infty(\Omega)\,.$$

By abuse of notations we will write $v = \partial u / \partial x_i$ *or* u_{x_i}*.*

We will say that u *is weakly differentiable if all weak partial derivatives,* $u_{x_1}, \ldots, u_{x_n}$*, exist.*

Remark 1.27 *(i) If such a weak derivative exists it is unique (a.e.), as a consequence of Theorem 1.24.*

(ii) All the usual rules of differentiation are easily generalized to the present context of weak differentiability.

(iii) In a similar way we can introduce the higher derivatives.

(iv) If a function is C^1*, then the usual notion of derivative and the weak one coincide.*

(v) The advantage of this notion of weak differentiability will be obvious when defining Sobolev spaces. We can compute many more derivatives of functions than one can usually do. However not all measurable functions can be differentiated in this way. In particular a discontinuous function of $\mathbb{R}$ *cannot be differentiated in the weak sense (see Example 1.29).*

Example 1.28 *Let* $\Omega=\mathbb{R}$ *and the function* $u(x)=|x|$. *Its weak derivative is then given by*

$$u'(x)=\begin{cases} +1 & \text{if } x>0 \\ -1 & \text{if } x<0\,. \end{cases}$$

Example 1.29 *(**Dirac mass**). Let*

$$H(x)=\begin{cases} +1 & \text{if } x>0 \\ 0 & \text{if } x\leq 0\,. \end{cases}$$

We will show that H *has no weak derivative. Let* $\Omega=(-1,1)$. *Assume, for the sake of contradiction, that* $H'=\delta\in L^1_{loc}(-1,1)$ *and let us prove that we reach a contradiction. Let* $\varphi\in C_0^\infty(0,1)$ *be arbitrary and extend it to* $(-1,0)$ *by* $\varphi\equiv 0$. *We therefore have by definition that*

$$\begin{aligned} \int_{-1}^{1}\delta(x)\,\varphi(x)\;dx &= -\int_{-1}^{1}H(x)\,\varphi'(x)\;dx=-\int_0^1\varphi'(x)\;dx \\ &= \varphi(0)-\varphi(1)=0\,. \end{aligned}$$

We therefore find

$$\int_0^1\delta(x)\,\varphi(x)\,dx=0,\ \forall\varphi\in C_0^\infty(0,1)$$

which combined with Theorem 1.24, leads to $\delta=0$ *a.e. in* $(0,1)$. *With an analogous reasoning we would get that* $\delta=0$ *a.e. in* $(-1,0)$ *and consequently* $\delta=0$ *a.e. in* $(-1,1)$. *Let us show that we already reached the desired contradiction. Indeed if this were the case we would have, for every* $\varphi\in C_0^\infty(-1,1)$,

$$\begin{aligned} 0 &= \int_{-1}^{1}\delta(x)\,\varphi(x)\;dx=-\int_{-1}^{1}H(x)\,\varphi'(x)\;dx \\ &= -\int_0^1\varphi'(x)\;dx=\varphi(0)-\varphi(1)=\varphi(0)\;. \end{aligned}$$

This would imply that $\varphi(0)=0$, *for every* $\varphi\in C_0^\infty(-1,1)$, *which is clearly absurd. Thus* H *is not weakly differentiable.*

Remark 1.30 *By weakening even more the notion of derivative (for example, by not requiring anymore that* v *is in* L^1_{loc}*), the theory of distributions can give a meaning at* $H'=\delta$, *it is then called the* Dirac mass. *We will however not need this theory in the sequel, except, but only marginally, in the exercises of Section 3.5.*

Definition 1.31 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$.*

(i) We let $W^{1,p}(\Omega)$ be the set of functions $u : \Omega \to \mathbb{R}$, $u \in L^p(\Omega)$, whose weak partial derivatives $u_{x_i} \in L^p(\Omega)$ for every $i = 1, ..., n$. We endow this space with the following norm

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty$$

$$\|u\|_{W^{1,\infty}} = \max\left\{\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}\right\} \text{ if } p = \infty\,.$$

In the case $p = 2$ the space $W^{1,2}(\Omega)$ is sometimes denoted by $H^1(\Omega)$.

(ii) We define $W^{1,p}\left(\Omega;\mathbb{R}^N\right)$ to be the set of maps $u : \Omega \to \mathbb{R}^N$, $u = \left(u^1, ..., u^N\right)$, with $u^i \in W^{1,p}(\Omega)$ for every $i = 1, ..., N$.

(iii) If $1 \leq p < \infty$, the set $W_0^{1,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ functions in $W^{1,p}(\Omega)$. By abuse of language, we will often say, if Ω is bounded, that $u \in W_0^{1,p}(\Omega)$ is such that $u \in W^{1,p}(\Omega)$ and $u = 0$ on $\partial\Omega$. If $p = 2$, the set $W_0^{1,2}(\Omega)$ is sometimes denoted by $H_0^1(\Omega)$.

(iv) We will also write $u \in u_0 + W_0^{1,p}(\Omega)$ meaning that $u, u_0 \in W^{1,p}(\Omega)$ and $u - u_0 \in W_0^{1,p}(\Omega)$.

(v) We let $W_0^{1,\infty}(\Omega) = W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega)$.

(vi) Analogously we define the Sobolev spaces with higher derivatives as follows. If $k > 0$ is an integer we let $W^{k,p}(\Omega)$ to be the set of functions $u : \Omega \to \mathbb{R}$, whose weak partial derivatives $D^a u \in L^p(\Omega)$, for every multi-index $a \in \mathcal{A}_m$, $0 \leq m \leq k$. The norm will then be

$$\|u\|_{W^{k,p}} = \begin{cases} \left(\displaystyle\sum_{0 \leq |a| \leq k} \|D^a u\|_{L^p}^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\[2ex] \displaystyle\max_{0 \leq |a| \leq k}\left(\|D^a u\|_{L^\infty}\right) & \text{if } p = \infty\,. \end{cases}$$

(vii) If $1 \leq p < \infty$, $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ and $W_0^{k,\infty}(\Omega) = W^{k,\infty}(\Omega) \cap W_0^{k,1}(\Omega)$.

If $p = 2$, the spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ are sometimes respectively denoted by $H^k(\Omega)$ and $H_0^k(\Omega)$.

Remark 1.32 *(i) By abuse of notations we will write* $W^{0,p} = L^p$.

(ii) Roughly speaking, we can say that $W^{1,p}$ *is an extension of* C^1 *similar to that of* L^p *as compared to* C^0.

(iii) Note that if Ω *is bounded, then*

$$C^1\left(\overline{\Omega}\right) \underset{\neq}{\subset} W^{1,\infty}(\Omega) \underset{\neq}{\subset} W^{1,p}(\Omega) \underset{\neq}{\subset} L^p(\Omega)$$

for every $1 \leq p < \infty$.

Example 1.33 *The following cases are discussed in Exercise 1.4.1.*

(i) Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ *and* $\psi(x) = |x|^{-s}$, *for* $s > 0$. *We then have*

$$\psi \in L^p \ \Leftrightarrow\ sp < n \text{ and } \psi \in W^{1,p} \ \Leftrightarrow\ (s+1)\,p < n\,.$$

(ii) Let $\Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1/2\right\}$ *and* $\psi(x) = |\log|x||^s$ *where* $0 < s < 1/2$. *We have that* $\psi \in W^{1,2}(\Omega)$, $\psi \in L^p(\Omega)$ *for every* $1 \leq p < \infty$, *but* $\psi \notin L^\infty(\Omega)$.

(iii) Let $\Omega = \left\{x \in \mathbb{R}^2 : |x| < 1\right\}$. *We have that* $u(x) = x/|x| \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$ *for every* $1 \leq p < 2$. *Similarly in higher dimensions, namely we will establish that* $u(x) = x/|x| \in W^{1,p}(\Omega; \mathbb{R}^n)$ *for every* $1 \leq p < n$.

Theorem 1.34 *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $1 \leq p \leq \infty$ *and* $k \geq 1$ *an integer.*

(i) $W^{k,p}(\Omega)$ *equipped with its norm* $\|\cdot\|_{k,p}$ *is a Banach space which is separable if* $1 \leq p < \infty$ *and reflexive if* $1 < p < \infty$.

(ii) $W^{1,2}(\Omega)$ *is a Hilbert space when endowed with the following scalar product*

$$\langle u; v\rangle_{W^{1,2}} = \int_\Omega u(x)\,v(x)\ dx + \int_\Omega \langle \nabla u(x)\,; \nabla v(x)\rangle\ dx\,.$$

(iii) The $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *functions are dense in* $W^{k,p}(\Omega)$ *provided* $1 \leq p < \infty$. *Moreover, if* Ω *is a bounded domain with Lipschitz boundary (cf. Definition 1.40), then* $C^\infty\left(\overline{\Omega}\right)$ *is also dense in* $W^{k,p}(\Omega)$ *provided* $1 \leq p < \infty$.

(iv) $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$, *whenever* $1 \leq p < \infty$.

Remark 1.35 *(i) Note that as for the case of* L^p *the space* $W^{k,p}$ *is reflexive only when* $1 < p < \infty$ *and hence* $W^{1,1}$ *is not reflexive; as already said, this is the main source of difficulties in the minimal surface problem.*

(ii) The density result is due to Meyers and Serrin, see Section 7.6 in Gilbarg-Trudinger [49], Section 5.3 in Evans [43] or Theorem 3.16 in Adams [1].

(iii) In general, we have $W_0^{1,p}(\Omega) \underset{\neq}{\subset} W^{1,p}(\Omega)$, *but when* $\Omega = \mathbb{R}^n$ *both coincide (see Corollary 3.19 in Adams [1]).*

We will now give a simple characterization of $W^{1,p}$ which will turn out to be particularly helpful when dealing with regularity problems (Chapter 4).

Theorem 1.36 *Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p \leq \infty$ and $u \in L^p(\Omega)$. The following properties are then equivalent.*

(i) $u \in W^{1,p}(\Omega)$*;*

(ii) there exists a constant $c = c(u, \Omega, p)$ so that

$$\left| \int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx \right| \leq c \|\varphi\|_{L^{p'}}, \ \forall \varphi \in C_0^\infty(\Omega), \ \forall i = 1, 2, ..., n$$

(recalling that $1/p + 1/p' = 1$);

(iii) there exists a constant $c = c(u, \Omega, p)$ so that for every open set $\omega \subset \overline{\omega} \subset \Omega$, with $\overline{\omega}$ compact, and for every $h \in \mathbb{R}^n$ with $|h| < \operatorname{dist}(\omega, \Omega^c)$ (where $\Omega^c = \mathbb{R}^n \setminus \Omega$), then

$$\left(\int_\omega |u(x+h) - u(x)|^p\, dx \right)^{\frac{1}{p}} \leq c\,|h| \ \text{ if } 1 < p < \infty$$

$$|u(x+h) - u(x)| \leq c\,|h| \ \text{ for almost every } x \in \omega \text{ if } p = \infty\,.$$

Furthermore one can choose $c = \|\nabla u\|_{L^p}$ in (ii) and (iii).

Remark 1.37 *(i) As a consequence of the theorem, it can easily be proved that if Ω is bounded and open then*

$$C^{0,1}\left(\overline{\Omega}\right) \subset W^{1,\infty}(\Omega)$$

where $C^{0,1}\left(\overline{\Omega}\right)$ has been defined in Section 1.2, and the inclusion is, in general, strict. If, however, the set Ω is also convex (or sufficiently regular, see Theorem 5.8.4 in Evans [43]), then these two sets coincide (as usual, up to the choice of a representative in $W^{1,\infty}(\Omega)$). In other words we can say that the set of Lipschitz functions over $\overline{\Omega}$ can be identified, if Ω is convex, with the space $W^{1,\infty}(\Omega)$.

(ii) The theorem is false when $p = 1$. We then only have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). The functions satisfying (ii) or (iii) are then called functions of bounded variations.

Proof. We will prove the theorem only when $n = 1$ and $\Omega = (a, b)$. For the more general case see, for example, Proposition IX.3 in Brézis [14] or Theorem 5.8.3 and 5.8.4 in Evans [43].

(i) $\Rightarrow$ (ii). This follows from Hölder inequality and the fact that u has a weak derivative; indeed

$$\left| \int_a^b u(x)\, \varphi'(x)\; dx \right| = \left| \int_a^b u'(x)\, \varphi(x)\; dx \right| \le \|u'\|_{L^p} \|\varphi\|_{L^{p'}} .$$

(ii) $\Rightarrow$ (i). Let F be a linear functional defined by

$$F(\varphi) = \langle F; \varphi \rangle = \int_a^b u(x)\, \varphi'(x)\; dx,\ \varphi \in C_0^\infty(a,b) . \tag{1.6}$$

Note that, by (ii), it is continuous over $C_0^\infty(a,b)$. Since $C_0^\infty(a,b)$ is dense in $L^{p'}(a,b)$ (note that we used here the fact that $p \neq 1$ and hence $p' \neq \infty$), we can extend it, by continuity (or appealing to Hahn-Banach theorem), to the whole $L^{p'}(a,b)$; we have therefore defined a continuous linear operator F over $L^{p'}(a,b)$. From Riesz theorem (Theorem 1.13) we find that there exists $v \in L^p(a,b)$ so that

$$F(\varphi) = \langle F; \varphi \rangle = \int_a^b v(x)\, \varphi(x)\; dx,\ \forall \varphi \in L^{p'}(a,b) . \tag{1.7}$$

Combining (1.6) and (1.7) we get

$$\int_a^b (-v(x))\, \varphi(x)\; dx = -\int_a^b u(x)\, \varphi'(x)\; dx,\ \forall \varphi \in C_0^\infty(a,b)$$

which exactly means that $u' = -v \in L^p(a,b)$ and hence $u \in W^{1,p}(a,b)$.

(iii) $\Rightarrow$ (ii). Let $\varphi \in C_0^\infty(a,b)$ and let $\omega \subset \overline{\omega} \subset (a,b)$ with $\overline{\omega}$ compact and such that $\operatorname{supp}\varphi \subset \omega$. Let $h \in \mathbb{R}$ so that $|h| < \mathit{dist}(\omega, (a,b)^c)$. We have then, appealing to (iii),

$$\left| \int_a^b [u(x+h) - u(x)]\, \varphi(x)\, dx \right| \le \left(\int_\omega |u(x+h) - u(x)|^p\, dx \right)^{\frac{1}{p}} \|\varphi\|_{L^{p'}}$$

$$\le \begin{cases} c\,|h|\, \|\varphi\|_{L^{p'}} & \text{if } 1 < p < \infty \\ c\,|h|\, \|\varphi\|_{L^1} & \text{if } p = \infty . \end{cases} \tag{1.8}$$

We know, by hypothesis, that $\varphi \equiv 0$ on $(a, a+h)$ and $(b-h, b)$ if $h > 0$ and we therefore find (letting $\varphi \equiv 0$ outside (a,b))

$$\int_a^b u(x+h)\, \varphi(x)\, dx = \int_{a+h}^{b+h} u(x+h)\, \varphi(x)\, dx = \int_a^b u(x)\, \varphi(x-h)\, dx . \tag{1.9}$$

Since a similar argument holds for $h < 0$, we deduce from (1.8) and (1.9) that, if $1 < p \leq \infty$,

$$\left| \int_a^b u(x) \left[\varphi(x-h) - \varphi(x) \right] dx \right| \leq c\,|h| \, \|\varphi\|_{L^{p'}} \,.$$

Letting $|h|$ tend to zero, we get

$$\left| \int_a^b u\varphi' dx \right| \leq c\, \|\varphi\|_{L^{p'}} \,,\ \forall \varphi \in C_0^\infty(a,b)$$

which is exactly (ii).

(i) $\Rightarrow$ (iii). From Lemma 1.38 below, we have for every $x \in \omega$

$$u(x+h) - u(x) = \int_x^{x+h} u'(t)\; dt = h \int_0^1 u'(x+sh)\; ds$$

and hence

$$|u(x+h) - u(x)| \leq |h| \int_0^1 |u'(x+sh)|\; ds \,.$$

Let $1 < p < \infty$ (the conclusion is obvious if $p = \infty$), we have from Hölder inequality that

$$|u(x+h) - u(x)|^p \leq |h|^p \int_0^1 |u'(x+sh)|^p \, ds$$

and hence after integration

$$\begin{aligned} \int_\omega |u(x+h) - u(x)|^p \, dx &\leq |h|^p \int_\omega \int_0^1 |u'(x+sh)|^p \, ds dx \\ &= |h|^p \int_0^1 \int_\omega |u'(x+sh)|^p \, dx ds \,. \end{aligned}$$

Since $\omega + sh \subset (a,b)$, we find

$$\int_\omega |u'(x+sh)|^p \, dx = \int_{\omega+sh} |u'(y)|^p \, dy \leq \|u'\|_{L^p}^p$$

and hence

$$\left(\int_\omega |u(x+h) - u(x)|^p \, dx \right)^{\frac{1}{p}} \leq \|u'\|_{L^p} \, |h|$$

which is the claim. ∎

In the proof of Theorem 1.36, we have used a result that, roughly speaking, says that functions in $W^{1,p}$ are continuous and are primitives of functions in L^p.

Lemma 1.38 *Let $u \in W^{1,p}(a,b)$, $1 \le p \le \infty$. Then there exists a function $\widetilde{u} \in C([a,b])$ such that $u = \widetilde{u}$ a.e. and*

$$\widetilde{u}(x) - \widetilde{u}(y) = \int_y^x u'(t)\, dt,\ \forall x,y \in [a,b]\ .$$

Remark 1.39 *(i) As already repeated, we will ignore the difference between u and $\widetilde{u}$ and we will say that if $u \in W^{1,p}(a,b)$ then $u \in C([a,b])$ and u is the primitive of u', i.e.*

$$u(x) - u(y) = \int_y^x u'(t)\, dt\,.$$

(ii) Lemma 1.38 is a particular case of Sobolev imbedding theorem (cf. below). It gives a non trivial result, in the sense that it is not, a priori, obvious that a function $u \in W^{1,p}(a,b)$ is continuous. We can therefore say that

$$C^1([a,b]) \subset W^{1,p}(a,b) \subset C([a,b]),\ 1 \le p \le \infty\,.$$

(iii) The inequality (1.12) in the proof of the lemma below shows that if $u \in W^{1,p}(a,b)$, $1 < p < \infty$, then $u \in C^{0,1/p'}([a,b])$ and hence u is Hölder continuous with exponent $1/p'$. We have already seen in Remark 1.37 that if $p = \infty$, then $C^{0,1}([a,b])$ and $W^{1,\infty}(a,b)$ can be identified.

Proof. We divide the proof into two steps.

Step 1. Let $c \in (a,b)$ be fixed and define

$$v(x) = \int_c^x u'(t)\, dt,\ x \in [a,b]\ . \tag{1.10}$$

We will show that $v \in C([a,b])$ and

$$\int_a^b v(x)\varphi'(x)\, dx = -\int_a^b u'(x)\varphi(x)\, dx,\ \forall \varphi \in C_0^\infty(a,b)\ . \tag{1.11}$$

Indeed we have

$$\begin{aligned} \int_a^b v(x)\varphi'(x)\, dx &= \int_a^b \left(\int_c^x u'(t)\, dt\right)\varphi'(x)\, dx \\ &= \int_a^c dx \int_c^x u'(t)\varphi'(x)\, dt + \int_c^b dx \int_c^x u'(t)\varphi'(x)\, dt \end{aligned}$$

which combined with Fubini theorem (which allows to permute the integrals), leads to

$$\begin{aligned}\int_a^b v(x)\,\varphi'(x)\,dx &= -\int_a^c u'(t)\,dt\int_a^t \varphi'(x)\,dx+\int_c^b u'(t)\,dt\int_t^b \varphi'(x)\,dx\\ &= -\int_a^c u'(t)\,\varphi(t)\,dt+\int_c^b u'(t)\,(-\varphi(t))\,dt\\ &= -\int_a^b u'(t)\,\varphi(t)\,dt\end{aligned}$$

which is exactly (1.11). The fact that v is continuous follows from the observation that if $x \geq y$, then

$$\begin{aligned}|v(x)-v(y)| \leq \int_y^x |u'(t)|\,dt &\leq \left(\int_y^x |u'(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_y^x 1^{p'}dt\right)^{\frac{1}{p'}}\\ &\leq |x-y|^{\frac{1}{p'}}\cdot\|u'\|_{L^p}\end{aligned}\tag{1.12}$$

where we have used Hölder inequality. If $p = 1$, the inequality (1.12) does not imply that v is continuous; the continuity of v follows from classical results of Lebesgue integrals, see Exercise 1.3.7.

Step 2. We now are in a position to conclude. Since from (1.11), we have

$$\int_a^b v\varphi'dx = -\int_a^b u'\varphi\,dx,\ \forall\varphi\in C_0^\infty(a,b)$$

and we know that $u \in W^{1,p}(a,b)$ (and hence $\int u\varphi'dx = -\int u'\varphi\,dx$), we deduce

$$\int_a^b (v-u)\,\varphi'dx = 0,\ \forall\varphi\in C_0^\infty(a,b)\ .$$

Applying Exercise 1.3.6, we find that $v - u = \gamma$ a.e., γ denoting a constant, and since v is continuous, we have that $\widetilde{u} = v - \gamma$ has all the desired properties. ■

We are now in a position to state the main results concerning Sobolev spaces. They give some inclusions between these spaces, as well as some compact imbeddings. These results generalize to $\mathbb{R}^n$ what has already been seen in Lemma 1.38 for the one dimensional case. Before stating these results we need to define what kind of regularity will be assumed on the boundary of the domains $\Omega \subset \mathbb{R}^n$ that we will consider. When $\Omega = (a,b) \subset \mathbb{R}$, there was no restriction. We will assume, for the sake of simplicity, that $\Omega \subset \mathbb{R}^n$ is bounded. The following definition expresses in precise terms the intuitive notion of regular boundary (C^∞, C^k or Lipschitz).

Definition 1.40 *(i) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that Ω is a* bounded open set with C^k, $k \geq 1$, boundary *if for every $x \in \partial\Omega$, there exist a neighborhood $U \subset \mathbb{R}^n$ of x and a one-to-one and onto map $H : Q \to U$, where*

$$Q = \{x \in \mathbb{R}^n : |x_j| < 1,\ j = 1, 2, ..., n\}$$

$$H \in C^k\left(\overline{Q}\right),\ H^{-1} \in C^k\left(\overline{U}\right),\ H\left(Q_+\right) = U \cap \Omega,\ H\left(Q_0\right) = U \cap \partial\Omega$$

with $Q_+ = \{x \in Q : x_n > 0\}$ and $Q_0 = \{x \in Q : x_n = 0\}$.

(ii) If H is in $C^{k,\alpha}$, $0 < \alpha \leq 1$, we will say that Ω is a bounded open set with $C^{k,\alpha}$ boundary.

(iii) If H is only in $C^{0,1}$, we will say that Ω is a bounded open set with Lipschitz boundary.

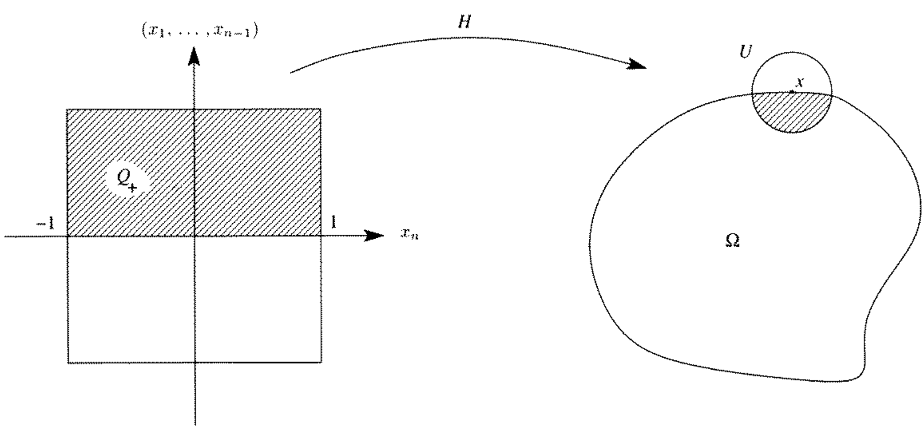

Figure 1.1: regular boundary

Remark 1.41 *Every polyhedron has Lipschitz boundary, while the unit ball in $\mathbb{R}^n$ has a C^∞ boundary.*

In the next two theorems (see for references Theorems 5.4 and 6.2 in Adams [1]) we will write some inclusions between spaces; they have to be understood up to a choice of a representative.

Theorem 1.42 ***(Sobolev imbedding theorem)****. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary.*

Case 1. If $1 \le p < n$ *then*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

for every $q \in [1, p^*]$ *where*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \text{ i.e. } p^* = \frac{np}{n-p}.$$

More precisely, for every $q \in [1, p^*]$ *there exists* $c = c(\Omega, p, q)$ *so that*

$$\|u\|_{L^q} \le c \|u\|_{W^{1,p}} .$$

Case 2. If $p = n$ *then*

$$W^{1,n}(\Omega) \subset L^q(\Omega), \text{ for every } q \in [1, \infty).$$

More precisely, for every $q \in [1, \infty)$ *there exists* $c = c(\Omega, p, q)$ *so that*

$$\|u\|_{L^q} \le c \|u\|_{W^{1,n}} .$$

Case 3. If $p > n$ *then*

$$W^{1,p}(\Omega) \subset C^{0,\alpha}\left(\overline{\Omega}\right), \text{ for every } \alpha \in [0, 1 - n/p].$$

In particular, there exists a constant $c = c(\Omega, p)$ *so that*

$$\|u\|_{L^\infty} \le c \|u\|_{W^{1,p}} .$$

The above theorem gives, not only imbeddings, but also compactness of these imbeddings under further restrictions.

Theorem 1.43 ***(Rellich-Kondrachov Theorem)****. Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with Lipschitz boundary.*

Case 1. If $1 \le p < n$ *then the imbedding of* $W^{1,p}$ *in* L^q *is compact, for every* $q \in [1, p^*)$*. This means that any bounded set of* $W^{1,p}$ *is precompact (i.e., its closure is compact) in* L^q *for every* $1 \le q < p^*$ *(the result is false if* $q = p^*$*).*

Case 2. If $p = n$ *then the imbedding of* $W^{1,n}$ *in* L^q *is compact, for every* $q \in [1, \infty)$*.*

Case 3. If $p > n$ *then the imbedding of* $W^{1,p}$ *in* $C^{0,\alpha}\left(\overline{\Omega}\right)$ *is compact, for every* $0 \le \alpha < 1 - n/p$*.*

In particular in all cases (i.e., $1 \le p \le \infty$*) the imbedding of* $W^{1,p}(\Omega)$ *in* $L^p(\Omega)$ *is compact.*

Remark 1.44 *(i) Let us examine the theorems when $\Omega = (a, b) \subset \mathbb{R}$. Only cases 2 and 3 apply and in fact we have an even better result (cf. Lemma 1.38), namely*

$$C^1([a, b]) \subset W^{1,p}(a, b) \subset C^{0,1/p'}([a, b]) \subset C([a, b])$$

for every $p \geq 1$ (hence even when $p = 1$ we have that functions in $W^{1,1}$ are continuous). However, the imbedding is compact only when $p > 1$.

(ii) In higher dimension, $n \geq 2$, the case $p = n$ cannot be improved, in general. The functions in $W^{1,n}$ are in general not continuous and not even bounded (cf. Example 1.33).

(iii) If Ω is unbounded, for example $\Omega = \mathbb{R}^n$, we must be more careful, in particular, the compactness of the imbeddings is lost (see the bibliography for more details).

(iv) If we consider $W_0^{1,p}$ instead of $W^{1,p}$ then the same imbeddings are valid, but no restriction on the regularity of $\partial\Omega$ is anymore required.

(v) Similar imbeddings can be obtained if we replace $W^{1,p}$ by $W^{k,p}$.

(vi) Recall that $W^{1,\infty}(\Omega)$, when Ω is bounded and convex, is identified with $C^{0,1}(\overline{\Omega})$.

(vii) We now try to summarize the results when $n = 1$. If we denote by $I = (a, b)$, we have, for $p \geq 1$,

$$\begin{array}{rcl}\mathcal{D}(I) & = & C_0^\infty(I) \subset \cdots \subset W^{2,p}(I) \subset C^1(\overline{I}) \subset W^{1,p}(I) \\ & \subset & C(\overline{I}) \subset L^\infty(I) \subset \cdots \subset L^2(I) \subset L^1(I)\end{array}$$

and furthermore C_0^∞ is dense in L^1, equipped with its norm.

Theorems 1.42 and 1.43 will not be proved; they have been discussed in the one dimensional case in Lemma 1.38. Concerning the compactness of the imbedding when $n = 1$, it is a consequence of Ascoli-Arzela theorem (see Exercise 1.4.4 for more details).

Before proceeding further it is important to understand the significance of Theorem 1.43. We are going to formulate it for sequences, since it is in this framework that we will use it. The corollary says that if a sequence converges weakly in $W^{1,p}$, it, in fact, converges strongly in L^p.

Corollary 1.45 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $1 \leq p < \infty$. If*

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega)$$

(this means that u_ν, $u \in W^{1,p}(\Omega)$, $u_\nu \rightharpoonup u$ in L^p and $\nabla u_\nu \rightharpoonup \nabla u$ in L^p). Then

$$u_\nu \to u \text{ in } L^p(\Omega) .$$

If $p = \infty$, $u_\nu \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}$, then $u_\nu \to u$ in L^∞.

Example 1.46 *Let $I = (0, 2\pi)$ and $u_\nu(x) = (1/\nu)\cos\nu x$. We have already seen that $u'_\nu \overset{*}{\rightharpoonup} 0$ in $L^\infty(0, 2\pi)$ and hence*

$$u_\nu \overset{*}{\rightharpoonup} 0 \text{ in } W^{1,\infty}(0, 2\pi) .$$

It is clear that we also have

$$u_\nu \to 0 \text{ in } L^\infty(0, 2\pi) .$$

The last theorem that we will often use is (see Corollary IX.19 in Brézis [14]):

Theorem 1.47 ***(Poincaré inequality).*** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 \leq p \leq \infty$. Then there exists $\gamma = \gamma(\Omega, p) > 0$ so that*

$$\|u\|_{L^p} \leq \gamma \|\nabla u\|_{L^p} , \ \forall u \in W_0^{1,p}(\Omega)$$

or equivalently

$$\|u\|_{W^{1,p}} \leq \gamma \|\nabla u\|_{L^p} , \ \forall u \in W_0^{1,p}(\Omega) .$$

Remark 1.48 *(i) We need to impose a condition of the type $u = 0$ on $\partial\Omega$ (which comes from the hypothesis $u \in W_0^{1,p}$) to avoid constant functions u (which imply $\nabla u = 0$), otherwise the inequality would be trivially false.*

(ii) Sometimes the Poincaré inequality appears under the following form (see Theorem 5.8.1 in Evans [43]). If $1 \leq p \leq \infty$, if $\Omega \subset \mathbb{R}^n$ is a bounded connected open set, with Lipschitz boundary, and if we denote by

$$u_\Omega = \frac{1}{\operatorname{meas}\Omega} \int_\Omega u(x)\, dx$$

then there exists $\gamma = \gamma(\Omega, p) > 0$ so that

$$\|u - u_\Omega\|_{L^p} \leq \gamma \|\nabla u\|_{L^p} , \ \forall u \in W^{1,p}(\Omega) .$$

(iii) In the case $n = 1$, that will be discussed in the proof, we will not really use that $u(a) = u(b) = 0$, but only $u(a) = 0$ (or equivalently $u(b) = 0$). The theorem remains thus valid under this weaker hypothesis.

(iv) We will often use Poincaré inequality under the following form. If $u_0 \in W^{1,p}(\Omega)$ and $u \in u_0 + W_0^{1,p}(\Omega)$, then there exist $\gamma_1, \gamma_2 > 0$ so that

$$\|\nabla u\|_{L^p} \geq \gamma_1 \|u\|_{W^{1,p}} - \gamma_2 \|u_0\|_{W^{1,p}} .$$

Proof. We will prove the inequality only when $n = 1$ and $\Omega = (a, b)$. Since $u \in W_0^{1,p}(a, b)$, we have $u \in C([a, b])$ and $u(a) = u(b) = 0$.

We will prove that, for every $1 \leq p \leq \infty$,

$$\|u\|_{L^p} \leq (b - a) \|u'\|_{L^p} . \tag{1.13}$$

Since $u(a)=0$, we have

$$|u(x)| = |u(x) - u(a)| = \left| \int_a^x u'(t)\, dt \right| \le \int_a^b |u'(t)|\, dt = \|u'\|_{L^1} \ .$$

From this inequality we immediately get that (1.13) is true for $p=\infty$. When $p=1$, we have after integration that

$$\|u\|_{L^1} = \int_a^b |u(x)|\, dx \le (b-a)\,\|u'\|_{L^1} \ .$$

So it remains to prove (1.13) when $1<p<\infty$. Applying Hölder inequality, we obtain

$$|u(x)| \le \left(\int_a^b 1^{p'} \right)^{\frac{1}{p'}} \left(\int_a^b |u'|^p \right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p'}}\, \|u'\|_{L^p}$$

and hence

$$\begin{aligned} \|u\|_{L^p} &= \left(\int_a^b |u|^p \, dx \right)^{\frac{1}{p}} \\ &\le \left((b-a)^{\frac{p}{p'}}\, \|u'\|_{L^p}^p \int_a^b dx \right)^{\frac{1}{p}} = (b-a)\,\|u'\|_{L^p} \ . \end{aligned}$$

This concludes the proof of the theorem when $n=1$. ■

1.4.1 Exercises

Exercise 1.4.1 *Let* $1 \le p < \infty$, $R>0$ *and* $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. *Let for* $f \in C^\infty(0,+\infty)$ *and for* $x \in B_R$

$$u(x) = f(|x|) \ .$$

(i) Show that $u \in L^p(B_R)$ *if and only if*

$$\int_0^R r^{n-1} |f(r)|^p \, dr < \infty \, .$$

(ii) Assume that

$$\lim_{r \to 0} \left[r^{n-1} |f(r)| \right] = 0 \, .$$

Prove that $u \in W^{1,p}(B_R)$ *if and only if* $u \in L^p(B_R)$ *and*

$$\int_0^R r^{n-1} |f'(r)|^p \, dr < \infty \, .$$

(iii) Discuss all the cases of Example 1.33.

Exercise 1.4.2 *Let $AC([a,b])$ be the space of absolutely continuous functions on $[a,b]$. This means that a function $u \in AC([a,b])$, if for every $\epsilon > 0$ there exists $\delta > 0$ so that for every disjoint union of intervals $(a_k, b_k) \subset (a,b)$ the following implication is true*

$$\sum_k |b_k - a_k| < \delta \;\Rightarrow\; \sum_k |u(b_k) - u(a_k)| < \epsilon\,.$$

(i) Prove that $W^{1,1}(a,b) \subset AC([a,b]) \subset C([a,b])$, up to the usual selection of a representative.

(ii) The converse $AC([a,b]) \subset W^{1,1}(a,b)$ is also true (see Section 2.2 in Buttazzo-Giaquinta-Hildebrandt [17] or Section 9.3 in De Barra [37]).

Exercise 1.4.3 *Let $u \in W^{1,p}(a,b)$, $1 < p < \infty$ and $a < y < x < b$. Show that*

$$u(x) - u(y) = o\left(|x-y|^{1/p'}\right)$$

where $o(t)$ stands for a function $f = f(t)$ so that $f(t)/t$ tends to 0 as t tends to 0.

Exercise 1.4.4 *(Corollary 1.45 in dimension $n = 1$). Prove that if $1 < p < \infty$, then*

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}(a,b) \;\Rightarrow\; u_\nu \to u \text{ in } L^p(a,b)$$

and even $u_\nu \to u$ in $L^\infty(a,b)$.

Exercise 1.4.5 *Show that if $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $1 < p < \infty$ and if there exists a constant $\gamma > 0$ so that*

$$\|u_\nu\|_{W^{1,p}} \le \gamma$$

then there exist a subsequence $\{u_{\nu_i}\}$ and $u \in W^{1,p}(\Omega)$ such that

$$u_{\nu_i} \rightharpoonup u \text{ in } W^{1,p}.$$

Exercise 1.4.6 *Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and*

$$u_\nu(x,y) = \frac{1}{\sqrt{\nu}}(1-y)^\nu \sin \nu x\,.$$

Prove that $u_\nu \to 0$ in L^∞, $\|\nabla u_\nu\|_{L^2} \le \gamma$, for some constant $\gamma > 0$. Deduce that a subsequence (one can even show that the whole sequence) converges weakly to 0 in $W^{1,2}(\Omega)$.

Exercise 1.4.7 *Let $u \in W^{1,p}(\Omega)$ and $\varphi \in W_0^{1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p > 1$. Show that*

$$\int_\Omega u_{x_i}\varphi\,dx = -\int_\Omega u\varphi_{x_i}\,dx,\; i = 1,...,n\,.$$

1.5 Convex analysis

In this final section we recall the most important results concerning convex functions.

Definition 1.49 *(i) The set $\Omega \subset \mathbb{R}^n$ is said to be* convex *if for every $x, y \in \Omega$ and every $\lambda \in [0,1]$ we have $\lambda x + (1-\lambda)\, y \in \Omega$.*

(ii) Let $\Omega \subset \mathbb{R}^n$ be convex. The function $f : \Omega \to \mathbb{R}$ is said to be convex if for every $x, y \in \Omega$ and every $\lambda \in [0,1]$, the following inequality holds

$$f\left(\lambda x + (1-\lambda)\, y\right) \le \lambda f(x) + (1-\lambda)\, f(y)\,.$$

We now give some criteria equivalent to the convexity.

Theorem 1.50 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $f \in C^1(\mathbb{R}^n)$.*

(i) The function f is convex if and only if

$$f(x) \ge f(y) + \langle \nabla f(y)\,; x - y\rangle\,,\ \forall x, y \in \mathbb{R}^n$$

where $\langle .;.\rangle$ denotes the scalar product in $\mathbb{R}^n$.

(ii) If $f \in C^2(\mathbb{R}^n)$, then f is convex if and only if its Hessian, $\nabla^2 f$, is positive semi definite.

The following inequality will be important (and will be proved in a particular case in Exercise 1.5.2).

Theorem 1.51 ***(Jensen inequality)****. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u \in L^1(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ be convex, then*

$$f\left(\frac{1}{\operatorname{meas}\Omega}\int_\Omega u(x)\, dx\right) \le \frac{1}{\operatorname{meas}\Omega}\int_\Omega f(u(x))\, dx\,.$$

We now need to introduce the notion of duality, also known as Legendre transform, for convex functions. It will be convenient to accept in the definitions functions that are allowed to take the value $+\infty$ (a function that takes only finite values, will be called finite).

Definition 1.52 ***(Legendre transform)****. Let $f : \mathbb{R}^n \to \mathbb{R}$ (or $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$).*

(i) The Legendre transform*, or* dual*, of f is the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f^*(x^*) = \sup_{x \in \mathbb{R}^n} \left\{\langle x; x^*\rangle - f(x)\right\}$$

(in general, f^* takes the value $+\infty$ even if f takes only finite values) where $\langle .; . \rangle$ denotes the scalar product in $\mathbb{R}^n$.

(ii) The bidual *of f is the function $f^{**} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$f^{**}(x) = \sup_{x^* \in \mathbb{R}^n} \{\langle x; x^* \rangle - f^*(x^*)\}.$$

Let us see some simple examples that will be studied in Exercise 1.5.4.

Example 1.53 *(i) Let $n = 1$ and $f(x) = |x|^p/p$, where $1 < p < \infty$. We then find*

$$f^*(x^*) = \frac{1}{p'} |x^*|^{p'}$$

where p' is, as usual, defined by $1/p + 1/p' = 1$.

(ii) Let $n = 1$ and $f(x) = \left(x^2 - 1\right)^2$. We then have

$$f^{**}(x) = \begin{cases} \left(x^2 - 1\right)^2 & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| < 1. \end{cases}$$

(iii) Let $n = 1$ and

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1) \\ +\infty & \text{otherwise.} \end{cases}$$

We immediately find that

$$f^*(x^*) = \sup_{x \in (0,1)} \{x x^*\} = \begin{cases} x^* & \text{if } x^* \geq 0 \\ 0 & \text{if } x^* \leq 0 \end{cases}$$

f is often called the indicator function of $(0,1)$, and f^* the support function. We also have

$$f^{**}(x) = \begin{cases} 0 & \text{if } x \in [0,1] \\ +\infty & \text{otherwise} \end{cases}$$

and hence f^{**} is the indicator function of $[0,1]$.

(iv) Let $X \in \mathbb{R}^{2\times 2}$, where $\mathbb{R}^{2\times 2}$ is the set of 2×2 real matrices which will be identified with $\mathbb{R}^4$, and let $f(X) = \det X$, then

$$f^*(X^*) \equiv +\infty \text{ and } f^{**}(X) \equiv -\infty.$$

We now gather some properties of the Legendre transform (for a proof see the exercises).

Theorem 1.54 *Let $f : \mathbb{R}^n \to \mathbb{R}$ (or $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$).*

(i) The function f^* is convex (even if f is not).

(ii) The function f^{**} is convex and $f^{**} \leq f$. If, furthermore, f is convex and finite then $f^{**} = f$. More generally, if f is finite but not necessarily convex, then f^{**} is its convex envelope (which means that it is the largest convex function that is smaller than f).

(iii) The following identity always holds: $f^{***} = f^*$.

(iv) If $f \in C^1(\mathbb{R}^n)$, convex and finite, then

$$f(x) + f^*(\nabla f(x)) = \langle \nabla f(x) ; x \rangle, \ \forall x \in \mathbb{R}^n.$$

(v) If $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and if

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = +\infty$$

then $f^* \in C^1(\mathbb{R}^n)$. Moreover if $f \in C^1(\mathbb{R}^n)$ and

$$f(x) + f^*(x^*) = \langle x^* ; x \rangle$$

then

$$x^* = \nabla f(x) \ \text{ and } \ x = \nabla f^*(x^*).$$

We finally conclude with a theorem that allows to compute the convex envelope without using duality (see Theorem 2.2.9 in [31] or Corollary 17.1.5 in Rockafellar [87]).

Theorem 1.55 ***(Carathéodory theorem)****. Let $f : \mathbb{R}^n \to \mathbb{R}$ then*

$$f^{**}(x) = \inf \left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) : x = \sum_{i=1}^{n+1} \lambda_i x_i \,, \ \lambda_i \geq 0 \ \text{and} \ \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

1.5.1 Exercises

Exercise 1.5.1 *Prove Theorem 1.50 when* $n = 1$.

Exercise 1.5.2 *Prove Jensen inequality, when* $f \in C^1$.

Exercise 1.5.3 *Let* $f(x) = \sqrt{1 + x^2}$. *Compute* f^*.

Exercise 1.5.4 *Establish (i), (ii) and (iv) of Example 1.53.*

Exercise 1.5.5 *Prove (i), (iii) and (iv) of Theorem 1.54. For proofs of (ii) and (v) see the bibliography in the corrections of the present exercise and the exercise below.*

Exercise 1.5.6 *Show (v) of Theorem 1.54 under the further restrictions that* $n = 1$, $f \in C^2(\mathbb{R})$ *and*

$$f''(x) > 0, \ \forall x \in \mathbb{R}\,.$$

Prove in addition that $f^* \in C^2(\mathbb{R})$.

Exercise 1.5.7 *Let* $f \in C^1(\mathbb{R})$ *be convex,* $p \geq 1$, $\alpha_1 > 0$ *and*

$$|f(x)| \leq \alpha_1 (1 + |x|^p)\,, \ \forall x \in \mathbb{R}\,. \tag{1.14}$$

Show that there exist $\alpha_2, \alpha_3 > 0$, *so that*

$$|f'(x)| \leq \alpha_2 \left(1 + |x|^{p-1}\right), \ \forall x \in \mathbb{R} \tag{1.15}$$

$$|f(x) - f(y)| \leq \alpha_3 \left(1 + |x|^{p-1} + |y|^{p-1}\right) |x - y|\,, \ \forall x, y \in \mathbb{R}\,. \tag{1.16}$$

Note that (1.15) always implies (1.14) independently of the convexity of f.

Chapter 2

Classical methods

2.1 Introduction

In this chapter we study the model problem

$$(P) \quad \inf \left\{ I(u) : u \in C^1([a,b]),\ u(a) = \alpha,\ u(b) = \beta \right\}$$

where $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$ and

$$I(u) = \int_a^b f(x, u(x), u'(x))\, dx\,.$$

Before describing the results that we will obtain, it might be useful to recall the analogy with minimizations in $\mathbb{R}^n$, namely

$$\inf \{F(x) : x \in X \subset \mathbb{R}^n\}\,.$$

The methods that we call *classical* consist in finding $\overline{x} \in X$ satisfying $F'(\overline{x}) = 0$, and then analyze the higher derivatives of F so as to determine the nature of the critical point $\overline{x}$: absolute minimizer or maximizer, local minimizer or maximizer or saddle point.

In Section 2.2 we derive the *Euler-Lagrange equation* (analogous to $F'(\overline{x}) = 0$ in $\mathbb{R}^n$) that should satisfy any $C^2([a,b])$ minimizer, $\overline{u}$, of (P),

$$(E) \quad \frac{d}{dx}\left[f_\xi\left(x, \overline{u}(x), \overline{u}'(x)\right)\right] = f_u\left(x, \overline{u}(x), \overline{u}'(x)\right),\ x \in [a,b]$$

where for $f = f(x,u,\xi)$ we let $f_\xi = \partial f / \partial \xi$ and $f_u = \partial f / \partial u$.

In general (as in the case of $\mathbb{R}^n$), the solutions of (E) are not necessarily minima of (P); they are merely stationary points of I (cf. below for a more precise definition). However if $(u, \xi) \to f(x, u, \xi)$ is convex for every $x \in [a, b]$, then every solution of (E) is automatically a minimizer of (P).

In Section 2.3 we show that any minimizer $\overline{u}$ of (P) satisfies a different form of the Euler-Lagrange equation. Namely for every $x \in [a, b]$ the following differential equation holds:

$$\frac{d}{dx}\left[f\left(x, \overline{u}\left(x\right), \overline{u}'\left(x\right)\right) - \overline{u}'\left(x\right) f_\xi\left(x, \overline{u}\left(x\right), \overline{u}'\left(x\right)\right)\right] = f_x\left(x, \overline{u}\left(x\right), \overline{u}'\left(x\right)\right) .$$

This rewriting of the equation turns out to be particularly useful when f does not depend explicitly on the variable x. Indeed we then have a first integral of (E) which is

$$f\left(\overline{u}\left(x\right), \overline{u}'\left(x\right)\right) - \overline{u}'\left(x\right) f_\xi\left(\overline{u}\left(x\right), \overline{u}'\left(x\right)\right) = \text{constant}, \ \forall x \in [a, b] .$$

In Section 2.4, we will present the *Hamiltonian formulation* of the problem. Roughly speaking the idea is that the solutions of (E) are also solutions (and conversely) of

$$(H) \begin{cases} u'\left(x\right) = H_v\left(x, u\left(x\right), v\left(x\right)\right) \\ \\ v'\left(x\right) = -H_u\left(x, u\left(x\right), v\left(x\right)\right) \end{cases}$$

where $v\left(x\right) = f_\xi\left(x, u\left(x\right), u'\left(x\right)\right)$ and H is the Legendre transform of f, namely

$$H\left(x, u, v\right) = \sup_{\xi \in \mathbb{R}} \left\{v\,\xi - f\left(x, u, \xi\right)\right\} .$$

In classical mechanics f is called the *Lagrangian* and H the *Hamiltonian*.

In Section 2.5, we will study the relationship between the solutions of (H) with those of a partial differential equation known as *Hamilton-Jacobi equation*

$$(HJ) \quad S_x\left(x, u\right) + H\left(x, u, S_u\left(x, u\right)\right) = 0, \ \forall \left(x, u\right) \in [a, b] \times \mathbb{R} .$$

Finally, in Section 2.6, we will present the *fields theories* introduced by Weierstrass and Hilbert which allow, in certain cases, to decide if a solution of (E) is a (local or global) minimizer of (P).

We conclude this Introduction with some comments. The methods presented in this chapter can easily be generalized to vector valued functions of the form $u : [a, b] \longrightarrow \mathbb{R}^N$, with $N > 1$, to different boundary conditions, to integral constraints, or to higher derivatives. These extensions will be considered in the exercises at the end of each section. However, except Section 2.2, the remaining part of the chapter does not generalize easily and completely to the multi dimensional case, $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, with $n > 1$; let alone the considerably harder case where $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$, with $n, N > 1$.

Moreover the classical methods suffer of two main drawbacks. The first one is that they assume, implicitly, that the solutions of (P) are regular (C^1, C^2 or sometimes piecewise C^1); this is, in general, difficult (or even often false in the case of (HJ)) to prove. However the main drawback is that they rely on the fact that we can solve either of the equations (E), (H) or (HJ), which is, usually, not the case. The main interest in the classical methods is, when they can be carried completely, that we have an essentially explicit solution. The advantage of the direct methods presented in the next two chapters is that they do not assume any solvability of such equations.

We recall, once more, that our presentation is only a brief one and we have omitted several important classical conditions such as Legendre, Weierstrass, Weierstrass-Erdmann or Jacobi conditions. The fields theories as well as all the sufficient conditions for the existence of local minima have only been, very briefly, presented. We refer for more developments to the following books: Akhiezer [2], Bliss [12], Bolza [13], Buttazzo-Giaquinta-Hildebrandt [17], Carathéodory [19], Cesari [20], Courant [25], Courant-Hilbert [26], Gelfand-Fomin [46], Giaquinta-Hildebrandt [48], Hestenes [56], Pars [82], Rund [90], Troutman [95] or Weinstock [97].

2.2 Euler-Lagrange equation

The main result of this chapter is

Theorem 2.1 *Let $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$, $f = f(x,u,\xi)$, and*

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \right\} = m$$

where $X = \left\{ u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta \right\}$.

Part 1. If (P) admits a minimizer $\overline{u} \in X \cap C^2([a,b])$, then necessarily

$$(E) \qquad \frac{d}{dx}\left[f_\xi(x, \overline{u}(x), \overline{u}'(x)) \right] = f_u(x, \overline{u}(x), \overline{u}'(x)),\ x \in (a,b)$$

or in other words

$$\begin{gathered} f_{\xi\xi}(x, \overline{u}(x), \overline{u}'(x))\, \overline{u}''(x) + f_{u\xi}(x, \overline{u}(x), \overline{u}'(x))\, \overline{u}'(x) \\ + f_{x\xi}(x, \overline{u}(x), \overline{u}'(x)) = f_u(x, \overline{u}(x), \overline{u}'(x)) \end{gathered}$$

where we denote by $f_\xi = \partial f / \partial \xi$, $f_u = \partial f / \partial u$, $f_{\xi\xi} = \partial^2 f / \partial \xi^2$, $f_{x\xi} = \partial^2 f / \partial x \partial \xi$ and $f_{u\xi} = \partial^2 f / \partial u \partial \xi$.

Part 2. Conversely if $\overline{u}$ satisfies (E) and if $(u,\xi) \to f(x,u,\xi)$ is convex for every $x \in [a,b]$ then $\overline{u}$ is a minimizer of (P).

Part 3. If moreover the function $(u,\xi) \to f(x,u,\xi)$ is strictly convex for every $x \in [a,b]$ then the minimizer of (P), if it exists, is unique.

Remark 2.2 *(i) One should immediately draw the attention to the fact that this theorem does not state any existence result.*

(ii) As will be seen below it is not always reasonable to expect that the minimizer of (P) is $C^2([a,b])$ or even $C^1([a,b])$.

(iii) If $(u,\xi) \to f(x,u,\xi)$ is not convex (even if $\xi \to f(x,u,\xi)$ is convex for every $(x,u) \in [a,b] \times \mathbb{R}$) then a solution of (E) is not necessarily an absolute minimizer of (P). It can be a local minimizer, a local maximizer.... It is often said that such a solution of (E) is a stationary point of I.

(iv) The theorem easily generalizes, for example (see the exercises below), to the following cases:

- u is a vector, i.e. $u : [a,b] \to \mathbb{R}^N$, $N > 1$, the Euler-Lagrange equations are then a system of ordinary differential equations;

- $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $n > 1$, the Euler-Lagrange equation is then a single partial differential equation;

- $f = f\left(x,u,u',u'',...,u^{(n)}\right)$, the Euler-Lagrange equation is then an ordinary differential equation of $(2n)$th order;

- other type of boundary conditions such as $u'(a) = \alpha,\ u'(b) = \beta$;

- integral constraints of the form $\int_a^b g(x,u(x),u'(x))\,dx = 0$.

Proof. *Part 1.* Since $\overline{u}$ is a minimizer among all elements of X, we have

$$I(\overline{u}) \le I(\overline{u} + hv)$$

for every $h \in \mathbb{R}$ and every $v \in C^1([a,b])$ with $v(a) = v(b) = 0$. In other words, setting $\Phi(h) = I(\overline{u} + hv)$, we have that $\Phi \in C^1(\mathbb{R})$ and that $\Phi(0) \le \Phi(h)$ for every $h \in \mathbb{R}$. We therefore deduce that

$$\Phi'(0) = \left.\frac{d}{dh} I(\overline{u} + hv)\right|_{h=0} = 0$$

and hence

$$\int_a^b \left[f_\xi\left(x,\overline{u}(x),\overline{u}'(x)\right)v'(x) + f_u\left(x,\overline{u}(x),\overline{u}'(x)\right)v(x)\right]dx = 0\,. \tag{2.1}$$

Let us mention that the above integral form is called the *weak form of the Euler-Lagrange equation*. Integrating by parts (2.1) we obtain that the following identity holds for every $v \in C^1([a,b])$ with $v(a) = v(b) = 0$

$$\int_a^b \left[-\frac{d}{dx}\left[f_\xi\left(x, \overline{u}(x), \overline{u}'(x)\right)\right] + f_u\left(x, \overline{u}(x), \overline{u}'(x)\right)\right] v(x)\, dx = 0\,.$$

Applying the fundamental lemma of the calculus of variations (Theorem 1.24) we have indeed obtained the Euler-Lagrange equation (E).

Part 2. Let $\overline{u}$ be a solution of (E) with $\overline{u}(a) = \alpha$, $\overline{u}(b) = \beta$. Since $(u, \xi) \to f(x, u, \xi)$ is convex for every $x \in [a,b]$, we get from Theorem 1.50 that

$$f(x, u, u') \geq f(x, \overline{u}, \overline{u}') + f_u(x, \overline{u}, \overline{u}')(u - \overline{u}) + f_\xi(x, \overline{u}, \overline{u}')(u' - \overline{u}')$$

for every $u \in X$. Integrating the above inequality we get

$$\begin{aligned} I(u) &\geq I(\overline{u}) \\ &\quad + \int_a^b \left[f_u(x, \overline{u}, \overline{u}')(u - \overline{u}) + f_\xi(x, \overline{u}, \overline{u}')(u' - \overline{u}')\right] dx\,. \end{aligned}$$

Integrating by parts the second term in the integral, bearing in mind that $u(a) - \overline{u}(a) = u(b) - \overline{u}(b) = 0$, we get

$$\begin{aligned} I(u) &\geq I(\overline{u}) \\ &\quad + \int_a^b \left[f_u(x, \overline{u}, \overline{u}') - \frac{d}{dx} f_\xi(x, \overline{u}, \overline{u}')\right](u - \overline{u})\, dx\,. \end{aligned}$$

Using (E) we get indeed that $I(u) \geq I(\overline{u})$, which is the claimed result.

Part 3. Let u, $v \in X$ be two solutions of (P) (recall that m denote the value of the minimum) and let us show that they are necessarily equal. Define

$$w = \frac{1}{2}u + \frac{1}{2}v$$

and observe that $w \in X$. Appealing to the convexity of $(u, \xi) \to f(x, u, \xi)$, we obtain

$$\frac{1}{2} f(x, u, u') + \frac{1}{2} f(x, v, v') \geq f\left(x, \frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u' + \frac{1}{2}v'\right) = f(x, w, w')$$

and hence

$$m = \frac{1}{2} I(u) + \frac{1}{2} I(v) \geq I(w) \geq m\,.$$

We therefore get

$$\int_a^b \left[\frac{1}{2} f(x, u, u') + \frac{1}{2} f(x, v, v') - f\left(x, \frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u' + \frac{1}{2}v'\right)\right] dx = 0\,.$$

Since the integrand is, by strict convexity of f, positive unless $u = v$ and $u' = v'$ we deduce that $u \equiv v$, as wished. ■

We now consider several particular cases and examples that are arranged in an order of increasing difficulty.

Case 2.3 $\mathbf{f}\,(x, u, \xi) = \mathbf{f}\,(\xi)$.

This is the simplest case. The Euler-Lagrange equation is

$$\frac{d}{dx}\left[f'\left(u'\right)\right] = 0, \text{ i.e. } f'\left(u'\right) = \text{constant}.$$

Note that

$$\overline{u}\,(x) = \frac{\beta - \alpha}{b - a}\,(x - a) + \alpha \tag{2.2}$$

is a solution of the equation and furthermore satisfies the boundary conditions $\overline{u}\,(a) = \alpha$, $\overline{u}\,(b) = \beta$. It is therefore a stationary point of I. It is not, however, always a minimizer of (P) as will be seen in the second and third examples.

1. **f is convex**.

   If f is convex the above $\overline{u}$ is indeed a minimizer. This follows from the theorem but it can be seen in a more elementary way (which is also valid even if $f \in C^0(\mathbb{R})$). From Jensen inequality (cf. Theorem 1.51) it follows that for any $u \in C^1([a, b])$ with $u\,(a) = \alpha,\ u\,(b) = \beta$

$$\begin{aligned} \frac{1}{b-a}\int_a^b f\left(u'\left(x\right)\right)\,dx &\geq f\left(\frac{1}{b-a}\int_a^b u'\left(x\right)\,dx\right) = f\left(\frac{u\left(b\right)-u\left(a\right)}{b-a}\right) \\ &= f\left(\frac{\beta-\alpha}{b-a}\right) = f\left(\overline{u}'\left(x\right)\right) \\ &= \frac{1}{b-a}\int_a^b f\left(\overline{u}'\left(x\right)\right)\,dx \end{aligned}$$

   which is the claim.

2. **f is non convex**.

   If f is non convex then (P) has, in general, no solution and therefore the above $\overline{u}$ is not necessarily a minimizer (in the particular example below it is a maximizer of the integral). Consider $f\,(\xi) = e^{-\xi^2}$ and

$$(P) \quad \inf_{u \in X}\left\{I\,(u) = \int_0^1 f\left(u'\left(x\right)\right)\,dx\right\} = m$$

where $X = \left\{u \in C^1([0,1]) : u(0) = u(1) = 0\right\}$. We have from (2.2) that $\overline{u} \equiv 0$ (and it is clearly a maximizer of I in the class of admissible functions X), however (P) has no minimizer as we will now show. Assume for a moment that $m = 0$, then, clearly, no function $u \in X$ can satisfy

$$\int_0^1 e^{-\left(u'(x)\right)^2} dx = 0$$

and hence (P) has no solution. Let us now show that $m = 0$. Let $n \in \mathbb{N}$ and define

$$u_n(x) = n\left(x - \frac{1}{2}\right)^2 - \frac{n}{4}$$

then $u_n \in X$ and

$$I(u_n) = \int_0^1 e^{-4n^2(x-1/2)^2} dx = \frac{1}{2n}\int_{-n}^{n} e^{-y^2} dy \to 0, \text{ as } n \to \infty.$$

Thus $m = 0$, as claimed.

3. **Solutions of (P) are not necessarily $\mathbf{C}^1$.**

We now show that solutions of (P) are not necessarily C^1 even in the present simple case (another example with a similar property will be given in Exercise 2.2.8). Let $f(\xi) = \left(\xi^2 - 1\right)^2$

$$(P) \qquad \inf_{u \in X}\left\{I(u) = \int_0^1 f(u'(x))\, dx\right\} = m$$

where $X = \left\{u \in C^1([0,1]) : u(0) = u(1) = 0\right\}$. We associate to (P) the following problem

$$(P_{\text{piec}}) \qquad \inf_{u \in X_{\text{piec}}}\left\{I(u) = \int_0^1 f(u'(x))\, dx\right\} = m_{\text{piec}}$$

$$X_{\text{piec}} = \left\{u \in C^1_{\text{piec}}([0,1]) : u(0) = u(1) = 0\right\}.$$

This last problem has clearly

$$v(x) = \begin{cases} x & \text{if } x \in [0, 1/2] \\ 1 - x & \text{if } x \in (1/2, 1] \end{cases}$$

as a solution since v is *piecewise* C^1 and satisfies $v(0) = v(1) = 0$ and $I(v) = 0$; thus $m_{\text{piec}} = 0$. Assume for a moment that we already proved that not only $m_{\text{piec}} = 0$ but also $m = 0$. This readily implies that (P), contrary to ($\mathrm{P_{piec}}$), has no solution. Indeed $I(u)=0$ implies that $|u'|=1$ almost everywhere and no function $u\in X$ can satisfy $|u'|=1$ (since by continuity of the derivative we should have either $u'=1$ everywhere or $u'=-1$ everywhere and this is incompatible with the boundary data).

We now show that $m=0$. We will give a direct argument now and a more elaborate one in Exercise 2.2.6. Consider the following sequence

$$u_n(x)=\begin{cases} x & \text{if } x\in\left[0,\frac{1}{2}-\frac{1}{n}\right] \\ -2n^2\left(x-\frac{1}{2}\right)^3-4n\left(x-\frac{1}{2}\right)^2-x+1 & \text{if } x\in\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right] \\ 1-x & \text{if } x\in\left(\frac{1}{2},1\right]. \end{cases}$$

Observe that $u_n\in X$ and

$$I(u_n)=\int_0^1 f(u_n'(x))\,dx=\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} f(u_n'(x))\,dx\le\frac{4}{n}\to 0\,.$$

This implies that indeed $m=0$.

We can also make the further observation that the Euler-Lagrange equation is

$$\frac{d}{dx}\left[u'\left((u')^2-1\right)\right]=0.$$

It has $\overline{u}\equiv 0$ as a solution. However, since $m=0$, it is not a minimizer ($I(0)=1$).

Case 2.4 $\mathbf{f}(x,u,\xi)=\mathbf{f}(x,\xi)$.

The Euler-Lagrange equation is

$$\frac{d}{dx}\left[f_\xi(x,u')\right]=0,\ \text{i.e.}\ f_\xi(x,u')=\text{constant}.$$

The equation is already harder to solve than the preceding one and, in general, it has not as simple a solution as in (2.2).

