Răzvan Diaconescu

Institution-independent Model Theory

BIRKHÄUSER

Studies in Universal Logic

BIRKHÄUSER

Studies in Universal Logic

This series is devoted to the universal approach to logic and the development of a general theory of logics. It covers topics such as global set-ups for fundamental theorems of logic and frameworks for the study of logics, in particular logical matrices, Kripke structures, combination of logics, categorical logic, abstract proof theory, consequence operators, and algebraic logic. It includes also books with historical and philosophical discussions about the nature and scope of logic. Three types of books will appear in the series: graduate textbooks, research monographs, and volumes with contributed papers.

Răzvan Diaconescu

Institution-independent Model Theory

Birkhäuser
Basel · Boston · Berlin

Author:

Răzvan Diaconescu
Institute of Mathematics „Simion Stoilow“
Calea Griviței 21
București 010702
Romania
Email: razvan.diaconescu@imar.ro

2000 Mathematical Subject Classification: 03-02, 03B99, 03C20, 03C52, 03C95

Library of Congress Control Number: 2008924875

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 978-3-7643-8707-5 Birkhäuser Verlag AG, Basel · Boston · Berlin

Basel · Boston · Berlin
P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany

ISBN 978-3-7643-8707-5 e-ISBN 978-3-7643-8708-2

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

To my parents, Elena and Ştefan

Preface

This is a book about doing model theory without an underlying logical system. It teaches us how to live without concrete models, sentences, satisfaction and so on. Our approach is based upon the theory of institutions, which has witnessed a vigorous and systematic development over the past two decades and which provides an ideal framework for true abstract model theory. The concept of institution formalizes the intuitive notion of logical system into a mathematical object. Thus our model theory without underlying logical systems and based upon institution theory may be called ‘institution-independent model theory’.

Institution-independent model theory has several advantages. One is its generality, since it can be easily applied to a multitude of logical systems, conventional or less conventional, many of the latter kind getting a proper model theory for the first time through this approach. This is important especially in the context of the recent high proliferation of logics in computing science, especially in the area of formal specification. Then there is the advantage of illuminating the model theoretic phenomena and its subtle network of causality relationships, thus leading to a deeper understanding which produces new fundamental insights and results even in well worked traditional areas of model theory.

In this way we study well established topics in model theory but also some newly emerged important topics. The former category includes methods (in fact much of model theory can be regarded as a collection of sometimes overlapping methods) such as (elementary) diagrams, ultraproducts, saturated models and studies about preservation, axiomatizability, interpolation, definability, and possible worlds semantics. The latter category includes methods of doing model theory ‘by translation’, and Grothendieck institutions, which is a recent successful model theoretic framework for multi-logic heterogeneous environments. The last two chapters (14 and 15) digress from the main topic of the book in that they present some applications of institution-independent model theory to specification and programming and Chap. 13 shows how to integrate proof theoretic concepts to institution-independent model theory (including a general approach to completeness).

This book is far from being a complete encyclopedia of institution-independent model theory. While several important concepts and results have not been treated here, we believe they can be approached successfully with institutions in the style promoted by our work. Most of all, this book shows *how* to do things rather than provides an exhaustive account of all model theory that can be done institution-independently. It can be used by any working user of model theory but also as a resource for learning model theory.

From the philosophical viewpoint, the institution-independent approach to model theory is based upon a non-essentialist, groundless, perspective on logic and model theory, directly influenced by the doctrine of *śunyata* of the Madhyamaka Prasangika school within Mahayana Buddhism. The interested reader may find more about this connection in the essay [54]. This has been developed mainly at Nalanda monastic university about 2000 years ago by Arya Nāgārjuna and its successors and has been continued to our days by all traditions of Tibetan Buddhism. The relationship between Madhyamaka Prasangika thinking and various branches of modern science is surveyed in [176].

I am grateful to a number of people who supported in various ways the project of institution-independent model theory in general and the writing of this book in particular. I was extremely fortunate to be first the student and later a close friend and collaborator of late Professor Joseph Goguen who together with Rod Burstall introduced institutions. He strongly influenced this work in many ways and at many levels, from philosophical to technical aspects, and was one of the greatest promoters of the non-essentialist approach to science. Andrzej Tarlecki was the true pioneer of doing model theory in an abstract institutional setting. Till Mossakowski made a lot of useful comments on several preliminary drafts of this book and supported this activity in many other ways too. Grigore Roşu and Marc Aiguier made valuable contributions to this area. Lutz Schröder made several comments and gave some useful suggestions. Achim Blumensath read very carefully a preliminary draft of this book and helped to correct a series of errors. I am indebted to Hans-Jürgen Hoenhke for encouragement and managerial support. Special thanks go to the former students of the Informatics Department of "Şcoala Normală Superioară" of Bucharest, namely Marius Petria, Daniel Găină, Andrei Popescu, Mihai Codescu, Traian Şerbănuţă and Cristian Cucu. They started as patient students of institution-independent model theory only to become important contributors to this area. Finally, Jean-Yves Béziau greatly supported the publication and dissemination of this book. I acknowledge financial support for writing this book from the CNCSIS grants GR202/2006 and GR54/2007.

Ploieşti,

December 2007 *Răzvan Diaconescu*

Contents

Chapter 1

Introduction

Model theory is in essence the mathematical study of semantics, or meaning, of logic systems. As it has a multitude of applications to various areas of classical mathematics, and of logic, but also to many areas of informatics and computing science, there are various perspectives on model theory which differ slightly. A rather classical viewpoint is formulated in [32]:

Model theory = logic + universal algebra.

A rather different and more radical perspective which reflects the success of model theoretic methods in some areas of classical mathematics is given in [99]:

Model theory = algebraic geometry - fields.

From a formal specification viewpoint, in a similar tone, one may say that

Model theory = logical semantics - specification.

Each such viewpoint implies a specific way in developing the key concepts and the main model theory methods; it also puts different emphasis on results. For example while forcing is a very important method for the applications of model theory to conventional logic, it plays a very little role in computing science. On the other hand, formal specification theory requires a much more abstract view on model theory than the conventional one. The institution theory of Goguen and Burstall [30, 75] arose out of this necessity.

Institutions. The theory of institutions is a categorical abstract model theory which formalizes the intuitive notion of a logical system, including syntax, semantics, and the satisfaction relation between them. Institutions constitute a model-oriented meta-theory on logics similarly to how the theory of rings and modules constitute a meta-theory for classical linear algebra. Another analogy can be made with universal algebra versus particular algebraic structures such as groups, rings, modules, etc., or with mathematical analysis over Banach spaces versus real analysis.

The notion of institution was introduced by Goguen and Burstall in the late 1970s [30] (with the seminal journal paper [75] being printed rather late) in response to the population explosion of specification logics with the original intention of providing a proper abstract framework for specification of, and reasoning about, software systems. Since then institutions have become a major tool in development of the theory of specification, mainly because they provide a language-independent framework applicable to a wide variety of particular specification logics. It became standard in the field to have a logic system captured as the institution underlying a particular language or system, such that all language/system constructs and features can be rigorously explained as mathematical entities and to separate all aspects that depend on the details of the particular logic system from those that are general and independent of this logic system by basing the latter on an arbitrary institution. All well-designed specification formalisms follow this path, including for example CASL [10] and CafeOBJ [57].

Recently institutions have also been applied to computing science fields other than formal specification; these include ontologies and cognitive semantics [73], concurrency [138], and quantum computing [31].

Institution-independent model theory. This means the development of model theory in the very abstract setting of arbitrary institutions, free of any commitment to a particular logic system. In this way we gain another level of abstraction and generality and a deeper understanding of model theoretic phenomena, not hindered by the largely irrelevant details of a particular logic system, but guided by structurally clean causality. The latter aspect is based upon the fact that concepts come naturally as presumed features that "a logic" might exhibit or not and are defined at the most appropriate level of abstraction; hypotheses are kept as general as possible and introduced on a by-need basis, and thus results and proofs are modular and easy to track down regardless of their depth. Access to highly non-trivial results is also considerably facilitated, which is contrary to the impression of some people that such general abstract approaches produce results that are trivial. As Béziau explains in [20]:

> "This impression is generally due to the fact that these people have a concrete-oriented mind, and that something which is not specified [n.a. concretely] has no meaning for them, and therefore universal logic [n.a. institution-independent model theory in our case] appears as a logical abstract nonsense. They are like someone who understands perfectly what is Felix, his cat, but for whom the concept of cat is a meaningless abstraction. This psychological limitation is in fact a strong defect because, ... [n.a. as this book shows], what is trivial is generally the specific part, not the universal one [n.a. the institution-independent one] which requires what is the fundamental capacity of human thought: abstraction."

The continuous interplay between the specific and the general in institution-independent model theory brings a large array of new results for particular non-conventional logics, unifies several known results, produces new results in well-studied conventional areas, reveals previously unknown causality relations, and dismantles some which are usually assumed as natural.

Institution-independent model theory also provides a clear and efficient framework for doing logic and model theory 'by translation (or borrowing)' via a general theory of mappings (homomorphisms) between institutions. For example, a certain property P which holds in an institution I' can be also established in another institution I provided that we can define a mapping $I \to I'$ which 'respects' P.

Institution-independent model theory can be regarded as a form of 'universal model theory', part of the so-called 'universal logic', a recent trend in logic promoted by Bèziau and others [21].

Other abstract model theories. Only two major abstract approaches to logic have a model theoretic nature and are therefore comparable to the institution-independent model theory.

The so-called "abstract model theory" developed by Barwise and others [12, 13] however keeps a strong commitment to conventional concrete systems of logic by explicitly extending them and retaining many of their features, hence one may call this framework "half-abstract model theory". In this context even the remarkable Lindström characterization of first order logic by some of its properties should be rather considered as a first order logic result rather than as a true abstract model theoretic one.

Another framework is given by the so-called "categorical model theory" best represented by the works on sketches [63, 88, 181] or on satisfaction as cone injectivity [5, 6, 7, 120, 118, 116]. The former just develops another language for expressing (possibly infinitary) first order logic realities. While the latter considers models as objects of abstract categories, it lacks the multi-signature aspect of institutions given by the signature morphism and the model reducts, which leads to severe methodological limitations. Moreover in these categorical model theory frameworks, the satisfaction of sentences by the models is usually defined rather than being axiomatized.

By contrast to the two abstract model theoretic approaches mentioned above, institutions capture directly the essence of logic systems by axiomatizing the satisfaction relationship between models and sentences without any initial commitment to a particular logic system and by emphasizing propertly the multi-signature aspect of logics.

Book content. The book consists of four parts.

In the first part we introduce the basic institution theory including the concept of institution and institution morphisms, and several model theoretic fundamental concepts such as model amalgamation, (elementary) diagrams, inclusion systems, and free models. We develop an 'internal logic' for abstract institutions, which includes a semantic treatment to Boolean connectives, quantifiers, atomic sentences, substitutions, and elementary homomorphisms, all of them in an institution-independent setting.

The second part is the core of our institution-independent model theoretic study because it develops the main model theory methods and results in an institution-independent setting.

The first method considered in this part is that of ultraproducts. Based upon the well-established concept of categorical filtered products, we develop an ultraproduct fundamental theorem in an institution-independent setting and explore some of its immediate consequences, such as ultrapower embeddings and compactness.

The chapter on saturated models starts by developing sufficient conditions for directed co-limits of homomorphisms to retain the elementarity. This rather general version of Tarski's elementary chain theorem is a prerequisite for a general result about existence of saturated models, later used for developing other important results. We also develop the complementary result on uniqueness of saturated models. Here the necessary concept of cardinality of a model is handled categorically with the help of elementary extensions, a concept given by the method of diagrams. We develop an important application for the uniqueness of saturated models, namely a generalized version of the remarkable Keisler-Shelah result in first order model theory, "two models are elementarily equivalent if and only if they have isomorphic ultrapowers".

A good application of the existence result for saturated models is seen in the preservation results, such as "a theory has a set of universal axioms if and only if its class of models is closed under 'sub-models'". We develop a generic preservation-by-saturation theorem. Such preservation results might lead us straight to their axiomatizability versions. One way is to assume the Keisler-Shelah property for the institution and to use a direct consequence of the fundamental ultraproducts theorem which may concisely read as "a class of models is elementary if and only if it is closed under elementary equivalence and ultraproducts".

Another method to reach an important class of axiomatizability results is by expressing the satisfaction of Horn sentences as categorical injectivity. This leads to general quasi-variety theorems such as "a class of models is closed under products and 'sub-models' if and only if it is axiomatizable by a set of (universal) Horn sentences" and variety theorems such as "a class of models is closed under products and 'sub-models' and 'homomorphic images' if and only if it is axiomatizable by a set of (universal) 'atoms'".

All axiomatizability results presented here are collected under the abstract concept of 'Birkhoff institution'.

The next topic is interpolation. The institution-independent approach brings several significant upgrades to the conventional formulation. We develop here three main methods for obtaining the interpolation property, the first two having rather complementary application domains. The first one is based upon a semantic approach to interpolation and exploits the Birkhoff-style axiomatizability properties of the institution (captured by the above mentioned concept of Birkhoff institution), while the second, inspired by the conventional methods of first order logic, is via Robinson consistency. The third one is a borrowing method across institutions.

We next treat definability, again with rather two complementary methods, via Birkhoff-style axiomatizability and via interpolation. While the latter represents a generalization of Beth's theorem of conventional first order model theory, the former reveals a causality relationship between axiomatizability and definability.

The final chapter of the second part of the book is devoted to possible worlds (Kripke) semantics and to extensions of the satisfaction relation of abstract institutions to modal satisfaction. By applying the general ultraproducts method to possible worlds semantics, we develop the preservation of modal satisfaction by ultraproducts together with its semantic compactness consequence.

The third part of the book is devoted to special modern topics in institution theory, such as Grothendieck constructions on systems of institutions with applications to heterogeneous multi-logic frameworks, and an extension of institutions with proof theoretic concepts. For the Grothendieck institutions we develop a systematic study of lifting of important properties such as theory co-limits, model amalgamation, and interpolation, from the level of the 'local' institutions to the 'global' Grothendieck institution. We present a rather striking application of the interpolation result for Grothendieck institutions, which leads for example to a quite surprising interpolation property in the Horn fragment of conventional first order logic. The chapter on proof theory for institutions introduces the concept of proof in a simple way that suits the model theory, explores proof theoretic versions of compactness and internal logic, and presents general soundness results for institutions with proofs. The final part of this chapter develops a general sound and complete Birkhoff-style proof system with applications significantly wider than that of the Horn institutions.

The last part presents a few of the multitude of applications of institution-independent model theory to computing science, especially in the areas of formal specification and logic programming. This includes structured specifications over arbitrary institutions, the lifting of a complete calculus from the base institution to structured specifications, Herbrand theorems and modularization for logic programming, and semantics of logic programming with pre-defined types.

The concepts introduced and the results obtained are systematically illustrated in the main text by their applications to the model theory of conventional logic (which includes first order logic but also fragments and extensions of it). There are only two reasons for doing this. The first is to build a bridge between our approach and the conventional model theory culture. The second reason has to do with keeping the material within reasonable size. Otherwise, while conventional (first-order) model theory has been historically the framework for the development of the main concepts and methods of model theory, one of the main messages of this book is that these do not depend on that framework. Any other concrete logic or model theory could be used as a benchmark example in this book, and in fact we do this systematically in the exercise sections with several less conventional logics.

How to use this book. The material of this book can be used in various ways by various audiences both from logic and computing science. Students and researchers of logic can use material of the first two parts (up to Chap. 11 included) as an institution-independent introduction to model theory. Working logicians and model theorists will find in this monograph a novel view and a new methodological approach to model theory. Computer scientists may use the material of the first part as an introduction to institution theory, and material from the third and the fourth parts for an advanced approach to topics from the semantics of formal specification and logic programming. Also, institution-independent model theory constitutes a powerful tool for workers in formal specification to perform a systematic model theoretic analysis of the logic underlying the particular system they employ.

Each section comes with a number of exercises. While some of them are meant to help the reader accommodate the concepts introduced, others contain quite important results and applications. In fact, in order to keep the book within a reasonable size, much of the knowledge had to be exiled to the exercise sections.

Chapter 2

Categories

Institution-independent model theory as a categorical abstract model theory relies heavily on category theory. This preliminary chapter gives a brief overview of the categorical concepts and results used by this book. The reader without enough familiarity with category theory is advised to use one of the textbooks on category theory available in the literature. [111] and [26] are among standard references for category theory. A reference for indexed categories discussing many examples from the model theory of algebraic specification is [174], while [101] contains a rather compact presentation of fibred category theory.

2.1 Basic Concepts

Categories

A *category* $\mathbb{C}$ consists of

- a class $|\mathbb{C}|$ of *objects*,
- a class of *arrows* (sometimes also called 'morphisms' or 'homomorphisms'), denoted just as $\mathbb{C}$,
- two maps $dom, cod : \ \mathbb{C} \to |\mathbb{C}|$ giving the *domain* and *codomain* of each arrow such that for each pair of objects A and B, $\mathbb{C}(A,B) = \{f \in \mathbb{C} \mid dom(f) = A, cod(f) = B\}$ is a *set*,
- for all objects A,B,C, a *composition* map $\_;\_ : \ \mathbb{C}(A,B) \times \mathbb{C}(B,C) \to \mathbb{C}(A,C)$,
- an *identity* arrow map $1 : \ |\mathbb{C}| \to \mathbb{C}$ such that $1_A \in \mathbb{C}(A,A)$ for each $A \in |\mathbb{C}|$,

such that the (arrow) composition $\_;\_$ is associative and with identity arrows as left and right identities.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f;g"'] & B \arrow[d, "g"] \arrow[dr, "g;h"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
$$

Notice that we prefer to use the diagrammatic notation $f;g$ for composition of arrows in categories, rather than the alternative set theoretic one $g \circ f$ used in many category theory works.

Categories arise everywhere in mathematics. A most typical example is that of sets (as objects) and functions (as arrows) with the usual (functional) composition. We denote this category by $\mathbb{S}et$. Notice that $|\mathbb{S}et|$, the collection of all sets, is *not* a set, it is a proper *class*.

The arrows of a category in general reflect the structure of objects in the sense of preserving that structure. However, obviously this should not always be the case. One can go further by saying that, in reality, a particular category is determined only by its arrows, the objects being a derived rather than a primary concept.

A category $\mathbb{C}$ is *small* when its class of objects $|\mathbb{C}|$ is a set. Note that this implies that $\mathbb{C}$, the class of arrows, is also a set.

$\mathbb{C}$ is *connected* when there exists only one equivalence class for the equivalence generated by the relation on objects given by "there exists an arrow $A \to B$".

Isomorphisms. An arrow $f : A \to B$ is an *isomorphism* when there exists an arrow $g : B \to A$ such that $f;g = 1_A$ and $g;f = 1_B$. The *inverse* g is denoted as f^{-1}. Two objects A and B are *isomorphic*, and we denote this by $A \cong B$, when there exists an isomorphism $f : A \to B$. Isomorphisms in $\mathbb{S}et$ are precisely the bijective (injective and surjective) functions. However this is not true in general; structure preserving mappings that are bijective are not necessarily isomorphisms. A simple counterexample is given by the category of partial orders (objects) with order-preserving functions as arrows.

Monoids are exactly the categories with only one object. Then *groups* are exactly the monoids for which all elements (arrows) are isomorphisms.

Being isomorphic is an equivalence relation on objects; the equivalence classes of $\cong$ are called *isomorphism classes*.

Epis and monos. A family of arrows $\{f_i : A \to B\}_{i \in I}$ is *epimorphic* when for each pair of parallel arrows $g_1, g_2 : B \to C$, $f_i;g_1 = f_i;g_2$ for each $i \in I$ implies $g_1 = g_2$, and it is *monomorphic* when for each pair of parallel arrows $g_1, g_2 : C \to A$, $g_1;f_i = g_2;f_i$ for each $i \in I$ implies $g_1 = g_2$. An arrow $f : A \to B$ is *epi/mono* when it is epimorphic/monomorphic as a (singleton) family, i.e., $\{f\}$ is epimorphic/monomorphic.

In $\mathbb{S}et$ epis are exactly the surjective functions and the monos are exactly the injective ones. Note that while, in general, whenever arrows appear as functions with additional structure, the injectivity (respectively surjectivity) of the underlying function is a sufficient condition for a function to be mono (respectively epi), the converse is not true. For example, the inclusion $\mathbb{Z} \to \mathbb{Q}$ of integers into the rationals is epi in the category of rings but it is not surjective. This is also an example of an arrow which is both epi and mono but is not an isomorphism.

An arrow $f : A \to B$ is a *retract* to $g : B \to A$ when $g;f = 1_B$. Notice that each retract is epi. The converse, which is not true in general, is one of the categorical formulations of the Axiom of Choice. Note that $\mathbb{S}et$ has the Axiom of Choice in this sense.

An object A is *injective* with respect to an arrow h when for each arrow $f: dom(h) \to A$ there exists an arrow g such that $h;g = f$. A is simply *injective* when it is injective with respect to all mono arrows.

$$\begin{tikzcd}
\bullet \arrow[rr, "h"] \arrow[dr, "f"'] & & \bullet \arrow[dl, dotted, "g"] \\
& A &
\end{tikzcd}$$

Dually, an object A is *projective* with respect to an arrow h when for each arrow $f: A \to cod(h)$ there exists an arrow g such that $g;h = f$. A is simply *projective* when it is projective with respect to all epi arrows. Note that in $\mathbb{S}et$ all objects (sets) are both injective and projective.

$$\begin{tikzcd}
\bullet \arrow[rr, "h"] & & \bullet \\
& A \arrow[ul, dotted, "g"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Functors

A *functor* $\mathcal{U}: \mathbb{C} \to \mathbb{C}'$ between categories $\mathbb{C}$ and $\mathbb{C}'$ maps

- objects to objects, $|\mathcal{U}|: |\mathbb{C}| \to |\mathbb{C}'|$, and
- arrows to arrows, $\mathcal{U}_{A,B}: \mathbb{C}(A,B) \to \mathbb{C}'(\mathcal{U}(A), \mathcal{U}(B))$ for all objects $A, B \in |\mathbb{C}|$

such that

– $\mathcal{U}(1_A) = 1_{\mathcal{U}(A)}$ for each object $A \in |\mathbb{C}|$, and

– $\mathcal{U}(f;g) = \mathcal{U}(f);\mathcal{U}(g)$ for all composable arrows $f, g \in \mathbb{C}$.

Most of the time we will denote $|\mathcal{U}|$ and $\mathcal{U}_{A,B}$ simply by $\mathcal{U}$. The application of functors (to either objects or arrows) can also be written in a "diagrammatic" way as $f\mathcal{U}$ rather than the more classical $\mathcal{U}(f)$. Sometimes it is even convenient to use subscripts or superscripts for the application of functors to objects or arrows.

A simple example is the power-set functor $\mathcal{P}: \mathbb{S}et \to \mathbb{S}et$ which maps each set S to the set of its subsets $\{X \mid X \subseteq S\}$ and maps each function $f: S \to S'$ to the function $\mathcal{P}(f): \mathcal{P}(S) \to \mathcal{P}(S')$ such that $\mathcal{P}(f)(X) = f(X) = \{f(x) \mid x \in X\}$.

Another example is given by 'cartesian product with A'. For any fixed set A, let $A \times -: \mathbb{S}et \to \mathbb{S}et$ be the functor mapping each set B to $A \times B = \{(a,b) \mid a \in A, b \in B\}$ and each function $f: B \to C$ to $(A \times f): A \times B \to A \times C$ defined by $(A \times f)(a,b) = (a, f(b))$.

$$\begin{tikzcd}
A \times B \arrow[r, "(a,b)\mapsto b"] \arrow[d, "A\times f"'] & B \arrow[d, "f"] \\
A \times C \arrow[r, "(a,c)\mapsto c"'] & C
\end{tikzcd}$$

A third example is that of 'hom-functors'. For any category $\mathbb{C}$ and any object $A \in |\mathbb{C}|$, the hom-functor $\mathbb{C}(A,-):\ \mathbb{C} \to \mathbb{S}et$ maps any object $B \in |\mathbb{C}|$ to the set of arrows $\mathbb{C}(A,B)$ and each arrow $f:\ B \to B'$ to the function $\mathbb{C}(A,f):\ \mathbb{C}(A,B) \to \mathbb{C}(A,B')$ defined by $\mathbb{C}(A,f)(g) = g;f$.

Each preorder-preserving function between two preorders $(P,\leq) \to (Q,\leq)$ is another example of a functor. In fact, functors between preorders are precisely the monotonic functions.

A functor $\mathcal{U}:\ \mathbb{C} \to \mathbb{C}'$ is *full* when for each objects A and B, the mapping on arrows $\mathcal{U}_{A,B}:\ \mathbb{C}(A,B) \to \mathbb{C}'(\mathcal{U}(A),\mathcal{U}(B))$ is surjective and is *faithful* when $\mathcal{U}_{A,B}$ is injective. Note that both functors of the first and of the second example are faithful but not full.

Functors can be composed in the obvious way and each category has an identity functor with respect to functor composition. By discarding the foundational issues (for the interested reader we recommend [95] or [111]), let $\mathbb{C}at$ be the 'quasi-category' of categories (as objects) and functors (as arrows).

$\mathbb{C} \subseteq \mathbb{C}'$ is a *subcategory* (of $\mathbb{C}'$) when $|\mathbb{C}| \subseteq |\mathbb{C}'|$, $\mathbb{C}(A,B) \subseteq \mathbb{C}'(A,B)$ for all $A,B \in |\mathbb{C}|$, and the composition in $\mathbb{C}$ is a restriction of the composition in $\mathbb{C}'$. A subcategory $\mathbb{C} \subseteq \mathbb{C}'$ is *broad* when $|\mathbb{C}| = |\mathbb{C}'|$.

Natural transformations

Fixing categories $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{C}at(\mathbb{A},\mathbb{B})$ can be regarded as a category with functors as objects and *natural transformations* as arrows. A natural transformation $\tau:\ \mathcal{S} \Rightarrow \mathcal{T}$ between functors $\mathcal{S},\mathcal{T}:\ \mathbb{A} \to \mathbb{B}$ is a map $|\mathbb{A}| \to \mathbb{B}$ such that $\tau(A) \in \mathbb{B}(\mathcal{S}(A),\mathcal{T}(A))$ for each $A \in |\mathbb{A}|$ and the following diagram commutes (in $\mathbb{B}$)

$$\begin{array}{ccc} \mathcal{S}(A) & \xrightarrow{\tau(A)} & \mathcal{T}(A) \\ {\scriptstyle \mathcal{S}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{T}(f)} \\ \mathcal{S}(B) & \xrightarrow[\tau(B)]{} & \mathcal{T}(B) \end{array}$$

for each arrow $f \in \mathbb{A}(A,B)$. The classical notation for the component $\tau(A)$ is τ_A, however the diagrammatic notation $A\tau$ is also frequently used.

A simple example is generated by considering a function $A \xrightarrow{f} A'$ which determines a natural transformation $nt(f):\ (A \times -) \Rightarrow (A' \times -)$ given by $nt(f)_B = f \times 1_B$ for each set B, where $(f \times 1_B)(a,b) = (f(a),b)$ for each $(a,b) \in A \times B$.

An additional example is given by the natural transformation $\mathbb{C}(f,-):\ \mathbb{C}(A,-) \Rightarrow \mathbb{C}(B,-)$ for each arrow $B \xrightarrow{f} A$ in a category $\mathbb{C}$. For each $D \in |\mathbb{C}|$, $\mathbb{C}(f,-)_D = \mathbb{C}(f,D):\ \mathbb{C}(A,D) \to \mathbb{C}(B,D)$ where $\mathbb{C}(f,D)(g) = f;g$.

The composition of natural transformations is defined component-wise, i.e., $A(\sigma;\tau) = A\sigma;A\tau$ where $\sigma:\ \mathcal{R} \Rightarrow \mathcal{S}:\ \mathbb{A} \to \mathbb{B}$ and $\tau:\ \mathcal{S} \Rightarrow \mathcal{T}:\ \mathbb{A} \to \mathbb{B}$. This is called the *'vertical' composition* of natural transformations.

Given the natural transformations $\tau:\ \mathcal{S} \Rightarrow \mathcal{T}:\ \mathbb{A} \to \mathbb{B}$ and $\tau':\ \mathcal{S}' \Rightarrow \mathcal{T}':\ \mathbb{B} \to \mathbb{C}$

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, bend left, "\mathcal{S}"] \arrow[r, bend right, "\mathcal{T}"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbb{B} \arrow[r, bend left, "\mathcal{S}'"] \arrow[r, bend right, "\mathcal{T}'"'] \arrow[r, phantom, "\Downarrow\tau'"] & \mathbb{C}
\end{tikzcd}
$$

we may define their 'horizontal' composition $\tau\tau' : \mathcal{S};\mathcal{S}' \Rightarrow \mathcal{T};\mathcal{T}'$ by

$$A(\tau\tau') = (A\mathcal{S})\tau';(A\tau)\mathcal{T}' = (A\tau)\mathcal{S}';(A\mathcal{T})\tau'.$$

When τ, respectively τ', is an identity natural transformation we may replace it in notation by $\mathcal{S}$, respectively $\mathcal{S}'$.

Basic categorical constructions

The *opposite* $\mathbb{C}^{\mathrm{op}}$ of a category $\mathbb{C}$ is just reversing the arrows and the arrow composition. This means $|\mathbb{C}^{\mathrm{op}}| = |\mathbb{C}|$, $\mathbb{C}^{\mathrm{op}}(A,B) = \mathbb{C}(B,A)$. Identities in $|\mathbb{C}^{\mathrm{op}}|$ are the same as in $\mathbb{C}$.

Given a functor $\mathcal{U} : \mathbb{C}' \to \mathbb{C}$, for any object $A \in |\mathbb{C}|$, the *comma category* $A/\mathcal{U}$ has arrows $f : A \to \mathcal{U}(B)$ as objects (sometimes denoted as (f,B)) and $h \in \mathbb{C}'(B,B')$ with $f;\mathcal{U}(h) = f'$ as arrows $(f,B) \to (f',B')$.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f'"'] & \mathcal{U}(B) \arrow[d, "\mathcal{U}(h)"] \\
& \mathcal{U}(B')
\end{tikzcd}
$$

When $\mathbb{C} = \mathbb{C}'$ and $\mathcal{U}$ is the identity functor, the category $A/\mathcal{U}$ is denoted by $A/\mathbb{C}$. $\mathbb{C}/A$ is just $(A/\mathbb{C}^{\mathrm{op}})^{\mathrm{op}}$.

Given a class $\mathcal{D} \subseteq \mathbb{C}$ of arrows of a category $\mathbb{C}$ we say that $\mathbb{C}$ is *$\mathcal{D}$-well-powered* when for each object $A \in |\mathbb{C}|$ the isomorphism classes of $\{(B,f) \in |\mathbb{C}/A| \mid f \in \mathcal{D}\}$ form a set (rather than a proper class). Dually, $\mathbb{C}$ is *$\mathcal{D}$-co-well-powered* when for each $A \in |\mathbb{C}|$ the isomorphism classes of $\{(f,B) \in |A/\mathbb{C}| \mid f \in \mathcal{D}\}$ form a set.

2.2 Limits and Co-limits

An object 0 is *initial* in a category $\mathbb{C}$ when for each object $A \in |\mathbb{C}|$ there exists a unique arrow in $\mathbb{C}(0,A)$. Dually, an object 1 is *final* in $\mathbb{C}$ when it is initial in $\mathbb{C}^{\mathrm{op}}$, which means that for each object $A \in |\mathbb{C}|$ there exists a unique arrow in $\mathbb{C}(A,1)$.

In $\mathbb{S}et$, the empty set $\emptyset$ is initial and each singleton set $\{*\}$ is final. In $\mathbb{G}rp$, the category of groups, the trivial groups (with only one element) are both initial and final.

Given a functor $\mathcal{U} : \mathbb{A} \to \mathbb{X}$, for each $X \in |\mathbb{X}|$, a *universal arrow from X to $\mathcal{U}$* is just an initial object in the comma category $X/\mathcal{U}$. Notice that universal arrows are unique up to isomorphism.

For any categories J and $\mathbb{C}$, the *diagonal* functor $\Delta : \mathbb{C} \to \mathbb{C}at(J,\mathbb{C})$ maps any $A \in |\mathbb{C}|$ to the functor $A\Delta : J \to \mathbb{C}$ such that $(A\Delta)(j) = A$ for each object $j \in |J|$ and

$(A\Delta)(u) = 1_A$ for each arrow $u \in J$, and maps any $f \in \mathbb{C}(A,B)$ to the natural transformation $f\Delta : A\Delta \Rightarrow B\Delta$ with $(f\Delta)_j = f$ for each $j \in |J|$.

Co-limits. For any functor $D : J \to \mathbb{C}$, a *co-cone* to D is just an object of the comma category D/Δ, while a *co-limit* of D is a universal arrow from D to the diagonal functor Δ. As universal arrows, co-limits of functors are unique up to isomorphism. A co-limit $\mu : D \Rightarrow A\Delta$ of D may be therefore denoted as $\mu : D \Rightarrow A$ (by omitting the diagonal functor from the notation). More explicitly, a co-limit of D consists of a family of arrows $\{\mu_i\}_{i\in|J|}$ such that $\mu_i = D(u);\mu_j$ for each $u \in J(i,j)$ which behaves like a lowest upper bound for D, i.e., for any family $\{\nu_i\}_{i\in|J|}$ such that $\nu_i = D(u);\nu_j$ for each $u \in J(i,j)$, there exists a unique arrow f such that $\mu_i;f = \nu_i$ for each $i \in |J|$.

$$\begin{tikzcd}
D(i) \arrow[rr, "D(u)"] \arrow[dr, "\mu_i"] \arrow[ddr, bend right, "\nu_i"'] & & D(j) \arrow[dl, "\mu_j"'] \arrow[ddl, bend left, "\nu_j"] \\
& A \arrow[d, "f"'] & \\
& B &
\end{tikzcd}$$

We may denote the vertex A by $Colim(D)$.

Limits. Limits are dual to co-limits. For any functor $D : J \to \mathbb{C}$, a *limit* $\mu : A \Rightarrow D$ of D is the 'greatest lower bound' of the *cones* over D, i.e. $\mu = \{\mu_i\}_{i\in|J|}$ such that $\mu_i;D(u) = \mu_j$ for each $u \in J(i,j)$ and for any family $\{\nu_i\}_{i\in|J|}$ with the same property, there exists a unique arrow f such that $f;\mu_i = \nu_i$ for each $i \in |J|$.

$$\begin{tikzcd}
D(i) \arrow[rr, "D(u)"] & & D(j) \\
& A \arrow[ul, "\mu_i"'] \arrow[ur, "\mu_j"] & \\
& B \arrow[u, "f"'] \arrow[uul, bend left, "\nu_i"] \arrow[uur, bend right, "\nu_j"'] &
\end{tikzcd}$$

We may denote the vertex A by $Lim(D)$.

Diagrams as functors. The functors $D : J \to \mathbb{C}$ for which we have considered limits and co-limits are often called *categorical diagrams* (in $\mathbb{C}$), or just *diagrams* for short. Such a diagram D may be denoted $\left(D(i)\xrightarrow{D(u)}D(j)\right)_{(i\xrightarrow{u}j)\in J}$. Note that the meaning of the functoriality of D, that $D(u;u') = D(u);D(u')$, is the commutativity of D regarded as a diagram in $\mathbb{C}$.

Products and co-products. When J is discrete (has no arrows except the identities), J-limits are called *products* and J-co-limits are called *co-products*; when J is a finite set then the corresponding products or co-products are referred to as finite. The product of two objects A and B is denoted by $A \times B$ and their co-product by $A+B$. Notice that when $J = \emptyset$, then the products are the final objects and the co-products are initial objects. The product of a family $\{A_i\}_{i \in I}$ of objects is denoted by $\prod_{i \in I} A_i$.

In $\mathbb{S}et$ the categorical products are just cartesian products, while co-products $A+B$ are disjoint unions $A \uplus B$ which can be defined as $\{(a,1) \mid a \in A\} \cup \{(b,2) \mid b \in B\}$.

Pullbacks. When J is the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ with three objects and two non-identity arrows, J-limits are called *pullbacks*.

In $\mathbb{S}et$, the pullback square

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "f"] \\ B \arrow[r, "g"'] & A \end{tikzcd}$$

of $C \xrightarrow{f} A \xleftarrow{g} B$ can be defined by $D = \{(b,c) \in B \times C \mid g(b) = f(c)\}$, $k(b,c) = b$, and $h(b,c) = c$.

For any arrow f, the pullback of a span $\bullet \xrightarrow{f} \bullet \xleftarrow{f} \bullet$ is called the *kernel of* f. The kernel of any function $f: A \to B$ is $\{(a,a') \in A \times A \mid f(a) = f(a')\}$.

Pushouts. When J is the category $\bullet \longleftarrow \bullet \longrightarrow \bullet$ with three objects and two non-identity arrows, J-co-limits are called *pushouts*.

In $\mathbb{S}et$, the pushout of any *span* of functions $B \xleftarrow{f} A \xrightarrow{g} C$ always exists and is given by the quotient of the disjoint union $B \uplus C$ which identifies all the elements $f(a)$ and $g(a)$ for each $a \in A$.

Equalizers and co-equalizers. When J is the category with two objects and a pair of parallel arrows between these objects, then J-limits are called *equalizers* and J-co-limits are called *co-equalizers*.

$$\begin{tikzcd} \bullet \arrow[r, "eq"] & \bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \bullet \arrow[r, "coeq"] \arrow[dr, "k"'] & \bullet \arrow[d, dotted, "k'"] \\ \bullet \arrow[u, dotted, "h'"] \arrow[ur, "h"'] & & & \bullet \end{tikzcd}$$

In $\mathbb{S}et$, the equalizer of any pair of parallel arrows $f, g: A \to B$ is just the subset inclusion $\{a \mid f(a) = g(a)\} \subseteq A$. The co-equalizer k is the quotient of B by the equivalence generated by $\{(f(a), g(a)) \mid a \in A\}$.

Directed co-limits. When J is a directed partially ordered set (i.e., for each $i, i' \in |J|$ there exists $j \in |J|$ such that $i \le j$ and $i' \le j$), then J-co-limits are called *directed* co-limits. For the special case when J is a total order, the J-co-limits are called *inductive* co-limits.

In $\mathbb{S}et$, directed co-limits can be thought of as a generalized kind of union. For any directed diagram of sets $(A_i \xrightarrow{f_{i,j}} A_j)_{(i\le j)\in(J,\le)}$ its co-limit is given by the quotient of the disjoint union $\uplus\{A_i \mid i \in |J|\}$ which identifies the elements a_i and $f_{i,j}(a_i)$.

A category that has all J-(co-)limits is called *J-(co-)complete*. Also, by *small* (co-)-limits we mean all J-(co-)limits for all J that are small categories.

Theorem 2.1. *In any category the following conditions are equivalent:*

1. *the category has finite (co-)limits,*
2. *the category has finite (co-)products and (co-)equalizers, and*
3. *the category has a final (initial) object and pullbacks (pushouts).*

Lifting, creation, preservation, reflection of (co-)limits

Limits and co-limits, respectively, in base categories determine 'pointwise' limits and co-limits, respectively, in corresponding functor categories.

Proposition 2.2. *If the category $\mathbb{B}$ has J-(co-)limits, then for any category $\mathbb{A}$, the category $\mathbb{C}at(\mathbb{A},\mathbb{B})$ of functors $\mathbb{A} \to \mathbb{B}$ has small J-(co-)limits (which can be calculated separately in $\mathbb{B}$ for each object $A \in |\mathbb{A}|$).*

A functor $\mathcal{U}\colon\ \mathbb{C} \to \mathbb{C}'$ *preserves* a (co-)limit of a functor $D\colon\ J \to \mathbb{C}$ when $\mu\mathcal{U}$ is a (co-)limit of $D;\mathcal{U}$. Note that in $\mathbb{S}et$ the 'product with A', $A \times -$, preserves all co-limits.

The functor $\mathcal{U}$ *lifts (uniquely)* a (co-)limit μ' of $D;\mathcal{U}$ for any functor $D\colon\ J \to \mathbb{C}$, if there exists a (unique) (co-)limit μ of D such that $\mu\mathcal{U} = \mu'$. Notice that if $\mathcal{U}$ lifts J-(co-)limits and $\mathbb{C}'$ has J-(co-)limits, then $\mathbb{C}$ has J-(co-)limits which are preserved by $\mathcal{U}$.

Stronger than lifting is the following notion. The functor $\mathcal{U}$ *creates* a (co-)limit μ' of $D;\mathcal{U}$, when there exists a unique (co-)cone μ of D such that that $\mu\mathcal{U} = \mu'$ and such that μ is a (co-)limit. For example the forgetful functor $\mathbb{G}rp \to \mathbb{S}et$ creates all limits.

Proposition 2.3. *If the functor $\mathcal{U}\colon\ \mathbb{C}' \to \mathbb{C}$ preserves J-(co-)limits, then for each object $A \in |\mathbb{C}|$, the forgetful functor $A/\mathcal{U} \to \mathbb{C}'$ creates J-(co-)limits.*

The functor $\mathcal{U}$ *reflects* (co-)limits of a functor $D\colon\ J \to \mathbb{C}$ if μ is a (co-)limit of D whenever $\mu\mathcal{U}$ is a (co-)limit of $D;\mathcal{U}$.

Co-limits of final functors

A functor $L\colon\ J' \to J$ is called *final* if for each object $j \in |J|$ the comma category j/L is non-empty and connected. Consequently, a subcategory $J' \subseteq J$ is final when the corresponding inclusion functor is final.

For example, for each natural number n, $(n \to n+1 \to n+2 \to \dots)$ is a final subcategory of $\omega = (0 \to 1 \to 2 \to \dots)$. More generally, for each directed poset $(P, \leq)$ and each $p \in P$, $\{p' \in P \mid p \leq p'\}$ is final in $(P, \leq)$.

Theorem 2.4. *For each final functor $L\colon\ J' \to J$ and each functor $D\colon\ J \to \mathbb{C}$ when a co-limit $\mu'\colon\ L;D \Rightarrow Colim(L;D)$ exists, there exists a co-limit $\mu\colon\ D \Rightarrow Colim(D)$ and the canonical arrow $h\colon\ Colim(L;D) \to Colim(D)$ (given by the universal property of the co-limit of $L;D$) is an isomorphism.*

Finitely presented objects

An object A in a category $\mathbb{C}$ is *finitely presented* if and only if the hom-functor $\mathbb{C}(A,-)$: $\mathbb{C} \to \mathbb{S}et$ preserves directed co-limits. This is equivalent to the following condition:

- for any arrow $f\colon\ A \to C$ to the vertex of a co-limiting co-cone $\mu\colon\ D \Rightarrow C$ of a directed diagram $D\colon\ (J, \leq) \to \mathbb{C}$, there exists $i \in J$ and an arrow $f_i\colon\ A \to D(i)$ such that $f = f_i;\mu_i$, and
- for any two arrows f_i and f_j as above, there exists $k > i, j$ such that $f_i;D_{i,k} = f_j;D_{j,k}$.

$$
\begin{tikzcd}
 & & & D(j) \arrow[dl, "D_{j,k}"'] \arrow[dddl, bend left, "\mu_j"] \\
 & & D(k) \arrow[dd, "\mu_k"] & \\
A \arrow[r, "f_i"] \arrow[uurrr, bend left, "f_j"] \arrow[urr, bend left, "f_k"] \arrow[drr, bend right, "f"'] & D(i) \arrow[ur, "D_{i,k}"] \arrow[dr, "\mu_i"] & & \\
 & & C &
\end{tikzcd}
$$

In $\mathbb{S}et$ the finitely presented objects are precisely the finite sets. In the category of groups $\mathbb{G}rp$, the finitely presented groups are exactly the quotients of finitely generated groups by finitely generated congruences.

A category is *locally presentable* when each object is a directed co-limit of finitely presented objects. $\mathbb{S}et$ is locally presentable because each set is the (directed) co-limit of its finite subsets.

Stability under pushouts/pullbacks

A class of arrows $\mathcal{S} \subseteq \mathbb{C}$ in a category $\mathbb{C}$ is *stable under pushouts* if for any pushout square in $\mathbb{C}$

$$
\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, "u'"'] & \bullet
\end{tikzcd}
$$

$u' \in \mathcal{S}$ whenever $u \in \mathcal{S}$. *Stability under pullbacks* in $\mathbb{C}$ is stability under pushouts in $\mathbb{C}^{\mathrm{op}}$.

In general, epis are stable under pushouts and monos under pullbacks. In $\mathbb{S}et$, monos (injective functions) are stable under pushouts too. Injective functions $f: A \to B$ such that $B \setminus f(A)$ is finite are also stable under pushouts.

Weak limits and co-limits

These are weaker variants of the concepts of limits and co-limits, respectively, obtained by dropping the uniqueness requirement from the universal property of the limits and co-limits, respectively. For example, in $\mathbb{S}et$ for any two sets A and B, any super-set C of their disjoint union, i.e., $A \uplus B \subseteq C$, is a weak co-product for A and B. Obviously, weak limits and co-limits, respectively, are no longer unique up to isomorphism.

2.3 Adjunctions

Adjoint functors are a core concept of category theory. Mathematical practice abounds with examples of adjoint functors.

Proposition 2.5. *For any functor $\mathcal{U}: \mathbb{A} \to \mathbb{X}$ the following conditions are equivalent:*

1. *For each object $X \in \mathbb{X}$ there exists a universal arrow from X to $\mathcal{U}$.*
2. *There exists a functor $\mathcal{F}: \mathbb{X} \to \mathbb{A}$ and a bijection $\varphi_{X,A}: \mathbb{A}(\mathcal{F}(X),A) \to \mathbb{X}(X, \mathcal{U}(A))$ indexed by $|\mathbb{X}| \times |\mathbb{A}|$ and natural in X and A.*
3. *There exists a functor $\mathcal{F}: \mathbb{X} \to \mathbb{A}$ and natural transformations $\eta: 1_{\mathbb{X}} \Rightarrow \mathcal{F};\mathcal{U}$ (called the* unit*) and $\varepsilon: \mathcal{U};\mathcal{F} \Rightarrow 1_{\mathbb{A}}$ (called the* co-unit*) such that the following* triangular equations *hold: $\eta\mathcal{F};\mathcal{F}\varepsilon = 1_{\mathcal{F}}$ and $\mathcal{U}\eta;\varepsilon\mathcal{U} = 1_{\mathcal{U}}$.*

If the conditions above hold, then $\mathcal{U}$ is called a *right adjoint*, and the functor $\mathcal{F}$ is called a *left adjoint* to $\mathcal{U}$. The tuple $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$ is called an *adjunction* from (the category) $\mathbb{X}$ to (the category) $\mathbb{A}$.

Very often the notion of adjunction is used in the following "freeness" form. Given an adjunction $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$, for any object $X \in |\mathbb{X}|$ there exists an object $\mathcal{F}(X)$, called $\mathcal{U}$-*free over* A and an arrow $\eta_X: X \to \mathcal{U}(\mathcal{F}(X))$ such that for each object $A \in |\mathbb{A}|$ and arrow $h: X \to \mathcal{U}(A)$, there exists a unique arrow $h': \mathcal{F}(X) \to A$ such that $h = \eta_X;\mathcal{U}(h')$.

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "h"'] & & \mathcal{U}(\mathcal{F}(X)) \arrow[dl, "\mathcal{U}(h')"] \\
& \mathcal{U}(A) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(X) \arrow[d, "h'"] \\
A
\end{tikzcd}$$

When a category $\mathbb{C}$ has J-(co-)limits, then these are adjoints to the diagonal functor $\Delta: \mathbb{C} \to \mathbb{C}at(J,\mathbb{C})$. More precisely, *Lim* is a right adjoint to Δ, while *Colim* is a left adjoint to Δ.

The forgetful functor $\mathbb{G}rp \to \mathbb{S}et$ is right adjoint, its left adjoint constructing the groups freely generated by sets.

Galois connections. Let $(P,\leq)$ and $(Q,\leq)$ be preorders. Two preorder preserving functions $L\colon\ (P,\leq)\to(Q,\leq)^{\mathrm{op}}$ and $R\colon\ (Q,\leq)^{\mathrm{op}}\to(P,\leq)$ constitute an adjunction when $L(p)\geq q$ if and only if $p\leq R(q)$ for all $p\in P$ and $q\in Q$. Notice that triangular equations mean $L(p)\geq L(R(L(p)))\geq L(p)$ and $R(q)\leq R(L(R(q)))\leq R(q)$. The pair (L,R) is called a *Galois connection* between $(P,\leq)$ and $(Q,\leq)$.

Persistent adjunctions. Given an adjunction $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$, the object $\mathcal{F}(X)$ is called *persistently* $\mathcal{U}$-free when the unit component η_X is an isomorphism, and is called *strongly persistently* $\mathcal{U}$-free when η_X is identity. We can easily see that an object of $\mathbb{A}$ is persistently free if and only if it is strongly persistently free. An adjunction such that for each object X of $\mathbb{X}$, $\mathcal{F}(X)$ is [strongly] persistently $\mathcal{U}$-free, is called a *[strongly] persistent adjunction* .

Composition of adjunctions. Given two adjunctions $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$ from $\mathbb{X}$ to $\mathbb{A}$, and $(\mathcal{U}',\mathcal{F}',\eta',\varepsilon')$ from $\mathbb{A}$ to $\mathbb{A}'$, note that $(\mathcal{U}';\mathcal{U},\mathcal{F};\mathcal{F}',\eta;\mathcal{F}\eta'\mathcal{U},\mathcal{U}'\varepsilon\mathcal{F}';\varepsilon')$ is an adjunction from $\mathbb{X}$ to $\mathbb{A}'$. This is called the *composition* of the two adjunctions. Adjunctions thus form a 'quasi-category' $\mathbb{A}dj$ with categories as objects and adjunctions as arrows.

The following is one of the most useful properties of adjoint functors.

Proposition 2.6. *Right adjoints preserve all limits and, dually, left adjoints preserve all co-limits.*

Special adjunctions

Categorical equivalences. The following equivalent conditions define a functor $\mathcal{U}\colon\ \mathbb{X}\to\mathbb{X}'$ as an *equivalence of categories*:

Proposition 2.7. *For any functor $\mathcal{U}\colon\ \mathbb{X}\to\mathbb{X}'$ the following conditions are equivalent:*

- *$\mathcal{U}$ belongs to an adjunction with unit and co-unit being natural isomorphisms, and*
- *$\mathcal{U}$ is full and faithful and each object $A'\in|\mathbb{X}'|$ is isomorphic to $\mathcal{U}(A)$ for some object $A\in|\mathbb{X}|$.*

We say that $\mathbb{X}$ is a *skeleton* of $\mathbb{X}'$ when all isomorphisms in $\mathbb{X}$ are identities.

Cartesian closed categories. A category $\mathbb{C}$ is *cartesian closed* when it has all finite products, denoted $-\times-$, and for each object A the product functor $-\times A\colon\ \mathbb{C}\to\mathbb{C}$ has a right adjoint $[A,-]$. If we denote the co-unit of this adjunction by ev^A, it means that for each pair of objects A and B, and for each arrow $f:\ C\times A\to B$, there exists a unique arrow $f':\ C\to[A,B]$ such that $f=(f'\times 1_A);ev^A_B$,

$$\begin{tikzcd}
{[A,B]\times A} \arrow[rr, "ev^A_B"] & & B \\
& C\times A \arrow[ul, "f'\times 1_A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

In examples the co-unit components ev_B^A play the role of 'evaluation maps'. We have that $\mathbb{S}et$ is cartesian closed where $[A,B]$ is the set of all functions $A \to B$, and $ev_B^A(f,a) = f(a)$. $\mathbb{C}at$ is also cartesian closed with $[A,B]$ being the category $\mathbb{C}at(A,B)$ of the functors $A \to B$ and with the natural transformations between them as arrows.

2.4 2-categories

A *2-category* $\mathbb{C}$ is an ordinary category whose objects are called *0-cells*, whose arrows are called *1-cells*, and in addition to ordinary objects and arrows, for each pair of 1-cells $\mathcal{S}, \mathcal{T}$ there is a set $\mathbb{C}(\mathcal{S}, \mathcal{T})$ of *2-cells* (denoted by $\mathcal{S} \Rightarrow \mathcal{T}$) together with two compositions for the 2-cells:

- a 'vertical' one $\sigma;\tau : \ \mathcal{S} \Rightarrow \mathcal{T}$

$$\begin{tikzcd}
A \arrow[r, bend left=60, "\mathcal{S}"{name=S}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, "\mathcal{T}"'{name=T}] & B
\arrow[from=S, to=M, phantom, "\Downarrow\sigma"] \arrow[from=M, to=T, phantom, "\Downarrow\tau"]
\end{tikzcd}$$

  and

- a 'horizontal' one (denoted by simple juxtaposition) $\tau\tau' : \ \mathcal{S};\mathcal{S}' \Rightarrow \mathcal{T};\mathcal{T}'$

$$\begin{tikzcd}
A \arrow[r, bend left, "\mathcal{S}"{name=S}] \arrow[r, bend right, "\mathcal{T}"'{name=T}] & B \arrow[r, bend left, "\mathcal{S}'"{name=S2}] \arrow[r, bend right, "\mathcal{T}'"'{name=T2}] & C
\arrow[from=S, to=T, phantom, "\Downarrow\tau"] \arrow[from=S2, to=T2, phantom, "\Downarrow\tau'"]
\end{tikzcd}$$

such that every identity arrow for the first composite is also an identity for the second composition, $1_{\mathcal{S};\mathcal{T}} = 1_{\mathcal{S}}1_{\mathcal{T}}$ for all composable 1-cells $\mathcal{S}$ and $\mathcal{T}$, and such that the following *Interchange Law* holds: given three categories and four natural transformations

$$\begin{tikzcd}
A \arrow[r, bend left=60, ""{name=S}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""'{name=T}] & B \arrow[r, bend left=60, ""{name=S2}] \arrow[r, ""{name=M2}] \arrow[r, bend right=60, ""'{name=T2}] & C
\arrow[from=S, to=M, phantom, "\Downarrow\sigma"] \arrow[from=M, to=T, phantom, "\Downarrow\tau"]
\arrow[from=S2, to=M2, phantom, "\Downarrow\sigma'"] \arrow[from=M2, to=T2, phantom, "\Downarrow\tau'"]
\end{tikzcd}$$

the 'vertical' compositions and the 'horizontal' compositions are related by

$$(\sigma;\tau)(\sigma';\tau') = (\sigma\sigma');(\tau\tau').$$

Evidently any category is trivially a 2-category without proper 2-cells. The typical non-trivial example of a 2-category is $\mathbb{C}at$ with categories as 0-cells, functors as 1-cells, and natural transformations as 2-cells.

Adjunctions, natural transformations, (co-)limits

The concept of adjunction can be defined abstractly in any 2-category: $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$ is an *adjunction* if $\mathcal{U} : A \to X$ and $\mathcal{F} : X \to A$ are 1-cells, $\eta : 1_X \Rightarrow \mathcal{F}; \mathcal{U}$ and $\varepsilon : \mathcal{U}; \mathcal{F} \Rightarrow 1_{\mathbb{A}}$ are 2-cells such that the *triangular equations* are satisfied:

$$\eta\mathcal{F}; \mathcal{F}\varepsilon = 1_{\mathcal{F}} \quad \text{and} \quad \mathcal{U}\eta; \varepsilon\mathcal{U} = 1_{\mathcal{U}}.$$

The proper mappings between 2-categories are 2-functors. A *2-functor* $F : \mathbb{C} \to \mathbb{C}'$ between 2-categories $\mathbb{C}$ and $\mathbb{C}'$ maps 0-cells to 0-cells, 1-cells to 1-cells, and 2-cells to 2-cells, such that $F(\mathcal{S}) : F(A) \to F(B)$ for any 1-cell $\mathcal{S} : A \to B$, and $F(\sigma) : F(\mathcal{S}) \Rightarrow F(\mathcal{T})$ for any 2-cell $\sigma : \mathcal{S} \Rightarrow \mathcal{T}$, and such that it preserves both the 'vertical' and the 'horizontal' compositions.

A *2-natural transformation* $\tau : F \Rightarrow G$ between 2-functors $F, G : \mathbb{A} \to \mathbb{B}$ maps any object A of $|\mathbb{A}|$ to a 1-cell $A\tau : F(A) \to G(A)$ such that $(A\tau)G(\sigma) = F(\sigma)(B\tau)$ for each 2-cell $\sigma : f \Rightarrow f' : A \to B$.

$$\begin{tikzcd}
F(A) \arrow[rr, "A\tau"] \arrow[d, bend right, "F(f)"'] \arrow[d, bend left, "F(f')"] \arrow[d, phantom, "\overset{F(\sigma)}{\Rightarrow}"] & & G(A) \arrow[d, bend right, "G(f)"'] \arrow[d, bend left, "G(f')"] \arrow[d, phantom, "\overset{G(\sigma)}{\Rightarrow}"] \\
F(B) \arrow[rr, "B\tau"'] & & G(B)
\end{tikzcd}$$

Lax natural transformations relax the commutativity of the natural transformation square above to the existence of 2-cells. Therefore a lax natural transformation τ between 2-functors F and G maps any object $A \in |\mathbb{A}|$ to $A\tau : F(A) \to G(A)$ and any 1-cell $u : A \to B$ to $u\tau : A\tau; G(u) \Rightarrow F(u); B\tau$ such that $(F(\sigma)(B\tau)); f'\tau = f\tau; ((A\tau)G(\sigma))$ for each 2-cell $\sigma : f \Rightarrow f' : A \to B$ and

$$\begin{tikzcd}
F(A) \arrow[r, "F(u)"] \arrow[d, "A\tau"'] \arrow[dr] & F(B) \arrow[r, "F(v)"] \arrow[d, "B\tau"] \arrow[dr] \arrow[dl, phantom, "u\tau \Uparrow"] & F(C) \arrow[d, "C\tau"] \arrow[dl, phantom, "v\tau \Uparrow"] \\
G(A) \arrow[r, "G(u)"'] & G(B) \arrow[r, "G(v)"'] & G(C)
\end{tikzcd}$$

$(u; v)\tau = (u\tau)(G(v)); F(u)(v\tau)$ for each $u : A \to B$ and $v : B \to C$.

2-categorical limits and co-limits can be defined similarly to the conventional limits and co-limits as universal arrows from/to a diagonal functor. However, in the 2-categorical framework, different concepts of natural transformations determine different concepts of (co-)limits. Therefore, when we employ 2-natural transformations we get the concepts of *2-(co-)limit* as a final (initial) *2-(co-)cone*, and when we employ lax natural transformations we get the concepts of *lax (co-)limit* as a final/initial *lax cone/co-cone*.

2.5 Indexed Categories and Fibrations

An *indexed category* is a functor $B: \ I^{\mathrm{op}} \to \mathbb{C}at$; sometimes we denote $B(i)$ as B_i (or B^i) for an index $i \in |I|$ and $B(u)$ as B^u for an index morphism $u \in I$. Given an indexed category $B: \ I^{\mathrm{op}} \to \mathbb{C}at$, let $B^\sharp$ be the *Grothendieck category* having $\langle i, \Sigma \rangle$, with $i \in |I|$ and $\Sigma \in |B^i|$, as objects and $\langle u, \varphi \rangle: \ \langle i, \Sigma \rangle \to \langle i', \Sigma' \rangle$, with $u \in I(i, i')$ and $\varphi: \ \Sigma \to B^u(\Sigma')$, as arrows. The composition of arrows in $B^\sharp$ is defined by $\langle u, \varphi \rangle ; \langle u', \varphi' \rangle = \langle u; u', \varphi; (B^u(\varphi')) \rangle$.

Proposition 2.8. *The Grothendieck category $B^\sharp$ of an indexed category $B: \ I^{\mathrm{op}} \to \mathbb{C}at$ is the vertex of the lax co-limit $\mu: \ B \rightsquigarrow B^\sharp$ of B in $\mathbb{C}at$, where*

- *for each index $i \in |I|$, $\mu^i: \ B^i \to B^\sharp$ is the canonical inclusion of categories, and*
- *for each index morphism $u \in I(i, j)$, $\mu^u: \ B^u ; \mu^i \Rightarrow \mu^j$ is defined by $\mu^u_b = \langle u, 1_{B^u(b)} \rangle$ for each object $b \in |B^j|$.*

$$\begin{tikzcd} B^i \arrow[dr, "\mu^i"'] & & B^j \arrow[ll, "B^u"'] \arrow[dl, bend right] \arrow[dl, bend left, "\mu^j"] \arrow[dl, phantom, "\Swarrow \mu^u"] \\ & B^\sharp & \end{tikzcd}$$

Grothendieck constructions in 2-categories. Prop. 2.8 allows us to internalize the concept of Grothendieck construction to any 2-category. Given a (1-)functor $B: \ I^{\mathrm{op}} \to V$, where V is an arbitrary 2-category, a *Grothendieck construction* for B is a lax co-limit $\mu: \ B \rightsquigarrow B^\sharp$. Then the vertex $B^\sharp$ is called the *Grothendieck object* associated to B. We say that a 2-category V *admits Grothendieck constructions* when each (1-)functor $B: \ I^{\mathrm{op}} \to V$ has a lax co-limit.

Notice also that any 2-functor $B: \ I^* \to \mathbb{C}at$, where I^* is the *2-dimensional opposite* changing the direction of 2-cells both horizontally and vertically, induces a canonical 2-category structure on the Grothendieck category $B^\sharp$ of the (1)-functor $B: \ I^{\mathrm{op}} \to \mathbb{C}at$.

Fibrations

Given a functor $p: \ \mathbb{B} \to I$, an object/arrow $f \in \mathbb{B}$ is said to be *above* an object/arrow $u \in I$ when $p(f) = u$. An arrow above an identity is called *vertical*. Every object $i \in |I|$ determines a *fibre* category $\mathbb{B}_i$ consisting of objects above i and vertical morphisms above 1_i. An arrow $f \in \mathbb{B}(A,C)$ is called *cartesian* over an arrow $u \in I$ when f is above u and every $f' \in \mathbb{B}(A',C)$ with $p(f') = v; u$ uniquely determines a $g \in \mathbb{B}(A',A)$ above v such that $f' = g; f$. p is called a *fibred category* or *fibration* when for every $A \in |\mathbb{B}|$ and $u \in I(i, p(A))$ there is a cartesian arrow (called *cartesian lifting* or *critical lifts* in [1]) with codomain A above u.

Each indexed category $B: \ I^{\mathrm{op}} \to \mathbb{C}at$ naturally determines a fibration $p: \ B^\sharp \to I$ as the index projection, i.e., $p(\langle i, \Sigma \rangle) = i$, such that for each index i, the fibre $B^\sharp_i$ is B^i and $\langle u, \varphi \rangle \in B^\sharp$ is cartesian over u when φ is isomorphism. Notice that for each index morphism $u\colon\ i \to i'$ and $\langle i', \Sigma'\rangle \in B^{\sharp}$, $\langle u, 1_{B^u(\Sigma')}\rangle\colon\ \langle i, B^u(\Sigma')\rangle \to \langle i', \Sigma'\rangle$ is a cartesian lifting of u with codomain $\langle i', \Sigma'\rangle$.

Conversely, if $p\colon\ \mathbb{B} \to I$ is a fibration, for each $u \in I(i,i')$ and $A \in \mathbb{B}_{i'}$, we chose a cartesian lifting $\overline{u}\colon\ u^*(A) \to A$ (called the *distinguished cartesian morphism* corresponding to u and A). Such choice determines a functor $u^*\colon\ \mathbb{B}_{i'} \to \mathbb{B}_i$ called a *inverse image* functor. Notice that two inverse image functors corresponding to the same u are naturally isomorphic, $(u;v)^* \cong v^*;u^*$ for each $u,v \in I$, and $(1_i)^* \cong 1_{\mathbb{B}_i}$ for each $i \in |I|$. When these natural isomorphisms are identities we say that the fibration is *split*.

Proposition 2.9. *The fibred category given by the forgetful functor from a Grothendieck category to its category of indices is split. Conversely, each split fibration is a Grothendieck category and each fibration is equivalent to a Grothendieck category.*

Cartesian functors are "morphisms of fibrations". Given fibrations $p\colon\ \mathbb{B} \to I$ and $p'\colon\ \mathbb{B}' \to I$, a cartesian functor $U\colon\ \mathbb{B} \to \mathbb{B}'$ commutes with the fibrations, i.e., $U;p' = p$, and preserves the cartesian arrows, i.e., maps any cartesian arrow for p to a cartesian arrow for p'.

Limits/co-limits in Grothendieck/fibred categories can be obtained from (co-)limits in the "local" categories or fibres.

Theorem 2.10. *Given an indexed category $B\colon\ I^{\mathrm{op}} \to \mathbb{C}at$, then for each category J the Grothendieck category $B^{\sharp}$ has*

- *J-limits when I has J-limits, B^i has J-limits for each index i, and B^u preserves J-limits for each index morphism u, and*
- *J-co-limits when I has J-co-limits, B^i has J-co-limits for each index i, and B^u has a left adjoint for each index morphism u.*

Chapter 3

Institutions

In this chapter we first give a model theoretic presentation of classical first order logic with equality and show the invariance of the satisfaction relation between models and sentences with respect to the change of notation. This is our first example of an institution. We then introduce the abstract concept of institution and illustrate it by a list of examples from logic and computing science. The next section introduces morphisms and comorphisms of institutions, which are mappings preserving the structure of institution with rather complementary meaning in the actual situations. The final section of this chapter, which is intended for the more category theoretic minded readers, provides a more categorical definition for the concept of institution, which eases considerably our access to the structural properties of categories of institutions. As an application we prove the existence of limits of institutions.

3.1 From concrete logic to Institutions

Perhaps the most representative concrete logic system is first order logic. Here we present it in its many-sorted variants and in a particularly structured way which will serve our goal to capture it as an institution.

Many-sorted first order logic with equality (FOL)

Signatures. A (many-sorted) *signature* in **FOL** is a tuple (S,F,P) where

- S is the set of *sort* symbols,
- $F = \{F_{w\to s} \mid w \in S^*, s \in S\}$ is a family of sets of (S-sorted) *operation* symbols such that $F_{w\to s}$ denotes the set of operations with *arity* w and *sort* s (in particular, when the arity w is empty, $F_{\to s}$ denotes the set of *constants* of sort s), and
- $P = \{P_w \mid w \in S^*\}$ is a family of sets of (S-sorted) relation symbols where P_w denotes the set of relations with *arity* w.

We may sometimes omit the word 'symbol' and simply refer to sort symbols as sorts, to operation symbols as operations, and to relation symbols as relations.

That a symbol of operation σ belongs to some $F_{w\to s}$ for some arity w and some sort s may be unprecisely but compactly denoted by '$\sigma \in F$'. The same may be of course applied for the relation symbols.

When P is empty, then we may write (S,F) rather than $(S,F,\emptyset)$ and we call this an *algebraic signature*.

The fact that the sets $F_{w\to s}$ (or P_w) are not required to be disjoint reflect the possibility of the so-called *overloading* of symbols. A simple example is given by the following choice for a signature (S,F,P) for specifying natural and integer numbers:

- $S = \{\mathsf{N}, \mathsf{Z}\}$ (with N denoting the natural numbers and Z the integers),
- $F_{\mathsf{NN}\to\mathsf{N}} = \{+\}$, $F_{\mathsf{ZZ}\to\mathsf{Z}} = \{+,-\}$, $F_{\mathsf{Z}\to\mathsf{Z}} = F_{\mathsf{NN}\to\mathsf{Z}} = \{-\}$, and $F_{w\to s} = \emptyset$ otherwise,
- $P_{\mathsf{NN}} = P_{\mathsf{ZZ}} = \{\leq\}$ and $P_w = \emptyset$ otherwise.

Models. Given a **FOL** signature (S,F,P), a *model* M interprets:

- each sort symbol s as a set M_s, called the *carrier set of sort* s,
- each operation symbol $\sigma \in F_{w\to s}$ as a function $M_{\sigma\colon w\to s}\colon M_w \to M_s$, where M_w stands for $M_{s_1} \times \cdots \times M_{s_n}$ for $w = s_1 \ldots s_n$ with $s_1, \ldots, s_n \in S$, and
- each relation symbol $\pi \in P_w$ as a subset $M_{\pi\colon w} \subseteq M_w$.

The models of algebraic signatures are called *algebras*.

In order to simplify notation we will often write M_σ instead of $M_{\sigma\colon w\to s}$ and M_π instead of $M_{\pi\colon w}$.

The presentation of many results could be nicely simplified if we assumed that M_s is non-empty for each sort s. Alternatively, this can be achieved by default if one assumes that there is at least one constant for each sort. In this book we will tacitly assume this in all many-sorted situations in which non-empty sorts are necessary.

An (S,F,P)-*model homomorphism* $h\colon M \to M'$ is an indexed family of functions $\{h_s\colon M_s \to M'_s\}_{s\in S}$ such that

- h is an F-algebra homomorphism $M \to M'$, i.e., $h_s(M_\sigma(m)) = M'_\sigma(h_w(m))$ for each $\sigma \in F_{w\to s}$ and each $m \in M_w$,[1]

$$\begin{tikzcd} M_w \arrow[r, "M_\sigma"] \arrow[d, "h_w"'] & M_s \arrow[d, "h_s"] \\ M'_w \arrow[r, "M'_\sigma"'] & M'_s \end{tikzcd}$$

  and

- $h_w(m) \in M'_\pi$ if $m \in M_\pi$ (i.e. $h_w(M_\pi) \subseteq M'_\pi$) for each relation $\pi \in P_w$ and each $m \in M_w$.

[1] $h_w\colon M_w \to M'_w$ is the canonical component-wise extension of h, i.e., $h_w(m_1 \ldots m_n) = h_{s_1}(m_1) \ldots h_{s_n}(m_n)$ where $w = s_1 \ldots s_n$ and $m_i \in M_{s_i}$.

Fact 3.1. *For any signature* (S,F,P)*, the* (S,F,P)*-model homomorphisms form a category under the obvious composition (component-wise as many-sorted functions). The category of* (S,F,P)*-models is denoted by* $\mathsf{Mod}(S,F,P)$*.*

Sentences. An F-term t of sort s is a syntactic structure $\sigma(t_1 \dots t_n)$ where $\sigma \in F_{s_1 \dots s_n \to s}$ is an operation symbol and $t_1, \dots, t_n$ are F-terms of sorts $s_1, \dots, s_n$. By T_F let us denote the set of F-terms.

Given a signature (S,F,P), the set of (S,F,P)-*sentences* is the least set containing the (quantifier-free) atoms and which is closed under Boolean connectives and quantification as follows:

- An *equation* is an equality $t = t'$ between F-terms t and t' of the same sort. A *relational atom* is an expression $\pi(t_1, \dots, t_n)$ where $\pi \in P$ and $(t_1, \dots, t_n) \in (T_F)_w$ is any list of F-terms for the arity w of π (i.e., $w = s_1 \dots s_n$ where s_k is the sort of t_k for $1 \le k \le n$). An atom is either an equation or a relational atom.
- For ρ_1 and ρ_2 any (S,F,P)-sentences, let $\rho_1 \wedge \rho_2$ be their conjunction which is also an (S,F,P)-sentence. Other Boolean connectives are the disjunction $(\rho_1 \vee \rho_2)$, implication $(\rho_1 \Rightarrow \rho_2)$, negation $(\neg\rho)$, and equivalence $(\rho_1 \Leftrightarrow \rho_2)$.
- Any finite set X of *variables* for the sorts S can be added to the signature as new constants. The fact that a variable $x \in X$ has the sort $s \in S$ is denoted by $x : s$. Then $(\forall X)\rho$ is an (S,F,P)-sentence for each $(S, F \uplus X, P)$-sentence ρ. A similar definition can be applied to the existential quantification, denoted $(\exists X)$.

Signature morphisms. A *signature morphism* $\varphi = (\varphi^{\mathrm{st}}, \varphi^{\mathrm{op}}, \varphi^{\mathrm{rl}}) : (S,F,P) \to (S',F',P')$ consists of

- a function between the sets of sorts $\varphi^{\mathrm{st}} : S \to S'$,
- a family of functions between the sets of operation symbols $\varphi^{\mathrm{op}} = \{\varphi^{\mathrm{op}}_{w \to s} : F_{w \to s} \to F'_{\varphi^{\mathrm{st}}(w) \to \varphi^{\mathrm{st}}(s)}\}_{w \in S^*, s \in S}$,[2] and
- a family of functions between the sets of relation symbols $\varphi^{\mathrm{rl}} = \{\varphi^{\mathrm{rl}}_w : P_w \to P'_{\varphi^{\mathrm{st}}(w)}\}_{w \in S^*}$.

Model reducts. Given a signature morphism $\varphi : (S,F,P) \to (S',F',P')$, the *reduct* $M' \upharpoonright_\varphi$ of a (S',F',P')-model M' is an (S,F,P)-model which is defined as follows:

- $(M' \upharpoonright_\varphi)_s = M'_{\varphi^{\mathrm{st}}(s)}$ for each sort $s \in S$,
- $(M' \upharpoonright_\varphi)_\sigma = M'_{\varphi^{\mathrm{op}}_{w \to s}(\sigma)}$ for each operation symbol $\sigma \in F_{w \to s}$, and
- $(M' \upharpoonright_\varphi)_\pi = M'_{\varphi^{\mathrm{rl}}_w(\pi)}$ for each relation symbol $\pi \in P_w$.

The *reduct* $h' \upharpoonright_\varphi$ *of a model homomorphism* is also defined by $(h' \upharpoonright_\varphi)_s = h'_{\varphi(s)}$ for each sort $s \in S$.

[2] Here $(\_)^{\mathrm{op}}$ should not be confused with the similar notation for the opposite of a category; also for any string of sorts $w = s_1 \dots s_n$, by $\varphi^{\mathrm{st}}(w)$ we mean the string of sorts $\varphi^{\mathrm{st}}(s_1) \dots \varphi^{\mathrm{st}}(s_n)$.

Fact 3.2. *For each signature morphism* $\varphi : (S,F,P) \to (S',F',P')$*, the model reduct* $\_\!\upharpoonright_\varphi$ *is a functor* $\mathsf{Mod}(S',F',P') \to \mathsf{Mod}(S,F,P)$.

Moreover, Mod *becomes a functor* $\mathbb{S}ig \to \mathbb{C}at^{\mathrm{op}}$*, with* $\mathsf{Mod}(\varphi)(M) = M\!\upharpoonright_\varphi$ *for each signature morphism* φ.

Sentence translations. The *sentence translation* $\mathsf{Sen}(\varphi) : \mathsf{Sen}(S,F,P) \to \mathsf{Sen}(S', F',P')$ along φ is defined inductively on the structure of the sentences by replacing the symbols from (S,F,P) with symbols from (S',F',P') as defined by φ. At the level of terms, this defines a function $T_F \to T_{F'}$ which we may denote by φ^{tm}, or simply by φ. This can be formally defined by

$$\varphi^{\mathrm{tm}}(\sigma(t_1,\dots,t_n)) = \varphi^{\mathrm{op}}(\sigma)(\varphi^{\mathrm{tm}}(t_1),\dots,\varphi^{\mathrm{tm}}(t_n)).$$

Then

- $\mathsf{Sen}(\varphi)(t = t') = (\varphi^{\mathrm{tm}}(t) = \varphi^{\mathrm{tm}}(t'))$ for equations,
- $\mathsf{Sen}(\varphi)(\pi(t_1,\dots,t_n)) = \varphi^{\mathrm{rl}}(\pi)(\varphi^{\mathrm{tm}}(t_1),\dots,\varphi^{\mathrm{tm}}(t_n))$ for relational atoms,
- $\mathsf{Sen}(\varphi)(\rho_1 \wedge \rho_2) = \mathsf{Sen}(\varphi)(\rho_1) \wedge \mathsf{Sen}(\varphi)(\rho_2)$ and similarly for all other Boolean connectives, and
- $\mathsf{Sen}(\varphi)((\forall X)\rho) = (\forall X^\varphi)\mathsf{Sen}(\varphi')(\rho)$ for each finite set of variables X, each $(S, F \uplus X, P)$-sentence ρ, and where $X^\varphi = \{(x : \varphi^{\mathrm{st}}(s)) \mid (x : s) \in X\}$, and $\varphi' : (S, F \uplus X, P) \to (S', F' \uplus X^\varphi, P')$ extends φ canonically.

Fact 3.3. Sen *is a functor* $\mathbb{S}ig \to \mathbb{S}et$.

Satisfaction. First let us note that each term t of sort s gets interpreted by any (S,F,P) model M as an element $M_t \in M_s$ by

$$M_t = M_\sigma(M_{t_1},\dots,M_{t_n})$$

where $t = \sigma(t_1,\dots,t_n)$.

The *satisfaction* between models and sentences is the Tarskian satisfaction defined inductively on the structure of sentences. Given a fixed arbitrary signature (S,F,P),

- for equations: $M \models t = t'$ if $M_t = M_{t'}$,
- for relational atoms $M \models \pi(t)$ if $M_t \in M_\pi$,[3]
- $M \models \rho_1 \wedge \rho_2$ if and only if $M \models \rho_1$ *and* $M \models \rho_2$,
- $M \models \neg\rho$ if and only if $M \not\models \rho$,
- $M \models \rho_1 \vee \rho_2$ if and only if $M \models \rho_1$ *or* $M \models \rho_2$,
- $M \models \rho_1 \Rightarrow \rho_2$ if and only if $M \models \rho_2$ whenever $M \models \rho_1$,
- $M \models (\forall X)\rho$ if $M' \models \rho$ *for each expansion* M' of M along the signature inclusion $(S,F,P) \hookrightarrow (S, F \uplus X, P)$ (i.e., M is the reduct of M'), and
- $M \models (\exists X)\rho$ if and only if $M \models \neg(\forall X)\neg\rho$.

[3] $M_t = (M_{t_1},\dots,M_{t_n})$ for $t = t_1 \dots t_n$ string of F-terms.

The result below shows that, in first order logic, satisfaction is an invariant with respect to changes of signatures.

Proposition 3.4. *For any signature morphism* $\varphi : (S,F,P) \to (S',F',P')$*, any* (S',F',P')*-model* M'*, and any* (S,F,P)*-sentence* ρ*,*

$$M'\restriction_\varphi \models \rho \ \textit{ if and only if } \ M' \models \mathsf{Sen}(\varphi)(\rho).$$

Proof. We prove this by induction on the structure of the sentences. First notice that for any (S,F,P)-term t, $M'_{\varphi^{\mathrm{tm}}(t)} = (M'\restriction_\varphi)_t$. The satisfaction condition for atoms follows immediately, while the preservation of the satisfaction condition by Boolean connectives can also be checked very easily.

Now we show that the satisfaction condition is preserved by quantification too. We consider only the case of universal quantification, since existential quantification can be treated similarly. Consider an (S,F,P)-sentence $(\forall X)\rho$.

$$\begin{tikzcd}
(S,F,P) \arrow[r, "\varphi"] \arrow[d] & (S',F',P') \arrow[d] \\
(S,F \uplus X,P) \arrow[r, "\varphi'"'] & (S',F' \uplus X^\varphi,P')
\end{tikzcd}$$

The conclusion follows by noticing the canonical bijection between the expansions M'' of M' to $(S',F' \uplus X^\varphi,P')$ and the expansions N of $M'\restriction_\varphi$ to $(S,F \uplus X,P)$ given by $M''\restriction_{\varphi'} = N$, and by using the satisfaction condition $M'' \models \mathsf{Sen}(\varphi')(\rho)$ if and only if $N \models \rho$ which is given by the induction hypothesis. $\square$

Institutions

The above presentation of **FOL** just shows that it is an institution. Below is the definition of the abstract concept of institution.

An *institution* $I = (\mathbb{S}ig^I, \mathsf{Sen}^I, \mathsf{Mod}^I, \models^I)$ consists of

$$\begin{tikzcd}
& \mathbb{S}ig^I \arrow[dl, "(\mathsf{Mod}^I)^{\mathrm{op}}"'] \arrow[dr, "\mathsf{Sen}^I"] & \\
\mathbb{C}at^{\mathrm{op}} & \models & \mathbb{S}et
\end{tikzcd}$$

1. a category $\mathbb{S}ig^I$, whose objects are called *signatures*,
2. a functor $\mathsf{Sen}^I : \mathbb{S}ig^I \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $\mathsf{Mod}^I : (\mathbb{S}ig^I)^{\mathrm{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *homomorphisms*, and
4. a relation $\models_\Sigma \subseteq |\mathsf{Mod}^I(\Sigma)| \times \mathsf{Sen}^I(\Sigma)$ for each $\Sigma \in |\mathbb{S}ig^I|$, called Σ-*satisfaction*,

such that for each morphism $\varphi\colon \Sigma \to \Sigma'$ in $\mathbb{S}ig^I$, the *satisfaction condition*

$$M' \models_{\Sigma'} \mathsf{Sen}^I(\varphi)(e) \text{ if and only if } \mathsf{Mod}^I(\varphi)(M') \models_{\Sigma} e$$

holds for each $M' \in |\mathsf{Mod}^I(\Sigma')|$ and $e \in \mathsf{Sen}^I(\Sigma)$. The satisfaction condition can be graphically represented by the following commutative diagram:

$$\begin{tikzcd}
\Sigma \arrow[d, "\varphi"'] & \mathsf{Mod}^I(\Sigma) \arrow[r, dash, "\models^I_\Sigma"] & \mathsf{Sen}^I(\Sigma) \arrow[d, "\mathsf{Sen}^I(\varphi)"] \\
\Sigma' & \mathsf{Mod}^I(\Sigma') \arrow[u, "\mathsf{Mod}^I(\varphi)"] \arrow[r, dash, "\models^I_{\Sigma'}"'] & \mathsf{Sen}^I(\Sigma')
\end{tikzcd}$$

The meaning of the satisfaction condition of institutions is that

> Truth is invariant under change of notation (and under extension of the context).

We may denote the reduct functor $\mathsf{Mod}^I(\varphi)$ by $\_\restriction_\varphi$ and the sentence translation $\mathsf{Sen}^I(\varphi)$ simply by $\varphi(\_)$. When $M = M'\restriction_\varphi$ we say that M is a φ-*reduct* of M and that M' is an φ-*expansion of* M. When φ is clear (such as an inclusion), we may even write $M\restriction_\Sigma$ rather than $M\restriction_\varphi$. Also, when there is no danger of ambiguity, we may skip the superscripts from the notations of the entities of the institution; for example $\mathbb{S}ig^I$ may be simply denoted $\mathbb{S}ig$.

Corollary 3.5. **FOL** *is an institution.*

Closure under isomorphisms. In this book we assume that all institutions are *closed under isomorphisms*, meaning that the satisfaction relation is invariant with respect to model isomorphisms, i.e., for isomorphic Σ-models $M \cong N$, $M \models_\Sigma \rho$ if and only if $M \models_\Sigma \rho$ for any Σ-sentence ρ.

Although this a very natural property from a model theoretic perspective, it evidently should not be expected in general at the level of abstract institutions.

Exercises

3.1. Give an example of an institution that is *not* closed under isomorphisms.

3.2 Examples of institutions

This section is devoted to examples of institutions. The reader is invited to complete the missing details, including a proof of the satisfaction condition for each of the examples presented.

Sub-institutions. Many examples of institutions are obtained as 'sub-institutions' of given institutions. A *sub-institution* $I' = (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', \models')$ of $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is obtained by narrowing either the category of the signatures, the sentences, and/or the class of models of I. We may express this formally as follows:

- $\mathbb{S}ig'$ is a sub-category of $\mathbb{S}ig$,
- for each signature $\Sigma \in |\mathbb{S}ig'|$ we have that
  - $\mathsf{Sen}'(\Sigma) \subseteq \mathsf{Sen}(\Sigma)$, and
  - $\mathsf{Mod}'(\Sigma)$ is a sub-category of $\mathsf{Mod}(\Sigma)$,
- for each signature morphism $\varphi \in \mathbb{S}ig'$ we have that
  - $\mathsf{Sen}'(\varphi)$ is the restriction of $\mathsf{Sen}(\varphi)$, and
  - $\mathsf{Mod}'(\varphi)$ is the restriction of $\mathsf{Mod}(\varphi)$,
- for each signature $\Sigma \in |\mathbb{S}ig'|$, the satisfaction relation $\models'_\Sigma$ is the restriction of the satisfaction relation $\models_\Sigma$.

Below we give several rather well known examples of 'sub-institutions' of **FOL**.

Single-sorted logic ($\mathbf{FOL}^{\mathbf{1}}$). This is the 'sub-institution' of **FOL** determined by the single-sorted signatures for a fixed sort. Evidently, the name of this sort does not matter since different choices give rise to 'isomorphic sub-institutions'.

Note that in $\mathbf{FOL}^{\mathbf{1}}$ the arities of the operation and of the relation symbols are essentially natural numbers rather than strings of sort symbols. Also the set of sorts S may be omitted from the notation of signatures, therefore the $\mathbf{FOL}^{\mathbf{1}}$-signatures are pairs (F, P) of families F of sets of operation symbols and of families P of sets of relation symbols.

$\mathbf{FOL}^{\mathbf{1}}$ is the version of first order logic used mainly in conventional logic, while the more general many-sorted version **FOL** is used mainly in computing science.

Propositional logic (PL). This can be seen as the 'sub-institution' of **FOL** obtained by restricting the signatures to those with an empty set of sort symbols. This means that **PL** signatures consist only of sets (of zero arity relation symbols), therefore $\mathbb{S}ig^{\mathbf{PL}}$ is just $\mathbb{S}et$, for each set P the set of P-sentences consists of the Boolean expressions formed with variables from P, and the model functor is the contravariant power set functor $\mathcal{P} : \mathbb{S}et \to \mathbb{C}at^{\mathrm{op}}$ (the category of P-models is the partial order $(\mathcal{P}(P), \subseteq)$ regarded as a category). Note that a P-model $M \subseteq P$ satisfies $\pi \in P$ when $\pi \in M$.

While **PL** can be seen as a 'sub-institution' of **FOL**, evidently it cannot be seen as a 'sub-institution' of the single-sorted version $\mathbf{FOL}^{\mathbf{1}}$.

Positive first order logic ($\mathbf{FOL}^{+}$). Sentences are restricted only to those constructed by means of $\wedge, \vee, \forall, \exists$, but not negation. Here $\vee$ and $\exists$ are no longer reducible to $\wedge$ and $\forall$ and vice versa.

Universal sentences in FOL (UNIV). A *universal sentence* for a **FOL** signature (S,F,P) is a sentence of the form $(\forall X)\rho$ where ρ is a sentence formed without quantifiers.

Horn clause logic (HCL). A *(universal) Horn sentence* for a **FOL** signature (S,F,P) is a (universal) sentence of the form $(\forall X)(H \Rightarrow C)$, where H is a finite conjunction of (relational or equational) atoms and C is a (relational or equational) atom, and $H \Rightarrow C$ is the implication of C by H. In the tradition of logic programming, universal Horn sentences are known as *Horn clauses*. We may often write Horn clauses as $(\forall X)H \Rightarrow C$ by omitting the brackets around $H \Rightarrow C$. Thus **HCL** has the same signatures and models as **FOL** but only universal Horn sentences as sentences.

Equational logic. The institution **FOEQL** of *first order equational logic* is obtained from **FOL** by discarding both the relation symbols and their interpretation in models.

The institution **EQL** of *equational logic* is obtained by restricting the sentences of **FOEQL** only to universally quantified equations.

The institution **CEQL** of *conditional equational logic* is obtained as the 'intersection' between **FOEQL** and **HCL**.

EQLN is the minimal extension of **EQL** with negation, allowing sentences obtained from atoms and negations of atoms through only one round of quantification, either universal or existential. More precisely, all sentences have the form $(QX)t_1 \pi t_2$ where $Q \in \{\forall, \exists\}$, $\pi \in \{=, \neq\}$, and t_1 and t_2 are terms with variables X.

Relational logic (REL). This is obtained as the sub-institution of **FOL** determined by those signatures without non-constant operation symbols. Many older works have developed conventional classical logic in **REL** rather than **FOL**.

$(\Pi \cup \Sigma)^0_n$. This is the fragment of **FOL** containing only sentences of the form $Q\rho$ where Q consists of (at most) n alternated quantifiers (universal and existential) and ρ is atomic.

Second order logic (SOL). This is obtained as the extension of **FOL** which allows quantification over sorts, operations, and relation symbols. This differs slightly from the usual presentations of second order logic in the literature which do not consider quantifications over the sorts.

Infinitary logic ($\mathbf{FOL}_{\infty,\omega}$, $\mathbf{FOL}_{\alpha,\omega}$). These are infinitary extensions of **FOL**. $\mathbf{FOL}_{\infty,\omega}$ allows conjunctions of arbitrary sets of sentences, while $\mathbf{FOL}_{\alpha,\omega}$ admits conjunction of sets of sentences with cardinal smaller than α.

Infinitary Horn clause logic ($\mathbf{HCL}_{\infty}$). This is the infinitary extension of **HCL** obtained by allowing the hypotheses parts H of Horn clauses $(\forall X)H \Rightarrow C$ to consist of infinitary conjunctions of atoms.

Partial algebra (PA)

A *partial* algebraic signature is a tuple (S, TF, PF) such that $(S, TF \cup PF)$ is an algebraic signature. Then TF is the set of *total* operations and PF is the set of *partial* operations. A *morphism of* **PA** *signatures* $\varphi\colon (S, TF, PF) \to (S', TF', PF')$ is just a morphism of algebraic signatures $(S, TF \cup PF) \to (S', TF' \cup PF')$ such that $\varphi(TF) \subseteq TF'$.

A *partial algebra A* for a **PA** signature (S, TF, PF) is just like an ordinary algebra but interpreting the operations of PF as partial rather than total functions, which means that A_σ might be *undefined* for some arguments. A *partial algebra homomorphism* $h\colon A \to B$ is a family of (total) functions $\{h_s\colon A_s \to B_s\}_{s \in S}$ indexed by the set of sorts S of the signature such that $h_w(A_\sigma(a)) = B_\sigma(h_s(a))$ for each operation $\sigma\colon w \to s$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined.

The sentences have three kinds of atoms: *definedness* $\mathsf{def}(\_)$, *strong* equality $\stackrel{s}{=}$, and *existence* equality $\stackrel{e}{=}$. The definedness $\mathsf{def}(t)$ of a term t holds in a partial algebra A when the interpretation A_t of t is defined. The strong equality $t \stackrel{s}{=} t'$ holds when both terms are undefined or both of them are defined *and* are equal. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and are equal. The sentences are formed from these atoms by means of Boolean connectives and quantification over *total* (first order) variables.

QE(**PA**). A *(universal) quasi-existence equation* is an infinitary Horn sentence in the infinitary extension $\mathbf{PA}_{\infty,\omega}$ of **PA** of the form

$$(\forall X) \bigwedge_{i \in I} (t_i \stackrel{e}{=} t'_i) \Rightarrow (t \stackrel{e}{=} t').$$

Let *QE*(**PA**) be the sub-institution of the infinitary extension $\mathbf{PA}_{\infty,\omega}$ of **PA** which restricts the sentences only to quasi-existence equations, $QE_1(\mathbf{PA})$ the institution of the quasi-existence equations $(\forall X) \bigwedge_{i \in I} (t_i \stackrel{e}{=} t'_i) \Rightarrow (t \stackrel{e}{=} t')$ that have *either t or t'* 'already defined' (i.e., they occur as subterms of the terms of the equations in the premise or are formed only from total operation symbols), and $QE_2(\mathbf{PA})$ the institution of the quasi-existence equations that have *both t and t'* 'already defined'.

Modal (first order) logic (MFOL)

In Chap. 11 we will undertake a deeper institution-independent study of modal institutions, while here we present only the standard extension of **FOL** with modalities and Kripke semantics.

The **MFOL** *signatures* are tuples (S, S_0, F, F_0, P, P_0) where

- (S, F, P) is a **FOL** signature, and
- (S_0, F_0, P_0) is a sub-signature of (S, F, P) of *rigid* symbols.

Signature morphisms $\varphi\colon (S, S_0, F, F_0, P, P_0) \to (S', S'_0, F', F'_0, P', P'_0)$ are just **FOL** signature morphisms $\varphi\colon (S, F, P) \to (S', F', P')$ which preserve the rigid symbols, i.e., $\varphi(S_0) \subseteq S'_0$, $\varphi(F_0) \subseteq F'_0$, $\varphi(P_0) \subseteq P'_0$.

A **MFOL** *model* (W,R) for a signature (S,S_0,F,F_0,P,P_0), called a *Kripke model*, consists of

- a family $W = \{W^i\}_{i\in I_W}$ of 'possible worlds', which are (S,F,P)-models in **FOL**, indexed by a set I_W, and such that for all rigid symbols x, $W^i_x = W^j_x$ for all $i,j \in I_W$, and
- an 'accessibility' binary relation $R \subseteq I_W \times I_W$ between the possible worlds.

The *reduct* $(W',R')\upharpoonright_\varphi$ of a Kripke model along a signature morphism φ is defined as (W,R') where $I_{W'} = I_W$ and $W^i = (W')^i\upharpoonright_\varphi$ for each $i \in I_W$.

A Kripke model (W,R) is T when R is reflexive, *S4* when it is T and R is transitive, and is $S5$ when it is $S4$ and R is symmetric.

Homomorphisms between Kripke models preserve their mathematical structure. Thus a *Kripke model homomorphism* $h\colon\ (W,R) \to (W',R')$ consists of

- a function $h\colon\ I_W \to I_{W'}$ which preserves the accessibility relation, i.e., $\langle i,\, j\rangle \in R$ implies $\langle h(i),\, h(j)\rangle \in R'$, and
- for each $i \in I_W$ an S-sorted function $\{h^i_s\colon\ W^i_s \to {W'}^{h(i)}_s\}_{s\in S}$, which is an (S,F,P)-model homomorphism $W^i \to {W'}^{h(i)}$, and such that for each rigid sort s_0 we have that $h^i_{s_0} = h^j_{s_0}$ for any $i,j \in I_W$. (Notice the overloading of 'h' in this definition!)

The (S,S_0,F,F_0,P,P_0)-*sentences* are expressions formed from **FOL** (S,F,P)-atoms by closing under usual Boolean connectives, universal and existential first order quantifications by rigid variables (i.e., quantifications by rigid new constants), and unary modal connectives $\Box$ (necessity) and $\Diamond$ (possibility).

The satisfaction of **MFOL** sentences by the Kripke models, $(W,R) \models \rho$ is defined by $(W,R) \models^i \rho$ for each $i \in I_W$, where $\models^i$ is defined by induction on the structure of the sentences as follows:

- $(W,R) \models^i \rho$ iff $W^i \models^{\mathbf{FOL}} \rho$ for each atom ρ and each $i \in I_W$,
- $(W,R) \models^i \rho_1 \wedge \rho_2$ iff $(W,R) \models^i \rho_1$ and $(W,R) \models^i \rho_2$; and similarly for the other Boolean connectives,
- $(W,R) \models^i \Box\rho$ iff $(W,R) \models^j \rho$ for each j such that $\langle i,\, j\rangle \in R$,
- $\Diamond\rho$ abbreviates $\neg\Box\neg\rho$,
- $(W,R) \models^i (\forall X)\rho$ when $(W',R) \models^i \rho$ for each expansion (W',R) of (W,R) to a Kripke $(S,F\uplus X,P)$-model and $(W,R) \models^i (\exists X)\rho$ if and only if $(W,R) \models^i \neg(\forall X)\neg\rho$.

Modal propositional logic (MPL). This is the sub-institution of **MFOL** determined by the signatures with an empty set of sort symbols (and therefore empty sets of operation symbols) and empty sets of rigid relation symbols. Most of the conventional modal logic studies are concerned with this institution.

Intuitionistic logic

Heyting algebras. A *Heyting algebra* A is a bounded lattice which is cartesian closed as category. In other words A is a partial order $(A, \leq)$ with a greatest element $\top$ and a least one $\bot$ and such that any two elements $a, b \in A$

- have a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$, and
- there exists a greatest element x such that $a \wedge x \leq b$; this element is denoted $a \Rightarrow b$.

From these axioms a series of important properties can be derived, such as the distributivity of the lattice. Also each element a has a *pseudo-complement* $\neg a$ defined as $a \Rightarrow \bot$. However, while it is possible to show that $a \leq \neg\neg a$, in general we do not have that $a = \neg\neg a$. This shows that Heyting algebras are more general than Boolean algebras. A famous class of examples of Heyting algebras that are not Boolean algebras come from general topology; the set of open sets of any topological space ordered by (sub)set inclusion forms a Heyting algebra.

Intuitionistic propositional logic (IPL). The institution of intuitionistic propositional logic generalizes (classical) propositional logic (**PL**) by considering models to be valuations of the propositional variables to arbitrary Heyting algebras rather than the two-elements Boolean algebra $\{\bot, \top\}$. More precisely, **IPL** has the same signatures as **PL**, i.e., plain sets P, and for any set P, a P-model M is just a function $M : P \to A$ where A is any Heyting algebra. If $f : P \to P'$ is a signature morphism, then the reduct of any P'-model M' is just $f;M'$. **IPL** and **PL** share the same sentences.

The function M can be extended from P to $\mathsf{Sen}(P)$ by $M(\rho_1 \wedge \rho_2) = M(\rho_1) \wedge M(\rho_2)$, $M(\rho_1 \vee \rho_2) = M(\rho_1) \vee M(\rho_2)$, $M(\neg\rho) = \neg M(\rho)$, $M(\rho_1 \Rightarrow \rho_2) = M(\rho_1) \Rightarrow M(\rho_2)$, etc. The satisfaction relation is defined by

$$M \models_P \rho \text{ if and only if } M(\rho) = \top.$$

Preorder algebra (POA)

The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered algebras* which are interpretations of signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e., monotonic) function. A *preordered algebra homomorphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra homomorphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e., $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first order quantification.

Horn preordered algebra (HPOA). This is the sub-institution of **POA** whose sentences are the universal Horn sentences $(\forall X)H \Rightarrow C$ formed over equational and preorder atoms.

Multialgebras (MA)

The category of the **MA** signatures is just that of the algebraic signatures. Multialgebras generalize algebras by nondeterministic operations returning a *set* of all possible outputs for the operation rather than a single value. Hence multialgebra operations are interpreted as functions from the carrier to the powerset of the carrier. Therefore each term $t = \sigma(t_1 \dots t_n)$ is interpreted in any multialgebra M by $M_t = \bigcup\{M_\sigma(m_1 \dots m_n) \mid m_1 \in M_{t_1}, \dots, m_n \in M_{t_n}\}$.

Given a signature (S,F), a *multialgebra homomorphism* $h\colon\ M \to N$ consists of an S-indexed family of functions $\{h_s\colon\ M_s \to N_s \mid s \in S\}$ such that for each operation symbol $\sigma \in F_{s_1 \dots s_n \to s}$ and each $m_k \in M_{s_k}$ for $1 \le k \le n$ we have

$$h_s(M_\sigma(m_1,\dots,m_n)) \subseteq N_\sigma(h_{s_1}(m_1),\dots,h_{s_n}(m_n)).$$

The sentences have two kinds of atoms: set inclusion $\prec$ and (deterministic) element equality $\doteq$. The set inclusion $t \prec t'$ holds in a multialgebra M if and only if $M_t \subseteq M_{t'}$, i.e., the term t is "more deterministic" than t'. The element equality $t \doteq t'$ states that the terms t and t' are deterministic and must return the same element. (This means that M_t and $M_{t'}$ are both singleton sets containing the same element.) Full sentences are built from these atoms by using Boolean connectives and first order quantification in the manner of **FOL**.

Membership algebra (MBA)

A **MBA** signature is a tuple (S,K,F,kind) where S is a set of *sorts*, K is a set of *kinds*, (K,F) is an algebraic signature, and $\mathrm{kind}\colon\ S \to K$ is a function. A *morphism of* **MBA** *signatures* $\varphi\colon\ (S,K,F,\mathrm{kind}) \to (S',K',F',\mathrm{kind}')$ consists of functions $\varphi^{\mathrm{st}}\colon\ S \to S'$, $\varphi^{\mathrm{k}}\colon\ K \to K'$ such that the following diagram commutes

$$\begin{tikzcd}
S \arrow[r, "\mathrm{kind}"] \arrow[d, "\varphi^{\mathrm{st}}"'] & K \arrow[d, "\varphi^{\mathrm{k}}"] \\
S' \arrow[r, "\mathrm{kind}'"'] & K'
\end{tikzcd}$$

and a family of functions $\{\varphi^{\mathrm{op}}_{w \to s} \mid w \in K^*, s \in K\}$ such that $(\varphi^{\mathrm{k}}, \varphi^{\mathrm{op}})\colon\ (K,F) \to (K',F')$ is an algebraic signature morphism.

Given a membership algebraic signature (S,K,F,kind), an (S,K,F,kind)-*algebra* A is a (K,F)-algebra together with a set $A_s \subseteq A_{\mathrm{kind}(s)}$ for each sort $s \in S$ such that $A_s \subseteq A_{\mathrm{kind}(s)}$ for each sort s. A (S,K,F,kind)-*algebra homomorphism* $A \to B$ is a (K,F)-algebra homomorphism such that $h_{\mathrm{kind}(s)}(A_s) \subseteq B_s$ for each sort s.

Sentences for membership algebra have two types of atoms, atomic equations $t = t'$ for t,t' any F-terms of the same kind, and *atomic membership* $t\colon\ s$ where s is a sort and t is an F-term of kind(s). A membership algebra A satisfies an equation $t = t'$ when $A_t = A_{t'}$ and satisfies a membership atom $t :\ s$ when $A_t \in A_s$. Full sentences are formed from atoms by iteration of Boolean connectives and first order quantification.

Higher Order Logic (HOL)

For any set S of *sorts*, let $\overrightarrow{S}$ be the set of S-*types* defined as the least set such that $S \subseteq \overrightarrow{S}$ and $s_1 \to s_2 \in \overrightarrow{S}$ when $s_1, s_2 \in \overrightarrow{S}$. A **HOL** *signature* (S,F) consists of a set S of sorts and a family of sets of constants $F = \{F_s \mid s \in \overrightarrow{S}\}$. A *morphism of* **HOL** *signatures* $\varphi :\ (S,F) \to (S',F')$ consists of a function $\varphi^{\mathrm{st}} :\ S \to S'$ and a family of functions $\{\varphi^{\mathrm{op}}_s :\ F_s \to F'_{\varphi^{\mathrm{type}}(s)} \mid s \in \overrightarrow{S}\}$ where $\varphi^{\mathrm{type}} :\ \overrightarrow{S} \to \overrightarrow{S'}$ is the canonical extension of φ^{st} such that $\varphi^{\mathrm{type}}(s_1 \to s_2) = \varphi^{\mathrm{type}}(s_1) \to \varphi^{\mathrm{type}}(s_2)$.

Given a signature (S,F), an (S,F)-*model* interprets each sort $s \in S$ as a set M_s and each operation symbol $\sigma \in F_s$ as an element $M_\sigma \in M_s$, where for each type s_1, s_2, $M_{s_1 \to s_2} = [M_{s_1} \to M_{s_2}] = \{f \text{ function } \mid f :\ M_{s_1} \to M_{s_2}\}$. An (S,F)-*model homomorphism* $h :\ M \to N$ interprets each S-type s as a function $h_s :\ M_s \to N_s$ such that $h(M_\sigma) = N_\sigma$ for each $\sigma \in F$ and such that the diagram

$$\begin{tikzcd}
M_s \arrow[r, "f"] \arrow[d, "h_s"'] & M_{s'} \arrow[d, "h_{s'}"] \\
N_s \arrow[r, "h_{s\to s'}(f)"'] & N_{s'}
\end{tikzcd}$$

commutes for all types s and s' and each $f \in M_{s \to s'}$.

For any **HOL** signature (S,F), each operation symbol σ of type s is a term of type s, and (tt') is a term of type s_2 whenever t is a term of type $s_1 \to s_2$ and t' is a term of type s_1. A **HOL** (S,F)-*equation* consists of a pair $t_1 = t_2$ of terms of the same type. A **HOL** (S,F)-sentence is obtained from equations by iteration of the usual Boolean connectives and of *higher order (universal or existential) quantification* which is defined similarly to the quantification in **FOL**. Note however that because of the 'higher order' types, the constants in **HOL** denote higher order rather than first order entities.

The interpretation of operation symbols by models can be extended to terms by defining $M_{(tt')} = M_t(M_{t'})$ for each term t of of type $s_1 \to s_2$ and each term t' of type s_1. A model M satisfies the equation $t = t'$ when $M_t = M_{t'}$. This satisfaction relation can be extended in an obvious manner from equations to any sentences.

Henkin semantics. The institution of higher order logic with Henkin semantics, denoted **HNK**, extends the **HOL** models by relaxing the condition $M_{s \to s'} = [M_s \to M_{s'}]$ to $M_{s \to s'} \subseteq [M_s \to M_{s'}]$.

Automata (AUT)

Given a set V (of 'input symbols'), a *V-automaton* A consists of

- a set A_{state} of 'states' with some 'initial' state $A_0 \in A_{\text{state}}$ and with some 'final' states, and
- a *transition function* $A_t : V \times A_{\text{state}} \to A_{\text{state}}$.

A *homomorphism of V-automata* $h : A \to B$ consists of a function $h : A_{\text{state}} \to B_{\text{state}}$ such that $h(A_0) = B_0$, $h(a)$ is final whenever a is final, and

$$\begin{tikzcd} V \times A_{\text{state}} \arrow[r, "A_t"] \arrow[d, "1_V \times h"'] & A_{\text{state}} \arrow[d, "h"] \\ V \times B_{\text{state}} \arrow[r, "B_t"'] & B_{\text{state}} \end{tikzcd}$$

commutes. The transition function extends canonically by iteration to $A_t^* : V^* \times A_{\text{state}} \to A_{\text{state}}$. A word (or string) $w \in V^*$ is *recognized* by a V-automaton if and only if $A_t^*(w, A_0)$ is a final state.

The institution **AUT** of automata has $\mathbb{S}et$ as its category of signatures, automata as models, and strings of input symbols as sentences. A string w is satisfied by an automaton A when A recognizes w.

Exercises

3.2. Define a version of *weak* second order logic as an institution which has second order variables ranging over finite subsets of the models.

3.3. Extend the definition of **IPL** to an institution of 'intuitionistic first order logic'.

3.4. Contraction Algebras
A *contraction algebraic signature* (S, F, q) consists of an algebraic signature (S, F) and a real number $0 < q < 1$. $\varphi : (S, F, q) \to (S', F', q')$ is a *morphism of contraction algebraic signatures* if $\varphi : (S, F) \to (S', F')$ is an algebraic signature morphism and $q' \leq q$.