We now give an important example known as *Weierstrass example*. Let $f(x,\xi)=x\xi^2$ (note that $\xi\to f(x,\xi)$ is convex for every $x\in[0,1]$ and even strictly convex if $x\in(0,1]$) and

$$(P)\quad \inf_{u\in X}\left\{I(u)=\int_0^1 f(x,u'(x))\,dx\right\}=m$$

where $X = \left\{u \in C^1([0,1]) : u(0) = 1,\ u(1) = 0\right\}$. We will show that (P) has no C^1 or piecewise C^1 solution (not even in any Sobolev space). The Euler-Lagrange equation is

$$(xu')' = 0 \Rightarrow u' = \frac{c}{x} \Rightarrow u(x) = c\log x + d,\ x \in (0,1)$$

where c and d are constants. Observe first that such a u cannot satisfy simultaneously $u(0) = 1$ and $u(1) = 0$.

We associate to (P) the following problem

$$(P_{\text{piec}}) \qquad \inf_{u \in X_{\text{piec}}} \left\{ I(u) = \int_0^1 f(x, u'(x))\, dx \right\} = m_{\text{piec}}$$

$$X_{\text{piec}} = \left\{u \in C^1_{\text{piec}}([0,1]) : u(0) = 1,\ u(1) = 0\right\}.$$

We next prove that neither (P) nor (P_{piec}) have a minimizer. For both cases it is sufficient to establish that $m_{\text{piec}} = m = 0$. Let us postpone for a moment the proof of these facts and show the claim. If there exists a piecewise C^1 function v satisfying $I(v) = 0$, this would imply that $v' = 0$ almost everywhere in $(0,1)$. Since the function $v \in X_{\text{piec}}$, it should be continuous and $v(1)$ should be equal to 0, we would then deduce that $v \equiv 0$, which does not verify the other boundary condition, namely $v(0) = 1$. Hence neither (P) nor (P_{piec}) have a minimizer.

We first prove that $m_{\text{piec}} = 0$. Let $n \in \mathbb{N}$ and consider the sequence

$$u_n(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{n}\right] \\[2ex] \frac{-\log x}{\log n} & \text{if } x \in \left(\frac{1}{n}, 1\right]. \end{cases}$$

Note that u_n is piecewise C^1, $u_n(0) = 1$, $u_n(1) = 0$ and

$$I(u_n) = \frac{1}{\log n} \to 0, \text{ as } n \to \infty,$$

hence $m_{\text{piec}} = 0$.

We finally prove that $m = 0$. This can be done in two different ways. A more sophisticated argument is given in Exercise 2.2.6 and it provides an interesting continuity argument. A possible approach is to consider the following sequence

$$u_n(x) = \begin{cases} \frac{-n^2}{\log n}x^2 + \frac{n}{\log n}x + 1 & \text{if } x \in \left[0, \frac{1}{n}\right] \\[2ex] \frac{-\log x}{\log n} & \text{if } x \in \left(\frac{1}{n}, 1\right]. \end{cases}$$

We easily have $u_n \in X$ and since

$$u_n'(x) = \begin{cases} \frac{n}{\log n}(1 - 2nx) & \text{if } x \in \left[0, \frac{1}{n}\right] \\[2ex] \frac{-1}{x\log n} & \text{if } x \in \left(\frac{1}{n}, 1\right] \end{cases}$$

we deduce that

$$0 \leq I(u_n) = \frac{n^2}{\log^2 n} \int_0^{1/n} x(1-2nx)^2 \, dx + \frac{1}{\log^2 n} \int_{1/n}^1 \frac{dx}{x} \to 0, \text{ as } n \to \infty.$$

This indeed shows that $m = 0$.

Case 2.5 $\mathbf{f}(x,u,\xi) = \mathbf{f}(u,\xi)$.

Although this case is a lot harder to treat than the preceding ones it has an important property that is not present in the most general case when $f = f(x,u,\xi)$. The Euler-Lagrange equation is

$$\frac{d}{dx}\left[f_\xi(u(x), u'(x))\right] = f_u(u(x), u'(x)), \; x \in (a,b)$$

and according to Theorem 2.7 below, it has a *first integral* that is given by

$$f(u(x), u'(x)) - u'(x) f_\xi(u(x), u'(x)) = \text{constant}, x \in (a,b).$$

1. **Poincaré-Wirtinger inequality**.

   We will show, in several steps, that

   $$\int_a^b u'^2 \, dx \geq \left(\frac{\pi}{b-a}\right)^2 \int_a^b u^2 \, dx$$

   for every u satisfying $u(a) = u(b) = 0$. By a change of variable we immediately reduce the study to the case $a = 0$ and $b = 1$. We will also prove in Theorem 6.1 a slightly more general inequality known as *Wirtinger inequality* which states that

   $$\int_{-1}^1 u'^2 \, dx \geq \pi^2 \int_{-1}^1 u^2 \, dx$$

   among all u satisfying $u(-1) = u(1)$ and $\int_{-1}^1 u \, dx = 0$.

   We start by writing the problem under the above formalism and we let $\lambda \geq 0$, $f_\lambda(u,\xi) = \left(\xi^2 - \lambda^2 u^2\right)/2$ and

   $$(P_\lambda) \quad \inf_{u \in X}\left\{I_\lambda(u) = \int_0^1 f_\lambda(u(x), u'(x)) \, dx\right\} = m_\lambda$$

   where $X = \left\{u \in C^1([0,1]) : u(0) = u(1) = 0\right\}$. Observe that $\xi \to f_\lambda(u,\xi)$ is convex while $(u,\xi) \to f_\lambda(u,\xi)$ is not. The Euler-Lagrange equation and its first integral are

   $$u'' + \lambda^2 u = 0 \;\text{ and } u'^2 + \lambda^2 u^2 = \text{constant}.$$

We will show in Exercise 2.2.7, Example 2.23 and Theorem 6.1 the following facts.

- If $\lambda \leq \pi$ then $m_\lambda = 0$, which implies, in particular,

$$\int_0^1 u'^2\, dx \geq \pi^2 \int_0^1 u^2\, dx\,.$$

Moreover if $\lambda < \pi$, then $u_0 \equiv 0$ is the only minimizer of (P_λ). If $\lambda = \pi$, then (P_λ) has infinitely many minimizers which are all of the form $u_\alpha(x) = \alpha \sin \pi x$ with $\alpha \in \mathbb{R}$.

- If $\lambda > \pi$ then $m_\lambda = -\infty$, which implies that (P_λ) has no solution. To see this fact it is enough to choose u_α as above and to observe that since $\lambda > \pi$, then

$$I_\lambda(u_\alpha) = \alpha^2 \int_0^1 \left[\pi^2 \cos^2(\pi x) - \lambda^2 \sin^2(\pi x)\right] dx \to -\infty \text{ as } \alpha \to \infty.$$

2. **Brachistochrone**.

The function, under consideration, is $f(u,\xi) = \sqrt{1+\xi^2}/\sqrt{u}$, here (compared with Chapter 0) we take $g = 1/2$, and

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_0^b f(u(x), u'(x))\, dx \right\} = m$$

where $X = \left\{u \in C^1([0,b]) : u(0) = 0,\ u(b) = \beta \text{ and } u(x) > 0,\ \forall x \in (0,b]\right\}$. The Euler-Lagrange equation and its first integral are

$$\left[\frac{u'}{\sqrt{u}\sqrt{1+u'^2}}\right]' = -\frac{\sqrt{1+u'^2}}{2\sqrt{u^3}}$$

$$\frac{\sqrt{1+u'^2}}{\sqrt{u}} - u'\left[\frac{u'}{\sqrt{u}\sqrt{1+u'^2}}\right] = \text{constant}.$$

This leads (μ being a positive constant) to

$$u\left(1+u'^2\right) = 2\mu\,.$$

The solution is a *cycloid* and it is given in implicit form by

$$u(x) = \mu\left(1 - \cos\theta^{-1}(x)\right)$$

where

$$\theta(t) = \mu(t - \sin t)\,.$$

Note that $u(0) = 0$. It therefore remains to choose μ so that $u(b) = \beta$.

3. **Minimal surfaces of revolution**.

   This example will be treated in Chapter 5. Let us briefly present it here. The function under consideration is $f(u,\xi) = 2\pi u\sqrt{1+\xi^2}$ and the minimization problem (which corresponds to minimization of the area of a surface of revolution) is

$$(P) \qquad \inf_{u\in X}\left\{ I(u) = \int_a^b f(u(x), u'(x))\, dx \right\} = m$$

   where $X = \left\{u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta,\ u > 0\right\}$ and $\alpha,\ \beta > 0$. The Euler-Lagrange equation and its first integral are

$$\left[\frac{u'u}{\sqrt{1+u'^2}}\right]' = \sqrt{1+u'^2} \quad \Leftrightarrow \quad u''u = 1 + u'^2$$

$$u\sqrt{1+u'^2} - u'\frac{u'u}{\sqrt{1+u'^2}} = \lambda = \text{constant.}$$

   This leads to

$$u'^2 = \frac{u^2}{\lambda^2} - 1.$$

   The solutions, if they exist (this depends on a, b, α and β, see Exercise 5.2.3), are of the form (μ being a constant)

$$u(x) = \lambda\cosh\left(\frac{x}{\lambda} + \mu\right).$$

We conclude this section with a generalization of a classical example.

Example 2.6 ***Fermat principle***. *The function is* $f(x,u,\xi) = g(x,u)\sqrt{1+\xi^2}$ *and*

$$(P) \qquad \inf_{u\in X}\left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \right\} = m$$

where $X = \left\{u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta\right\}$. *Therefore the Euler-Lagrange equation is*

$$\frac{d}{dx}\left[g(x,u)\frac{u'}{\sqrt{1+u'^2}}\right] = g_u(x,u)\sqrt{1+u'^2}.$$

Observing that

$$\frac{d}{dx}\left[\frac{u'}{\sqrt{1+u'^2}}\right] = \frac{u''}{(1+u'^2)^{3/2}}$$

we get

$$g(x,u)u'' + [g_x(x,u)u' - g_u(x,u)]\left(1+u'^2\right) = 0.$$

2.2.1 Exercises

Exercise 2.2.1 *Generalize Theorem 2.1 to the case where* $u : [a,b] \to \mathbb{R}^N$*,* $N \geq 1$*.*

Exercise 2.2.2 *Generalize Theorem 2.1 to the case where* $u : [a,b] \to \mathbb{R}$ *and*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f\left(x, u(x), ..., u^{(n)}(x)\right) dx \right\}$$

where $X = \left\{ u \in C^n([a,b]) : u^{(j)}(a) = \alpha_j,\ u^{(j)}(b) = \beta_j,\ 0 \leq j \leq n-1 \right\}$*.*

Exercise 2.2.3 *(i) Find the appropriate formulation of Theorem 2.1 when* $u : [a,b] \to \mathbb{R}$ *and*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x)\, u'(x))\, dx \right\}$$

where $X = \left\{ u \in C^1([a,b]) : u(a) = \alpha \right\}$*, i.e. we leave one of the end points free.*

(ii) Similar question, when we leave both end points free; i.e. when we minimize I *over* $C^1([a,b])$*.*

Exercise 2.2.4 ***(Lagrange multiplier)****. Generalize Theorem 2.1 in the following case where* $u : [a,b] \to \mathbb{R}$*,*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \right\},$$

$$X = \left\{ u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta,\ \int_a^b g(x, u(x), u'(x))\, dx = 0 \right\}$$

where $g \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$*.*

Exercise 2.2.5 ***(Second variation of*** I***)****. Let* $f \in C^3([a,b] \times \mathbb{R} \times \mathbb{R})$ *and*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \right\}$$

where $X = \left\{ u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta \right\}$*. Let* $\overline{u} \in X \cap C^2([a,b])$ *be a minimizer for (P). Show that the following inequality*

$$\int_a^b \left[f_{\mathrm{uu}}(x, \overline{u}, \overline{u}')\, v^2 + 2 f_{u\xi}(x, \overline{u}, \overline{u}')\, v v' + f_{\xi\xi}(x, \overline{u}, \overline{u}')\, v'^2 \right] dx \geq 0$$

holds for every $v \in C_0^1(a,b)$ *(i.e.* $v \in C^1([a,b])$ *and* v *has compact support in* (a,b)*). Setting*

$$P(x) = f_{\xi\xi}(x, \overline{u}, \overline{u}')\,,\ Q(x) = f_{\mathrm{uu}}(x, \overline{u}, \overline{u}') - \frac{d}{dx}\left[f_{u\xi}(x, \overline{u}, \overline{u}')\right]$$

rewrite the above inequality as

$$\int_a^b \left[P(x)\, v'^2 + Q(x)\, v^2\right] dx \geq 0\,.$$

Exercise 2.2.6 *Show (cf. Case 2.3 and Case 2.4)*

(i) If $X = \left\{u \in C^1([0,1]) : u(0) = u(1) = 0\right\}$*, then*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 \left((u'(x))^2 - 1 \right)^2 dx \right\} = m = 0\,.$$

(ii) If $X = \left\{u \in C^1([0,1]) : u(0) = 1,\ u(1) = 0\right\}$*, then*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 x\,(u'(x))^2\, dx \right\} = m = 0\,.$$

Exercise 2.2.7 *Show (cf. Poincaré-Wirtinger inequality), using Poincaré inequality (cf. Theorem 1.47), that for* $\lambda \geq 0$ *small enough then*

$$(P_\lambda) \quad \inf_{u \in X} \left\{ I_\lambda(u) = \frac{1}{2} \int_0^1 \left[(u'(x))^2 - \lambda^2 (u(x))^2 \right] dx \right\} = m_\lambda = 0\,.$$

Deduce that $u \equiv 0$ *is the unique solution of* (P_λ) *for* $\lambda \geq 0$ *small enough.*

Exercise 2.2.8 *Let* $f(u,\xi) = u^2(1-\xi)^2$ *and*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_{-1}^1 f(u(x), u'(x))\, dx \right\} = m$$

where $X = \left\{u \in C^1([-1,1]) : u(-1) = 0,\ u(1) = 1\right\}$*. Show that (P) has no solution in* X*. Prove, however, that*

$$\overline{u}(x) = \begin{cases} 0 & \text{if } x \in [-1,0] \\ x & \text{if } x \in (0,1] \end{cases}$$

is a solution of (P) among all piecewise C^1 *functions.*

Exercise 2.2.9 *Let* $X = \left\{u \in C^1([0,1]) : u(0) = 0,\ u(1) = 1\right\}$ *and*

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_0^1 |u'(x)|\, dx \right\} = m\,.$$

Show that (P) has infinitely many solutions.

Exercise 2.2.10 *Let* $p \geq 1$ *and* $a \in C^0(\mathbb{R})$*, with* $a(u) \geq a_0 > 0$*. Let* A *be defined by*

$$A'(u) = [a(u)]^{1/p}\,.$$

Show that a minimizer (which is unique if $p > 1$*) of*

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_a^b a(u(x))\, |u'(x)|^p\, dx \right\}$$

where $X = \left\{u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta\right\}$ *is given by*

$$u(x) = A^{-1}\left[\frac{A(\beta) - A(\alpha)}{b - a}(x - a) + A(\alpha)\right].$$

2.3 Second form of the Euler-Lagrange equation

The next theorem gives a different way of expressing the Euler-Lagrange equation, this new equation is sometimes called *DuBois-Reymond equation*. It turns out to be useful when f does not depend explicitly on x, as already seen in some of the above examples.

Theorem 2.7 *Let* $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$*,* $f = f(x, u, \xi)$*, and*

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \right\} = m$$

where $X = \left\{u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta\right\}$*. Let* $u \in X \cap C^2([a,b])$ *be a minimizer of (P) then for every* $x \in [a,b]$ *the following equation holds*

$$\frac{d}{dx}\left[f(x, u(x), u'(x)) - u'(x)\, f_\xi(x, u(x), u'(x))\right] = f_x(x, u(x), u'(x))\,. \quad (2.3)$$

Proof. We will give two different proofs of the theorem. The first one is very elementary and uses the Euler-Lagrange equation . The second one is more involved but has several advantages that we do not discuss now.

Proof 1. Observe first that for any $u \in C^2([a,b])$ we have, by straight differentiation,

$$\begin{aligned}&\frac{d}{dx}\left[f(x,u,u') - u' f_\xi(x,u,u')\right]\\ =\ & f_x(x,u,u') + u'\left[f_u(x,u,u') - \frac{d}{dx}\left[f_\xi(x,u,u')\right]\right].\end{aligned}$$

By Theorem 2.1 we know that any solution u of (P) satisfies the Euler-Lagrange equation

$$\frac{d}{dx}\left[f_\xi(x,u(x),u'(x))\right] = f_u(x,u(x),u'(x))$$

hence combining the two identities we have the result.

Proof 2. We will use a technique known as *variations of the independent variables* and that we will encounter again in Chapter 5; the classical derivation of Euler-Lagrange equation can be seen as a technique of *variations of the dependent variables.*

Let $\epsilon \in \mathbb{R}$, $\varphi \in C_0^\infty(a,b)$, $\lambda = \left(2\|\varphi'\|_{L^\infty}\right)^{-1}$ and

$$\xi(x,\epsilon) = x + \epsilon\lambda\varphi(x) = y\,.$$

Observe that for $|\epsilon| \leq 1$, then $\xi(.,\epsilon) : [a,b] \to [a,b]$ is a diffeomorphism with $\xi(a,\epsilon) = a$, $\xi(b,\epsilon) = b$ and $\xi_x(x,\epsilon) > 0$. Let $\eta(.,\epsilon) : [a,b] \to [a,b]$ be its inverse, i.e.

$$\xi(\eta(y,\epsilon),\epsilon) = y\,.$$

Since $\xi_x(\eta(y,\epsilon),\epsilon)\,\eta_y(y,\epsilon) = 1$ and $\xi_x(\eta(y,\epsilon),\epsilon)\,\eta_\epsilon(y,\epsilon) + \xi_\epsilon(\eta(y,\epsilon),\epsilon) = 0$, we find ($O(t)$ stands for a function f so that $|f(t)/t|$ is bounded in a neighborhood of $t = 0$)

$$\begin{aligned}\eta_y(y,\epsilon) &= 1 - \epsilon\lambda\varphi'(y) + O\left(\epsilon^2\right)\\ \eta_\epsilon(y,\epsilon) &= -\lambda\varphi(y) + O(\epsilon)\,.\end{aligned}$$

Set for u a minimizer of (P)

$$u^\epsilon(x) = u(\xi(x,\epsilon))\,.$$

Note that, performing also a change of variables $y = \xi(x,\epsilon)$,

$$\begin{aligned}I(u^\epsilon) &= \int_a^b f\left(x, u^\epsilon(x), (u^\epsilon)'(x)\right)dx\\ &= \int_a^b f\left(x, u(\xi(x,\epsilon)), u'(\xi(x,\epsilon))\,\xi_x(x,\epsilon)\right)dx\\ &= \int_a^b f\left(\eta(y,\epsilon), u(y), u'(y)/\eta_y(y,\epsilon)\right)\eta_y(y,\epsilon)\,dy\,.\end{aligned}$$

Denoting by $g(\epsilon)$ the last integrand, we get

$$g'(\epsilon) = \eta_{y\epsilon} f + \left[f_x \eta_\epsilon - \frac{\eta_{y\epsilon}}{\eta_y^2} u' f_\xi \right] \eta_y$$

which leads to

$$g'(0) = \lambda \left[-f_x \varphi + (u' f_\xi - f) \varphi' \right].$$

Since by hypothesis u is a minimizer of (P) and $u^\epsilon \in X$ we have $I(u^\epsilon) \geq I(u)$ and hence

$$\begin{aligned} 0 &= \left. \frac{d}{d\epsilon} I(u^\epsilon) \right|_{\epsilon=0} = \lambda \int_a^b \{ -f_x(x, u(x), u'(x)) \varphi(x) \\ &\quad + [u'(x) f_\xi(x, u(x), u'(x)) - f(x, u(x), u'(x))] \varphi'(x) \} \, dx \\ &= \lambda \int_a^b \{ -f_x(x, u(x), u'(x)) \\ &\quad + \frac{d}{dx} [-u'(x) f_\xi(x, u(x), u'(x)) + f(x, u(x), u'(x))] \Big\} \varphi(x) \, dx . \end{aligned}$$

Appealing, once more, to Theorem 1.24 we have indeed obtained the claim. ■

2.3.1 Exercises

Exercise 2.3.1 *Generalize Theorem 2.7 to the case where $u : [a, b] \to \mathbb{R}^N$, $N \geq 1$.*

Exercise 2.3.2 *Let*

$$f(x, u, \xi) = f(u, \xi) = \frac{1}{2}\xi^2 - u .$$

Show that $u \equiv 1$ is a solution of (2.3), but not of the Euler-Lagrange equation (E).

2.4 Hamiltonian formulation

Recall that we are considering functions $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x, u, \xi)$, and

$$I(u) = \int_a^b f(x, u(x), u'(x)) \, dx .$$

The Euler-Lagrange equation is

$$(E) \qquad \frac{d}{dx} [f_\xi(x, u, u')] = f_u(x, u, u'), \; x \in [a, b] .$$

We have seen in the preceding sections that a minimizer of I, if it is sufficiently regular, is also a solution of (E). The aim of this section is to show that, in certain cases, solving (E) is equivalent to finding stationary points of a different functional, namely

$$J(u,v) = \int_a^b \left[u'(x)\, v(x) - H(x, u(x), v(x)) \right] dx$$

whose Euler-Lagrange equations are

$$(H) \quad \begin{cases} u'(x) = H_v(x, u(x), v(x)) \\ v'(x) = -H_u(x, u(x), v(x))\ . \end{cases}$$

The function H is called the *Hamiltonian* and it is defined as the Legendre transform of f, which is defined as

$$H(x,u,v) = \sup_{\xi \in \mathbb{R}} \{ v\,\xi - f(x,u,\xi) \} .$$

Sometimes the system (H) is called the *canonical form* of the Euler-Lagrange equation.

We start our analysis with a lemma.

Lemma 2.8 *Let $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$, $f = f(x,u,\xi)$, such that*

$$f_{\xi\xi}(x,u,\xi) > 0, \text{ for every } (x,u,\xi) \in [a,b] \times \mathbb{R} \times \mathbb{R} \tag{2.4}$$

$$\lim_{|\xi| \to \infty} \frac{f(x,u,\xi)}{|\xi|} = +\infty, \text{ for every } (x,u) \in [a,b] \times \mathbb{R}. \tag{2.5}$$

Let

$$H(x,u,v) = \sup_{\xi \in \mathbb{R}} \{ v\,\xi - f(x,u,\xi) \} . \tag{2.6}$$

Then $H \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$ and

$$H_x(x,u,v) = -f_x(x,u,H_v(x,u,v)) \tag{2.7}$$

$$H_u(x,u,v) = -f_u(x,u,H_v(x,u,v)) \tag{2.8}$$

$$H(x,u,v) = v\, H_v(x,u,v) - f(x,u,H_v(x,u,v)) \tag{2.9}$$

$$v = f_\xi(x,u,\xi) \;\Leftrightarrow\; \xi = H_v(x,u,v)\ . \tag{2.10}$$

Remark 2.9 *(i) The lemma remains partially true if we replace the hypothesis (2.4) by the weaker condition*

$$\xi \to f(x,u,\xi) \text{ is strictly convex.}$$

In general, however the function H is only C^1, as the following simple example shows

$$f(x,u,\xi) = \frac{1}{4}|\xi|^4 \text{ and } H(x,u,v) = \frac{3}{4}|v|^{4/3}.$$

(See also Example 2.13.)

(ii) The lemma remains also valid if the hypothesis (2.5) does not hold but then, in general, H is not anymore finite everywhere as the following simple example suggests. Consider the strictly convex function

$$f(x,u,\xi) = f(\xi) = \sqrt{1+\xi^2}$$

and observe that

$$H(v) = \begin{cases} -\sqrt{1-v^2} & \text{if } |v| \le 1 \\ +\infty & \text{if } |v| > 1. \end{cases}$$

(iii) The same proof leads to the fact that if $f \in C^k$, $k \ge 2$, then $H \in C^k$.

Proof. We divide the proof into several steps.

Step 1. Fix $(x,u) \in [a,b] \times \mathbb{R}$. From the definition of H and from (2.5) we deduce that there exists $\xi = \xi(x,u,v)$ such that

$$\begin{cases} H(x,u,v) = v\,\xi - f(x,u,\xi) \\ v = f_\xi(x,u,\xi)\ . \end{cases} \tag{2.11}$$

Step 2. The function H is easily seen to be continuous. Indeed let (x,u,v), $(x',u',v') \in [a,b] \times \mathbb{R} \times \mathbb{R}$, using (2.11) we find $\xi = \xi(x,u,v)$ such that

$$H(x,u,v) = v\,\xi - f(x,u,\xi).$$

Appealing to the definition of H we also have

$$H(x',u',v') \ge v'\,\xi - f(x',u',\xi).$$

Combining the two facts we get

$$H(x,u,v) - H(x',u',v') \le (v-v')\,\xi + f(x',u',\xi) - f(x,u,\xi),$$

since the reverse inequality is obtained similarly, we deduce the continuity of H from the one of f (in fact only in the variables (x,u)).

Step 3. The inverse function theorem, the fact that $f \in C^2$ and the inequality (2.4) imply that $\xi \in C^1$. But, as an exercise, we will establish this fact again. First let us prove that ξ is continuous (in fact locally Lipschitz). Let $R > 0$ be fixed. From (2.5) we deduce that we can find $R_1 > 0$ so that

$$|\xi(x,u,v)| \leq R_1 \,, \text{ for every } x \in [a,b]\,,\ |u|\,,\ |v| \leq R\,.$$

Since f_ξ is C^1, we can find $\gamma_1 > 0$ so that

$$\left|f_\xi(x,u,\xi) - f_\xi\left(x',u',\xi'\right)\right| \leq \gamma_1\left(|x-x'| + |u-u'| + \left|\xi-\xi'\right|\right) \tag{2.12}$$

for every $x,\ x' \in [a,b]\,,\ |u|\,,\ |u'| \leq R,\ |\xi|\,,\ \left|\xi'\right| \leq R_1$.

From (2.4), we find that there exists $\gamma_2 > 0$ so that

$$f_{\xi\xi}(x,u,\xi) \geq \gamma_2, \text{ for every } x \in [a,b]\,,\ |u| \leq R,\ |\xi| \leq R_1$$

and we thus have, for every $x \in [a,b]\,,\ |u| \leq R,\ |\xi|\,,\ \left|\xi'\right| \leq R_1$,

$$\left|f_\xi(x,u,\xi) - f_\xi\left(x,u,\xi'\right)\right| \geq \gamma_2\left|\xi-\xi'\right|\,. \tag{2.13}$$

Let $x,\ x' \in [a,b]\,,\ |u|\,,\ |u'| \leq R,\ |v|\,,\ |v'| \leq R$. By definition of ξ we have

$$f_\xi(x,u,\xi(x,u,v)) = v$$

$$f_\xi(x',u',\xi(x',u',v')) = v'\,,$$

which leads to

$$\begin{aligned} & f_\xi(x,u,\xi(x',u',v')) - f_\xi(x,u,\xi(x,u,v)) \\ = \quad & f_\xi(x,u,\xi(x',u',v')) - f_\xi(x',u',\xi(x',u',v')) + v' - v\,. \end{aligned}$$

Combining this identity with (2.12) and (2.13) we get

$$\gamma_2\left|\xi(x,u,v) - \xi(x',u',v')\right| \leq \gamma_1\left(|x-x'| + |u-u'|\right) + |v-v'|$$

which, indeed, establishes the continuity of ξ.

We now show that ξ is in fact C^1. From the equation $v = f_\xi(x,u,\xi)$ we deduce that

$$\begin{cases} f_{x\xi}(x,u,\xi) + f_{\xi\xi}(x,u,\xi)\,\xi_x = 0 \\ f_{u\xi}(x,u,\xi) + f_{\xi\xi}(x,u,\xi)\,\xi_u = 0 \\ f_{\xi\xi}(x,u,\xi)\,\xi_v = 1\,. \end{cases}$$

Since (2.4) holds and $f \in C^2$, we deduce that $\xi \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$.

Step 4. We therefore have that the functions

$$(x,u,v) \to \xi(x,u,v)\,,\ f_x(x,u,\xi(x,u,v))\,,\ f_u(x,u,\xi(x,u,v))$$

are C^1. We then immediately obtain (2.7), (2.8), and thus $H \in C^2$. Indeed we have, differentiating (2.11),

$$\begin{cases} H_x = v\xi_x - f_x - f_\xi\,\xi_x = (v - f_\xi)\,\xi_x - f_x = -f_x \\ \\ H_u = v\xi_u - f_u - f_\xi\,\xi_u = (v - f_\xi)\,\xi_u - f_u = -f_u \\ \\ H_v = \xi + v\xi_v - f_\xi\,\xi_v = (v - f_\xi)\,\xi_v + \xi = \xi \end{cases}$$

and in particular

$$\xi = H_v(x,u,v)\,.$$

This achieves the proof of the lemma. ■

The main theorem of the present section is:

Theorem 2.10 *Let f and H be as in the above lemma. Let $(u,v) \in C^2([a,b]) \times C^2([a,b])$ satisfy for every $x \in [a,b]$*

$$(H)\ \begin{cases} u'(x) = H_v(x,u(x),v(x)) \\ \\ v'(x) = -H_u(x,u(x),v(x))\ . \end{cases}$$

Then u verifies

$$(E)\quad \frac{d}{dx}\left[f_\xi(x,u(x),u'(x))\right] = f_u(x,u(x),u'(x))\,,\ \forall x \in [a,b]\ .$$

Conversely if $u \in C^2([a,b])$ satisfies (E) then (u,v) are solutions of (H) where

$$v(x) = f_\xi(x,u(x),u'(x))\,,\ \forall x \in [a,b]\ .$$

Remark 2.11 *The same remarks as in the lemma apply also to the theorem.*

Proof. *Part 1.* Let (u,v) satisfy (H). Using (2.10) and (2.8) we get

$$u' = H_v(x,u,v) \ \Leftrightarrow\ v = f_\xi(x,u,u')$$

$$v' = -H_u(x,u,v) = f_u(x,u,u')$$

and thus u satisfies (E).

Part 2. Conversely by (2.10) and since $v = f_\xi(x,u,u')$ we get the first equation

$$u' = H_v(x,u,v)\,.$$

Moreover since $v = f_\xi(x, u, u')$ and u satisfies (E), we have

$$v' = \frac{d}{dx}[v] = \frac{d}{dx}[f_\xi(x, u, u')] = f_u(x, u, u').$$

The second equation follows then from the combination of the above identity and (2.8). ■

Example 2.12 *The present example is motivated by classical mechanics. Let $m > 0$, $g \in C^1([a, b])$ and $f(x, u, \xi) = (m/2)\xi^2 - g(x)u$. The integral under consideration is*

$$I(u) = \int_a^b f(x, u(x), u'(x))\, dx$$

and the associated Euler-Lagrange equation is

$$mu''(x) = -g(x),\ x \in (a, b).$$

The Hamiltonian is then

$$H(x, u, v) = \frac{v^2}{2m} + g(x)u$$

while the associated Hamiltonian system is

$$\begin{cases} u'(x) = v(x)/m \\ \\ v'(x) = -g(x)\ . \end{cases}$$

Example 2.13 *We now generalize the preceding example. Let $p > 1$ and $p' = p/(p-1)$,*

$$f(x, u, \xi) = \frac{1}{p}|\xi|^p - g(x, u) \ \text{ and } H(x, u, v) = \frac{1}{p'}|v|^{p'} + g(x, u).$$

The Euler-Lagrange equation and the associated Hamiltonian system are

$$\frac{d}{dx}\left[|u'|^{p-2}u'\right] = -g_u(x, u)$$

and

$$\begin{cases} u' = |v|^{p'-2}v \\ \\ v' = -g_u(x, u)\ . \end{cases}$$

Example 2.14 *Consider the simplest case where $f(x,u,\xi) = f(\xi)$ with $f'' > 0$ (or more generally f is strictly convex) and $\lim_{|\xi|\to\infty} f(\xi)/\xi = +\infty$. The Euler-Lagrange equation and its integrated form are*

$$\frac{d}{dx}\left[f'(u')\right] = 0 \Rightarrow f'(u') = \lambda = \textit{constant}.$$

The Hamiltonian is given by

$$H(v) = f^*(v) = \sup_{\xi}\{v\xi - f(\xi)\} \ .$$

The associated Hamiltonian system is

$$\begin{cases} u' = f^{*\prime}(v) \\ \\ \quad v' = 0\,. \end{cases}$$

We find trivially that (λ and μ denoting some constants) $v' = \lambda$ and hence (compare with Case 2.3)

$$u(x) = f^{*\prime}(\lambda)\,x + \mu.$$

Example 2.15 *We now look for the slightly more involved case where $f(x,u,\xi) = f(x,\xi)$ with the appropriate hypotheses. The Euler-Lagrange equation and its integrated form are*

$$\frac{d}{dx}\left[f_\xi(x,u')\right] = 0 \Rightarrow f_\xi(x,u') = \lambda = \textit{constant}.$$

The Hamiltonian of f, is given by

$$H(x,v) = \sup_{\xi}\{v\xi - f(x,\xi)\}\,.$$

The associated Hamiltonian system is

$$\begin{cases} u'(x) = H_v(x,v(x)) \\ \\ \qquad v' = 0\,. \end{cases}$$

The solution is then given by $v = \lambda = \textit{constant}$ and $u'(x) = H_v(x,\lambda)$.

Example 2.16 *We consider next the more difficult case where $f(x,u,\xi) = f(u,\xi)$ with the hypotheses of the theorem. The Euler-Lagrange equation and its integrated form are*

$$\frac{d}{dx}\left[f_\xi(u,u')\right] = f_u(u,u') \Rightarrow f(u,u') - u' f_\xi(x,u') = \lambda = \textit{constant}.$$

The Hamiltonian of f, is given by

$$H(u,v)=\sup_{\xi}\{v\xi-f(u,\xi)\}\ \ \text{with } v=f_{\xi}(u,\xi)\,.$$

The associated Hamiltonian system is

$$\begin{cases} u'(x)=H_v(u(x),v(x)) \\ \\ v'(x)=-H_u(u(x),v(x))\ . \end{cases}$$

The Hamiltonian system also has a first integral given by

$$\frac{d}{dx}\left[H(u(x),v(x))\right]=H_u(u,v)\,u'+H_v(u,v)\,v'\equiv 0.$$

In physical terms we can say that if the Lagrangian f is independent of the variable x (which is here the time), the Hamiltonian H is constant along the trajectories.

2.4.1 Exercises

Exercise 2.4.1 *Generalize Theorem 2.10 to the case where $u:[a,b]\to\mathbb{R}^N$, $N\geq 1$.*

Exercise 2.4.2 *Consider a mechanical system with N particles whose respective masses are m_i and positions at time t are $u_i(t)=(x_i(t),y_i(t),z_i(t))\in\mathbb{R}^3$, $1\leq i\leq N$. Let*

$$T(u')=\frac{1}{2}\sum_{i=1}^{N}m_i\left|u_i'\right|^2=\frac{1}{2}\sum_{i=1}^{N}m_i\left(x_i'^2+y_i'^2+z_i'^2\right)$$

be the kinetic energy and denote by $U=U(t,u)$ the potential energy. Finally let

$$L(t,u,u')=T(u')-U(t,u)$$

be the Lagrangian. Let also H be the associated Hamiltonian. With the help of the preceding exercise show the following results.

(i) Write the Euler-Lagrange equations. Find the associated Hamiltonian system.

(ii) Show that, along the trajectories (i.e. when $v=L_{\xi}(t,u,u')$), the Hamiltonian can be written as (in mechanical terms it is the total energy of the system)

$$H(t,u,v)=T(u')+U(t,u)\ .$$

Exercise 2.4.3 *Let $f(x,u,\xi)=\sqrt{g(x,u)}\sqrt{1+\xi^2}$. Write the associated Hamiltonian system and find a first integral of this system when g does not depend explicitly on x.*

2.5 Hamilton-Jacobi equation

We now discuss the connection between finding stationary points of the functionals I and J considered in the preceding sections and solving a first order partial differential equation known as *Hamilton-Jacobi equation*. This equation also plays an important role in the fields theories developed in the next section (cf. Exercise 2.6.3).

Let us start with the main theorem.

Theorem 2.17 *Let $H \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$, $H = H(x,u,v)$. Assume that there exists $S \in C^2([a,b] \times \mathbb{R})$, $S = S(x,u)$, a solution of the* Hamilton-Jacobi equation

$$S_x + H(x,u,S_u) = 0,\ \forall (x,u) \in [a,b] \times \mathbb{R}\,, \tag{2.14}$$

where $S_x = \partial S/\partial x$ and $S_u = \partial S/\partial u$. Assume also that there exists $u \in C^1([a,b])$ a solution of

$$u'(x) = H_v(x,u(x),S_u(x,u(x)))\,,\ \forall x \in [a,b]\,. \tag{2.15}$$

Setting

$$v(x) = S_u(x,u(x)) \tag{2.16}$$

then $(u,v) \in C^1([a,b]) \times C^1([a,b])$ is a solution of

$$\begin{cases} u'(x) = H_v(x,u(x),v(x)) \\ \\ v'(x) = -H_u(x,u(x),v(x))\,. \end{cases} \tag{2.17}$$

Moreover if there is a one parameter family $S = S(x,u,\alpha)$, $S \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$, solving (2.14) for every $(x,u,\alpha) \in [a,b] \times \mathbb{R} \times \mathbb{R}$, then any solution of (2.15) satisfies

$$\frac{d}{dx}[S_\alpha(x,u(x),\alpha)] = 0,\ \forall (x,\alpha) \in [a,b] \times \mathbb{R}\,,$$

where $S_\alpha = \partial S/\partial \alpha$.

Remark 2.18 *(i) If the Hamiltonian does not depend explicitly on x then every solution $S^*(u,\alpha)$ of*

$$H(u,S_u^*) = \alpha,\ \forall (u,\alpha) \in \mathbb{R} \times \mathbb{R} \tag{2.18}$$

leads immediately to a solution of (2.14), setting

$$S(x,u,\alpha) = S^*(u,\alpha) - \alpha x\,.$$

(ii) It is, in general, a difficult task to solve (2.14) and an extensive bibliography on the subject exists, cf. Evans [43], Lions [69].

Proof. *Step 1.* We differentiate (2.16) to get

$$v'(x) = S_{xu}(x, u(x)) + u'(x) S_{\mathrm{uu}}(x, u(x)), \ \forall x \in [a, b] \ .$$

Differentiating (2.14) with respect to u we find, for every $(x, u) \in [a, b] \times \mathbb{R}$,

$$S_{xu}(x, u) + H_u(x, u, S_u(x, u)) + H_v(x, u, S_u(x, u)) S_{\mathrm{uu}}(x, u) = 0 \, .$$

Combining the two identities (the second one evaluated at $u = u(x)$) and (2.15) with the definition of v, we have

$$v'(x) = -H_u(x, u(x), S_u(x, u(x))) = -H_u(x, u(x), v(x))$$

as wished.

Step 2. Since S is a solution of the Hamilton-Jacobi equation, we have, for every $(x, u, \alpha) \in [a, b] \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} & \frac{d}{d\alpha}[S_x(x, u, \alpha) + H(x, u, S_u(x, u, \alpha))] \\ = \quad & S_{x\alpha}(x, u, \alpha) + H_v(x, u, S_u(x, u, \alpha)) S_{u\alpha}(x, u, \alpha) = 0 \, . \end{aligned}$$

Since this identity is valid for every u, it is also valid for $u = u(x)$ satisfying (2.15) and thus

$$S_{x\alpha}(x, u(x), \alpha) + u'(x) S_{u\alpha}(x, u(x), \alpha) = 0 \, .$$

This last identity can be rewritten as

$$\frac{d}{dx}[S_\alpha(x, u(x), \alpha)] = 0$$

which is the claim. ■

The above theorem admits a converse.

Theorem 2.19 ***(Jacobi Theorem)****. Let $H \in C^1([a, b] \times \mathbb{R} \times \mathbb{R})$, $S = S(x, u, \alpha)$ be $C^2([a, b] \times \mathbb{R} \times \mathbb{R})$ and solving (2.14), for every $(x, u, \alpha) \in [a, b] \times \mathbb{R} \times \mathbb{R}$, with*

$$S_{u\alpha}(x, u, \alpha) \neq 0, \ \forall (x, u, \alpha) \in [a, b] \times \mathbb{R} \times \mathbb{R} \, .$$

If $u = u(x)$ satisfies

$$\frac{d}{dx}[S_\alpha(x, u(x), \alpha)] = 0, \ \forall (x, \alpha) \in [a, b] \times \mathbb{R} \tag{2.19}$$

then u necessarily verifies

$$u'(x) = H_v(x, u(x), S_u(x, u(x), \alpha)), \ \forall (x, \alpha) \in [a, b] \times \mathbb{R} \, .$$

Thus if $v(x) = S_u(x, u(x), \alpha)$, then $(u, v) \in C^1([a, b]) \times C^1([a, b])$ is a solution of (2.17).

Proof. Since (2.19) holds we have, for every $(x,\alpha) \in [a,b] \times \mathbb{R}$,

$$0 = \frac{d}{dx}\left[S_\alpha\left(x,u\left(x\right),\alpha\right)\right] = S_{x\alpha}\left(x,u\left(x\right),\alpha\right) + S_{u\alpha}\left(x,u\left(x\right),\alpha\right)u'\left(x\right) .$$

From (2.14) we obtain, for every $(x,u,\alpha) \in [a,b] \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} 0 &= \frac{d}{d\alpha}\left[S_x\left(x,u,\alpha\right) + H\left(x,u,S_u\left(x,u,\alpha\right)\right)\right] \\ &= S_{x\alpha}\left(x,u,\alpha\right) + H_v\left(x,u,S_u\left(x,u,\alpha\right)\right)S_{u\alpha}\left(x,u,\alpha\right) . \end{aligned}$$

Combining the two identities (the second one evaluated at $u = u(x)$), with the hypothesis $S_{u\alpha}(x,u,\alpha) \neq 0$, we get

$$u'\left(x\right) = H_v\left(x,u\left(x\right),S_u\left(x,u\left(x\right),\alpha\right)\right),\ \forall\left(x,\alpha\right) \in [a,b] \times \mathbb{R}$$

as wished. We still need to prove that $v' = -H_u$. Differentiating v we have, for every $(x,\alpha) \in [a,b] \times \mathbb{R}$,

$$\begin{aligned} v'\left(x\right) &= S_{xu}\left(x,u\left(x\right),\alpha\right) + u'\left(x\right)S_{\mathrm{uu}}\left(x,u\left(x\right),\alpha\right) \\ &= S_{xu}\left(x,u\left(x\right),\alpha\right) + H_v\left(x,u\left(x\right),S_u\left(x,u\left(x\right),\alpha\right)\right)S_{\mathrm{uu}}\left(x,u\left(x\right),\alpha\right) . \end{aligned}$$

Appealing, once more, to (2.14) we obtain, for every $(x,u,\alpha) \in [a,b] \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} 0 &= \frac{d}{du}\left[S_x\left(x,u,\alpha\right) + H\left(x,u,S_u\left(x,u,\alpha\right)\right)\right] \\ &= S_{xu}\left(x,u,\alpha\right) + H_u\left(x,u,S_u\left(x,u,\alpha\right)\right) + H_v\left(x,u,S_u\left(x,u,\alpha\right)\right)S_{\mathrm{uu}}\left(x,u,\alpha\right) . \end{aligned}$$

Combining the two identities (the second one evaluated at $u = u(x)$) we infer the result, namely

$$v'\left(x\right) = -H_u\left(x,u\left(x\right),S_u\left(x,u\left(x\right),\alpha\right)\right) = -H_u\left(x,u\left(x\right),v\left(x\right)\right) .$$

This achieves the proof of the theorem. ■

Example 2.20 *Let $g \in C^1(\mathbb{R})$ with $g(u) \geq g_0 > 0$. Let*

$$H\left(u,v\right) = \frac{1}{2}v^2 - g\left(u\right)$$

be the Hamiltonian associated to

$$f\left(u,\xi\right) = \frac{1}{2}\xi^2 + g\left(u\right) .$$

The Hamilton-Jacobi equation and its reduced form are given by

$$S_x + \frac{1}{2}\left(S_u\right)^2 - g\left(u\right) = 0 \text{ and } \frac{1}{2}\left(S_u^*\right)^2 = g\left(u\right) .$$

Therefore a solution of the equation is given by

$$S = S(x,u) = S(u) = \int_0^u \sqrt{2g(s)}ds\,.$$

We next solve

$$u'(x) = H_v(u(x), S_u(u(x))) = S_u(u(x)) = \sqrt{2g(u(x))}$$

which has a solution given implicitly by

$$\int_{u(0)}^{u(x)} \frac{ds}{\sqrt{2g(s)}} = x\,.$$

Setting $v(x) = S_u(u(x))$, we have indeed found a solution of the Hamiltonian system

$$\begin{cases} u'(x) = H_v(u(x), v(x)) = v(x) \\ \\ v'(x) = -H_u(u(x), v(x)) = g'(u(x))\ . \end{cases}$$

Note also that such a function u solves

$$u''(x) = g'(u(x))$$

which is the Euler-Lagrange equation associated to the Lagrangian f.

2.5.1 Exercises

Exercise 2.5.1 *Write the Hamilton-Jacobi equation when $u \in \mathbb{R}^N$, $N \geq 1$, and generalize Theorem 2.19 to this case.*

Exercise 2.5.2 *Let $f(x,u,\xi) = f(u,\xi) = \sqrt{g(u)}\sqrt{1+\xi^2}$. Solve the Hamilton-Jacobi equation and find the stationary points of*

$$I(u) = \int_a^b f(u(x), u'(x))\ dx\,.$$

Exercise 2.5.3 *Same exercise as the preceding one with $f(x,u,\xi) = f(u,\xi) = a(u)\xi^2/2$ where $a(u) \geq a_0 > 0$. Compare the result with Exercise 2.2.10.*

2.6 Fields theories

As already said we will only give a very brief account on the fields theories and we refer to the bibliography for more details. These theories are conceptually important but often difficult to manage for specific examples.

Let us recall the problem under consideration

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \right\} = m$$

where $X = \left\{ u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta \right\}$. The Euler-Lagrange equation is

$$(E) \quad \frac{d}{dx}\left[f_\xi(x, u, u') \right] = f_u(x, u, u'),\ x \in (a,b)\ .$$

We will try to explain the nature of the theory, starting with a particularly simple case. We have seen in Section 2.2 that a solution of (E) is not, in general, a minimizer for (P). However (cf. Theorem 2.1) if $(u, \xi) \to f(x, u, \xi)$ is convex for every $x \in [a,b]$ then any solution of (E) is necessarily a minimizer of (P). We first show that we can, sometimes, recover this result under the only assumption that $\xi \to f(x, u, \xi)$ is convex for every $(x, u) \in [a,b] \times \mathbb{R}$.

Theorem 2.21 *Let $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$. If there exists $\Phi \in C^3([a,b] \times \mathbb{R})$ with $\Phi(a, \alpha) = \Phi(b, \beta)$ such that*

$$(u, \xi) \to \widetilde{f}(x, u, \xi) \text{ is convex for every } x \in [a,b]$$

where

$$\widetilde{f}(x, u, \xi) = f(x, u, \xi) + \Phi_u(x, u)\,\xi + \Phi_x(x, u)\ ;$$

then any solution $\overline{u}$ of (E) is a minimizer of (P).

Remark 2.22 *We should immediately point out that in order to have $(u, \xi) \to \widetilde{f}(x, u, \xi)$ convex for every $x \in [a,b]$ we should, at least, have that $\xi \to f(x, u, \xi)$ is convex for every $(x, u) \in [a,b] \times \mathbb{R}$. If $(u, \xi) \to f(x, u, \xi)$ is already convex, then choose $\Phi \equiv 0$ and apply Theorem 2.1.*

Proof. Define

$$\varphi(x, u, \xi) = \Phi_u(x, u)\,\xi + \Phi_x(x, u)\ .$$

Observe that the two following identities (the first one uses that $\Phi(a, \alpha) = \Phi(b, \beta)$ and the second one is just straight differentiation)

$$\int_a^b \frac{d}{dx}\left[\Phi(x, u(x)) \right] dx = \Phi(b, \beta) - \Phi(a, \alpha) = 0$$

$$\frac{d}{dx}\left[\varphi_\xi(x, u, u') \right] = \varphi_u(x, u, u'),\ x \in [a,b]$$

hold for any $u \in X = \left\{u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta\right\}$. The first identity expresses that the integral is *invariant*, while the second one says that $\varphi(x,u,u')$ satisfies the Euler-Lagrange equation identically (it is then called a *null Lagrangian*).

With the help of the above observations we immediately obtain the result by applying Theorem 2.1 to $\widetilde{f}$. Indeed we have that $(u,\xi) \to \widetilde{f}(x,u,\xi)$ is convex,

$$I(u) = \int_a^b \widetilde{f}(x,u(x),u'(x))\,dx = \int_a^b f(x,u(x),u'(x))\,dx$$

for every $u \in X$ and any solution $\overline{u}$ of (E) also satisfies

$$(\widetilde{E}) \qquad \frac{d}{dx}\left[\widetilde{f}_\xi(x,\overline{u},\overline{u}')\right] = \widetilde{f}_u(x,\overline{u},\overline{u}'),\ x \in (a,b)\ .$$

This concludes the proof. ■

With the help of the above elementary theorem we can now fully handle the Poincaré-Wirtinger inequality.

Example 2.23 ***(Poincaré-Wirtinger inequality).*** *Let $\lambda \geq 0$, $f_\lambda(u,\xi) = \left(\xi^2 - \lambda^2 u^2\right)/2$ and*

$$(P_\lambda) \qquad \inf_{u \in X}\left\{I_\lambda(u) = \int_0^1 f_\lambda(u(x),u'(x))\,dx\right\} = m_\lambda$$

where $X = \left\{u \in C^1([0,1]) : u(0) = u(1) = 0\right\}$. Observe that $\xi \to f_\lambda(u,\xi)$ is convex while $(u,\xi) \to f_\lambda(u,\xi)$ is not. The Euler-Lagrange equation is

$$(E_\lambda) \quad u'' + \lambda^2 u = 0,\ x \in (0,1)\ .$$

Note that $u_0 \equiv 0$ is a solution of (E_λ). Define, if $\lambda < \pi$,

$$\Phi(x,u) = \frac{\lambda}{2}\tan\left[\lambda\left(x - \frac{1}{2}\right)\right]u^2,\ (x,u) \in [0,1] \times \mathbb{R}$$

and observe that Φ satisfies all the properties of Theorem 2.21. The function $\widetilde{f}$ is then

$$\widetilde{f}(x,u,\xi) = \frac{1}{2}\xi^2 + \lambda\tan\left[\lambda\left(x - \frac{1}{2}\right)\right]u\xi + \frac{\lambda^2}{2}\tan^2\left[\lambda\left(x - \frac{1}{2}\right)\right]u^2\,.$$

It is easy to see that $(u,\xi) \to \widetilde{f}(x,u,\xi)$ is convex and therefore applying Theorem 2.21 we have that, for every $0 \leq \lambda < \pi$,

$$I_\lambda(u) \geq I_\lambda(0),\ \forall u \in X\,.$$

An elementary passage to the limit leads to Poincaré-Wirtinger inequality

$$\int_0^1 u'^2 dx \geq \pi^2 \int_0^1 u^2 dx,\ \forall u \in X\,.$$

For a different proof of a slightly more general form of Poincaré-Wirtinger inequality see Theorem 6.1.

The way of proceeding, in Theorem 2.21, is, in general, too naive and can be done only locally; in fact one needs a similar but more subtle theory.

Definition 2.24 *Let $D \subset \mathbb{R}^2$ be a domain. We say that $\Phi : D \to \mathbb{R}$, $\Phi = \Phi(x,u)$, is an* exact field for f covering D *if there exists $S \in C^1(D)$ satisfying*

$$\begin{array}{rcl} S_u(x,u) & = & f_\xi(x,u,\Phi(x,u)) = p(x,u) \\ S_x(x,u) & = & f(x,u,\Phi(x,u)) - p(x,u)\Phi(x,u) = h(x,u)\ . \end{array}$$

Remark 2.25 *(i) If $f \in C^2$, then a necessary condition for Φ to be exact is that $p_x = h_u$. Conversely if D is simply connected and if $p_x = h_u$ then such an S exists.*

(ii) In the case where $u : [a,b] \to \mathbb{R}^N$, $N > 1$, we have to add to the preceding remark, not only that $(p_i)_x = h_{u_i}$, but also $(p_i)_{u_j} = (p_j)_{u_i}$, for every $1 \leq i,j \leq N$.

We start with an elementary result that is a first justification for defining such a notion.

Proposition 2.26 *Let $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$, $f = f(x,u,\xi)$, and*

$$I(u) = \int_a^b f(x,u(x),u'(x))\,dx\,.$$

Let $\Phi : D \to \mathbb{R}^2$, $\Phi = \Phi(x,u)$ be a C^1 exact field for f covering D, $[a,b] \times \mathbb{R} \subset D$. Then any solution $u \in C^2([a,b])$ of

$$u'(x) = \Phi(x,u(x)) \tag{2.20}$$

solves the Euler-Lagrange associated to the functional I, namely

$$(E) \quad \frac{d}{dx}\left[f_\xi(x,u(x),u'(x))\right] = f_u(x,u(x),u'(x))\,,\ x \in [a,b]\,. \tag{2.21}$$

Proof. By definition of Φ and using the fact that $p = f_\xi$, we have, for any $(x,u) \in D$,

$$h_u = f_u(x,u,\Phi) + f_\xi(x,u,\Phi)\Phi_u - p_u\Phi - p\Phi_u = f_u(x,u,\Phi) - p_u\Phi$$

and hence

$$f_u(x,u,\Phi) = h_u(x,u) + p_u(x,u)\,\Phi(x,u)\ .$$

We therefore get for every $x \in [a,b]$

$$\begin{aligned}\frac{d}{dx}\left[f_\xi\left(x,u,u'\right)\right] - f_u\left(x,u,u'\right) &= \frac{d}{dx}\left[p(x,u)\right] - \left[h_u(x,u) + p_u(x,u)\,\Phi(x,u)\right] \\ &= p_x + p_u u' - h_u - p_u\Phi = p_x - h_u = 0\end{aligned}$$

since we have that $u' = \Phi$ and $p_x = h_u$, Φ being exact. Thus we have reached the claim. ■

The next theorem is the main result of this section and was established by Weierstrass and Hilbert.

Theorem 2.27 ***(Hilbert Theorem)****. Let $f \in C^2([a,b]\times\mathbb{R}\times\mathbb{R})$ with $\xi \to f(x,u,\xi)$ convex for every $(x,u) \in [a,b]\times\mathbb{R}$. Let $D \subset \mathbb{R}^2$ be a domain and $\Phi : D \to \mathbb{R}^2$, $\Phi = \Phi(x,u)$, be an exact field for f covering D. Assume that there exists $u_0 \in C^1([a,b])$ satisfying*

$$(x,u_0(x)) \in D,\ \forall x \in [a,b]$$

$$u_0'(x) = \Phi(x,u_0(x))\,,\ \forall x \in [a,b]$$

then u_0 is a minimizer for I, i.e.

$$I(u) = \int_a^b f\left(x,u(x),u'(x)\right)\,dx \geq I(u_0)\,,\ \forall u \in X$$

where

$$X = \left\{\begin{array}{c} u \in C^1([a,b]) : u(a) = u_0(a)\,,\ u(b) = u_0(b) \\ \textit{with } (x,u(x)) \in D,\ \forall x \in [a,b]\end{array}\right\}.$$

Remark 2.28 *(i) Observe that according to the preceding proposition we have that such a u_0 is necessarily a solution of the Euler-Lagrange equation.*

(ii) As already mentioned it might be very difficult to construct such exact fields. Moreover, in general, D does not contain the whole of $[a,b]\times\mathbb{R}$ and, consequently, the theorem will provide only local minima. The construction of such fields is intimately linked with the so called Jacobi condition concerning conjugate points (see the bibliography for more details).

Proof. Denote by E the *Weierstrass function* defined by

$$E(x,u,\eta,\xi) = f(x,u,\xi) - f(x,u,\eta) - f_\xi(x,u,\eta)(\xi-\eta)$$

or in other words

$$f(x,u,\xi) = E(x,u,\eta,\xi) + f(x,u,\eta) + f_\xi(x,u,\eta)(\xi-\eta)\ .$$

Since $\xi \to f(x, u, \xi)$ is convex, then the function E is always non negative.

Note also that since $u_0'(x) = \Phi(x, u_0(x))$ then

$$E(x, u_0(x), \Phi(x, u_0(x)), u_0'(x)) = 0, \ \forall x \in [a, b] \ .$$

Now using the definition of exact field we get that, for every $u \in X$,

$$\begin{aligned} & f(x, u(x), \Phi(x, u(x))) + f_\xi(x, u(x), \Phi(x, u(x)))(u'(x) - \Phi(x, u(x))) \\ = \ & f(x, u, \Phi) - p\Phi + pu' = S_x + S_u u' = \frac{d}{dx}[S(x, u(x))] \ . \end{aligned}$$

Combining these facts we have obtained that

$$\begin{aligned} I(u) &= \int_a^b f(x, u(x), u'(x)) \, dx \\ &= \int_a^b \left\{ E(x, u(x), \Phi(x, u(x)), u'(x)) + \frac{d}{dx}[S(x, u(x))] \right\} dx \\ &\geq \int_a^b \frac{d}{dx}[S(x, u(x))] \, dx = S(b, u(b)) - S(a, u(a)) \ . \end{aligned}$$

Since $E(x, u_0, \Phi(x, u_0), u_0') = 0$ we have that

$$I(u_0) = S(b, u_0(b)) - S(a, u_0(a)) \ .$$

Moreover since $u_0(a) = u(a)$, $u_0(b) = u(b)$ we deduce that $I(u) \geq I(u_0)$ for every $u \in X$. This achieves the proof of the theorem.

The quantity

$$\int_a^b \frac{d}{dx}[S(x, u(x))] \, dx$$

is called *invariant Hilbert integral.* ■

2.6.1 Exercises

Exercise 2.6.1 *Generalize Theorem 2.21 to the case where* $u : [a, b] \to \mathbb{R}^N$, $N \geq 1$.

Exercise 2.6.2 *Generalize Hilbert Theorem (Theorem 2.27) to the case where* $u : [a, b] \to \mathbb{R}^N$, $N \geq 1$.

Exercise 2.6.3 *(The present exercise establishes the connection between exact field and Hamilton-Jacobi equation). Let* $f = f(x, u, \xi)$ *and* $= H(x, u, v)$ *be as in Theorem 2.10.*

(i) Show that if there exists an exact field Φ *covering* D*, then*

$$S_x + H\left(x, u, S_u\right) = 0,\ \forall\left(x, u\right) \in D$$

where

$$\begin{array}{rcl} S_u\left(x, u\right) & = & f_\xi\left(x, u, \Phi\left(x, u\right)\right) \\ S_x\left(x, u\right) & = & f\left(x, u, \Phi\left(x, u\right)\right) - S_u\left(x, u\right)\Phi\left(x, u\right)\ . \end{array}$$

(ii) Conversely if the Hamilton-Jacobi equation has a solution for every $\left(x, u\right) \in D$*, prove that*

$$\Phi\left(x, u\right) = H_v\left(x, u, S_u\left(x, u\right)\right)$$

is an exact field for f *covering* D*.*

Chapter 3

Direct methods

3.1 Introduction

In this chapter we will study the problem

$$(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx : u \in u_0 + W_0^{1,p}(\Omega) \right\} = m$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded open set;
- $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $f = f(x, u, \xi)$;
- $u \in u_0 + W_0^{1,p}(\Omega)$ means that $u, u_0 \in W^{1,p}(\Omega)$ and $u - u_0 \in W_0^{1,p}(\Omega)$ (which roughly means that $u = u_0$ on $\partial\Omega$).

This is the fundamental problem of the calculus of variations. We will show that the problem (P) has a solution $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ provided the two following main hypotheses are satisfied.

(H1) *Convexity*: $\xi \to f(x, u, \xi)$ is convex for every $(x, u) \in \overline{\Omega} \times \mathbb{R}$;

(H2) *Coercivity*: there exist $p > q \geq 1$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$f(x, u, \xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3, \ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

The *Dirichlet integral* which has as integrand

$$f(x, u, \xi) = \frac{1}{2} |\xi|^2$$

satisfies both hypotheses.

However the *minimal surface* problem whose integrand is given by

$$f(x, u, \xi) = \sqrt{1 + |\xi|^2}$$

satisfies (H1) but verifies (H2) only with $p = 1$. Therefore this problem will require a special treatment (see Chapter 5).

It is interesting to compare the generality of the result with those of the preceding chapter. The main drawback of the present analysis is that we prove existence of minima only in Sobolev spaces. In the next chapter we will see that, under some extra hypotheses, the solution is in fact more regular (for example it is C^1, C^2 or C^∞).

We now describe the content of the present chapter. In Section 3.2 we consider the model case, namely the Dirichlet integral. Although this is just an example of the more general theorem obtained in Section 3.3 we will fully discuss the particular case because of its importance and to make easier the understanding of the method. Recall that the origin of the direct methods goes back to Hilbert while treating the Dirichlet integral and to Lebesgue and Tonelli. Let us briefly describe the two main steps in the proof.

Step 1 (Compactness). Let $u_\nu \in u_0 + W_0^{1,p}(\Omega)$ be a minimizing sequence of (P), this means that

$$I(u_\nu) \to \inf\{I(u)\} = m, \text{ as } \nu \to \infty.$$

It will be easy invoking (H2) and Poincaré inequality (cf. Theorem 1.47) to obtain that there exists $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ and a subsequence (still denoted u_ν) so that u_ν converges weakly to $\overline{u}$ in $W^{1,p}$, i.e.

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p}, \text{ as } \nu \to \infty.$$

Step 2 (Lower semicontinuity). We will then show that (H1) implies the (sequential) weak lower semicontinuity of I, namely

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \;\Rightarrow\; \liminf_{\nu\to\infty} I(u_\nu) \geq I(\overline{u}) \,.$$

Since $\{u_\nu\}$ was a minimizing sequence we deduce that $\overline{u}$ is a minimizer of (P).

In Section 3.4 we will derive the *Euler-Lagrange equation* associated to (P). Since the solution of (P) is only in a Sobolev space, we will be able to write only a weak form of this equation.

In Section 3.5 we will say some words about the considerably harder case where the unknown function u is a vector, i.e. $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$, with $n, N > 1$.

In Section 3.6 we will explain briefly what can be done, in some cases, when the hypothesis (H1) of convexity fails to hold.

The interested reader is referred for further reading to the book of the author [31] or to Buttazzo [15], Buttazzo-Giaquinta-Hildebrandt [17], Cesari [20], Ekeland-Témam [41], Giaquinta [47], Giusti [51], Ioffe-Tihomirov [62], Morrey [75], Struwe [92] and Zeidler [99].

3.2 The model case: Dirichlet integral

The main result is

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $u_0 \in W^{1,2}(\Omega)$. The problem*

$$(D) \quad \inf \left\{ I(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx : u \in u_0 + W_0^{1,2}(\Omega) \right\} = m$$

has one and only one solution $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$.

Furthermore $\overline{u}$ satisfies the weak form of Laplace equation, namely

$$\int_\Omega \langle \nabla \overline{u}(x) ; \nabla \varphi(x) \rangle \, dx = 0, \ \forall \varphi \in W_0^{1,2}(\Omega) \tag{3.1}$$

where $\langle .;. \rangle$ denotes the scalar product in $\mathbb{R}^n$.

Conversely if $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$ satisfies (3.1) then it is a minimizer of (D).

Remark 3.2 *(i) We should again emphasize the very weak hypotheses on u_0 and Ω and recall that $u \in u_0 + W_0^{1,2}(\Omega)$ means that $u = u_0$ on $\partial\Omega$ (in the sense of Sobolev spaces).*

(ii) If the solution $\overline{u}$ turns out to be more regular, namely in $W^{2,2}(\Omega)$ then (3.1) can be integrated by parts and we get

$$\int_\Omega \Delta \overline{u}(x) \, \varphi(x) \, dx = 0, \ \forall \varphi \in W_0^{1,2}(\Omega)$$

which combined with the fundamental lemma of the calculus of variations (Theorem 1.24) leads to $\Delta \overline{u} = 0$ a.e. in Ω. This extra regularity of $\overline{u}$ (which will turn out to be even $C^\infty(\Omega)$) will be proved in Section 4.3.

(iii) As we already said, the above theorem was proved by Hilbert, Lebesgue and Tonelli, but it was expressed in a different way since Sobolev spaces did not exist then. Throughout the 19th century there were several attempts to establish a theorem of the above kind, notably by Dirichlet and Riemann.

Proof. The proof is surprisingly simple.

Part 1 (Existence). We divide, as explained in the Introduction, the proof into three steps.

Step 1 (Compactness). We start with the observation that since $u_0 \in u_0 + W_0^{1,2}(\Omega)$ we have

$$0 \leq m \leq I(u_0) < \infty.$$

Let $u_\nu \in u_0 + W_0^{1,2}(\Omega)$ be a minimizing sequence of (D), this means that

$$I(u_\nu) \to \inf\{I(u)\} = m, \text{ as } \nu \to \infty.$$

Observe that by Poincaré inequality (cf. Theorem 1.47) we can find constants $\gamma_1, \gamma_2 > 0$ so that

$$\sqrt{2I(u_\nu)} = \|\nabla u_\nu\|_{L^2} \geq \gamma_1 \|u_\nu\|_{W^{1,2}} - \gamma_2 \|u_0\|_{W^{1,2}} \ .$$

Since u_ν is a minimizing sequence and $m < \infty$ we deduce that there exists $\gamma_3 > 0$ so that

$$\|u_\nu\|_{W^{1,2}} \leq \gamma_3 \, .$$

Applying Exercise 1.4.5 we deduce that there exists $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$ and a subsequence (still denoted u_ν) so that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,2}, \text{ as } \nu \to \infty.$$

Step 2 (Lower semicontinuity). We now show that I is (sequentially) weakly lower semicontinuous; this means that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,2} \ \Rightarrow \ \liminf_{\nu \to \infty} I(u_\nu) \geq I(\overline{u}) \ .$$

This step is independent of the fact that $\{u_\nu\}$ is a minimizing sequence. We trivially have that

$$\begin{aligned} |\nabla u_\nu|^2 &= |\nabla \overline{u}|^2 + 2\langle \nabla \overline{u}; \nabla u_\nu - \nabla \overline{u}\rangle + |\nabla u_\nu - \nabla \overline{u}|^2 \\ &\geq |\nabla \overline{u}|^2 + 2\langle \nabla \overline{u}; \nabla u_\nu - \nabla \overline{u}\rangle \, . \end{aligned}$$

Integrating this expression we have

$$I(u_\nu) \geq I(\overline{u}) + \int_\Omega \langle \nabla \overline{u}; \nabla u_\nu - \nabla \overline{u}\rangle \, dx \, .$$

To conclude we show that the second term in the right hand side of the inequality tends to 0. Indeed since $\nabla \overline{u} \in L^2$ and $\nabla u_\nu - \nabla \overline{u} \rightharpoonup 0$ in L^2 this implies, by definition of weak convergence in L^2, that

$$\lim_{\nu \to \infty} \int_\Omega \langle \nabla \overline{u}; \nabla u_\nu - \nabla \overline{u}\rangle \, dx = 0 \, .$$

Therefore returning to the above inequality we have indeed obtained that

$$\liminf_{\nu\to\infty} I(u_\nu) \geq I(\overline{u}) \; .$$

Step 3. We now combine the two steps. Since $\{u_\nu\}$ was a minimizing sequence (i.e. $I(u_\nu) \to \inf\{I(u)\} = m$) and for such a sequence we have lower semicontinuity (i.e. $\liminf I(u_\nu) \geq I(\overline{u})$) we deduce that $I(\overline{u}) = m$, i.e. $\overline{u}$ is a minimizer of (D).

Part 2 (Uniqueness). Assume that there exist $\overline{u}, \overline{v} \in u_0 + W_0^{1,2}(\Omega)$ so that

$$I(\overline{u}) = I(\overline{v}) = m$$

and let us show that this implies $\overline{u} = \overline{v}$. Denote by $\overline{w} = (\overline{u} + \overline{v})/2$ and observe that $\overline{w} \in u_0 + W_0^{1,2}(\Omega)$. The function $\xi \to |\xi|^2$ being convex, we can infer that $\overline{w}$ is also a minimizer since

$$m \leq I(\overline{w}) \leq \frac{1}{2} I(\overline{u}) + \frac{1}{2} I(\overline{v}) = m \, ,$$

which readily implies that

$$\int_\Omega \left[\frac{1}{2} |\nabla \overline{u}|^2 + \frac{1}{2} |\nabla \overline{v}|^2 - \left| \frac{\nabla \overline{u} + \nabla \overline{v}}{2} \right|^2 \right] dx = 0 \, .$$

Appealing once more to the convexity of $\xi \to |\xi|^2$, we deduce that the integrand is non negative, while the integral is 0. This is possible only if

$$\frac{1}{2} |\nabla \overline{u}|^2 + \frac{1}{2} |\nabla \overline{v}|^2 - \left| \frac{\nabla \overline{u} + \nabla \overline{v}}{2} \right|^2 = 0 \text{ a.e. in } \Omega \, .$$

We now use the strict convexity of $\xi \to |\xi|^2$ to obtain that $\nabla \overline{u} = \nabla \overline{v}$ a.e. in Ω, which combined with the fact that the two functions agree on the boundary of Ω (since $\overline{u}, \overline{v} \in u_0 + W_0^{1,2}(\Omega)$) leads to the claimed uniqueness $\overline{u} = \overline{v}$ a.e. in Ω.

Part 3 (Euler-Lagrange equation). Let us now establish (3.1). Let $\epsilon \in \mathbb{R}$ and $\varphi \in W_0^{1,2}(\Omega)$ be arbitrary. Note that $\overline{u} + \epsilon\varphi \in u_0 + W_0^{1,2}(\Omega)$, which combined with the fact that $\overline{u}$ is the minimizer of (D) leads to

$$\begin{aligned} I(\overline{u}) \quad &\leq \quad I(\overline{u} + \epsilon\varphi) = \int_\Omega \frac{1}{2} |\nabla \overline{u} + \epsilon \nabla \varphi|^2 \, dx \\ &= \quad I(\overline{u}) + \epsilon \int_\Omega \langle \nabla \overline{u}; \nabla \varphi \rangle \, dx + \epsilon^2 I(\varphi) \; . \end{aligned}$$

The fact that ϵ is arbitrary leads immediately to (3.1), which expresses nothing else than

$$\left. \frac{d}{d\epsilon} I(\overline{u} + \epsilon\varphi) \right|_{\epsilon=0} = 0 \, .$$

Part 4 (Converse). We finally prove that if $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$ satisfies (3.1) then it is necessarily a minimizer of (D). Let $u \in u_0 + W_0^{1,2}(\Omega)$ be any element and set $\varphi = u - \overline{u}$. Observe that $\varphi \in W_0^{1,2}(\Omega)$ and

$$\begin{aligned} I(u) &= I(\overline{u} + \varphi) = \int_\Omega \frac{1}{2} |\nabla \overline{u} + \nabla \varphi|^2 \, dx \\ &= I(\overline{u}) + \int_\Omega \langle \nabla \overline{u}; \nabla \varphi \rangle \, dx + I(\varphi) \geq I(\overline{u}) \end{aligned}$$

since the second term is 0 according to (3.1) and the last one is non negative. This achieves the proof of the theorem. ■

3.2.1 Exercises

Exercise 3.2.1 *Let Ω be as in the theorem and $g \in L^2(\Omega)$. Show that*

$$(P) \quad \inf \left\{ I(u) = \int_\Omega \left[\frac{1}{2} |\nabla u(x)|^2 - g(x) u(x) \right] dx : u \in W_0^{1,2}(\Omega) \right\} = m$$

has a unique solution $\overline{u} \in W_0^{1,2}(\Omega)$ which satisfies in addition

$$\int_\Omega \langle \nabla \overline{u}(x); \nabla \varphi(x) \rangle \, dx = \int_\Omega g(x) \varphi(x) \, dx, \ \forall \varphi \in W_0^{1,2}(\Omega) \, .$$

3.3 A general existence theorem

The main theorem of the present chapter is the following.

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $f \in C^0\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\right)$, $f = f(x, u, \xi)$, satisfy*

(H1) $\xi \to f(x, u, \xi)$ is convex for every $(x, u) \in \overline{\Omega} \times \mathbb{R}$;

(H2) there exist $p > q \geq 1$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$f(x, u, \xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3, \ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

Let

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W_0^{1,p}(\Omega) \right\} = m$$

where $u_0 \in W^{1,p}(\Omega)$ with $I(u_0) < \infty$. Then there exists $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ a minimizer of (P).

Furthermore if $(u, \xi) \to f(x, u, \xi)$ is strictly convex for every $x \in \overline{\Omega}$, then the minimizer is unique.

Remark 3.4 *(i) The hypotheses of the theorem are nearly optimal, in the sense that the weakening of any of them leads to a counterexample to the existence of minima (cf. below). The only hypothesis that can be slightly weakened is the continuity of f (see the above mentioned literature).*

(ii) The proof will show that uniqueness holds under a slightly weaker condition, namely that $(u, \xi) \to f(x, u, \xi)$ is convex and either $u \to f(x, u, \xi)$ is strictly convex or $\xi \to f(x, u, \xi)$ is strictly convex.

(iii) The theorem remains valid in the vectorial case, where $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$, with $n, N > 1$. However the hypothesis (H1) is then far from being optimal (cf. Section 3.5).

(iv) This theorem has a long history and we refer to [31] for details. The first one that noticed the importance of the convexity of f is Tonelli.

Before proceeding with the proof of the theorem, we discuss several examples, emphasizing the optimality of the hypotheses.

Example 3.5 *(i) The Dirichlet integral considered in the preceding section enters, of course, in the framework of the present theorem; indeed we have that*

$$f(x, u, \xi) = f(\xi) = \frac{1}{2} |\xi|^2$$

satisfies all the hypotheses of the theorem with $p = 2$.

(ii) The natural generalization of the preceding example is

$$f(x, u, \xi) = \frac{1}{p} |\xi|^p + g(x, u)$$

where g is continuous and non negative and $p > 1$.

Example 3.6 *The minimal surface problem has an integrand given by*

$$f(x, u, \xi) = f(\xi) = \sqrt{1 + |\xi|^2}$$

that satisfies all the hypotheses of the theorem but (H2), this hypothesis is only verified with $p = 1$. We will see in Chapter 5 that this failure may lead to non existence of minima for the corresponding (P). The reason why $p = 1$ is not allowed is that the corresponding Sobolev space $W^{1,1}$ is not reflexive (see Chapter 1).

Example 3.7 *This example is of the minimal surface type but easier, it also shows that all the hypotheses of the theorem are satisfied, except (H2) that is* *true with* $p = 1$. *This weakening of (H2) leads to the following counterexample. Let* $n = 1$, $f(x, u, \xi) = f(u, \xi) = \sqrt{u^2 + \xi^2}$ *and*

$$(P) \quad \inf\left\{ I(u) = \int_0^1 f(u(x), u'(x))\, dx : u \in X \right\} = m$$

where $X = \left\{ u \in W^{1,1}(0,1) : u(0) = 0,\ u(1) = 1 \right\}$. *Let us prove that (P) has no solution. We first show that* $m = 1$ *and start by observing that* $m \geq 1$ *since*

$$I(u) \geq \int_0^1 |u'(x)|\, dx \geq \int_0^1 u'(x)\, dx = u(1) - u(0) = 1\,.$$

To establish that $m = 1$ *we construct a minimizing sequence* $u_\nu \in X$ *(*ν *being an integer) as follows*

$$u_\nu(x) = \begin{cases} 0 & \text{if } x \in \left[0, 1 - \frac{1}{\nu}\right] \\ 1 + \nu(x-1) & \text{if } x \in \left(1 - \frac{1}{\nu}, 1\right]\,. \end{cases}$$

We therefore have $m = 1$ *since*

$$\begin{aligned} 1 \quad &\leq \quad I(u_\nu) = \int_{1-\frac{1}{\nu}}^1 \sqrt{(1 + \nu(x-1))^2 + \nu^2}\, dx \\ &\leq \quad \frac{1}{\nu}\sqrt{1 + \nu^2} \longrightarrow 1, \text{ as } \nu \to \infty\,. \end{aligned}$$

Assume now, for the sake of contradiction, that there exists $\overline{u} \in X$ *a minimizer of (P). We should then have, as above,*

$$\begin{aligned} 1 \quad &= \quad I(\overline{u}) = \int_0^1 \sqrt{\overline{u}^2 + \overline{u}'^2}\, dx \geq \int_0^1 |\overline{u}'|\, dx \\ &\geq \quad \int_0^1 \overline{u}'\, dx = \overline{u}(1) - \overline{u}(0) = 1\,. \end{aligned}$$

This implies that $\overline{u} = 0$ *a.e. in* $(0,1)$. *Since elements of* X *are continuous we have that* $\overline{u} \equiv 0$ *and this is incompatible with the boundary data. Thus (P) has no solution.*

Example 3.8 *(**Weierstrass example**). We have seen this example in Section 2.2. Recall that* $n = 1$, $f(x, u, \xi) = f(x, \xi) = x\xi^2$ *and*

$$(P) \quad \inf\left\{ I(u) = \int_0^1 f(x, u'(x))\, dx : u \in X \right\} = m_X$$

where $X = \left\{u \in W^{1,2}(0,1) : u(0) = 1,\ u(1) = 0\right\}$*. All the hypotheses of the theorem are verified with the exception of (H2) that is satisfied only with* $\alpha_1 = 0$*. This is enough to show that (P) has no minimizer in* X*. Indeed we have seen in Exercise 2.2.6 that (P) has no solution in* $Y = X \cap C^1([0,1])$ *and that the corresponding value of the infimum, let us denote it by* m_Y*, is* 0*. Since trivially* $0 \le m_X \le m_Y$ *we deduce that* $m_X = 0$*. Now assume, by absurd hypothesis, that (P) has a solution* $\overline{u} \in X$*, we should then have* $I(\overline{u}) = 0$*, but since the integrand is non negative we deduce that* $\overline{u}' = 0$ *a.e. in* $(0,1)$*. Since elements of* X *are continuous we have that* $\overline{u}$ *is constant, and this is incompatible with the boundary data. Hence (P) has no solution.*

Example 3.9 *The present example (cf. Poincaré-Wirtinger inequality) shows that we cannot allow, in general, that* $q = p$ *in (H2). Let* $n = 1$*,* $\lambda > \pi$ *and*

$$f(x,u,\xi) = f(u,\xi) = \frac{1}{2}\left(\xi^2 - \lambda^2 u^2\right).$$

We have seen in Section 2.2 that if

$$(P) \quad \inf\left\{I(u) = \int_0^1 f(u(x), u'(x))\,dx : u \in W_0^{1,2}(0,1)\right\} = m$$

then $m = -\infty$*, which means that (P) has no solution.*

Example 3.10 ***(Bolza example)****. We now show that, as a general rule, one cannot weaken (H1) either. One such example has already been seen in Section 2.2 where we had* $f(x,u,\xi) = f(\xi) = e^{-\xi^2}$ *(which satisfies neither (H1) nor (H2)). Let* $n = 1$*,*

$$f(x,u,\xi) = f(u,\xi) = \left(\xi^2 - 1\right)^2 + u^4$$

$$(P) \quad \inf\left\{I(u) = \int_0^1 f(u(x), u'(x))\,dx : u \in W_0^{1,4}(0,1)\right\} = m\,.$$

Assume for a moment that we already proved that $m = 0$ *and let us show that (P) has no solution, using an argument by contradiction. Let* $\overline{u} \in W_0^{1,4}(0,1)$ *be a minimizer of (P), i.e.* $I(\overline{u}) = 0$*. This implies that* $\overline{u} = 0$ *and* $|\overline{u}'| = 1$ *a.e. in* $(0,1)$*. Since the elements of* $W^{1,4}$ *are continuous we have that* $\overline{u} \equiv 0$ *and hence* $\overline{u}' \equiv 0$ *which is clearly absurd.*

So let us show that $m = 0$ *by constructing an appropriate minimizing sequence. Let* $u_\nu \in W_0^{1,4}$ *(*$\nu \ge 2$ *being an integer) defined on each interval* $[k/\nu, (k+1)/\nu]$*,* $0 \le k \le \nu - 1$*, as follows*

$$u_\nu(x) = \begin{cases} x - \frac{k}{\nu} & \text{if } x \in \left[\frac{2k}{2\nu}, \frac{2k+1}{2\nu}\right] \\ -x + \frac{k+1}{\nu} & \text{if } x \in \left(\frac{2k+1}{2\nu}, \frac{2k+2}{2\nu}\right]\,. \end{cases}$$

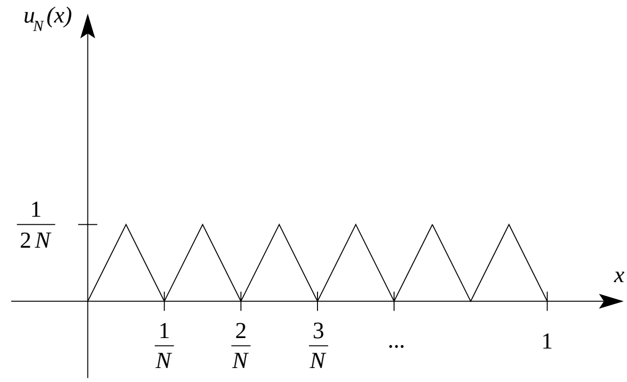

Figure 3.1: minimizing sequence

Observe that $|u_\nu'| = 1$ *a.e. and* $|u_\nu| \leq 1/(2\nu)$ *leading therefore to the desired convergence, namely*

$$0 \leq I(u_\nu) \leq \frac{1}{(2\nu)^4} \longrightarrow 0, \text{ as } \nu \to \infty\,.$$

Proof. We will not prove the theorem in its full generality. We refer to the literature and in particular to Theorem 3.4.1 in [31] for a general proof; see also the exercises below. We will prove it under the stronger following hypotheses. We will assume that $f \in C^1\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\right)$, instead of C^0, and

(H1+) $(u, \xi) \to f(x, u, \xi)$ is convex for every $x \in \overline{\Omega}$;

(H2+) there exist $p > 1$ and $\alpha_1 > 0$, $\alpha_3 \in \mathbb{R}$ such that

$$f(x, u, \xi) \geq \alpha_1 |\xi|^p + \alpha_3,\ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

(H3) there exists a constant $\beta \geq 0$ so that for every $(x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$

$$|f_u(x, u, \xi)|\,,\ |f_\xi(x, u, \xi)| \leq \beta\left(1 + |u|^{p-1} + |\xi|^{p-1}\right)$$

where $f_\xi = \left(f_{\xi_1}, ..., f_{\xi_n}\right)$, $f_{\xi_i} = \partial f / \partial \xi_i$ and $f_u = \partial f / \partial u$.

Once these hypotheses are made, the proof is very similar to that of Theorem 3.1. Note also that the function $f(x, u, \xi) = f(\xi) = |\xi|^2/2$ satisfies the above stronger hypotheses.

Part 1 (Existence). The proof is divided into three steps.

Step 1 (Compactness). Recall that by assumption on u_0 and by (H2+) we have

$$-\infty < m \leq I(u_0) < \infty.$$

Let $u_\nu \in u_0 + W_0^{1,p}(\Omega)$ be a minimizing sequence of (P), i.e.

$$I(u_\nu) \to \inf\{I(u)\} = m, \text{ as } \nu \to \infty.$$

We therefore have from (H2+) that for ν large enough

$$m + 1 \geq I(u_\nu) \geq \alpha_1 \|\nabla u_\nu\|_{L^p}^p - |\alpha_3| \operatorname{meas}\Omega$$

and hence there exists $\alpha_4 > 0$ so that

$$\|\nabla u_\nu\|_{L^p} \leq \alpha_4 \,.$$

Appealing to Poincaré inequality (cf. Theorem 1.47) we can find constants $\alpha_5, \alpha_6 > 0$ so that

$$\alpha_4 \geq \|\nabla u_\nu\|_{L^p} \geq \alpha_5 \|u_\nu\|_{W^{1,p}} - \alpha_6 \|u_0\|_{W^{1,p}}$$

and hence we can find $\alpha_7 > 0$ so that

$$\|u_\nu\|_{W^{1,p}} \leq \alpha_7 \,.$$

Applying Exercise 1.4.5 (it is only here that we use the fact that $p > 1$) we deduce that there exists $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ and a subsequence (still denoted u_ν) so that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p}, \text{ as } \nu \to \infty.$$

Step 2 (Lower semicontinuity). We now show that I is (sequentially) weakly lower semicontinuous; this means that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \;\Rightarrow\; \liminf_{\nu \to \infty} I(u_\nu) \geq I(\overline{u}) \,. \tag{3.2}$$

This step is independent of the fact that $\{u_\nu\}$ is a minimizing sequence. Using the convexity of f and the fact that it is C^1 we get

$$\begin{gathered} f(x, u_\nu, \nabla u_\nu) \geq \\ f(x, \overline{u}, \nabla\overline{u}) + f_u(x, \overline{u}, \nabla\overline{u})(u_\nu - \overline{u}) + \langle f_\xi(x, \overline{u}, \nabla\overline{u}); \nabla u_\nu - \nabla\overline{u}\rangle \,. \end{gathered} \tag{3.3}$$

Before proceeding further we need to show that the combination of (H3) and $\overline{u} \in W^{1,p}(\Omega)$ leads to

$$f_u(x, \overline{u}, \nabla\overline{u}) \in L^{p'}(\Omega) \text{ and } f_\xi(x, \overline{u}, \nabla\overline{u}) \in L^{p'}(\Omega; \mathbb{R}^n) \tag{3.4}$$

where $1/p + 1/p' = 1$ (i.e. $p' = p/(p-1)$). Indeed let us prove the first statement, the other one being shown similarly. We have (β_1 being a constant)

$$\begin{aligned} \int_\Omega |f_u(x, \overline{u}, \nabla\overline{u})|^{p'} dx &\leq \beta^{p'} \int_\Omega \left(1 + |\overline{u}|^{p-1} + |\nabla\overline{u}|^{p-1}\right)^{\frac{p}{p-1}} dx \\ &\leq \beta_1 \left(1 + \|\overline{u}\|_{W^{1,p}}^p\right) < \infty \,. \end{aligned}$$

Using Hölder inequality and (3.4) we find that for $u_\nu \in W^{1,p}(\Omega)$

$$f_u(x, \overline{u}, \nabla\overline{u})(u_\nu - \overline{u}), \ \langle f_\xi(x, \overline{u}, \nabla\overline{u}); \nabla u_\nu - \nabla\overline{u}\rangle \in L^1(\Omega).$$

We next integrate (3.3) to get

$$\begin{aligned} I(u_\nu) \geq I(\overline{u}) &+ \int_\Omega f_u(x, \overline{u}, \nabla\overline{u})(u_\nu - \overline{u})\, dx \\ &+ \int_\Omega \langle f_\xi(x, \overline{u}, \nabla\overline{u}); \nabla u_\nu - \nabla\overline{u}\rangle\, dx. \end{aligned} \tag{3.5}$$

Since $u_\nu - \overline{u} \rightharpoonup 0$ in $W^{1,p}$ (i.e. $u_\nu - \overline{u} \rightharpoonup 0$ in L^p and $\nabla u_\nu - \nabla\overline{u} \rightharpoonup 0$ in L^p) and (3.4) holds, we deduce, from the definition of weak convergence in L^p, that

$$\lim_{\nu\to\infty}\int_\Omega f_u(x, \overline{u}, \nabla\overline{u})(u_\nu - \overline{u})\, dx = \lim_{\nu\to\infty}\int_\Omega \langle f_\xi(x, \overline{u}, \nabla\overline{u}); \nabla u_\nu - \nabla\overline{u}\rangle\, dx = 0.$$

Therefore returning to (3.5) we have indeed obtained that

$$\liminf_{\nu\to\infty} I(u_\nu) \geq I(\overline{u}).$$

Step 3. We now combine the two steps. Since $\{u_\nu\}$ was a minimizing sequence (i.e. $I(u_\nu) \to \inf\{I(u)\} = m$) and for such a sequence we have lower semicontinuity (i.e. $\liminf I(u_\nu) \geq I(\overline{u})$) we deduce that $I(\overline{u}) = m$, i.e. $\overline{u}$ is a minimizer of (P).

Part 2 (Uniqueness). The proof is almost identical to the one of Theorem 3.1 and Theorem 2.1. Assume that there exist $\overline{u}, \overline{v} \in u_0 + W_0^{1,p}(\Omega)$ so that

$$I(\overline{u}) = I(\overline{v}) = m$$

and we prove that this implies $\overline{u} = \overline{v}$. Denote by $\overline{w} = (\overline{u} + \overline{v})/2$ and observe that $\overline{w} \in u_0 + W_0^{1,p}(\Omega)$. The function $(u, \xi) \to f(x, u, \xi)$ being convex, we can infer that $\overline{w}$ is also a minimizer since

$$m \leq I(\overline{w}) \leq \frac{1}{2}I(\overline{u}) + \frac{1}{2}I(\overline{v}) = m,$$

which readily implies that

$$\int_\Omega \left[\frac{1}{2}f(x, \overline{u}, \nabla\overline{u}) + \frac{1}{2}f(x, \overline{v}, \nabla\overline{v}) - f\left(x, \frac{\overline{u} + \overline{v}}{2}, \frac{\nabla\overline{u} + \nabla\overline{v}}{2}\right)\right] dx = 0.$$

The convexity of $(u, \xi) \to f(x, u, \xi)$ implies that the integrand is non negative, while the integral is 0. This is possible only if

$$\frac{1}{2}f(x, \overline{u}, \nabla\overline{u}) + \frac{1}{2}f(x, \overline{v}, \nabla\overline{v}) - f\left(x, \frac{\overline{u} + \overline{v}}{2}, \frac{\nabla\overline{u} + \nabla\overline{v}}{2}\right) = 0 \text{ a.e. in } \Omega.$$

We now use the strict convexity of $(u, \xi) \to f(x, u, \xi)$ to obtain that $\overline{u} = \overline{v}$ and $\nabla\overline{u} = \nabla\overline{v}$ a.e. in Ω, which implies the desired uniqueness, namely $\overline{u} = \overline{v}$ a.e. in Ω. ■

3.3.1 Exercises

Exercise 3.3.1 *Prove Theorem 3.3 under the hypotheses (H1+), (H2) and (H3).*

Exercise 3.3.2 *Prove Theorem 3.3 if*

$$f(x,u,\xi) = g(x,u) + h(x,\xi)$$

where

(i) $h \in C^1\left(\overline{\Omega}\times\mathbb{R}^n\right)$, $\xi \to h(x,\xi)$ *is convex for every* $x\in\overline{\Omega}$, *and there exist* $p>1$ *and* $\alpha_1>0$, $\beta,\alpha_3\in\mathbb{R}$ *such that*

$$h(x,\xi)\geq \alpha_1|\xi|^p+\alpha_3,\ \forall(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n$$

$$|h_\xi(x,\xi)|\leq\beta\left(1+|\xi|^{p-1}\right),\ \forall(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n\,;$$

(ii) $g\in C^0\left(\overline{\Omega}\times\mathbb{R}\right)$, $g\geq 0$ *and either of the following three cases hold*

Case 1: $p>n$. *For every* $R>0$, *there exists* $\gamma=\gamma(R)$ *such that*

$$|g(x,u)-g(x,v)|\leq\gamma|u-v|$$

for every $x\in\overline{\Omega}$ *and every* $u,v\in\mathbb{R}$ *with* $|u|,|v|\leq R$.

Case 2: $p=n$. *There exist* $q\geq 1$ *and* $\gamma>0$ *so that*

$$|g(x,u)-g(x,v)|\leq\gamma\left(1+|u|^{q-1}+|v|^{q-1}\right)|u-v|$$

for every $x\in\overline{\Omega}$ *and every* $u,v\in\mathbb{R}$.

Case 3: $p<n$. *There exist* $q\in[1,np/(n-p))$ *and* $\gamma>0$ *so that*

$$|g(x,u)-g(x,v)|\leq\gamma\left(1+|u|^{q-1}+|v|^{q-1}\right)|u-v|$$

for every $x\in\overline{\Omega}$ *and every* $u,v\in\mathbb{R}$.

Exercise 3.3.3 *Prove Theorem 3.3 in the following framework. Let* $\alpha,\beta\in\mathbb{R}^N$, $N>1$ *and*

$$(P)\quad \inf_{u\in X}\left\{I(u)=\int_a^b f(x,u(x),u'(x))\,dx\right\}=m$$

where $X=\left\{u\in W^{1,p}\left((a,b);\mathbb{R}^N\right): u(a)=\alpha,\ u(b)=\beta\right\}$ *and*

(i) $f\in C^1\left([a,b]\times\mathbb{R}^N\times\mathbb{R}^N\right)$, $(u,\xi)\to f(x,u,\xi)$ *is convex for every* $x\in[a,b]$;

(ii) there exist $p>q\geq 1$ *and* $\alpha_1>0$, $\alpha_2,\alpha_3\in\mathbb{R}$ *such that*

$$f(x,u,\xi)\geq\alpha_1|\xi|^p+\alpha_2|u|^q+\alpha_3,\ \forall(x,u,\xi)\in[a,b]\times\mathbb{R}^N\times\mathbb{R}^N\,;$$

(iii) for every $R > 0$, there exists $\beta = \beta(R)$ such that

$$|f_u(x,u,\xi)| \le \beta(1 + |\xi|^p) \text{ and } |f_\xi(x,u,\xi)| \le \beta\left(1 + |\xi|^{p-1}\right)$$

for every $x \in [a,b]$ and every $u, \xi \in \mathbb{R}^N$ with $|u| \le R$.

3.4 Euler-Lagrange equations

We now derive the Euler-Lagrange equation associated to (P). The way of proceeding is identical to that of Section 2.2, but we have to be more careful. Indeed we assumed there that the minimizer $\overline{u}$ was C^2, while here we only know that it is in the Sobolev space $W^{1,p}$.

Theorem 3.11 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $p \ge 1$ and $f \in C^1\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\right)$, $f = f(x,u,\xi)$, satisfy*

(H3) there exists $\beta \ge 0$ so that for every $(x,u,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$

$$|f_u(x,u,\xi)|, \; |f_\xi(x,u,\xi)| \le \beta\left(1 + |u|^{p-1} + |\xi|^{p-1}\right)$$

where $f_\xi = \left(f_{\xi_1}, ..., f_{\xi_n}\right)$, $f_{\xi_i} = \partial f / \partial \xi_i$ and $f_u = \partial f / \partial u$.

Let $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ be a minimizer of

$$(P) \quad \inf\left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx : u \in u_0 + W_0^{1,p}(\Omega) \right\} = m$$

where $u_0 \in W^{1,p}(\Omega)$, then $\overline{u}$ satisfies the weak form of the Euler-Lagrange equation

$$(E_w) \quad \int_\Omega \left[f_u(x, \overline{u}, \nabla\overline{u})\, \varphi + \langle f_\xi(x, \overline{u}, \nabla\overline{u}) ; \nabla\varphi \rangle \right] dx = 0, \; \forall \varphi \in W_0^{1,p}(\Omega) \; .$$

Moreover if $f \in C^2\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\right)$ and $\overline{u} \in C^2\left(\overline{\Omega}\right)$ then $\overline{u}$ satisfies the Euler-Lagrange equation

$$(E) \quad \sum_{i=1}^n \frac{\partial}{\partial x_i}\left[f_{\xi_i}(x, \overline{u}, \nabla\overline{u}) \right] = f_u(x, \overline{u}, \nabla\overline{u}), \; \forall x \in \overline{\Omega} \, .$$

Conversely if $(u,\xi) \to f(x,u,\xi)$ is convex for every $x \in \overline{\Omega}$ and if $\overline{u}$ is a solution of either (E_w) or (E) then it is a minimizer of (P).

Remark 3.12 *(i) A more condensed way of writing (E) is*

$$(E) \quad \operatorname{div}\left[f_{\xi}\left(x, \overline{u}, \nabla \overline{u}\right)\right] = f_{u}\left(x, \overline{u}, \nabla \overline{u}\right), \ \forall x \in \overline{\Omega}\,.$$

(ii) The hypothesis (H3) is necessary for giving a meaning to (E_w); more precisely for ensuring that $f_u \varphi$, $\langle f_\xi ; \nabla \varphi \rangle \in L^1(\Omega)$. It can be weakened, but only slightly by the use of Sobolev imbedding theorem (see Exercise 3.4.1).

(iii) Of course any solution of (E) is a solution of (E_w). The converse is true only if $\overline{u}$ is sufficiently regular.

(iv) In the statement of the theorem we do not need hypothesis (H1) or (H2) of Theorem 3.3. Therefore we do not use the convexity of f (naturally for the converse we need the convexity of f). However we require that a minimizer of (P) does exist.

(v) The theorem remains valid in the vectorial case, where $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$, with $n, N > 1$. The Euler-Lagrange equation becomes now a system of partial differential equations and reads as follows

$$(E) \quad \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[f_{\xi_i^j}\left(x, \overline{u}, \nabla \overline{u}\right)\right] = f_{u^j}\left(x, \overline{u}, \nabla \overline{u}\right), \ \forall x \in \overline{\Omega}, \ j = 1, ..., N$$

where $f : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ and

$$u = \left(u^1, ..., u^N\right) \in \mathbb{R}^N, \ \xi = \left(\xi_i^j\right)_{1 \le i \le n}^{1 \le j \le N} \in \mathbb{R}^{N \times n} \text{ and } \nabla u = \left(\frac{\partial u^j}{\partial x_i}\right)_{1 \le i \le n}^{1 \le j \le N}.$$

(vi) In some cases one can be interested in an even weaker form of the Euler-Lagrange equation. More precisely if we choose the test functions φ in (E_w) to be in $C_0^\infty(\Omega)$ instead of in $W_0^{1,p}(\Omega)$ then one can weaken the hypothesis (H3) and replace it by

(H3') there exist $p \ge 1$ and $\beta \ge 0$ so that for every $(x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$

$$\left|f_u(x, u, \xi)\right|, \ \left|f_\xi(x, u, \xi)\right| \le \beta\left(1 + |u|^p + |\xi|^p\right).$$

The proof of the theorem remains almost identical. The choice of the space where the test function φ belongs depends on the context. If we want to use the solution, $\overline{u}$, itself as a test function then we are obliged to choose $W_0^{1,p}(\Omega)$ as the right space (see Section 4.3) while some other times (see Section 4.2) we can actually limit ourselves to the space $C_0^\infty(\Omega)$.

Proof. The proof is divided into four steps.

Step 1 (Preliminary computation). From the observation that

$$f(x, u, \xi) = f(x, 0, 0) + \int_0^1 \frac{d}{dt}\left[f(x, tu, t\xi)\right] dt, \ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

and from (H3), we find that there exists $\gamma_1 > 0$ so that

$$|f(x,u,\xi)| \leq \gamma_1 \left(1 + |u|^p + |\xi|^p\right), \ \forall (x,u,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n. \tag{3.6}$$

In particular we deduce that

$$|I(u)| < \infty, \ \forall u \in W^{1,p}(\Omega) .$$

Step 2 (Derivative of I). We now prove that for every $u, \varphi \in W^{1,p}(\Omega)$ and every $\epsilon \in \mathbb{R}$ we have

$$\lim_{\epsilon \to 0} \frac{I(u + \epsilon\varphi) - I(u)}{\epsilon} = \int_\Omega \left[f_u(x,u,\nabla u)\varphi + \langle f_\xi(x,u,\nabla u); \nabla\varphi \rangle \right] dx. \tag{3.7}$$

We let

$$g(x,\epsilon) = f(x, u(x) + \epsilon\varphi(x), \nabla u(x) + \epsilon\nabla\varphi(x))$$

so that

$$I(u + \epsilon\varphi) = \int_\Omega g(x,\epsilon)\, dx.$$

Since $f \in C^1$ we have, for almost every $x \in \Omega$, that $\epsilon \to g(x,\epsilon)$ is C^1 and therefore there exists $\theta \in [-|\epsilon|, |\epsilon|]$, $\theta = \theta(x)$, such that

$$g(x,\epsilon) - g(x,0) = g_\epsilon(x,\theta)\,\epsilon$$

where

$$g_\epsilon(x,\theta) = f_u(x, u + \theta\varphi, \nabla u + \theta\nabla\varphi)\varphi + \langle f_\xi(x, u + \theta\varphi, \nabla u + \theta\nabla\varphi); \nabla\varphi \rangle .$$

The hypothesis (H3) implies then that we can find $\gamma_2 > 0$ so that, for every $\theta \in [-1,1]$,

$$\left| \frac{g(x,\epsilon) - g(x,0)}{\epsilon} \right| = |g_\epsilon(x,\theta)| \leq \gamma_2 \left(1 + |u|^p + |\varphi|^p + |\nabla u|^p + |\nabla\varphi|^p\right) \equiv G(x) .$$

Note that since $u, \varphi \in W^{1,p}(\Omega)$, we have $G \in L^1(\Omega)$.

We now observe that, since $u, \varphi \in W^{1,p}(\Omega)$, we have from (3.6) that the functions $x \to g(x,0)$ and $x \to g(x,\epsilon)$ are both in $L^1(\Omega)$.

Summarizing the results we have that

$$\begin{aligned}
\frac{g(x,\epsilon) - g(x,0)}{\epsilon} &\in L^1(\Omega), \\
\left| \frac{g(x,\epsilon) - g(x,0)}{\epsilon} \right| &\leq G(x), \text{ with } G \in L^1(\Omega) \\
\frac{g(x,\epsilon) - g(x,0)}{\epsilon} &\to g_\epsilon(x,0) \text{ a.e. in } \Omega .
\end{aligned}$$

Applying Lebesgue dominated convergence theorem we deduce that (3.7) holds.

Step 3 (Derivation of (E_{w}) *and (E)).* The conclusion of the theorem follows from the preceding step. Indeed since $\overline{u}$ is a minimizer of (P) then

$$I(\overline{u}+\epsilon\varphi)\geq I(\overline{u}),\ \forall\varphi\in W_0^{1,p}(\Omega)$$

and thus

$$\lim_{\epsilon\to 0}\frac{I(\overline{u}+\epsilon\varphi)-I(\overline{u})}{\epsilon}=0$$

which combined with (3.7) implies $(\mathrm{E_w})$.

To get (E) it remains to integrate by parts (using Exercise 1.4.7) and to find

$$(E_w)\ \int_\Omega\left[f_u(x,\overline{u},\nabla\overline{u})-\operatorname{div}f_\xi(x,\overline{u},\nabla\overline{u})\right]\varphi\,dx=0,\ \forall\varphi\in W_0^{1,p}(\Omega)\,.$$

The fundamental lemma of the calculus of variations (Theorem 1.24) implies the claim.

Step 4 (Converse). Let $\overline{u}$ be a solution of $(\mathrm{E_w})$ (note that any solution of (E) is necessarily a solution of $(\mathrm{E_w})$). From the convexity of f we deduce that for every $u\in u_0+W_0^{1,p}(\Omega)$ the following holds

$$\begin{aligned}f(x,u,\nabla u)\quad\geq\quad & f(x,\overline{u},\nabla\overline{u})+f_u(x,\overline{u},\nabla\overline{u})(u-\overline{u})\\ & +\langle f_\xi(x,\overline{u},\nabla\overline{u});(\nabla u-\nabla\overline{u})\rangle\,.\end{aligned}$$

Integrating, using $(\mathrm{E_w})$ and the fact that $u-\overline{u}\in W_0^{1,p}(\Omega)$ we immediately get that $I(u)\geq I(\overline{u})$ and hence the theorem. ■

We now discuss some examples.

Example 3.13 *In the case of Dirichlet integral we have*

$$f(x,u,\xi)=f(\xi)=\frac{1}{2}|\xi|^2$$

which satisfies (H3). The equation (E_w) *is then*

$$\int_\Omega\langle\nabla\overline{u}(x);\nabla\varphi(x)\rangle\,dx=0,\ \forall\varphi\in W_0^{1,2}(\Omega)$$

while (E) is $\Delta\overline{u}=0$.

Example 3.14 *Consider the generalization of the preceding example, where*

$$f(x,u,\xi)=f(\xi)=\frac{1}{p}|\xi|^p\,.$$

The equation (E) is known as the p-Laplace equation (so called, since when $p=2$ *it corresponds to Laplace equation)*

$$\operatorname{div}\left[|\nabla\overline{u}|^{p-2}\nabla\overline{u}\right]=0,\ in\ \Omega\,.$$

Example 3.15 *The minimal surface problem has an integrand given by*

$$f(x,u,\xi) = f(\xi) = \sqrt{1+|\xi|^2}$$

that satisfies (H3) with $p=1$, since

$$|f_\xi(\xi)| = \left(\sum_{i=1}^{n}\left|\frac{\xi_i}{\sqrt{1+|\xi|^2}}\right|^2\right)^{1/2} \leq 1\,.$$

The equation (E) is the so called minimal surface equation

$$\operatorname{div}\frac{\nabla\overline{u}}{\sqrt{1+|\nabla\overline{u}|^2}} = 0,\ in\ \Omega$$

and can be rewritten as

$$\left(1+|\nabla\overline{u}|^2\right)\Delta\overline{u} - \sum_{i,j=1}^{n}\overline{u}_{x_i}\overline{u}_{x_j}\overline{u}_{x_ix_j} = 0,\ in\ \Omega\,.$$

Example 3.16 *Let $f(x,u,\xi) = f(u,\xi) = g(u)|\xi|^2$, with $0 \leq g(u), |g'(u)| \leq g_0$. We then have*

$$|f_u(u,\xi)| = |g'(u)||\xi|^2,\ |f_\xi(u,\xi)| = 2|g(u)||\xi| \leq 2g_0|\xi|$$

We see that if $g'(u) \neq 0$, then f does not satisfy (H3) but only the above (H3'). We are therefore authorized to write only

$$\int_\Omega \left[f_u(\overline{u},\nabla\overline{u})\varphi + \langle f_\xi(\overline{u},\nabla\overline{u});\nabla\varphi\rangle\right]dx = 0,\ \forall\varphi\in C_0^\infty(\Omega)$$

or more generally the equation should hold for any $\varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Let us now recall two examples from Section 2.2, showing that, without any hypotheses of convexity of the function f, the converse part of the theorem is false.

Example 3.17 ***(Poincaré-Wirtinger inequality)***. *Let $\lambda > \pi$, $n = 1$ and*

$$f(x,u,\xi) = f(u,\xi) = \frac{1}{2}\left(\xi^2 - \lambda^2u^2\right)$$

$$(P)\quad \inf\left\{I(u) = \int_0^1 f(u(x),u'(x))\,dx : u\in W_0^{1,2}(0,1)\right\} = m\,.$$

Note that $\xi \to f(u,\xi)$ is convex while $(u,\xi) \to f(u,\xi)$ is not. We have seen that $m = -\infty$ and therefore (P) has no minimizer; however the Euler-Lagrange equation

$$u'' + \lambda^2 u = 0 \text{ in } [0,1]$$

has $u \equiv 0$ as a solution. It is therefore not a minimizer.

Example 3.18 *Let $n = 1$, $f(x,u,\xi) = f(\xi) = \left(\xi^2 - 1\right)^2$, which is non convex, and*

$$(P) \quad \inf\left\{ I(u) = \int_0^1 f(u'(x))\, dx : u \in W_0^{1,4}(0,1) \right\} = m\,.$$

We have seen that $m = 0$. The Euler-Lagrange equation is

$$(E) \quad \frac{d}{dx}\left[\overline{u}'\left(\overline{u}'^2 - 1\right)\right] = 0$$

and its weak form is (note that f satisfies (H3))

$$(E_w) \quad \int_0^1 \overline{u}'\left(\overline{u}'^2 - 1\right)\varphi'\, dx = 0,\ \forall \varphi \in W_0^{1,4}(0,1)\ .$$

It is clear that $\overline{u} \equiv 0$ is a solution of (E) and (E_w), but it is not a minimizer of (P) since $m = 0$ and $I(0) = 1$. The present example is also interesting for another reason. Indeed the function

$$v(x) = \begin{cases} x & \text{if } x \in [0,1/2] \\ 1 - x & \text{if } x \in (1/2,1] \end{cases}$$

is clearly a minimizer of (P) which is not C^1; it satisfies (E_w) but not (E).

3.4.1 Exercises

Exercise 3.4.1 *(i) Show that the theorem remains valid if we weaken the hypothesis (H3), for example, as follows: if $1 \le p < n$, replace (H3) by:*
there exist $\beta > 0$, $1 \le s_1 \le (np - n + p)/(n - p)$, $1 \le s_2 \le (np - n + p)/n$, $1 \le s_3 \le n(p-1)/(n-p)$ so that the following hold, for every $(x,u,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$,

$$|f_u(x,u,\xi)| \le \beta\left(1 + |u|^{s_1} + |\xi|^{s_2}\right),\ |f_\xi(x,u,\xi)| \le \beta\left(1 + |u|^{s_3} + |\xi|^{p-1}\right)\ .$$

(ii) Find, with the help of Sobolev imbedding theorem, other ways of weakening (H3) and keeping the conclusions of the theorem valid.

Exercise 3.4.2 *Let*

$$f(x,u,\xi)=\frac{1}{p}|\xi|^p+g(x,u)\ .$$

Find growth conditions (depending on p and n) on g that improve (H3) and still allow to derive, as in the preceding exercise, (E_w).

Exercise 3.4.3 *Let $n=2$, $\Omega=(0,\pi)^2$, $u=u(x,t)$, $u_t=\partial u/\partial t$, $u_x=\partial u/\partial x$ and*

$$(P)\quad \inf\left\{I(u)=\frac{1}{2}\iint_\Omega \left(u_t^2-u_x^2\right)\,dx\,dt : u\in W_0^{1,2}(\Omega)\right\}=m\,.$$

(i) Show that $m=-\infty$.

(ii) Prove, formally, that the Euler-Lagrange equation associated to (P) is the wave equation $u_{tt}-u_{xx}=0$.

3.5 The vectorial case

The problem under consideration is

$$(P)\quad \inf\left\{I(u)=\int_\Omega f(x,u(x),\nabla u(x))\,dx : u\in u_0+W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\}=m$$

where $n,N>1$ and

- $\Omega\subset\mathbb{R}^n$ is a bounded open set;
- $f:\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\longrightarrow\mathbb{R}$, $f=f(x,u,\xi)$;
- $u=\left(u^1,...,u^N\right)\in\mathbb{R}^N$, $\xi=\left(\xi_i^j\right)_{1\le i\le n}^{1\le j\le N}\in\mathbb{R}^{N\times n}$ and $\nabla u=\left(\frac{\partial u^j}{\partial x_i}\right)_{1\le i\le n}^{1\le j\le N}$;
- $u\in u_0+W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)$ means that $u^j,u_0^j\in W^{1,p}(\Omega)$, $j=1,...,N$, and $u-u_0\in W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)$ (which roughly means that $u=u_0$ on $\partial\Omega$).

All the results of the preceding sections apply to the present context when $n,N>1$. However, while for $N=1$ (or analogously when $n=1$) Theorem 3.3 was almost optimal, it is now far from being so. The vectorial case is intrinsically more difficult. For example the Euler-Lagrange equations associated to (P) are then a system of partial differential equations, whose treatment is considerably harder than that of a single partial differential equation.

We will present one extension of Theorem 3.3; it will not be the best possible result, but it has the advantage of giving some flavours of what can be done. For the sake of clarity we will essentially consider only the case $n=N=2$; but we will, in a remark, briefly mention what can be done in the higher dimensional case.

Theorem 3.19 *Let $n = N = 2$ and $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. Let $f : \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}$, $f = f(x,u,\xi)$, and $F : \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \times \mathbb{R} \longrightarrow \mathbb{R}$, $F = F(x,u,\xi,\delta)$, be continuous and satisfying*

$$f(x,u,\xi) = F(x,u,\xi,\det\xi)\,,\ \forall (x,u,\xi) \in \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$$

where $\det\xi$ denotes the determinant of the matrix ξ. Assume also that

(H1$_{vect}$) $(\xi,\delta) \to F(x,u,\xi,\delta)$ is convex for every $(x,u) \in \overline{\Omega} \times \mathbb{R}^2$;

(H2$_{vect}$) there exist $p > \max[q,2]$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$F(x,u,\xi,\delta) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3,\ \forall (x,u,\xi,\delta) \in \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \times \mathbb{R}\,.$$

Let $u_0 \in W^{1,p}\left(\Omega;\mathbb{R}^2\right)$ be such that $I(u_0) < \infty$, then (P) has at least one solution.

Remark 3.20 *(i) It is clear that from the point of view of convexity the theorem is more general than Theorem 3.3. Indeed if $\xi \to f(x,u,\xi)$ is convex then choose $F(x,u,\xi,\delta) = f(x,u,\xi)$ and therefore (H1$_{vect}$) and (H1) are equivalent. However (H1$_{vect}$) is more general since, for example, functions of the form*

$$f(x,u,\xi) = |\xi|^4 + (\det\xi)^4$$

can be shown to be non convex, while

$$F(x,u,\xi,\delta) = |\xi|^4 + \delta^4$$

is obviously convex as a function of (ξ,δ).

(ii) The theorem is however slightly weaker from the point of view of coercivity. Indeed in (H2$_{vect}$) we require $p > 2$, while in (H2) of Theorem 3.3 we only asked that $p > 1$.

(iii) Similar statements and proofs hold for the general case $n, N > 1$. For example when $n = N = 3$ we ask that there exists a function

$$F : \overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R} \longrightarrow \mathbb{R}, F = F(x,u,\xi,\eta,\delta)$$

so that

$$f(x,u,\xi) = F(x,u,\xi,\mathrm{adj}_2\xi,\det\xi)\,,\ \forall (x,u,\xi) \in \overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$$

where $\mathrm{adj}_2\xi$ denotes the matrix of cofactors of ξ (i.e. all the 2×2 minors of the matrix ξ). (H1$_{vect}$) becomes then: $(\xi,\eta,\delta) \to F(x,u,\xi,\eta,\delta)$ is convex for every $(x,u) \in \overline{\Omega} \times \mathbb{R}^3$; while (H2$_{vect}$) should hold for $p > \max[q,3]$.

(iv) When $n, N > 1$ the function f should be of the form

$$f(x, u, \xi) = F(x, u, \xi, \mathrm{adj}_2\xi, \mathrm{adj}_3\xi, ..., \mathrm{adj}_s\xi)$$

where $s = \min[n, N]$ and $\mathrm{adj}_r\xi$ denotes the matrix of all $r \times r$ minors of the matrix $\xi \in \mathbb{R}^{N\times n}$. The hypothesis (H1$_{vect}$) requires then that the function F be convex for every (x, u) fixed. The hypothesis (H2$_{vect}$) should then hold for $p > \max[q, s]$.

(v) A function f that can be written in terms of a convex function F as in the theorem is called polyconvex. *The theorem is due to Morrey (see also Ball [7] for important applications of such results to non linear elasticity). We refer for more details to [31].*

Let us now see two examples.

Example 3.21 *Let $n = N = 2$, $p > 2$ and*

$$f(x, u, \xi) = f(\xi) = \frac{1}{p}|\xi|^p + h(\det\xi)$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is non negative and convex (for example $h(\det\xi) = (\det\xi)^2$). All hypotheses of the theorem are clearly satisfied. It is also interesting to compute the Euler-Lagrange equations associated. To make them simple consider only the case $p = 2$ and set $u = u(x, y) = \left(u^1(x, y), u^2(x, y)\right)$. The system is then

$$\begin{cases} \Delta u^1 + \left[h'(\det\nabla u)\, u_y^2\right]_x - \left[h'(\det\nabla u)\, u_x^2\right]_y = 0 \\ \\ \Delta u^2 - \left[h'(\det\nabla u)\, u_y^1\right]_x + \left[h'(\det\nabla u)\, u_x^1\right]_y = 0\,. \end{cases}$$

Example 3.22 *Another important example coming from applications is the following: let $n = N = 3$, $p > 3$, $q \geq 1$ and*

$$f(x, u, \xi) = f(\xi) = \alpha|\xi|^p + \beta|\mathrm{adj}_2\xi|^q + h(\det\xi)$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is non negative and convex and $\alpha, \beta > 0$.

The key ingredient in the proof of the theorem is the following lemma that is due to Morrey and Reshetnyak.

Lemma 3.23 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, $p > 2$ and*

$$u^\nu = (\varphi^\nu, \psi^\nu) \rightharpoonup u = (\varphi, \psi) \text{ in } W^{1,p}\left(\Omega; \mathbb{R}^2\right);$$

then

$$\det\nabla u^\nu \rightharpoonup \det\nabla u \text{ in } L^{p/2}(\Omega).$$

Remark 3.24 *(i) At first glance the result is a little surprising. Indeed we have seen in Chapter 1 (in particular Exercise 1.3.3) that if two sequences, say* $(\varphi^\nu)_x$ *and* $(\psi^\nu)_y$, *converge weakly respectively to* φ_x *and* ψ_y, *then, in general, their product* $(\varphi^\nu)_x (\psi^\nu)_y$ *does not converge weakly to* $\varphi_x \psi_y$. *Writing*

$$\det \nabla u^\nu = (\varphi^\nu)_x (\psi^\nu)_y - (\varphi^\nu)_y (\psi^\nu)_x$$

we see that both terms $(\varphi^\nu)_x (\psi^\nu)_y$ *and* $(\varphi^\nu)_y (\psi^\nu)_x$ *do not, in general, converge weakly to* $\varphi_x \psi_y$ *and* $\varphi_y \psi_x$ *but, according to the lemma, their difference, which is* $\det \nabla u^\nu$, *converges weakly to their difference, namely* $\det \nabla u$. *We therefore have a non linear function, the determinant, that has the property to be weakly continuous. This is a very rare event (see for more details [30] or Theorem 4.2.6 in [31]).*

(ii) From Hölder inequality we see that, whenever $p \geq 2$ *and* $u \in W^{1,p}$ *then* $\det \nabla u \in L^{p/2}$.

(iii) The lemma is false if $1 \leq p \leq 2$ *but remains partially true if* $p > 4/3$; *this will be seen from the proof and from Exercise 3.5.5.*

(iv) The lemma generalizes to the case where $n, N > 1$ *and we obtain that any minor has this property (for example when* $n = N = 3$, *then any* 2×2 *minor and the determinant are weakly continuous). Moreover they are the only non linear functions which have the property of weak continuity.*

Proof. We have to show that for every $v \in \left(L^{p/2}\right)' = L^{p/(p-2)}$

$$\lim_{\nu \to \infty} \iint_\Omega \det \nabla u^\nu (x,y)\; v(x,y)\; dxdy = \iint_\Omega \det \nabla u (x,y)\; v(x,y)\; dxdy\,. \tag{3.8}$$

The proof will be divided into three steps. Only the first one carries the important information, namely that the determinant has a divergence structure; the two last steps are more technical. We also draw the attention on a technical fact about the exponent p. The first step can also be proved if $p > 4/3$ (cf. also Exercise 3.5.5). The second, in fact, requires that $p \geq 2$ and only the last one fully uses the strict inequality $p > 2$. However, in order not to burden the proof too much, we will always assume that $p > 2$.

Step 1. We first prove (3.8) under the further hypotheses that $v \in C_0^\infty(\Omega)$ and $u^\nu, u \in C^2\left(\overline{\Omega}; \mathbb{R}^2\right)$.

We start by proving a preliminary result. If we let $v \in C_0^\infty(\Omega)$ and $w \in C^2\left(\overline{\Omega}; \mathbb{R}^2\right)$, $w = (\varphi, \psi)$, we always have

$$\iint_\Omega \det \nabla w \, v \, dxdy = -\iint_\Omega \left[\varphi \psi_y v_x - \varphi \psi_x v_y\right] dxdy\,. \tag{3.9}$$

Indeed, using the fact that $\varphi, \psi \in C^2$, we obtain

$$\det \nabla w = \varphi_x \psi_y - \varphi_y \psi_x = \left(\varphi \psi_y\right)_x - \left(\varphi \psi_x\right)_y \tag{3.10}$$

and therefore, since $v \in C_0^\infty(\Omega)$, we have

$$\iint_\Omega \det \nabla w \, v \, dxdy = \iint_\Omega \left[\left(\varphi \psi_y\right)_x v - \left(\varphi \psi_x\right)_y v \right] dxdy$$

and hence (3.9) after integration by parts.

The result (3.8) then easily follows. Indeed from Rellich theorem (Theorem 1.43) we have, since $\varphi^\nu \rightharpoonup \varphi$ in $W^{1,p}$ and $p > 2$, that $\varphi^\nu \to \varphi$ in L^∞. Combining this observation with the fact that $\psi_x^\nu, \psi_y^\nu \rightharpoonup \psi_x, \psi_y$ in L^p we deduce (cf. Exercise 1.3.3) that

$$\varphi^\nu \psi_x^\nu, \varphi^\nu \psi_y^\nu \rightharpoonup \varphi \psi_x, \varphi \psi_y \text{ in } L^p. \tag{3.11}$$

Since $v_x, v_y \in C_0^\infty \subset L^{p'}$ we deduce from (3.9), applied to $w = u^\nu$, and from (3.11) that

$$\lim_{\nu \to \infty} \iint_\Omega \det \nabla u^\nu \, v \, dxdy = -\iint_\Omega \left[\varphi \psi_y v_x - \varphi \psi_x v_y \right] dxdy.$$

Using again (3.9), applied to $w = u$, we have indeed obtained the claimed result (3.8).

Step 2. We now show that (3.8) still holds under the further hypothesis $v \in C_0^\infty(\Omega)$, but considering now the general case, i.e. $u^\nu, u \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$.

In fact (3.9) continues to hold under the weaker hypothesis that $v \in C_0^\infty(\Omega)$ and $w \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$; of course the proof must be different, since this time we only know that $w \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$. Let us postpone for a moment the proof of this fact and observe that if (3.9) holds for $w \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$ then, with exactly the same argument as in the previous step, we get (3.8) under the hypotheses $v \in C_0^\infty(\Omega)$ and $u^\nu, u \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$.

We now prove the above claim and we start by regularizing $w \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$ appealing to Theorem 1.34. We therefore find for every $\epsilon > 0$, a function $w^\epsilon = (\varphi^\epsilon, \psi^\epsilon) \in C^2\left(\overline{\Omega}; \mathbb{R}^2\right)$ so that

$$\|w - w^\epsilon\|_{W^{1,p}} \leq \epsilon \text{ and } \|w - w^\epsilon\|_{L^\infty} \leq \epsilon.$$

Since $p \geq 2$ we can find (cf. Exercise 3.5.4) a constant α_1 (independent of ϵ) so that

$$\|\det \nabla w - \det \nabla w^\epsilon\|_{L^{p/2}} \leq \alpha_1 \epsilon. \tag{3.12}$$

It is also easy to see that we have, for α_2 a constant (independent of ϵ),

$$\left\|\varphi \psi_y - \varphi^\epsilon \psi_y^\epsilon\right\|_{L^p} \leq \alpha_2 \epsilon, \ \|\varphi \psi_x - \varphi^\epsilon \psi_x^\epsilon\|_{L^p} \leq \alpha_2 \epsilon \tag{3.13}$$

since, for example, the first inequality follows from

$$\left\|\varphi \psi_y - \varphi^\epsilon \psi_y^\epsilon\right\|_{L^p} \leq \|\varphi\|_{L^\infty} \left\|\psi_y - \psi_y^\epsilon\right\|_{L^p} + \left\|\psi_y^\epsilon\right\|_{L^p} \|\varphi - \varphi^\epsilon\|_{L^\infty}.$$

Returning to (3.9) we have

$$\begin{aligned}
&\iint_\Omega \det \nabla w\, v\, dxdy + \iint_\Omega \left[\varphi\psi_y v_x - \varphi\psi_x v_y\right] dxdy \\
= \quad &\iint_\Omega \det \nabla w^\epsilon\, v\, dxdy + \iint_\Omega \left[\varphi^\epsilon\psi_y^\epsilon v_x - \varphi^\epsilon\psi_x^\epsilon v_y\right] dxdy \\
&+ \iint_\Omega \left(\det \nabla w - \det \nabla w^\epsilon\right) v\, dxdy \\
&+ \iint_\Omega \left[\left(\varphi\psi_y - \varphi^\epsilon\psi_y^\epsilon\right) v_x - \left(\varphi\psi_x - \varphi^\epsilon\psi_x^\epsilon\right) v_y\right] dxdy\,.
\end{aligned}$$

Appealing to (3.9) which has already been proved to hold for $w^\epsilon = (\varphi^\epsilon, \psi^\epsilon) \in C^2$, to Hölder inequality, to (3.12) and to (3.13) we find that, α_3 being a constant independent of ϵ,

$$\begin{aligned}
&\left|\iint_\Omega \det \nabla w\, v\, dxdy + \iint_\Omega \left[\varphi\psi_y v_x - \varphi\psi_x v_y\right] dxdy\right| \\
\leq \quad &\alpha_3 \epsilon \left[\|v\|_{\left(L^{p/2}\right)'} + \|v_x\|_{L^{p'}} + \|v_y\|_{L^{p'}}\right].
\end{aligned}$$

Since ϵ is arbitrary we have indeed obtained that (3.9) is also valid for $w \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$.