(A, d) is a *(S, F, q)-contraction algebra* when A is an (S, F)-algebra, d gives a complete metric space (A_s, d_s) for each sort $s \in S$ such that d_s is bounded by 1, and

$$d(A_\sigma(a_1 \ldots a_n), A_\sigma(b_1 \ldots b_n)) \leq q \cdot max\{d(a_k, b_k) \mid k \in \{1, \ldots, n\}\}.$$

A *homomorphism of contraction algebras* $h : (A, d) \to (A', d')$ is just an (S, F)-algebra homomorphism $A \to A'$ such that $d'(h(a), h(b)) \leq d(a, b)$ for all elements $a, b \in A$.

For each algebraic signature (S, F) let T_F^ω be the S-sorted set of (possibly) infinite F-terms. Show that for any contraction algebra (A, d) there exists a unique mapping $T_F^\omega \to A$ mapping each (possibly) infinite term t to an element A_t of A such that $A_{\sigma(t_1 \ldots t_n)} = A_\sigma(A_{t_1} \ldots A_{t_n})$ for each infinite term $\sigma(t_1 \ldots t_n)$.

An *(S, F, q)-approximation equation $t \approx_\varepsilon t'$* consists of a pair of (possibly) infinite terms t and t' and a real number $0 \leq \varepsilon < 1$. A contraction algebra A satisfies $t \approx_\varepsilon t'$ if and only if $d(A_t, A_{t'}) \leq \varepsilon$. Full 'approximation' sentences are formed from atomic approximation equalities by iteration of

Boolean connectives and quantification. These data define an institution **CA** of contraction algebras and approximation sentences.

3.5. Linear Algebra

The institution **LA** has the category $\mathbb{C}Rng$ of commutative rings as the category of signatures such that for each commutative ring R the category of R-models is R-Mod the category of R-modules, an R-sentence is a linear system of equations with coefficients from R, and the satisfaction relation is defined by the existence of solutions for the system of equations.

3.6. HOL with λ-abstraction

The institution $\mathbf{HOL}_\lambda$ has become quite popular for computer-assisted theorem proving. It adds λ-abstraction and products to **HOL**. Signatures and signature morphisms are similar to those of **HOL**. The only difference is in the definition of the set of higher types: a type Ω of truth-values and products are added. Thus $\overrightarrow{S}$ is defined to be the least set such that

- $S \uplus \{\Omega\} \subseteq \overrightarrow{S}$,
- $s_1 \to s_2 \in \overrightarrow{S}$ and $s_1 \times s_2 \in \overrightarrow{S}$ when $s_1, s_2 \in \overrightarrow{S}$.

For each $\mathbf{HOL}_\lambda$-signature (S,F),

- each operation symbol σ of type s is a term of type s,
- $(t\ t')$ is a term of type s_2 whenever t is a term of type $s_1 \to s_2$ and t' is a term of type s_1,
- $\langle t_1, t_2\rangle$ is a term of type $s_1 \times s_2$ when t_1 is a term of type s_1 and t_2 is a term of type s_2,
- for any finite list $X = \langle x_1{:}s_1, \ldots, x_n{:}s_n\rangle$ of typed variables and any $(S, F \uplus X)$-term t of type s, $\lambda X.t$ is an (S,F)-term of type $(((s_1 \times s_2) \times \ldots) \times s_n) \to s$,
- $t_1 = t_2$ is a term of type Ω for terms t_1, t_2 of the same type.

A $\mathbf{HOL}_\lambda$-model (also called *standard model*) interprets Ω as a two-element set $\{\bot, \top\}$, $\_ \times \_$ as a cartesian product, and is otherwise like a **HOL**-model. The interpretation M_t of a term t in a model M is defined as in **HOL** for the cases σ and $(t\ t')$. $M_{\langle t_1,t_2\rangle}$ is just $\langle M_{t_1}, M_{t_2}\rangle$. $M_{\lambda X.t}$ is the function that, for any $(S, F \uplus X)$-expansion M' of M, maps the tuple $\langle\langle\langle M'_{x_1}, M'_{x_2}\rangle, \ldots\rangle, M'_{x_n}\rangle$ to M'_t. $M_{t_1 = t_2}$ is $\top$, if $M_{t_1} = M_{t_2}$, and $\bot$ otherwise.

A (S,F)-*sentence* ρ is just a (S,F)-term of type Ω. It holds in a model M if $M_\rho = \top$.

3.7. HNK with λ-abstraction

$\mathbf{HNK}_\lambda$ is a generalization of $\mathbf{HOL}_\lambda$, much in the same way as **HNK** is a generalization of **HOL**. However, there is an additional requirement for models. Let a Σ-*frame* be like a $\mathbf{HOL}_\lambda$-model of signature Σ, but with the relaxed condition that $M_{s_1 \to s_2}$ may be any subset of $[M_{s_1} \to M_{s_2}]$. A Σ-frame is a Σ-*general model*, if every Σ-term has an interpretation in it (note that the interpretations of λ-abstractions require the existence of certain functions in the model). The model functor of $\mathbf{HNK}_\lambda$ uses general models instead of standard models.

3.8. Categorical Equational Logic

For any category $\mathbb{A}$, an (unconditional) $\mathbb{A}$-*equation* $(\forall B) l = r$ is a pair of parallel arrows $l, r : C \to B$ in $\mathbb{A}$. An $\mathbb{A}$-*model* is simply any object of $\mathbb{A}$, and a homomorphism of $\mathbb{A}$-models is an arrow of $\mathbb{A}$. An $\mathbb{A}$-model A *satisfies* the equation $(\forall B) l = r$ when $l; h = r; h$ for each arrow $h : B \to A$.

For each right adjoint $U : \mathbb{A}' \to \mathbb{A}$ with $F : \mathbb{A} \to \mathbb{A}'$ as left adjoint, the following satisfaction condition holds:

$$U(A') \models (\forall B) l = r \text{ if and only if } A' \models (\forall F(B)) F(l) = F(r)$$

for each $\mathbb{A}'$-model A' and each $\mathbb{A}$-equation $(\forall B) l = r$.

This defines the institution Cat**EQL** of *categorical equational logic* with categories as signatures and adjunctions as signature morphisms.

3.9. Institution of the signature morphisms
For any institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ we define

- $\mathbb{S}ig^{\rightarrow}$ to be the category of functors $(\bullet \rightarrow \bullet) \rightarrow \mathbb{S}ig$,
- $\mathsf{Sen}^{\rightarrow}(\varphi) = \mathsf{Sen}(\Sigma)$ for each signature morphism $\varphi \in \mathbb{S}ig(\Sigma, \Sigma')$,
- $\mathsf{Mod}^{\rightarrow}(\varphi) = \mathsf{Mod}(\Sigma')$ for each signature morphism $\varphi \in \mathbb{S}ig(\Sigma, \Sigma')$, and
- for each signature morphism $\varphi : \Sigma \rightarrow \Sigma'$, for each Σ'-model M', and each Σ-sentence ρ, $M' \models^{\rightarrow}_{\varphi} \rho$ if and only if $M' \models_{\Sigma'} \varphi(\rho)$.

Then $(\mathbb{S}ig^{\rightarrow}, \mathsf{Sen}^{\rightarrow}, \mathsf{Mod}^{\rightarrow}, \models^{\rightarrow})$ is an institution.

3.10. [162] **Extended models**
A *pre-institution* [160] I consists of the same data as an institution, but without the requirement of the Satisfaction Condition. An *extended model* of a signature Σ_1 is a pair (φ, N), where $\varphi : \Sigma_1 \rightarrow \Sigma_2$ is a signature morphism and N is a Σ_2-model. The extended model (φ, N) satisfies a Σ_1-sentence ρ if and only if $N \models^{I}_{\Sigma_2} \varphi(\rho)$.

The extended models, together with the I-signatures and I-sentences form an institution.

3.11. [74] **Charters**
A *charter* consists of

- an adjunction $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$ between a category of signatures $\mathbb{S}ig$ and a category Syn of "syntactic systems", with $\mathcal{U} : Syn \rightarrow \mathbb{S}ig$ the right adjoint and $\mathcal{F}$ the left adjoint,
- a "ground object" $G \in |Syn|$ (in which other syntactic systems are interpreted), and
- a "base" functor $B : Syn \rightarrow \mathbb{S}et$ (extracting the sentence component from the syntactic system) such that $B(G) = \{\mathbf{true}, \mathbf{false}\}$.

An institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is *chartable* when there exists a charter $(\mathbb{S}ig, Syn, \mathcal{U}, \mathcal{F}, B, G)$ such that

- $|\mathsf{Mod}(\Sigma)| = Syn(\Sigma, \mathcal{U}(G))$ and $\mathsf{Mod}(\varphi)(M') = \varphi; M'$,
- $Sen = \mathcal{F}; B$, and
- for each Σ-model $M : \Sigma \rightarrow \mathcal{U}(G)$

$$M \models_{\Sigma} e \text{ if and only if } B(M^{\sharp})(e) = \mathbf{true}$$

  where $M^{\sharp}$ is the unique arrow $\mathcal{F}(\Sigma) \rightarrow G$ such that $M = \eta_{\Sigma}; \mathcal{U}(M^{\sharp})$.

PL is chartable by taking Syn as the category of (unsorted) $(\neg, \wedge, \vee, \Rightarrow)$-algebras (where $\neg$ is an unary operation and $\wedge$,$\vee$, and $\Rightarrow$ are binary operation symbols), $\mathcal{U} = B$ is the forgetful functor $\mathsf{Mod}(\neg, \wedge, \vee, \Rightarrow) \rightarrow \mathbb{S}et$, and G is the canonical $(\neg, \wedge, \vee, \Rightarrow)$-algebraic structure on $\{\mathbf{true}, \mathbf{false}\}$ interpreting the Boolean connectors as usually.

Show that other institutions are chartable too.

3.3 Morphisms and Comorphisms

Let us look into the way the institution **EQL** can be obtained by forgetting the relational part and by discarding all sentences but equations in **FOL**. This is a three-fold process. Firstly, there is a forgetful functor between the categories of signatures "forgetting" the relations, i.e., mapping each **FOL** signature (S, F, P) to the algebraic signature (S, F). On the sentences side, each (S, F)-equation can be regarded as an (S, F, P)-sentence; this gives a family of translation functions between sets of sentences. On the models side, each (S,F,P)-model can be regarded as an (S,F)-algebra by forgetting the interpretations of the relation symbols; this gives a family of functors between categories of models. Notice that the satisfaction of sentences by models is invariant with respect to this mapping **FOL** $\to$ **EQL**.

Institution morphisms. Such structure preserving mappings from a more complex to a simpler institution can be formalized by the general concept of *institution morphism* $(\Phi,\alpha,\beta): I' \to I$ consisting of

$$\begin{tikzcd}
 & \mathbb{S}ig' \arrow[dl, bend right=40, "\mathsf{Sen}'"'] \arrow[dl, bend left=20, ""{name=A}] \arrow[d, "\Phi"] \arrow[dr, bend left=40, "(\mathsf{Mod}')^{\mathrm{op}}"] \arrow[dr, bend right=20, ""{name=B}] & \\
\mathbb{S}et & \mathbb{S}ig \arrow[l, "\mathsf{Sen}"] \arrow[r, "\mathsf{Mod}^{\mathrm{op}}"'] & \mathbb{C}at^{\mathrm{op}}
\arrow[from=1-2, to=2-1, phantom, bend right=10, "\alpha\Nwarrow"]
\arrow[from=1-2, to=2-3, phantom, bend left=10, "\Swarrow\beta"]
\end{tikzcd}$$

1. a functor $\Phi: \mathbb{S}ig' \to \mathbb{S}ig$, called the *signature functor*,
2. a natural transformation $\alpha: \Phi;\mathsf{Sen} \Rightarrow \mathsf{Sen}'$, called the *sentence transformation*, and
3. a natural transformation $\beta: \mathsf{Mod}' \Rightarrow \Phi^{\mathrm{op}};\mathsf{Mod}$, called the *model transformation*

such that the following *satisfaction condition* holds:

$$M' \models'_{\Sigma'} \alpha_{\Sigma'}(e) \text{ if and only if } \beta_{\Sigma'}(M') \models_{\Phi(\Sigma')} e$$

for any signature $\Sigma' \in |\mathbb{S}ig'|$, for any Σ'-model M', and any $\Phi(\Sigma')$-sentence e.

Although institution morphisms are suitable to formalize 'forgetful' mappings between more complex institutions to simpler ones, there are also other kinds of examples of institution morphisms. Some of them can be found among the exercises.

The composition of institution morphisms $(\Phi',\alpha',\beta'): I'' \to I'$ and $(\Phi,\alpha,\beta): I' \to I$ is $(\Phi';\Phi, \Phi'\alpha;\alpha', \beta';\Phi'^{\mathrm{op}}\beta): I'' \to I$. Under this composition, institutions and institution morphisms form the *category* $\mathbb{I}ns$ *of institution morphisms* . This can established by routine calculations which are left as an exercise for the reader.

Institution modifications. The category $\mathbb{I}ns$ has a 2-dimension too, given by the institution *modifications*. An institution modification between institution morphisms $(\Phi,\alpha,\beta) \Rightarrow (\Phi',\alpha',\beta')$ consists of

1. a natural transformation $\tau: \Phi \Rightarrow \Phi'$, called the *signature transformation*,
2. a modification $\omega: \beta \Rightarrow \beta';\tau\mathsf{Mod}$, called the *model transformation*, i.e., for each $\Sigma' \in |\mathbb{S}ig'|$, a natural transformation $\omega_{\Sigma'}: \beta_{\Sigma'} \Rightarrow \beta'_{\Sigma'};\mathsf{Mod}(\tau_{\Sigma'})$.

This makes $\mathbb{I}ns$ a 2-category with institutions as 0-cells, institution morphisms as 1-cells, and their modifications as 2-cells. Routine calculations, left as exercise to the reader, show that the horizontal composition of institution morphisms and the vertical composition of modifications satisfy the 2-category Interchange Laws (see Sect. 2.4).

Comorphisms. This relationship between **FOL** and **EQL** can be also looked at from the opposite direction, by emphasizing the "embedding" rather than the "forgetful" aspect. Each algebraic signature (S,F) can be regarded as a **FOL** signature $(S,F,\emptyset)$ without relation symbols. This determines an "embedding" functor from the category of algebraic signatures to the category of **FOL** signatures. On the sentence side, each (S,F)-equation is an $(S,F,\emptyset)$-sentence, and each $(S,F,\emptyset)$-model is just an (S,F)-algebra. The satisfaction of sentences by models is invariant with respect to this embedding of **EQL** into **FOL**. Such an embedding relationship between institutions is formalized by the concept of *institution comorphism* $(\Phi,\alpha,\beta) : \ I \to I'$ consisting of

$$\begin{tikzcd}
& \mathbb{S}ig' \arrow[dl, "\mathsf{Sen}'"'] \arrow[dr, "(\mathsf{Mod}')^{\mathrm{op}}"] & \\
\mathbb{S}et & \mathbb{S}ig \arrow[u, "\Phi"] \arrow[l, bend right] \arrow[l, bend left, "\mathsf{Sen}"] \arrow[l, phantom, "\alpha\Uparrow"] \arrow[r, bend left] \arrow[r, bend right, "\mathsf{Mod}^{\mathrm{op}}"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbb{C}at^{\mathrm{op}}
\end{tikzcd}$$

1. a functor $\Phi : \ \mathbb{S}ig \to \mathbb{S}ig'$,
2. a natural transformation $\alpha : \ \mathsf{Sen} \Rightarrow \Phi;\mathsf{Sen}'$, and
3. a natural transformation $\beta : \ \Phi^{\mathrm{op}};\mathsf{Mod}' \Rightarrow \mathsf{Mod}$

such that the following *satisfaction condition* holds:

$$M' \models'_{\Phi(\Sigma)} \alpha_\Sigma(e) \ \text{ if and only if } \ \beta_\Sigma(M') \models_\Sigma e$$

for any signature $\Sigma \in |\mathbb{S}ig|$, for any $\Phi(\Sigma)$-model M', and any Σ-sentence e. The category of institutions and their comorphisms is denoted by $co\mathbb{I}ns$.

Category theoretic thinking promotes the idea that the arrows are the primary concept rather than the objects. It is even possible to define the concept of category only by means of arrows, the objects being assimilated to the identity arrows. Institutions serve as a clear example for this since both $\mathbb{I}ns$ and $co\mathbb{I}ns$ have institutions as objects but have different classes of arrows, both classes having the same level of preservation of institutional structure. Therefore we should never refer to the 'category of institutions', which does not make sense, instead we should refer to the 'category of institution morphisms' or to the 'category of institution comorphisms'.

The adjoint relationship. Often the forgetful nature of many functors can be captured formally by the concept of right adjoint functor. For example, the embedding of the algebraic signatures into the **FOL** signatures is in fact a left adjoint to the forgetful functor from the **FOL** signatures to the algebraic signatures. The following general theorem shows that the 'embedding' comorphism **EQL** $\to$ **FOL** and the 'forgetful' morphism **FOL** $\to$ **EQL** determine each other, their interdependency being caused by the adjunction between their categories of signatures.

Theorem 3.6. *An adjunction* $(\Phi,\overline{\Phi},\zeta,\overline{\zeta})$ *between the categories of signatures*[4] *of institutions* I *and* I' *determines a canonical bijection between institution morphisms* $(\Phi,\alpha,\beta) : I' \to I$ *and institution comorphisms* $(\overline{\Phi},\overline{\alpha},\overline{\beta}) : I \to I'$ *given by the following equalities:*

– $\overline{\alpha} = \zeta\mathsf{Sen};\overline{\Phi}\alpha$ *and* $\overline{\beta} = \overline{\Phi}^{\mathrm{op}}\beta;\zeta^{\mathrm{op}}\mathsf{Mod}$*, and*

– $\alpha = \Phi\overline{\alpha};\overline{\zeta}\mathsf{Sen}'$ *and* $\beta = \overline{\zeta}^{\mathrm{op}}\mathsf{Mod}';\Phi^{\mathrm{op}}\overline{\beta}$*.*

The proof of this theorem follows by routine calculations, which are left as an exercise for the reader.

An institution morphism or comorphism is called *adjoint* when this is part of a morphism-comorphism duality determined by an adjunction between the categories of signatures. Notice that the composition of institution adjoints is still an adjoint. Let $e\mathbb{I}ns$ denote the category of pairs of institution adjoint morphism-comorphism.

Equivalence of institutions. As in the case of categories, the equivalence concept for institutions captures the fact that the institutions are the 'same', while being weaker than isomorphism. This concept is also an example of an adjoint institution morphism. An institution morphism (Φ,α,β) is an *equivalence of institutions* when

- Φ is an equivalence of categories,
- α_Σ has an inverse up to semantic equivalence, denoted α'_Σ, such that α' is a natural transformation, and
- β_Σ is an equivalence of categories, such that its inverse up to isomorphism and the corresponding isomorphism natural transformations are natural in Σ.

Institution encodings

There is a class of comorphisms, very useful in applications, which are generally not adjoints. Rather than giving the flavor of an 'embedding', they are in fact 'encodings' of more complex institutions into simpler ones. We give now a couple of examples.

Encoding relations as operations in FOL. This example formalizes the basic intuition in logic that relations can be simulated by (pseudo-)Boolean-valued operations. We may map each **FOL** signature (S,F,P) to an algebraic signature $(S \uplus \{\mathbf{b}\}, F \uplus \overline{P} \uplus \{\mathbf{true}\})$ where $\mathbf{b}$ is a (new) sort, **true** is a (new) constant of sort $\mathbf{b}$, and for each arity $w \in S^*$, $\overline{P}_{w\to s} = P_w$ if $s = \mathbf{b}$ and $\overline{P}_{w\to s} = \emptyset$ otherwise. This determines an institution comorphism **FOL** $\to$ **FOEQL** which

– maps each relational atom $\pi(t)$ to the equation $\pi(t) = \mathbf{true}$, and

– maps each $(S \uplus \{\mathbf{b}\}, F \uplus \overline{P} \uplus \{\mathbf{true}\})$-algebra A to the (S,F,P)-model $\beta(A)$ maintaining the interpretations of the sorts and F-operations of A but $\beta(A)_\pi = A_\pi^{-1}(A_{\mathbf{true}})$ for each relation symbol π.

[4] $\Phi : \mathbb{S}ig' \to \mathbb{S}ig$ is the right adjoint, $\overline{\Phi}$ is the left adjoint, ζ is the unit, and $\overline{\zeta}$ is the co-unit of the adjunction.

We leave to the reader the task of developing the details of the definition of this comorphism and of its satisfaction condition.

Encoding modalities in relational logic. Let $\mathbf{REL^1}$ be the single-sorted variant of **REL**. We may build a comorphism $(\Phi, \alpha, \beta) : \mathbf{MPL} \to \mathbf{REL^1}$ as follows:

- Each **MPL**-signature, i.e., set P, gets mapped to the single-sorted relational signature without constants $(\emptyset, \overline{P})$ where
  - $\overline{P}_1 = P$, $\overline{P}_2 = \{r\}$, and $\overline{P}_n = \emptyset$ for $n \notin \{1,2\}$,
- each $(\emptyset, \overline{P})$-model M gets mapped to the Kripke P-model $\beta(M) = (W, R)$ with I_W being the carrier set of M, $R = M_r$, and $W^i = \{\pi \mid i \in M_\pi\}$ for each $i \in I_W$, and
- for each P-sentence ρ, $\alpha(\rho) = (\forall x)\alpha_x(\rho)$ where $\alpha_x : \mathsf{Sen}^{\mathbf{MPL}}(P) \to \mathsf{Sen}^{\mathbf{REL^1}}(\{x\}, \overline{P})$ is such that
  - $\alpha_x(\pi) = \pi(x)$ for each $\pi \in P$,
  - α_x commutes with the Boolean connectives, i.e., $\alpha_x(\rho_1 \wedge \rho_2) = \alpha_x(\rho_1) \wedge \alpha_x(\rho_2)$, etc., and
  - $\alpha_x(\Box\rho) = (\forall y)(r(x,y) \Rightarrow \alpha_y(\rho))$.

This example shows that modal propositional logic is a 'fragment' of ordinary (single-sorted) first order logic.

Exercises

3.12. Morphism FOL → REL
Each **FOL** signature (S, F, P) can be mapped to a **FOL** signature $(S, C(F), \overline{F} \cup P)$ without non-constant operation symbols, where $C(F)$ is the set of constants of F, $\overline{F}_s = \emptyset$ for each sort $s \in S$, and $\overline{F}_{ws} = F_{w \to s}$ when w is non-empty.

This determines a non-adjoint institution morphism **FOL** → **REL**. (*Hint:* For each (S, F, P)-model M and each $\sigma \in \overline{F}_{ws}$ where w is non-empty, $\beta(M)_\sigma = \{\langle m, M_\sigma(m)\rangle \mid m \in M_w\}$ and $\alpha(\sigma(x,y)) = (\sigma(x) = y)$ for each $\sigma \in \overline{F}$.)

3.13. Morphism PA → FOL
There exists a forgetful institution morphism **PA** → **FOL** which forgets the partial operations. Is this an adjoint morphism?

3.14. Morphism FOL → MFOL
There exists an institution morphism **FOL** → **MFOL** which maps any **FOL** signature (S, F, P) to the **MFOL** signature (S, S, F, F, P, P), such that α erases the modalities $\Box$ and $\Diamond$ from the sentences, and $\beta(M) = (W, R)$ such that $I_W = \{*\}$, $W^* = M$, $R = \{\langle *, *\rangle\}$.

3.15. Morphism POA → FOL
There exists a forgetful institution morphism **POA** → **FOL** which forgets the preorder structure both syntactically and semantically.

3.16. Morphism PL → IPL
There exists a canonical adjoint institution morphism **PL** → **IPL** which regards the standard two-elements Boolean algebra as a Heyting algebra.

3.17. Morphism MA → POA
Each multialgebra operation determines an ordinary algebra operation on the powerset of the carrier of the multialgebra. This determines a preordered algebra.

1. Adjust the concept of homomorphism of multialgebras such that the mapping of multialgebras to preorder algebras is functorial.
2. By mapping each preorder atom $t \leq t'$ to its corresponding inclusion sentence $t \prec t'$ and each equation $t = t'$ to the conjunction of the inclusions $t \prec t'$ and $t' \prec t$ we obtain an adjoint morphism of institutions **MA** → **POA**.

3.18. Morphism FOL → MA
Each **FOL**-model can be canonically regarded as a 'deterministic' multialgebra, i.e., in which all operations are deterministic. This determines an institution morphism **FOL** → **MA** which at the level of sentences maps both deterministic equations $t \doteq t'$ and inclusions $t \prec t'$ to equations $t = t'$.

3.19. Morphism CA → FOL
There exists a forgetful institution morphism **CA** → **FOL** mapping each contraction algebra to its underlying algebra and each equation $t = t'$ to the approximation equation $t \approx_0 t'$.

3.20. Morphism and comorphism MBA → FOL
Each membership algebraic signature (S, K, F, kind) determines a **FOL** signature (K, F, P) where $P = \{\_ :\ s \mid s \in S\}$ such that $\_ :\ s \in P_{\text{kind}(s)}$ for each sort s. This determines both an institution morphism and an institution comorphism **MBA** → **FOL** . (*Hint:* $\mathsf{Mod}^{\mathbf{MBA}}(S, K, F, \text{kind})$ is canonically isomorphic to $\mathsf{Mod}^{\mathbf{FOL}}(K, F, P)$ by mapping each (S, K, F, kind)-algebra A to the (K, F, P)-model with $A_{(\_ :\ s)} = A_s$, and $\mathsf{Sen}^{\mathbf{FOL}}(K, F, P)$ is canonically isomorphic to $\mathsf{Sen}^{\mathbf{MBA}}(S, K, F, \text{kind})$ by mapping each atomic relation $t :\ s$ to the atomic membership $t :\ s$ and by mapping equations to themselves.)

3.21. Comorphism AUT → FOL¹
Any set V determines a $\mathbf{FOL^1}$ signature $(F = V \uplus \{0\}, P = \{final\})$ such that $F_0 = \{0\}$, $F_1 = V$, and $P_1 = \{final\}$. This can be extended to a functor $\mathbb{S}et \to \mathbb{S}ig^{\mathbf{FOL^1}}$ which constitutes the signature functor Φ for a comorphism **AUT** → $\mathbf{FOL^1}$.

3.22. Comorphism HNK → HOL
The inclusion of model categories $\mathsf{Mod}^{\mathbf{HOL}}(S, F) \subseteq \mathsf{Mod}^{\mathbf{HNK}}(S, F)$ determines a canonical comorphism **HNK** → **HOL**. Note this does not have the flavor either of an 'embedding' or of an 'encoding'.

3.23. Comorphism FOEQL → HNK
Each algebraic signature (S, F) can be regarded as a **HOL**-signature by defining the type of σ as $s_1 \to (s_2 \to \ldots (s_n \to s) \ldots)$ for each operation symbol $\sigma \in F_{s_1 \ldots s_n \to s}$. Then each (S, F)-term $\sigma(t_1 \ldots t_n)$ can be mapped to its 'Polish prefix translation', the **HOL** (S, F)-term $\alpha(\sigma(t_1, \ldots, t_n)) = (\ldots (\sigma \alpha(t_1)) \ldots \alpha(t_n))$.

This determines a canonical institution comorphism **FOEQL** → **HNK**. By using the encoding of relations as operations, this can be extended to an institution comorphism **FOL** → **HNK**.

3.24. Comorphism HOL → HOL$_\lambda$
There is a 'natural embedding' comorphism from **HOL** to $\mathbf{HOL}_\lambda$, and also a similar comorphism from **HNK** to $\mathbf{HNK}_\lambda$, such that the translation α on the sentences is defined as follows:

- any equation $t = t'$ is mapped to the term $t = t'$ of Ω,
- $(\forall X)\rho$ is mapped to $\lambda X.\rho = \lambda X.true$,

- $\neg e$ is mapped to $e = \mathit{false}$,
- $e_1 \wedge e_2$ is mapped to $\langle e_1, e_2\rangle = \langle \mathit{true}, \mathit{true}\rangle$,

where *true* abbreviates $\lambda x{:}\Omega.x = \lambda x{:}\Omega.x$ and *false* abbreviates $(\forall x{:}\Omega)x$.

3.25. [19] **Comorphism PL → WPL**
Let *weak propositional logic* (denoted **WPL**) designate a variant of **PL**, where the sentences are the same as in **PL**, but the models are valuations $M : \mathsf{Sen}(P) \to \{0,1\}$ of *all* sentences that respect the usual truth table semantics of all the Boolean connectives except negation, for which they respect only one half of the usual condition:

- $M(\rho_1 \wedge \rho_2) = 1$ if and only if both $M(\rho_1) = 1$ and $M(\rho_2) = 1$, $M(\rho_1 \vee \rho_2) = 0$ if and only if both $M(\rho_1) = 0$ and $M(\rho_2) = 0$, $M(\rho_1 \Rightarrow \rho_2) = 1$ if and only if $M(\rho_1) = 0$ or $M(\rho_2) = 1$, and
- $M(\neg\rho) = 0$ if $M(\rho) = 1$.

There exists a comorphism **PL** → **WPL** such that the sentence translations are defined by $\alpha_P(\pi) = \pi$ for $\pi \in P$, $\alpha_P(\rho_1 \bowtie \rho_2) = \alpha_P(\rho_1) \bowtie \alpha_P(\rho_1)$ for $\bowtie \in \{\wedge, \vee, \Rightarrow\}$, and $\alpha_P(\neg\rho) = \alpha_P(\rho) \Rightarrow \neg\alpha_P(\rho)$ and such that the models are translated by $\beta_P(M') = \{\pi \mid M'(\pi) = 1\}$.

3.26. ***S*-sorted FOL**
For any fixed set S, let $\mathbf{FOL}^S = (\mathbb{S}ig^S, \mathsf{Sen}^S, \mathsf{Mod}^S, \models)$ be the institution of *S-sorted first order logic* defined as the sub-institution of **FOL** determined by the subcategory $\mathbb{S}ig^S$ of the signatures with S-sorted operation and relation symbols. (A signature in $\mathbb{S}ig^S$ is just a **FOL** signature (S, F, P), and a signature morphism φ in $\mathbb{S}ig^S$ is identity on the sort symbols, i.e., $\varphi^{\mathrm{st}} = 1_S$.)

1. Each function $u : S \to S'$ determines a canonical 'forgetful' adjoint institution morphism $(\Phi^u, \alpha^u, \beta^u) : \mathbf{FOL}^{S'} \to \mathbf{FOL}^S$ such that for each signature (S', F', P') of S'-sorted operation and relation symbols, $\Phi^u(S', F', P') = (S, F, P)$ with $F_{w\to s} = F'_{u(w)\to u(s)}$ and $P_w = P'_{u(w)}$ for each arity $w \in S^*$ and each sort symbol $s \in S$.
2. Describe the institution comorphism $(\overline{\Phi^u}, \overline{\alpha^u}, \overline{\beta^u}) : \mathbf{FOL}^S \to \mathbf{FOL}^{S'}$ adjoint to $(\Phi^u, \alpha^u, \beta^u)$. Show that $\overline{\alpha^u}$ is a bijection when u is injective.

3.27. Exercise 3.9 continued
For each institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ there exists a forgetful adjoint institution morphism $(\mathbb{S}ig^{\to}, \mathsf{Sen}^{\to}, \mathsf{Mod}^{\to}, \models^{\to}) \to (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ which maps each signature morphism $\varphi : \Sigma \to \Sigma'$ to its domain signature Σ.

3.28. [129] **Charter morphisms (Ex. 3.11 continued)**
Define the concept of *charter morphism* and show that there is a functor from the category of charters to $\mathbb{I}ns$. Does this have a left adjoint?

3.29. For any adjoint pair of institution morphism (Φ, α, β) and institution comorphism $(\overline{\Phi}, \overline{\alpha}, \overline{\beta})$ between the institutions I and I' corresponding to an adjunction $(\Phi, \overline{\Phi}, \zeta, \overline{\zeta})$ between their categories of signatures, the following squares commute:

$$\begin{tikzcd}
\mathsf{Sen}(\Sigma) \arrow[r, "\mathsf{Sen}(\varphi)"] \arrow[d, "\overline{\alpha}_\Sigma"'] & \mathsf{Sen}(\Phi(\Sigma')) \arrow[d, "\alpha_{\Sigma'}"] \\
\mathsf{Sen}'(\overline{\Phi}(\Sigma)) \arrow[r, "\mathsf{Sen}'(\overline{\varphi})"'] & \mathsf{Sen}'(\Sigma')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Mod}(\Sigma) & \mathsf{Mod}(\Phi(\Sigma')) \arrow[l, "\mathsf{Mod}(\varphi)"'] \\
\mathsf{Mod}'(\overline{\Phi}(\Sigma)) \arrow[u, "\overline{\beta}_\Sigma"] & \mathsf{Mod}'(\Sigma') \arrow[l, "\mathsf{Mod}'(\overline{\varphi})"] \arrow[u, "\beta_{\Sigma'}"']
\end{tikzcd}$$

for each signature morphism $\varphi : \Sigma \to \Phi(\Sigma')$ and $\overline{\varphi} : \overline{\Phi}(\Sigma) \to \Sigma'$ such that $\varphi = \zeta_\Sigma ; \Phi(\overline{\varphi})$.

3.4 Institutions as Functors

The definition of the concept of institution given above supports very well model theoretic intuitions. In this section we give an alternative more categorical definition for institutions.

Rooms. A *room* is a triple (S,M,R) such that S is a set, M is a category, and R is a function $|M| \to [S \to 2]$ where (as usual) $|M|$ is the class of the objects of M and $[S \to 2] = \mathbb{S}et(S,2) = \{f : S \to 2 \mid f \text{ function }\}$.

A *rooms morphism* $(s,m) : (S',M',R') \to (S,M,R)$ consists of a function $s : S \to S'$ and a functor $m : M' \to M$ such that the diagram below commutes:

$$\begin{array}{ccc} |M'| & \xrightarrow{R'} & [S' \to 2] \\ {\scriptstyle |m|}\downarrow & & \downarrow{\scriptstyle (s;-)} \\ |M| & \xrightarrow[R]{} & [S \to 2] \end{array}$$

where $|m|$ is the 'discretization' of m, i.e., the mapping on the objects given by m, and $(s;-)(f) = s;f$ for each function $f : S' \to 2$.

Let $\mathbb{R}oom$ be the category of rooms and their morphisms.

Proposition 3.7. *$\mathbb{R}oom$ has all small limits.*

Proof. Because $\mathbb{S}et^{\mathrm{op}}$ has all small limits (since $\mathbb{S}et$ has all small co-limits) and the (contravariant) homomorphism-functor $\mathbb{S}et(-,2) : \mathbb{S}et^{\mathrm{op}} \to \mathbb{S}et$ preserves them, by Proposition 2.3 we obtain that the comma-category $A/\mathsf{Sen}(-,2)$ has all small limits. Moreover, it is easy to see that for each function $f : A \to B$, the induced functor $B/\mathbb{S}et(-,2) \to A/\mathbb{S}et(-,2)$ preserves these limits.

This means that the indexed category $(|-|)/\mathbb{S}et(-,2) : \mathbb{C}at^{\mathrm{op}} \to \mathbb{C}at$ mapping each category M to $|M|/\mathbb{S}et(-,2)$ satisfies the hypotheses of the limit part of Thm. 2.10. It follows that its Grothendieck category $((|-|)/\mathbb{S}et(-,2))^\sharp$, which is just $\mathbb{R}oom$, has all small limits. □

Institutions as functors. Let $\mathbb{S}ig$ be any category and $I : \mathbb{S}ig^{\mathrm{op}} \to \mathbb{R}oom$ a functor. If we write

- $I(\Sigma) = (\mathsf{Sen}(\Sigma), \mathsf{Mod}(\Sigma), \models_\Sigma)$ for each object $\Sigma \in |\mathbb{S}ig|$, and
- $I(\varphi) = (\mathsf{Sen}(\varphi), \mathsf{Mod}(\varphi))$ for each arrow $\varphi \in \mathbb{S}ig$,

then it is easy to see that $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is an institution. The converse is also true, institutions are exactly the functors $\mathbb{S}ig^{\mathrm{op}} \to \mathbb{R}oom$.

Any functor $\Phi : \mathbb{S}ig' \to \mathbb{S}ig$ induces a canonical functor

$$(\Phi^{\mathrm{op}};-) : \mathbb{C}at((\mathbb{S}ig')^{\mathrm{op}}, \mathbb{R}oom) \to \mathbb{C}at(\mathbb{S}ig^{\mathrm{op}}, \mathbb{R}oom).$$

This gives an indexed category $\mathbb{C}at((-)^{\mathrm{op}}, \mathbb{R}oom) : \mathbb{C}at^{\mathrm{op}} \to \mathbb{C}at$.

Fact 3.8. *The category $\mathbb{I}ns$ of the institution morphisms is the Grothendieck category $\mathbb{C}at((-)^{\mathrm{op}}, \mathbb{R}oom)^\sharp$.*

Now we have already collected all necessary ingredients for showing easily the completeness property of $\mathbb{I}ns$.

Corollary 3.9. *$\mathbb{I}ns$ has all small limits.*

Proof. Because $\mathbb{R}oom$ has small limits (cf. Prop. 3.7), by Prop. 2.2 we have that each $\mathbb{C}at(\mathbb{S}ig^{\mathrm{op}}, \mathbb{R}oom)$ has all small limits. Note that each functor $(\Phi^{\mathrm{op}}; -)$ preserves these limits. Since $\mathbb{C}at$ has all small limits we can apply now the limit part of Thm. 2.10. $\square$

Exercises

3.30. [172, 81]

1. $\mathbb{R}oom$ has small co-limits.
2. $co\mathbb{I}ns$ has small limits.

3.31. [46] $\mathbb{R}oom$ can be enriched with a 2-categorical structure given by the natural transformations $m_1 \Rightarrow m_2$ between the 'model components' of room homomorphisms $(s_1, m_1), (s_2, m_2) \colon (S', M', R') \to (S, M, R)$. Then the 2-categorical structure on $\mathbb{I}ns$ given by the institution modifications arises as a Grothendieck 2-categorical construction.

3.32. [134] Each institution comorphism $(\Phi, \alpha, \beta) \colon (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', \models')$ determines a *span* of institution morphisms

$$(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \longleftarrow (\mathbb{S}ig, \Phi; \mathsf{Sen}', \Phi^{\mathrm{op}}; \mathsf{Mod}', \models) \longrightarrow (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', \models').$$

In a category $\mathbb{C}$, two spans $A \xleftarrow{f_1} B_1 \xrightarrow{g_1} A'$ and $A \xleftarrow{f_2} B_2 \xrightarrow{g_2} A'$ are *equivalent* when there exists an isomorphism $i \colon B_1 \to B_2$ such that $f_1 = i; f_2$ and $g_1 = i; g_2$. In any category with pullbacks, equivalence classes of spans can be composed as follows:

$$[A \xleftarrow{f} B \xrightarrow{g} A']; [A' \xleftarrow{f'} B' \xrightarrow{g'} A''] = [A \xleftarrow{h;f} B_0 \xrightarrow{h';g'} A'']$$

where $B \xleftarrow{h} B_0 \xrightarrow{h'} B'$ is a pullback of $B \xrightarrow{g} A' \xleftarrow{f'} B'$. This yields a category $span(\mathbb{C})$ having the same objects as $\mathbb{C}$ but (equivalence class of) spans as arrows. Show that the construction of a span of morphisms from an institution comorphism is *functorial*, i.e. it yields a functor $co\mathbb{I}ns \to span(\mathbb{I}ns)$.

Notes. The origins of institution theory are within the theory of algebraic specification, the seminal work being [75].

FOL was first presented as an institution in [75]. There are many approaches to partial algebra, two classical references being [29, 152], however it has been organized as the institution **PA** presented here in [133]. Preorder algebras (**POA**) are used for formal specification and verifications of algorithms [56], for automatic generation of case analysis [56], and in general for reasoning about transitions between states of systems. **POA** constitutes an unlabeled form of Meseguer's rewriting logic [124], but the latter fails to be an institution. The institution of multialgebras has been studied in [110]. Our multialgebra homomorphisms are called 'weak' homomorphisms in the literature, for alternative notions of multialgebra homomorphisms see [180]. Membership algebra has been introduced by [125]. Standard modal logic was first captured as an institution in [166]. Higher order logic with Henkin semantics has been introduced and studied in [33, 92], a recent book on the topic being [8]. Here, in the main text we consider a simplified variant close to the work of [126], while some exercises consider a more sophisticated version containing products and λ-abstraction in a variant very close to the original one. Contraction algebras have been introduced in [40] in the context of the extension of logic programming to infinite terms. The institution *Cat*$\mathbf{EQL}$ of categorical equational logic is a slightly more abstract version of the institution of the so-called 'category-based equational logic' of [71, 42].

Formal intuitionistic logic was developed by seminal works [70, 96, 105] while Heyting algebras emerged from the so-called 'closure algebras' or 'Browerian algebras' investigated by McKinsey and Tarski. A categorical approach to intuitionistic logic can be found in [109].

Due to its abstract definition, institutions may accommodate examples which might appear as 'non-logical', at least in the conventional sense. While some of them are only mildly 'non-logical' (automata, linear algebra), much less conventional examples appear in myriad ways including abstract constructions on (already existing) institutions.

A very brief list of logics from formal specification theory that have been captured as institutions but have not been presented here includes polymorphic [162], temporal [66], process [66], behavioral [23], coalgebraic [34], object-oriented [76] logics.

Institution morphisms were introduced in [75], while comorphisms were studied later under the name of "plain map" in [123] or "representation" in [171, 173]. The literature studies many other types of mappings between institutions, each of them playing a specific role in applications. The name "comorphism" was introduced by [81]. The duality between institution morphisms and comorphisms was established in [9]. In [44, 46, 57] institution adjoints are called "embeddings". Notice also that institution adjoint morphism or comorphism are *not* adjunctions in the 2-categorical sense. Equivalences of institutions have been introduced in [137].

The presentation of institutions as functors was given already in [75] and the 2-categorical structure of the category of institutions has been studied in [46].

Completeness of $\mathbb{I}ns$ was first obtained by Tarlecki in [168] and of $co\mathbb{I}ns$ in [172]. Both results have been re-done by Roşu using Kan extensions [154]. Cocompleteness fails for both $\mathbb{I}ns$ and $co\mathbb{I}ns$ due to foundational issues (see [81] for a counterexample originally due to Tarlecki) but it can still be recovered under the condition that the categories of the signatures are small.

Chapter 4

Theories and Models

In this chapter we develop some fundamental institution theoretic concepts that play an important role for our institution-independent approach to model theory.

A simple abstract construction is that of the (theory) presentations of a given base institution as the signatures for a new set of institutions. This is very useful for institution encodings supporting the transfer of model theoretic properties between institutions.

Theory co-limits are especially useful in formal specification theory since they provide support for advanced modularization techniques of software systems. They are also required within the context of some institution encodings.

Model amalgamation, here introduced as a limit preservation property of the model functor, is the institutional property which is required by almost all institution-independent model theoretic developments. Even the satisfaction condition of institution capturing logics with quantifiers may rely upon a form of model amalgamation. The institution theoretic concept of model amalgamation is a rather basic property of institutions which should not be confused with the existence of a common (elementary) extension of models, a much harder property playing an important role in conventional model theory.

The method of diagrams pervades a large part of conventional model theory and, in its abstract institution-independent form, also of the results presented in this book. At the level of abstract institutions this appears as a coherence property between syntactic and semantic sides of the institution, which gets a simple categorical formulation. As a simple consequence of the method of diagrams, we develop a general result about existence of limits and co-limits of models.

The concepts of 'sub-model' and 'quotient model' are handled by the so-called 'inclusion systems', which constitutes a categorical abstraction of their basic factorization property. Another rather different application domain for inclusion systems are the signature morphisms, which is especially relevant for the studies of modularization properties of formal specification.

The last topic of this chapter is that of free constructions of models along signature morphisms, called 'liberality' in institution theory. Liberality is intimately related to good computational properties of the actual institutions and it plays a crucial role for the semantics of abstract data types and of logic programming. In its simple form, liberality means the existence of initial models for theories, a property which holds for Horn theories.

4.1 Theories and Presentations

A Galois connection between syntax and semantics. Let Σ be a signature in an institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$. Then

- for each set of Σ-sentences E, let $E^* = \{M \in \mathsf{Mod}(\Sigma) \mid M \models_\Sigma e \text{ for each } e \in E\}$, and
- for each class $\mathbb{M}$ of Σ-models, let $\mathbb{M}^* = \{e \in \mathsf{Sen}(\Sigma) \mid M \models_\Sigma e \text{ for each } M \in \mathbb{M}\}$.

For any individual sentence or model X, by X^* we mean $\{X\}^*$. These two functions, denoted "$(-)^*$", form what is known as a *Galois connection* (see Sect. 2.3), in that they satisfy the following easy-to-check properties for any collections E, E' of Σ-sentences and collections $\mathbb{M}, \mathbb{M}'$ of Σ-models:

1. $E \subseteq E'$ implies $E'^* \subseteq E^*$.
2. $\mathbb{M} \subseteq \mathbb{M}'$ implies $\mathbb{M}'^* \subseteq \mathbb{M}^*$.
3. $E \subseteq E^{**}$.
4. $\mathbb{M} \subseteq \mathbb{M}^{**}$.

Closed classes of models $\mathbb{M} = \mathbb{M}^{**}$ are called *elementary* and closed sets of sentences $E = E^{**}$ are called *theories*.

The above properties 1–4 imply quite immediately the following properties:

5. $E^* = E^{***}$.
6. $\mathbb{M}^* = \mathbb{M}^{***}$.
7. There is a dual (i.e., inclusion reversing) isomorphism between the closed collections of sentences and the closed collections of models.

When E and E' are sets of sentences, $E^* \subseteq E'^*$ is denoted by $E \models E'$. Two sentences e and e' of the same signature are *semantically equivalent* (denoted as $e \models\!\mid e'$) if they are satisfied by the same class of models, i.e., $\{e\} \models \{e'\}$ and $\{e'\} \models \{e\}$.

Two models M and M' of the same signature are *elementarily equivalent* (denoted as $M \equiv M'$) if they satisfy the same set of sentences, i.e., $M \models \rho$ if and only if $M' \models \rho$ for each sentence ρ of the signature.

Presentations. A theory E is *presented by* a set of sentences E_0 if $E_0 \subseteq E$ and $E_0 \models E$, and is *finitely presented* if there exists a finite E_0 which presents E. A *presentation* is thus a pair (Σ, E) consisting of a signature Σ and a set E of Σ-sentences.

A *presentation morphism* $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is a signature morphism such that $\varphi(E) \subseteq E'^{**}$. A presentation morphism between theories is called *theory morphism*. Note therefore that a theory morphism $\varphi\colon\ (\Sigma,E)\to(\Sigma',E')$ is a signature morphism such that $\varphi(E)\subseteq E'$.

Proposition 4.1. *In any institution I, the presentation morphisms, respectively the theory morphisms, form a category (denoted $\mathbb{P}res^I$, respectively $\mathbb{T}h^I$) with the composition inherited from the category of the signatures $\mathbb{S}ig^I$. Moreover, $\mathbb{P}res^I$ and $\mathbb{T}h^I$ are equivalent categories.*

Proof. That composition of presentation morphisms is a presentation morphism follows by simple calculations using the observation that $\varphi(E^{**})\subseteq\varphi(E)^{**}$ for each signature morphism $\varphi\colon\ \Sigma\to\Sigma'$ and each set E of Σ-sentences.

The equivalence between $\mathbb{T}h$ and $\mathbb{P}res$ is defined by the forgetful inclusion functor $\mathbb{T}h\hookrightarrow\mathbb{P}res$ and the functor $\mathbb{P}res\to\mathbb{T}h$ mapping each presentation (Σ,E) to its semantic closure (Σ,E^{**}). $\square$

In other logic or model theory works our presentations are called ‘theories’ and our theories are called ‘closed theories’. Here we prefer to stick to the original institutional terminology. One thing Prop. 4.1 tells us is that in general there is no difference between working with presentations or with theories. However the formulation of some few results require theories rather than presentations.

The institution of the presentations. The model functor Mod of an institution can be extended from the category of its signatures $\mathbb{S}ig$ to a model functor from the category of its presentations $\mathbb{P}res$, by mapping a presentation (Σ,E) to the full subcategory $\mathsf{Mod}^{\mathrm{pres}}(\Sigma,E)$ of $\mathsf{Mod}(\Sigma)$ consisting of all Σ-models satisfying E. The correctness of the definition of $\mathsf{Mod}^{\mathrm{pres}}$ is guaranteed by the satisfaction condition of the base institution; this is easy to check. This leads to the *institution of presentations* $I^{\mathrm{pres}}=(\mathbb{S}ig^{\mathrm{pres}},\mathsf{Sen}^{\mathrm{pres}},\mathsf{Mod}^{\mathrm{pres}},\models^{\mathrm{pres}})$ over the base institution $I=(\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models)$ where

– $\mathbb{S}ig^{\mathrm{pres}}$ is the category $\mathbb{P}res$ of presentations of I,

– $\mathsf{Sen}^{\mathrm{pres}}(\Sigma,E)=\mathsf{Sen}(\Sigma)$, and

– for each (Σ,E)-model M and Σ-sentence e, $M\models^{\mathrm{pres}}_{(\Sigma,E)} e$ if and only if $M\models_{\Sigma} e$.

This construction is very useful for institution encodings. Often, comorphisms encoding ‘complex’ institutions into ‘simpler’ ones map a signature of the ‘complex’ institution to a *presentation* of the ‘simpler’ institution. As comorphisms usually correspond to embeddings, from the point of view of the structural complexity of institutions this is a quite expected cost, since such difference of complexity has to show up somewhere. The rest of this section is devoted to examples of such encodings. The reader is invited to complete the definitions given and to check all the details of each of these examples, including their satisfaction condition.

Encoding many-sorted logic into single-sorted logic

This is a comorphism $(\Phi,\alpha,\beta): \ \mathbf{FOL} \to (\mathbf{FOL^1})^{\text{pres}}$ defined as follows:

- A many-sorted signature (S,F,P) gets mapped to the single-sorted presentation $((\overline{F},\overline{P}\cup\{(-:\ s) \mid s\in S\}),\Gamma_{(S,F,P)})$ where
  - for each natural number n, $\overline{F}_n = \{\sigma \in F_{w\to s} \mid |ws| = n\}$ and $\overline{P}_n = \{\sigma \in P_w \mid |w| = n\}$ (here by $|w|$ we denote the length of the string w),
  - $\Gamma_{(S,F,P)} = \{(\forall x_1 \dots x_n)\bigwedge_{i\le n}(x_i:\ s_i) \Rightarrow (\sigma(x_1\dots x_n):\ s) \mid \sigma \in F_{s_1\dots s_n\to s}\}$.
- On the sentence side:
  - any equation $t = t'$ gets mapped to itself,
  - α commutes with the Boolean connectives, i.e., $\alpha(\rho_1\wedge\rho_2) = \alpha(\rho_1)\wedge\alpha(\rho_2)$, etc.,
  - any sentence of the form $(\forall x)\rho$ gets mapped to $(\forall x)(x:\ s)\wedge\alpha(\rho)$ with s being the sort of the variable x.
- On the models side, for each (S,F,P)-model M
  - $\beta(M)_s = M_{(-:\ s)}$ for each sort s,
  - for each operation symbol $\sigma\in F$, $\beta(M)_\sigma$ is the restriction of M_σ to $\beta(M)_w$, and
  - $\beta(M)_\pi = \beta(M)_w \cap M_\pi$ for each relation symbol $\pi\in P$.

This comorphism may give an insight into why and how the single-sorted approach of conventional mathematical practice works in spite of the fact that mathematical realities constitute a many-sorted heterogeneous rather than a single-sorted homogeneous framework.

Encoding operations as relations in FOL

This is a comorphism $(\Phi,\alpha,\beta): \ \mathbf{FOL} \to \mathbf{REL}^{\text{pres}}$ defined as follows:

- Each **FOL** signature (S,F,P) gets mapped to a **REL**-presentation $((S,C(F),\overline{F}\uplus P),rel_{(S,F,P)})$ where
  - $C(F)$ is the set of the constants of F,
  - $\overline{F}_s = \emptyset$ for each sort $s\in S$ and $\overline{F}_{ws} = F_{w\to s}$ when w is non-empty, and
  - $rel_{(S,F,P)} =$
    $\{((\forall X)(\exists y)\sigma(X,y))\wedge((\forall X)(\forall y)(\forall y')\sigma(X,y)\wedge\sigma(X,y')\Rightarrow(y=y')) \mid \sigma\in\overline{F}\}$.
- On the sentence side:
  - $x = y$ gets mapped to itself when both x and y are constants,
  - $\alpha(\sigma(t_1,\dots,t_n) = y) = (\exists\{x_1,\dots,x_n\})(\sigma(x_1,\dots,x_n)\wedge\bigwedge_{1\le i\le n}\alpha(t_i = x_i))$ for each operation symbol σ, appropriate list of terms $t_1,\dots,t_n$ and $x_1,\dots,x_n$ (new) constants,
  - $\alpha(t_1 = t_2) = (\exists y)(\alpha(t_1 = y)\wedge\alpha(t_2 = y))$ for any terms t_1 and t_2 of the same sort and y (new) constant,

- $\alpha(\pi(t_1,\dots,t_n)) = (\exists\{x_1,\dots,x_n\})(\pi(x_1,\dots,x_n) \wedge \bigwedge_{i\leq n}\alpha(t_i = x_i))$ for each relational atom $\pi(t_1,\dots,t_n)$, and
- α commutes with the Boolean connectives and with the quantifiers.

- On the models side, for each $((S,C(F),\overline{F} \uplus P), rel_{(S,F,P)})$-model M,
  - for each relation symbol $\sigma \in \overline{F}$, $\beta(M)_\sigma(m) = y$ if and only if $\langle m, y\rangle \in M_\sigma$.

This encoding goes in a sense opposite to the encoding of the relations as operations presented in Sect. 3.3, that one being quite exceptional since it does map the signatures to signatures rather than to proper presentations. In that case the difference in structural complexity, which is rather slight, still shows up at the level of the signatures.

Encoding partial operations as total operations

The so-called '*operational encoding*' of **PA** into **FOL** is a comorphism **PA** $\to$ **FOL**$^{\mathrm{pres}}$ defined as follows:

- Each **PA**-signature (S,TF,PF) gets mapped to the **FOL**-presentation $((S,TF \cup PF,\{D_s\}_{s\in S}),\Gamma_{(S,TF,PF)})$ where
  - for each sort $s \in S$, D_s is a relation symbol of arity s,

  and $\Gamma_{(S,TF,PF)}$ consists of the Horn sentences
  - $(\forall X)D_s(\sigma(X)) \Rightarrow D_w(X)$ for each $\sigma \in (TF \cup PF)_{w\to s}$, and
  - $(\forall X)D_w(X) \Rightarrow D_s(\sigma(X))$ for each $\sigma \in TF_{w\to s}$

  (where $D_w(X)$ denotes $\bigwedge_{(x:\ s)\in X} D_s(x)$).

- On the sentence side:
  - $\alpha(t \stackrel{e}{=} t') = (D_s(t) \wedge (t = t'))$,
  - α commutes with the Boolean connectives, and
  - $\alpha((\forall X)\rho) = (\forall X)\alpha(\rho)$ for each sentence ρ.

- Each (total) $((S,TF \cup PF,D),\Gamma_{(S,TF,PF)})$-model M gets mapped to the partial (S,TF,PF)-algebra $\beta(M)$ such that
  - $\beta(M)_s = M_{D_s}$ for each sort s, and
  - for each operation $\sigma :\ s_1 \dots s_n \to s$, $\beta(M)_\sigma$ is the 'restriction' of M_σ to $M_{D_{s_1}} \times \cdots \times M_{D_{s_n}}$ and 'co-restriction' to M_{D_s}. (Note that if $\sigma \in PF$ this restriction may be partial in order to give results in M_{D_s}.)

Encoding partial operations as relations

Another comorphism **PA** $\to$ **FOL**$^{\mathrm{pres}}$, which may be called the '*relational encoding*' of **PA** into **FOL**, encodes the partial operations as relations as follows:

- Each **PA**-signature (S,TF,PF) gets mapped to the **FOL** presentation $((S,TF,\overline{PF}),$ $\Gamma_{(S,TF,PF)})$ such that $\overline{PF}_{ws} = PF_{w\to s}$ for each $w \in S^*$ and $s \in S$, and

$$\Gamma_{(S,TF,PF)} = \{(\forall X \uplus \{y,z\})\sigma(X,y) \wedge \sigma(X,z) \Rightarrow (y = z) \mid \sigma \in PF\}.$$

- Each $(S,TF,\overline{PF})$-model M gets mapped to the partial (S,TF,PF)-algebra $\beta(M)$ such that
  - $\beta(M)_x = M_x$ for each $x \in S$ or $x \in TF$,
  - for each $\sigma \in PF$, if $(m,m_0) \in M_\sigma$ then $\beta(M)_\sigma(m) = m_0$.
- α commutes with the quantifiers and the Boolean connectives, and

$$\alpha(t \stackrel{e}{=} t') = (\exists X \uplus \{x_0\})bind(t,x_0) \wedge bind(t',x_0)$$

  where for each (S,TF,PF)-term t and variable x, $bind(t,x)$ is a (finite) conjunction of atoms defined by

$$bind(\sigma(t_1 \dots t_n),x) = \bigwedge_{1 \le i \le n} bind(t_i,x_i) \wedge \begin{cases} \sigma(x_1,\dots,x_n) = x & \text{when } \sigma \in TF \\ \sigma(x_1,\dots,x_n,x) & \text{when } \sigma \in PF \end{cases}$$

  and X is the set of the new constants introduced by $bind(t,x_0)$ and $bind(t',x_0)$.

The proof of the Satisfaction Condition relies upon the fact that

$$M \models (\exists X \uplus \{x_0\})bind(t,x_0) \text{ if and only if } \beta(M)_t = M'_{x_0}$$

where M' is the unique expansion of M that satisfies $bind(t,x_0)$.

Exercises

4.1. In any institution, for any signature Σ

- $(\bigcup_{i\in I} E_i)^* = \bigcap_{i\in I} E_i^*$ for each family of sets of Σ-sentences $\{E_i\}_{i\in I}$, and
- $(\bigcup_{i\in I} \mathbb{M}_i)^* = \bigcap_{i\in I} \mathbb{M}_i^*$ for each family of classes of Σ-models $\{\mathbb{M}_i\}_{i\in I}$.

4.2. For a fixed signature, any (possibly infinite) intersection of theories is a theory.

4.3. Strong theory morphisms
A theory morphism $\varphi\colon (\Sigma,E) \to (\Sigma',E')$ is *strong* when $E' = \varphi(E)^{**}$. Strong theory morphisms are closed under composition.

4.4. Given a signature morphism $\varphi\colon \Sigma \to \Sigma'$ in any institution

- for each E_1 and E_2 sets of Σ-sentences, $E_1 \models_\Sigma E_2$ implies $\varphi(E_1) \models_{\Sigma'} \varphi(E_2)$,
- for each set E of Σ-sentences, $\varphi(E)^* = \mathsf{Mod}(\varphi)^{-1}(E^*)$, and
- $(\Sigma,\varphi^{-1}(E'))$ is theory when (Σ',E') is theory.

4.5. [113] **Semantic topology**
Recall that a *topology* (X,τ) consists of a set X and a set τ of subsets of X such that $\emptyset, X \in \tau$ and τ is closed under finite intersections and (possibly infinite) unions. Then for each signature Σ of any institution, the class of $|\mathsf{Mod}(\Sigma)|$ of all Σ-models admits a natural *semantic topology*

$$\tau_\Sigma = \{\bigcup_{i\in I} E_i^* \mid \{E_i\}_{i\in I} \text{ family of finite sets of } \Sigma\text{-sentences}\}.$$

Recall also that given two topologies (X,τ) and (X',τ') a function $f\colon X \to X'$ is *continuous* when $f^{-1}(U') \in \tau$ for all $U' \in \tau'$. Then $\mathsf{Mod}(\varphi)\colon (|\mathsf{Mod}(\Sigma')|,\tau_{\Sigma'}) \to (|\mathsf{Mod}(\Sigma)|,\tau_\Sigma)$ is continuous for each signature morphism $\varphi\colon \Sigma \to \Sigma'$.

4.6. For any institution morphism $(\Phi,\alpha,\beta)\colon I' \to I$, $(\Phi(\Sigma'),\alpha_{\Sigma'}^{-1}(E'))$ is a theory in I for each theory (Σ',E') of I'.

4.7. In **AUT** any set of sentences is a theory. (*Hint:* each language can be represented as a (possibly infinite) intersection of regular languages.)

4.8. In any institution I the forgetful functor $\mathbb{P}res \to \mathbb{S}ig$ determines a canonical institution adjoint morphism $I^{\mathrm{pres}} \to I$. Moreover

- $(I^{\mathrm{pres}})^{\mathrm{pres}} \to I^{\mathrm{pres}}$ is an equivalence of institutions, and
- $(-)^{\mathrm{pres}}\colon \mathbb{I}ns \to \mathbb{I}ns$ is a functor mapping each institution morphism (Φ,α,β) to the institution morphism $(\Phi^{\mathrm{pres}},\alpha^{\mathrm{pres}},\beta^{\mathrm{pres}})$ such that $\Phi^{\mathrm{pres}}(\Sigma',E') = (\Phi(\Sigma'),\alpha_{\Sigma'}^{-1}(E'^{**}))$ and α^{pres} and β^{pres} being the restrictions of α and β.

4.9. Comorphism POA $\to$ FOL$^{\mathrm{pres}}$
There exists a comorphism **POA** $\to$ **FOL**$^{\mathrm{pres}}$ mapping each algebraic signature (S,F) to the **FOL**-presentation $((S,F,\{\leq_s\}_{s\in S}), pre_{(S,F)})$ such that

- for each sort symbol $s \in S$ the arity of $\leq_s$ is ss, and
- $pre_{(S,F)}$ contains the preorder axioms for each $\leq_s$ and all axioms stating that the preorder functoriality of the operations of F.

4.10. Comorphism IPL $\to$ (FOEQL^1)$^{\mathrm{pres}}$
There exists a comorphism $(\Phi,\alpha,\beta)\colon$ **IPL** $\to$ (**FOEQL**^{1})$^{\mathrm{pres}}$ such that:

- Let (H,E) be the single-sorted equational clause presentation of the Heyting algebras with $H_0 = \{\top,\bot\}$, $H_1 = \{\neg\}$, and $H_2 = \{\wedge,\vee,\Rightarrow\}$ (otherwise $H_n = \emptyset$). Each set (= **IPL**-signature) P gets mapped to the presentation $(H \uplus P, E)$ where P are added to H as constants.
- $\alpha_P(\rho) = (\rho = \top)$ for each **IPL**-signature P and each P-sentence ρ.
- For each **IPL**-signature P and each $(H\uplus P,E)$-algebra A, $\beta_P(A) = M$ where $M\colon P \to A\restriction_H$ is defined by $M(\pi) = A_\pi$ for each $\pi \in P$.

4.11. Comorphism HNK $\to$ FOEQL$^{\mathrm{pres}}$
There exists a comorphism $(\Phi,\alpha,\beta)\colon$ **HNK** $\to$ **FOEQL**$^{\mathrm{pres}}$ such that

- Each **HNK**-signature (S,F) gets mapped to the presentation $((\overrightarrow{S},\overrightarrow{F}),\Gamma_{(S,F)})$ where
  - $\overrightarrow{S}$ is the set of all S-types,
  - $\overrightarrow{F}_s = F_s$ for each $s \in \overrightarrow{S}$, $\overrightarrow{F}_{[(s\to s')s]\to s'} = \{\mathsf{app}_{s,s'}\}$ for all $s,s' \in \overrightarrow{S}$ and $\overrightarrow{F}_{w\to s} = \emptyset$ otherwise,
  - $\Gamma_{(S,F)} = \{(\forall f)(\forall g)((\forall x)\mathsf{app}_{s,s'}(f,x) = \mathsf{app}_{s,s'}(g,x)) \Rightarrow (f = g) \mid s,s' \in \overrightarrow{S}\}$.

- $\beta_{(S,F)}(M) = \overline{M}$ where $\overline{M}$ is the inductively (on the structure of the types) defined **HNK**-model such that there exists an isomorphism $\mathrm{fun}^M : M \to \overline{M}$ (here $\overline{M}$ is canonically regarded as a **FOL** $((\overrightarrow{S}, \overrightarrow{F}), \Gamma_{(S,F)})$-model with app interpreted as ordinary functional application) with fun^M_s being identities for $s \in S$.
  Then $\beta_{(S,F)}$ is an equivalence of categories with an 'inverse' $\overline{\beta}_{(S,F)}$ such that $\overline{\beta}_{(S,F)};\beta_{(S,F)} = 1$ and $\mathrm{fun} : 1 \xrightarrow{\cong} \beta_{(S,F)};\overline{\beta}_{(S,F)}$.
- α is defined as the canonical extension of the mapping on the terms α^{tm} defined by $\alpha^{\mathrm{tm}}(tt') = \mathsf{app}(\alpha^{\mathrm{tm}}(t), \alpha^{\mathrm{tm}}(t'))$.

4.12. Comorphism $\mathbf{HOL}_\lambda \to \mathbf{HOL}^{\mathrm{pres}}$
There is an 'encoding' comorphism from $\mathbf{HOL}_\lambda$ to $\mathbf{HOL}^{\mathrm{pres}}$. (*Hints*: A $\mathbf{HOL}_\lambda$-signature (S,F) is mapped to a **HOL**-presentation that extends (S,F) with an axiomatization of Ω, product types and pairing functions. λ-abstraction is coded in an innermost way by appropriate existential quantification over functions. $\lambda x{:}s.t$ is just coded as f, where $\exists f{:}s \to s'.\forall x{:}s.f(x) = t \wedge \cdots$ is added at an appropriate place.)

This 'encoding' comorphism can also be modified into a comorphism from $\mathbf{HNK}_\lambda$ to $\mathbf{HNK}^{\mathrm{pres}}$. (*Hint*: It must additionally be ensured that all λ-terms have a denotation. This can be expressed by appropriate existential statements.)

4.13. Comorphism $\mathbf{LA} \to (\mathbf{FOEQL^1})^{\mathrm{pres}}$
Let $\mathbf{FOEQL^1}$ be the single-sorted variant of **FOEQL**. There exists an institution comorphism $\mathbf{LA} \to (\mathbf{FOEQL^1})^{\mathrm{pres}}$ mapping each commutative ring $R = (|R|, +, -, \times, 0)$ to the presentation (F_R, E_R) where

- $(F_R)_0 = \{0\}$, $(F_R)_1 = |R| \uplus \{-\}$, $(F_R)_2 = \{+\}$, and $(F_R)_n = \emptyset$ otherwise, and
- E_R consists of the axioms for the commutative group for $\{+,-,0\}$ and $(\forall x)\ r(r'(x)) = (r \times r')(x)$, $(\forall x)\ (r+r')(x) = r(x) + r'(x)$, $(\forall x)\ (-r)(x) = -r(x)$, and $(\forall x) 0(x) = 0$ for each $r, r' \in |R|$ elements of the ring R.

4.14. [136] **Comorphism $\mathbf{WPL} \to \mathbf{PL}^{\mathrm{pres}}$ (see Ex. 3.25)**
For each set P (of propositional variables) let us consider $\mathsf{Sen}(P)$ as a **PL** signature and let Γ_P be the specification of the **WPL** semantics, i.e., $\Gamma_P = \{[\rho_1 \bowtie \rho_2] \Leftrightarrow ([\rho_1] \bowtie [\rho_2]) \mid \bowtie \in \{\wedge, \vee, \Rightarrow\}$ and $\rho_1, \rho_2 \in \mathsf{Sen}(P)\} \cup \{[\rho] \Rightarrow \neg[\neg\rho] \mid \rho \in \mathsf{Sen}(P)\}$, where, in order to avoid confusion, by $[\rho]$ we denote the **WPL**-sentence ρ regarded as a propositional variable of the **PL** signature $\mathsf{Sen}(P)$. The mapping of P to $(\mathsf{Sen}(P), \Gamma_P)$ determines a comorphism $\mathbf{WPL} \to \mathbf{PL}^{\mathrm{pres}}$.

4.15. [110] **Comorphism $\mathbf{PA} \to \mathbf{MA}^{\mathrm{pres}}$**
This comorphism is defined by mapping each **PA** signature (S, TF, PF) to a **MA** presentation $((S, TF \cup PF), \Gamma_{(S,TF,PF)})$ such that

- $\Gamma_{(S,TF,PF)} = \{(\forall y)(\forall X)(y \doteq y) \wedge (y \prec \sigma(X) \Rightarrow \sigma(X) \doteq \sigma(X)) \mid \sigma \in TF \cup PF\}$,
- $\alpha(t \overset{e}{=} t') = (t \doteq r')$, α commutes with the Boolean connectives and $\alpha((\forall X)\rho) = (\forall X)((X \doteq X) \wedge \alpha(\rho))$,
- $\beta(A)_s = A_s$ for each sort $s \in S$ and $\beta(A)_\sigma(a_1, \ldots, a_n) = a$ when $A_\sigma(a_1, \ldots, a_n) = \{a\}$, otherwise it is undefined.

4.16. [110] **Comorphism $\mathbf{MBA} \to \mathbf{MA}^{\mathrm{pres}}$**
Each **MBA** signature (S, K, F, kind) can be mapped to the **MA** presentation $((K, F \cup \{p_s \mid s \in S\}), \Gamma_{(S,K,F,\mathrm{kind})})$, where p_s are constants of sort $\mathrm{kind}(s)$ and

$$\Gamma_{(S,K,F,\mathrm{kind})} = \{(\forall X)(X \doteq X) \Rightarrow (\sigma(X) \doteq \sigma(X)) \mid \sigma \in F\}.$$

This determines a comorphism $\mathbf{MBA} \to \mathbf{MA}^{\mathrm{pres}}$.

4.2 Theory (co-)limits

The following simple but very useful important result shows that limits and co-limits of presentations exist in dependence on limits and co-limits of signatures.

Proposition 4.2. *In any institution, the forgetful functor $U : \mathbb{P}res \to \mathbb{S}ig$ lifts limits and co-limits. Moreover, the forgetful functor $\mathbb{T}h \to \mathbb{S}ig$ lifts them uniquely.*

Proof. Consider a functor $D : J \to \mathbb{P}res$. When $\mu : D;U \Rightarrow \Sigma$ is a co-limit co-cone in $\mathbb{S}ig$, then by a simple checking we obtain that $\mu : D \Rightarrow (\Sigma, (\bigcup_{i\in|J|} \mu_i(E_i)))$ is a co-limit co-cone in $\mathbb{P}res$, where $D(i) = (\Sigma_i, E_i)$ is the presentation corresponding to i for each $i \in |J|$.

$$\begin{tikzcd}
\Sigma_i \arrow[rr, "D(u)"] \arrow[dr, "\mu_i"'] & & \Sigma_j \arrow[dl, "\mu_j"] \\
& \Sigma &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\Sigma_i, E_i) \arrow[rr, "D(u)"] \arrow[dr, "\mu_i"'] & & (\Sigma_j, E_j) \arrow[dl, "\mu_j"] \\
& (\Sigma, \bigcup_{i\in|J|} \mu_i(E_i)) &
\end{tikzcd}$$

Similarly, when $\mu : \Sigma \Rightarrow D;U$ is a limit cone in $\mathbb{S}ig$, then $\mu : (\Sigma, \bigcap_{i\in|J|} \mu_i^{-1}(E_i)) \Rightarrow D$ is a limit co-cone in $\mathbb{P}res$. $\square$

Corollary 4.3. *In any institution, the category $\mathbb{P}res$ of its presentations, respectively $\mathbb{T}h$ of its theories, has whatever limits or co-limits its category $\mathbb{S}ig$ of the signatures has.*

Limits and co-limits of FOL signatures

We can apply Prop. 4.2 through Cor. 4.3 to show that **FOL** has small limits and co-limits of presentations or theories by proving that the category $\mathbb{S}ig^{\mathbf{FOL}}$ of **FOL**-signatures has small limits and co-limits. The arguments of the proof of the result below can be repeated with some adjustments in form to many other many-sorted institutions.

Proposition 4.4. *The category of* **FOL** *signatures has small limits and co-limits.*

Proof. Given a any set S, because $\mathbb{S}et$ has all small limits and co-limits, cf. Prop. 2.2, the functor categories $\mathbb{C}at(S^* \times S, \mathbb{S}et)$ and $\mathbb{C}at(S^*, \mathbb{S}et)$ have small limits and co-limits too. So do their products $\mathbb{C}at(S^* \times S, \mathbb{S}et) \times \mathbb{C}at(S^*, \mathbb{S}et)$ (by calculating (co-)limits componentwise).

Each function $f : S \to S'$ determines a functor $\mathbb{C}at(S'^* \times S', \mathbb{S}et) \times \mathbb{C}at(S'^*, \mathbb{S}et) \to \mathbb{C}at(S^* \times S, \mathbb{S}et) \times \mathbb{C}at(S^*, \mathbb{S}et)$ by composition to the left with $(f^* \times f, f^*)$. This functor has

- a left adjoint mapping each (F,P) to (F',P') such that $F'_{w'\to s'} = \uplus\{F_{w\to s} \mid f(ws) = w's'\}$ and $P'_{w'} = \uplus\{P_w \mid f(w) = w'\}$, and
- a right adjoint mapping each (F,P) to (F'',P'') such that $F''_{w'\to s'} = \prod\{F_{w\to s} \mid f(ws) = w's'\}$ and $P''_{w'} = \prod\{P_w \mid f(w) = w'\}$.

From Prop. 2.6 we know that a right adjoint preserves all limits, thus the hypotheses of Thm. 2.10 are fulfilled for the indexed category $\mathbb{S}et^{\mathrm{op}} \to \mathbb{C}at$ mapping each set S to $\mathbb{C}at(S'^* \times S', \mathbb{S}et) \times \mathbb{C}at(S'^*, \mathbb{S}et)$. It follows that its Grothendieck category, which is exactly $\mathbb{S}ig^{\mathbf{FOL}}$, has small limits and co-limits. $\square$

The above proof is highly conceptual. It is also useful to understand in more concrete terms how the limits and the co-limits of **FOL** signatures are constructed.

Limits. For any small category J, the limit cone $\mu\colon\ (S,F,P) \Rightarrow (D\colon\ J \to \mathbb{S}ig)$ (where $D(u)\colon\ (S_i,F_i,P_i) \to (S_j,F_j,P_j)$ for each $u \in J(i,j)$) is defined by

1. μ^{st} is the limit of $D;(-)^{\mathrm{st}}\colon\ J \to \mathbb{S}et$

$$\begin{tikzcd} S_i \arrow[rr, "D(u)^{\mathrm{st}}"] & & S_j \\ & S \arrow[ul, "\mu_i^{\mathrm{st}}"] \arrow[ur, "\mu_j^{\mathrm{st}}"'] & \end{tikzcd}$$

2. Each arity $w \in S^*$ and each sort $s \in S$ determine a functor $J \to \mathbb{S}et$ mapping each arrow $u \in J(i,j)$ to $D(u)^{\mathrm{op}}_{\mu_i^{\mathrm{st}}(w)\to\mu_i^{\mathrm{st}}(s)}\colon\ (F_i)_{\mu_i^{\mathrm{st}}(w)\to\mu_i^{\mathrm{st}}(s)} \to (F_j)_{\mu_j^{\mathrm{st}}(w)\to\mu_j^{\mathrm{st}}(s)}$. Let $\{(\mu_i^{\mathrm{op}})_{w\to s}\}_{i\in|J|}$ be the limit cone of this functor.

$$\begin{tikzcd} (F_i)_{\mu_i^{\mathrm{st}}(w)\to\mu_i^{\mathrm{st}}(s)} \arrow[rr, "D(u)^{\mathrm{op}}_{\mu_i^{\mathrm{st}}(w)\to\mu_i^{\mathrm{st}}(s)}"] & & (F_j)_{\mu_j^{\mathrm{st}}(w)\to\mu_j^{\mathrm{st}}(s)} \\ & F_{w\to s} \arrow[ul, "(\mu_i^{\mathrm{op}})_{w\to s}"] \arrow[ur, "(\mu_j^{\mathrm{op}})_{w\to s}"'] & \end{tikzcd}$$

3. For each arity $w \in S^*$, $\{(\mu_i^{\mathrm{rl}})_w\colon\ P_w \to (P_i)_{\mu_i^{\mathrm{st}}(w)}\}_{i\in|J|}$ is the limit cone for the functor $J \to \mathbb{S}et$ mapping each arrow $u \in J(i,j)$ to $D(u)^{\mathrm{rl}}_{\mu_i^{\mathrm{st}}(w)}\colon\ (P_i)_{\mu_i^{\mathrm{st}}(w)} \to (P_j)_{\mu_j^{\mathrm{st}}(w)}$.

Co-limits. For any small category J, the co-limit co-cone $\mu\colon\ (D\colon\ J \to \mathbb{S}ig) \Rightarrow (S,F,P)$ (where $D(u)\colon\ (S_i,F_i,P_i) \to (S_j,F_j,P_j)$ for each $u \in J(i,j)$) is defined by

1. μ^{st} is the co-limit of $D;(-)^{\mathrm{st}}\colon\ J \to \mathbb{S}et$

$$\begin{tikzcd} S_i \arrow[rr, "D(u)^{\mathrm{st}}"] \arrow[dr, "\mu_i^{\mathrm{st}}"'] & & S_j \arrow[dl, "\mu_j^{\mathrm{st}}"] \\ & S & \end{tikzcd}$$

2. For each arity $w \in S^*$ and each sort $s \in S$, let $(F_i')_{w\to s} = \biguplus_{\mu_i^{\mathrm{st}}(w_i s_i)=ws}(F_i)_{w_i\to s_i}$. For each arrow $u \in J(i,j)$ let $D(u)^{\mathrm{op}}_{w\to s}\colon\ (F_i')_{w\to s} \to (F_j')_{w\to s}$ be the 'disjoint union' of all functions

$$(F_i)_{w_i\to s_i} \xrightarrow{D(u)^{\mathrm{op}}_{w_i\to s_i}} (F_j)_{D(u)^{\mathrm{st}}(w_i)\to D(u)^{\mathrm{st}}(s_i)} \longrightarrow \biguplus_{\mu_j^{\mathrm{st}}(w_j s_j)=ws}(F_j)_{w_j\to s_j}.$$

Then we define $\{(\mu_i)^{\mathrm{op}}_{w\to s}\}_{i\in|J|}$ to be the co-limit co-cone for the functor $J \to \mathbb{S}et$ mapping each u to $D(u)^{\mathrm{op}}_{w\to s}$.

$$\begin{tikzcd}
(F_i')_{w\to s} \arrow[rr, "(D(u)^{\mathrm{op}})_{w\to s}"] \arrow[dr, "(\mu_i^{\mathrm{op}})_{w\to s}"'] & & (F_j')_{w\to s} \arrow[dl, "(\mu_j^{\mathrm{op}})_{w\to s}"] \\
& F_{w\to s} &
\end{tikzcd}$$

For each w_i and s_i we define $(\mu_i^{\mathrm{op}})_{w_i\to s_i}$ as the restriction of $(\mu_i^{\mathrm{op}})_{w\to s}$ to $(F_i)_{w_i\to s_i}$.

3. For each $i \in |J|$ and arity $w_i \in S_i^*$ we define $(\mu_i^{\mathrm{rl}})_{w_i}$ in the same way we have defined $(\mu_i^{\mathrm{op}})_{w_i\to s_i}$ in the item above.

Exercises

4.17. The category of **CA** signatures has small co-limits but only finite limits.

4.18. The category of **HOL/HNK** signatures does not have all pushouts, but it has pushouts of sort-preserving signature morphisms. It also has small co-products.

4.19. Weak co-amalgamation for sentences
In **FOL** the sentence functor weakly preserves pullbacks, i.e., any pullback of signature morphisms gets mapped by $\mathsf{Sen}^{\mathbf{FOL}}$ to a weak pullback in $\mathbb{S}et$.

4.20. [55] **Finitely presented signatures**
A **FOL** signature (S,F,P) is finitely presented (as an object of $\mathbb{S}ig^{\mathbf{FOL}}$) if and only if S, F, and P are finite. (F 'finite' means that $\{(w,s) \mid F_{w\to s} \neq \emptyset\}$ is finite and each non-empty $F_{w\to s}$ is also finite and the same for P.)

4.21. [59] **Finitary sentences**
A sentence ρ for a signature Σ of an institution is *finitary* when there exists a signature morphism $\varphi\colon\ \Sigma_0 \to \Sigma$ such that Σ_0 is finitely presented and there exists a Σ_0-sentence ρ_0 such that $\rho = \varphi(\rho_0)$. Then any **FOL**-sentence is finitary. Give an example of a $\mathbf{FOL}_{\infty,\omega}$-sentence which is *not* finitary.

4.22. [55] **Finitely presented theories**
Assume an institution with finitary sentences. Then for each finitely presented theory (Σ,E) (i.e., it is a finitely presented object in the category $\mathbb{T}h$ of theories),

- Σ is a finitely presented signature, and
- E can be presented by a finite set of sentences.

4.3 Model Amalgamation

Model amalgamation in institutions. In any institution, a commuting square of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

is an *amalgamation square* if and only if for each Σ_1-model M_1 and a Σ_2-model M_2 such that $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$, there exists a unique Σ'-model M', called the *amalgamation* of M_1 and M_2, such that $M'\upharpoonright_{\theta_1} = M_1$ and $M'\upharpoonright_{\theta_2} = M_2$. The amalgamation M' may be denoted by $M_1 \otimes_{\varphi_1,\varphi_2} M_2$ or simply by $M_1 \otimes M_2$.

In the absence of uniqueness of the amalgamation M', we say that this is a *weak amalgamation* square.

Note that from a categorical viewpoint, the model amalgamation property means that

$$\begin{tikzcd} |\mathsf{Mod}(\Sigma)| & |\mathsf{Mod}(\Sigma_1)| \arrow[l, "\mathsf{Mod}(\varphi_1)"'] \\ |\mathsf{Mod}(\Sigma_2)| \arrow[u, "\mathsf{Mod}(\varphi_2)"] & |\mathsf{Mod}(\Sigma')| \arrow[l, "\mathsf{Mod}(\theta_2)"] \arrow[u, "\mathsf{Mod}(\theta_1)"'] \end{tikzcd}$$

is a pullback in $\mathbb{C}lass$, the (quasi-)category of classes.[1]

In order to have model amalgamation, it is necessary that the corresponding square of signature morphisms does not collapse entities of Σ_1 and Σ_2 which do not come from Σ (via φ_1 and φ_2). On the other hand, for ensuring the uniqueness of the amalgamation it is necessary that Σ' does not contain entities which do not come from either Σ_1 or Σ_2. Therefore the primary candidates for model amalgamation are the pushout squares of signature morphisms. An institution has *model amalgamation* if and only if each pushout of signatures is an amalgamation square.

Model amalgamation in FOL

Modulo some adjustments the result below can be replicated to a multitude of actual institutions.

Proposition 4.5. **FOL** *has model amalgamation.*

Proof. The key to this proof is to define a 'very big' global **FOL** (hyper-)signature Ω such that for each **FOL** signature Σ, the class of Σ-models coincides with the class of signature morphisms $\Sigma \to \Omega$. Moreover, given a **FOL** signature morphism $\varphi : \Sigma \to \Sigma'$, the reduct $M'\upharpoonright_{\varphi}$ of any Σ'-model M' appears as $\varphi;M'$.

[1] This is the 'extension' of $\mathbb{S}et$ having classes as objects.

Suppose such a hyper-signature Ω exists. Then for any pushout (θ_1, θ_2) of a span of signature morphisms (φ_1, φ_2),

$$
\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"'] \arrow[ddr, bend left, "M_1"] & \\
\Sigma_2 \arrow[r, "\theta_2"] \arrow[drr, bend right, "M_2"'] & \Sigma' \arrow[dr, "M'"] & \\
& & \Omega
\end{tikzcd}
$$

$M_1\!\upharpoonright_{\varphi_1} = M_2\!\upharpoonright_{\varphi_2}$ just means that $\varphi_1;M_1 = \varphi_2;M_2$. Let $M' : \Sigma' \to \Omega$ be the unique signature morphism such that $\theta_k;M' = M_k$ for $k \in \{1,2\}$. Then M' is the unique amalgamation of M_1 and M_2.

Now let us turn our attention to $\Omega = (S^\Omega, F^\Omega, P^\Omega)$ which is defined as follows:

- $S^\Omega = |\mathbb{S}et|$, i.e., the class of all sets,
- for any sets $s_1,\ldots,s_n,s$, $F^\Omega_{s_1\ldots s_n \to s} = \mathbb{S}et(s_1 \times \cdots \times s_n, s)$, i.e., the set of all functions $s_1 \times \cdots \times s_n \to s$, and
- for any sets $s_1,\ldots,s_n$, $P^\Omega_{s_1\ldots s_n} = \mathcal{P}(s_1 \times \cdots \times s_n)$, i.e., all subsets of $s_1 \times \cdots \times s_n$.

We can notice immediately that (S,F,P)-models are just signature morphisms $(S,F,P) \to \Omega$. $\square$

Extended model amalgamation. The concept of model amalgamation is most often used in the form presented above, for squares of signature morphisms. However, sometimes other forms of model amalgamation are necessary, e.g., for co-cones over other types of diagrams of signature morphisms.

Given a diagram $D : J \to \mathbb{S}ig$, let us call a family $(M_i)_{i\in|J|}$ *consistent with* D when

- M_i is a $D(i)$-model for each $i \in |J|$, and
- $M_j\!\upharpoonright_{D(u)} = M_i$ for each arrow $u \in J(i,j)$.

We say that a co-cone μ over a diagram $D : J \to \mathbb{S}ig$ of signature morphisms has *model amalgamation* when for each family of models $(M_i)_{i\in|J|}$ *consistent with* D, there exists a unique model M such that $M\!\upharpoonright_{\mu_i} = M_i$ for each $i \in |J|$. When we drop the uniqueness requirement, we say that μ has *weak model amalgamation.*

Ordinary model amalgamation, as originally introduced in this section, is thus J-model amalgamation for J being a span of arrows $\bullet \longleftarrow \bullet \longrightarrow \bullet$.

An institution has *J-model amalgamation* for a category J when all co-limits of all diagrams $J \to \mathbb{S}ig$ have model amalgamation. This terminology can be also extended to classes $\mathbb{J}$ of categories J. For example, when $\mathbb{J}$ consists of all directed, respectively total posets, we talk about *directed*, respectively *inductive*, *model amalgamation*.

The proof of Prop. 4.5 can be extended without any problem to all small co-limits of signatures (which exist by virtue of Prop. 4.4).

Proposition 4.6. **FOL** *has J-model amalgamation for all small categories J.*

Exact institutions

In most situations the kind of amalgamation which is needed is at the level of the models only, however there are results which rely upon a form of amalgamation for model homomorphisms.

Amalgamation for model homomorphisms means that Mod maps any pushout of signatures to a pullback of categories (of models) rather than to a pullback of classes (of models). We called this property the *semi-exactness* of the institution. The terminology introduced for model amalgamation can be extended to exactness. Thus an institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is

- *semi-exact* when the model functor $\mathsf{Mod} : \mathbb{S}ig^{\mathrm{op}} \to \mathbb{C}at$ preserves pullbacks,
- *directed/inductive-exact* when Mod preserves directed/inductive limits,
- *(J)-exact* when Mod preserves all (J) small limits, and
- *weakly* J-exact when Mod preserves weak J-limits.[2]

FOL exactness. Prop. 4.5 can be refined to model homomorphisms.

Proposition 4.7. **FOL** *is exact.*

Proof. We have to show that $\mathsf{Mod}^{\mathbf{FOL}} : (\mathbb{S}ig^{\mathbf{FOL}})^{\mathrm{op}} \to \mathbb{C}at$ preserves all small limits. Let us consider the case of the pullbacks, other limits being handled similarly.

We re-use the idea underlying the proof of Proposition 4.5 by changing the hyper-signature Ω in order to capture model homomorphisms as follows.

- $S^{\Omega} = \mathbb{S}et$, i.e., the class of all functions,
- for all functions $s_1, \ldots, s_n, s$, $F^{\Omega}_{s_1 \ldots s_n \to s}$ is the sub-set of $\mathbb{S}et(dom(s_1) \times \cdots \times dom(s_n), dom(s)) \times \mathbb{S}et(cod(s_1) \times \cdots \times cod(s_n), cod(s))$ of all pairs which satisfy the homomorphism property for operations, i.e., $\langle M_\sigma, N_\sigma \rangle \in F^{\Omega}_{s_1 \ldots s_n \to s}$ if and only if $M_\sigma ; s = (s_1 \times \cdots \times s_n) ; N_\sigma$ and
- for any functions $s_1, \ldots, s_n$, $P^{\Omega}_{s_1 \ldots s_n}$ is the subset of $\mathcal{P}(dom(s_1) \times \cdots \times dom(s_n)) \times \mathcal{P}(cod(s_1) \times \cdots \times cod(s_n))$ of all pairs that satisfy the homomorphism property for relations, i.e., $\langle M_\pi, N_\pi \rangle \in P^{\Omega}_{s_1 \ldots s_n}$ if and only if $(s_1 \times \cdots \times s_n)(M_\pi) \subseteq N_\pi$.

$\square$

Model amalgamation for theories

Given a weak amalgamation square in an institution

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\
\Sigma_2 \arrow[r, "\theta_2"'] & \Sigma'
\end{tikzcd}$$

[2]A weak universal property, such as adjunction, limits, etc., is the same as the ordinary universal property except that only the existence part is required while uniqueness is not required.

if $M_1 \models_{\Sigma_1} E_1$ and $M_2 \models_{\Sigma_2} E_2$ and $M_1\restriction_{\varphi_1} = M_2\restriction_{\varphi_2}$, then by the satisfaction condition $M_1 \otimes_{\varphi_1,\varphi_2} M_2 \models_{\Sigma'} \theta_1(E_1) \cup \theta_2(E_2)$.

This argument works also for all co-cones of diagrams of presentation morphisms. By recalling how co-limits of presentations are constructed on top of co-limits of signatures (Prop. 4.2), the above argument shows that any model amalgamation property of an institution can be lifted from the level of the signatures to the level of the presentations or theories. Moreover, this can be extended easily to model homomorphisms too. These considerations are collected by the following result.

Theorem 4.8. *If the institution I is J-exact, then the institution of its presentations I^{pres} is J-exact too.*

By Prop. 4.7 and Thm. 4.8 it follows that

Corollary 4.9. *The institution* $\mathbf{FOL}^{\mathrm{pres}}$ *(of* $\mathbf{FOL}$ *presentations) is exact.*

Model amalgamation for institution mappings

The transfer of institutional properties along institution mappings, usually comorphisms, relies sometimes upon a form of model amalgamation of the respective institution mapping.

Exact comorphisms. An institution comorphism $(\Phi,\alpha,\beta) : I \to I'$ is *exact* if for each I-signature morphism $\varphi : \Sigma_1 \to \Sigma_2$ the naturality square below

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma_1) & \mathsf{Mod}'(\Phi(\Sigma_1)) \arrow[l, "\beta_{\Sigma_1}"'] \\
\mathsf{Mod}(\Sigma_2) \arrow[u, "\mathsf{Mod}(\varphi)"] & \mathsf{Mod}'(\Phi(\Sigma_2)) \arrow[l, "\beta_{\Sigma_2}"] \arrow[u, "\mathsf{Mod}'(\Phi(\varphi))"']
\end{tikzcd}$$

is a pullback. When discarding the model homomorphisms from the above (i.e., the diagram above is a pullback of classes of models rather than categories of models), we say that (Φ,α,β) *has model amalgamation*. This means that for any $\Phi(\Sigma_1)$-model M'_1 and any Σ_2-model M_2, if $\beta_{\Sigma_1}(M'_1) = M_2\restriction_{\varphi}$, then there exists a unique $\Phi(\Sigma_2)$-model M'_2 such that $\beta_{\Sigma_2}(M'_2) = M_2$ and $M'_2\restriction_{\Phi(\varphi)} = M'_1$. If we drop the uniqueness requirement on M'_2, then we say that (Φ,α,β) *has weak model amalgamation*.

Notice that the exactness of the institution comorphism $\mathbf{EQL} \to \mathbf{FOL}$ holds trivially because the model translation functors $\beta_{(S,F)}$ are isomorphisms for all algebraic signatures (S,F).

Exact morphisms. A similar definition can be formulated for *exact institution morphisms*. However, in the actual institutions, comorphisms rather than morphisms interact better with model amalgamation. For example, while the comorphism $\mathbf{EQL} \to \mathbf{FOL}$ is trivially exact, its adjoint (forgetful) institution morphism $\mathbf{FOL} \to \mathbf{EQL}$ does *not* have model amalgamation.

Exercises

4.23. Conservative signature morphisms
A signature morphism $\varphi : \Sigma \to \Sigma'$ is *conservative* when each Σ-model has at least a φ-expansion. Given a weak amalgamation square in an institution

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma' \arrow[d, "\theta'"] \\ \Sigma_1 \arrow[r, "\varphi_1"'] & \Sigma'_1 \end{tikzcd}$$

show that φ_1 is conservative when φ is conservative.

4.24. In the commuting diagram below

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[d] & \Sigma_1 \arrow[d] \arrow[ddr, bend left] & \\ \Sigma_2 \arrow[r] \arrow[drr, bend right] & \Sigma'' \arrow[dr, "\varphi"] & \\ & & \Sigma' \end{tikzcd}$$

if $[\Sigma,\Sigma_1,\Sigma_2,\Sigma'']$ is an amalgamation square, then $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ is a weak amalgamation square if and only if φ is conservative.

4.25. Categorical equational logic *Cat***EQL** (see Ex. 3.8) trivially is exact.

4.26. For any semi-exact institution I, the institution $I^{\to}$ of its signature morphisms (see Ex. 3.9) is semi-exact.

4.27. A method to prove model amalgamation properties of institutions I is to 'borrow' them from another institution I' via a comorphism $I \to I'$ with the following properties:

1. the signature translation functor preserves the respective co-limits of signatures, and
2. the model translation functor has a left inverse.

Apply this method to obtain model amalgamation properties for various institutions presented in this book.

4.28. The institution **IPL** is exact.

4.29. The institutions $\mathbf{FOL^1}$, **LA** and **AUT** are semi-exact but they are not exact.

4.30. The institution **CA** of contraction algebras does *not* have model amalgamation. Explain why. However model amalgamation holds for the pushout squares for which the contraction parameter q is fixed for all signatures.

4.31. While **HOL** has model amalgamation, **HNK** has only weak model amalgamation.

4.32. Any chartable institution has model amalgamation. (see Ex. 3.11)

4.33. The sub-institutions of **FOL** obtained by restricting the model homomorphisms to those which are injective, respectively surjective, are exact.

4.34. The comorphism **FOL** $\to$ **FOEQL** encoding relations as operations (see Sect. 3.3) does not have model amalgamation but it has weak model amalgamation.

4.35. Let us assume that for an institution morphism $(\Phi,\alpha,\beta)\colon\ I' \to I$, the signature translation functor $\Phi\colon\ \mathbb{S}ig' \to \mathbb{S}ig$ has a left adjoint $\overline{\Phi}$, is full and surjective on objects, and that I' is semi-exact. If (Φ,α,β) is exact, then its adjoint institution comorphism $(\overline{\Phi},\overline{\alpha},\overline{\beta})$ is exact too.

4.36. Study the model amalgamation properties of the following comorphisms which have been introduced above in the book (either in the main text or in exercises): **MPL** $\to$ $\mathbf{REL^1}$, **PA** $\to$ $\mathbf{FOL}^{\mathrm{pres}}$, **FOL** $\to$ $\mathbf{REL}^{\mathrm{pres}}$, **POA** $\to$ $\mathbf{FOL}^{\mathrm{pres}}$, **FOL** $\to$ $\mathbf{PA}^{\mathrm{pres}}$, **FOL** $\to$ $(\mathbf{FOL^1})^{\mathrm{pres}}$, and **HNK** $\to$ $\mathbf{FOEQL}^{\mathrm{pres}}$.

4.4 The method of Diagrams

This is one of the most useful methods of model theory. As we will see below, ‘diagrams’ here are used with a different meaning than the categorical diagrams. For this reason we prefer to use ‘elementary diagrams’ for the model theoretic concept.

Elementary diagrams in FOL

Each model M of a signature (S,F,P) determines an extension of signatures $\iota\ :\ \ (S,F,P) \hookrightarrow (S,F_M,P)$ where

- $(F_M)_{w\to s} = F_{w\to s}$ for any non-empty arity w and any sort $s \in S$, and
- $(F_M)_{\to s} = F_{\to s} \cup M_s$ for any sort $s \in S$.

The second step is to note that M can be canonically expanded to an (S,F_M,P)-model M_M by interpreting the new constants of $(F_M)_{\to s}$ by the corresponding elements of M_s, i.e., $(M_M)_a = a$ for each $a \in M$. Let E_M be the set of all atoms (either equational or relational) satisfied by M_M.

The presentation $((S,F_M,P),E_M)$, called the *elementary diagram of M* has the crucial categorical property that it *axiomatizes the class of homomorphisms from M*.

Proposition 4.10. *There exists a natural isomorphism*

$$i\colon\ \mathsf{Mod}((S,F_M,P),E_M) \to M/\mathsf{Mod}(S,F,P).$$

Proof. The isomorphism i maps each (S,F_M,P)-model N satisfying E_M to the (S,F,P)-model homomorphism $h_N\colon\ M \to N\!\upharpoonright_\iota$ such that $h_N(a) = N_a$ for each element $a \in M$. Let us check that h_N is indeed a model homomorphism.

- For each operation $\sigma \in F_{w\to s}$ and for each $m \in M_w$, $(\sigma(m) = M_\sigma(m)) \in E_M$, which implies $N \models \sigma(m) = M_\sigma(m)$, which means $N_{M_\sigma(m)} = N_{\sigma(m)}$. But $N_{M_\sigma(m)} = (h_N)_s(M_\sigma(m))$ and $N_{\sigma(m)} = N_\sigma(N_m) = (N\!\upharpoonright_\iota)_\sigma((h_N)_w(m))$, which implies $(h_N)_s(M_\sigma(m)) = (N\!\upharpoonright_\iota)_\sigma((h_N)_w(m))$.
- For each relation $\pi \in P_w$ and for each $m \in M_w$, if $m \in M_\pi$, then $\pi(m) \in E_M$, which implies $N \models \pi(m)$. But this means $(h_N)_w(m) = N_m \in N_\pi$.

The inverse isomorphism i^{-1} maps any (S,F,P)-model homomorphism $h\colon\ M \to N$ to the (S,F_M,P)-model $i^{-1}(h) = N_h$ where $(N_h)\upharpoonright_\iota = N$ and $(N_h)_a = h(a)$ for each $a \in M$. We have to check that $N_h \models E_M$. First let us notice that h is also an (S,F_M,P)-model homomorphism $M_M \to N_h$.

- Consider an equation $t = t'$ in E_M. By induction on the structure of the terms, we can prove that $(N_h)_t = h((M_M)_t)$ and $(N_h)_{t'} = h((M_M)_{t'})$. Since $(M_M)_t = (M_M)_{t'}$ we deduce that $(N_h)_t = (N_h)_{t'}$, which means $N_h \models t = t'$.
- Now consider a relational atom $\pi(t) \in E_M$ for a relation symbol $\pi \in P_w$. Because $(N_h)_t = h((M_M)_t)$, $(M_M)_t \in (M_M)_\pi$ and h is an (S,F_M,P)-model homomorphism, we deduce that $(N_h)_t \in (N_h)_\pi$ which means $N_h \models \pi(t)$.

We have analyzed i and i^{-1} on models only. They also work as expected on model homomorphisms. □

Changing model homomorphisms. In order to maintain the isomorphic relationship between the category of homomorphisms $M/\mathrm{Mod}(S,F,P)$ and the category of the models of the elementary diagram $\mathrm{Mod}((S,F_M,P),E_M)$, any change of the concept of model homomorphism induces a change of the concept of elementary diagram. Note that considering other model homomorphisms between **FOL** models means in fact working with another institution. For example, if we impose some condition which shrinks the class of model homomorphisms, then consequently the elementary diagram should get bigger in order that the class of its models shrinks too.

Below we give a list of several possibilities for model homomorphisms between **FOL** models obtained by imposing some additional conditions on the standard **FOL** model homomorphisms. In all cases elementary diagrams do exist as shown in the right-hand side column of the table. All entries of the table can be checked similarly to the proof of Prop. 4.10.

model homomorphisms	E_M
all	all atoms in M_M^*
injective	all atoms and negations of atomic equations in by M_M^*
closed	all atoms and negations of atomic relations in M_M^*
closed and injective	all atoms and negations of atoms in M_M*
elementary embeddings	M_M^*

A **FOL**-model homomorphism $h\colon\ M \to N$

- is *closed* when $M_\pi = h^{-1}(N_\pi)$ for each relation symbol π of the signature, and
- is an *elementary embedding* when $M_M \equiv N_h$ where $N_h = i_{\Sigma,M}(h)$. (Note that because $M_M \models m \neq m'$ for all $m,m' \in M$ which are different, h is also injective.)

In some other model theoretic works, in the context of **FOL** models, the terminology ‘elementary diagram’ is reserved only for the last institution in the table above. Here we use this terminology in a much wider sense, which is partly justified by the different level of abstraction, as will be seen in the following paragraph. However, when there is no danger of ambiguity we may also refer to the elementary diagrams just as ‘diagrams’.

Institution-independent elementary diagrams

The isomorphism between the category of model homomorphisms from M (for whatever concept of model homomorphism we employ) and the class of models of the 'elementary diagram' of M is a purely categorical property which can be formulated at an institution-independent level.

An institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ has *elementary diagrams* if and only if for each signature Σ and each Σ-model M, there exists a signature Σ_M and a signature morphism $\iota_\Sigma(M) : \Sigma \to \Sigma_M$, functorial in Σ and M, and a set E_M of Σ_M-sentences such that $\mathsf{Mod}(\Sigma_M, E_M)$ and the comma category $M/\mathsf{Mod}(\Sigma)$ are naturally isomorphic, i.e., the following diagram commutes by the isomorphism $i_{\Sigma,M}$ natural in Σ and M.

$$\begin{tikzcd} \mathsf{Mod}(\Sigma_M, E_M) \arrow[r, "i_{\Sigma,M}"] \arrow[dr, "\mathsf{Mod}(\iota_\Sigma(M))"'] & (M/\mathsf{Mod}(\Sigma)) \arrow[d, "\text{forgetful}"] \\ & \mathsf{Mod}(\Sigma) \end{tikzcd}$$

The signature morphism $\iota_\Sigma(M) : \Sigma \to \Sigma_M$ is called the *elementary extension of* Σ *via* M and the set E_M of Σ_M-sentences is called the *elementary diagram* of the model M. For each model homomorphism $h : M \to N$ let N_h denote $i^{-1}_{\Sigma,M}(N)$.

The "functoriality" of ι means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ-model homomorphism $h : M \to M'\restriction_\varphi$, there exists a presentation morphism $\iota_\varphi(h) : (\Sigma_M, E_M) \to (\Sigma'_{M'}, E_{M'})$ such that

$$\begin{tikzcd} \Sigma \arrow[r, "\iota_\Sigma(M)"] \arrow[d, "\varphi"'] & \Sigma_M \arrow[d, "\iota_\varphi(h)"] \\ \Sigma' \arrow[r, "\iota_{\Sigma'}(M')"'] & \Sigma'_{M'} \end{tikzcd}$$

commutes and $\iota_\varphi(h);\iota_{\varphi'}(h') = \iota_{\varphi;\varphi'}(h;h'\restriction_\varphi)$ and $\iota_{1_\Sigma}(1_M) = 1_{\Sigma_M}$.

The "naturality" of i means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ-model homomorphism $h : M \to M'\restriction_\varphi$ the following diagram commutes:

$$\begin{tikzcd} \mathsf{Mod}(\Sigma_M, E_M) \arrow[r, "i_{\Sigma,M}"] & M/\mathsf{Mod}(\Sigma) \\ \mathsf{Mod}(\Sigma'_{M'}, E'_{M'}) \arrow[r, "i_{\Sigma',M'}"'] \arrow[u, "\mathsf{Mod}(\iota_\varphi(h))"] & M'/\mathsf{Mod}(\Sigma') \arrow[u, "h/\mathsf{Mod}(\varphi)=h;(-)\restriction_\varphi"'] \end{tikzcd}$$

The reader is invited to check the above functoriality and naturality properties of the elementary diagrams for **FOL** and its sub-institutions presented above.

An institution with elementary diagrams ι may be denoted by $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \iota)$.

A more categorical formulation

The elementary diagrams of an institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \iota)$ can be expressed more compactly as a functor $\iota : \mathsf{Mod}^\sharp \to \mathbb{P}res^\to$ from the Grothendieck category $\mathsf{Mod}^\sharp$ determined by the model functor Mod to the category $\mathbb{P}res$ of the presentations of the institution such that

$$\begin{tikzcd}
\mathsf{Mod}^\sharp \arrow[r, "\iota"] \arrow[d, "\text{(fibration) projection}"'] & \mathbb{P}res^\to \arrow[d, "dom"] \\
\mathbb{S}ig \arrow[r, "\text{left adjoint}"'] & \mathbb{P}res
\end{tikzcd}$$

commutes, where $\mathbb{P}res^\to$ is the category of presentation morphisms (i.e., the functors $(\bullet \longrightarrow \bullet) \to \mathbb{P}res$), and dom is the functor projecting on the domain of the presentation morphisms, and such that the following functors $\mathsf{Mod}^\sharp \to \mathbb{C}at^\to$ are isomorphic:

$$\begin{tikzcd}
\mathsf{Mod}^\sharp \arrow[d, "\iota"'] & \mathsf{Mod}^\sharp \arrow[dd, "-/\mathsf{Mod}(-)"] \\
\mathbb{P}res^\to \arrow[d, "\mathsf{Mod}^\to"'] \arrow[r, phantom, "\cong"] & \\
\mathbb{C}at^\to & \mathbb{C}at^\to
\end{tikzcd}$$

where

- $\mathsf{Mod}^\to((\Sigma, E) \xrightarrow{\varphi} (\Sigma', E')) = \mathsf{Mod}(\Sigma', E') \xrightarrow{\mathsf{Mod}(\varphi)} \mathsf{Mod}(\Sigma, E)$ and
- $(-/\mathsf{Mod}(-))(\langle \Sigma, M \rangle) = M/\mathsf{Mod}(\Sigma) \to \mathsf{Mod}(\Sigma)$.