Step 3. We are finally in a position to prove the lemma, removing the last unnecessary hypothesis ($v \in C_0^\infty(\Omega)$). We want (3.8) to hold for $v \in L^{p/(p-2)}$. This is obtained by regularizing the function as in Theorem 1.13. This means, for every $\epsilon > 0$ and $v \in L^{p/(p-2)}$, that we can find $v^\epsilon \in C_0^\infty(\Omega)$ so that

$$\|v - v^\epsilon\|_{L^{p/(p-2)}} \leq \epsilon\,. \tag{3.14}$$

We moreover have

$$\iint_\Omega \det \nabla u^\nu\, v\, dxdy = \iint_\Omega \det \nabla u^\nu\, \left(v - v^\epsilon\right) dxdy + \iint_\Omega \det \nabla u^\nu\, v^\epsilon\, dxdy\,.$$

Using, once more, Hölder inequality we find

$$\left|\iint_\Omega \left(\det \nabla u^\nu - \det \nabla u\right) v\right|$$

$$\leq \|v - v^\epsilon\|_{L^{p/(p-2)}} \|\det \nabla u^\nu - \det \nabla u\|_{L^{p/2}} + \left|\iint_\Omega \left(\det \nabla u^\nu - \det \nabla u\right) v^\epsilon\right|.$$

The previous step has shown that

$$\lim_{\nu\to\infty} \left|\iint_\Omega \left(\det \nabla u^\nu - \det \nabla u\right) v^\epsilon\right| = 0$$

while (3.14), the fact that $u^\nu \rightharpoonup u$ in $W^{1,p}$ and Exercise 3.5.4 show that we can find $\gamma > 0$ so that

$$\|v - v^\epsilon\|_{L^{p/(p-2)}} \|\det \nabla u^\nu - \det \nabla u\|_{L^{p/2}} \leq \gamma\epsilon\,.$$

Since ϵ is arbitrary we have indeed obtained that (3.8) holds for $v \in L^{p/(p-2)}$ and for $u^\nu \rightharpoonup u$ in $W^{1,p}$. The lemma is therefore proved. ■

We can now proceed with the proof of Theorem 3.19.

Proof. We will prove the theorem under the further following hypotheses (for a general proof see Theorem 4.2.10 in [31])

$$f(x, u, \xi) = g(x, u, \xi) + h(x, \det \xi)$$

where g satisfies (H1) and (H2), with $p > 2$, of Theorem 3.3 and $h \in C^1\left(\overline{\Omega} \times \mathbb{R}\right)$, $h \geq 0$, $\delta \to h(x, \delta)$ is convex for every $x \in \overline{\Omega}$ and there exists $\gamma > 0$ so that

$$|h_\delta(x, \delta)| \leq \gamma\left(1 + |\delta|^{(p-2)/2}\right)\,. \tag{3.15}$$

The proof is then identical to the one of Theorem 3.3, except the second step (the weak lower semicontinuity), that we discuss now. We have to prove that

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,p} \;\Rightarrow\; \liminf_{\nu\to\infty} I(u_\nu) \geq I(\overline{u})$$

where $I(u) = G(u) + H(u)$ with

$$G(u) = \int_\Omega g(x, u(x), \nabla u(x))\, dx,\;\; H(u) = \int_\Omega h(x, \det \nabla u(x))\, dx\,.$$

We have already proved in Theorem 3.3 that

$$\liminf_{\nu\to\infty} G(u_\nu) \geq G(\overline{u})$$

and therefore the result will follow if we can show

$$\liminf_{\nu\to\infty} H(u_\nu) \geq H(\overline{u})\,.$$

Since h is convex and C^1 we have

$$h(x, \det \nabla u_\nu) \geq h(x, \det \nabla \overline{u}) + h_\delta(x, \det \nabla \overline{u})(\det \nabla u_\nu - \det \nabla \overline{u})\,. \tag{3.16}$$

We know that $\overline{u} \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$, which implies that $\det \nabla \overline{u} \in L^{p/2}(\Omega)$, and hence using (3.15) we deduce that

$$h_\delta(x, \det \nabla \overline{u}) \in L^{p/(p-2)} = L^{(p/2)'}, \tag{3.17}$$

since we can find a constant $\gamma_1 > 0$ so that

$$\begin{aligned} |h_\delta(x, \det \nabla \overline{u})|^{p/(p-2)} &\leq \left[\gamma\left(1 + |\det \nabla \overline{u}|^{(p-2)/2}\right)\right]^{p/(p-2)} \\ &\leq \gamma_1 \left(1 + |\det \nabla \overline{u}|^{p/2}\right). \end{aligned}$$

Returning to (3.16) and integrating we get

$$H(u_\nu) \geq H(\overline{u}) + \int_\Omega h_\delta(x, \det \nabla \overline{u})(\det \nabla u_\nu - \det \nabla \overline{u})\, dx\,.$$

Since $u_\nu - \overline{u} \rightharpoonup 0$ in $W^{1,p}$, $p > 2$, we have from Lemma 3.23 that $\det \nabla u_\nu - \det \nabla \overline{u} \rightharpoonup 0$ in $L^{p/2}$ which combined with (3.17) and the definition of weak convergence in $L^{p/2}$ lead to

$$\lim_{\nu\to\infty} \int_\Omega h_\delta(x, \det \nabla \overline{u})(\det \nabla u_\nu - \det \nabla \overline{u})\, dx = 0\,.$$

We have therefore obtained that

$$\liminf_{\nu\to\infty} H(u_\nu) \geq H(\overline{u})$$

and the proof is complete. ■

3.5.1 Exercises

The exercises will focus on several important analytical properties of the determinant. Although we will essentially deal only with the two dimensional case, most results, when properly adapted, remain valid in the higher dimensional cases.

We will need in some of the exercises the following definition.

Definition 3.25 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u_\nu, u \in L^1_{loc}(\Omega)$. We say that u_ν* converges in the sense of distributions *to u, and we denote it by $u_\nu \rightharpoonup u$ in $\mathcal{D}'(\Omega)$, if*

$$\lim_{\nu\to\infty} \int_\Omega u_\nu \varphi\, dx = \int_\Omega u\varphi\, dx,\ \forall \varphi \in C_0^\infty(\Omega)\,.$$

Remark 3.26 *(i) If Ω is bounded we then have the following relations*

$$u_\nu \overset{*}{\rightharpoonup} u \text{ in } L^\infty \Rightarrow u_\nu \rightharpoonup u \text{ in } L^1 \Rightarrow u_\nu \rightharpoonup u \text{ in } \mathcal{D}'\,.$$

(ii) The definition can be generalized to u_ν and u that are not necessarily in $L^1_{loc}(\Omega)$, but are merely what is known as "distributions", cf. Exercise 3.5.6.

Exercise 3.5.1 *Show that* $f(\xi) = (\det \xi)^2$*, where* $\xi \in \mathbb{R}^{2\times 2}$*, is not convex.*

Exercise 3.5.2 *Show that if* $\Omega \subset \mathbb{R}^2$ *is a bounded open set with Lipschitz boundary and if* $u \in v + W_0^{1,p}(\Omega;\mathbb{R}^2)$*, with* $p \geq 2$*, then*

$$\iint_\Omega \det \nabla u \, dxdy = \iint_\Omega \det \nabla v \, dxdy \, .$$

Suggestion: Prove first the result for $u, v \in C^2(\overline{\Omega};\mathbb{R}^2)$ *with* $u = v$ *on* $\partial\Omega$*.*

Exercise 3.5.3 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded open set with Lipschitz boundary,* $u_0 \in W^{1,p}(\Omega;\mathbb{R}^2)$*, with* $p \geq 2$*, and*

$$(P) \quad \inf \left\{ I(u) = \iint_\Omega \det \nabla u(x) \; dx : u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^2) \right\} = m \, .$$

Write the Euler-Lagrange equation associated to (P). Is the result totally surprising?

Exercise 3.5.4 *Let* $u, v \in W^{1,p}(\Omega;\mathbb{R}^2)$*, with* $p \geq 2$*. Show that there exists* $\alpha > 0$ *(depending only on p) so that*

$$\|\det \nabla u - \det \nabla v\|_{L^{p/2}} \leq \alpha \left(\|\nabla u\|_{L^p} + \|\nabla v\|_{L^p} \right) \|\nabla u - \nabla v\|_{L^p} \; .$$

Exercise 3.5.5 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded open set with Lipschitz boundary. We have seen in Lemma 3.23 that, if* $p > 2$*, then*

$$u^\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^2) \Rightarrow \det \nabla u^\nu \rightharpoonup \det \nabla u \text{ in } L^{p/2}(\Omega) \; .$$

(i) Show that the result is, in general, false if $p = 2$*. To achieve this goal choose, for example,* $\Omega = (0,1)^2$ *and*

$$u^\nu(x,y) = \frac{1}{\sqrt{\nu}} (1-y)^\nu (\sin \nu x, \cos \nu x) \; .$$

(ii) Show, using Rellich theorem (Theorem 1.43), that if $u^\nu, u \in C^2(\overline{\Omega};\mathbb{R}^2)$ *and if* $p > 4/3$ *(so in particular for* $p = 2$*), then*

$$u^\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^2) \Rightarrow \det \nabla u^\nu \rightharpoonup \det \nabla u \text{ in } \mathcal{D}'(\Omega) \; .$$

(iii) This last result is false if $p \leq 4/3$*, see Dacorogna-Murat [34].*

Exercise 3.5.6 *Let* $\Omega = \left\{ x \in \mathbb{R}^2 : |x| < 1 \right\}$ *and* $u(x) = x/|x|$*.*

(i) Show that $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ *for every* $1 \leq p < 2$ *(observe, however, that* $u \notin W^{1,2}$ *and* $u \notin C^0$*).*

(ii) Let $u^{\nu}(x) = x/(|x| + 1/\nu)$. *Show that* $u^{\nu} \rightharpoonup u$ *in* $W^{1,p}$, *for any* $1 \leq p < 2$.

(iii) Let $\delta_{(0,0)}$ *be the Dirac mass at* $(0,0)$, *which means*

$$\left\langle \delta_{(0,0)}; \varphi \right\rangle = \varphi(0,0), \ \forall \varphi \in C_0^{\infty}(\Omega) .$$

Prove that

$$\det \nabla u^{\nu} \rightharpoonup \pi \delta_{(0,0)} \text{ in } \mathcal{D}'(\Omega) .$$

3.6 Relaxation theory

Recall that the problem under consideration is

$$(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W_0^{1,p}(\Omega) \right\} = m$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary;
- $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $f = f(x, u, \xi)$, is continuous, uniformly in u with respect to ξ;
- $u_0 \in W^{1,p}(\Omega)$ with $I(u_0) < \infty$.

Before stating the main theorem, let us recall some facts from Section 1.5.

Remark 3.27 *The convex envelope of* f, *with respect to the variable* ξ, *will be denoted by* f^{**}. *It is the largest convex function (with respect to the variable* ξ*) which is smaller than* f. *In other words*

$$g(x, u, \xi) \leq f^{**}(x, u, \xi) \leq f(x, u, \xi), \ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

for every convex function g *(*$\xi \rightarrow g(x, u, \xi)$ *is convex),* $g \leq f$. *We have two ways of computing this function.*

(i) From the duality theorem (Theorem 1.54) we have, for every $(x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$,

$$f^{*}(x, u, \xi^{*}) = \sup_{\xi \in \mathbb{R}^n} \left\{ \langle \xi; \xi^{*} \rangle - f(x, u, \xi) \right\}$$

$$f^{**}(x, u, \xi) = \sup_{\xi^{*} \in \mathbb{R}^n} \left\{ \langle \xi; \xi^{*} \rangle - f^{*}(x, u, \xi^{*}) \right\} .$$

(ii) From Carathéodory theorem (Theorem 1.55) we have, for every $(x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$,

$$f^{**}(x, u, \xi) = \inf \left\{ \sum_{i=1}^{n+1} \lambda_i f(x, u, \xi_i) : \xi = \sum_{i=1}^{n+1} \lambda_i \xi_i \, , \ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{n+1} \lambda_i = 1 \right\} .$$

We have seen in Theorem 3.3 that the existence of minimizers of (P) depend strongly on the two hypotheses (H1) and (H2). We now briefly discuss the case where (H1) does not hold, i.e. the function $\xi \to f(x, u, \xi)$ is not anymore convex. We have seen in several examples that in general (P) will have no minimizers. We present here a way of defining a "generalized" solution of (P). The main theorem (without proof) is the following.

Theorem 3.28 *Let Ω, f, f^{**} and u_0 be as above. Let $p > 1$ and α_1 be such that*

$$0 \leq f(x, u, \xi) \leq \alpha_1 (1 + |u|^p + |\xi|^p), \ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

Finally let

$$(\overline{P}) \quad \inf \left\{ \overline{I}(u) = \int_\Omega f^{**}(x, u(x), \nabla u(x))\, dx : u \in u_0 + W_0^{1,p}(\Omega) \right\} = \overline{m}\,.$$

Then

(i) $\overline{m} = m$;

(ii) for every $u \in u_0 + W_0^{1,p}(\Omega)$, *there exists* $u_\nu \in u_0 + W_0^{1,p}(\Omega)$ *so that*

$$u_\nu \rightharpoonup u \text{ in } W^{1,p} \text{ and } I(u_\nu) \to \overline{I}(u), \text{ as } \nu \to \infty\,.$$

If, in addition, there exist $\alpha_2 > 0$, $\alpha_3 \in \mathbb{R}$ such that

$$f(x, u, \xi) \geq \alpha_2 |\xi|^p + \alpha_3, \ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

then $(\overline{P})$ has at least one solution $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$.

Remark 3.29 *(i) If f satisfies*

$$f(x, u, \xi) \geq \alpha_2 |\xi|^p + \alpha_3, \ \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

then its convex envelope f^{**} satisfies the same inequality since $\xi \to \alpha_2 |\xi|^p + \alpha_3 \equiv h(\xi)$ is convex and $h \leq f$. This observation implies that f^{**} verifies (H1) and (H2) of Theorem 3.3 and therefore the existence of a minimizer of $(\overline{P})$ is guaranteed.

(ii) Theorem 3.28 allows therefore to define $\overline{u}$ as a generalized solution of (P), even though (P) may have no minimizer in $W^{1,p}$.

(iii) The theorem has been established by L.C. Young, Mac Shane and as stated by Ekeland (see Theorem 10.3.7 in Ekeland-Témam [41], Corollary 3.13 in Marcellini-Sbordone [71] or [31]). It is false in the vectorial case (see Example 3.31 below). However the author in [29] (see Theorem 5.2.1 in [31]) has shown that a result in the same spirit can be proved.

We conclude this section with two examples.

Example 3.30 *Let us return to Bolza example (Example 3.10). We here have* $n = 1$,

$$f(x, u, \xi) = f(u, \xi) = \left(\xi^2 - 1\right)^2 + u^4$$

$$(P) \quad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x))\, dx : u \in W_0^{1,4}(0,1) \right\} = m\,.$$

We have already shown that $m = 0$ *and that (P) has no solution. An elementary computation (cf. Example 1.53 (ii)) shows that*

$$f^{**}(u, \xi) = \begin{cases} f(u, \xi) & \text{if } |\xi| \geq 1 \\ u^4 & \text{if } |\xi| < 1\,. \end{cases}$$

Therefore $\overline{u} \equiv 0$ *is a solution of*

$$\left(\overline{P}\right) \quad \inf \left\{ \overline{I}(u) = \int_0^1 f^{**}(u(x), u'(x))\, dx : u \in W_0^{1,4}(0,1) \right\} = \overline{m} = 0.$$

The sequence $u_\nu \in W_0^{1,4}$ *(*$\nu \geq 2$ *being an integer) constructed in Example 3.10 satisfies the conclusions of the theorem, i.e.*

$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,4} \text{ and } I(u_\nu) \to \overline{I}(\overline{u}) = 0, \text{ as } \nu \to \infty\,.$$

Example 3.31 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded open set with Lipschitz boundary. Let* $u_0 \in W^{1,4}\left(\Omega; \mathbb{R}^2\right)$ *be such that*

$$\iint_\Omega \det \nabla u_0(x)\, dx \neq 0\,.$$

Let, for $\xi \in \mathbb{R}^{2\times 2}$, $f(\xi) = (\det \xi)^2$,

$$(P) \quad \inf \left\{ I(u) = \iint_\Omega f(\nabla u(x))\, dx : u \in u_0 + W_0^{1,4}\left(\Omega; \mathbb{R}^2\right) \right\} = m$$

$$(\overline{P}) \quad \inf \left\{ \overline{I}(u) = \iint_\Omega f^{**}(\nabla u(x))\, dx : u \in u_0 + W_0^{1,4}\left(\Omega; \mathbb{R}^2\right) \right\} = \overline{m}\,.$$

We will show that Theorem 3.28 is false, by proving that $m > \overline{m}$. *Indeed it is easy to prove (cf. Exercise 3.6.3) that* $f^{**}(\xi) \equiv 0$, *which therefore implies that* $\overline{m} = 0$. *Let us show that* $m > 0$. *Indeed by Jensen inequality (cf. Theorem 1.51) we have, for every* $u \in u_0 + W_0^{1,4}\left(\Omega; \mathbb{R}^2\right)$,

$$\iint_\Omega (\det \nabla u(x))^2\, dx \geq \operatorname{meas} \Omega \left(\frac{1}{\operatorname{meas} \Omega} \iint_\Omega \det \nabla u(x)\, dx \right)^2.$$

Appealing to Exercise 3.5.2 we therefore find that

$$\iint_{\Omega} (\det \nabla u(x))^2 \, dx \geq \frac{1}{\operatorname{meas}\Omega} \left(\iint_{\Omega} \det \nabla u_0(x) \, dx \right)^2 .$$

The right hand side being strictly positive, by hypothesis, we have indeed found that $m > 0$, which leads to the claimed counterexample.

3.6.1 Exercises

Exercise 3.6.1 *Let $n = 1$ and*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(u'(x)) \, dx \right\} = m$$

where $X = \left\{ u \in W^{1,\infty}(0,1) : u(0) = \alpha, \ u(1) = \beta \right\}$.

(i) Assume that there exist $\lambda \in [0,1]$, $a, b \in \mathbb{R}$ such that

$$\begin{cases} \beta - \alpha = \lambda a + (1-\lambda) b \\ \\ f^{**}(\beta - \alpha) = \lambda f(a) + (1-\lambda) f(b) \,. \end{cases}$$

Show then that (P) has a solution, independently of wether f is convex or not (compare the above relations with Theorem 1.55). Of course if f is convex, the above hypothesis is always true, it suffices to choose $\lambda = 1/2$ and $a = b = \beta - \alpha$.

(ii) Can we apply the above considerations to $f(\xi) = e^{-\xi^2}$ (cf. Section 2.2) and $\alpha = \beta = 0$?

(iii) What happens when $f(\xi) = \left(\xi^2 - 1\right)^2$?

Exercise 3.6.2 *Apply the theorem to $\Omega = (0,1)^2$,*

$$f(x, u, \xi) = f(\xi) = \left((\xi_1)^2 - 1 \right)^2 + (\xi_2)^4$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and

$$(P) \quad \inf \left\{ I(u) = \iint_{\Omega} f(\nabla u(x,y)) \, dxdy : u \in W_0^{1,4}(\Omega) \right\} = m \,.$$

Exercise 3.6.3 *Let $f(\xi) = (\det \xi)^2$, where $\xi \in \mathbb{R}^{2 \times 2}$. Show that $f^{**}(\xi) \equiv 0$.*

Chapter 4

Regularity

4.1 Introduction

We are still considering the problem

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx : u \in u_0 + W_0^{1,p}(\Omega) \right\} = m$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded open set;
- $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $f = f(x, u, \xi)$;
- $u \in u_0 + W_0^{1,p}(\Omega)$ means that $u, u_0 \in W^{1,p}(\Omega)$ and $u - u_0 \in W_0^{1,p}(\Omega)$.

We have shown in Chapter 3 that, under appropriate hypotheses on f, u_0 and Ω, (P) has a minimizer $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$.

The question that we will discuss now is to determine whether, in fact, the minimizer $\overline{u}$ is not more *regular*, for example $C^1\left(\overline{\Omega}\right)$. More precisely if the data f, u_0 and Ω are sufficiently regular, say C^∞, does $\overline{u} \in C^\infty$? This is one of the 23 problems of Hilbert that were mentioned in Chapter 0.

The case $n = 1$ will be discussed in Section 4.2. We will obtain some general results. We will then turn our attention to the higher dimensional case. This is a considerably harder problem and we will treat only the case of the Dirichlet integral in Section 4.3. We will in Section 4.4 give, without proofs, some general theorems.

We should also point out that all the regularity results that we will obtain here are about solutions of the Euler-Lagrange equation and therefore not only minimizers of (P).

The problem of regularity, including the closely related ones concerning regularity for elliptic partial differential equations, is a difficult one that has attracted many mathematicians. We quote only a few of them: Agmon, Bernstein, Calderon, De Giorgi, Douglis, E. Hopf, Leray, Liechtenstein, Morrey, Moser, Nash, Nirenberg, Rado, Schauder, Tonelli, Weyl and Zygmund.

In addition to the books that were mentioned in Chapter 3 one can consult those by Gilbarg-Trudinger [49] and Ladyzhenskaya-Uraltseva [66].

4.2 The one dimensional case

Let us restate the problem. We consider

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x), u'(x))\, dx \right\} = m$$

where $X = \left\{ u \in W^{1,p}(a,b) : u(a) = \alpha,\ u(b) = \beta \right\}$, $f \in C^0([a,b] \times \mathbb{R} \times \mathbb{R})$, $f = f(x,u,\xi)$.

We have seen that if f satisfies

(H1) $\xi \to f(x,u,\xi)$ is convex for every $(x,u) \in [a,b] \times \mathbb{R}$;

(H2) there exist $p > q \geq 1$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$f(x,u,\xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3,\ \forall (x,u,\xi) \in [a,b] \times \mathbb{R} \times \mathbb{R};$$

then (P) has a solution $\overline{u} \in X$.

If, furthermore, $f \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ and verifies (cf. Remark 3.12)

(H3') for every $R > 0$, there exists $\alpha_4 = \alpha_4(R)$ such that

$$|f_u(x,u,\xi)|,\ |f_\xi(x,u,\xi)| \leq \alpha_4 (1 + |\xi|^p),\ \forall (x,u,\xi) \in [a,b] \times [-R,R] \times \mathbb{R}.$$

then any minimizer $\overline{u} \in X$ satisfies the weak form of the Euler-Lagrange equation

$$(E_w) \quad \int_a^b [f_u(x,\overline{u},\overline{u}')\, v + f_\xi(x,\overline{u},\overline{u}')\, v']\, dx = 0,\ \forall v \in C_0^\infty(a,b)\ .$$

We will show that under some strengthening of the hypotheses, we have that if $f \in C^\infty$ then $\overline{u} \in C^\infty$. These results are, in part, also valid if $u : [a,b] \to \mathbb{R}^N$, for $N > 1$.

We start with a very elementary result that will illustrate our purpose.

Proposition 4.1 *Let $g \in C^{\infty}([a,b] \times \mathbb{R})$ satisfy*

(H2) there exist $2 > q \geq 1$ and $\alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$g(x,u) \geq \alpha_2 |u|^q + \alpha_3,\ \forall (x,u) \in [a,b] \times \mathbb{R}.$$

Let

$$f(x,u,\xi) = \frac{1}{2}\xi^2 + g(x,u) \ .$$

Then there exists $\overline{u} \in C^{\infty}([a,b])$, a minimizer of (P). If, in addition, $u \to g(x,u)$ is convex for every $x \in [a,b]$, then the minimizer is unique.

Proof. The existence (and uniqueness, if g is convex) of a solution $\overline{u} \in W^{1,2}(a,b)$ follows from Theorem 3.3. We also know from Theorem 3.11 that it satisfies the weak form of the Euler-Lagrange equation

$$\int_a^b \overline{u}' v' \, dx = -\int_a^b g_u(x,\overline{u})\, v \, dx,\ \forall v \in C_0^{\infty}(a,b) \ . \tag{4.1}$$

To prove further regularity of $\overline{u}$, we start by showing that $\overline{u} \in W^{2,2}(a,b)$. This follows immediately from (4.1) and from the definition of weak derivative. Indeed since $\overline{u} \in W^{1,2}$, we have that $\overline{u} \in L^{\infty}$ and thus $g_u(x,\overline{u}) \in L^2$, leading to

$$\left| \int_a^b \overline{u}' v' \, dx \right| \leq \|g_u(x,\overline{u})\|_{L^2} \|v\|_{L^2} \ ,\ \forall v \in C_0^{\infty}(a,b) \ . \tag{4.2}$$

Theorem 1.36 implies then that $\overline{u} \in W^{2,2}$. We can then integrate by parts (4.1), bearing in mind that $v(a) = v(b) = 0$, and using the fundamental lemma of the calculus of variations (cf. Theorem 1.24), we deduce that

$$\overline{u}''(x) = g_u(x,\overline{u}(x)),\ \text{a.e. } x \in (a,b) \ . \tag{4.3}$$

We are now in a position to start an iteration process. Since $\overline{u} \in W^{2,2}(a,b)$ we deduce that (cf. Theorem 1.42) $\overline{u} \in C^1([a,b])$ and hence the function $x \to g_u(x,\overline{u}(x))$ is $C^1([a,b])$, g being C^{∞}. Returning to (4.3) we deduce that $\overline{u}'' \in C^1$ and hence $\overline{u} \in C^3$. From there we can infer that $x \to g_u(x,\overline{u}(x))$ is C^3, and thus from (4.3) we obtain that $\overline{u}'' \in C^3$ and hence $\overline{u} \in C^5$. Continuing this process we have indeed established that $\overline{u} \in C^{\infty}([a,b])$. ■

We will now generalize the argument of the proposition and we start with a lemma.

Lemma 4.2 *Let $f \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ satisfy (H1), (H2) and (H3'). Then any minimizer $\overline{u} \in W^{1,p}(a,b)$ of (P) is in fact in $W^{1,\infty}(a,b)$ and the Euler-Lagrange equation holds almost everywhere, i.e.*

$$\frac{d}{dx}\left[f_{\xi}(x,\overline{u},\overline{u}')\right] = f_u(x,\overline{u},\overline{u}') \ ,\ a.e.\ x \in (a,b) \ .$$

Proof. We know from Remark 3.12 that the following equation holds

$$(E_w) \quad \int_a^b \left[f_u\left(x,\overline{u},\overline{u}'\right)v + f_\xi\left(x,\overline{u},\overline{u}'\right)v'\right]\,dx = 0,\ \forall v \in C_0^\infty\left(a,b\right). \tag{4.4}$$

We then divide the proof into two steps.

Step 1. Define

$$\varphi\left(x\right) = f_\xi\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right) \text{ and } \psi\left(x\right) = f_u\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right).$$

We easily see that $\varphi \in W^{1,1}\left(a,b\right)$ and that $\varphi'\left(x\right) = \psi\left(x\right)$, for almost every $x \in \left(a,b\right)$, which means that

$$\frac{d}{dx}\left[f_\xi\left(x,\overline{u},\overline{u}'\right)\right] = f_u\left(x,\overline{u},\overline{u}'\right), \text{ a.e. } x \in \left(a,b\right). \tag{4.5}$$

Indeed since $\overline{u} \in W^{1,p}\left(a,b\right)$, and hence $\overline{u} \in L^\infty\left(a,b\right)$, we deduce from (H3') that $\psi \in L^1\left(a,b\right)$. We also have from (4.4) that

$$\int_a^b \psi\left(x\right)v\left(x\right)\,dx = -\int_a^b \varphi\left(x\right)v'\left(x\right)\,dx,\ \forall v \in C_0^\infty\left(a,b\right).$$

Since $\varphi \in L^1\left(a,b\right)$ (from (H3')), we have by definition of weak derivatives the claim, namely $\varphi \in W^{1,1}\left(a,b\right)$ and $\varphi' = \psi$ a.e..

Step 2. Since $\varphi \in W^{1,1}\left(a,b\right)$, we have that $\varphi \in C^0\left(\left[a,b\right]\right)$ which means that there exists a constant $\alpha_5 > 0$ so that

$$\left|\varphi\left(x\right)\right| = \left|f_\xi\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right)\right| \le \alpha_5,\ \forall x \in \left[a,b\right]. \tag{4.6}$$

Since $\overline{u}$ is bounded (and even continuous), let us say $\left|\overline{u}\left(x\right)\right| \le R$ for every $x \in \left[a,b\right]$, we have from (H1) that

$$f\left(x,u,0\right) \ge f\left(x,u,\xi\right) - \xi f_\xi\left(x,u,\xi\right),\ \forall\left(x,u,\xi\right) \in \left[a,b\right]\times\left[-R,R\right]\times\mathbb{R}.$$

Combining this inequality with (H2) we find that there exists $\alpha_6 \in \mathbb{R}$ such that, for every $\left(x,u,\xi\right) \in \left[a,b\right]\times\left[-R,R\right]\times\mathbb{R}$,

$$\xi f_\xi\left(x,u,\xi\right) \ge f\left(x,u,\xi\right) - f\left(x,u,0\right) \ge \alpha_1\left|\xi\right|^p + \alpha_6.$$

Using (4.6) and the above inequality we find

$$\alpha_1\left|\overline{u}'\right|^p + \alpha_6 \le \overline{u}' f_\xi\left(x,\overline{u},\overline{u}'\right) \le \left|\overline{u}'\right|\left|f_\xi\left(x,\overline{u},\overline{u}'\right)\right| \le \alpha_5\left|\overline{u}'\right|, \text{ a.e. } x \in \left(a,b\right)$$

which implies, since $p > 1$, that $\left|\overline{u}'\right|$ is uniformly bounded. Thus the lemma. ■

Theorem 4.3 *Let $f \in C^{\infty}([a,b] \times \mathbb{R} \times \mathbb{R})$ satisfy (H2), (H3') and*

$$\text{(H1')} \quad f_{\xi\xi}(x,u,\xi) > 0,\ \forall (x,u,\xi) \in [a,b] \times \mathbb{R} \times \mathbb{R}\ .$$

Then any minimizer of (P) is in $C^{\infty}([a,b])$.

Remark 4.4 *(i) Note that (H1') is more restrictive than (H1). This stronger condition is usually, but not always as will be seen in Theorem 4.5, necessary to get higher regularity.*

(ii) Proposition 4.1 is, of course, a particular case of the present theorem.

(iii) The proof will show that if $f \in C^k$, $k \geq 2$, then the minimizer is also C^k.

Proof. We will propose a different proof in Exercise 4.2.1. The present one is more direct and uses Lemma 2.8.

Step 1. We know from Lemma 4.2 that $x \to \varphi(x) = f_{\xi}(x, \overline{u}(x), \overline{u}'(x))$ is in $W^{1,1}(a,b)$ and hence it is continuous. Appealing to Lemma 2.8 (and the remark following this lemma), we have that if

$$H(x,u,v) = \sup_{\xi \in \mathbb{R}} \{v\,\xi - f(x,u,\xi)\}$$

then $H \in C^{\infty}([a,b] \times \mathbb{R} \times \mathbb{R})$ and, for every $x \in [a,b]$, we have

$$\varphi(x) = f_{\xi}(x, \overline{u}(x), \overline{u}'(x)) \ \Leftrightarrow\ \overline{u}'(x) = H_v(x, \overline{u}(x), \varphi(x))\ .$$

Since H_v, $\overline{u}$ and φ are continuous, we infer that $\overline{u}'$ is continuous and hence $\overline{u} \in C^1([a,b])$. We therefore deduce that $x \to f_u(x, \overline{u}(x), \overline{u}'(x))$ is continuous, which combined with the fact that (cf. (4.5))

$$\frac{d}{dx}[\varphi(x)] = f_u(x, \overline{u}(x), \overline{u}'(x)),\ \text{a.e. } x \in (a,b)$$

(or equivalently, by Lemma 2.8, $\varphi' = -H_u(x, \overline{u}, \varphi)$) leads to $\varphi \in C^1([a,b])$.

Step 2. Returning to our Hamiltonian system

$$\begin{cases} \overline{u}'(x) = H_v(x, \overline{u}(x), \varphi(x)) \\ \\ \varphi'(x) = -H_u(x, \overline{u}(x), \varphi(x)) \end{cases}$$

we can start our iteration. Indeed since H is $C^{\infty}$and $\overline{u}$ and φ are C^1 we deduce from our system that, in fact, $\overline{u}$ and φ are C^2. Returning to the system we get that $\overline{u}$ and φ are C^3. Finally we get that $\overline{u}$ is C^{∞}, as wished. ■

We conclude the section by giving an example where we can get further regularity without assuming the non degeneracy condition $f_{\xi\xi} > 0$.

Theorem 4.5 *Let $g \in C^1([a,b] \times \mathbb{R})$ satisfy*

(H2) there exist $p > q \geq 1$ and $\alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$g(x,u) \geq \alpha_2 |u|^q + \alpha_3,\ \forall (x,u) \in [a,b] \times \mathbb{R}.$$

Let

$$f(x,u,\xi) = \frac{1}{p}|\xi|^p + g(x,u)\ .$$

Then there exists $\overline{u} \in C^1([a,b])$, with $|\overline{u}'|^{p-2}\overline{u}' \in C^1([a,b])$, a minimizer of (P) and the Euler-Lagrange equation holds everywhere, i.e.

$$\frac{d}{dx}\left[|\overline{u}'(x)|^{p-2}\overline{u}'(x)\right] = g_u(x,\overline{u}(x)),\ \forall x \in [a,b]\ .$$

Moreover if $1 < p \leq 2$, then $\overline{u} \in C^2([a,b])$.

If, in addition, $u \to g(x,u)$ is convex for every $x \in [a,b]$, then the minimizer is unique.

Remark 4.6 *The result cannot be improved in general, cf. Exercise 4.2.2.*

Proof. The existence (and uniqueness, if g is convex) of a solution $\overline{u} \in W^{1,p}(a,b)$ follows from Theorem 3.3. According to Lemma 4.2 we know that $\overline{u} \in W^{1,\infty}(a,b)$ and since $x \to g_u(x,\overline{u}(x))$ is continuous, we have that the Euler-Lagrange equation holds everywhere, i.e.

$$\frac{d}{dx}\left[|\overline{u}'(x)|^{p-2}\overline{u}'(x)\right] = g_u(x,\overline{u}(x)),\ x \in [a,b]\ .$$

We thus have that $|\overline{u}'|^{p-2}\overline{u}' \in C^1([a,b])$. Call $v \equiv |\overline{u}'|^{p-2}\overline{u}'$. We may then infer that

$$\overline{u}' = |v|^{\frac{2-p}{p-1}}v\,.$$

Since the function $t \to |t|^{\frac{2-p}{p-1}}t$ is continuous if $p > 2$ and C^1 if $1 < p \leq 2$, we obtain, from the fact that $v \in C^1([a,b])$, the conclusions of the theorem. ■

4.2.1 Exercises

Exercise 4.2.1 *With the help of Lemma 4.2, prove Theorem 4.3 in the following manner.*

(i) First show that $\overline{u} \in W^{2,\infty}(a,b)$, by proving (iii) of Theorem 1.36.

(ii) Conclude, using the following form of the Euler-Lagrange equation

$$\begin{aligned}\frac{d}{dx}\left[f_\xi(x,\overline{u},\overline{u}')\right] &= f_{\xi\xi}(x,\overline{u},\overline{u}')\,\overline{u}'' + f_{u\xi}(x,\overline{u},\overline{u}')\,\overline{u}' + f_{x\xi}(x,\overline{u},\overline{u}') \\ &= f_u(x,\overline{u},\overline{u}')\ .\end{aligned}$$

Exercise 4.2.2 *Let $p > 2q > 2$ and*

$$f(x,u,\xi) = f(u,\xi) = \frac{1}{p}|\xi|^p + \frac{\lambda}{q}|u|^q \quad \text{where } \lambda = \frac{qp^{q-1}(p-1)}{(p-q)^q}$$

$$\overline{u}(x) = \frac{p-q}{p}|x|^{p/(p-q)}$$

(note that if, for example, $p = 6$ and $q = 2$, then $f \in C^\infty(\mathbb{R}^2)$).

(i) Show that $\overline{u} \in C^1([-1,1])$ but $\overline{u} \notin C^2([-1,1])$.

(ii) Find some values of p and q so that

$$|\overline{u}'|^{p-2}\overline{u}', |\overline{u}|^{q-2}\overline{u} \in C^\infty([-1,1]),$$

although $\overline{u} \notin C^2([-1,1])$.

(iii) Show that $\overline{u}$ is the unique minimizer of

$$(P) \quad \inf_{u \in W^{1,p}(-1,1)} \left\{ I(u) = \int_{-1}^{1} f(u(x), u'(x))\, dx : u(-1) = u(1) = \frac{p-q}{p} \right\}.$$

4.3 The model case: Dirichlet integral

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $u_0 \in W^{1,2}(\Omega)$ and consider the problem

$$(P) \quad \inf\left\{ I(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx : u \in u_0 + W_0^{1,2}(\Omega) \right\}.$$

We have seen in Section 3.2 that there exists a unique minimizer $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$ of (P). Furthermore $\overline{u}$ satisfies the weak form of Laplace equation, namely

$$(E_w) \quad \int_\Omega \langle \nabla \overline{u}(x); \nabla\varphi(x)\rangle\, dx = 0,\ \forall \varphi \in W_0^{1,2}(\Omega)$$

where $\langle .;. \rangle$ denotes the scalar product in $\mathbb{R}^n$.

We will now show that $\overline{u} \in C^\infty(\Omega)$ and that it satisfies Laplace equation

$$\Delta u(x) = 0,\ \forall x \in \Omega\,.$$

We speak then of *interior regularity*. If, in addition, Ω is a bounded open set with C^∞ boundary and $u_0 \in C^\infty(\overline{\Omega})$ one can show that in fact $\overline{u} \in C^\infty(\overline{\Omega})$; we speak then of *regularity up to the boundary*.

Dirichlet integral has been studied so much that besides the books that we have quoted in the introduction of the present chapter, one can find numerous references where this problem is discussed. We quote here only two of them, namely Brézis [14] and John [63].

In this section we will treat only the problem of interior regularity, we refer to the literature for the question of the regularity up to the boundary. We will, however, give two different proofs. The first one is really specific to the present problem but gives, in some sense, a sharper result than the second one, which however applies to more general problems than the present context.

Theorem 4.7 ***(Weyl lemma)****. Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{loc}(\Omega)$ satisfy*

$$\int_\Omega u(x)\,\Delta v(x)\,dx = 0,\ \forall v \in C_0^\infty(\Omega) \tag{4.7}$$

then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

Remark 4.8 *(i) The function u being defined only almost everywhere, we have to interpret the result, as usual, up to a change of the function on a set of measure zero.*

(ii) Note that a solution of the weak form of Laplace equation

$$(E_w) \qquad \int_\Omega \langle \nabla u(x)\,;\nabla\varphi(x)\rangle\,dx = 0,\ \forall\varphi \in W_0^{1,2}(\Omega)$$

satisfies (4.7). The converse being true if, in addition, $u \in W^{1,2}(\Omega)$. Therefore (4.7) can be seen as a "very weak" form of Laplace equation and a solution of this equation as a "very weak" solution of $\Delta u = 0$.

Proof. Let $x \in \Omega$ and $R > 0$ sufficiently small so that

$$B_R(x) = \{y \in \mathbb{R}^n : |y - x| < R\} \subset \Omega\,.$$

Let $\sigma_{n-1} = \operatorname{meas}(\partial B_1(0))$ (i.e. $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi$,...). The idea is to show that if

$$\overline{u}(x) = \frac{1}{\sigma_{n-1}R^{n-1}}\int_{\partial B_R(x)} u\,d\sigma \tag{4.8}$$

then $\overline{u}$ is independent of R, $\overline{u} \in C^0(\Omega)$ and $\overline{u} = u$ a.e. in Ω. A classical result (cf. Exercise 4.3.2) allows to conclude that in fact $\overline{u} \in C^\infty(\Omega)$ and hence $\Delta\overline{u} = 0$ in Ω, as claimed.

These statements will be proved in three steps.

Step 1. We start by making an appropriate choice of the function v in (4.7). Let R be as above and choose $\epsilon \in (0, R)$ and $\varphi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (\epsilon, R)$. Define then

$$v(y) = \varphi(|x - y|)$$

and observe that $v \in C_0^\infty(B_R(x))$. An easy computation gives, for $r = |x - y|$,

$$\Delta v = \varphi''(r) + \frac{n-1}{r}\varphi'(r) = r^{1-n}\frac{d}{dr}\left[r^{n-1}\varphi'(r)\right].$$

From now on we assume that $n \geq 2$, the case $n = 1$ is elementary and is discussed in Exercise 4.3.1. We next let

$$\psi(r) = \frac{d}{dr}\left[r^{n-1}\varphi'(r)\right]. \tag{4.9}$$

Note that $\psi \in C_0^\infty(\epsilon, R)$ and

$$\int_\epsilon^R \psi(r)\, dr = 0. \tag{4.10}$$

Remark that the converse is also true, namely that given $\psi \in C_0^\infty(\epsilon, R)$ satisfying (4.10) we can find $\varphi \in C_0^\infty(\epsilon, R)$ verifying (4.9).

Step 2. Let $\psi \in C_0^\infty(\epsilon, R)$, satisfying (4.10), be arbitrary. Define then φ and v as above and use such a v in (4.7). We get, since $v \equiv 0$ on $\Omega \setminus B_R(x)$,

$$\begin{aligned} 0 &= \int_\Omega u\Delta v\, dy = \int_{B_R(x)} u\Delta v\, dy = \int_\epsilon^R \psi(r)\, r^{1-n}\int_{\partial B_r(x)} u\, d\sigma\, dr \\ &= \int_\epsilon^R \psi(r)\, w(r)\, dr \end{aligned}$$

where we have set

$$w(r) = r^{1-n}\int_{\partial B_r(x)} u\, d\sigma.$$

We can use Corollary 1.25 to deduce that

$$w(r) = \text{constant, a.e. } r \in (\epsilon, R).$$

We denote this constant by $\sigma_{n-1}\overline{u}(x)$ and we use the fact that ϵ is arbitrary to write

$$w(r) = \sigma_{n-1}\overline{u}(x), \text{ a.e. } r \in (0, R). \tag{4.11}$$

Step 3. We now conclude with the proof of the theorem. We first observe that (4.11) is nothing else than (4.8) with the fact that $\overline{u}$ is independent of R. We next integrate (4.8) and get

$$\overline{u}(x) = \frac{1}{\operatorname{meas} B_R(x)}\int_{B_R(x)} u(z)\, dz. \tag{4.12}$$

From (4.12) we deduce that $\overline{u} \in C^0(\Omega)$. Indeed let $x, y \in \Omega$ and R be sufficiently small so that $\overline{B}_R(x) \cup \overline{B}_R(y) \subset \Omega$. We then have that (denoting by ω_n the measure of the unit ball)

$$|\overline{u}(x) - \overline{u}(y)| = \frac{1}{\omega_n R^n} \left| \int_{B_R(x)} u(z)\, dz - \int_{B_R(y)} u(z)\, dz \right| \leq \frac{1}{\omega_n R^n} \int_O |u(z)|\, dz\,,$$

where $O = (B_R(x) \cup B_R(y)) \setminus (B_R(x) \cap B_R(y))$. Appealing to the fact that $u \in L^1(B_R(x) \cup B_R(y))$ and to Exercise 1.3.7, we deduce that $\overline{u}$ is indeed continuous.

It therefore remains to prove that $\overline{u} = u$ a.e. in Ω. This follows from Lebesgue theorem and the fact that $u \in L^1_{\text{loc}}(\Omega)$. Indeed letting R tend to 0 in (4.12) we have that for almost every $x \in \Omega$ the right hand side of (4.12) is $u(x)$. The theorem has therefore been established. ■

We now present a second proof that uses the so called *difference quotients*, introduced by Nirenberg.

Theorem 4.9 *Let $k \geq 0$ be an integer, $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^{k+2} boundary, $f \in W^{k,2}(\Omega)$ and*

$$(P') \quad \inf \left\{ I(u) = \int_\Omega \left[\frac{1}{2} |\nabla u(x)|^2 - f(x)\, u(x) \right] dx : u \in W_0^{1,2}(\Omega) \right\}.$$

Then there exists a unique minimizer $\overline{u} \in W^{k+2,2}(\Omega)$ of (P'). Furthermore there exists a constant $\gamma = \gamma(\Omega, k) > 0$ so that

$$\|\overline{u}\|_{W^{k+2,2}} \leq \gamma \|f\|_{W^{k,2}}\,. \tag{4.13}$$

In particular if $k = \infty$, then $\overline{u} \in C^\infty\left(\overline{\Omega}\right)$.

Remark 4.10 *(i) Problem (P) and (P') are equivalent. If in (P) the boundary datum $u_0 \in W^{k+2,2}(\Omega)$, then choose $f = \Delta u_0 \in W^{k,2}(\Omega)$.*

(ii) A similar result as (4.13) can be obtained in Hölder spaces (these are then known as Schauder estimates), under appropriate regularity hypotheses on the boundary and when $0 < a < 1$, namely

$$\|\overline{u}\|_{C^{k+2,a}} \leq \gamma \|f\|_{C^{k,a}}\,.$$

If $1 < p < \infty$, it can also be proved that

$$\|\overline{u}\|_{W^{k+2,p}} \leq \gamma \|f\|_{W^{k,p}}\,;$$

these are then known as Calderon-Zygmund estimates and are considerably harder to obtain than those for $p = 2$.

(iii) Both above results are however false if $a = 0$, $a = 1$ or $p = \infty$ (see Exercise 4.3.3) and if $p = 1$ (see Exercise 4.3.4). This is another reason why, when dealing with partial differential equations or the calculus of variations, Sobolev spaces and Hölder spaces are more appropriate than C^k spaces.

Proof. We know from the theory developed in Chapter 3 (in particular, Exercise 3.2.1) that (P') has a unique solution $\overline{u} \in W_0^{1,2}(\Omega)$ which satisfies in addition

$$\int_\Omega \langle \nabla \overline{u}(x); \nabla v(x) \rangle \, dx = \int_\Omega f(x) v(x) \, dx, \ \forall v \in W_0^{1,2}(\Omega) \, .$$

We will only show the interior regularity of $\overline{u}$, more precisely we will show that $f \in W^{k,2}(\Omega)$ implies that $\overline{u} \in W_{\mathrm{loc}}^{k+2,2}(\Omega)$. To show the sharper result (4.13) we refer to the literature (see Theorem 8.13 in Gilbarg-Trudinger [49]).

The claim is then equivalent to proving that $\varphi \overline{u} \in W^{k+2,2}(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. We let $u = \varphi \overline{u}$ and notice that $u \in W^{1,2}(\mathbb{R}^n)$ and that it is a weak solution of

$$\begin{aligned} \Delta u &= \Delta(\varphi \overline{u}) = \varphi \Delta \overline{u} + \overline{u} \Delta \varphi + 2 \langle \nabla \overline{u}; \nabla \varphi \rangle \\ &= -\varphi f + \overline{u} \Delta \varphi + 2 \langle \nabla \overline{u}; \nabla \varphi \rangle \equiv g \, . \end{aligned}$$

Since $f \in W^{k,2}(\Omega)$, $\overline{u} \in W_0^{1,2}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ we have that $g \in L^2(\mathbb{R}^n)$. We have therefore transformed the problem into showing that any $u \in W^{1,2}(\mathbb{R}^n)$ which satisfies

$$\int_{\mathbb{R}^n} \langle \nabla u(x); \nabla v(x) \rangle \, dx = \int_{\mathbb{R}^n} g(x) v(x) \, dx, \ \forall v \in W^{1,2}(\mathbb{R}^n) \tag{4.14}$$

is in fact in $W^{k+2,2}(\mathbb{R}^n)$ whenever $g \in W^{k,2}(\mathbb{R}^n)$. We will prove this claim in two steps. The first one deals with the case $k = 0$, while the second one will handle the general case.

Step 1. We here show that $g \in L^2(\mathbb{R}^n)$ implies $u \in W^{2,2}(\mathbb{R}^n)$. To achieve this goal we use the method of *difference quotients*. We introduce the following notations, for $h \in \mathbb{R}^n$, $h \neq 0$, we let

$$(D_h u)(x) = \frac{u(x+h) - u(x)}{|h|} \, .$$

It easily follows from Theorem 1.36 that

$$\nabla(D_h u) = D_h(\nabla u) \, , \ \|D_{-h} u\|_{L^2(\mathbb{R}^n)} \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}$$

$$\|D_h u\|_{L^2(\mathbb{R}^n)} \leq \gamma \Rightarrow u \in W^{1,2}(\mathbb{R}^n)$$

where γ denotes a constant independent of h. Returning to (4.14), we choose

$$v(x) = (D_{-h}(D_h u))(x) = \frac{2u(x) - u(x+h) - u(x-h)}{|h|^2}$$

and observe that, since $u \in W^{1,2}(\mathbb{R}^n)$, $v \in W^{1,2}(\mathbb{R}^n)$. We therefore find

$$\int_{\mathbb{R}^n} \langle \nabla u(x); \nabla (D_{-h}(D_h u))(x) \rangle \, dx = \int_{\mathbb{R}^n} g(x)(D_{-h}(D_h u))(x) \, dx . \quad (4.15)$$

Let us express differently the left hand side of the above identity and write

$$\begin{aligned}
&\int_{\mathbb{R}^n} \langle \nabla u; \nabla (D_{-h}(D_h u)) \rangle \, dx \\
= \;& \frac{1}{|h|^2} \int_{\mathbb{R}^n} \langle \nabla u(x); 2\nabla u(x) - \nabla u(x+h) - \nabla u(x-h) \rangle \, dx \\
= \;& \frac{2}{|h|^2} \int_{\mathbb{R}^n} \left[|\nabla u(x)|^2 - \langle \nabla u(x); \nabla u(x+h) \rangle \right] dx \\
= \;& \frac{1}{|h|^2} \int_{\mathbb{R}^n} |\nabla u(x+h) - \nabla u(x)|^2 \, dx
\end{aligned}$$

where we used, for passing from the first to the second identity and from the second to the third one, respectively

$$\int_{\mathbb{R}^n} [\langle \nabla u(x); \nabla u(x+h) \rangle] \, dx = \int_{\mathbb{R}^n} [\langle \nabla u(x); \nabla u(x-h) \rangle] \, dx$$

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx = \int_{\mathbb{R}^n} |\nabla u(x+h)|^2 \, dx .$$

Returning to (4.15) we just found that

$$\begin{aligned}
\int_{\mathbb{R}^n} \langle \nabla u(x); \nabla (D_{-h}(D_h u))(x) \rangle \, dx &= \frac{1}{|h|^2} \int_{\mathbb{R}^n} |\nabla u(x+h) - \nabla u(x)|^2 \, dx \\
&= \int_{\mathbb{R}^n} |(D_h \nabla u)(x)|^2 \, dx \\
&= \int_{\mathbb{R}^n} g(x)(D_{-h}(D_h u))(x) \, dx .
\end{aligned}$$

Applying Cauchy-Schwarz inequality and the properties of the operator D_h we get

$$\|D_h \nabla u\|_{L^2}^2 \le \|g\|_{L^2} \|D_{-h}(D_h u)\|_{L^2} \le \|g\|_{L^2} \|D_h \nabla u\|_{L^2}$$

and hence

$$\|D_h \nabla u\|_{L^2} \le \|g\|_{L^2} .$$

Using again the properties of the operator D_h we have indeed obtained that $\nabla u \in W^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ and hence $u \in W^{2,2}(\mathbb{R}^n)$.

Step 2. (The present step, contrary to the preceding one, relies heavily on the special form of the equation). Let now $g \in W^{1,2}(\mathbb{R}^n)$ and let us show that $u \in W^{3,2}(\mathbb{R}^n)$. The general case $g \in W^{k,2}$ implying that $u \in W^{k+2,2}$ follows by repeating the argument. The idea is simple, it consists in applying the previous step to $u_{x_i} = \partial u/\partial x_i$ and observing that since $\Delta u = g$, then $\Delta u_{x_i} = g_{x_i}$. Indeed it is elementary to see that we have, for every $i = 1, ..., n$,

$$\int_{\mathbb{R}^n} \langle \nabla u_{x_i}(x); \nabla v(x) \rangle \, dx = \int_{\mathbb{R}^n} g_{x_i}(x) v(x) \, dx, \; \forall v \in W^{1,2}(\mathbb{R}^n) \,. \tag{4.16}$$

To prove this, it is sufficient to establish it for $v \in C_0^\infty(\mathbb{R}^n)$ (since $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,2}(\mathbb{R}^n)$). We have, using (4.14), that

$$\begin{aligned}\int_{\mathbb{R}^n} \langle \nabla u_{x_i}; \nabla v \rangle \, dx &= \int_{\mathbb{R}^n} \left\langle (\nabla u)_{x_i}; \nabla v \right\rangle dx = -\int_{\mathbb{R}^n} \left\langle \nabla u; (\nabla v)_{x_i} \right\rangle dx \\ &= -\int_{\mathbb{R}^n} \langle \nabla u; \nabla v_{x_i} \rangle \, dx = -\int_{\mathbb{R}^n} g v_{x_i} \, dx = \int_{\mathbb{R}^n} g_{x_i} v \, dx \,.\end{aligned}$$

Since $g \in W^{1,2}$, we have that $g_{x_i} \in L^2$ and hence by the first step applied to (4.16) we get that $u_{x_i} \in W^{2,2}$. Since this holds for every $i = 1, ..., n$, we have indeed obtained that $u \in W^{3,2}$. This concludes the proof of the theorem. ■

4.3.1 Exercises

Exercise 4.3.1 *Prove Theorem 4.7 when $n = 1$.*

Exercise 4.3.2 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\sigma_{n-1} = \operatorname{meas}(\partial B_1(0))$ (i.e. $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi$,...). Let $u \in C^0(\Omega)$ satisfy the mean value formula, which states that*

$$u(x) = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{\partial B_r(x)} u \, d\sigma$$

for every $x \in \Omega$ and for every $r > 0$ sufficiently small so that

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\} \subset \Omega \,.$$

Show that $u \in C^\infty(\Omega)$.

Exercise 4.3.3 *We show here that if $f \in C^0$, then, in general, there is no solution $u \in C^2$ of $\Delta u = f$. Let $\Omega = \left\{x \in \mathbb{R}^2 : |x| < 1/2\right\}$ and for $0 < \alpha < 1$, define*

$$u(x) = u(x_1, x_2) = \begin{cases} x_1 x_2 \left|\log|x|\right|^\alpha & \text{if } 0 < |x| \le 1/2 \\ 0 & \text{if } x = 0 \,. \end{cases}$$

Show that

$$u_{x_1x_1}, u_{x_2x_2} \in C^0(\Omega), \; u_{x_1x_2} \notin L^\infty(\Omega)$$

which implies that $\Delta u = u_{x_1x_1} + u_{x_2x_2} \in C^0$, *while* $u \notin C^2$, *in fact* u *is not even in* $W^{2,\infty}$.

Exercise 4.3.4 *The present exercise (in the spirit of the preceding one) will give an example of a function* $u \notin W^{2,1}$, *with* $\Delta u \in L^1$. *Let*

$$\Omega = \left\{ x \in \mathbb{R}^2 : 0 < |x| < 1/2 \right\}$$

$$u(x) = u(x_1, x_2) = \log|\log|x||, \textit{ if } x \in \Omega .$$

Show that $u \notin W^{2,1}(\Omega)$ *while* $\Delta u \in L^1(\Omega)$.

4.4 Some general results

The generalization of the preceding section to integrands of the form $f = f(x, u, \nabla u)$ is a difficult task. We will give here, without proof, a general theorem and we refer for more results to the literature. The next theorem can be found in Morrey [75] (Theorem 1.10.4).

Theorem 4.11 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set and* $f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $f = f(x, u, \xi)$. *Let* $f_x = (f_{x_1}, ..., f_{x_n})$, $f_\xi = \left(f_{\xi_1}, ..., f_{\xi_n}\right)$ *and similarly for the higher derivatives. Let* f *satisfy, for every* $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}^n$,

$$(C) \quad \begin{cases} \alpha_1 V^p - \alpha_2 \le f(x, u, \xi) \le \alpha_3 V^p \\[2ex] |f_\xi|, \; |f_{x\xi}|, \; |f_u|, \; |f_{xu}| \le \alpha_3 V^{p-1}, \; |f_{u\xi}|, \; |f_{\mathrm{uu}}| \le \alpha_3 V^{p-2} \\[2ex] \alpha_4 V^{p-2} |\lambda|^2 \le \displaystyle\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, u, \xi) \lambda_i \lambda_j \le \alpha_5 V^{p-2} |\lambda|^2 \end{cases}$$

where $p \ge 2$, $V^2 = 1 + u^2 + |\xi|^2$ *and* $\alpha_i > 0$, $i = 1, ..., 5$, *are constants.*

Then any minimizer of

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W_0^{1,p}(\Omega) \right\}$$

is in $C^\infty(D)$, *for every* $D \subset \overline{D} \subset \Omega$.

Remark 4.12 *(i) The last hypothesis in (C) implies a kind of uniform convexity of* $\xi \to f(x, u, \xi)$; *it guarantees the uniform ellipticity of the Euler-Lagrange equation. The example of the preceding section, obviously, satisfies (C).*

(ii) For the regularity up to the boundary, we refer to the literature.

The proof of such a theorem relies on the De Giorgi-Nash-Moser theory. In the course of the proof, one transforms the nonlinear Euler-Lagrange equation into an elliptic linear equation with bounded measurable coefficients. Therefore to obtain the desired regularity, one needs to know the regularity of solutions of such equations and this is precisely the famous theorem that is stated below (see Giaquinta [47] Theorem 2.1 of Chapter II). It was first established by De Giorgi, then simplified by Moser and also proved, independently but at the same time, by Nash.

Theorem 4.13 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $v \in W^{1,2}(\Omega)$ be a solution of*

$$\sum_{i,j=1}^{n} \int_{\Omega} \left[a_{ij}(x)\, v_{x_i}(x)\, \varphi_{x_j}(x) \right] dx = 0,\ \forall \varphi \in W_0^{1,2}(\Omega)$$

where $a_{ij} \in L^\infty(\Omega)$ and, denoting by $\gamma > 0$ a constant,

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \lambda_i \lambda_j \geq \gamma\, |\lambda|^2\,,\ a.e.\ in\ \Omega\ and\ \forall \lambda \in \mathbb{R}^n\,.$$

Then there exists $0 < \alpha < 1$ so that $v \in C^{0,\alpha}(D)$, for every $D \subset \overline{D} \subset \Omega$.

Remark 4.14 *It is interesting to try to understand, formally, the relationship between the last two theorems, for example in the case where $f = f(x,u,\xi) = f(\xi)$. The coefficients $a_{ij}(x)$ and the function v in Theorem 4.13 are, respectively, $f_{\xi_i \xi_j}(\nabla u(x))$ and u_{x_i} in Theorem 4.11. The fact that $v = u_{x_i} \in W^{1,2}$ is proved by the method of difference quotients presented in Theorem 4.9.*

The two preceding theorems do not generalize to the vectorial case $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$, with $n, N > 1$. In this case only partial regularity can, in general, be proved. We give here an example of such a phenomenon due to Giusti-Miranda (see Giaquinta [47] Example 3.2 of Chapter II).

Example 4.15 *Let n, an integer, be sufficiently large, $\Omega \subset \mathbb{R}^n$ be the unit ball and $u_0(x) = x$. Let*

$$f(x,u,\xi) = f(u,\xi) = \sum_{i,j=1}^{n} \left(\xi_i^j\right)^2 + \left[\sum_{i,j=1}^{n} \left(\delta_{ij} + \frac{4}{n-2} \frac{u^i u^j}{1+|u|^2} \right) \xi_i^j \right]^2$$

where ξ_i^j stands for $\partial u^j / \partial x_i$ and δ_{ij} is the Kronecker symbol (i.e., $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$). Then $\overline{u}(x) = x/|x|$ is the unique minimizer of

$$(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(u(x), \nabla u(x))\, dx : u \in u_0 + W_0^{1,2}(\Omega; \mathbb{R}^n) \right\}.$$

Chapter 5

Minimal surfaces

5.1 Introduction

We start by explaining informally the problem under consideration. We want to find among all surfaces $\Sigma \subset \mathbb{R}^3$ (or more generally in $\mathbb{R}^{n+1}$, $n \geq 2$) with prescribed boundary, $\partial \Sigma = \Gamma$, where Γ is a Jordan curve, one that is of minimal area.

Unfortunately the formulation of the problem in more precise terms is delicate. It depends on the kind of surfaces we are considering. We will consider two types of surfaces: parametric and nonparametric surfaces. The second ones are less general but simpler from the analytical point of view.

We start with the formulation for nonparametric (hyper)surfaces (this case is easy to generalize to $\mathbb{R}^{n+1}$). These are of the form

$$\Sigma = \left\{ v\left(x\right) = \left(x, u\left(x\right)\right) \in \mathbb{R}^{n+1} : x \in \overline{\Omega} \right\}$$

with $u : \overline{\Omega} \to \mathbb{R}$ and where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The surface Σ is therefore the graph of the function u. The fact that $\partial \Sigma$ is prescribed reads now as $u = u_0$ on $\partial \Omega$, where u_0 is a given function. The area of such surface is given by

$$\operatorname{Area}\left(\Sigma\right) = I\left(u\right) = \int_\Omega f\left(\nabla u\left(x\right)\right) dx$$

where, for $\xi \in \mathbb{R}^n$, we have set

$$f\left(\xi\right) = \sqrt{1 + \left|\xi\right|^2}\,.$$

The problem is then written in the usual form

$$(P) \quad \inf \left\{ I\left(u\right) = \int_\Omega f\left(\nabla u\left(x\right)\right) dx : u \in u_0 + W_0^{1,1}\left(\Omega\right) \right\}.$$

As already seen in Chapter 3, even though the function f is strictly convex and $f(\xi) \geq |\xi|^p$ with $p = 1$, we cannot use the direct methods of the calculus of variations, since we are lead to work, because of the coercivity condition $f(\xi) \geq |\xi|$, in a non reflexive space $W^{1,1}(\Omega)$. In fact, in general, there is no minimizer of (P) in $u_0 + W_0^{1,1}(\Omega)$. We therefore need a different approach to deal with this problem.

Before going further we write the associated Euler-Lagrange equation to (P)

$$(E) \quad \operatorname{div}\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right] = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right] = 0$$

or equivalently

$$(E) \quad Mu \equiv \left(1+|\nabla u|^2\right)\Delta u - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0\,.$$

The last equation is known as the *minimal surface equation*. If $n = 2$ and $u = u(x,y)$, it reads as

$$Mu = \left(1+u_y^2\right)u_{xx} - 2u_x u_y u_{xy} + \left(1+u_x^2\right)u_{yy} = 0\,.$$

Therefore any $C^2\left(\overline{\Omega}\right)$ minimizer of (P) should satisfy the equation (E) and conversely, since the integrand f is convex. Moreover, since f is strictly convex, the minimizer, if it exists, is unique.

The equation (E) is equivalent (see Section 5.2) to the fact that the *mean curvature* of Σ, denoted by H, vanishes everywhere.

It is clear that the above problem is, geometrically, too restrictive. Indeed if any surface can be locally represented as a graph of a function (i.e., a non-parametric surface), it is not the case globally. We are therefore lead to consider more general ones known as the *parametric surfaces*. These are sets $\Sigma \subset \mathbb{R}^{n+1}$ so that there exist a domain (i.e. an open and connected set) $\Omega \subset \mathbb{R}^n$ and a map $v : \overline{\Omega} \to \mathbb{R}^{n+1}$ such that

$$\Sigma = v\left(\overline{\Omega}\right) = \left\{v(x) : x \in \overline{\Omega}\right\}.$$

For example, when $n = 2$ and $v = v(x,y) \in \mathbb{R}^3$, if we denote by $v_x \times v_y$ the normal to the surface (where $a \times b$ stands for the vectorial product of $a, b \in \mathbb{R}^3$ and $v_x = \partial v/\partial x$, $v_y = \partial v/\partial y$) we find that the area is given by

$$\operatorname{Area}(\Sigma) = J(v) = \iint_{\Omega} |v_x \times v_y| \, dxdy\,.$$

More generally if $n \geq 2$, we define (cf. Theorem 4.4.10 in Morrey [75])

$$g(\nabla v) = \left[\sum_{i=1}^{n+1} \left(\frac{\partial \left(v^1, ..., v^{i-1}, v^{i+1}, ..., v^{n+1} \right)}{\partial \left(x_1, ..., x_n \right)} \right)^2 \right]^{1/2}$$

where $\partial \left(u^1, ..., u^n \right) / \partial \left(x_1, ..., x_n \right)$ stands for the determinant of the $n \times n$ matrix $\left(\partial u^i / \partial x_j \right)_{1 \leq i,j \leq n}$. In the terminology of Section 3.5 such a function g is polyconvex but not convex. The area for such a surface is therefore given by

$$\operatorname{Area}(\Sigma) = J(v) = \int_\Omega g(\nabla v(x))\, dx\,.$$

The problem is then, given Γ, to find a parametric surface that minimizes

$$(Q) \quad \inf \left\{ \operatorname{Area}(\Sigma) : \partial \Sigma = \Gamma \right\}.$$

It is clear that problem (Q) is more general than (P). It is however a more complicated problem than (P) for several reasons besides the geometrical ones. Contrary to (P) it is a vectorial problem of the calculus of variations and the Euler-Lagrange equations associated to (Q) form now a system of $(n + 1)$ partial differential equations. Moreover, although, as for (P), any minimizer is a solution of these equations, it is not true in general, contrary to what happens with (P), that every solution of the Euler-Lagrange equations is necessarily a minimizer of (Q). Finally uniqueness is also lost for (Q) in contrast with what happens for (P).

We now come to the definition of *minimal surfaces*. A minimal surface will be a solution of the Euler-Lagrange equations associated to (Q), it will turn out that it has (see Section 5.2) zero mean curvature. We should draw the attention to the misleading terminology (this confusion is not present in the case of nonparametric surfaces): a minimal surface is not necessarily a *surface of minimal area*, while the converse is true, namely, a surface of minimal area is a minimal surface.

The problem of finding a minimal surface with prescribed boundary is known as *Plateau problem*.

We now describe the content of the present chapter. In most part we will only consider the case $n = 2$. In Section 5.2 we will recall some basic facts about surfaces, mean curvature and isothermal coordinates. We will then give several examples of minimal surfaces. In Section 5.3 we will outline some of the main ideas of the method of Douglas, as revised by Courant and Tonelli, for solving Plateau problem. This method is valid only when $n = 2$, since it uses strongly the notion and properties of conformal mappings. In Section 5.4 we briefly, and without proofs, mention some results of regularity, uniqueness and non uniqueness of minimal surfaces. In the final section we come back to the case of nonparametric surfaces and we give some existence results.

We now briefly discuss the historical background of the problem under consideration. The problem in nonparametric form was formulated and the equation (E) of minimal surfaces was derived by Lagrange in 1762. It was immediately understood that the problem was a difficult one. The more general Plateau problem (the name was given after the theoretical and experimental work of the physicist Plateau) was solved in 1930 simultaneously and independently by Douglas and Rado. One of the first two Fields medals was awarded to Douglas in 1936 for having solved the problem. Before that many mathematicians have contributed to the study of the problem: Ampère, Beltrami, Bernstein, Bonnet, Catalan, Darboux, Enneper, Haar, Korn, Legendre, Lie, Meusnier, Monge, Müntz, Riemann, H.A. Schwarz, Serret, Weierstrass, Weingarten and others. Immediately after the work of Douglas and Rado, we can quote Courant, Mac Shane, Morrey, Morse, Tonelli and many others since then. It is still a very active field.

We conclude this introduction with some comments on the bibliography. We should first point out that we gave many results without proofs and the ones that are given are only sketched. It is therefore indispensable in this chapter, even more than in the others, to refer to the bibliography. There are several excellent books but, due to the nature of the subject, they are difficult to read. The most complete to which we will refer constantly are those of Dierkes-Hildebrandt-Küster-Wohlrab [39] and Nitsche [78]. As a matter of introduction, interesting for a general audience, one can consult Hildebrandt-Tromba [58]. We refer also to the monographs of Almgren [4], Courant [24], Federer [45], Gilbarg-Trudinger [49] (for the nonparametric surfaces), Giusti [50], Morrey [75], Osserman [80] and Struwe [91].

5.2 Generalities about surfaces

We now introduce the different types of surfaces that we will consider. We will essentially limit ourselves to surfaces of $\mathbb{R}^3$, although in some instances we will give some generalizations to $\mathbb{R}^{n+1}$. Besides the references that we already mentioned, one can consult books of differential geometry such as that of Hsiung [61].

Definition 5.1 *(i) A set $\Sigma \subset \mathbb{R}^3$ will be called a* parametric surface *(or more simply a surface) if there exist a domain (i.e. an open and connected set) $\Omega \subset \mathbb{R}^2$ and a (non constant) continuous map $v : \overline{\Omega} \to \mathbb{R}^3$ such that*

$$\Sigma = v\left(\overline{\Omega}\right) = \left\{ v\left(x, y\right) \in \mathbb{R}^3 : \left(x, y\right) \in \overline{\Omega} \right\}.$$

(ii) We say that Σ *is a* nonparametric surface *if*

$$\Sigma = \left\{ v\,(x,y) = (x,y,u\,(x,y)) \in \mathbb{R}^3 : (x,y) \in \overline{\Omega} \right\}$$

with $u : \overline{\Omega} \to \mathbb{R}$ *continuous and where* $\Omega \subset \mathbb{R}^2$ *is a domain.*

(iii) A parametric surface is said to be regular *of class* C^m, *(*$m \geq 1$ *an integer) if, in addition,* $v \in C^m\left(\Omega; \mathbb{R}^3\right)$ *and* $v_x \times v_y \neq 0$ *for every* $(x,y) \in \Omega$ *(where* $a \times b$ *stands for the vectorial product of* $a, b \in \mathbb{R}^3$ *and* $v_x = \partial v / \partial x$, $v_y = \partial v / \partial y$*). We will in this case write*

$$e_3 = \frac{v_x \times v_y}{|v_x \times v_y|}\,.$$

Remark 5.2 *(i) In many cases we will restrict our attention to the case where* Ω *is the unit disk and* $\Sigma = v\left(\overline{\Omega}\right)$ *will then be called a surface of the type of the disk.*

(ii) In the sequel we will let

$$\mathcal{M}\left(\overline{\Omega}\right) = C^0\left(\overline{\Omega}; \mathbb{R}^3\right) \cap W^{1,2}\left(\Omega; \mathbb{R}^3\right)\,.$$

(iii) For a regular surface the area *will be defined as*

$$J\,(v) = \operatorname{Area}\,(\Sigma) = \iint_{\Omega} |v_x \times v_y|\, dxdy\,.$$

It can be shown, following Mac Shane and Morrey, that if $v \in \mathcal{M}\left(\overline{\Omega}\right)$, *then the above formula still makes sense (see Nitsche [78] pages 195-198).*

(iv) In the case of nonparametric surface $v\,(x,y) = (x,y,u\,(x,y))$ *we have*

$$\operatorname{Area}\,(\Sigma) = J\,(v) = I\,(u) = \iint_{\Omega} \sqrt{1 + u_x^2 + u_y^2}\, dxdy\,.$$

Note also that, for a nonparametric surface, we always have $|v_x \times v_y|^2 = 1 + u_x^2 + u_y^2 \neq 0$.

We now introduce the different notions of curvatures.

Definition 5.3 *Let* Σ *be a regular surface of class* C^m, $m \geq 2$, *we let*

$$\begin{aligned} &E = |v_x|^2\,,\ F = \langle v_x; v_y \rangle\,,\ G = |v_y|^2\,,\ e_3 = \frac{v_x \times v_y}{|v_x \times v_y|} \\ &L = \langle e_3; v_{xx} \rangle\,,\ M = \langle e_3; v_{xy} \rangle\,,\ N = \langle e_3; v_{yy} \rangle \end{aligned}$$

where $\langle .; . \rangle$ *denotes the scalar product in* $\mathbb{R}^3$.

(i) The mean curvature *of* Σ*, denoted by* H*, at a point* $p \in \Sigma$ $(p = v(x,y))$ *is given by*

$$H = \frac{1}{2} \frac{EN - 2FM + GL}{EG - F^2}.$$

(ii) The Gaussian curvature *of* Σ*, denoted by* K*, at a point* $p \in \Sigma$ $(p = v(x,y))$ *is by definition*

$$K = \frac{LN - M^2}{EG - F^2}.$$

(iii) The principal curvatures*,* k_1 *and* k_2*, are defined as*

$$k_1 = H + \sqrt{H^2 - K} \text{ and } k_2 = H - \sqrt{H^2 - K}$$

so that $H = (k_1 + k_2)/2$ *and* $K = k_1 k_2$*.*

Remark 5.4 *(i) We always have* $H^2 \geq K$*.*

(ii) For a nonparametric surface $v(x,y) = (x, y,\ u(x,y))$*, we have*

$$\begin{aligned} E &= 1 + u_x^2,\ F = u_x u_y,\ G = 1 + u_y^2,\ EG - F^2 = 1 + u_x^2 + u_y^2 \\ e_3 &= \frac{(-u_x, -u_y, 1)}{\sqrt{1 + u_x^2 + u_y^2}},\ L = \frac{u_{xx}}{\sqrt{1 + u_x^2 + u_y^2}}, \\ M &= \frac{u_{xy}}{\sqrt{1 + u_x^2 + u_y^2}},\ N = \frac{u_{yy}}{\sqrt{1 + u_x^2 + u_y^2}} \end{aligned}$$

and hence

$$H = \frac{\left(1 + u_y^2\right) u_{xx} - 2 u_x u_y u_{xy} + \left(1 + u_x^2\right) u_{yy}}{2\left(1 + u_x^2 + u_y^2\right)^{3/2}} \text{ and } K = \frac{u_{xx} u_{yy} - u_{xy}^2}{\left(1 + u_x^2 + u_y^2\right)^2}.$$

(iii) For a nonparametric surface in $\mathbb{R}^{n+1}$ *given by* $x_{n+1} = u(x_1, \ldots, x_n)$*, we have that the mean curvature is defined by (cf. (A.14) in Gilbarg-Trudinger [49])*

$$\begin{aligned} H &= \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{u_{x_i}}{\sqrt{1 + |\nabla u|^2}} \right] \\ &= \frac{1}{n} \left(1 + |\nabla u|^2\right)^{-\frac{3}{2}} \left[\left(1 + |\nabla u|^2\right) \Delta u - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} \right]. \end{aligned}$$

In terms of the operator M *defined in the introduction of the present chapter, we can write*

$$Mu = n \left(1 + |\nabla u|^2\right)^{\frac{3}{2}} H.$$

(iv) Note that we always have (see Exercise 5.2.1)

$$|v_x \times v_y| = \sqrt{EG - F^2}\,.$$

We are now in a position to define the notion of minimal surface.

Definition 5.5 *A regular surface of class C^2 is said to be* minimal *if $H = 0$ at every point.*

We next give several examples of minimal surfaces, starting with the nonparametric ones.

Example 5.6 *The first minimal surface that comes to mind is naturally the plane, defined parametrically by (α, β, γ being constants)*

$$\Sigma = \left\{ v\,(x,y) = (x, y, \alpha x + \beta y + \gamma) : (x,y) \in \mathbb{R}^2 \right\}.$$

We trivially have $H = 0$.

Example 5.7 *Scherk surface is a minimal surface in nonparametric form given by*

$$\Sigma = \left\{ v\,(x,y) = (x, y, u\,(x,y)) : |x|\,, |y| < \frac{\pi}{2} \right\}$$

where

$$u\,(x,y) = \log\cos y - \log\cos x\,.$$

We now turn our attention to minimal surfaces in parametric form.

Example 5.8 *Catenoids defined, for $(x,y) \in \mathbb{R}^2$, by*

$$v\,(x,y) = (x, w\,(x)\cos y, w\,(x)\sin y) \;\; \text{with } w\,(x) = \lambda \cosh \frac{x+\mu}{\lambda},$$

where $\lambda \neq 0$ and μ are constants, are minimal surfaces. We will see that they are the only minimal surfaces of revolution (here around the x axis).

Example 5.9 *The helicoid given, for $(x,y) \in \mathbb{R}^2$, by*

$$v\,(x,y) = (y\cos x, y\sin x, ax)$$

with $a \in \mathbb{R}$ is a minimal surface (see Exercise 5.2.2).

Example 5.10 *Enneper surface defined, for $(x,y) \in \mathbb{R}^2$, by*

$$v\,(x,y) = \left(x - \frac{x^3}{3} + xy^2, -y + \frac{y^3}{3} - yx^2, x^2 - y^2 \right)$$

is a minimal surface (see Exercise 5.2.2).

As already said we have the following characterization for surfaces of revolution.

Proposition 5.11 *The only regular minimal surfaces of revolution of the form*

$$v(x,y) = (x, w(x)\cos y, w(x)\sin y) \ ,$$

are the catenoids, *i.e.*

$$w(x) = \lambda \cosh \frac{x+\mu}{\lambda}$$

where $\lambda \neq 0$ *and* μ *are constants.*

Proof. We have to prove that Σ given parametrically by v is minimal if and only if

$$w(x) = \lambda \cosh((x+\mu)/\lambda) \ .$$

Observe first that

$$v_x = (1, w'\cos y, w'\sin y) \, , \ v_y = (0, -w\sin y, w\cos y) \, , \ E = 1+w'^2, \ F = 0, \ G = w^2$$

$$v_x \times v_y = w(w', -\cos y, -\sin y) \, , \ e_3 = \frac{w}{|w|} \frac{(w', -\cos y, -\sin y)}{\sqrt{1+w'^2}}$$

$$v_{xx} = w''(0, \cos y, \sin y) \, , \ v_{xy} = w'(0, -\sin y, \cos y) \, , \ v_{yy} = -w(0, \cos y, \sin y)$$

$$L = \frac{w}{|w|} \frac{-w''}{\sqrt{1+w'^2}}, \ M = 0, \ N = \frac{|w|}{\sqrt{1+w'^2}}.$$

Since Σ is a regular surface, we must have $|w| > 0$ (because $|v_x \times v_y|^2 = EG - F^2 > 0$). We therefore deduce that (recalling that $|w| > 0$)

$$H = 0 \Leftrightarrow EN + GL = 0 \Leftrightarrow |w| \left(ww'' - \left(1 + w'^2\right)\right) = 0$$

$$\Leftrightarrow ww'' = 1 + w'^2 \, . \tag{5.1}$$

Any solution of the differential equation necessarily satisfies

$$\frac{d}{dx} \left[\frac{w(x)}{\sqrt{1+w'^2(x)}} \right] = 0 \, .$$

The solution of this last differential equation (see the corrections of Exercise 5.2.3) being either $w \equiv$ constant (which however does not satisfy (5.1)) or of the form $w(x) = \lambda \cosh((x+\mu)/\lambda)$, we have the result. ■

We now turn our attention to the relationship between minimal surfaces and surfaces of minimal area.

Theorem 5.12 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain.*

Part 1. Let $\Sigma_0 = v\left(\overline{\Omega}\right)$ where $v \in C^2\left(\overline{\Omega}; \mathbb{R}^3\right)$, $v = v(x, y)$, with $v_x \times v_y \neq 0$ in $\overline{\Omega}$. If

$$\operatorname{Area}(\Sigma_0) \leq \operatorname{Area}(\Sigma)$$

among all regular surfaces Σ of class C^2 with $\partial\Sigma = \partial\Sigma_0$, then Σ_0 is a minimal surface.

Part 2. Let $\mathcal{S}_\Omega$ be the set of nonparametric surfaces of the form $\Sigma_u = \left\{(x, y, u(x, y)) : (x, y) \in \overline{\Omega}\right\}$ with $u \in C^2\left(\overline{\Omega}\right)$ and let $\Sigma_{\overline{u}} \in \mathcal{S}_\Omega$. The two following assertions are then equivalent.

(i) $\Sigma_{\overline{u}}$ is a minimal surface, which means

$$M\overline{u} = \left(1 + \overline{u}_y^2\right)\overline{u}_{xx} - 2\overline{u}_x\overline{u}_y\overline{u}_{xy} + \left(1 + \overline{u}_x^2\right)\overline{u}_{yy} = 0\,.$$

(ii) For every $\Sigma_u \in \mathcal{S}_\Omega$ with $u = \overline{u}$ on $\partial\Omega$

$$\operatorname{Area}(\Sigma_{\overline{u}}) \leq \operatorname{Area}(\Sigma_u) = I(u) = \iint_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dxdy\,.$$

Moreover, $\Sigma_{\overline{u}}$ is, among all surfaces of $\mathcal{S}_\Omega$ with $u = \overline{u}$ on $\partial\Omega$, the only one to have this property.

Remark 5.13 *(i) The converse of Part 1, namely that if Σ_0 is a minimal surface then it is of minimal area, is, in general, false. The claim of Part 2 is that the converse is true when we restrict our attention to nonparametric surfaces.*

(ii) This theorem is easily extended to $\mathbb{R}^{n+1}$, $n \geq 2$.

Proof. We will only prove Part 2 of the theorem and we refer to Exercise 5.2.4 for Part 1. Let

$$v(x, y) = (x, y, u(x, y))\,,\ (x, y) \in \overline{\Omega}$$

we then have

$$J(v) = \iint_\Omega |v_x \times v_y|\, dxdy = \iint_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dxdy \equiv I(u)\ .$$

(ii) $\Rightarrow$ (i). We write the associated Euler-Lagrange equation. Since $\overline{u}$ is a minimizer we have

$$I(\overline{u}) \leq I(\overline{u} + \epsilon\varphi)\,,\ \forall\varphi \in C_0^\infty(\Omega)\,,\ \forall\epsilon \in \mathbb{R}$$

and hence

$$\frac{d}{d\epsilon} I\left(\overline{u}+\epsilon\varphi\right)\bigg|_{\epsilon=0} = \iint_{\Omega} \frac{\overline{u}_x\varphi_x+\overline{u}_y\varphi_y}{\sqrt{1+\overline{u}_x^2+\overline{u}_y^2}}\,dxdy=0,\ \forall\varphi\in C_0^{\infty}\left(\Omega\right).$$

Since $\overline{u}\in C^2\left(\overline{\Omega}\right)$ we have, after integration by parts and using the fundamental lemma of the calculus of variations (Theorem 1.24),

$$\frac{\partial}{\partial x}\left[\frac{\overline{u}_x}{\sqrt{1+\overline{u}_x^2+\overline{u}_y^2}}\right]+\frac{\partial}{\partial y}\left[\frac{\overline{u}_y}{\sqrt{1+\overline{u}_x^2+\overline{u}_y^2}}\right]=0 \text{ in } \overline{\Omega} \tag{5.2}$$

or equivalently

$$M\overline{u}=\left(1+\overline{u}_y^2\right)\overline{u}_{xx}-2\overline{u}_x\overline{u}_y\overline{u}_{xy}+\left(1+\overline{u}_x^2\right)\overline{u}_{yy}=0 \text{ in } \overline{\Omega}\,. \tag{5.3}$$

This just asserts that $H=0$ and hence $\Sigma_{\overline{u}}=\left\{(x,y,\overline{u}\,(x,y)):(x,y)\in\overline{\Omega}\right\}$ is a minimal surface.

(i) $\Rightarrow$ (ii). We start by noting that the function $\xi\rightarrow f\left(\xi\right)=\sqrt{1+\left|\xi\right|^2}$, where $\xi\in\mathbb{R}^2$, is strictly convex. So let $\Sigma_{\overline{u}}=\left\{(x,y,\overline{u}\,(x,y)):(x,y)\in\overline{\Omega}\right\}$ be a minimal surface. Since $H=0$, we have that $\overline{u}$ satisfies (5.2) or (5.3). Let $\Sigma_u=\left\{(x,y,u\,(x,y)):(x,y)\in\overline{\Omega}\right\}$ with $u\in C^2\left(\overline{\Omega}\right)$ and $u=\overline{u}$ on $\partial\Omega$. We want to show that $I\left(\overline{u}\right)\leq I\left(u\right)$. Since f is convex, we have

$$f\left(\xi\right)\geq f\left(\eta\right)+\left\langle\nabla f\left(\eta\right);\xi-\eta\right\rangle,\ \forall\xi,\eta\in\mathbb{R}^2$$

and hence

$$f\left(u_x,u_y\right)\geq f\left(\overline{u}_x,\overline{u}_y\right)+\frac{1}{\sqrt{1+\overline{u}_x^2+\overline{u}_y^2}}\left\langle\left(\overline{u}_x,\overline{u}_y\right);\left(u_x-\overline{u}_x,u_y-\overline{u}_y\right)\right\rangle.$$

Integrating the above inequality and appealing to (5.2) and to the fact that $u=\overline{u}$ on $\partial\Omega$ we readily obtain the result.

The uniqueness follows from the strict convexity of f. ■

We next introduce the notion of *isothermal coordinates* (sometimes also called *conformal parameters*). This notion will help us to understand the method of Douglas that we will discuss in the next section.

Let us start with an informal presentation. Among all the parametrizations of a given curve the arc length plays a special role; for a given surface the isothermal coordinates play a similar role. They are given by $E=\left|v_x\right|^2=G=\left|v_y\right|^2$ and $F=\left\langle v_x;v_y\right\rangle=0$, which means that the tangent vectors are orthogonal and have equal norms. In general and contrary to what happens for curves, we can only *locally* find such a system of coordinates (i.e., with $E=G$ and $F=0$), according to the result of Korn, Lichtenstein and Chern [21].

Remark 5.14 *(i) Note that for a nonparametric surface*

$$\Sigma = \left\{ v(x,y) = (x, y, u(x,y)) , \ \ (x,y) \in \overline{\Omega} \right\}$$

we have $E = G = 1 + u_x^2 = 1 + u_y^2$ *and* $F = u_x u_y = 0$ *only if* $u_x = u_y = 0$.

(ii) Enneper surface (Example 5.10) is globally parametrized with isothermal coordinates.

One of the remarkable aspects of minimal surfaces is that they can be globally parametrized by such coordinates as the above Enneper surface. We have the following result that we will not use explicitly. We will, in part, give an idea of the proof in the next section (for a proof, see Nitsche [78], page 175).

Theorem 5.15 *Let* $\Omega = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \right\}$. *Let* Σ *be a minimal surface of the type of the disk (i.e., there exists* $\widetilde{v} \in C^2\left(\overline{\Omega}; \mathbb{R}^3\right)$ *so that* $\Sigma = \widetilde{v}\left(\overline{\Omega}\right)$*) such that* $\partial\Sigma = \Gamma$ *is a Jordan curve. Then there exists a global isothermal representation of the surface* Σ. *This means that* $\Sigma = \left\{ v(x,y) : (x,y) \in \overline{\Omega} \right\}$ *with* v *satisfying*

(i) $v \in C\left(\overline{\Omega}; \mathbb{R}^3\right) \cap C^\infty\left(\Omega; \mathbb{R}^3\right)$ *and* $\Delta v = 0$ *in* Ω;

(ii) $E = G > 0$ *and* $F = 0$ *(i.e.,* $|v_x|^2 = |v_y|^2 > 0$ *and* $\langle v_x; x_y \rangle = 0$*);*

(iii) v *maps the boundary* $\partial\Omega$ *topologically onto the Jordan curve* Γ.

Remark 5.16 *The second result asserts that* Σ *is a regular surface (i.e.,* $v_x \times v_y \neq 0$*) since* $|v_x \times v_y| = \sqrt{EG - F^2} = E = |v_x|^2 > 0$.

To conclude we point out the deep relationship between isothermal coordinates of minimal surfaces and harmonic functions (see also Theorem 5.15) which is one of the basic facts in the proof of Douglas.

Theorem 5.17 *Let* $\Sigma = \left\{ v(x,y) \in \mathbb{R}^3 : (x,y) \in \overline{\Omega} \right\}$ *be a regular surface (i.e.* $v_x \times v_y \neq 0$*) of class* C^2 *globally parametrized by isothermal coordinates; then*

$$\Sigma \text{ is a minimal surface} \quad \Leftrightarrow \quad \Delta v = 0 \text{ (i.e., } \Delta v^1 = \Delta v^2 = \Delta v^3 = 0\text{).}$$

Proof. We will show that if $E = G = |v_x|^2 = |v_y|^2$ and $F = 0$, then

$$\Delta v = 2EHe_3 = 2H\, v_x \times v_y \tag{5.4}$$

where H is the mean curvature and $e_3 = (v_x \times v_y) / |v_x \times v_y|$. The result will readily follow from the fact that $H = 0$.

Since $E = G$ and $F = 0$, we have

$$H = \frac{L+N}{2E} \Rightarrow L + N = 2EH\,. \tag{5.5}$$

We next prove that $\langle v_x; \Delta v\rangle = \langle v_y; \Delta v\rangle = 0$. Using the equations $E = G$ and $F = 0$, we have, after differentiation of the first one by x and the second one by y,

$$\langle v_x; v_{xx}\rangle = \langle v_y; v_{xy}\rangle \text{ and } \langle v_x; v_{yy}\rangle + \langle v_y; v_{xy}\rangle = 0\,.$$

This leads, as wished, to $\langle v_x; \Delta v\rangle = 0$ and in a similar way to $\langle v_y; \Delta v\rangle = 0$. Therefore Δv is orthogonal to v_x and v_y and thus parallel to e_3, which means that there exists $a \in \mathbb{R}$ so that $\Delta v = ae_3$. We then deduce that

$$a = \langle e_3; \Delta v\rangle = \langle e_3; v_{xx}\rangle + \langle e_3; v_{yy}\rangle = L + N\,. \tag{5.6}$$

Combining (5.5) and (5.6), we immediately get (5.4) and the theorem then follows. ■

5.2.1 Exercises

Exercise 5.2.1 *(i) Let $a, b, c \in \mathbb{R}^3$ show that*

$$|a \times b|^2 = |a|^2 |b|^2 - (\langle a; b\rangle)^2$$

$$(a \times b) \times c = \langle a; c\rangle\, b - \langle b; c\rangle\, a\,.$$

(ii) Deduce that $|v_x \times v_y| = \sqrt{EG - F^2}$.
(iii) Show that

$$\begin{aligned} L &= \langle e_3; v_{xx}\rangle = -\langle e_{3x}; v_x\rangle\,, \\ M &= \langle e_3; v_{xy}\rangle = -\langle e_{3x}; v_y\rangle = -\langle e_{3y}; v_x\rangle\,, \\ N &= \langle e_3; v_{yy}\rangle = -\langle e_{3y}; v_y\rangle\,. \end{aligned}$$

Exercise 5.2.2 *Show that the surfaces in Example 5.9 and Example 5.10 are minimal surfaces.*

Exercise 5.2.3 *Let Σ be a surface (of revolution) given by*

$$v(x, y) = (x, w(x)\cos y, w(x)\sin y)\,, \quad x \in (0,1)\,, \quad y \in (0, 2\pi)\,, \quad w \geq 0\,.$$

(i) Show that

$$\text{Area}(\Sigma) = I(w) = 2\pi \int_0^1 w(x)\sqrt{1 + (w'(x))^2}\, dx.$$

(ii) Consider the problem (where $\alpha > 0$)

$$(P_\alpha) \qquad \inf \{ I(w) : w(0) = w(1) = \alpha \} .$$

Prove that any $C^2([0,1])$ minimizer is necessarily of the form

$$w(x) = a \cosh \frac{2x-1}{2a} \quad \textit{with} \quad a \cosh \frac{1}{2a} = \alpha .$$

Discuss the existence of such solutions, as function of α.

Exercise 5.2.4 *Prove the first part of Theorem 5.12.*

5.3 The Douglas-Courant-Tonelli method

We now present the main ideas of the method of Douglas, as modified by Courant and Tonelli, for solving Plateau problem in $\mathbb{R}^3$. For a complete proof, we refer to Courant [24], Dierkes-Hildebrandt-Küster-Wohlrab [39], Nitsche [78] or for a slightly different approach to a recent article of Hildebrandt-Von der Mosel [59].

Let $\Omega = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \right\}$ and $\Gamma \subset \mathbb{R}^3$ be a rectifiable (i.e., of finite length) Jordan curve. Let $w_i \in \partial\Omega$ $(w_i \neq w_j)$ and $p_i \in \Gamma$ $(p_i \neq p_j)$ $i = 1, 2, 3$ be fixed. The set of admissible surfaces will then be

$$\mathcal{S} = \left\{ \begin{array}{lll} \Sigma = v\left(\overline{\Omega}\right) & \text{where} & v : \overline{\Omega} \to \Sigma \subset \mathbb{R}^3 \text{ so that} \\ & & \text{(S1)} \;\; v \in \mathcal{M}\left(\overline{\Omega}\right) = C^0\left(\overline{\Omega}; \mathbb{R}^3\right) \cap W^{1,2}\left(\Omega; \mathbb{R}^3\right) \\ & & \text{(S2)} \;\; v : \partial\Omega \to \Gamma \text{ is weakly monotonic and onto} \\ & & \text{(S3)} \;\; v(w_i) = p_i, \quad i = 1, 2, 3 \end{array} \right\} .$$

Remark 5.18 *(i) The set of admissible surfaces is then the set of parametric surfaces of the type of the disk with parametrization in $\mathcal{M}\left(\overline{\Omega}\right)$. The condition weakly monotonic in (S2) means that we allow the map v to be constant on some parts of $\partial\Omega$; thus v is not necessarily a homeomorphism of $\partial\Omega$ onto Γ. However the minimizer of the theorem will have the property to map the boundary $\partial\Omega$ topologically onto the Jordan curve Γ. The condition (S3) may appear a little strange, it will help us to get compactness (see the proof below).*

(ii) A first natural question is to ask if $\mathcal{S}$ is non empty. If the Jordan curve Γ is rectifiable then $\mathcal{S} \neq \emptyset$ (see for more details Dierkes-Hildebrandt-Küster-Wohlrab [39] pages 232-234 and Nitsche [78], pages 253-257).

(iii) Recall from the preceding section that for $\Sigma \in \mathcal{S}$ we have

$$\operatorname{Area}(\Sigma) = J(v) = \iint_\Omega |v_x \times v_y| \, dxdy .$$

The main result of this chapter is then

Theorem 5.19 *Under the above hypotheses there exists $\Sigma_0 \in \mathcal{S}$ so that*

$$\operatorname{Area}(\Sigma_0) \leq \operatorname{Area}(\Sigma), \qquad \forall\, \Sigma \in \mathcal{S}.$$

Moreover there exists $\overline{v}$ satisfying (S1), (S2) and (S3), such that $\Sigma_0 = \overline{v}\left(\overline{\Omega}\right)$ and

(i) $\overline{v} \in C^{\infty}\left(\Omega; \mathbb{R}^3\right)$ with $\Delta \overline{v} = 0$ in Ω,

(ii) $E = |\overline{v}_x|^2 = G = |\overline{v}_y|^2$ and $F = \langle \overline{v}_x; \overline{v}_y \rangle = 0$.

(iii) $\overline{v}$ maps the boundary $\partial\Omega$ topologically onto the Jordan curve Γ.

Remark 5.20 *(i) The theorem asserts that Σ_0 is of minimal area. To solve completely Plateau problem, we still must prove that Σ_0 is a regular surface (i.e. $\overline{v}_x \times \overline{v}_y \neq 0$ everywhere); we will then be able to apply Theorem 5.12 to conclude. We will mention in the next section some results concerning this problem. We also have a regularity result, namely that $\overline{v}$ is C^{∞} and harmonic, as well as a choice of isothermal coordinates ($E = G$ and $F = 0$).*

(ii) The proof uses properties of conformal mappings in a significant way and hence cannot be generalized as such to $\mathbb{R}^{n+1}$, $n \geq 2$. The results of De Giorgi, Federer, Fleming, Morrey, Reifenberg (cf. Giusti [50], Morrey [75]) and others deal with such a problem

Proof. We will only give the main ideas of the proof. It is divided into four steps.

Step 1. Let $\Sigma \in \mathcal{S}$ and define

$$\begin{aligned} \operatorname{Area}(\Sigma) &= J(v) = \iint_{\Omega} |v_x \times v_y| \; dxdy \\ D(v) &\equiv \frac{1}{2} \iint_{\Omega} \left(|v_x|^2 + |v_y|^2\right) dxdy\,. \end{aligned}$$

We then trivially have

$$J(v) \leq D(v) \tag{5.7}$$

since we know that

$$|v_x \times v_y| = \sqrt{EG - F^2} \leq \frac{1}{2}(E+G) = \frac{1}{2}\left(|v_x|^2 + |v_y|^2\right). \tag{5.8}$$

Furthermore, we have equality in (5.8) (and hence in (5.7)) if and only if $E = G$ and $F = 0$ (i.e., the parametrization is given by isothermal coordinates). We then consider the minimization problems

$$\begin{array}{llll} (D) & d & = & \inf\left\{D(v) : \ v \text{ satisfies (S1), (S2), (S3)}\right\} \\ (A) & a & = & \inf\left\{\operatorname{Area}(\Sigma) : \ \Sigma \in \mathcal{S}\right\}. \end{array}$$

We will show, in Step 2, that there exists a minimizer $\overline{v}$ of (D), whose components are harmonic functions, which means that $\Delta\overline{v} = 0$. Moreover, this $\overline{v}$ verifies $E = G$ and $F = 0$ (cf. Step 3) and hence, according to Theorem 5.17, $\Sigma_0 = \overline{v}\left(\overline{\Omega}\right)$ solves Plateau problem (up to the condition $\overline{v}_x \times \overline{v}_y \neq 0$). Finally we will show, in Step 4, that in fact $a = d = D(\overline{v})$ and thus, since $E = G$, $F = 0$ and (5.7) holds, we will have found that Σ_0 is also of minimal area.

Step 2. We now show that (D) has a minimizer. This does not follow from the results of the previous chapters; it would be so if we had chosen a fixed parametrization of the boundary Γ. Since $\mathcal{S} \neq \emptyset$, we can find a minimizing sequence $\{v_\nu\}$ so that

$$D(v_\nu) \to d. \tag{5.9}$$

Any such sequence $\{v_\nu\}$ will not, in general, converge. The idea is to replace v_ν by a harmonic function $\widetilde{v}_\nu$ such that $v_\nu = \widetilde{v}_\nu$ on $\partial\Omega$. More precisely, we define $\widetilde{v}_\nu$ as the minimizer of

$$D(\widetilde{v}_\nu) = \min\{D(v) : \ v = v_\nu \text{ on } \partial\Omega\}. \tag{5.10}$$

Such a $\widetilde{v}_\nu$ exists and its components are harmonic (cf. Chapter 2). Combining (5.9) and (5.10), we still have

$$D(\widetilde{v}_\nu) \to d.$$

Without the hypotheses (S2), (S3), this new sequence $\{\widetilde{v}_\nu\}$ does not converge either. The condition (S3) is important, since (see Exercise 5.3.1) Dirichlet integral is invariant under any conformal transformation from Ω onto Ω; (S3) allows to select a unique one. The hypothesis (S2) and the Courant-Lebesgue lemma imply that $\{\widetilde{v}_\nu\}$ is a sequence of equicontinuous functions (see Courant [24] page 103, Dierkes-Hildebrandt-Küster-Wohlrab [39] pages 235-237 or Nitsche [78], page 257). It follows from Ascoli-Arzela theorem (Theorem 1.3) that, up to a subsequence,

$$\widetilde{v}_{\nu_k} \to \overline{v} \text{ uniformly.}$$

Harnack theorem (see, for example, Gilbarg-Trudinger [49], page 21), a classical property of harmonic functions, implies that $\overline{v}$ is harmonic, satisfies (S1), (S2), (S3) and

$$D(\overline{v}) = d.$$

Step 3. We next show that this map $\overline{v}$ verifies also $E = G$ (i.e., $|\overline{v}_x|^2 = |\overline{v}_y|^2$) and $F = 0$ (i.e., $\langle \overline{v}_x; \overline{v}_y \rangle = 0$). We will use, in order to establish this fact, the technique of variations of the independent variables that we have already encountered in Section 2.3, when deriving the second form of the Euler-Lagrange equation.

Since the proof of this step is a little long, we subdivide it into three substeps.

Step 3.1. Let $\lambda, \mu \in C^{\infty}\left(\overline{\Omega}\right)$, to be chosen later, and let $\epsilon \in \mathbb{R}$ be sufficiently small so that the map

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \varphi^{\epsilon}(x,y) = \begin{pmatrix} \varphi_1^{\epsilon}(x,y) \\ \varphi_2^{\epsilon}(x,y) \end{pmatrix} = \begin{pmatrix} x + \epsilon\lambda(x,y) \\ y + \epsilon\mu(x,y) \end{pmatrix}$$

is a diffeomorphism from $\overline{\Omega}$ onto a simply connected domain $\overline{\Omega}^{\epsilon} = \varphi^{\epsilon}\left(\overline{\Omega}\right)$. We denote its inverse by ψ^{ϵ} and we find that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \psi^{\epsilon}(x',y') = \begin{pmatrix} \psi_1^{\epsilon}(x',y') \\ \psi_2^{\epsilon}(x',y') \end{pmatrix} = \begin{pmatrix} x' - \epsilon\lambda(x',y') + o(\epsilon) \\ y' - \epsilon\mu(x',y') + o(\epsilon) \end{pmatrix}$$

where $o(t)$ stands for a function $f = f(t)$ so that $f(t)/t$ tends to 0 as t tends to 0. We therefore have

$$\varphi^{\epsilon}(\psi^{\epsilon}(x',y')) = (x',y') \text{ and } \psi^{\epsilon}(\varphi^{\epsilon}(x,y)) = (x,y)$$

moreover the Jacobian is given by

$$\det \nabla\varphi^{\epsilon}(x,y) = 1 + \epsilon\left(\lambda_x(x,y) + \mu_y(x,y)\right) + o(\epsilon)\ . \tag{5.11}$$

We now change the independent variables and write

$$u^{\epsilon}(x',y') = \overline{v}(\psi^{\epsilon}(x',y'))\ .$$

We find that

$$\begin{aligned} u_{x'}^{\epsilon}(x',y') &= \overline{v}_x(\psi^{\epsilon}(x',y'))\frac{\partial}{\partial x'}\psi_1^{\epsilon}(x',y') + \overline{v}_y(\psi^{\epsilon}(x',y'))\frac{\partial}{\partial x'}\psi_2^{\epsilon}(x',y') \\ &= \overline{v}_x(\psi^{\epsilon}) - \epsilon\left[\overline{v}_x(\psi^{\epsilon})\lambda_x(\psi^{\epsilon}) + \overline{v}_y(\psi^{\epsilon})\mu_x(\psi^{\epsilon})\right] + o(\epsilon) \end{aligned}$$

and similarly

$$u_{y'}^{\epsilon}(x',y') = \overline{v}_y(\psi^{\epsilon}) - \epsilon\left[\overline{v}_x(\psi^{\epsilon})\lambda_y(\psi^{\epsilon}) + \overline{v}_y(\psi^{\epsilon})\mu_y(\psi^{\epsilon})\right] + o(\epsilon)\ .$$

This leads to

$$\begin{aligned} & \left|u_{x'}^{\epsilon}(x',y')\right|^2 + \left|u_{y'}^{\epsilon}(x',y')\right|^2 \\ = \; & \left|\overline{v}_x(\psi^{\epsilon})\right|^2 + \left|\overline{v}_y(\psi^{\epsilon})\right|^2 \\ & -2\epsilon\left[\left|\overline{v}_x(\psi^{\epsilon})\right|^2\lambda_x(\psi^{\epsilon}) + \left|\overline{v}_y(\psi^{\epsilon})\right|^2\mu_y(\psi^{\epsilon})\right] \\ & -2\epsilon\left[\langle\overline{v}_x(\psi^{\epsilon});\overline{v}_y(\psi^{\epsilon})\rangle\left(\lambda_y(\psi^{\epsilon}) + \mu_x(\psi^{\epsilon})\right)\right] + o(\epsilon)\ . \end{aligned}$$

Integrating this identity and changing the variables, letting $(x', y') = \varphi^{\epsilon}(x, y)$, in the right hand side we find (recalling (5.11)) that

$$\iint_{\Omega^{\epsilon}} \left[\left| u^{\epsilon}_{x'}(x', y') \right|^2 + \left| u^{\epsilon}_{y'}(x', y') \right|^2 \right] dx'dy' = \iint_{\Omega} \left[\left| \overline{v}_x(x, y) \right|^2 + \left| \overline{v}_y(x, y) \right|^2 \right] dxdy$$
$$- \epsilon \iint_{\Omega} \left[\left(\left| \overline{v}_x \right|^2 - \left| \overline{v}_y \right|^2 \right) \left(\lambda_x - \mu_y \right) + 2 \left\langle \overline{v}_x ; \overline{v}_y \right\rangle \left(\lambda_y + \mu_x \right) \right] dxdy + o(\epsilon) \, . \tag{5.12}$$

Step 3.2. We now use Riemann theorem to find a conformal mapping

$$\alpha^{\epsilon} : \Omega \to \Omega^{\epsilon}$$

which is also a homeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}^{\epsilon}$. We can also impose that the mapping verifies

$$\alpha^{\epsilon}(w_i) = \varphi^{\epsilon}(w_i)$$

where w_i are the points that enter in the definition of $\mathcal{S}$.

We finally let

$$v^{\epsilon}(x, y) = u^{\epsilon} \circ \alpha^{\epsilon}(x, y) = \overline{v} \circ \psi^{\epsilon} \circ \alpha^{\epsilon}(x, y)$$

where u^{ϵ} is as in Step 3.1.

Since $\overline{v} \in \mathcal{S}$, we deduce that $v^{\epsilon} \in \mathcal{S}$. Therefore using the conformal invariance of the Dirichlet integral (see Exercise 5.3.1), we find that

$$\begin{aligned} D(v^{\epsilon}) &= \frac{1}{2} \iint_{\Omega} \left[\left| v^{\epsilon}_x(x, y) \right|^2 + \left| v^{\epsilon}_y(x, y) \right|^2 \right] dxdy \\ &= \frac{1}{2} \iint_{\Omega^{\epsilon}} \left[\left| u^{\epsilon}_{x'}(x', y') \right|^2 + \left| u^{\epsilon}_{y'}(x', y') \right|^2 \right] dx'dy' \end{aligned}$$

which combined with (5.12) leads to

$$\begin{aligned} D(v^{\epsilon}) &= D(\overline{v}) - \frac{\epsilon}{2} \iint_{\Omega} \left[\left(\left| \overline{v}_x \right|^2 - \left| \overline{v}_y \right|^2 \right) \left(\lambda_x - \mu_y \right) \right] dxdy \\ &\quad - \epsilon \iint_{\Omega} \left[\left\langle \overline{v}_x ; \overline{v}_y \right\rangle \left(\lambda_y + \mu_x \right) \right] dxdy + o(\epsilon) \, . \end{aligned}$$

Since $v^{\epsilon}, \overline{v} \in \mathcal{S}$ and $\overline{v}$ is a minimizer of the Dirichlet integral, we find that

$$\iint_{\Omega} \left[\left(\left| \overline{v}_x \right|^2 - \left| \overline{v}_y \right|^2 \right) \left(\lambda_x - \mu_y \right) + 2 \left\langle \overline{v}_x ; \overline{v}_y \right\rangle \left(\lambda_y + \mu_x \right) \right] dxdy = 0. \tag{5.13}$$

Step 3.3. We finally choose in an appropriate way the functions $\lambda, \mu \in C^{\infty}\left(\overline{\Omega}\right)$ that appeared in the previous steps. We let $\sigma, \tau \in C_0^{\infty}(\Omega)$ be arbitrary, we then choose λ and μ so that

$$\begin{cases} \lambda_x - \mu_y = \sigma \\ \\ \lambda_y + \mu_x = \tau \end{cases}$$

(this is always possible; find first λ satisfying $\Delta\lambda = \sigma_x + \tau_y$ then choose μ such that $\left(\mu_x, \mu_y\right) = (\tau - \lambda_y, \lambda_x - \sigma)$). Returning to (5.13) we find

$$\iint_\Omega \{(E-G)\,\sigma + 2F\tau\}\,dxdy = 0,\ \forall\sigma,\tau \in C_0^\infty(\Omega)\ .$$

The fundamental lemma of the calculus of variations (Theorem 1.24) implies then $E = G$ and $F = 0$. Thus, up to the condition $\overline{v}_x \times \overline{v}_y \neq 0$, Plateau problem is solved (cf. Theorem 5.17).

We have still to prove (iii) in the statement of the theorem. However this follows easily from (i), (ii) and (S2) of the theorem, cf. Dierkes-Hildebrandt-Küster-Wohlrab [39] page 248.

Step 4. We let $\Sigma_0 = \overline{v}\left(\overline{\Omega}\right)$ where $\overline{v}$ is the element that has been found in the previous steps and satisfies in particular

$$d = \inf\{D(v) : v \text{ satisfies (S1), (S2), (S3)}\} = D(\overline{v})\ .$$

To conclude the proof of the theorem it remains to show that

$$a = \inf\{\operatorname{Area}(\Sigma) :\ \Sigma \in \mathcal{S}\} = \operatorname{Area}(\Sigma_0) = D(\overline{v}) = d\,.$$

We already know, from the previous steps, that

$$a \leq \operatorname{Area}(\Sigma_0) = D(\overline{v}) = d$$

and we therefore wish to show the reverse inequality.

A way of proving this claim is by using a result of Morrey (see Dierkes-Hildebrandt-Küster-Wohlrab [39] page 252 and for a slightly different approach see Courant [24] and Nitsche [78]) which asserts that for any $\epsilon > 0$ and any v satisfying (S1), (S2), (S3), we can find v^ϵ verifying (S1), (S2), (S3) so that

$$D(v^\epsilon) - \epsilon \leq \operatorname{Area}(\Sigma)$$

where $\Sigma = v\left(\overline{\Omega}\right)$. Since $d \leq D(v^\epsilon)$ and ϵ is arbitrary, we obtain that $d \leq a$; the other inequality being trivial we deduce that $a = d$.

This concludes the proof of the theorem. ■

5.3.1 Exercises

Exercise 5.3.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and*

$$\varphi(x,y) = (\lambda(x,y), \mu(x,y))$$

be a conformal mapping from $\overline{\Omega}$ onto $\overline{B}$. Let $v \in C^1\left(\overline{B}; \mathbb{R}^3\right)$ and $w = v \circ \varphi$; show that

$$\iint_\Omega \left[|w_x|^2 + |w_y|^2\right] dxdy = \iint_B \left[|v_\lambda|^2 + |v_\mu|^2\right] d\lambda d\mu \,.$$

5.4 Regularity, uniqueness and non uniqueness

We are now going to give some results without proofs. The first ones concern the regularity of the solution found in the previous section.

We have seen how to find a minimal surface with a C^∞ parametrization. We have seen, and we will see it again below, that several minimal surfaces may exist. The next result gives a regularity result for all such surfaces with given boundary (see Nitsche [78], page 274).

Theorem 5.21 *If Γ is a Jordan curve of class $C^{k,\alpha}$ (with $k \geq 1$ an integer and $0 < \alpha < 1$) then every solution of Plateau problem (i.e., a minimal surface whose boundary is Γ) admits a $C^{k,\alpha}\left(\overline{\Omega}\right)$ parametrization.*

However the most important regularity result concerns the existence of a *regular surface* (i.e., with $v_x \times v_y \neq 0$) which solves Plateau problem? We have seen in Section 5.3, that the method of Douglas does not answer this question. A result in this direction is the following (see Nitsche [78], page 334).

Theorem 5.22 *(i) If Γ is an analytical Jordan curve and if its total curvature does not exceed 4π then any solution of Plateau problem is a regular minimal surface.*

(ii) If a solution of Plateau problem is of minimal area then the result remains true without any hypothesis on the total curvature of Γ.

Remark 5.23 *The second part of the theorem allows, a posteriori, to assert that the solution found in Section 5.3 is a regular minimal surface (i.e., $\overline{v}_x \times \overline{v}_y \neq 0$), provided Γ is analytical.*

We now turn our attention to the problem of uniqueness of minimal surfaces. Recall first (Theorem 5.12) that we have uniqueness when restricted to nonparametric surfaces. For general surfaces we have the following uniqueness result (see Nitsche [78], page 351).

Theorem 5.24 *Let Γ be an analytical Jordan curve with total curvature not exceeding 4π, then Plateau problem has a unique solution.*

We now give a non uniqueness result (for more details we refer to Dierkes-Hildebrandt-Küster-Wohlrab [39] and to Nitsche [78]).

Example 5.25 *(**Enneper surface**, see Example 5.10). Let $r \in \left(1, \sqrt{3}\right)$ and*

$$\Gamma_r = \left\{ \left(r\cos\theta - \frac{1}{3}r^3\cos 3\theta, -r\sin\theta - \frac{1}{3}r^3\sin 3\theta, r^2\cos 2\theta \right) : \theta \in [0, 2\pi) \right\}.$$

We have seen (Example 5.10) that

$$\Sigma_r = \left\{ \left(rx + r^3xy^2 - \frac{r^3}{3}x^3, -ry - r^3x^2y + \frac{r^3}{3}y^3, r^2\left(x^2 - y^2\right) \right) : x^2 + y^2 \leq 1 \right\}$$

is a minimal surface and that $\partial\Sigma_r = \Gamma_r$. It is possible to show (cf. Nitsche [78], page 338) that Σ_r is not of minimal area if $r \in \left(1, \sqrt{3}\right)$; therefore it is distinct from the one found in Theorem 5.19.

5.5 Nonparametric minimal surfaces

We now discuss the case of nonparametric surfaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (in the present section we do not need to limit ourselves to the case $n = 2$). The surfaces that we will consider will be of the form

$$\Sigma = \left\{ (x, u(x)) = (x_1, ..., x_n, u(x_1, ..., x_n)) : x \in \overline{\Omega} \right\}.$$

The area of such a surface is given by

$$I(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx\,.$$

As already seen in Theorem 5.12, we have that any $C^2\left(\overline{\Omega}\right)$ minimizer of

$$(P) \quad \inf\left\{ I(u) : u = u_0 \text{ on } \partial\Omega \right\}$$

satisfies the minimal surface equation

$$\begin{aligned}(E) \quad Mu \quad &\equiv \quad \left(1 + |\nabla u|^2\right)\Delta u - \sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_ix_j} = 0 \\ &\Leftrightarrow \quad \sum_{i=1}^n \frac{\partial}{\partial x_i}\left[\frac{u_{x_i}}{\sqrt{1 + |\nabla u|^2}} \right] = 0\end{aligned}$$

and hence Σ has mean curvature that vanishes everywhere. The converse is also true; moreover we have uniqueness of such solutions. However, the existence of a minimizer still needs to be proved because Theorem 5.19 does not deal with this case. The techniques for solving such a problem are much more analytical than the previous ones.

Before proceeding with the existence theorems we would like to mention a famous related problem known as *Bernstein problem*. The problem is posed in the whole space $\mathbb{R}^n$ (i.e., $\Omega = \mathbb{R}^n$) and we seek for $C^2(\mathbb{R}^n)$ solutions of the minimal surface equation

$$(E) \quad \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right) = 0 \text{ in } \mathbb{R}^n$$

or of its equivalent form $Mu = 0$. In terms of regular surfaces, we are searching for a nonparametric surface (defined over the whole of $\mathbb{R}^n$) in $\mathbb{R}^{n+1}$ which has vanishing mean curvature. Obviously the function $u(x) = \langle a; x\rangle + b$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, which in geometrical terms represents a hyperplane, is a solution of the equation. The question is to know if this is the only one.

In the case $n = 2$, Bernstein has shown that, indeed, this is the only C^2 solution (the result is known as Bernstein theorem). Since then several authors found different proofs of this theorem. The extension to higher dimensions is however much harder. De Giorgi extended the result to the case $n = 3$, Almgren to the case $n = 4$ and Simons to $n = 5, 6, 7$. In 1969, Bombieri, De Giorgi and Giusti proved that when $n \geq 8$, there exists a nonlinear $u \in C^2(\mathbb{R}^n)$ (and hence the surface is not a hyperplane) satisfying equation (E). For more details on Bernstein problem, see Giusti [50], Chapter 17 and Nitsche [78], pages 429-430.

We now return to our problem in a bounded domain. We start by quoting a result of Jenkins and Serrin; for a proof see Gilbarg-Trudinger [49], page 297.

Theorem 5.26 ***(Jenkins-Serrin).*** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$, $0 < \alpha < 1$, boundary and let $u_0 \in C^{2,\alpha}\left(\overline{\Omega}\right)$. The problem $Mu = 0$ in Ω with $u = u_0$ on $\partial\Omega$ has a solution for every u_0 if and only if the mean curvature of $\partial\Omega$ is everywhere non negative.*

Remark 5.27 *(i) We now briefly mention a related result due to Finn and Osserman. It roughly says that if Ω is a non convex domain, there exists a continuous u_0 so that the problem $Mu = 0$ in Ω with $u = u_0$ on $\partial\Omega$ has no C^2 solution. Such a u_0 can even have arbitrarily small norm $\|u_0\|_{C^0}$.*

(ii) The above theorem follows several earlier works that started with Bernstein (see Nitsche [78], pages 352-358).

We end the present chapter with a simple theorem whose ideas contained in the proof are used in several different problems of partial differential equations.

Theorem 5.28 ***(Korn-Müntz).*** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$, $0 < \alpha < 1$, boundary and consider the problem*

$$\begin{cases} Mu = \left(1 + |\nabla u|^2\right) \Delta u - \sum\limits_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 & \text{in } \Omega \\ \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists $\epsilon > 0$ so that, for every $u_0 \in C^{2,\alpha}\left(\overline{\Omega}\right)$ with $\|u_0\|_{C^{2,\alpha}} \leq \epsilon$, the above problem has a (unique) $C^{2,\alpha}\left(\overline{\Omega}\right)$ solution.

Remark 5.29 *It is interesting to compare this theorem with the preceding one. We see that we have not made any assumption on the mean curvature of $\partial\Omega$; but we require that the $C^{2,\alpha}$ norm of u_0 to be small. We should also observe that the above mentioned result of Finn and Osserman shows that if Ω is non convex and if $\|u_0\|_{C^0} \leq \epsilon$, this is, in general, not sufficient to get existence of solutions. Therefore we cannot, in general, replace the condition $\|u_0\|_{C^{2,\alpha}}$ small, by $\|u_0\|_{C^0}$ small.*

Proof. The proof is divided into three steps. We write

$$Mu = 0 \Leftrightarrow \Delta u = N(u) \equiv \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} - |\nabla u|^2 \Delta u\,. \tag{5.14}$$

From estimates of the linearized equation $\Delta u = f$ (Step 1) and estimates of the nonlinear part $N(u)$ (Step 2), we will be able to conclude (Step 3) with the help of Banach fixed point theorem.

Step 1. Let us recall the classical Schauder estimates concerning Poisson equation (see Theorem 6.6 and page 103 in Gilbarg-Trudinger [49]). If $\Omega \subset \mathbb{R}^n$ is a bounded domain of $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary and if

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{5.15}$$

we can then find a constant $C = C(\Omega) > 0$ so that the (unique) solution u of (5.15) satisfies

$$\|u\|_{C^{2,\alpha}} \leq C\left(\|f\|_{C^{0,\alpha}} + \|\varphi\|_{C^{2,\alpha}}\right). \tag{5.16}$$

Step 2. We now estimate the nonlinear term N. We will show that we can find a constant, still denoted by $C = C(\Omega) > 0$, so that for every $u, v \in C^{2,\alpha}\left(\overline{\Omega}\right)$ we have

$$\|N(u) - N(v)\|_{C^{0,\alpha}} \leq C\left(\|u\|_{C^{2,\alpha}} + \|v\|_{C^{2,\alpha}}\right)^2 \|u - v\|_{C^{2,\alpha}}\,. \tag{5.17}$$

From

$$\begin{aligned} N(u) - N(v) &= \sum_{i,j=1}^{n} u_{x_i} u_{x_j} \left(u_{x_i x_j} - v_{x_i x_j} \right) \\ &\quad + \sum_{i,j=1}^{n} v_{x_i x_j} \left(u_{x_i} u_{x_j} - v_{x_i} v_{x_j} \right) \\ &\quad + |\nabla u|^2 (\Delta v - \Delta u) + \Delta v \left(|\nabla v|^2 - |\nabla u|^2 \right) \end{aligned}$$

we deduce that (5.17) holds from Proposition 1.10 and its proof (cf. Exercise 1.2.1).

Step 3. We are now in a position to show the theorem. We define a sequence $\{u_\nu\}_{\nu=1}^{\infty}$ of $C^{2,\alpha}\left(\overline{\Omega}\right)$ functions in the following way

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega \\ u_1 = u_0 & \text{on } \partial\Omega \end{cases} \tag{5.18}$$

and by induction

$$\begin{cases} \Delta u_{\nu+1} = N(u_\nu) & \text{in } \Omega \\ u_{\nu+1} = u_0 & \text{on } \partial\Omega\,. \end{cases} \tag{5.19}$$

The previous estimates will allow us to deduce that for $\|u_0\|_{C^{2,\alpha}} \leq \epsilon$, ϵ to be determined, we have

$$\|u_{\nu+1} - u_\nu\|_{C^{2,\alpha}} \leq K \|u_\nu - u_{\nu-1}\|_{C^{2,\alpha}} \tag{5.20}$$

for some $K < 1$. Banach fixed point theorem will then imply that $u_\nu \to u$ in $C^{2,\alpha}$ and hence

$$\begin{cases} \Delta u = N(u) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

which is the claimed result.

We now establish (5.20), which amounts to find the appropriate $\epsilon > 0$. We start by choosing $0 < K < 1$ and we then choose $\epsilon > 0$ sufficiently small so that

$$2C^2\epsilon\left(1 + \frac{C^4\epsilon^2}{1-K}\right) \leq \sqrt{K} \tag{5.21}$$

where C is the constant appearing in Step 1 and Step 2 (we can consider, without loss of generality, that they are the same).

We therefore only need to show that if $\|u_0\|_{C^{2,\alpha}} \leq \epsilon$, we have indeed (5.20). Note that for every $\nu \geq 2$, we have

$$\begin{cases} \Delta\left(u_{\nu+1}-u_\nu\right) = N\left(u_\nu\right) - N\left(u_{\nu-1}\right) & \text{in } \Omega \\ \\ u_{\nu+1}-u_\nu = 0 & \text{on } \partial\Omega\,. \end{cases}$$

From Step 1 and Step 2, we find that, for every $\nu \geq 2$,

$$\begin{aligned} &\|u_{\nu+1}-u_\nu\|_{C^{2,\alpha}} \leq C\left\|N\left(u_\nu\right) - N\left(u_{\nu-1}\right)\right\|_{C^{0,\alpha}} \\ &\leq C^2\left(\|u_\nu\|_{C^{2,\alpha}} + \|u_{\nu-1}\|_{C^{2,\alpha}}\right)^2 \|u_\nu - u_{\nu-1}\|_{C^{2,\alpha}}\ . \end{aligned} \tag{5.22}$$

Similarly for $\nu = 1$, we have

$$\|u_2-u_1\|_{C^{2,\alpha}} \leq C\left\|N\left(u_1\right)\right\|_{C^{0,\alpha}} \leq C^2\left\|u_1\right\|_{C^{2,\alpha}}^3\ . \tag{5.23}$$

From now on, since all norms will be $C^{2,\alpha}$ norms, we will denote them simply by $\|\cdot\|$. From (5.22), we deduce that it is enough to show

$$C^2\left(\|u_\nu\| + \|u_{\nu-1}\|\right)^2 \leq K,\ \nu \geq 2 \tag{5.24}$$

to obtain (5.20) and thus the theorem. To prove (5.24), it is sufficient to show that

$$\|u_\nu\| \leq C\epsilon\left(1+\frac{C^4\epsilon^2}{1-K}\right),\ \nu \geq 2\,. \tag{5.25}$$

The inequality (5.22) will then follow from the choice of ϵ in (5.21). We will prove (5.25) by induction. Observe that by Step 1 and from (5.18), we have

$$\|u_1\| \leq C\|u_0\| \leq C\epsilon\left(\leq C\epsilon\left(1+\frac{C^4\epsilon^2}{1-K}\right)\right)\,. \tag{5.26}$$

We now show (5.25) for $\nu = 2$. We have, from (5.23) and from (5.26), that

$$\|u_2\| \leq \|u_1\| + \|u_2-u_1\| \leq \|u_1\|\left(1+C^2\|u_1\|^2\right) \leq C\epsilon\left(1+C^4\epsilon^2\right)$$

and hence since $K \in (0,1)$, we deduce the inequality (5.25). Suppose now, from the hypothesis of induction, that (5.25) is valid up to the order ν and let us prove that it holds true for $\nu+1$. We trivially have that

$$\|u_{\nu+1}\| \leq \|u_1\| + \sum_{j=1}^{\nu}\|u_{j+1}-u_j\|\ . \tag{5.27}$$

We deduce from (5.22) and (5.24) (since (5.25) holds for every $1 \leq j \leq \nu$) that

$$\|u_{j+1} - u_j\| \leq K \|u_j - u_{j-1}\| \leq K^{j-1} \|u_2 - u_1\| \; .$$

Returning to (5.27), we find

$$\|u_{\nu+1}\| \leq \|u_1\| + \|u_2 - u_1\| \sum_{j=1}^{\nu} K^{j-1} \leq \|u_1\| + \frac{1}{1-K} \|u_2 - u_1\| \; .$$

Appealing to (5.23) and then to (5.26), we obtain

$$\|u_{\nu+1}\| \leq \|u_1\| \left(1 + \frac{C^2 \|u_1\|^2}{1-K}\right) \leq C\epsilon \left(1 + \frac{C^4 \epsilon^2}{1-K}\right),$$

which is exactly (5.25). The theorem then follows. ■

5.5.1 Exercises

Exercise 5.5.1 *Let $u \in C^2(\mathbb{R}^2)$, $u = u(x,y)$, be a solution of the minimal surface equation*

$$Mu = \left(1 + u_y^2\right) u_{xx} - 2u_x u_y u_{xy} + \left(1 + u_x^2\right) u_{yy} = 0 \, .$$

Show that there exists a convex function $\varphi \in C^2(\mathbb{R}^2)$, so that

$$\varphi_{xx} = \frac{1 + u_x^2}{\sqrt{1 + u_x^2 + u_y^2}}, \; \varphi_{xy} = \frac{u_x u_y}{\sqrt{1 + u_x^2 + u_y^2}}, \; \varphi_{yy} = \frac{1 + u_y^2}{\sqrt{1 + u_x^2 + u_y^2}} \, .$$

Deduce that

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = 1 \, .$$

Chapter 6

Isoperimetric inequality

6.1 Introduction

Let $A \subset \mathbb{R}^2$ be a bounded open set whose boundary, ∂A, is a sufficiently regular, simple closed curve. Denote by $L(\partial A)$ the length of the boundary and by $M(A)$ the measure (the area) of A. The isoperimetric inequality states that

$$[L(\partial A)]^2 - 4\pi M(A) \geq 0\,.$$

Furthermore, equality holds if and only if A is a disk (i.e., ∂A is a circle).