Elementary homomorphisms

Recall that a **FOL**-model homomorphism $h : M \to N$ is by definition an *elementary embedding* if and only if M_M and N_h are elementarily equivalent (they satisfy exactly the same sentences). Note that by involving M_M and N_h we have used the diagrams of **FOL**.

In the same way a concept of 'elementary embedding' can be defined in any abstract institution provided it has diagrams.

Fact 4.11. *In any institution with diagrams, the diagram of any model M has an initial model, denoted M_M.*

A model homomorphism $h : M \to N$ is *elementary* when $N_h = i^{-1}_{\Sigma,M}(N) \models M^*_M$.

Fact 4.12. *For each elementary homomorphism $h : M \to N$, $M^* \subseteq N^*$.*

We say that an institution with diagrams is *elementary* when each model homomorphism is elementary. For example $E(\mathbf{FOL})$, the sub-institution of **FOL** with 'elementary embeddings' as model homomorphism is elementary.

Fact 4.13. *An institution is elementary if and only if $M^*_M = E^{**}_M$ for each model M.*

It is now the moment to ask: do the elementary homomorphisms of an institution (with elementary diagrams) form a (sub-)institution, and if yes, does it have elementary diagrams too? At this stage it is not possible to answer properly this question, however this will get a positive answer in the next chapter.

Morphisms of institutions with elementary diagrams

Given a model M for a **FOL** signature (S,F,P), notice that the forgetful institution morphism **FOL** $\to$ **EQL**

- maps the elementary extension $(S,F,P) \hookrightarrow (S,F_M,P)$ to the elementary extension $(S,F) \hookrightarrow (S,F_M)$ of algebraic signatures which corresponds to the (S,F)-algebra underlying M, and
- the diagram of the (S,F)-algebra underlying M is the restriction of the diagram of M to all equations.

This situation suggests that the forgetful institution morphism **FOL** $\to$ **EQL** is a 'morphism of elementary diagrams' between the system of diagrams of **FOL** and that of **EQL**.

In general, a *morphism of institutions with diagrams* $(\Phi,\alpha,\beta) : \ (I',\iota') \to (I,\iota)$ is an institution morphism such that

$$\begin{array}{ccc} (\mathsf{Mod}')^\sharp & \xrightarrow{\iota'} & (\mathbb{P}res')^{\to} \\ {\scriptstyle\beta^\sharp}\downarrow & & \downarrow{\scriptstyle\Phi^{\to}} \\ (\mathsf{Mod})^\sharp & \xrightarrow[\iota]{} & (\mathbb{P}res)^{\to} \end{array}$$

commutes, where

- for each signature $\Sigma' \in |\mathbb{S}ig'|$ and each Σ'-model M', the functor $\beta^\sharp$ maps $\langle \Sigma', M' \rangle$ to $\langle \Phi(\Sigma'), \beta'_\Sigma(M') \rangle$, and
- the functor $\Phi^{\to}$ maps each presentation morphism $\varphi : \ (\Sigma'_1, E'_1) \to (\Sigma'_2, E'_2)$ to $\Phi(\varphi) : \ (\Phi(\Sigma'_1), \alpha^{-1}_{\Sigma'_1}(E'^{**}_1)) \to (\Phi(\Sigma'_2), \alpha^{-1}_{\Sigma'_2}(E'^{**}_2))$.

More concretely, this means that $\Phi(\iota'_{\Sigma'}(M')) = \iota_{\Phi(\Sigma')}(\beta_{\Sigma'}(M'))$ (which implies $\Phi(\Sigma'_{M'}) = (\Phi(\Sigma'))_{\beta_{\Sigma'}(M')}$) and $E_{\beta_{\Sigma'}(M')} \models\!\!\!\!\mid \alpha^{-1}_{\Sigma'_{M'}}(E^{**}_{M'})$ for each signature $\Sigma' \in |\mathbb{S}ig'|$ and each Σ'-model M'.

The category of institutions with elementary diagrams is denoted as $\mathbb{EDI}ns$.

A dual concept of 'comorphism of institutions with elementary diagrams' can be defined similarly.

Limits and co-limits of models

In the presence of diagrams, limits and co-limits of models can be obtained from corresponding limits and co-limits of signatures. This is an important consequence of the existence of diagrams because in the actual institutions, limits, and especially co-limits of models are much more difficult to establish than (co-)limits of signatures.

Theorem 4.14. *Consider an institution with diagrams and initial models of presentations. Then, for each signature Σ, the category of Σ-models has J-(co-)limits whenever the category of signatures $\mathbb{S}ig$ has J-(co-)limits.*

Proof. Limits: Let J be a category such that $\mathbb{S}ig$ has J-limits, and consider a J-diagram $M: J \to \mathsf{Mod}(\Sigma)$ of Σ-models. Let us denote $M(i)$ by M^i for each index $i \in |J|$, $M(u)$ by M^u for each index morphism $u \in J$, and let $\Sigma \xrightarrow{\iota_\Sigma(M^i)} (\Sigma_{M^i}, E_{M^i})$ be the diagram of M^i.

$$\begin{tikzcd}
 & & (\Sigma_{M^i}, E_{M^i}) \arrow[dd, "\iota_\Sigma(M^u)"] \\
\Sigma \arrow[r, "\varphi"] \arrow[urr, bend left, "\iota_\Sigma(M^i)"] \arrow[drr, bend right, "\iota_\Sigma(M^j)"'] & (\Sigma', E') \arrow[ur, "\phi^i"] \arrow[dr, "\phi^j"'] & \\
 & & (\Sigma_{M^j}, E_{M^j})
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M^i \arrow[dd, "M^u"] \\
N \arrow[ur, "\mu^i"] \arrow[dr, "\mu^j"'] & \\
 & M^j
\end{tikzcd}
\qquad
\begin{tikzcd}
i \arrow[dd, "u"] \\
\\
j
\end{tikzcd}$$

Let $\phi: \Sigma' \Rightarrow \Sigma_M$ be the limit cone where $\Sigma_M: J \to \mathbb{S}ig^I$ is defined by

- $(\Sigma_M)(i) = \Sigma_{M^i}$ for each index $i \in |J|$, and
- $(\Sigma_M)(u) = \iota_\Sigma(M^u)$ for each index morphism $u \in J$.

Let $N = (0_{\Sigma',E'})\upharpoonright_\varphi$ where

- $0_{\Sigma',E'}$ is the initial model of (Σ', E'),
- $((\Sigma', E') \xrightarrow{\phi^i} (\Sigma_{M^i}, E_{M^i}))_{i \in |J|}$ is the limit of the diagram of elementary diagrams $((\Sigma_{M^i}, E_{M^i}) \xrightarrow{\iota_\Sigma(M^u)} (\Sigma_{M^j}, E_{M^j}))_{u \in J}$, and
- $\varphi: \Sigma \to \Sigma'$ is the unique signature morphism such that $\varphi;\phi^i = \iota_\Sigma(M^i)$ for each index $i \in |J|$ (cf. Prop. 4.2 and Cor. 4.3), $(\Sigma' \xrightarrow{\phi^i} \Sigma_{M^i})_{i \in |J|}$ is the co-limit of $(\Sigma_{M^i} \xrightarrow{\iota_\Sigma(M^u)} \Sigma_{M^j})_{u \in J}$.

Because $E_{M^i} \models \phi^i(E')$ we have that $(M^i)_{M^i}\upharpoonright_{\phi^i} \models E'$. Let $\nu^i: 0_{\Sigma',E'} \to (M^i)_{M^i}\upharpoonright_{\phi^i}$ be the unique (Σ', E')-model homomorphism. We define $\mu_i = \nu^i\upharpoonright_\varphi$ for each $i \in |J|$. That $\mu: N \Rightarrow M$ is a cone follows from the fact that $\nu^i; i^{-1}_{\Sigma,M^i}(M^u) = \nu^j$ for each $i \xrightarrow{u} j \in J$, which follows from the uniqueness part of the initiality property of $0_{\Sigma',E'}$.

For any other cone $\mu': N' \Rightarrow M$, let $\iota_\Sigma(N'): \Sigma \to \Sigma_{N'}$ be the elementary extension of Σ via N'. Notice that $\{\iota_\Sigma(\mu'^i)\}_{i \in |J|}$ is a cone $\Sigma_{N'} \Rightarrow \Sigma_M$. Therefore let $\varphi': \Sigma_{N'} \to \Sigma'$ be the unique signature morphism such that $\varphi';\phi^i = \iota_\Sigma(\mu'_i)$ for each $i \in |J|$.

For each $i \in |J|$, $\iota_\Sigma(\mu'_i)$ is a presentation morphism $(\Sigma_{N'}, E_{N'}) \to (\Sigma_{M^i}, E_{M^i})$, which implies $E_{M^i} \models \phi^i(\varphi'(E_{N'}))$. By the satisfaction condition this means $0_{\Sigma',E'}\upharpoonright_{\varphi'} \models E_{N'}$. Let $h = h'\upharpoonright_{\iota_\Sigma(N')}$ where h' is the unique model homomorphism $(N')_{N'} \to 0_{\Sigma',E'}\upharpoonright_{\varphi'}$. Then h is the unique model homomorphism such that $\mu' = h;\mu$. That $\mu'_i = h;\mu_i$ for each $i \in |J|$

follows from the uniqueness part of the initiality property of $(N')_{N'}$. The uniqueness of h follows by the isomorphism $i_{\Sigma,N'}$ and from the initiality property of $(N')_{N'}$.

Co-limits: The proof of the co-limit part follows ideas similar to the proof of the limit part. We therefore give here only the sketch of the proof for the co-limit part.

$$\begin{tikzcd}
i \arrow[dd, "u"'] & M^i \arrow[dd, "M^u"'] \arrow[dr, "\mu^i"] & & & & (\Sigma_{M^i}, E_{M^i}) \arrow[dd, "\iota_\Sigma(M^u)"'] \arrow[dr, "\phi^i"] \arrow[drr, bend left, "\iota_\Sigma(\mu'^i)"] & & \\
& & N & & \Sigma \arrow[ur, "\iota_\Sigma(M^i)"] \arrow[dr, "\iota_\Sigma(M^j)"'] & & (\Sigma', E') \arrow[r, "\varphi'"] & (\Sigma_{N'}, E_{N'}) \\
j & M^j \arrow[ur, "\mu^j"'] & & & & (\Sigma_{M^j}, E_{M^j}) \arrow[ur, "\phi^j"'] \arrow[urr, bend right, "\iota_\Sigma(\mu'^j)"'] & &
\end{tikzcd}$$

We take the co-limit $(\,(\Sigma_{M^i}, E_{M^i}) \xrightarrow{\phi^i} (\Sigma', E')\,)_{i\in|J|}$ of $(\,(\Sigma_{M^i}, E_{M^i}) \xrightarrow{\iota_\Sigma(M^u)} (\Sigma_{M^j}, E_{M^j})\,)_{u\in J}$ and define $N = 0_{\Sigma',E'}\restriction_{\varphi}$ where $\varphi = \iota_\Sigma(M^i);\phi^i$. Then the co-limit $\mu\colon\ M \Rightarrow N$ is defined by $\mu_i = i_{\Sigma,M^i}(0_{\Sigma',E'}\restriction_{\phi^i})$ for each $i \in |J|$. □

Limits and co-limits of FOL models. Let us apply Thm. 4.14 above to obtain the existence of limits and co-limits of **FOL** models. The method illustrated by the proof of Cor. 4.15 may be also applied to other actual institutions.

Corollary 4.15. *The category of models of any* **FOL** *signature has small limits and co-limits.*

Proof. Let us consider the sub-institution **AFOL** of the atoms of **FOL**, which restricts the sentences to (equational or relational) atoms only. Obviously, **AFOL** inherits the **FOL** diagrams, but unlike **FOL**, it has initial models for all its presentations (a result which we anticipate and is given by Cor. 4.28 below). The category of signatures has small limits and co-limits (cf. Prop. 4.4), therefore by Thm. 4.14 the category of models of any signature has small limits and co-limits. □

Exercises

4.37. The standard elementary diagrams of **FOL** can be defined slightly differently than the ordinary way, such that the elementary extension adds to the given signature only the elements which are not interpretations of constants.

4.38. A **FOL** model homomorphism $h\colon\ M \to N$ is *strong* when $N_\pi = h(M_\pi)$ for each relation symbol π of the signature.

The sub-institution of infinitary first order logic $\mathbf{FOL}_{\infty,\omega}$ where the model homomorphisms are restricted to the strong ones has elementary diagrams with the same elementary extensions as in **FOL** but such that for each (S,F,P)-model M the elementary diagram E_M consists of the **FOL** elementary diagram plus

- $\{\neg\pi(m) \in M_M^* \mid \pi \in P\}$, and
- all sentences of the form

$$(\forall X)(\pi(X) \Rightarrow \bigvee_{m \in M_w} (X = m))$$

for each relation symbol π of arity w, and where $X = m$ means $\bigwedge_{1 \leq k \leq n}(x_k = m_k)$ for $X = x_1 \ldots x_n$ and $m = m_1 \ldots m_n$.

4.39. Borrowing elementary diagrams
Let I' be an institution with elementary diagrams ι' and let $(\Phi, \alpha, \beta)\colon\ I \to I'$ be an institution comorphism such that

1. Φ is full and faithful,
2. β_Σ are isomorphisms (for each model M let M' denote $\beta_\Sigma^{-1}(M)$),
3. for each Σ-model M in I:
   (a) there exists a signature Σ_M in I such that $\Phi(\Sigma)_{M'} = \Phi(\Sigma_M)$, and
   (b) for each sentence $\rho' \in E_{M'}$ there exists a Σ_M-sentence ρ such that $\rho' \models\!\mid \alpha_{\Sigma_M}(\rho)$.

Then the institution I has elementary diagrams ι defined by

- $\iota_\Sigma(M)$ is the unique signature morphism such that $\Phi(\iota_\Sigma(M)) = \Phi(\Sigma_M)$, and
- $E_M = \{\rho \mid \text{there exists } \rho' \in E_{M'} \text{ such that } \alpha_{\Sigma_{M'}}(\rho) \models\!\mid \rho'\}$.

Then the diagrams of **EQL** are 'borrowed' from **FOL** as above.

4.40. The table below gives the elementary diagrams of several institutions:

I	Σ	Σ_M	M_M	E_M
PA	(S,TF,PF)	(S,TF_M,PF) with $(TF_M)_{\to s} = TF_{\to s} \cup M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t \stackrel{e}{=} t' \mid M_M \models t \stackrel{e}{=} t'\}$
POA	(S,F)	(S,F_M) with $(F_M)_{\to s} = F_{\to s} \cup M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t = t' \mid M_M \models t = t'\} \cup$ $\{t \leq t' \mid M_M \models t \leq t'\}$
MBA	(S,K,F,kind)	(S,K,F_M,kind) with $(F_M)_{\to k} = F_{\to k} \cup M_k$ for $k \in K$	$(M_M)_m = m$ for $m \in M$	$\{t = t' \mid M_M \models t = t'\} \cup$ $\{(t : s) \mid M_M \models (t : s)\}$
MA	(S,F)	(S,F_M) with $(F_M)_{\to s} = F_{\to s} \cup M_s$ for $s \in S$	$(M_M)_m = \{m\}$ for $m \in M$	$\{m \doteq m \mid m \in M\} \cup$ $\{x \prec \sigma(m) \mid \sigma \in F_{w \to s}$ and $m \in M_w$ and $x \in M_\sigma(m)\}$
CA	(S,F,q)	(S,F_M,q) with $(F_M)_{\to s} = F_{\to s} \cup M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t \approx_\varepsilon t' \mid M_M \models t \approx_\varepsilon t'\}$
HNK	(S,F)	(S,F_M) with $(F_M)_s = F_s \cup M_s$ for $s \in S$	$(M_M)_m = m$ for $m \in M$	$\{t = t' \mid M_M \models t = t'\}$

4.41. Elementary diagrams in intuitionistic logic
IPL has diagrams for the model homomorphisms of the form $h\colon\ (M\colon\ P\to A)\to(N\colon\ P\to B)$ where $h\colon\ A\to B$ are monotone (i.e., order preserving) functions which preserve the upper bound $\top$ and such that $M;h=N$.

$$\begin{tikzcd} P \arrow[r, "M"] \arrow[dr, "N"'] & A \arrow[d, "h"] \\ & B \end{tikzcd}$$

(*Hint:* For any P-model $M\colon\ P\to A$ its elementary extension $P\to P\uplus A$ is given by adding the elements of the Heyting algebra A to P. The diagram of M consists of all sentences of the form ρ and $\rho_1\Rightarrow\rho_2$ with $\rho,\rho_1,\rho_2\in P\uplus A$ which are satisfied by the canonical expansion $M_M\colon\ P\uplus A\to A$ of M.)

A model homomorphism is elementary if and only if it is also a *homomorphism of Heyting algebras*, i.e., it preserves the interpretations of $\wedge$, $\vee$, $\Rightarrow$, $\top$, and $\bot$. Moreover, the sub-institution of the **IPL** elementary homomorphisms has diagrams.

4.42. Let Cat_+**EQL** be the subinstitution of Cat**EQL** determined by categories with binary co-products. Then Cat_+**EQL** has empty elementary diagrams. (*Hint:* For any object A in a category $\mathbb{C}$ having binary co-products, the elementary extension of $\mathbb{C}$ via A is the left adjoint to the forgetful functor $A/\mathbb{C}\to\mathbb{C}$.)

4.43. Elementary diagrams for presentations
(a) For each institution I with elementary diagrams the institution I^{pres} of its presentations has elementary diagrams such that the (original) diagrams of I are 'borrowed' from those of I^{pres} along the canonical embedding comorphism $I\to I^{\mathrm{pres}}$.
(b) As an application to (a), for any institution with elementary diagrams and initial models for presentations, the categories of models of presentations have all (co-)limits of the category of signatures.

4.44. Preserving carriers
A signature morphism $\varphi\colon\ \Sigma\to\Sigma'$ *preserves carriers* when

$$\begin{tikzcd} \Sigma \arrow[r, "\iota_\Sigma(M)"] \arrow[d, "\varphi"'] & \Sigma_M \arrow[d, "\iota_\varphi(1_M)"] \\ \Sigma' \arrow[r, "\iota_{\Sigma'}(M')"'] & \Sigma'_{M'} \end{tikzcd}$$

is a pushout of signature morphisms for all Σ'-models M' and Σ-models M for which $M'\upharpoonright_\varphi=M$.
Then signature morphisms preserving carriers are closed under composition.
In **FOL** all signature morphisms which are bijective on sorts preserve the carriers.

4.45. Study the model amalgamation properties of $E(\mathbf{FOL})$, i.e., the sub-institution of **FOL** with elementary embeddings as model homomorphisms.

4.5 Inclusion Systems

The standard inclusion system of $\mathbb{S}et$. Each function $f\colon A \to B$ can be factored as a composite between a surjection and an inclusion, i.e., $f = e_f;i_f$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e_f"'] & & B \arrow[dl, "i_f"] \\ & f(A) & \end{tikzcd}$$

as follows:

- $f(A) = \{f(a) \mid a \in A\}$,
- $e_f(a) = f(a)$ for each $a \in A$, and
- $i_f(b) = b$ for each $b \in f(A)$.

It is easy to see that this factorization is *unique*, that is for any other factorization $f = e'_f;i'_f$ with e'_f surjection and i'_f inclusion we necessarily have $e'_f = e_f$ and $i'_f = i_f$. The existence and the uniqueness of such a factorization is a consequence of the nature of the surjective functions and of the inclusions. For the uniqueness it is especially important that inclusions are unique in the sense that there exists at most one inclusion between any two given sets.

This factorization phenomenon may be found in various forms in many other categories, including categories of models. It constitutes an important conceptual device in model theory.

Categorical inclusion systems. The factorization property of functions presented above can be expressed at the level of abstract categories. In this book this will be used in the following ways, for categories of models in institutions, and for the categories of signatures of institutions.

$\langle I, \mathcal{E}\rangle$ is an *inclusion system* for a category $\mathbb{C}$ if I and $\mathcal{E}$ are two sub-categories with $|I| = |\mathcal{E}| = |\mathbb{C}|$ such that

1. I is a partial order, and
2. every arrow f in $\mathbb{C}$ can be factored uniquely as $f = e_f;i_f$ with $e_f \in \mathcal{E}$ and $i_f \in I$.

The arrows of I are called *abstract inclusions*, and the arrows of $\mathcal{E}$ are called *abstract surjections*. The domain of the inclusion i_f in the factorization of f is called the *image of* f and is denoted as $\mathrm{Im}(f)$ or $f(A)$ when $dom(f) = A$.

The following property is a useful technical device in many proofs.

Lemma 4.16 (Diagonal-fill). *Given an inclusion system* $\langle I, \mathcal{E}\rangle$ *in a category* $\mathbb{C}$*, if* $f, g \in \mathbb{C}$*,* $e \in \mathcal{E}$*,* $i \in I$*, and* $f;i = e;g$ *then there exists an unique* $h \in \mathbb{C}$ *such that* $e;h = f$ *and* $h;i = g$*.*

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \arrow[dl, "h"'] \\ \bullet \arrow[r, "i"'] & \bullet \end{tikzcd}$$

Proof. Let us factor $f = e_f;i_f$ and $g = e_g;i_g$. Then $e;e_g;i_g = e;g = f;i = e_f;i_f;i$. By the uniqueness of the factorization of $e;g = f;i$ it follows that $e_f = e;e_g$ and $i_g = i_f;i$ and also that $dom(i_f) = cod(e_g)$. Then $h = e_g;i_f$.

$$\begin{tikzcd} \bullet \arrow[rr, "e"] \arrow[dr, "e_f"] \arrow[dd, "f"'] & & \bullet \arrow[dl, "e_g"'] \arrow[dd, "g"] \\ & \bullet \arrow[dl, "i_f"'] \arrow[dr, "i_g"] & \\ \bullet \arrow[rr, "i"'] & & \bullet \end{tikzcd}$$

The uniqueness of h follows by noticing that each inclusion is mono and because $h;i = g$. □

Epic inclusion systems. The abstract surjections of some inclusion systems need not necessarily be surjective in the ordinary set-theoretic sense. Consider for example the trivial inclusion system for $\mathbb{S}et$ where each function is an abstract surjection and the abstract inclusions are just the identities.An inclusion system $\langle I, \mathcal{E}\rangle$ is *epic* when all abstract surjections are epis. Therefore the standard inclusion system of $\mathbb{S}et$ presented above is epic, while the trivial one is not.

Unions. An inclusion system $\langle I, \mathcal{E}\rangle$ and *has unions* when I has finite least upper bounds (denoted $\cup$). Note that the standard inclusion system of $\mathbb{S}et$ has unions which are exactly the usual unions of sets, while the trivial inclusion system of $\mathbb{S}et$ evidently does not have unions.

Inclusive functors. A functor $\mathcal{U}\colon \langle I, \mathcal{E}\rangle \to \langle I', \mathcal{E}'\rangle$ (between the underlying categories of the inclusion systems) is *inclusive* when it preserves the inclusions, i.e., $\mathcal{U}(I) \subseteq I'$. Inclusion systems and inclusive functors form a category denoted $\mathbb{IS}$.

Model inclusions and quotients

Closed and strong model homomorphisms. The category of models for a **FOL** signature (S, F, P) admits two meaningful epic inclusion systems which inherit the conventional inclusion system of the category of sets and functions. Before discussing them, we have to define some special classes of model homomorphisms.

A model homomorphism $h \colon M \to N$

- is *closed* when $M_\pi = h^{-1}(N_\pi)$ for each relation symbol $\pi \in P$, and
- is *strong* when $h(M_\pi) = N_\pi$ for each relation symbol $\pi \in P$.

For each model homomorphism $M \to N$ that is a set inclusion for each sort $s \in S$, let us say that M is a *submodel* of N.

Inclusion systems for FOL models.

Fact 4.17. *For any* **FOL** *signature* (S,F,P)*, the category of* (S,F,P)*-models admits the following two inclusion systems:*

inclusion system	abstract surjections	abstract inclusions
closed	surjective homomorphisms	closed sub-models
strong	strong surjective homomorphisms	sub-models

Moreover, for each signature morphism $\varphi \colon (S,F,P) \to (S',F',P')$*, the model reduct functor* $\mathsf{Mod}(\varphi)$ *is inclusive between both the closed and the strong inclusion systems of* $\mathsf{Mod}(S',F',P')$ *and* $\mathsf{Mod}(S,F,P)$

The difference between these two inclusion systems can easily be understood when we try to factor a model homomorphism $h \colon M \to M'$:

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "e"'] & & M' \\ & h(M) \arrow[ur, "i"'] & \end{tikzcd}$$

Then in both inclusion systems $e;i$ is the unique factorization of h as (many-sorted) function and $h(M)_s = h_s(M_s)$ for each sort s. Also, in both inclusion systems the interpretation of the operation symbols is canonically defined by $h(M)_\sigma(m) = M'_\sigma(m)$ for each operation symbol $\sigma \in F_{w\to s}$ and each $m \in h(M)_w$. It is easy to see that for the carriers and the operations there is no other possibility. However, the difference between the two inclusion systems occurs at the level of the interpretations of the relation symbols for $h(M)$. Given $\pi \in P$, we should have $e(M_\pi) \subseteq h(M)_\pi$ and $i(h(M)_\pi) \subseteq M'_\pi$. This means

$$e(M_\pi) \subseteq h(M)_\pi \subseteq i^{-1}(M'_\pi).$$

For the closed inclusion system the interpretation of the relations is defined 'maximally' with respect to i, while in the second situation they are defined 'minimally' with respect to e.

Congruences. Several types of abstract surjections for model homomorphisms correspond to several types of *congruences*. Given a model M for a **FOL** signature (S,F,P), an *S-sorted equivalence* relation $\sim$ on M consists of an equivalence relation $\sim_s$ on M_s for each sort s. It is

- an *F-congruence* when for each operation symbol $\sigma \in F_{w \to s}$, $M_\sigma(m) \sim_s M_\sigma(m')$ for all $m, m' \in M_w$ with $m \sim_w m'$,[3]
- a *P-congruence* when for each relation symbol $\pi \in P_w$, $m \sim_w m'$ and $m \in M_\pi$ implies $m' \in M_\pi$ for each $m, m' \in M_w$, and
- an (F,P)*-congruence* when it is both an F-congruence and a P-congruence.

Quotient models. Given an F-congruence $\sim$ on M, the *quotient* $M/_\sim$ (of the model M by the congruence $\sim$) is defined by

- $(M/_\sim)_s = \{m/_\sim \mid m \in M_s\}$ is the set of equivalence classes for $\sim_s$ for each sort $s \in S$,
- $(M/_\sim)_\sigma(m/_\sim) = M_\sigma(m)/_\sim$ for each operation $\sigma \in F_{w \to s}$ and each $m \in M_w$, and
- $(M/_\sim)_\pi = \{m/_\sim \mid m \in M_\pi\}$ for each relation symbol $\pi \in P$.

The homomorphism $M \to M/_\sim$ mapping each m to its congruence class $m/_\sim$ is called a *quotient homomorphism.*

Fact 4.18. *Any quotient homomorphism $M \to M/_\sim$ is strong surjective. Moreover when $\sim$ is an (F,P)-congruence it is also closed.*

It is also easy to see that each closed abstract surjection is strong too.

Kernels. Given a model homomorphism $f\colon M \to N$, its *kernel* is defined by

$$=_f = \{(a,a') \mid f(a) = f(a')\}$$

Fact 4.19. *The kernel of any homomorphism f is an F-congruence. Moreover, it is an (F,P)-congruence when f is closed.*

The universal property of quotients. Model quotients admit the following universal property:

Proposition 4.20. *Let $q\colon M \to M'$ be a surjective (S,F,P)-model homomorphism for a signature (S,F,P). Then for each model homomorphism $f\colon M \to N$, if $=_q \subseteq =_f$, then there exists an unique model homomorphism $f'\colon M' \to N$ such that $q;f' = f$.*

$$\begin{tikzcd} M \arrow[rr, "q"] \arrow[dr, "f"'] & & M' \arrow[dl, "f'"] \\ & N & \end{tikzcd}$$

Moreover, f' is strong when f is strong and it is closed when f is closed.

[3] For each $w = s_1 \dots s_n$ any list of sorts, $m_1 \dots m_n \sim_w m'_1 \dots m'_n$ when $m_1 \sim_{s_1} m'_1, \dots, m_n \sim_{s_n} m'_n$.

Proof. f' is defined by $f'(m/_{=_q}) = f(m)$ for each $m \in M$. This definition is correct since $=_q \subseteq =_f$. The fact that f is an F-algebra homomorphism implies that f' is an F-algebra homomorphism. Also, the fact that f is a P-model homomorphism implies that f' is a P-model homomorphism. The uniqueness of f' follows from the fact that q is surjective.

Simple calculations show that f being strong, respectively, closed implies f' is strong, respectively, closed. □

Corollary 4.21. *For each strong surjective model homomorphism $f : M \to N$, $M/_{=_f}$ and N are isomorphic, i.e., $M/_{=_f} \cong N$.*

Proof. Assume that f is surjective and q is the quotient $M \to M/_{=_f}$ in Proposition 4.20 above. $\sim = =_f$ implies that f' is injective, while f surjective implies that f' is surjective. Therefore f' is a bijection, which makes it immediately an F-algebra isomorphism. When f is strong, f' is also strong, which means that for each relation symbol $\pi \in P$, $f'((M/_{=_f})_\pi) = N_\pi$. This implies that the inverse f'^{-1} is also a P-model homomorphism, hence f' is an (S,F,P)-model isomorphism. □

Signature inclusions in FOL

Fact 4.22. *The category of* **FOL** *signatures admits the inclusion systems given by the table below:*

inclusion system	abstract surjections $\varphi : (S,F,P) \to (S',F',P')$	abstract inclusions $(S,F,P) \hookrightarrow (S',F',P')$
closed	$\varphi^{\mathrm{st}} : S \to S'$ surjective	$S \subseteq S'$ $F_{w\to s} = F'_{w\to s}$ for $w \in S^*$ $P_w = P'_w$ for $s \in S$
strong	$\varphi^{\mathrm{st}} : S \to S'$ surjective $F'_{w'\to s'} = \bigcup_{\varphi^{\mathrm{st}}(ws)=w's'} \varphi^{\mathrm{op}}(F_{w\to s})$ $P'_{w'} = \bigcup_{\varphi^{\mathrm{st}}(w)=w'} \varphi^{\mathrm{rl}}(P_w)$	$S \subseteq S'$ $F_{w\to s} \subseteq F'_{w\to s}$ for $w \in S^*$ $P_w \subseteq P'_w$ for $s \in S$

We can also note that the closed inclusion system does not have unions but the strong one has them:

Fact 4.23. *In the strong inclusion system the union of signatures $(S,F,P) = (S_1,F_1,P_1) \cup (S_2,F_2,P_2)$ is given by*

- $S = S_1 \cup S_2$,
- *for each $w \in S^*$ and $s \in S$, $F_{w\to s} = (F'_1)_{w\to s} \cup (F'_2)_{w\to s}$ where $(F'_k)_{w\to s} = (F_k)_{w\to s}$ when $w \in S_k^*$, $s \in S_k$ and $(F'_k)_{w\to s} = \emptyset$ otherwise, and*
- *for each $w \in S^*$, $P_w = (P'_1)_w \cup (P'_2)_w$ where $(P'_k)_w = (P_k)_w$ when $w \in S_k^*$ and $(P'_k)_w = \emptyset$ otherwise.*

Inclusive institutions. Both the strong and the closed inclusion systems make **FOL** an 'inclusive' institution. An institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is called *inclusive* when Sen is an inclusive functor, i.e., the category of signatures comes equipped with an inclusion system such that $\mathsf{Sen}(\Sigma) \subseteq \mathsf{Sen}(\Sigma')$ whenever $\Sigma \hookrightarrow \Sigma'$ is inclusion of signatures.

Theory inclusions

In general, theories of institutions inherit inclusion systems from the category of signatures in two different ways similar to the ways model homomorphisms in **FOL** inherit the conventional inclusion system of $\mathbb{S}et$, the category of sets and functions.

A theory morphism $\varphi\colon\ (\Sigma, E) \to (\Sigma', E')$

- is *closed* when $E = \varphi^{-1}(E')$, and
- is *strong* when $E' = \varphi(E)^{**}$.

Given an inclusion system for the category of the signatures, we may factor each theory morphism $\varphi\colon\ (\Sigma, E) \to (\Sigma', E')$ through the inclusion system of the signatures as $\varphi = e_\varphi; i_\varphi$

$$\begin{tikzcd}
(\Sigma, E) \arrow[rr, "\varphi"] \arrow[dr, "e_\varphi"'] & & (\Sigma', E') \\
& (\varphi(\Sigma), E'') \arrow[ur, "i_\varphi"'] &
\end{tikzcd}$$

In order to get a factorization of φ in the category of theories, we have only to fix the theory E'', which can be defined either

- 'maximally' with respect to the signature (abstract) inclusion i_φ by letting $E'' = i_\varphi^{-1}(E')$, or
- 'minimally' with respect to e_φ by letting $E'' = e_\varphi(E)^{**}$.

Hence

Proposition 4.24. *In any inclusive institution, the inclusion system of signatures lifts to theories in two different ways:*

inclusion system	abstract surjections $\varphi\colon\ (\Sigma, E) \to (\Sigma', E')$	abstract inclusions $(\Sigma, E) \hookrightarrow (\Sigma', E')$
closed	$\varphi\colon\ S \to S'$ abstract surjection	$\Sigma \subseteq \Sigma'$ abstract inclusion $E = E' \cap \mathsf{Sen}(\Sigma)$
strong	$\varphi\colon\ S \to S'$ abstract surjection $E' = \varphi(E)^{**}$	$\Sigma \subseteq \Sigma'$ abstract inclusion

Fact 4.25. *When the signatures admit an inclusion system with unions, the strong inclusion system of theories has unions by letting*

$$(\Sigma, E) \cup (\Sigma', E') = (\Sigma \cup \Sigma', (E \cup E')^{**}).$$

Exercises

4.46. In any inclusion system the class of the abstract inclusions determines the class of abstract surjections, in the sense that if $\langle I, \mathcal{E}\rangle$ and $\langle I', \mathcal{E}'\rangle$ are two inclusion systems for the same category and if $I \subseteq I'$, then $\mathcal{E}' \subseteq \mathcal{E}$.

4.47. In any inclusion system
(a) each abstract inclusion is a mono,
(b) each co-equalizer is an abstract surjection,
(c) an arrow is both an abstract inclusion and an abstract surjection if and only if it is an identity,
(d) if $f;g$ is an abstract surjection, then g is an abstract surjection.

4.48. In any category with an inclusion system

- the abstract surjections are stable under pushouts,
- if the category has pullbacks
  - for each $\langle i, g\rangle$ where i is an abstract inclusion, there exists a unique pullback $\langle g', i'\rangle$ such that i' is an abstract inclusion

$$\begin{tikzcd} \bullet \arrow[r, "i"] & \bullet \\ \bullet \arrow[u, "g'"] \arrow[r, "i'"'] & \bullet \arrow[u, "g"'] \end{tikzcd}$$

    Consequently, each span of abstract inclusions $\bullet \longrightarrow \bullet \longleftarrow \bullet$ has a unique pullback of abstract inclusions.
  - By virtue of the previous item, we may define the 'intersection' $A \cap B$ of any two objects in an inclusion system with unions. Then the following distributivity laws hold:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \ \text{ and } \ A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

4.49. In any inclusion system for a category with small limits, each small co-cone $\{i_k : N_k \to M\}_{k \in I}$ of inclusions has a limiting co-cone $\{i'_k : N \to N_k\}_{k \in I}$ of inclusions.

4.50. (a) The forgetful functor $\mathbb{IS} \to \mathbb{C}at$ mapping inclusion systems to their underlying categories has a left adjoint and creates small products.
(b) The category $\mathbb{IS}$ of inclusion systems is cartesian closed.

4.51. Generated closed sub-models
Given an (S,F,P)-model M for a **FOL** signature (S,F,P), an arbitrary intersection of [closed] sub-models of M is a submodel of M.
For any S-sorted set $\{X_s\}_{s \in S} \subseteq \{M_s\}_{s \in S}$ we say that N *is the [closed] submodel of* M *generated by* X when N is the least [closed] submodel containing X.

4.52. Intersection of congruences
For any model of any **FOL** signature (S,F,P), an arbitrary intersection of F-congruences is a congruence but only the intersection of a *non-empty* family of (F,P)-congruences is an (F,P)-congruence.

4.53. In the **FOL** model, products preserve closed and strong models homomorphisms. For any family $\{f_i : M_i \to N_i \mid i \in I\}$ of model homomorphisms for a fixed **FOL** signature, the product $\prod_{i \in I} f_i : \prod_{i \in I} M_i \to \prod_{i \in I} N_i$ is closed, respectively strong, when f_i is closed, respectively strong, for each $i \in I$.

4.54. Amalgamation of homomorphisms
The sub-institutions of **FOL** determined by the closed, respectively strong, model homomorphisms are exact.

4.55. Inclusion system for preordered algebras
For any algebraic signature (S,F), the category of preordered (S,F)-algebras admits an inclusion system in which the abstract inclusions are *closed* preordered subalgebras, i.e., preodered subalgebras $M \hookrightarrow N$ such that $m \leq_M m'$ if and only if $m \leq_N m'$, and the abstract surjections are just preordered algebra homomorphisms which are (component-wise) surjective functions.

4.56. Preorder algebra congruences
A **POA**-*congruence* (preordered algebra congruence) on a preordered algebra for a signature (S,F) is a pair $(\sim, \sqsubseteq)$ such that

- $\sim$ is an F-congruence on M,
- $\sqsubseteq$ is a(n S-sorted) preorder on M compatible with the operations and which contains $M_{\leq}$, i.e., $M_{\leq} \subseteq \sqsubseteq$, and
- $a' \sim a, a \sqsubseteq b, b \sim b'$ implies $a' \sim b'$ for all elements a, a', b, b' of M.

The **POA**-*kernel* $ker(h)$ of a preordered algebra homomorphism $h\colon\ M \to N$ is $(=_h, \leq_h)$ where $a =_h b$ is defined by $h(a) = h(b)$ and $a \leq_h b$ by $h(a) \leq h(b)$.

Define the quotient of preordered algebras by **POA**-congruences. Extend Proposition 4.20 to a universal property for preordered algebra quotients.

4.57. Inclusion systems for partial algebra
Let (S,TF,PF) be a **PA** signature. A homomorphism $h\colon\ A \to B$ of partial algebras is

- *full* if whenever $B_\sigma(h(a)) \in h(A)$, then there exists $a' \in A_w$ such that $A_\sigma(a')$ is defined and $h(a') = h(a)$,
- *closed* when $A_\sigma(a)$ is defined if $B_\sigma(h(a))$ is defined

for each $\sigma \in PF_{w\to s}$.
The category of (S,TF,PF)-partial algebras admits the following inclusion systems:

abstract surjections	*abstract inclusions*
epi model homomorphisms	closed inclusions (S_c)
surjective homomorphisms	full inclusions (S_f)
full surjective homomorphisms	(plain) inclusions (S_w)

4.58. Full FOL model homomorphisms
Let (S,F,P) be a **FOL** signature. An (S,F,P)-model homomorphism $h\colon\ M \to N$ is *full* when $h(M_\pi) = N_\pi \cap h(M_w)$ for each relation symbol $\pi \in P$. Then

- any full surjective model homomorphism is strong, and
- any closed model homomorphism is full.

4.59. Which of the institutions **MBA**, **MA**, **CA**, **IPL**, **HOL**, and **HNK**, admit non-trivial inclusion systems for their categories of models?

4.60. Let (Σ,E), (Σ',E'), and (Σ'',E'') be theories in an arbitrary institution and $\varphi\colon\ \Sigma \to \Sigma'$ and $\phi\colon\ \Sigma' \to \Sigma''$ be signature morphisms such that $\varphi;\phi$ is a theory morphism $(\Sigma,E) \to (\Sigma'',E'')$. Then

- if φ is a strong theory morphism $(\Sigma,E) \to (\Sigma',E')$, then ϕ is a theory morphism $(\Sigma',E') \to (\Sigma'',E'')$, and

- if ϕ is a closed theory morphism $(\Sigma', E') \to (\Sigma'', E'')$, then φ is a theory morphism $(\Sigma, E) \to (\Sigma', E')$.

4.61. The strong inclusion system of **FOL** signatures is epic.

4.62. When the inclusion system of signatures is epic, both the closed and the strong inclusion systems of theories are epic too.

4.6 Free Models

In this section we first study the existence of initial models for Horn theories. Here we develop this result for the special case of first order logic, later in the book (Sect. 8.3), in the context of quasi-varieties, we will present a much more general version of this result.

Next, this time in an institution-independent framework, we show that the existence of initial models of theories can be extended to the existence of free models along theory morphisms.

Initial models of Horn theories in FOL

Recall that a *universal Horn sentence* for a signature (S,F,P) is a sentence of the form $(\forall X)H \Rightarrow C$, where H is a finite conjunction of (relational or equational) atoms, C is a (relational or equational) atom, and $H \Rightarrow C$ is the implication of C by H.

For each (S,F,P)-model M and for each set Γ of universal Horn (S,F,P)-sentences, we define the model M_Γ by

– Let

$$=_\Gamma = \bigcap\{=_h \mid h : \ M \to N \ \text{ model homomorphism and } \ N \models \Gamma\}.$$

Since any intersection of F-congruences is an F-congruence, $=_\Gamma$ is an F-congruence too.

– As (S,F)-algebra, let M_Γ be the quotient $M/_{=_\Gamma}$.

– For each relation symbol $\pi \in P$ let

$$(M_\Gamma)_\pi = \{m/_{=_\Gamma} \mid h(m) \in N_\pi \ \text{ for each } \ h : \ M \to N \ \text{ such that } \ N \models \Gamma\}.$$

We notice easily that the quotient mapping $q_\Gamma : \ M \to M_\Gamma$ defined by $q_\Gamma(m) = m/_{=_\Gamma}$ is a model homomorphism.

However note also that M_Γ is *not* the quotient $M/_{=_\Gamma}$ (as defined in Sect. 4.5) of the (S,F,P)-model M by $=_\Gamma$. The reason is that they differ on the interpretations of the relation symbols; we have that $(M/_{=_\Gamma})_\pi \subseteq (M_\Gamma)_\pi$ but this is a strict inclusion in general.

Proposition 4.26. *Let* Γ *be any set of universal Horn* (S,F,P)*-sentences.*

1. *For each* (S,F,P)*-model homomorphism* $h\colon\ M \to N$ *such that* $N \models \Gamma$ *there exists a unique model homomorphism* $h_\Gamma\colon\ M_\Gamma \to N$ *such that* $q_\Gamma;h_\Gamma = h$.

$$\begin{tikzcd} M \arrow[r, "q_\Gamma"] \arrow[dr, "h"'] & M_\Gamma \arrow[d, "h_\Gamma"] \\ & N \end{tikzcd}$$

2. $M_\Gamma \models \Gamma$.

Proof. 1. This follows from the universality Prop. 4.20 because $(=_{q_\Gamma}) = (=_\Gamma) \subseteq (=_h)$. Note that $h_\Gamma(m/_{=_\Gamma}) = h(m)$ for each $m \in M$.

2. Let $(\forall X)H \Rightarrow C$ be any universal Horn sentence in Γ. Consider any expansion M'_Γ of M_Γ to $(S,F\cup X,P)$ such that $M'_\Gamma \models H$. Let M' be any expansion of M to $(S,F\cup X,P)$ such that $q_\Gamma\colon\ M' \to M'_\Gamma$ is an $(S,F\cup X,P)$-model homomorphism (which means that for each $x \in X$ we choose an element M'_x of $(M'_\Gamma)_x$).

For any model homomorphism $h\colon\ M \to N$ such that $N \models \Gamma$ let N' be the expansion of N to $(S,F\cup X,P)$ such that $h\colon\ M' \to N'$ is an $(S,F\cup X,P)$-model homomorphism (defined by $N'_x = h(M'_x)$). Then $h_\Gamma\colon\ M'_\Gamma \to N'$ becomes an $(S,F\cup X,P)$-model homomorphism too. Because $M' \models H$ and model homomorphisms preserve the satisfaction of the atoms (the reader is requested to check this) we have that $N' \models H$, which implies $N' \models C$ (because $N \models (\forall X)H \Rightarrow C$).

When C is an equational atom $t = t'$, $N' \models t = t'$ means that $h(M'_t) = h(M'_{t'})$ which, written differently, means $(M'_t, M'_{t'}) \in =_h$. Since h is arbitrarily chosen, this implies $M'_t =_\Gamma M'_{t'}$. Thus $M'_\Gamma \models t = t'$.

When C is a relational atom $\pi(t)$ (for t an appropriate list of terms), $N' \models \pi(t)$ means that $h(M'_t) = N'_t \in N_\pi$ which, since h is arbitrarily chosen, implies $M'_t/_{=_\Gamma} \in (M_\Gamma)_\pi$. But this means that $(M'_\Gamma)_t \in (M_\Gamma)_\pi$ which is the same with $M'_\Gamma \models \pi(t)$. □

Initial models of FOL signatures. In order to obtain that each Horn theory in **FOL** has initial models, we should apply Prop. 4.26 for M being the initial (S,F,P)-model.

Proposition 4.27. *For any* **FOL***-signature* (S,F,P) *there exists an initial* (S,F,P)*-model* $0_{(S,F,P)}$ *defined by*

- *for each sort* $s \in S$*, let* $(0_{(S,F,P)})_s = (T_F)_s$ *be the set of all* F*-terms of sort* s*,*
- *for each operation symbol* $\sigma \in F_{w\to s}$*,* $(0_{(S,F,P)})_\sigma$ *is defined by* $(0_{(S,F,P)})_\sigma(t_1,\ldots,t_n) = \sigma(t_1,\ldots,t_n)$ *for each list of terms* $(t_1,\ldots,t_n) \in (T_F)_w$*.*
- *for each relation symbol* $\pi \in P_w$*,* $(0_{(S,F,P)})_\pi = \emptyset$*.*

Proof. For each (S,F,P)-model M, there exists a unique model homomorphism $h\colon 0_{(S,F,P)} \to M$ defined by

$$h_s(\sigma(t)) = M_\sigma(h_w(t))$$

for each operation symbol $\sigma \in F_{w\to s}$ and each list of terms $t \in (T_F)_w$. □

Corollary 4.28. *For any set Γ of universal Horn (S,F,P)-sentences, the model $0_\Gamma = (0_{(S,F,P)})_\Gamma$ is the initial Γ-model, i.e., the initial model in $\mathsf{Mod}((S,F,P),\Gamma)$.*

Liberal theory morphisms

In any institution $(\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models)$, a theory morphism $\varphi\colon (\Sigma,E) \to (\Sigma',E')$ is *liberal* if and only if the reduct functor $\mathsf{Mod}^{\mathrm{pres}}(\varphi)\colon \mathsf{Mod}^{\mathrm{pres}}(\Sigma',E') \to \mathsf{Mod}^{\mathrm{pres}}(\Sigma,E)$ has a left-adjoint $(\_)^\varphi$.

In other words, for each (Σ,E)-model M there exists a (Σ',E')-model M^φ and a Σ-model homomorphism $\eta_M\colon M \to (M^\varphi)\upharpoonright_\varphi$

$$\begin{tikzcd}
M \models_\Sigma E & M \arrow[r, "\eta_M"] \arrow[d, "h"'] & (M^\varphi)\upharpoonright_\varphi \arrow[dl, "h'\upharpoonright_\varphi"] & M^\varphi \arrow[dl, "\text{there exists a unique } h'"] \\
M' \models_{\Sigma'} E' & M'\upharpoonright_\varphi & M' &
\end{tikzcd}$$

such that for each (Σ',E')-model M' and for each Σ-model homomorphism $h\colon M \to M'\upharpoonright_\varphi$, there exists a unique Σ'-model homomorphism $h'\colon M^\varphi \to M'$ such that $\eta_M;h'\upharpoonright_\varphi = h$.

Note that by composition of adjunctions (see Sect. 2.3), the composition of liberal theory morphisms is liberal. An institution is *liberal* if and only if each theory morphism is liberal.

In any institution with initial signatures which are mapped by the model functor to the terminal category, it is immediate to notice that the existence of initial models for a theory (Σ,E) is the same as the liberality of the unique presentation morphism $(\Sigma,\emptyset) \to (\Sigma,E)$ from the initial Σ-presentation to (Σ,E). The results below show that liberality can be established in general from initiality of theory models.

Proposition 4.29. *Let $(\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models,\iota)$ be an institution with elementary diagrams such that each theory has an initial model. Then*

1. *for each theory (Σ,E), the forgetful functor $\mathsf{Mod}^{\mathrm{pres}}(\Sigma,E) \to \mathsf{Mod}(\Sigma)$ has a left adjoint, and*
2. *if in addition the institution has pushouts of signatures and is semi-exact, then for each signature morphism φ the reduct functor $\mathsf{Mod}(\varphi)$ has a left adjoint.*

Proof. For each presentation (Σ,E), we denote its initial model by $0_{\Sigma,E}$.

1. Consider a presentation (Σ,E) and let M be a Σ-model. Let $\iota_\Sigma(M)\colon \Sigma \to \Sigma_M$ be the elementary extension of Σ via M and let $E' = \iota_\Sigma(M)(E)$. We show that $M' = (0_{\Sigma_M,E_M\cup E'})\upharpoonright_{\iota_\Sigma(M)}$ is the free (Σ,E)-model over M with the universal arrow $\eta_M = (M_M \to 0_{\Sigma_M,E_M\cup E'})\upharpoonright_{\iota_\Sigma(M)}\colon M \to M'$.

We have to prove that for each model homomorphism $h\colon M \to N$ with $N \models_\Sigma E$, there exists a unique $h'\colon M' \to N$ such that $\eta_M;h' = h$. Let $N_h = i^{-1}_{\Sigma,M}(h)$. Then $N_h \models_{\Sigma_M} E'$ because $N_h\upharpoonright_{\iota_\Sigma(M)} = N$ and $N \models_\Sigma E$. Let h'' be the unique model homomorphism $h''\colon 0_{\Sigma_M,E_M\cup E'} \to N_h$. Let h' be $h''\upharpoonright_{\iota_\Sigma(M)}$. Then $\eta_M;h' = (M_M \to N_h)\upharpoonright_{\iota_\Sigma(M)} = h$.

The uniqueness of h' follows by the bijection between $(M/\mathsf{Mod}(\Sigma))(\eta_M, h)$ and $\mathsf{Mod}^{\mathrm{pres}}(\Sigma_M, E_M)(0_{\Sigma_M, E_M \cup E'}, N_h)$.

2. Let $\varphi\colon\ \Sigma \to \Sigma'$ be a signature morphism and let M be a Σ-model. Consider the pushout of signatures

$$\begin{tikzcd} \Sigma_M \arrow[r, "\varphi'"] & \Sigma'' \\ \Sigma \arrow[u, "\iota_\Sigma(M)"] \arrow[r, "\varphi"'] & \Sigma' \arrow[u, "\iota'"'] \end{tikzcd}$$

We define M^φ to be $(0_{\Sigma'', \varphi'(E_M)})\upharpoonright_{\iota'}$ and the universal arrow $\eta_M\colon\ M \to (M^\varphi)\upharpoonright_\varphi$ to be $(M_M \to (0_{\Sigma'', \varphi'(E_M)})\upharpoonright_{\varphi'})\upharpoonright_{\iota_\Sigma(M)}$.

For proving the universal property of η_M, consider $h\colon\ M \to N\upharpoonright_\varphi$ with N any Σ'-model. Let $M_h = i^{-1}_{\Sigma,M}(h)$. Notice that $M_h\upharpoonright_{\iota_\Sigma(M)} = N\upharpoonright_\varphi$. Because the institution has model amalgamation, let $N \otimes_{\varphi, \iota_\Sigma(M)} M_h$ be the amalgamation of N and M_h. Notice that $N \otimes_{\varphi, \iota_\Sigma(M)} M_h \models \varphi'(E_M)$ because $(N \otimes_{\varphi, \iota_\Sigma(M)} M_h)\upharpoonright_{\varphi'} = M_h \models E_M$. Therefore there exists a unique model homomorphism $h''\colon\ 0_{\Sigma'', \varphi'(E_M)} \to N \otimes_{\varphi, \iota_\Sigma(M)} M_h$. Let $h' = h''\upharpoonright_{\iota'}$. We have that $h'\colon\ M^\varphi \to N$ and $\eta_M; h'\upharpoonright_\varphi = \eta_M; h''\upharpoonright_{\iota'}\upharpoonright_\varphi = \eta_M; h''\upharpoonright_{\varphi'}\upharpoonright_{\iota_\Sigma(M)} = (M_M \to M_h)\upharpoonright_{\iota_\Sigma(M)} = h$.

The uniqueness of h' follows from the uniqueness of h'' and of the amalgamation property for model homomorphisms given by the assumption that the institution is semi-exact. $\square$

Corollary 4.30. *A semi-exact institution with elementary diagrams and pushouts of signatures is liberal when each theory has an initial model.*

Conversely, if the institution has initial signatures and is finitely exact, each theory has an initial model whenever the institution is liberal.

Proof. The second part of this corollary has been already proved above. For the first part let us consider a theory morphism $\varphi\colon\ (\Sigma, E) \to (\Sigma', E')$.

$$\begin{tikzcd} \mathsf{Mod}(\Sigma) & \mathsf{Mod}(\Sigma') \arrow[l, "\mathsf{Mod}(\varphi)"'] \\ \mathsf{Mod}^{\mathrm{pres}}(\Sigma, E) \arrow[u, "\text{forgetful subcategory}"] & \mathsf{Mod}^{\mathrm{pres}}(\Sigma', E') \arrow[l, "\mathsf{Mod}^{\mathrm{pres}}(\varphi)"] \arrow[u, "\text{forgetful subcategory}"'] \end{tikzcd}$$

By Proposition 4.29, both $\mathsf{Mod}(\varphi)\colon\ \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$ and the forgetful functor $\mathsf{Mod}^{\mathrm{pres}}(\Sigma', E') \to \mathsf{Mod}(\Sigma')$ have left-adjoints. By composition of adjunctions, the composite functor $\mathsf{Mod}^{\mathrm{pres}}(\Sigma', E') \to \mathsf{Mod}(\Sigma)$ has a left-adjoint.

The proof of the first part is resumed by substituting the category D by $\mathsf{Mod}^{\mathrm{pres}}(\Sigma', E')$, the category C by $\mathsf{Mod}(\Sigma)$, and the category C' by $\mathsf{Mod}^{\mathrm{pres}}(\Sigma, E)$ in the following simple categorical lemma (whose proof is left as exercise for the reader).

Lemma 4.31. *Let $C' \hookrightarrow C$ be a full subcategory and consider a functor $D \to C'$. If the composite functor $D \to C$ has a left-adjoint F, then the restriction of F to C' is a left-adjoint to $D \to C'$.*

$$\begin{tikzcd} C & \\ C' \arrow[u] & D \arrow[l] \arrow[ul] \end{tikzcd}$$

□

A concrete application of Cor. 4.30 is the following:

Corollary 4.32. *The institution* **HCL** *is liberal.*

Liberal institution mappings

It is also useful to consider free models also across institution morphisms or comorphisms.

An institution morphism $(\Phi, \alpha, \beta): I' \to I$ is *liberal* when the model translations $\beta_{\Sigma'}: \mathsf{Mod}'(\Sigma') \to \mathsf{Mod}(\Phi(\Sigma'))$ have left adjoints for all I'-signatures Σ'. Similarly, an institution comorphism (Φ, α, β) is *liberal* when β_Σ has a left adjoint for each I-signature Σ.

Persistently liberal institution (co-)morphisms. Especially useful for the transfer of institutional properties across institution mappings, is the case when these adjunctions corresponding to the model translations β_Σ are persistent, which means that the left-adjoint to β_Σ is also a left-inverse (up to isomorphism) to β_Σ. In many actual situations, persistently liberal institution comorphisms determine useful 'representations' of a more complex institution into a simpler one.

Encoding relations as operations. A first example is given by the comorphism **FOL** $\to$ **FOEQL** discussed in Sect. 3.3. For each **FOL**-signature (S, F, P), the adjunction between $\mathsf{Mod}^{\mathbf{FOL}}(S,F,P)$ and $\mathsf{Alg}(S \uplus \{\mathbf{b}\}, F \uplus \overline{P} \uplus \{\mathbf{true}\})$ is persistently liberal, with the free $(S \uplus \{\mathbf{b}\}, F \uplus \overline{P} \uplus \{\mathbf{true}\})$-algebra M' over a model M interpreting 'freely' the non-true values by $M'_{\mathbf{b}} = \{M'_{\mathbf{true}}\} \uplus \{\pi(m) \mid \pi \in P, m \notin M_\pi\}$. Hence

Fact 4.33. *The encoding of relations as operations* **FOL** $\to$ **FOEQL** *is persistently liberal.*

Exercises

4.63. Give a counterexample showing that **FOL** is not liberal.

4.64. Γ-congruences
Let Γ be a set of universal Horn sentences for an algebraic signature (S,F). A congruence $\equiv$ on an algebra A is a Γ-*congruence* if and only if for any sentence $(\forall X)H \Rightarrow (t = t')$ in Γ and for each expansion A' of A to $(S, F \cup X)$, $A'_t \equiv A'_{t'}$ if $A'_{t_1} \equiv A'_{t_2}$ for all $t_1 = t_2$ in H.

Then $=_\Gamma$ is the least Γ-congruence.

4.65. Liberality in PA
In **PA** each morphism between presentations of universally quantified (possibly conditional) existence equations is liberal.

4.66. Liberality in POA
The institution **HPOA** (of Horn **POA**-sentences) is liberal. (*Hint:* Extend the concept of Γ-congruence of Ex. 4.64 to **POA**-congruences of Ex. 4.56 and show that for each **POA**-algebra M, the quotient $q_\Gamma : M \to M/_{(=_\Gamma, \leq_\Gamma)}$ is the free preordered algebra satisfying Γ, where $(=_\Gamma, \leq_\Gamma)$ is the least Γ-**POA**-congruence.)

4.67. Give a counterexample showing that in the institution **MA** of multialgebras not all sets of atoms have initial models.

4.68. (a) Give a counterexample showing in general, in **HOL**, signatures do not admit initial models.
(b) On the other hand, all **HNK**-signatures which have at least one constant for each type have initial models. (*Hint:* Consider the comorphism $(\Phi, \alpha, \beta) :$ **HNK** $\to$ **FOEQL**$^{\text{pres}}$ of Ex. 4.11. Then for each **HNK**-signature Σ, the **FOEQL**-presentation $\Phi(\Sigma)$ has initial models, one of them being just the term model.)

4.69. Each **LA**-signature morphism is liberal.

4.70. Each **CA**-signature has initial algebras. (S, F, q) in **CA** has an initial algebra. (*Hint:* For any **CA**-signature (S, F, q) the S-sorted set T_F^ω of (possibly) infinite terms can be organized as a contraction (S, F, q)-algebra with the distance between two terms t and t' being $q^{\alpha(t,t')}$, where $\alpha(t,t')$ is the minimum depth at which t and t' differ.)

4.71. Liberality of comorphism FOL $\to$ (FOL^1)$^{\text{pres}}$
The encoding of many-sorted logic into single-sorted logic described in Sect. 4.1 is a liberal comorphism. (*Hint:* For each **FOL**-signature (S, F, P) and any (S, F, P)-model M, we first take the disjoint union $\uplus_{s\in S} M_s$. Then we take the free $\overline{F}$-algebra over $\uplus_{s\in S} M_s$ where $\overline{F}$ is the single-sorted variant of F. Then we take take its quotient under the congruence generated by the pairs $\langle \sigma(m), m' \rangle$ for which $M_\sigma(m) = m'$ for all $\sigma \in F$. The final step is to organize this quotient $\overline{F}$-algebra as an $(\overline{F}, \overline{P} \cup \{(- :\ s) \mid s \in S\})$-model; this is done in a canonical way.)

4.72. Institution representations
An *institution representation* $I \to I'$ is just a persistently liberal institution comorphism $I \to I'^{\text{p}}$ from I to the presentations of I'. Institution representations compose and form a category. (*Hint:* For any institution representation $I \to I'$, the induced institution comorphism $I^{\text{pres}} \to I'^{\text{pres}}$ is persistently liberal.)

4.73. [133] **Creating liberality along institution comorphisms**
Persistently liberal institution comorphisms $(\Phi,\alpha,\beta)\colon\ I \to I'$ create liberality in the sense that any theory morphism $\varphi\colon\ (\Sigma_1,E_1) \to (\Sigma_2,E_2)$ is liberal if $\Phi(\varphi)\colon\ (\Phi(\Sigma_1),\alpha(E_1)) \to (\Phi(\Sigma_2),\alpha(E_2))$ is liberal.

Apply this for the following comorphisms:

- $\mathbf{PA} \to \mathbf{FOL}^{\mathrm{pres}}$ (the operational encoding introduced in Sect. 4.1),
- $\mathbf{POA} \to \mathbf{FOL}^{\mathrm{pres}}$ of Ex. 4.9,
- $\mathbf{MBA} \to \mathbf{FOL}$ of Ex. 3.20,
- $\mathbf{AUT} \to \mathbf{FOL^1}$ of Ex. 3.21,
- $\mathbf{IPL} \to (\mathbf{FOEQL^1})^{\mathrm{pres}}$ of Ex. 4.10, and
- $\mathbf{LA} \to (\mathbf{FOEQL^1})^{\mathrm{pres}}$ of Ex. 4.13,

and deduce corresponding liberality results for **PA**, **POA**, **MBA**, **AUT**, **IPL**, and **LA**.

4.74. Comorphism $\mathbf{EQL} \to \mathit{Cat}\mathbf{EQL}$
Construct a canonical institution comorphism $\mathbf{EQL} \to \mathit{Cat}\mathbf{EQL}$ (see Ex. 3.8) by mapping

- each algebraic signature (S,F) to the category $\mathsf{Alg}(S,F)$ of (S,F)-algebras, and
- each (S,F)-equation $(\forall X)t = t'$ to the $\mathsf{Alg}(S,F)$-equation $(\forall T_{(S,F)}(X))t^\sharp = t'^\sharp$ where $t^\sharp$, $t'^\sharp$ are the unique extensions of t, t' to (S,F)-algebra homomorphisms $T_{(S,F)}(\{*\}) \to T_{(S,F)}(\{X\})$ from the (S,F)-algebra free over the singleton set $\{*\}$ to the (S,F)-algebra free over the set X.

4.75. Model pushouts [48]
In any liberal institution with elementary diagrams the category of models of any theory has pushouts. Moreover if the institution is also exact and has initial signatures, then the category of models of any theory has finite co-limits. (*Hint:* the pushout of model homomorphisms is the same with the universal arrow to a canonical functor between comma categories of models.)

Notes. Both the 'operational' and the 'relational' encoding comorphisms $\mathbf{PA} \to \mathbf{FOL}^{\mathrm{pres}}$ appear in [133]. Encoding modalities in relational logic is known in modal logic literature under the name of 'standard translation'. The ideas behind the comorphism $\mathbf{HNK} \to \mathbf{FOEQL}^{\mathrm{pres}}$ appear in [126].

Co-limits of theories have been playing a very important role in algebraic specification [75, 58]; one could say that the search for an institution-independent approach to compositionality of specification theories was one of the origins of institutions. By contrast, theory limits seem to be much less important in applications.

Institution theory is the only model theory that first properly identified [161] and then gradually realized the importance [58] of the model amalgamation (exactness) properties of logics. Since then semi-exactness has been intensively used as a basic institutional property by various works in algebraic specification. In practice very often the weak version of exactness suffices. This has been already considered in several works [44, 173] and is especially important for the case of the multi-paradigm or heterogeneous institutions obtained by a Grothendieck construction on institutions [46]. Model amalgamation has been extended to arbitrary co-cones in works such as [163].

The model amalgamation proof for **FOL** is similar in flavor to the functorial semantics of [112], and appears in the form we have presented here in [163].

The method of diagrams pervades much of conventional model theory [32]. The institution-independent method of diagrams used here was developed in [48] and has been used in [48, 87, 86] etc. A quite different institution-independent version of the method of diagrams has been used for developing quasi-variety theorems and existence of free models within the context of the so-called 'abstract algebraic institutions' [169, 170]. Elementary homomorphisms have been introduced in [86]. The existence of limits and co-limits of models via elementary diagrams has been obtained in [48].

Inclusion systems and inclusive institutions were introduced in [58] for the institution-independent study of structuring specifications, however they were defined in a stronger version corresponding to our epic inclusion systems with unions. In [58, 81] they provide the underlying mathematical concept for module imports, which are the most fundamental structuring constructs. Inclusions of models is used in [155, 48] for an institution-independent approach to quasi-varieties of models. Mathematically, inclusion systems capture categorically the concept of set-theoretic 'inclusion' in a way reminiscent of the well-known factorization systems [26]; however in many applications the former are more convenient than the latter. In [38] the original definition of [58] has been weakened to what they called 'weak inclusion systems', which are just our inclusion systems.

Our (F,P)-congruences are elsewhere called 'closed' congruences.

Liberality has played a central role in institution theory from its beginning [75]. This was due to the traditional important role played in algebraic specification by initial algebra semantics. Free models along presentation morphisms provide semantics for initial denotation modules in structured algebraic specifications [75]. Our institution comorphisms $I \to (I')^{\mathrm{pres}}$ have been studied in [130, 81] under the name of 'simple theoroidal comorphisms'.

Chapter 5

Internal Logic

In many institutions the satisfaction relation between models and sentences is defined by induction on the structure of the sentences. Usually sentences are formed from 'atomic' sentences, which constitute the starting building blocks, by applying iteratively constructs such as quantifiers and connectives. The connectives may be Boolean or potentially of another kind, such as modal for example. The definition of a satisfaction relation in these institutions can be seen as a two-layered process. At the base level, one defines satisfaction of the 'atomic' sentences. Then the induction step consists of a definition of satisfaction for the quantified sentences and of sentences formed by Boolean (or another kind of) connectives on the basis of satisfaction of the components. This Tarskian process of determining the actual satisfaction between models and sentences is a common pattern for a multitude of institutions and has an institution-independent nature. The uniform treatment at a general institution-independent level of the semantics of Boolean connectives, quantifiers, and to some extent even of the atomic sentences, is the gate to institution-independent model theory and constitutes the main topic of this chapter.

The general approach to atomic sentences based on a simple form of categorical injectivity leads to a general uniform semantic approach to Horn sentences at an institution-independent level, their satisfaction being equivalent to categorical injectivity. Later the equivalence between the semantics of Horn sentences and injectivity will prove very useful within the context of axiomatizability results.

Many important results in model theory rely upon quantification being first order. First order quantifiers are handled at the institution-independent level by the concept of '(quasi-)representable' signature morphisms. Although (quasi-)representability is a property of the signature morphisms, we will see that in reality it is a semantic concept because its definition involves the models of the institution.

Auxiliary related topics of this chapter include substitutions and a deepening of the study of elementary homomorphisms.

5.1 Logical Connectives

Boolean connectives

Given a signature Σ in an institution, a Σ-sentence ρ' is a *semantic*

- *negation* of ρ when $\rho'^* = \overline{\rho^*}$,
- *conjunction* of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = \rho_1^* \cap \rho_2^*$,
- *disjunction* of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = \rho_1^* \cup \rho_2^*$,
- *implication* of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = \overline{\rho_1^*} \cup \rho_2^*$, and
- *equivalence*[1] of the Σ-sentences ρ_1 and ρ_2 when $\rho'^* = (\rho_1^* \cap \rho_2^*) \cup (\overline{\rho_1^*} \cap \overline{\rho_2^*})$

where $\overline{e^*}$ denotes $\mathsf{Mod}(\Sigma)| \setminus e^*$.

A more informal way to express these connectives is by using them at a meta-level. For example ρ' is the negation of ρ when for each Σ-model M, $M \models \rho'$ if and only if $M \not\models \rho$. Or ρ' is the conjunction of ρ_1 and ρ_2 when for each Σ-model M, $M \models \rho'$ if and only if $M \models \rho_1$ *and* $M \models \rho_2$.

Fact 5.1. *Negations, conjunctions, disjunctions, implications, and equivalences of sentences are unique up to semantical equivalence.*

An institution *has (semantic) negation* when each sentence of the institution has a negation. It has *(semantic) conjunctions* when each two sentences (of the same signature) have a conjunction. Similar definitions can be formulated for disjunctions, implications, and equivalences. Distinguished negations are usually denoted by $\neg\_$, distinguished conjunctions by $\_\wedge\_$, distinguished disjunctions by $\_\vee\_$, distinguished implications by $\_\Rightarrow\_$, and distinguished equivalences by $\_\Leftrightarrow\_$.

When they exist, the semantic Boolean connectives are inter-definable as shown by the following easy result.

Fact 5.2. *In any institution having the corresponding Boolean connectives we have that*

- disjunction: $\rho_1 \vee \rho_2 \models\!\mid \neg(\neg\rho_1 \wedge \neg\rho_2)$,
- implication: $\rho_1 \Rightarrow \rho_2 \models\!\mid \neg\rho_1 \vee \rho_2$, *and*
- equivalence: $\rho_1 \Leftrightarrow \rho_2 \models\!\mid (\rho_1 \Rightarrow \rho_2) \wedge (\rho_2 \Rightarrow \rho_1)$.

An institution which has all semantic Boolean connectives is called a *Boolean complete institution*.

The following gives the situation of the semantic Boolean connectives in some institutions (the reader is invited to check this table by herself/himself):

[1] Not to be confused with the semantical equivalence *relation* $\models\!\mid$ between sentences.

institution	$\wedge$	$\neg$	$\vee$	$\Rightarrow$	$\Leftrightarrow$
FOL, **PL**, **HOL**, **HNK**	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
FOL$^+$	$\surd$		$\surd$		
EQL, **HCL**					
EQLN		$\surd$			
MFOL, **MPL**	$\surd$				
IPL	$\surd$				

Abstract logical connectives

The abstract logical connectives may be regarded as a generalization of the Boolean connectives.

Given an institution I, for any ordinal n, a *(semantic logical) connective c of arity n* consists of a family $\{c_\Sigma\}_{\Sigma\in|\mathbb{S}ig|}$ of functions $c_\Sigma\colon\ \mathcal{P}(|\mathsf{Mod}(\Sigma)|)^n \to \mathcal{P}(|\mathsf{Mod}(\Sigma)|)$.[2] A connective c is *Boolean* if it is a (possibly) derived Boolean operation of the Boolean algebra $(\mathcal{P}(|\mathsf{Mod}(\Sigma)|),\cap,\cup,\neg,\emptyset,|\mathsf{Mod}(\Sigma)|)$.

A sentence ρ is a *c-connection* of sentences ρ_i, $i \leq n$, when $\rho^* = c_\Sigma(\rho_1^*,\dots,\rho_n^*)$; in this case we may denote ρ by $c(\rho_1,\dots,\rho_n)$.

The institution *has the connective c* when for all sentences $\rho_1,\dots,\rho_n$ there exists a sentence ρ such that $\rho \models\!\mid c(\rho_1,\dots,\rho_n)$.

A signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ *preserves c* when

$$\varphi(c(\rho_1,\dots,\rho_n)) \models\!\mid c(\varphi(\rho_1),\dots,\varphi(\rho_n)).$$

Although in general the preservation of the semantic connectives is not guaranteed, from the satisfaction condition and from the definition of the semantic Boolean connectives we have the following:

Fact 5.3. *In any institution all the Boolean connectives which exist in that institution, are preserved by all signature morphisms.*

Exercises

5.1. Weak propositional logic (**WPL**, see Ex. 3.25) does have all semantic Boolean connectives apart from negation.

5.2. [55] **Finitary sentences (Ex. 4.21 continued)**
(a) In any institution the negation of a finitary sentence is finitary.
(b) If the category of signatures has binary co-products, then any finite logical connection of finitary sentences is finitary too, provided that the signature morphisms preserve the respective connective. For example, in any institution with conjunctions and with binary co-products of signatures, the conjunctions of finitary sentences are still finitary.

[2] c_Σ is a function on classes rather than sets.

5.2 Quantifiers

Let us first recall the semantics of quantifiers in a concrete institution such as **FOL**. Given a **FOL**-signature (S,F,P) and a set X of new constants for S, let ρ' be an $(S,F\uplus X,P)$-sentence and M be an (S,F,P)-model. Then

$$M \models (\exists X)\rho' \text{ if and only if } M' \models \rho' \text{ for some } (S,F\uplus X,P)\text{-expansion } M' \text{ of } M.$$

General institution-independent quantifiers are defined similarly to the above by abstracting from **FOL** signature inclusions $(S,F,P) \hookrightarrow (S,F\uplus X,P)$ to any signature morphisms $\chi\colon \Sigma \to \Sigma'$ in any arbitrary institution.

– A Σ-sentence ρ is a *(semantic) existential* χ*-quantification* of a Σ'-sentence ρ' when $\rho^* = (\rho'^*)\upharpoonright_\chi$; "distinguished existential quantification" is usually written as $(\exists\chi)\rho'$,

– A Σ-sentence ρ is a *(semantic) universal* χ*-quantification* of a Σ'-sentence ρ' when $\rho^* = \overline{\overline{\rho'^*}\upharpoonright_\chi}$ (where by $\overline{\mathbb{M}}$ we denote the complement of the class of models $\mathbb{M}$); "distinguished universal quantification" is usually written as $(\forall\chi)\rho'$.

A more informal way to express semantic existential/universal quantifiers, which uses meta-level 'all' and 'some', is as follows:

- $M \models_\Sigma (\exists\chi)\rho'$ when there *exists* a χ-expansion M' of M such that $M' \models_{\Sigma'} \rho'$, and
- $M \models_\Sigma (\forall\chi)\rho'$ when $M' \models_{\Sigma'} \rho'$ for *all* χ-expansions M' of M.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the finitary signature extensions with constants. For a class $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, we say that the institution has universal/existential $\mathcal{D}$*-quantification* when for each $\chi\colon \Sigma \to \Sigma'$ in $\mathcal{D}$, each Σ'-sentence has a universal/existential χ-quantification. The table below shows the situation of internal quantification in some institutions.

institution	$\mathcal{D}$	$\forall$	$\exists$
FOL	finitary sign. extensions with constants	$\surd$	$\surd$
SOL	finitary sign. extensions	$\surd$	$\surd$
PA	finitary sign. extensions with total constants	$\surd$	$\surd$
EQL, **HCL**	finitary sign. extensions with constants	$\surd$	
MFOL	finitary sign. extensions with rigid constants	$\surd$	
HOL, **HNK**	finitary sign. extensions	$\surd$	$\surd$

Generally, one may consider quantification only up to what the respective concept of signature supports. For example **FOL** signatures support quantifications only up to second order. Quantifications higher than second order require thus another concept of signature involving higher-order types, such an example being given by **HOL** or **HNK**.

Conservative quantifications

Let us say that an institution *has* false when for each signature Σ there exists a Σ-sentence false_Σ such that $\mathsf{false}_\Sigma^* = \emptyset$. For example, if the institution has negation and conjunctions, then $\mathsf{false}_\Sigma \models\!\!\mid \neg\rho \wedge \rho$ for each Σ-sentences ρ.

In any institution with false, $M \models (\forall\chi)\mathsf{false}$ for all models M which do not admit a χ-expansion. This indicates that quantification behaves 'well' only for the signature morphisms χ for which each model admits at least one χ-expansion. These signature morphisms are called *conservative*.

Conservative signature morphisms in FOL. The following gives a characterization of conservative **FOL** signature morphisms and it is based on our fundamental assumption that each **FOL** signature has non-empty sorts, i.e., it has at least one constant for each sort.

Fact 5.4. *A* **FOL** *signature morphism* φ *is conservative if and only if it is injective, i.e.,* φ^{st}*,* φ^{op}*, and* φ^{rl} *are injective.*

Finitary quantifications

Many important model theory results rely upon the finiteness of the quantifications. In many of the examples presented the quantifications are finitary in the sense that the signature extensions considered add only a finite number of syntactic entities, which are usually constants.

Finitary signature morphisms. A signature morphism $\chi\colon \Sigma \to \Sigma'$ is *finitary* when for each co-limit $(\mu_i)_{i\in I}$ of a directed diagram $(f_{i,j})_{(i<j)\in(I,\leq)}$ of Σ-models

$$\begin{tikzcd} A_i \arrow[rr, "f_{i,j}"] \arrow[dr, "\mu_i"'] & & A_j \arrow[dl, "\mu_j"] \\ & A & \end{tikzcd}$$

and for each χ-expansion A' of A

- there exists an index $i \in I$ and a χ-expansion $\mu'_i\colon A'_i \to A'$ of μ_i, and
- any two different expansions as above can be 'unified' in the sense that for any χ-expansions μ'_i and μ'_k as above there exists an index $j \in I$ with $i,k < j$, a χ-expansion μ'_j as above and $f'_{i,j}, f'_{k,j}$ χ-expansions of $f_{i,j}, f_{k,j}$ such that the following commutes

$$\begin{tikzcd} A'_i \arrow[r, "f'_{i,j}"] \arrow[dr, "\mu'_i"'] & A'_j \arrow[d, "\mu'_j"] & A'_k \arrow[l, "f'_{k,j}"'] \arrow[dl, "\mu'_k"] \\ & A' & \end{tikzcd}$$

The following is a standard example.

Proposition 5.5. *In* **FOL** *each signature extension with a finite number of constants is finitary.*

Proof. This is based on the remark that directed co-limits of **FOL** models are created on top of the corresponding directed co-limits of the underlying carrier sets (see Prop. 6.5 below for a proof of this fact). Then we have just to note that expansions of models M along signature extensions $\Sigma \hookrightarrow \Sigma \uplus X$ are just functions $X \to M$ and use the fact that X being finite is a finitely presented object in the category of $\mathbb{S}et$. □

Accessibility

For each set of sentences E and each set O of logical operators (such as logical connectives or quantification), let $O(E)$ be the least set of 'internal' sentences closed under O and containing E. In general, the actual institution does not necessarily have all sentences of $O(E)$.

A sentence ρ of the institution is *(semantically) accessible from E by O* when ρ is semantically equivalent to a sentence from $O(E)$. For example:

- in **FOL** each sentence is accessible from (equational and relational) atoms by $\neg$, $\wedge$, and universal quantification by signature extensions with a finite number of constants,
- in **EQL** each sentence is accessible from equations by **FOL** universal quantification,
- in **HCL** each sentence is accessible from atoms by $\wedge$, implication, and universal quantification by signature extensions with a finite number of constants, and
- in **PA** each sentence is accessible from existence equations by $\neg$, $\wedge$, and universal quantification by signature extensions with a finite number of total constants.

In the list above only **HCL** does not have all sentences from $O(E)$.

Exercises

5.3. In each institution
(a) $(\exists\chi)\rho \models\!\mid \neg(\forall\chi)\neg\rho$, and
(b) $(\forall\chi)\rho \Rightarrow (\rho_1 \wedge \rho_2) \models\!\mid ((\forall\chi)\rho \Rightarrow \rho_1) \wedge ((\forall\chi)\rho \Rightarrow \rho_2)$.

5.4. Translation of internal quantifiers
Given a pushout of signature morphisms in an institution with model amalgamation, a Σ'-model M' and a Σ_1-sentence ρ

$$\begin{tikzcd} \Sigma \arrow[r, "X"] \arrow[d, "\theta"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma' \arrow[r, "X'"'] & \Sigma'_1 \end{tikzcd}$$

the following satisfaction condition holds:

$$M' \models_{\Sigma'} (\forall X')\theta_1(\rho) \text{ iff } M'\restriction_\theta \models_\Sigma (\forall X)\rho.$$

This provides foundations for extending the sentence functor Sen of the original institution with 'internally quantified' sentences. If $X \in \mathcal{D}$ with $\mathcal{D}$ stable under pushouts, then $\mathsf{Sen}(\theta)((\forall X)\rho)$ may be defined as $(\forall X')\theta_1(\rho)$. Note that translation of quantifiers is one of the situations which requires the stability of $\mathcal{D}$ under pushouts.

5.5. In any institution with weak model amalgamation let $\mathcal{D}$ be a class of conservative signature morphism which is stable under pushouts. Then

$$(Q\chi_1)\rho_1(c)(Q\chi_2)\rho_2 \models\!\!\!\dashv (Q\chi_1;\chi_2)\theta_1(\rho_1)(c)\theta_2(\rho_2)$$

where $Q \in \{\forall, \exists\}$, $(c) \in \{\wedge, \vee\}$ and the following is a pushout square of signature morphisms of $\mathcal{D}$:

$$\begin{tikzcd} \Sigma \arrow[r, "\chi_1"] \arrow[d, "\chi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

5.6. (a) Let $\chi_2 = \chi_1;\chi$ be signature morphisms such that χ is conservative. Then $(\forall\chi_2)\chi(\rho) \models\!\!\!\dashv (\forall\chi_1)\rho$.
(b) In the extension of **FOL** with infinitary quantifications, each sentence is semantically equivalent to a **FOL** sentence.

5.7. Generalization Rule
For each signature morphism $\chi\colon \Sigma \to \Sigma'$ and each set E of Σ-sentences

$$E \models_\Sigma (\forall\chi)e \text{ if and only if } \chi(E) \models_{\Sigma'} e.$$

5.8. Stability under pushouts of finitary signature morphisms
In any semi-exact institution the finitary signature morphisms are stable under pushouts along those signature morphisms for which their model reducts preserve directed co-limits of models.

5.9. Finite models
(a) In any institution with elementary diagrams a model is *finite* when its elementary diagram E_M is finite. If the institution has finite conjunctions and existential quantification over elementary extensions along finite models, any two elementary equivalent finite models are homomorphically related. (*Hint:* For a model M consider the sentence $(\exists\iota_\Sigma(M)) \wedge E_M$.)
(b) In any finite **FOL**-signature, any two elementary equivalent models with finite carriers are isomorphic. (*Hint:* The sub-institution of **FOL** determined by the closed and injective model homomorphisms admits a system of elementary diagrams such that a model is finite whenever its signature is finite and it has finite carrier sets.)

5.3 Substitutions

First order substitutions in FOL. Given a **FOL** signature (S,F,P) and two sets of new constants X and Y, called *first order variables*, a *first order* (S,F,P)*-substitution from* X *to* Y consists of a mapping $\psi\colon X \to T_F(Y)$ of the variables X with F-terms over Y.

On the semantics side, each first order (S,F,P)-substitution $\psi\colon X \to T_F(Y)$ determines a functor

$$\mathsf{Mod}(\psi)\colon \mathsf{Mod}^{\mathbf{FOL}}(S,F\cup Y,P) \to \mathsf{Mod}^{\mathbf{FOL}}(S,F\cup X,P)$$

defined by

– $\mathsf{Mod}(\psi)(M)_x = M_x$ for each sort $x \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$, and

– $\mathsf{Mod}(\psi)(M)_x = M_{\psi(x)}$, i.e., the evaluation of the term $\psi(x)$ in M, for each $x \in X$.

On the syntax side, ψ determines a sentence translation function

$$\mathsf{Sen}(\psi) : \mathsf{Sen}^{\mathbf{FOL}}(S, F \cup X, P) \to \mathsf{Sen}^{\mathbf{FOL}}(S, F \cup Y, P)$$

which in essence replaces all symbols from X with the corresponding $(F \cup Y)$-terms according to ψ. This can be formally defined as follows:

– $\mathsf{Sen}(\psi)(t = t')$ is defined as $\psi^{\mathrm{tm}}(t) = \psi^{\mathrm{tm}}(t')$ for each $(S, F \cup X, P)$-equation $t = t'$, where $\psi^{\mathrm{tm}} : T_F(X) \to T_F(Y)$ is the unique extension of ψ to an F-homomorphism (ψ^{tm} replaces the variables $x \in X$ with $\psi(x)$ in each $(F \cup X)$-term t).

– $\mathsf{Sen}(\psi)(\pi(t_1, \ldots, t_n))$ is defined as $\pi(\psi^{\mathrm{tm}}(t_1), \ldots, \psi^{\mathrm{tm}}(t_n))$ for each $(S, F \cup X, P)$-relational atom $\pi(t_1, \ldots, t_n)$.

– $\mathsf{Sen}(\psi)(\rho_1 \wedge \rho_2)$ is defined as $\mathsf{Sen}(\psi)(\rho_1) \wedge \mathsf{Sen}(\psi)(\rho_2)$ for each conjunction $\rho_1 \wedge \rho_2$ of $(S, F \cup X, P)$-sentences, and similarly for the case of any other Boolean connectives.

– $\mathsf{Sen}(\psi)((\forall Z)\rho) = (\forall Z)\mathsf{Sen}(\psi_Z)(\rho)$ for each $(S, F \cup X \cup Z, P)$-sentence ρ, where ψ_Z is the trivial extension of ψ to an $(S, F \cup Z, P)$-substitution.

Note that we have extended the notation used for the models functor Mod and for the sentence functor Sen from the signatures to the first order substitutions. This notational extension is justified by the satisfaction condition given by Prop. 5.6 below.

Proposition 5.6. *For each* **FOL**-*signature* (S, F, P) *and each* (S, F, P)-*substitution* $\psi : X \to T_F(Y)$,

$$\mathsf{Mod}(\psi)(M) \models \rho \quad \textit{if and only if} \quad M \models \mathsf{Sen}(\psi)(\rho)$$

for each $(S, F \cup Y, P)$-*model* M *and each* $(S, F \cup X, P)$-*sentence* ρ.

Proof. By noticing that $\mathsf{Mod}(\psi)(M)_t = M_{\psi^{\mathrm{tm}}(t)}$ for each $(F \cup X)$-term t, and by a straightforward induction on the structure of the sentences. $\square$

General substitutions. The satisfaction condition property expressed in Prop. 5.6 permits the definition of a general concept of substitution by abstracting

- **FOL** signatures (S, F, P) to signatures Σ in arbitrary institutions, and
- sets of first order variables X for (S, F, P) to signature morphisms $\Sigma \to \Sigma_1$.

For any signature Σ of an institution, and any signature morphisms $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$, a Σ-*substitution* $\psi : \chi_1 \to \chi_2$ consists of a pair $(\mathsf{Sen}(\psi), \mathsf{Mod}(\psi))$, where

– $\mathsf{Sen}(\psi) : \mathsf{Sen}(\Sigma_1) \to \mathsf{Sen}(\Sigma_2)$ is a function, and

– $\mathsf{Mod}(\psi) : \mathsf{Mod}(\Sigma_2) \to \mathsf{Mod}(\Sigma_1)$ is a functor such that both of them preserve Σ, i.e., the following diagrams commute:

$$\begin{tikzcd}[column sep=small]
\mathsf{Sen}(\Sigma_1) \arrow[rr, "\mathsf{Sen}(\psi)"] & & \mathsf{Sen}(\Sigma_2) \\
& \mathsf{Sen}(\Sigma) \arrow[ul, "\mathsf{Sen}(\chi_1)"] \arrow[ur, "\mathsf{Sen}(\chi_2)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathsf{Mod}(\Sigma_1) \arrow[dr, "\mathsf{Mod}(\chi_1)"'] & & \mathsf{Mod}(\Sigma_2) \arrow[ll, "\mathsf{Mod}(\psi)"'] \arrow[dl, "\mathsf{Mod}(\chi_2)"] \\
& \mathsf{Mod}(\Sigma) &
\end{tikzcd}$$

and such that the following satisfaction condition holds:

$$\mathsf{Mod}(\psi)(M_2) \models \rho_1 \text{ if and only if } M_2 \models \mathsf{Sen}(\psi)(\rho_1)$$

for each Σ_2-model M_2 and each Σ_1-sentence ρ_1.

Note that we have again extended the notations Mod and Sen from the model and the sentence functors of the institution to the model and the sentence components of substitutions.

Fact 5.7. *The Σ-substitutions come equipped with a natural composition satisfying the category axioms by inheriting the composition of the function and functor components. Let Subst*(Σ) *denote this* category of Σ-substitutions.

$\mathcal{D}$-substitutions. In actual situations one usually considers only substitutions between signature morphisms which are used in quantifications. The main motivation for this practice is proof theoretic. We have already seen that often the class of the signature morphisms used for quantifications is a special subclass of all signature morphisms of the institution. Therefore, for any class $\mathcal{D}$ of signature morphisms in an institution, let us say that a *$\mathcal{D}$-substitution* is just a substitution between signature morphisms in $\mathcal{D}$.

Equivalent substitutions. Since general substitutions are a semantic concept, semantical equivalence on substitutions is more meaningful than the strict equality. In other words, what really matters about substitution is their semantic effect.

The substitutions $\psi, \psi' : \chi_1 \to \chi_2$ are *equivalent* when $\mathsf{Mod}(\psi)(M_2) = \mathsf{Mod}(\psi')(M_2)$ for each Σ_2-model M_2. The translation on models determines the translation on sentences up to semantical equivalence.

Fact 5.8. *If* ψ *and* ψ' *are equivalent substitutions, then* $\mathsf{Sen}(\psi)(\rho_1) \models\!\!\mid \mathsf{Sen}(\psi')(\rho_1)$ *for each* Σ_1*-sentence* ρ_1.

Derived FOL signatures

The so-called concept of ‘derived signature’, from algebraic specification languages, yields a less conventional example of general substitution in **FOL**. In order to explain this, we have to introduce some auxiliary notions.

Arity of terms. For any string k of elements let $[k]$ be the substring retaining from left to right one copy of each element. This can be formally defined by $[kx] = [k]$ when x does occur in k, and $[k]x$ otherwise. For any **FOL** signature (S,F,P) and X a set of new constants (i.e., first order variables), let $var\colon\ T_F(X) \to X^*$ (where X^* denotes here the set of strings with elements from X) be the function collecting the variables occurring in a term, which is defined by $var(x) = x$ if $x \in X$, and $var(\sigma(t_1,\ldots,t_n)) = var(t_1)\ldots var(t_n)$. Then the *arity* of a term $t \in T_F(X)$, denoted $ar(t)$, is defined to be the string of sorts corresponding to the string of variables $[var(t)]$.

For example, if x is a variable of sort N and -1 is a constant of sort Z, then the arity of the term $(x+(-1)) * x$ is N.

Terms as operations. Each term t with arity w and sort s gets interpreted in any model M as a function $M_t\colon\ M_w \to M_s$ as follows. For each $m \in M_w$ let M_m be the expansion of M to $(S, F \cup [var(t)], P)$ such that $M_{[var(t)]} = m$. Then $M_t(m)$ is defined as $(M_m)_t$.

Derived signatures. For any **FOL** signature (S,F,P), let $(S,T(F),\Pi(F,P))$ be its *derived signature* defined by

- for each arity $w \in S^*$ and sort $s \in S$, $T(F)_{w\to s} = \{t \in (T_F(X_w))_s \mid ar(t)$ is a permutation of $w\}$, where $w = s_1 \ldots s_n$, $X_w = \{x_i \mid 1 \le i \le n\}$, and $(X_w)_{s'} = \{x_i \mid 1 \le i \le n$ and $s_i = s'\}$ for each sort $s' \in S$, and
- for each arity $w \in S^*$, $\Pi(F,P)_w = \{\pi(t) \mid \pi \in P_w$ and t appropriate string of terms such that the concatenated arities of t form a permutation of $w\}$.

The elements of $T(F)$ are called *derived F-operations*, while the elements of $\Pi(F,P)$ are called *derived (F,P)-relations*.

There is a canonical signature morphism $\varphi_{(S,F,P)}\colon\ (S,F,P) \to (S,T(F),\Pi(F,P))$ mapping each operation symbol $\sigma \in F_{s_1\ldots s_n \to s}$ to $\sigma(x_1,\ldots,x_n)$ and each relation symbol $\pi \in P_{s_1\ldots s_n}$ to $\pi(x_1,\ldots,x_n)$.

Note that $\mathsf{Mod}(\varphi_{(S,F,P)})\colon\ \mathsf{Mod}(S,T(F),\Pi(F,P)) \to \mathsf{Mod}(S,F,P)$ *is* a retract to the functor $\mathsf{Mod}(\varphi_{(S,F,P)})^{-1}\colon\ \mathsf{Mod}(S,F,P) \to \mathsf{Mod}(S,T(F),\Pi(F,P))$ which is defined on any model M by

- $\mathsf{Mod}(\varphi_{(S,F,P)})^{-1}(M)_t = M_t$ for any $t \in T(F)$, and
- $\mathsf{Mod}(\varphi_{(S,F,P)})^{-1}(M)_{\pi(t)} = M_t^{-1}(M_\pi)$ for any $\pi(t) \in \Pi(F,P)$.

On the sentence side, $\mathsf{Sen}(\varphi_{(S,F,P)})\colon\ \mathsf{Sen}(S,F,P) \to \mathsf{Sen}(S,T(F),\Pi(F,P))$ *has* a retract, denoted $\mathsf{Sen}(\varphi_{(S,F,P)})^{-1}$, which extends the canonical interpretation of any $T(F)$-term t by the the F-term $(T_F)_t$, to a functor $\mathsf{Sen}(S,T(F),\Pi(F,P)) \to \mathsf{Sen}(S,F,P)$ preserving the relation symbols, the Boolean connectives, and the quantifications.

By induction on the structure of the sentences we can easily show the satisfaction condition for $\mathsf{Mod}(\varphi_{(S,F,P)})^{-1}$ and $\mathsf{Sen}(\varphi_{(S,F,P)})^{-1}$, which shows that

Fact 5.9. *The construction above yields an (S,F,P)-substitution $\varphi_{(S,F,P)} \to 1_{(S,F,P)}$, denoted $\varphi^{-1}_{(S,F,P)}$, with $\mathsf{Mod}(\varphi^{-1}_{(S,F,P)}) = \mathsf{Mod}(\varphi_{(S,F,P)})^{-1}$, and $\mathsf{Sen}(\varphi^{-1}_{(S,F,P)}) = \mathsf{Sen}(\varphi_{(S,F,P)})^{-1}$.*

Second order substitutions in FOL

Given a **FOL** signature (S,F,P) and two signature extensions $(S,F,P) \hookrightarrow (S \cup S_1, F \cup F_1, P \cup P_1)$ and $(S,F,P) \hookrightarrow (S \cup S_2, F \cup F_2, P \cup P_2)$, a *second order* (S,F,P)*-substitution* $\psi\colon\ (S_1,F_1,P_1) \to (S_2,F_2,P_2)$ is a signature morphism $(S \cup S_1, F \cup F_1, P \cup P_1) \to (S \cup S_2, T(F \cup F_2), \Pi(F \cup F_2, P \cup P_2))$ which in addition 'preserves (S,F,P)', i.e., the following diagram commutes:

$$\begin{tikzcd}
(S \cup S_1, F \cup F_1, P \cup P_1) \arrow[r, "\psi"] & (S \cup S_2, T(F \cup F_2), \Pi(F \cup F_2, P \cup P_2)) \\
(S,F,P) \arrow[u] \arrow[r] & (S \cup S_2, F \cup F_2, P \cup P_2) \arrow[u, "\varphi_{(S \cup S_2, F \cup F_2, P \cup P_2)}"']
\end{tikzcd}$$

This means that ψ maps each operation of F_1 to a derived $(F \cup F_2)$-operation and each relation of P_1 to a derived $(F \cup F_2, P \cup P_2)$-relation.

Fact 5.10. *Each second order substitution $\psi\colon\ (S_1,F_1,P_1) \to (S_2,F_2,P_2)$ determines a general (S,F,P)-substitution in* **FOL** *between the signature extensions $(S,F,P) \to (S \cup S_1, F \cup F_1, P \cup P_1)$ and $(S,F,P) \to (S \cup S_2, F \cup F_2, P \cup P_2)$ defined by the (S,F,P)-substitution composition $\psi;\varphi^{-1}_{(S \cup S_2, F \cup F_2, P \cup P_2)}$ between the signature morphism ψ regarded as an (S,F,P)-substitution and the $(S \cup S_2, F \cup F_2, P \cup P_2)$-substitution regarded as an (S,F,P)-substitution.*

Note that first order substitution in **FOL** are special cases of second order substitutions when $S_1 = S_2 = P_1 = P_2 = \emptyset$ and F_1 and F_2 contain only constants.

Exercises

5.10. Substituting relations by sentences
Let (S,F,P) be a **FOL** signature and P_1 a set of new relation symbols for S. Each mapping ψ of relation symbols $\pi \in (P_1)_w$ to sentences $\psi(\pi) \in \mathsf{Sen}(S, F \cup X, P)$ where $X = \{x_1, \dots, x_n\}$ such x_i is a new constant of sort s_i where $w = s_1 \dots s_n$, can be extended to a mapping $\mathsf{Sen}(S,F,P \cup P_1) \to \mathsf{Sen}(S,F,P)$ by replacing each relational atom $\pi(t_1,\dots,t_n)$ with $\psi(\pi)(t_1,\dots,t_n)$. This determines a general substitution in **FOL** between $(S,F,P) \hookrightarrow (S,F,P \cup P_1)$ and $1_{(S,F,P)}$.

5.11. Substitution rule
In any institution, for any substitution $\psi\colon\ \chi \to \chi'$ and any sentence ρ,

$$(\forall\chi)\rho \models (\forall\chi')\mathsf{Sen}(\psi)(\rho).$$

5.12. Institution of substitutions
For each signature Σ of an institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$, let $(Subst(\Sigma), \mathsf{Sen}, \mathsf{Mod}, \models)$ denote the *institution of* Σ*-substitutions*. Its signatures are the signature morphisms $\Sigma \to \bullet$ of the original institution and its signature morphisms are the Σ-substitutions.

Then each signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ determines canonically a functor $Subst(\varphi)$: $Subst(\Sigma) \to Subst(\Sigma')$. This construction further determines a functor $\mathbb{S}ig^{\mathrm{op}} \to \mathbb{I}ns$.

5.13. The composition of second order substitutions in **FOL** yields a second order substitution.

5.14. For any signatures $(S \cup S_1, F \cup F_1)$ and $(S \cup S_2, F \cup F_2)$ in **HOL/HNK** each pair consisting of a mapping $\psi^{\mathrm{st}}: S_1 \to \overrightarrow{(S \cup S_2)}$ and of a family of mappings $\{\psi^{\mathrm{op}}_s : (F_1)_s \to (T_{F \cup F_2})_{\psi^{\mathrm{type}}(s)} \mid s \in \overrightarrow{(S \cup S_1)}\}$ determines a general (S,F)-substitution in **HOL/HNK** between $(S,F) \hookrightarrow (S \cup S_1, F \cup F_1)$ and $(S,F) \hookrightarrow (S \cup S_2, F \cup F_2)$.

5.4 Representable Signature Morphisms

Quasi-representable signature morphisms

Many important results in model theory rely upon quantifications by signature extensions with constants, usually called *first order quantifications*. The signature extensions with constants are be characterized by the following general property.

In any institution, a signature morphism $\chi : \Sigma \to \Sigma'$ is *quasi-representable* when for each Σ'-model M', the canonical functor determined by the reduct functor $\mathsf{Mod}(\chi)$ is an isomorphism (of comma categories)

$$M'/\mathsf{Mod}(\Sigma') \cong (M'\restriction_\chi)/\mathsf{Mod}(\Sigma).$$

This means that each Σ-model homomorphism $h : M'\restriction_\chi \to N$ admits a unique χ-expansion $h' : M' \to N'$.

Fact 5.11. *In* **FOL** *each signature extension with constants* $(S,F,P) \hookrightarrow (S, F \uplus X, P)$ *is quasi-representable. Given any* (S,F,P)*-model homomorphism* $h : M \to N$*, any* $(S, F \uplus X, P)$*-expansion* M' *of* M *determines* uniquely *a* $(S, F \uplus X, P)$*-expansion* $h' : M' \to N'$ *of* h *by defining* $N'_x = h(M'_x)$ *for each* $x \in X$.

Note that in **FOL** quasi-representability fails when we extend signature morphisms with relation or non-constant operation symbols.

Quasi-representability of signature extensions with constants holds in various institutions in ways similar to Fact 5.11. For example, it also works in the institution $E(\mathbf{FOL})$ of the **FOL** elementary embeddings. However, in some cases quasi-representability goes beyond extensions with constants. An example is given by the restriction of **FOL** to strong model homomorphisms (recall that $h : M \to N$ is strong when $h(M_\pi) = N_\pi$ for each relation symbol π). In this institution any signature extension with constants or relation symbols is quasi-representable.

Structural properties of quasi-representability. The following gives a list of basic structural properties of quasi-representable signature morphisms which are useful for many results relying upon the quasi-representability property.

Proposition 5.12. *In any institution*

1. *The quasi-representable signature morphisms are closed under composition.*
2. *If the institution is semi-exact, then quasi-representable signature morphisms are stable under pushouts.*

3. *If the institution is directed-exact, then any directed co-limit of quasi-representable signature morphisms consists of quasi-representable signature morphisms.*
4. *If φ and $\varphi;\chi$ are quasi-representable, then χ is quasi-representable.*

Proof. 1. That composition of quasi-representable morphisms is quasi-representable follows immediately from the definition.

2. Consider a pushout of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\chi"] \arrow[d, "\theta"'] & \Sigma' \arrow[d, "\theta'"] \\ \Sigma_1 \arrow[r, "\chi_1"'] & \Sigma'_1 \end{tikzcd}$$

such that χ is quasi-representable. We have to show that χ_1 is quasi-representable.

Consider a Σ_1-model homomorphism $h_1 : M'_1\upharpoonright_{\chi_1} \to N_1$. Let $h : M \to N$ be its θ-reduct. $M = M'_1\upharpoonright_{\chi_1}\upharpoonright_\theta = M'_1\upharpoonright_{\theta'}\upharpoonright_\chi$. Because χ is quasi-representable, let $h' : M'_1\upharpoonright_{\theta'} \to N'$ be the unique χ-expansion of h. By the semi-exactness of the institution, the unique amalgamation h'_1 of h_1 and h' is the unique χ_1-expansion $M'_1 \to N'_1$ of h_1.

3. Let $(\varphi_{i,j})_{(i<j)\in(I,\leq)}$ be a directed diagram of quasi-representable signature morphisms and let $(\theta_i)_{i\in I}$ be its co-limit.

$$\begin{tikzcd} \Sigma_i \arrow[rr, "\varphi_{i,j}"] \arrow[dr, "\theta_i"'] & & \Sigma_j \arrow[dl, "\theta_j"] \\ & \Sigma & \end{tikzcd}$$

For each $i \in I$ we show that θ_i is quasi-representable. Let $M\upharpoonright_{\theta_i} \xrightarrow{h_i} N_i$ be a Σ_i-homomorphism for some Σ-model M. For each $j \in I$, let $M_j = M\upharpoonright_{\theta_j}$. Notice that $M_j\upharpoonright_{\varphi_{i,j}} = M_i$ when $j > i$. For each $j > i$, because $\varphi_{i,j}$ is quasi-representable, let $h_j : M_j \to N_j$ be the unique $\varphi_{i,j}$-expansion of h_i. By the uniqueness of expansion for quasi-representable signature morphisms, we can show that $h_{j'}\upharpoonright_{\varphi_{j,j'}} = h_j$ for each $i \leq j < j'$.

Now let $(J,\leq)$ be the sub-poset of $(I,\leq)$ determined by the elements $\{j \mid i \leq j\}$. Because $(J,\leq)$ is a final sub-poset of $(I,\leq)$, by Thm. 2.4 we deduce that $(\theta_i)_{i\in J}$ is a co-limit of $(\varphi_{j,j'})_{(j<j')\in(J,\leq)}$. Because the institution is directed-exact, let $h : M \to N$ be the unique Σ-homomorphism such that $h\upharpoonright_{\theta_j} = h_j$ for each $j \in J$. Then h is the unique θ_i-expansion $M \to N$ of h_i.

4. Let $\varphi : \Sigma \to \Sigma'$ and $\chi : \Sigma' \to \Sigma''$ be signature morphisms. Consider any Σ'-model homomorphism $h' : M''\upharpoonright_\chi \to N'$. We show that the unique $(\varphi;\chi)$-expansion of $h'\upharpoonright_\varphi$ to a Σ''-model homomorphism $h'' : M'' \to N''$ constitutes the unique χ-expansion of h' to a Σ''-model homomorphism $M'' \to N''$.

That $h' = h''\upharpoonright_\chi$ follows by the quasi-representability of φ because $h'\upharpoonright_\varphi = (h''\upharpoonright_\chi)\upharpoonright_\varphi$ and h' and $h''\upharpoonright_\chi$ both have $M''\upharpoonright_\chi$ as their domain.

The uniqueness of h'' as χ-expansion of h' follows by the uniqueness of h'' as $(\varphi;\chi)$-expansion of $h'\upharpoonright_\varphi$. $\square$

Quasi-representable signature morphisms in FOL. We know that the **FOL** signature extensions with constants are quasi-representable. Below we give a complete description of the quasi-representability in **FOL**.

Proposition 5.13. *A* **FOL** *signature morphism is quasi-representable if and only if it is bijective on sort symbols, relation symbols, and non-constant operation symbols.*

Consequently, a **FOL** *signature morphism is conservative and quasi-representable if and only if it is an injective extension with constants.*

Proof. Consider such a **FOL**-signature morphism $\chi : \Sigma \to \Sigma'$. Then there exists a signature Σ_0 and injective extensions with constants $\varphi : \Sigma_0 \to \Sigma$ and $\varphi' : \Sigma_0 \to \Sigma'$ such that the following triangle commutes:

$$\begin{tikzcd} & \Sigma_0 \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ \Sigma \arrow[rr, "\chi"'] & & \Sigma' \end{tikzcd}$$

Because both φ and φ' are (injective) extensions with constants, they are quasi-representable, hence by Prop. 5.12 (4.) χ is quasi-representable too.

Conversely, let us assume that χ is quasi-representable. If one of χ^{st}, χ^{op} restricted to non-constant operation symbols, or χ^{rl}, is not surjective, respectively not injective, then we can find a Σ-homomorphism $h : M \to N$ and a χ-expansion M' of M such that h has more than one, respectively does not have any, χ-expansion $h' : M' \to N'$. We leave the details of this argument to the reader.