This is one of the oldest problems in mathematics. A variant of this inequality is known as Dido problem (who is said to have been a Phoenician princess). Several more or less rigorous proofs were known since the times of the Ancient Greeks; the most notable attempt for proving the inequality is due to Zenodorus, who proved the inequality for polygons. There are also significant contributions by Archimedes and Pappus. To come closer to us one can mention, among many, Euler, Galileo, Legendre, L'Huilier, Riccati or Simpson. A special tribute should be paid to Steiner who derived necessary conditions through a clever argument of symmetrization. The first proof that agrees with modern standards is due to Weierstrass. Since then, many proofs were given, notably by Blaschke, Bonnesen, Carathéodory, Edler, Frobenius, Hurwitz, Lebesgue, Liebmann, Minkowski, H.A. Schwarz, Sturm, and Tonelli among others. We refer to Porter [86] for an interesting article on the history of the inequality.

We will give here the proof of Hurwitz as modified by H. Lewy and Hardy-Littlewood-Polya [55]. In particular we will show that the isoperimetric inequality is equivalent to Wirtinger inequality that we have already encountered in a weaker form (cf. Poincaré-Wirtinger inequality). This inequality reads as

$$\int_{-1}^{1} u'^2 \, dx \geq \pi^2 \int_{-1}^{1} u^2 \, dx, \ \forall u \in X$$

where $X = \left\{ u \in W^{1,2}(-1,1) : u(-1) = u(1) \text{ and } \int_{-1}^{1} u \, dx = 0 \right\}$. It also states that equality holds if and only if $u(x) = \alpha \cos \pi x + \beta \sin \pi x$, for any $\alpha, \beta \in \mathbb{R}$.

In Section 6.3 we discuss the generalization to $\mathbb{R}^n$, $n \geq 3$, of the isoperimetric inequality. It reads as follows

$$[L(\partial A)]^n - n^n \omega_n [M(A)]^{n-1} \geq 0$$

for every bounded open set $A \subset \mathbb{R}^n$ with sufficiently regular boundary, ∂A; and where ω_n is the measure of the unit ball of $\mathbb{R}^n$, $M(A)$ stands for the measure of A and $L(\partial A)$ for the $(n-1)$ measure of ∂A. Moreover, if A is sufficiently regular (for example, convex), there is equality if and only if A is a ball.

The inequality in higher dimensions is considerably harder to prove; we will discuss, briefly, in Section 6.3 the main ideas of the proof. When $n = 3$, the first complete proof was the one of H.A. Schwarz. Soon after there were generalizations to higher dimensions and other proofs notably by A. Aleksandrov, Blaschke, Bonnesen, H. Hopf, Liebmann, Minkowski and E. Schmidt.

Finally numerous generalizations of this inequality have been studied in relation to problems of mathematical physics, see Bandle [9], Payne [83] and Polya-Szegö [85] for more references.

There are several articles and books devoted to the subject, we recommend the review article of Osserman [81] and the books by Berger [10], Blaschke [11], Federer [45], Hardy-Littlewood-Polya [55] (for the two dimensional case) and Webster [96]. The book of Hildebrandt-Tromba [58] also has a chapter on this matter.

6.2 The case of dimension 2

We start with the key result for proving the isoperimetric inequality; but before that we introduce the following notation, for any $p \geq 1$,

$$W_{\text{per}}^{1,p}(a,b) = \left\{ u \in W^{1,p}(a,b) : u(a) = u(b) \right\}.$$

Theorem 6.1 ***(Wirtinger inequality).*** *Let*

$$X = \left\{ u \in W_{per}^{1,2}(-1,1) : \int_{-1}^{1} u(x) \, dx = 0 \right\}$$

then

$$\int_{-1}^{1} u'^2 \, dx \geq \pi^2 \int_{-1}^{1} u^2 \, dx, \ \forall u \in X \, .$$

Furthermore equality holds if and only if $u(x) = \alpha \cos \pi x + \beta \sin \pi x$*, for any* $\alpha, \beta \in \mathbb{R}$.

Remark 6.2 *(i) It will be implicitly shown below that Wirtinger inequality is equivalent to the isoperimetric inequality.*

(ii) More generally we have if

$$X = \left\{ u \in W_{per}^{1,2}(a,b) : \ \int_a^b u(x) \, dx = 0 \right\}$$

that

$$\int_a^b u'^2 \, dx \geq \left(\frac{2\pi}{b-a} \right)^2 \int_a^b u^2 \, dx, \ \forall u \in X \, .$$

(iii) The inequality can also be generalized (cf. Croce-Dacorogna [28]) to

$$\left(\int_a^b |u'|^p \, dx \right)^{1/p} \geq \alpha(p,q,r) \left(\int_a^b |u'|^q \, dx \right)^{1/q}, \ \forall u \in X$$

for some appropriate $\alpha(p,q,r)$ *(in particular* $\alpha(2,2,2) = 2\pi/(b-a)$*) and where*

$$X = \left\{ u \in W_{per}^{1,p}(a,b) : \ \int_a^b |u(x)|^{r-2} u(x) \, dx = 0 \right\}.$$

(iv) We have seen in Example 2.23 a weaker form of the inequality, known as Poincaré-Wirtinger inequality, namely

$$\int_0^1 u'^2 \, dx \geq \pi^2 \int_0^1 u^2 \, dx, \ \forall u \in W_0^{1,2}(0,1) \, .$$

This inequality can be inferred from the theorem by setting

$$u(x) = -u(-x) \ \text{ if } x \in (-1,0) \, .$$

Proof. An alternative proof, more in the spirit of Example 2.23, is proposed in Exercise 6.2.1. The proof given here is, essentially, the classical proof of Hurwitz. We divide the proof into two steps.

Step 1. We start by proving the theorem under the further restriction that $u \in X \cap C^2[-1,1]$. We express u in Fourier series

$$u(x) = \sum_{n=1}^{\infty} [a_n \cos n\pi x + b_n \sin n\pi x] \, .$$

Note that there is no constant term since $\int_{-1}^{1} u(x)\,dx = 0$. We know at the same time that

$$u'(x) = \pi \sum_{n=1}^{\infty} \left[-na_n \sin n\pi x + nb_n \cos n\pi x \right] .$$

We can now invoke *Parseval formula* to get

$$\begin{aligned} \int_{-1}^{1} u^2\,dx &= \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \\ \int_{-1}^{1} u'^2\,dx &= \pi^2 \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) n^2. \end{aligned}$$

The desired inequality follows then at once

$$\int_{-1}^{1} u'^2\,dx \geq \pi^2 \int_{-1}^{1} u^2\,dx, \ \forall u \in X \cap C^2 .$$

Moreover equality holds if and only if $a_n = b_n = 0$, for every $n \geq 2$. This implies that equality holds if and only if $u(x) = \alpha \cos \pi x + \beta \sin \pi x$, for any $\alpha, \beta \in \mathbb{R}$, as claimed.

Step 2. We now show that we can remove the restriction $u \in X \cap C^2[-1,1]$. By the usual density argument we can find for every $u \in X$ a sequence $u_\nu \in X \cap C^2[-1,1]$ so that

$$u_\nu \to u \text{ in } W^{1,2}(-1,1) .$$

Therefore, for every $\epsilon > 0$, we can find ν sufficiently large so that

$$\int_{-1}^{1} u'^2\,dx \geq \int_{-1}^{1} u_\nu'^2\,dx - \epsilon \ \text{ and } \int_{-1}^{1} u_\nu^2\,dx \geq \int_{-1}^{1} u^2\,dx - \epsilon .$$

Combining these inequalities with Step 1 we find

$$\int_{-1}^{1} u'^2\,dx \geq \pi^2 \int_{-1}^{1} u^2\,dx - \left(\pi^2 + 1\right)\epsilon .$$

Letting $\epsilon \to 0$ we have indeed obtained the inequality.

We still need to see that equality in X holds if and only if $u(x) = \alpha \cos \pi x + \beta \sin \pi x$, for any $\alpha, \beta \in \mathbb{R}$. This has been proved in Step 1 only if $u \in X \cap C^2[-1,1]$. This property is established in Exercise 6.2.2. ■

We get as a direct consequence of the theorem

Corollary 6.3 *The following inequality holds*

$$\int_{-1}^{1} \left(u'^2 + v'^2\right) \, dx \geq 2\pi \int_{-1}^{1} uv' \, dx, \ \forall u, v \in W^{1,2}_{per}(-1,1) \, .$$

Furthermore equality holds if and only if

$$\left(u(x) - r_1\right)^2 + \left(v(x) - r_2\right)^2 = r_3^2, \ \forall x \in [-1,1]$$

where $r_1, r_2, r_3 \in \mathbb{R}$ *are constants.*

Proof. We first observe that if we replace u by $u - r_1$ and v by $v - r_2$ the inequality remains unchanged, therefore we can assume that

$$\int_{-1}^{1} u \, dx = \int_{-1}^{1} v \, dx = 0$$

and hence that $u, v \in X = \left\{u \in W^{1,2}_{\text{per}}(-1,1) : \ \int_{-1}^{1} u(x) \, dx = 0\right\}$. We write the inequality in the equivalent form

$$\int_{-1}^{1} \left(u'^2 + v'^2 - 2\pi uv'\right) \, dx = \int_{-1}^{1} \left(v' - \pi u\right)^2 \, dx + \int_{-1}^{1} \left(u'^2 - \pi^2 u^2\right) \, dx \geq 0 \, .$$

From Theorem 6.1 we deduce that the second term in the above inequality is non negative while the first one is trivially non negative; thus the inequality is established.

We now discuss the equality case. If equality holds we should have

$$v' = \pi u \ \text{ and } \int_{-1}^{1} \left(u'^2 - \pi^2 u^2\right) \, dx = 0$$

which implies from Theorem 6.1 that

$$u(x) = \alpha \cos \pi x + \beta \sin \pi x \text{ and } v(x) = \alpha \sin \pi x - \beta \cos \pi x \, .$$

Since we can replace u by $u - r_1$ and v by $v - r_2$, we have that

$$\left(u(x) - r_1\right)^2 + \left(v(x) - r_2\right)^2 = r_3^2, \ \forall x \in [-1,1]$$

as wished. ■

We are now in a position to prove the isoperimetric inequality in its analytic form; we postpone the discussion of its geometric meaning for later.

Theorem 6.4 ***(Isoperimetric inequality)***. *Let for* $u, v \in W_{per}^{1,1}(a,b)$

$$\begin{aligned} L(u,v) &= \int_a^b \sqrt{u'^2 + v'^2}\, dx \\ M(u,v) &= \frac{1}{2}\int_a^b (uv' - vu')\, dx = \int_a^b uv'\, dx\,. \end{aligned}$$

Then

$$[L(u,v)]^2 - 4\pi M(u,v) \geq 0,\ \forall u,v \in W_{per}^{1,1}(a,b)\,.$$

Moreover, among all $u, v \in W_{per}^{1,1}(a,b) \cap C^1([a,b])$*, equality holds if and only if*

$$(u(x) - r_1)^2 + (v(x) - r_2)^2 = r_3^2,\ \forall x \in [a,b]$$

where $r_1, r_2, r_3 \in \mathbb{R}$ *are constants.*

Remark 6.5 *The uniqueness holds under fewer regularity hypotheses that we do not discuss here. We, however, point out that the very same proof for the uniqueness is valid for* $u, v \in W_{per}^{1,1}(a,b) \cap C_{piec}^1([a,b])$*.*

Proof. We divide the proof into two steps.

Step 1. We first prove the theorem under the further restriction that $u, v \in W_{\text{per}}^{1,1}(a,b) \cap C^1([a,b])$. We will also assume that

$$u'^2(x) + v'^2(x) > 0,\ \forall x \in [a,b]\,.$$

This hypothesis is unnecessary and can be removed, see Exercise 6.2.3.

We start by reparametrizing the curve by a multiple of its arc length, namely

$$\begin{cases} y = \eta(x) = -1 + \frac{2}{L(u,v)} \displaystyle\int_a^x \sqrt{u'^2 + v'^2}\, dx \\ \varphi(y) = u\left(\eta^{-1}(y)\right) \text{ and } \psi(y) = v\left(\eta^{-1}(y)\right)\,. \end{cases}$$

It is easy to see that $\varphi, \psi \in W_{\text{per}}^{1,2}(-1,1) \cap C^1([-1,1])$ and

$$\sqrt{\varphi'^2(y) + \psi'^2(y)} = \frac{L(u,v)}{2},\ \forall y \in [-1,1]\,.$$

We therefore have

$$\begin{aligned} L(u,v) &= \int_{-1}^1 \sqrt{\varphi'^2(y) + \psi'^2(y)}\, dy = \left(2\int_{-1}^1 \left[\varphi'^2(y) + \psi'^2(y)\right] dy\right)^{1/2} \\ M(u,v) &= \int_{-1}^1 \varphi(y)\,\psi'(y)\, dy\,. \end{aligned}$$

We, however, know from Corollary 6.3 that

$$\int_{-1}^{1} \left(\varphi'^2 + \psi'^2\right) dx \geq 2\pi \int_{-1}^{1} \varphi\psi'\, dx,\ \forall \varphi, \psi \in W_{\text{per}}^{1,2}\left(-1,1\right)$$

which implies the claim

$$\left[L\left(u,v\right)\right]^2 - 4\pi M\left(u,v\right) \geq 0,\ \forall u,v \in W_{\text{per}}^{1,1}\left(a,b\right) \cap C^1\left(\left[a,b\right]\right).$$

The uniqueness in the equality case follows also from the corresponding one in Corollary 6.3.

Step 2. We now remove the hypothesis $u,v \in W_{\text{per}}^{1,1}\left(a,b\right) \cap C^1\left(\left[a,b\right]\right)$. As before, given $u,v \in W_{\text{per}}^{1,1}\left(a,b\right)$, we can find $u_\nu, v_\nu \in W_{\text{per}}^{1,1}\left(a,b\right) \cap C^1\left(\left[a,b\right]\right)$ so that

$$u_\nu, v_\nu \to u, v \text{ in } W^{1,1}\left(a,b\right) \cap L^\infty\left(a,b\right).$$

Therefore, for every $\epsilon > 0$, we can find ν sufficiently large so that

$$\left[L\left(u,v\right)\right]^2 \geq \left[L\left(u_\nu,v_\nu\right)\right]^2 - \epsilon \ \text{ and } M\left(u_\nu,v_\nu\right) \geq M\left(u,v\right) - \epsilon$$

and hence, combining these inequalities with Step 1, we get

$$\left[L\left(u,v\right)\right]^2 - 4\pi M\left(u,v\right) \geq \left[L\left(u_\nu,v_\nu\right)\right]^2 - 4\pi M\left(u_\nu,v_\nu\right) - \left(1+4\pi\right)\epsilon \geq -\left(1+4\pi\right)\epsilon\,.$$

Since ϵ is arbitrary, we have indeed obtained the inequality. ■

We now briefly discuss the geometrical meaning of the inequality obtained in Theorem 6.4. Any bounded open set A, whose boundary ∂A is a closed curve which possesses a parametrization $u,v \in W_{\text{per}}^{1,1}\left(a,b\right)$ so that its length and area are given by

$$\begin{aligned} L\left(\partial A\right) &= L\left(u,v\right) = \int_a^b \sqrt{u'^2 + v'^2}\, dx \\ M\left(A\right) &= M\left(u,v\right) = \frac{1}{2}\int_a^b \left(uv' - vu'\right) dx = \int_a^b uv'\, dx \end{aligned}$$

will therefore satisfy the isoperimetric inequality

$$\left[L\left(\partial A\right)\right]^2 - 4\pi M\left(A\right) \geq 0\,.$$

This is, of course, the case for any simple closed smooth curve, whose interior is A.

One should also note that very wild sets A can be allowed. Indeed sets A that can be approximated by sets A_ν that satisfy the isoperimetric inequality and which are so that

$$L\left(\partial A_\nu\right) \to L\left(\partial A\right) \quad \text{and} \quad M\left(A_\nu\right) \to M\left(A\right),\ \text{as } \nu \to \infty$$

also verify the inequality.

6.2.1 Exercises

Exercise 6.2.1 *Prove Theorem 6.1 in an analogous manner as that of Example 2.23.*

Exercise 6.2.2 *Let*

$$X=\left\{u\in W_{per}^{1,2}(-1,1):\ \int_{-1}^{1}u(x)\,dx=0\right\}$$

and consider

$$(P)\ \inf\left\{I(u)=\int_{-1}^{1}\left(u'^2-\pi^2u^2\right)\,dx:u\in X\right\}=m\,.$$

We have seen in Theorem 6.1 that $m=0$ and the minimum is attained in $X\cap C^2[-1,1]$ if and only if $u(x)=\alpha\cos\pi x+\beta\sin\pi x$, for any $\alpha,\beta\in\mathbb{R}$.

Show that these are the only minimizers in X (and not only in $X\cap C^2[-1,1]$).

Suggestion: Show that any minimizer of (P) is $C^2[-1,1]$. Conclude.

Exercise 6.2.3 *Prove Step 1 of Theorem 6.4 for any $u,v\in W_{per}^{1,1}(a,b)\cap C^1([a,b])$.*

6.3 The case of dimension n

The above proof does not generalize to $\mathbb{R}^n$, $n\geq 3$. A completely different and harder proof is necessary to deal with this case.

Before giving a sketch of the classical proof based on Brunn-Minkowski theorem, we want to briefly mention an alternative proof. The inequality $L^n-n^n\omega_nM^{n-1}\geq 0$ ($L=L(\partial A)$ and $M=M(A)$) is equivalent to the minimization of L for fixed M together with showing that the minimizers are given by spheres. We can then write the associated Euler-Lagrange equation, with a Lagrange multiplier corresponding to the constraint that M is fixed (see Exercise 6.3.2). We then obtain that for ∂A to be a minimizer it must have constant mean curvature (we recall that a minimal surface is a surface with vanishing mean curvature, see Chapter 5). The question is then to show that the sphere is, among all compact surfaces with constant mean curvature, the only one to have this property. This is the result proved by Aleksandrov, Hopf, Liebmann, Reilly and others (see Hsiung [61], page 280, for a proof). We immediately see that this result only partially answers the problem. Indeed we have only found a necessary condition that the minimizer should satisfy. Moreover this method requires a strong regularity on the minimizer.

We now turn our attention to the proof of the isoperimetric inequality. We will need several definitions and intermediate results.

Definition 6.6 *(i) For $A, B \subset \mathbb{R}^n$, $n \geq 1$, we define*

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

(ii) For $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, we let

$$d(x, A) = \inf\{|x - a| : a \in A\}.$$

Example 6.7 *(i) If $n = 1$, $A = [a, b]$, $B = [c, d]$, we have*

$$A + B = [a + c,\ b + d]\ .$$

(ii) If we let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, we get

$$B_R + B_S = B_{R+S}\,.$$

Proposition 6.8 *Let $A \subset \mathbb{R}^n$, $n \geq 1$ be compact and $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. The following properties then hold.*

(i) $A + \overline{B}_R = \{x \in \mathbb{R}^n : d(x, A) \leq R\}$.

(ii) If A is convex, then $A + B_R$ is also convex.

Proof. (i) Let $x \in A + \overline{B}_R$ and

$$X = \{x \in \mathbb{R}^n : d(x, A) \leq R\}.$$

We then have that $x = a + b$ for some $a \in A$ and $b \in \overline{B}_R$, and hence

$$|x - a| = |b| \leq R$$

which implies that $x \in X$. Conversely, since A is compact, we can find, for every $x \in X$, an element $a \in A$ so that $|x - a| \leq R$. Letting $b = x - a$, we have indeed found that $x \in A + \overline{B}_R$.

(ii) Trivial. ■

We now examine the meaning of the proposition in a simple example.

Example 6.9 *If A is a rectangle in $\mathbb{R}^2$, we find that $A + \overline{B}_R$ is given by the figure below. Anticipating, a little, on the following results we see that we have*

$$M\left(A + \overline{B}_R\right) = M(A) + RL(\partial A) + R^2\pi$$

where $L(\partial A)$ is the perimeter of ∂A.

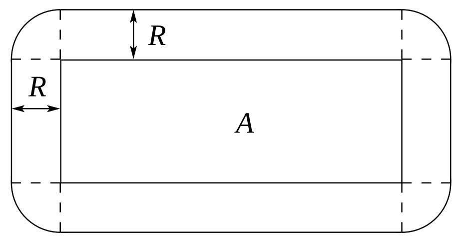

Figure 6.1: $A+B_R$

Definition 6.10 *We now define the meaning of $L(\partial A)$ and $M(A)$ for $A\subset\mathbb{R}^n$, $n\geq 2$, a compact set. $M(A)$ will denote the Lebesgue measure of A. The quantity $L(\partial A)$ will be given by the* Minkowski-Steiner formula

$$L(\partial A)=\liminf_{\epsilon\to 0}\frac{M\left(A+\overline{B}_\epsilon\right)-M(A)}{\epsilon}$$

where $B_\epsilon=\{x\in\mathbb{R}^n:|x|<\epsilon\}$.

Remark 6.11 *(i) The first natural question that comes to mind is to know if this definition of $L(\partial A)$ corresponds to the usual notion of $(n-1)$ measure of ∂A. This is the case if A is "sufficiently regular". This is a deep result that we will not prove and that we will, not even, formulate precisely (see Federer [45] for a thorough discussion on this matter and the remark below when A is convex). One can also try, with the help of drawings such as the one in Figure 7.1, to see that, indeed, the above definition corresponds to some intuitive notion of the area of ∂A.*

(ii) When $A\subset\mathbb{R}^n$ is convex, the above limit is a true limit and we can show (cf. Berger [10], Sections 12.10.6 and 9.12.4.6) that

$$M\left(A+\overline{B}_\epsilon\right)=M(A)+L(\partial A)\,\epsilon+\sum_{i=2}^{n-1}L_i(A)\,\epsilon^i+\omega_n\epsilon^n$$

where $L_i(A)$ are some (continuous) functions of A and ω_n is the measure of the unit ball in $\mathbb{R}^n$ given by

$$\omega_n=\frac{2\pi^{n/2}}{n\Gamma(n/2)}=\begin{cases}\pi^k/k! & \text{if } n=2k\\ 2^{k+1}\pi^k/1.3.5....(2k+1) & \text{if } n=2k+1.\end{cases}$$

Example 6.12 *If $A = \overline{B}_R$, we find the well known formula for the area of the sphere $S_R = \partial B_R$*

$$\begin{aligned} L(S_R) &= \lim_{\epsilon \to 0} \frac{M\left(\overline{B}_R + \overline{B}_\epsilon\right) - M\left(\overline{B}_R\right)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\left[(R+\epsilon)^n - R^n\right]\omega_n}{\epsilon} = nR^{n-1}\omega_n \end{aligned}$$

where ω_n is as above.

We are now in a position to state the theorem that plays a central role in the proof of the isoperimetric inequality.

Theorem 6.13 ***(Brunn-Minkowski theorem).*** *Let $A, B \subset \mathbb{R}^n$, $n \geq 1$, be compact, then the following inequality holds*

$$[M(A+B)]^{1/n} \geq [M(A)]^{1/n} + [M(B)]^{1/n}.$$

Remark 6.14 *(i) The same proof establishes that the function $A \to (M(A))^{1/n}$ is concave. We thus have*

$$[M(\lambda A + (1-\lambda)B)]^{1/n} \geq \lambda [M(A)]^{1/n} + (1-\lambda)[M(B)]^{1/n}$$

for every compact $A, B \subset \mathbb{R}^n$ and for every $\lambda \in [0,1]$.

(ii) One can even show that the function is strictly concave. This implies that the inequality in the theorem is strict unless A and B are homothetic.

Example 6.15 *Let $n = 1$.*

(i) If $A = [a,b]$, $B = [c,d]$, we have $A + B = [a+c,\ b+d]$ and

$$M(A+B) = M(A) + M(B).$$

(ii) If $A = [0,1]$, $B = [0,1] \cup [2,3]$, we find $A + B = [0,4]$ and hence

$$M(A+B) = 4 > M(A) + M(B) = 3.$$

We will prove Theorem 6.13 at the end of the section. We are now in a position to state and to prove the isoperimetric inequality.

Theorem 6.16 ***(Isoperimetric inequality).*** *Let $A \subset \mathbb{R}^n$, $n \geq 2$, be a compact set, $L = L(\partial A)$, $M = M(A)$ and ω_n be as above, then the following inequality holds*

$$L^n - n^n \omega_n M^{n-1} \geq 0.$$

Furthermore equality holds, among all convex sets, if and only if A is a ball.

Remark 6.17 *(i) The proof that we will give is also valid in the case $n = 2$. However it is unduly complicated and less precise than the one given in the preceding section.*

(ii) Concerning the uniqueness that we will not prove below (cf. Berger [10], Section 12.11), we should point out that it is a uniqueness only among convex sets. In dimension 2, we did not need this restriction; since for a non convex set A, its convex hull has larger area and smaller perimeter. In higher dimensions this is not true anymore. In the case $n \geq 3$, one can still obtain uniqueness by assuming some regularity of the boundary ∂A, in order to avoid "hairy" spheres (i.e., sets that have zero n and $(n-1)$ measures but non zero lower dimensional measures).

Proof. (Theorem 6.16). Let $A \subset \mathbb{R}^n$ be compact, we have from the definition of L (see Minkowski-Steiner formula) and from Theorem 6.13 that

$$\begin{aligned} L(\partial A) &= \liminf_{\epsilon \to 0} \frac{M\left(A + \overline{B}_\epsilon\right) - M(A)}{\epsilon} \\ &\geq \liminf_{\epsilon \to 0} \left[\frac{\left[(M(A))^{1/n} + (M(B_\epsilon))^{1/n}\right]^n - M(A)}{\epsilon} \right]. \end{aligned}$$

Since $M(B_\varepsilon) = \epsilon^n \omega_n$, we get

$$\begin{aligned} L(\partial A) &\geq M(A) \liminf_{\epsilon \to 0} \frac{\left[1 + \epsilon \left(\frac{\omega_n}{M(A)}\right)^{1/n}\right]^n - 1}{\epsilon} \\ &= M(A) \cdot n \left(\frac{\omega_n}{M(A)}\right)^{1/n} \end{aligned}$$

and the isoperimetric inequality follows. ■

We conclude the present section with an idea of the proof of Brunn-Minkowski theorem (for more details see Berger [10], Section 11.8.8, Federer [45], page 277 or Webster [96] Theorem 6.5.7). In Exercise 6.3.1 we will propose a proof of the theorem valid in the case $n = 1$. Still another proof in the case of $\mathbb{R}^n$ can be found in Pisier [84].

Proof. (Theorem 6.13). The proof is divided into four steps.

Step 1. We first prove an elementary inequality. Let $u_i > 0$, $\lambda_i \geq 0$ with $\Sigma_{i=1}^n \lambda_i = 1$, then

$$\prod_{i=1}^n u_i^{\lambda_i} \leq \sum_{i=1}^n \lambda_i u_i \,. \tag{6.1}$$

This is a direct consequence of the fact that the logarithm function is concave and hence

$$\log\left(\sum_{i=1}^{n}\lambda_i u_i\right)\geq\sum_{i=1}^{n}\lambda_i\log u_i=\log\left(\prod_{i=1}^{n}u_i^{\lambda_i}\right).$$

Step 2. Let $\mathcal{F}$ be the family of all open sets A of the form

$$A=\prod_{i=1}^{n}\left(a_i,b_i\right).$$

We will now prove the theorem for $A,B\in\mathcal{F}$. We will even show that for every $\lambda\in[0,1]$, $A,B\in\mathcal{F}$ we have

$$\left[M\left(\lambda A+(1-\lambda)B\right)\right]^{1/n}\geq\lambda\left[M\left(A\right)\right]^{1/n}+(1-\lambda)\left[M\left(B\right)\right]^{1/n}. \tag{6.2}$$

The theorem follows from (6.2) by setting $\lambda=1/2$. If we let

$$A=\prod_{i=1}^{n}\left(a_i,b_i\right)\ \text{ and }B=\prod_{i=1}^{n}\left(c_i,d_i\right)$$

we obtain

$$\lambda A+(1-\lambda)B=\prod_{i=1}^{n}\left(\lambda a_i+(1-\lambda)c_i,\lambda b_i+(1-\lambda)d_i\right).$$

Setting, for $1\leq i\leq n$,

$$u_i=\frac{b_i-a_i}{\lambda\left(b_i-a_i\right)+(1-\lambda)\left(d_i-c_i\right)},\ v_i=\frac{d_i-c_i}{\lambda\left(b_i-a_i\right)+(1-\lambda)\left(d_i-c_i\right)} \tag{6.3}$$

we find that

$$\lambda u_i+(1-\lambda)v_i=1,\ 1\leq i\leq n, \tag{6.4}$$

$$\frac{M\left(A\right)}{M\left(\lambda A+(1-\lambda)B\right)}=\prod_{i=1}^{n}u_i,\ \frac{M\left(B\right)}{M\left(\lambda A+(1-\lambda)B\right)}=\prod_{i=1}^{n}v_i. \tag{6.5}$$

We now combine (6.1), (6.4) and (6.5) to deduce that

$$\begin{aligned}\frac{\lambda\left[M\left(A\right)\right]^{1/n}+(1-\lambda)\left[M\left(B\right)\right]^{1/n}}{\left[M\left(\lambda A+(1-\lambda)B\right)\right]^{1/n}} &= \lambda\prod_{i=1}^{n}u_i^{1/n}+(1-\lambda)\prod_{i=1}^{n}v_i^{1/n}\\ &\leq \lambda\sum_{i=1}^{n}\frac{u_i}{n}+(1-\lambda)\sum_{i=1}^{n}\frac{v_i}{n}\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\lambda u_i+(1-\lambda)v_i\right)=1\end{aligned}$$

and hence the result.

Step 3. We now prove (6.2) for any A and B of the form

$$A=\bigcup_{\mu=1}^{M}A_{\mu}\,,\ B=\bigcup_{\nu=1}^{N}B_{\nu}$$

where A_μ, $B_\nu\in\mathcal{F}$, $A_\nu\cap A_\mu=B_\nu\cap B_\mu=\emptyset$ if $\mu\neq\nu$. The proof is then achieved through induction on $M+N$. Step 2 has proved the result when $M=N=1$. We assume now that $M>1$. We then choose $i\in\{1,...,n\}$ and $a\in\mathbb{R}$ such that if

$$A^{+}=A\cap\{x\in\mathbb{R}^{n}:x_{i}>a\}\,,\ A^{-}=A\cap\{x\in\mathbb{R}^{n}:x_{i}<a\}$$

then A^+ and A^- contain at least one of the A_μ, $1\leq\mu\leq M$, i.e. the hyperplane $\{x_i=a\}$ separates at least two of the A_μ (see Figure 7.2).

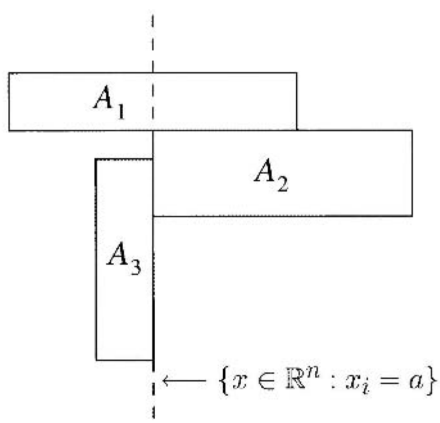

Figure 6.2: separating hyperplane

We clearly have

$$M\left(A^{+}\right)+M\left(A^{-}\right)=M\left(A\right). \tag{6.6}$$

We next choose $b\in\mathbb{R}$ (such a b exists by an argument of continuity) so that if

$$B^{+}=B\cap\{x\in\mathbb{R}^{n}:x_{i}>b\}\,,\ B^{-}=B\cap\{x\in\mathbb{R}^{n}:x_{i}<b\}$$

then

$$\frac{M\left(A^{+}\right)}{M\left(A\right)}=\frac{M\left(B^{+}\right)}{M\left(B\right)}\ \text{ and }\ \frac{M\left(A^{-}\right)}{M\left(A\right)}=\frac{M\left(B^{-}\right)}{M\left(B\right)}. \tag{6.7}$$

We let

$$A_\mu^\pm = A^\pm \cap A_\mu \quad \text{and} \quad B_\nu^\pm = B^\pm \cap B_\nu$$

provided these intersections are non empty; and we deduce that

$$A^\pm = \bigcup_{\mu=1}^{M^\pm} A_\mu^\pm \quad \text{and} \quad B^\pm = \bigcup_{\nu=1}^{N^\pm} B_\nu^\pm \,.$$

By construction we have $M^+ < M$ and $M^- < M$, while $N^+,\ N^- \le N$. If $\lambda \in [0,1]$, we see that $\lambda A^+ + (1-\lambda)\, B^+$ and $\lambda A^- + (1-\lambda)\, B^-$ are separated by $\{x : x_i = \lambda a + (1-\lambda)\, b\}$ and thus

$$M\left(\lambda A + (1-\lambda)\, B\right) = M\left(\lambda A^+ + (1-\lambda)\, B^+\right) + M\left(\lambda A^- + (1-\lambda)\, B^-\right) \,.$$

Applying the hypothesis of induction to A^+, B^+ and A^-, B^-, we deduce that

$$\begin{aligned} M\left(\lambda A + (1-\lambda)\, B\right) \;\; \geq \;\; & \left[\lambda \left[M\left(A^+\right)\right]^{1/n} + (1-\lambda) \left[M\left(B^+\right)\right]^{1/n}\right]^n \\ & + \left[\lambda \left[M\left(A^-\right)\right]^{1/n} + (1-\lambda) \left[M\left(B^-\right)\right]^{1/n}\right]^n . \end{aligned}$$

Using (6.7) we obtain

$$\begin{aligned} M\left(\lambda A + (1-\lambda)\, B\right) \;\; \geq \;\; & \frac{M\left(A^+\right)}{M\left(A\right)} \left[\lambda \left[M\left(A\right)\right]^{1/n} + (1-\lambda) \left[M\left(B\right)\right]^{1/n}\right]^n \\ & + \frac{M\left(A^-\right)}{M\left(A\right)} \left[\lambda \left[M\left(A\right)\right]^{1/n} + (1-\lambda) \left[M\left(B\right)\right]^{1/n}\right]^n . \end{aligned}$$

The identity (6.6) and the above inequality imply then (6.2).

Step 4. We now show (6.2) for any compact set, concluding thus the proof of the theorem. Let $\epsilon > 0$, we can then approximate the compact sets A and B, by A_ϵ and B_ϵ as in Step 3, so that

$$\left|M\left(A\right) - M\left(A_\epsilon\right)\right| \,,\;\; \left|M\left(B\right) - M\left(B_\epsilon\right)\right| \le \epsilon \,, \tag{6.8}$$

$$\left|M\left(\lambda A + (1-\lambda)\, B\right) - M\left(\lambda A_\epsilon + (1-\lambda)\, B_\epsilon\right)\right| \le \epsilon \,. \tag{6.9}$$

Applying (6.2) to A_ϵ, B_ϵ, using (6.8) and (6.9), we obtain, after passing to the limit as $\epsilon \to 0$, the claim

$$\left[M\left(\lambda A + (1-\lambda)\, B\right)\right]^{1/n} \geq \lambda \left[M\left(A\right)\right]^{1/n} + (1-\lambda) \left[M\left(B\right)\right]^{1/n} .$$

■

6.3.1 Exercises

Exercise 6.3.1 *Let $A, B \subset \mathbb{R}$ be compact,*

$$\overline{a} = \min\{a : a \in A\} \quad \text{and} \quad \overline{b} = \max\{b : b \in B\}.$$

Prove that

$$(\overline{a} + B) \cup \left(\overline{b} + A\right) \subset A + B$$

and deduce that

$$M(A) + M(B) \leq M(A + B).$$

Exercise 6.3.2 *Denote by $\mathcal{A}$ the set of bounded open sets $A \subset \mathbb{R}^3$ whose boundary ∂A is the image of a bounded smooth domain $\Omega \subset \mathbb{R}^2$ by a $C^2\left(\overline{\Omega}; \mathbb{R}^3\right)$ map v, $v = v(x, y)$, with $v_x \times v_y \neq 0$ in $\overline{\Omega}$. Denote by $L(\partial A)$ and $M(A)$ the area of the boundary ∂A and the volume of A respectively.*

Show that if there exists $A_0 \in \mathcal{A}$ so that

$$L(\partial A_0) = \inf_{A \in \mathcal{A}} \{L(\partial A) : M(A) = M(A_0)\}$$

then ∂A_0 has constant mean curvature.

Chapter 7

Solutions to the Exercises

7.1 Chapter 1: Preliminaries

7.1.1 Continuous and Hölder continuous functions

Exercise 1.2.1. (i) We have

$$\|uv\|_{C^{0,\alpha}} = \|uv\|_{C^0} + [uv]_{C^{0,\alpha}} \ .$$

Since

$$\begin{aligned} [uv]_{C^{0,\alpha}} &\leq \sup \frac{|u(x)v(x) - u(y)v(y)|}{|x-y|^\alpha} \\ &\leq \|u\|_{C^0} \sup \frac{|v(x) - v(y)|}{|x-y|^\alpha} + \|v\|_{C^0} \sup \frac{|u(x) - u(y)|}{|x-y|^\alpha} \end{aligned}$$

we deduce that

$$\begin{aligned} \|uv\|_{C^{0,\alpha}} &\leq \|u\|_{C^0} \|v\|_{C^0} + \|u\|_{C^0} [v]_{C^{0,\alpha}} + \|v\|_{C^0} [u]_{C^{0,\alpha}} \\ &\leq 2 \|u\|_{C^{0,\alpha}} \|v\|_{C^{0,\alpha}} \ . \end{aligned}$$

(ii) The inclusion $C^{k,\alpha} \subset C^k$ is obvious. Let us show that $C^{k,\beta} \subset C^{k,\alpha}$. We will prove, for the sake of simplicity, only the case $k = 0$. Observe that

$$\sup_{\substack{x,y \in \overline{\Omega} \\ 0<|x-y|<1}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\alpha} \right\} \leq \sup_{\substack{x,y \in \overline{\Omega} \\ 0<|x-y|<1}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\beta} \right\} \leqslant [u]_{C^{0,\beta}} \ .$$

Since

$$\sup_{\substack{x,y \in \overline{\Omega} \\ |x-y|\geq 1}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\alpha} \right\} \leq \sup_{x,y \in \overline{\Omega}} \{|u(x)| - u(y)\} \leq 2 \|u\|_{C^0}$$

we get

$$\begin{aligned}\|u\|_{C^{0,\alpha}} &= \|u\|_{C^0} + [u]_{C^{0,\alpha}} \\ &\leq \|u\|_{C^0} + \max\left\{2\|u\|_{C^0}, [u]_{C^{0,\beta}}\right\} \leq 3\|u\|_{C^{0,\beta}} .\end{aligned}$$

(iii) We now assume that Ω is bounded and convex and $k = 0$ (for the sake of simplicity) and let us show that $C^1\left(\overline{\Omega}\right) \subset C^{0,1}\left(\overline{\Omega}\right)$. From the mean value theorem, we have that for every $x, y \in \overline{\Omega}$ we can find $z \in [x, y] \subset \overline{\Omega}$ (i.e., z is an element of the segment joining x to y) so that

$$u(x) - u(y) = \langle \nabla u(z); x - y \rangle .$$

We have, indeed, obtained that

$$|u(x) - u(y)| \leq |\nabla u(z)| \, |x - y| \leq \|u\|_{C^1} |x - y| .$$

7.1.2 L^p spaces

Exercise 1.3.1. (i) **Hölder inequality**. Let $a, b > 0$ and $1/p + 1/p' = 1$, with $1 < p < \infty$. Since the function $f(x) = \log x$ is concave, we have that

$$\log\left(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}\right) \geqslant \frac{1}{p}\log a^p + \frac{1}{p'}\log b^{p'} = \log ab$$

and hence

$$\frac{1}{p}a^p + \frac{1}{p'}b^{p'} \geq ab .$$

Choose then

$$a = \frac{|u|}{\|u\|_{L^p}}, \; b = \frac{|v|}{\|v\|_{L^{p'}}}$$

and integrate to get the inequality for $1 < p < \infty$. The cases $p = 1$ or $p = \infty$ are trivial.

Minkowski inequality. The cases $p = 1$ and $p = \infty$ are obvious. We therefore assume that $1 < p < \infty$. Use Hölder inequality to get

$$\begin{aligned}\|u + v\|_{L^p}^p &= \int_\Omega |u + v|^p \leq \int_\Omega |u| \, |u + v|^{p-1} + \int_\Omega |v| \, |u + v|^{p-1} \\ &\leq \|u\|_{L^p} \left\| |u + v|^{p-1} \right\|_{L^{p'}} + \|v\|_{L^p} \left\| |u + v|^{p-1} \right\|_{L^{p'}} .\end{aligned}$$

The result then follows, since

$$\left\| |u + v|^{p-1} \right\|_{L^{p'}} = \|u + v\|_{L^p}^{p-1} .$$

(ii) Use Hölder inequality with $\alpha = (p+q)/q$ and hence $\alpha' = (p+q)/p$ to obtain

$$\begin{aligned}\int_\Omega |uv|^{pq/p+q} &\leq \left(\int_\Omega |u|^{pq\alpha/(p+q)}\right)^{1/\alpha}\left(\int_\Omega |v|^{pq\alpha'/(p+q)}\right)^{1/\alpha'}\\ &\leq \left(\int_\Omega |u|^p\right)^{q/(p+q)}\left(\int_\Omega |v|^q\right)^{p/(p+q)}.\end{aligned}$$

(iii) The inclusion $L^\infty(\Omega) \subset L^p(\Omega)$ is trivial. The other inclusions follow from Hölder inequality. Indeed we have

$$\begin{aligned}\int_\Omega |u|^q &= \int_\Omega (|u|^q \cdot 1) \leq \left(\int_\Omega |u|^{q\cdot p/q}\right)^{q/p}\left(\int_\Omega 1^{p/(p-q)}\right)^{(p-q)/p}\\ &\leq (\operatorname{meas}\Omega)^{(p-q)/p}\left(\int_\Omega |u|^p\right)^{q/p}\end{aligned}$$

and hence

$$\|u\|_{L^q} \leq (\operatorname{meas}\Omega)^{(p-q)/pq}\|u\|_{L^p}$$

which gives the desired inclusion.

If, however, the measure is not finite the result is not valid as the simple example $\Omega = (1,\infty)$, $u(x) = 1/x$ shows; indeed we have $u \in L^2$ but $u \notin L^1$.

Exercise 1.3.2. A direct computation leads to

$$\|u_\nu\|_{L^p}^p = \int_0^1 |u_\nu(x)|^p\,dx = \int_0^{1/\nu} \nu^{\alpha p}dx = \nu^{\alpha p-1}.$$

We therefore have that $u_\nu \to 0$ in L^p provided $\alpha p - 1 < 0$. If $\alpha = 1/p$, let us show that $u_\nu \rightharpoonup 0$ in L^p. We have to prove that for every $\varphi \in L^{p'}(0,1)$, the following convergence holds

$$\lim_{\nu\to\infty}\int_0^1 u_\nu(x)\,\varphi(x)\,dx = 0\,.$$

By a density argument, and since $\|u_\nu\|_{L^p} = 1$, it is sufficient to prove the result when φ is a step function, which means that there exist $0 = a_0 < a_1 < \cdots < a_N = 1$ so that $\varphi(x) = \alpha_i$ whenever $x \in (a_i, a_{i+1})$, $0 \leq i \leq N-1$. We hence find, for ν sufficiently large, that

$$\int_0^1 u_\nu(x)\,\varphi(x)\,dx = \alpha_0\int_{a_0}^{1/\nu} \nu^{1/p}dx = \alpha_0\nu^{(1/p)-1} \to 0\,.$$

Exercise 1.3.3. (i) We have to show that for every $\varphi \in L^\infty$, then

$$\lim_{\nu\to\infty}\int_\Omega (u_\nu v_\nu - uv)\,\varphi = 0\,.$$

Rewriting the integral we have

$$\int_\Omega (u_\nu v_\nu - uv)\,\varphi = \int_\Omega u_\nu\,(v_\nu - v)\,\varphi + \int_\Omega (u_\nu - u)\,v\varphi\,.$$

Since $(v_\nu - v) \to 0$ in $L^{p'}$ and $\|u_\nu\|_{L^p} \le K$, we deduce from Hölder inequality that the first integral tends to 0. The second one also tends to 0 since $u_\nu - u \rightharpoonup 0$ in L^p and $v\varphi \in L^{p'}$ (this follows from the hypotheses $v \in L^{p'}$ and $\varphi \in L^\infty$).

Let us show that the result is, in general, false if $v_\nu \rightharpoonup v$ in $L^{p'}$ (instead of $v_\nu \to v$ in $L^{p'}$). Choose $p = p' = 2$ and $u_\nu(x) = v_\nu(x) = \sin \nu x$, $\Omega = (0, 2\pi)$. We have that $u_\nu, v_\nu \rightharpoonup 0$ in L^2 but the product $u_\nu v_\nu$ does not tend to 0 weakly in L^2 (since $u_\nu(x)v_\nu(x) = \sin^2 \nu x \rightharpoonup 1/2 \neq 0$ in L^2).

(ii) We want to prove that $\|u_\nu - u\|_{L^2} \to 0$. We write

$$\int_\Omega |u_\nu - u|^2 = \int_\Omega u_\nu^2 - 2\int_\Omega uu_\nu + \int_\Omega u^2.$$

The first integral tends to $\int u^2$ since $u_\nu^2 \rightharpoonup u^2$ in L^1 (choosing $\varphi(x) \equiv 1 \in L^\infty$ in the definition of weak convergence). The second one tends to $-2\int u^2$ since $u_\nu \rightharpoonup u$ in L^2 and $u \in L^2$. The claim then follows.

Exercise 1.3.4. (i) The case $p = \infty$ is trivial. So assume that $1 \le p < \infty$. We next compute

$$\begin{aligned} u_\nu(x) &= \int_{-\infty}^{+\infty} \varphi_\nu\,(x-y)\,u\,(y)\;dy = \nu \int_{-\infty}^{+\infty} \varphi\,(\nu\,(x-y))\,u\,(y)\;dy \\ &= \int_{-\infty}^{+\infty} \varphi\,(z)\,u\left(x - \frac{z}{\nu}\right)\,dz\,. \end{aligned}$$

We therefore find

$$\begin{aligned} |u_\nu\,(x)| &\le \int_{-\infty}^{+\infty} \varphi(z)\left|u\left(x-\frac{z}{\nu}\right)\right|\,dz \\ &= \int_{-\infty}^{+\infty} |\varphi(z)|^{1/p'}\left[|\varphi(z)|^{1/p}\left|u\left(x-\frac{z}{\nu}\right)\right|\right]\,dz\,. \end{aligned}$$

Hölder inequality leads to

$$|u_\nu\,(x)| \le \left(\int_{-\infty}^{+\infty} \varphi(z)\,dz\right)^{1/p'} \left(\int_{-\infty}^{+\infty} \varphi(z)\left|u\left(x-\frac{z}{\nu}\right)\right|^p dz\right)^{1/p}.$$

Since $\int \varphi = 1$, we have, after interchanging the order of integration,

$$\begin{aligned} \|u_\nu\|_{L^p}^p &= \int_{-\infty}^{+\infty} |u_\nu\,(x)|^p\,dx \le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left\{\varphi\,(z)\left|u\left(x-\frac{z}{\nu}\right)\right|^p dz\right\} dx \\ &\le \int_{-\infty}^{+\infty} \left\{\varphi\,(z)\int_{-\infty}^{+\infty}\left|u\left(x-\frac{z}{\nu}\right)\right|^p dx\right\} dz \le \|u\|_{L^p}^p\;. \end{aligned}$$

(ii) The result follows, since φ is C^∞ and

$$u'_\nu(x) = \int_{-\infty}^{+\infty} \varphi'_\nu(x-y)\, u(y)\; dy\,.$$

(iii) Let $K \subset \mathbb{R}$ be a fixed compact. Since u is continuous, we have that for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, K) > 0$ so that

$$|y| \le \delta \Rightarrow |u(x-y) - u(x)| \le \epsilon,\ \forall x \in K\,.$$

Since $\varphi = 0$ if $|x| > 1$, $\int \varphi = 1$, and hence $\int \varphi_\nu = 1$, we find that

$$\begin{aligned} u_\nu(x) - u(x) &= \int_{-\infty}^{+\infty} [u(x-y) - u(x)]\, \varphi_\nu(y)\; dy \\ &= \int_{-1/\nu}^{1/\nu} [u(x-y) - u(x)]\, \varphi_\nu(y)\; dy\,. \end{aligned}$$

Taking $x \in K$ and $\nu > 1/\delta$, we deduce that $|u_\nu(x) - u(x)| \le \epsilon$, and thus the claim.

(iv) Since $u \in L^p(\mathbb{R})$ and $1 \le p < \infty$, we deduce (see Theorem 1.13) that for every $\epsilon > 0$, there exists $\overline{u} \in C_0(\mathbb{R})$ so that

$$\|u - \overline{u}\|_{L^p} \le \epsilon\,. \tag{7.1}$$

Define then

$$\overline{u}_\nu(x) = (\varphi_\nu * \overline{u})(x) = \int_{-\infty}^{+\infty} \varphi_\nu(x-y)\, \overline{u}(y)\; dy\,.$$

Since $u - \overline{u} \in L^p$, it follows from (i) that

$$\|u_\nu - \overline{u}_\nu\|_{L^p} \le \|u - \overline{u}\|_{L^p} \le \epsilon. \tag{7.2}$$

Moreover, since $\operatorname{supp} \overline{u}$ is compact and $\varphi = 0$ if $|x| > 1$, we find that there exists a compact set K so that $\operatorname{supp} \overline{u}, \operatorname{supp} \overline{u}_\nu \subset K$ (for every ν). From (iii) we then get that $\|\overline{u}_\nu - \overline{u}\|_{L^p} \to 0$. Combining (7.1) and (7.2), we deduce that

$$\begin{aligned} \|u_\nu - u\|_{L^p} &\le \|u_\nu - \overline{u}_\nu\|_{L^p} + \|\overline{u}_\nu - \overline{u}\|_{L^p} + \|\overline{u} - u\|_{L^p} \\ &\le 2\epsilon + \|\overline{u}_\nu - \overline{u}\|_{L^p} \end{aligned}$$

which is the claim, since ϵ is arbitrary.

Exercise 1.3.5. We adopt the same hypotheses and notations of Theorem 1.22. Step 1 remains unchanged and we modify Step 2 as follows.

We define

$$\begin{aligned} v_\nu(x) &= \int_0^x u_\nu(t)\,dt = \int_0^x u(\nu t)\,dt = \frac{1}{\nu}\int_0^{\nu x} u(s)\,ds \\ &= \frac{1}{\nu}\int_{[\nu x]}^{\nu x} u(s)\,ds = \frac{1}{\nu}\int_0^{\nu x-[\nu x]} u(s)\,ds \end{aligned}$$

where $[a]$ stands for the integer part of $a \geq 0$ and where we have used the periodicity of u and the fact that $\overline{u} = \int_0^1 u = 0$.

We therefore find that

$$\|v_\nu\|_{L^\infty} \leq \frac{1}{\nu}\int_0^{\nu x-[\nu x]} |u(s)|\,ds \leq \frac{1}{\nu}\int_0^1 |u(s)|\,ds = \frac{1}{\nu}\|u\|_{L^1} \leq \frac{1}{\nu}\|u\|_{L^p}\,. \tag{7.3}$$

Recall that we have to show that

$$\lim_{\nu\to\infty}\int_0^1 u_\nu(x)\,\varphi(x)\,dx = 0,\ \forall\varphi \in L^{p'}(0,1)\,. \tag{7.4}$$

Let $\epsilon > 0$ be arbitrary. Since $\varphi \in L^{p'}(0,1)$ and $1 < p \leq \infty$, which implies $1 \leq p' < \infty$ (i.e., $p' \neq \infty$), we have from Theorem 1.13 that there exists $\psi \in C_0^\infty(0,1)$ so that

$$\|\varphi - \psi\|_{L^{p'}} \leq \epsilon\,. \tag{7.5}$$

We now compute

$$\begin{aligned} \int_0^1 u_\nu(x)\,\varphi(x)\,dx &= \int_0^1 u_\nu(x)\,[\varphi(x)-\psi(x)]\,dx + \int_0^1 u_\nu(x)\,\psi(x)\,dx \\ &= \int_0^1 u_\nu(x)\,[\varphi(x)-\psi(x)]\,dx - \int_0^1 v_\nu(x)\,\psi'(x)\,dx \end{aligned}$$

where we have used integration by parts, the fact that $v_\nu' = u_\nu$ and $\psi \in C_0^\infty(0,1)$. Using Hölder inequality, (1.1), (7.3) and (7.5), we obtain that

$$\left|\int_0^1 u_\nu(x)\,\varphi(x)\,dx\right| \leq \epsilon\|u\|_{L^p} + \|v_\nu\|_{L^\infty}\|\psi'\|_{L^1} \leq \epsilon\|u\|_{L^p} + \frac{1}{\nu}\|u\|_{L^p}\|\psi'\|_{L^1}\,.$$

Let $\nu \to \infty$, we hence obtain

$$0 \leq \limsup_{\nu\to\infty}\left|\int_0^1 u_\nu\varphi\,dx\right| \leq \epsilon\|u\|_{L^p}\,.$$

Since ϵ is arbitrary, we immediately have (7.4) and thus the result.

Exercise 1.3.6. (i) Let $f \in C_0^\infty(\Omega)$ with $\int_\Omega f(x)\,dx = 1$, be a fixed function. Let $w \in C_0^\infty(\Omega)$ be arbitrary and

$$\psi(x) = w(x) - \left[\int_\Omega w(y)\,dy\right] f(x)\,.$$

We therefore have $\psi \in C_0^\infty(\Omega)$ and $\int \psi = 0$ which leads to

$$\begin{aligned} 0 &= \int_\Omega u(x)\,\psi(x)\;dx = \int_\Omega u(x)\,w(x)\;dx - \int_\Omega f(x)\,u(x)\;dx \cdot \int_\Omega w(y)\;dy \\ &= \int_\Omega \left[u(x) - \int_\Omega u(y)\,f(y)\,dy\right] w(x)\;dx\,. \end{aligned}$$

Appealing to Theorem 1.24, we deduce that $u(x) = \int u(y)\,f(y)\;dy =$ constant a.e.

(ii) Let $\psi \in C_0^\infty(a,b)$, with $\int_a^b \psi = 0$, be arbitrary and define

$$\varphi(x) = \int_a^x \psi(t)\;dt\,.$$

Note that $\psi = \varphi'$ and $\varphi \in C_0^\infty(a,b)$. We may thus apply (i) and get the result.

Exercise 1.3.7. Let, for $\nu \in \mathbb{N}$,

$$u_\nu(x) = \min\{|u(x)|\,,\nu\}\,.$$

The monotone convergence theorem implies that, for every $\epsilon > 0$, we can find ν sufficiently large so that

$$\int_\Omega |u(x)|\;dx \le \int_\Omega u_\nu(x)\;dx + \frac{\epsilon}{2}\,.$$

Choose then $\delta = \epsilon/2\nu$. We therefore deduce that, if $\operatorname{meas} E \le \delta$, then

$$\int_E |u(x)|\;dx = \int_E u_\nu(x)\;dx + \int_E [|u(x)| - u_\nu(x)]\;dx \le \nu \operatorname{meas} E + \frac{\epsilon}{2} \le \epsilon\,.$$

For a more general setting see, for example, Theorem 5.18 in De Barra [37].

7.1.3 Sobolev spaces

Exercise 1.4.1. Let $\sigma_{n-1} = \operatorname{meas}(\partial B_1(0))$ (i.e. $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi$,...).

(i) The result follows from the following observation

$$\|u\|_{L^p}^p = \int_{B_R} |u(x)|^p\,dx = \sigma_{n-1}\int_0^R r^{n-1}\,|f(r)|^p\,dr.$$

(ii) We find, if $x \neq 0$, that

$$u_{x_i} = f'(|x|)\,\frac{x_i}{|x|} \;\Rightarrow\; |\nabla u(x)| = |f'(|x|)|\,.$$

Assume, for a moment, that we already proved that u is weakly differentiable in B_R, then

$$\|\nabla u\|_{L^p}^p = \sigma_{n-1}\int_0^R r^{n-1}\,|f'(r)|^p\,dr\,,$$

which is the claim.

Let us now show that u_{x_i}, as above, is indeed the weak derivative (with respect to x_i) of u. We have to prove that, for every $\varphi \in C_0^\infty(B_R)$,

$$\int_{B_R} u\varphi_{x_i}\,dx = -\int_{B_R}\varphi u_{x_i}\,dx\,. \tag{7.6}$$

Let $\epsilon > 0$ be sufficiently small and observe that (recall that $\varphi = 0$ on ∂B_R)

$$\begin{aligned}\int_{B_R} u\varphi_{x_i}dx &= \int_{B_R\setminus B_\epsilon} u\varphi_{x_i}dx + \int_{B_\epsilon} u\varphi_{x_i}dx\\ &= -\int_{B_R\setminus B_\epsilon}\varphi u_{x_i}dx - \int_{\partial B_\epsilon} u\varphi\frac{x_i}{|x|}d\sigma + \int_{B_\epsilon} u\varphi_{x_i}dx\\ &= -\int_{B_R}\varphi u_{x_i}dx + \int_{B_\epsilon}\varphi u_{x_i}dx + \int_{B_\epsilon} u\varphi_{x_i}dx - \int_{\partial B_\epsilon} u\varphi\frac{x_i}{|x|}d\sigma\,.\end{aligned}$$

Since the elements φu_{x_i} and $u\varphi_{x_i}$ are both in $L^1(B_R)$, we deduce (this follows from Hölder inequality if $p > 1$ or from standard properties of integrals if $p \geq 1$, see Exercise 1.3.7) that

$$\lim_{\epsilon\to 0}\int_{B_\epsilon}\varphi u_{x_i}dx = \lim_{\epsilon\to 0}\int_{B_\epsilon} u\varphi_{x_i}dx = 0\,.$$

Moreover, by hypothesis, we have the claim (i.e. (7.6)), since

$$\left|\int_{\partial B_\epsilon} u\varphi\frac{x_i}{|x|}d\sigma\right| \leq \sigma_{n-1}\|\varphi\|_{L^\infty}\,\epsilon^{n-1} f(\epsilon) \to 0,\ \text{as } \epsilon \to 0\,.$$

(iii) 1) The first example follows at once and gives

$$\psi \in L^p \;\Leftrightarrow\; sp < n \text{ and } \psi \in W^{1,p} \;\Leftrightarrow\; (s+1)\,p < n\,.$$

2) We find, for every $0 < s < 1/2$ and $p \geq 1$, that

$$\int_0^{1/2} r\,|\log r|^{sp}\,dr < \infty,\ \int_0^{1/2} r^{-1}\,|\log r|^{2(s-1)}\,dr = \frac{|\log 2|^{2s-1}}{1-2s} < \infty\,.$$

The first one guarantees that $\psi \in L^p(B_R)$ and the second one that $\psi \in W^{1,2}(B_R)$. The fact that $\psi \notin L^\infty(B_R)$ is obvious.

3) We have, denoting by δ_{ij} the Kronecker symbol, that

$$\frac{\partial u^i}{\partial x_j} = \frac{\delta_{ij}|x|^2 - x_i x_j}{|x|^3} \implies |\nabla u|^2 = \frac{n-1}{|x|^2}\,.$$

We therefore find

$$\int_\Omega |\nabla u(x)|^p\, dx = (n-1)^{p/2}\,\sigma_{n-1}\int_0^1 r^{n-1-p} dr\,.$$

This quantity is finite if and only if $p \in [1, n)$.

Exercise 1.4.2. The inclusion $AC([a,b]) \subset C([a,b])$ is easy. Indeed by definition any function in $AC([a,b])$ is uniformly continuous in (a,b) and therefore can be continuously extended to $[a,b]$.