The second conclusion now follows because a **FOL** signature morphism is conservative if and only if it is injective. $\square$

Representable signature morphisms

Consider a quasi-representable signature morphism $\chi : \Sigma \to \Sigma'$ and assume that $\mathsf{Mod}(\Sigma')$ has an initial model $0_{\Sigma'}$. We have the following canonical isomorphisms:

$$\mathsf{Mod}(\Sigma') \cong 0_{\Sigma'}/\mathsf{Mod}(\Sigma') \cong (0_{\Sigma'}\restriction_\chi)/\mathsf{Mod}(\Sigma).$$

This situation shows that the Σ'-models M' can be 'represented' isomorphically by Σ-model homomorphisms $0_{\Sigma'}\restriction_\chi \to M'\restriction_\chi$.

A signature morphism $\chi : \Sigma \to \Sigma'$ is *representable* if and only if there exists a Σ-model M_χ (called the *representation of* χ) and an isomorphism i_χ of categories such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Mod}(\Sigma') \arrow[r, "i_\chi"] \arrow[dr, "\mathsf{Mod}(\chi)"'] & (M_\chi/\mathsf{Mod}(\Sigma)) \arrow[d, "\text{forgetful}"] \\ & \mathsf{Mod}(\Sigma) \end{tikzcd}$$

Fact 5.14. *A signature morphism* $\chi: \Sigma \to \Sigma'$ *is representable if and only if it is quasi-representable and* $\mathsf{Mod}(\Sigma')$ *has an initial model.*

For example, since **FOL** has initial models of signatures, in **FOL** representable and quasi-representable signature morphisms are the same concept. Given a set X of first order variables (i.e., new constants) for a **FOL** signature (S,F,P), the representation of the signature inclusion $(S,F,P) \hookrightarrow (S,F \uplus X,P)$ is given by the model of the $(F \cup X)$-terms $T_F(X)$, which is the free (S,F,P)-model over X. This is due to the fact that $(S,F \uplus X,P)$-models M are in canonical bijection with valuations of variables from X to the carrier sets of M. By the freeness property of $T_F(X)$, these valuations are in canonical bijection with (S,F,P)-model homomorphisms $T_F(X) \to M$.

By Fact 5.14, examples that fall between representability and quasi-representability can be found only in institutions which do not have initial models of signatures. Examples include **MFOL** and **HOL**. A special class of institutions without initial models for signatures arises by narrowing the class of model homomorphisms in institutions; examples include the sub-institution $E(\mathbf{FOL})$ of **FOL** elementary embeddings, and the sub-institution of strong **FOL**-model homomorphisms. In all examples mentioned above the signature extensions with constants are quasi-representable but in general they are not representable.

Finitary representable signature morphisms. The following gives a characterization for the combination of two classes of signature morphisms.

Fact 5.15. *A representable signature morphism* $\chi: \Sigma \to \Sigma'$ *is finitary if and only if its representation* M_χ *is finitely presented.*

For example, by the fact above we can see that any **FOL**-signature extension of (S,F,P) with finite set of constants X is finitary representable by noticing that $T_F(X)$ is finitely presented as a (S,F,P)-model.

Representable substitutions

The **FOL** phenomenon that each first order (S,F,P)-substitution $\psi: X \to T_F(Y)$ (of variables X with F-terms over Y) can be extended uniquely to a model homomorphism $h_\psi: T_F(X) \to T_F(Y)$ is a reflection of the more general fact that substitutions between representable signature morphisms can be 'represented' as model homomorphisms.

Proposition 5.16. *Any substitution* $\psi: \chi_1 \to \chi_2$ *between representable signature morphisms* $\chi_1: \Sigma \to \Sigma_1$ *and* $\chi_2: \Sigma \to \Sigma_2$ *determines canonically a* Σ*-model homomorphism* $M_\psi: M_{\chi_1} \to M_{\chi_2}$ *between the representations of the signature morphisms* χ_1 *and* χ_2*. Moreover, the mapping of the substitutions* ψ *to the model homomorphisms* M_ψ *is functorial and faithful modulo substitution equivalence.*

Proof. We define $M_\psi = (i_{\chi_2}^{-1}; \mathsf{Mod}(\psi); i_{\chi_1})(1_{M_{\chi_2}})$. Then by the functoriality of $\mathsf{Mod}(\psi)$ for each $f: M_{\chi_2} \to M$, regarded as $f: 1_{M_{\chi_2}} \to f$ in $M_{\chi_2}/\mathsf{Mod}(\Sigma)$, we have that $f: M_\psi \to (i_{\chi_2}^{-1}; \mathsf{Mod}(\psi); i_{\chi_1})(f)$, which implies that $(i_{\chi_2}^{-1}; \mathsf{Mod}(\psi); i_{\chi_1})(f) = M_\psi; f$.

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma_2) \arrow[r, "i_{\chi_2}", "\sim"'] \arrow[d, "\mathsf{Mod}(\psi)"'] & M_{\chi_2}/\mathsf{Mod}(\Sigma) \arrow[d, "M_\psi/\mathsf{Mod}(\Sigma)"] \\
\mathsf{Mod}(\Sigma_1) \arrow[r, "\sim", "i_{\chi_1}"'] & M_{\chi_1}/\mathsf{Mod}(\Sigma)
\end{tikzcd}$$

The functoriality of the mapping of Σ-substitutions to Σ-model homomorphisms follows by simple calculations by using that $\mathsf{Mod}(1_{\chi_1}) = 1_{\mathsf{Mod}(\Sigma_1)}$ and $\mathsf{Mod}(\psi');\mathsf{Mod}(\psi) = \mathsf{Mod}(\psi;\psi')$ for any composable substitutions ψ and ψ'.

If ψ and ψ' are equivalent substitutions then $M_\psi = M_{\psi'}$ since by its definition M_ψ are uniquely determined by the model translations $\mathsf{Mod}(\psi)$. □

The opposite property is that each model homomorphism $h\colon\ M_{\chi_1} \to M_{\chi_2}$ determines a unique equivalence class of substitutions $\psi_h\colon\ \chi_1 \to \chi_2$ such that $h = M_{\psi_h}$. Thus we say that an institution *has representable $\mathcal{D}$-substitutions* for a class $\mathcal{D}$ of signature morphisms when for each signature Σ the functor from the category of the Σ-$\mathcal{D}$-substitutions between representable signature morphisms to the category of Σ-models is full.

Although a general criterion for an institution to have representable substitutions is not to be expected, this property can be established rather easily for some particular institutions. The following is a rather typical example.

Proposition 5.17. **FOL** *has all representable substitutions.*

Proof. Let $\chi_1 :\ \Sigma \to \Sigma_1$ and $\chi_2 :\ \Sigma \to \Sigma_2$ be representable signature morphisms in **FOL** and let $h\colon\ M_{\chi_1} \to M_{\chi_2}$ be a Σ-model homomorphism.

By Prop. 5.13 we may assume that $\Sigma = (S, F \cup X, P)$, $\Sigma_1 = (S, F \cup X_1, P)$ and $\Sigma_2 = (S, F \cup X_2, P)$ where X, X_1, X_2 are sets of constants and χ_1 and χ_2 keep (S,F,P) invariant but on X manifest as functions $f_1 :\ X \to X_1$ and $f_2 :\ X \to X_2$. Then $M_{\chi_1} = T_F(X_1)\upharpoonright_{\chi_1}$ and $M_{\chi_2} = T_F(X_2)\upharpoonright_{\chi_2}$. Note that because h is a $(S, F \cup X, P)$-homomorphism we have that $h(f_1(x)) = f_2(x)$ for each $x \in X$.

The desired substitution ψ is defined as the first order substitution given by the restriction $h\colon\ X_1 \to T_F(X_2)$. Although ψ appears as a substitution between $(S,F,P) \hookrightarrow (S, F \cup X_1, P)$ and $(S,F,P) \hookrightarrow (S, F \cup X_2, P)$, the condition $h(f_1(x)) = f_2(x)$ ensures that ψ is a substitution $\chi_1 \to \chi_2$. Finally, it is easy to notice that $M_\psi = h$. □

Exercises

5.15. Representable signature morphisms in HNK
In **HNK** the signature extensions with constants $\chi :\ \Sigma \to \Sigma'$, although in general are not representable, they are however quasi-representable. Moreover, χ is representable whenever Σ' has at least a constant operation symbol for each type.

5.16. In any institution the quasi-representable signature morphisms preserve the epi model homomorphisms, i.e., the model homomorphism reduct $h\upharpoonright_\chi$ is epi when h is epi and χ is quasi-representable.

5.17. Quasi-representable presentation morphisms
In any institution I for each presentation (Σ, E) and each quasi-representable signature morphism $\chi : \Sigma \to \Sigma'$, the presentation morphism $\chi : (\Sigma, E) \to (\Sigma', \chi(E))$ is quasi-representable (as signature morphism in I^{pres}). For the non-liberal institutions this constitutes another source of examples which fall between representability and quasi-representability.

5.18. Quasi-representability along institution comorphisms
Any exact institution comorphism (Φ, α, β) preserves quasi-representable signature morphisms in the sense that $\Phi(\chi)$ is quasi-representable when χ is quasi-representable.

5.19. In **IPL** each morphism of signatures is representable. (*Hint:* Consider the comorphism **IPL** $\to$ $(\mathbf{FOEQL}^1)^{\mathrm{pres}}$ of Ex. 4.10 and use the combined conclusions of Exercises 5.17 and 5.18.)

5.20. Liberal representable signature morphisms
In any institution with binary co-products of models for each signature each representable signature morphism is liberal.

5.21. Finitary quasi-representable signature morphisms
Any quasi-representable signature morphism $\chi : \Sigma \to \Sigma'$ determines a canonical functor $\mathsf{Mod}(\chi)^{-1}$: $\mathsf{Mod}(\Sigma) \to \mathbb{S}et$ mapping each Σ-model M to $\{M' \in |\mathsf{Mod}(\Sigma')| \mid M'\restriction_\chi = M\}$. Then χ is finitary if and only if $\mathsf{Mod}(\chi)^{-1}$ preserves the directed co-limits.

5.22. In any institution the finitary quasi-representable signature morphisms are closed under composition.

5.23. Co-products of substitutions
In an institution with representable substitutions which has pushouts of signatures, is semi-exact and its categories of models have finite co-products, for each signature Σ, the category of the Σ-substitutions modulo substitution equivalence between representable signature morphisms has finite co-products.

5.24. Representable substitutions for presentations
Consider a liberal institution I and a class $\mathcal{D}$ of representable signature morphisms such that

- for all presentations (Σ, E) the units of the adjunctions determined by the forgetful functors $\mathsf{Mod}^{\mathrm{pres}}(\Sigma, E) \to \mathsf{Mod}(\Sigma)$ are epi, and
- the representations M_χ of the signature morphisms $\chi \in \mathcal{D}$ are projective.

Let $\mathcal{D}^{\mathrm{pres}}$ be the class of strong presentation morphisms $\chi : (\Sigma, E) \to (\Sigma', E')$ for which $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$. Then the institution I^{pres} of I presentations has representable $\mathcal{D}^{\mathrm{pres}}$-substitutions.

$\mathbf{AFOL}^{\mathrm{pres}}$ (where **AFOL** is the atomic sub-institution of **FOL**) has representable $\mathcal{D}^{\mathrm{pres}}$-substitutions for $\mathcal{D}$ the class of **FOL** signature extensions with a finite number of constants.

5.5 Satisfaction by Injectivity

Basic sentences

In this chapter we have already introduced the semantics of Boolean connectives and quantifiers at a general institution-independent level. The case of the atoms is a bit different because their nature depends to some extent on the actual institution, hence the concept of an atom can only be approximated at an institution-independent level.

Basic sentences. In any **FOL**-signature (S,F,P) let E be a set of atoms. Recall from Sect. 4.6 that E has an initial model 0_E constructed as follows: on the quotient $(T_F)_{=_E}$ of the term model T_F by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(0_E)_\pi = \{(t_1/_{=_E}, \ldots, t_n/_{=_E}) \mid \pi(t_1,\ldots,t_n) \in E\}$.

Fact 5.18. *For each set E of* **FOL***-atoms and for each model M, $M \models E$ if and only if there exists a model homomorphism $0_E \to M$.*

The categorical characterization of atomic satisfaction above can serve as a first institution-independent approximation for the concept of atom.

In any institution, a set E of Σ-sentences is *basic* if there exists a Σ-model M_E such that for each Σ-model M,

$$M \models_\Sigma E \text{ if and only if there exists a model homomorphism } M_E \to M.$$

Often the model M_E is the initial model of E; we have already seen this in Fact 5.18. One may think that the existence of an initial model implies that the respective set of sentences is basic. This is not true, and a simple counterexample in **FOL** is given by the negation of an (S,F)-equation $t_1 \neq t_2$ which has the term model T_F as its initial model but is not basic.

The fact that being basic covers significantly more than atomic sentences is shown by the fact that existentially quantified atoms are also basic. More generally we have the following:

Fact 5.19. *Basic sentences are closed under quasi-representable existential quantification. Moreover $M_{(\exists\chi)\rho'} = M_{\rho'}\upharpoonright_\chi$.*

Epi basic sentences. The concept of epi basic sentence constitutes a better institution-independent capture of the actual atoms.

For a basic set E of sentences, when for each model $M \models E$ the model homomorphism $M_E \to M$ is unique, we say that E is *epi basic*.

Fact 5.20. *All sets E of* **FOL** *atoms are epi basic, with M_E being 0_E, the initial model of E.*

Directly from the definition we have that epi basic sets of sentences always admit initial models. This is one of the important consequences of being epi basic.

Corollary 5.21. *For any epi basic set of sentences E, the model M_E is the initial model which satisfies E.*

Note that existential quantifications of **FOL** atoms are *not* epic basic.

Finitary basic sentences. A basic set of sentences E is *finitary* if the model M_E is finitely presented in the category $\mathsf{Mod}(\Sigma)$ of Σ-models.

Proposition 5.22. *All finite sets of* **FOL** *atoms are finitary basic.*

Proof. We have to prove that for each set E of Σ-atoms in **FOL**, their initial model 0_E is finitely presented.

Let us first suppose that $E = \{t = t'\}$ and consider a model homomorphism $h : (T_F)/_{=_E} \to A$ where the $(\mu_i)_{\in I}$ is a co-limit of a directed diagram $(f_{i,j})_{(i<j)\in(I,\leq)}$ of Σ-model homomorphisms.

$$\begin{tikzcd} & A_i \arrow[r, "f_{i,j}"] \arrow[dr, "\mu_i"'] & A_j \arrow[d, "\mu_j"] \\ (T_F)/_{=_E} \arrow[rr, "h"'] & & A \end{tikzcd}$$

Because of h we have that $A_t = A_{t'}$. If for each $i \in I$ we have that $(A_i)_t \neq (A_i)_{t'}$, then we may consider the family $(\nu_i)_{i\in I}$ of functions $\nu_i : A_i \to \{t, t'\}$ defined by $\nu_i(a) = t$ if and only if there exists $i < j$ such that $f_{i,j}(a) = (A_j)_t$. Then $(\nu_i)_{i\in I}$ is a co-cone for $(f_{i,j})_{(i<j)\in(I,\leq)}$ considered as functions and $\nu_i((A_i)_t) = t$ and $\nu_i((A_i)_{t'}) = t'$ for each $i \in I$. Since $(\mu_i)_{\in I}$ is a co-limit of $(f_{i,j})_{(i<j)\in(I,\leq)}$ considered as functions (forgetful functors from **FOL** models to their carrier sets preserve directed co-limits cf. Prop. 6.5 below) we should get a function $f : A \to \{t, t'\}$ such that $\mu_i; f = \nu_i$ for each $i \in I$, which is not possible because we would have $f(A_t) = \nu_i((A_i)_t) = t$ and $f(A_{t'}) = \nu_i((A_i)_{t'}) = t'$. Thus there exists $i \in I$ such that $(A_i)_t = (A_i)_{t'}$, a fact that gives the desired unique Σ-model homomorphism $(T_F)/_{=_E} \to A_i$.

Now let us suppose that $E = \{\pi(t)\}$ where $\pi(t)$ is a relational atom. Then 0_E is just the term model T_F as (S,F)-algebra and with $(0_E)_\pi = \{t\}$ and with the interpretations of all other relation symbols being $\emptyset$. Because of h we have that $A \models \pi(t)$ and thus for at least one $i \in I$ we have that $A_i \models \pi(t)$. This gives the desired unique model homomorphism $0_E \to A_i$.

For the general case we just have to notice that for any sets E_1 and E_2 of basic sentences, $E_1 \cup E_2$ is still basic with $M_{E_1 \cup E_2}$ being the co-product of models $M_{E_1} + M_{E_2}$ and that in any category co-products of finitely presented objects is still finitely presented (see Ex. 5.25). $\square$

Basic elementary diagrams. The intuition that semantically the elementary diagrams have the nature of atomic sentences is confirmed by the following result.

Proposition 5.23. *In any institution with elementary diagrams with quasi-representable elementary extensions, the elementary diagrams are epi basic.*

Proof. Let A be a Σ-model. Then $M_{E_A} = A_A$, the initial model of the elementary diagram (Σ_A, E_A) of A.

Let N' be any Σ_A-model. If $N' \models E_A$, then because A_A is the initial (Σ_A, E_A)-model we have that there exists a unique model homomorphism $A_A \to N'$.

Conversely, if there exists a model homomorphism $h' : A_A \to N'$ let $h = h' \restriction_{\iota_\Sigma(A)} : A \to N = N' \restriction_{\iota_\Sigma(A)}$. Let $N'' = i_{\Sigma,A}^{-1}(h)$. Then $N'' \models E_A$. Because $\iota_\Sigma(A)$ is quasi-representable and $A_A\upharpoonright_{\iota_\Sigma(A)} = A$, there exists a unique $\iota_\Sigma(A)$-expansion of h to a Σ_A-model homomorphism from A_A, therefore $N'' = N'$, hence $N' \models E_A$. $\square$

That elementary diagrams are epi basic shows that even the epi basic sets of sentences, although having the semantic appearance of atoms, do not necessarily consist of atoms only. Consider for example the sub-institution of injective homomorphisms in **FOL**. Recall (Sect. 4.4) that in this institution the elementary diagram of a model M consists of all atoms and *negations* of equations satisfied by M_M. Perhaps this tells us that from a semantic perspective the concept of atom as the most primitive constituent of sentences should be considered according to the actual concept of model homomorphism.

Satisfaction by injectivity. Recall that a model M is injective with respect to a model homomorphism $h\colon\ A \to B$ when for each homomorphism $f\colon\ A \to M$ there exists a homomorphism $g\colon\ B \to M$ such that $h;g = f$.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

Let us denote this by $M \models^{\text{inj}} h$. For each homomorphism h let $Inj(h)$ be the class of models injective with respect to h, and for each class of homomorphisms H let $Inj(H) = \bigcap_{h\in H} Inj(h)$.

The semantics of the basic sentences constitutes a simple example of satisfaction by injectivity.

Fact 5.24. *Let Σ be a signature with initial model 0_Σ. For any basic set E of Σ-sentences and any Σ-model M,*

$$M \models e \ \textit{ if and only if } \ M \models^{\text{inj}} (0_\Sigma \to M_E),$$

where $0_\Sigma \to M_E$ is the unique model homomorphism given by the initiality of 0_Σ.

General Horn sentences. In any institution, for a designated class $\mathcal{D}$ of signature morphisms, a *$\mathcal{D}$-universal Horn sentence* is a sentence semantically equivalent to $(\forall\chi)(E \Rightarrow E')$ where

- $\chi\colon\ \Sigma \to \Sigma'$ is a representable signature morphism in $\mathcal{D}$,
- E is a set of epic basic Σ'-sentences, and
- E' is a basic set of Σ'-sentences.

A universal Horn sentence $(\forall\chi)(E \Rightarrow E')$ is *finitary* when E, and χ are finitary.

Note that the general concept of finitary Horn sentence defined above covers more sentences than some of the actual concepts of Horn sentence in institutions. This is due to the fact that the basic, and even the epi basic, sentences are usually more than the actual atoms in institutions. For example $(\forall X)(\exists Y)t = t'$ is a general finitary Horn sentence in **FOL** but it is not a **HCL**-sentence.

As for the **HCL**-sentences, we will rather omit the second pair of brackets from the notation of the general Horn sentences, which means that we write $(\forall\chi)E \Rightarrow E'$ rather than $(\forall\chi)(E \Rightarrow E')$.

Satisfaction of Horn sentences is injectivity

For the rest of this section let us assume that in our institutions the basic sets of sentences are closed under finite unions. For example this happens whenever there exists co-products of models by letting $M_{E\cup E'} = M_E + M_{E'}$. The following extends the encoding of satisfaction of sentences as injectivity from basic sentences to Horn sentences.

Proposition 5.25. *In any institution, for any universal Horn sentence $(\forall\chi)E \Rightarrow E'$ there exists a model homomorphism h such that for each Σ-model M,*

$$M \models^{\mathrm{inj}} h \ \textit{ if and only if } \ M \models (\forall\chi)E \Rightarrow E'.$$

Proof. Because $E \cup E' \models E$, $M_{E\cup E'} \models E$. Therefore there exists $h' : M_E \to M_{E\cup E'}$. We let $h = h' \upharpoonright_\chi$.

First let us assume that the model M is injective with respect to $h = h' \upharpoonright_\chi$. Consider an arbitrary Σ'-model M' such that $M' \upharpoonright_\chi = M$ and $M' \models E$. This implies that there exists a model homomorphism $f' : M_E \to M'$. Because M is injective with respect to h, there exists a Σ-model homomorphism g such that $h;g = f' \upharpoonright_\chi$. Because χ is quasi-representable, we get $g' : M_{E\cup E'} \to M'$ such that $g' \upharpoonright_\chi = g$. This means that $M' \models_{\Sigma'} E \cup E'$, which implies $M' \models_{\Sigma'} E'$.

$$\begin{tikzcd} M_E\upharpoonright_\chi \arrow[r, "h"] \arrow[dr, "f"'] & M_{E\cup E'}\upharpoonright_\chi \arrow[d, "g"] \\ & M = M'\upharpoonright_\chi \end{tikzcd} \qquad \begin{tikzcd} M_E \arrow[r, "h'"] \arrow[dr, "f'"'] & M_{E\cup E'} \arrow[d, "g'"] \\ & M' \end{tikzcd}$$

Conversely, assume that $M \models (\forall\chi)E \Rightarrow E'$. Because χ is quasi-representable, each Σ-model homomorphism $f : M_E \upharpoonright_\chi \to M$ admits an expansion to an Σ'-model homomorphism $f' : M_E \to M'$. This implies that $M' \models E$, therefore $M' \models E \cup E'$, which guarantees the existence of a model homomorphism $g' : M_{E\cup E'} \to M'$. Because E is epi basic $h';g' = f'$, which implies $h;g' \upharpoonright_\chi = f$. $\square$

Corollary 5.26. *For each Σ-model M and each basic set of Σ'-sentences E',*

$$M \models^{\mathrm{inj}} (0_{\Sigma'} \to M_{E'})\upharpoonright_\chi \ \textit{ if and only if } \ M \models_\Sigma (\forall\chi)E'.$$

Injectivity is satisfaction of Horn sentences

We have seen that the satisfaction of Horn sentences can be expressed as categorical injectivity with respect to a model homomorphism which has the flavor of a model quotient and actually is a model quotient in many actual situations.

We show now that the opposite is also true, the injectivity with respect to a 'quotient' model homomorphism can be expressed as the satisfaction of a Horn sentence. This result is based on the following diagram-based abstract concept of model 'quotient'.

ι-conservative model homomorphisms. Given an institution with elementary diagrams ι, a Σ-model homomorphism $h\colon A \to B$ is ι-*conservative* when the presentation morphism $\iota_\Sigma(h)\colon (\Sigma_A, \iota_\Sigma(h)^{-1}(E_B^{**})) \to (\Sigma_B, E_B)$ is *conservative*, which means that each $(\Sigma_A, \iota_\Sigma(h)^{-1}(E_B^{**}))$-model has a $\iota_\Sigma(h)$-expansion to a (Σ_B, E_B)-model.

$$\begin{tikzcd} & \Sigma \arrow[dl, "\iota_\Sigma(A)"'] \arrow[dr, "\iota_\Sigma(B)"] & \\ (\Sigma_A, E_A) \arrow[rr, "\iota_\Sigma(h)"'] & & (\Sigma_B, E_B) \end{tikzcd}$$

The reader is invited to check the following.

Fact 5.27. *Consider* **FOL** *with its standard system of elementary diagrams. Then each surjective model homomorphism is* ι-*conservative.*

Proposition 5.28. *In any institution with elementary diagrams* ι, *let* h *be a* Σ-*model homomorphism* $h\colon A \to B$. *Then*

1. *For any model* M, $M \models^{\mathrm{inj}} h$ *implies* $M \models_\Sigma (\forall \iota_\Sigma(A))E_A \Rightarrow \iota_\Sigma(h)^{-1}(E_B^{**})$.
2. *If* h *is* ι-*conservative, then for each model* M, $M \models_\Sigma (\forall \iota_\Sigma(A))E_A \Rightarrow \iota_\Sigma(h)^{-1}(E_B^{**})$ *implies* $M \models^{\mathrm{inj}} h$.

Consequently, if the elementary extensions are representable then the satisfaction by injectivity with respect to any ι-*conservative* h *is the same as satisfaction of a general Horn sentence.*

Proof. 1. Consider a model M such that $M \models^{\mathrm{inj}} h$. Let M' be any Σ_A-model such that $M'\upharpoonright_{\iota_\Sigma(A)} = M$ and $M' \models E_A$. Let $f = i_{\Sigma,A}(M')$. By the injectivity of M, let g be such that $h;g = f$ and let $M'' = i_{\Sigma,B}^{-1}(g)$. Notice that by the naturality of i, $M''\upharpoonright_{\iota_\Sigma(h)} = M'$. Since $M'' \models E_B^{**}$, by the satisfaction condition we deduce that $M' \models \iota_\Sigma(h)^{-1}(E_B^{**})$.

$$\begin{tikzcd} \Sigma \arrow[r, "\iota_\Sigma(A)"] \arrow[dr, "\iota_\Sigma(B)"'] & \Sigma_A \arrow[d, "\iota_\Sigma(h)"] \\ & \Sigma_B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\ & M \end{tikzcd}$$

2. Consider a model M such that $M \models_\Sigma (\forall \iota_\Sigma(A))E_A \Rightarrow \iota_\Sigma(h)^{-1}(E_B^{**})$ and a model homomorphism $f\colon A \to M$. Let $M_f = i_{\Sigma,A}^{-1}(f)$. Then $M_f \models \iota_\Sigma(h)^{-1}(E_B^{**})$ since $M_f \models E_A$. The existence of a (Σ_B, E_B)-model M' with $M'\upharpoonright_{\iota_\Sigma(h)} = M_f$ is granted by the fact that h is ι-conservative. Now let $g = i_{\Sigma,B}(M')$. By the naturality of i, we have $h;g = f$.

Finally, the last conclusion follows because E_A is epi basic (cf. Prop. 5.23) and by the following lemma:

Lemma 5.29. *If all elementary extensions are quasi-representable, then* $\iota_\Sigma(h)^{-1}(E_B^{**})$ *is basic for any* ι*-conservative h. Moreover,* $M_{\iota_\Sigma(h)^{-1}(E_B^{**})} = (B_B)\restriction_{\iota_\Sigma(h)}$.

Proof. Consider any Σ-model M.

On the one hand, if $M \models \iota_\Sigma(h)^{-1}(E_B^{**})$, then because h is ι-conservative, there exists a $\iota_\Sigma(h)$-expansion M' of M to a (Σ_B, E_B)-model. Let f be the unique (Σ_B, E_B)-model homomorphism $B_B \to M'$. Then $f\restriction_{\iota_\Sigma(h)} : (B_B)\restriction_{\iota_\Sigma(h)} \to M$.

Conversely, if we have a Σ_A-model homomorphism $(B_B)\restriction_{\iota_\Sigma(h)} \to M$, then because $\iota_\Sigma(h)$ is quasi-representable (by Prop. 5.12 since both $\iota_\Sigma(A)$ and $\iota_\Sigma(B)$ are quasi-representable) we can expand it (uniquely) to a Σ_B-model homomorphism $B_B \to M'$. Therefore $M' \models E_B$ because E_B is basic and $M_{E_B} = B_B$ (cf. Prop. 5.23) which by the Satisfaction Condition implies that $M = M'\restriction_{\iota_\Sigma(h)} \models \iota_\Sigma(h)^{-1}(E_B^{**})$. □

□

From Cor. 5.26 and Lemma 5.29 we get easily the following specialized variant of Prop. 5.28.

Corollary 5.30. *In any institution with quasi-representable elementary diagrams, for any representable signature morphism* $\chi : \Sigma \to \Sigma'$ *and any* Σ*-model B,*

$$M \models^{\mathrm{inj}} (M_\chi \xrightarrow{h} B) \text{ if and only if } M \models_\Sigma (\forall \iota_\Sigma(M_\chi))\iota_\Sigma(h)^{-1}(E_B^{**})$$

for each ι*-conservative* Σ*-model homomorphism* $h : M_\chi \to B$.

Exercises

5.25. Unions of finitary basic sentences
Finite co-products of finitely presented objects are still finitely presented. If finite co-products of models exist then the union of finitary basic sets of sentences is still finitary basic.

5.26. Finitary basic sentences are closed under finitary quasi-representable existential quantifications. (*Hint:* The model reducts corresponding to the finitary quasi-representable signature morphisms preserve the finitely presented models.)

5.27. Representable presentation morphisms
Let $\varphi : \Sigma \to \Sigma'$ be a quasi-representable signature morphism in an institution I. If E' is epi basic then each presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is representable (as a signature morphism of I^{pres}).

5.28. Preservation of Horn sentences
A sentence ρ is *preserved by a limit* $(M_i \xrightarrow{\mu_i} M)_{i\in|J|}$ of a diagram of models $(M_i \xrightarrow{f_u} M_j)_{u\in J}$ of models when $M_i \models \rho$ for each $i \in |J|$ implies $M \models \rho$.
In any institution:

1. Small products of models preserve Horn sentences.
2. Small limits of models preserve all Horn sentences $(\forall\chi)E \Rightarrow E'$ for which E' is *epi* basic.
3. Directed co-limits of models preserve the finitary Horn sentences.

5.29. Basic sentences modulo presentations
Let I be a liberal institution.

1. Each set of sentences which is (epi) basic in I is (epi) basic in the institution of its presentations I^{pres} too.
2. If each sentence of I is preserved by directed co-limits, then any finitary basic sets of sentences in I is finitary basic in I^{pres} too.

5.30. Borrowing basic sentences along comorphisms
Any persistently liberal institution comorphism (Φ,α,β) 'borrows' the (finitary) epi basic sentences, i.e., E is (finitary) epi basic when $\alpha(E)$ is so. This can be applied in conjunction with the results of Ex. 5.29 for the comorphisms of Ex. 4.73 for showing that finite sets of existence equations in **PA** or of any atoms of several other institutions (such as **POA**, **AUT**, **MBA**) are finitary epi basic.

5.31. [35] **HNK** has atoms that are not basic. In the signature (S,F) defined by $S = \{s,s'\}$ and $F_{s\to s} = \{f\}$, $F_{(s\to s)\to s'} = \{\sigma_1,\sigma_2\}$, for other types x, F_x being empty, the atom $\sigma_1(f) = \sigma_2(f)$ is *not* basic. (*Hint:* Consider the model M defined by M_s empty, and $M_{s'}$, $M_{s\to s}$ and $M_{(s\to s)\to s'}$ containing only one element. Then $M \models \sigma_1(f) = \sigma_2(f)$. For any other model N satisfying $\sigma_1(f) = \sigma_2(f)$ but such that $N_{\sigma_1} \neq N_{\sigma_2}$ there exists no model homomorphism $M \to N$. Deduce from here that $\sigma_1(f) = \sigma_2(f)$ cannot be basic.)

5.32. In the institution **IPL** considered with elementary homomorphisms of models (see Ex. 4.41) all sentences are epi basic. (*Hint:* For any P-sentence ρ the model M_ρ is given by the free Heyting algebra over P satisfying $\rho = \top$.) Moreover, if P is finite, then each P-sentence is finitary epi basic.

5.33. Institution of injectivity
The following defines a 'hyper-institution' **INJ**:

1. $\mathbb{S}ig^{\mathbf{INJ}}$ is the 'hyper-category' of the adjunctions, i.e., the signatures are categories and the signature morphisms are adjunctions $(\mathcal{U},\mathcal{F},\eta,\varepsilon)\colon\ \mathbb{A}\to\mathbb{B}$ where $\mathcal{U}\colon\ \mathbb{B}\to\mathbb{A}$ is the right adjoint and $\mathcal{F}$ is the left adjoint.
2. $\mathsf{Sen}^{\mathbf{INJ}}(\mathbb{A}) = arr(\mathbb{A})$ (for each category $\mathbb{A}$ the class of all arrows of A), and $\mathsf{Sen}^{\mathbf{INJ}}(\mathcal{U},\mathcal{F},\eta,\varepsilon) = \mathcal{F}$ for adjunctions,
3. $\mathsf{Mod}^{\mathbf{INJ}}(\mathbb{A}) = \mathbb{A}$ for each category $\mathbb{A}$ and $\mathsf{Mod}^{\mathbf{INJ}}(\mathcal{U},\mathcal{F},\eta,\varepsilon) = \mathcal{U}$ for adjunctions, and
4. $A \models^{\mathbf{INJ}} f$ if and only if $A \models^{\mathrm{inj}} f$.

There exists an institution comorphism from the sub-institution of Cat**EQL** for which the categories have co-equalizers to **INJ** mapping each categorical equation $(\forall B)l = r$ to a designated co-equalizer.

5.6 Elementary Homomorphisms

Let us now recall the concept of elementary homomorphism introduced in Sect. 4.4. In any institution with elementary diagrams ι, a Σ-model homomorphism $h\colon\ M \to N$ is elementary when $N_h \models M^*_M$ where N_h is the canonical expansion determined by h of N to a Σ_M-model, i.e., $N_h = i_{\Sigma,M}(N)$.

This diagram-based definition of elementary homomorphism does not support some desirable structural properties of elementary homomorphisms, such as closure under composition. In this section we provide an alternative concept of elementary homomorphism which we show to coincide with the diagram-based one under some general 'normality' condition on the system of diagrams. This involves representable signature morphisms. The most important gain of the identification between these two different perspectives on elementary homomorphisms is good structural properties which can be summed up by the fact that elementary homomorphisms of the institution form themselves an institution with elementary diagrams.

$\mathcal{D}$-elementary homomorphisms. Given a class $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, a Σ-model homomorphism $h : A \to B$ is *$\mathcal{D}$-elementary* when $A'^* \subseteq B'^*$ for each $\mathcal{D}$-expansion $h' : A' \to B'$ of h.

In the actual institutions, $\mathcal{D}$ is typically the class of all signature extensions with constants. Notice that in the case of **FOL**, and in fact in all institutions with finitary sentences, elementarity with respect to signature extensions with an arbitrary number of constants is equivalent to elementarity with respect to extensions adding *finite* numbers of constants.

The following applies to the cases when $\mathcal{D}$ contains all signature extensions with constants.

Fact 5.31. *In any institution with elementary diagrams such that $\mathcal{D}$ contains all elementary extensions, any $\mathcal{D}$-elementary homomorphism is elementary.*

Structural properties of elementary homomorphisms. We now give some general conditions on $\mathcal{D}$ which ensure that the $\mathcal{D}$-elementary homomorphisms form a sub-institution of the original institution.

Proposition 5.32. *Let $\mathcal{D}$ be a class of signature morphisms.*

1. *If each morphism in $\mathcal{D}$ is quasi-representable, then $\mathcal{D}$-elementary homomorphisms are closed under composition.*
2. *If the institution is weakly semi-exact and $\mathcal{D}$ is stable under pushouts, then $\mathcal{D}$-elementary homomorphisms are preserved by any model reduct functor.*
3. *$\mathcal{D}$-elementary homomorphisms are closed under $\mathcal{D}$-expansions.*

Proof. 1. Let $f : A \to B$ and $g : B \to C$ be $\mathcal{D}$-elementary homomorphism and let $h' : A' \to C'$ be a χ-expansion of $f;g$ for $\chi \in \mathcal{D}$. Then f and A' determine a unique χ-expansion $f' : A' \to B'$ of f, and g and B' determine a unique χ-expansion $g' : B' \to C''$ of g. Therefore $f';g'$ is the unique χ-expansion of $f;g$, hence $C'' = C'$ and $f';g' = h'$. $A'^* \subseteq B'^*$ because f is $\mathcal{D}$-elementary and $B'^* \subseteq C'^*$ because g is $\mathcal{D}$-elementary, hence $A'^* \subseteq C'^*$.

2. Let $h_1 : A_1 \to B_1$ be a $\mathcal{D}$-elementary Σ_1-model homomorphism and $\varphi : \Sigma \to \Sigma_1$ be any signature morphism. In order to prove that $h_1\upharpoonright_\varphi$ is $\mathcal{D}$-elementary, we consider $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$. Because $\mathcal{D}$ is stable under pushouts, in the pushout square below we

have that $\chi_1 \in \mathcal{D}$.

$$\begin{tikzcd}
\Sigma \arrow[r, "\chi"] \arrow[d, "\varphi"'] & \Sigma' \arrow[d, "\varphi'"] \\
\Sigma_1 \arrow[r, "\chi_1"'] & \Sigma'_1
\end{tikzcd}$$

Let $h' : A' \to B'$ be a χ-expansion of $h_1\restriction_\varphi$. By the weak semi-exactness, there exists $h'_1 : A'_1 \to B'_1$ such that $h'_1\restriction_{\varphi'} = h'$ and $h'_1\restriction_{\chi_1} = h_1$. Because h_1 is $\mathcal{D}$-elementary and $\chi_1 \in \mathcal{D}$, $A'^*_1 \subseteq B'^*_1$. By the Satisfaction Condition this implies that $A'^* \subseteq B'^*$.

3. Immediate from the definition. □

Corollary 5.33. *Under the conditions of 1 and 2 of Prop. 5.32, the $\mathcal{D}$-elementary homomorphisms form a sub-institution of the original institution.*

Normal elementary diagrams

We have already seen that $\mathcal{D}$-elementary homomorphisms are elementary under the natural assumption that the elementary extensions belong to $\mathcal{D}$. This was rather easy. Now we focus on the opposite inclusion, which is not so immediate.

For any class $\mathcal{D}$ of representable signature morphisms, the elementary diagrams of an institution are *$\mathcal{D}$-normal* if for each $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ (represented by M_χ), there exists a signature morphism $\varphi : \Sigma' \to \Sigma_{M_\chi}$ such that the diagrams below commute:

$$\begin{tikzcd}
& \Sigma \arrow[dl, "\iota_\Sigma(M_\chi)"'] \arrow[d, "\chi"] \\
\Sigma_{M_\chi} & \Sigma' \arrow[l, "\varphi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
M_\chi/\mathsf{Mod}(\Sigma) & \mathsf{Mod}(\Sigma_{M_\chi}, E_{M_\chi}) \arrow[l, "i_{\Sigma,M_\chi}"'] \arrow[d, "\text{forgetful}"] \\
\mathsf{Mod}(\Sigma') \arrow[u, "i_\chi"] & \mathsf{Mod}(\Sigma_{M_\chi}) \arrow[l, "\mathsf{Mod}(\varphi)"]
\end{tikzcd}$$

In actual institutions, for the choice of $\mathcal{D}$ above supporting that $\mathcal{D}$-elementary homomorphisms are elementary (i.e., $\mathcal{D}$ consisting of signature extensions with constants), the elementary diagrams are also $\mathcal{D}$-normal. The following **FOL** case is rather typical.

Proposition 5.34. **FOL** *has $\mathcal{D}$-normal diagrams for $\mathcal{D}$ consisting of all signature extensions with constants.*

Proof. Given a **FOL** signature extension with constants $\chi : (S,F,P) \hookrightarrow (S,F \uplus X,P)$, represented by the free (term) model $T_F(X)$, the desired φ such that $\chi;\varphi = \iota_{(S,F,P)}(T_F(X))$ is the signature inclusion $(S,F \uplus X,P) \hookrightarrow (S,F \uplus T_F(X),P)$ given by the set inclusion $X \hookrightarrow T_F(X)$.

In order to see that the corresponding condition on the model categories holds, let N be a $(S,F \uplus T_F(X),P)$-model that satisfying $E_{T_F(X)}$. Then

$$i_{(S,F,P),T_F(X)}(N) = T_F(X) \xrightarrow{h} N\restriction_{\iota_{(S,F,P)}(T_F(X))}$$

where $h(t) = N_t$ for all $t \in T_F(X)$. On the other hand,

$$i_\chi(N{\upharpoonright}_\varphi) = T_F(X) \xrightarrow{g} N{\upharpoonright}_{\chi;\varphi} = N{\upharpoonright}_{\iota_{(S,F,P)}(T_F(X))}$$

where g is the unique homomorphism such that $g(x) = N_x$ for each $x \in X$. Thus, by the freeness of $T_F(X)$, $g = h$. Hence the functors $\mathsf{Mod}(\varphi); i_\chi$ and $i_{(S,F,P),T_F(X)}$ coincide on models. That they coincide on model homomorphisms follows at once from $\chi;\varphi = \iota_{(S,F,P)}(T_F(X))$. □

The following is the main motivation for the normality condition for elementary diagrams.

Proposition 5.35. *In any institution with $\mathcal{D}$-normal elementary diagrams, any elementary homomorphism is $\mathcal{D}$-elementary.*

Proof. Consider $h\colon\ A \to B$ an elementary Σ-homomorphism and let $(\chi\colon\ \Sigma \to \Sigma') \in \mathcal{D}$ and $h'\colon\ A' \to B'$ be a χ-expansion of h. We have to show that $A^* \subseteq B^*$.

We have that

$$i^{-1}_{\Sigma,M_\chi}(i_\chi(A' \xrightarrow{h'} B')) = i^{-1}_{\Sigma,M_\chi}(M_\chi \xrightarrow{i_\chi(A')} A) \xrightarrow{i^{-1}_{\Sigma,M_\chi}(h)} i^{-1}_{\Sigma,M_\chi}(M_\chi \xrightarrow{i_\chi(B')} B).$$

By the naturality of i, the diagram below commutes.

$$\begin{tikzcd}
M_\chi/\mathsf{Mod}(\Sigma) & \mathsf{Mod}(\Sigma_{M_\chi}, E_{M_\chi}) \arrow[l, "i_{\Sigma,M_\chi}"'] \\
A/\mathsf{Mod}(\Sigma) \arrow[u, "i_\chi(A');_"] & \mathsf{Mod}(\Sigma_A, E_A) \arrow[l, "i_{\Sigma,A}"] \arrow[u, "\mathsf{Mod}(\iota_\Sigma(i_\chi(A')))"']
\end{tikzcd}$$

When we apply this to $(A \xrightarrow{1_A} A) \xrightarrow{h} (A \xrightarrow{h} B)$, regarded as arrows in $A/\mathsf{Mod}(\Sigma)$, we get that

$$(A_A \xrightarrow{i^{-1}_{\Sigma,A}(h)} B_h){\upharpoonright}_{\iota_\Sigma(i_\chi(A'))} = i^{-1}_{\Sigma,M_\chi}(i_\chi(A' \xrightarrow{h'} B')).$$

Because the diagrams are normal

$$(A_A \xrightarrow{i^{-1}_{\Sigma,A}(h)} B_h){\upharpoonright}_{\iota_\Sigma(i_\chi(A'))}{\upharpoonright}_\varphi = A' \xrightarrow{h'} B'$$

Because h is elementary, $A_A^* \subseteq B_h^*$. From this by the satisfaction condition applied twice we obtain that $A'^* \subseteq B'^*$. □

Corollary 5.36. *In any institution with $\mathcal{D}$-normal elementary diagrams such that $\mathcal{D}$ contains all elementary extensions, the notions of elementary homomorphism and $\mathcal{D}$-elementary homomorphisms coincide.*

The following sums up the developments of this section.

Corollary 5.37. *In any weakly semi-exact institution I with $\mathcal{D}$-normal elementary diagrams for $\mathcal{D}$ a class of quasi-representable signature morphisms which is stable under pushouts, and such that it contains all elementary extensions, the elementary homomorphisms form a (sub-)institution $E(I)$ which has elementary diagrams. $E(I)$ is called the* elementary sub-institution *of I.*

Proof. By Corollary 5.36 elementary homomorphism and $\mathcal{D}$-elementary homomorphisms coincide in I. Thus by Corollary 5.33, elementary homomorphisms form a sub-institution. The elementary extensions in $E(I)$ are inherited from I, and for each Σ-model M, its elementary diagram in $E(I)$ is given by M^*_M. □

In particular we have the following.

Corollary 5.38. *In* **FOL** *the elementary embeddings form an institution.*

Exercises

5.34. Taking the elementary sub-institution is an idempotent operation on institutions, i.e., $E(E(I)) = E(I)$.

Notes. The institution-independent semantics of Boolean connectives is rather folklore of institution theory, perhaps this was introduced first time in [169]. . Abstract logical connectives have been introduced here. The institution-independent semantics of quantifiers has been introduced first time by [170] and is used extensively in [47]. A rather different institution-independent approach to connectives, which treats Boolean, modal connectives but also quantifiers in an uniform manner is give by the so-called ‘connector algebras’ of [3].

Although quantification by signature extensions is well known in conventional mathematical logic [165, 98] it is quite rare in the usual presentations of conventional logic or model theory.

The institution-independent concept of substitution has been introduced in [49]. A rather different approach has been developed within the framework of the ‘context institutions’ of [145]. Derived signatures have been used in algebraic specification for defining the instantiations of parameters in parameterized modules. The so-called ‘views’ of OBJ [82, 56] are just an implementation of second order substitutions.

The institution-independent approach on first order quantifiers via representable signature morphisms has been developed in [47] which also introduced the concept of finitary representable signature morphisms as an abstract categorical treatment for finitary first order quantification. General finitary signature morphisms are introduced here. Prop. 5.13 has been proved in [37].

Satisfaction by injectivity is a well-known concept in categorical universal algebra and it has been intensively used in the general study of Birkhoff axiomatizability in arbitrary categories [6]. According to [142] injectivity was first used to represent satisfaction in [11]. In [6] the injectivity is extended to arbitrary cones which covers the satisfaction of all first order formulæ, however this leads to enormous conceptual and proof complexity without actually going beyond the boundaries of first order satisfaction. The same satisfaction power, and even much more, can be achieved only by basic sentences, internal quantification and logical connectives, but in a much simpler framework. This is due to the advantage of using the multi-signature framework based on institutions as opposed to the other more rigid single-signature categorical abstract model theoretic frameworks.

The institution-independent concept of elementary homomorphism, due to [86], unifies various concepts of model embeddings from the literature, such as elementary embeddings from conventional model theory [32] for **FOL**, elementary embeddings of partial algebras [29], $L_{\infty,\omega}$- and $L_{\alpha,\omega}$-elementary embeddings from infinitary model theory [104, 118, 97], the existentially closed embeddings of [97] for $(\Pi \cup \Sigma)^0_1$, the Σ^0_n-extensions [32] for $(\Pi \cup \Sigma)^0_n$. $\mathcal{D}$-elementary homomorphisms have been introduced by [86], which also proved their equivalence to (ordinary) elementary homomorphisms under the normality condition on the elementary diagrams.

Chapter 6

Model Ultraproducts

The ultraproduct construction on models is one of the most important devices used by 'first order model theory', which is that part of model theory relying upon 'first order' quantifiers (handled by representable signature morphisms) and finiteness at various syntactic levels such as arities of symbols, atoms, quantification, and logical connectives.

The main result presented in this chapter is a fundamental ultraproducts theorem which pervades all application of the method of ultraproducts. We develop it here in a modular manner, different combinations of its various parts being applicable to a great variety of institutions.

Some immediate applications of the method of ultraproducts presented in this chapter include a general ultrapower embedding theorem, compactness results, and a general isomorphism criterion for finitely sized models.

6.1 Filtered Products

Ultraproducts of models is a special case of filtered products of models. In this section we first illustrate the filtered products construction for the particular example of the **FOL** models and only after this do we introduce the general concept of filtered products in arbitrary categories.

Filters. For each non-empty set I we denote the set of all subsets of I by $\mathcal{P}(I)$. A *filter F over I* is defined to be a set $F \subseteq \mathcal{P}(I)$ such that

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$, and
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

A filter F is *proper* when F is not $\mathcal{P}(I)$ and it is an *ultrafilter* when $X \in F$ if and only if $(I \setminus X) \notin F$ for each $X \in \mathcal{P}(I)$. Notice that ultrafilters are proper filters. We will always assume that all our filters are proper.

Filtered products in FOL

Given a **FOL** signature Σ, let $\{M_i\}_{i\in I}$ be a family of models and let F be a filter over I. Let M denote the product of models $\prod_{i\in I} M_i$. For sort s of Σ, for each element $m \in M_s$ let $m = (m_i)_{i\in I}$ with $m_i \in (M_i)_s$ for each $i \in I$. By defining the equivalence $\sim_F$ on M by

$$m \sim_F m' \text{ if and only if } \{i \mid m_i = m'_i\} \in F$$

(which is correctly defined because F is a filter), we construct the *filtered product* $\prod_F M_i$ *of* $\{M_i\}_{i\in I}$ *modulo* F by

- $(\prod_F M_i)_s = M_s/_{\sim_F}$ for each sort s of Σ,
- $(\prod_F M_i)_\sigma(m/_{\sim_F}) = M_\sigma(m)/_{\sim_F}$ for each operation σ of Σ and each list m of arguments for M_σ, and
- $(\prod_F M_i)_\pi = \{m/_{\sim_F} \mid \{i \in I \mid m_i \in (M_i)_\pi\} \in F\}$ for each relation π of Σ and each list m of arguments for M_π.

Routine calculations based on the filter property of F give the correctness of the definition of the filtered product $\prod_F M_i$.

For each $J \in F$, let $\mu_J : \prod_{i\in J} M_i \to \prod_F M_i$ be the model homomorphism such that $\mu_J(m) = m'/_{\sim_F}$ for each m' such that $m = \{m'_i\}_{i\in J}$. Because F is a filter, μ_J is well defined and is a model homomorphism. The reader is invited to check this.

Then the family $(\mu)_{J\in F}$ form a co-cone $M_F \Rightarrow \prod_F M_i$, where $M_F : F \to \mathsf{Mod}(\Sigma)$ is the functor mapping each $(J \subset J')$ to the canonical projection $p_{J',J} : \prod_{i\in J'} M_i \to \prod_{i\in J} M_i$.

Proposition 6.1. *$\mu : M_F \Rightarrow \prod_F M_i$ is a co-limit of $M_F : F \to \mathsf{Mod}(\Sigma)$.*

Proof. Let $\nu : M_F \Rightarrow N$ be another co-cone over M_F.

$$\begin{tikzcd}
\prod_{i\in J'} M_i \arrow[rr, "p_{J',J}"] \arrow[dr, "\mu_{J'}"] \arrow[ddr, bend right, "\nu_{J'}"'] & & \prod_{i\in J} M_i \arrow[dl, "\mu_J"'] \arrow[ddl, bend left, "\nu_J"] \\
& M/_{\sim_F} \arrow[d, "h"'] & \\
& N &
\end{tikzcd}$$

There exists a unique many-sorted function $h : \prod_F M_i \to N$ such that $h_s(m/_{\sim_F}) = \nu_I(m)$ for each $m \in M_s$ for each sort s. Notice that the definition of h is correct because for each $m \sim_F m'$,

$$\nu_I(m) = (p_{I,J};\nu_J)(m) = (p_{I,J};\nu_J)(m') = \nu_I(m')$$

where $J = \{i \mid m_i = m'_i\}$.

We prove that h is a model homomorphism $\prod_F M_i \to N$. For each operation σ and each list of arguments m for M_σ, we successively have $h((\prod_F M_i)_\sigma(m/_{\sim_F})) = h((\prod_F M_i)_\sigma(\mu_I(m))) = h(\mu_I(M_\sigma(m))) = \nu_I(M_\sigma(m)) = N_\sigma(\nu_I(m)) = N_\sigma(h(\mu_I(m))) =$

$N_\sigma(h(m/_{\sim_F}))$, therefore h commutes with the interpretations of the operations. For each relation symbol π, let w be its arity, and assume that $m/_{\sim_F} \in (\prod_F M_i)_\pi$ for some $m \in M_w$. Then the set $J = \{i \in I \mid m_i \in (M_i)_\pi\} \in F$. We have that $h(m/_{\sim_F}) = (\mu_I;h)(m) = (\mu_J;h)(p_{I,J}(m)) = \nu_J(p_{I,J}(m)) \in N_\pi$ since $p_{I,J}(m) = (m_i)_{i\in J} \in (\prod_{i\in J} M_i)_\pi$. Therefore h preserves the interpretations of the relation symbols too. $\square$

Prop. 6.1 has the merit that it gives a purely categorical description of filtered products of **FOL** models. This indicates that filtered products can be defined at the level of abstract categories.

Categorical filtered products

Consider a filter F over the set of indices I and a family of objects $\{A_i\}_{i\in I}$ in any category $\mathbb{C}$ with small products. These determine a functor $A_F : F^{\mathrm{op}} \to \mathbb{C}$ mapping each subset inclusion $J \subset J'$ of F to the canonical projection $p_{J',J} : \prod_{i\in J'} A_i \to \prod_{i\in J} A_i$.

A *filtered product of* $\{A_i\}_{i\in I}$ *modulo* F is a co-limit $\mu : A_F \Rightarrow \prod_F A_i$ of the functor A_F.

$$\begin{tikzcd}
\prod_{i\in J'} A_i \arrow[rr, "p_{J',J}"] \arrow[dr, "\mu_{J'}"'] & & \prod_{i\in J} A_i \arrow[dl, "\mu_J"] \\
& \prod_F A_i &
\end{tikzcd}$$

Obviously, as co-limits of diagrams of products, filtered products are unique up to isomorphisms.

Note that the co-limits defining filtered products are directed. Therefore a sufficient condition for the existence of filtered products, which applies to many institutions, is the existence of small products and of directed co-limits of models. Note however that this is not a necessary condition because only co-limits over diagrams of projections are involved. We will see examples when directed co-limits of models do not exist in general but some filtered products exist.

If F is an ultrafilter then filtered products modulo F are called *ultraproducts*. When $A_i = A$ for all $i \in I$, then a filtered product is called *filtered power*. Filtered powers corresponding to ultrafilters are called *ultrapowers*. Note that a (direct) product $\prod_{i\in I} A_i$ is the same as the filtered product $\prod_{\{I\}} A_i$.

Filter reductions. Let F be a filter over I and $I' \subseteq I$. The *reduction of* F *to* I' is denoted by $F|_{I'}$ and defined as $\{I' \cap X \mid X \in F\}$.

Fact 6.2. *The reduction of any filter is still a filter.*

A class $\mathcal{F}$ of filters is *closed under reductions* if and only if $F|_J \in \mathcal{F}$ for each $F \in \mathcal{F}$ and $J \in F$. Examples of classes of filters closed under reductions include the class of all filters, the class of all ultrafilters, the class of $\{\{I\} \mid I \text{ set}\}$, etc.

The following is a useful property of filter reductions which will be used in several situations.

Proposition 6.3. *Let F be a filter over I and $\{A_i\}_{i\in I}$ a family of objects in a category $\mathbb{C}$. For each $J \in F$, the filtered products $\prod_{F|_J} A_i$ and $\prod_F A_i$ are isomorphic.*

Proof. Note that $(F|_J, \supseteq) \subseteq (F, \supseteq)$ is a final functor since for each $J' \in F$ we have that $J' \cap J \in F|_J$. The conclusion follows directly from Thm. 2.4. $\square$

Exercises

6.1. Filtered products in PL
In **PL**, for any filter F over a set I, and for each family $\{M_i\}_{i\in I}$ of models, its filtered product modulo F is $\bigcup_{J\in F} \bigcap_{i\in J} M_i$.

6.2. Let Σ be the **FOL** signature having only one sort and only one binary relation symbol R. Let $\{M_i\}_{i\in I}$ be a family of models and F be a filter over I. Prove that $(\prod_F M_i)_R$ is reflexive, symmetric, or transitive when $(M_i)_R$ is reflexive, symmetric, respectively transitive for each $i \in I$.

6.3. The class of all ultrafilters is closed under reductions.

6.4. Borrowing filtered products along institution comorphisms
For any persistently liberal institution comorphism $I \to I'$ the institution I has the limits and the co-limits of models that I' has. This leads to the existence of filtered products of models in several institutions (such as **POA**, **PA**, **AUT**, **MBA**, **LA**, etc) via the examples of Ex. 4.73 and to the existence of direct products of models in **HNK** via the comorphism of Ex. 4.11.

6.5. [35] In general **HNK** does not have directed co-limits of models. However, it has co-limits of directed 'injective' diagrams, i.e., diagrams consisting of injective model homomorphisms.

6.6. **IPL** has direct products and directed co-limits of models.

6.7. **MFOL** has filtered products of Kripke models.

6.8. The categories of multialgebras (**MA**) do have direct products and directed co-limits.

6.9. [85] The categories of contraction algebras (**CA**) do have direct products and directed co-limits.

6.2 Fundamental Theorem

In the first part of this section we present some properties of signature morphisms which express rather abstractly various actual finiteness properties which underly the method of ultraproducts.

Preservation of filtered products (of models)

Consider a functor $G\colon\ \mathbb{C}' \to \mathbb{C}$ and F a filter over a set I. Then G *preserves the fil-*

tered product $\mu' : B_F \Rightarrow \prod_F B_i$ (for $\{B_i\}_{i \in I}$ a family of objects in $\mathbb{C}'$), if $\mu' G : B_F ; G \Rightarrow \prod_F G(B_i)$ is also a filtered product in $\mathbb{C}$ of $\{G(B_i)\}_{i \in I}$.

$$\begin{tikzcd}
\prod_{i\in J'} B_i \arrow[r, "p_{J',J}"] \arrow[dr, "\mu_{J'}"'] & \prod_{i\in J} B_i \arrow[d, "\mu_J"] \\
& \prod_F B_i
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_{i\in J'} G(B_i) \arrow[r, "p_{J',J}"] \arrow[dr, "G(\mu_{J'})"'] & \prod_{i\in J} G(B_i) \arrow[d, "G(\mu_J)"] \\
& \prod_F G(B_i)
\end{tikzcd}$$

For any class $\mathcal{F}$ of filters, we say a functor *preserves $\mathcal{F}$-filtered products* if it preserves all filtered products modulo F for each filter $F \in \mathcal{F}$.

In many institutions, when the role of G is played by model reduct functors, the preservation of filtered products is an immediate consequence of preservation of (direct) product and of directed co-limits. In the following we analyze these two conditions.

Preservation of direct products. Since right adjoint functors preserve all limits (see Prop. 2.6), one way to see that model reducts preserve direct products is to invoke liberality of the signature morphisms. According to Prop. 4.29 a sufficient set of conditions for this is the existence of signature pushouts, semi-exactness, existence of elementary diagrams, and existence of initial models of presentations. At a first glance the latter condition might seem quite strong, however it is not since we need only the sub-institution of those sentences which are involved by the elementary diagrams. As we know, in the case of the standard concepts of model homomorphisms, these are the atomic sentences of the institution. **FOL** is a typical case, since for the standard concept of model homomorphism we thus need that only (sets of) atoms have initial models, a property which is easy to establish (see Cor. 4.28).

Note that restricted concepts of model homomorphisms may break the argument above. Consider for example the injective **FOL**-model homomorphisms. Recall that the corresponding elementary diagrams consist of (equational and relational) atoms plus negations of equational atoms. Arbitrary sets of atoms and negations of equational atoms do not necessarily have an initial model, even when homomorphisms are all injective. Moreover, in this case even the existence of (direct) products of models is lost. This shows that categorical filtered products require appropriate model homomorphisms which guarantee good structural properties for the categories of models.

An important special case is when the signature morphism is representable. In this case we can have a general preservation result for direct products of models based on the following slightly more general result.

Proposition 6.4. *All model reduct functors corresponding to representable signature morphisms create limits of models.*

Proof. Let $\chi : \Sigma \to \Sigma'$ be a representable signature morphism. The proposition holds by the general categorical argument that the forgetful functor from a comma category to the base category creates all limits (Prop. 2.3) applied to the forgetful $M_\chi/\mathsf{Mod}(\Sigma) \to \mathsf{Mod}(\Sigma)$. $\square$

Preservation of directed co-limits. In many institutions this property usually holds in a stronger form: the model reduct functors lift, and sometimes even create, directed co-limits.

This property relies upon the finiteness of the arities of the symbols of the signatures, as shown by the following typical example.

Proposition 6.5. *In* **FOL** *all model reduct functors lift directed co-limits. Moreover, for the signature morphisms that are surjective on the sorts, the corresponding model functors create directed co-limits.*

Proof. Let us first consider a simpler case, when the model reduct functor is the forgetful functor from models to their underlying set carriers. Let $(h_{i,j})_{(i\leq j)\in(J,\leq)}$ be a directed diagram of (S,F,P)-models, and let $((\mu_i)_s)_{s\in S})_{i\in J}$ be the co-limit of the corresponding diagram of underlying sets.

$$\begin{tikzcd} M_i \arrow[rr, "h_{i,j}"] \arrow[dr, "\mu_i"'] & & M_j \arrow[dl, "\mu_j"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} (M_i)_s \arrow[rr, "(h_{i,j})_s"] \arrow[dr, "(\mu_i)_s"'] & & (M_j)_s \arrow[dl, "(\mu_j)_s"] \\ & M_s & \end{tikzcd}$$

Then there exists a *unique* way one can interpret the operations F and relations P on the sets $\{M_s\}_{s\in S}$ such that μ_i become (S,F,P)-model homomorphisms:

- For each $\sigma \in F_{w\to s}$ and each tuple of elements $(m_1,\ldots,m_k)\in M_w$, define

  $$M_\sigma(m_1,\ldots,m_k) = \mu_j((M_j)_\sigma(m_1^j,\ldots,m_k^j))$$

  where j and $m_1^j \ldots m_k^j$ are such that $m_1 = \mu_j(m_1^j),\ldots,m_k = \mu_j(m_k^j)$. This is possible because $(\mu_i)_{i\in J}$ is already the co-limit of the underlying set carriers (hence each m_i can be written as $\mu_{j_i}(m_i^{j_i})$ for some $j_i \in J$) and because the length k of the arity w is finite (hence by the directedness of $(J,\leq)$ we can find a common j for all m_i, $1\leq i\leq k$). Also, the correctness of the definition is guaranteed by the directedness of $(J,\leq)$ and because $h_{i,j}$ are F-homomorphisms. It is easy to check that this definition of M_σ makes all μ_i's F-homomorphisms.
- For each $\pi \in P$,

  $$M_\pi = \bigcup\{\mu_i((M_i)_\pi) \mid i \in J\}$$

  This makes all μ_i P-homomorphisms. Also M_π is the smallest with this property, which guarantees the co-limit property of the co-cone $(\mu_i)_{i\in J}$ in the category of P-homomorphisms.

Now it is easy to check that $(\mu_i)_{i\in J}$ is also a co-limit in the category F-homomorphisms, hence μ is a co-limit in $\mathsf{Mod}^{\mathbf{FOL}}(S,F,P)$.

In the second part of the proof, we consider *any* **FOL**-signature morphism φ : $(S,F,P)\to(S',F',P')$. Let $(h'_{i,j})_{(i\leq j)\in(J,\leq)}$ be a directed diagram of (S',F',P')-models, and let $(h_{i,j})_{(i\leq j)\in(J,\leq)}$ be its φ-reduct. Let μ be a co-limit of $(h_{i,j})_{(i\leq j)\in(J,\leq)}$. By the first part of the proof, we know that the underlying (many-sorted) function of μ forms a co-limit for the diagram consisting of the underlying functions of $(h_{i,j})_{(i\leq j)\in(J,\leq)}$.

$$\begin{tikzcd} M'_i \arrow[rr, "h'_{i,j}"] \arrow[dr, "\mu'_i"'] & & M'_j \arrow[dl, "\mu'_j"] \\ & M' & \end{tikzcd} \qquad \begin{tikzcd} M_i \arrow[rr, "h_{i,j}"] \arrow[dr, "\mu_i"'] & & M_j \arrow[dl, "\mu_j"] \\ & M & \end{tikzcd}$$

There exists a co-cone μ' for the diagram consisting of the underlying functions of $(h'_{i,j})_{(i\leq j)\in(J\leq)}$ such that $(\mu'_i)_{\varphi(s)} = (\mu_i)_s$ for each $i\in J$ and $s\in S$. If $\varphi^{\mathrm{st}}:\ S\to S'$ is surjective then μ' is unique and it is also a co-limit. If φ^{st} is not surjective, then μ' may not be unique but can be fixed to be a co-limit. For the final step we may just apply the first part of the proof to μ' and obtain it as a co-limit of $(h'_{i,j})_{(i\leq j)\in(J\leq)}$. The fact that $M'_{\varphi(x)} = M_x$, for x operation or relation symbol in (S,F,P), can be established easily from the definitions of $M'_{\varphi(x)}$ as follows from the first part of the proof.

□

In most situations the preservation of directed co-limits of models is required only for the model reducts corresponding to a special class of signature morphisms, which are often quasi-representable. A general preservation result applicable to these situations can be obtained from the following:

Proposition 6.6. *All model reduct functors corresponding to quasi-representable signature morphisms create directed co-limits of models.*

Proof. Let $\chi:\ \Sigma\to\Sigma'$ be a quasi-representable signature morphism, let $(f'_{i,j})_{(i<j)\in(I,\leq)}$ be a directed diagram of Σ'-models, and let $(f_{i,j})_{(i<j)\in(I,\leq)}$ be its χ-reduct. Consider a co-limit μ of $(f_{i,j})_{(i<j)\in(I,\leq)}$.

$$\begin{tikzcd} A'_i \arrow[rr, "f'_{i,j}"] \arrow[dr, "\mu'_i"'] & & A'_j \arrow[dl, "\mu'_j"] \\ & B_i = B_j & \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[rr, "f_{i,j}"] \arrow[dr, "\mu_i"'] & & A_j \arrow[dl, "\mu_j"] \\ & A & \end{tikzcd}$$

Because χ is quasi-representable, for each $i\in I$, there exists a unique χ-expansion $\mu'_i:\ A'_i\to B_i$ of μ_i. By the uniqueness property of quasi-representability and because the diagram is directed we can show that $B_i = B_j$ for all $i,j\in I$, and that $\mu'_i = f'_{i,j};\mu'_j$ for all $(i\leq j)\in(I,\leq)$. By a similar argument we can further show that μ' is a co-limit of $(f'_{i,j})_{(i<j)\in(I,\leq)}$ □

The following is a consequence of Prop. 6.6 and we will use it later.

Corollary 6.7. *The model reducts corresponding to finitary quasi-representable signature morphisms preserve the finitely presented models, i.e., if χ is finitary quasi-representable, then $M'\upharpoonright_\chi$ is finitely presented whenever M' is finitely presented.*

Proof. Let us consider a quasi-representable signature morphism $\chi: \Sigma \to \Sigma'$, a finitely presented Σ'-model M', and a Σ-model homomorphism $h: M'\upharpoonright_\chi \to A$ to the vertex of a co-limit $(A_i \xrightarrow{\mu_i} A)_{i\in J}$ of a directed diagram $(A_i \xrightarrow{f_{i,j}} A_j)_{(i<j)\in(J,\leq)}$.

We prove that there exists $j \in J$ and $g_j: M'\upharpoonright_\chi \to A_j$ such that $g_j;\mu_j = h$. By the quasi-representability of χ let $h': M' \to A'$ be the unique χ-expansion of h. Because χ is finitary there exists $i \in J$ and $\mu'_i: A'_i \to A'$ a χ-expansion of μ_i. A'_i and the quasi-representability of χ determines a directed diagram of Σ'-models $(A'_j \xrightarrow{f'_{j,k}} A'_k)_{(i\leq j<k)\in(J,\leq)}$ and a co-cone $(\mu'_j)_{i\leq j\in J}$ such that $f'_{j,k}\upharpoonright_\chi = f_{j,k}$ and $\mu'_j\upharpoonright_\chi = \mu_j$. The χ-reduct of this diagram is a final sub-diagram of $(f_{j,k})_{(j<k)\in(J,\leq)}$, and because the model reduct corresponding to χ creates directed co-limits (cf. Prop. 6.6) we obtain that $(\mu'_j)_{i\leq j\in J}$ is the co-limit of $(f'_{j,k})_{(i\leq j<k)\in(J,\leq)}$. Because M' is finitely presented there exists $i \leq j$ and $g'_j: M' \to A'_j$ such that $g'_j;\mu'_j = h'$. Let $g_j = g'_j\upharpoonright_\chi$. Then $g_j;\mu_j = (g'_j;\mu'_j)\upharpoonright_\chi = h'\upharpoonright_\chi = h$.

Now we prove that for any $g_i: M'\upharpoonright_\chi \to A_i$ and $g_k: M'\upharpoonright_\chi \to A_k$ such that $g_i;\mu_i = h = g_j;\mu_j$ there exists $i,k < j$ such $g_i;f_{i,j} = g_k;f_{k,j}$. By the quasi-representability property for χ let

- $g'_i: M' \to A'_i$ and $g'_k: M' \to A'_k$ be the χ-expansions of g_i and g_k, respectively, and
- $\mu'_i: A'_i \to A'$ and $\mu'_k: A'_k \to A'$ be the χ-expansions of μ_i and μ_k, respectively.

Note that by the uniqueness of the χ-expansions of $g_i;\mu_i = h = g_j;\mu_j$ as a Σ'-homomorphism from M' we have that μ'_i and μ'_k have the same codomain A'. By the uniqueness part of the fact that χ is finitary there exists $i,k < l \in J$ and χ-expansions $f'_{i,l}$, $f'_{k,l}$ and μ'_l of $f_{i,l}$, $f_{k,l}$ and μ_l, respectively, such that $\mu'_i = f'_{i,l};\mu'_l$ and $\mu'_k = f'_{k,l};\mu'_l$. Thus

$$g'_i;f'_{i,l};\mu'_l = g'_k;f'_{k,l};\mu'_l = h'.$$

By the quasi-representability of χ we have that $f'_{i,l}$ and $f'_{k,l}$ have the same codomain; let us denote it by A'_l. Then again by the quasi-representability of χ and by the directedness of $(J,\leq)$, A'_l determines a unique χ-expansion $(f'_{j1,j2})_{(l\leq j1<j2)\in(J,\leq)}$ of the final sub-diagram $(f_{j1,j2})_{(l\leq j1<j2)\in(J,\leq)}$ together with a unique χ-expansion $(\mu'_j)_{(l\leq j)\in(J,\leq)}$ of the co-limiting co-cone $(\mu_j)_{(l\leq j)\in(J,\leq)}$. By the uniqueness part of the fact that M' is finitely presented there exists $l \leq j$ such that $g'_i;f'_{i,l};f'_{l,j} = g'_k;f'_{k,l};f'_{l,j}$. From this by reduction by χ we obtain $g_i;f_{i,j} = g_k;f_{k,j}$. □

Lifting of filtered products (of models)

Let $\mathcal{F}$ be a class of filters closed under reductions. A functor $G: \mathbb{C}' \to \mathbb{C}$ *lifts $\mathcal{F}$-filtered products* when for each $F \in \mathcal{F}$, each filtered product $\mu: A_F \Rightarrow \prod_F A_i$ (for $\{A_i\}_{i\in I}$ a family of objects in $\mathbb{C}$), and for each object B in $\mathbb{C}'$ such that $G(B) = \prod_F A_i$,

– there exists $J \in F$ and $\{B_i\}_{i\in J}$ a family of objects in $\mathbb{C}'$ such that $G(B_i) = A_i$ for each $i \in J$ and such that

– there exists a filtered product $\mu' : B_{F|_J} \Rightarrow B$ such that $G(\mu'_{J'}) = \mu_{J'}$ for each $J' \in F|_J$.

When $J = I$ we say that G *lifts completely* the respective filtered product. Note that in this case the closure of $\mathcal{F}$ under reductions is redundant.

In essence, this lifting property means that each $\mathcal{F}$-filtered product construction of $G(B)$ can be established as the image by G of an $\mathcal{F}$-filtered product construction of B by means of a filter reduction.

Unlike preservation of filtered products, in the actual institutions the lifting of filtered products is more delicate, holding for a narrower class of signature morphisms. However, in general we can establish the following classes of signature morphisms for which their model reduct functors lift filtered products.

Proposition 6.8. *In any institution the model reduct functor corresponding to a signature morphism* χ

1. *lifts $\mathcal{F}$-filtered products if it is finitary representable and $\mathcal{F}$ is closed under reductions, or*
2. *lifts completely $\mathcal{F}$-filtered products if it is* projectively representable*, i.e., it is representable such that its representation M_χ is projective, and all projections of model products are epis.*[1]

Proof. Consider a filter $F \in \mathcal{F}$ over a set I and a χ-expansion A' of a filtered product $\prod_F A_i$ of Σ-models. Let μ be the co-limiting co-cone of the filtered product.

1. Because χ is finitary, there exists $J \in F$ and $\mu'_J : A'_J \to A'$ such that $\mu'_J\restriction_\chi = \mu_J$.

By the quasi-representability of χ, for each $i \in J$ let $p'_i : A'_J \to A'_i$ be the unique χ-expansion of $p_i : \prod_{j\in J} A_j \to A_i$. Then, because χ is representable, the $\{p'_i\}_{i\in J}$ form a product cone.

By using the fact that χ is representable, A'_J and $\{A'_i\}_{i\in J}$ determine unique χ-expansions $(p'_{J',J''})_{(J''\subseteq J')\in F|_J}$ of $(p_{J',J''})_{(J''\subseteq J')\in F|_J}$ and $(\mu'_{J'})_{J'\in F|_J}$ of $(\mu_{J'})_{J'\in F|_J}$. Because representable signature morphisms create directed co-limits (cf. Prop. 6.6), and because $(p_{J',J''})_{(J''\subseteq J')\in F|_J}$ is a final sub-diagram of $(p_{J',J''})_{(J''\subseteq J')\in F}$, by Thm. 2.4 we have thus obtained a lifting of the original filtered product.

2. First we show that $\mu_I : \prod_{i\in I} A_i \to \prod_F A_i$ is epi. Let $f, g : \prod_F A_i \to \bullet$ such that $\mu_I; f = \mu_I; g$. Because the projections $p_{I,J'} : \prod_{i\in I} A_i \to \prod_{i\in J'} A_i$ are epis, it follows that for each $J' \in F$, we have that $\mu_{J'}; f = \mu_{J'}; g$. Because $(\mu_{J'})_{J'\in F}$ is a co-limiting co-cone, it is an epimorphic family, therefore $f = g$.

Now, by using the fact that μ_I is epi, because M_χ is projective, there exists $b : M_\chi \to \prod_{i\in I} A_i$ such that $b; \mu_I = i_\chi(A')$. For each $i \in I$, let $A'_i = i_\chi^{-1}(b; p_{I,i})$, where $p_{I,i}$ is the projection $\prod_{i\in I} A_i \to A_i$. A routine check shows that $\{A'_i\}_{i\in I}$ determines a complete lifting of the original filtered product. $\square$

Although in most situations we will use Prop. 6.8 for lifting of filtered products, the representability of the signature morphism is not always a necessary condition for this. A remarkable counterexample which will be seen below (Cor. 11.15) is given by the lifting of filtered products of Kripke models in modal institutions.

[1] For any model product $\prod_{i\in I} M_i$, the canonical projections $\prod_{i\in I} M_i \to M_i$ are epis.

The Fundamental Theorem

Sentences preserved by filtered factors/products. The following notions of preservation by filtered factors and by filtered products are dual to each other.

For a signature Σ in an institution, a Σ-sentence e is

- *preserved by $\mathcal{F}$-filtered factors* if $\prod_F A_i \models_\Sigma e$ implies $\{i \in I \mid A_i \models_\Sigma e\} \in F$, and
- *preserved by $\mathcal{F}$-filtered products* if $\{i \in I \mid A_i \models_\Sigma e\} \in F$ implies $\prod_F A_i \models_\Sigma e$

for each filter $F \in \mathcal{F}$ over a set I and for each family $\{A_i\}_{i\in I}$ of Σ-models.

When $\mathcal{F}$ is the class of all ultrafilters, preservation by $\mathcal{F}$-filtered factors, respectively products, is called *preservation by ultrafactors*, respectively *ultraproducts*.

Theorem 6.9 (Fundamental ultraproducts theorem). *In any institution:*

1. *The basic sentences are preserved by all filtered products.*
2. *The finitary basic sentences are preserved by all filtered products and all filtered factors.*

For any class $\mathcal{F}$ of filters closed under reductions:

3. *The sentences preserved by $\mathcal{F}$-filtered products are closed under existential χ-quantification, when χ preserves $\mathcal{F}$-filtered products.*
4. *The sentences preserved by $\mathcal{F}$-filtered factors are closed under existential χ-quantification, when χ lifts $\mathcal{F}$-filtered products.*
5. *The sentences preserved by $\mathcal{F}$-filtered factors and the sentences preserved by $\mathcal{F}$-filtered products are both closed under conjunction.*
6. *The sentences preserved by $\mathcal{F}$-filtered products are closed under infinite conjunctions.*
7. *If a sentence is preserved by $\mathcal{F}$-filtered factors then its negation is preserved by $\mathcal{F}$-filtered products.*

And finally, if we further assume that $\mathcal{F}$ contains only ultrafilters:

8. *If a sentence is preserved by $\mathcal{F}$-filtered products then its negation is preserved by $\mathcal{F}$-filtered factors.*
9. *The sentences preserved by both $\mathcal{F}$-filtered products and factors are closed under negation.*

Proof. 1. Let F be any filter over I and let $\{A_i\}_{i\in I}$ be a family of Σ-models for a signature Σ.

Let e be a basic sentence and consider $J = \{i \in I \mid A_i \models_\Sigma e\}$. There exists a model homomorphism $M_e \to A_i$ for each $i \in J$, therefore by the universal property of the products, there exists a model homomorphism $M_e \to \prod_{i\in J} A_i$. By composing it with $\mu_J : \prod_{i\in J} A_i \to \prod_F A_i$, we get a model homomorphism $M_e \to \prod_F A_i$, which implies that $\prod_F A_i \models e$.

2. Consider a finitary basic Σ-sentence e. By 1 we have to prove only that e is preserved by filtered factors. If $\prod_F A_i \models e$, then there exists a model homomorphism $M_e \to \prod_F A_i$. Since M_e is finitely presented, there exists a model homomorphism $M_e \to \prod_{i\in J} A_i$ for some non-empty $J \in F$, which, by the product projections, means that $A_i \models e$ for all $i \in J$. Therefore $\{i \in I \mid A_i \models_\Sigma e\} \in F$ because $J \subseteq \{i \in I \mid A_i \models_\Sigma e\}$.

3. Let $\chi : \ \Sigma \to \Sigma'$ be a signature morphism which preserves $\mathcal{F}$-filtered products. Let e' be a Σ'-sentence preserved by $\mathcal{F}$-filtered products, and let e be an existential χ-quantification of e'. Consider a filter $F \in \mathcal{F}$ over a set I, and let $\{A_i\}_{i\in I}$ be a family of Σ-models such that $J = \{i \in I \mid A_i \models_\Sigma e\} \in F$. We have to prove that $\prod_F A_i \models_\Sigma e$.

For each $i \in J$ let B_i be a Σ'-model such that $B_i\upharpoonright_\chi = A_i$ and $B_i \models_{\Sigma'} e'$. Because $F|_J \in \mathcal{F}$ and because e' is preserved by $\mathcal{F}$-filtered products we have that $\prod_{F|_J} B_i \models_{\Sigma'} e'$. Because χ preserves $\mathcal{F}$-filtered products, we have that $(\prod_{F|_J} B_i)\upharpoonright_\chi = \prod_{F|_J} A_i$, which implies that $\prod_{F|_J} A_i \models_\Sigma e$. By Prop. 6.3, we have that $\prod_{F|_J} A_i \cong \prod_F A_i$, which shows that $\prod_F A_i \models e$.

4. Let $\chi : \ \Sigma \to \Sigma'$ be a signature morphism which lifts $\mathcal{F}$-filtered products. Let e' be a Σ'-sentence preserved by $\mathcal{F}$-filtered factors, and let e be an existential χ-quantification of e'. Consider a filter $F \in \mathcal{F}$ over a set I, and let $\{A_i\}_{i\in I}$ be a family of Σ-models such that $\prod_F A_i \models_\Sigma e$. We have to prove that $\{i \in I \mid A_i \models_\Sigma e\} \in F$.

Let B be a χ-expansion of $\prod_F A_i$ such that $B \models_{\Sigma'} e'$. Because χ lifts $\mathcal{F}$-filtered products, there exists $J \in F$ such that for each $i \in J$ there exists a Σ'-model B_i such that $B_i\upharpoonright_\chi = A_i$ and such that $\prod_{F|_J} B_i = B$. Because e' is preserved by $\mathcal{F}$-filtered factors and $\mathcal{F}$ is closed under reductions, $J' = \{i \in J \mid B_i \models_{\Sigma'} e'\} \in F|_J \subseteq F$. But $J' \subseteq \{i \in I \mid A_i \models_\Sigma e\}$, therefore $\{i \in I \mid A_i \models_\Sigma e\} \in F$ because F is a filter.

5. This follows from:

- $\{i \in I \mid A_i \models e' \wedge e''\} = \{i \in I \mid A_i \models e'\} \cap \{i \in I \mid A_i \models e''\}$,
- $J' \cap J'' \in F$ if and only if $J', J'' \in F$, and
- $\prod_F A_i \models e' \wedge e''$ if and only if $\prod_F A_i \models e'$ and $\prod_F A_i \models e''$

for any $F \in \mathcal{F}$ filter over a set I, and $\{A_i\}_{i\in I}$ any family of Σ-models.

6. Given a signature Σ, for each family $\{e_l\}_{l\in L}$ of Σ-sentences preserved by $\mathcal{F}$-filtered products, assume that $\{i \in I \mid A_i \models e_l \text{ for each } l \in L\} \in F$, where $F \in \mathcal{F}$ is any filter over a set I and $\{A_i\}_{i\in I}$ is any family of Σ-models. Then for each $l \in L$, $\{i \in I \mid A_i \models e_l\} \supseteq \{i \in I \mid A_i \models e_l \text{ for each } l \in L\} \in F$, thus $\{i \in I \mid A_i \models e_l\} \in F$, therefore $\prod_F A_i \models e_l$ for each $l \in L$.

7. Let e be the negation of a Σ-sentence e' for a signature Σ such that e' is preserved by $\mathcal{F}$-filtered factors. Let F be any filter in $\mathcal{F}$ over a set I and let $\{A_i\}_{i\in I}$ be a family of models such that $J = \{j \in I \mid A_j \models e\} \in F$.

We have to prove that $\prod_F A_i \models e$. If we assume the contrary, it means that $\prod_F A_i \models e'$. Since e' is preserved by $\mathcal{F}$-filtered factors, $J' = \{j \in I \mid A_j \models e'\} \in F$. Because F is a proper filter $J \cap J' \in F$ is not empty, hence we can find j such that $A_j \models e$ and $A_j \models e'$, which is impossible.

8. Let e be the negation of e' such that e' is preserved by $\mathcal{F}$-filtered products. Let F be any ultrafilter in $\mathcal{F}$ over a set I and let $\{A_i\}_{i\in I}$ be a family of models such that $\prod_F A_i \models e$. If $\{j \in I \mid A_j \models e\} \notin F$ then its complement $\{j \in I \mid A_j \models e'\}$ belongs to F

(because F is a ultrafilter). Because e' is preserved by $\mathcal{F}$-filtered products, this would imply $\prod_F A_i \models e'$ which contradicts $\prod_F A_i \models e$. This means $\{j \in I \mid A_j \models e\} \in F$.

9. From 7 and 8. □

6.3 Łoś Institutions

Łoś sentences. A sentence is a *Łoś-sentence* when it is preserved by all ultrafactors and all ultraproducts.

Corollary 6.10. *In any institution I, the sentences accessible from the finitary basic sentences by Boolean connectives and χ-quantification for which χ preserves and lifts ultraproducts, are Łoś-sentences.*

Proof. For any class $\mathcal{D}$ of signature morphisms which preserve and lift ultraproducts and is stable under pushouts, let $I(\mathcal{D})$ be the extension of I by closing the finitary basic sentences of I to Boolean connectives and universal and existential $\mathcal{D}$-quantification. By taking $\mathcal{F}$ to be the class of all ultrafilters in Thm. 6.9, we have that any sentence in $I(\mathcal{D})$ is a Łoś-sentence. By taking $\mathcal{D}$ to be the class of signature morphisms involved in the semantic quantifications supported by I and which preserve and lift ultraproducts, we may note that each sentence of I is semantically equivalent to a sentence of $I(\mathcal{D})$. □

Cor. 6.10 can be further specialized by using Prop. 6.6, 6.4 and 6.8:

Corollary 6.11. *In any institution, any sentence which is accessible from the finitary basic sentences by*

- *Boolean connectives,*
- *finitary representable quantification, and*
- *projectively representable quantification (assuming that the institution has epi model projections)*

is a Łoś-sentence.

Łoś-institutions. An institution is a *Łoś-institution* if and only if it has all ultraproducts of models and all its sentences are Łoś-sentences. Note that the condition on the existence of ultraproducts requires the existence of direct products but not necessarily of other filtered products.

Since each **FOL**-sentence is accessible by finitary representable quantification and Boolean connectives from equations and relational atoms, which are finitary basic, we have the following instance of Cor. 6.11.

Corollary 6.12. **FOL** *is a Łoś-institution.*

Note that for the above corollary, instead of finitary representable quantification, we could have alternatively used the argument of projectively representable quantifications.

Filtered power embedding

Let F be filter over a set I. In any institution with F-filtered products of models, for each model M there is a canonical model homomorphism d_M^F from M to its filtered power $\prod_F M$ defined by

$$d_M^F = (M \xrightarrow{\delta_M^I} \prod_{i\in I} M \xrightarrow{\mu_I} \prod_F M)$$

where $\delta_M^I : M \to \prod_{i\in I} M$ is the diagonal ($\delta_M^I ; p_{I,i} = 1_M$ for each $i \in I$) and $\mu^I : \prod_{i\in I} M \to \prod_F M$ is the 'top' component of the filtered product co-cone $\mu : M_F \Rightarrow \prod_F M$.

Proposition 6.13. *In any institution with elementary diagrams ι, for any filter F over an arbitrary set I such that*

1. *the institution has F-filtered products of models which are preserved by the elementary extensions, and*
2. *all sentences are preserved by F-filtered products,*

the canonical homomorphism $d_M^F : M \to \prod_F M$ is elementary for any model M.

Proof. Let M_M be the initial model of the elementary diagram of a Σ-model M. We have to prove that $i_{\Sigma,M}^{-1}(d_M^F) \models M_M^*$.

Because the sentences of the elementary diagram E_M are preserved by all F-filtered products, $\prod_F M_M \models E_M$. Because $\mathsf{Mod}(\iota_\Sigma(M))$ preserves F-filtered products, we have that $d_{M_M}^F\restriction_{\iota_\Sigma(M)} = d_M^F$. Thus

$$d_M^F = i_{\Sigma,M}(d_{M_M}^F) : \; i_{\Sigma,M}(M_M) \to i_{\Sigma,M}(\prod_F M_M).$$

Because $i_{\Sigma,M}(M_M) = 1_M$, we have that $i_{\Sigma,M}(\prod_F M_M) = d_M^F$. Now because all sentences are preserved by F-filtered products, $\prod_F M_M \models M_M^*$, therefore d_M^F is elementary. $\square$

The condition of preservation of filtered products by the elementary extensions in the above proposition is fulfilled trivially in all institutions for which all signature morphisms preserve (direct) products and directed co-limits of models. We have seen in Sect. 6.2 that this is a quite common situation. Alternatively we may use the general argument that representable signature morphisms preserve all filtered products (cf. Prop. 6.4 and 6.6) and note that elementary extensions are usually representable.

By Cor. 6.12 we obtain the following instance of Prop. 6.13.

Corollary 6.14. *Any* **FOL***-model can be elementarily embedded in any of its ultrapowers.*

Exercises

6.10. Several concrete institutions (such as **HCL**, **POA**, **PA**, **AUT**, **MBA**, **LA**, **IPL**, etc.) can be established as Łoś institutions by virtue of Cor. 6.11.

6.11. Ultraproducts in HNK
HNK has ultraproducts of models. (*Hint:* Consider comorphism **HNK** $\to$ **FOEQL**$^{\text{pres}}$ of Ex. 4.11 and use the fact that (cf. Ex. 6.4) **HNK** has direct products of models and that (cf. Cor. 6.12) **FOL** is a Łoś institution.) Note that **HNK** is an example of an institution that has direct products and ultraproducts but does not necessarily have all filtered products.

6.12. Borrowing the Łoś property
For any persistently liberal institution comorphism $(\Phi, \alpha, \beta): I \to I'$, for any I-sentence ρ, if $\alpha(\rho)$ is a Łoś sentence then ρ is a Łoś sentence too.

This can be applied to the examples of comorphisms of Ex. 4.73 for obtaining the Łoś property for several concrete institutions (such as **POA**, **PA**, **AUT**, **MBA**, **IPL**, **LA**, etc.). By this borrowing method **HNK** can also be established as a Łoś institution. (*Hint:* Use the comorphism of Ex. 4.11 and refer also to the result of Ex. 6.11.)

6.13. Horn sentences are preserved by filtered products
Let $(\forall\chi)E \Rightarrow E'$ be a finitary universal Horn sentence for a signature Σ in an arbitrary institution that has filtered products of models. For each family of Σ-models $\{M_i\}_{i\in I}$ and filter F over I, show that the filtered product $\prod_F M_i$ satisfies $(\forall\chi)E \Rightarrow E'$ when M_i satisfies $(\forall\chi)E \Rightarrow E'$ for each $i \in I$.

Apply the above for showing that in **HCL** any model can be elementarily embedded in any of its filtered powers.

6.14. Σ_1^1-sentences
In any institution, e is a Σ_1^1-*sentence* if it is an existential χ-quantification of a Łoś sentence, where χ is any filtered product preserving signature morphism. For example any second order existential quantification of a **FOL** sentence is a Σ_1^1-sentence. In any institution each Σ_1^1-sentence is preserved by ultraproducts.

6.4 Compactness

If for each set of sentences E and each sentence e, $E \models e$ implies the existence of a finite subset $E_f \subseteq E$ such that $E_f \models e$, then we say that the institution is *compact*.

A set of sentences E for a signature Σ is *consistent* if E^* is not empty. An institution is *model compact* or *m-compact* for short, if each set of sentences is consistent when all its finite subsets are consistent.

Compactness versus model compactness. The significance of consistency and of the distinction between compactness and m-compactness depends on the actual institution. For example, consistency has real significance in **FOL**, while in **EQL** or **HCL** it is a trivial property since each set of sentences is consistent. Therefore in some institutions compactness and m-compactness are not necessarily the same concept. For example, any institution in which each set of sentences is consistent is trivially m-compact, but it is not necessarily compact. Below is a simple (counter)example.

Proposition 6.15. **HCL**$_\infty$ *(infinitary Horn clause logic) is model compact but it is not compact.*

Proof. That $\mathbf{HCL}_\infty$ is m-compact follows from the fact that each presentation in $\mathbf{HCL}_\infty$ is consistent since it has an initial model. This can be established in the same way as for **HCL** since in Prop. 4.26 we have not used the fact that the Horn sentences are finitary.

Now let us show that $\mathbf{HCL}_\infty$ is not compact. For this we consider a signature without sorts, consisting only of an infinite set P of relation symbols of empty arity. Note that models of this signature consist of subsets of P. Obviously each element of P is an atom. Let us pick a $\pi \in P$ and consider $E = P \setminus \{\pi\}$. Then $E \cup \{\wedge E \Rightarrow \pi\} \models \pi$. However for any finite part $\Gamma \subseteq E \cup \{\wedge E \Rightarrow \pi\}$ we can find a model M for Γ which is not a model for π. This model M is the subset of P defined by $M = \Gamma \setminus \{\wedge E \Rightarrow \pi\}$. Then $M \models \Gamma$ but obviously $M \not\models \pi$. $\square$

The following establishes a general relationship between compactness and m-compactness.

Proposition 6.16.

– *Each compact institution having* false[2] *is m-compact.*

– *Each m-compact institution having negations is compact.*

Proof. Let E be any set of Σ-sentences. If the institution is compact and has false, then E is consistent when all its finite subsets are consistent, otherwise $E \models \mathsf{false}$ which implies the existence of a finite subset $E \supseteq E_f \models \mathsf{false}$ which gives an inconsistent finite subset of E.

Conversely, in any institution that is m-compact and has negations, if $E \models e$ and each finite subset $E \supseteq E_f \not\models e$, then there exists a model $M_f \models E_f$ such that $M_f \models \neg e$, which implies that $E \cup \{\neg e\}$ is consistent, which is a contradiction. This means that there exists a finite subset $E_f \models e$, therefore the institution is compact. $\square$

Compactness by ultraproducts

Theorem 6.17. *In any institution, let E be a set of sentences preserved by ultraproducts. Let I be the set of all finite subsets of E. Consider a model A_i for each finite subset $i \in I$. Then there exists an ultraproduct $\prod_U A_i$ such that $\prod_U A_i \models E$.*

Proof. A set $S \subseteq \mathcal{P}(I)$ *has the finite intersection property* if $J_1 \cap J_2 \cap \cdots \cap J_n \neq \emptyset$ for all $J_1, J_2, \ldots, J_n \in S$. We will use the following classical Ultrafilter Lemma (its proof can be found for example in [32]).

Lemma 6.18. *If $S \subseteq \mathcal{P}(I)$ has the finite intersection property, then there exists an ultrafilter U over I such that $S \subseteq U$.*

Let $S = \{\{i \in I \mid \rho \in i\} \mid \rho \in E\}$. S has the finite intersection property because

$$\{\rho_1, \rho_2, \ldots, \rho_n\} \in \{i \in I \mid \rho_1 \in i\} \cap \{i \in I \mid \rho_2 \in i\} \cap \cdots \cap \{i \in I \mid \rho_n \in i\}.$$

By the Ultrafilter Lemma 6.18, let U be an ultrafilter such that $S \subseteq U$.

[2] For each signature Σ there exists a Σ-sentence false_Σ which is not satisfied by any Σ-model.

For each $\rho \in E$, we have that $\{i \in I \mid \rho \in i\} \subseteq \{i \in I \mid A_i \models \rho\}$. This means that $\{i \in I \mid A_i \models \rho\} \in U$. Because ρ is preserved by ultraproducts, it implies that $\prod_U A_i \models \rho$. Because $\rho \in E$ is arbitrary, it follows that $\prod_U A_i \models E$. $\square$

Corollary 6.19. *Any institution in which each sentence is preserved by ultraproducts is m-compact.*

Corollary 6.20. *Let E be a set of sentences preserved by ultraproducts, and let e be a sentence preserved by ultrafactors such that $E \models e$. Then there exists a finite subset $E' \subseteq E$ such that $E' \models e$.*

Proof. Let us assume the contrary, i.e., that for each finite $i \subseteq E$, $i \not\models e$. This means that there exist models A_i such that $A_i \models i$ but $A_i \not\models e$.

Let I be the set of all finite subsets of E. By Thm. 6.17, there exists an ultraproduct such that $\prod_U A_i \models E$. Therefore $\prod_U A_i \models e$. Because e is preserved by ultrafactors, $\{i \in I \mid A_i \models e\} \in U$. But $\{i \in I \mid A_i \models e\} = \emptyset$ which is a contradiction since as an ultrafilter U is a proper filter. $\square$

Corollary 6.21. *Any Łoś-institution is (m-)compact.*

Cor. 6.21 constitutes a great source of examples of compact and m-compact institutions. The following well-known example is obtained via Cor. 6.12.

Corollary 6.22. **FOL** *is (m-)compact.*

Exercises

6.15. [113] **Logical compactness versus topological compactness**
Recall that a topology (X, τ) is *compact* when for each family $\{U_i\}_{i \in I}$ such that $U_i \in \tau$ for each $i \in I$ and such that $\bigcup_{i \in I} U_i = X$, there exists a finite subset $J \in I$ such that $\bigcup_{i \in J} U_i = X$.

An institution with negation is (m-)compact if all its semantic topologies (see Ex. 4.5) are compact. Moreover, if the institution has finite conjunctions too, then it is (m-)compact if and only if all its semantic topologies are compact.

6.16. Maximally consistent sets
(a) We say that a set of sentences E for a signature Σ in an arbitrary institution is *maximally consistent* if and only if for any other consistent set E', $E \subseteq E'$ implies $E = E'$. In any institution with negation, a set E of sentences (for a given signature) is maximally consistent only if for each sentence e exactly one of e and $\neg e$ belong to E.
(b) By (a), in any institution with negation, for any signature morphism $\varphi \colon \Sigma \to \Sigma'$ and each maximally consistent set of Σ'-sentences E', $\varphi^{-1}(E')$ is maximally consistent.

6.17. Lindenbaum Theorem
We say that an institution has the *Lindenbaum Property* if and only if each consistent set of sentences can be extended to a maximal consistent set of sentences. Each m-compact institution has the Lindenbaum Property. (*Hint:* Let β be the cardinal of $\mathsf{Sen}(\Sigma)$ and arrange $\mathsf{Sen}(\Sigma) = \{e_\alpha\}_{\alpha < \beta}$. Define $E_0 = E$, and for each successor ordinal $\alpha + 1$ define $E_{\alpha+1} = E_\alpha \cup \{e_\alpha\}$ if $E_\alpha \cup \{e_\alpha\}$ is consistent, otherwise $E_{\alpha+1} = E_\alpha$, and for each limit ordinal α' define $E_{\alpha'} = \bigcup_{\alpha < \alpha'} E_\alpha$. Then $E_\beta = \bigcup_{\alpha < \beta} E_\alpha$ is the desired maximally consistent set.)

6.18. In any institution show that if $E \models e$ where E is a set of Σ_1^1-sentences and e is preserved by ultrafactors, then there exists a finite subset $E' \subseteq E$ such that $E' \models e$.

6.19. Finitely presented theories (Ex. 4.22 continued)
In any compact institution with conservative signature morphisms, a theory (Σ, E) is finitely presented if Σ is a finitely presented signature and (Σ, E) can be presented by a finite set of sentences.

6.5 Finitely Sized Models

The method of ultraproducts can be used to prove that elementary equivalence and the finiteness of the size of the models is a sufficient condition for two models to be isomorphic. This is the main topic of this section.

Although saturated models (introduced in Chap. 7 below) provides a more general framework for such types of results in which the finiteness condition on the size of the models can be relaxed to a much softer condition, for finitely sized models this isomorphism result can be achieved by using ultrapower embeddings within the much simpler minded framework of this section.

Dense signature morphisms. A signature morphism $\chi : \Sigma \to \Sigma'$ is *dense* if and only if for any couple of parallel Σ-model homomorphisms $f, g : M \to N$ if for each χ-expansion M' of M there exists χ-expansions f', respectively g', of f, respectively g, such that $f', g' : M' \to N'$ then $f = g$. In concrete terms the density of a signature morphism means that it affects all sorts of the source signature. The following is a typical example.

Proposition 6.23. *In* **FOL** *any signature extension $\chi : \Sigma \to \Sigma'$ with constants such that χ adds at least one new element for each sort of Σ, is dense.*

Proof. Let us use the following important remark.

Fact 6.24. *If χ is representable (represented by M_χ) then it is dense if and only if $\mathsf{Mod}^{\mathbf{FOL}}(\Sigma)(M_\chi, N)$ is epimorphic for each Σ-model N.*

Let us write $\Sigma' = \Sigma \uplus X$ where X is a (new) set of constants for Σ. Then $M_\chi = T_\Sigma(X)$ where $T_\Sigma(X)$ is the term Σ-model over X. Because X contains at least one element for each sort, the set of the Σ-homomorphisms $T_\Sigma(X) \to N$ is epimorphic for each Σ-model N since for each element $n \in N$ there exists at least one function $f : X \to N$ such that n belongs to the image of f. $\square$

Finitely sized models. Finitely sized models capture the situation when the carrier sets of a models have only a finite set of elements. This is stronger than being finitely presented and weaker than being finite in the sense that the signature is also finite. It is rather easy to show that elementary equivalence implies isomorphism for the latter case (see Ex. 5.9).

In any institution with elementary diagrams ι, a Σ-model M *is finitely sized* if and only if

- the elementary extension $\iota_\Sigma(M)$ is finitary, and
- it has a finite number of $\iota_\Sigma(M)$-expansions.

Fact 6.25. *For any* **FOL** *signature, a model is finitely sized if and only if it has a finite number of elements.*

Note that under our basic assumption of non-emptiness of the sorts (i.e., that each sort has at least one constant), having a finite number of elements implies also that the number of the sorts of the signature is also finite. However no other finiteness restriction is implied, such as on the number of operation or relation symbols.

Proposition 6.26. *Let us consider an institution with elementary diagrams* ι *such that all elementary extensions are dense and quasi-representable. For any finitely sized model* N *and any ultrafilter* U *over a set* I*, the canonical diagonal homomorphism*

$$d_N^U : \ N \xrightarrow{\delta_N^I} \prod_{i\in I} N \xrightarrow{\mu_I} \prod_U N$$

is epi.

Proof. Consider two model homomorphisms $f, g : \ \prod_U N \to A$ such that $d_N^U ; f = d_N^U ; g$. We have to prove $f = g$. Because $\iota_\Sigma(N)$ is dense and quasi-representable, it is enough to prove that for each $\iota_\Sigma(N)$-expansion N'_U of $\prod_U N$, both f and g expand to homomorphisms $N'_U \to A'$.

Because the $\iota_\Sigma(N)$ is finitary, there exists $J \in U$ and $\mu'_J : \ N'_J \to N'_U$ a χ-expansion of μ_J.

$$\begin{tikzcd}
N'_J \arrow[dr, "\mu'_J"'] & & \prod_{i\in J} N \arrow[rr, "p_{J,J'}"] \arrow[dr, "\mu_J"'] & & \prod_{i\in J'} N' \arrow[dl, "\mu_{J'}"] \\
& N'_U & & \prod_U N &
\end{tikzcd}$$

Because χ is quasi-representable, for each $K \subseteq J$ in U let us denote

- by $p'_{J,K}$ the unique $\iota_\Sigma(N)$-expansion of the projection $p_{J,K}$ to a Σ_N-homomorphism $N'_J \to N'_K$, and
- by μ'_K the unique $\iota_\Sigma(N)$-expansion of μ_k from N'_K. By the uniqueness property of the quasi-representability we have that $\mu'_K : \ N'_K \to N'_U$.

Let us assume that there exists a $\iota_\Sigma(N)$-expansion N' of N, $J \supseteq K \in U$, and $(\delta_N^K)'$ a $\iota_\Sigma(N)$-expansion of the diagonal $\delta_N^K : \ N \to \prod_{i\in K} N$.

$$\begin{tikzcd}
N' \arrow[r, "(\delta_N^K)'"] \arrow[dr, "(\delta_N^J)'"'] & N'_K \arrow[r, "\mu'_K"] & N'_U \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & A' \\
& N'_J \arrow[u, "p'_{J,K}"] \arrow[ur, "\mu'_J"'] & &
\end{tikzcd}$$

Then because $\delta_N^K ; \mu_K = \delta_N^I ; \mu_I = d_N^U$, by the uniqueness property of the quasi-representability we have that $(\delta_N^J)' ; \mu'_K$ is the unique $\iota_\Sigma(N)$-expansion of d_N^U to a Σ_N-homomorphism from N'. Let $f' : \ N'_U \to A'_f$, respectively $g' : \ N'_U \to A'_g$, be the unique $\iota_\Sigma(N)$-expansions of

f, respectively g, to Σ_N-homomorphisms from N'_U. Then both $(\delta_N^J)';\mu'_K;f'$ and $(\delta_N^J)';\mu'_K;g'$ are $\iota_\Sigma(N)$-expansions from N' of $d_N^U;f = d_N^U;g$. By the uniqueness property of the quasi-representability we have that $A'_f = A'_g$, which proves the proposition.

Now let us show that there exists a $\iota_\Sigma(N)$-expansion N' of N, $J \supseteq K \in U$, and $(\delta_N^K)'$ a $\iota_\Sigma(N)$-expansion of $\delta_N^K : N \to \prod_{i\in K} N$. For each $i \in J$ let $p'_{J,i} : N'_J \to N'_i$ be the unique $\iota_\Sigma(N)$-expansion of the projection $p_{J,i} : \prod_{i\in J} N \to N$ to a Σ_N-homomorphism from N'_J. For each $\iota_\Sigma(N)$-expansion N' of N let $J(N') = \{i \in J \mid N'_i = N'\}$. Assume we showed that there exists N' such that $J(N') \in U$ and let us define $K = J(N')$ and $(\delta_N^K)' : N' \to N''_K$ be the unique $\iota_\Sigma(N)$-expansion of (δ_N^K) to a Σ_N-homomorphism from N'. Then both $p'_{J,K}$ and $p'_{J,i};(\delta_N^K)'$ are $\iota_\Sigma(N)$-expansions of the same Σ-homomorphism $p_{J,K} = p_{J,i};\delta_N^K$, therefore they are equal. This means $N''_K = N'_K$ and $(\delta_N^K)' : N' \to N'_K$.

If there exists no expansion N' such that $J(N') \in U$, then because U is an ultrafilter let $I \setminus J(N') \in U$ for each $\iota_\Sigma(N)$-expansion N' of N. Because the $\iota_\Sigma(N)$-expansions of N form a finite set, we have that $\bigcap_{N'}(I \setminus J(N')) \in U$. Hence $I \setminus \bigcup_{N'} J(N') \in U$. But $\bigcup_{N'} J(N') = J$ because for each $i \in J$ we have that $i \in J(N'_i)$, therefore $I \setminus J \in U$ which because U as an ultrafilter is a proper filter, contradicts the fact that $J \in U$. $\square$

Note that the conditions of the elementary extensions of Prop. 6.26 above are fulfilled immediately in all institutions where the elementary extensions add the elements of the model as new constants to the signature. We know that this is a rather typical situation. The assumption of non-empty sorts plays again a role here ensuring the density of the elementary extensions.

Corollary 6.27. *Consider an institution with elementary diagrams ι such that*

1. *the elementary extensions are dense and quasi-representable,*
2. *it has ultraproducts of models which are preserved by the elementary extensions,*
3. *each sentence is preserved by ultraproducts of models, and*
4. *each elementary homomorphism which is an epi is an isomorphism.*

Then any finitely sized model is isomorphic to any of its ultrapowers.