Let us now discuss the second inclusion, namely $W^{1,1}(a,b) \subset AC([a,b])$. Let $u \in W^{1,1}(a,b)$. We know from Lemma 1.38 that

$$u(b_k) - u(a_k) = \int_{a_k}^{b_k} u'(t)\, dt\,.$$

We therefore find

$$\sum_k |u(b_k) - u(a_k)| \le \sum_k \int_{a_k}^{b_k} |u'(t)|\, dt\,.$$

Let $E = \cup_k (a_k, b_k)$. A classical property of Lebesgue integral (see Exercise 1.3.7) asserts that if $u' \in L^1$, then, for every $\epsilon > 0$, there exists $\delta > 0$ so that

$$\operatorname{meas} E = \sum_k |b_k - a_k| < \delta \;\Rightarrow\; \int_E |u'| < \epsilon\,.$$

The claim then follows.

Exercise 1.4.3. This follows from Hölder inequality, since

$$\begin{aligned}
|u(x) - u(y)| &\le \int_y^x |u'(t)|\, dt \le \left(\int_y^x |u'(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_y^x 1^{p'} dt\right)^{\frac{1}{p'}} \\
&\le \left(\int_y^x |u'(t)|^p\, dt\right)^{\frac{1}{p}} |x-y|^{\frac{1}{p'}}
\end{aligned}$$

and by the properties of Lebesgue integrals (see Exercise 1.3.7) the quantity $\left(\int_y^x |u'(t)|^p\,dt\right)^{\frac{1}{p}}$ tends to 0 as $|x-y|$ tends to 0.

Exercise 1.4.4. Observe first that if $v \in W^{1,p}(a,b)$, $p>1$ and $y>x$, then

$$\begin{aligned} |v(x)-v(y)| &= \left|\int_x^y v'(z)\,dz\right| \\ &\le \left(\int_x^y |v'(z)|^p\,dz\right)^{1/p}\left(\int_x^y dz\right)^{1/p'} \le \|v'\|_{L^p}\,|x-y|^{1/p'}. \end{aligned} \tag{7.7}$$

Let us now show that if $u_\nu \rightharpoonup u$ in $W^{1,p}$, then $u_\nu \to u$ in L^∞. Without loss of generality, we can take $u \equiv 0$. Assume, for the sake of contradiction, that $u_\nu \nrightarrow 0$ in L^∞. We can therefore find $\epsilon>0$, $\{\nu_i\}$ so that

$$\|u_{\nu_i}\|_{L^\infty} \ge \epsilon,\ \nu_i \to \infty. \tag{7.8}$$

From (7.7) we have that the subsequence $\{u_{\nu_i}\}$ is equicontinuous (note also that by Theorem 1.42 and Theorem 1.20 (iii) we have $\|u_{\nu_i}\|_{L^\infty} \le c'\|u_{\nu_i}\|_{W^{1,p}} \le c$) and hence from Ascoli-Arzela theorem, we find, up to a subsequence,

$$u_{\nu_{i_j}} \to v \text{ in } L^\infty. \tag{7.9}$$

We, however, must have $v=0$ since (7.9) implies $u_{\nu_{i_j}} \rightharpoonup v$ in L^p and by uniqueness of the limits (we already know that $u_{\nu_{i_j}} \rightharpoonup u = 0$ in L^p) we deduce that $v=0$ a.e., which contradicts (7.8).

Exercise 1.4.5. Follows immediately from Theorem 1.20.

Exercise 1.4.6. It is clear that $u_\nu \to 0$ in L^∞. We also find

$$\frac{\partial u_\nu}{\partial x} = \sqrt{\nu}\,(1-y)^\nu \cos(\nu x)\,,\ \frac{\partial u_\nu}{\partial y} = -\sqrt{\nu}\,(1-y)^{\nu-1}\sin(\nu x)$$

which implies that, there exists a constant $K>0$ independent of ν, such that

$$\iint_\Omega |\nabla u_\nu(x,y)|^2\,dxdy \le K\,.$$

Apply Exercise 1.4.5 to get the result.

Exercise 1.4.7. Since $u \in W^{1,p}(\Omega)$, we have that it is weakly differentiable and therefore

$$\int_\Omega \left[u_{x_i}\psi + u\psi_{x_i}\right]\,dx = 0,\ \forall \psi \in C_0^\infty(\Omega)\,.$$

Let $\varphi \in W_0^{1,p'}(\Omega)$ and $\epsilon>0$ be arbitrary. We can then find $\psi \in C_0^\infty(\Omega)$ so that

$$\|\psi-\varphi\|_{L^{p'}} + \|\nabla\psi - \nabla\varphi\|_{L^{p'}} \le \epsilon\,.$$

We hence obtain, appealing to the two above relations, that

$$\begin{aligned}\left|\int_\Omega \left[u_{x_i}\varphi + u\varphi_{x_i}\right] dx\right| &\leq \int_\Omega \left[\left|u_{x_i}\right| \left|\varphi-\psi\right| + \left|u\right| \left|\varphi_{x_i}-\psi_{x_i}\right|\right] dx \\ &\leq \left\|u\right\|_{W^{1,p}} \left[\left\|\psi-\varphi\right\|_{L^{p'}} + \left\|\varphi_{x_i}-\psi_{x_i}\right\|_{L^{p'}}\right] \leq \epsilon \left\|u\right\|_{W^{1,p}} .\end{aligned}$$

Since ϵ is arbitrary, we have indeed obtained that

$$\int_\Omega u_{x_i}\varphi \, dx = -\int_\Omega u\varphi_{x_i} \, dx, \; i = 1, ..., n .$$

7.1.4 Convex analysis

Exercise 1.5.1. (i) We first show that if f is convex then, for every $x, y \in \mathbb{R}$,

$$f(x) \geq f(y) + f'(y)(x-y) .$$

Apply the inequality of convexity

$$\frac{1}{\lambda}\left[f(y+\lambda(x-y)) - f(y)\right] \leq f(x) - f(y)$$

and let $\lambda \to 0$ to get the result.

We next show the converse. Let $\lambda \in [0,1]$ and apply the above inequality to find

$$\begin{aligned} f(x) &\geq f(\lambda x + (1-\lambda) y) + (1-\lambda) f'(\lambda x + (1-\lambda) y)(x-y) \\ f(y) &\geq f(\lambda x + (1-\lambda) y) - \lambda f'(\lambda x + (1-\lambda) y)(x-y) .\end{aligned}$$

Multiplying the first inequality by λ, the second one by $(1-\lambda)$ and summing the two of them, we have indeed obtained the desired convexity inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) .$$

(ii) We next show that f is convex if and only if, for every $x, y \in \mathbb{R}$,

$$\left[f'(x) - f'(y)\right](x-y) \geq 0 . \tag{7.10}$$

Once this will be established, we will get the claimed result, namely that if $f \in C^2(\mathbb{R})$ then f is convex if and only if $f''(x) \geq 0$ for every $x \in \mathbb{R}$. We start by assuming that f is convex and hence apply (i) to get

$$\begin{aligned} f(x) &\geq f(y) + f'(y)(x-y) \\ f(y) &\geq f(x) + f'(x)(y-x) .\end{aligned}$$

Combining the two inequalities we have immediately (7.10).

Let us now show the converse and assume that (7.10) holds. Let $\lambda \in (0,1)$, $x, y \in \mathbb{R}$ and define

$$z = \frac{y-x}{\lambda} + x \;\Leftrightarrow\; y = x + \lambda(z-x)$$

$$\varphi(\lambda) = f(x + \lambda(z-x)) \; .$$

Observe that

$$\begin{aligned} \varphi'(\lambda) - \varphi'(0) &= [f'(x + \lambda(z-x)) - f'(x)](z-x) \\ &= \frac{1}{\lambda}[f'(x + \lambda(z-x)) - f'(x)](x + \lambda(z-x) - x) \geq 0 \end{aligned}$$

since (7.10) holds. Therefore, integrating the inequality, we find

$$\varphi(\lambda) \geq \varphi(0) + \lambda\varphi'(0)$$

and thus, returning to the definition of y, we find

$$f(y) \geq f(x) + f'(x)(y-x)$$

which is equivalent by (i) to the convexity of f.

Exercise 1.5.2. Since f is convex, we have, for every $\alpha, \beta \in \mathbb{R}$,

$$f(\alpha) \geq f(\beta) + f'(\beta)(\alpha - \beta) \; .$$

Choose then $\alpha = u(x)$ and $\beta = (1/\operatorname{meas}\Omega)\int_\Omega u(x)\, dx$, and integrate to get the inequality.

Exercise 1.5.3. We easily find that

$$f^*(x^*) = \begin{cases} -\sqrt{1-x^{*2}} & \text{if } |x^*| \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Note, in passing, that $f(x) = \sqrt{1+x^2}$ is strictly convex over $\mathbb{R}$.

Exercise 1.5.4. (i) We have that

$$f^*(x^*) = \sup_{x\in\mathbb{R}} \left\{ xx^* - \frac{|x|^p}{p} \right\} .$$

The supremum is, in fact, attained at a point y where

$$x^* = |y|^{p-2}\, y \;\Leftrightarrow\; y = |x^*|^{p'-2}\, x^*.$$

Replacing this value in the definition of f^* we have obtained that

$$f^*(x^*) = \frac{|x^*|^{p'}}{p'}.$$

(ii) We do not compute f^*, but instead use Theorem 1.55. We let

$$g(x) = \begin{cases} \left(x^2-1\right)^2 & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| < 1 \end{cases}$$

and we wish to show that $f^{**} = g$. We start by observing that g is convex, $0 \leq g \leq f$, and therefore according to Theorem 1.54 (ii) we must have

$$g \leq f^{**} \leq f.$$

First consider the case where $|x| \geq 1$; the functions g and f coincide there and hence $f^{**}(x) = g(x)$, for such x. We next consider the case $|x| < 1$. Choose in Theorem 1.55

$$x_1 = 1,\ x_2 = -1,\ \lambda_1 = \frac{1+x}{2},\ \lambda_2 = \frac{1-x}{2}$$

to get immediately that $f^{**}(x) = g(x) = 0$. We have therefore proved the claim.

(iv) This is straightforward since clearly

$$f^*(\xi^*) = \sup_{\xi \in \mathbb{R}^{2\times 2}} \{\langle \xi; \xi^* \rangle - \det \xi\} \equiv +\infty$$

and therefore

$$f^{**}(\xi) = \sup_{\xi^* \in \mathbb{R}^{2\times 2}} \{\langle \xi; \xi^* \rangle - f^*(\xi^*)\} \equiv -\infty.$$

Exercise 1.5.5. (i) Let $x^*, y^* \in \mathbb{R}^n$ and $\lambda \in [0,1]$. It follows from the definition that

$$\begin{aligned} f^*(\lambda x^* + (1-\lambda) y^*) &= \sup_{x \in \mathbb{R}^n} \{\langle x; \lambda x^* + (1-\lambda) y^* \rangle - f(x)\} \\ &= \sup_x \{\lambda (\langle x; x^* \rangle - f(x)) + (1-\lambda)(\langle x; y^* \rangle - f(x))\} \\ &\leq \lambda \sup_x \{\langle x; x^* \rangle - f(x)\} + (1-\lambda) \sup_x \{\langle x; y^* \rangle - f(x)\} \\ &\leq \lambda f^*(x^*) + (1-\lambda) f^*(y^*). \end{aligned}$$

(ii) For this part we can refer to Theorem I.10 in Brézis [14], Theorem 2.2.5 in [31] or Theorem 12.2 (coupled with Corollaries 10.1.1 and 12.1.1) in Rockafellar [87].

(iii) Since $f^{**} \leq f$, we find that $f^{***} \geq f^{*}$. Furthermore, by definition of f^{**}, we find, for every $x \in \mathbb{R}^n$, $x^* \in \mathbb{R}^n$,

$$\langle x; x^* \rangle - f^{**}(x) \leq f^*(x^*) \ .$$

Taking the supremum over all x in the left hand side of the inequality, we get $f^{***} \leq f^*$, and hence the claim.

(iv) By definition of f^*, we have

$$f^*(\nabla f(x)) = \sup_y \{\langle y; \nabla f(x) \rangle - f(y)\} \geq \langle x; \nabla f(x) \rangle - f(x)$$

and hence

$$f(x) + f^*(\nabla f(x)) \geq \langle x; \nabla f(x) \rangle \ .$$

We next show the opposite inequality. Since f is convex, we have

$$f(y) \geq f(x) + \langle y - x; \nabla f(x) \rangle \ ,$$

which means that

$$\langle x; \nabla f(x) \rangle - f(x) \geq \langle y; \nabla f(x) \rangle - f(y) \ .$$

Taking the supremum over all y, we have indeed obtained the opposite inequality and thus the proof is complete.

(v) We refer to the bibliography, in particular to Mawhin-Willem [72], page 35, Rockafellar [87] (Theorems 23.5, 26.3 and 26.5 as well as Corollary 25.5.1) and for the second part to [31] or Theorem 23.5 in Rockafellar [87]; see also the exercise below.

Exercise 1.5.6. We divide the proof into three steps.

Step 1. We know that

$$f^*(v) = \sup_\xi \{\xi v - f(\xi)\}$$

and since $f \in C^1$ and satisfies

$$\lim_{|\xi| \to \infty} \frac{f(\xi)}{|\xi|} = +\infty \tag{7.11}$$

we deduce that there exists $\xi = \xi(v)$ such that

$$f^*(v) = \xi v - f(\xi) \text{ and } v = f'(\xi) \ . \tag{7.12}$$

Step 2. Since f satisfies (7.11), we have

$$\text{Image}\,[f'(\mathbb{R})] = \mathbb{R}\,.$$

Indeed, f being convex, we have

$$f(0) \geq f(\xi) - \xi f'(\xi) \;\Rightarrow\; \lim_{\xi \to \pm\infty} f'(\xi) = \pm\infty\,.$$

Moreover, since $f'' > 0$, we have that f' is strictly increasing and therefore invertible and we hence obtain

$$\xi = f'^{-1}(v)\ .$$

The hypotheses on f clearly imply that $\xi = f'^{-1}$ is $C^1(\mathbb{R})$.

Step 3. We now conclude that

$$f^{*\prime} = f'^{-1}. \tag{7.13}$$

Indeed, we have from (7.12) that

$$\begin{aligned} f^{*\prime}(v) &= \xi(v) + \xi'(v)\,v - f'(\xi(v))\,\xi'(v) \\ &= \xi(v) + \xi'(v)\,[v - f'(\xi(v))] = \xi(v)\ . \end{aligned}$$

We therefore have that f^* is $C^1(\mathbb{R})$ and (7.13). Furthermore, since $f'' > 0$, we deduce that f^* is as regular as f (so, in particular f^* is C^2).

Exercise 1.5.7. (i) Let $h > 0$. Use the convexity of f and (1.14) to write

$$\begin{aligned} hf'(x) &\leq f(x+h) - f(x) \leq \alpha_1(1 + |x+h|^p) + \alpha_1(1 + |x|^p) \\ -hf'(x) &\leq f(x-h) - f(x) \leq \alpha_1(1 + |x-h|^p) + \alpha_1(1 + |x|^p)\ . \end{aligned}$$

We can therefore find $\widetilde{\alpha}_1 > 0$, so that

$$|f'(x)| \leq \frac{\widetilde{\alpha}_1(1 + |x|^p + |h|^p)}{h}\,.$$

Choosing $h = 1 + |x|$, we can surely find $\alpha_2 > 0$ so that (1.15) is satisfied, i.e.,

$$|f'(x)| \leq \alpha_2\left(1 + |x|^{p-1}\right),\ \forall x \in \mathbb{R}\,.$$

The inequality (1.16) is then a consequence of (1.15) and of the mean value theorem.

(ii) Note that the convexity of f is essential in the above argument. Indeed, taking, for example, $f(x) = x + \sin x^2$, we find that f satisfies (1.14) with $p = 1$, but it does not verify (1.15).

(iii) Of course if f' satisfies (1.15), we have by straight integration

$$f(x) = f(0) + \int_0^x f'(s)\ ds$$

that f verifies (1.14), even if f is not convex.

7.2 Chapter 2: Classical methods

7.2.1 Euler-Lagrange equation

Exercise 2.2.1. The proof is almost identical to that of the theorem. The Euler-Lagrange equation becomes then a system of ordinary differential equations, namely, if $u = (u_1, ..., u_N)$ and $\xi = (\xi_1, ..., \xi_N)$, we have

$$\frac{d}{dx}\left[f_{\xi_i}\left(x,\overline{u},\overline{u}'\right)\right] = f_{u_i}\left(x,\overline{u},\overline{u}'\right),\ i = 1, ..., N.$$

Exercise 2.2.2. We proceed as in the theorem. We let

$$X = \left\{u \in C^n\left([a,b]\right) : u^{(j)}(a) = \alpha_j,\ u^{(j)}(b) = \beta_j,\ 0 \le j \le n-1\right\}$$

If $\overline{u} \in X \cap C^{2n}([a,b])$ is a minimizer of (P) we have $I(\overline{u} + \epsilon v) \ge I(\overline{u})$, $\forall \epsilon \in \mathbb{R}$ and $\forall v \in C_0^\infty(a,b)$. Letting $f = f(x, u, \xi_1, ..., \xi_n)$ and using the fact that

$$\frac{d}{d\epsilon} I(\overline{u} + \epsilon v)\Big|_{\epsilon=0} = 0$$

we find

$$\int_a^b \left\{ f_u\left(x,\overline{u},...,\overline{u}^{(n)}\right) v + \sum_{i=1}^n f_{\xi_i}\left(x,\overline{u},...,\overline{u}^{(n)}\right) v^{(i)} \right\} dx = 0,\ \forall v \in C_0^\infty(a,b).$$

Integrating by parts and appealing to the fundamental lemma of the calculus of variations (Theorem 1.24) we find

$$\sum_{i=1}^n (-1)^{i+1} \frac{d^i}{dx^i}\left[f_{\xi_i}\left(x,\overline{u},...,\overline{u}^{(n)}\right)\right] = f_u\left(x,\overline{u},...,\overline{u}^{(n)}\right).$$

Exercise 2.2.3. (i) Let

$$X_0 = \left\{v \in C^1([a,b]) : v(a) = 0\right\}.$$

Let $\overline{u} \in X \cap C^2([a,b])$ be a minimizer for (P), since $I(\overline{u} + \epsilon v) \ge I(\overline{u})$, $\forall v \in X_0$, $\forall \epsilon \in \mathbb{R}$ we deduce as above that

$$\int_a^b \left\{f_u\left(x,\overline{u},\overline{u}'\right) v + f_\xi\left(x,\overline{u},\overline{u}'\right) v'\right\} dx = 0,\ \forall v \in X_0.$$

Integrating by parts (bearing in mind that $v(a) = 0$) we find

$$\int_a^b \left\{\left[f_u - \frac{d}{dx} f_\xi\right] v\right\} dx + f_\xi\left(b, \overline{u}(b), \overline{u}'(b)\right) v(b) = 0,\ \forall v \in X_0.$$

Using the fundamental lemma of the calculus of variations and the fact that $v(b)$ is arbitrary we find

$$\begin{cases} \dfrac{d}{dx}\left[f_\xi\left(x,\overline{u},\overline{u}'\right)\right] = f_u\left(x,\overline{u},\overline{u}'\right),\ \forall x \in [a,b] \\ \\ f_\xi\left(b,\overline{u}\left(b\right),\overline{u}'\left(b\right)\right) = 0\,. \end{cases}$$

We sometimes say that $f_\xi\left(b,\overline{u}\left(b\right),\overline{u}'\left(b\right)\right) = 0$ is a *natural* boundary condition.

(ii) The proof is completely analogous to the preceding one and we find, in addition to the above conditions, that

$$f_\xi\left(a,\overline{u}\left(a\right),\overline{u}'\left(a\right)\right) = 0\,.$$

Exercise 2.2.4. Let $\overline{u} \in X \cap C^2\left([a,b]\right)$ be a minimizer of (P). Recall that

$$X = \left\{u \in C^1\left([a,b]\right) : u\left(a\right) = \alpha,\ u\left(b\right) = \beta,\ \int_a^b g\left(x,u\left(x\right),u'\left(x\right)\right)\,dx = 0\right\}.$$

We assume that there exists $w \in C_0^\infty\left(a,b\right)$ such that

$$\int_a^b \left[g_\xi\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right)w'\left(x\right) + g_u\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right)w\left(x\right)\right]\,dx \neq 0;$$

this is always possible if

$$\frac{d}{dx}\left[g_\xi\left(x,\overline{u},\overline{u}'\right)\right] \neq g_u\left(x,\overline{u},\overline{u}'\right)\,.$$

By homogeneity we choose one such w so that

$$\int_a^b \left[g_\xi\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right)w'\left(x\right) + g_u\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right)w\left(x\right)\right]\,dx = 1\,.$$

Let $v \in C_0^\infty\left(a,b\right)$ be arbitrary, w as above and define for $\epsilon, h \in \mathbb{R}$

$$\begin{aligned} F\left(\epsilon,h\right) &= I\left(\overline{u}+\epsilon v+hw\right) = \int_a^b f\left(x,\overline{u}+\epsilon v+hw,\overline{u}'+\epsilon v'+hw'\right)\,dx \\ G\left(\epsilon,h\right) &= \int_a^b g\left(x,\overline{u}+\epsilon v+hw,\overline{u}'+\epsilon v'+hw'\right)\,dx\,. \end{aligned}$$

Observe that $G\left(0,0\right) = 0$ and that by hypothesis

$$G_h\left(0,0\right) = \int_a^b \left[g_\xi\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right)w'\left(x\right) + g_u\left(x,\overline{u}\left(x\right),\overline{u}'\left(x\right)\right)w\left(x\right)\right]\,dx = 1\,.$$

Applying the implicit function theorem we can find $\epsilon_0 > 0$ and a function $t \in C^1([-\epsilon_0, \epsilon_0])$ with $t(0) = 0$ such that

$$G(\epsilon, t(\epsilon)) = 0, \ \forall \epsilon \in [-\epsilon_0, \epsilon_0]$$

which implies that $\overline{u} + \epsilon v + t(\epsilon) w \in X$. Note also that

$$G_\epsilon(\epsilon, t(\epsilon)) + G_h(\epsilon, t(\epsilon)) t'(\epsilon) = 0, \ \forall \epsilon \in [-\epsilon_0, \epsilon_0]$$

and hence

$$t'(0) = -G_\epsilon(0,0) .$$

Since we know that

$$F(0,0) \leq F(\epsilon, t(\epsilon)), \ \forall \epsilon \in [-\epsilon_0, \epsilon_0]$$

we deduce that

$$F_\epsilon(0,0) + F_h(0,0) t'(0) = 0$$

and thus letting $\lambda = F_h(0,0)$ be the Lagrange multiplier we find

$$F_\epsilon(0,0) - \lambda G_\epsilon(0,0) = 0$$

or in other words

$$\int_a^b \{[f_\xi(x, \overline{u}, \overline{u}') v' + f_u(x, \overline{u}, \overline{u}') v] - \lambda [g_\xi(x, \overline{u}, \overline{u}') v' + g_u(x, \overline{u}, \overline{u}') v]\} \, dx = 0 .$$

Appealing once more to the fundamental lemma of the calculus of variations and to the fact that $v \in C_0^\infty(a,b)$ is arbitrary we get

$$\frac{d}{dx}[f_\xi(x, \overline{u}, \overline{u}')] - f_u(x, \overline{u}, \overline{u}') = \lambda \left\{ \frac{d}{dx}[g_\xi(x, \overline{u}, \overline{u}')] - g_u(x, \overline{u}, \overline{u}') \right\} .$$

Exercise 2.2.5. Let $v \in C_0^1(a,b)$, $\epsilon \in \mathbb{R}$ and set $\varphi(\epsilon) = I(\overline{u} + \epsilon v)$. Since $\overline{u}$ is a minimizer of (P) we have $\varphi(\epsilon) \geq \varphi(0)$, $\forall \epsilon \in \mathbb{R}$, and hence we have that $\varphi'(0) = 0$ (which leads to the Euler-Lagrange equation) and $\varphi''(0) \geq 0$. Computing the last expression we find

$$\int_a^b \left\{ f_{uu} v^2 + 2 f_{u\xi} v v' + f_{\xi\xi} v'^2 \right\} dx \geq 0, \ \forall v \in C_0^1(a,b) .$$

Noting that $2vv' = \left(v^2\right)'$ and recalling that $v(a) = v(b) = 0$, we find

$$\int_a^b \left\{ f_{\xi\xi} v'^2 + \left(f_{uu} - \frac{d}{dx} \, f_{u\xi} \right) v^2 \right\} dx \geq 0, \ \forall v \in C_0^1(a,b) .$$

Exercise 2.2.6. i) Setting

$$u_1(x) = \begin{cases} x & \text{if } x \in [0, 1/2] \\ 1 - x & \text{if } x \in (1/2, 1] \end{cases}$$

we find that $I(u_1) = 0$. Observe however that $u_1 \notin X$ where

$$X = \left\{u \in C^1([0,1]) : u(0) = u(1) = 0\right\}.$$

Let $\epsilon > 0$. Since $u_1 \in W_0^{1,\infty}(0,1)$, we can find $v \in C_0^\infty(0,1)$ (hence in particular $v \in X$) such that

$$\|u_1 - v\|_{W^{1,4}} \leq \epsilon .$$

Note also that since $f(\xi) = \left(\xi^2 - 1\right)^2$ we can find $K > 0$ such that

$$|f(\xi) - f(n)| \leq K\left(1 + |\xi|^3 + |\eta|^3\right)|\xi - \eta| .$$

Combining the above inequalities with Hölder inequality we get ($\widetilde{K}$ denoting a constant not depending on ϵ)

$$\begin{aligned} 0 &\leq m \leq I(v) = I(v) - I(u_1) \leq \int_0^1 |f(v') - f(u_1')|\, dx \\ &\leq K\int_0^1 \left\{\left(1 + |v'|^3 + |u_1'|^3\right)|v' - u_1'|\right\} \\ &\leq K\left(\int_0^1 \left(1 + |v'|^3 + |u_1'|^3\right)^{4/3}\right)^{3/4}\left(\int_0^1 |v' - u_1'|^4\right)^{1/4} \\ &\leq \widetilde{K}\,\|u_1 - v\|_{W^{1,4}} \leq \widetilde{K}\epsilon . \end{aligned}$$

Since ϵ is arbitrary we deduce the result, i.e. $m = 0$.

ii) The argument is analogous to the preceding one and we skip the details. We first let

$$\begin{aligned} X &= \left\{v \in C^1([0,1]) : v(0) = 1,\ v(1) = 0\right\} \\ X_{\text{piec}} &= \left\{v \in C^1_{\text{piec}}([0,1]) : v(0) = 1,\ v(1) = 0\right\} \end{aligned}$$

where C^1_{piec} stands for the set of piecewise C^1 functions. We have already proved that

$$(P_{\text{piec}}) \quad \inf_{u \in X_{\text{piec}}} \left\{I(u) = \int_0^1 x\left(u'(x)\right)^2 dx\right\} = 0$$

We need to establish that $m = 0$ where

$$(P) \quad \inf_{u \in X} \left\{I(u) = \int_0^1 x\left(u'(x)\right)^2 dx\right\} = m$$

We start by observing that for any $\epsilon > 0$ and $u \in X_{\text{piec}}$, we can find $v \in X$ such that

$$\|u - v\|_{W^{1,2}} \leq \epsilon\,.$$

It is an easy matter (exactly as above) to show that if

$$I(u) = \int_0^1 x\,(u'(x))^2\,dx$$

then we can find a constant K so that

$$0 \leq I(v) \leq I(u) + K\epsilon\,.$$

Taking the infimum over all elements $v \in X$ and $u \in X_{\text{piec}}$ we get that

$$0 \leq m \leq K\epsilon$$

which is the desired result since ϵ is arbitrary.

Exercise 2.2.7. Let $u \in C^1([0,1])$ with $u(0) = u(1) = 0$. Invoking Poincaré inequality (cf. Theorem 1.47), we can find a constant $c > 0$ such that

$$\int_0^1 u^2 dx \leq c \int_0^1 u'^2 dx\,.$$

We hence obtain that $m_\lambda = 0$ if λ is small (more precisely $\lambda^2 \leq 1/c$). Observe that $u_0 \equiv 0$ satisfies $I_\lambda(u_0) = m_\lambda = 0$. Furthermore it is the unique solution of (P_λ) since, by inspection, it is the only solution (if $\lambda^2 < \pi^2$) of the Euler-Lagrange equation

$$\begin{cases} u'' + \lambda^2 u = 0 \\ u(0) = u(1) = 0\,. \end{cases}$$

The claim then follows.

Exercise 2.2.8. Let

$$X_{\text{piec}} = \left\{u \in C^1_{\text{piec}}([-1,1]) : u(-1) = 0,\ u(1) = 1\right\}$$

and

$$\overline{u}(x) = \begin{cases} 0 & \text{if } x \in [-1,0] \\ x & \text{if } x \in (0,1]\,. \end{cases}$$

It is then obvious to see that

$$\inf_{u \in X_{\text{piec}}} \left\{ I(u) = \int_{-1}^1 f(u(x), u'(x))\,dx \right\} = I(\overline{u}) = 0\,.$$

Note also that the only solution in X_{piec} is $\overline{u}$.

Since any element in X_{piec} can be approximated arbitrarily closed by an element of X (see the argument of Exercise 2.2.6) we deduce that $m = 0$ in

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_{-1}^{1} f(u(x), u'(x))\, dx \right\} = m\,.$$

To conclude it is sufficient to observe that if for a certain $u \in C^1([-1,1])$ we have $I(u) = 0$, then either $u \equiv 0$ or $u' \equiv 1$, and both possibilities are incompatible with the boundary data.

Another possibility of showing that $m = 0$ is to consider the sequence

$$u_n(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ -n^2 x^3 + 2nx^2 & \text{if } x \in \left(0, \frac{1}{n}\right] \\ x & \text{if } x \in \left(\frac{1}{n}, 1\right] \end{cases}$$

and observe that $u_n \in X$ and that

$$I(u_n) = \int_0^{1/n} f(u_n(x), u_n'(x))\, dx \to 0.$$

This proves that $m = 0$, as wished.

Exercise 2.2.9. Note first that by Jensen inequality $m \geq 1$, where

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 |u'(x)|\, dx \right\} = m$$

and $X = \left\{ u \in C^1([0,1]) : u(0) = 0,\ u(1) = 1 \right\}$. Let $n \geq 1$ be an integer and observe that u_n defined by $u_n(x) = x^n$ belongs to X and satisfies $I(u_n) = 1$. Therefore u_n is a solution of (P) for every n. In fact any $u \in X$ with $u' \geq 0$ in $[0,1]$ is a minimizer of (P).

Exercise 2.2.10. Set $v(x) = A(u(x))$. We then have, using Jensen inequality,

$$\begin{aligned} I(u) &= \int_a^b a(u(x))\, |u'(x)|^p\, dx \\ &= \int_a^b \left| (a(u(x)))^{\frac{1}{p}} u'(x) \right|^p dx = \int_a^b |v'(x)|^p\, dx \\ &\geq (b-a) \left| \frac{1}{b-a} \int_a^b v'(x)\, dx \right|^p = (b-a) \left| \frac{v(b) - v(a)}{b-a} \right|^p \end{aligned}$$

and hence

$$I(u) \geq (b-a) \left| \frac{A(\beta) - A(\alpha)}{b-a} \right|^p .$$

Setting

$$\overline{v}(x) = \frac{A(\beta) - A(\alpha)}{b-a}(x-a) + A(\alpha) \quad \text{and } \overline{u}(x) = A^{-1}(\overline{v}(x))$$

we have from the preceding inequality

$$I(u) \geq (b-a)\left|\frac{A(\beta) - A(\alpha)}{b-a}\right|^p = \int_a^b |\overline{v}'(x)|^p \, dx = I(\overline{u})$$

as claimed.

7.2.2 Second form of the Euler-Lagrange equation

Exercise 2.3.1. Write $f_\xi = \left(f_{\xi_1}, ..., f_{\xi_N}\right)$ and start by the simple observation that for any $u \in C^2\left([a,b];\mathbb{R}^N\right)$

$$\begin{aligned} & \frac{d}{dx}\left[f(x,u,u') - \langle u'; f_\xi(x,u,u')\rangle\right] \\ = \quad & f_x(x,u,u') + \sum_{i=1}^N u_i' \left[f_{u_i}(x,u,u') - \frac{d}{dx}\left[f_{\xi_i}(x,u,u')\right]\right] . \end{aligned}$$

Since the Euler-Lagrange system (see Exercise 2.2.1) is given by

$$\frac{d}{dx}\left[f_{\xi_i}(x,\overline{u},\overline{u}')\right] = f_{u_i}(x,\overline{u},\overline{u}'), \; i = 1, ..., N$$

we obtain

$$\frac{d}{dx}\left[f(x,\overline{u},\overline{u}') - \langle \overline{u}'; f_\xi(x,\overline{u},\overline{u}')\rangle\right] = f_x(x,\overline{u},\overline{u}') \; .$$

Exercise 2.3.2. The second form of the Euler-Lagrange equation is

$$\begin{aligned} 0 \quad &= \quad \frac{d}{dx}\left[f(u(x), u'(x)) - u'(x) f_\xi(u(x), u'(x))\right] = \frac{d}{dx}\left[-u(x) - \frac{1}{2}(u'(x))^2\right] \\ &= \quad -u'(x) - u''(x)u'(x) = -u'(x)\left[u''(x) + 1\right] , \end{aligned}$$

and it is satisfied by $u \equiv 1$. However $u \equiv 1$ does not verify the Euler-Lagrange equation, which is in the present case

$$u''(x) = -1 \, .$$

7.2.3 Hamiltonian formulation

Exercise 2.4.1. The proof is a mere repetition of that of Theorem 2.10 and we skip the details. We just state the result. Let $u = (u_1, ..., u_N)$ and $\xi = (\xi_1, ..., \xi_N)$. We assume that $f \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $f = f(x,u,\xi)$, and that it verifies

$$D^2 f(x,u,\xi) = \left(f_{\xi_i \xi_j}\right)_{1 \le i,j \le N} > 0, \text{ for every } (x,u,\xi) \in [a,b] \times \mathbb{R}^N \times \mathbb{R}^N$$

$$\lim_{|\xi| \to \infty} \frac{f(x,u,\xi)}{|\xi|} = +\infty, \text{ for every } (x,u) \in [a,b] \times \mathbb{R}^N.$$

If we let

$$H(x,u,v) = \sup_{\xi \in \mathbb{R}^N} \{\langle v; \xi \rangle - f(x,u,\xi)\}$$

then $H \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$ and, denoting by

$$H_v(x,u,v) = \left(H_{v_1}(x,u,v), ..., H_{v_N}(x,u,v)\right)$$

and similarly for $H_u(x,u,v)$, we also have

$$H_x(x,u,v) = -f_x(x,u,H_v(x,u,v))$$

$$H_u(x,u,v) = -f_u(x,u,H_v(x,u,v))$$

$$H(x,u,v) = \langle v; H_v(x,u,v) \rangle - f(x,u,H_v(x,u,v))$$

$$v = f_\xi(x,u,\xi) \quad \Leftrightarrow \quad \xi = H_v(x,u,v)\ .$$

The Euler-Lagrange system is

$$\frac{d}{dx}\left[f_{\xi_i}(x,u,u')\right] = f_{u_i}(x,u,u')\,,\ i = 1, ..., N.$$

and the associated Hamiltonian system is given, for every $i = 1, ..., N$, by

$$\begin{cases} u_i' = H_{v_i}(x,u,v) \\ \\ v_i' = -H_{u_i}(x,u,v)\,. \end{cases}$$

Exercise 2.4.2. i) The Euler-Lagrange equations are, for $i = 1..., N$,

$$\begin{cases} m_i x_i'' = -U_{x_i}(t,u) \\ \\ m_i y_i'' = -U_{y_i}(t,u) \\ \\ m_i z_i'' = -U_{z_i}(t,u)\,. \end{cases}$$

In terms of the Hamiltonian, if we let $u_i = (x_i, y_i, z_i)$, $\xi_i = (\xi_i^x, \xi_i^y, \xi_i^z)$ and $v_i = (v_i^x, v_i^y, v_i^z)$, for $i = 1..., N$, we find

$$\begin{aligned} H(t,u,v) &= \sup_{\xi\in\mathbb{R}^{3N}}\left\{\sum_{i=1}^{N}\left[\langle v_i;\xi_i\rangle - \frac{1}{2}m_i\left|\xi_i\right|^2\right] + U(t,u)\right\} \\ &= \sum_{i=1}^{N}\frac{\left|v_i\right|^2}{2m_i} + U(t,u)\ . \end{aligned}$$

ii) Note that along the trajectories we have $v_i = m_i u_i'$, i.e.

$$v_i^x = m_i x_i',\ v_i^y = m_i y_i',\ v_i^z = m_i z_i'$$

and hence

$$H(t,u,v) = \frac{1}{2}\sum_{i=1}^{N} m_i\left|u_i'\right| + U(t,u)\,.$$

Exercise 2.4.3. Although the hypotheses of Theorem 2.10 are not satisfied in the present context; the procedure is exactly the same and leads to the following analysis. The Hamiltonian is

$$H(x,u,v) = \begin{cases} -\sqrt{g(x,u) - v^2} & \text{if } v^2 \le g(x,u) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

We therefore have, provided $v^2 < g(x,u)$, that

$$\begin{cases} u' = H_v = \dfrac{v}{\sqrt{g(x,u) - v^2}} \\ \\ v' = -H_u = \dfrac{1}{2}\dfrac{g_u}{\sqrt{g(x,u) - v^2}}. \end{cases}$$

We hence obtain that $2vv' = g_u u'$ and thus

$$\left[v^2(x) - g(x,u(x))\right]' + g_x(x,u(x)) = 0.$$

If $g(x,u) = g(u)$, we get (c being a constant)

$$v^2(x) = c + g(u(x))\,.$$

7.2.4 Hamilton-Jacobi equation

Exercise 2.5.1. We state without proofs the results (they are similar to the case $N = 1$ and we refer to Gelfand-Fomin [46] page 88, if necessary). Let $H \in C^1\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $H = H(x,u,v)$ and $u = (u_1, ..., u_N)$. The Hamilton-Jacobi equation is

$$S_x + H(x,u,S_u) = 0,\ \forall (x,u,\alpha) \in [a,b] \times \mathbb{R}^N \times \mathbb{R}^N,$$

where $S = S(x,u,\alpha)$ and $S_u = (S_{u_1}, ..., S_{u_N})$. Jacobi Theorem reads then as follows. Let $S \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$ be a solution of the Hamilton-Jacobi equation and

$$\det\left(S_{u\alpha}(x,u,\alpha)\right) \neq 0,\ \forall (x,u,\alpha) \in [a,b] \times \mathbb{R}^N \times \mathbb{R}^N,$$

where $S_{u\alpha} = \left(\partial^2 S / \partial\alpha_i \partial u_j\right)_{1 \le i,j \le N}$. If $u \in C^1\left([a,b]; \mathbb{R}^N\right)$ satisfies

$$\frac{d}{dx}\left[S_{\alpha_i}(x,u(x),\alpha)\right] = 0,\ \forall (x,\alpha) \in [a,b] \times \mathbb{R}^N,\ i = 1, ..., N$$

and if $v(x) = S_u(x,u(x),\alpha)$ then

$$\begin{cases} u'(x) = H_v(x,u(x),v(x)) \\ \\ v'(x) = -H_u(x,u(x),v(x))\ . \end{cases}$$

Exercise 2.5.2. The procedure is formal because the hypotheses of Theorem 2.19 are not satisfied. We have seen in Exercise 2.4.3 that

$$H(u,v) = \begin{cases} -\sqrt{g(u) - v^2} & \text{if } v^2 \le g(u) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The Hamilton-Jacobi equation (it is called in this context: *eikonal equation*) is then

$$S_x - \sqrt{g(u) - S_u^2} = 0 \ \Leftrightarrow\ S_x^2 + S_u^2 = g(u)\ .$$

Its reduced form is then, for $\alpha > 0$, $g(u) - (S_u^*)^2 = \alpha^2$ and this leads to

$$S^*(u,\alpha) = \int_{u_0}^{u} \sqrt{g(s) - \alpha^2}\, ds\,.$$

We therefore get

$$S(x,u,\alpha) = \alpha x + \int_{u_0}^{u} \sqrt{g(s) - \alpha^2}\, ds\,.$$

The equation $S_\alpha = \beta$ (where β is a constant) reads as

$$x - \alpha \int_{u_0}^{u(x)} \frac{ds}{\sqrt{g(s) - \alpha^2}} = \beta$$

which implies

$$1 - \frac{\alpha u'(x)}{\sqrt{g(u(x)) - \alpha^2}} = 0\,.$$

Note that, indeed, any such $u = u(x)$ and

$$v = v(x) = S_u(x, u(x), \alpha) = \sqrt{g(u(x)) - \alpha^2}$$

satisfy

$$\begin{cases} u'(x) = H_v(x, u(x), v(x)) = \frac{\sqrt{g(u(x)) - \alpha^2}}{\alpha} \\ v'(x) = -H_u(x, u(x), v(x)) = \frac{g'(u(x))u'(x)}{2\sqrt{g(u(x)) - \alpha^2}}\,. \end{cases}$$

Exercise 2.5.3. The Hamiltonian is easily seen to be

$$H(u, v) = \frac{v^2}{2a(u)}\,.$$

The Hamilton-Jacobi equation and its reduced form are given by

$$S_x + \frac{(S_u)^2}{2a(u)} = 0 \quad \text{and} \quad \frac{(S_u^*)^2}{2a(u)} = \frac{\alpha^2}{2}\,.$$

Therefore, defining A by $A'(u) = \sqrt{a(u)}$, we find

$$S^*(u, \alpha) = \alpha A(u) \quad \text{and } S(x, u, \alpha) = -\frac{\alpha^2}{2}x + \alpha A(u)\,.$$

Hence, according to Theorem 2.19 (note that $S_{u\alpha} = \sqrt{a(u)} > 0$) the solution is given implicitly by

$$S_\alpha(x, u(x), \alpha) = -\alpha x + A(u(x)) \equiv \beta = \text{constant}\,.$$

Since A is invertible we find (compare with Exercise 2.2.10)

$$u(x) = A^{-1}(\alpha x + \beta)\,.$$

7.2.5 Fields theories

Exercise 2.6.1. Let $f \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$. Assume that there exists $\Phi \in C^3\left([a,b] \times \mathbb{R}^N\right)$ satisfying $\Phi(a,\alpha) = \Phi(b,\beta)$. Suppose also that

$$\widetilde{f}(x,u,\xi) = f(x,u,\xi) + \langle \Phi_u(x,u);\xi\rangle + \Phi_x(x,u)$$

is such that $(u,\xi) \to \widetilde{f}(x,u,\xi)$ is convex. The claim is then that any solution $\overline{u}$ of the Euler-Lagrange system

$$\frac{d}{dx}\left[f_{\xi_i}(x,\overline{u},\overline{u}')\right] = f_{u_i}(x,\overline{u},\overline{u}'),\ i = 1,...,N$$

is a minimizer of

$$(P) \quad \inf_{u\in X}\left\{I(u) = \int_a^b f(x,u(x),u'(x))\,dx\right\} = m$$

where $X = \left\{u \in C^1\left([a,b];\mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$.
The proof is exactly as the one dimensional one and we skip the details.
Exercise 2.6.2. The procedure is very similar to the one of Theorem 2.27. An exact field $\Phi = \Phi(x,u)$ covering a domain $D \subset \mathbb{R}^{N+1}$ is a map $\Phi : D \to \mathbb{R}^N$ so that there exists $S \in C^1\left(D;\mathbb{R}^N\right)$ satisfying

$$\begin{aligned} S_{u_i}(x,u) &= f_{\xi_i}(x,u,\Phi(x,u)),\ i = 1,...,N \\ S_x(x,u) &= f(x,u,\Phi(x,u)) - \langle S_u(x,u);\Phi(x,u)\rangle\ . \end{aligned}$$

The Weierstrass function is defined, for $u,\eta,\xi \in \mathbb{R}^N$, as

$$E(x,u,\eta,\xi) = f(x,u,\xi) - f(x,u,\eta) - \langle f_\xi(x,u,\eta);(\xi-\eta)\rangle\ .$$

The proof is then identical to the one dimensional case.
Exercise 2.6.3. (i) We have by definition

$$\begin{cases} S_u(x,u) = f_\xi(x,u,\Phi(x,u)) \\ \\ S_x(x,u) = -\left[S_u(x,u)\Phi(x,u) - f(x,u,\Phi(x,u))\right]\ . \end{cases}$$

We therefore have immediately from Lemma 2.8

$$S_x(x,u) = -H(x,u,S_u(x,u))\ .$$

(ii) Using again Lemma 2.8 we obtain

$$\begin{cases} H(x,u,S_u) = S_u(x,u)\Phi(x,u) - f(x,u,\Phi(x,u)) \\ \\ S_u(x,u) = f_\xi(x,u,\Phi(x,u))\ . \end{cases}$$

Since S is a solution of Hamilton-Jacobi equation, we get $S_x = -H(x,u,S_u)$ as wished.

7.3 Chapter 3: Direct methods

7.3.1 The model case: Dirichlet integral

Exercise 3.2.1. The proof is almost completely identical to that of Theorem 3.1; only the first step is slightly different. So let $\{u_\nu\}$ be a minimizing sequence

$$I(u_\nu) \to m = \inf\left\{I(u) : u \in W_0^{1,2}(\Omega)\right\}.$$

Since $I(0) < +\infty$, we have that $m < +\infty$. Consequently we have from Hölder inequality that

$$\begin{aligned} m+1 &\geq I(u_\nu) = \frac{1}{2}\int_\Omega |\nabla u_\nu|^2 \, dx - \int_\Omega g(x) u_\nu(x) \, dx \\ &\geq \int_\Omega \frac{1}{2}|\nabla u_\nu|^2 - \|g\|_{L^2}\|u_\nu\|_{L^2} = \frac{1}{2}\|\nabla u_\nu\|_{L^2}^2 - \|g\|_{L^2}\|u_\nu\|_{L^2} . \end{aligned}$$

Using Poincaré inequality (cf. Theorem 1.47) we can find constants (independent of ν) $\gamma_k > 0$, $k = 1, ..., 5$, so that

$$m+1 \geq \gamma_1 \|u_\nu\|_{W^{1,2}}^2 - \gamma_2 \|u_\nu\|_{W^{1,2}} \geq \gamma_3 \|u_\nu\|_{W^{1,2}}^2 - \gamma_4$$

and hence, as wished,

$$\|u_\nu\|_{W^{1,2}} \leq \gamma_5 .$$

7.3.2 A general existence theorem

Exercise 3.3.1. As in the preceding exercise it is the compactness proof in Theorem 3.3 that has to be modified, the remaining part of the proof is essentially unchanged. Let therefore $\{u_\nu\}$ be a minimizing sequence, i.e. $I(u_\nu) \to m$. We have from (H2) that for ν sufficiently large

$$m+1 \geq I(u_\nu) \geq \alpha_1 \|\nabla u_\nu\|_{L^p}^p - |\alpha_2| \|u_\nu\|_{L^q}^q - |\alpha_3| \operatorname{meas}\Omega .$$

From now on we will denote by $\gamma_k > 0$ constants that are independent of ν. Since by Hölder inequality we have

$$\|u_\nu\|_{L^q}^q = \int_\Omega |u_\nu|^q \leq \left(\int_\Omega |u_\nu|^p\right)^{q/p} \left(\int_\Omega dx\right)^{(p-q)/p} = (\operatorname{meas}\Omega)^{(p-q)/p} \|u_\nu\|_{L^p}^q$$

we deduce that we can find constants γ_1 and γ_2 such that

$$\begin{aligned} m+1 &\geq \alpha_1 \|\nabla u_\nu\|_{L^p}^p - \gamma_1 \|u_\nu\|_{L^p}^q - \gamma_2 \\ &\geq \alpha_1 \|\nabla u_\nu\|_{L^p}^p - \gamma_1 \|u_\nu\|_{W^{1,p}}^q - \gamma_2 . \end{aligned}$$

Invoking Poincaré inequality (cf. Theorem 1.47) we can find $\gamma_3, \gamma_4, \gamma_5$, so that

$$m + 1 \geq \gamma_3 \left\| u_\nu \right\|_{W^{1,p}}^p - \gamma_4 \left\| u_0 \right\|_{W^{1,p}}^p - \gamma_1 \left\| u_\nu \right\|_{W^{1,p}}^q - \gamma_5$$

and hence, γ_6 being a constant,

$$m + 1 \geq \gamma_3 \left\| u_\nu \right\|_{W^{1,p}}^p - \gamma_1 \left\| u_\nu \right\|_{W^{1,p}}^q - \gamma_6 \,.$$

Since $1 \leq q < p$, we can find γ_7, γ_8 so that

$$m + 1 \geq \gamma_7 \left\| u_\nu \right\|_{W^{1,p}}^p - \gamma_8$$

which, combined with the fact that $m < \infty$, leads to the claim, namely

$$\left\| u_\nu \right\|_{W^{1,p}} \leq \gamma_9 \,.$$

Exercise 3.3.2. This time it is the lower semicontinuity step in Theorem 3.3 that has to be changed. Let therefore $u_\nu \rightharpoonup \overline{u}$ in $W^{1,p}$ and let

$$I(u) = I_1(u) + I_2(u)$$

where

$$I_1(u) = \int_\Omega h(x, \nabla u(x))\, dx\,,\ I_2(u) = \int_\Omega g(x, u(x))\, dx\,.$$

It is clear that, by the proof of the theorem, $\liminf I_1(u_\nu) \geq I_1(\overline{u})$. The result will therefore be proved if we can show

$$\lim_{i \to \infty} I_2(u_\nu) = I_2(\overline{u})\,.$$

Case 1: $p > n$. From Rellich theorem (Theorem 1.43) we obtain that $u_\nu \to \overline{u}$ in L^∞; in particular there exists $R > 0$ such that $\left\| u_\nu \right\|_{L^\infty}, \left\| \overline{u} \right\|_{L^\infty} \leq R$. The result then follows since

$$\left| I_2(u_\nu) - I_2(\overline{u}) \right| \leq \int_\Omega \left| g(x, u_\nu) - g(x, \overline{u}) \right| dx \leq \gamma \left\| u_\nu - \overline{u} \right\|_{L^\infty} \operatorname{meas} \Omega.$$

Case 2: $p = n$. The estimate

$$\left| I_2(u_\nu) - I_2(\overline{u}) \right| \leq \gamma \int_\Omega \left(1 + \left| u_\nu \right|^{q-1} + \left| \overline{u} \right|^{q-1} \right) \left| u_\nu - \overline{u} \right| dx\,,$$

combined with Hölder inequality gives us

$$\left| I_2(u_\nu) - I_2(\overline{u}) \right| \leq \gamma \left(\int_\Omega \left(1 + \left| u_\nu \right|^{q-1} + \left| \overline{u} \right|^{q-1} \right)^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \left(\int_\Omega \left| u_\nu - \overline{u} \right|^q \right)^{\frac{1}{q}}.$$

Since we have from Rellich theorem, that $u_\nu \to \overline{u}$ in L^q, $\forall q \in [1,\infty)$ we obtain the desired convergence.

Case 3: $p < n$. The same argument as in Case 2 leads to the result, the difference being that Rellich theorem gives now $u_\nu \to \overline{u}$ in L^q, $\forall q \in [1, np/(n-p))$.

Exercise 3.3.3. We here have weakened the hypothesis (H3) in the proof of the theorem. We used this hypothesis only in the lower semicontinuity part of the proof, so let us establish this property under the new condition. So let $u_\nu \rightharpoonup \overline{u}$ in $W^{1,p}\left((a,b);\mathbb{R}^N\right)$. Using the convexity of $(u,\xi) \to f(x,u,\xi)$ we find

$$\int_a^b f(x,u_\nu,u_\nu')\,dx \geq \int_a^b f(x,\overline{u},\overline{u}')\,dx$$

$$+\int_a^b \left[\langle f_u(x,\overline{u},\overline{u}');u_\nu-\overline{u}\rangle + \langle f_\xi(x,\overline{u},\overline{u}');u_\nu'-\overline{u}'\rangle\right]dx\,.$$

Since, by Rellich theorem, $u_\nu \to \overline{u}$ in L^∞, to pass to the limit in the second term of the right hand side of the inequality we need only to have $f_u(x,\overline{u},\overline{u}') \in L^1$. This is ascertained by the hypothesis $|f_u(x,u,\xi)| \leq \beta(1+|\xi|^p)$. Similarly to pass to the limit in the last term we need to have $f_\xi(x,\overline{u},\overline{u}') \in L^{p'}$, $p' = p/(p-1)$; and this is precisely true because of the hypothesis $|f_\xi(x,u,\xi)| \leq \beta\left(1+|\xi|^{p-1}\right)$.

7.3.3 Euler-Lagrange equations

Exercise 3.4.1. We have to prove that for $\overline{u} \in W^{1,p}$, the following expression is meaningful

$$\int_\Omega \{f_u(x,\overline{u},\nabla\overline{u})\,\varphi + \langle f_\xi(x,\overline{u},\nabla\overline{u});\nabla\varphi\rangle\}\,dx = 0,\ \forall\varphi \in W_0^{1,p}\,.$$

Case 1: $p > n$. We have from Sobolev imbedding theorem (cf. Theorem 1.42) that $\overline{u},\varphi \in C\left(\overline{\Omega}\right)$. We therefore only need to have $f_u(x,\overline{u},\nabla\overline{u}) \in L^1$ and $f_\xi(x,\overline{u},\nabla\overline{u}) \in L^{p'}$, where $p' = p/(p-1)$. This is true if we assume that for every $R > 0$, there exists $\beta = \beta(R)$ so that $\forall(x,u,\xi)$ with $|u| \leq R$ the following inequalities hold

$$|f_u(x,u,\xi)| \leq \beta(1+|\xi|^p)\,,\ |f_\xi(x,u,\xi)| \leq \beta\left(1+|\xi|^{p-1}\right)\,.$$

Case 2: $p = n$. This time we have $\overline{u},\varphi \in L^q$, $\forall q \in [1,\infty)$. We therefore have to ascertain that $f_u \in L^r$ for a certain $r > 1$ and $f_\xi \in L^{p'}$. To guarantee this claim we impose that there exist $\beta > 0$, $p > s_2 \geq 1$, $s_1 \geq 1$ such that

$$|f_u(x,u,\xi)| \leq \beta(1+|u|^{s_1}+|\xi|^{s_2})\,,\ |f_\xi(x,u,\xi)| \leq \beta\left(1+|u|^{s_1}+|\xi|^{p-1}\right)\,.$$

Case 3: $p < n$. We now only have $\overline{u}, \varphi \in L^q$, $\forall q \in [1, np/(n-p)]$. We therefore should have $f_u \in L^{q'}$, $q' = q/(q-1)$, and $f_\xi \in L^{p'}$. This happens if there exist $\beta > 0$, $1 \le s_1 \le (np - n + p)/(n-p)$, $1 \le s_2 \le (np - n + p)/n$, $1 \le s_3 \le n(p-1)/(n-p)$ so that

$$|f_u(x, u, \xi)| \le \beta \left(1 + |u|^{s_1} + |\xi|^{s_2}\right), \ |f_\xi(x, u, \xi)| \le \beta \left(1 + |u|^{s_3} + |\xi|^{p-1}\right).$$

Exercise 3.4.2. Use the preceding exercise to deduce the following growth conditions on $g \in C^1\left(\overline{\Omega} \times \mathbb{R}\right)$.

Case 1: $p > n$. No growth condition is imposed on g.

Case 2: $p = n$. There exist $\beta > 0$ and $s_1 \ge 1$ such that

$$|g_u(x, u)| \le \beta \left(1 + |u|^{s_1}\right), \ \forall (x, u) \in \overline{\Omega} \times \mathbb{R}.$$

Case 3: $p < n$. There exist $\beta > 0$ and $1 \le s_1 \le (np - n + p)/(n-p)$, so that

$$|g_u(x, u)| \le \beta \left(1 + |u|^{s_1}\right), \ \forall (x, u) \in \overline{\Omega} \times \mathbb{R}.$$

Exercise 3.4.3. (i) Let N be an integer and

$$u_N(x, t) = \sin Nx \sin t.$$

We obviously have $u_N \in W_0^{1,2}(\Omega)$ (in fact $u_N \in C^\infty\left(\overline{\Omega}\right)$ and $u_N = 0$ on $\partial\Omega$). An elementary computation shows that $\lim I(u_N) = -\infty$.

(ii) The second part is elementary.

It is also clear that for the wave equation it is not reasonable to impose an initial condition (at $t = 0$) and a final condition (at $t = \pi$).

7.3.4 The vectorial case

Exercise 3.5.1. Let

$$\xi_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \ \xi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

We therefore have the desired contradiction, namely

$$\frac{1}{2} f(\xi_1) + \frac{1}{2} f(\xi_2) = \frac{1}{2} (\det \xi_1)^2 + \frac{1}{2} (\det \xi_2)^2 = 0 < f\left(\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2\right) = 1.$$

Exercise 3.5.2. We divide the discussion into two steps.

Step 1. Let $u, v \in C^2\left(\overline{\Omega}; \mathbb{R}^2\right)$ with $u = v$ on $\partial\Omega$. Write $u = u(x, y) = (\varphi(x, y), \psi(x, y))$ and $v = v(x, y) = (\alpha(x, y), \beta(x, y))$. Use the fact that

$$\det \nabla u = \varphi_x \psi_y - \varphi_y \psi_x = \left(\varphi \psi_y\right)_x - \left(\varphi \psi_x\right)_y$$

and the divergence theorem to get

$$\iint_{\Omega} \det \nabla u \, dxdy = \int_{\partial\Omega} \left(\varphi\psi_y\nu_1 - \varphi\psi_x\nu_2\right) \, d\sigma$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal to $\partial\Omega$. Since $\varphi = \alpha$ on $\partial\Omega$, we have, applying twice the divergence theorem,

$$\begin{aligned}\iint_{\Omega} \det \nabla u \, dxdy &= \iint_{\Omega} \left[\left(\alpha\psi_y\right)_x - \left(\alpha\psi_x\right)_y\right] \, dxdy \\ &= \iint_{\Omega} \left[\left(\alpha_x\psi\right)_y - \left(\alpha_y\psi\right)_x\right] \, dxdy = \int_{\partial\Omega} \left(\alpha_x\psi\nu_2 - \alpha_y\psi\nu_1\right) \, d\sigma \, .\end{aligned}$$

Since $\psi = \beta$ on $\partial\Omega$, we obtain, using again the divergence theorem, that

$$\iint_{\Omega} \det \nabla u \, dxdy = \iint_{\Omega} \left[\left(\alpha_x\beta\right)_y - \left(\alpha_y\beta\right)_x\right] \, dxdy = \iint_{\Omega} \det \nabla v \, dxdy \, .$$

Step 2. We first regularize v, meaning that for every $\epsilon > 0$ we find $v^\epsilon \in C^2\left(\overline{\Omega}; \mathbb{R}^2\right)$ so that

$$\|v - v^\epsilon\|_{W^{1,p}} \leq \epsilon \, .$$

Since $u - v \in W_0^{1,p}\left(\Omega; \mathbb{R}^2\right)$, we can find $w^\epsilon \in C_0^\infty\left(\Omega; \mathbb{R}^2\right)$ so that

$$\|(u - v) - w^\epsilon\|_{W^{1,p}} \leq \epsilon \, .$$

Define $u^\epsilon = v^\epsilon + w^\epsilon$ and observe that $u^\epsilon, v^\epsilon \in C^2\left(\overline{\Omega}; \mathbb{R}^2\right)$, with $u^\epsilon = v^\epsilon$ on $\partial\Omega$, and

$$\|u - u^\epsilon\|_{W^{1,p}} = \|(u - v) - w^\epsilon + (v - v^\epsilon)\|_{W^{1,p}} \leq 2\epsilon \, .$$

Using Exercise 3.5.4 below, we deduce that there exists α_1 (independent of ϵ) so that

$$\|\det \nabla u - \det \nabla u^\epsilon\|_{L^{p/2}} \, , \; \|\det \nabla v - \det \nabla v^\epsilon\|_{L^{p/2}} \leq \alpha_1\epsilon \, .$$

Combining Step 1 with the above estimates we obtain that there exists a constant α_2 (independent of ϵ) such that

$$\left|\iint_{\Omega} \left(\det \nabla u - \det \nabla v\right) \, dxdy\right| \leq \left|\iint_{\Omega} \left(\det \nabla u^\epsilon - \det \nabla v^\epsilon\right) \, dxdy\right|$$

$$+ \left|\iint_{\Omega} \left(\det \nabla u - \det \nabla u^\epsilon\right) \, dxdy\right| + \left|\iint_{\Omega} \left(\det \nabla v^\epsilon - \det \nabla v\right) \, dxdy\right| \leq \alpha_2\epsilon \, .$$

Since ϵ is arbitrary we have indeed the result.

Exercise 3.5.3. Let $u \in C^2\left(\overline{\Omega};\mathbb{R}^2\right)$, $u(x,y) = (\varphi(x,y), \psi(x,y))$, be a minimizer of (P) and let $v \in C_0^\infty\left(\Omega;\mathbb{R}^2\right)$, $v(x,y) = (\alpha(x,y), \beta(x,y))$, be arbitrary. Since $I(u+\epsilon v) \geq I(u)$ for every ϵ, we must have

$$\begin{aligned}
0 &= \left.\frac{d}{d\epsilon} I(u+\epsilon v)\right|_{\epsilon=0} \\
&= \left.\frac{d}{d\epsilon}\left\{\iint_\Omega \left[(\varphi_x+\epsilon\alpha_x)\left(\psi_y+\epsilon\beta_y\right)-\left(\varphi_y+\epsilon\alpha_y\right)(\psi_x+\epsilon\beta_x)\right]\,dxdy\right\}\right|_{\epsilon=0} \\
&= \iint_\Omega \left[\left(\psi_y\alpha_x-\psi_x\alpha_y\right)+\left(\varphi_x\beta_y-\varphi_y\beta_x\right)\right]\,dxdy\,.
\end{aligned}$$

Integrating by parts, we find that the right hand side vanishes identically. The result is not surprising in view of Exercise 3.5.2, which shows that $I(u)$ is in fact constant.

Exercise 3.5.4. The proof is divided into two steps.

Step 1. It is easily proved that the following algebraic inequality holds

$$|\det A - \det B| \leq \alpha\,(|A|+|B|)\,|A-B|\,,\ \forall A, B \in \mathbb{R}^{2\times 2}$$

where α is a constant.