Moreover, if in addition the institution

5. *has finite conjunctions, and*
6. *has existential $\mathcal{D}$-quantification such that*
7. *each finitary elementary extension belongs to $\mathcal{D}$,*

then two elementary equivalent finitely sized models are related by an elementary homomorphism.

Proof. The first part follows immediately from Prop. 6.13 and 6.26.

For the second part, let $M \equiv N$ be finitely sized models. For each finite $E \subseteq M_M^*$, because $M_M \models E$, we have that $M \models (\exists \iota_\Sigma(M)) \wedge E$. Because $M \equiv N$, we have that $N \models (\exists \iota_\Sigma(M)) \wedge E$. Therefore there exists N_E a $\iota_\Sigma(M)$-expansion of N such that $N_E \models E$.

By Thm. 6.17, there exists an ultrafilter U on the set of all finite subsets E of M^*_M such that $\prod_U N_E \models M^*_M$. This shows there exists an elementary homomorphism $M \to (\prod_U N_E)\restriction_{\iota_\Sigma(M)}$. But $(\prod_U N_E)\restriction_{\iota_\Sigma(M)} = \prod_U N$ by the preservation of ultraproducts by the model reducts corresponding to elementary extensions. By the first part of the proposition we also have that $N \cong \prod_U N$, therefore there exists an elementary homomorphism $M \to N$. $\square$

A typical concrete application of Cor. 6.27 is the following.

Corollary 6.28. *Any* **FOL** *finitely sized model is isomorphic to any of its ultrapowers. Any two elementary equivalent finitely sized* **FOL***-models are isomorphic.*

Proof. Only two points may need a bit of additional explanation. The first one concerns the fact that in **FOL**, the elementary embedding homomorphisms which are also epis, are in fact isomorphisms. In **FOL** the elementary embeddings are injective and closed, and the injective closed epis are isomorphisms.

The second point concerns how to derive the isomorphism between two elementary equivalent finitely sized models from the final conclusion of Cor. 6.27. This follows easily by cardinality reasons because **FOL** elementary embeddings are injective. $\square$

Exercises

6.20. The compositions between dense and conservative signature morphisms are dense.

6.21. Derive results similar to Cor. 6.28 of examples of institutions presented in this book other than **FOL**.

Notes. The filtered product construction from conventional model theory (see Chap. 4 of [32]) has been introduced in [115] and probably defined categorically for the first time in [120]. Categorical filtered products have been intensively used in categorical logic and model theory works such as [6] or [116, 117]. The equivalence between the category theoretic and the set theoretic definitions of the filtered products appears in [84]. Filtered products are sometimes known under the name of *reduced products*, such as in [32].

The fundamental ultraproducts theorem is the foundation for the method of ultraproducts in conventional model theory [32] and has been stated for the first time in [115]. [14] is an exposition of that part of conventional model theory that can reached using only ultraproducts. A rather different abstract model theoretic approach to the fundamental ultraproducts theorem based on satisfaction by injectivity is given in [5]. Our approach originates from [47] and contrasts [5] by making essential use of concepts central to institution theory, such as signature morphisms and model reducts. This multi-signature framework, very characteristic to institution theory, leads to higher generality and simpler proofs. Although in the literature there are quite established concepts of lifting of co-limits [1], there seems to be no standard notion corresponding to our concept of lifting of filtered products.

The institution-independent results on compactness by ultraproducts essentially constitute a generalization of similar ultraproduct-based compactness results from conventional model theory [32]. Finitely sized models have been introduced in an institution-independent setting and their isomorphism criterion has been developed in [146].

Chapter 7

Saturated Models

A lot of deep results in model theory can be reached by the method of saturated models. Two of the most useful properties of saturated models are their existence and their uniqueness. The existence means that each model can be elementarily extended to a saturated model, while uniqueness holds when the model is 'sufficiently' small. The main topic of this chapter is the investigation of frameworks supporting these two properties, and of some important applications.

The existence property of saturated models requires that directed co-limits of diagrams of elementary homomorphisms are still elementary. This is treated in the first section of this chapter. This important preservation property of elementary homomorphisms, which is due to Tarski in the conventional concrete setting of **FOL**, will be used for several results in other chapters too.

An important class of applications can be developed in conjunction with the method of ultraproducts. In the last section we show that for certain ultrafilters, the corresponding ultraproducts of models are always saturated. Assuming the Generalized Continuum Hypothesis, this leads to one of the most beautiful applications of saturated models to first order model theory, the Keisler-Shelah isomorphism theorem saying that "two models are elementary equivalent if and only if they have isomorphic ultrapowers". Apart from its theoretical significance it has several applications, such as to axiomatizability and to interpolation.

7.1 Elementary Co-limits

For this section, we assume an arbitrary institution with a designated (sub-)category $\mathcal{D}$ of quasi-representable signature morphisms such that $\mathcal{D}$ is stable under pushouts.

Sentences preserved/reflected by directed co-limits of models

The preservation and the reflection of sentences by directed co-limits of models have the flavor of dual properties.

We say that a Σ-sentence ρ is *$\mathcal{D}$-preserved, respectively reflected, by directed co-limits of models* if for each directed diagram of $\mathcal{D}$-elementary Σ-model homomorphisms $(f_{i,j})_{(i<j)\in(I,\leq)}$ with co-limit $(\mu_i)_{i\in I}$,

$$\begin{tikzcd} A_i \arrow[rr, "f_{i,j}"] \arrow[dr, "\mu_i"'] & & A_j \arrow[dl, "\mu_j"] \\ & A & \end{tikzcd}$$

for each $i \in |I|$, $A_i \models \rho$ implies $A \models \rho$, respectively $A \models \rho$ implies $A_i \models \rho$.

Theorem 7.1. *The set of sentences $\mathcal{D}$-preserved by directed co-limits:*

1. *contains all basic sentences,*
2. *is closed under (possibly infinite) conjunctions and disjunctions,*
3. *is closed under existential $\mathcal{D}$-quantifications, and*
4. *is closed under finitary universal $\mathcal{D}$-quantifications.*

Proof. 1. Consider ρ a basic Σ-sentence. If $A_i \models \rho$ then there exists a homomorphism $M_\rho \to A_i$, which implies there exists a homomorphism $M_\rho \to A_i \xrightarrow{\mu_i} A$. Hence $A \models \rho$.

2. Consider ρ_1 and ρ_2 sentences $\mathcal{D}$-preserved by directed co-limits. If $A_i \models \rho_1 \wedge \rho_2$ then $A_i \models \rho_1$ and $A_i \models \rho_2$. Hence $A \models \rho_1$ and $A \models \rho_2$, which just means $A \models \rho_1 \wedge \rho_2$. The proof for disjunctions is similar.

3. Consider $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ and a Σ'-sentence ρ' which is $\mathcal{D}$-preserved by directed co-limits, and assume that $A_i \models (\exists\chi)\rho'$ for some fixed i. Then there exists a χ-expansion A'_i of A_i such that $A'_i \models \rho'$.

Because χ is quasi-representable and $(I,\leq)$ is directed, we can χ-expand the sub-diagram $(f_{j,k})_{(i\leq j<k)\in(I,\leq)}$ of the original diagram $(f_{j,k})_{(j<k)\in(I,\leq)}$ to $(A'_j \xrightarrow{f'_{j,k}} A'_k)_{(i\leq j<k)\in(I,\leq)}$. Because quasi-representable signature morphisms create directed co-limits (Prop. 6.6) and $(f_{j,k})_{(i\leq j<k)\in(I,\leq)}$ is a final sub-diagram of $(f_{j,k})_{(j<k)\in(I,\leq)}$, we have that the unique χ-expansion $(A'_j \xrightarrow{\mu'_j} A')_{(i\leq j)\in(I,\leq)}$ of $(\mu_j)_{(i\leq j)\in(I,\leq)}$ is a co-limiting co-cone for $(f'_{j,k})_{(i\leq j<k)\in(I,\leq)}$. Because each $f_{j,k}$ is elementary we have that $A'_j \models \rho'$ for each $j \geq i$. By the induction hypothesis, $A' \models \rho'$. Therefore $A \models (\exists\chi)\rho'$.

4. Consider $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ finitary and a Σ'-sentence ρ' such that ρ' is $\mathcal{D}$-preserved by directed co-limits and assume that $A_i \models (\forall\chi)\rho'$.

Consider A' any χ-expansion of A. Because χ is finitary, we can find $i < j$ and a χ-expansion $A'_j \xrightarrow{\mu'_j} A'$ of μ_j. Because $f_{i,j}$ is $\mathcal{D}$-elementary, it follows that $A_j \models$

$(\forall\chi)\rho'$, hence $A'_j \models \rho'$. By an argument similar to 3, by χ-expanding $(f_{j,k})_{(i\leq j<k)\in(I,\leq)}$ and $(\mu_j)_{(i\leq j)\in(I,\leq)}$, by the induction hypothesis we obtain that $A' \models \rho'$. $\square$

The following extends Thm. 7.1 to institutions with negations:

Theorem 7.2. *If the institution has negations, the set of sentences $\mathcal{D}$-[preserved and reflected] by directed co-limits*

1. *contains all finitary basic sentences,*
2. *is closed under (possibly infinite) conjunctions and negations, and*
3. *is closed under finitary $\mathcal{D}$-quantifications.*

Proof. 1. Consider ρ a finitary basic Σ-sentence. That $A_i \models \rho$ implies $A \models \rho$ follows from 1 of Thm. 7.1. Therefore let us suppose $A \models \rho$ and let us show that $A_i \models \rho$.

Because M_ρ is finitely presented and $A \models \rho$ (which means there exists homomorphism $M_\rho \to A$), there exists $i < j \in I$ and $h : M_\rho \to A_j$. This means $A_j \models \rho$. Because $f_{i,j} : A_i \to A_j$ is $\mathcal{D}$-elementary and because the institution has negations, it follows that $A_i \models \rho$.

2. Similarly to 2 of Thm. 7.1.

3. Because of negations, existential and universal quantifications are inter-definable. Therefore let us focus on the existential quantification. Let $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ be a finitary signature morphism and ρ' be a Σ'-sentence which is $\mathcal{D}$-[preserved and reflected] by directed co-limits.

That $A_i \models (\exists\chi)\rho'$ implies $A \models (\exists\chi)\rho'$ follows from 3 of Thm. 7.1. Therefore let us suppose $A \models (\exists\chi)\rho'$ and let us show that $A_i \models (\exists\chi)\rho'$. We have that $A \models \neg(\forall\chi)\neg\rho'$, which means $A \not\models (\forall\chi)\neg\rho'$. By 2 we have that $\neg\rho'$ is $\mathcal{D}$-[preserved and reflected] by directed co-limits. By 4 of Thm. 7.1, we have that $(\forall\chi)\neg\rho'$ is $\mathcal{D}$-preserved by directed co-limits, hence $A_i \models (\forall\chi)\neg\rho'$ would imply $A \models (\forall\chi)\neg\rho'$ which contradicts $A \not\models (\forall\chi)\neg\rho'$. Therefore $A_i \not\models (\forall\chi)\neg\rho'$, which means $A_i \models \neg(\forall\chi)\neg\rho' \models\!\!\!\!\!\dashv (\exists\chi)\rho'$. $\square$

Elementary co-limit theorem

Proposition 7.3. *Assume that all sentences of the institution are $\mathcal{D}$-preserved by directed co-limits of $\mathcal{D}$-elementary homomorphisms. Then, for each signature Σ, any co-limit of a directed diagram of $\mathcal{D}$-elementary Σ-homomorphisms is $\mathcal{D}$-elementary.*

Proof. Let $(f_{i,j})_{(i\leq j)\in(I,\leq)}$ be a directed diagram of Σ-homomorphisms such that each $f_{i,j}$ is $\mathcal{D}$-elementary and let $(\mu_i)_{i\in I}$ be its co-limit.

$$\begin{tikzcd} A_i \arrow[rr, "f_{i,j}"] \arrow[dr, "\mu_i"'] & & A_j \arrow[dl, "\mu_j"] \\ & A & \end{tikzcd}$$

For each $k \in I$, in order to prove that μ_k is $\mathcal{D}$-elementary, let $\mu'_k : A'_k \to A'$ be a χ-expansion of μ_k (for some $\chi \in \mathcal{D}$) and let ρ' be a Σ'-sentence such that $A'_k \models \rho'$. We have to show that $A' \models \rho'$.

As in Theorem 7.1 we can χ-expand the final sub-diagram $(f_{i,j})_{(k \leq i < j) \in (I,\leq)}$ and its co-limit $(\mu_i)_{(k \leq i) \in (I,\leq)}$ to $(f'_{i,j})_{(k \leq i < j) \in (I,\leq)}$ with co-limit $(\mu'_i)_{(k \leq i) \in (I,\leq)}$. Because ρ' is $\mathcal{D}$-preserved by directed co-limits we obtain that $A' \models \rho'$. □

Prop. 7.3 helps to express Theorems 7.1 and 7.2 in the following form.

Corollary 7.4. *Assume the institution satisfies one of the following:*

1. *each sentence is accessible from the finitary basic ones by (possibly infinite) conjunctions, disjunctions, universal $\mathcal{D}$-quantifications, and finitary existential $\mathcal{D}$-quantifications, or*
2. *the institution has negations and each sentence is accessible from the basic ones by (possibly infinite) conjunctions, negations, and finitary $\mathcal{D}$-quantifications.*

Then any co-limit of a directed diagram of $\mathcal{D}$-elementary homomorphisms is $\mathcal{D}$-elementary.

When in addition the institution has $\mathcal{D}$-normal elementary diagrams such that all elementary extensions are in $\mathcal{D}$, by virtue of Cor. 5.36, in the above Cor. 7.4 we may replace '$\mathcal{D}$-elementary' just by 'elementary'.

A typical concrete instance of Cor. 7.4 is obtained by taking $\mathcal{D}$ to the class of all **FOL**-signature extensions with constants.

Corollary 7.5. *In* **FOL***,* **EQLN***,* **FOL**$^+$*, and* **EQL***, the class of elementary homomorphisms is closed under directed co-limits.*

Cor. 7.5 shows that the closure of elementary homomorphisms under directed co-limits holds when the institution either has all negations (such as **FOL**, **EQLN**), or no negation at all (such as **FOL**$^+$, **EQL**), and it may fail on intermediate cases (such as **HCL**).

7.2 Existence of Saturated Models

In this section we introduce the concept of a saturated model and develop the fundamental existence theorem for saturated models. We start by a brief survey of some basic set theoretic notions required by the concept of saturated models.

Some set theory

For a gentle introduction to (axiomatic) set theory we recommend [178].

Ordinals. We skip the formal lists of axioms for set theories such as Zermelo, Zermelo-Fraenkel, Bernays, or Bernays-Morse which can be found in the rather rich set theory literature, and just recall that from the point of view of formal set theory:

- $0 = \emptyset$,
- $n+1 = n \cup \{n\}$ for each natural number n,
- $\omega = \{0, 1, 2, \ldots\}$ the set of all natural numbers,
- $\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, \omega\}$, etc.

All these are examples of ordinals. Formally, an *ordinal* is a set X such that X and each member of X is $\in$*-transitive*, i.e., every member of x is a subset of x. Although this definition might seem quite artificial, it has become fairly standard in the literature. The underlying intuition is that an ordinal is a special kind of ordering (by the membership relation $\in$), its definition as a certain kind of set being just a trick.

One of the important properties of ordinals is that they are well ordered, i.e., totally ordered and any non-empty class of ordinals has a least element. Ordinals support the following *Principle of Transfinite (or Ordinal) Induction:*

$$[(\forall\alpha)((\forall\beta < \alpha) \Rightarrow P(\beta)) \Rightarrow P(\alpha)] \Rightarrow (\forall\alpha)P(\alpha)$$

for each property P on ordinals.

For each ordinal α, let $\alpha + 1 = \alpha \cup \{\alpha\}$ be its *successor ordinal*. If α is neither a successor ordinal nor 0, we say that α is a *limit ordinal*.

Cardinals. Cardinal numbers are essentially equivalence classes, or representatives of equivalence classes, of sets under the bijection relation. For each set X, let $card(X)$ denote its *cardinality*. An ordering between cardinals can be defined by $card(X) \leq card(Y)$ if and only if there exists an injective function $X \to Y$.

When we take the point of view of cardinals as representatives of equivalence classes, we can formally define *cardinals* the smallest ordinals α which are in bijective correspondence to $card(\alpha)$. For example, ω is a cardinal while $\omega + 1$ is not. In fact infinite cardinals are always limit ordinals.

Basic arithmetic operations can be defined by

- $\alpha + \beta = card(\alpha \uplus \beta)$,
- $\alpha \times \beta = card(\alpha \times \beta)$, and
- $\alpha^\beta = card\{f \text{ function } \mid f : \alpha \to \beta\}$.

For each ordinal α the least cardinal greater than α is denoted by α^+. The *Generalized Continuum Hypothesis* (abbreviated *GCH*) states that for every infinite cardinal α, $2^\alpha = \alpha^+$.

The following is a list of well-known cardinal arithmetic properties which will be used in this chapter. More on cardinal arithmetic can be found in [100].

Proposition 7.6 (Cardinal arithmetic).

- *if* $\omega \leq \alpha$ *then* $\alpha \times \alpha = \alpha$,
- *if* $2 \leq \alpha \leq \beta$ *and* $\omega \leq \beta$ *then* $\alpha^\beta = 2^\beta$,
- *if* $\alpha \leq \beta$ *then* $\alpha^\beta \leq \beta^+$ *(requires GCH).*

Saturated models

Chains. In any category $\mathbb{C}$, for any ordinal λ, a λ*-chain* is a (commutative) diagram $\lambda \to \mathbb{C}$, written $(A_i \xrightarrow{f_{i,j}} A_j)_{i<j\leq\lambda}$, such that for each limit ordinal $\zeta \leq \lambda$, $(f_{i,\zeta})_{i<\zeta}$ is the co-limit of $(f_{i,j})_{i<j<\zeta}$. Note that the commutativity of the chain, which is implicit by functoriality, just means that $f_{i,j};f_{j,k} = f_{i,k}$ for all $i < j < k \leq \lambda$.

For any class of arrows $\mathcal{D} \subseteq \mathbb{C}$, a $(\lambda, \mathcal{D})$-*chain* is any λ-chain $(f_{i,j})_{i<j\leq\lambda}$ such that $f_{i,i+1} \in \mathcal{D}$ for each $i < \lambda$. We say that an *arrow* h *is a* $(\lambda, \mathcal{D})$-*chain* if there exists a $(\lambda, \mathcal{D})$-chain $(f_{i,j})_{i<j\leq\lambda}$ such that $h = f_{0,\lambda}$.

λ-small signature morphisms. A signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ is λ-*small* for a cardinal λ when for each λ-chain $(M_i \xrightarrow{f_{i,j}} M_j)_{i<j\leq\lambda}$ of Σ-homomorphisms and each φ-expansion M' of M_λ, there exists $i < \lambda$ and a φ-expansion $M'_i \xrightarrow{f'_{i,\lambda}} M'$ of $f_{i,\lambda}$.

Fact 7.7. *Finitary signature morphisms are λ-small for each infinite cardinal λ.*

$(\lambda, \mathcal{D})$-saturated models. For each signature morphism $\chi\colon\ \Sigma \to \Sigma'$, a Σ-model M χ-*realizes* a set E' of Σ'-sentences, if there exists a χ-expansion M' of M which satisfies E'. It χ-*realizes* E' finitely if it realizes every finite part of E'.

A Σ-model M is $(\lambda, \mathcal{D})$-*saturated* for λ a cardinal and $\mathcal{D}$ a class of signature morphisms when for each ordinal $\alpha < \lambda$ and each $(\alpha, \mathcal{D})$-chain $(\Sigma_i \xrightarrow{\varphi_{i,j}} \Sigma_j)_{i<j\leq\alpha}$ with $\Sigma_0 = \Sigma$, for each $(\chi\colon\ \Sigma_\alpha \to \Sigma') \in \mathcal{D}$, each $\varphi_{0,\alpha}$-expansion of M χ-realizes any set of sentences if and only if it χ-realizes it finitely.

An immediate example of saturated models is given by finitely sized models in **FOL**.

Proposition 7.8. *Let $\mathcal{D}$ be the class of* **FOL** *signature extensions with a finite number of constants, and let λ be an infinite cardinal.*

A. *Any* **FOL** *model which has a finite number of elements is $(\lambda, \mathcal{D})$-saturated.*

B. *For each $(\lambda, \mathcal{D})$-saturated* **FOL**-*model M and for each sort s, if M_s is infinite then $card(M_s) \geq \lambda$.*

Proof. A. Let $\varphi\colon\ \Sigma_0 \to \Sigma_\alpha$ be a $(\alpha, \mathcal{D})$-chain for $\alpha < \lambda$, and let M_α be a φ-expansion of a Σ_0-model M. Let $(\chi\colon\ \Sigma_\alpha \to \Sigma') \in \mathcal{D}$. Assume that M_α realizes finitely a set E' of Σ'-sentences and for each finite $i \subseteq E'$ let M^i be the χ-expansion of M_α such that $M^i \models_{\Sigma'} i$. By Thm. 6.17, there exists an ultrafilter U on the set $\mathcal{P}_\omega(E')$ of the finite subsets of E' such that $\prod_U M^i \models E'$.

But $(\prod_U M^i)\upharpoonright_\chi = \prod_U (M^i\upharpoonright_\chi) = \prod_U M_\alpha$. Because M_α is finitely sized, by Cor. 6.28, $\prod_U M_\alpha$ is isomorphic to M_α, hence M_α χ-realizes E'.

B. Let Σ be the signature of M. Let us assume that $card(M_s) < \lambda$ for some sort s for which M_s is infinite. Then we take the $(card(M_s), \mathcal{D})$-chain given by the signature extension with constants $\Sigma \hookrightarrow \Sigma \uplus M_s$, and let χ be the extension of $\Sigma \uplus M_s$ with one new constant x. Consider $E = \{x \neq m \mid m \in M_s\}$. Then the $(\Sigma \uplus M_s)$-expansion M' of M such that $M'_m = m$ for each $m \in M_s$, finitely realizes E but does not χ-realize E, which contradicts the fact that M is $(\lambda, \mathcal{D})$-saturated. $\square$

Existence theorem

Let us say that an institution *has $\mathcal{D}$-saturated models* if for any cardinal λ and for each Σ-model M there exists a Σ-homomorphism $M \to N$ such that $M \equiv N$ and N is $(\lambda, \mathcal{D})$-saturated.

Theorem 7.9 (Existence of saturated models). *Consider an institution and a class $\mathcal{D}$ of signature morphisms such that*

1. *$M \equiv N$ if there exists a model homomorphism $M \to N$,*
2. *it has finite conjunctions and existential $\mathcal{D}$-quantifications,*
3. *it has inductive co-limits of signatures and is inductive-exact,*
4. *for each signature Σ, the category of Σ-models has inductive co-limits,*
5. *for each signature morphism $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ and set E' of Σ'-sentences, if M χ-realizes E' finitely then there exists a model homomorphism $M \to N$ such that N χ-realizes E',*
6. *for each signature morphism $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ and each Σ-model M, the class of χ-expansions of M form a set, and*
7. *each signature morphism from $\mathcal{D}$ is quasi-representable, the category $\mathbb{S}ig$ of signatures is $\mathcal{D}$-co-well-powered, and for each ordinal λ there exists a cardinal α such that each morphism that is a $(\lambda, \mathcal{D})$-chain is α-small.*

Then the institution has $\mathcal{D}$-saturated models.

Proof. First we prove that there exists a Σ-homomorphism $h : M \to N$ such that for each $(\lambda, \mathcal{D})$-chain $\varphi : \Sigma \to \Sigma'$, each $(\chi : \Sigma' \to \Sigma'') \in \mathcal{D}$, each φ-expansion M' of M, and each set E'' of Σ''-sentences finitely realized by M', N' χ-realizes E'', where $h' : M' \to N'$ is the unique φ-expansion of h. (The existence of h' is guaranteed by the fact that φ is quasi-representable, which follows by ordinal induction from the condition that all signature morphisms in $\mathcal{D}$ are quasi-representable and that the institution is inductive-exact, by applying Prop. 5.12 [for inductive co-limits rather than the more general directed co-limits].)

$$\begin{array}{ccc}
\Sigma & & M \xrightarrow{h} N \\
\varphi\big\downarrow & & \\
\Sigma' & & M' \xrightarrow{h'} N' \\
\chi\big\downarrow & & \\
\Sigma'' & & M'' \xrightarrow{h''} N''
\end{array}$$

For fixed Σ and M, by (φ, M', χ, E'') let us denote tuples where $\varphi : \Sigma \to \Sigma'$ is a $(\lambda, \mathcal{D})$-chain, M' is a φ-expansion of M, $(\chi : \Sigma' \to \Sigma'') \in \mathcal{D}$, and E'' is a set of Σ''-sentences which is χ-realized finitely by M'. Two such tuples $(\varphi^1, M'^1, \chi^1, E^1)$ and

$(\varphi^2, M'^2, \chi^2, E^2)$ are *isomorphic* when there exists a natural isomorphism $\theta: \varphi^1;\chi^1 \Rightarrow \varphi^2;\chi^2$

$$\begin{array}{ccccccccc}
\Sigma = \Sigma^1_0 \ldots & \longrightarrow & \ldots \Sigma^1_i & \xrightarrow{\varphi^1_{i,j}} & \Sigma^1_j & \longrightarrow \cdots \longrightarrow & \Sigma^1_\lambda = \Sigma'^1 & \xrightarrow{\chi_1} & \Sigma''^1 \\
{\scriptstyle 1_\Sigma = \theta_0}\downarrow & & {\scriptstyle \theta_i}\downarrow & & \downarrow{\scriptstyle \theta_j} & & \downarrow{\scriptstyle \theta_\lambda = \theta'} & & \downarrow{\scriptstyle \theta''} \\
\Sigma = \Sigma^2_0 \ldots & \longrightarrow & \ldots \Sigma^2_i & \xrightarrow[\varphi^2_{i,j}]{} & \Sigma^2_j & \longrightarrow \cdots \longrightarrow & \Sigma^2_\lambda = \Sigma'^2 & \xrightarrow[\chi_2]{} & \Sigma''^2
\end{array}$$

such that $M'^2\restriction_{\theta'} = M'^1$ and $\theta''(E^1) = E^2$. By the conditions of the theorem ($\mathbb{S}ig$ being $\mathcal{D}$-co-well-powered and all χ-expansions of a model forming a set), the isomorphism classes of tuples (φ, M', χ, E'') form a set; let us denote it by $L(M)$. If k is the cardinal of $L(M)$, we may consider $\{(\varphi^i, M'^i, \chi^i, E^i) \mid i < k\}$ a complete system of independent representatives for $L(M)$.

Now, by ordinal induction we construct a chain of Σ-homomorphisms $(M_i \xrightarrow{h_{i,j}} M_j)_{i<j\leq k}$ as follows:

– $M_0 = M$,

– for each successor ordinal $j+1$ let $M'^j \xrightarrow{h'_{0,j}} M'_j$ be the unique φ^j-expansion of $M \xrightarrow{h_{0,j}} M_j$. Because M'^j χ^j-realizes E^j finitely, we have that M'_j χ^j-realizes E^j finitely too. By condition 5 there exists $M'_j \xrightarrow{f'} P'$ such that P' χ^j-realizes E^j. Then we define $M_{j+1} = P'\restriction_{\varphi^j}$ and $h_{j,j+1} = f'\restriction_{\varphi^j}$, and

– for each limit ordinal we take the co-limit of the chain before j.

Let $N = M_k$ and $h = h_{0,k}$. Keeping the above notation, consider (φ, M', χ, E''). If $j < k$ is the isomorphism class of (φ, M', χ, E''), we may assume without any loss of generality that $(\varphi, M', \chi, E'') = (\varphi^j, M'^j, \chi^j, E^j)$. We have to show that N' χ^j-realizes E^j.

This holds because we have that M'_{j+1} χ^j-realizes E^j (where M'_{j+1} is the unique φ^j-expansion of M_{j+1} determined by $M'^j\restriction_{\varphi_j} = M \xrightarrow{h_{0,j+1}} M_{j+1}$.). Let M''_{j+1} be a χ^j-expansion of M'_{j+1} such that $M''_{j+1} \models E^j$. Because $h''_{j+1,k}: M''_{j+1} \to N''$, the unique $(\varphi^j;\chi^j)$-expansion of $h_{j+1,k}$, preserves satisfaction $N'' \models E^j$, hence N' χ^j-realizes E^j.

In the second part of the proof we assume the conclusion of the first part and consider a cardinal α such that each $(\lambda, \mathcal{D})$-chain is α-small. By ordinal induction we construct an α-chain $(N_i \xrightarrow{f_{i,j}} N_j)_{i<j\leq\alpha}$ such that $N_0 = M$ and each $f_{j,j+1}$ has the property of h above. We want to show that N_α is $(\lambda, \mathcal{D})$-saturated, therefore the desired model homomorphism is $f_{0,\alpha}: M \to N_\alpha$.

Assume N'_α χ-realizes E'' finitely, where $(\varphi, N'_\alpha, \chi, E'') \in L(N_\alpha)$. We have to prove that N'_α χ-realizes E''. Because φ is α-small, there exists $j < \alpha$ and a φ-expansion

$f'_{j,\alpha} : N'_j \to N'_\alpha$ of $f_{j,\alpha}$. By quasi-representability this determines expansions $f'_{j,j+1}$: $N'_j \to N'_{j+1}$ and $f'_{j+1,\alpha} : N'_{j+1} \to N'_\alpha$. Notice that by conditions 1 and 2, N'_j χ-realizes E'' finitely because N'_α does. Recall that $f_{j,j+1}$ has the property of h from the first part of the proof, therefore we have that N'_{j+1} χ-realizes E''. Let N''_{j+1} be χ-expansion of N'_{j+1} such that $N''_{j+1} \models E''$. By quasi-representability we lift $f'_{j+1,\alpha}$ to $f''_{j+1,\alpha} : N''_{j+1} \to N''_\alpha$ and because model homomorphisms preserve satisfaction we get that $N''_\alpha \models E''$. Hence N'_α χ-realizes E''. □

In the following we discuss the applicability of the existence Thm. 7.9 by making an analysis of its underlying conditions, often illustrating using **FOL** as a typical example.

Model homomorphisms preserve elementary equivalence. Typically, one considers elementary homomorphisms as model homomorphisms of the considered institution. In the case of **FOL** these are the elementary embeddings. In general, in any institution with negation we have that $M \equiv N$ for each elementary homomorphism $h : M \to N$.

Inductive-exactness. The existence of inductive co-limits of signatures actually implies that we have to allow infinitely large signatures.

The inductive-exactness property of models is also a straightforward property in many institutions, a special case of exactness. It is however a bit more delicate in model homomorphisms when the considered model homomorphisms are elementary. Thus, in an institution that is inductive-exact on model homomorphisms, let us consider a chain $\varphi = (\varphi_{i,j})_{i<j\leq\lambda}$ of signature morphisms and a family of model homomorphisms $(h_i)_{i\leq\lambda}$ such that $h_j \restriction_{\varphi_{i,j}} = h_i$ for $i < j$. We have to establish that h_λ is elementary whenever h_i is elementary for each $i < \lambda$.

In actual situations this is in general very easy. Let us consider the case of **FOL**. In order to make this argument as simple as possible, we may consider only signature morphisms which are (arbitrarily large) extensions with constants. Then the elementarity of h_λ follows trivially because it is an expansion of any h_i along a representable signature morphism (cf. Prop. 5.32).

Therefore, in applications, in addition to the restriction on model homomorphisms to be elementary we have also to restrict the signature morphisms to the signature extensions with constants.

The category of models has inductive co-limits. Because we have already established that the model homomorphisms are the elementary ones, this condition is handled by one of Theorems 7.1 or 7.2 via Cor. 7.4. For the **FOL** example this means Cor. 7.5.

Compactness. The 5th condition can be regarded as a form of compactness. Let us see how it works for the special case of **FOL**. For each finite $i \subseteq E$, let A_i be the χ-expansion of A such that $A_i \models i$. By compactness Thm. 6.17, there exists an ultrafilter on $\mathcal{P}_\omega(E)$ (the set of all finite subsets of E) such that $\prod_U A_i \models E$. Then $\prod_U A_i \restriction_\chi = \prod_U A$. By Cor. 6.14, A can be elementarily embedded into $\prod_U A$.

The same argument can be invoked when the role of **FOL** is played by any other Łoś institution such that signature morphisms preserve products and directed co-limits.

Expansions of a model form a set. This is evidently fulfilled in any institution where models consist of interpretations of the symbols of the signatures in set theoretic universes, for those signature morphisms which do not add new sorts. Note that signature extensions with constants meet the latter condition.

Conditions on $\mathcal{D}$. Since at the concrete level we have established that we work only with signature extensions with constants as signature morphisms, *all* signature morphisms are quasi-representable.

Let $\mathcal{D}$ be the class of signature extensions with a *finite* number of constants. For each signature Σ, there exists only a set of isomorphism classes of finitary extensions of Σ with constants, hence $\mathcal{D}$ is co-well-powered.

Finally, for each ordinal λ, each $(\lambda,\mathcal{D})$-chain is λ^+-small. This is a special case of the following general result:

Proposition 7.10. *In any inductive-exact institution, if each signature morphism of $\mathcal{D}$ is finitary and quasi-representable, then for each infinite ordinal λ, each $(\lambda,\mathcal{D})$-chain of signature morphisms is λ^+-small.*

Proof. Consider a $(\lambda,\mathcal{D})$-chain of signature morphisms $\varphi\colon\ \Sigma \to \Sigma'$ and consider a λ^+-chain of Σ-model homomorphisms $(M_i \xrightarrow{h_{i,j}} M_j)_{i<j\leq\lambda^+}$. Let $M^\lambda_{\lambda^+}$ be a φ-expansion of M_{λ^+}.

For each $0 \leq i < j \leq \lambda$, let $\varphi_{i,j}\colon\ \Sigma_i \to \Sigma_j$ be the segment in the chain φ determined by i and j.

By transfinite induction on $\alpha \leq \lambda$ we define an increasing sequence of ordinals strictly bounded by λ^+, $\{i_\alpha\}_{\alpha\leq\lambda}$ and an inductive diagram $(M^\alpha_j \xrightarrow{h^\alpha_{j,k}} M^\alpha_k)_{i_\alpha\leq j<k\leq\lambda^+}$ in $\mathsf{Mod}(\Sigma_\alpha)$ such that $h^\alpha_{j,k}\restriction_{\varphi_{\beta,\alpha}} = h^\beta_{j,k}$ for all $0 \leq \beta < \alpha \leq \lambda$ and $i_\beta \leq j < k$ as follows:

- $i_0 = 0$ and $h^0_{j,k} = h_{j,k}$ for all $j < k \leq \lambda^+$.
- Assume $\alpha = \beta + 1$ is a successor ordinal. We first notice that $(h^\beta_{i,\lambda^+})_{i_\beta\leq i<\lambda^+}$ is a co-limit of $(h^\beta_{j,k})_{i_\beta\leq j<k<\lambda^+}$. This is so because $h^\beta_{j,k}\restriction_{\varphi_{0,\beta}} = h_{j,k}$, $(h_{i,\lambda^+})_{i_\beta\leq i<\lambda^+}$ is a co-limit of $(h_{j,k})_{i_\beta\leq j<k<\lambda^+}$ (since $(h_{j,k})_{i_\beta\leq j<k<\lambda^+}$ is a final sub-diagram of $(h_{j,k})_{0\leq j<k<\lambda^+}$; see Thm. 2.4), $\varphi_{0,\beta}$ is quasi-representable (by an argument similar to the argument that φ is quasi-representable used by the proof of Thm. 7.9, and because quasi-representable signature morphisms create directed co-limits (cf. Prop. 6.6)). But $M^\alpha_{\lambda^+} = M^\lambda_{\lambda^+}\restriction_{\varphi_{\alpha,\lambda}}$ is a $\varphi_{\beta,\alpha}$-expansion of $M^\beta_{\lambda^+}$. Because $\alpha = \beta + 1$, $\varphi_{\beta,\alpha} \in \mathcal{D}$, hence it is finitary. Therefore there exists $i_\beta \leq i_\alpha < \lambda^+$ and a $\varphi_{\beta,\alpha}$-expansion $h^\alpha_{i_\alpha,\lambda^+}\colon\ M^\alpha_{i_\alpha} \to M^\alpha_{\lambda^+}$ of $h^\beta_{i_\alpha,\lambda^+}$. By the quasi-representability of $\varphi_{\beta,\alpha}$,

by transfinite induction, this further determines a unique $\varphi_{\beta,\alpha}$-expansion $(h^{\alpha}_{j,k})_{i_\alpha \leq j<k\leq\lambda^+}$ of $(h^{\beta}_{j,k})_{i_\alpha\leq j<k\leq\lambda^+}$.

- Assume α is a limit ordinal. Then we define $i_\alpha = sup\{i_\beta \mid \beta < \alpha\}$. We have that $i_\alpha < \lambda^+$ because $\alpha \leq \lambda$ and $i_\beta < \lambda^+$ for each $\beta < \alpha$. (This holds because one can prove that $\bigcup_{\beta<\alpha} i_\beta$ is ordinal and hence $i_\alpha = \bigcup_{\beta<\alpha} i_\beta$, therefore we have $card(i_\alpha) = card(\bigcup_{\beta<\alpha} i_\beta) \leq card(\alpha)\times card(\alpha) \leq \lambda\times\lambda = \lambda < \lambda^+$.) For all $i_\alpha \leq j < k \leq \lambda^+$ by inductive-exactness let $h^{\alpha}_{j,k}$ be the amalgamation of $(h^{\beta}_{j,k})_{\beta<\alpha}$, i.e., the unique Σ_α-model homomorphism such that $h^{\alpha}_{j,k}\restriction_{\varphi_{\beta,\alpha}} = h^{\beta}_{j,k}$.

□

Based on the above analysis of the conditions underlying Thm. 7.9 we obtain the following typical concrete existence of saturated models.

Corollary 7.11. **FOL** *has $\mathcal{D}$-saturated models for $\mathcal{D}$ the class of signature extensions with a finite number of constants.*

Borrowing saturated models along institution comorphisms

Although the hypotheses of the existence Thm. 7.9 require existential quantifiers and conjunctions, existence of saturated models can, by the following rather general borrowing result, be easily extended to sub-institutions with less expressive power of sentences.

Proposition 7.12. *Let $(\Phi,\alpha,\beta)\colon\ I \to I'$ be an institution comorphism and $\mathcal{D} \subseteq \mathbb{S}ig$, $\mathcal{D}' \subseteq \mathbb{S}ig'$ be classes of signature morphisms such that*

1. (Φ,α,β) *is conservative and has weak model amalgamation, and*
2. Φ *preserves inductive co-limits and $\Phi(\mathcal{D}) \subseteq \mathcal{D}'$.*

Then I has $\mathcal{D}$-saturated models whenever I' has $\mathcal{D}'$-saturated models.

Proof. We first show that β maps $(\lambda,\mathcal{D}')$-saturated models to $(\lambda,\mathcal{D})$-saturated models. Consider M' a $(\lambda,\mathcal{D}')$-saturated $\Phi(\Sigma)$-model. Let $M = \beta_\Sigma(M')$ and consider a $(k,\mathcal{D})$-chain $\varphi\colon\ \Sigma\to\Sigma_k$ for $k<\lambda$, $(\chi\colon\ \Sigma_k\to\overline{\Sigma})\in\mathcal{D}$, M_k a φ-expansion of M, and E a set of $\overline{\Sigma}$-sentences such that M_k χ-realizes E finitely.

By condition 2 we have that $\Phi(\varphi)$ is a $(k,\mathcal{D}')$-chain $\Phi(\Sigma)\to\Phi(\Sigma_k)$. By the weak model amalgamation property of (Φ,α,β) let M'_k be a $\Phi(\varphi)$-expansion of M' such that $\beta_{\Sigma_k}(M'_k) = M_k$. By the weak model amalgamation again we can note that M'_k $\Phi(\chi)$-realizes $\alpha_{\overline{\Sigma}}(E)$ finitely. Because M' is $(\lambda,\mathcal{D}')$-saturated we have that M'_k $\Phi(\chi)$-realizes $\alpha_{\overline{\Sigma}}(E)$. Let $\overline{M'}$ be the $\Phi(\chi)$-expansion of M'_k such that $\overline{M'}\models\alpha_{\overline{\Sigma}}(E)$. Then $\beta_{\overline{\Sigma}}(\overline{M'})\models E$. By the naturality of β we have that $\beta_{\overline{\Sigma}}(\overline{M'})\restriction_\chi = M_k$, hence M_k χ-realizes E.

Now that we have established that β maps $(\lambda,\mathcal{D}')$-saturated models to $(\lambda,\mathcal{D})$-saturated models, we may proceed to the final part of the proof. For any Σ-model M, by the conservativeness of (Φ,α,β) there exists a $\Phi(\Sigma)$-model M' such that $\beta_\Sigma(M') = M$. Because I' has $\mathcal{D}'$-saturated models, let $h'\colon\ M'\to N'$ be such that $M'\equiv N'$ and N' is $(\lambda,\mathcal{D}')$-saturated. Then $\beta(N')$ is $(\lambda,\mathcal{D})$-saturated and $M\equiv\beta(N')$. □

An example of use of the above borrowing result is the following.

Corollary 7.13. **EQL** *and* **HCL** *have $\mathcal{D}$-saturated models for the usual $\mathcal{D}$ consisting of the signature extensions with a finite number of constants.*

Exercises

7.1. Let $\mathcal{D}$ be the class of $\mathbf{FOL^1}$ signature extensions with a finite number of constants. Consider the $\mathbf{FOL^1}$ signature having only one binary relation symbol $<$. The model $\mathbb{Q}$ of the rational numbers interpreting $<$ as the 'strictly less than' relation is $(\omega, \mathcal{D})$-saturated but it is not $(\lambda, \mathcal{D})$-saturated for cardinals $\lambda > \omega$.
(*Hint:* For each finite n the elementary equivalence relation between the expansions of $\mathbb{Q}$ with n constants determines a finite partition, i.e., has a finite number of equivalence classes, whose cardinal is less than $n!$. Each such equivalence class is determined by the mutual position of the constants with respect to $<$.)

7.2. In any semi-exact institution, the model reduct functors preserve the $(\lambda, \mathcal{D})$-saturated models if $\mathcal{D}$ is stable under pushouts. (*Hint:* $(\lambda, \mathcal{D})$-chains are stable under pushouts.)

7.3. Establish the existence of saturated models in several concrete institutions presented as examples in this book (such as **PA**, **POA**, etc.) as instances of the general institution-independent Thm. 7.9.

7.4. Saturated models for presentations
For any class $\mathcal{D}$ of signature morphisms in an institution I, let $\mathcal{D}^{\mathrm{pres}}$ denote the class of the presentation morphisms $\chi : (\Sigma, E) \to (\Sigma', E')$ for which $\chi \in \mathcal{D}$.

1. Any model of the institution I^{pres} of the presentations of I is $(\lambda, \mathcal{D}^{\mathrm{pres}})$-saturated if it is $(\lambda, \mathcal{D})$-saturated in I.
2. I^{pres} has $\mathcal{D}^{\mathrm{pres}}$-saturated models if I has $\mathcal{D}$-saturated models.
3. The following two institutions have $\mathcal{D}$-saturated models which are 'borrowed' by Prop. 7.12:
   (a) **HNK** has $\mathcal{D}$-saturated models for $\mathcal{D}$ the class of signature extensions with a finite number of constants (*Hint:* use the comorphism $\mathbf{HNK} \to \mathbf{FOEQL}^{\mathrm{pres}}$ of Ex. 4.11.)
   (b) **IPL** has $\mathcal{D}$-saturated models for $\mathcal{D}$ the class of signature extensions with a finite number of symbols (*Hint:* use the comorphism $\mathbf{IPL} \to \mathbf{FOEQL}^{\mathrm{pres}}$ of Ex. 4.10.)

7.3 Uniqueness of Saturated Models

The uniqueness property of saturated models is subject to a set of conditions which are introduced and discussed in the first part of this section. The most important one, in the sense that it is the only one with a special significance, is a limit to the 'size' of the models. A general concept of size may be defined when the institution has elementary diagrams. Another condition is that the elementary diagrams satisfy a certain rather natural property. Another property required is that sentences are finitary; in concrete terms this means that they can contain only a finite number of symbols.

Simple elementary diagrams. The elementary diagrams ι of an institution are *simple* when for each signature Σ and all Σ-models A, B, for each $\iota_\Sigma(B)$-expansion A' of A, the following is a pushout square of signature morphisms.

$$
\begin{tikzcd}
\Sigma \arrow[r, "\iota_\Sigma(B)"] \arrow[d, "\iota_\Sigma(A)"'] & \Sigma_B \arrow[d, "\iota_{\Sigma_B}(A')"] \\
\Sigma_A \arrow[r, "\iota_{\iota_\Sigma(B)}(1_A)"'] & (\Sigma_B)_{A'}
\end{tikzcd}
$$

Fact 7.14. *In the rather common situation when the elementary extensions just add the elements of the model as new constants to its signature, the elementary diagrams are simple because the above diagram is in fact a diagram of the form*

$$
\begin{tikzcd}
\Sigma \arrow[r] \arrow[d] & \Sigma \uplus |B| \arrow[d] \\
\Sigma \uplus |A| \arrow[r] & \Sigma \uplus |B| \uplus |A|
\end{tikzcd}
$$

where $|A|$, respectively $|B|$, denote the underlying carrier sets of A and B, respectively.

Sizes of models. Let M be a model in an institution with elementary diagrams ι. For any cardinal number λ, we say that M has $\mathcal{D}$*-size* λ when $\iota_\Sigma(M) = \varphi_{0,\lambda}$ for some $(\lambda, \mathcal{D})$-chain $(\varphi_{i,j})_{i<j\leq\lambda}$. Note that this concept of 'size' is a relation between models and cardinals rather than a function from models to cardinals.

Fact 7.15. *Let $\mathcal{D}$ be the class of the* **FOL** *signature extensions with a finite number of constants. An infinite* **FOL***-model M has $\mathcal{D}$-size λ if and only if $card(|M|) \leq \lambda$, where $card(|M|)$ denotes the cardinality of its set $|M|$ of elements ($|M| = \uplus_{s\in S} M_s$ where S is the set of the sorts of Σ). Consequently, any finitely sized model (see Sect. 6.5) has $\mathcal{D}$-size λ for any finite cardinal λ.*

By Prop. 7.8 we can further establish the following:

Corollary 7.16. *For any infinite cardinal λ, for each λ-saturated* **FOL** *model M of $\mathcal{D}$-size λ such that M_s is infinite for at least one sort, $card(|M|) = \lambda$.*

Finitary sentences. In any institution a Σ-sentence ρ is *finitary* if and only if it can be written as $\varphi(\rho_0)$ where $\varphi\colon \Sigma_0 \to \Sigma$ is a signature morphism such that Σ_0 is a finitely presented signature and ρ_0 is a Σ_0-sentence. An institution *has finitary sentences* when all its sentences are finitary. This concept is a categorical expression of the fact that a sentence contain only a finite number of symbols. This is illustrated by the following typical example.

Fact 7.17. *A* **FOL** *signature* (S,F,P) *is finitely presented if and only if* S, F, *and* P *are finite. (Here* F *'finite' means that* $\{(w,s) \mid F_{w\to s} \neq \emptyset\}$ *is finite and each non-empty* $F_{w\to s}$ *is also finite.) Consequently,* **FOL** *has finitary sentences.*

Here we have to warn the reader about some possible terminology confusion which may arise in relation to the term 'finitary' when used in conjunction with basic sentences. Therefore by 'finitary basic' sentences we will always mean that the respective model M_E is finitely presented and not that the set of sentences is 'finitary' and 'basic'.

Uniqueness theorem

Theorem 7.18. *Assume that the institution*

1. *has pushouts and inductive co-limits of signatures,*
2. *is semi-exact and inductive-exact on models,*
3. *has simple elementary diagrams* ι,
4. *has existential* $\mathcal{D}$*-quantification for a (sub)category* $\mathcal{D}$ *of signature morphisms which is stable under pushouts,*
5. *has negations and finite conjunctions, and*
6. *has finitary sentences.*

Then any two elementary equivalent $(\lambda,\mathcal{D})$*-saturated* Σ*-models of* $\mathcal{D}$*-size* λ *are isomorphic.*

Proof. Let M,N be Σ-models satisfying the hypotheses of the theorem. We consider the following pushout of signature morphisms:

$$\begin{tikzcd}
\Sigma \arrow[r, "\iota_\Sigma(M)"] \arrow[d, "\iota_\Sigma(N)"'] & \Sigma_M \arrow[d, "\phi_M"] \\
\Sigma_N \arrow[r, "\phi_N"'] & \Sigma''
\end{tikzcd}$$

and construct elementarily equivalent Σ''-expansions, M'' of M_M and N'' of N_N.

Suppose we have already constructed M'' and N''. Let $M' = M''\!\upharpoonright_{\phi_N}$ and $N' = N''\!\upharpoonright_{\phi_M}$. Because the elementary diagrams are simple and pushouts are unique up to isomorphism, we may assume without any loss of generality that $\Sigma'' = (\Sigma_N)_{M'}, \phi_M = \iota_{\iota_\Sigma(N)}(1_M)$ and $\phi_N = \iota_{\Sigma_N}(M')$.

Because $M'_{M'}\!\upharpoonright_{\phi_N} = M'$ and $M'_{M'}\!\upharpoonright_{\phi_M} = M_M$ (which follows from the naturality of i and because $M'_{M'} = i^{-1}_{\Sigma_N,M'}(1_{M'})$ and $M_M = i^{-1}_{\Sigma,M}(1_M)$), by the uniqueness part of the semi-exactness we get that $M'' = M'_{M'}$.

But $N'' \models E_{M'}$ (because $M'' \equiv N''$), hence we get a model homomorphism $h: M'' \to N''$. Similarly we get another Σ''-homomorphism $h': N'' \to M''$. By the initiality of M''

and N'' we have that $h;h' = 1_{M''}$ and $h';h = 1_{N''}$. Thus we have that $M'' \cong N''$, hence by reduction to Σ we obtain also that $M \cong N$, which proves the theorem.

Now let us come back to the construction of M'' and N''. Since both M and N have the same $\mathcal{D}$-size λ, $\iota_\Sigma(M) = \varphi_M^{0,\lambda}$ and $\iota_\Sigma(N) = \varphi_N^{0,\lambda}$ where $(\Sigma_M^i \xrightarrow{\varphi_M^{i,j}} \Sigma_M^j)_{i<j\leq\lambda}$ and $(\Sigma_N^i \xrightarrow{\varphi_N^{i,j}} \Sigma_N^j)_{i<j\leq\lambda}$ are $(\lambda, \mathcal{D})$-chains of signature morphisms.

By ordinal induction we define another $(\lambda, \mathcal{D})$-chain $(\Sigma^i \xrightarrow{\phi^{i,j}} \Sigma^j)_{i<j\leq\lambda}$ such that $\Sigma^0 = \Sigma$ and for each $j < \lambda$

- $\Sigma^j \xrightarrow{\gamma_M^j} \Sigma'^j_M \xleftarrow{\theta'^j_M} \Sigma_M^{j+1}$ is the pushout of $\Sigma^j \xleftarrow{\theta_M^j} \Sigma_M^j \xrightarrow{\varphi_M^{j,j+1}} \Sigma_M^{j+1}$,
- $\Sigma^j \xrightarrow{\gamma_N^j} \Sigma'^j_N \xleftarrow{\theta'^j_N} \Sigma_N^{j+1}$ is the pushout of $\Sigma^j \xleftarrow{\theta_N^j} \Sigma_N^j \xrightarrow{\varphi_N^{j,j+1}} \Sigma_N^{j+1}$,
- $\Sigma'^j_M \xrightarrow{\psi_M^j} \Sigma^{j+1} \xleftarrow{\psi_N^j} \Sigma'^j_M$ is the pushout of $\Sigma'^j_M \xleftarrow{\gamma_M^j} \Sigma^j \xrightarrow{\gamma_N^j} \Sigma'^j_N$,
- $\theta_M^{j+1} = \theta'^j_M;\psi_M^j$ and $\theta_N^{j+1} = \theta'^j_N;\psi_N^j$, and
- $\phi^{j,j+1} = \gamma_M^j;\psi_M^j = \gamma_N^j;\psi_N^j$.

$$
\begin{tikzcd}
 & & \Sigma_M^{j+1} \arrow[dr, "\theta'^j_M"] & & \\
 & \Sigma_M^j \arrow[ur, "\varphi_M^{j,j+1}"] \arrow[dr, "\theta_M^j"] & & \Sigma'^j_M \arrow[dr, "\psi_M^j"] & \\
\Sigma \arrow[ur, "\varphi_M^{0,j}"] \arrow[dr, "\varphi_N^{0,j}"'] & & \Sigma^j \arrow[ur, "\gamma_M^j"] \arrow[dr, "\gamma_N^j"'] \arrow[rr, "\phi^{j,j+1}"] & & \Sigma^{j+1} \\
 & \Sigma_N^j \arrow[ur, "\theta_N^j"'] \arrow[dr, "\varphi_N^{j,j+1}"'] & & \Sigma'^j_N \arrow[ur, "\psi_N^j"'] & \\
 & & \Sigma_N^{j+1} \arrow[ur, "\theta'^j_N"'] & &
\end{tikzcd}
$$

Because $\mathcal{D}$ is stable under pushouts, $\gamma_M^j, \gamma_N^j, \psi_M^j, \psi_N^j \in \mathcal{D}$, and because $\mathcal{D}$ is closed under compositions, $\phi^{j,j+1} = \gamma_M^j;\psi_M^j \in \mathcal{D}$.

For each limit ordinal $k \leq \lambda$, we define $\Sigma_M^k \xrightarrow{\theta_M^k} \Sigma^k$ as the unique signature morphism such that $\varphi_M^{i,k};\theta_M^k = \theta_M^i;\phi_M^{i,k}$ for each $i < k$. θ_N^k is defined similarly. Therefore θ_M and θ_N appear as natural transformations between λ-chains, $\theta_M : \varphi_M \Rightarrow \phi$ and $\theta_N : \varphi_N \Rightarrow \phi$. Let $\phi_M = \theta_M^\lambda$, $\phi_N = \theta_N^\lambda$, and $\Sigma'' = \Sigma^\lambda$.

It is clear from the construction that for each $j \leq \lambda$ we have that $\Sigma_M^j \xrightarrow{\theta_M^j} \Sigma^j \xleftarrow{\theta_N^j} \Sigma_N^j$ is a pushout for $\Sigma_M^j \xleftarrow{\varphi_M^{0,j}} \Sigma \xrightarrow{\varphi_N^{0,j}} \Sigma_N^j$.

In the second part of the proof, by ordinal induction we define for each $j \leq \lambda$, Σ^j-models $M^j \equiv N^j$ such that $M^j\upharpoonright_{\phi^{i,j}} = M^i$, $N^j\upharpoonright_{\phi^{i,j}} = N^i$ for each $i \leq j$, and $M^j\upharpoonright_{\theta^j_M} = M_M\upharpoonright_{\varphi^{i,j}_M}$ and $N^j\upharpoonright_{\theta^j_N} = N_N\upharpoonright_{\varphi^{i,j}_N}$, as follows.

- $M^0 = M$ and $N^0 = N$.
- For each successor ordinal $j+1$ let M'^j be the (unique) amalgamation $M_M\upharpoonright_{\varphi^{j+1,\lambda}_M} \otimes M^j$. For each finite $E' \subseteq (M'^j)^*$, we have that $M^j \models (\exists\gamma^j_M) \wedge E'$ (which is a sentence of the institution because the institution has finite conjunctions and existential $\mathcal{D}$-quantification and $\gamma^j_M \in \mathcal{D}$). Because $M^j \equiv N^j$ we deduce that $N^j \models (\exists\gamma^j_M) \wedge E'$. Because N is λ-saturated, $(\phi^{i,k})_{0\leq i<k\leq j}$ is a $(j,\mathcal{D})$-chain with $j < \lambda$, $\gamma^j_M \in \mathcal{D}$, and $N^j\upharpoonright_{\phi^{0,j}} = N$, there exists a γ^j_M-expansion N'^j of N^j such that $N'^j \models (M'^j)^*$. Because the institution has negations, this means $N'^j \equiv M'^j$. Now we define N^{j+1} to be the (unique) amalgamation $N_N\upharpoonright_{\varphi^{j+1,\lambda}_N} \otimes N'^j$. As for N'^j, but now using the saturation of M, we obtain the existence of $M^{j+1} \equiv N^{j+1}$ such that $M^{j+1}\upharpoonright_{\psi^j_M} = M'^j$.
- For each limit ordinal j, by the inductive-exactness property M^j and N^j are the unique Σ^j-models such that $M^j\upharpoonright_{\phi^{i,j}} = M^i$ and $N^j\upharpoonright_{\phi^{i,j}} = N^i$ for each $i < j$. In order to prove $M^j \equiv N^j$, we use the fact that the institution has finitary sentences. For any Σ^j-sentence ρ^j there exists $i < j$ such that $\rho^j = \phi^{i,j}(\rho^i)$ for some Σ^i-sentence ρ^i. Then by the Satisfaction Condition and because $M^i \equiv N^i$, $M^j \models \rho^j$ iff $M^i \models \rho^i$ iff $N^i \models \rho^i$ iff $N^j \models \rho^j$. That $M^j\upharpoonright_{\theta^j_M} = M_M\upharpoonright_{\varphi^{j,\lambda}_M}$ and $N^j\upharpoonright_{\theta^j_N} = N_N\upharpoonright_{\varphi^{j,\lambda}_N}$ follow by the uniqueness part of the inductive-exactness property by noticing that for each $i < j$, $(M^j\upharpoonright_{\theta^j_M})\upharpoonright_{\varphi^{i,j}_M} = (M_M\upharpoonright_{\varphi^{j,\lambda}_M})\upharpoonright_{\varphi^{i,j}_M}$ and $(N^j\upharpoonright_{\theta^j_N})\upharpoonright_{\varphi^{i,j}_N} = (N_N\upharpoonright_{\varphi^{j,\lambda}_N})\upharpoonright_{\varphi^{i,j}_N}$.

We finalize this proof by taking M'' as M^λ and N'' as N^λ. $\square$

The following is a typical concrete instance of the uniqueness Thm. 7.18.

Corollary 7.19. *In* **FOL**, *saturated models are unique up to isomorphisms.*

Because finitely sized **FOL** models are saturated (cf. Prop. 7.8) this can be further applied to give an alternative proof of Cor. 6.28 which we state again below.

Corollary 7.20. *In* **FOL** *any two finitely sized elementary equivalent models are isomorphic.*

Exercises

7.5. Establish the uniqueness of saturated models in several concrete institutions presented as examples in this book (such as **PA**, **POA**, etc.) as instances of the general institution-independent Thm. 7.18.

7.6. In **FOL** any two elementary equivalent models admit a common elementary extension.

7.4 Saturated Ultraproducts

For this section we assume the Generalized Continuum Hypothesis.

Good ultrafilters. Let $(P, \leq)$ and $(P', \leq)$, respectively, be partial orders with binary least upper bounds $\vee$ and greatest lower bounds $\wedge$, respectively. A function $f\colon P \to P'$ is

- *anti-monotonic* if $x < y$ implies $f(x) > f(y)$, and
- *anti-additive* if $f(x \vee y) = f(x) \wedge f(y)$.

For any functions $f, g\colon P \to P'$, $f \leq g$ if $f(x) \leq g(x)$ for all $x \in P$.

An ultrafilter U is λ-*good* for a cardinal λ if for each $\alpha < \lambda$ and each anti-monotonic function $f\colon \mathcal{P}_\omega(\alpha) \to U$ there exists an anti-additive function $g\colon \mathcal{P}_\omega(\alpha) \to U$ such that $g \leq f$.

Countably incomplete ultrafilters. An ultrafilter U over I is *countably incomplete* if there exists an ω-chain $I = I_0 \supset I_1 \supset \cdots \supset I_n \supset \ldots$ such that $I_n \in U$ and $I_\omega = \bigcap_{n \in \omega} I_n = \emptyset$.

The proof of the following theorem consists of combinatorial set-theoretic arguments, and can be found in [32].

Theorem 7.21. *For any set I of cardinality λ, there exists a λ^+-good countably incomplete ultrafilter over I.*

Stable sentence functors. A sentence functor Sen is $\mathcal{D}$-*stable* for a class $\mathcal{D}$ of signature morphisms when for each $\chi\colon \Sigma \to \Sigma'$ in $\mathcal{D}$ we have $card(\mathsf{Sen}(\Sigma')) \leq card(\mathsf{Sen}(\Sigma))$.

Stability of sentence functors is a rather common property of institution, as shown by the following typical example.

Proposition 7.22. *The* **FOL** *sentence functor is $\mathcal{D}$-stable for $\mathcal{D}$ the class of all signature extensions with a finite number of constants.*

Proof. We show that for each $(\chi\colon \Sigma \to \Sigma') \in \mathcal{D}$ we have that $card(\mathsf{Sen}(\Sigma)) = card(\mathsf{Sen}(\Sigma'))$. From this follows that $\mathsf{Sen}^{\mathbf{FOL}}$ is $\mathcal{D}$-stable.

On the one hand because $\Sigma \subseteq \Sigma'$ we have that $\mathsf{Sen}(\Sigma) \subseteq \mathsf{Sen}(\Sigma')$ hence $card(\mathsf{Sen}(\Sigma)) \leq card(\mathsf{Sen}(\Sigma'))$. On the other hand the function $\mathsf{Sen}(\Sigma') \to \mathsf{Sen}(\Sigma)$ which maps each Σ'-sentence ρ' to $(\exists\chi)\rho'$ is an injection, hence $card(\mathsf{Sen}(\Sigma')) \leq card(\mathsf{Sen}(\Sigma))$. □

The following shows that stability of the sentence functor guarantees a limit to the growth of the cardinality of the sets of sentences.

Proposition 7.23. *Consider an institution with finitary sentences and with a class $\mathcal{D}$ of signature morphisms such that the sentence functor is $\mathcal{D}$-stable. Then for each $(\alpha, \mathcal{D})$-chain $\varphi\colon \Sigma \to \Sigma'$ we have that $card(\mathsf{Sen}(\Sigma')) \leq card(\alpha) \times card(\mathsf{Sen}(\Sigma))$.*

Proof. Let us denote the segment of the chain φ between i and j by $\varphi_{i,j}\colon \Sigma_i \to \Sigma_j$. Then $\Sigma = \Sigma_0$ and $\Sigma' = \Sigma_\alpha$. We prove the proposition by transfinite induction on α.

If α is a successor ordinal $\beta+1$ we have that $card(\mathsf{Sen}(\Sigma_\alpha)) = card(\mathsf{Sen}(\Sigma_{\beta+1})) \leq card(\mathsf{Sen}(\Sigma_\beta)) \leq card(\beta) \times card(\mathsf{Sen}(\Sigma_0)) = card(\beta+1) \times card(\mathsf{Sen}(\Sigma_0))$.

If α is a limit ordinal then $\alpha = \cup_{\beta<\alpha}\beta$. Because the institution has finitary sentences, by Lemma 7.24 below we obtain that $card(\mathsf{Sen}(\Sigma_\alpha)) \leq card(\biguplus_{\beta<\alpha}\mathsf{Sen}(\Sigma_\beta))$. By the induction hypothesis $card(\mathsf{Sen}(\Sigma_\beta)) \leq card(\beta) \times card(\mathsf{Sen}(\Sigma_0)) \leq card(\alpha \times \mathsf{Sen}(\Sigma_0))$. Therefore, we have $card(\mathsf{Sen}(\Sigma_\alpha)) \leq card(\alpha \times \alpha \times \mathsf{Sen}(\Sigma_0))$ and finally $card(\mathsf{Sen}(\Sigma_\alpha)) \leq card(\alpha) \times card(\mathsf{Sen}(\Sigma_0))$.

Lemma 7.24. *Consider an institution with finitary sentences. Then for each limit ordinal* α *and* $(\alpha,\mathcal{D})$*-chain* $(\Sigma_i \xrightarrow{\varphi_{i,j}} \Sigma_j)_{i<j\leq\alpha}$ *we have that* $card(\mathsf{Sen}(\Sigma_\alpha)) \leq card(\biguplus_{\eta<\alpha}\mathsf{Sen}(\Sigma_\beta))$.

Proof. We define an injection ι from $\mathsf{Sen}(\Sigma_\alpha)$ to $\biguplus_{\beta<\alpha}\mathsf{Sen}(\Sigma_\beta)$. For each $\rho \in \mathsf{Sen}(\Sigma_\alpha)$ there exists a finitely presented signature Σ_ρ, a sentence $\rho^{fin} \in \mathsf{Sen}(\Sigma_\rho)$ and a signature morphism $\psi_\rho : \Sigma_\rho \to \Sigma_\alpha$ such that $\psi_\rho(\rho^{fin}) = \rho$. Because Σ_ρ is finitely presented and Σ_α is an inductive co-limit, there exists an ordinal β such that ψ_ρ factors through Σ_β. Let $\phi_\rho : \Sigma_\rho \to \Sigma_\beta$ such that $\phi_\rho;\varphi_{\beta,\alpha} = \psi_\rho$. We define $\iota(\rho)$ to be $\phi_\rho(\rho^{fin})$. Because $\varphi_{\beta,\alpha}(\iota(\rho)) = \rho$ we get immediately that ι is an injection. □

□

Ultraproducts that are saturated

The following gives sufficient conditions for ultraproducts to be saturated.

Theorem 7.25. *Consider a Łoś institution with finitary sentences and with a class* $\mathcal{D}$ *of signature morphisms such that*

1. *it has finite conjunctions and existential* $\mathcal{D}$*-quantifications,*
2. *the sentence functor* Sen *is* $\mathcal{D}$*-stable,*
3. *the model reduct functors corresponding to signature morphisms in* $\mathcal{D}$ *preserve ultraproducts of models,*
4. *each signature morphism lifts completely ultraproducts.*

For any infinite cardinal λ *and each countably incomplete* λ*-good ultrafilter* U *over* I*, for any signature* Σ*, if* $card(\mathsf{Sen}(\Sigma)) < \lambda$*, then for any family* $\{A_i\}_{i\in I}$ *of* Σ*-models, the ultraproduct* $\prod_U A_i$ *is* $(\lambda,\mathcal{D})$*-saturated.*

Proof. Consider an $(\alpha,\mathcal{D})$-chain $(\Sigma_i \xrightarrow{\varphi_{i,j}} \Sigma_j)_{i<j\leq\alpha}$ with $\alpha < \lambda$ such that $\Sigma_0 = \Sigma$, a $\varphi_{0,\alpha}$-expansion A^α of $\prod_U A_i$, $(\chi : \Sigma_\alpha \to \Sigma') \in \mathcal{D}$, and a set E of Σ'-sentences such that A^α χ-realizes E finitely.

Because each signature morphism lifts completely ultraproducts, for each $i \in I$, there exists a $\varphi_{0,\alpha}$-expansion A_i^α of A_i such that $\prod_U A_i^\alpha = A_\alpha$.

Because U is countably incomplete, there exists an ω-chain $I = I_0 \supset I_1 \supset \cdots \supset I_n \supset \ldots$ such that $I_n \in U$ and $I_\omega = \bigcap_{n\in\omega} I_n = \emptyset$. We define $f : \mathcal{P}_\omega(E) \to U$ (recall that $\mathcal{P}_\omega(E)$ is the set of all *finite* subsets E' of E)

– $f(\emptyset) = I$, and
– $f(E') = I_n \cap \{i \mid A_i^\alpha \models (\exists\chi) \wedge E'\}$ where n is the cardinality of E'.

f is well defined because $\prod_U A_i^\alpha = A^\alpha \models (\exists\chi) \wedge E'$ and sentences are preserved by ultrafactors (hence $\{i \in I \mid A_i^\alpha \models (\exists\chi) \wedge E'\} \in U$).

f is also anti-monotonic because for each $E_1 \subset E_2 \subseteq E$, $I_{n_1} > I_{n_2}$ (where n_1 respectively n_2 are the cardinalities of E_1 respectively E_2), and $\{i \mid A_i^\alpha \models (\exists\chi) \wedge E_1\} \supseteq \{i \mid A_i^\alpha \models (\exists\chi) \wedge E_2\}$. Because U is λ-good and the cardinality of E is less than $\lambda \times \lambda = \lambda$ (see Propositions 7.23 and 7.6) there exists an anti-additive function $g : \mathcal{P}_\omega(E) \to U$ such that $g \leq f$. For each $i \in I$ let $E_i = \{\rho \in E \mid i \in g(\rho)\}$.

If the cardinality of E_i is greater than n, then $i \in I_n$. In order to see this, consider $\{\rho_1, \ldots, \rho_n\} \subseteq E_i$. This means $i \in g(\rho_k)$ for all $k \leq n$. As g is anti-additive, we have that $i \in \bigcap_{k \leq n} g(\rho_k) = g(\{\rho_1, \ldots, \rho_n\}) \subseteq f(\{\rho_1, \ldots, \rho_n\}) \subseteq I_n$.

Because $\bigcap_{n \in \omega} I_n = \emptyset$, for each $i \in I$, E_i is finite. Otherwise if E_i were infinite we would have that $i \in I_n$ for all $n \in \omega$, which contradicts $\bigcap_{n \in \omega} I_n = \emptyset$.

Because each E_i is finite, we have that $i \in \bigcap_{\rho \in E_i} g(\rho) = g(\bigcup_{\rho \in E_i} \{\rho\}) = g(E_i) \subseteq f(E_i)$. This means that $A_i^\alpha \models (\exists\chi) \wedge E_i$. Let A_i' be the χ-expansion of A_i^α such that $A_i' \models E_i$.

Finally, we show that $\prod_U A_i' \models E$. Because χ preserves ultraproducts, from $\prod_U A_i' \models E$ we obtain that $A_\alpha = \prod_U A_i^\alpha$ χ-realizes E. For each $\rho \in E$, we have that $g(\rho) \subseteq \{i \mid A_i' \models \rho\}$. Because $g(\rho) \in U$, we deduce that $\{i \mid A_i' \models \rho\} \in U$, hence $\prod_U A_i' \models \rho$ because ρ is preserved by ultraproducts. $\square$

The conditions of Thm. 7.25 that need some additional explanation are perhaps the last two ones.

The condition about lifting completely the ultraproducts is fulfilled by all projectively representable signature morphisms in institutions for which the model projections are epis (cf. Prop. 6.8). This means we may have to narrow the class of signature morphisms of the original institution. For example, in the case of **FOL**, a solution is provided by considering only the signature extensions with constants. In this case, the typical choice for $\mathcal{D}$ is the class of all signature extensions with a finite number of constants.

The condition about preservation of ultraproducts of models by the model reduct functors can be resolved more generally for any filtered products by Propositions 6.6 and 6.4.

It is important to note that such narrowing of the class of signature morphisms does not hinder applicability of Thm. 7.25, since the only signature morphisms of the institution that are involved in this result are the $(\alpha, \mathcal{D})$-chains (for $\alpha < \lambda$), other signature morphisms being irrelevant for this result. This situation bears some similarity to how the existence Thm. 7.9 is applied to actual situations.

Another remark is that in some institutions Thm. 7.25 together with Prop. 6.13 (and its **FOL** Cor. 6.14, each model can be elementarily embedded in any of its ultrapowers) may provide an alternative way to reach essentially the existence of saturated models (Thm. 7.9). The costs are however quite high: the Łoś property for the institution, and especially the rather difficult result on the existence of good countably incomplete ultrafilters (Thm. 7.21).

Keisler-Shelah isomorphism theorem

The following is one of the most important applications of the uniqueness property of saturated models and of saturated ultraproducts.

Corollary 7.26. *Consider an institution which satisfies the hypotheses of Theorems 7.18 and 7.25 and such that each model M has a $\mathcal{D}$-size such that if M has a $\mathcal{D}$-size λ, then each ultrapower $\prod_U M$ for an ultrafilter U over I has $\mathcal{D}$-size $\lambda^{card(I)}$.*

Then any two elementarily equivalent models have isomorphic ultrapowers (for the same ultrafilter).

Proof. Let $M \equiv N$ be elementarily equivalent Σ-models. Consider a cardinal λ such that both M and N have $\mathcal{D}$-size λ^+ and such that $card(\mathsf{Sen}(\Sigma)) \leq \lambda$. Let U be a countably incomplete λ^+-good ultrafilter over λ. Then both $\prod_U M$ and $\prod_U N$ have $\mathcal{D}$-size $(\lambda^+)^\lambda = \lambda^+$ (cf. Prop. 7.6 on cardinal arithmetic). By Thm. 7.25 both ultrapowers are $(\lambda^+, \mathcal{D})$-saturated. By the uniqueness Thm. 7.18 they are therefore isomorphic. □

Corollary 7.27. *In* **FOL***, any two elementarily equivalent models have isomorphic ultrapowers.*

Proof. While hypotheses of Theorems 7.25 and 7.18 in the framework of **FOL** have been discussed above, if we define the sizes of models by their cardinality, then the specific condition about sizes of Cor. 7.26 holds obviously since each ultrapower $\prod_U M$ is the quotient of the power $\prod_{i \in I} M$. □

Keisler-Shelah institutions. An institution with ultraproducts of models satisfying the conclusion of Cor. 7.26 is called a *Keisler-Shelah institution*.

A counterexample. In the sub-institution of **FOL** that restricts sentences to those that do not use the equality symbol, consider the signature $\Sigma = (\{s\}, \{\sigma : \ s \to s\}, \emptyset)$ and two models of this signature A and B defined as follows:

- $A_s = \{0,1\}$; $A_\sigma(0) = 0$ and $A_\sigma(1) = 1$,
- $B_s = \{0,1\}$; $B_\sigma(0) = 1$ and $B_\sigma(1) = 0$.

It is clear that $A \equiv B$ but A and B are not isomorphic. Because A and B are finite, each of them is isomorphic to any of their ultrapowers, hence for each ultrafilter U, ultrapowers $\prod_U A$ and $\prod_U B$ cannot be isomorphic.

This counterexample for the Keisler-Shelah property exploits an institution where the syntactic power (given by the sentences) is not enough to enforce a semantic property (isomorphism of models). The concordance between these aspects is ensured in our results by the existence of elementary diagrams. In the absence of elementary diagrams the uniqueness of saturated models (Thm. 7.18), which is one of the main conditions for the Keisler-Shelah property, is no longer guaranteed.

Exercises

7.7. Establish the Keisler-Shelah property in concrete institutions presented as examples in this book (such as **PA**, **POA**, etc.) by one of the methods below:

1. directly the general result of Cor. 7.26, or
2. by 'borrowing' it from **FOL** via various institution comorphisms presented in this book.

Which of the two methods suggested above does apply to **HNK**?

Notes. The $\mathbf{FOL^1}$ special case of Cor. 7.5 for chains instead of any directed co-limit was proved by Tarski and Vaught in [175] and received high notoriety in conventional model theory under the name 'Elementary Chain Theorem', while Theorems 7.1 and 7.2 are due to [86].

The concept of a saturated model can be traced back to the η_α-sets of [91]. A good reference for cardinal arithmetic is [100]. The theory of saturated models at the institution-independent level was developed in [59] where both the existence and the uniqueness Theorems 7.9 and 7.18 appear. The $\mathbf{FOL^1}$ instances of these results were proved in [128].

Our definition of countably incomplete ultrafilters is formulated slightly differently but equivalently to the standard one in [32]. The existence of saturated ultraproducts (Thm. 7.25) is due to [59] and generalizes the corresponding $\mathbf{FOL^1}$ result which can be traced back to [103]. The Keisler-Shelah isomorphism theorem in $\mathbf{FOL^1}$ (Cor. 7.27) was proved in [164] without assuming GCH.

Chapter 8

Preservation and Axiomatizability

Axiomatizability results express a rather subtle relationship between semantics and syntax. They give complete characterizations of certain classes of theories in purely semantic terms, formulated as closure properties of classes of models under some categorical operators. Perhaps the most famous example is the Birkhoff Variety theorem of equational logic: a class of algebras for a signature is closed under products, sub-algebras, and homomorphic images if and only if it is the class of algebras of an equational theory.

Axiomatizability results have been traditionally considered to have mostly theoretical significance. However they do have important applications such as interpolation and definability. Some of these have been discovered and understood properly only relatively recently. After developing several general Birkhoff-style axiomatizability results we continue this chapter by giving an abstract formulation for Birkhoff-style axiomatizability which captures uniformly all results of this chapter and much more.

Preservation results are half way to axiomatizability results in the sense that, assuming a theory we can establish, it can be presented by a certain kind of sentences whenever it is 'preserved' by some semantic operations. A typical example is the following: a **FOL** theory can be presented by a set of universal sentences if and only if it is 'preserved' by sub-models. Some axiomatizability results can be obtained via their preservation correspondents.

8.1 Preservation by Saturation

In this section we develop a general preservation result as an application of saturated models.

The framework. For this section we consider

- an institution with elementary diagrams $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \iota)$ together
- with a sub-functor $\mathsf{Sen}^0 \subseteq \mathsf{Sen}$ (i.e., $\mathsf{Sen}_0 \colon\ \mathbb{S}ig \to \mathbb{S}et$ such that $\mathsf{Sen}^0(\Sigma) \subseteq \mathsf{Sen}(\Sigma)$ and $\varphi(\mathsf{Sen}^0(\Sigma)) \subseteq \mathsf{Sen}^0(\Sigma')$ for each signature morphism $\varphi \colon\ \Sigma \to \Sigma'$), and
- a sub-category $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms,

such that

- for each Σ-model M, its elementary diagram $E_M \subseteq \mathsf{Sen}^0(\Sigma_M)$,
- $I^0 = (\mathbb{S}ig, \mathsf{Sen}^0, \mathsf{Mod}, \models)$ has finite conjunctions and finite disjunctions, and
- Sen^0 is closed under existential $\mathcal{D}$-quantification.

Universal and existential sentences in FOL. As a typical example for this framework we may take I to be **FOL** and $\mathsf{Sen}^0(\Sigma)$ to be the set of all *existential Σ-sentences* which are existential quantifications of sentences accessible from the atoms only by Boolean connectives. Note that existential sentences in **FOL** are indeed closed under conjunctions and disjunctions (Ex. 5.5 above gives a general institution-independent version of this argument). Recall from Sect. 3.2 that *universal sentences* are the negations of the existential sentences, which means that they are universal quantifications of sentences accessible from the atoms only by Boolean connectives.

Sen^0**-extensions.** For any Σ-models M and N, let us establish the notation

$$M[\mathsf{Sen}^0]N \text{ if and only if } M^* \cap \mathsf{Sen}^0(\Sigma) \subseteq N^* \cap \mathsf{Sen}^0(\Sigma).$$

We say that *M is a Sen^0-submodel of N* when there exists a Σ-model homomorphism $h \colon\ M \to N$ such that $M_M[\mathsf{Sen}^0]N_h$. Recall that by N_h we mean $i^{-1}_{\Sigma,M}(h)$, the mapping of h by the canonical isomorphism $M/\mathsf{Mod}(\Sigma) \to \mathsf{Mod}(\Sigma_M, E_M)$. Alternatively we may say that *N is a Sen^0-extension of M*. Let us denote this relation by $M \xrightarrow{\mathsf{Sen}^0} N$.

In particular situations it is often possible to express the Sen^0-extension relationship in purely semantic terms. The following is a typical example.

Proposition 8.1. *In* **FOL***, let* $\mathsf{Exist} \subseteq \mathsf{Sen}^{\mathbf{FOL}}$ *be the sub-functor of the existential sentences. Then* $M \xrightarrow{\mathsf{Exist}} N$ *if and only if there exists a closed injective model homomorphism* $M \to N$.

Proof. Consider $M \xrightarrow{\mathsf{Exist}} N$ for a **FOL**-signature (S,F,P). By definition there exists a model homomorphism $h \colon\ M \to N$ such that $M_M[\mathsf{Exist}]N_h$. Then

- h is injective because for all $m_1 \neq m_2 \in M$, $M_M \models \neg(m_1 = m_2)$ implies $N_h \models \neg(m_1 = m_2)$ which means $h(m_1) \neq h(m_2)$,
- h is closed because for each relation symbol $\pi \in P_w$ and each string of elements $m \in M_w$, if $h(m) \in N_\pi$ then $N_h \models \pi(m)$, which implies $M_M \models \pi(m)$ (otherwise if $M_M \models \neg\pi(m)$ then $N_h \models \neg\pi(m)$) which means $m \in M_\pi$.

Now we show that any closed injective model homomorphism $h\colon\ M \to N$ implies that $M \xrightarrow{\mathsf{Exist}} N$.

The first step is to note that for each quantifier-free Σ_M-sentence ρ (i.e., ρ is accessible from the Σ_M-atoms only by Boolean connectives) $M_M \models \rho$ if and only if $N_h \models \rho$. (This is achieved easily by induction on the structure of the sentences by making use of the fact that h is injective and closed.)

Let $(\exists X)\rho'$ be an existential $(S,F,P)_M$-sentence such that $M_M \models (\exists X)\rho'$. Let M' be the expansion of M_M such that $M' \models \rho'$ and let ρ'_M be the result of replacing each $x \in X$ with M'_x in ρ'. Then $M_M \models \rho'_M$ and hence $N_h \models \rho'_M$. But $N_h \models \rho'_M$ means that $N' \models \rho'$ for the expansion N' of N_h such that $N'_x = h(M'_x)$, therefore $N_h \models (\exists X)\rho'$. $\square$

Preservation by saturation

We say that a set of sentences is *preserved by* Sen^0*-extensions* when for any two models M and N, $M \xrightarrow{\mathsf{Sen}^0} N$ and $M \models E$ implies $N \models E$. Dually, *E is preserved by* Sen^0*-submodels* when $N \models E$ implies $M \models E$.

For any set of sentences Γ let $\neg\Gamma$ denote $\{\neg\rho \mid \rho \in \Gamma\}$.

Theorem 8.2 (Preservation by saturation). *In addition to the framework of this section let us also assume the following conditions:*

1. *I has inductive weak model amalgamation,*
2. *I is compact and Boolean complete,*
3. Sen^0 *consists of finitary sentences,*
4. *each model has a $\mathcal{D}$-size,*
5. *I has $\mathcal{D}$-saturated models.*

Then for any consistent Σ-theory E

- *E is preserved by* Sen^0*-extensions if and only if $E \models\!\mid E \cap \mathsf{Sen}^0(\Sigma)$, and*
- *E is preserved by* Sen^0*-submodels if and only if $E \models\!\mid E \cap \neg\mathsf{Sen}^0(\Sigma)$.*

Proof. Let us first prove that for any Σ-models M and N

(1) if M has $\mathcal{D}$-size λ, N is $(\lambda^+,\mathcal{D})$-saturated, and $M[\mathsf{Sen}^0]N$, then $M \xrightarrow{\mathsf{Sen}^0} N$.

Let $(\Sigma_i \xrightarrow{\varphi_{i,j}} \Sigma_j)_{0\le i<j\le\lambda}$ be a $(\lambda,\mathcal{D})$-chain such that $\iota_\Sigma(M) = \varphi_{0,\lambda}$. By transfinite induction we define $(N_i)_{0\le i\le\lambda}$ such that $M_i[\mathsf{Sen}^0]N_i$ and $N_i\restriction_{\varphi_{0,i}} = N$, where $M_i = (M_M)\restriction_{\varphi_{i,\lambda}}$. If this were achieved, then $M_M[\mathsf{Sen}^0]N_\lambda$, and from this by noticing successively that $N_\lambda \models E_M$ (as $E_M \subseteq \mathsf{Sen}^0(\Sigma_M)$) and that $N_\lambda = N_h$ for $h = i_{\Sigma,M}(N_\lambda)\colon\ M \to N$, we obtain that $M \xrightarrow{\mathsf{Sen}^0} N$.

For any successor ordinal $\alpha+1 < \lambda$, let Γ be an arbitrarily finite subset of $M^*_{\alpha+1} \cap \mathsf{Sen}^0(\Sigma_{\alpha+1})$. Because I^0 has finite conjunctions we get that $\wedge\Gamma \in M^*_{\alpha+1} \cap \mathsf{Sen}^0(\Sigma_{\alpha+1})$.

Because Sen^0 is closed under existential $\mathcal{D}$-quantification we get that $(\exists\varphi_{\alpha,\alpha+1}) \wedge \Gamma \in \mathsf{Sen}^0(\Sigma_\alpha)$. Because $M_\alpha \models (\exists\varphi_{\alpha,\alpha+1}) \wedge \Gamma$ we get that $(\exists\varphi_{\alpha,\alpha+1}) \wedge \Gamma \in M_\alpha^* \cap \mathsf{Sen}^0(\Sigma_\alpha)$. Because $M_\alpha[\mathsf{Sen}^0]N_\alpha$ (cf. induction hypothesis), we obtain that $N_\alpha \models (\exists\varphi_{\alpha,\alpha+1}) \wedge \Gamma$, therefore since Γ was chosen arbitrary, N_α $\varphi_{\alpha,\alpha+1}$-realizes finitely $M_{\alpha+1}^* \cap \mathsf{Sen}^0(\Sigma_{\alpha+1})$. Because $\alpha+1 \leq \lambda$ and N is $(\lambda^+, \mathcal{D})$-saturated, it follows that N_α $\varphi_{\alpha,\alpha+1}$-realizes $M_{\alpha+1}^* \cap \mathsf{Sen}^0(\Sigma_{\alpha+1})$, so let $N_{\alpha+1}$ be such that $N_{\alpha+1}\restriction_{\varphi_{\alpha,\alpha+1}} = N_\alpha$ and $N_{\alpha+1} \models M_{\alpha+1}^* \cap \mathsf{Sen}^0(\Sigma_{\alpha+1})$.

For β a limit ordinal, let N_β be an amalgamation of $(N_\alpha)_{0\leq\alpha<\beta}$. Let $\rho_\beta \in M_\beta^* \cap \mathsf{Sen}^0(\Sigma_\beta)$. Because Sen^0 has finitary sentences, there exist $\alpha < \beta$ and $\rho_\alpha \in \mathsf{Sen}^0(\Sigma_\alpha)$ such that $\varphi_{\alpha,\beta}(\rho_\alpha) = \rho_\beta$. Then by the Satisfaction Condition $M_\alpha \models \rho_\alpha$ and by the induction hypothesis $N_\alpha \models \rho_\alpha$ too. By the Satisfaction Condition, this time in the other direction, we finally get that $N_\beta \models \rho_\beta$.

Now let us prove that

(2) if $M[\mathsf{Sen}^0]N$ then there exists N' such that $M \xrightarrow{\mathsf{Sen}^0} N' \equiv N$.

Let λ be a $\mathcal{D}$-size for M. Because I has $\mathcal{D}$-saturated models there exists a homomorphism $N \to N'$ such that N' is $(\lambda^+, \mathcal{D})$-saturated and $N' \equiv N$. By (1) we also have that $M \xrightarrow{\mathsf{Sen}^0} N'$.

Now we proceed to the proof of the conclusions of the theorem. We focus only on the hard part.

First let us consider a consistent Σ-theory E preserved by Sen^0-extensions. By the following (whose proof will be given later)

Lemma 8.3. *In any m-compact Boolean complete institution, for any consistent theory E and set Δ of sentences closed under finite disjunctions, the following are equivalent:*

- *$E \cap \Delta \models\!\!\!\!\dashv E$, and*
- *for all models M,N, $M \models E$ and $N \models M^* \cap \Delta$ implies $N \models E$.*

Because $\mathsf{Sen}^0(\Sigma)$ is closed under finite disjunctions it is enough to show that for arbitrary Σ-models, if $M \models E$ and $N \models M^* \cap \mathsf{Sen}^0(\Sigma)$ then $N \models E$.

From $N \models M^* \cap \mathsf{Sen}^0(\Sigma)$ we have $M[\mathsf{Sen}^0]N$, hence by (2) there exists N' such that $M \xrightarrow{\mathsf{Sen}^0} N' \equiv N$, therefore because E is preserved by Sen^0-extensions $N' \models E$, and thus $N \models E$.

The submodel part of the conclusion can be shown in a similar manner by setting the Δ of Lemma 8.3 to $\neg\mathsf{Sen}^0(\Sigma)$. Let M,N be models such that $M \models E$ and $N \models M^* \cap \neg\mathsf{Sen}^0(\Sigma)$. From the latter we get $M[\neg\mathsf{Sen}^0]N$ which means $N[\mathsf{Sen}^0]M$. By (2) there exists a model M' such that $N \xrightarrow{\mathsf{Sen}^0} M' \equiv M$. Thus $M' \models E$ and by the hypothesis that E is preserved by submodels we get that $N \models E$. This concludes the proof of this theorem. Now we present a proof of Lemma 8.3.

Proof of Lemma 8.3. We prove only the hard part, that the second item implies the first one. Let N be a model such that $N \models E \cap \Delta$. We want to prove that $N \models E$.

Let us first show that $E \cup \Gamma$ is consistent, where $\Gamma = N^* \cap \neg\Delta$. If $E \cup \Gamma$ were inconsistent, then by compactness there would exist $E_0 \subseteq E$ and $\Gamma_0 \subseteq \Gamma$ both finite such that $E_0 \cup \Gamma_0$ is inconsistent. Then $E \models \neg(\wedge\Gamma_0)$ (otherwise the consistency assumption for E is contradicted). If $\Gamma_0 = \{\neg\delta_k \mid 1 \leq k \leq n\}$ then $E \models \delta_1 \vee \cdots \vee \delta_n$. But $\delta_1 \vee \cdots \vee \delta_n = \delta \in \Delta$ since Δ is closed under finite disjunctions. Hence $\delta \in E \cap \Delta$, which implies $N \models \delta$. Therefore there exists $1 \leq k \leq n$ for which $N \models \delta_k$. Because $\neg\delta_k \in \Gamma_0 \subseteq N^*$ we have reached a contradiction.

As $E \cup \Gamma$ is consistent let M be one of its models. If we showed that $N \models M^* \cap \Delta$, then by the assumption of the second item we would get that $N \models E$. Therefore if there were $\delta \in M^* \cap \Delta$ such that $N \not\models \delta$, then $N \models \neg\delta$, hence $\neg\delta \in \Gamma$. Then $M \models \neg\delta$ which contradicts $\delta \in M^*$. $\square$

From the conditions of the preservation Thm. 8.2, except the condition on existence of saturated models all others are rather common conditions. In applications the existence of saturated models is handled by Thm. 7.9. The following is a typical concrete instance.

Corollary 8.4. *A* **FOL** *theory is preserved by closed sub-models, respectively extensions, if and only if it is presented by a set of universal, respectively existential, sentences.*

Proof. **FOL** has model amalgamation (Prop. 4.6), is compact (Cor. 6.22), has only finitary sentences (Fact 7.17), each model has a $\mathcal{D}$-size given by the cardinality of its carrier sets (Fact 7.15) for $\mathcal{D}$ the class of the signature extensions with a finite number of constants, and has $\mathcal{D}$-saturated models (Cor. 7.11). The conclusion follows by Thm. 8.2 and by Prop. 8.1 when taking Sen^0 to be Exist, the existential sentences. $\square$

Exercises

8.1. Elementary diagrams for Sen^0-submodels
For each signature Σ, the Sen^0-submodels form a subcategory $\mathsf{Mod}^0(\Sigma)$ of $\mathsf{Mod}(\Sigma)$. Moreover, the resulting sub-institution $(\mathbb{S}ig, \mathsf{Sen}^0, \mathsf{Mod}^0, \models)$ has elementary diagrams. (*Hint:* The elementary diagram of M is $(\Sigma_M, M_M^* \cap \mathsf{Sen}^0(\Sigma_M))$.)

8.2. The result of Prop. 8.1 can be changed in various ways by weakening the requirements on the model homomorphisms as follows. Let Sen^0 consist of the existential quantifications of sentences accessible by disjunction and conjunction from a class of sentences B (which is a parameter of the problem). For any **FOL** models M and N (of the same signature) $M \xrightarrow{\mathsf{Sen}^0} N$ if and only if there exists a homomorphism $h\colon M \to N$ which is

- just plain, when B consists of all the atoms,
- injective, when B consists of all the atoms and the negations of equational atoms, and
- closed, when B consists of all the atoms and the negations of relational atoms.

By instantiating the general preservation Thm. 8.2, formulate variants of the preservation results of Cor. 8.4 corresponding to the three situations above.

8.3. Preservation in PA
A **PA** sentence is existential, respectively universal, when it is an existential, respectively universal, quantification of a sentence which is accessible by Boolean connectives from existence equations. A **PA** theory is preserved by closed sub-algebras (see Ex. 4.57), respectively extensions, if and only if it is presented by a set of universal, respectively existential, sentences.

Provide variants of this result which correspond to the three situations from Ex. 8.2.

8.2 Axiomatizability by Ultraproducts

Recall that a class of models $\mathbb{M}$ for a signature is called *elementary* when it is closed, i.e., $\mathbb{M}^{**} = \mathbb{M}$. In other words elementary classes of models are the classes of models of theories.

Theorem 8.5. *In any institution that has negation and conjunction and whose sentences are preserved by ultraproducts, a class of models of a given signature is elementary if and only if it is closed under ultraproducts and elementary equivalence.*

Proof. The implication that any elementary class of models is closed under elementary equivalence and ultraproducts follows immediately from the hypothesis.

For the opposite implication, consider a class of models $\mathbb{M}$ closed under ultraproducts and elementary equivalence. Let $E = \mathbb{M}^*$. We prove that $\mathbb{M} = E^*$.

For any $B \in E^*$ let $\mathcal{P}_\omega(B^*)$ be the set of the finite subsets of $\{B\}^*$. For each $i \in \mathcal{P}_\omega(B^*)$, there exists $A_i \in \mathbb{M}$ such that $A_i \models i$. (Otherwise for all $A \in \mathbb{M}$, $A \models \neg \wedge i$ which implies that $\neg \wedge i \in E$, which further implies that $B \models \neg \wedge i$ which contradicts the fact that $B \models i$.) By the compactness Thm. 6.17, there exists an ultrafilter U over $\mathcal{P}_\omega(B^*)$ such that $\prod_U A_i \models \{B\}^*$. This implies that $\prod_U A_i \equiv B$ (otherwise if there exists a sentence e such that $\prod_U A_i \models e$ but $B \not\models e$, then $B \models \neg e$ which implies $\prod_U A_i \models \neg e$ contradicting $\prod_U A_i \models e$). Because $\mathbb{M}$ is closed under ultraproducts and elementary equivalence, we have that $B \in \mathbb{M}$. □

Finitely elementary classes. A class of models of a signature is *finitely elementary* when it is the class of models of a finite presentation.

Corollary 8.6. *Under the hypotheses of Thm. 8.5 above, the following are equivalent for a class $\mathbb{M}$ of Σ-models:*

- $\mathbb{M}$ *is finitely elementary, and*
- *both* $\mathbb{M}$ *and its complementary* $|\mathsf{Mod}(\Sigma)| \setminus \mathbb{M}$ *are elementary.*

Proof. If E is a finite set of Σ-sentences, then the complement of E^* is $(\neg \wedge E)^*$.

For the opposite implication, consider E^* an elementary class of models such that its complement is also elementary. We show that there exists $E_0 \subseteq E$ finite such that $E^* = E_0^*$. If we assume the contrary, then for each $E_0 \subseteq E$ finite there exists a model A in the complement of E^* such that $A \models E_0$. Because each sentence is preserved by ultraproducts, by the compactness Thm. 6.17, there exists an ultraproduct $\prod_U A_i$ over $\mathcal{P}_\omega(E)$ such that $\prod_U A_i \models E$ and for each $i \in \mathcal{P}_\omega(E)$, $A_i \models i$ but $A_i \notin E^*$. Because the complement of E^* is closed under ultraproducts, we also get that $\prod_U A_i \notin E^*$ which contradicts $\prod_U A_i \models E$. □

Axiomatizability in Keisler-Shelah institutions

Institutions admitting the Keisler-Shelah property permit a purely algebraic characterization of elementary equivalence as a corollary of Thm. 8.5.

Ultraradicals. Let us say that a model A is an *ultraradical* of a model B if B is an ultrapower of A.

Corollary 8.7. *In any Keisler-Shelah institution satisfying the conditions of Thm.8.5, the following are equivalent: for any class of models of a signature, the class*

- *is elementary,*
- *is closed under ultraproducts and ultraradicals,*
- *is closed under ultraproducts and its complement is closed under ultrapowers.*

A concrete example is given by **FOL** which we already know is a Łoś-institution (Cor. 6.12) and has the Keisler-Shelah property (Cor. 7.27).

Corollary 8.8. *In* **FOL***, a class of models of a given signature*

- *is elementary if and only if it is closed under ultraproducts and under ultraradicals, and*
- *is finitely elementary if and only if both it and its complement are closed under ultraproducts.*

Universal axiomatizability

The Keisler-Shelah property makes it possible to convert the general preservation result of Thm. 8.2 into an axiomatizability result.

Corollary 8.9 (Universal axiomatizability). *Further to the framework of Sect. 8.1 and the conditions of the preservation Thm. 8.2 let us also assume that*

1. *the institution has ultraproducts of models which are preserved by (the model reducts corresponding to) the elementary extensions,*
2. *all sentences of the institution are preserved by ultraproducts, and*
3. *the institution has the Keisler-Shelah property.*

Then the following are equivalent for a non-empty class of models of a signature:

- *it is closed under ultraproducts and* Sen^0*-submodels, and*
- *it is the class of models of a* $\neg\mathsf{Sen}^0$*-theory.*

Proof. We focus only on the hard implication. Let us consider a class $\mathbb{M}$ of models that is closed under ultraproducts and Sen^0-submodels.

First we show that $\mathbb{M}$ is just elementary. By Thm. 8.5 it is enough to show that $\mathbb{M}$ is closed under elementary equivalence. Let $M \equiv N \in \mathbb{M}$. By the Keisler-Shelah property there exists an ultrafilter such that $\prod_U M \cong \prod_U N$. As $\mathbb{M}$ is closed under ultraproducts, $\prod_U N \in \mathbb{M}$, hence $\prod_U M \in \mathbb{M}$ too. By Prop. 6.13, there exists an elementary homomorphism $M \to \prod_U M$. Because obviously each elementary homomorphism is a Sen^0-submodel too, and $\mathbb{M}$ is closed under Sen^0-submodels, we get that $M \in \mathbb{M}$.

Now we consider the theory $\mathbb{M}^*$. Because $\mathbb{M}$ is elementary we have that $\mathbb{M}^{**} = \mathbb{M}$. Note that $\mathbb{M}^*$ is preserved by Sen^0-submodels because $\mathbb{M}^{**} = \mathbb{M}$ and $\mathbb{M}$ is closed under Sen^0-submodels. By Thm. 8.2 we therefore have that $\mathbb{M}^* \cap \neg\mathsf{Sen}^0(\Sigma) \models\!\!\!\mid \mathbb{M}^*$. Because $\mathbb{M}^{**} = \mathbb{M}$, we have that $\mathbb{M} = (\mathbb{M}^* \cap \neg\mathsf{Sen}^0(\Sigma))^*$. $\square$

By taking Sen^0 to be the functor Exist of the existential sentences in **FOL**, we get the following concrete universal axiomatizability result.

Corollary 8.10. *A class of* **FOL** *models is the class of models of a universal theory (i.e., a theory presented by universal sentences) if and only if it is closed under ultraproducts and closed submodels.*

Exercises

8.4. Develop instances of the universal axiomatizability Cor. 8.9 in **FOL**, different from Cor. 8.10, based upon the preservation results of Ex. 8.2. Develop similar universal axiomatizability results in **PA** and other concrete institutions presented in the book.

8.3 Quasi-varieties and Initial Models

In this section we establish an important connection between existence of initial models for theories and the closure of the class of models of the theory to products and 'submodels'.

Subobjects in categories with inclusion systems. We have already introduced and used several notions of 'submodel', such as plain **FOL** submodels or closed **FOL** submodels (see Sect. 4.5). Both the simple and the closed concepts of **FOL** submodels are examples of the following general concept of 'subobject'.

In any category $\mathbb{C}$ with an inclusion system $\langle I, \mathcal{E}\rangle$, we say that an object A is an *I-subobject* of another object B if there exists an abstract inclusion $(A \hookrightarrow B) \in I$. When the inclusion system is fixed then we may simply say 'subobject' instead of 'I-subobject'.

An object A of $\mathbb{C}$ is *I-reachable* if and only if it has no I-subobjects which are different from A. The same as above, when $\langle I, \mathcal{E}\rangle$ is fixed we may simply say 'reachable' rather than 'I-reachable'. By varying the inclusion system of a category, one obtains different notions of reachability. For example, in the category of the Σ-models for a **FOL** signature, a reachable model in the strong inclusion system is reachable in the closed inclusion system too, but the other way around is not true.

Fact 8.11. *In any category $\mathbb{C}$ with a given inclusion system and which has an initial object $0_{\mathbb{C}}$*

- *each object A is reachable if and only if the unique arrow $0_{\mathbb{C}} \to A$ is an abstract surjection, and*
- *each object has exactly one reachable subobject.*

Quotient objects in categories with inclusion systems. The concept of quotient object can be seen as dual to that of subobject. In any category $\mathbb{C}$ with an inclusion system $\langle I, \mathcal{E}\rangle$, an object B is an *$\mathcal{E}$-quotient representation* of A if there exists an abstract surjection $A \to B$. An *$\mathcal{E}$-quotient* of A is an isomorphism class of $\mathcal{E}$-quotient representations. As for subobject, when the inclusion system is fixed we may simply say 'quotient' instead of '$\mathcal{E}$-quotient'.

We say that the inclusion system is $\langle I, \mathcal{E}\rangle$ *co-well-powered* if the category $\mathbb{C}$ is $\mathcal{E}$-co-well-powered. We may recall from Sect. 2.1 that this means the class of $\mathcal{E}$-quotients of each object is a *set*.

Quasi-varieties and varieties. In any category $\mathbb{C}$ with a given inclusion system and with small products, a class of objects of $\mathbb{C}$ closed under isomorphisms

- is a *quasi-variety* if it is closed under small products and subobjects, and
- is a *variety* if it is a quasi-variety closed under quotient representations.

Initial models of quasi-varieties

The existence of initial models of quasi-varieties can be obtained at the very general level of abstract categories with inclusion systems.

Proposition 8.12. *Consider a category $\mathbb{C}$ with an initial object $0_{\mathbb{C}}$, small products, and with a co-well-powered epic inclusion system. Each quasi-variety Q of $\mathbb{C}$ has a reachable initial object.*

Proof. Let $\{A_i \mid i \in I\}$ be the class of all reachable subobjects of all objects of Q. Then we consider a subclass of indices $I' \subseteq I$ such that there are no isomorphic objects in $\{A_i \mid i \in I'\}$ and for each $i \in I$ there exists $j \in I'$ such that $A_i \simeq A_j$. I' is a *set* because the inclusion system of $\mathbb{C}$ is co-well-powered and because we know that for each reachable object B the unique arrow $0_{\mathbb{C}} \to B$ is abstract surjection (Fact 8.11). Let 0_Q be the reachable subobject of the product $\prod_{j\in I'} A_j$ (see Fact 8.11). We prove that 0_Q is initial in Q.

$$0_Q \longrightarrow \prod_{j\in I'} A_j \xrightarrow{p_j} A_j \cong A_i \longrightarrow A.$$

For each object A of Q, there exists $i \in I$ such that A_i is a reachable subobject of A. Then there exists $j \in I'$ such that A_i is isomorphic to A_j, therefore there exists an arrow $\prod_{j\in I'} A_j \to A$. Because 0_Q is a subobject of the product $\prod_{j\in I'} A_j$, there exists an arrow $0_Q \to A$. Because 0_Q is reachable, the unique arrow $0_{\mathbb{C}} \to 0_Q$ is abstract surjection, which is also epi because the inclusion system is epic. This implies the uniqueness of the arrow $0_Q \to A$. $\square$

Prop. 8.12 provides a rather simple way for showing the existence of initial models of Horn theories. The example below extends the corresponding **FOL** result of Cor. 4.28 to infinitary Horn sentences. Recall that an infinitary universal Horn (S,F,P)-sentence is a sentence of the form $(\forall X)H \Rightarrow C$ where X is a set of first order variables (i.e., new constants), H is the conjunction of any set of $(S,F\uplus X,P)$-atoms and C is an (S,F,P)-atom.

Corollary 8.13. *For any* **FOL** *signature* (S,F,P)*, any set of infinitary universal Horn* (S,F,P)*-sentences has an initial model.*

Proof. Let Γ be a set of infinitary universal Horn (S,F,P)-sentences. Because there exists the initial (S,F,P)-model (cf. Prop. 4.27), by Prop. 8.12 it is enough to show that Γ^* is a quasi-variety. For this we consider the closed inclusion system for the categories of **FOL** models. Recall from Sect. 4.5 that in the closed inclusion system the abstract surjections are the surjective homomorphisms and the abstract inclusions are the closed submodels.

Let $(A_i)_{i\in I}$ be a family of (S,F,P)-models satisfying Γ. We have to prove that the product $\prod_{i\in I} A_i$ satisfies each sentence $(\forall X)H \Rightarrow C$ of Γ. Let A' be any $(S,F\uplus X,P)$-expansion of A which satisfies H. Each projection $p_i\colon\ \prod_{i\in I} A_i \to A_i$ lifts uniquely to $p'_i\colon\ A' \to A'_i$ and $(p'_i)_{i\in I}$ is a product cone. Then each $A'_i \models H$ and since each $A_i \models (\forall X)H \Rightarrow C$ we have that each $A'_i \models C$. Then $A' = \prod_{i\in I} A'_i \models C$.

Let $B \hookrightarrow A$ be a closed submodel of a model A which satisfies Γ. For any $(\forall X)H \Rightarrow C$ in Γ, let B' be a $(S,F\uplus X,P)$-expansion of B such that $B' \models H$. Let A' be the $(S,F\uplus X,P)$-expansion of A such that $A'_x = B'_x$ for each $x \in X$. This gives a closed submodel $B' \hookrightarrow A'$ for $(S,F\uplus X,P)$. Then $A' \models H$ and consequently $A' \models C$. Since B' is a closed submodel of A' we get that $B' \models C$. □

Note that the proof of the existence of initial models of Horn theories given by Cor. 8.13 is simpler than the proof provided by Cor. 4.28 in the sense that it avoids construction of the congruence $=_\Gamma$ and of the quotient of the initial (S,F,P)-model by $=_\Gamma$.

Liberality via quasi-varieties. Cor. 4.30 showed that the existence of initial models of theories is the essential factor for the liberality of institutions. By using Prop. 8.12 it can be reformulated as follows:

Corollary 8.14. *Consider a semi-exact institution with pushouts of signatures and with elementary diagrams such that for each signature the category of models has an initial model, small products, and a co-well-powered epic inclusion system. If the class of models of each theory is a quasi-variety, then the institution is liberal.*

The following typical concrete instance of Cor. 8.14 has already been obtained as Cor. 4.32, now being obtained via the existence of initial models of quasi-varieties.

Corollary 8.15. *The institution* **HCL** *is liberal.*

Equivalence between quasi-varieties and existence of initial models. We have seen (Prop. 8.12) that quasi-varieties have initial models and that this holds in the very general setting of abstract categories. The following establishes the equivalence between the class of models of a theory being quasi-variety and the theory having initial models. This rather remarkable result needs some model theoretic infrastructure.

Theorem 8.16. *Consider an institution with elementary diagrams* ι *such that*

1. *for each signature* Σ *the category of* Σ*-models*
   (a) *has an initial object* 0_Σ,
   (b) *has small products, and*
   (c) *has a co-well-powered epic inclusion system,*

   and

2. *the model reduct functors corresponding to the elementary extensions*
   (a) *preserve the abstract inclusions and the abstract surjections and*
   (b) *reflect identities, i.e., if* $\mathsf{Mod}(\iota_\Sigma(M))(h)$ *is identity then* h *is identity.*

Each theory has a reachable initial model if and only if the class of models of each theory is a quasi-variety.

Proof. Because of Prop. 8.12 we have to prove only one implication, that if a theory (Σ, E) has reachable initial models, then its class of models form a quasi-variety.

We first show the preservation by submodels. Let (Σ, E) be a theory and consider $N \hookrightarrow M$ a submodel of a (Σ, E)-model M. We prove that $N \models_\Sigma E$. Let $h\colon N_N \to M_N$ be the (Σ_N, E_N)-homomorphism $i^{-1}_{\Sigma,N}(N \hookrightarrow M)$. Let us factor $h = e_h; i_h$ in the inclusion system of $\mathsf{Mod}(\Sigma_N)$ with e_h abstract surjection and i_h abstract inclusion. Because the reduct functor $\mathsf{Mod}(\iota_\Sigma(N))$ preserves both the abstract inclusions and the abstract surjections, $N \hookrightarrow M$ gets factored as $e_h\restriction_{\iota_\Sigma(N)}; i_h\restriction_{\iota_\Sigma(N)}$ in the inclusion system of $\mathsf{Mod}(\Sigma)$. Because $B \hookrightarrow A$ is abstract inclusion, we deduce that $e_h\restriction_{\iota_\Sigma(N)} = 1_N$, which, by the condition that $\mathsf{Mod}(\iota_\Sigma(B))$ reflects identities, means that e_h is identity. Therefore $h = i_h$ which means h is abstract inclusion.

By the Satisfaction Condition $M \models_\Sigma E$ implies that $M_N \models_{\Sigma_N} E'$, where $E' = \iota_\Sigma(N)(E)$. Let $f\colon 0_{\Sigma_N, E_N \cup E'} \to M_N$ be the unique model homomorphism from the initial $(\Sigma_N, E_N \cup E')$-model. Because $h\colon N_N \to M_N$ is abstract inclusion and $N_N \to 0_{\Sigma_N, E_N \cup E'}$ is abstract surjection (by the reachability of $0_{\Sigma_N, E_N \cup E'}$), by factoring f in the inclusion system of $\mathsf{Mod}(\Sigma_N)$ it follows that N_N and $0_{\Sigma_N, E_N \cup E'}$ are isomorphic. This means that $N_N \models_{\Sigma_N} E'$, which by the Satisfaction Condition implies $N \models_\Sigma E$.

For the preservation by products, consider $(N \xrightarrow{p_i} M_i)_{i \in I}$ a product of Σ-models for a signature Σ such that $M_i \models E$ for each $i \in I$. We have to prove that $N \models E$. By the canonical isomorphism $i_{\Sigma,N}\colon \mathsf{Mod}(\Sigma_N, E_N) \to N/\mathsf{Mod}(\Sigma)$ and because the forgetful functor $N/\mathsf{Mod}(\Sigma) \to \mathsf{Mod}(\Sigma)$ reflects the products, we have that $(N_N \xrightarrow{(p_i)_N} (M_i)_N)_{i \in I}$ is a product in $\mathsf{Mod}(\Sigma_N, E_N)$, where $(p_i)_N = i^{-1}_{\Sigma,N}(p_i\colon 1_N \to p_i)$ for each $i \in I$.

$$\begin{tikzcd}
N_N \arrow[r, "(p_i)_N"] & (M_i)_N \\
0_{\Sigma_N, E_N \cup E'} \arrow[u, "\cong"] \arrow[ur] &
\end{tikzcd}$$

By the Satisfaction Condition $(M_i)_N \models E'$ for each $i \in I$, where $E' = \iota_\Sigma(N)(E)$. Therefore we get a unique Σ_N-model homomorphism $0_{\Sigma_N, E_N \cup E'} \to (M_i)_N$ for each $i \in I$. By the universal property of products, this gives a Σ_N-model homomorphism $0_{\Sigma_N, E_N \cup E'} \to N_N$. Because we already have a homomorphism $N_N \to 0_{\Sigma_N, E_N \cup E'}$, by the universal property of the initial objects N_N and $0_{\Sigma_N, E_N \cup E'}$, we have that N_N and $0_{\Sigma_N, E_N \cup E'}$ are isomorphic. This implies that $N_N \models_{\Sigma_N} E'$, which by the Satisfaction Condition gives that $N \models_\Sigma E$. □

Because concrete elementary extensions are usually signature extensions with constants, which are quasi-representable, the condition that model reducts corresponding to elementary extensions reflects the identities can be handled easily in the applications of Thm. 8.16 by the following rather general fact.

Fact 8.17. *The model reducts corresponding to quasi-representable signature morphisms reflect identities.*

The other conditions of Thm. 8.16 are also rather easy to check in the applications as suggested by the following typical example.

Corollary 8.18. *In* **FOL***, both in the case of the strong and of the closed inclusion systems for categories of models, a theory has a reachable initial model if and only if the class of its models forms a quasi-variety.*

Exercises

8.5. Any intersection of quasi-varieties is a quasi-variety. Any intersection of varieties is a variety.

8.4 Quasi-Variety Theorem

In Cor. 8.13 we have seen that the **FOL**-models of infinitary Horn sentences form quasi-varieties. In this section we will see that this holds more generally in institutions. We will also establish the more difficult opposite implication, which constitutes the axiomatizability result for quasi-varieties, that each quasi-variety is the class of models of a set of Horn sentences.

The framework. The general concept of a Horn sentence as a sentence of the form $(\forall\chi)E \Rightarrow E'$ with χ being a representable signature morphism from a designated class $\mathcal{D}$ of signature morphisms, E being a set of epi basic sentences, and E' being a set of basic sentences, is too lax for the purpose of this section mainly because basic sentences capture significantly more than the atoms of the institutions (recall that existentially quantified atoms are also basic in **FOL** and other institutions). Although epi basic sentences might constitute a better abstract capture for the atoms of concrete institutions, we do not have any guarantee that in each situation each epi basic sentence is ‘atomic’. The solution to this problem is to consider a designated sub-class of the class of general Horn sentences as a parameter for our framework. Therefore for this section we introduce a framework consisting of the following additional data for the institution:

- a class $\mathcal{D}$ of representable signature morphisms,
- a system of elementary diagrams ι for the institution such that each elementary extension $\iota_\Sigma(M) \in \mathcal{D}$,
- a sub-functor Horn of Sen, i.e., Horn : $\mathbb{S}ig \to \mathbb{S}et$ such that $\mathsf{Horn}(\Sigma) \subseteq \mathsf{Sen}(\Sigma)$ and $\varphi(\mathsf{Horn}(\Sigma)) \subseteq \mathsf{Horn}(\Sigma')$ for each signature morphism $\varphi\colon\ \Sigma \to \Sigma'$, such that each sentence of Horn is (semantically equivalent to) a $\mathcal{D}$-universal Horn sentence, and
- for each signature Σ the category of the Σ-models has products and a designated co-well-powered inclusion system.

A typical example is to consider $\mathcal{D}$ the class of all **FOL**-signature extensions with constants and for each **FOL**-signature Σ the set $\mathsf{Horn}(\Sigma)$ to be the set of all infinitary Horn sentences $(\forall X)H \Rightarrow C$ (with X being a set of variables, H [the conjunction of] a set of **FOL**-atoms, and C a **FOL**-atom). The finitary variant of this, i.e., when H is a finite conjunction of atoms, is also an example.

Models of Horn sentences form quasi-varieties

Theorem 8.19. *In any institution such that*

(QP1) *the abstract surjections are preserved by $\mathcal{D}$-reducts (i.e., the model reducts corresponding to signature morphisms of $\mathcal{D}$), and*

(QP2) *for each* Horn*-sentence $(\forall\chi)E \Rightarrow E'$ the canonical model homomorphism $M_E \to M_{E\cup E'}$ is an abstract surjection,*

the models of any Horn*-sentence form a quasi-variety.*

Proof. From Prop. 5.25 we know that for each Horn sentence $(\forall\chi)E \Rightarrow E'$ there exists a model homomorphism h such that for each model M,

$$M \models (\forall\chi)E \Rightarrow E' \text{ if and only if } M \models^{\mathrm{inj}} h \text{ (i.e., } M \text{ is injective with respect to } h) \ .$$

Moreover, the above model homomorphism h is a χ-reduct of the canonical model homomorphism $M_E \to M_{E\cup E'}$, hence by $(QP1-2)$ it is an abstract surjection.

The conclusion follows by the general categorical result below:

Lemma 8.20. *For each abstract surjection e of an inclusion system in a category with products, the class $Inj(e)$ of the objects injective with respect to e form a quasi-variety.*

Proof. Consider a family of objects $\{A_j\}_{j\in J} \subseteq Inj(e)$ and let $\{A \xrightarrow{p_j} A_j\}_{j\in J}$ be their product. We prove that $A \in Inj(e)$. Let $f\colon\ dom(f) \to A$. For each index $j \in J$, let g_j be such that $e;g_j = f;p_j$. Let g be the unique arrow such that $g;p_j = g_j$ for each index $j \in J$. For each $j \in J$, $e;g;p_j = e;g_j = f;p_j$. By the uniqueness aspect of the universal property of products, we obtain that $e;g = f$.

$$
\begin{tikzcd}
 & \bullet \arrow[d, "g"] \arrow[dr, "g_j"] & \\
\bullet \arrow[ur, "e"] \arrow[r, "f"'] & A \arrow[r, "p_j"'] & A_j
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[dl, "h"'] \arrow[d, "\exists g"] \\
A \arrow[r, "i"'] & B
\end{tikzcd}
$$

Now consider a subobject $i\colon\ A \hookrightarrow B$ for $B \in Inj(e)$. We prove that $A \in Inj(e)$ too. Let $f\colon\ dom(f) \to A$. Because $B \in Inj(e)$, there exists g such that $e;g = f;i$. By Diagonal-fill Lemma 4.16 there exists h such that $e;h = f$. □

Thus we have completed the proof of the theorem. □

The FOL example. For the **FOL** models the above Thm. 8.19 gives two different instances corresponding to the two inclusion systems for model categories. One of them has already been proved in Cor. 8.13. The difference between these two is determined by the condition $(QP2)$, and each of them corresponds to a different choice for the subfunctor Horn.

Corollary 8.21. *For any* **FOL** *signature*

1. *the models of any set of infinitary Horn sentences form a quasi-variety for the closed inclusion system, and*
2. *the models of any set of infinitary Horn sentences* $(\forall X)H \Rightarrow C$ *for which the conclusion* C *is an* equational *atom form a quasi-variety for the strong inclusion system.*

Note that condition $(QP2)$ rules out the possibility of extending Horn to sentences of the form $(\forall X)H \Rightarrow C$ with C being an existentially quantified atom.

Each quasi-variety is axiomatizable by Horn sentences

The following constitutes the more difficult implication of the equivalence between quasi-varieties and classes of models of Horn sentences.

Theorem 8.22 (Quasi-variety). *In any institution such that*

(QA1) *the inclusion systems of the model categories are epic,*

(QA2) *each abstract surjection (of models) is* ι*-conservative, and*

(QA3) *for any abstract surjection (of models)* $h\colon\ A \to B$*, the 'internal' sentence* $(\forall \iota_\Sigma(A))E_A \Rightarrow \iota_\Sigma(h)^{-1}(E_B^{**})$ *is semantically equivalent to a set of* Horn*-sentences,*

any quasi-variety is the class of models of a set of Horn*-sentences.*

Proof. From Prop. 5.28 we know that for each ι-conservative model homomorphism h,

$$M \models^{\text{inj}} h \ \text{ if and only if } \ M \models (\forall \iota_\Sigma(A))E_A \Rightarrow \iota_\Sigma(h)^{-1}(E_B^{**}).$$

By $(QA1-3)$ the problem is reduced to the following categorical lemma:

Lemma 8.23. *In any category with products and with a co-well-powered epic inclusion system, for each quasi-variety Q there exists a class E of surjections such that $Q = Inj(E)$.*

Proof. Let us define $E = \{e \text{ abstract surjection } \mid Q \subseteq Inj(e)\}$. We notice immediately that $Q \subseteq Inj(E)$, therefore we have to prove only that $Inj(E) \subseteq Q$. Consider $A \in Inj(E)$. We prove that $A \in Q$.

Because the inclusion system is co-well-powered we can chose a 'complete' *set* $\{A \xrightarrow{e^j} M_j \in Q\}_{j \in J}$ of quotient representatives of A in Q in the sense that for each quotient representative $e : A \to cod(e) \in Q$ there exists an isomorphism γ and some $j \in J$ such that $e;\gamma = e^j$.

Let $\{M \xrightarrow{p_j} M_j\}_{j \in J}$ be the product of $\{M_j\}_{j \in J}$. Notice that $M \in Q$ since Q as a quasi-variety is closed under products. By the universal property of the products let $f : A \to M$ be the unique arrow such that $f;p_j = e^j$ for each $j \in J$, and let us factor $f = e_f;i_f$ such that e_f is abstract surjection and i_f is abstract inclusion. We prove that $Q \subseteq Inj(e_f)$.

$$\begin{tikzcd}
 & g(A) \arrow[dr, "i_g"] & \\
M_l \arrow[ur, "\gamma"] & A \arrow[l, "e^l"'] \arrow[u, "e_g"'] \arrow[r, "g"] \arrow[dl, "f"'] \arrow[d, "e_f"] & B \\
M \arrow[u, "p_l"] & f(A) \arrow[l, "i_f"] \arrow[ur, "i_f;p_l;\gamma;i_g"'] &
\end{tikzcd}$$

Consider any $B \in Q$ and $g : A \to B$. Let $g = e_g;i_g$ with e_g abstract surjection and i_g abstract inclusion. Then $g(A) \in Q$ since Q is closed under subobjects. Because $\{e^j\}_{j \in J}$ is 'complete', there exists $l \in J$ and an isomorphism γ such that $e_g = e^l;\gamma$. Notice that $e_f;(i_f;p_l;\gamma;i_g) = g$, hence $B \models^{\text{inj}} e_f$, which means $B \in Inj(e_f)$.

Therefore $Q \subseteq Inj(e_f)$, which by the definition of E implies $e_f \in E$. Because $A \in Inj(E)$ this means that $A \models^{\text{inj}} e_f$. If we consider $1_A : A \to A$, then we get m such that $e_f;m = 1_A$ which, because e_f is epi (the inclusion system being epic), implies that e_f is an isomorphism. Because $f(A) \in Q$ (since $M \in Q$ and Q is closed under subobjects), we deduce $A \in Q$. □

Thus we have completed the proof of the theorem. □

Now we can put together both implications given by the Theorems 8.19 and 8.22.

Corollary 8.24. *In any institution satisfying $(QP1-2)$ and $(QA1-3)$ a class of models of a signature is a quasi-variety if and only if it is the class of models of a set of* Horn-*sentences.*

While the conditions $(QA1-2)$ do not really narrow the applicability of Quasi-variety Thm. 8.22 and its Corollary 8.24, the conjunction between $(QA3)$ and $(QP2)$ may eliminate some apparent possible applications as illustrated by the following example.

Quasi-varieties in FOL. In the example of **FOL** models, the conjunction between $(QA3)$ and $(QP2)$ eliminates an apparent instance of Cor. 8.24 corresponding to the choice of the inclusion systems for the categories of models. Hence the only possible choice remains that of the closed inclusion systems. Note that the condition $(QA3)$ holds by the semantical equivalence given by the result below (its proof is left to the reader).

Lemma 8.25. *Let Σ be a signature in* **FOL***, which is considered with its standard system of elementary diagrams (see Sect. 4.4), and $h\colon A \to B$ be a surjective Σ-model homomorphism. Then*

$$(\forall \iota_\Sigma(A))E_A \Rightarrow \iota_\Sigma(h)^{-1}(E_B^{**}) \models\!\mid \{(\forall \iota_\Sigma(A))E_A \Rightarrow \rho \mid B_B \models \iota_\Sigma(h)(\rho),\ \rho \ \textit{atom}\}.$$

Therefore the **FOL** instance of Cor. 8.24 is as follows.

Corollary 8.26. *For any* **FOL** *signature a class of models is a quasi-variety for the closed inclusion system if and only if it is the class of models of a set of infinitary Horn sentences.*

Exercises

8.6. Axiomatizability for quasi-varieties of partial algebras

(a) As an instance of Cor. 8.24, a class of partial algebras is

axiomatizable by	iff it is closed under
QE_2-sentences	products and (plain) subalgebras
QE-sentences	products and closed subalgebras

(*Hint:* Use Ex. 4.57.)

(b) A result similar to (a) for QE_1 fails on the condition $(QP2)$.

(c) In **PA** each morphism between presentations of universal quasi-existence equations is liberal.

8.5 Birkhoff Variety Theorem

In the framework of the previous Sect. 8.4, instead of the sentence subfunctor Horn let us consider

- a sentence subfunctor $\mathsf{UA}\colon \mathbb{S}ig \to \mathbb{S}et$ such that each UA-sentence is semantically equivalent to a sentence of the form $(\forall\chi)E'$ with $\chi \in \mathcal{D}$ and E' being a set of basic sentences.

A typical example for UA is given by the universally quantified **FOL**-atoms.

Models of 'universal atoms' form varieties

Proposition 8.27. *In any institution satisfying $(QP1-2)$ (of Thm. 8.19) and such that*

(VP) *any abstract surjection $f\colon A \to B$ in $\mathsf{Mod}(\Sigma)$ can be χ-expanded to a Σ'-model homomorphism $A' \to B'$ for each signature morphism $\chi \in \mathcal{D}$ and each χ-expansion B' of B,*

the models of any UA*-sentence form a variety.*

Proof. The models of any UA-sentence form a quasi-variety because the subfunctor UA fulfills the conditions required by the subfunctor Horn in Thm. 8.19. (Note that here in the condition $(QP2)$ we should read UA instead of Horn.) We therefore need only to prove that UA-sentences are 'preserved' by the abstract surjections.

Let $h\colon M \to N$ be an abstract surjection and $M \models (\forall\chi)E'$ where $\chi \in \mathcal{D}$ and E' is basic. For any χ-expansion N' of N by (VP) we get a χ-expansion $h'\colon M' \to N'$ of h. Then $M' \models E'$ and because E' is basic we have that $N' \models E'$ too. This shows $N' \models (\forall\chi)E'$. $\square$

The condition (VP) is easy to check in the applications as shown by the following example.

The FOL example. Let us consider any of the closed or the strong inclusion systems in **FOL**, and $\mathcal{D}$ the class of signature extensions with constants. For both inclusion systems considered the abstract surjections are also surjective as functions. We have that for each $(\chi\colon \Sigma \to \Sigma \uplus X) \in \mathcal{D}$ and each surjective Σ-model homomorphism $h\colon M \to N$, for any χ-expansion N' of N for each $x \in X$ let us pick any $M'_x \in N'_x$. This lifts h to a Σ'-homomorphism $M' \to N'$.

This gives the following continuation of Cor. 8.21 to varieties.

Corollary 8.28. *For any* **FOL** *signature*

1. *the models of any set of universally quantified atoms form a variety for the closed inclusion system, and*
2. *the models of any set of universally quantified equations form a variety for the strong inclusion system.*

Each variety is axiomatizable by 'universal atoms'

The following constitutes the more difficult implication of the equivalence between varieties and classes of models of 'universal atoms'.

Theorem 8.29 (Birkhoff variety). *In any institution satisfying $(QA2)$ (of Thm. 8.22) and such that*

(VA1) *for each model A, $i_{\iota_\Sigma(A)}\colon \mathsf{Mod}(\Sigma_A) \to M_{\iota_\Sigma(A)}/\mathsf{Mod}(\Sigma)$ maps the initial (Σ_A, E_A)-model A_A to an abstract surjection $M_{\iota_\Sigma(A)} \to A$, and*

(VA2) *for any abstract surjection (of models) $h\colon A \to B$, the 'internal' sentence $(\forall\iota_\Sigma(A))\iota_\Sigma(h)^{-1}(E_B^{**})$ is semantically equivalent to a set of UA-sentences,*

any variety is the class of models of a set of UA-sentences.

Proof. For a given signature Σ let us consider the class of all representations of the signature morphisms $\Sigma \to \Sigma'$ which belong to $\mathcal{D}$, i.e.,

$$\mathcal{K} = \{M_\chi \mid (\chi\colon \Sigma \to \Sigma') \in \mathcal{D}\}.$$

Because of $(VA1)$ we can apply Lemma 8.30 below, and obtain that for each variety V there exists a class E of abstract surjections with domains in $\mathcal{K}$ such that $V = Inj(E)$.

By $(QA2)$ each abstract surjection is ι-conservative, therefore by Cor. 5.30, for each $(h:\ M_\chi \to B) \in E$,

$$M \models^{\mathrm{inj}} h \ \text{ if and only if } \ M \models_\Sigma (\forall \iota_\Sigma(M_\chi))\iota_\Sigma(h)^{-1}(E_B^{**}).$$

Now by $(VA2)$ we obtain the conclusion of the Variety Theorem.

Given a class $\mathcal{K} \subseteq |\mathbb{C}|$ of objects in a category $\mathbb{C}$ with an inclusion system, f is a $\mathcal{K}$*-surjection* when it is an abstract surjection and $dom(f) \in \mathcal{K}$.

Lemma 8.30. *In a category with products and a co-well-powered inclusion system, let $\mathcal{K}$ be a class of objects such that for each object A of the category there exists an abstract surjection $A' \to A$ with $A' \in \mathcal{K}$. Then for each variety V there exists a class E of $\mathcal{K}$-surjections such that $V = Inj(E)$.*

Proof. Let us define $E = \{e\ \mathcal{K}\text{-surjection} \mid V \subseteq Inj(e)\}$. We notice immediately that $V \subseteq Inj(E)$, therefore we have to prove only that $Inj(E) \subseteq V$. Consider $A \in Inj(E)$. We will prove that $A \in V$.

There exists an object $A' \in \mathcal{K}$ and an abstract surjection f_A such that $f_A:\ A' \to A$. Similarly to the argument in the proof of Lemma 8.23, because the inclusion system is co-well-powered we can chose a 'complete' *set* $(e^j:\ A' \to M_j \in V)_{j \in J}$ of quotient representatives of A' in V in the sense that for each quotient representative $e:\ A' \to B \in V$ there exists an isomorphism γ and some $j \in J$ such that $e;\gamma = e^j$.

Let $(p_j:\ M \to M_j)_{j \in J}$ be the product of $(M_j)_{j \in J}$. Notice that $M \in V$ since V as a variety is closed under products. By the universal property of the products let $f:\ A' \to M$ be the unique arrow such that $f;p_j = e^j$ for each $j \in J$, and let us factor $f = e_f;i_f$ such that e_f is abstract surjection and i_f is abstract inclusion. Similarly to the proof of Lemma 8.23, we can prove that $e_f \in E$.

$$\begin{tikzcd}
 & M_j & \\
A' \arrow[ur, "e^j"] \arrow[r, "e_f"'] \arrow[rr, bend left, "f"] \arrow[dr, "f_A"'] & f(A') \arrow[r, "i_f"'] \arrow[d, "m"] & M \arrow[ul, "p_j"'] \\
 & A &
\end{tikzcd}$$

$e_f \in E$ and $A \in Inj(E)$ imply that there exists m such that $f_A = e_f;m$. Because both e_f and f_A are abstract surjections, we deduce that m is an abstract surjection too. Because V is a variety we have successively that $M \in V$, $f(A') \in V$, and $A \in V$. □

Thus we have completed the proof of the theorem. □

Now we can put together both implications given by Prop. 8.27 and Thm. 8.29.

Corollary 8.31. *In any institution satisfying $(QP1-2)$, $(QA2)$, (VP), and $(VA1-2)$ a class of models of a signature is a variety if and only if it is the class of models of a set* UA*-sentences.*

Conditions $(VA1-2)$ can be checked rather easily in concrete applications as suggested by the following example.

Varieties in FOL. For each (S,F,P)-model A, if we denote by $|A|$ the (many-sorted) carrier set of A, then $i_{\iota_{(S,F,P)}(A)}(A_A)$ is the unique extension of the identity function $A \to A$ to an (S,F,P)-model homomorphism $T_F(|A|) \to A$ from the free (term) (S,F,P)-model $T_F(|A|)$. This is surjective but not strong, hence it is an abstract surjection only for the closed inclusion system for models. This eliminates the **FOL** variant corresponding to the strong inclusion systems from the possible instances of Cor. 8.31.

Condition $(VA2)$ is fulfilled by the **FOL** example with UA being the universally quantified atoms because of the semantic equivalence below (its rather simple proof is left to the reader).

Lemma 8.32. *Let Σ be a signature in* **FOL***, which is considered with its standard system of elementary diagrams (see Sect. 4.4), and $h\colon\ A \to B$ be a surjective Σ-model homomorphism. Then*

$$(\forall\iota_\Sigma(A))\iota_\Sigma(h)^{-1}(E_B^{**}) \models\!\!\mid \{(\forall\iota_\Sigma(A))\rho \mid B_B \models \iota_\Sigma(h)(\rho),\ \rho\ \ atom\}.$$

Therefore we can now formulate the following:

Corollary 8.33. *For any* **FOL** *signature a class of models is a variety for the closed inclusion system if and only if it is the class of models of a set of universally quantified atoms.*

Exercises

8.7. Axiomatizability for varieties of partial algebras
As an instance of the general Birkhoff variety Thm. 8.29, we establish that for any **PA** signature each class of models that is closed under products, closed submodels, and epi homomorphic images (see also Ex. 4.57) is the class of models of a set of universally quantified existence equations. However the corresponding preservation result fails because not every universally quantified existence equation is preserved by any epi homomorphism. At the general level this failure is reflected as a failure of the condition *(VP)* for the epi homomorphisms of partial algebras.

8.6 General Birkhoff Axiomatizability

We have already proved a series of axiomatizability results. In this section we show how they can be captured uniformly by introducing the new concept of 'Birkhoff institution'.

First let us recall some facts about relations and establish a couple of useful notations.

Application of relations. Given a binary relation $R \subseteq A \times B$, for each $A' \subseteq A$ let

$$R(A') = \{b \mid \langle a,b\rangle \in R, a \in A'\}.$$

Let us also recall

- that the composition of binary relations $R \subseteq A \times B$ and $R' \subseteq B \times C$ is defined by

  $$R;R' = \{\langle a, c\rangle \mid \text{ there exists } b \text{ such that } \langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in R'\}$$

  and
- that the inverse R^{-1} of a binary relation $R \subseteq A \times B$ is defined by

  $$R^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in R\}.$$

Relations induced by classes of arrows. Any class of arrows H in a category $\mathbb{C}$ determines a (class) relation $\xrightarrow{H} \subseteq |\mathbb{C}| \times |\mathbb{C}|$ by

$$a \xrightarrow{H} b \text{ if there exists an arrow } h\colon\ a \to b \in H.$$

The inverse $(\xrightarrow{H})^{-1}$ is denoted by $\xleftarrow{H}$.

Relations induced by classes of arrows. Any class of arrows H in a category $\mathbb{C}$ determines a (class) relation $\xrightarrow{H} \subseteq |\mathbb{C}| \times |\mathbb{C}|$ by

$$a \xrightarrow{H} b \text{ if there exists an arrow } h\colon\ a \to b \in H.$$

The inverse $(\xrightarrow{H})^{-1}$ is denoted by $\xleftarrow{H}$.

Quasi-variety theorem revisited. We reformulate the conclusion of Thm. 8.22:

Theorem 8.34 (Quasi-variety). *Under the conditions* $(QP1-2)$ *(of Thm. 8.19) and* $(QA1-3)$ *(of Thm. 8.22), for any class of* Σ*-models* $\mathbb{M}$*,*

$$(\mathbb{M}^* \cap \mathsf{Horn}(\Sigma))^* = \xleftarrow{I} (\mathrm{P}\mathbb{M})$$

where I *is the class of abstract inclusions for the* Σ*-models, and* $\mathrm{P}\mathbb{M}$ *is the class of all (small) products of models of* $\mathbb{M}$*.*

Proof. By the following general categorical lemma (whose proof is left as an exercise for the reader)

Lemma 8.35. *For any inclusion system* $\langle I, \mathcal{E}\rangle$ *in a category* $\mathbb{C}$ *with small products, the* products preserve the abstract inclusions, *i.e., if* $\{f_i\colon\ M_i \to N_i\}_{i\in I}$ *are abstract inclusions then there exists a product* $\prod_{i\in I} f_i\colon\ \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ *which is abstract inclusion.*

We have that $\mathrm{P}(\xleftarrow{I} \mathbb{M}) \subseteq \xleftarrow{I} (\mathrm{P}\mathbb{M})$ which means that $\mathrm{P}(\xleftarrow{I} (\mathrm{P}\mathbb{M})) \subseteq \xleftarrow{I} (\mathrm{PP}(\mathbb{M})) = \xleftarrow{I} (\mathrm{P}\mathbb{M})$. Also $\xleftarrow{I} (\xleftarrow{I} (\mathrm{P}\mathbb{M})) = \xleftarrow{I} (P\mathbb{M})$. Therefore $\xleftarrow{I} (\mathrm{P}\mathbb{M})$ is the least quasi-variety containing $\mathbb{M}$. By Thm. 8.19 we know that $(\mathbb{M}^* \cap \mathsf{Horn}(\Sigma))^*$ is a quasi-variety, which obviously contains $\mathbb{M}$, hence $\xleftarrow{I} (\mathrm{P}\mathbb{M}) \subseteq (\mathbb{M}^* \cap \mathsf{Horn}(\Sigma))^*$. By Thm. 8.22 we know that there exists a set $E \subseteq \mathsf{Horn}(\Sigma)$ such that $\xleftarrow{I} (\mathrm{P}\mathbb{M}) = E^*$. Since $E \subseteq \mathbb{M}^*$ we obtain that $(\mathbb{M}^* \cap \mathsf{Horn}(\Sigma))^* \subseteq E^* = \xleftarrow{I} (\mathrm{P}\mathbb{M})$. Thus $(\mathbb{M}^* \cap \mathsf{Horn}(\Sigma))^* = \xleftarrow{I} (\mathrm{P}\mathbb{M})$. $\square$

Birkhoff variety theorem revisited. We reformulate the conclusion of Thm. 8.29:

Theorem 8.36 (Birkhoff variety). *Under the conditions* $(QP1-2)$ *(of Thm. 8.19),* $(QA2)$ *(of Thm. 8.22),* (VP) *(of Prop. 8.27),* $(VA1-2)$ *(of Thm. 8.29) and under the following additional conditions:*

(VA3) *the* products preserve the abstract surjections, *i.e., if* $\{f_i : M_i \to N_i\}_{i\in I}$ *are abstract surjections of models, then there exists a product* $\prod_{i\in I} f_i : \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ *which is an abstract surjection,*

for any class of Σ*-models* $\mathbb{M}$

$$(\mathbb{M}^* \cap \mathsf{UA}(\Sigma))^* = \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))$$

where $\langle I, \mathcal{E}\rangle$ *is the inclusion system for the* Σ*-models and* $\mathrm{P}\mathbb{M}$ *is the class of all (small) products of models of* $\mathbb{M}$.

Proof. Lemma 8.35 means that $\mathrm{P}(\stackrel{I}{\leftarrow} \mathbb{N}) \subseteq \stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{N})$, the condition $(VA3)$ means that $\mathrm{P}(\stackrel{\mathcal{E}}{\to} \mathbb{N}) \subseteq \stackrel{\mathcal{E}}{\to} (\mathrm{P}\mathbb{N})$ for any class $\mathbb{N}$ of Σ-models, and Lemma 8.37 below gives us that $\stackrel{I}{\leftarrow} (\stackrel{\mathcal{E}}{\to} \mathbb{N}) \subseteq \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathbb{N}))$. We can use these for the following calculations:

- $\stackrel{\mathcal{E}}{\to} (\stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))) = \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))$,
- $\stackrel{I}{\leftarrow} (\stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))) \subseteq \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))) = \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))$, and
- $\mathrm{P}(\stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))) \subseteq \stackrel{\mathcal{E}}{\to} (\mathrm{P}(\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))) \subseteq \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{PP}\mathbb{M})) = \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} (\mathrm{P}\mathbb{M}))$.

These show that $\stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} \mathrm{P}\mathbb{M})$ is a variety and by an argument similar to that from the proof of Thm. 8.34 above we obtain the conclusion by using Prop. 8.27 and Thm. 8.29.

Lemma 8.37. *For any* Σ*-model* N,

$$\stackrel{I}{\leftarrow} (\stackrel{\mathcal{E}}{\to} N) \subseteq \stackrel{\mathcal{E}}{\to} (\stackrel{I}{\leftarrow} N).$$

Proof. Consider $N \in \mathbb{N}$ and $(e : N \to A) \in \mathcal{E}$, $(i : B \to A) \in I$. By condition $(VA1)$, $i_{\iota_\Sigma(B)}(B_B) : M_{\iota_\Sigma(B)} \to B$ is an abstract surjection.

$$\begin{tikzcd}
B \arrow[rr, "i"] & & A & & A' \\
& f(M_{\iota_\Sigma(B)})_{i_f} \arrow[ul, "g"'] \arrow[dr] & & & \\
0_{\Sigma_B}\restriction_{\iota_\Sigma(B)} = M_{\iota_\Sigma(B)} \arrow[uu, "i_{\iota_\Sigma(B)}(B_B)"] \arrow[ur, "e_f"] \arrow[rr, "f"'] & & N \arrow[uu, "e"'] & 0_{\Sigma_B} \arrow[uur, "h'"] \arrow[r, "f'"'] & N' \arrow[uu, "e'"']
\end{tikzcd}$$

Recall from basic assumptions introduced at the beginning of Sect. 8.4 that $\iota_\Sigma(B) \in \mathcal{D}$ and that all signature morphisms from $\mathcal{D}$ are representable. Since $M_{\iota_\Sigma(B)}$ is the reduct of the initial Σ_B-model 0_{Σ_B}, this means we can find a $\iota_\Sigma(B)$-expansion $h' : 0_{\Sigma_B} \to A'$ of $i_{\iota_\Sigma(B)}(B_B); i$. Because $e \in \mathcal{E}$, by the condition (VP) we can also expand e to $e' : N' \to A'$. Let $f' : 0_{\Sigma_B} \to N'$ be the unique Σ_B-model homomorphism given by the initiality property of 0_{Σ_B} and f be its $\iota_\Sigma(B)$-reduct. By the uniqueness aspect of the initiality property of 0_{Σ_B} we have that $h' = f';e'$. This leads to $i_{\iota_\Sigma(B)}(B_B);i = f;e$.

Now let us factor $f = e_f;i_f$ in the inclusion system for the Σ-models. By the Diagonal-fill Lemma 4.16 there exists g such that $g;i = i_f;e$. Because both $i_{\iota_\Sigma(B)}(B_B)$ and e_f are abstract surjection, $g \in \mathcal{E}$ too. This shows that $B \in \xrightarrow{\mathcal{E}} (\xleftarrow{I} N)$. $\square$

$\square$

This revised version of the Birkhoff Variety Theorem introduces the new condition $(VA3)$. This cannot be proved at the general level of abstract inclusion systems, but it can be established very easily in concrete applications. For example in the case of the **FOL** models $(VA3)$ holds for the closed inclusion system (recall that this is the inclusion system for which the Birkhoff Variety Theorem can be established; see Cor. 8.33) by the simple fact that products of surjective functions are still surjective.

Axiomatizability by ultraproducts revisited. Thm. 8.5 and its Cor. 8.7 can be reformulated as follows:

Theorem 8.38. *Consider an institution with negation and conjunctions and such that sentences are preserved by ultraproducts. Then for each class $\mathbb{M}$ of Σ-models:*

– $\mathbb{M}^{**} = \equiv(\mathrm{Up}(\mathbb{M}))$ *where $\equiv$ is the elementary equivalence relation, and* $\mathrm{Up}(\mathbb{M})$ *is the class of all ultraproducts of models of $\mathbb{M}$, and*

– *if in addition the institution has the Keisler-Shelah property and* Up *is idempotent (i.e.,* $\mathrm{Up};\mathrm{Up} = \mathrm{Up}$*) then*

$$\mathbb{M}^{**} = \mathrm{Ur}^{-1}(\mathrm{Up}(\mathbb{M}))$$

where Ur *is the 'ultraradical' relation on models defined by $\langle A, B\rangle \in \mathrm{Ur}$ if and only if $A \cong B$ or A is an ultraradical of B.*

Proof. The first part follows immediately by an inspection of the proof of Thm. 8.5. For the second part, it is therefore enough to show that $\mathrm{Up};\mathrm{Ur}^{-1} = \mathrm{Up};\equiv$.

$\mathrm{Ur}^{-1}(\mathrm{Up}\mathbb{M}) \subseteq \equiv(\mathrm{Up}\mathbb{M})$ follows immediately from $\mathrm{Ur}^{-1}(\mathbb{M}) \subseteq \equiv(\mathbb{M})$ which holds by virtue of the hypothesis that the sentences are preserved by ultraproducts. For the other inclusion, $\equiv(\mathbb{N}) \subseteq \mathrm{Ur}^{-1}(\mathrm{Up}(\mathbb{N}))$ because the institution is Keisler-Shelah, therefore $\equiv(\mathrm{Up}\mathbb{M}) \subseteq \mathrm{Ur}^{-1}(\mathrm{Up}\mathrm{Up}\mathbb{M}) = \mathrm{Ur}^{-1}(\mathrm{Up}\mathbb{M})$. $\square$

The condition that Up is idempotent is rather hard to establish at the general level. However with some effort it can be established in concrete institutions such as **FOL**. Here we skip this argument.

Universal axiomatizability revisited. We reformulate the conclusion of Cor. 8.9.

Theorem 8.39. *Under the framework and the hypotheses of Cor. 8.9 if we also assume that*

1. *the institution is a Łoś-institution,*
2. *each elementary extension lifts completely ultraproducts, and*
3. *as in Thm. 8.38, the ultraproduct construction is idempotent (i.e.,* $\mathrm{Up};\mathrm{Up} = \mathrm{Up}$*),*
4. *the* Sen^0*-submodels are preserved by expansions along elementary extensions,*

for any class of Σ*-models* $\mathbb{M}$

$$(\mathbb{M}^* \cap \neg\mathsf{Sen}^0(\Sigma))^* = \overset{\mathsf{Sen}^0}{\longleftarrow} (\mathrm{Up}\mathbb{M}).$$

Proof. As for the proofs of Theorems 8.34 and 8.36 we have only to prove that $\overset{\mathsf{Sen}^0}{\longleftarrow}$ $(\mathrm{Up}\mathbb{M})$ is closed under Sen^0-submodels and ultraproducts.

The closure of $\overset{\mathsf{Sen}^0}{\longleftarrow}$ $(\mathrm{Up}\mathbb{M})$ under $\overset{\mathsf{Sen}^0}{\longleftarrow}$ follows from the transitivity of $\overset{\mathsf{Sen}^0}{\longrightarrow}$. Let $f\colon\ M \to N$ such that $M_M[\mathsf{Sen}^0]N_f$ and $g\colon\ N \to P$ such that $N_N[\mathsf{Sen}^0]P_g$. We show that $M_M[\mathsf{Sen}^0]P_{f;g}$. For this let us note that $P_{f;g} = P_g\upharpoonright_{\iota_\Sigma(f)}$ and that $N_N\upharpoonright_{\iota_\Sigma(f)} = N_f$. For any $\rho \in M_M^* \cap \mathsf{Sen}^0(\Sigma)$, by the Satisfaction Condition we have that $P_{f;g} \models \rho$ iff $P_g \models \iota_\Sigma(f)(\rho)$ and $N_f \models \rho$ iff $N_N \models \iota_\Sigma(f)(\rho)$. Since $N_f \models M_M^* \cap \mathsf{Sen}^0(\Sigma_M)$ we deduce that $\iota_\Sigma(f)(\rho) \in N_N^* \cap \mathsf{Sen}^0(\Sigma_N)$ which means that $P_g \models \iota_\Sigma(f)(\rho)$ and thus $P_{f;g} \models \rho$.

The closure of $\overset{\mathsf{Sen}^0}{\longleftarrow}$ $(\mathrm{Up}\mathbb{M})$ under ultraproducts follows from $\mathrm{Up}(\overset{\mathsf{Sen}^0}{\longleftarrow} \mathbb{N}) \subseteq \overset{\mathsf{Sen}^0}{\longleftarrow}$ $(\mathrm{Up}\mathbb{N})$ for any class $\mathbb{N}$ of Σ-models and from the idempotency of Up. In order to prove the former property, consider $\{h_i\colon\ M_i \to N_i \in \mathbb{N}\}_{i\in I}$ such that $(M_i)_{M_i}[\mathsf{Sen}^0](N_i)_{h_i}$ for each $i \in I$. Consider the ultraproducts $M = \prod_U M_i$, respectively $N = \prod_U N_i$, with $(\prod_{i\in J} M_i \xrightarrow{\mu_J} M)_{J\in U}$, respectively $(\prod_{i\in J} N_i \xrightarrow{\nu_J} N)_{J\in U}$, their corresponding co-limits and let $h\colon\ M \to N$ be the canonical model homomorphism such that $(\prod_{i\in J} h_i);\nu_J = \mu_j;h$. We show that $M_M[\mathsf{Sen}^0]N_h$ meaning that $M. \overset{\mathsf{Sen}^0}{\longleftarrow} N$.

Because the elementary extensions lift completely ultraproducts, let M_i' be $\iota_\Sigma(M)$-expansions of M_i and μ_J' be $\iota_\Sigma(M)$-expansions of μ_J such that $M_M = \prod_U M_i'$. Each M_i' also determines a unique expansion of h_i to $h_i\colon\ M_i' \to N_i'$. The expansions N_i' induce a complete lifting of the ultraproduct co-cone for N denoted by $(\prod_{i\in J} N_i' \xrightarrow{\nu_J'} N')_{J\in U}$. Let $h'\colon\ M_M \to N'$ be the unique $\iota_\Sigma(M)$-expansion of h to a Σ_M-model homomorphism. Because $h\colon\ M \to N$ lifts uniquely to a Σ_M-model homomorphism from M_M we deduce that $N' = N_h$. Now let $\rho \in M_M^* \cap \mathsf{Sen}^0(\Sigma_M)$. Because ρ is preserved by ultrafactors there exists $J \in U$ such that $M_i' \models \rho$ for each $i \in J$. Because $h_i'\colon\ M_i' \to N_i'$ is a Sen^0-submodel (since Sen^0 submodels are closed under expansions along elementary extensions) we have that $N_i' \models \rho$ for each $\in J$. Because ρ is preserved by ultraproducts we have that $N_h \models \rho$. □

The following concrete instance of Thm. 8.39 shows that its conditions are rather easy to establish.

Corollary 8.40. *For any* **FOL** *signature* Σ *and any class of* Σ*-models* $\mathbb{M}$*,*

$$(\mathbb{M}^* \cap \mathsf{Univ}(\Sigma))^* = \overset{S_c}{\leftarrow} (\mathrm{Up}\mathbb{M}))$$

where S_c *is the class of the closed injective model homomorphisms and* Univ *is the functor of the universal sentences.*

Proof. Sen^0 is the subfunctor Exist of the existential sentences such as in Prop. 8.1 and Corollaries 8.4 and 8.10. Elementary extensions lift completely ultraproducts because as signature extensions with constants they meet the conditions of Prop. 6.8. Finally, by Prop. 8.1 the Sen^0-submodels are precisely the closed injective homomorphisms which are obviously preserved by expansions along signature extensions with constants. $\square$

Birkhoff institutions

The way we have developed and presented the axiomatizability results in this section follows a certain pattern. One starts with an arbitrary class of models $\mathbb{M}$. One the one hand one considers the models of the sentences of a certain kind which are satisfied by all models in $\mathbb{M}$, and on the other hand one takes the closure of $\mathbb{M}$ first under a class of filtered products, and afterwards under some relations defined in terms of certain classes of model homomorphisms. These two operations give the same result; this is the respective axiomatizability result. (In the literature the latter closure operations are called *axiomatizable hulls*.)

The definition. The pattern for axiomatizability results discussed above is captured formally by the concept of Birkhoff institution.

$(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathcal{F}, \mathcal{B})$ is a *Birkhoff institution* when

- $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is an institution such that for each signature $\Sigma \in |\mathbb{S}ig|$ the category $\mathsf{Mod}(\Sigma)$ of Σ-models has $\mathcal{F}$-filtered products,
- $\mathcal{F}$ is a class of filters with $\{\{*\}\} \in \mathcal{F}$, and
- $\mathcal{B}_\Sigma \subseteq |\mathsf{Mod}(\Sigma)| \times |\mathsf{Mod}(\Sigma)|$ is a binary relation for each signature $\Sigma \in |\mathbb{S}ig|$, which is closed under isomorphisms, i.e., $(\mathcal{B}_\Sigma; \cong_\Sigma) = \mathcal{B}_\Sigma = (\cong_\Sigma; \mathcal{B}_\Sigma)$,

such that

$$\mathbb{M}^{**} = \mathcal{B}_\Sigma^{-1}(\mathcal{F}\mathbb{M})$$

for each signature Σ and each class of Σ-models $\mathbb{M} \subseteq |\mathsf{Mod}(\Sigma)|$, and where $\mathcal{F}\mathbb{M}$ is the class of all F-filtered products of models from $\mathbb{M}$ for all filters $F \in \mathcal{F}$. Note that each $\mathcal{B}_\Sigma$ is reflexive by the closure under isomorphisms.

Examples. Based on the results we have already developed we can now present a list of Birkhoff institutions obtained around **FOL** by varying the style of the sentences. The second part of the list contains some example of Birkhoff institutions not developed in this book, but which are known in the literature.

institution	$\mathcal{B}$	$\mathcal{F}$	*source*
FOL	$\equiv$	all ultrafilters	Thm. 8.38
FOL	*ultraradicals*	all ultrafilters	Thm. 8.38
PL	$=$	all ultrafilters	Thm. 8.38
UNIV	$\xrightarrow{S_c}$	all ultrafilters	Cor. 8.40
$\mathbf{HCL}_\infty$	$\xrightarrow{S_c}$	$\{\{I\} \mid I \text{ set}\}$	Thm. 8.34
universal **FOL**-atoms	$\xleftarrow{H_r};\xrightarrow{S_c}$	$\{\{I\} \mid I \text{ set}\}$	Thm. 8.29
EQL	$\xleftarrow{H_r};\xrightarrow{S_w}$	$\{\{I\} \mid I \text{ set}\}$	Thm. 8.34
universal $\mathbf{FOL}_{\infty,\omega}$ sentences	$\xrightarrow{S_c}$	$\{\{\{*\}\}\}$	[6]
HCL	$\xrightarrow{S_c}$	all filters	[6]
$\forall\vee$ (universal disjunctions of atoms)	$\xleftarrow{H_s};\xrightarrow{S_c}$	all ultrafilters	[6]
$\forall\vee_\infty$ (univ. infinitary disj. of atoms)	$\xleftarrow{H_s};\xrightarrow{S_c}$	$\{\{\{*\}\}\}$	[6]
$\forall\exists$ (universal-existential sentences)	*sandwiches* ([32])	all ultrafilters	[6]

where H_r denotes the class of surjective, H_s the class of strong surjective, S_w the class of injective, and S_c the class of closed injective model homomorphisms.

Exercises

8.8. In the categories of **FOL** models the ultraproduct construction is idempotent, i.e., Up; Up = Up.

8.9. Birkhoff institutions of partial algebras
The following institutions of partial algebras arise as Birkhoff institutions according to the following table:

institution	$\mathcal{B}$	$\mathcal{F}$
UNIV(**PA**)	$\xrightarrow{S_c}$	all ultrafilters
QE_2(**PA**)	$\xrightarrow{S_w}$	$\{\{I\} \mid I \text{ set}\}$
QE(**PA**)	$\xrightarrow{S_c}$	$\{\{I\} \mid I \text{ set}\}$

where S_w and S_c are the classes of plain, respectively closed, injective homomorphisms and where **UNIV**(**PA**) is the institution of the 'universal' sentences in **PA** (see Ex. 8.3).

Notes. Thm. 8.5 and Cor. 8.6 are institution-independent generalizations of basic axiomatizability results in first order logic of [68] (see also [32]). Our general preservation-by-saturation Thm. 8.2 generalizes and extends its first-order logic Cor. 8.4 which can be found in [32]. Its axiomatizability consequence Cor. 8.10 can also be found in [32] while Cor. 8.9 constitutes its institution-independent generalization. The ultraradicals have been introduced and used in [150] which also used the ultraradical formulation given by Cor. 8.8.

Similar quasi-variety concepts to ours have been formulated and results obtained within the framework of factorization systems (see [169, 170] or [6] for a very general approach), however the inclusion systems framework leads to greater simplicity. Thm. 8.16 generalizes a well-known result from universal algebra [84] and conventional model theory of first-order logic [119]. A similar institution-independent result has been obtained by Tarlecki [169] within the framework of the so-called 'abstract algebraic institutions'. However, the concept of abstract algebraic institution provides a set of conditions much more complex than our framework, the greater simplicity of our approach leading also to simpler and somehow different proofs. Within the same setting [170] develops an institution-independent approach to the quasi-variety theorem related to ours, however Birkhoff Variety Thm. 8.29 seems to have no previous institution-independent variant.

Both quasi-variety and Birkhoff variety theorems have rather old roots in universal algebra; the former had been discovered by Mal'cev [119] while the latter by Birkhoff back in 1935 (see [24]). Lemmas 8.23 and 8.30 are inclusion system versions of well-known Birkhoff-like axiomatizability results for satisfaction by injectivity originally developed within the framework of factorization systems [142, 7]. They appeared in the current form as axiomatizability results for the so-called 'inclusive equational logic' of [155].

Birkhoff institutions were introduced in [50]. A more complete list of Birkhoff sub-institutions of first-order logic can be obtained by using results from [6]. Examples of Birkhoff institutions in the context of less conventional logics arise in the context of Birkhoff-style axiomatizability results for these logics. For example, a large list of Birkhoff institutions based on partial algebra can also be obtained from [6]. Moreover, the very general axiomatizability results of [6] can be applied for obtaining Birkhoff institutions out of recent algebraic specification logics.

Chapter 9

Interpolation

Interpolation is one of the most important topics of logic and model theory. Below is a very simple example. Consider the following semantic deduction in **PL** (propositional logic):

$$p_1 \wedge q \models p_2 \vee q$$

where p_1, p_2, q are propositional symbols (i.e., relation symbols of zero arity). The simplest justification for this deduction is by factoring it as

$$p_1 \wedge q \models q \models p_2 \vee q$$

which meets the intuition that p_1 is not involved in establishing the truth of $p_2 \vee q$. In general, the so-called 'Craig interpolation' (abbreviated CI) property can be formulated as follows:

> if $\rho_1 \models \rho_2$ for two sentences, then there exists a sentence ρ, called the *interpolant* of ρ_1 and ρ_2, that uses logical symbols that appear both in ρ_1 and ρ_2 and such that $\rho_1 \models \rho \models \rho_2$.

An equivalent expression of the above property assumes $\rho_1 \models \rho_2$ in the *union signature* $\Sigma_1 \cup \Sigma_2$, and asks for ρ to be in the *intersection signature* $\Sigma_1 \cap \Sigma_2$, where Σ_i is the signature of ρ_i.

Interpolation squares. If we naturally generalize the inclusion square

$$\begin{tikzcd}
\Sigma_1 \cap \Sigma_2 \arrow[r] \arrow[d] & \Sigma_1 \arrow[d] \\
\Sigma_2 \arrow[r] & \Sigma_1 \cup \Sigma_2
\end{tikzcd}$$

to any commuting square of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

and replace sentences ρ_1, ρ_2, and ρ with *sets of sentences* E_1, E_2, and E, we get the following form of CI:

If $\theta_1(E_1) \models_{\Sigma'} \theta_2(E_2)$, then there exists an *interpolant* $E \subseteq \mathsf{Sen}(\Sigma)$ such that $E_1 \models_{\Sigma_1} \varphi_1(E)$ and $\varphi_2(E) \models_{\Sigma_2} E_2$.

A commuting square satisfying the above property is called a *Craig Interpolation square*.

Quasi-compactness and single sentence formulation of CI. In some situations the existence of infinite conjunctions may replace compactness as a working hypothesis. Fact 9.1 below is such an example. We say that an institution is *quasi-compact* if and only if it is compact or it has infinite conjunctions.

Fact 9.1. *In a compact institution, if E_2 is finite, then the interpolant E can be chosen to be finite too. Consequently, in quasi-compact institutions having finite conjunctions, the CI formulation given above is equivalent to the formulation considering single sentences rather than sets of sentences.*

In fact, it is the potential absence of conjunctions which motivates us to consider sets of sentences rather than single sentences in the formulation of interpolation.

$(\mathcal{L}, \mathcal{R})$-interpolation. CI squares can be found mostly among pushout squares since these are the squares of signature morphisms which constitute the accurate generalization of intersection-union squares of signatures. However, in many institutions only *some* pushout squares of signature morphisms have the CI property. For example, while in $\mathbf{FOL}^1$ (the unsorted version of **FOL**) *all* pushout squares have the CI property, this is not the case in **FOL**. Also, in **EQL** and **HCL**, not all pushout squares have the CI property.

It is often convenient to capture such classes of CI squares by restricting independently φ_1 and φ_2 to belong to certain classes of signature morphisms. Therefore, for any classes of signature morphisms $\mathcal{L}, \mathcal{R}$, we say that the institution has the *Craig $(\mathcal{L}, \mathcal{R})$-Interpolation property* if each pushout square of signature morphism of the form

$$\begin{tikzcd} \bullet \arrow[r, "\mathcal{L}"] \arrow[d, "\mathcal{R}"'] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

is a CI square.

The list below anticipates some of the concrete $(\mathcal{L},\mathcal{R})$-interpolation properties obtained in this chapter. But before presenting this list let us establish the following notation for **FOL** signature morphisms.

(xyz)**-morphisms of signatures.** Let us define the following syntactic properties for signature morphisms. A **FOL** signature morphism φ is an *(xyz)-morphism*, with $x,z \in \{i,s,b,*\}$ and $y \in \{i,s,b,e,*\}$ (where i stands for 'injective', s for 'surjective', b for 'bijective', e for 'encapsulated', and $*$ for 'all') when the sort component φ^{st} has the property x, the operation component φ^{op} has the property y, and the relation component φ^{rl} has the property z. Note that by 'injectivity', respectively, 'surjectivity' of φ^{op} we mean that for all arities w and sorts s, $\varphi^{\mathrm{op}}_{w\to s}$ is injective, respectively, surjective. The same applies of course to φ^{rl}, the relation symbols component. That $\varphi^{\mathrm{op}}_{w\to s}$ is encapsulated means that no 'new' operation symbol, i.e., outside the image of φ, is allowed to have the sort in the image of φ. In other words, if $\varphi\colon (S,F,P) \to (S',F',P')$ and $\sigma' \in F'_{w'\to s'}$ with $s' \in \varphi(s)$ then there exists $\sigma \in F_{w\to s}$ such that $\varphi(\sigma) = \sigma'$.

For example, an $(ss*)$-morphism of signatures is surjective on the sorts and on the operations, while a (bis)-morphism of signatures is bijective on the sorts, is injective on the operations, and is surjective on the relations.

This notational convention can be extended to other institutions too, such as for example **PA**, **EQL** or $\mathbf{FOL^1}$. In the case of **EQL**, because we do not have relation symbols, the last component is missing. The same applies to $\mathbf{FOL^1}$, in this case the first component (i.e., the sort component) is missing.

The list. Below is the above mentioned list of $(\mathcal{L},\mathcal{R})$-interpolation properties:

institution	$\mathcal{L}$	$\mathcal{R}$	*reference*
$\mathbf{FOL^1}$	$**$	$**$	Cor. 9.9 or 9.18
FOL	$i**$	$***$	Cor. 9.15 or 9.18
	$***$	$i**$	Cor. 9.9 or 9.18
EQL	$**$	ii	Cor. 9.8
	ie	$**$	Cor. 9.13
HCL	$***$	iii	Cor. 9.8
	$ie*$	$***$	Cor. 9.13
SOL	$***$	iii	Cor. 9.5

Summary of the chapter. In this chapter we develop two direct methods for obtaining interpolation results, one of them based on the Birkhoff-style axiomatizability properties of institutions, and the other one based on Robinson consistency. Although these two methods have quite complementary application domains, interpolation in **FOL** appears as an application of both of them, the first method calling for the Keisler-Shelah property (Cor. 7.27).

A third method to establish interpolation which is presented here is an indirect one, which 'borrows' interpolation along institution comorphisms.

Another topic of this chapter refers to an extension of the Craig interpolation concept to the so-called 'Craig-Robinson interpolation' which is the interpolation concept appropriate for several applications of interpolation such as definability and semantics of structured specifications.

Exercises

9.1. Composition of interpolation squares
CI squares can be composed both 'horizontally' and 'vertically': in any institution, consider the commuting squares of signature morphisms

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[r, "\phi_1"] \arrow[d, "\varphi'_1"] & \Sigma'_1 \arrow[d, "\phi'_1"] \\
\Sigma_2 \arrow[r, "\varphi'_2"'] \arrow[d, "\phi_2"'] & \Sigma' \arrow[r, "\phi"'] \arrow[d, "\phi"] & \Sigma'' \\
\Sigma'_2 \arrow[r, "\phi'_2"'] & \Sigma'' &
\end{tikzcd}$$

Then
(1a) $[\Sigma,\Sigma'_1,\Sigma_2,\Sigma'']$ is a CI square if $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ and $[\Sigma_1,\Sigma'_1,\Sigma',\Sigma'']$ are CI squares.
(1b) $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ is a CI square if $[\Sigma,\Sigma'_1,\Sigma_2,\Sigma'']$ is a CI square and ϕ_1 is conservative.
(2a) $[\Sigma,\Sigma_1,\Sigma'_2,\Sigma'']$ is a CI square if $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ and $[\Sigma_2,\Sigma',\Sigma'_2,\Sigma'']$ are CI squares.
(2b) $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ is a CI square if $[\Sigma,\Sigma_1,\Sigma'_2,\Sigma'']$ is a CI square and ϕ_2 is conservative.

9.1 Semantic interpolation

By using the Galois connection between sets of sentences and classes of models given by the satisfaction relation, we may shift the interpolation concept from sets of sentences to classes of models. This translation is based upon the following observations for any commuting square of signature morphisms in an arbitrary institution:

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\
\Sigma_2 \arrow[r, "\theta_2"'] & \Sigma'
\end{tikzcd}$$

E_1 a set of Σ_1-sentences, E_2 a set of Σ_2-sentence, and a E set of Σ-sentences:

$$\begin{array}{rlcll}
\theta_1(E_1) \models_{\Sigma'} \theta_2(E_2) & \text{iff} & \theta_1(E_1)^* \subseteq \theta_2(E_2)^* & \text{iff} & \mathsf{Mod}(\theta_1)^{-1}(E_1^*) \subseteq \mathsf{Mod}(\theta_2)^{-1}(E_2^*), \\
E_1 \models_{\Sigma_1} \varphi_1(E) & \text{iff} & E_1^* \subseteq \varphi_1(E)^* & \text{iff} & E_1^* \subseteq \mathsf{Mod}(\varphi_1)^{-1}(E^*), \\
\varphi_2(E) \models_{\Sigma_2} E_2 & \text{iff} & \varphi_2(E)^* \subseteq E_2^* & \text{iff} & \mathsf{Mod}(\varphi_2)^{-1}(E^*) \subseteq E_2^*.
\end{array}$$

If we abstract

- E_1^* to any class $\mathbb{M}_1$ of Σ_1-models,

– E_2^* to any class $\mathbb{M}_2$ of Σ_2-models, and
– E^* to any class $\mathbb{M}$ of Σ-models,

then we may say that $\mathbb{M}$ is the *semantic interpolant* for $\mathbb{M}_1$ and $\mathbb{M}_2$ satisfying $\mathsf{Mod}(\theta_1)^{-1}(\mathbb{M}_1) \subseteq \mathsf{Mod}(\theta_2)^{-1}(\mathbb{M}_2)$ when

– $\mathbb{M}_1 \subseteq \mathsf{Mod}(\varphi_1)^{-1}(\mathbb{M})$ and
– $\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{M}) \subseteq \mathbb{M}_2$.

We may therefore state the following 'principle of semantic interpolation':

> *The existence of a syntactic interpolant for E_1 and E_2 is equivalent to the existence of a semantic interpolant $\mathbb{M}$ for E_1^* and E_2^* which is elementary, in this case the syntactic interpolant being $\mathbb{M}^*$.*

Fact 9.2. $\mathbb{M}$ *is a semantic interpolant for* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *satisfying* $\mathsf{Mod}(\theta_1)^{-1}(\mathbb{M}_1) \subseteq \mathsf{Mod}(\theta_2)^{-1}(\mathbb{M}_2)$ *if and only if*

$$\mathbb{M}_1\!\restriction_{\varphi_1} \subseteq \mathbb{M} \subseteq |\mathsf{Mod}(\Sigma)| \setminus (|\mathsf{Mod}(\Sigma_2)| \setminus \mathbb{M}_2)\!\restriction_{\varphi_2}.$$

Moreover if the square of signature morphisms is a weak amalgamation square, then semantic interpolants always exist.

Therefore when we shift the interpolation problem from sets of sentences to classes of models, weak amalgamation is sufficient. However, when shifting back to sets of sentences, we need to express the semantic interpolants $\mathbb{M}$ as elementary classes of models E^*. Semantic operators and their fixed points constitute a useful device to achieve this.

Semantic operators. Given a signature Σ, a *semantic Σ-operator* is just a mapping of Σ-classes of Σ-models $\mathcal{U}_\Sigma : \mathcal{P}(|\mathsf{Mod}(\Sigma)|) \to \mathcal{P}(|\mathsf{Mod}(\Sigma)|)$. It is a *closure operator* when it has the following additional properties:

- *reflexivity:* $\mathbb{M} \subseteq \mathcal{U}_\Sigma(\mathbb{M})$,
- *monotonicity:* $\mathbb{M} \subseteq \mathbb{M}'$ implies $\mathcal{U}_\Sigma(\mathbb{M}) \subseteq \mathcal{U}_\Sigma(\mathbb{M}')$,
- *idempotency:* $\mathcal{U}_\Sigma(\mathcal{U}_\Sigma(\mathbb{M})) = \mathcal{U}_\Sigma(\mathbb{M})$, and
- *closure under isomorphisms:* if $\mathbb{M}$ is closed under isomorphisms, then $\mathcal{U}_\Sigma(\mathbb{M})$ is also closed under isomorphisms.

Two simple examples of semantic closure operators are

- the *isomorphic closure* operator Iso defined by $\mathrm{Iso}(\mathbb{M}) = \{M \mid M \cong N \text{ for some } N \in \mathbb{M}\}$, and
- the *elementary closure* operator $(-)^{**}$ mapping each class of models $\mathbb{M}$ to $\mathbb{M}^{**}$.

Fixed points of semantic operators. A class $\mathbb{M}$ of Σ-models is a *fixed point* for a semantic operator $\mathcal{U}_\Sigma$ when $\mathcal{U}_\Sigma(\mathbb{M}) = \mathrm{Iso}(\mathbb{M})$. Let $Fixed(\mathcal{U}_\Sigma)$ be the class of all fixed points of $\mathcal{U}_\Sigma$.

The following is a rather abstract generic result which gives a set of sufficient conditions for the existence of a 'good' semantic interpolant. This result will be used later on several different occasions for producing more concrete interpolation results by giving various meanings to the parameters $\mathcal{U}$ and $\mathcal{V}$.

Theorem 9.3. *For any weak amalgamation square of signature morphisms*

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

and pairs of semantic operators $\mathcal{U} = \langle \mathcal{U}_\Sigma, \mathcal{U}_{\Sigma_1} \rangle$ *and* $\mathcal{V} = \langle \mathcal{V}_\Sigma, \mathcal{V}_{\Sigma_2} \rangle$ *such that*

1. $\mathcal{U}_\Sigma; \mathcal{V}_\Sigma; \mathcal{U}_\Sigma = \mathcal{U}_\Sigma; \mathcal{V}_\Sigma$,
2. $\mathcal{V}$ *are closure operators,*
3. φ_1 *preserves fixed points of* $\mathcal{U}$ *(i.e.,* $Fixed(\mathcal{U}_{\Sigma_1})\restriction_{\varphi_1} \subseteq Fixed(\mathcal{U}_\Sigma)$*),*
4. $\mathsf{Mod}(\varphi_1); \mathrm{Iso}; \mathcal{V}_\Sigma = \mathrm{Iso}; \mathsf{Mod}(\varphi_1); \mathcal{V}_\Sigma$, *and*
5. $\mathsf{Mod}(\varphi_2)^{-1}(\mathcal{V}_\Sigma(\mathbb{N})) \subseteq \mathcal{V}_{\Sigma_2}(\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{N}))$ *for each* $\mathbb{N} \subseteq |\mathsf{Mod}(\Sigma)|$,

all classes of models $\mathbb{M}_1 \in Fixed(\mathcal{U}_{\Sigma_1})$ *and* $\mathbb{M}_2 \in Fixed(\mathcal{V}_{\Sigma_2})$ *which are closed under isomorphisms and such that* $\mathsf{Mod}(\theta_1)^{-1}(\mathbb{M}_1) \subseteq \mathsf{Mod}(\theta_2)^{-1}(\mathbb{M}_2)$ *have a semantic interpolant* $\mathbb{M}$ *in* $Fixed(\mathcal{U}_\Sigma) \cap Fixed(\mathcal{V}_\Sigma)$ *which is closed under isomorphisms.*

Proof. The semantic interpolant $\mathbb{M}$ is defined as $\mathcal{V}_\Sigma(\mathbb{M}_1\restriction_{\varphi_1})$.

- $\mathbb{M}$ is closed under isomorphisms because

  $$\mathcal{V}_\Sigma(\mathbb{M}_1\restriction_{\varphi_1}) = \mathcal{V}_\Sigma((\mathrm{Iso}\mathbb{M}_1)\restriction_{\varphi_1}) = \mathcal{V}_\Sigma(\mathrm{Iso}(\mathbb{M}_1\restriction_{\varphi_1}))$$

  and because of the property of closure under isomorphisms of $\mathcal{V}_\Sigma$ as closure operator.

Let us now show that $\mathbb{M} \in Fixed(\mathcal{U}_\Sigma) \cap Fixed(\mathcal{V}_\Sigma)$.

- Because $\mathcal{V}_\Sigma$ is idempotent (as a closure operator)

  $$\mathcal{V}_\Sigma(\mathbb{M}) = \mathcal{V}_\Sigma^2(\mathbb{M}_1\restriction_{\varphi_1}) = \mathcal{V}_\Sigma(\mathbb{M}_1\restriction_{\varphi_1}) = \mathbb{M}$$

  hence $\mathbb{M} \in Fixed(\mathcal{V}_\Sigma)$. From $\mathcal{V}_\Sigma(\mathbb{M}) = \mathbb{M}$ it follows that $\mathbb{M}$ is closed under isomorphisms.

- Because $\mathbb{M}_1 \in \mathit{Fixed}(\mathcal{U}_{\Sigma_1})$, we have that $\mathbb{M}_1 \restriction_{\varphi_1} \in \mathit{Fixed}(\mathcal{U}_{\Sigma_1}) \restriction_{\varphi_1} \subseteq \mathit{Fixed}(\mathcal{U}_\Sigma)$ (the inclusion holds because φ_1 preserves fixed points of $\mathcal{U}$). Therefore

$$\begin{aligned}\mathcal{U}_\Sigma(\mathbb{M}) &= \mathcal{U}_\Sigma(\mathcal{V}_\Sigma(\mathbb{M}_1 \restriction_{\varphi_1})) = \mathcal{U}_\Sigma(\mathcal{V}_\Sigma((\mathrm{Iso}\mathbb{M}_1)\restriction_{\varphi_1})) = \mathcal{U}_\Sigma(\mathcal{V}_\Sigma(\mathrm{Iso}(\mathbb{M}_1 \restriction_{\varphi_1})))\\ &= \mathcal{U}_\Sigma(\mathcal{V}_\Sigma(\mathcal{U}_\Sigma(\mathbb{M}_1 \restriction_{\varphi_1}))) = \mathcal{V}_\Sigma(\mathcal{U}_\Sigma(\mathbb{M}_1 \restriction_{\varphi_1})) = \mathcal{V}_\Sigma(\mathrm{Iso}(\mathbb{M}_1 \restriction_{\varphi_1}))\\ &= \mathcal{V}_\Sigma((\mathrm{Iso}\mathbb{M}_1)\restriction_{\varphi_1}) = \mathcal{V}_\Sigma(\mathbb{M}_1 \restriction_{\varphi_1}) = \mathbb{M}\end{aligned}$$

  hence $\mathbb{M} \in \mathit{Fixed}(\mathcal{U}_\Sigma)$.

We now show that $\mathbb{M}$ is a semantic interpolant.

- $\mathbb{M}_1 \subseteq \mathsf{Mod}(\varphi_1)^{-1}(\mathbb{M})$ is the same thing as $\mathbb{M}_1 \restriction_{\varphi_1} \subseteq \mathbb{M} = \mathcal{V}_\Sigma(\mathbb{M}_1 \restriction_{\varphi_1})$. This inclusion holds by the reflexivity of $\mathcal{V}_\Sigma$ as a closure operator.
- By taking $\mathbb{M}_1 \restriction_{\varphi_1}$ in the role of $\mathbb{N}$ in condition 5, we obtain that $\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{M}) \subseteq \mathcal{V}_{\Sigma_2}(\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{M}_1 \restriction_{\varphi_1}))$. Because of weak amalgamation, by applying Fact 9.2, we obtain that $\mathbb{M}_1 \restriction_{\varphi_1} \subseteq \mathbb{M} \subseteq |\mathsf{Mod}(\Sigma)| \setminus (|\mathsf{Mod}(\Sigma_2)| \setminus \mathbb{M}_2)\restriction_{\varphi_2}$ which means $\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{M}_1 \restriction_{\varphi_1}) \subseteq \mathbb{M}_2$. By using the monotonicity of the closure operator $\mathcal{V}_{\Sigma_2}$, that $\mathbb{M}_2 \in \mathit{Fixed}(\mathcal{V}_{\Sigma_2})$, and that $\mathbb{M}_2$ is closed under isomorphisms, we obtain that

$$\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{M}) \subseteq \mathcal{V}_{\Sigma_2}(\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{M}_1 \restriction_{\varphi_1})) \subseteq \mathcal{V}_{\Sigma_2}(\mathbb{M}_2) = \mathrm{Iso}(\mathbb{M}_2) = \mathbb{M}_2.$$

□

Higher-order interpolation. An immediate application of the general semantic interpolation Thm. 9.3 is the following general result which gives significant interpolation results in institutions having enough higher-order expressive power.

Corollary 9.4. *In any institution with universal $\mathcal{R}$-quantification for a class $\mathcal{R}$ of signature morphisms, any weak amalgamation square*

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

for which $\varphi_2 \in \mathcal{R}$ is a Craig interpolation square.

Proof. In Thm. 9.3 let us take

- $\mathcal{U}$ to be identities, and
- $\mathcal{V}$ to be elementary closures, i.e., $\mathcal{V}(\mathbb{M}) = \mathbb{M}^{**}$.

Since for this setting of $\mathcal{U}$ and $\mathcal{V}$ the first four conditions of Thm. 9.3 are rather easy or even trivial, let us focus on the last condition, that $\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{N}^{**}) \subseteq \mathsf{Mod}(\varphi_2)^{-1}(\mathbb{N})^{**}$ for each $\mathbb{N} \subseteq |\mathsf{Mod}(\Sigma)|$. Consider a Σ_2-model N_2 such that $N_2\restriction_{\varphi_2} \in \mathbb{N}^{**}$ and let $\rho_2 \in \mathsf{Mod}(\varphi_2)^{-1}(\mathbb{N})^{*}$. We need to show that $N_2 \models_{\Sigma_2} \rho_2$.

But $\rho_2 \in \mathsf{Mod}(\varphi_2)^{-1}(\mathbb{N})^{*}$ means $\mathsf{Mod}(\varphi_2)^{-1}(\mathbb{N}) \models \rho_2$ which implies $\mathbb{N} \models (\forall\varphi_2)\rho_2$. By the hypothesis we have that $(\forall\varphi_2)\rho_2$ is a Σ-sentence of the institution. Because $N_2\restriction_{\varphi_2} \in \mathbb{N}^{**}$, we get that $N_2\restriction_{\varphi_2} \models (\forall\varphi_2)\rho_2$, hence $N_2 \models \rho_2$.

By the conclusion of Thm. 9.3 we get a semantic interpolant $\mathbb{M}$, closed under isomorphisms, and such that $\mathbb{M}^{**} = \mathrm{Iso}(\mathbb{M})$ (as a fixed point for $\mathcal{V}$), which means $\mathbb{M}^{**} = \mathbb{M}$. Hence we get an elementary semantic interpolant $\mathbb{M}$, which by the principle of semantic interpolation implies the existence of a syntactic interpolant. □

The following are instances of Cor. 9.4. (Recall that **SOL** is the 'second-order' extension of **FOL** admitting quantifiers over *any* signature extensions with a finite number of symbols.)

Corollary 9.5. *The institutions* **FOL**, **HCL**, **EQL**, **SOL** *have Craig* $(\mathbb{S}ig, \mathcal{R})$*-interpolation where* $\mathcal{R}$ *is*

- *the class of all signature extensions with constants in the case of* **FOL**, **HCL** *and* **EQL**, *and*
- *is the class of* (iii)*-morphisms of signatures in the case of* **SOL**.

Proof. In order to apply Cor. 9.4 we have to establish that the considered institutions admit universal $\mathcal{R}$-quantification.

In any of the considered institutions let $\varphi : \Sigma \to \Sigma'$ be a signature morphism in $\mathcal{R}$ and let ρ' be a Σ'-sentence. We have to prove that $(\forall\varphi)\rho'$ is semantically equivalent to a Σ-sentence, the problem being when φ is an extension of Σ with an infinite number of symbols.

Because ρ' is finitary, there exists a sub-signature $\Sigma_0 \subseteq \Sigma'$ such that Σ_0 has a finite number of sorts, operation, and relation symbols and ρ' is a Σ_0-sentence. Then the square of signature extensions

$$\begin{array}{ccc} \Sigma\cap\Sigma_0 & \xrightarrow{\varphi_0} & \Sigma_0 \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma' \end{array}$$

is a weak amalgamation square. This is justified by the fact that $\Sigma\cup\Sigma_0$ is the pushout of the span of signature extensions $\Sigma \leftarrow \Sigma\cap\Sigma_0 \to \Sigma_0$, the institutions are semi-exact, and the signature inclusion $\Sigma\cup\Sigma_0 \subseteq \Sigma'$ is conservative (cf. Fact 5.4). By using this weak amalgamation property it is easy to see that $(\forall\varphi)\rho' \models\!\!\!| (\forall\varphi_0)\rho'$. □

The interpolation properties for **FOL**, **EQL**, **HCL** given by Cor. 9.5 are rather weak due to the fact that in all these cases $\mathcal{R}$ is quite narrow. Later in the section we will prove much stronger interpolation results for these institutions. On the other hand, the interpolation property for **SOL** given by Cor. 9.5 is rather substantial. This difference is caused by the possibility of higher-order quantifications in **SOL** which is missing in **FOL**, **EQL** or **HCL**.

Exercises

9.2. Interpolation in HNK
The institution of higher order logic with Henkin semantics (**HNK**) has Craig ($\mathbb{S}ig^{\mathbf{HNK}}, (bi)$)-interpolation. (*Hint:* From Cor. 9.4.)

9.2 Interpolation by Axiomatizability

In this section we derive a couple of general interpolation results for Birkhoff institutions from the abstract semantic interpolation Thm. 9.3. For this we need the following concept of lifting relations.

Lifting relations. Let $\varphi\colon \Sigma_1 \to \Sigma_2$ be a signature morphism and $\mathcal{R} = \langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ with $\mathcal{R}_1 \subseteq |\mathsf{Mod}(\Sigma_1)| \times |\mathsf{Mod}(\Sigma_1)|$ and $\mathcal{R}_2 \subseteq |\mathsf{Mod}(\Sigma_2)| \times |\mathsf{Mod}(\Sigma_2)|$ be a pair of binary relations. We say that φ *lifts* $\mathcal{R}$ if and only if for each $M_2 \in |\mathsf{Mod}(\Sigma_2)|$ and $N_1 \in |\mathsf{Mod}(\Sigma_1)|$ if $\langle M_2\restriction_\varphi, N_1 \rangle \in \mathcal{R}_1$, then there exists $N_2 \in |\mathsf{Mod}(\Sigma_2)|$ such that $N_2\restriction_\varphi = N_1$ and $\langle M_2, N_2 \rangle \in \mathcal{R}_2$.

$$\begin{array}{ccc} M_2\restriction_\varphi & \overset{\mathcal{R}_1}{\text{———}} & N_1 = N_2\restriction_\varphi \\ \\ M_2 & \underset{\mathcal{R}_2}{\text{————}} & (\exists)N_2 \end{array}$$

The 'right' interpolation theorem

The first interpolation by the Birkhoff axiomatizability theorem, presented below, relies upon the properties of the morphisms on the right-hand side of the interpolation squares.

Theorem 9.6. *In a Birkhoff institution* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathcal{F}, \mathcal{B})$*, any weak amalgamation square*

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

such that

1. $\mathsf{Mod}(\varphi_1)$ *preserves* $\mathcal{F}$*-filtered products (of models), and*
2. φ_2 *lifts* $\mathcal{B}$

is a Craig Interpolation square.

Proof. We apply Thm. 9.3 by setting the semantic operators $\mathcal{U}$ and $\mathcal{V}$ as follows (we omit the signature subscripts from the notation of the operators):

- $\mathcal{U}(\mathbb{M}) = \mathcal{F}\mathbb{M}$, and
- $\mathcal{V}(\mathbb{M}) = (\mathcal{B}^{-1})^{+}(\mathbb{M})$, where $(\mathcal{B}^{-1})^{+}$ is the transitive closure of $\mathcal{B}^{-1}$.

The hypotheses of Thm. 9.3 can be checked as follows:

1. On the one hand, $\mathcal{V}(\mathcal{U}(\mathbb{M})) \subseteq \mathcal{U}(\mathcal{V}(\mathcal{U}(\mathbb{M})))$ follows by the fact that $X \subseteq \mathcal{F}X$ for all X because $\{\{*\}\} \in \mathcal{F}$, which is one of the hypotheses of Birkhoff institutions. On the other hand,

$$\mathcal{U}(\mathcal{V}(\mathcal{U}(\mathbb{M}))) \subseteq \mathcal{V}(\mathcal{U}(\mathcal{V}(\mathcal{U}(\mathbb{M})))) = \mathcal{V}(\mathcal{U}(\mathbb{M})).$$

   The inclusion holds because $\mathcal{B}$ is reflexive. The equality holds because $\mathcal{V}(\mathcal{U}(\mathbb{N})) = \mathbb{N}^{**}$ for each class $\mathbb{N}$ of models, which follows from the fact that $\mathcal{B}^{-n}(\mathcal{F}\mathbb{N}) = \mathbb{N}^{**}$ for each natural number $n \geq 1$. This can be shown by induction on n. For $n = 1$ we have that $\mathcal{B}^{-1}(\mathcal{F}\mathbb{N}) = \mathbb{N}^{**}$ by the definition of Birkhoff institutions. For the induction step, we successively have

$$\mathcal{B}^{-(n+1)}(\mathcal{F}\mathbb{N}) = \mathcal{B}^{-1}(\mathcal{B}^{-n}(\mathcal{F}\mathbb{N})) = \mathcal{B}^{-1}(\mathbb{N}^{**}) \subseteq \mathcal{B}^{-1}(\mathcal{F}\mathbb{N}^{**}) = \mathbb{N}^{****} = \mathbb{N}^{**}.$$

   Since $\mathbb{N}^{**} = \mathcal{B}^{-n}(\mathcal{F}\mathbb{N}) \subseteq \mathcal{B}^{-(n+1)}(\mathcal{F}\mathbb{N})$ we obtain the conclusion for the induction step, that $\mathcal{B}^{-(n+1)}(\mathcal{F}\mathbb{N}) = \mathbb{N}^{**}$.

2. $\mathcal{V}$ are closure operators by the transitivity of $(\mathcal{B}^{-1})^{+}$, the reflexivity of $\mathcal{B}$, and because $\mathcal{B}$ is closed under isomorphism.

3. Consider any $\mathbb{M}_1 \in \mathit{Fixed}(\mathcal{U}_{\Sigma_1})$. This means $\mathrm{Iso}(\mathbb{M}_1) = \mathcal{F}\mathbb{M}_1$. Then

$$\mathcal{F}(\mathbb{M}_1\!\restriction_{\varphi_1}) = \mathrm{Iso}((\mathcal{F}\mathbb{M}_1)\!\restriction_{\varphi_1}) = \mathrm{Iso}(\mathrm{Iso}(\mathbb{M}_1)\!\restriction_{\varphi_1}) = \mathrm{Iso}(\mathbb{M}_1\!\restriction_{\varphi_1}).$$

   The first equality holds because φ_1 preserves all filtered products, the second by the hypothesis that $\mathbb{M}_1$ is a fixed point for $\mathcal{F} = \mathcal{U}$, and the third by a simple calculation with isomorphisms. Hence $\mathbb{M}_1\!\restriction_{\varphi_1} \in \mathit{Fixed}(\mathcal{U}_{\Sigma})$.

4. This condition holds because for each $\mathbb{M}_1 \subseteq \mathsf{Mod}(\Sigma_1)$,

$$\mathcal{B}^{-1}(\mathrm{Iso}(\mathbb{M}_1\!\restriction_{\varphi_1})) = \mathcal{B}^{-1}(\mathbb{M}_1\!\restriction_{\varphi_1}) \subseteq \mathcal{B}^{-1}((\mathrm{Iso}\mathbb{M}_1)\!\restriction_{\varphi_1}) \subseteq \mathcal{B}^{-1}(\mathrm{Iso}(\mathbb{M}_1\!\restriction_{\varphi_1}))$$

   by using the fact that $\mathcal{B}$ is closed under isomorphisms and that the reduct $\mathsf{Mod}(\varphi_1)$ as a functor preserves isomorphisms.

5. This condition just means that φ_2 lifts $\mathcal{B}^{+}$ (the transitive closure of $\mathcal{B}$) which is a consequence of the fact that φ_2 lifts $\mathcal{B}$.

Now consider a set E_1 of Σ_1-sentences and E_2 a set of Σ_2-sentences such that $\theta_1(E_1) \models \theta_2(E_2)$. By setting $\mathbb{M}_1 = E_1^*$ and $\mathbb{M}_2 = E_2^*$ in the statement of Thm. 9.3, according to its conclusion we obtain a semantic interpolant closed under isomorphisms $\mathbb{M} \subseteq |\mathsf{Mod}(\Sigma)|$ such that $\mathbb{M} \in \mathit{Fixed}(\mathcal{U}_\Sigma) \cap \mathit{Fixed}(\mathcal{V}_\Sigma)$. This means $\mathbb{M} = \mathcal{B}^{-1}(\mathcal{F}\mathbb{M})$ hence $\mathbb{M}^{**} = \mathbb{M}$. Thus the desired interpolant is $E = \mathbb{M}^*$. □

Apart from the fundamental axiomatizability framework of a Birkhoff institution, from the hypotheses of Thm. 9.6 only the lifting condition sets substantial limits to its applicability. The other conditions can usually be handled as follows:

- Any pushout square of signature morphisms in an institution with weak model amalgamation is a weak amalgamation square. Therefore we need the basic assumption that the institution has weak model amalgamation.
- It is common that in institutions in which the signatures contain only symbols with finite arities, filtered products of models are preserved by the model reducts corresponding to *any* signature morphism. For the case of **FOL** and related institutions this has been shown in Sect. 6.2.

The lifting condition. Let us now focus on the 'interesting' condition underlying Thm. 9.6, that φ_2 lifts $\mathcal{B}$. This condition has to be handled at the level of concrete applications. Below we give an important example.

Proposition 9.7. *In* **FOL**, *each* (iii)*-morphism of signatures lifts* $\mathcal{B}$ *for each* $\mathcal{B} \in \{\stackrel{S_w}{\to}, \stackrel{S_c}{\to}, \stackrel{H_r}{\leftarrow}, \stackrel{H_s}{\leftarrow}\}$. *Consequently, each* (iii)*-morphism of signatures lifts* $\stackrel{H}{\leftarrow};\stackrel{S}{\to}$ *for each* $H \in \{H_r, H_s\}$ *and each* $S \in \{S_w, S_c\}$.

Proof. Let $\varphi : (S,F,P) \to (S',F',P')$ be a (iii)-morphism of **FOL**-signatures.

Let $h : M'\!\upharpoonright_\varphi \to N$ be an injective (S,F,P)-model homomorphism, respectively, let $h : N \to M'\!\upharpoonright_\varphi$ be a surjective (S,F,P)-model homomorphism.

We define the (S',F',P')-model N' and a model homomorphism $h' : M' \to N'$, respectively, $h' : N' \to M'$ as follows:

– $N'_{\varphi^{\mathrm{st}}(s)} = N_s$ for each $s \in S$ (this is correctly defined because φ^{st} is injective),

– $N'_{s'} = M'_{s'}$ for each $s' \notin \varphi^{\mathrm{st}}(S)$.

This determines an injective S'-sorted function $h' : M' \to N'$, respectively, a surjective S'-sorted function $h' : N' \to M'$ which is an expansion of the S-sorted function $h : M'\!\upharpoonright_\varphi \to N$, respectively, of $h : N \to M'\!\upharpoonright_\varphi$, and such that $h'_{s'}$ is an identity function for each $s' \in S' \setminus \varphi^{\mathrm{st}}(S)$.

- $N'_{\varphi^{\mathrm{op}}_{w\to s}(\sigma)} = N_\sigma$ for each $\sigma \in F_{w\to s}$ (this is correctly defined because $\varphi^{\mathrm{op}}_{w\to s}$ is injective),
- for each $\sigma' \in F'_{w\to s}$ such that $\sigma' \notin \varphi^{\mathrm{op}}(F)$,
  - for the case of $h : M'\!\upharpoonright_\varphi \to N$ injective, let

    * $N'_{\sigma'}(h_w(x)) = h_s(M'_{\sigma'}(x))$ if $x \in M'_w$ (this is correctly defined because h is injective),
    * otherwise let $N'_{\sigma'}(x)$ be any element of N'_s

  and
  - for the case of $h\colon\ N \to M'\restriction_\varphi$ surjective, let $N'_{\sigma'}(x)$ be any element of N'_s such that $h_s(N'_{\sigma'}(x)) = M'_{\sigma'}(h_w(x))$.

- $N'_{\varphi^{\mathrm{rl}}(\pi)} = N_\pi$ for each $\pi \in P$ (this is correctly defined because φ^{rl} is injective),
- for each $\pi' \in P'_w$ such that $\pi' \notin \varphi^{\mathrm{rl}}(P)$:
  - $N'_{\pi'} = h'_w(M'_{\pi'})$ for the injective homomorphism case, and
  - $N'_{\pi'} = (h'_w)^{-1}(M'_{\pi'})$ for the surjective homomorphism case.

Note that $N'\restriction_\varphi = N$ and that h' is injective, respectively, surjective (S', F', P')-model homomorphism such that $h'\restriction_\varphi = h$. Also h' is closed when h is closed. Moreover, in the case of $h\colon\ N \to M'\restriction_\varphi$ surjective, h' is strong when h is strong. □

Instances of Thm. 9.6. Based upon some of the axiomatizability results listed at the end of Sect. 8.6, by the interpolation Thm. 9.6 and the lifting Prop. 9.7 we have the following interpolation results:

Corollary 9.8. *The institutions* **UNIV**, *of the universal* $\mathbf{FOL}_{\infty,\omega}$*-sentences,* **HCL**, $\mathbf{HCL}_\infty$, *of universal* **FOL***-atoms,* **EQL**, $\forall\vee$, $\forall\vee_\infty$ *have Craig* $(\mathbb{S}ig, (iii))$*-interpolation.*

The counterexample below shows that the injectivity condition on the signature morphisms from $\mathcal{R}$ is necessary.

A counterexample. In **EQL** consider the pushout square of signature morphisms

$$\begin{array}{ccc} \Sigma = \{f, g\} & \xrightarrow{\varphi_1} & \Sigma_1 = \{f, g, h\} \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 = \{k\} & \xrightarrow[\theta_2]{} & \Sigma' = \{k, h\} \end{array}$$

such that all the signatures involved contain only one sort and one constant (not shown in the diagram) and only unary operations as shown in the diagram. Let

- $E_1 = \{(\forall x)g(x) = h(f(x)), (\forall x)f(g(x)) = h(g(x))\}$ and
- $E_2 = \{(\forall x)k(k(x)) = k(x)\}$.

It is easy to see that $\theta_1(E_1) \models_{\Sigma'} \theta_2(E_2)$. Any interpolant E would contain only equations containing the unary operations f and g which are consequences of E_1. Since there is no way to get rid off h, E may contain only reflexive equations $(\forall X)t = t$ but in this case $\varphi_2(E) \not\models E_2$. This shows that an interpolant for E_1 and E_2 does not exist.

Interpolation by the Keisler-Shelah property. In situations when $\mathcal{B}$ is rather weakly defined, the lifting condition can be rather difficult to establish. The cost is thus shifted from the axiomatizability property to the lifting condition on φ_2. A typical example is given by **FOL**, regarded as a Birkhoff institution with $\mathcal{B}$ the elementary equivalence relation $\equiv$, and $\mathcal{F}$ the class of all ultrafilters (cf. Thm. 8.38). A solution to this problem is given by the Keisler-Shelah property (cf. Cor. 7.27) via Cor. 8.8 which says that a class of **FOL**-models is elementary if and only if it is closed under ultraproducts and under ultraradicals. This provides a characterization of elementary equivalence $\equiv$ strong enough for supporting an easy applicability of the interpolation Thm. 9.6.

Corollary 9.9. **FOL** *has Craig* $(\mathbb{S}ig^{\mathbf{FOL}}, (i**))$*-interpolation.*

Proof. We use the Birkhoff axiomatizability characterization of elementary classes in **FOL**,

$$\mathbb{M}^{**} = \mathrm{Ur}^{-1}(\mathrm{Up}(\mathbb{M}))$$

given by Thm. 8.38.

Let us show that each **FOL** signature morphism φ which lifts isomorphisms, also lifts the ultraradical relation Ur. If $M'\restriction_\varphi(\mathrm{Ur})N$ then either $M'\restriction_\varphi \cong N$ or N is an ultrapower of $M'\restriction_\varphi$. The first case is immediate by the hypothesis on φ. So assume $\prod_U(M'\restriction_\varphi) = N$ for some ultrafilter U. Then $(\prod_U M')\restriction_\varphi = \prod_U(M'\restriction_\varphi)$ because all **FOL**-signature morphisms preserve all filtered products, and by the $\cong$-lifting there exists N' a φ-expansion of N such that $\prod_U M' \cong N'$, hence $M'(\mathrm{Ur})N'$.

The conclusion of this corollary follows by:

Fact 9.10. *A* **FOL** *signature morphism lifts isomorphisms if and only if it is* $(i**)$.

$\square$

The 'left' interpolation theorem

The second interpolation by axiomatizability theorem presented below shifts the reliance upon the lifting property of the signature morphisms from those on the 'right-hand side' to those on the 'left-hand side' of the interpolation squares of signature morphisms. One consequence of this is that the lifting condition on $\mathcal{B}$ rather becomes a lifting condition on its inverse $\mathcal{B}^{-1}$.

Theorem 9.11. *In a Birkhoff institution* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathcal{F}, \mathcal{B})$*, any weak amalgamation square*

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

such that

1. $\mathsf{Mod}(\varphi_1)$ *preserves* $\mathcal{F}$*-filtered products (of models), and*

2. *φ_1 lifts $\mathcal{B}^{-1}$ and isomorphisms,*

is a Craig Interpolation square.

Proof. We apply the abstract semantic interpolation Thm. 9.3 by setting the semantic operators $\mathcal{U}$ and $\mathcal{V}$ as follows:

- $\mathcal{U}$ to be the elementary closure operators, i.e., $\mathcal{U}(\mathbb{M}) = \mathbb{M}^{**}$, and
- $\mathcal{V}$ to be the identities.

Because the hypotheses 1,2 and 5 of Thm. 9.3 are rather easy or trivial to check, we focus on the remaining ones.

3. Let $\mathbb{M}_1 \in \mathit{Fixed}(\mathcal{U}_{\Sigma_1})$ which means that $\mathbb{M}_1^{**} = \mathrm{Iso}(\mathbb{M}_1)$. We have to show that $(\mathbb{M}_1 \restriction_{\varphi_1})^{**} = \mathrm{Iso}(\mathbb{M}_1 \restriction_{\varphi_1})$. We have the following chain of equalities

$$
\begin{array}{lll}
(\mathbb{M}_1 \restriction_{\varphi_1})^{**} & = \mathcal{B}_\Sigma^{-1}(\mathcal{F}(\mathbb{M}_1 \restriction_{\varphi_1})) & \text{(by the Birkhoff institution condition)} \\
& = \mathcal{B}_\Sigma^{-1}(\mathcal{F}\mathrm{Iso}(\mathbb{M}_1 \restriction_{\varphi_1})) & \text{(filtered products are defined up to iso-morphisms)} \\
& = \mathcal{B}_\Sigma^{-1}(\mathcal{F}((\mathrm{Iso}\mathbb{M}_1) \restriction_{\varphi_1})) & (\varphi_1 \text{ lifts isomorphisms}) \\
& = \mathcal{B}_\Sigma^{-1}(\mathcal{F}(\mathbb{M}_1^{**} \restriction_{\varphi_1})) & (\mathbb{M}_1 \in \mathit{Fixed}(\mathcal{U}_{\Sigma_1})) \\
& = \mathcal{B}_\Sigma^{-1}(\mathrm{Iso}((\mathcal{F}\mathbb{M}_1^{**}) \restriction_{\varphi_1})) & (\mathsf{Mod}(\varphi_1) \text{ preserves } \mathcal{F}\text{-filtered products}) \\
& = \mathcal{B}_\Sigma^{-1}((\mathcal{F}\mathbb{M}_1^{**}) \restriction_{\varphi_1}) & (\mathcal{B} \text{ is closed under isomorphisms}) \\
& \subseteq \mathcal{B}_\Sigma^{-1}(\mathcal{B}_{\Sigma_1}^{-1}(\mathcal{F}\mathbb{M}_1^{**})) \restriction_{\varphi_1}) & (\mathcal{B}\text{'s are reflexive}) \\
& = \mathcal{B}_\Sigma^{-1}(\mathbb{M}_1^{****} \restriction_{\varphi_1}) & \text{(by the Birkhoff institution condition)} \\
& = \mathcal{B}_\Sigma^{-1}(\mathbb{M}_1^{**} \restriction_{\varphi_1}) & \\
& \subseteq (\mathcal{B}_{\Sigma_1}^{-1}(\mathbb{M}_1^{**})) \restriction_{\varphi_1} & (\varphi_1 \text{ lifts } \mathcal{B}^{-1}) \\
& \subseteq (\mathcal{B}_{\Sigma_1}^{-1}(\mathcal{F}\mathbb{M}_1^{**})) \restriction_{\varphi_1} & (\{\{*\}\} \in \mathcal{F}) \\
& = \mathbb{M}_1^{****} \restriction_{\varphi_1} & \text{(by the Birkhoff institution condition)} \\
& = \mathbb{M}_1^{**} \restriction_{\varphi_1} & \\
& = (\mathrm{Iso}(\mathbb{M}_1)) \restriction_{\varphi_1} & (\mathbb{M}_1 \in \mathit{Fixed}(\mathcal{U}_{\Sigma_1})) \\
& \subseteq \mathrm{Iso}(\mathbb{M}_1 \restriction_{\varphi_1}) & (\mathsf{Mod}(\varphi_1) \text{ as functor preserves iso-morphisms})
\end{array}
$$

   Since we work only with institutions closed under isomorphisms, we also have that $(\mathbb{M}_1 \restriction_{\varphi_1})^{**} \subseteq \mathrm{Iso}(\mathbb{M}_1 \restriction_{\varphi_1})$, hence $(\mathbb{M}_1 \restriction_{\varphi_1})^{**} = \mathrm{Iso}(\mathbb{M}_1 \restriction_{\varphi_1})$.

4. This condition holds because $\mathsf{Mod}(\varphi_1);\mathrm{Iso} = \mathrm{Iso};\mathsf{Mod}(\varphi_1)$ which just means that φ_1 lifts isomorphisms.

The conclusion of Thm. 9.3 tells us that for any sets E_1 of Σ_1-sentences and E_2 of Σ_2-sentences such that $\theta_1(E_1) \models \theta_2(E_2)$ there exists a semantic interpolant $\mathbb{M}$ closed under isomorphisms and such that $\mathbb{M}^{**} = \mathrm{Iso}(\mathbb{M})$. By the closure of $\mathbb{M}$ under isomorphisms this means that $\mathbb{M}^{**} = \mathbb{M}$, hence $\mathbb{M}$ is elementary. By the principle of semantic interpolation, the desired (syntactic) interpolant for E_1 and E_2 is just $E = \mathbb{M}^*$. □

For obtaining concrete instances of Thm. 9.11 above we follow the same path as in the case of the first interpolation by axiomatizability Thm. 9.6. Therefore we have to establish classes of signature morphisms that lift various concrete relations.

The lifting condition. The following result establishes lifting of the inverses of the relations considered by Prop. 9.7.

Proposition 9.12. *In* **FOL**, *each* $(ie*)$*-morphism of signatures lifts* $\mathcal{B}^{-1}$ *for each* $\mathcal{B} \in \{\xrightarrow{S_w}, \xrightarrow{S_c}, \xleftarrow{H_s}\}$ *and each* (iei)*-morphism lifts* $\xrightarrow{H_r}$. *Consequently, each* $(ie*)$*-morphism of signatures lifts* $\xleftarrow{S};\xrightarrow{H_s}$ *and each* (iei)*-morphism lifts* $\xleftarrow{S};\xrightarrow{H_r}$ *for each* $S \in \{S_w, S_c\}$.

Proof. Let $\varphi : (S,F,P) \to (S',F',P')$ be an $(ie*)$-morphism of **FOL**-signatures.

Let $h : N \to M'\upharpoonright_\varphi$ be an injective (S,F,P)-model homomorphism, respectively, let $h : M'\upharpoonright_\varphi \to N$ be a surjective (S,F,P)-model homomorphism.

We define the (S',F',P')-model N' and an injective model homomorphism $h' : N' \to M'$, respectively, a surjective model homomorphism $h' : M' \to N'$ as follows:

- $N'_{\varphi(s)} = N_s$ for each $s \in S$ (this is correctly defined because φ^{st} is injective),
- for each $s' \notin \varphi^{\mathrm{st}}(S)$ let $N'_{s'} = M'_{s'}$ for the injective case, respectively, let $N'_{s'} = \{*\}$ be a singleton set for the surjective case.

This determines an injective S'-sorted function $h' : N' \to M'$, respectively, a surjective S'-sorted function $h' : M' \to N'$, which is an expansion of the S-sorted function $h : N \to M'\upharpoonright_\varphi$, respectively of, $h : M'\upharpoonright_\varphi \to N$.

For each operation symbol $\sigma' \in F'_{w'\to s'}$ and for each list of arguments $x \in N'_{w'}$ we define $N'_{\sigma'}(x)$ to be

- $N'_{\sigma'}(x) = h^{-1}(M'_{\sigma'}(h(x)))$ in the injective case. This is correctly defined because of the encapsulation condition e on the operations, because of the injectivity of h, and furthermore it makes h' an F'-homomorphism.
- $N'_{\sigma'}(x) = \begin{cases} N_\sigma(x) & \text{when } \sigma' = \varphi(\sigma), \\ * & \text{when } \sigma' \notin \varphi(F) \end{cases}$
  in the surjective case. This definition is correct because of the encapsulation condition e on the operations, because of the surjectivity of h, and it furthermore makes h' an F'-homomorphism.

For each relation symbol $\pi' \in P'$ we define $N'_{\pi'}$ to be

- $N'_{\pi'} = h^{-1}(M'_{\pi'})$ in the injective case, and
- $N'_{\pi'} = \begin{cases} N_\pi & \text{when } \pi' = \varphi(\pi) \text{ and } h \in H_r, \\ h(M'_{\pi'}) & \text{when } \pi' \notin \varphi(P) \text{ or } h \in H_s \end{cases}$
  in the surjective case.

Note that $N'\upharpoonright_\varphi = N$, $h'\upharpoonright_\varphi = h$, h' is closed when h is closed in the injective case, and h' is strong if h is strong in the surjective case. □

Instances of Thm. 9.11. Based upon some of the axiomatizability results listed at the end of Sect. 8.6, by the interpolation Thm. 9.11 and the lifting Prop. 9.12, and also because each $(i**)$-morphism of signatures lifts isomorphisms of **FOL**-models (cf. Fact 9.10) we have the following interpolation results:

Corollary 9.13. *The institutions below have Craig $(\mathcal{L}, \mathbb{S}ig)$-interpolation*

institution	$\mathcal{L}$
UNIV	$ie*$
universal $\mathbf{FOL}_{\infty,\omega}$*-sentences*	
HCL, $\mathbf{HCL}_\infty$, $\forall\vee$, *and* $\forall\vee_\infty$	
universal **FOL**-*atoms*	iei
EQL	ie

Exercises

9.3. Interpolation in PL
In propositional logic (**PL**) each pushout square of signatures is a CI square. (*Hint:* **PL** is a Birkhoff institution with $\mathcal{F}$ the class of all ultrafilters and $\mathcal{B}$ the identity relation.)

9.4. Interpolation for partial algebra
By the general axiomatizability results of this section, through the axiomatizability results for partial algebras of the Exercises 8.4, 8.6 and 8.7, formulate and prove interpolation results for partial algebras.

9.5. Given a weakly semi-exact institution I, let C be the class of the conservative signature morphisms φ for which $\mathsf{Sen}^I(\varphi)$ is surjective. Then I has both the Craig $(C, \mathbb{S}ig^I)$ and $(\mathbb{S}ig^I, C)$-interpolation properties.

9.3 Interpolation by Consistency

In this section we develop another method for obtaining interpolation properties of institutions, via Robinson consistency.

Robinson consistency. Recall that a set of sentences E for a signature Σ in an arbitrary institution is *consistent* if it has models, i.e., E^* is not empty.

A commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is a *Robinson Consistency square* (abbreviated *RC* square) if and only if all theories $E_i \subseteq \mathsf{Sen}(\Sigma_i)$, $i \in \{1,2\}$, with 'inter-consistent reducts', i.e., $\varphi_1^{-1}(E_1) \cup \varphi_2^{-1}(E_2)$ is consistent, have 'inter-consistent Σ'-translations', i.e., $\theta_1(E_1) \cup \theta_2(E_2)$ is consistent.

Note that Robinson consistency has substance in institutions where consistency is not a property of each set of sentences (unlike in **HCL**or **EQL**, where it is, see Cor. 4.28).

Robinson consistency versus Craig interpolation

The method to obtain Craig interpolation by Robinson consistency relies upon the following equivalence between these two properties.

Theorem 9.14. *In any quasi-compact institution with negation and conjunctions, each commuting square of signature morphisms is a Robinson Consistency square if and only if it is a Craig Interpolation square.*

Proof. CI implies RC: Let us assume the existence of theories E_1 and E_2 such that $\theta_1(E_1) \cup \theta_2(E_2)$ is inconsistent while $\varphi_1^{-1}(E_1) \cup \varphi_2^{-1}(E_2)$ is consistent and reach a contradiction.

By quasi-compactness there are finite subsets $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$ such that $\theta_1(E_1') \cup \theta_2(E_2')$ is inconsistent. This implies that $\theta_1(\wedge E_1') \models_{\Sigma'} \theta_2(\neg \wedge E_2')$. By CI and compactness there exists a finite set of Σ-sentences E_0 such that $\wedge E_1' \models \varphi_1(E_0)$ and $\varphi_2(E_0) \models \neg \wedge E_2'$.

On the one hand $\wedge E_1' \models \varphi_1(E_0)$ implies that $E_1 \models \varphi_1(E_0)$ which means that $\wedge E_0 \in \varphi_1^{-1}(E_1)$. On the other hand, $\varphi_2(E_0) \models \neg \wedge E_2'$ implies $\wedge E_2' \models \varphi_2(\neg \wedge E_0)$, which further implies $E_2 \models \varphi_2(\neg \wedge E_0)$, which means $\neg \wedge E_0 \in \varphi_2^{-1}(E_2)$.

But from $\wedge E_0 \in \varphi_1^{-1}(E_1)$ and $\neg \wedge E_0 \in \varphi_2^{-1}(E_2)$ we derive the inconsistency of $\varphi_1^{-1}(E_1) \cup \varphi_2^{-1}(E_2)$ contradicting thus the assumption that $\varphi_1^{-1}(E_1) \cup \varphi_2^{-1}(E_2)$ is consistent.