Step 2. We therefore deduce that

$$|\det \nabla u - \det \nabla v|^{p/2} \leq \alpha^{p/2}\,(|\nabla u|+|\nabla v|)^{p/2}\,|\nabla u-\nabla v|^{p/2}\,.$$

Hölder inequality implies then

$$\iint_\Omega |\det \nabla u - \det \nabla v|^{p/2}\,dxdy$$

$$\leq \alpha^{p/2}\left(\iint_\Omega (|\nabla u|+|\nabla v|)^p\,dxdy\right)^{1/2}\left(\iint_\Omega |\nabla u-\nabla v|^p\,dxdy\right)^{1/2}.$$

We therefore obtain that

$$\|\det \nabla u - \det \nabla v\|_{L^{p/2}} = \left(\iint_\Omega |\det \nabla u - \det \nabla v|^{p/2}\,dxdy\right)^{2/p}$$

$$\leq \alpha\left(\iint_\Omega (|\nabla u|+|\nabla v|)^p\,dxdy\right)^{1/p}\left(\iint_\Omega |\nabla u-\nabla v|^p\,dxdy\right)^{1/p}$$

and hence the claim.

Exercise 3.5.5. For more details concerning this exercise see [31] page 158.

(i) We have seen (Exercise 1.4.6) that the sequence $u^\nu \rightharpoonup 0$ in $W^{1,2}$ (we have shown this only up to a subsequence, but it is not difficult to see that the whole sequence has this property). An elementary computation gives

$$\det \nabla u^\nu = -\nu (1-y)^{2\nu-1}.$$

Let us show that $\det \nabla u^\nu \rightharpoonup 0$ in L^1 does not hold. Indeed, let $\varphi \equiv 1 \in L^\infty(\Omega)$. It is not difficult to see that

$$\lim_{\nu\to\infty} \iint_\Omega \det \nabla u^\nu (x,y)\, dxdy \neq 0,$$

and thus the result.

(ii) Note first that by Rellich theorem (Theorem 1.43) we have that if $u^\nu \rightharpoonup u$ in $W^{1,p}$ then $u^\nu \to u$ in L^q, $\forall q \in [1, 2p/(2-p))$ provided $p < 2$ and $\forall q \in [1,\infty)$ if $p = 2$ (the case $p > 2$ has already been considered in Lemma 3.23). Consequently if $p > 4/3$, we have $u^\nu \to u$ in L^4. Let therefore $u^\nu = u^\nu(x,y) = (\varphi^\nu(x,y), \psi^\nu(x,y))$ and $v \in C_0^\infty(\Omega)$. Since $\det \nabla u^\nu = \varphi_x^\nu \psi_y^\nu - \varphi_y^\nu \psi_x^\nu = \left(\varphi^\nu \psi_y^\nu\right)_x - \left(\varphi^\nu \psi_x^\nu\right)_y$ (this is allowed since $u^\nu \in C^2$) we have, after integrating by parts,

$$\iint_\Omega \det \nabla u^\nu v\, dxdy = \iint_\Omega \left(\varphi^\nu \psi_x^\nu v_y - \varphi^\nu \psi_y^\nu v_x\right) dxdy.$$

However we know that $\psi^\nu \rightharpoonup \psi$ in $W^{1,4/3}$ (since $u^\nu \rightharpoonup u$ in $W^{1,p}$ and $p > 4/3$) and $\varphi^\nu \to \varphi$ in L^4, we therefore deduce that $\left(\varphi^\nu \psi_x^\nu, \varphi^\nu \psi_y^\nu\right) \rightharpoonup \left(\varphi\psi_x, \varphi\psi_y\right)$ in L^1 (Exercise 1.3.3). Passing to the limit and integrating by parts once more we get the claim, namely

$$\lim_{\nu\to\infty} \iint_\Omega \det \nabla u^\nu v\, dxdy = \iint_\Omega \left(\varphi\psi_x v_y - \varphi\psi_x v_y\right) dxdy = \iint_\Omega \det \nabla u\, v\, dxdy.$$

Exercise 3.5.6. (i) Let $x = (x_1, x_2)$, we then find

$$\nabla u = \begin{pmatrix} \dfrac{x_2^2}{|x|^3} & -\dfrac{x_1 x_2}{|x|^3} \\ -\dfrac{x_1 x_2}{|x|^3} & \dfrac{x_1^2}{|x|^3} \end{pmatrix} \Rightarrow |\nabla u|^2 = \frac{1}{|x|^2}.$$

We therefore deduce (cf. Exercise 1.4.1) that $u \in L^\infty$ and $u \in W^{1,p}$ provided $p \in [1,2)$, but, however, $u \notin W^{1,2}$ and $u \notin C^0$.

(ii) Since

$$\iint_\Omega |u^\nu(x) - u(x)|^q\, dx = 2\pi \int_0^1 \frac{r}{(\nu r+1)^q}\, dr = \frac{2\pi}{\nu^2} \int_1^{\nu+1} \frac{s-1}{s^q}\, ds$$

we deduce that $u^\nu \to u$ in L^q, for every $q \geq 1$; however the convergence $u^\nu \to u$ in L^∞ does not hold. We next show that $u^\nu \rightharpoonup u$ in $W^{1,p}$ if $p \in [1,2)$. We will show this only up to a subsequence (it is not difficult to see that the whole sequence has this property). We readily have

$$\nabla u^\nu = \frac{1}{|x|\left(|x|+1/\nu\right)^2}\begin{pmatrix} x_2^2+\dfrac{|x|}{\nu} & -x_1x_2 \\ -x_1x_2 & x_1^2+\dfrac{|x|}{\nu}\end{pmatrix}$$

and thus

$$|\nabla u^\nu| = \frac{\left(|x|^2+\dfrac{2|x|}{\nu}+\dfrac{2}{\nu^2}\right)^{1/2}}{\left(|x|+1/\nu\right)^2}.$$

We therefore find, if $1 \leq p < 2$, that, γ denoting a constant independent of ν,

$$\begin{aligned}\iint_\Omega |\nabla u^\nu|^p\,dx_1dx_2 &= 2\pi\int_0^1 \frac{\left(\left(r+1/\nu\right)^2+1/\nu^2\right)^{p/2}}{\left(r+1/\nu\right)^{2p}}r\,dr \\ &\leq 2\pi\int_0^1 \frac{2^{p/2}\left(r+1/\nu\right)^p}{\left(r+1/\nu\right)^{2p}}r\,dr = 2^{(2+p)/2}\pi\nu^p\int_0^1\frac{r\,dr}{\left(\nu r+1\right)^p} \\ &\leq 2^{(2+p)/2}\pi\nu^{p-2}\int_1^{\nu+1}\frac{(s-1)\,ds}{s^p} \leq \gamma\,.\end{aligned}$$

This implies that, up to the extraction of a subsequence, $u^\nu \rightharpoonup u$ in $W^{1,p}$, as claimed.

(iii) A direct computation gives

$$\det\nabla u^\nu = |\det\nabla u^\nu| = \frac{1}{\nu\left(|x|+1/\nu\right)^3}$$

and hence

$$\begin{aligned}\iint_\Omega |\det\nabla u^\nu|\,dx_1dx_2 &= 2\pi\nu^2\int_0^1\frac{r\,dr}{\left(\nu r+1\right)^3} = 2\pi\int_1^{\nu+1}\frac{(s-1)\,ds}{s^3} \\ &= 2\pi\left[\frac{1}{2s^2}-\frac{1}{s}\right]_1^{\nu+1}.\end{aligned}$$

We therefore have

$$\lim_{\nu\to\infty}\iint_\Omega |\det\nabla u^\nu|\,dx_1dx_2 = \pi\,. \tag{7.14}$$

Observe that if $\Omega_\delta = \left\{x \in \mathbb{R}^2 : |x| < \delta\right\}$, then for every fixed $\delta > 0$, we have

$$\det\nabla u^\nu = |\det\nabla u^\nu| \to 0 \text{ in } L^\infty\left(\Omega\diagdown\Omega_\delta\right). \tag{7.15}$$

So let $\epsilon > 0$ be arbitrary and let $\varphi \in C^\infty(\Omega)$. We can therefore find $\delta = \delta(\epsilon) > 0$ such that

$$x \in \Omega_\delta \Rightarrow |\varphi(x) - \varphi(0)| < \epsilon. \tag{7.16}$$

We then combine (7.14), (7.15) and (7.16) to get the result. Indeed let $\varphi \in C_0^\infty(\Omega)$ and obtain

$$\begin{aligned}\iint_\Omega \det \nabla u^\nu \varphi \, dx &= \varphi(0) \iint_\Omega \det \nabla u^\nu dx + \iint_{\Omega_\delta} \det \nabla u^\nu (\varphi(x) - \varphi(0)) \, dx \\ &\quad + \iint_{\Omega - \Omega_\delta} \det \nabla u^\nu (\varphi(x) - \varphi(0)) \, dx \,.\end{aligned}$$

This leads to the following estimate

$$\left| \iint_\Omega \det \nabla u^\nu \varphi \, dx - \varphi(0) \iint_\Omega \det \nabla u^\nu dx \right|$$

$$\leq \sup_{x \in \Omega_\delta} [|\varphi(x) - \varphi(0)|] \iint_\Omega |\det \nabla u^\nu| \, dx + 2 \|\varphi\|_{L^\infty} \iint_{\Omega - \Omega_\delta} |\det \nabla u^\nu| \, dx \,.$$

Keeping first fixed ϵ, and thus δ, we let $\nu \to \infty$ and obtain

$$\lim_{\nu \to \infty} \left| \iint_\Omega \det \nabla u^\nu \varphi \, dx - \pi \varphi(0) \right| \leq \pi \epsilon \,,$$

ϵ being arbitrary, we have indeed obtained the result.

7.3.5 Relaxation theory

Exercise 3.6.1. (i) Let

$$\overline{u}(x) = \begin{cases} ax + \alpha & \text{if } x \in [0, \lambda] \\ b(x-1) + \beta & \text{if } x \in [\lambda, 1] \,. \end{cases}$$

Note that $\overline{u}(0) = \alpha$, $\overline{u}(1) = \beta$ and $\overline{u}$ is continuous at $x = \lambda$ since $\beta - \alpha = \lambda a + (1-\lambda) b$, hence $\overline{u} \in X$. Since $f^{**} \leq f$ and f^{**} is convex, we have appealing to Jensen inequality that, for any $u \in X$,

$$\begin{aligned} I(u) &= \int_0^1 f(u'(x)) \, dx \geq \int_0^1 f^{**}(u'(x)) \, dx \\ &\geq f^{**}\left(\int_0^1 u'(x) \, dx \right) = f^{**}(\beta - \alpha) = \lambda f(a) + (1-\lambda) f(b) = I(\overline{u}) \,. \end{aligned}$$

Hence $\overline{u}$ is a minimizer of (P).

(ii) The preceding result does not apply to $f(\xi) = e^{-\xi^2}$ and $\alpha = \beta = 0$. Indeed we have $f^{**}(\xi) \equiv 0$ and we therefore cannot find $\lambda \in [0,1]$, $a, b \in \mathbb{R}$ so that

$$\begin{cases} \lambda a + (1-\lambda) b = 0 \\ \lambda e^{-a^2} + (1-\lambda) e^{-b^2} = 0 . \end{cases}$$

In fact we should need $a = -\infty$ and $b = +\infty$. Recall that in Section 2.2 we already saw that (P) has no solution.

(iii) If $f(\xi) = \left(\xi^2 - 1\right)^2$, we then find

$$f^{**}(\xi) = \begin{cases} \left(\xi^2 - 1\right)^2 & \text{if } |\xi| \geq 1, \\ \\ 0 & \text{if } |\xi| < 1 . \end{cases}$$

Therefore if $|\beta - \alpha| \geq 1$ choose in (i) $\lambda = 1/2$ and $a = b = \beta - \alpha$. However if $|\beta - \alpha| < 1$, choose $a = 1$, $b = -1$ and $\lambda = (1 + \beta - \alpha)/2$. In conclusion, in both cases, we find that problem (P) has $\overline{u}$ as a minimizer.

Exercise 3.6.2. If we set $\xi = (\xi_1, \xi_2)$, we easily have that

$$f^{**}(\xi) = \begin{cases} f(\xi) & \text{if } |\xi_1| \geq 1, \\ \\ \xi_2^4 & \text{if } |\xi_1| < 1 . \end{cases}$$

From the Relaxation theorem (cf. Theorem 3.28) we find $m = 0$ (since $\overline{u} \equiv 0$ is such that $\overline{I}(\overline{u}) = 0$). However no function $\overline{u} \in W_0^{1,4}(\Omega)$ can satisfy $I(\overline{u}) = 0$, hence (P) has no solution.

Exercise 3.6.3. It is easy to see that

$$f^*(\xi^*) = \sup_{\xi \in \mathbb{R}^{2\times 2}} \left\{ \langle \xi ; \xi^* \rangle - (\det \xi)^2 \right\} = \begin{cases} 0 & \text{if } \xi^* = 0 \\ \\ +\infty & \text{if } \xi^* \neq 0 \end{cases}$$

and therefore

$$f^{**}(\xi) = \sup_{\xi^* \in \mathbb{R}^{2\times 2}} \left\{ \langle \xi ; \xi^* \rangle - f^*(\xi^*) \right\} \equiv 0 .$$

7.4 Chapter 4: Regularity

7.4.1 The one dimensional case

Exercise 4.2.1. (i) We first show that $\overline{u} \in W^{2,\infty}(a,b)$, by proving (iii) of Theorem 1.36. Observe first that from (H1') and the fact that $\overline{u} \in W^{1,\infty}(a,b)$, we can find a constant $\gamma_1 > 0$ such that, for every $z \in \mathbb{R}$ with $|z| \leq \|\overline{u}'\|_{L^\infty}$,

$$f_{\xi\xi}(x, \overline{u}(x), z) \geq \gamma_1 > 0, \ \forall x \in [a,b] . \tag{7.17}$$

We have to prove that we can find a constant $\alpha > 0$ so that

$$|\overline{u}'(x+h) - \overline{u}'(x)| \leq \alpha |h|, \text{ a.e. } x \in \omega$$

for every open set $\omega \subset \overline{\omega} \subset (a,b)$ and every $h \in \mathbb{R}$ satisfying $|h| < \operatorname{dist}(\omega, (a,b)^c)$. Using (7.17) we have

$$\begin{aligned}
&\gamma_1 |\overline{u}'(x+h) - \overline{u}'(x)| \\
\leq\ & \left| \int_{\overline{u}'(x)}^{\overline{u}'(x+h)} f_{\xi\xi}(x, \overline{u}(x), z)\, dz \right| \\
\leq\ & |f_\xi(x, \overline{u}(x), \overline{u}'(x+h)) - f_\xi(x, \overline{u}(x), \overline{u}'(x))| \\
\leq\ & |f_\xi(x+h, \overline{u}(x+h), \overline{u}'(x+h)) - f_\xi(x, \overline{u}(x), \overline{u}'(x))| \\
& + |f_\xi(x, \overline{u}(x), \overline{u}'(x+h)) - f_\xi(x+h, \overline{u}(x+h), \overline{u}'(x+h))| \,.
\end{aligned}$$

Now let us evaluate both terms in the right hand side of the inequality. Since we know from Lemma 4.2 that $x \to f_u(x, \overline{u}(x), \overline{u}'(x))$ is in $L^\infty(a,b)$ and the Euler-Lagrange equation holds we deduce that $x \to \varphi(x) = f_\xi(x, \overline{u}(x), \overline{u}'(x))$ is in $W^{1,\infty}(a,b)$. Therefore applying Theorem 1.36 to φ, we can find a constant $\gamma_2 > 0$, such that

$$\begin{aligned}
|\varphi(x+h) - \varphi(x)| &= |f_\xi(x+h, \overline{u}(x+h), \overline{u}'(x+h)) - f_\xi(x, \overline{u}(x), \overline{u}'(x))| \\
&\leq \gamma_2 |h| \,.
\end{aligned}$$

Similarly since $\overline{u} \in W^{1,\infty}$ and $f \in C^\infty$, we can find constant $\gamma_3, \gamma_4 > 0$, such that

$$\begin{aligned}
&|f_\xi(x, \overline{u}(x), \overline{u}'(x+h)) - f_\xi(x+h, \overline{u}(x+h), \overline{u}'(x+h))| \\
\leq\ & \gamma_3 (|h| + |\overline{u}(x+h) - \overline{u}(x)|) \leq \gamma_4 |h| \,.
\end{aligned}$$

Combining these two inequalities we find

$$|\overline{u}'(x+h) - \overline{u}'(x)| \leq \frac{\gamma_2 + \gamma_4}{\gamma_1} |h|$$

as wished; thus $\overline{u} \in W^{2,\infty}(a,b)$.

(ii) Since $\overline{u} \in W^{2,\infty}(a,b)$, and the Euler-Lagrange equation holds, we get that, for almost every $x \in (a,b)$,

$$\begin{aligned}
\frac{d}{dx}[f_\xi(x, \overline{u}, \overline{u}')] &= f_{\xi\xi}(x, \overline{u}, \overline{u}')\, \overline{u}'' + f_{u\xi}(x, \overline{u}, \overline{u}')\, \overline{u}' + f_{x\xi}(x, \overline{u}, \overline{u}') \\
&= f_u(x, \overline{u}, \overline{u}') \,.
\end{aligned}$$

Since (H1') holds and $\overline{u} \in C^1([a,b])$, we deduce that there exists $\gamma_5 > 0$ such that

$$f_{\xi\xi}\left(x, \overline{u}\left(x\right), \overline{u}'\left(x\right)\right) \geq \gamma_5 > 0,\ \forall x \in [a,b]\ .$$

The Euler-Lagrange equation can then be rewritten as

$$\overline{u}'' = \frac{f_u\left(x, \overline{u}, \overline{u}'\right) - f_{x\xi}\left(x, \overline{u}, \overline{u}'\right) - f_{u\xi}\left(x, \overline{u}, \overline{u}'\right)\overline{u}'}{f_{\xi\xi}\left(x, \overline{u}, \overline{u}'\right)}$$

and hence $\overline{u} \in C^2([a,b])$. Returning to the equation we find that the right hand side is then C^2, and hence $\overline{u} \in C^3$. Iterating the process we conclude that $\overline{u} \in C^\infty([a,b])$, as claimed.

Exercise 4.2.2. (i) We have

$$\overline{u}' = |x|^{\frac{p}{p-q}-2}\,x \ \text{ and } \overline{u}'' = \left(\frac{p}{p-q} - 1\right)|x|^{\frac{p}{p-q}-2} = \frac{q}{p-q}\,|x|^{\frac{2q-p}{p-q}}$$

which implies, since $p > 2q > 2$, that $\overline{u} \in C^1([-1,1])$ but $\overline{u} \notin C^2([-1,1])$.

(ii) We find that

$$\begin{aligned} \left|\overline{u}'\right|^{p-2}\overline{u}' &= |x|^{\frac{q(p-2)}{p-q}}\,|x|^{\frac{2q-p}{p-q}}\,x = |x|^{\frac{p(q-1)}{p-q}}\,x \\ \left|\overline{u}\right|^{q-2}\overline{u} &= \left(\frac{p-q}{p}\right)^{q-1}|x|^{\frac{p(q-1)}{p-q}}\,. \end{aligned}$$

If we choose, for instance, $\frac{p(q-1)}{p-q} = 4$ (which is realized, for example, if $p = 8$ and $q = 10/3$), then $\left|\overline{u}'\right|^{p-2}\overline{u}'$, $\left|\overline{u}\right|^{q-2}\overline{u} \in C^\infty([-1,1])$, although $\overline{u} \notin C^2([-1,1])$.

(iii) Since the function $(u,\xi) \to f(u,\xi)$ is strictly convex and satisfies all the hypotheses of Theorem 3.3 and Theorem 3.11, we have that (P) has a unique minimizer and that it should be the solution of the Euler-Lagrange equation

$$\left(\left|u'\right|^{p-2}u'\right)' = \lambda\,|u|^{q-2}\,u\,.$$

A direct computation shows that, indeed, $\overline{u}$ is a solution of this equation and therefore it is the unique minimizer of (P).

7.4.2 The model case: Dirichlet integral

Exercise 4.3.1. We have to show that if $u \in L^1_{\text{loc}}(a,b)$ and

$$\int_a^b u\left(x\right)\varphi''\left(x\right)\,dx = 0,\ \forall \varphi \in C_0^\infty\left(a,b\right)$$

then, up to changing u on a set of measure zero, $u(x) = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$. This follows exactly as in Exercise 1.3.6.

Exercise 4.3.2. This is a classical result.

(i) We start by choosing $\psi \in C_0^\infty(0,1)$, $\psi \geq 0$, so that

$$\int_0^1 r^{n-1}\psi(r)\,dr = \frac{1}{\sigma_{n-1}} \tag{7.18}$$

where $\sigma_{n-1} = \operatorname{meas}(\partial B_1(0))$. We then let $\psi \equiv 0$ outside of $(0,1)$ and we define for every $\epsilon > 0$

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\psi\left(\frac{|x|}{\epsilon}\right).$$

(ii) Let $\Omega_\epsilon = \left\{x \in \mathbb{R}^n : \overline{B_\epsilon(x)} \subset \Omega\right\}$. Let $x \in \Omega_\epsilon$, the function $y \to \varphi_\epsilon(x-y)$ has then its support in Ω since $\operatorname{supp}\varphi_\epsilon \subset \overline{B_\epsilon(0)}$. We therefore have

$$\begin{aligned}\int_{\mathbb{R}^n} u(y)\,\varphi_\epsilon(x-y)\,dy &= \int_{\mathbb{R}^n} u(x-y)\,\varphi_\epsilon(y)\,dy = \frac{1}{\epsilon^n}\int_{|y|<\epsilon} u(x-y)\,\psi\left(\frac{|y|}{\epsilon}\right)dy \\ &= \int_{|z|<1} u(x-\epsilon z)\,\psi(|z|)\;dz = \int_0^1\int_{|y|=1} u(x-\epsilon r y)\,\psi(r)\,r^{n-1}dr d\sigma(y)\,.\end{aligned} \tag{7.19}$$

We next use the mean value formula

$$u(x) = \frac{1}{\sigma_{n-1}r^{n-1}}\int_{\partial B_r(x)} u\,d\sigma = \frac{1}{\sigma_{n-1}}\int_{|y|=1} u(x+ry)\;d\sigma(y)\,. \tag{7.20}$$

Returning to (7.19) and using (7.18) combined with (7.20), we deduce that

$$u(x) = \int_{\mathbb{R}^n} u(y)\,\varphi_\epsilon(x-y)\;dy\,.$$

Since $\varphi_\epsilon \in C_0^\infty(\mathbb{R}^n)$, we immediately get that $u \in C^\infty(\Omega_\epsilon)$. Since ϵ is arbitrary, we find that $u \in C^\infty(\Omega)$, as claimed.

Exercise 4.3.3. Let $V(r) = |\log r|^\alpha$ and

$$u(x_1,x_2) = x_1 x_2 V(|x|)\,.$$

A direct computation shows that

$$u_{x_1} = x_2 V(|x|) + \frac{x_1^2 x_2}{|x|}V'(|x|) \quad \text{and} \quad u_{x_2} = x_1 V(|x|) + \frac{x_1 x_2^2}{|x|}V'(|x|)$$

while

$$\begin{aligned} u_{x_1x_1} &= \frac{x_1^3 x_2}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3}\left(2x_1^2 + 3x_2^2\right) V'(|x|) \\ u_{x_2x_2} &= \frac{x_2^3 x_1}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3}\left(2x_2^2 + 3x_1^2\right) V'(|x|) \\ u_{x_1x_2} &= \frac{x_1^2 x_2^2}{|x|^2} V''(|x|) + \frac{x_1^4 + x_1^2 x_2^2 + x_2^4}{|x|^3} V'(|x|) + V(|x|)\ . \end{aligned}$$

We therefore get that

$$u_{x_1x_1}, u_{x_2x_2} \in C^0(\Omega)\,,\ u_{x_1x_2} \notin L^\infty(\Omega)\,.$$

Exercise 4.3.4. Let $V(r) = \log|\log r|$. A direct computation shows that

$$u_{x_1} = \frac{x_1}{|x|} V'(|x|) \quad \text{and} \quad u_{x_2} = \frac{x_2}{|x|} V'(|x|)$$

and therefore

$$\begin{aligned} u_{x_1x_1} &= \frac{x_1^2}{|x|^2} V''(|x|) + \frac{x_2^2}{|x|^3} V'(|x|) \\ u_{x_2x_2} &= \frac{x_2^2}{|x|^2} V''(|x|) + \frac{x_1^2}{|x|^3} V'(|x|) \\ u_{x_1x_2} &= \frac{x_1 x_2}{|x|^2} V''(|x|) - \frac{x_1 x_2}{|x|^3} V'(|x|)\ . \end{aligned}$$

This leads to

$$\Delta u = V''(|x|) + \frac{V'(|x|)}{|x|} = \frac{-1}{|x|^2 \left|\log|x|\right|^2} \in L^1(\Omega)$$

while $u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2} \notin L^1(\Omega)$. Summarizing the results we indeed have that $u \notin W^{2,1}(\Omega)$ while $\Delta u \in L^1(\Omega)$. We also observe (compare with Example 1.33 (ii)) that, trivially, $u \notin L^\infty(\Omega)$ while $u \in W^{1,2}(\Omega)$, since

$$\iint_\Omega |\nabla u|^2\, dx = 2\pi \int_0^{1/2} \frac{dr}{r\,|\log r|^2} = \frac{2\pi}{\log 2}\,.$$

A much more involved example due to Ornstein [79] produces a u such that

$$u_{x_1x_1}, u_{x_2x_2} \in L^1(\Omega)\,,\ u_{x_1x_2} \notin L^1(\Omega)\ .$$

7.5 Chapter 5: Minimal surfaces

7.5.1 Generalities about surfaces

Exercise 5.2.1. (i) Elementary.

(ii) Apply (i) with $a = v_x$, $b = v_y$ and the definition of E, F and G.

(iii) Since $e_3 = (v_x \times v_y) / |v_x \times v_y|$, we have

$$\langle e_3; v_x \rangle = \langle e_3; v_y \rangle = 0 \,.$$

Differentiating with respect to x and y, we deduce that

$$\begin{aligned} 0 &= \langle e_3; v_{xx} \rangle + \langle e_{3x}; v_x \rangle = \langle e_3; v_{xy} \rangle + \langle e_{3y}; v_x \rangle \\ &= \langle e_3; v_{xy} \rangle + \langle e_{3x}; v_y \rangle = \langle e_3; v_{yy} \rangle + \langle e_{3y}; v_y \rangle \end{aligned}$$

and the result follows from the definition of L, M and N.

Exercise 5.2.2. (i) We have

$$v_x = (-y \sin x, y \cos x, a) \,, \ v_y = (\cos x, \sin x, 0) \,,$$

$$e_3 = \frac{(-a \sin x, a \cos x, -y)}{\sqrt{a^2 + y^2}},$$

$$v_{xx} = (-y \cos x, -y \sin x, 0) \,, \ v_{xy} = (-\sin x, \cos x, 0) \,, \ v_{yy} = 0$$

and hence

$$E = a^2 + y^2, \ F = 0, \ G = 1, \ L = N = 0, \ M = \frac{a}{\sqrt{a^2 + y^2}}$$

which leads to $H = 0$, as wished.

(ii) A straight computation gives

$$v_x = \left(1 - x^2 + y^2, -2xy, 2x\right) \,, \ v_y = \left(2xy, -1 + y^2 - x^2, -2y\right) \,,$$

$$e_3 = \frac{\left(2x, 2y, x^2 + y^2 - 1\right)}{(1 + x^2 + y^2)},$$

$$v_{xx} = (-2x, \ -2y, 2) \,, \ v_{xy} = (2y, -2x, 0) \,, \ v_{yy} = (2x, 2y, -2)$$

and hence

$$E = G = \left(1 + x^2 + y^2\right)^2 \,, \ F = 0, \ L = -2, \ N = 2, \ M = 0$$

which shows that, indeed, $H = 0$.

Exercise 5.2.3. (i) Since $|v_x \times v_y|^2 = w^2 \left(1 + w'^2\right)$, we obtain the result.

(ii) Let $f(w,\xi) = w\sqrt{1+\xi^2}$. Observe that the function f is not convex over $(0,+\infty)\times\mathbb{R}$; although the function $\xi \to f(w,\xi)$ is strictly convex, whenever $w > 0$. We will therefore only give necessary conditions for existence of minimizers of (P_α) and hence we write the Euler-Lagrange equation associated to (P_α), namely

$$\frac{d}{dx}\left[f_\xi(w,w')\right] = f_w(w,w') \;\Leftrightarrow\; \frac{d}{dx}\left[\frac{ww'}{\sqrt{1+w'^2}}\right] = \sqrt{1+w'^2}\,. \tag{7.21}$$

Invoking Theorem 2.7, we find that any minimizer w of (P_α) satisfies

$$\frac{d}{dx}\left[f(w,w') - w' f_\xi(w,w')\right] = 0 \;\Leftrightarrow\; \frac{d}{dx}\left[\frac{w}{\sqrt{1+w'^2}}\right] = 0$$

which implies, if we let $a > 0$ be a constant,

$$w'^2 = \frac{w^2}{a^2} - 1\,. \tag{7.22}$$

Before proceeding further, let us observe the following facts.

1) The function $w \equiv a$ is a solution of (7.22) but not of (7.21) and therefore it is irrelevant for our analysis.

2) To $a = 0$ corresponds $w \equiv 0$, which is also not a solution of (7.21) and moreover does not satisfy the boundary conditions $w(0) = w(1) = \alpha > 0$.

3) Any solution of (7.22) must verify $w^2 \geq a^2$ and, since $w(0) = w(1) = \alpha > 0$, thus verifies $w \geq a > 0$.

We can therefore search for solutions of (7.22) the form

$$w(x) = a\cosh\frac{f(x)}{a}$$

where f satisfies, when inserted into the equation, $f'^2 = 1$, which implies that either $f' \equiv 1$ or $f' \equiv -1$, since f is C^1. Thus the solution of the differential equation is of the form

$$w(x) = a\cosh\frac{x+\mu}{a}\,.$$

Since $w(0) = w(1)$, we deduce that $\mu = -1/2$. Finally since $w(0) = w(1) = \alpha$, every solution C^2 of (P_α) must be of the form

$$w(x) = a\cosh\left(\frac{2x-1}{2a}\right) \quad\text{and}\quad a\cosh\frac{1}{2a} = \alpha\,.$$

Summarizing, we see that depending on the values of α, the Euler-Lagrange equation (7.21) may have 0, 1 or 2 solutions (in particular for α small, (7.21) has no C^2 solution satisfying $w(0) = w(1) = \alpha$ and hence (P_α) also has no C^2 minimizer).

Exercise 5.2.4. By hypothesis there exist a bounded smooth domain $\Omega \subset \mathbb{R}^2$ and a map $v \in C^2\left(\overline{\Omega}; \mathbb{R}^3\right)$ ($v = v(x, y)$, with $v_x \times v_y \neq 0$ in $\overline{\Omega}$) so that $\Sigma_0 = v\left(\overline{\Omega}\right)$. Let $e_3 = (v_x \times v_y) / |v_x \times v_y|$. We then let for $\epsilon \in \mathbb{R}$ and $\varphi \in C_0^\infty(\Omega)$

$$v^\epsilon(x, y) = v(x, y) + \epsilon\varphi(x, y)\, e_3\,.$$

Finally let $\Sigma^\epsilon = v^\epsilon\left(\overline{\Omega}\right)$. Since Σ_0 is of minimal area and $\partial\Sigma^\epsilon = \partial\Sigma_0$, we should have

$$\iint_\Omega |v_x \times v_y|\, dxdy \leq \iint_\Omega \left|v_x^\epsilon \times v_y^\epsilon\right|\, dxdy\,. \tag{7.23}$$

Let $E^\epsilon = |v_x^\epsilon|^2$, $F^\epsilon = \left\langle v_x^\epsilon; v_y^\epsilon \right\rangle$, $G^\epsilon = \left|v_y^\epsilon\right|^2$, $E = |v_x|^2$, $F = \langle v_x; v_y \rangle$ and $G = |v_y|^2$. We therefore get

$$\begin{aligned}
E^\epsilon &= |v_x + \epsilon\varphi e_{3x} + \epsilon\varphi_x e_3|^2 = E + 2\epsilon\left[\varphi_x \langle v_x; e_3\rangle + \varphi\langle v_x; e_{3x}\rangle\right] + O\left(\epsilon^2\right) \\
F^\epsilon &= F + \epsilon\left[\varphi_x \langle v_y; e_3\rangle + \varphi_y \langle v_x; e_3\rangle + \varphi\langle v_y; e_{3x}\rangle + \varphi\langle v_x; e_{3y}\rangle\right] + O\left(\epsilon^2\right) \\
G^\epsilon &= G + 2\epsilon\left[\varphi_y \langle v_y; e_3\rangle + \varphi\langle v_y; e_{3y}\rangle\right] + O\left(\epsilon^2\right)
\end{aligned}$$

where $O(t)$ stands for a function f so that $|f(t)/t|$ is bounded in a neighborhood of $t = 0$. Appealing to the definition of L, M, N, Exercise 5.2.1 and to the fact that $\langle v_x; e_3\rangle = \langle v_y; e_3\rangle = 0$, we obtain

$$\begin{aligned}
E^\epsilon G^\epsilon - (F^\epsilon)^2 &= (E - 2\epsilon L\varphi)(G - 2\epsilon\varphi N) - (F - 2\epsilon\varphi M)^2 + O\left(\epsilon^2\right) \\
&= EG - F^2 - 2\epsilon\varphi\left[EN - 2FM + GL\right] + O\left(\epsilon^2\right) \\
&= \left(EG - F^2\right)\left[1 - 4\epsilon\varphi H\right] + O\left(\epsilon^2\right)\,.
\end{aligned}$$

We therefore conclude that

$$\left|v_x^\epsilon \times v_y^\epsilon\right| = |v_x \times v_y|\,(1 - 2\epsilon\varphi H) + O\left(\epsilon^2\right)$$

and hence

$$\operatorname{Area}(\Sigma^\epsilon) = \operatorname{Area}(\Sigma_0) - 2\epsilon \iint_\Omega \varphi H\, |v_x \times v_y|\, dxdy + O\left(\epsilon^2\right)\,. \tag{7.24}$$

Using (7.23) and (7.24) (i.e., we perform the derivative with respect to ϵ) we get

$$\iint_\Omega \varphi H\, |v_x \times v_y|\, dxdy = 0,\ \forall\varphi \in C_0^\infty(\Omega)\,.$$

Since $|v_x \times v_y| > 0$ (due to the fact that Σ_0 is a regular surface), we deduce from the fundamental lemma of the calculus of variations (Theorem 1.24) that $H = 0$.

7.5.2 The Douglas-Courant-Tonelli method

Exercise 5.3.1. We have

$$w_x = v_\lambda \lambda_x + v_\mu \mu_x \, , \; w_y = v_\lambda \lambda_y + v_\mu \mu_y$$

and thus

$$\begin{array}{rcl} |w_x|^2 & = & |v_\lambda|^2 \lambda_x^2 + 2\lambda_x \mu_x \langle v_\lambda ; v_\mu \rangle + \mu_x^2 |v_\mu|^2 \\ |w_y|^2 & = & |v_\lambda|^2 \lambda_y^2 + 2\lambda_y \mu_y \langle v_\lambda ; v_\mu \rangle + \mu_y^2 |v_\mu|^2 . \end{array}$$

Since $\lambda_x = \mu_y$ and $\lambda_y = -\mu_x$, we deduce that

$$|w_x|^2 + |w_y|^2 = \left[|v_\lambda|^2 + |v_\mu|^2 \right] \left[\lambda_x^2 + \lambda_y^2 \right]$$

and thus

$$\iint_\Omega \left[|w_x|^2 + |w_y|^2 \right] dxdy = \iint_\Omega \left[|v_\lambda|^2 + |v_\mu|^2 \right] \left[\lambda_x^2 + \lambda_y^2 \right] dxdy \, .$$

Changing variables in the second integral, bearing in mind that

$$\lambda_x \mu_y - \lambda_y \mu_x = \lambda_x^2 + \lambda_y^2 \, ,$$

we get the result, namely

$$\iint_\Omega \left[|w_x|^2 + |w_y|^2 \right] dxdy = \iint_B \left[|v_\lambda|^2 + |v_\mu|^2 \right] d\lambda d\mu \, .$$

7.5.3 Nonparametric minimal surfaces

Exercise 5.5.1. Set

$$f = \frac{1 + u_x^2}{\sqrt{1 + u_x^2 + u_y^2}}, \; g = \frac{u_x u_y}{\sqrt{1 + u_x^2 + u_y^2}}, \; h = \frac{1 + u_y^2}{\sqrt{1 + u_x^2 + u_y^2}} \, .$$

A direct computation shows that

$$f_y = g_x \;\; \text{and} \;\; g_y = h_x \, ,$$

since

$$Mu = \left(1 + u_y^2 \right) u_{xx} - 2 u_x u_y u_{xy} + \left(1 + u_x^2 \right) u_{yy} = 0 \, .$$

Setting

$$\varphi(x, y) = \int_0^x \int_0^y g(s, t) \; dtds + \int_0^x \int_0^t f(s, 0) \; dsdt + \int_0^y \int_0^t h(0, s) \; dsdt$$

we find that

$$\varphi_{xx} = f,\ \varphi_{xy} = g,\ \varphi_{yy} = h$$

and hence that

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = 1\,.$$

The fact that φ is convex follows from the above identity, $\varphi_{xx} > 0$, $\varphi_{yy} > 0$ and Theorem 1.50.

7.6 Chapter 6: Isoperimetric inequality

7.6.1 The case of dimension 2

Exercise 6.2.1. One can consult Hardy-Littlewood-Polya [55], page 185, for more details. Let $u \in X$ where

$$X = \left\{ u \in W^{1,2}(-1,1) : u(-1) = u(1) \text{ with } \int_{-1}^{1} u = 0 \right\}.$$

Define

$$z(x) = u(x+1) - u(x)$$

and note that $z(-1) = -z(0)$, since $u(-1) = u(1)$. We deduce that we can find $\alpha \in (-1, 0]$ so that $z(\alpha) = 0$, which means that $u(\alpha+1) = u(\alpha)$. We denote this common value by a (i.e. $u(\alpha+1) = u(\alpha) = a$). Since $u \in W^{1,2}(-1,1)$ it is easy to see that the function $v(x) = (u(x) - a)^2 \cot(\pi(x-\alpha))$ vanishes at $x = \alpha$ and $x = \alpha + 1$ (this follows from Hölder inequality, see Exercise 1.4.3). We therefore have (recalling that $u(-1) = u(1)$)

$$\begin{aligned} & \int_{-1}^{1} \left\{ u'^2 - \pi^2 (u-a)^2 - (u' - \pi(u-a)\cot\pi(x-\alpha))^2 \right\} dx \\ = \quad & \pi \left[(u(x) - a)^2 \cot(\pi(x-\alpha)) \right]_{-1}^{1} = 0\,. \end{aligned}$$

Since $\int_{-1}^{1} u = 0$, we get from the above identity that

$$\int_{-1}^{1} \left(u'^2 - \pi^2 u^2 \right) dx = 2\pi^2 a^2 + \int_{-1}^{1} (u' - \pi(u-a)\cot\pi(x-\alpha))^2\, dx$$

and hence Wirtinger inequality follows. Moreover we have equality in Wirtinger inequality if and only if $a = 0$ and,c denoting a constant,

$$u' = \pi u \cot\pi(x-\alpha) \quad \Leftrightarrow \quad u = c\sin\pi(x-\alpha)\,.$$

Exercise 6.2.2. Since the minimum in (P) is attained by $u \in X$, we have, for any $v \in X \cap C_0^\infty(-1,1)$ and any $\epsilon \in \mathbb{R}$, that

$$I(u+\epsilon v) \geq I(u) .$$

Therefore the Euler-Lagrange equation is satisfied, namely

$$\int_{-1}^{1} \left(u'v' - \pi^2 uv\right) dx = 0, \; \forall v \in X \cap C_0^\infty(-1,1) . \tag{7.25}$$

Let us transform it in a more classical way and choose a function $f \in C_0^\infty(-1,1)$ with $\int_{-1}^1 f = 1$ and let $\varphi \in C_0^\infty(-1,1)$ be arbitrary. Set

$$v(x) = \varphi(x) - \left(\int_{-1}^{1} \varphi \, dx\right) f(x) \text{ and } \lambda = -\frac{1}{\pi^2} \int_{-1}^{1} \left(u'f' - \pi^2 uf\right) dx .$$

Observe that $v \in X \cap C_0^\infty(-1,1)$. Use (7.25), the fact that $\int_{-1}^1 f = 1$, $\int_{-1}^1 v = 0$ and the definition of λ to get, for every $\varphi \in C_0^\infty(-1,1)$,

$$\begin{aligned}
&\int_{-1}^{1} \left[u'\varphi' - \pi^2 (u-\lambda)\varphi\right] dx \\
= &\int \left[u'\left(v' + f'\int\varphi\right) - \pi^2 u\left(v + f\int\varphi\right)\right] + \pi^2\lambda\int\varphi \\
= &\int \left(u'v' - \pi^2 uv\right) + \left[\int\varphi\right]\left[\pi^2\lambda + \int\left(u'f' - \pi^2 uf\right)\right] = 0 .
\end{aligned}$$

The regularity of u (which is a minimizer of (P) in X) then follows (as in Proposition 4.1) at once from the above equation. Since we know (from Theorem 6.1) that among smooth minimizers of (P) the only ones are of the form $u(x) = \alpha\cos\pi x + \beta\sin\pi x$, we have the result.

Exercise 6.2.3. We divide the proof into two steps.

Step 1. We start by introducing some notations. Since we will work with fixed u, v, we will drop the dependence on these variables in $L = L(u,v)$ and $M = M(u,v)$. However we will need to express the dependence of L and M on the intervals (α, β),where $a \leq \alpha < \beta \leq b$, and we will therefore let

$$\begin{aligned}
L(\alpha,\beta) &= \int_\alpha^\beta \sqrt{u'^2 + v'^2} \, dx \\
M(\alpha,\beta) &= \int_\alpha^\beta uv' \, dx .
\end{aligned}$$

So that in these new notations

$$L(u,v) = L(a,b) \text{ and } M(u,v) = M(a,b) .$$

We next let

$$O = \left\{ x \in (a,b) : u'^2 (x) + v'^2 (x) > 0 \right\}.$$

The case where $O = (a,b)$ has been considered in Step 1 of Theorem 6.4. If O is empty the result is trivial, so we will assume from now on that this is not the case. Since the functions u' and v' are continuous, the set O is open. We can then find (see Theorem 6.59 in [57] or Theorem 9 of Chapter 1 in [37])

$$a \le a_i < b_i < a_{i+1} < b_{i+1} \le b, \ \forall i \ge 1$$

$$O = \bigcup_{i=1}^{\infty} (a_i, b_i) \ .$$

In the complement of O, O^c, we have $u'^2 + v'^2 = 0$, and hence

$$L(b_i, a_{i+1}) = M(b_i, a_{i+1}) = 0\,. \tag{7.26}$$

Step 2. We then change the parametrization on every (a_i, b_i). We choose a multiple of the arc length, namely

$$\begin{cases} y = \eta(x) = -1 + 2\dfrac{L(a,x)}{L(a,b)} \\[2ex] \varphi(y) = u\left(\eta^{-1}(y)\right) \text{ and } \psi(y) = v\left(\eta^{-1}(y)\right) \ . \end{cases}$$

Note that this is well defined, since $(a_i, b_i) \subset O$. We then let

$$\alpha_i = -1 + 2\frac{L(a,a_i)}{L(a,b)} \quad \text{and } \beta_i = -1 + 2\frac{L(a,b_i)}{L(a,b)}$$

so that

$$\beta_i - \alpha_i = 2\frac{L(a_i,b_i)}{L(a,b)} \ .$$

Furthermore, since $L(b_i, a_{i+1}) = 0$, we get

$$\beta_i = \alpha_{i+1} \text{ and } \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i] = [-1, 1] \ .$$

We also easily find that, for $y \in (\alpha_i, \beta_i)$,

$$\sqrt{\varphi'^2(y) + \psi'^2(y)} = \frac{L(a,b)}{2} = \frac{L(a_i,b_i)}{\beta_i - \alpha_i}$$

$$\varphi(\alpha_i) = u(a_i)\,,\ \psi(\alpha_i) = v(a_i)\,,\ \varphi(\beta_i) = u(b_i)\,,\ \psi(\beta_i) = v(b_i)\ .$$

In particular we have that $\varphi, \psi \in W^{1,2}(-1,1)$, with $\varphi(-1) = \varphi(1)$ and $\psi(-1) = \psi(1)$, and

$$L(a_i, b_i) = \frac{2}{L(a,b)} \int_{\alpha_i}^{\beta_i} \left(\varphi'^2(y) + \psi'^2(y)\right) dy \tag{7.27}$$

$$M(a_i, b_i) = \int_{\alpha_i}^{\beta_i} \varphi(y)\, \psi'(y)\, dy\,. \tag{7.28}$$

We thus obtain, using (7.26), (7.27) and (7.28),

$$L(a,b) = \sum_{i=1}^{\infty} L(a_i, b_i) = \frac{2}{L(a,b)} \int_{-1}^{1} \left(\varphi'^2(y) + \psi'^2(y)\right) dy$$

$$M(a,b) = \sum_{i=1}^{\infty} M(a_i, b_i) = \int_{-1}^{1} \varphi(y)\, \psi'(y)\, dy\,.$$

We therefore find, invoking Corollary 6.3, that

$$\begin{aligned} [L(u,v)]^2 - 4\pi M(u,v) &= [L(a,b)]^2 - 4\pi M(a,b) \\ &= 2\int_{-1}^{1} \left(\varphi'^2 + \psi'^2\right) dy - 4\pi \int_{-1}^{1} \varphi\psi'\, dy \geq 0 \end{aligned}$$

as wished.

7.6.2 The case of dimension n

Exercise 6.3.1. We clearly have

$$C = (\overline{a} + B) \cup \left(\overline{b} + A\right) \subset A + B\,.$$

It is also easy to see that $(\overline{a} + B) \cap \left(\overline{b} + A\right) = \left\{\overline{a} + \overline{b}\right\}$. Observe then that

$$M(C) = M(\overline{a} + B) + M\left(\overline{b} + A\right) - M\left[(\overline{a} + B) \cap \left(\overline{b} + A\right)\right] = M(A) + M(B)$$

and hence

$$M(A) + M(B) \leq M(A + B)\ .$$

Exercise 6.3.2. (i) We adopt the same notations as those of Exercise 5.2.4. By hypothesis there exist a bounded smooth domain $\Omega \subset \mathbb{R}^2$ and a map $v \in C^2\left(\overline{\Omega}; \mathbb{R}^3\right)$ ($v = v(x,y)$, with $v_x \times v_y \neq 0$ in $\overline{\Omega}$) so that $\partial A_0 = v\left(\overline{\Omega}\right)$.

From the divergence theorem it follows that

$$M(A_0) = \frac{1}{3} \iint_{\Omega} \langle v; v_x \times v_y \rangle\, dxdy\,. \tag{7.29}$$

Let then $\epsilon \in \mathbb{R}$, $\varphi \in C_0^\infty(\Omega)$ and

$$v^\epsilon(x,y) = v(x,y) + \epsilon\varphi(x,y)\,e_3$$

where $e_3 = (v_x \times v_y) / |v_x \times v_y|$.

We next consider $\partial A^\epsilon = \left\{v^\epsilon(x,y) : (x,y) \in \overline{\Omega}\right\} = v^\epsilon\left(\overline{\Omega}\right)$. We have to evaluate $M(A^\epsilon)$ and we start by computing

$$\begin{aligned} v_x^\epsilon \times v_y^\epsilon &= (v_x + \epsilon(\varphi_x e_3 + \varphi e_{3x})) \times \left(v_y + \epsilon\left(\varphi_y e_3 + \varphi e_{3y}\right)\right) \\ &= v_x \times v_y + \epsilon\left[\varphi\left(e_{3x} \times v_y + v_x \times e_{3y}\right)\right] \\ &\quad +\epsilon\left[\varphi_x e_3 \times v_y + \varphi_y v_x \times e_3\right] + O\left(\epsilon^2\right) \end{aligned}$$

(where $O(t)$ stands for a function f so that $|f(t)/t|$ is bounded in a neighborhood of $t=0$) which leads to

$$\begin{aligned} \left\langle v^\epsilon; v_x^\epsilon \times v_y^\epsilon\right\rangle &= \left\langle v + \epsilon\varphi e_3; v_x^\epsilon \times v_y^\epsilon\right\rangle \\ &= \langle v; v_x \times v_y\rangle + \epsilon\varphi\langle e_3; v_x \times v_y\rangle + \epsilon\left\langle v; \varphi\left(e_{3x} \times v_y + v_x \times e_{3y}\right)\right\rangle \\ &\quad +\epsilon\left\langle v; \varphi_x e_3 \times v_y + \varphi_y v_x \times e_3\right\rangle + O\left(\epsilon^2\right). \end{aligned}$$

Observing that $\langle e_3; v_x \times v_y\rangle = |v_x \times v_y|$ and returning to (7.29), we get after integration by parts that (recalling that $\varphi = 0$ on $\partial\Omega$).

$$\begin{aligned} M(A^\epsilon) - M(A_0) &= \frac{\epsilon}{3}\iint_\Omega \varphi\left\{|v_x \times v_y| + \left\langle v; e_{3x} \times v_y + v_x \times e_{3y}\right\rangle\right. \\ &\quad \left. - \left(\langle v; e_3 \times v_y\rangle\right)_x - \left(\langle v; v_x \times e_3\rangle\right)_y\right\} dxdy + O\left(\epsilon^2\right) \\ &= \frac{\epsilon}{3}\iint_\Omega \varphi\left\{|v_x \times v_y| - \langle v_x; e_3 \times v_y\rangle\right. \\ &\quad \left. - \langle v_y; v_x \times e_3\rangle\right\} dxdy + O\left(\epsilon^2\right). \end{aligned}$$

Since $\langle v_x; e_3 \times v_y\rangle = \langle v_y; v_x \times e_3\rangle = -|v_x \times v_y|$, we obtain that

$$M(A^\epsilon) - M(A_0) = \epsilon\iint_\Omega \varphi\,|v_x \times v_y|\,dxdy + O\left(\epsilon^2\right). \tag{7.30}$$

(ii) We recall from (7.24) in Exercise 5.2.4 that we have

$$L(\partial A^\epsilon) - L(\partial A_0) = -2\epsilon\iint_\Omega \varphi H\,|v_x \times v_y|\,dxdy + O\left(\epsilon^2\right). \tag{7.31}$$

Combining (7.30), (7.31), the minimality of A_0 and a Lagrange multiplier α, we get

$$\iint_\Omega (-2\varphi H + \alpha\varphi)\,|v_x \times v_y|\,dxdy = 0,\ \forall\varphi \in C_0^\infty(\Omega).$$

The fundamental lemma of the calculus of variations (Theorem 1.24) implies then that $H = \text{constant}$ (since ∂A_0 is a regular surface we have $|v_x \times v_y| > 0$).

Bibliography

[1] Adams R.A., *Sobolev spaces*, Academic Press, New York, 1975.

[2] Akhiezer N.I., *The calculus of variations*, Blaisdell, New York, 1962.

[3] Alibert J.J. and Dacorogna B., An example of a quasiconvex function that is not polyconvex in two dimensions, *Arch. Rational Mech. Anal.* **117** (1992), 155-166.

[4] Almgren F.J., *Plateau's problem. An invitation to varifold geometry*, W.A. Benjamin, New York, 1966.

[5] Ambrosio L., Fusco N. and Pallara D., *Functions of bounded variation and free discontinuity problems*, Oxford University Press, Oxford, 2000.

[6] Antman S.S., *Nonlinear problems of elasticity*, Springer, Berlin, 1995.

[7] Ball J.M., Convexity conditions and existence theorems in non linear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337-403.

[8] Ball J.M. and Mizel V., One dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equations, *Arch. Rational Mech. Anal.* **90** (1985), 325-388.

[9] Bandle C., *Isoperimetric inequalities and applications*, Pitman, London, 1980.

[10] Berger M., *Geometry I and II*, Springer, Berlin, 1987.

[11] Blaschke W., *Kreis und Kugel*, Chelsea, New York, 1949.

[12] Bliss G., *Lectures on the calculus of variations*, University of Chicago Press, Chicago, 1951.

[13] Bolza O., *Lectures on the calculus of variations*, Chelsea Publication, New York, 1946.

[14] Brézis H., *Analyse fonctionnelle, théorie et applications*, Masson, Paris, 1983.

[15] Buttazzo G., *Semicontinuity, relaxation and integral represention in the calculus of variations*, Pitman, Longman, London, 1989.

[16] Buttazzo G., Ferone V. and Kawohl B., Minimum problems over sets of concave functions and related questions, *Math. Nachrichten* **173** (1995), 71-89.

[17] Buttazzo G., Giaquinta M. and Hildebrandt S., *One dimensional variational problems*, Oxford University Press, Oxford, 1998.

[18] Buttazzo G. and Kawohl B., On Newton's problem of minimal resistance, *Math. Intell.* **15** (1992), 7-12.

[19] Carathéodory C., *Calculus of variations and partial differential equations of the first order*, Holden Day, San Francisco, 1965.

[20] Cesari L., *Optimization - Theory and applications*, Springer, New York, 1983.

[21] Chern S.S., An elementary proof of the existence of isothermal parameters on a surface, *Proc. Amer. Math. Soc.*, **6** (1955), 771-782.

[22] Ciarlet P., *Mathematical elasticity, Volume 1, Three dimensional elasticity*, North Holland, Amsterdam, 1988.

[23] Clarke F.H., *Optimization and nonsmooth analysis*, Wiley, New York, 1983.

[24] Courant R., *Dirichlet's principle, conformal mapping and minimal surfaces*, Interscience, New York, 1950.

[25] Courant R., *Calculus of variations*, Courant Institute Publications, New York, 1962.

[26] Courant R. and Hilbert D., *Methods of mathematical physics*, Wiley, New York, 1966.

[27] Crandall M.G., Ishii H. and Lions P.L., User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* **27** (1992), 1-67.

[28] Croce G. and Dacorogna B., On a generalized Wirtinger inequality, *Disc. and Cont. Dyn. Syst. Ser. A*, **9** (2003), 1329-1341.

[29] Dacorogna B., Quasiconvexity and relaxation of nonconvex variational problems, *J. of Funct. Anal.* **46** (1982), 102-118.

[30] Dacorogna B., *Weak continuity and weak lower semicontinuity of nonlinear functionals*, Springer, Berlin, 1982.

[31] Dacorogna B., *Direct methods in the calculus of variations*, Springer, Berlin, 1989.

[32] Dacorogna B. and Marcellini P., A counterexample in the vectorial calculus of variations, in *Material instabilities in continuum mechanics*, ed. Ball J.M., Oxford Sci. Publ., Oxford, 1988, 77-83.

[33] Dacorogna B. and Marcellini P., *Implicit partial differential equations*, Birkhäuser, Boston, 1999.

[34] Dacorogna B. and Murat F., On the optimality of certain Sobolev exponents for the weak continuity of determinants, *J. of Funct. Anal.* **105** (1992), 42-62.

[35] Dacorogna B. and Pfister C.E., Wulff theorem and best constant in Sobolev inequality, *J. Math. Pures Appl.* **71** (1992), 97-118.

[36] Dal Maso G., *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Appl., 8, Birkhäuser, Boston, 1993.

[37] De Barra G., *Measure theory and integration*, Wiley, New York, 1981.

[38] De Giorgi E., *Teoremi di semicontinuità nel calcolo delle variazioni*, Istituto Nazionale di Alta Matematica, Roma, 1968-1969.

[39] Dierkes U., Hildebrandt S., Küster A. and Wohlrab O., *Minimal surfaces I and II*, Springer, Berlin, 1992.

[40] Ekeland I., *Convexity methods in Hamiltonian mechanics*, Springer, Berlin, 1990.

[41] Ekeland I. and Témam R., *Analyse convexe et problèmes variationnels*, Dunod, Paris, 1974.

[42] Evans L.C., *Weak convergence methods for nonlinear partial differential equations*, Amer. Math. Soc., Providence, 1990.

[43] Evans L.C., *Partial differential equations*, Amer. Math. Soc., Providence, 1998.

[44] Evans L.C. and Gariepy R.F., *Measure theory and fine properties of functions*, Studies in Advanced Math., CRC Press, Boca Raton, 1992.

[45] Federer H., *Geometric measure theory*, Springer, Berlin, 1969.

[46] Gelfand I.M. and Fomin S.V., *Calculus of variations*, Prentice-Hall, Englewood, 1963.

[47] Giaquinta M., *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton University Press, Princeton, 1983.

[48] Giaquinta M. and Hildebrandt S., *Calculus of variations I and II*, Springer, Berlin, 1996.

[49] Gilbarg D. and Trudinger N.S., *Elliptic partial differential equations of second order*, Springer, Berlin, 1977.

[50] Giusti E., *Minimal surfaces and functions of bounded variations*, Birkhäuser, Boston, 1984.

[51] Giusti E., *Metodi diretti del calcolo delle variazioni*, Unione Matematica Italiana, Bologna, 1994.

[52] Goldstine H.H., *A history of the calculus of variations from the 17th to the 19th century*, Springer, Berlin, 1980.

[53] Hadamard J., Sur quelques questions du calcul des variations, *Bulletin Société Math. de France* **33** (1905), 73-80.

[54] Hadamard J., *Leçons sur le calcul des variations*, Hermann, Paris, 1910.

[55] Hardy G.H., Littlewood J.E. and Polya G., *Inequalities*, Cambridge University Press, Cambridge, 1961.

[56] Hestenes M.R., *Calculus of variations and optimal control theory*, Wiley, New York, 1966.

[57] Hewitt E. and Stromberg K., *Real and abstract analysis*, Springer, Berlin, 1965.

[58] Hildebrandt S. and Tromba A., *Mathematics and optimal form*, Scientific American Library, New York, 1984.

[59] Hildebrandt S. and Von der Mosel H., Plateau's problem for parametric double integrals: Existence and regularity in the interior, *Commun. Pure Appl. Math.* **56** (2003), 926–955.

[60] Hörmander L., *Notions of convexity*, Birkhäuser, Boston, 1994.

[61] Hsiung C.C., *A first course in differential geometry*, Wiley, New York, 1981.

[62] Ioffe A.D. and Tihomirov V.M., *Theory of extremal problems*, North Holland, Amsterdam, 1979.

[63] John F., *Partial differential equations*, Springer, Berlin, 1982.

[64] Kawohl B., Recent results on Newton's problem of minimal resistance, *Nonlinear analysis and applications* (Warsaw, 1994), Gakuto Internat. Ser. Math. Sci. Appl., **7** (1996), 249–259.

[65] Kinderlherer D. and Stampacchia G., *Introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

[66] Ladyzhenskaya O.A. and Uraltseva N.N., *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[67] Lebesgue H., *Leçons sur l'intégration et la recherche des fonctions primitives*, Gauthier-Villars, Paris, 1928.

[68] Lions J.L. and Magenes E., *Non-homogeneous boundary value problems and applications I,II,III*, Springer, Berlin, 1972.

[69] Lions P.L., *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Math. 69, Pitman, London, 1982.

[70] Marcellini P., Non convex integrals of the calculus of variations, in: *Methods of nonconvex analysis*, ed. Cellina A., Lecture Notes in Math. 1446, Springer, Berlin, 1990, 16-57.

[71] Marcellini P. and Sbordone C., Semicontinuity problems in the calculus of variations, *Nonlinear Anal., Theory, Methods and Applications* **4** (1980), 241-257.

[72] Mawhin J. and Willem M., *Critical point theory and Hamiltonian systems*, Springer, Berlin, 1989.

[73] Monna A.F., *Dirichlet's principle: a mathematical comedy of errors and its influence on the development of analysis*, Oosthoeck, Utrecht, 1975.

[74] Morrey C.B., Quasiconvexity and the lower semicontinuity of multiple integrals, *Pacific J. Math.* **2** (1952), 25-53.

[75] Morrey C.B., *Multiple integrals in the calculus of variations*, Springer, Berlin, 1966.

[76] Morse M., *The calculus of variations in the large*, Amer. Math. Soc., New York, 1934.

[77] Necas J., *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.

[78] Nitsche J.C., *Lecture on minimal surfaces*, Cambridge University Press, Cambridge, 1989.

[79] Ornstein D., A non-inequality for differential operators in the L^1 norm, *Arch. Rational Mech. Anal.* **11** (1962), 40-49.

[80] Osserman R., *A survey on minimal surfaces*, Van Nostrand, New York, 1969.

[81] Osserman R., The isoperimetric inequality, *Bull. Amer. Math. Soc.* **84** (1978), 1182-1238.

[82] Pars L., *An introduction to the calculus of variations*, Heinemann, London, 1962.

[83] Payne L., Isoperimetric inequalities and their applications, *SIAM Rev.* **9** (1967), 453-488.

[84] Pisier G., *The volume of convex bodies and Banach space geometry*, Cambridge University Press, Cambridge, 1989.

[85] Polya G. and Szegö G., *Isoperimetric inequalities in mathematical physics*, Princeton University Press, Princeton, 1951.

[86] Porter T.I., A history of the classical isoperimetric problem, in *Contributions to the calculus of variations (1931-1932)*, edited by Bliss G.A. and Graves L.M., University of Chicago Press, Chicago, 1933.

[87] Rockafellar R.T., *Convex Analysis*, Princeton University Press, Princeton, 1970.

[88] Rudin W., *Real and complex analysis*, McGraw-Hill, New York, 1966.

[89] Rudin W., *Functional analysis*, McGraw-Hill, New York, 1973.

[90] Rund H., *The Hamilton-Jacobi theory in the calculus of variations*, Van Nostrand, Princeton, 1966.

[91] Struwe M., *Plateau's problem and the calculus of variations*, Princeton University Press, Princeton, 1988.

[92] Struwe M., *Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems*, Springer, Berlin, 1990.

[93] Sverak V., Rank one convexity does not imply quasiconvexity, *Proc. Royal Soc. Edinburgh* **120A** (1992), 185-189.

[94] Tonelli L., *Fondamenti di calcolo delle variazioni I and II*, Zanichelli, Bologna, 1921.

[95] Troutman J.L., *Variational calculus with elementary convexity*, Springer, New York, 1983

[96] Webster R., *Convexity*, Oxford University Press, Oxford, 1994.

[97] Weinstock R., *Calculus of variations with applications to physics and engineering*, McGraw-Hill, New York, 1952.

[98] Young L.C., *Lectures on the calculus of variations and optimal control theory*, W.B. Saunders, Philadelphia, 1969.

[99] Zeidler E., *Nonlinear functional analysis and its applications, I, II, III, IV*, Springer, New York, 1985-1988.

[100] Ziemer W.P., *Weakly differentiable functions*, Springer, New York, 1989.

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