RC implies CI: Consider $E_i \subseteq \mathsf{Sen}(\Sigma_i)$, $i \in \{1,2\}$, such that $\theta_1(E_1) \models \theta_2(E_2)$. For each $e_2 \in E_2$ we have that $\theta_1(E_1) \cup \theta_2(\neg e_2)$ is inconsistent because $\theta_1(E_1) \models \theta_2(e_2)$. By RC we deduce that $\varphi_1^{-1}(E_1^{**}) \cup \varphi_2^{-1}(\{\neg e_2\}^{**})$ is inconsistent too. Thus there exists $\Gamma_1(e_2) \subseteq \varphi_1^{-1}(E_1^{**})$ and $\Gamma_2(e_2) \subseteq \varphi_2^{-1}(\{\neg e_2\}^{**})$, by quasi-compactness both finite, such that $\Gamma_1(e_2) \cup \Gamma_2(e_2)$ is inconsistent. We have that $\Gamma_1(e_2) \models_\Sigma \neg \wedge \Gamma_2(e_2)$. Because $\Gamma_1(e_2) \subseteq \varphi_1^{-1}(E_1^{**})$ we have that $E_1 \models \varphi_1(\Gamma_1(e_2))$. Because $\Gamma_2(e_2) \subseteq \varphi_2^{-1}(\{\neg e_2\}^{**})$ we have that $\varphi_2(\neg \wedge \Gamma_2(e_2)) \models e_2$ which means that $\varphi_2(\Gamma_1(e_2)) \models e_2$. Therefore the desired interpolant for E_1 and E_2 can be taken as $\bigcup\{\Gamma_1(e_2) \mid e_2 \in E_2\}$. $\square$

CI is generally an asymmetric property with respect to the reflection in the mirror of the considered squares of signature morphisms, while RC is a symmetric property. The equivalence between CI and RC given by Thm. 9.14 brings the symmetry of RC to CI. This allows the extension of CI properties of institutions as illustrated by the following extension of the interpolation property of **FOL** formulated by Cor. 9.9.

Corollary 9.15. **FOL** *has Craig* $(\mathbb{S}ig^{\mathbf{FOL}}, (i**))$ *and* $((i**), \mathbb{S}ig^{\mathbf{FOL}})$*-interpolation.*

Robinson consistency theorem

Below we give a set of sufficient conditions for Robinson consistency which contain the conditions underlying the equivalence between Robinson consistency and Craig interpolation. We need the following concept of lifting of isomorphisms.

Lifting isomorphisms. A span of signature morphisms $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ is said to *lift isomorphisms* if for each Σ_i-models M_i, $i \in \{1,2\}$, such that $M_1 \restriction_{\varphi_1} \cong M_2 \restriction_{\varphi_2}$ are isomorphic, there exists Σ_i-models N_i such that $M_i \cong N_i$ are isomorphic and $N_1 \restriction_{\varphi_1} = N_2 \restriction_{\varphi_2}$.

A commutative square of signature morphisms

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\
\Sigma_2 \arrow[r, "\theta_2"'] & \Sigma'
\end{tikzcd}$$

lifts isomorphisms if the span $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ lifts isomorphisms.

Theorem 9.16 (Robinson consistency). *In any institution with elementary diagrams ι such that*

1. *$M^* \subseteq N^*$ if there exists a model homomorphism $M \to N$,*
2. *it has pushouts of signatures and has weak model amalgamation,*
3. *it has universal χ-quantification for χ signature morphisms of the forms $\iota_\Sigma(h)$ and $\iota_\Sigma(M)$ for all Σ-model homomorphisms $h\colon\ M \to N$,*
4. *it has negations and finite conjunctions,*
5. *it has ω-co-limits*[1] *of models which are preserved by the model reduct functors, and*
6. *it is quasi-compact,*

any weak amalgamation square (and in particular any pushout square) which lifts isomorphisms

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\
\Sigma_2 \arrow[r, "\theta_2"'] & \Sigma'
\end{tikzcd}$$

is a Robinson Consistency square (and by Thm. 9.14 a Craig interpolation square too).

Proof. Let $E_i \subseteq \mathsf{Sen}(\Sigma_i)$ be theories. Denote $\Gamma_i = \varphi_i^{-1}(E_i)$, which are also theories. Assume $\Gamma_1 \cup \Gamma_2$ is consistent. We have to prove that $\theta_1(E_1) \cup \theta_2(E_2)$ is consistent too. It suffices to find Σ_i-models $M_i \models E_i$ such that $M_1 \restriction_{\varphi_1} = M_2 \restriction_{\varphi_2}$, and then we apply weak amalgamation to find the desired Σ'-model M' of $\theta_1(E_1) \cup \theta_2(E_2)$.

We construct inductively two chains of Σ_i-homomorphisms $\{A_i^n \xrightarrow{f_i^n} A_i^{n+1}\}_{n\in\omega}$ as follows:

(0.1) we find a Σ_1-model A_1^0 such that $A_1^0 \models E_1$ and $A_1^0 \restriction_{\varphi_1} \models \Gamma_2$,

[1] Here ω is the totally ordered set of the natural numbers.

(0.2) we find a Σ_2-model A_2^0 such that $A_2^0 \models E_2$ and a Σ-homomorphism $g^0 : A_1^0\upharpoonright_{\varphi_1} \to A_2^0\upharpoonright_{\varphi_2}$,

(n.1) for each natural number n, we find a Σ_1-homomorphism $f_1^n : A_1^n \to A_1^{n+1}$ and a Σ-homomorphism $h^n : A_2^n\upharpoonright_{\varphi_2} \to A_1^{n+1}\upharpoonright_{\varphi_1}$ such that $f_1^n\upharpoonright_{\varphi_1} = g^n;h^n$, and

(n.2) for each natural number n, we find a Σ_2-homomorphism $f_2^n : A_2^n \to A_2^{n+1}$ and a Σ-homomorphism $g^{n+1} : A_1^{n+1}\upharpoonright_{\varphi_1} \to A_2^{n+1}\upharpoonright_{\varphi_2}$ such that $f_2^n\upharpoonright_{\varphi_2} = h^n;g^{n+1}$.

We therefore get the following commutative diagram Dg in $\mathsf{Mod}(\Sigma)$:

$$\begin{tikzcd}
A_1^0\upharpoonright_{\varphi_1} \arrow[rr, "f_1^0\upharpoonright_{\varphi_1}"] \arrow[dr, "g^0"'] & & A_1^1\upharpoonright_{\varphi_1} \arrow[rr, "f_1^1\upharpoonright_{\varphi_1}"] \arrow[dr, "g^1"] & & A_1^2\upharpoonright_{\varphi_1} \arrow[dr, "g^2"] & & \cdots \\
& A_2^0\upharpoonright_{\varphi_2} \arrow[ur, "h^0"] \arrow[rr, "f_2^0\upharpoonright_{\varphi_2}"'] & & A_2^1\upharpoonright_{\varphi_2} \arrow[ur, "h^1"] \arrow[rr, "f_2^1\upharpoonright_{\varphi_2}"'] & & A_2^2\cdots &
\end{tikzcd}$$

Because the reduct functors preserve directed co-limits, the co-limits of $\{f_i^n\}_{n\in\omega}$ in $\mathsf{Mod}(\Sigma_i)$ (with vertexes denoted as N_i) are mapped by $\mathsf{Mod}(\varphi_i)$ to co-limits. Since both $\{f_i^n\upharpoonright_{\varphi_i}\}_{n\in\omega}$ are final sub-diagrams of Dg, it follows (by Thm. 2.4) that $N_1\upharpoonright_{\varphi_1} \cong N_2\upharpoonright_{\varphi_2} \cong colim(Dg)$. Because model homomorphisms preserve satisfaction and $A_i^0 \models E_i$ we have that $N_i \models E_i$. Because the commutative square of signature morphisms lifts isomorphisms we find $N_i \cong M_i$ (so $M_i \models E_i$ too) such that $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$.

Proof of (0.1): If A_1^0 did not exist, then $E_1 \cup \varphi_1(\Gamma_2)$ would be inconsistent. By quasi-compactness and finite conjunctions, $E_1 \cup \varphi_1(\gamma_2)$ would be inconsistent for some $\gamma_2 \in \Gamma_2$. This implies $E_1 \models \varphi_1(\neg\gamma_2)$, so $\neg\gamma_2 \in \Gamma_1$, making $\Gamma_1 \cup \Gamma_2$ inconsistent.

Proof of (0.2): By using elementary diagrams it suffices to find $B \models E_{(A_1^0\upharpoonright_{\varphi_1})}$ and $A_2^0 \models E_2$ such that $B\upharpoonright_{\iota_\Sigma(A_1^0\upharpoonright_{\varphi_1})} = A_2^0\upharpoonright_{\varphi_2}$ (define $g^0 = i_{\Sigma,A_1^0\upharpoonright_{\varphi_1}}(B)$).

For this it suffices to consider a pushout of $\iota_\Sigma(A_1^0\upharpoonright_{\varphi_1})$ and φ_2

$$\begin{tikzcd}
\Sigma \arrow[r, "\iota_\Sigma(A_1^0\upharpoonright_{\varphi_1})"] \arrow[d, "\varphi_2"'] & \Sigma_{(A_1^0\upharpoonright_{\varphi_1})} \arrow[d, "u"] \\
\Sigma_2 \arrow[r, "v"'] & \bullet
\end{tikzcd}$$

and find a model of $u(E_{(A_1^0\upharpoonright_{\varphi_1})}) \cup v(E_2)$. But if this set of sentences is inconsistent, then by quasi-compactness and finite conjunctions, we would find a $\Sigma_{(A_1^0\upharpoonright_{\varphi_1})}$-sentence e such that $E_{(A_1^0\upharpoonright_{\varphi_1})} \models e$ and $v(E_2) \models u(\neg e)$. The latter relation implies $E_2 \models \varphi_2((\forall \iota_\Sigma(A_1^0\upharpoonright_{\varphi_1}))\neg e)$, which by the definition of Γ_2 means $(\forall \iota_\Sigma(A_1^0\upharpoonright_{\varphi_1}))\neg e \in \Gamma_2$. Because $A_1^0\upharpoonright_{\varphi_1} \models \Gamma_2$, we have that $A_1^0\upharpoonright_{\varphi_1} \models (\forall \iota_\Sigma(A_1^0\upharpoonright_{\varphi_1}))\neg e$, which implies $(A_1^0\upharpoonright_{\varphi_1})_{A_1^0\upharpoonright_{\varphi_1}} \models \neg e$, contradicting the fact that $E_{(A_1^0\upharpoonright_{\varphi_1})} \models e$.

Proof of (n.1): We first show that it suffices to find a $(\Sigma_1)_{A_1^n}$-model $F_1^n \models E_{(A_1^n)}$ and a $\Sigma_{(A_2^n\upharpoonright_{\varphi_2})}$-model $H^n \models E_{(A_2^n\upharpoonright_{\varphi_2})}$ such that $F_1^n\upharpoonright_{\iota_{\varphi_1}(A_1^n\upharpoonright_{\varphi_1})} = H^n\upharpoonright_{\iota_\Sigma(g^n)}$.

Assuming F_1^n and H^n exist, we define $f_1^n = i_{\Sigma_1,A_1^n}(F_1^n)$ and $h^n = i_{\Sigma,A_2^n\upharpoonright_{\varphi_2}}(H^n)$ and let us prove that $f_1^n\upharpoonright_{\varphi_1} = g^n;h^n$. Note that by the functoriality of ι the diagram below commutes:

$$\begin{tikzcd}
(\Sigma_1)_{A_1^n} & \Sigma_{(A_1^n\upharpoonright_{\varphi_1})} \arrow[l, "\iota_{\varphi_1}(A_1^n\upharpoonright_{\varphi_1})"'] \arrow[r, "\iota_\Sigma(g^n)"] & \Sigma_{(A_2^n\upharpoonright_{\varphi_2})} \\
\Sigma_1 \arrow[u, "\iota_{\Sigma_1}(A_1^n)"] & \Sigma \arrow[l, "\varphi_1"] \arrow[u, "\iota_\Sigma(A_1^n\upharpoonright_{\varphi_1})"'] \arrow[ur, bend right, "\iota_\Sigma(A_2^n\upharpoonright_{\varphi_2})"'] &
\end{tikzcd}$$

and by the naturality of i the diagram below commutes:

$$\begin{tikzcd}
\mathsf{Mod}((\Sigma_1)_{A_1^n},E_{A_1^n}) \arrow[r, "\mathsf{Mod}(\iota_{\varphi_1}(A_1^n\upharpoonright_{\varphi_1}))"] \arrow[d, "i_{\Sigma_1,A_1^n}"'] & \mathsf{Mod}(\Sigma_{(A_1^n\upharpoonright_{\varphi_1})},E_{(A_1^n\upharpoonright_{\varphi_1})}) \arrow[d, "i_{\Sigma,(A_1^n\upharpoonright_{\varphi_1})}"'] & \mathsf{Mod}(\Sigma_{(A_2^n\upharpoonright_{\varphi_2})},E_{(A_2^n\upharpoonright_{\varphi_2})}) \arrow[l, "\mathsf{Mod}(\iota_\Sigma(g^n))"'] \arrow[d, "i_{\Sigma,(A_2^n\upharpoonright_{\varphi_2})}"] \\
A_1^n/\mathsf{Mod}(\Sigma_1) \arrow[r, "\mathsf{Mod}(\varphi_1)"'] & (A_1^n\upharpoonright_{\varphi_1})/\mathsf{Mod}(\Sigma) & (A_2^n\upharpoonright_{\varphi_2})/\mathsf{Mod}(\Sigma) \arrow[l, "g^n/\mathsf{Mod}(\Sigma)"]
\end{tikzcd}$$

Therefore $f_1^n\upharpoonright_{\varphi_1} = i_{\Sigma_1,A_1^n}(F_1^n)\upharpoonright_{\varphi_1} = i_{\Sigma,(A_1^n\upharpoonright_{\varphi_1})}(F_1^n\upharpoonright_{\iota_{\varphi_1}(A_1^n\upharpoonright_{\varphi_1})}) = i_{\Sigma,(A_1^n\upharpoonright_{\varphi_1})}(H^n\upharpoonright_{\iota_\Sigma(g^n)}) = g^n;i_{\Sigma,(A_2^n\upharpoonright_{\varphi_2})}(H^n) = g^n;h^n$.

Now in order to find F_1^n and H^n, it is enough to consider a pushout

$$\begin{tikzcd}
\Sigma_{(A_1^n\upharpoonright_{\varphi_1})} \arrow[r, "\iota_{\varphi_1}(A_1^n\upharpoonright_{\varphi_1})"] \arrow[d, "\iota_\Sigma(g^n)"'] & (\Sigma_1)_{A_1^n} \arrow[d, "u"] \\
\Sigma_{(A_2^n\upharpoonright_{\varphi_2})} \arrow[r, "v"'] & \bullet
\end{tikzcd}$$

and find a model for $u(E_{A_1^n})\cup v(E_{(A_2^n\upharpoonright_{\varphi_2})})$.

Suppose $u(E_{A_1^n})\cup v(E_{(A_2^n\upharpoonright_{\varphi_2})})$ is inconsistent. By an argument similar to the corresponding part of the proof of *(0.2)*, we find a $\Sigma_{(A_2^n\upharpoonright_{\varphi_2})}$-sentence e such that $E_{(A_2^n\upharpoonright_{\varphi_2})} \models e$ and $u(E_{A_1^n}) \models v(\neg e)$. The latter implies that $E_{A_1^n} \models \iota_{\varphi_1}(A_1^n\upharpoonright_{\varphi_1})((\forall\iota_\Sigma(g^n))\neg e)$. This shows that the initial $((\Sigma_1)_{A_1^n},E_{A_1^n})$-model $(A_1^n)_{A_1^n} \models (\forall\iota_\Sigma(g^n))\neg e$.

By the naturality of i we have that $((A_1^n)_{A_1^n})\upharpoonright_{\iota_{\varphi_1}(A_1^n\upharpoonright_{\varphi_1})} = (A_1^n\upharpoonright_{\varphi_1})_{A_1^n\upharpoonright_{\varphi_1}}$, therefore $(A_1^n\upharpoonright_{\varphi_1})_{A_1^n\upharpoonright_{\varphi_1}} \models (\forall\iota_\Sigma(g^n))\neg e$. Because $g^n:\ A_1^n\upharpoonright_{\varphi_1} \to A_2^n\upharpoonright_{\varphi_2}$, we have that $\iota_\Sigma(g^n): (\Sigma_{(A_1^n\upharpoonright_{\varphi_1})},E_{(A_1^n\upharpoonright_{\varphi_1})}) \to (\Sigma_{(A_2^n\upharpoonright_{\varphi_2})},E_{(A_2^n\upharpoonright_{\varphi_2})})$ is a presentation morphism, thus $(A_2^n\upharpoonright_{\varphi_2})_{A_2^n\upharpoonright_{\varphi_2}} \models \iota_\Sigma(g^n)(E_{(A_1^n\upharpoonright_{\varphi_1})})$.

Because $(\forall\iota_\Sigma(g^n))\neg e$ is a sentence of the institution, it is preserved by the unique model homomorphism $(A_1^n\upharpoonright_{\varphi_1})_{A_1^n\upharpoonright_{\varphi_1}} \to (A_2^n\upharpoonright_{\varphi_2})_{A_2^n\upharpoonright_{\varphi_2}}\upharpoonright_{\iota_\Sigma(g^n)}$ thus $(A_2^n\upharpoonright_{\varphi_2})_{A_2^n\upharpoonright_{\varphi_2}}\upharpoonright_{\iota_\Sigma(g^n)} \models (\forall\iota_\Sigma(g^n))\neg e$. This shows that $(A_2^n\upharpoonright_{\varphi_2})_{A_2^n\upharpoonright_{\varphi_2}} \models \neg e$ which is a contradiction with $E_{(A_2^n\upharpoonright_{\varphi_2})} \models e$.

Proof of (n.2): Similarly to the proof of *(n.1)*. □

The setup of Robinson consistency in Thm. 9.16 bears similarity to that of Thm. 7.9 on the existence of saturated models especially because the first condition requires that in the applications one has to consider the elementary homomorphisms as the model homomorphisms of the considered institution. Thus an important role in the applications of Robinson consistency Thm. 9.16 is played by instances of Cor. 5.37 establishing institutions with elementary diagrams for the elementary homomorphisms. In this context the 5th condition is handled as follows:

- The existence of ω-co-limits for models is handled by instances of the general result on directed co-limits of elementary homomorphisms given by Cor. 7.4.
- Since directed co-limits of elementary homomorphisms are obtained as co-limits using ordinary model homomorphisms (and afterwards the co-limiting co-cone is shown to consist of elementary homomorphisms), the preservation of ω-co-limits of elementary homomorphisms follows as a consequence of the arities of the symbols of the signatures being finite as in the typical example given by Prop. 6.5.

Note that the weak amalgamation condition is rather mild since it refers only to models and not to model homomorphisms.

In the applications the 3rd condition, on quantifiers, essentially requires that the institution has universal quantification for the class of signature extensions with constants. This is justified by the fact that usually both elementary extensions $\iota_\Sigma(M)$ and the signature morphisms of the form $\iota_\Sigma(h)$ for $h\colon\ M \to N$ elementary Σ-homomorphism are signature extensions with constants. In the case of $\iota_\Sigma(h)$ this is so because in the applications h is injective as an elementary homomorphism in an institution with negations. For establishing the universal quantification for signature extensions with an arbitrary number of constants, in institutions where quantification is defined only for signature extensions with a *finite* number of constants, one uses the same argument as in the proof of Cor. 9.5 which relies upon sentences being finitary.

Lifting of isomorphisms. The condition which narrows the class of RC squares is the lifting of isomorphisms. Let us see what it means in an actual situation.

Proposition 9.17. *A span of* **FOL** *signature morphisms* $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ *lifts isomorphisms if either* φ_1 *or* φ_2 *is an* $(i**)$*-morphism.*

Proof. Because of the symmetrical nature of the problem, let us assume without loss of generality that φ_1 is injective on sorts. Let M_i be Σ_i-models such that $M_1\restriction_{\varphi_1} \cong M_2\restriction_{\varphi_2}$. We have to find models N_1 and N_2 such that $M_i \cong N_i$ and $N_1\restriction_{\varphi_1} = N_2\restriction_{\varphi_2}$.

We define $N_2 = M_2$ and N_1 as follows:

- for each sort s_1 of Σ_1 which is not in the image of φ_1, let $(N_1)_{s_1} = (M_1)_{s_1}$,
- for each sort of Σ_1 of the form $\varphi_1(s)$, where s is a sort of Σ, let $(N_1)_{\varphi_1(s)} = (M_2)_{\varphi_2(s)}$, (This definition is correct because φ_1 is injective on the sorts.)
- for each sort s of Σ let $h_{\varphi_1(s)}\colon\ (M_1)_{\varphi_1(s)} \xrightarrow{\cong} (N_1)_{\varphi_1(s)}$ be the canonical bijection given by the isomorphism $(M_1)\restriction_{\varphi_1} \cong (M_2)\restriction_{\varphi_2}$),

- $h_{s_1} : (M_1)_{s_1} \xrightarrow{=} (N_1)_{s_1}$ is identity for each sort s_1 which is not in the image of φ_1, and
- N_1 is the unique Σ_1-model such that h defined as above is an isomorphism of Σ_1-models $M_1 \to N_1$.

□

We can now obtain again the **FOL** interpolation result of Cor. 9.15 but this time as an instance of Robinson consistency Thm. 9.16.

Corollary 9.18. **FOL** *has Craig* $(\mathbb{S}ig^{\mathbf{FOL}}, (i**))$ *and* $((i**), \mathbb{S}ig^{\mathbf{FOL}})$*-interpolation.*

A counterexample. The **FOL** interpolation result given by Cor. 9.15 or 9.18 is highly accurate in the sense that if none of the signature morphisms of a span $\Sigma_1 \xleftarrow{\varphi_1} \Sigma \xrightarrow{\varphi_2} \Sigma_2$ is $(i**)$ then the pushout of the span might fail to be a CI square. The following gives an example for this situation.

Consider the following pushout of **FOL** signatures containing only sorts and constants as shown in the diagram below:

$$\begin{tikzcd} \Sigma = \{a : s_1, b : s_2\} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 = \{a : s\} \arrow[d, "\theta_1"] \\ \Sigma_2 = \{a, b : s\} \arrow[r, "\theta_2"'] & \Sigma' = \{a : s\} \end{tikzcd}$$

Suppose that the semantic deduction $\theta_1(a = a) \models \theta_2(a = b)$ has an interpolant E in Σ. This means

$$a = a \models \varphi_1(E) \ \text{ and } \ \varphi_2(E) \models a = b.$$

Consider the models

model X	signature of X	X_s	X_{s_1}	X_{s_2}	X_a	X_b
M	Σ	–	$\{A,B\}$	$\{A,B\}$	A	B
N	Σ	–	$\{A,B\}$	$\{A,B\}$	B	B
N_1	Σ_1	$\{A,B\}$	–	–	B	–
M_2	Σ_2	$\{A,B\}$	–	–	A	B

Note that $N = N_1\restriction_{\varphi_1}$, $M = M_2\restriction_{\varphi_2}$ and that $M \cong N$ by the isomorphism which is identity on s_2 and swaps the elements of s_1.

Then $N_1 \models (a = a) \models \varphi_1(E)$ which by the satisfaction condition implies $N \models E$. Because $M \cong N$ we get that $M \models E$. By the satisfaction condition this implies $M_2 \models \varphi_2(E)$ which implies $M_2 \models (a = b)$ which is false. This shows that an interpolant E does not exist.

Exercises

9.6. [2] **Elementary amalgamation squares**
A commuting square of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

is an *elementary amalgamation square* if for each Σ_1-model M_1 and each Σ_2-model M_2 such that $M_1\upharpoonright_{\varphi_1} \equiv M_2\upharpoonright_{\varphi_2}$ there exists a unique Σ'-model M' such that $M'\upharpoonright_{\theta_1} \equiv M_1$ and $M'\upharpoonright_{\theta_2} \equiv M_2$.

In any institution with negation, a commuting square of signature morphisms is an elementary amalgamation square if and only if it is a Robinson consistency square.

9.7. In any institution with elementary diagrams such that each pushout of elementary extensions is a Robinson consistency square, any two elementary equivalent models can be "embedded" into a common model in the sense that for each $M_1 \equiv M_2$ there exists homomorphisms $M_1 \xrightarrow{h_1} M \xleftarrow{h_2} M_2$. (*Hint:* Consider the pushout of the span of elementary extensions along the models M_1 and M_2, and consider the theories $(M_1)^*_{M_1}$ and $(M_2)^*_{M_2}$.)

9.8. Interpolation in $\mathbf{FOL}_{\infty,\omega}$
$\mathbf{FOL}_{\infty,\omega}$ has Craig $(\mathbb{S}ig, (i**))$ and $((i**), \mathbb{S}ig)$-interpolation. (*Hint:* Use Robinson consistency Thm. 9.16.)

9.9. Robinson consistency in PA
Develop the Robinson consistency result for **PA** as an instance of Thm. 9.16. Derive a corresponding interpolation result for **PA**.

9.4 Craig-Robinson Interpolation

The Craig interpolation property can be strengthened by adding to the 'primary' premises E_1 a set Γ_2 (of Σ_2-sentences) as 'secondary' premises. In any institution we say that a commuting square of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

is a *Craig-Robinson Interpolation square* (abbreviated *CRI* square) when for each set E_1 of Σ_1-sentences and each sets E_2 and Γ_2 of Σ_2-sentences, if $\theta_1(E_1) \cup \theta_2(\Gamma_2) \models_{\Sigma'} \theta_2(E_2)$, then there exists a set E of Σ-sentences such that $E_1 \models_{\Sigma_1} \varphi_1(E)$ and $\Gamma_2 \cup \varphi_2(E) \models_{\Sigma_2} E_2$.

Also the $\langle \mathcal{L}, \mathcal{R} \rangle$-interpolation concept can be extended in a straightforward way from Craig interpolation to Craig-Robinson interpolation.

Craig-Robinson versus Craig interpolation. By taking Γ_2 to be the empty set $\emptyset$ we can see that

Fact 9.19. *Any CRI square is also a CI square.*

The opposite implication does not hold in general. The following gives a sufficient condition when CI and CRI are equivalent interpolation concepts.

Proposition 9.20. *In any institution that has implications and is quasi-compact, a commuting square of signature morphisms is a Craig-Robinson Interpolation square if and only if it is a Craig Interpolation square.*

Proof. We focus only on the non-trivial part, that CI implies CRI. Consider $E_1 \subseteq \mathsf{Sen}(\Sigma_1)$ and $E_2, \Gamma_2 \subseteq \mathsf{Sen}(\Sigma_2)$ such that $\theta_1(E_1) \cup \theta_2(\Gamma_2) \models \theta_2(E_2)$.

First we notice that without loss of generality we may assume that E_2 consists of only one sentence e, i.e., $E_2 = \{e\}$. Indeed, if we assumed that CRI property holds for each $e \in E_2$, let E_e be the interpolant corresponding to each $e \in E_2$. Then $\bigcup_{e \in E_2} E_e$ is an interpolant corresponding to E_2.

Because we may assume that $E_2 = \{e\}$, then by the quasi-compactness assumption, we may assume without loss of generality that E_1 and Γ_2 are finite.

Let $\Gamma_2 \Rightarrow e$ denote $\gamma_1 \Rightarrow (\cdots \Rightarrow (\gamma_n \Rightarrow e))$ where $\Gamma_2 = \{\gamma_1, \ldots, \gamma_n\}$. Then we have that $\theta_1(E_1) \models \theta_2(\Gamma_2 \Rightarrow e)$. By CI there exists $E \subseteq \mathsf{Sen}(\Sigma)$ such that $E_1 \models \varphi_1(E)$ and $\varphi_2(E) \models \Gamma_2 \Rightarrow e$. But the latter is equivalent to $\varphi_2(E) \cup \Gamma_2 \models e$. $\square$

Prop. 9.20 gives the possibility to extend CI properties to CRI properties in institutions as illustrated by the following example.

Corollary 9.21. **FOL** *has Craig-Robinson* $(\mathbb{S}ig, (i**))$ *and* $((i**), \mathbb{S}ig)$*-interpolation.*

Proof. By Cor. 9.15, 9.18, and 9.9 **FOL** has the corresponding Craig interpolation properties, has implications and is compact (cf. Cor. 6.22). $\square$

Although one may get the feeling that CRI codes a form of implication and therefore it is expected only in institutions having semantic implications, it is not so. Later (in Sect. 12.3) we will see that institutions without semantic implications such as **EQL** and **HCL** may have CRI for a wide class of pushout squares of signature morphisms.

Extending interpolation

Sometimes interpolation properties can be established in two stages. At the first stage we establish it for a particular class of commuting squares of signature morphisms. At the second stage we extend them to a larger class of squares of signature morphisms by the general method formulated by Thm. 9.24 below. This technique uses Craig-Robinson interpolation and also requires the introduction of the following concept.

Logical kernels. A signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ has a *logical kernel*, denoted lk_φ, when there exists a set of Σ-sentences $lk_\varphi \subseteq \mathsf{Sen}(\Sigma)$ such that

any Σ-model M has a φ-expansion if and only if $M \models lk_\varphi$.

Fact 9.22. *Any logical kernel is a tautology in the target signature, i.e.,* $\models_{\Sigma'} \varphi(lk_\varphi)$.

The following is a typical example for logical kernels.

Fact 9.23. *Any $(i**)$-morphism of signatures $\varphi\colon\ (S,F,P) \to (S',F',P')$ in* **FOL** *has the logical kernel*

$$\begin{array}{rl} lk_\varphi = & \{(\forall X)\pi(X) \Leftrightarrow \pi'(X) \mid \varphi^{\mathrm{rl}}(\pi) = \varphi^{\mathrm{rl}}(\pi')\} \cup \\ & \{(\forall X)\sigma(X) = \sigma'(X) \mid \varphi^{\mathrm{op}}(\sigma) = \varphi^{\mathrm{op}}(\sigma')\}. \end{array}$$

Theorem 9.24 (Extending interpolation). *In a semi-exact institution consider classes of signature morphisms $\mathcal{L}_0, \mathcal{L}, \mathcal{R}_0, \mathcal{R}, \mathcal{E} \subseteq \mathbb{S}ig$ such that*

1. *each signature morphism $\phi \in \mathcal{L}$, respectively, $\phi \in \mathcal{R}$ can be factored as $\phi = i;\varphi$ such that $\varphi \in \mathcal{E}$ and $i \in \mathcal{L}_0$, respectively, $i \in \mathcal{R}_0$, and*
2. *each $\varphi \in \mathcal{E}$ is a retract and has a logical kernel.*

If the institution has the Craig-Robinson $(\mathcal{L}_0, \mathcal{R}_0)$-interpolation property then it also has the Craig-Robinson $(\mathcal{L}, \mathcal{R})$-interpolation property.

Proof. Consider a pushout $\Sigma_1 \xrightarrow{\theta_1} \Sigma' \xleftarrow{\theta_2} \Sigma_2$ of an arbitrary span of signature morphisms $\Sigma_1 \xleftarrow{\phi_1 \in \mathcal{L}} \Sigma \xrightarrow{\phi_2 \in \mathcal{R}} \Sigma_2$ and let $E_1 \subseteq \mathsf{Sen}(\Sigma_1)$ and $\Gamma_2, E_2 \subseteq \mathsf{Sen}(\Sigma_2)$ such that

$$\theta_1(E_1) \cup \theta_2(\Gamma_2) \models \theta_2(E_2).$$

Let $\phi_1 = i_1;\varphi_1$ and $\phi_2 = i_2;\varphi_2$ such that $i_1 \in \mathcal{L}_0$ and $i_2 \in \mathcal{R}_0$ and $\varphi_1, \varphi_2 \in \mathcal{E}$. Let $\Sigma'_1 \xrightarrow{i'_1} \Sigma'' \xleftarrow{i'_2} \Sigma'_2$ be a pushout of $\langle i_1, i_2\rangle$. By the universal property of pushouts, let φ be the unique signature morphism $\Sigma'' \to \Sigma'$ such that $i'_1;\varphi = \varphi_1;\theta_1$ and $i'_2;\varphi = \varphi_2;\theta_2$.

$$\begin{tikzcd}
\Sigma \arrow[r, "i_1"'] \arrow[rr, bend left, "\phi_1"] \arrow[d, "i_2"] \arrow[dd, bend right, "\phi_2"'] & \Sigma'_1 \arrow[r, "\varphi_1"'] \arrow[d, "i'_1"] & \Sigma_1 \arrow[dd, "\theta_1"] \\
\Sigma'_2 \arrow[r, "i'_2"'] \arrow[d, "\varphi_2"] & \Sigma'' \arrow[dr, "\varphi"] & \\
\Sigma_2 \arrow[rr, "\theta_2"'] & & \Sigma'
\end{tikzcd}$$

Consider $\overline{\varphi_1}$ a (mono) left inverse to φ_1 and $\overline{\varphi_2}$ a (mono) left inverse to φ_2 and define $E'_1 = \overline{\varphi_1}(E_1) \cup lk_{\varphi_1}$, $\Gamma'_2 = \overline{\varphi_2}(\Gamma_2) \cup lk_{\varphi_2}$, and $E'_2 = \overline{\varphi_2}(E_2)$.

We will show that $i'_1(E'_1) \cup i'_2(\Gamma'_2) \models i'_2(E'_2)$. Consider a Σ''-model M'' such that $M'' \models i'_1(E'_1) \cup i'_2(E'_2)$.

Define $M'_1 = M''\upharpoonright_{i'_1}$ and $M'_2 = M''\upharpoonright_{i'_2}$. By the satisfaction condition $M'_1 \models \overline{\varphi_1}(E_1) \cup lk_{\varphi_1}$ and $M'_2 \models \overline{\varphi_1}(\Gamma_2) \cup lk_{\varphi_2}$.

Because of the logical kernel property, there exists a φ_1-expansion M_1 of M'_1 and a φ_2-expansion M_2 of M'_2. By the satisfaction condition $M_1 \models E_1$ and $M_2 \models \Gamma_2$. Notice also that $M_1\upharpoonright_{\phi_1} = M_2\upharpoonright_{\phi_2} (= M''\upharpoonright_{i_k;i'_k})$. Therefore let the Σ'-model M' be the amalgamation $M_1 \otimes M_2$ of M_1 and M_2.

By the uniqueness part of the semi-exactness property of the institution we have that $M'\upharpoonright_{\varphi} = M''$ because $M'\upharpoonright_{\varphi}\upharpoonright_{i'_1} = M''\upharpoonright_{i'_1}$ and $M'\upharpoonright_{\varphi}\upharpoonright_{i'_2} = M''\upharpoonright_{i'_2}$. Hence $M' \models \varphi(i'_1(E'_1) \cup i'_2(\Gamma'_2)) \models\!\mid \theta_1(E_1) \cup \theta_2(\Gamma_2) \models \theta_2(E_2) = \varphi(i'_2(E'_2))$. By the satisfaction condition this implies $M'' \models i'_2(E'_2)$. This shows that $i'_1(E'_1) \cup i'_2(\Gamma'_2) \models i'_2(E'_2)$.

By the hypothesis we can find an interpolant $E \subseteq \mathsf{Sen}(\Sigma)$ such that $E'_1 \models i_1(E)$ and $\Gamma'_2 \cup i_2(E) \models E'_2$. We will show that E is an interpolant for the original interpolation problem too.

We have that $E'_1 \models i_1(E)$ implies that $\varphi_1(E'_1) \models \varphi_1(i_1(E))$ which means $E_1 \cup \varphi_1(lk_{\varphi_1}) \models \phi_1(E)$. Because $\models \varphi_1(lk_{\varphi_1})$ we deduce $E_1 \models \phi_1(E)$.

We also have that $\Gamma'_2 \cup i_2(E) \models E'_2$ implies $\varphi_2(\Gamma'_2) \cup \varphi_2(i_2(E)) \models \varphi_2(E'_2)$ which means $\Gamma_2 \cup \varphi_2(lk(\varphi_2)) \cup \phi_2(E) \models E_2$. Because $\models \varphi_2(lk_{\varphi_2})$ we deduce that $\Gamma_2 \cup \phi_2(E) \models E_2$. $\square$

In the following we illustrate the applicability of the extension Thm. 9.24 by a concrete case.

Interpolation in infinitary second order logic. Let $\mathbf{SOL}_{\infty,\omega}$ be the extension of second order logic **SOL** which allows infinite conjunctions of sentences.

Proposition 9.25. $\mathbf{SOL}_{\infty,\omega}$ *has Craig* $(\mathbb{S}ig^{\mathbf{SOL}}, \mathcal{R})$ *and* $(\mathcal{R}, \mathbb{S}ig^{\mathbf{SOL}})$*-interpolation for* $\mathcal{R}$ *the class of* $(i{*}{*})$*-morphisms of signatures* $\varphi\colon\ \Sigma \to \Sigma'$ *for which* Σ' *is finitely presented (i.e. the sets of sort, operation, and relation symbols are all finite).*

Proof. By the equivalence between Craig interpolation and Robinson consistency (Thm. 9.14) and because of the symmetry of Robinson consistency we need only to show that $\mathbf{SOL}_{\infty,\omega}$ has Craig $(\mathbb{S}ig^{\mathbf{SOL}}, \mathcal{R})$-interpolation. By Cor. 9.4 we have that $\mathbf{SOL}_{\infty,\omega}$ has Craig $(\mathbb{S}ig^{\mathbf{SOL}}, \mathcal{R}_0)$-interpolation for $\mathcal{R}_0$ the class of the (iii)-morphism of signatures $\varphi\colon\ \Sigma \to \Sigma'$ for which Σ' is finitely presented. We use the extension Thm. 9.24 to lift interpolation from $\mathcal{R}_0$ to $\mathcal{R}$.

By Prop. 9.20 we get that $\mathbf{SOL}_{\infty,\omega}$ has Craig-Robinson $(\mathbb{S}ig^{\mathbf{SOL}}, \mathcal{R}_0)$-interpolation. Note that the quasi-compactness condition (also needed by Thm. 9.14) is fulfilled because $\mathbf{SOL}_{\infty,\omega}$ has infinite conjunctions (**SOL** cannot be established compact because of its second order quantifications). In order to apply the extension Thm. 9.24 we have to set the class $\mathcal{E}$ of signature morphisms to the class of the (bss)-morphisms. By Fact 9.23 the morphisms of $\mathcal{E}$ have logical kernels and since they are (bss) they are also retracts. It remains to show that each signature morphism $(\phi\colon\ (S,F,P) \to (S',F',P')) \in \mathcal{R}$ can be factored as $\phi = i;\varphi$ with $i \in \mathcal{R}_0$ and $\varphi \in \mathcal{E}$. This is illustrated by the diagram

$$\begin{tikzcd}
(S,F,P) \arrow[rr, "\phi\in\mathcal{R}"] \arrow[dr, "i\in\mathcal{R}_0"'] & & (S',F',P') \\
& (S',\phi(F)+F',\phi(P)+P') \arrow[ur, "\varphi\in\mathcal{E}"'] &
\end{tikzcd}$$

where for each arity w and sort s

$$(\phi(F)+F')_{\phi(w)\to\phi(s)} = F_{w\to s} \uplus F'_{\phi(w)\to\phi(s)} \text{ and } (\phi(P)+P')_{\phi(w)} = P_w \uplus P'_{\phi(w)}.$$

□

Exercises

9.10. Symmetric Birkhoff institutions
A Birkhoff institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathcal{F}, \mathcal{B})$ is *symmetric* when $\mathcal{B}$ is symmetric. Extend Thm. 9.6 to Craig-Robinson interpolation for symmetric Birkhoff institutions.

9.11. [53] **Lifting interpolation to presentations**
For any institution I and a class $\mathcal{S} \subseteq \mathbb{S}ig$ of signature morphisms let $\mathcal{S}^{\text{pres}}$ be the class of presentation morphisms φ such that $\varphi \in \mathcal{S}$ (as a signature morphism). The institution I^{pres} of the presentations of I has the Craig-Robinson $(\mathcal{L}^{\text{pres}}, \mathcal{R}^{\text{pres}})$-interpolation if I has the Craig-Robinson $(\mathcal{L}, \mathcal{R})$-interpolation.

9.5 Borrowing Interpolation

The method of borrowing interpolation properties along institution comorphisms requires some special interpolation properties of the comorphism.

Left interpolation property for comorphisms. For a fixed class $\mathcal{S} \subseteq \mathbb{S}ig$ of signature morphisms, we say that an institution comorphism $(\Phi, \alpha, \beta) : I \to I'$ has the *Craig $\mathcal{S}$-left Interpolation property* when for each $(\varphi_1 : \Sigma \to \Sigma_1) \in \mathcal{S}$, for each set E_1 of Σ_1-sentences and each set E_2 of $\Phi(\Sigma)$-sentences such that $\alpha_{\Sigma_1}(E_1) \models' \Phi(\varphi_1)(E_2)$, there exists a set of Σ-sentences E such that $E_1 \models \varphi_1(E)$ and $\alpha_\Sigma(E) \models' E_2$.

$$\begin{tikzcd}
\mathsf{Sen}(\Sigma) \arrow[r, "\mathsf{Sen}(\varphi_1)"] \arrow[d, "\alpha_\Sigma"'] & \mathsf{Sen}(\Sigma_1) \arrow[d, "\alpha_{\Sigma_1}"] \\
\mathsf{Sen}'(\Phi(\Sigma)) \arrow[r, "\mathsf{Sen}'(\Phi(\varphi_1))"'] & \mathsf{Sen}'(\Phi(\Sigma_1))
\end{tikzcd}$$

Prop. 9.26 below gives a class of examples of left interpolation properties for comorphisms. Before presenting this result let us introduce the following model amalgamation concept.

(Φ,β)**-amalgamation.** For any $(\Phi,\alpha,\beta): I \to I'$ we say that a signature morphism $\varphi_1 : \Sigma \to \Sigma_1$ in I *has model* (Φ,β)*-amalgamation* when for each Σ_1-model M_1 and each $\Phi(\Sigma)$-model M' with $M_1\restriction_{\varphi_1} = \beta_\Sigma(M')$ there exists a unique $\Phi(\Sigma_1)$-model M'_1 such that $\beta_{\Sigma_1}(M'_1) = M_1$ and $M'_1\restriction_{\Phi(\varphi_1)} = M'$. As usual, if we drop the uniqueness requirement on M'_1 we have the weak version of the concept, called *weak* (Φ,β)*-amalgamation*.

For example if β's are isomorphisms the (Φ,β)-amalgamation is immediately fulfilled. This is the case of many concrete situations, for example those listed in Cor. 9.27.

Proposition 9.26. *Any institution comorphism* $(\Phi,\alpha,\beta): I \to I'$ *such that* $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathcal{F}, \mathcal{B})$ *is a Birkhoff institution has the Craig* $\mathcal{S}$*-left interpolation property for* $\mathcal{S}$ *the class of signature morphisms which*

1. *have weak model* (Φ,β)*-amalgamation,*
2. *their corresponding model reducts preserve* $\mathcal{F}$*-filtered products, and*
3. *lift* $\mathcal{B}^{-1}$.

Proof. Consider $(\varphi_1 : \Sigma \to \Sigma_1) \in \mathcal{S}$ and let E_1 be a set of Σ_1-sentences and E_2 be a set of $\Phi(\Sigma)$-sentences such that $\alpha_{\Sigma_1}(E_1) \models' \Phi(\varphi_1)(E_2)$. We define the interpolant $E = \varphi_1^{-1}(E_1^{**})$. Since $E_1 \models \varphi_1(E)$ is immediate we have only to prove that $\alpha_\Sigma(E) \models E_2$.

Let M_2 be a model such that $M_2 \models \alpha_\Sigma(E)$. By the satisfaction condition for the institution comorphism we have that $\beta_\Sigma(M_2) \models E = \varphi_1^{-1}(E_1^{**})$. Thus

$$\beta_\Sigma(M_2) \in (\varphi_1^{-1}(E_1^{**}))^* = (E_1^*\restriction_{\varphi_1})^{**} = \mathcal{B}_\Sigma^{-1}(\mathcal{F}(E_1^*\restriction_{\varphi_1})).$$

The first equality holds by the satisfaction condition for φ_1 while the second by the Birkhoff institution condition for I. We have that

$$\begin{array}{lll}
\mathcal{B}_\Sigma^{-1}(\mathcal{F}(E_1^*\restriction_{\varphi_1})) & = \mathcal{B}_\Sigma^{-1}(\mathrm{Iso}((\mathcal{F}E_1^*)\restriction_{\varphi_1})) & (\mathsf{Mod}(\varphi_1) \text{ preserves } \mathcal{F}\text{-filtered products}) \\
 & = \mathcal{B}_\Sigma^{-1}((\mathcal{F}E_1^*)\restriction_{\varphi_1}) & (\mathcal{B} \text{ is closed under isomorphisms}) \\
 & = (\mathcal{B}_\Sigma^{-1}(\mathcal{F}E_1^*))\restriction_{\varphi_1} & (\varphi_1 \text{ lifts } \mathcal{B}^{-1}) \\
 & = E_1^{***}\restriction_{\varphi_1} & (\text{by the Birkhoff institution condition}) \\
 & = E_1^*\restriction_{\varphi_1} & (\text{by the Birkhoff institution condition})
\end{array}$$

Thus there exists a Σ_1-model M_1 such that $M_1 \models E_1$ and $\beta_\Sigma(M_2) = M_1\restriction_{\varphi_1}$. By the weak amalgamation hypothesis there exists a $\Phi(\Sigma_1)$-model M'_1 such that $\beta_{\Sigma_1}(M'_1) = M_1$ and $M'_1\restriction_{\Phi(\varphi_1)} = M_2$. By the satisfaction condition of the comorphism we get that $M'_1 \models \alpha_{\Sigma_1}(E_1)$ which by the interpolation hypothesis implies $M'_1 \models \Phi(\varphi_1)(E_2)$. By the satisfaction condition for $\Phi(\varphi_1)$ we obtain that $M_2 \models E_2$. $\square$

By using the lifting properties given by Prop. 9.12 we obtain the following instances of the general Prop. 9.26.

Corollary 9.27. *The following institution embeddings $I \to$ **FOL** have Craig $\mathcal{S}$-left interpolation according to the table*

institution I	$\mathcal{S}$
UNIV, **HCL**, $\forall\vee$	*ie*$*$
universal **FOL**-*atoms*	*iei*
EQL	*ie*

Right interpolation property for comorphisms. This is the reflection in the mirror of the left property. For a fixed class $\mathcal{S} \subseteq \mathbb{S}ig$ of signature morphisms, we say that an institution comorphism $(\Phi, \alpha, \beta) : I \to I'$ has the *Craig $\mathcal{S}$-right interpolation property* when for each $(\varphi_2 : \Sigma \to \Sigma_2) \in \mathcal{S}$, for each set E_1 of $\Phi(\Sigma)$-sentences and each set E_2 of Σ_2-sentences such that $\Phi(\varphi_2)(E_1) \models' \alpha_{\Sigma_2}(E_2)$, there exists a set of Σ-sentences E such that $E_1 \models' \alpha_\Sigma(E)$ and $\varphi(E) \models E_2$.

$$\begin{tikzcd}
\mathsf{Sen}(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathsf{Sen}(\varphi_2)"'] & \mathsf{Sen}'(\Phi(\Sigma)) \arrow[d, "\mathsf{Sen}'(\Phi(\varphi_2))"] \\
\mathsf{Sen}(\Sigma_2) \arrow[r, "\alpha_{\Sigma_2}"'] & \mathsf{Sen}'(\Phi(\Sigma_2))
\end{tikzcd}$$

The result below gives a class of examples of right interpolation properties for comorphisms.

Proposition 9.28. *Any institution comorphism $(\Phi, \alpha, \beta) : I \to I'$ such that*

- $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathcal{F}, \mathcal{B})$ *is a Birkhoff institution such that*
- β*'s preserve $\mathcal{F}$-filtered products, and*
- *each sentence of I' is preserved by $\mathcal{F}$-filtered products,*

has the Craig $\mathcal{S}$-right interpolation property for $\mathcal{S}$ the class of signature morphisms which

1. *have weak model (Φ, β)-amalgamation, and*
2. *lift $\mathcal{B}$.*

Proof. Let $(\varphi_2 : \Sigma \to \Sigma_2) \in \mathcal{S}$ and let $E_1 \subseteq \mathsf{Sen}'(\Phi(\Sigma))$ and $E_2 \subseteq \mathsf{Sen}(\Sigma_2)$ such that $\Phi(\varphi_2)(E_1) \models \alpha_{\Sigma_2}(E_2)$. The interpolant is defined as $E = \alpha_\Sigma^{-1}(E_1^{**})$. Obviously $E_1 \models \alpha_\Sigma(E)$. We have to prove that $\varphi_2(E) \models E_2$.

Consider a model M_2 such that $M_2 \models \varphi_2(E)$. By the satisfaction condition this means that $M_2 \restriction_{\varphi_2} \models E = \alpha_\Sigma^{-1}(E_1^{**})$. This means $M_2 \restriction_{\varphi_2} \in (\alpha_\Sigma^{-1}(E_1^{**}))^* = (\beta_\Sigma(E_1^*))^{**}$. (The last equality holds by the satisfaction condition for the institution comorphism.) We have that

$$\begin{array}{lll}
(\beta_\Sigma(E_1^*))^{**} & = \mathcal{B}_\Sigma^{-1}(\mathcal{F}\beta_\Sigma(E_1^*)) & \text{(by the Birkhoff institution condition)} \\
& = \mathcal{B}_\Sigma^{-1}(\mathrm{Iso}(\beta_\Sigma(\mathcal{F}E_1^*))) & \text{(β's preserve filtered products)} \\
& = \mathcal{B}_\Sigma^{-1}(\beta_\Sigma(\mathcal{F}E_1^*)) & \text{($\mathcal{B}$ is closed under isomorphisms)} \\
& = \mathcal{B}_\Sigma^{-1}(\beta_\Sigma(E_1^*)) & \text{(E_1 are preserved by $\mathcal{F}$-filtered products)}
\end{array}$$

Then there exists a Σ-model N such that $N \in \beta_\Sigma(E_1^*)$ and $\langle M_2\upharpoonright_{\varphi_2}, N\rangle \in \mathcal{B}_\Sigma$. We have the following:

- Because φ_2 lifts $\mathcal{B}$ there exists a Σ_2-model N_2 such that $N_2\upharpoonright_{\varphi_2} = N$ and $\langle M_2, N_2\rangle \in \mathcal{B}_{\Sigma_2}$.
- There exists a $\Phi(\Sigma)$-model N' such that $N = \beta_\Sigma(N')$ and such that $N' \models E_1$.

By the weak (Φ,β)-amalgamation for φ_2 there exists a $\Phi(\Sigma_2)$-model N_2' such that $\beta_{\Sigma_2}(N_2') = N_2$ and $N_2'\upharpoonright_{\Phi(\varphi_2)} = N'$. Because $N' \models E_1$, by the satisfaction condition the latter equality implies that $N_2' \models \Phi(\varphi_2)(E_1)$ which by the interpolation hypothesis implies that $N_2' \models \alpha_{\Sigma_2}(E_2)$. By the satisfaction condition for the comorphism this implies $N_2 \models E_2$. Because $\langle M_2, N_2\rangle \in \mathcal{B}_{\Sigma_2}$ we have that $M_2 \in \mathcal{B}_{\Sigma_2}^{-1}(N_2)$ which by the Birkhoff institution condition implies $M_2 \models E_2$. This shows that $\varphi_2(E) \models E_2$. □

By using the lifting properties given by Prop. 9.7 and the corresponding results about the preservation of sentences by classes of filtered products, we obtain the following instances of the general Prop. 9.28.

Corollary 9.29. *The following institution embeddings $I \to I'$ have Craig $\mathcal{S}$-interpolation according to the table*

I	I'	$\mathcal{S}$
UNIV, $\forall\vee$	**FOL**	*iii*
universal **FOL**-*atoms*	**HCL**	*iii*
EQL	**HCL**	*ii*

Borrowing interpolation along institution comorphisms

The following is the main result which can be used for borrowing interpolation properties along institution comorphisms.

Proposition 9.30. *Let $(\Phi,\alpha,\beta)\colon\ I \to I'$ be a conservative institution comorphism such that Φ preserves pushouts, and let $\mathcal{L},\mathcal{R} \subseteq \mathbb{S}ig$ be classes of signature morphisms such that I' has the Craig $(\Phi(\mathcal{L}),\Phi(\mathcal{R}))$-interpolation.*

If (Φ,α,β) has the Craig $\mathcal{L}$-left or $\mathcal{R}$-right interpolation, then I has Craig $(\mathcal{L},\mathcal{R})$-interpolation.

Proof. Consider a pushout of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

such that $\varphi_1 \in \mathcal{L}$ and $\varphi_2 \in \mathcal{R}$ and $E_1 \subseteq \mathsf{Sen}(\Sigma_1)$ and $E_2 \subseteq \mathsf{Sen}(\Sigma_2)$ such that $\theta_1(E_1) \models \theta_2(E_2)$.

The latter relation leads to $\alpha_{\Sigma'}(\theta_1(E_1)) \models' \alpha_{\Sigma'}(\theta_2(E_2))$ which by the naturality of α further leads us to the interpolation problem $\Phi(\theta_1)(\alpha_{\Sigma_1}(E_1)) \models' \Phi(\theta_2)(\alpha_{\Sigma_2}(E_2))$ for the pushout square of signature morphisms in I'

$$\begin{tikzcd} \Phi(\Sigma) \arrow[r, "\Phi(\varphi_1)"] \arrow[d, "\Phi(\varphi_2)"'] & \Phi(\Sigma_1) \arrow[d, "\Phi(\theta_1)"] \\ \Phi(\Sigma_2) \arrow[r, "\Phi(\theta_2)"'] & \Phi(\Sigma') \end{tikzcd}$$

By the Craig $(\Phi(\mathcal{L}),\Phi(\mathcal{R}))$-interpolation property of I', we find $E_0 \subseteq \mathsf{Sen}'(\Phi(\Sigma))$ such that

$$\alpha_{\Sigma_1}(E_1) \models' \Phi(\varphi_1)(E_0) \ \text{ and } \ \Phi(\varphi_2)(E_0) \models' \alpha_{\Sigma_2}(E_2).$$

Let us assume the Craig $\mathcal{L}$-left interpolation for the institution comorphism. Then we can find $E \subseteq \mathsf{Sen}(\Sigma)$ such that

$$E_1 \models \varphi_1(E) \ \text{ and } \ \alpha_\Sigma(E) \models' E_0.$$

By applying $\Phi(\varphi_2)$ to this we get that $\Phi(\varphi_2)(\alpha_\Sigma(E)) \models' \Phi(\varphi_2)(E_0)$. By the naturality of α we get $\alpha_{\Sigma_2}(\varphi_2(E)) \models' \Phi(\varphi_2)(E_0) \models' \alpha_{\Sigma_2}(E_2)$. Finally, by the conservativeness of the institution comorphism this can be simplified to $\varphi_2(E) \models E_2$.

The case when the institution comorphism has $\mathcal{R}$-right interpolation is handled similarly to the $\mathcal{L}$-left interpolation case by getting $E \subseteq \mathsf{Sen}(\Sigma)$ such that

$$E_0 \models' \alpha_\Sigma(E) \ \text{ and } \ \varphi_2(E) \models E_2.$$

□

Applications of Prop. 9.30. By the borrowing interpolation Prop. 9.30, by Prop. 9.26 and 9.28, and by the interpolation properties of **FOL** (given by Cor. 9.15,9.18, 9.9,9.18), we can obtain the following list of interpolation results for Birkhoff sub-institutions of **FOL**, results which have been previously obtained directly in Cor. 9.8 and 9.13.

institution	$\mathcal{L}$	$\mathcal{R}$	*directly obtained*
EQL	*ie*	$**$	Cor. 9.13
universal **FOL**-atoms	*iei*	$***$	Cor. 9.13
HCL	*ie*$*$	$***$	Cor. 9.13
UNIV	$***$	*iii*	Cor. 9.8
	ie$*$	$***$	Cor. 9.13
$\forall\vee$	$***$	*iii*	Cor. 9.8
	ie$*$	$***$	Cor. 9.13

Borrowing interpolation between institutions having the same expressive power. This property essentially means that the sentence translations α_Σ are surjective modulo semantical equivalence. In this case the interpolation properties for the comorphism can be established rather easily, leading to a rather easy transfer of interpolation from the target institution to the source institution.

Fact 9.31. *Any conservative institution morphism comorphism* $(\Phi,\alpha,\beta)\colon\ I \to I'$ *for which each* α_Σ *is surjective modulo semantical equivalence has both the Craig* $\mathbb{S}ig^I$*-left and right interpolation properties.*

Exercises

9.12. [53] **Interpolation in PA by borrowing**
PA has Craig-Robinson $(\mathbb{S}ig^{\mathbf{PA}},(i**))$ and $((i**),\mathbb{S}ig^{\mathbf{PA}})$-interpolation borrowed from **FOL** along the comorphism $\mathbf{PA} \to \mathbf{FOL}^{\mathrm{pres}}$ encoding partial operations as relations (see Sect. 4.1). (*Hint:* Use Ex. 9.11 and Fact 9.31.)

Apply this method for obtaining concrete interpolation results in other institutions such as **POA**, **MBA**, etc.

9.13. The institution comorphism $\mathbf{FOL} \to \mathbf{FOEQL}$ encoding relations as operations (see Sect. 3.3) has both Craig $(i**)$-left and right interpolation.

9.14. For each injective function $u\colon\ S \to S'$, the institution comorphism $(\overline{\Phi^u},\overline{\alpha^u},\overline{\beta^u})\colon\ \mathbf{FOL}^S \to \mathbf{FOL}^{S'}$ (see Ex. 3.26) has both the $\mathbb{S}ig^S$-left and right interpolation properties.

9.15. Interpolation in HNK
In any institution with pushouts of signatures, a commuting square of signatures

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\
\Sigma_2 \arrow[r, "\theta_2"'] & \Sigma'
\end{tikzcd}$$

is a *quasi-pushout* when the signature morphism $\psi\colon\ \Sigma'' \to \Sigma'$ from the vertex Σ'' of the pushout of φ_1 and φ_2 to Σ' is conservative.

1. Prove a relaxed variant of Prop. 9.30 which replaces the condition that the signature translation functor Φ preserve pushouts by the slightly more general condition that Φ maps pushouts to quasi-pushouts.
2. Apply this upgraded variant of Prop. 9.30 for 'borrowing' interpolation from **FOL** to **HNK** through the comorphism $\mathbf{HNK} \to \mathbf{FOEQL}^{\mathrm{pres}}$ of Ex. 4.11. (*Hint:* The sentence translations $\alpha_{(S,F)}$ of the comorphism $\mathbf{HNK} \to \mathbf{FOEQL}^{\mathrm{pres}}$ are bijective.)

9.16. (from [51], corrected) The embedding comorphism $\mathbf{FOEQL} \to \mathbf{POA}$ has both Craig (ie)-left and right interpolation (*Hint:* Use the encoding comorphism $\mathbf{POA} \to \mathbf{FOL}^{\mathrm{pres}}$ for translating the left and the right interpolation problems of the given comorphism to interpolation problems in **FOL**).

Give a counterexample for the (ii)-right interpolation by considering the signatures $\Sigma = (\{s_1,s_2\},\{a,b\colon\ \to s_1\})$ and Σ_2 which extends Σ with the operation $\sigma\colon\ s_1 \to s_2$. (*Hint:* Consider $E_1 = \{a \le b, (\forall x,y\colon\ s_2)(x \le y) \Rightarrow (x = y)\}$ and $E_2 = \{\sigma(a) = \sigma(b)\}$.)

Notes. The importance of interpolation in logic and model theory can be seen from [165, 32]. A recent monograph dedicated to interpolation in modal and intuitionistic logics is [69]. Interpolation also has numerous applications in computing science especially in formal specification theory [16, 58, 61, 22, 179, 28] but also in data bases (ontologies) [108], automated reasoning [141, 144], type checking [102], model checking [122], and structured theorem proving [4, 121].

The first pushout-based institution-independent formulation of Craig Interpolation appears in [168] but uses single sentences. This satisfied the need in formal specification theory to generalize interpolation from the conventional framework based on extensions of signatures to a framework involving arbitrary signature morphisms. The formulation of CI with *sets* of sentences comes from [58] under the influence of Rodenburg's work on equational interpolation [157]; in particular note that (cf. [157]) equational logic satisfies the formulation of CI with sets of sentences but not the single sentence version. The weak amalgamation square condition of Theorems 9.3 and 9.6 is weaker than the corresponding assumptions in the literature that the interpolation square is a pushout [169, 58, 28, 27, 62, 156]. The concept of semantic interpolation and the semantic interpolation Thm. 9.3 have been introduced, respectively, proved in [150]. The interpolation result for Birkhoff institutions (Thm. 9.6) has been developed in [50]. Its equational instance has been developed in an abstract setting in [156]. This work is also the source for our counterexample showing that the injectivity condition of φ_2 is necessary. The application of Theorem 9.6 to **FOL** interpolation by the Keisler-Shelah property (Cor. 9.9) has been noticed in [150].

That the equivalence between Robinson consistency and Craig interpolation relies upon (quasi-)compactness and the existence of negation and of conjunctions, other details of the actual institution being irrelevant, has been noticed within the framework of the so-called 'model-theoretic logics' by [139, 140]. A variant of Robinson consistency was defined for institutions in [168] following a variant of the corresponding property in $\mathbf{FOL^1}$. Our definition of Robinson consistency comes from [87] which follows another well-known definition. The first institution-independent proof of the equivalence between Robinson consistency and Craig interpolation appears in [160]. Robinson consistency (Thm. 9.16) is due to [87] where it has also been used to derive the **FOL** interpolation result of Cor. 9.18. This result extended to the limit the previously known interpolation properties of **FOL** which appeared in [27]. The counterexample showing the necessity to have injectivity on sorts at least for one signature morphism comes from [27]. The case of many-sorted interpolation shows that the classification of many-sorted logics as "inessential variations" of one-sorted logic [127] is certainly wrong.

Craig-Robinson interpolation plays an important role in specification language theory, see [16, 58, 62]. The name "Craig-Robinson" interpolation has been used for instances of this property in [165, 179, 62] and "strong Craig interpolation" has been used in [58]. Some of the ideas behind Thm. 9.24 come from [62].

The interpolation property for comorphisms was formulated in [51], and the borrowing method for interpolation was developed in [53].

Chapter 10

Definability

Definability theory provides answers to the question of to what extent implicit definitions can be made explicit. The inverse function of groups is a simple example.

The uniqueness of the inverse function in group theory. Let us extend the algebraic signature for monoids Σ consisting of

- one sort named g,
- one constant $0 : \ \to g$, and
- one binary operation symbol $\_+\_ : \ gg \to g$

with

- an operation symbol $- : \ g \to g$ (standing for inverses of elements)

and let E' be the set of $\Sigma \cup \{-\}$-sentences consisting of the usual associativity and identity monoid axioms plus

$$(\forall x)(x + (-x) = 0) \wedge ((-x) + x = 0).$$

Then E' is a presentation of group theory.

The uniqueness of the inverse operation $-$ in groups means that given any monoid M there exists *at most one* possibility to expand it to a group, i.e., a $(\Sigma \cup \{-\}, E')$-algebra. In model theoretic terminology we say that the operation $-$ is *implicitly defined by* E'. This is the semantic side of definability. Its syntactic side, a little less obvious, is that we can eliminate the inverse operation $-$ systematically from the group theory first-order sentences. The following example may provide insight into this process.

Let a be a *new* constant. Then the $\Sigma \cup \{a, -\}$-sentence $(\exists x) - x = -a + x$ is equivalent to the $\Sigma \cup \{a\}$-sentence $(\forall a_1, x_1)(\exists x)(a_1 + a = 0) \wedge (x_1 + x = 0) \Rightarrow (x_1 = a_1 + x)$ in group theory, i.e.,

$$\begin{aligned} E' \models (\forall a)(\quad & (\exists x)(-x = -a + x) \Leftrightarrow \\ & (\forall a_1, x_1)(\exists x)(a_1 + a = 0) \wedge (x_1 + x = 0) \Rightarrow (x_1 = a_1 + x)). \end{aligned}$$

The terminology used for this phenomenon of systematic elimination of the inverse operation $-$ from the group theory sentences is that $-$ is *explicitly defined by E'*.

Definability for signature morphisms. The syntactic framework for the definability example given by the uniqueness of the inverse operation in group theory discussed above is that of the signature extension $\varphi : (\{0,+\}) \hookrightarrow (\{0,+,-\})$. Implicit and explicit definability can be formulated generally for *any* signature morphism φ of any institution as follows:

Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism and E' be a Σ'-theory. Then φ

- is *defined implicitly* by E' if the reduct functor

$$\mathsf{Mod}(\Sigma', E') \longrightarrow \mathsf{Mod}(\Sigma') \xrightarrow{\mathsf{Mod}(\varphi)} \mathsf{Mod}(\Sigma)$$

is injective, and

- is *defined (finitely) explicitly* by E' if for each pushout square

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma' \arrow[d, "\theta'"] \\ \Sigma_1 \arrow[r, "\varphi_1"'] & \Sigma'_1 \end{tikzcd}$$

and each sentence $\rho \in \mathsf{Sen}(\Sigma'_1)$, there exists a (finite) set of sentences $E_\rho \subseteq \mathsf{Sen}(\Sigma_1)$ such that

$$E' \models_{\Sigma'} (\forall\theta')(\rho \Leftrightarrow \varphi_1(E_\rho)).$$

A signature morphism φ *has the (finite) definability property* if and only if a theory defines φ (finitely) explicitly whenever it defines φ implicitly.

In many model theory textbooks E_ρ is required to be a single sentence rather than a (finite) *set* of sentences. Note that when the institution has conjunctions the 'set of sentences' and the 'single sentence' formulations coincide. This situation is very similar to that of interpolation, where the concept of interpolant, which is meaningful for institutions not necessarily having conjunctions, is given by a set of sentences rather than by a single sentence.

The formulation of explicit definability does not really require that the institution have universal θ'-quantification and semantic equivalence ($\Leftrightarrow$) since the formula of explicit definability should be read in the sense of internal logic:

for any model M' of E' any θ'-expansion M'_1 of M' satisfies ρ if and only if it satisfies $\varphi_1(E_\rho)$.

The example about the explicit definability of the inverse operation in group theory presented above is an instance of the general concept of explicit definability by letting

- θ be the signature inclusion $\{0,+\} \hookrightarrow \{a,0,+\}$,
- ρ be $(\exists x) - x = -a + x$, and
- E_ρ be $(\forall a_1, x_1)(\exists x)(a_1 + a = 0) \wedge (x_1 + x = 0) \Rightarrow (x_1 = a_1 + x))$.

In institutions, it is common to have atomic sentences corresponding to some symbols in signatures. For example, in **FOL** for each relation symbol π we have the atom $\pi(X)$. Similarly, in **PA** for each partial operation symbol π we have the atom $\mathsf{def}(\pi(X))$. This means that explicit definability ensures a *uniform* elimination of the respective symbol π from the sentences. Although this uniformity cannot be expected at the abstract level, it can be established easily in concrete situations on the basis of such correspondences between symbols of signatures and atomic sentences.

Summary of the chapter. One of the most important aspects of definability theory is the relationship between implicit and explicit definability. While in many institutions, that explicit implies implicit definability is immediate, at the level of abstract institutions this is non-trivial. For this to hold, we show that it is sufficient to impose only a rather mild restriction on signature morphisms, which in actual (many-sorted) situations requires only surjectivity of the sorts mapping.

The core of this chapter however consists of the study of the other much more difficult and meaningful implication, that implicit implies explicit definability. In one section we develop a generic definability theorem based upon the Craig-Robinson interpolation property. In another section we develop another definability result which has a complementary range of applications with respect to the previous one and which is based upon general Birkhoff axiomatizability properties. The fact that this requires rather different conditions than the Birkhoff axiomatizability based result of Thm. 9.6 can be seen as a further indication of this complementarity.

Exercises

10.1. [148] **Composability of definability**
(a) In any institution the classes of signature morphisms which are defined implicitly/explicitly form a category. Moreover, if the institution is semi-exact, these classes of signature morphisms are also stable under pushouts.
(b) In any semi-exact institution with universal $\mathcal{D}$-quantification for a class $\mathcal{D}$ of signature morphisms that are stable under pushouts, for any pushout square of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma' \arrow[d, "\theta'"] \\ \Sigma_1 \arrow[r, "\varphi_1"'] & \Sigma'_1 \end{tikzcd}$$

such that $\theta \in \mathcal{D}$ and is conservative, φ has the definability property with respect to E' whenever φ_1 has the definability property with respect to $\theta'(E')$.

10.2. [148] **Borrowing definability**
Let $(\Phi,\alpha,\beta): I \to I'$ be an institution comorphism.
(a) *Borrowing implicit definability.* We say that an I-signature morphism $\varphi: \Sigma_1 \to \Sigma_2$ is (Φ,α,β)-*precise* whenever the function $\mathsf{Mod}'(\Phi(\Sigma_2)) \to \mathsf{Mod}'(\Phi(\Sigma_1)) \times \mathsf{Mod}(\Sigma_2)$ mapping each M'_2 to $\langle M'_2\restriction_{\Phi(\varphi)}, \beta_{\Sigma_2}(M'_2)\rangle$ is injective. We say that the comorphism (Φ,α,β) is *precise* when each I-signature morphism is (Φ,α,β)-precise.

For any (Φ,α,β)-precise signature morphism φ and theory E', $\Phi(\varphi)$ is defined implicitly by $\alpha(E')$ if φ is defined implicitly by an E'.
(b) *Borrowing explicit definability.* If

1. $(\Phi,\alpha,\beta): I \to I'$ is conservative,
2. Φ preserves pushouts, and
3. α is surjective modulo the semantic equivalence $\models\!\mid$,

then any I-signature morphism φ is defined (finitely) explicitly by a theory E' if $\Phi(\varphi)$ is defined (finitely) explicitly by $\alpha(E')$.
(c) Under the assumptions (b), any (Φ,α,β)-precise signature morphism φ has the definability property if $\Phi(\varphi)$ has the definability property.

10.1 Explicit implies implicit definability

In this section we give a rather mild sufficient condition for easy implication of the relationship between implicit and explicit definability. This condition can be formulated as a property of the signature morphism expressed by the following concept.

Tight signature morphisms

In any institution with model amalgamation and with elementary diagrams ι, a signature morphism $\varphi: \Sigma \to \Sigma'$ is ι-*tight* when for all Σ'-models M' and N' with a common φ-reduct denoted M or N, and for any pushout of signature morphisms as in the diagram,

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\iota_\Sigma(M)=\iota_\Sigma(N)"'] & \Sigma' \arrow[d, "\theta'"] \\ \Sigma_M = \Sigma_N \arrow[r, "\varphi_1"'] & \Sigma'_1 \end{tikzcd}$$

$M' \otimes M_M \equiv N' \otimes N_N$ implies $M' = N'$, where $M' \otimes M_M$ and $N' \otimes N_N$, respectively, are the unique (Σ'_1-model) amalgamations of M' with M_M (the initial model of the elementary diagram of M) and of N' with N_N, respectively.

As a typical example consider the situation when φ is a signature morphism in **FOL** adding one relation symbol π of arity w. Then the only possible difference between M' and N' could be found in the difference between M'_π and N'_π. But $M'_\pi = \{m \in M_w \mid M' \otimes M_M \models \pi(m)\} = \{n \in N_w \mid N' \otimes N_N \models \pi(n)\} = N'_\pi$.

The situation of this example is quite symptomatic for most of the actual institutions. $M' \otimes M_M$ is just the expansion of M' interpreting the elements of M by themselves.

Therefore $M' \otimes M_M \equiv N' \otimes N_N$ implies that each atom in the extended signature is satisfied either by none or by both models, which means that each symbol newly added by φ gets the same interpretation in M' and N'. This argument holds in all institutions in which models interpret the symbols of the signatures as sets and functions.

The following helps to characterize concretely the tight signature morphisms in institutions.

Proposition 10.1. *Let $\varphi\colon\ \Sigma \to \Sigma'$ be a ι-tight signature morphism in a semi-exact institution with elementary diagrams ι. Then any two Σ'-models that are isomorphic by a φ-expansion of an identity, are equal.*

Proof. Let $h\colon\ M' \to N'$ be a Σ'-isomorphism such that $h\upharpoonright_\varphi$ is identity. Let $M = M'\upharpoonright_\varphi$ and $N = N'\upharpoonright_\varphi$. For the diagram from the definition of tight signature morphisms consider the amalgamation $h \otimes 1_{M_M}$; this is also an isomorphism. Therefore $M' \otimes M_M$ and $N' \otimes N_N$ are isomorphic, hence they are elementarily equivalent. By the definition of φ being tight, we get that $M' = N'$. $\square$

Tight signature morphisms in FOL. The following characterization of tight signature morphisms in **FOL** can be replicated to other institutions too.

Corollary 10.2. *A* **FOL** *signature morphism φ is ι-tight (for the standard system of elementary diagrams ι) if and only if φ is an $(s**)$-morphism.*

Proof. The surjectivity on the sorts is necessary because otherwise, given a Σ'-model M' we may consider another Σ'-model N' which is like M' but interprets the sorts outside the image of the tight signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ differently but isomorphically to M'. This gives a Σ'-isomorphism between different Σ'-models expanding a Σ-identity, thus contradicting Prop. 10.1.

The surjectivity on the sorts is also sufficient. Consider the diagram from the definition of tight signature morphisms above. If $M' \otimes M_M \equiv N' \otimes N_N$ for Σ'-models M', N' with $(M =)M'\upharpoonright_\varphi = N'\upharpoonright_\varphi (= N)$, then for all operation symbols $(\sigma\colon\ w \to s)$ in Σ' and all $a \in M'_w = N'_w$ and $b \in M'_s = N'_s$, we have that $M' \otimes M_M \models \sigma(a) = b$ if and only if $N' \otimes N_N \models \sigma(a) = b$. This means that $M'_\sigma = N'_\sigma$. By replicating this argument to the relation symbols we have that $M_\pi = N_\pi$ for each relation symbol π of Σ' too. $\square$

Explicit implies implicit definability

Proposition 10.3. *In any institution having model amalgamation and elementary diagrams ι, each ι-tight signature morphism is defined implicitly whenever it is defined explicitly.*

Proof. Let $\varphi\colon\ \Sigma \to \Sigma'$ be a tight signature morphism which is explicitly defined by $E' \subseteq \mathsf{Sen}(\Sigma')$. We show that φ is defined implicitly by E'. Let M', N' be Σ'-models both satisfying E' and such that $M'\upharpoonright_\varphi = N'\upharpoonright_\varphi$ denoted as $M = N$.

It suffices to show that the model amalgamations $M' \otimes M_M$ and $N' \otimes N_N$ are elementarily equivalent for a pushout of signature morphisms

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi"] \arrow[d, "\iota_\Sigma(M)=\iota_\Sigma(N)"'] & \Sigma' \arrow[d, "\theta'"] \\
\Sigma_M = \Sigma_N \arrow[r, "\varphi_1"'] & \Sigma'_1
\end{tikzcd}$$

Let $M' \otimes M_M \models \rho$. Because φ is explicitly defined by E', there exists $E_\rho \subseteq \mathsf{Sen}(\Sigma_M)$ such that $E' \models (\forall\theta')(\varphi_1(E_\rho) \Leftrightarrow \rho)$. Since $M' \models E'$ we have $M' \models (\forall\theta')(\varphi_1(E_\rho) \Leftrightarrow \rho)$. Because $M' \otimes M_M$ is a θ'-expansion of M', we get that $M' \otimes M_M \models \varphi_1(E_\rho) \Leftrightarrow \rho$, which means that $M' \otimes M_M \models \varphi_1(E_\rho)$. By the Satisfaction Condition applied successively in both directions we get that $N_N = M_M \models E_\rho$ and that $N' \otimes N_N \models \varphi_1(E_\rho)$. Since $N' \models E'$ we have $N' \models (\forall\theta')(\varphi_1(E_\rho) \Leftrightarrow \rho)$, which further implies that $N' \otimes N_N \models \varphi_1(E_\rho) \Leftrightarrow \rho$. We have already shown that $N' \otimes N_N \models \varphi_1(E_\rho)$, thus $N' \otimes N_N \models \rho$.

Because the choice between M' and N' is immaterial, we may conclude that $M' \otimes M_M \equiv N' \otimes N_N$. □

The following sample concrete instance of Prop. 10.3 is based on the characterization of the tight signature morphisms in **FOL** given by Cor. 10.2.

Corollary 10.4. *In* **FOL** *any $(s**)$-morphism of signatures is defined implicitly whenever it is defined explicitly.*

Our usage of elementary diagrams here does involve only the elementary extensions $\iota_\Sigma(M) : \Sigma \to \Sigma_M$ and the existence of M_M as a 'canonical' $\iota_\Sigma(M)$-expansion of M. This framework is weaker than the full requirement of existence of elementary diagrams and can be fulfilled by institutions with a rather poor sentence functor, such as $QE_2(\mathbf{PA})$ for example. However, the sentence functor should be rich enough to give substance to the actual concept of elementary equivalence and thus allowing the existence of enough interesting tight signature morphisms. For example, in the sub-institution of **FOL** given by the empty sentence functor, a signature morphism φ is tight when the corresponding reduct $\mathsf{Mod}(\varphi)$ is injective (on models) meaning that it is an (sss)-morphism. In this example signature extensions (like any other signature morphisms in fact) are explicitly defined by the empty set of sentences but in general are not implicitly defined.

The rest of this chapter is dedicated to the hard implication of the relationship between implicit and explicit definability, i.e., implicit implies explicit definability, introduced above and called the *definability property*.

Exercises

10.3. A **PA** signature morphism is tight if and only if it is surjective on sorts.

10.2 Definability by Interpolation

The following result obtains the definability property as an application of interpolation, the causality relationship between interpolation and definability being well-known in conventional model theory.

Theorem 10.5. *In any (quasi-compact) institution with model amalgamation and having Craig-Robinson $(\mathcal{L},\mathcal{R})$-interpolation for classes $\mathcal{L}$ and $\mathcal{R}$ of signature morphisms which are stable under pushouts, any signature morphism in $\mathcal{L}\cap\mathcal{R}$ has the (finite) definability property.*

Proof. Let $(\varphi\colon\ \Sigma\to\Sigma')\in\mathcal{L}\cap\mathcal{R}$ be defined implicitly by $E'\subseteq \mathsf{Sen}(\Sigma')$. We consider the pushout of φ with an arbitrary signature morphism $\theta\colon\ \Sigma\to\Sigma_1$ and let ρ be any Σ_1'-sentence.

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma' \arrow[d, "\theta'"] \\ \Sigma_1 \arrow[r, "\varphi_1"'] & \Sigma_1' \end{tikzcd}$$

Now we consider the pushout of φ_1 with itself:

$$\begin{tikzcd} \Sigma_1 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_1"'] & \Sigma_1' \arrow[d, "\theta_1"] \\ \Sigma_1' \arrow[r, "\theta_2"'] & \Sigma'' \end{tikzcd}$$

Let us show that

$$\theta_1(\theta'(E'))\cup\theta_1(\rho)\cup\theta_2(\theta'(E'))\models_{\Sigma''}\theta_2(\rho).$$

Consider a Σ''-model $M''\models\theta_1(\theta'(E'))\cup\theta_1(\rho)\cup\theta_2(\theta'(E'))$. We have that

$$(M''\restriction_{\theta_1}\restriction_{\theta'})\restriction_{\varphi}=(M''\restriction_{\theta_1}\restriction_{\varphi_1})\restriction_{\theta}=(M''\restriction_{\theta_2}\restriction_{\varphi_1})\restriction_{\theta}=(M''\restriction_{\theta_2}\restriction_{\theta'})\restriction_{\varphi}.$$

By the Satisfaction Condition we have that $(M''\restriction_{\theta_1})\restriction_{\theta'}\models E'$ and $(M''\restriction_{\theta_2})\restriction_{\theta'}\models E'$. By the implicit definability of φ, we get that $(M''\restriction_{\theta_1})\restriction_{\theta'}=(M''\restriction_{\theta_2})\restriction_{\theta'}$. Since we also have $(M''\restriction_{\theta_1})\restriction_{\varphi_1}=(M''\restriction_{\theta_2})\restriction_{\varphi_1}$, by the uniqueness aspect of the model amalgamation property we get $M''\restriction_{\theta_1}=M''\restriction_{\theta_2}$. By the Satisfaction Condition $M''\models\theta_1(\rho)$ implies $M''\restriction_{\theta_2}=M''\restriction_{\theta_1}\models\rho$ which further implies $M''\models\theta_2(\rho)$.

Now because $\varphi\in\mathcal{L}\cap\mathcal{R}$ and $\mathcal{L}$ and $\mathcal{R}$ are stable under pushouts, we have that $\varphi_1\in\mathcal{L}\cap\mathcal{R}$, and by Craig-Robinson interpolation (and quasi-compactness) there exists (finite) $E_\rho\subseteq\mathsf{Sen}(\Sigma_1)$ such that $\theta'(E')\cup\{\rho\}\models\varphi_1(E_\rho)$ and $\theta'(E')\cup\varphi_1(E_\rho)\models\rho$, which just means that $\theta'(E')\models\rho\Leftrightarrow\varphi_1(E_\rho)$. By the Satisfaction Condition it follows that $E'\models(\forall\theta')(\rho\Leftrightarrow\varphi_1(E_\rho))$. □

Definability in FOL. The following **FOL** instance of the definability by interpolation Thm. 10.5 uses the interpolation result in **FOL** given by Cor. 9.15 or 9.18. Its second conclusion uses Cor. 10.4.

Corollary 10.6. *In* **FOL***, any $(i**)$-morphism of signatures has the finite definability property. Consequently, the equivalence between implicit and explicit definability holds in* **FOL** *for the $(b**)$-morphisms of signatures.*

Other instances of definability by interpolation Thm. 10.5 will be developed later in the book in Sect. 12.3.

Exercises

10.4. [148] **Definability by interpolation in PA**
Any $(i**)$-morphism of signatures has the finite definability property in **PA**. Prove this in two different ways:

1. directly, by Thm. 10.5, and
2. by borrowing along the relational encoding comorphism $\mathbf{PA} \to \mathbf{FOL}^{\mathrm{pres}}$ by using the result of Ex. 10.2 (*Hint:* A presentation morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is defined implicitly, respectively (finitely) explicitly, by E'' in the institution of presentations I^{pres} if and only if $\varphi\colon \Sigma \to \Sigma'$ is defined implicitly, respectively (finitely) explicitly, by $E' \cup E''$ in the base institution I. Consequently, φ has the (finite) definability property in the institution of presentations if and only if it has the (finite) definability property in the base institution.)

10.3 Definability by Axiomatizability

In this section we develop another method to obtain definability properties which relies upon the axiomatizability properties of the institution.

Weakly lifting relations. In Sect. 9.2 we have introduced a concept of lifting relations which have been used to derive concrete instances of the general interpolation by axiomatizability Thm. 9.6. For the purpose of this section we introduce a weaker variant of this concept.

Let $\varphi\colon \Sigma_1 \to \Sigma_2$ be a signature morphism and let $\mathcal{R} = \langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ such that $\mathcal{R}_1 \subseteq |\mathsf{Mod}(\Sigma_1)| \times |\mathsf{Mod}(\Sigma_1)|$ and $\mathcal{R}_2 \subseteq |\mathsf{Mod}(\Sigma_2)| \times |\mathsf{Mod}(\Sigma_2)|$ be a pair of binary relations.

We say that φ *lifts weakly* $\mathcal{R}$ if and only if for each $M_2, N_2 \in |\mathsf{Mod}(\Sigma_2)|$, if $(M_2\restriction_\varphi)\mathcal{R}_1(N_2\restriction_\varphi)$ then there exists $N_2' \in |\mathsf{Mod}(\Sigma_2)|$ such that $N_2'\restriction_\varphi = N_2\restriction_\varphi$ and $(M_2)\mathcal{R}_2(N_2')$.

$$M_2\restriction_\varphi \xrightarrow{\ \mathcal{R}_1\ } N_2\restriction_\varphi = N_2'\restriction_\varphi.$$

$$M_2 \xrightarrow[\ \mathcal{R}_2\ ]{\qquad} (\exists)N_2'.$$

Fact 10.7. *A signature morphism lifts weakly (a pair of relations) $\mathcal{R}$ whenever it lifts $\mathcal{R}$.*

Definability in Birkhoff institutions

Theorem 10.8. *Consider a (quasi-compact) Birkhoff institution with model amalgamation $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathcal{F}, \mathcal{B})$ and a class $\mathcal{S} \subseteq \mathbb{S}ig$ of signature morphisms which is stable under pushouts and such that for each $\varphi \in \mathcal{S}$*

1. $\mathsf{Mod}(\varphi)$ *preserves* $\mathcal{F}$*-filtered products (of models), and*
2. φ *lifts weakly* $\mathcal{B}^{-1}$.

Then any signature morphism in $\mathcal{S}$ *has the (finite) definability property.*

Proof. Let $\varphi \in \mathcal{S}$. If $\varphi : \Sigma \to \Sigma'$ is implicitly defined by E', then we show it is (finitely) explicitly defined by E' too. Therefore consider any pushout square of signature morphisms for the span $\Sigma_1 \xleftarrow{\theta} \Sigma \xrightarrow{\varphi} \Sigma'$

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma' \arrow[d, "\theta'"] \\ \Sigma_1 \arrow[r, "\varphi_1"'] & \Sigma'_1 \end{tikzcd}$$

and any Σ'_1-sentence ρ.

By the stability of $\mathcal{S}$ under pushouts we have that $\varphi_1 \in \mathcal{S}$, thus it lifts weakly $\mathcal{B}^{-1}$ and preserves filtered products. Let $\mathbb{M}'_1 = (\theta'(E') \cup \{\rho\})^*$ and let us define $E_\rho = (\mathbb{M}'_1 {\restriction}_{\varphi_1})^*$.

We first show $\theta'(E') \cup \{\rho\} \models \varphi_1(E_\rho)$. Consider M'_1 a model of $\theta'(E') \cup \rho$. This implies that $M'_1 {\restriction}_{\varphi_1} \in \mathbb{M}'_1 {\restriction}_{\varphi_1}$ and because E_ρ is satisfied by all models in $\mathbb{M}'_1 {\restriction}_{\varphi_1}$ we have that $M'_1 {\restriction}_{\varphi_1} \models E_\rho$. By the Satisfaction Condition we obtain that $M'_1 \models \varphi_1(E_\rho)$.

Now we show that $\theta'(E') \cup \varphi_1(E_\rho) \models \rho$. Consider M'_1 a Σ'_1-model satisfying $\theta'(E') \cup \varphi_1(E_\rho)$. By the Satisfaction Condition we have that $M'_1 {\restriction}_{\varphi_1} \models E_\rho = (\mathbb{M}'_1 {\restriction}_{\varphi_1})^*$ hence $M'_1 {\restriction}_{\varphi_1} \in (\mathbb{M}'_1 {\restriction}_{\varphi_1})^{**}$. By the Birkhoff institution condition this means $M'_1 {\restriction}_{\varphi_1} \in \mathcal{B}^{-1}_{\Sigma_1}(\mathcal{F}(\mathbb{M}'_1 {\restriction}_{\varphi_1}))$. By considering successively the following:

- $\mathcal{F}(\mathbb{M}'_1 {\restriction}_{\varphi_1}) = \cong_{\Sigma_1} ((\mathcal{F}\mathbb{M}'_1){\restriction}_{\varphi_1})$ (i.e., the isomorphic closure $\mathrm{Iso}((\mathcal{F}\mathbb{M}'_1){\restriction}_{\varphi_1})$ of $(\mathcal{F}\mathbb{M}'_1){\restriction}_{\varphi_1}$) because φ_1 preserves filtered products,
- $\cong_{\Sigma_1} ; \mathcal{B}^{-1}_{\Sigma_1} = \mathcal{B}^{-1}_{\Sigma_1}$ because $\mathcal{B}$ is closed under isomorphisms,
- $\mathcal{F}\mathbb{M}'_1 \subseteq \mathcal{B}^{-1}_{\Sigma'_1}(\mathcal{F}\mathbb{M}'_1)$ because $\mathcal{B}$ is reflexive, and
- $\mathcal{B}^{-1}_{\Sigma'_1}(\mathcal{F}\mathbb{M}'_1) = \mathbb{M}'_1$ because $\mathbb{M}'_1 = (\mathbb{M}'_1)^{**}$ and by the Birkhoff condition,

from $M'_1 {\restriction}_{\varphi_1} \in \mathcal{B}^{-1}_{\Sigma_1}(\mathcal{F}(\mathbb{M}'_1 {\restriction}_{\varphi_1}))$ we deduce that

$$M'_1 {\restriction}_{\varphi_1} \in \mathcal{B}^{-1}_{\Sigma_1}(\cong_{\Sigma_1} ((\mathcal{F}\mathbb{M}'_1){\restriction}_{\varphi_1})) = \mathcal{B}^{-1}_{\Sigma_1}((\mathcal{F}\mathbb{M}'_1){\restriction}_{\varphi_1}) \subseteq \mathcal{B}^{-1}_{\Sigma_1}(\mathbb{M}'_1 {\restriction}_{\varphi_1}).$$

This means that there exists a Σ'_1-model $N'_1 \in \mathbb{M}'_1$ thus $\theta'(E') \cup \{\rho\}$ and such that $\langle M'_1 {\restriction}_{\varphi_1}, N'_1 {\restriction}_{\varphi_1} \rangle \in \mathcal{B}_{\Sigma_1}$. Because φ_1 lifts $\mathcal{B}^{-1}$ there exists a Σ'_1-model P'_1 such that $P'_1 {\restriction}_{\varphi_1} = M'_1 {\restriction}_{\varphi_1}$ and $\langle P'_1, N'_1 \rangle \in \mathcal{B}_{\Sigma'_1}$.

Because $\{\{\{*\}\}\} \in \mathcal{F}$ we have that $\mathcal{B}^{-1}_{\Sigma'_1}(\mathbb{M}'_1) \subseteq \mathcal{B}^{-1}_{\Sigma'_1}(\mathcal{F}\mathbb{M}'_1) = \mathbb{M}'_1$. From $P'_1 \in \mathcal{B}^{-1}_{\Sigma'_1}(N'_1) \subseteq \mathcal{B}^{-1}_{\Sigma'_1}(\mathbb{M}'_1)$ we get that $P'_1 \in \mathbb{M}'_1$ which means that $P'_1 \models \theta'(E') \cup \{\rho\}$.

From $M'_1, P'_1 \models \theta'(E')$ we have that $M'_1 {\restriction}_{\theta'}, P'_1 {\restriction}_{\theta'} \models E'$ and because φ is implicitly defined by E' and $(M'_1 {\restriction}_{\theta'}){\restriction}_{\varphi} = M'_1 {\restriction}_{\varphi_1} {\restriction}_{\theta} = P'_1 {\restriction}_{\varphi_1} {\restriction}_{\theta} = (P'_1 {\restriction}_{\theta'}){\restriction}_{\varphi}$ we obtain $M'_1 {\restriction}_{\theta'} = P'_1 {\restriction}_{\theta'}$.

By the uniqueness aspect of the model amalgamation property, from $M_1'\restriction_{\varphi_1} = P_1'\restriction_{\varphi_1}$ and $M_1'\restriction_{\theta'} = P_1'\restriction_{\theta'}$ we get that $M_1' = P_1'$. Thus $M_1' \models \rho$.

We have therefore shown that $\theta'(E') \cup \{\rho\} \models \varphi_1(E_\rho)$ and $\theta'(E') \cup \varphi_1(E_\rho) \models \rho$. Moreover, when the institution is quasi-compact, E_ρ can be chosen finite. Thus $\theta'(E') \models \rho \Leftrightarrow \varphi_1(E_\rho)$, which implies that $E' \models (\forall\theta')(\rho \Leftrightarrow \varphi_1(E_\rho))$. □

Note that Thm. 10.8 above involves the lifting of the Birkhoff relation in an opposite direction from that of the interpolation by axiomatizability Thm. 9.6. This hints at the fact that the instances of the general definability result given by Thm. 10.8 may not be caused by interpolation properties. Otherwise the conditions underlying Thm. 10.8 bear strong similarity with those underlying Thm. 9.6. This means that for deriving instances in actual institutions, the core technical condition to be addressed is the lifting of the Birkhoff relation.

The lifting condition. The following result establishes weak lifting conditions required by Thm. 10.8 for some classes of **FOL** signature morphisms.

Proposition 10.9. *In* **FOL***, any* $(*e*)$*-morphism of signatures lifts weakly* $\xleftarrow{S_w}$ *and* $\xleftarrow{S_c}$.

Proof. Let us assume that $\varphi : (S,F,P) \to (S',F',P')$ is a $(*s*)$-morphism.

Let $h : N'\restriction_\varphi \to M'\restriction_\varphi$ be an injective (S,F,P)-model homomorphism. We define Q' to be the φ-expansion of $N'\restriction_\varphi$ such that

- $Q'_x = N'_x$ for each sort, operation, or relation symbol outside $\varphi(S,F,P)$, the image of φ,
- $Q'_{s'} = M'_{s'}$ for $s' \in S' \setminus \varphi(S)$; this determines a φ-expansion h' of h such that $h'_{s'} = 1_{M'_{s'}}$ for each $s' \in S' \setminus \varphi(S)$,
- $Q'_{\sigma'}(x_1,\ldots,x_n) = h'^{-1}(M_{\sigma'}(h'(x_1),\ldots,h'(x_n)))$ for $\sigma' \in F' \setminus \varphi(F)$, and
- $Q'_{\pi'} = h'^{-1}(M'_{\pi'})$ for $\pi' \in P' \setminus \varphi(P)$.

Then $h' : Q' \to M'$ is well defined and is an injective (S',F',P')-model homomorphism. Moreover, if $h : N'\restriction_\varphi \to M'\restriction_\varphi$ is closed, then $h' : Q' \hookrightarrow M'$ is closed too. □

Instances of Thm. 10.8. From the general definability Thm. 10.8 and from the lifting Prop. 10.9 and 9.12 (by noting that the composition of a signature morphism lifts weakly a relation R and lifts another relation R', then it lifts weakly their composition $R;R'$), by taking into consideration the quasi-compactness property of each institution and because in **FOL** the model reduct functors corresponding to any signature morphism preserves direct products and directed co-limits, we can now formulate a series of definability results for some of the Birkhoff institutions listed in Sect. 8.6.

Corollary 10.10. *We have the following table of definability results:*

institution	*signature morphism*	*definability property*
HCL	$*e*$	finite definability
$\mathbf{HCL}_\infty$	$*e*$	definability
UNIV	$*e*$	finite definability
universal $\mathbf{FOL}_{\infty,\omega}$-sentences	$*e*$	finite definability
universal **FOL**-atoms	iei	finite definability
EQL	ie	finite definability
$\forall\vee$	$ie*$	finite definability
$\forall\vee_\infty$	$ie*$	definability

Exercises

10.5. [148] **Definability in PA by axiomatizability** (Ex. 8.9 and 8.6 continued)
Any $(s**)$-morphism of **PA** signatures has the definability property in $QE_1(\mathbf{PA})$ and $QE_2(\mathbf{PA})$. In the case of $QE_1(\mathbf{PA})$, prove this in two different ways:

1. Directly by Thm. 10.8, and
2. By borrowing it from $\mathbf{HCL}_\infty$ along the relational encoding comorphism $\mathbf{PA} \to \mathbf{FOL}^{\mathrm{pres}}$ by using the result of Ex. 10.2 in the style of Ex. 10.4.

Notes. The material of this chapter is based on [148]. This includes the concept of definability for signature morphisms as well as the definability Theorems 10.5 and 10.8.

The definability by interpolation Thm. 10.5 is a generalization of the conventional concrete Beth definability theorem in $\mathbf{FOL}^{\mathbf{1}}$ of [18]. While traditional proofs of Beth's theorem use Craig interpolation and implication, the proof given in [148] uses only Craig-Robinson interpolation, being thus applicable to institutions without semantic implication.

The reformulation of definability for sets of sentences rather than single sentences of [148] settled the definability concept in a proper form applicable to institutions without conjunctions. The general definability by axiomatizability Thm. 10.8 and its instances in logics such as **HCL** owe much to this reformulation of definability.

Chapter 11

Possible Worlds

Recall that the first order modal logic institution **MFOL**, introduced as an example in Sect. 3.2, refines **FOL** by adding the modalities of 'necessity' and 'possibility'. While at the sentence level this means a couple of additional unary connectives ($\Box$ for 'necessity" and $\Diamond$ for 'possibility'), their semantics is much less straightforward because it requires the concept of 'possible worlds' semantics, which means that the models are Kripke models, i.e., collections of possible interpretations rather than single interpretation of the signatures. Possible worlds semantics even subtly 'alters' the standard semantics of the Boolean connectives and of the quantifiers.

The summary of this chapter. We first introduce possible worlds semantics and modal satisfaction at an institution-independent level by developing an *internal modal logic*. A typical example is **MFOL** which is obtained as an internal modal logic over the (sub-)institution of the atoms of **FOL**. Internal modal logic provides the basis for a general extension of the method of ultraproducts to possible worlds semantics. Possible worlds semantics allows only half of the ultraproducts fundamental theorem, fortunately the 'better' half, namely that each modal sentence is preserved by ultraproducts of Kripke models. This is a good example of a situation in which the sentences, although not Łoś sentences, are still preserved by ultraproducts. An important consequence of the preservation of modal sentences by ultraproducts of Kripke models is the (model) compactness of possible worlds semantics.

Before proceeding with the developments of this chapter it would be useful to get some insight in possible worlds semantics by means of a simple example in **MFOL**.

An example of possible worlds semantics. Recall from Sect. 3.2 that a Kripke model (W,R) for an **MFOL** signature (S,S_0,F,F_0,P,P_0) (with (S,F,P) being a **FOL** signature and (S_0,F_0,P_0) a sub-signature of 'rigid' symbols) consists of

- a non-empty family $W = \{W^i\}_{i\in I_W}$ of 'possible worlds', which are **FOL** (S,F,P)-models, together with

- an 'accessibility' relation $R \subseteq I_W \times I_W$ between the possible worlds such that the following sharing constraint holds:
- for each $i, i' \in I_W$ we have that $W^i_x = W^{i'}_x$ for each rigid [sort, operation, or relation] symbol x.

Let us consider a 'rigid' signature Σ for the natural numbers to which we add a 'flexible' unary operation symbol σ. The sentence

$$(\forall x)\Diamond(\forall y)x \leq \sigma(y)$$

which reads as "for all x it is possible that for all y, $x \leq \sigma(y)$" holds in the Kripke $\Sigma \cup \{\sigma\}$-model (W,R) for which

- the index set I_W of the Kripke model (W,R) is ω the set of the natural numbers,
- the accessibility relation R is the 'less than or equal' relation $\leq$ on the natural numbers, and
- for each natural number i, the possible world W^i is the expansion of the standard model of the natural numbers with the interpretation of σ given by $W^i_\sigma(n) = i + n$ for each natural number n.

Then for each i, we have that $(W,R) \models^i (\forall x)\Diamond(\forall y)x \leq \sigma(y)$ because for any natural number x we have that $(W,R) \models^{\max(i,x)} (\forall y)x \leq \sigma(y)$ (note that this means $W^{\max(i,x)} \models (\forall y)x \leq \sigma(y)$) and $\langle i, \max(i,x)\rangle \in R$. Although W^i may not satisfy $(\forall y)x \leq \sigma(y)$, this sentence is always 'possibly' satisfied at i, i.e., $W^i \models \Diamond(\forall y)x \leq \sigma(y)$, because there always exists j such that $\langle i, j\rangle \in R$ (i.e., $i \leq j$) and $W^j \models (\forall y)x \leq \sigma(y)$.

11.1 Internal Modal Logic

Internal Kripke models

The following definition abstracts the concept of the **MFOL** Kripke model to arbitrary institutions.

Internal Kripke models. The concept of a Kripke model can be defined internally to any 'base' institution $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ providing the base models of the Kripke models, the sharing parameter being handled by a 'forgetful' institution morphism $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta)$ to a 'domain' institution Δ providing the shared domains. More precisely, given a signature Σ in $\mathbb{S}ig$, a Σ-*Kripke model* (W,R) consists of

- a family of Σ-models $W : I_W \to |\mathsf{Mod}(\Sigma)|$ such that the sharing condition

  $$\beta^\Delta_\Sigma(W^i) = \beta^\Delta_\Sigma(W^{i'})$$

  holds for each $i, i' \in I_W$, and
- a binary 'accessibility' relation R on the index set I_W.

A Kripke model (W,R) is *T*, *S4*, or *S5*, respectively, when R is reflexive, T and transitive, or *S4* and symmetric, respectively.

MFOL Kripke models are a special case of the above general definition of internal Kripke models by considering the base institution to be $\mathbf{FOL}'$ which is like **FOL** but with signatures with marked rigid symbols, i.e., signatures of the form (S,S_0,F,F_0,P,P_0). Then $\mathsf{Mod}^{\mathbf{FOL}'}(S,S_0,F,F_0,P,P_0) = \mathsf{Mod}^{\mathbf{FOL}}(S,F,P)$ and $\mathsf{Sen}^{\mathbf{FOL}'}(S,S_0,F,F_0,P,P_0) = \mathsf{Sen}^{\mathbf{FOL}}(S,F,P)$ (although at this stage we do not need to care about sentences and satisfaction), and the domain institution Δ to be just **FOL**.

MPL Kripke models can be obtained by considering propositional logic **PL** as the base institution, and the institution with only one signature and one model, and without sentences, as the domain institution.

Homomorphisms of Kripke models. They preserve the mathematical structure of the Kripke models. Thus a *homomorphism of* Σ*-Kripke models* $(h^W,h^I) : (W,R) \to (W',R')$ consists of

- a function $h^I : I_W \to I_{W'}$ between the index sets which is a relation homomorphism, i.e., $\langle i, j\rangle \in R$ implies $\langle h^I(i), h^I(j)\rangle \in R'$; note this means that h^I is a $\mathbf{REL^1}$-model homomorphism $(I_W,R) \to (I_{W'},R')$ (in the single sorted relational signature with one binary relation symbol), and
- an I_W-indexed family of Σ-model homomorphisms $h^W = \{(h^W)^i : W^i \to W'^{h^I(i)}\}_{i\in I_W}$ such that $\beta^\Delta_\Sigma((h^W)^i) = \beta^\Delta_\Sigma((h^W)^{i'})$ for $i,i' \in I_W$.

Note that the family h^W can be regarded as a natural transformation $h^W : W \Rightarrow h^I;W'$ between functions (regarded as functors) $I_W \to |\mathsf{Mod}(\Sigma)|$.

When the context is clear we may omit the superscripts W and I from the notation of h^W and h^I, and simply use h instead.

Fact 11.1. *The* Σ*-Kripke models and their homomorphisms form a category denoted* $\mathsf{K\text{-}Mod}(\Sigma)$.

For the more category theory oriented readers the following characterization gives easier access to the structural properties of $\mathsf{K\text{-}Mod}(\Sigma)$.

Fact 11.2. $\mathsf{K\text{-}Mod}(\Sigma)$ *is the Grothendieck category of the indexed category* $\mathsf{Mod}^{\mathbf{REL^1}}(\emptyset,\{r : 2\}) \to \mathbb{C}at^{\mathrm{op}}$ *which maps each* $\mathbf{REL^1}$ *model* (I,R) *to the sub-category of* $\mathsf{Mod}(\Sigma)^I$ *of all families of* Σ*-model homomorphisms* $\{h^i\}_{i\in I}$ *such that* $\beta^\Delta_\Sigma(h^i) = \beta^\Delta_\Sigma(h^{i'})$ *for all* $i,i' \in I$*. (Recall that* $\mathbf{REL^1}$ *is the single-sorted variant of the relational logic* **REL***.)*

Reducing Kripke models. Given a signature morphism $\varphi : \Sigma \to \Sigma'$, each Σ'-Kripke model (W',R') can be reduced to the Σ-Kripke model $(W = W';\mathsf{Mod}(\varphi),R')$. This means that for each index $i \in I_W$ we have that $W^i = (W')^i\restriction_\varphi$.

By the naturality of β^Δ and by the sharing condition for (W',R), we obtain the sharing condition for the reduced Kripke model $W';\mathsf{Mod}(\varphi),R')$, hence:

Fact 11.3. *The reduct of a Σ'-Kripke model corresponding to a signature morphism $\varphi : \Sigma \to \Sigma'$ is a Σ-Kripke model.*

Similarly, each Kripke model homomorphism (h^W, h^I) can be reduced to $(h^W \mathsf{Mod}(\varphi),$
$h^I)$.

Fact 11.4. $\mathsf{K\text{-}Mod} : \mathbb{S}ig^{\mathrm{op}} \to \mathbb{C}at$ *is a functor.*

Model amalgamation for Kripke models. The model amalgamation properties of the base institution carries to the Kripke model functor.

Proposition 11.5. *Given an institution morphism* $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ *(from a 'base' institution to a 'domain' institution) any commuting square of signature morphisms in* $\mathbb{S}ig$

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

such that

1. *it is a model amalgamation square in the base institution, and*
2. Φ^Δ *maps it to a model amalgamation square in the domain institution,*

is a model amalgamation square with respect to the Kripke model functor $\mathsf{K\text{-}Mod}$.

Proof. Let (W_1, R_1) be a Kripke Σ_1-model and (W_2, R_2) be a Kripke Σ_2-model such that $(W_1, R_1)\restriction_{\varphi_1} = (W_2, R_2)\restriction_{\varphi_2}$. This means that $R_1 = R_2$ and $I_{W_1} = I_{W_2}$, and for each $i \in I_{W_1} = I_{W_2}$, $(W_1)^i\restriction_{\varphi_1} = (W_2)^i\restriction_{\varphi_2}$.

We define the Kripke Σ'-model (W', R') such that $R' = R_1 = R_2$, $I_{W'} = I_{W_1} = I_{W_2}$, and for each index $i \in I_{W'}$, W'^i is the amalgamation of $(W_1)^i$ and $(W_2)^i$. We can easily notice that $(W', R')\restriction_{\theta_1} = (W', R')\restriction_{\theta_2}$ and that (W', R') is the *unique* common expansion of (W_1, R_1) and (W_2, R_2). We still need to show the sharing condition for (W', R'), that for each $i, j \in I_{W'}$ we have that $\beta^\Delta_{\Sigma'}(W'^i) = \beta^\Delta_{\Sigma'}(W'^j)$.

Because

$$\begin{tikzcd} \Phi^\Delta(\Sigma) \arrow[r, "\Phi^\Delta(\varphi_1)"] \arrow[d, "\Phi^\Delta(\varphi_2)"'] & \Phi^\Delta(\Sigma_1) \arrow[d, "\Phi^\Delta(\theta_1)"] \\ \Phi^\Delta(\Sigma_2) \arrow[r, "\Phi^\Delta(\theta_2)"'] & \Phi^\Delta(\Sigma') \end{tikzcd}$$

is an amalgamation square in the domain institution Δ it is enough to show that $\beta^\Delta_{\Sigma'}(W'^i)\restriction_{\Phi^\Delta(\theta_k)} = \beta^\Delta_{\Sigma'}(W'^j)\restriction_{\Phi^\Delta(\theta_k)}$ for $k \in \{1, 2\}$. By the naturality of β^Δ this is equivalent to $\beta^\Delta_{\Sigma_k}(W'^i\restriction_{\theta_k}) = \beta^\Delta_{\Sigma_k}(W'^j\restriction_{\theta_k})$ which means $\beta^\Delta_{\Sigma_k}(W^i_k) = \beta^\Delta_{\Sigma_k}(W^j_k)$. This holds by the sharing condition for $(W_k, R' = R_k)$. $\square$

An instance of Prop. 11.5 using the model amalgamation property for **FOL** (cf. Prop. 4.5) gives the following model amalgamation property for **MFOL**.

Corollary 11.6. *Any commuting square of* **MFOL** *signature morphisms*

$$\begin{tikzcd}
(S,S_0,F,F_0,P,P_0) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & (S^1,S^1_0,F^1,F^1_0,P^1,P^1_0) \arrow[d, "\theta_1"] \\
(S^2,S^2_0,F^2,F^2_0,P^2,P^2_0) \arrow[r, "\theta_2"'] & (S',S'_0,F',F'_0,P',P'_0)
\end{tikzcd}$$

is an amalgamation square whenever both

$$\begin{tikzcd}
(S,F,P) \arrow[r] \arrow[d] & (S^1,F^1,P^1) \arrow[d] \\
(S^2,F^2,P^2) \arrow[r] & (S',F',P')
\end{tikzcd}
\qquad
\begin{tikzcd}
(S_0,F_0,P_0) \arrow[r] \arrow[d] & (S^1_0,F^1_0,P^1_0) \arrow[d] \\
(S^2_0,F^2_0,P^2_0) \arrow[r] & (S'_0,F'_0,P'_0)
\end{tikzcd}$$

are pushout squares of **FOL** *signature morphisms.*

Modal satisfaction

Possible worlds semantics implies a more refined treatment for the semantics of the Boolean connectives and of the quantifiers because it 'stratifies' the satisfaction relation by the 'possible worlds' of a Kripke model. Therefore, given a signature Σ, for each Σ-Kripke model (W,R) and each Σ- sentence ρ we define the *satisfaction of* ρ *in* (W,R) *at the possible world* $i \in I_W$, denoted $(W,R) \models^i \rho$. Then the 'global' satisfaction $(W,R) \models \rho$ is defined by $(W,R) \models^i \rho$ at *each* possible world $i \in I_W$.

The modal satisfaction of the Boolean connectives and of the quantifiers is defined by their standard internal logic semantics (see Sections 5.1 and 5.2) but applied to $\models^i$ rather than to $\models$. This makes all the difference, since for example the modal negation is not semantic in the sense that $(W,R) \models \neg\rho$ is not the same with $(W,R) \not\models \rho$ (in spite of the fact that $(W,R) \models^i \neg\rho$ is *defined* as $(W,R) \not\models^i \rho$). The same situation holds for most of the Boolean connectives or quantifiers, however there are some notable exceptions:

Fact 11.7. *Conjunctions and universal quantifiers are semantic with respect to the modal satisfaction, i.e.,*

- *for each* Σ*-Kripke model* (W,R) *and any* Σ*-sentences* ρ_1 *and* ρ_2*,* $(W,R) \models \rho_1 \wedge \rho_2$ *if and only if* $(W,R) \models \rho_1$ *and* $(W,R) \models \rho_2$*, and*
- *for each signature morphism* $\chi\colon\ \Sigma \to \Sigma'$*, each* Σ*-Kripke model* (W,R) *and each* Σ*-sentence* ρ*,* $(W,R) \models (\forall\chi)\rho$ *if and only if* $(W',R) \models \rho$ *for each* χ*-expansion* (W',R) *of* (W,R)*.*

The satisfaction of modalities 'necessity' and 'possibility' is defined by

$$(W,R) \models^i \Box\rho \ \text{ if and only if } \ (W,R) \models^j \rho \ \text{ for each } \ \langle i, j\rangle \in R,$$

$$(W,R) \models^i \Diamond\rho \ \text{ if and only if there exists } \ \langle i, j\rangle \in R \ \text{ such that } \ (W,R) \models^j \rho.$$

Note that the satisfaction of modalities, unlike the satisfaction of the Boolean connectives, really needs possible worlds semantics given by the concept of a Kripke model and the stratification of the satisfaction relation by the possible worlds. One can even say that possible worlds semantics was invented in order to meet the needs of the satisfaction of modalities.

Modal institutions. In order to complete the definition of a 'modal institution' on top of a 'base institution' we need to define a 'modal sentence' functor. Let $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ be an institution morphism (from a 'base' institution to a 'domain' institution). We extend Sen to a *modal sentence functor* $\mathsf{M\text{-}Sen} : \mathbb{S}ig \to \mathbb{S}et$ such that each $\mathsf{M\text{-}Sen}$ sentence is syntactically accessible from the sentences of the base institution by

- Boolean connectives,
- modalities ($\Box$ and $\Diamond$), and
- $\mathcal{D}$-quantifiers, for a class $\mathcal{D}$ of signature morphisms stable under pushouts and such that any pushout between any morphism from $\mathcal{D}$ and any other signature morphism

  (EB) is an amalgamation square in the base institution, and
  (ED) gets mapped by Φ^Δ to an amalgamation square in the domain institution.

Then we define a satisfaction relation between Kripke models and $\mathsf{M\text{-}Sen}$ sentences inductively on the structure of the sentences according to the internal modal satisfaction described above and by defining

$$(W,R) \models^i \rho \text{ if and only if } W^i \models \rho \text{ when } \rho \in \mathsf{Sen}(\Sigma).$$

The following result shows that this process builds indeed an institution.

Theorem 11.8. *For any institution morphism* $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ *(from a 'base' institution to a 'domain' institution), for any modal sentence functor constructed by a process described as above,* $(\mathbb{S}ig, \mathsf{M\text{-}Sen}, \mathsf{K\text{-}Mod}, \models)$ *is an institution.*

Proof. The satisfaction condition for $(\mathbb{S}ig, \mathsf{M\text{-}Sen}, \mathsf{K\text{-}Mod}, \models)$ follows from the fact that

$$(W',R') \models^i \varphi(\rho) \text{ if and only if } (W',R')\restriction_\varphi \models^i \rho$$

for each signature morphism $\varphi : \Sigma \to \Sigma'$, each $\rho \in \mathsf{M\text{-}Sen}(\Sigma)$, for each Σ'-Kripke model (W',R'), and for each $i \in I_{W'}$. This can be shown easily by induction on the structure of the sentence ρ in the manner we have established the satisfaction condition for **FOL** in Sect. 3.1. Note that when $\rho \in \mathsf{Sen}(\Sigma)$, this relation follows from the satisfaction condition of the base institution. Also, the induction step for quantifiers involves the model amalgamation property for the Kripke models given by Prop. 11.5. $\Box$

MFOL as modal institution. The **MFOL** sentences and their satisfaction by the **MFOL** Kripke models is an instance of the general process defined above as follows:

- We replace the base institution **FOL**$'$ used for defining the Kripke models with its 'atomic' sub-institution **AFOL**$'$ which has only the atoms as sentences. This is necessary because some of the Boolean connectives and of the quantifications obtained by internal modal logic will not be semantic, and thus semantically different from the Boolean connectives and the quantifiers of **FOL**$'$. Then the domain institution is **AFOL**, the atomic sub-institution of **FOL**, rather than **FOL**.
- We consider all sentences constructed from the atoms by iteratively applying Boolean and modal connectives and $\mathcal{D}$-quantifications for the signature extensions $(S, S_0, F, F_0, P, P_0) \hookrightarrow (S, S_0, F \uplus X, F_0 \uplus X, P, P_0)$ with a finite set of rigid constants X.

The conditions (EB) and (ED) hold easily because both squares involved in these conditions can be read as pushout squares of **FOL** signature morphisms and cf. Prop. 4.5 they are amalgamation squares. This leads to the following expected instance of Thm. 11.8.

Corollary 11.9. **MFOL** *is an institution.*

Exercises

11.1. For any signature Σ in any institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$, the category of Σ-Kripke models without sharing can be obtained as the pullback of the index projections for the Grothendieck categories

$$\begin{tikzcd}
\mathsf{K\text{-}Mod}(\Sigma) \arrow[r] \arrow[d] & \mathbb{C}at(-, \mathsf{Mod}(\Sigma))^{\sharp} \arrow[d] \\
((-)^2; \mathbb{C}at(-, \mathbf{2}))^{\sharp} \arrow[r] & \mathbb{S}et
\end{tikzcd}$$

where

- $\mathbb{C}at(-, \mathsf{Mod}(\Sigma)) : \mathbb{S}et^{\mathrm{op}} \to \mathbb{C}at$ maps each set I to the functor category $\mathbb{C}at(I, \mathsf{Mod}(\Sigma))$,
- $(-)^2 : \mathbb{S}et^{\mathrm{op}} \to \mathbb{S}et^{\mathrm{op}}$ maps each set I to its square product $I \times I$,
- $\mathbb{C}at(-, \mathbf{2}) : \mathbb{S}et^{\mathrm{op}} \to \mathbb{C}at$ maps each set J to the partial order of its subsets.

Extend this result to Kripke models with sharing by replacing $\mathbb{C}at(I, \mathsf{Mod}(\Sigma))$ with its subcategory consisting of the families of models satisfying the sharing.

11.2. Limits and co-limits of Kripke models
Consider an institution morphism $(\Phi^{\Delta}, \alpha^{\Delta}, \beta^{\Delta}) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$ from a base institution to a domain institution. Assume that for each signature Σ of the base institution, β^{Δ}_{Σ} preserves J-(co-)limits for a category J. Then $\mathsf{K\text{-}Mod}(\Sigma)$ has J-(co-)limits when $\mathsf{Mod}(\Sigma)$ has J-(co-)limits.

11.3. A quasi-representable signature morphism in the base institution is quasi-representable in the modal institution too. However this does not hold for representable signature morphisms.

11.4. For any commutative square of institution morphisms

$$\begin{tikzcd}
(\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', \models') \arrow[r, "(\Phi,\alpha,\beta)"] \arrow[d, "(\Phi^{\Delta'},\alpha^{\Delta'},\beta^{\Delta'})"'] & (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \arrow[d, "(\Phi^{\Delta},\alpha^{\Delta},\beta^{\Delta})"] \\
\Delta' \arrow[r] & \Delta
\end{tikzcd}$$

where the vertical arrows represent institution morphisms from base institutions to domain institutions, and for any natural transformation $\alpha^M : \Phi; \mathsf{M\text{-}Sen} \Rightarrow \mathsf{M\text{-}Sen}'$ between modal sentence functors over Sen and respectively Sen' which preserves the Boolean connectives, the quantifiers, and the modal operators, there exists a canonical institution morphism

$$(\Phi, \alpha^M, \beta^K) : (\mathbb{S}ig', \mathsf{M\text{-}Sen}', \mathsf{K\text{-}Mod}', \models') \to (\mathbb{S}ig, \mathsf{M\text{-}Sen}, \mathsf{K\text{-}Mod}, \models).$$

11.5. Prove the modal satisfaction relations given in the table below, for the following properties of the Kripke models:

	property of Kripke models
$\neg\Box\rho \models\!\!\!\dashv \Diamond\neg\rho$	–
$\models \Box\rho \Rightarrow \rho$	T
$\models \Box(\rho \Rightarrow \rho') \Rightarrow (\Box\rho \Rightarrow \Box\rho')$	–
$\rho \models \Box\rho$	–
$\models \Box\rho \Rightarrow \Box\Box\rho$	S4
$\models \Diamond\rho \Rightarrow \Box\Diamond\rho$	S5
$\models (\forall\chi)\Box\rho \Leftrightarrow \Box(\forall\chi)\rho$	–
$\models (\exists\chi)\Box\rho \Rightarrow \Box(\exists\chi)\rho$	–

11.2 Ultraproducts of Kripke models

The aim of this section is to develop an extension of the method of ultraproducts (introduced in Chap. 6) to possible worlds semantics and to modal satisfaction.

The framework. For this section we assume

- a class $\mathcal{F}$ of filters, and
- an institution morphism from a base institution to a domain institution $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models) \to \Delta$

such that the following two properties hold:

(FP) for each signature Σ the category of Σ-models $\mathsf{Mod}(\Sigma)$ has products and has $\mathcal{F}$-filtered products which are preserved by β^Δ_Σ, and

(LI) for any signature Σ, β^Δ_Σ *lifts isomorphisms*, i.e., if $\beta^\Delta_\Sigma(M) \cong N'$ there exists $N \cong M$ such that $N' = \beta^\Delta_\Sigma(N)$.

$$\beta^\Delta_\Sigma(M) \xrightarrow{\cong} N' = \beta^\Delta_\Sigma(N).$$

$$M \xrightarrow[\cong]{} (\exists)N.$$

While the assumption (FP) is expected since it constitutes the basis for the existence of filtered products of Kripke models, (LI) is rather technical but very easily satisfied in the applications. For example, it is obviously satisfied by the forgetful institution morphism $\mathbf{AFOL}' \to \mathbf{AFOL}$ which gives rise to **MFOL** as a modal institution.

Filtered products of Kripke models

Filtered products of Kripke models are obtained from the filtered products of models of the base institution as follows.

Proposition 11.10. *For each signature Σ, the category of Kripke models* $\mathsf{K}\text{-}\mathsf{Mod}(\Sigma)$ *has $\mathcal{F}$-filtered products.*

Proof. Let $F \in \mathcal{F}$ be any filter over a set I and let $\{(W_j, R_j) \mid j \in I\}$ be an I-indexed family of Kripke models for a fixed signature Σ. The filtered product modulo F of $\{(W_i, R_i) \mid i \in I\}$ is just the categorical filtered product in the category $\mathsf{K}\text{-}\mathsf{Mod}$ of Σ-Kripke models. This can be obtained by using the characterization of the category of the Kripke models as a Grothendieck category given by Fact 11.2 and by the general result of existence of limits and co-limits in Grothendieck categories given by Thm. 2.10. However it will also help to make the construction of the filtered products of Kripke models explicit.

For each $J \in F$ we denote the Kripke model product $\prod_{j\in J}(W_j, R_j)$ by (W_J, R_J). This product can be obtained in the following two steps:

- (I_{W_J}, R_J) is the product $\prod_{j\in J}(I_{W_j}, R_j)$ in the category of $\mathbf{REL}^{\mathbf{1}}$ models for a signature with one binary relation symbol; then if we write $k \in I_{W_J}$ as $(k_j)_{j\in J}$ with $k_j \in I_{W_j}$ for each $j \in J$, we have that

$$\langle k, k' \rangle \in R_J \ \text{ if and only if } \ \langle k_j, k'_j \rangle \in R_j \ \text{ for each } \ j \in J$$

- for each $k = (k_j)_{j\in J} \in I_{W_J}$ we have $W_J^k = \prod_{j\in J} W_j^{k_j}$.

Then for each $i \in J$ the canonical projection $p_{J,i} : \ (W_J, R_J) \to (W_i, R_i)$ is defined by

- $p_{J,i}(k) = k_i$ for each $k \in I_{W_J} = \prod_{i\in J} I_{W_i}$, and
- for each $k \in I_{W_J}$, $p^k_{J,i} : \ W_J^k \to W_i^{k_i}$ is the projection $\prod_{i\in J} W_i^{k_i} \to W_i^{k_i}$.

For each $J \subseteq J'$ where $J, J' \in F$, let $p_{J',J}$ denote the canonical projection $(W_{J'}, R_{J'}) \to (W_J, R_J)$. The filtered product (W_F, R_F) of $\{(W_i, R_i) \mid i \in I\}$ modulo F is the co-limit of the directed diagram made of all these projections $p_{J',J}$.

$$\begin{tikzcd}
(W_{J'}, R_{J'}) \arrow[rr, "p_{J',J}"] \arrow[dr, "\mu_{J'}"'] & & (W_J, R_J) \arrow[dl, "\mu_J"] \\
& (W_F, R_F) &
\end{tikzcd}$$

This co-limit is constructed in two steps. We first do the filtered product (I_{W_F}, R_F) of the family of $\mathbf{REL^1}$ models $\{(I_{W_i}, R_i) \mid i \in I\}$

$$\begin{tikzcd}
(I_{W_{J'}}, R_{J'}) \arrow[rr, "p_{J',J}"] \arrow[dr, "\mu_{J'}"'] & & (I_{W_J}, R_J) \arrow[dl, "\mu_J"] \\
& (I_{W_F}, R_F) &
\end{tikzcd}$$

Recall that $\mu_I(k) = \mu_I(k')$ if and only if $\{j \mid k_j = k'_j\} \in F$. At the second step, for each $i \in I_{W_F}$ we define W_F^i as the co-limit of the directed diagram constituted by the canonical projections $p_{k',k}:\ W_{J'}^{k'} \to W_J^k$ for each $J \subseteq J'$ in F, and each $k \in {\mu_J}^{-1}(i)$ and $k' \in {\mu_{J'}}^{-1}(i)$ with $p_{J',J}(k') = k$

$$\begin{tikzcd}
W_{J'}^{k'} \arrow[rr, "p_{k',k}"] \arrow[dr, "\mu_{k'}"'] & & W_J^k \arrow[dl, "\mu_k"] \\
& W_F^i &
\end{tikzcd}$$

By conditions (FP) and (LI) we can see that W_F^i can be chosen such that $\beta_\Sigma^\Delta(W_F^i) = \beta_\Sigma^\Delta(W_F^{i'})$ for each i and i' in I_{W_F}. $\square$

In this chapter the filtered product (W_F, R_F) will also be denoted as $\prod_F (W_i, R_i)$, notation corresponding to the standard notation for categorical filtered products.

Because Horn sentences are preserved by filtered products of **FOL** models we have that, if the accessibility relations $\{R_j\}_{j \in I}$ satisfy some properties expressed as Horn sentences (such as T, $S4$ or $S5$), then the accessibility relation R_F of the filtered product does satisfy the same properties. This extends the existence of filtered products in sub-categories of Kripke models determined by some Horn conditions on the accessibility relations. By a similar argument, in the case of ultraproducts this can be extended to sub-categories of Kripke models determined by any first-order conditions.

The following shows that for each $i \in I_{W_F}$, the (base) model W_F^i can be also presented as a filtered product of (base) models. This fact will be used in the proof of the modal fundamental Thm. 11.12 below.

Lemma 11.11. *For each $i \in I_{W_F}$, and each $(k_j)_{j \in I} \in \mu_I^{-1}(i)$, W_F^i is the filtered product modulo F of the family $\{W_j^{k_j} \mid j \in I\}$.*

Proof. For each $k \in \mu_I^{-1}(i)$ and each $J \in F$, let $k_J = p_{I,J}(k)$. Then the diagram formed by the projections $p_{k_{J'}, k_J}$ for all $J \subseteq J'$ in F is a final sub-diagram of the diagram defining W_F^i. The conclusion of the lemma now follows by the general categorical result of Thm. 2.4 showing that final sub-diagrams of directed diagrams give isomorphic co-limits. $\square$

Modal fundamental theorem

Sentences modally preserved by filtered factors/products. The method ultraproducts for possible worlds semantics requires the following refinement of the concept of preservation of sentences by filtered factors/products introduced in Sect. 6.2 for the 'single world' semantics.

Let $\mathcal{F}$ be a class of filters. For a signature Σ, a sentence ρ is

- *modally preserved by $\mathcal{F}$-filtered factors* when for each $i \in I_{W_F}$, $(W_F, R_F) \models^i \rho$ implies "there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \rho$ for each $j \in J$", and
- *modally preserved by $\mathcal{F}$-filtered products* when for each $i \in I_{W_F}$, "there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \rho$ for each $j \in J$" implies $(W_F, R_F) \models^i \rho$,

for each filter $F \in \mathcal{F}$ over a set I and for each family $\{(W_j, R_j)\}_{j \in I}$ of Σ-Kripke models, and where (W_F, R_F) as usual denotes the filtered product $\prod_F (W_j, R_j)$ and μ is the co-limiting co-cone of the filtered product (as in Prop. 11.10).

The following extends the ultraproducts fundamental Thm. 6.9 to possible worlds semantics.

Theorem 11.12 (Modal fundamental theorem). *1. Each sentence of the base institution which is preserved by $\mathcal{F}$-filtered products (in the base institution) is also modally preserved by $\mathcal{F}$-filtered products (of Kripke models).*

2. *Each sentence of the base institution which is preserved by $\mathcal{F}$-filtered factors (in the base institution) is also modally preserved by $\mathcal{F}$-filtered factors (of Kripke models).*
3. *The sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are closed under possibility $\Diamond$.*
4. *The sentences modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) are closed under possibility $\Diamond$.*
   Moreover if $\mathcal{F}$ is closed under reductions,
5. *The sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are closed under existential χ-quantification, when χ preserves $\mathcal{F}$-filtered products of models in the base institution (i.e., $\mathsf{Mod}(\chi)$ preserves $\mathcal{F}$-filtered products).*
6. *The sentences modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) are closed under existential χ-quantification, when χ lifts $\mathcal{F}$-filtered products of Kripke models (i.e., $\mathsf{K\text{-}Mod}(\chi)$ lifts $\mathcal{F}$-filtered products).*
7. *The sentences modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) and the sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are both closed under finite conjunctions.*
8. *The sentences modally preserved by $\mathcal{F}$-filtered products (of Kripke models) are closed under infinite conjunctions.*
9. *If a sentence is modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) then its negation is modally preserved by $\mathcal{F}$-filtered products (of Kripke models).*

And finally, if we further assume that $\mathcal{F}$ contains only ultrafilters,

10. *If a sentence is modally preserved by $\mathcal{F}$-filtered products (of Kripke models) then its negation is modally preserved by $\mathcal{F}$-filtered factors (of Kripke models).*

11. *The sentences modally preserved by both $\mathcal{F}$-filtered products and factors (of Kripke models) are closed under negation.*

Proof. Let F be any filter in $\mathcal{F}$ over set I, let $\{(W_j, R_j) \mid j \in I\}$ be a family of Kripke models, and let (W_F, R_F) be its filtered product modulo F. As usual, for any $k = (k_j)_{j \in J'} \in I_{W_{J'}} = \prod_{j \in J'} I_{W_j}$ and $J \subseteq J'$, by k_J we denote the tuple $(k_j)_{j \in J}$. Also, recall for any $J \in F$ its reduction to J is denoted by $F|_J$ and is defined as $\{J \cap X \mid X \in F\}$.

1. Assume that ρ is preserved by $\mathcal{F}$-filtered products in the base institution and let us fix $i \in I_{W_F}$. Let us assume that there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $W_j^{k_j} \models \rho$. Then we can find $k' \in \mu_I^{-1}(i)$ such that $k = k'_J$. By Lemma 11.11, W_F^i is the filtered product of $\{W_j^{k'_j}\}_{j \in I}$ modulo F, hence because ρ is preserved by $\mathcal{F}$-filtered products, $W_F^i \models \rho$.

2. Assume that ρ is preserved by $\mathcal{F}$-filtered factors in the base institution and let us fix $i \in I_{W_F}$. Let us assume that $W_F^i \models \rho$ and take arbitrary $k' \in \mu_I^{-1}(i)$. By Lemma 11.11, W_F^i is the filtered product of $\{W_j^{k'_j}\}_{j \in I}$ modulo F, hence there exists $J \in F$ such that $W_j^{k'_j} \models \rho$ for each $j \in J$. We can then take $k = k'_J$.

3. Assume ρ is modally preserved by $\mathcal{F}$-filtered products (of Kripke models) and fix $i \in I_{W_F}$. Let us assume that $(W_j, R_j) \models^{k_j} \Diamond\rho$ for each $j \in J$, for some $J \in F$ and some $k \in \mu_J^{-1}(i)$. Then, for each $j \in J$ there exists k'_j with $\langle k_j, k'_j \rangle \in R_j$ such that $(W_j, R_j) \models^{k'_j} \rho$. We define $i' = \mu_J((k'_j)_{j \in J})$ and we notice that $\langle i, i' \rangle \in R_F$. Because ρ is modally preserved by filtered products we deduce that $(W_F, R_F) \models^{i'} \rho$. Because $\langle i, i' \rangle \in R_F$ this means $(W_F, R_F) \models^i \Diamond\rho$.

4. Assume ρ is modally preserved by $\mathcal{F}$-filtered factors (of Kripke models) and fix $i \in I_{W_F}$. Let us assume that $(W_F, R_F) \models^i \Diamond\rho$. Then there exists i' with $\langle i, i' \rangle \in R_F$ such that $(W_F, R_F) \models^{i'} \rho$. This means that there exists $J' \in F$ and $l \in \mu_{J'}^{-1}(i)$ and $l' \in \mu_{J'}^{-1}(i')$ such that $\langle l, l' \rangle \in R_{J'}$. Because ρ is modally preserved by $\mathcal{F}$-filtered factors, there exists $J \in F$ and $k' \in \mu_J^{-1}(i')$ such that $(W_j, R_j) \models^{k'_j} \rho$ for each $j \in J$. Because $\mu_{J'}(l') = \mu_J(k') = i'$ there exists $J'' \subseteq J \cap J'$ in F such that $l'_{J''} = k'_{J''}$ denoted by k''. Let $k = l_{J''}$. Note that $k \in \mu_{J''}^{-1}(i)$. We have that $(W_j, R_j) \models^{k''_j = k'_j} \rho$ for each $j \in J''$ and since $\langle k, k'' \rangle \in R_{J''}$ we have that $(W_j, R_j) \models^{k_j} \Diamond\rho$ for each $j \in J''$.

5. Consider $(\exists\chi)\rho$ for signature morphism $\chi : \Sigma \to \Sigma'$ and a Σ'-sentence ρ modally preserved by $\mathcal{F}$-filtered products (of Kripke models). For an arbitrary fixed $i \in I_{W_F}$, we assume there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} (\exists\chi)\rho$ for each $j \in J$. We have to prove that $(W_F, R_F) \models^i (\exists\chi)\rho$.

For each $j \in J$ there exists a χ-expansion (W'_j, R_j) of (W_j, R_j) such that $(W'_j, R_j) \models^{k_j} \rho$. Because $F|_J \in \mathcal{F}$ and because ρ is preserved by $\mathcal{F}$-filtered products (of Kripke models), we have that $(W'_{F|_J}, R_{F|_J}) \models^i \rho$ where $(W'_{F|_J}, R_{F|_J})$ is the filtered product of $\{(W'_j, R_j) \mid j \in J\}$ modulo $F|_J$. Because χ preserves $\mathcal{F}$-filtered products of models in the base institution, it also preserves $\mathcal{F}$-filtered products of Kripke models, hence $(W'_{F|_J}, R_{F|_J})$ is a χ-expansion of $(W_{F|_J}, R_{F|_J})$. Therefore $(W_{F|_J}, R_{F|_J}) \models^i (\exists\chi)\rho$ and since by Prop. 6.3 $(W_{F|_J}, R_{F|_J}) \cong (W_F, R_F)$ we have that $(W_F, R_F) \models^i (\exists\chi)\rho$.

6. Consider $(\exists\chi)\rho$ for signature morphism $\chi : \Sigma \to \Sigma'$ and a Σ'-sentence ρ modally preserved by $\mathcal{F}$-filtered factors. Assume $(W_F, R_F) \models^i (\exists\chi)\rho$ for some $i \in I_{W_F}$. Then there exists a χ-expansion (W', R_F) of (W_F, R_F) such that $(W', R_F) \models^i \rho$. Because χ lifts filtered products of Kripke models, there exists $J \in F$ such that for each $j \in J$ there exists a χ-expansion (W'_j, R_j) of (W_j, R_j) such that (W', R_F) is the filtered product $\prod_{F|_J}(W'_j, R_j)$.

By hypothesis ρ is modally preserved by $\mathcal{F}$-filtered factors, hence there exists $J' \in F|_J$ and $k \in \mu_{J'}^{-1}(i)$ such that $(W'_j, R_j) \models^{k_j} \rho$ for each $j \in J'$. But this implies that $(W_j, R_j) \models^{k_j} (\exists\chi)\rho$ for each $j \in J'$.

7. The preservation by filtered products is immediate. Therefore we focus on the preservation by filtered factors.

Assume that $(W_F, R_F) \models^i \rho_1 \wedge \rho_2$. Then for each $l \in \{1,2\}$, there exists $J^l \in F$ and $k^l \in \mu_{J^l}^{-1}(i)$ such that $(W_j, R_j) \models^{k_j^l} \rho_l$ for each $j \in J^l$. Because $\mu_{J^1}(k^1) = \mu_{J^2}(k^2)$ there exists $J \subseteq J^1 \cap J^2$ in F such that $k_J^1 = k_J^2$; let us denote this by k. Note that $\mu_J(k) = i$. Then for each $j \in J$ we have that $(W_j, R_j) \models^{k_j} \rho_1 \wedge \rho_2$.

8. Immediate.

9. Let ρ be a sentence which is modally preserved by $\mathcal{F}$-filtered factors. For some $i \in I_{W_F}$ assume there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that for each $j \in J$ we have that $(W_j, R_j) \models^{k_j} \neg\rho$. We have to prove that $(W_F, R_F) \models^i \neg\rho$.

If we assume the contrary, it means that $(W_F, R_F) \models^i \rho$. Since ρ is modally preserved by $\mathcal{F}$-filtered factors, there exists $J' \in F$ and $k' \in \mu_{J'}^{-1}(i)$ such that for each $j \in J'$ we have that $(W_j, R_j) \models^{k'_j} \rho$. Because $\mu_J(k) = \mu_{J'}(k')$ we can find a non-empty $J'' \subseteq J \cap J'$ in F such that $k_{J''} = k'_{J''}$. Let us denote this by k''. For each $j \in J''$ we then have that $(W_j, R_j) \models^{k''_j} \neg\rho$ and $(W_j, R_j) \models^{k''_j} \rho$ which is a contradiction. This shows that $(W_F, R_F) \models^i \neg\rho$.

10. Let ρ be any sentence which is modally preserved by $\mathcal{F}$-filtered products and assume $(W_F, R_F) \models^i \neg\rho$. For any fixed $i \in I_{W_F}$ take an arbitrary $k \in \mu_I^{-1}(i)$. If $\{j \in I \mid (W_j, R_j) \models^{k_j} \neg\rho\} \notin F$ then its complement $\{j \in I \mid (W_j, R_j) \models^{k_j} \rho\}$ belongs to F (because F is an ultrafilter). Because ρ is preserved by ultraproducts, this would imply $(W_F, R_F) \models^i \rho$ which contradicts $(W_F, R_F) \models^i \neg\rho$, therefore $\{j \in I \mid (W_j, R_j) \models^{k_j} \neg\rho\} \in F$.

11. From 9 and 10. □

Corollary 11.13. *Each modal sentence which is accessible from the Łoś-sentences of the base institution by (modal) Boolean connectives, possibility $\Diamond$ and (modal) χ-quantifications for which χ preserves filtered products of models (in the base institution), and lifts filtered products of Kripke models*

- *is modally preserved by ultraproducts and ultrafactors, and*
- *is preserved by ultraproducts.*

Proof. In Thm. 11.12 we consider $\mathcal{F}$ to be the class of all ultrafilters. The first item follows immediately from the conclusions of Thm. 11.12.

The second item follows from the first one. To see this let us consider an ultrafilter U over a set I and let (W_U, R_U) be an ultraproduct of Kripke models $\prod_U (W_j, R_j)$ for a family $\{(W_j, R_j)\}_{j \in I}$ of Kripke models. Assume that $\{j \mid (W_j, R_j) \models \rho\} \in U$ and that $(W_U, R_U) \not\models \rho$. Then there exists $i \in I_{W_F}$ such that $(W_U, R_U) \not\models^i \rho$ which means $(W_U, R_U) \models^i \neg\rho$. Because ρ is preserved by ultrafactors, there exists $J \in U$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \neg\rho$ for each $j \in J$. Note that $J \cap \{j \mid (W_j, R_j) \models \rho\} \in U$. Then for any of its elements j we have both that $(W_j, R_j) \models^{k_j} \neg\rho$ and that $(W_j, R_j) \models^{k_j} \rho$ which is a contradiction. Hence $(W_U, R_U) \models \rho$. □

The (ordinary) preservation by ultrafactors cannot be established for the possible worlds semantics mainly because modal negation is not a semantic negation (in the sense of Sect. 5.1). This can be easily seen if one tries to replicate the argument for the preservation of sentences by ultraproducts of Cor. 11.13 above to the preservation by ultrafactors. However, preservation by ultraproducts is still sufficient to derive a series of important results, most notably model compactness.

Similarly to the correspondent result for ‘single world semantics’ (Cor. 6.10), the only conditions of Cor. 11.13 that in reality narrow the set of sentences which are preserved by ultraproducts refer to the quantifiers. Except lifting of filtered products of Kripke models, the other conditions refer to the level of the base institution to which the analysis provided in Sect. 6.2 applies. The key condition to be established remains the lifting of filtered products of Kripke models. For this we need the following concept.

$(\Phi^\Delta, \beta^\Delta)$-exact signature morphisms. A signature morphism $\chi : \Sigma \to \Sigma'$ is $(\Phi^\Delta, \beta^\Delta)$-*exact* when the square of the naturality of β^Δ for χ (square shown below) is a pullback.

$$\begin{array}{c}\Sigma \\ \downarrow \chi \\ \Sigma'\end{array} \qquad \begin{tikzcd} \mathsf{Mod}^\Delta(\Phi^\Delta(\Sigma)) & \mathsf{Mod}(\Sigma) \arrow[l, "\beta^\Delta_\Sigma"'] \\ \mathsf{Mod}^\Delta(\Phi^\Delta(\Sigma')) \arrow[u, "\mathsf{Mod}(\Phi^\Delta(\chi))"] & \mathsf{Mod}(\Sigma') \arrow[l, "\beta^\Delta_{\Sigma'}"] \arrow[u, "\mathsf{Mod}(\chi)"'] \end{tikzcd}$$

Lifting of filtered products of Kripke models. The result below reduces the lifting of filtered products of Kripke models to lifting of filtered products of models in the base institution.

Proposition 11.14. *A signature morphism χ lifts filtered products of Kripke models if it is $(\Phi^\Delta, \beta^\Delta)$-exact and lifts completely and preserves filtered products of models (in the base institution).*

Proof. Let (W_F, R_F) be the filtered product of a family of Σ-Kripke models $\{(W_j, R_j)\}_{j \in I}$ modulo a filter F over the set I and let (W', R_F) be a χ-expansion of (W_F, R_F).

Let $i \in I_{W_F}$ and $k \in \mu_I^{-1}(i)$. By Lemma 11.11, W_F^i is the filtered product modulo F of the family $\{W_j^{k_j} \mid j \in I\}$.

Because χ lifts completely filtered products of models (in the base institution), for each $j \in I$ let ${W'}_j^{k_j}$ be a χ-expansion of $W_j^{k_j}$ such that W'^i is the filtered product of $\{{W'}_j^{k_j} \mid j \in I\}$.

Because χ is $(\Phi^\Delta, \beta^\Delta)$-exact, for each $j \in I$ and each $l \in I_{W_j}$ let ${W'}_j^l$ be the unique Σ'-model such that $\beta^\Delta_{\Sigma'}({W'}_j^l) = \beta^\Delta_{\Sigma'}({W'}_j^{k_j})$ and ${W'}_j^l\restriction_\chi = W_j^l$.

Now we prove that (W', R_F) is the filtered product of $\{(W'_j, R_j)\}_{j \in J}$ modulo F. Consider an arbitrary $k' \in I_{W_I} = \prod_{j \in I} I_{W_J}$ and let $i' = \mu_I(k')$. By Lemma 11.11, it is enough to show that $W'^{i'}$ is the filtered product of $\{{W'}_j^{k'_j} \mid j \in I\}$ modulo F. This follows by the $(\Phi^\Delta, \beta^\Delta)$-exactness property of χ because

$$
\begin{aligned}
\beta^\Delta_{\Sigma'}(\textstyle\prod_F {W'}_j^{k'_j}) = {} & \textstyle\prod_F \beta^\Delta_{\Sigma'}({W'}_j^{k'_j}) && \text{(by } (FP)\text{)} \\
= {} & \textstyle\prod_F \beta^\Delta_{\Sigma'}({W'}_j^{k_j}) && \text{(by the sharing condition)} \\
= {} & \beta^\Delta_{\Sigma'}(\textstyle\prod_F {W'}_j^{k_j}) && \text{(by } (FP) \text{ and } (LI)\text{)} \\
= {} & \beta^\Delta_{\Sigma'}(W'^i) && \\
= {} & \beta^\Delta_{\Sigma'}(W'^{i'}) && \text{(by the sharing condition)}
\end{aligned}
$$

and because

$$
\begin{aligned}
(\textstyle\prod_F {W'}_j^{k'_j})\restriction_\chi = {} & \textstyle\prod_F ({W'}_j^{k'_j}\restriction_\chi) && \text{(because } \chi \text{ preserves filtered products)} \\
= {} & \textstyle\prod_F W_j^{k'_j} && \text{(by the definition of } W_j^{k'_j}\text{)} \\
= {} & \textstyle\prod_F W_j^{k'_j} && \text{(by Lemma 11.11)} \\
= {} & W'^{i'}\restriction_\chi && \text{(by the hypothesis that } (W_F, R_F)\restriction_\chi = (W', R_F)\text{).}
\end{aligned}
$$

□

Propositions 6.4, 6.6 and 6.8 allow a refinement of Prop. 11.14 to the following sufficient condition for the lifting of filtered products of Kripke models which is rather easily applicable to actual situations.

Corollary 11.15. *Assume that in the base institution all projections of model products are epis. Then a signature morphism lifts filtered products of Kripke models if it is* $(\Phi^\Delta, \beta^\Delta)$*-exact and projectively representable (in the base institution).*

As a side remark, note that possible worlds semantics via Cor. 11.15 provides a remarkable example of signature morphisms which are not representable but lift filtered products of models (Kripke models in this case). Cor. 11.15 permits the following reformulation of Cor. 11.13.

Corollary 11.16. *Assume that in the base institution all projections of model products are epis. Then the modal sentences preserved by ultraproducts*

- *contain all Łoś sentences of the base institution,*
- *are closed under (modal) Boolean connectives,*
- *are closed under modalities* $\Box$ *and* $\Diamond$*, and*

– *are closed under any quantification which is* $(\Phi^{\Delta}, \beta^{\Delta})$*-exact and projectively representable (in the base institution).*

An instance of Cor. 11.16 is given by modal first order logic **MFOL** considered as a modal institution (determined by the forgetful institution morphism $\mathbf{AFOL}' \to \mathbf{AFOL}$). For establishing this instance we have mainly to note that the quantifications, as injective signature morphism adding only (rigid) constants as new symbols, are projectively representable and $(\Phi^{\Delta}, \beta^{\Delta})$-exact.

Corollary 11.17. *Each sentence of* **MFOL** *is preserved by ultraproducts.*

Compactness

An immediate application of the preservation of sentences by ultraproducts is model compactness (cf. Cor. 6.19).

Corollary 11.18. *If each sentence of a 'modal' institution is accessible by the operations listed in Cor. 11.16, then the institution is m-compact.*

And below we can formulate an expected instance of this result.

Corollary 11.19. **MFOL** *is m-compact.*

Note that compactness of **MFOL** cannot be established from the model compactness by the general result given by Prop. 6.16 because **MFOL** has only modal negation, which is not a semantic negation as required by Prop. 6.16.

Exercises

11.6. Let us consider the following table of institution morphisms:

	base inst.	Φ^{Δ}	$\mathsf{Mod}^{\Delta}(\Sigma)$	*sharing constraint*
1.	**POA**	forgets non-constant operation symbols	$\mathsf{Mod}^{\mathbf{FOL}}(S,C,\emptyset)$	underlying carrier sets and interpretations of constants
2.	**POA**	identity	$\mathsf{Mod}^{\mathbf{FOL}}(S,F,\emptyset)$	underlying algebras
3.	**PA**	forgets non-constant total operation and all partial operation symbols	$\mathsf{Mod}^{\mathbf{FOL}}(S,C,\emptyset)$	underlying carrier sets and interpretations of total constants

The following classes of signature morphisms are $(\Phi^{\Delta}, \beta^{\Delta})$-exact for the corresponding entries in the above table.

Entry in table	signature morphism χ
1.	extensions with constants
2.	extensions with constants
3.	extensions with sorts and total constants

Notes. The material of this chapter is based upon [60]. Possible worlds semantics of modal logic [107] is one of the major developments in the area of non-classical logics. Most of the work in modal logic is focused on modal propositional logic **MPL**. For a historical overview we suggest [83], while [25] gives a rather complete state-of-the-art presentation of modal propositional logic .

Apart from its great influence in philosophy, logic, and linguistics, possible worlds semantics have been repeatedly applied to computing, in particular to the dynamic logic of programs [151, 89, 106], process algebra [93, 17] and the temporal logic's approach to concurrency [149, 65, 167].

Our current treatment of quantifiers for possible worlds semantics does support quantifications with rigid constants only over rigid sorts (called 'constant domains' by modal logicians). However quantifications with rigid constants over flexible sorts constitutes an important object of study in conventional modal logic. Since it is not difficult to extend our approach to capture this situation, we propose this as a research project for the interested reader.

Chapter 12

Grothendieck Institutions

Grothendieck institutions generalize the flattening Grothendieck construction from (indexed) categories, (see Sect. 2.5), to (indexed) institutions. Regarded from a fibration theoretic angle, Grothendieck institutions are just institutions for which their category of signatures is fibred. For example, the actual institutions with many-sorted signatures appear naturally as fibred institutions determined by the fibrations given by the functor mapping each signature to its set of sort symbols. In this sense, fibred institutions can be regarded as the reflection of many-sortedness at the level of abstract institutions.

For modeling heterogeneous multi-logic environments the flattening Grothendieck construction on a system of institutions related by institution morphisms (here called *indexed institution*) seems to be more adequate than the fibred institutions approach. A Grothendieck institution puts together a system of institutions into a single institution such that the individual identities of the component institutions and the relationships between them are fully retained.

Summary of the chapter. In this chapter we introduce the concepts of fibred and Grothendieck institutions and we show that they are equivalent.

The Grothendieck construction on institutions can be done in two variants, by institution morphisms or by institution comorphisms. We show that in the case when the institution morphisms or comorphisms correspond to adjunction situations between the categories of signatures of the institutions, the morphism-based and comorphism-based Grothendieck institutions are isomorphic.

An important class of problems posed by the Grothendieck, or fibred, institutions is that of lifting of model-theoretic properties from the 'local' level of index institutions, or fibres, to the 'global' level of the Grothendieck, or fibred, institution. We investigate the lifting of several important properties, such as theory co-limits, model amalgamation, and interpolation. An interesting application of the interpolation theorem for Grothendieck institutions is given by Craig-Robinson properties of institutions without implications, such as **EQL** and **HCL**.

The material of this chapter requires some familiarity with indexed categories and with fibrations. A concise introduction to these topics, presenting all concepts used by this chapter, can be found in Sect. 2.5.

12.1 Fibred and Grothendieck Institutions

Fibred institutions

FOL as fibred institution. For any set S, let the institution of *S-sorted first order logic* $\mathbf{FOL}^S = (\mathbb{S}ig^S, \mathsf{Sen}^S, \mathsf{Mod}^S, \models)$ be the sub-institution of **FOL** determined by fixing the set of sort symbols to S. The category of signatures $\mathbb{S}ig^S$ consists of all pairs (F,P) where F is an S-sorted set of operation symbols and P is an S-sorted set of relation symbols, morphisms of signatures in $\mathbb{S}ig^S$ being just morphism of signatures φ in first order logic which are identities on the sets S of sort symbols, i.e., $\varphi^{\mathrm{st}} = 1_S$. Then the (F,P)-sentences, respectively models, in $\mathbf{FOL}^S$ are the (S,F,P)-sentences, respectively models, in **FOL**. The satisfaction relation between models and sentences is of course inherited from **FOL**.

Fact 12.1. *Any function $u\colon\ S \to S'$ determines an institution morphism $(\Phi^u, \alpha^u, \beta^u)\ :\ \mathbf{FOL}^{S'} \to \mathbf{FOL}^S$ such that for each S'-sorted signature (F',P')*

- *$\Phi^u(F',P') = (F,P)$ with $F_{w\to s} = F'_{u(w)\to u(s)}$ and $P_w = P'_{u(w)}$ for each string of sort symbols $w \in S^*$ and each sort symbol $s \in S$. The canonical* **FOL** *signature morphism $(S,F,P) \to (S',F',P')$ thus determined is denoted by $\varphi^u_{(F',P')}$.*
- *$\alpha^u_{(F',P')}\colon\ \mathsf{Sen}^S(F,P) \to \mathsf{Sen}^{S'}(F',P')$ is defined as $\mathsf{Sen}^{\mathbf{FOL}}(\varphi^u_{(F',P')})$ and, informally, maps each (F,P)-sentence to itself but regarded as an (F',P')-sentence, and*
- *$\beta^u_{(F',P')}\colon\ \mathsf{Mod}^{S'}(F',P') \to \mathsf{Mod}^S(F,P)$ is defined as $\mathsf{Mod}^{\mathbf{FOL}}(\varphi^u_{(F',P')})$.*

This situation, common to all 'many-sorted' logics formalized as institutions, follows from the fact that $\mathbb{S}ig^{\mathbf{FOL}}$ is fibred over $\mathbb{S}et$ by the projection Π of each signature to its set of sorts (defined by $\Pi(S,F,P) = S$ on signatures and $\Pi(\phi) = \phi^{\mathrm{st}}$ on signature morphisms).

Fact 12.2. *The fibration $\Pi\colon\ \mathbb{S}ig^{\mathbf{FOL}} \to \mathbb{S}et$ is split. Moreover, a* **FOL** *signature morphism ϕ is cartesian when ϕ^{op} and ϕ^{rl} are bijections, and $\varphi^u_{(F',P')}$ is the distinguished cartesian lifting of u for each function $u\colon\ S \to S'$ and each* **FOL***-signature (S',F',P').*

Fibred institutions. By abstracting the forgetful functor $\Pi\colon\ \mathbb{S}ig^{\mathbf{FOL}} \to \mathbb{S}et$ above to any fibration, we can formulate the general concept of 'fibred institution' as follows.

Given a category I, a *fibred institution over the base I* is a tuple $(\Pi\colon\ \mathbb{S}ig \to I, \mathsf{Mod}, \mathsf{Sen}, \models)$ such that

- $\Pi\colon\ \mathbb{S}ig \to I$ is a fibred category, and
- $(\mathbb{S}ig, \mathsf{Mod}, \mathsf{Sen}, \models)$ is an institution.

Standard concepts from fibred category theory lift immediately to institutions. The fibred institution is *split* when the fibration Π is split. A *cartesian institution morphism* is an institution morphism between fibred institutions for which the signature mapping functor is a cartesian functor between the corresponding fibred categories of signatures.

Given a fibred institution $I = (\Pi : \mathbb{S}ig \to I, \mathsf{Mod}, \mathsf{Sen}, \models)$, for each object $i \in |I|$, the *fibre of I at i* is the institution $I^i = (\mathbb{S}ig^i, \mathsf{Mod}^i, \mathsf{Sen}^i, \models^i)$ where

- $\mathbb{S}ig^i$ is the fibre of Π at i, and
- Mod^i, Sen^i, and $\models^i$ are the restrictions of Mod, Sen, and respectively $\models$ to $\mathbb{S}ig^i$.

By applying this terminology to the **FOL** case, we can therefore say that **FOL** is fibred over $\mathbb{S}et$ with its fibre at a set S being the institution $\mathbf{FOL}^S$ of S-sorted first order logic.

The following generalizes Fact 12.1 to any fibred institution.

Proposition 12.3. *Given a fibred institution $I = (\Pi : \mathbb{S}ig \to I, \mathsf{Mod}, \mathsf{Sen}, \models)$, for each arrow $u \in I(i, j)$, any inverse image functor $\Phi^u : \mathbb{S}ig^j \to \mathbb{S}ig^i$ (with distinguished cartesian morphisms $\varphi^u_{\Sigma'} : \Phi^u(\Sigma') \to \Sigma'$) determines a canonical institution morphism $(\Phi^u, \alpha^u, \beta^u) : I^j \to I^i$ between the fibres of I, where for each signature Σ' in the fibre $\mathbb{S}ig^j$ at j, $\alpha^u_{\Sigma'} = \mathsf{Sen}(\varphi^u_{\Sigma'})$ and $\beta^u_{\Sigma'} = \mathsf{Mod}(\varphi^u_{\Sigma'})$.*

Proof. The naturality of α^u and β^u follow directly from the way the family of distinguished cartesian morphisms $\{\varphi^u_{\Sigma'}\}_{\Sigma' \in \mathbb{S}ig^j}$ determine the functor Φ^u, and by applying the sentence functor and the model functor, respectively, to the corresponding commutative diagrams.

$$
\begin{tikzcd}
\Phi^u(\Sigma') \arrow[r, "\varphi^u_{\Sigma'}"] \arrow[d, "\Phi^u(\theta)"'] & \Sigma' \arrow[d, "\theta"] \\
\Phi^u(\Sigma'_1) \arrow[r, "\varphi^u_{\Sigma'_1}"'] & \Sigma'_1
\end{tikzcd}
\quad
\begin{tikzcd}
\mathsf{Sen}^i(\Phi^u(\Sigma')) \arrow[r, "\alpha^u_{\Sigma'}"] \arrow[d, "\mathsf{Sen}^i(\Phi^u(\theta))"'] & \mathsf{Sen}^j(\Sigma') \arrow[d, "\mathsf{Sen}^j(\theta)"'] \\
\mathsf{Sen}^i(\Phi^u(\Sigma'_1)) \arrow[r, "\alpha^u_{\Sigma'_1}"'] & \mathsf{Sen}^j(\Sigma'_1)
\end{tikzcd}
\quad
\begin{tikzcd}
\mathsf{Mod}^i(\Phi^u(\Sigma')) & \mathsf{Mod}^j(\Sigma') \arrow[l, "\beta^u_{\Sigma'}"'] \\
\mathsf{Mod}^i(\Phi^u(\Sigma'_1)) \arrow[u, "\mathsf{Mod}^i(\Phi^u(\theta))"] & \mathsf{Mod}^j(\Sigma'_1) \arrow[l, "\beta^u_{\Sigma'_1}"] \arrow[u, "\mathsf{Mod}^j(\theta)"]
\end{tikzcd}
$$

The satisfaction condition for the institution morphism $(\Phi^u, \alpha^u, \beta^u)$ follows from the satisfaction condition of the fibred institution I applied for the distinguished cartesian morphisms. Consider a Σ'-model M' and a $\Phi^u(\Sigma')$-sentence ρ. Then $M' \models^j_{\Sigma'} \alpha^u_{\Sigma'}(\rho)$ means $M' \models_{\Sigma'} \mathsf{Sen}(\varphi^u_{\Sigma'})(\rho)$ which by the satisfaction condition of the fibred institution means $\mathsf{Mod}(\varphi^u_{\Sigma'})(M') \models_{\Phi^u(\Sigma')} \rho$ which finally means that $\beta^u_{\Sigma'}(M') \models^i_{\Phi^u(\Sigma')} \rho$. $\square$

Indexed and Grothendieck institutions

'Indexed institutions' lift the concept of indexed category to institutions.

The indexed institution determined by FOL. The institution morphisms $(\Phi^u, \alpha^u, \beta^u)$ provide an example of an indexed institution.

Fact 12.4. *The mapping of each function* $u\colon\ S \to S'$ *to the institution morphism* $(\Phi^u, \alpha^u, \beta^u)\colon\ \mathbf{FOL}^{S'} \to \mathbf{FOL}^{S}$ *is functorial.*

Let the functor between the opposite of $\mathbb{S}et$ to the (quasi-)category $\mathbb{I}ns$ of institution morphisms determined by the mapping above be denoted by $\mathbf{fol}\colon\ \mathbb{S}et^{\mathrm{op}} \to \mathbb{I}ns$.

Indexed institutions. Given a category I of indices, an *indexed institution* $\mathcal{J}$ is a functor $\mathcal{J}\colon\ I^{\mathrm{op}} \to \mathbb{I}ns$. For each index $i \in |I|$ we denote the institution $\mathcal{J}^i$ by $(\mathbb{S}ig^i, \mathsf{Mod}^i, \mathsf{Sen}^i, \models^i)$ and for each index morphism $u \in I$ we denote the institution morphism $\mathcal{J}^u$ by $(\Phi^u, \alpha^u, \beta^u)$.

Grothendieck institutions. The *Grothendieck institution* $\mathcal{J}^\sharp = (\mathbb{S}ig^\sharp, \mathsf{Sen}^\sharp, \mathsf{Mod}^\sharp, \models^\sharp)$ of an indexed institution $\mathcal{J}\colon\ I^{\mathrm{op}} \to \mathbb{I}ns$ is defined as follows:

1. Let $\mathbb{S}ig\colon\ I^{\mathrm{op}} \to \mathbb{C}at$ be the indexed institution mapping each index i to $\mathbb{S}ig^i$ and each index morphism u to Φ^u; then the category of the signatures of $\mathcal{J}^\sharp$ is the Grothendieck category $\mathbb{S}ig^\sharp$. Thus the signatures of $\mathcal{J}^\sharp$ consist of pairs $\langle i, \Sigma\rangle$ with i index and $\Sigma \in |\mathbb{S}ig^i|$ and signature morphisms $\langle u, \varphi\rangle\colon\ \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ consists of index morphisms $u\colon\ i \to i'$ and signature morphisms $\varphi\colon\ \Sigma \to \Phi^u(\Sigma')$.
2. The model functor $\mathsf{Mod}^\sharp\colon\ (\mathbb{S}ig^\sharp)^{\mathrm{op}} \to \mathbb{C}at$ is given by
   - $\mathsf{Mod}^\sharp(\langle i, \Sigma\rangle) = \mathsf{Mod}^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ig^i|$, and
   - $\mathsf{Mod}^\sharp(\langle u, \varphi\rangle) = \beta^u_{\Sigma'}; \mathsf{Mod}^i(\varphi)$ for each $\langle u, \varphi\rangle\colon\ \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$.
3. The sentence functor $\mathsf{Sen}^\sharp\colon\ \mathbb{S}ig^\sharp \to \mathbb{S}et$ is given by
   - $\mathsf{Sen}^\sharp(\langle i, \Sigma\rangle) = \mathsf{Sen}^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ig^i|$, and
   - $\mathsf{Sen}^\sharp(\langle u, \varphi\rangle) = \mathsf{Sen}^i(\varphi); \alpha^u_{\Sigma'}$ for each $\langle u, \varphi\rangle\colon\ \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$.
4. The satisfaction relation is given by

   $$M \models^\sharp_{\langle i,\Sigma\rangle} e \ \text{ if and only if } \ M \models^i_\Sigma e$$

   for each index $i \in |I|$, signature $\Sigma \in |\mathbb{S}ig^i|$, model $M \in |\mathsf{Mod}^\sharp(\langle i, \Sigma\rangle)|$, and sentence $e \in \mathsf{Sen}^\sharp(\langle i, \Sigma\rangle)$.

The following shows that the above construction gives an institution indeed.

Proposition 12.5. $\mathcal{J}^\sharp$ *is an institution. Moreover, for each index* $i \in |I|$ *there exists a canonical institution morphism* $(\Phi^i, \alpha^i, \beta^i)\colon\ \mathcal{J}^i \to \mathcal{J}^\sharp$ *mapping any signature* $\Sigma \in |\mathbb{S}ig^i|$ *to* $\langle i, \Sigma\rangle \in |\mathbb{S}ig^\sharp|$ *and such that the components of* α^i *and* β^i *are identities.*

Proof. Consider a signature morphism $\langle u, \varphi\rangle\colon\ \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$, a $\langle i', \Sigma'\rangle$-model M' and a $\langle i, \Sigma\rangle$-sentence e. Then
$M' \models^\sharp_{\langle i',\Sigma'\rangle} \mathsf{Sen}^\sharp(\langle u, \varphi\rangle)(e)$

if and only if	$M' \models^{i'}_{\Sigma'} \alpha^u_{\Sigma'}(\mathsf{Sen}^i(\varphi)(e))$	(by the definitions of $\models^\sharp$ and of $\mathsf{Sen}^\sharp$)
if and only if	$\beta^u_{\Sigma'}(M') \models^i_\Sigma \mathsf{Sen}^i(\varphi)(e)$	(by the satisfaction condition for $(\Phi^u, \alpha^u, \beta^u)$)
if and only if	$\mathsf{Mod}^i(\varphi)(\beta^u_{\Sigma'}(M')) \models^i_{\Phi^u(\Sigma')} e$	(by the satisfaction condition for φ)
if and only if	$\mathsf{Mod}^\sharp(\langle u, \varphi\rangle)(M') \models^\sharp_{\langle i', \Sigma'\rangle} e$	(by the definitions of $\models^\sharp$ and of $\mathsf{Mod}^\sharp$).

$\square$

Fact 12.6. *The Grothendieck institution $\mathcal{J}^\sharp$ of an indexed institution $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ is a split fibred institution $(\Pi : \mathbb{S}ig^\sharp \to I, \mathsf{Mod}^\sharp, \mathsf{Sen}^\sharp, \models^\sharp)$, where $\Pi : \mathbb{S}ig^\sharp \to I$ is the fibration projection from the Grothendieck category $\mathbb{S}ig^\sharp$ to its index category.*

On the other hand, cf. Prop. 12.3, each split fibred institution determines an indexed institution and consequently a Grothendieck institution. It is easy to see that the mappings from Grothendieck institutions to split fibred institutions and opposite are inverse to each other. For example **FOL** can be recovered as the Grothendieck institution **fol**$^\sharp$.

Fact 12.7. *For any category I, there exists a natural isomorphism between the category of split fibred institutions over I (with cartesian institution morphisms as arrows) and the category of I-indexed institutions (with natural transformation between the indexing functors as arrows).*

Recall from Sect. 3.3 that an institution morphism (Φ, α, β) is an *equivalence of institutions* when

- Φ is an equivalences of categories,
- α_Σ has an inverse up to semantic equivalence α'_Σ, which is natural in Σ, and
- β_Σ is an equivalence of categories, such that its inverse up to isomorphism and the corresponding isomorphism natural transformations are natural in Σ.

Because each fibred institution is equivalent to a split fibred institution, we have the following corollary.

Corollary 12.8. *Each fibred institution is equivalent to a Grothendieck institution.*

Comorphism-based Grothendieck institutions

Grothendieck institutions can be constructed using comorphisms instead of morphisms. As we will see below, comorphism-based Grothendieck institutions may be more friendly towards some model theoretic properties than the morphism-based ones.

Given a category I of indices, an *indexed comorphism-based institution* , in short called *indexed co-institution*, $\mathcal{J}$ is a functor $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$. (Recall that $co\mathbb{I}ns$ is the quasi-category having institutions as objects and institution comorphisms as arrows.) Its Grothendieck institution $\mathcal{J}^\sharp$ is defined as follows:

1. its category of signatures is $((\mathbb{S}ig; (\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ where $\mathbb{S}ig : I^{\mathrm{op}} \to \mathbb{C}at$ is the *indexed* category of signatures of the indexed co-institution $\mathcal{J}$, $(\_)^{\mathrm{op}} : \mathbb{C}at \to \mathbb{C}at$ is the 'opposite' functor, and $(\mathbb{S}ig; (\_)^{\mathrm{op}})^\sharp$ is its Grothendieck category; this means that

- signatures are pairs $\langle i, \Sigma\rangle$ for $i \in |I|$ index and $\Sigma \in |\mathbb{S}ig^i|$, and
- signature morphisms are pairs $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ where $u \in I(i', i)$ and $\varphi \in \mathbb{S}ig^{i'}(\Phi^u(\Sigma), \Sigma')$,

2. its model functor $\mathsf{Mod}^\sharp : (\mathbb{S}ig; (\_)^{\mathrm{op}})^\sharp \to \mathbb{C}at$ is given by
   - $\mathsf{Mod}^\sharp(\langle i, \Sigma\rangle) = \mathsf{Mod}^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ig^i|$, and
   - $\mathsf{Mod}^\sharp(\langle u, \varphi\rangle) = \mathsf{Mod}^{i'}(\varphi); \beta^u_\Sigma$ for each $\langle u, \varphi\rangle : \langle i', \Sigma'\rangle \to \langle i, \Sigma\rangle$,
3. its sentence functor $\mathsf{Sen}^\sharp : ((\mathbb{S}ig; (\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}} \to \mathbb{S}et$ is given by
   - $\mathsf{Sen}^\sharp(\langle i, \Sigma\rangle) = \mathsf{Sen}^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ig^i|$, and
   - $\mathsf{Sen}^\sharp(\langle u, \varphi\rangle) = \alpha^u_\Sigma; \mathsf{Sen}^{i'}(\varphi)$ for each $\langle u, \varphi\rangle : \langle i', \Sigma'\rangle \to \langle i, \Sigma\rangle$,
4. $M \models^\sharp_{\langle i,\Sigma\rangle} e$ if and only if $M \models^i_\Sigma e$ for each index $i \in |I|$, signature $\Sigma \in |\mathbb{S}ig^i|$, model $M \in |\mathsf{Mod}^\sharp(\langle i, \Sigma\rangle)|$, and sentence $e \in \mathsf{Sen}^\sharp(\langle i, \Sigma\rangle)$.

where $\mathcal{J}^i = (\mathbb{S}ig^i, \mathsf{Mod}^i, \mathsf{Sen}^i, \models^i)$ for each index $i \in |I|$ and $\mathcal{J}^u = (\Phi^u, \alpha^u, \beta^u)$ for $u \in I$ index morphism.

Routine calculations similar to those of Prop. 12.5 show that:

Proposition 12.9. *The comorphism-based Grothendieck institution $\mathcal{J}^\sharp$ is indeed an institution, i.e., the satisfaction condition holds.*

Adjoint-indexed institutions. These are indexed institutions $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ for which all institution morphisms $\mathcal{J}^u$ are adjoint morphisms for all index morphisms $u \in I$. An adjoint-indexed institution $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ is *coherent* when for each composable pair of index morphisms $u : i \to i'$ and $u' : i' \to i''$ the adjunction from $\mathbb{S}ig^i$ to $\mathbb{S}ig^{i''}$ corresponding to $u; u'$ is the composition of the adjunctions corresponding to u, respectively u'.

For example the $\mathbb{S}et$-indexed institution **fol** determined by the fibred institution **FOL** is coherent adjoint-indexed. For each function $u : S \to S'$, let $\overline{\Phi^u} : \mathbb{S}ig^S \to \mathbb{S}ig^{S'}$ map each S-sorted signature (F, P) to the S'-sorted signature (F^u, P^u) defined by $F^u_{w' \to s'} = \uplus_{u(ws)=w's'} F_{w \to s}$ and $P^u_{w'} = \uplus_{u(w)=w'} P_w$ for each string of sort symbols $w \in S^*$ and sort symbol $s \in S$.

Fact 12.10. *$\overline{\Phi^u}$ is a left adjoint to the 'forgetful' functor $\Phi^u : \mathbb{S}ig^{S'} \to \mathbb{S}ig^S$.*

Adjoint-indexed co-institutions are defined similarly to adjoint-indexed institutions. Notice that each adjoint-indexed institution $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ determines an adjoint-indexed coinstitution $\overline{\mathcal{J}} : (I^{\mathrm{op}})^{\mathrm{op}} \to co\mathbb{I}ns$ such that

- for each index $i \in I$, $\overline{\mathcal{J}}^i = \mathcal{J}^i$, and
- for each index morphism u, $\overline{\mathcal{J}}^u$ is the comorphism adjoint to the morphism $\mathcal{J}^u$ (as given by Thm. 3.6).

Therefore the duality relation between institution morphisms and comorphisms determines a similar duality relation between adjoint-indexed institutions and adjoint-indexed coinstitutions. For example, in the case of **FOL**, the $\mathbb{S}et$-indexed institution **fol** determines a $\mathbb{S}et^{\mathrm{op}}$-indexed coinstitution $\overline{\textbf{fol}}$.

The concept of a Grothendieck institution is invariant with respect to the duality between the concepts of institution morphism and institution comorphism:

Proposition 12.11. *For each dual pair of an adjoint-indexed institution $\mathcal{J}$ and an adjoint-indexed coinstitution $\overline{\mathcal{J}}$ their Grothendieck institutions $\mathcal{J}^\sharp$ and $\overline{\mathcal{J}}^\sharp$ are isomorphic.*

Proof. The isomorphism $\mathbb{S}ig^{\mathcal{J}^\sharp} \cong \mathbb{S}ig^{\overline{\mathcal{J}}^\sharp}$ maps each $\mathbb{S}ig^{\mathcal{J}^\sharp}$-signature morphism $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ to the $\mathbb{S}ig^{\overline{\mathcal{J}}^\sharp}$-signature morphism $\langle u, \overline{\varphi}\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ where $\varphi : \Sigma \to \Phi^u(\Sigma')$, $\overline{\varphi} : \overline{\Phi}^u(\Sigma) \to \Sigma'$ are such that $\varphi = \zeta_\Sigma ; \Phi^u(\overline{\varphi})$

$$\begin{tikzcd}
\Sigma \arrow[r, "\zeta"] \arrow[d, "\varphi"'] & \Phi^u(\overline{\Phi}^u(\Sigma)) \arrow[dl, "\Phi^u(\overline{\varphi})"] \\
\Phi^u(\Sigma') &
\end{tikzcd}$$

with ζ being the unit of the adjunction between $\mathbb{S}ig^i$ and $\mathbb{S}ig^{i'}$.

The conclusion of the proposition follows by the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{S}et & \mathbb{S}ig^{\mathcal{J}^\sharp} \arrow[l, "\mathsf{Sen}^{\mathcal{J}^\sharp}"'] \arrow[r, "\mathsf{Mod}^{\mathcal{J}^\sharp}"] \arrow[d, "\cong"] & \mathbb{C}at^{\mathrm{op}} \\
& \mathbb{S}ig^{\overline{\mathcal{J}}^\sharp} \arrow[ul, "\mathsf{Sen}^{\overline{\mathcal{J}}^\sharp}"] \arrow[ur, "\mathsf{Mod}^{\overline{\mathcal{J}}^\sharp}"'] &
\end{tikzcd}$$

which is obtained by routine calculations. □

Therefore **FOL** can be also obtained as a Grothendieck institution in two different ways: as the comorphism-based Grothendieck institution $\mathbf{fol}^\sharp$, and as the morphism-based Grothendieck institution $\overline{\mathbf{fol}}^\sharp$. In this case, the morphism-based Grothendieck construction seems rather simpler and more natural than the comorphism-based one.

Exercises

12.1. (a) The satisfaction condition of institution morphisms is a special case of the satisfaction condition of institutions. (*Hint:* For any institution morphism (Φ, α, β) consider the Grothendieck institution determined by the indexed institution $(\bullet \xrightarrow{u} \bullet) \to \mathbb{I}ns$ which maps u to (Φ, α, β).)
(b) The opposite also holds, the satisfaction condition of institutions is a special case of the satisfaction condition of institution morphisms. (*Hint:* Each institution is a trivially split fibred institution over its own category of signatures.)

12.2. (a) Let K be any 2-category and I_K be the Grothendieck 2-category for the 2-functor $\mathbb{C}at((-)^{\mathrm{op}}, K) : \mathbb{C}at^* \to \mathbb{C}at$ mapping each category S to $\mathbb{C}at(S^{\mathrm{op}}, K)$, and each functor Φ to $(\Phi^{\mathrm{op}}; -)$ (which maps each $I : S^{\mathrm{op}} \to K$ to $\Phi^{\mathrm{op}}; I$). Then the fibration $\Pi_K : I_K \to \mathbb{C}at$ creates Grothendieck constructions for each functor $\mathcal{J} : I^{\mathrm{op}} \to I_K$.
(b) Conclude that the 2-category of institutions $\mathbb{I}ns$ admits Grothendieck constructions with the Grothendieck institutions as the Grothendieck objects of $\mathbb{I}ns$. (*Hint:* $\mathbb{I}ns = I_{\mathbb{R}oom}$.)

12.3. The comorphism-based Grothendieck institutions are Grothendieck objects in the 2-category of $co\mathbb{I}ns$ of institution comorphisms.

12.2 Theory Co-limits and Model Amalgamation

In this section we study the lifting of a couple of important model theoretic properties from the 'local' level of the indexed institutions to the 'global' level of the corresponding Grothendieck institution.

Theory co-limits

Supporting co-limits. For any category J we say that an indexed co-institution $\mathcal{J}: I^{\mathrm{op}} \to co\mathbb{I}ns$ *supports* J*-co-limits* when

- the index category I is J-complete, i.e., has J-limits,
- the indexed category of signatures $\mathbb{S}ig: I^{\mathrm{op}} \to \mathbb{C}at$ of $\mathcal{J}$ is *locally* J*-co-complete*, i.e., $\mathbb{S}ig^i$ has all J-co-limits for each index $i \in |J|$, and
- for each index morphism u, the comorphism $\mathcal{J}^u$ preserves J-co-limits of signatures (meaning that the corresponding sentence translation functors Φ^u preserve pushouts).

Theorem 12.12. *The category of theories* $\mathbb{T}h^{\mathcal{J}^\sharp}$ *of a comorphism-based Grothendieck institution* $\mathcal{J}^\sharp$ *has* J*-co-limits if the indexed co-institution* $\mathcal{J}$ *supports* J*-co-limits.*

Proof. By the fundamental result that in any institution the forgetful functor from theories to signatures lifts co-limits (Prop. 4.2), we have only to show that the category of signatures of the Grothendieck institution $\mathcal{J}^\sharp$ has J-co-limits. But the category of signatures of $\mathcal{J}^\sharp$ is the opposite of the Grothendieck category $(\mathbb{S}ig;(-)^{\mathrm{op}})^\sharp$. The conclusion of the theorem now follows immediately from the general result on existence of limits in Grothendieck categories (Thm. 2.10). In the following we review the construction of co-limits in $\mathbb{S}ig^{\mathcal{J}^\sharp}$, the category of signatures of the comorphism-based Grothendieck institution $\mathcal{J}^\sharp$.

Let J be a small category and $F: J \to \mathbb{S}ig^{\mathcal{J}^\sharp}$ any functor. Let $K = F;\Pi$ where $\Pi: \mathbb{S}ig^{\mathcal{J}^\sharp} = ((\mathbb{S}ig;(-)^{\mathrm{op}})^\sharp)^{\mathrm{op}} \to I^{\mathrm{op}}$ is the projection mapping each $\langle i, \Sigma\rangle$ to i. Let us write $F(j) = \langle K_j, \Sigma_j\rangle$ for each index $j \in |J|$ and $F(u) = \langle K_u, \varphi_u\rangle$ for each index morphism $u \in J$.

Any co-cone $\nu: K \Rightarrow i$ determines a functor $F^\nu: J \to \mathbb{S}ig^i$ defined by $F^\nu(j) = \Phi^{\nu_j}(\Sigma_j)$ for each index $j \in |J|$ and by $F^\nu(u) = \Phi^{\nu_{j'}}(\varphi_u)$ for each index morphism $u \in J(j, j')$.

For each $u \in J(j, j')$ and $u' \in J(j', j'')$, from $F(u;u') = F(u);F(u')$ we have that

$$
\begin{tikzcd}
\Phi^{\nu_j}(\Sigma_j) \arrow[rr, "\Phi^{\nu_{j''}}(\varphi_{u;u'})"] \arrow[dr, "\Phi^{\nu_{j'}}(\varphi_u)"'] & & \Phi^{\nu_{j''}}(\Sigma_{j''}) \\
& \Phi^{\nu_{j'}}(\Sigma_{j'}) \arrow[ur, "\Phi^{\nu_{j''}}(\varphi_{u'})"'] &
\end{tikzcd}
$$

which shows that F^{ν} is indeed a functor.

The co-limit μ of F, $\mu : F \Rightarrow \langle i, \Sigma \rangle$, is defined by $\mu_j = \langle \nu_j, \theta_j \rangle : F(j) = \langle K_j, \Sigma_j \rangle \to \langle i, \Sigma \rangle$ where $\nu : F;\Pi = K \Rightarrow i$ is the co-limiting co-cone of $F;\Pi$ and $\theta : F^{\nu} \Rightarrow \Sigma$ is the co-limiting co-cone of F^{ν}.

$$
\begin{tikzcd}
K_j \arrow[d, "K_u"'] \arrow[dr, "\nu_j"] & \\
K_{j'} \arrow[r, "\nu_{j'}"'] & i
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi^{\nu_j}(\Sigma_j) \arrow[d, "\Phi^{\nu_{j'}}(\varphi_u)"'] \arrow[dr, "\theta_j"] & \\
\Phi^{\nu_{j'}}(\Sigma_{j'}) \arrow[r, "\theta_{j'}"'] & \Sigma
\end{tikzcd}
$$

For any other co-cone $\mu' : F \Rightarrow \langle i', \Sigma' \rangle$, let $\mu'_j = \langle \nu'_j, \theta'_j \rangle$ for each index $j \in |J|$. Then $\nu' : K \Rightarrow i'$ is a co-cone. Let $v : i' \to i$ (in I^{op}) be the unique arrow such that $\nu' = \nu; v$.

Because Φ^v preserves J-co-limits, $\theta\Phi^v$ is a co-limit for $F^{\nu};\Phi^v$. Note that $F^{\nu'} = F^{\nu};\Phi^v$. Since θ' is a co-cone $F^{\nu'} \Rightarrow \Sigma'$, let $\varphi : \Phi^v(\Sigma) \to \Sigma'$ be the unique arrow such that $\theta\Phi^v;\varphi = \theta'$. Then $\langle v, \varphi \rangle$ is the unique morphism of Grothendieck signatures $\langle i, \Sigma \rangle \to \langle i', \Sigma' \rangle$ such that $\mu; \langle v, \varphi \rangle = \mu'$. $\square$

Co-limits of FOL signatures. As an application let us check Thm. 12.12 on **FOL** regarded as the Grothendieck institution $\mathbf{fol}^{\sharp}$ where $\mathbf{fol} : \mathbb{S}et^{\mathrm{op}} \to \mathbb{I}ns$ is the coherent adjoint-indexed institution determined by the sorting fibration for the **FOL** signatures. Let $\overline{\mathbf{fol}} : \mathbb{S}et \to co\mathbb{I}ns$ be its adjoint-indexed co-institution. $\overline{\mathbf{fol}}$ supports small co-limits because

- $\mathbb{S}et$ has all small co-limits (which means that the index category $\mathbb{S}et^{\mathrm{op}}$ has all small limits),
- for any set S, the category $\mathbb{S}ig^S$ of the S-sorted signatures has small co-limits which can be calculated 'pointwise' (i.e., separately for each arity) as small co-limits of sets of operation or relation symbols, and
- for each function $u : S \to S'$, the functor $\overline{\Phi^u} : \mathbb{S}ig^S \to \mathbb{S}ig^{S'}$ preserves all co-limits since it is a left adjoint to Φ^u.

The reader may compare the argument above for the existence of co-limits of **FOL** signatures to the proof of Prop. 4.3. The difference between them is that the proof of Prop. 4.3 is based upon co-limits in Grothendieck categories while the proof of Thm. 12.12 is based upon limits in Grothendieck categories. One may wonder how it is possible that in this case co-limits in Grothendieck categories can be obtained by limits in Grothendieck categories, since in principle the former requires stronger hypotheses than the latter (i.e., existence of adjoints compared to preservation of limits; see Thm. 2.10). The answer to this apparent paradox lies in the particularity of the example, since **FOL** appears as a comorphism-based Grothendieck institution with the signature translation functors in the indexed co-institution being *left-adjoint* functors. Hence the left-adjoint property still exists in the background of this **FOL** framework.

Model amalgamation

Model amalgamation in a Grothendieck institution can be treated in a manner similar to theory co-limits by reducing the problem to model amalgamation properties at the 'local' level of the component institutions and at the level of the indexed co-institution. We treat here only the semi-exactness property. Its weak version can be handled in the same way.

Local semi-exactness. An indexed coinstitution $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ is *locally (semi-)exact* if and only if the institution $\mathcal{J}^i$ is (semi-)exact for each index $i \in I$. The following shows that this is a necessary condition for the semi-exactness of the Grothendieck institution.

Proposition 12.13. *Let $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ be a co-institution which supports pushouts. Then the semi-exactness of the Grothendieck institution $\mathcal{J}^\sharp$ implies the local semi-exactness of the indexed co-institution $\mathcal{J}$.*

Proof. For each index i, the model functor Mod^i is the restriction $\mathsf{Mod}^\sharp(\langle i, -\rangle)$ of the model functor of the Grothendieck institution to $\mathbb{S}ig^i$ regarded as a sub-category of $((\mathbb{S}ig;(-)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ (the category of signatures of the Grothendieck institution).

$$\begin{tikzcd}
(\mathbb{S}ig^i)^{\mathrm{op}} \arrow[r] \arrow[dr, "\mathsf{Mod}^i"'] & (\mathbb{S}ig;(-)^{\mathrm{op}})^\sharp \arrow[d, "\mathsf{Mod}^\sharp"] \\
& \mathbb{C}at
\end{tikzcd}$$

Because the comorphisms $\mathcal{J}^u$ preserve the pushouts of signatures, by a simple calculation we can establish that the canonical injection $\mathbb{S}ig^i \to ((\mathbb{S}ig;(-)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ preserves pushouts too. Therefore we have that Mod^i preserves pullbacks as a composition of two preserving pullback functors. $\square$

Exactness of the institution comorphisms. Recall from Sect. 4.3 that an institution comorphism $(\Phi,\alpha,\beta) : I \to I'$ is *exact* if for each I-signature morphism $\varphi : \Sigma_1 \to \Sigma_2$ the naturality square

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma_1) & \mathsf{Mod}'(\Phi(\Sigma_1)) \arrow[l, "\beta_{\Sigma_1}"'] \\
\mathsf{Mod}(\Sigma_2) \arrow[u, "\mathsf{Mod}(\varphi)"] & \mathsf{Mod}'(\Phi(\Sigma_2)) \arrow[l, "\beta_{\Sigma_2}"] \arrow[u, "\mathsf{Mod}'(\Phi(\varphi))"']
\end{tikzcd}$$

is a pullback.

Proposition 12.14. *If the Grothendieck institution of an indexed co-institution $\mathcal{J}$ which supports pushouts is semi-exact, then each institution comorphism $\mathcal{J}^u = (\Phi^u, \alpha^u, \beta^u)$ is exact.*

Proof. Consider an index morphism $u\colon\ i' \to i$ and an arbitrary signature morphism $\varphi\colon\ \Sigma_1 \to \Sigma_2$ in $\mathcal{J}^i$.

Then the square

$$\begin{tikzcd}
\langle i, \Sigma_1\rangle \arrow[r, "\langle 1_i, \varphi\rangle"] \arrow[d, "\langle u, 1_{\Phi(\Sigma_1)}\rangle"'] & \langle i, \Sigma_2\rangle \arrow[d, "\langle u, 1_{\Phi^u(\Sigma_2)}\rangle"] \\
\langle i', \Phi^u(\Sigma_1)\rangle \arrow[r, "\langle 1_{i'}, \Phi^u(\varphi)\rangle"'] & \langle i', \Phi^u(\Sigma_2)\rangle
\end{tikzcd}$$

is a pushout in the category of signatures of the Grothendieck institution. Because the Grothendieck institution is semi-exact, this pushout is mapped by the (Grothendieck) model functor to a pullback square giving the exactness of the institution comorphism $\mathcal{J}^u$. $\square$

Semi-exactness of indexed co-institutions. An indexed co-institution $\mathcal{J}\colon\ I^{\mathrm{op}} \to co\mathbb{I}ns$ is *semi-exact* if and only if for each pullback

$$\begin{tikzcd}
i & j1 \arrow[l, "u1"'] \\
j2 \arrow[u, "u2"] & k \arrow[l, "v2"] \arrow[u, "v1"']
\end{tikzcd}$$

in I and each signature Σ in I^i, the square

$$\begin{tikzcd}
\mathsf{Mod}^i(\Sigma) & \mathsf{Mod}^{j1}(\Phi^{u1}(\Sigma)) \arrow[l, "\beta^{u1}_{\Sigma}"'] \\
\mathsf{Mod}^{j2}(\Phi^{u2}(\Sigma)) \arrow[u, "\beta^{u2}_{\Sigma}"] & \mathsf{Mod}^k(\Phi^{vi}(\Phi^{ui}(\Sigma))) \arrow[l, "\beta^{v2}_{\Phi^{u2}(\Sigma)}"] \arrow[u, "\beta^{v1}_{\Phi^{u1}(\Sigma)}"']
\end{tikzcd}$$

is a pullback.

Proposition 12.15. *If the Grothendieck institution $\mathcal{J}^\sharp$ of an indexed co-institution $\mathcal{J}\colon\ I^{\mathrm{op}} \to co\mathbb{I}ns$ which supports pushouts is semi-exact, then $\mathcal{J}$ is also semi-exact.*

Proof. Consider $\langle v1, v2\rangle$ a pullback of $\langle u1, u2\rangle$ in the index category I. Note (by the co-limit construction in Grothendieck categories) that the square

$$\begin{tikzcd}
\langle i, \Sigma \rangle \arrow[r, "\langle u1, 1_{\Phi^{u1}(\Sigma)} \rangle"] \arrow[d, "\langle u2, 1_{\Phi^{u2}(\Sigma)} \rangle"'] & \langle j1, \Phi^{u1}(\Sigma) \rangle \arrow[d, "\langle v1, 1_{\Phi^{v1}(\Phi^{u1}(\Sigma))} \rangle"] \\
\langle j2, \Phi^{u2}(\Sigma) \rangle \arrow[r, "\langle v2, 1_{\Phi^{v2}(\Phi^{u2}(\Sigma))} \rangle"'] & \langle k, \Phi^{vi}(\Phi^{ui}(\Sigma)) \rangle
\end{tikzcd}$$

is a pushout in the category of signatures $((\mathbb{S}ig; (-)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ of the Grothendieck institution. Because the Grothendieck institution is semi-exact, the Grothendieck model functor $\mathsf{Mod}^\sharp$ maps this pushout square to a pullback square, which is precisely the square giving the semi-exactness of the indexed coinstitution $\mathcal{J}$. $\square$

The sufficient theorem. We have seen that the semi-exactness, the local semi-exactness of the indexed coinstitution, and the exactness of all its comorphisms are necessary conditions for the semi-exactness of the corresponding Grothendieck institution. The following establishes that these conditions are also sufficient.

Theorem 12.16. *Let $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ be an indexed coinstitution which supports pushouts. Then the Grothendieck institution $\mathcal{J}^\sharp$ is semi-exact if and only if*

1. *the indexed coinstitution $\mathcal{J}$ is locally semi-exact,*
2. *the indexed coinstitution $\mathcal{J}$ is semi-exact, and*
3. *all institution comorphisms are exact.*

Proof. The 'necessary' part of this theorem holds by Propositions 12.13, 12.15, and 12.14.

For the 'sufficient' part, we consider an arbitrary pushout of signatures in the Grothendieck institution

$$\begin{tikzcd}
\langle i_0, \Sigma_0 \rangle \arrow[r, "\langle u1, \varphi_1 \rangle"] \arrow[d, "\langle u2, \varphi_2 \rangle"'] & \langle i_1, \Sigma_1 \rangle \arrow[d, "\langle v1, \theta_1 \rangle"] \\
\langle i_2, \Sigma_2 \rangle \arrow[r, "\langle v2, \theta_2 \rangle"'] & \langle i, \Sigma \rangle
\end{tikzcd}$$

By the construction of co-limits of signatures in comorphism-based Grothendieck institutions, given by Thm. 12.12, we have that

$$\begin{tikzcd}
i_0 & i_1 \arrow[l, "u1"'] \\
i_2 \arrow[u, "u2"] & i \arrow[l, "v2"] \arrow[u, "v1"']
\end{tikzcd}$$

is a pullback in the index category I.

The main idea behind this proof is that the given pushout square of signatures in the Grothendieck institution can be expressed as the following composition of four pushout squares:

$$
\begin{tikzcd}
\langle i_0, \Sigma_0 \rangle \arrow[r, "\langle u1, 1_{\Phi^{u1}(\Sigma_0)} \rangle"] \arrow[d, "\langle u2, 1_{\Phi^{u2}(\Sigma_0)} \rangle"'] & \langle i_1, \Phi^{u1}(\Sigma_0) \rangle \arrow[r, "\langle 1_{i_1}, \varphi_1 \rangle"] \arrow[d, "\langle v1, 1_{\Phi^{v1}(\Phi^{u1}(\Sigma_0))} \rangle"] & \langle i_1, \Sigma_1 \rangle \arrow[d, "\langle v1, 1_{\Phi^{v1}(\Sigma_1)} \rangle"] \\
\langle i_2, \Phi^{u2}(\Sigma_0) \rangle \arrow[r, "\langle v2, 1_{\Phi^{v2}(\Phi^{u2}(\Sigma_0))} \rangle"'] \arrow[d, "\langle 1_{i_2}, \varphi_2 \rangle"'] & \langle i, \Phi^{vi}(\Phi^{ui}(\Sigma_0)) \rangle \arrow[r, "\langle 1_i, \Phi^{v1}(\varphi_1) \rangle"'] \arrow[d, "\langle 1_i, \Phi^{v2}(\varphi_2) \rangle"] & \langle i, \Phi^{v1}(\Sigma_1) \rangle \arrow[d, "\langle 1_i, \theta_1 \rangle"] \\
\langle i_2, \Sigma_2 \rangle \arrow[r, "\langle v2, 1_{\Phi^{v2}(\Sigma_2)} \rangle"'] & \langle i, \Phi^{v2}(\Sigma_2) \rangle \arrow[r, "\langle 1_i, \theta_2 \rangle"'] & \langle i, \Sigma \rangle
\end{tikzcd}
$$

Then the Grothendieck model functor

- maps the up-left pushout square to a pullback square because the indexed co-institution is semi-exact,
- maps the down-right pushout square to a pullback square because the indexed institution is locally semi-exact, and
- maps the up-right and down-left pushout squares to pullback squares because the institution comorphisms $(\Phi^{v1}, \alpha^{v1}, \beta^{v1})$ and $(\Phi^{v2}, \alpha^{v2}, \beta^{v2})$ are exact.

Therefore, the Grothendieck model functor maps the original pushout square of signatures in the Grothendieck institution to a pullback square obtained as the composition of the four pullback squares resulting from mapping the four component pushout squares. $\square$

Exercises

12.4. Apply the model amalgamation Thm. 12.16 on the example of **FOL** as the comorphism-based Grothendieck institution $\overline{\mathbf{fol}}^\sharp$.

12.5. The institution comorphism **FOL** $\to$ **FOEQL** encoding relations as operations (see Sect. 3.3) preserves pushouts of signatures although it is not an adjoint institution comorphism.

12.6. The Grothendieck institution determined by the forgeful institution morphism **POA** $\to$ **FOL** has small co-limits.

12.7. Give a counterexample showing that even if the index category I is J-co-complete, the comorphism-based Grothendieck institution has J-co-limits of theories, and the institution comorphisms $\mathcal{J}^u$ preserve J-co-limits, the indexed coinstitution $\mathcal{J}$ is not necessarily locally J-co-complete.

12.8. Let $\mathcal{J}\colon\ I^{\mathrm{op}} \to \mathbb{I}ns$ be an adjoint-indexed (morphism-based) institution such that I is J-co-complete for a small category J, and the indexed category of signatures $\mathbb{S}ig$ of $\mathcal{J}$ is locally J-co-complete. Then the category of theories $\mathbb{T}h^{\mathcal{J}^\sharp}$ of the (morphism-based) Grothendieck institution $\mathcal{J}^\sharp$ has J-co-limits.

12.9. The Grothendieck institution determined by the forgetful institution morphism from **POA** to **FOL** is *not* semi-exact due only to the failure of the exactness of the embedding institution comorphism **FOL** $\to$ **POA**.

12.10. [46, 44] **Liberality in Grothendieck institutions**
An indexed institution $\mathcal{J}: I^{\mathrm{op}} \to \mathbb{I}ns$ is *locally liberal* if and only if the institution $\mathcal{J}^i$ is liberal for each index $i \in I$. The Grothendieck institution $\mathcal{J}^\sharp$ of an indexed institution $\mathcal{J}: I^{\mathrm{op}} \to \mathbb{I}ns$ is liberal if and only if $\mathcal{J}$ is liberal and each institution morphism $\mathcal{J}^u$ is liberal for each index morphism $u \in I$.

12.11. [39] **Grothendieck inclusion systems**
The category $\mathbb{IS}$ of inclusion systems can be endowed with a 2-categorical structure in which the 2-cells are inclusion natural transformation (i.e., such that all their components are inclusions) between inclusive functors. An adjunction in $\mathbb{IS}$ is thus just an ordinary adjunction (in $\mathbb{C}at$) such that all the components of the unit and of the counit of the adjunction are inclusions. An *enriched indexed inclusion system* is a functor $B: \langle I, \mathcal{E}\rangle \to \mathbb{IS}^{\mathrm{op}}$ from the opposite of the underlying category of an inclusion system 'of indices' to the category of inclusions systems and inclusive functors.

An enriched indexed inclusion system is *invertible* when each inclusion system morphism B^u has a $\mathbb{IS}$-left-adjoint $[-]^u$. It is $\mathcal{E}$-invertible when the $\mathbb{IS}$-left-adjoint to B^u exists for $u \in \mathcal{E}$ (and not necessarily for all index morphisms u).

For any $\mathcal{E}$-invertible enriched indexed inclusion system $B: \langle I, \mathcal{E}\rangle \to \mathbb{IS}^{\mathrm{op}}$ the Grothendieck category $B^\sharp$ of $B^{\mathrm{op}}; (\mathbb{IS} \to \mathbb{C}at): \langle I, \mathcal{E}\rangle^{\mathrm{op}} \to \mathbb{C}at$ can be endowed with an inclusion system $\langle I^\sharp, \mathcal{E}^\sharp\rangle$ such that $\langle u, \varphi\rangle: \langle j, \Sigma\rangle \to \langle j', \Sigma'\rangle$ is

- *abstract inclusion* iff both u and φ are abstract inclusions, and
- *abstract surjection* iff u is abstract surjection and $\Sigma' = [\varphi(\Sigma)]^u$.

Show that the strong inclusion systems of **FOL**-models and of theories (see Sect. 4.5) are instances of this Grothendieck inclusion system construction. What about the strong inclusion system of the **FOL**-signatures?

12.12. [39] For any invertible enriched indexed inclusion system $B: \langle I, \mathcal{E}\rangle \to \mathbb{IS}^{\mathrm{op}}$, the Grothendieck inclusion system $\langle I^\sharp, \mathcal{E}^\sharp\rangle$ (of Ex. 12.11) has unions if

- the inclusion system of indices $\langle I, \mathcal{E}\rangle$ has unions, and
- for each index j the 'local' inclusion system $B^j = \langle I^j, \mathcal{E}^j\rangle$ has unions.

12.13. [39] In addition to the conditions of Ex. 12.11 if the inclusion system of the indices $\langle I, \mathcal{E}\rangle$ is epic, $B^j = \langle I^j, \mathcal{E}^j\rangle$ is epic for each index j, and B^u are faithful for $u \in \mathcal{E}$, then the inclusion system $\langle I^\sharp, \mathcal{E}^\sharp\rangle$ defined in Ex. 12.11 is epic too.

12.14. [39] For any pair of functors $F, G: \langle I, \mathcal{E}\rangle \to \mathbb{IS}^{\mathrm{op}}$ (from the underlying category of an inclusion system $\langle I, \mathcal{E}\rangle$), a $\mathbb{IS}$-*lax natural transformation* $\mu: F \Rightarrow G$ is a lax natural transformation such that

- for any object j of $\langle I, \mathcal{E}\rangle$, the functor $\mu^j: F(j) \to G(j)$ is inclusive, and
- for any $u \in I$, the natural transformation μ^u is abstract inclusion (for the inclusion system of the corresponding functor category; see Ex. 4.50).

$\mathbb{IS}$-*lax co-cone* and $\mathbb{IS}$-*lax colimits*, respectively, are just lax co-cone and lax colimits, respectively, which are $\mathbb{IS}$-lax as natural transformations.

For any $\mathcal{E}$-invertible $\mathbb{IS}$-enriched indexed inclusion system $B: \langle I, \mathcal{E}\rangle \to \mathbb{IS}^{\mathrm{op}}$, the Grothendieck inclusion system $\langle I^\sharp, \mathcal{E}^\sharp\rangle$ defined by Ex. 12.11 is the $\mathbb{IS}$-lax co-limit of B.

12.15. [39] **Closed inclusion systems on Grothendieck categories**
For any indexed category $B : \langle I, \mathcal{E}\rangle \to \mathbb{C}at^{\mathrm{op}}$ (functor from the underlying category of an inclusion system $\langle I, \mathcal{E}\rangle$ to the opposite of $\mathbb{C}at$), its Grothendieck category $B^\sharp$ admits an inclusion system such that $\langle u, \varphi\rangle : \langle j, \Sigma\rangle \to \langle j', \Sigma'\rangle$

- is abstract inclusion if and only if $u \in I$ and φ is identity, and
- is abstract surjection if and only if $u \in \mathcal{E}$.

Show that the closed inclusion systems of **FOL**-signatures, of **FOL**-models, and of theories (see Sect. 4.5) are instances of this general construction.

12.3 Interpolation

The interpolation problem for Grothendieck institutions is treated similarly to the model amalgamation problem by isolating a set of three sufficient and necessary conditions. These conditions are similar in flavor to those underlying the model amalgamation. We need the following interpolation concept for indexed co-institutions.

Interpolation squares of institution comorphisms. A commuting square of institution comorphisms

$$\begin{tikzcd} I \arrow[r, "(\Phi_1,\alpha_1,\beta_1)"] \arrow[d, "(\Phi_2,\alpha_2,\beta_2)"'] & I_1 \arrow[d, "(\Phi'_1,\alpha'_1,\beta'_1)"] \\ I_2 \arrow[r, "(\Phi'_2,\alpha'_2,\beta'_2)"'] & I' \end{tikzcd}$$

is a *Craig Interpolation square* if for each I-signature Σ, for each set E_1 of $\Phi_1(\Sigma)$-sentences and for each set E_2 of $\Phi_2(\Sigma)$-sentences, if $(\alpha'_1)_{\Phi_1(\Sigma)}(E_1) \models' (\alpha'_2)_{\Phi_2(\Sigma)}(E_2)$, then there exists a set E of Σ-sentences such that $E_1 \models^{I_1} (\alpha_1)_\Sigma(E)$ and $(\alpha_2)_\Sigma(E) \models^{I_2} E_2$.

$$\begin{tikzcd} \mathsf{Sen}(\Sigma) \arrow[r, "(\alpha_1)_\Sigma"] \arrow[d, "(\alpha_2)_\Sigma"'] & \mathsf{Sen}^1(\Phi_1(\Sigma)) \arrow[d, "(\alpha'_1)_{\Phi_1(\Sigma)}"] \\ \mathsf{Sen}^2(\Phi_2(\Sigma)) \arrow[r, "(\alpha'_2)_{\Phi_2(\Sigma)}"'] & \mathsf{Sen}'(\Phi'_k(\Phi_k(\Sigma))) \end{tikzcd}$$

Interpolation in Grothendieck institutions

The theorem below, giving a set of necessary and sufficient conditions for interpolation in Grothendieck institutions, involves the concept of left/right interpolation for institution comorphisms introduced in Sect. 9.5.

Theorem 12.17. *Let $\mathcal{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ be an indexed coinstitution which supports pushouts such that*

- *there are fixed classes of index morphisms $\mathcal{L}, \mathcal{R} \subseteq I$ containing all identities, and*
- *for each index $i \in |I|$ there are fixed classes of signature morphisms $\mathcal{L}^i, \mathcal{R}^i \subseteq \mathbb{S}ig^i$ containing all identities,*

such that

- *$\mathcal{L}$ and $\mathcal{R}$ are stable under pullbacks,*
- *$\Phi^u(\mathcal{R}^i) \subseteq \mathcal{R}^j$ for each index morphism $u : j \to i$ in $\mathcal{L}$, and*
- *$\Phi^u(\mathcal{L}^i) \subseteq \mathcal{L}^j$ for each index morphism $u : j \to i$ in $\mathcal{R}$.*

Let $\mathcal{L}^\sharp$, and $\mathcal{R}^\sharp$, be the classes of signature morphisms $\langle u : j \to i, \varphi\rangle$ of the Grothendieck institution such that $u \in \mathcal{L}$, respectively $u \in \mathcal{R}$, and $\varphi \in \mathcal{L}^j$, respectively $\varphi \in \mathcal{R}^j$.

Then the Grothendieck institution $\mathcal{J}^\sharp$ has the Craig $(\mathcal{L}^\sharp, \mathcal{R}^\sharp)$-interpolation property if and only if

1. *for each index i the institution $\mathcal{J}^i$ has the $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation property,*
2. *each pullback square of index morphisms*

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\mathcal{L}"'] \\ \cdot \arrow[u, "\mathcal{R}"] & \cdot \arrow[l] \arrow[u] \end{tikzcd}$$

   determines a Craig interpolation square of institution comorphisms,
3. *for each $u : j \to i$ in $\mathcal{L}$ the institution comorphism $\mathcal{J}^u = (\Phi^u, \alpha^u, \beta^u)$ has the Craig $\mathcal{R}^i$-right interpolation property, and*
4. *for each $u : j \to i$ in $\mathcal{R}$ the institution comorphism $\mathcal{J}^u$ has the Craig $\mathcal{L}^i$-left interpolation property.*

Proof. For the 'sufficient' part, we consider an arbitrary pushout of signatures in the Grothendieck institution

$$\begin{tikzcd} \langle i_0, \Sigma_0\rangle \arrow[r, "\langle u1, \varphi_1\rangle"] \arrow[d, "\langle u2, \varphi_2\rangle"'] & \langle i_1, \Sigma_1\rangle \arrow[d, "\langle v1, \theta_1\rangle"] \\ \langle i_2, \Sigma_2\rangle \arrow[r, "\langle v2, \theta_2\rangle"'] & \langle i, \Sigma\rangle \end{tikzcd}$$

such that $u1 \in \mathcal{L}$, $\varphi_1 \in \mathcal{L}^{i_1}$, and $u2 \in \mathcal{R}$, $\varphi_2 \in \mathcal{R}^{i_2}$.

As in the proof of Thm. 12.16, by the construction of signature co-limits in comorphism-based Grothendieck institutions given by Thm. 12.12 we have that

$$\begin{tikzcd} i_0 & i_1 \arrow[l, "u1"'] \\ i_2 \arrow[u, "u2"] & i \arrow[l, "v2"] \arrow[u, "v1"'] \end{tikzcd}$$

is a pullback in the index category I and that the original pushout square of signatures can be expressed as the following composition of four pushout squares:

$$\begin{tikzcd}
\langle i_0,\Sigma_0\rangle \arrow[r, "\langle u1, 1_{\Phi^{u1}(\Sigma_0)}\rangle"] \arrow[d, "\langle u2, 1_{\Phi^{u2}(\Sigma_0)}\rangle"'] & \langle i_1, \Phi^{u1}(\Sigma_0)\rangle \arrow[r, "\langle 1_{i_1}, \varphi_1\rangle"] \arrow[d, "\langle v1, 1_{\Phi^{v1}(\Phi^{u1}(\Sigma_0))}\rangle"] & \langle i_1, \Sigma_1\rangle \arrow[d, "\langle v1, 1_{\Phi^{v1}(\Sigma_1)}\rangle"] \\
\langle i_2, \Phi^{u2}(\Sigma_0)\rangle \arrow[r, "\langle v2, 1_{\Phi^{v2}(\Phi^{u2}(\Sigma_0))}\rangle"'] \arrow[d, "\langle 1_{i_2}, \varphi_2\rangle"'] & \langle i, \Phi^{vi}(\Phi^{ui}(\Sigma_0))\rangle \arrow[r, "\langle 1_i, \Phi^{v1}(\varphi_1)\rangle"'] \arrow[d, "\langle 1_i, \Phi^{v2}(\varphi_2)\rangle"] & \langle i, \Phi^{v1}(\Sigma_1)\rangle \arrow[d, "\langle 1_i, \theta_1\rangle"] \\
\langle i_2, \Sigma_2\rangle \arrow[r, "\langle v2, 1_{\Phi^{v2}(\Sigma_2)}\rangle"'] & \langle i, \Phi^{v2}(\Sigma_2)\rangle \arrow[r, "\langle 1_i, \theta_2\rangle"'] & \langle i, \Sigma\rangle
\end{tikzcd}$$

Note that by the stability hypothesis we have that $v1 \in \mathcal{R}$ and $v2 \in \mathcal{L}$. We have the following:

- The up-left pushout square is a CI square by applying the fact that the corresponding square of institution comorphisms is a CI square and by considering the signature Σ_0.
- The down-right pushout square is a CI square because it is a CI square in the institution $\mathcal{J}^i$ as a pushout square of a signature morphism in $\mathcal{L}^i$ with a signature morphism in $\mathcal{R}^i$. Here we have to notice that $\Phi^{v1}(\varphi_1) \in \mathcal{L}^i$ because $\varphi_1 \in \mathcal{L}^{i_1}$ and $v1 \in \mathcal{R}$, and that $\Phi^{v2}(\varphi_2) \in \mathcal{R}^i$ because $\varphi_2 \in \mathcal{R}^{i_2}$ and $v2 \in \mathcal{L}$.
- The up-right pushout square is a CI square because $\varphi_1 \in \mathcal{L}^{i_1}$ and $v1 \in \mathcal{R}$ which allow us to apply the assumption that $(\Phi^{v1}, \alpha^{v1}, \beta^{v1})$ has the Craig $\mathcal{L}^{i_1}$-left interpolation property.
- The down-left pushout square is a CI square by an argument symmetrical to the argument of the item above.

Therefore all four components of the big pushout square in the Grothendieck institution are CI squares. By a simple calculation (see Ex. 9.1) we have that both the 'horizontal' and the 'vertical' compositions of CI squares get CI squares:

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & {}
\end{tikzcd}$$

This completes the proof of the 'sufficient' part of the theorem.

For the 'necessary' part, we have only to notice the following:

- For each index i, by considering 1_i as an index morphism, any Craig $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation square in $\mathcal{J}^i$ is a Craig $(\mathcal{L}^\sharp, \mathcal{R}^\sharp)$-interpolation square in the Grothendieck institution.

- For $\langle v1, v2\rangle$ a pullback of $\langle u1, u2\rangle$ in the index category I, by the co-limit construction on signatures in Grothendieck institutions, for each signature Σ in $|\mathbb{S}ig^{i_0}|$ the square

$$\begin{tikzcd} \langle i_0, \Sigma\rangle \arrow[r, "{\langle u1, 1_{\Phi^{u1}(\Sigma)}\rangle}"] \arrow[d, "{\langle u2, 1_{\Phi^{u2}(\Sigma)}\rangle}"'] & \langle i1, \Phi^{u1}(\Sigma)\rangle \arrow[d, "{\langle v1, 1_{\Phi^{v1}(\Phi^{u1}(\Sigma))}\rangle}"] \\ \langle i2, \Phi^{u2}(\Sigma)\rangle \arrow[r, "{\langle v2, 1_{\Phi^{v2}(\Phi^{u2}(\Sigma))}\rangle}"'] & \langle i, \Phi^{vi}(\Phi^{ui}(\Sigma))\rangle \end{tikzcd}$$

  is pushout in the category of signatures $((\mathbb{S}ig;(-)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ of the Grothendieck institution. Therefore, these are CI squares in the Grothendieck institution if and only if the square of index morphisms determines a CI square of institution comorphisms.

- For each $u : j \to i$ in $\mathcal{L}$ and each signature morphism $\varphi : \Sigma_1 \to \Sigma_2$ in $\mathcal{R}^i$, the square below

$$\begin{tikzcd} \langle i, \Sigma_1\rangle \arrow[r, "{\langle u, 1_{\Phi^u(\Sigma_1)}\rangle}"] \arrow[d, "{\langle 1_i, \varphi\rangle}"'] & \langle j, \Phi^u(\Sigma_1)\rangle \arrow[d, "{\langle 1_j, \Phi^u(\varphi)\rangle}"] \\ \langle i, \Sigma_2\rangle \arrow[r, "{\langle u, 1_{\Phi^u(\Sigma_2)}\rangle}"'] & \langle j, \Phi^u(\Sigma_2)\rangle \end{tikzcd}$$

  is a pushout in the category of signatures of the Grothendieck institution. Moreover, these squares are CI squares if and only if $(\Phi^u, \alpha^u, \beta^u)$ has the Craig $\mathcal{R}^i$-right interpolation property.

- By replacing $\mathcal{L}$ by $\mathcal{R}$, $\mathcal{R}^i$ by $\mathcal{L}^i$, and 'right' by 'left' in the argument above, we can deduce its symmetrical conclusion.

$\square$

Craig-Robinson interpolation by Grothendieck interpolation

We have seen (Prop. 9.20) that Craig-Robinson interpolation can be obtained from Craig interpolation when the institution has implications and it is quasi-compact (i.e., it is compact or has infinite conjunctions). The requirement on implications does not allow lifting Craig interpolation to Craig-Robinson interpolation by Prop. 9.20 in institutions such as **EQL** or **HCL**. However the Grothendieck interpolation Thm. 12.17 can be used to avoid the existence of implications, as shown by the following Craig-Robinson interpolation theorem.

Theorem 12.18. *Consider a conservative institution comorphism* $(\Phi, \alpha, \beta) : I \to I'$ *and classes* $\mathcal{L}^i, \mathcal{R}^i$ *of signature morphisms in* I *such that*

1. *I and I' have pushouts of signatures and Φ preserves pushouts,*
2. *the institution comorphism (Φ, α, β) has Craig $\mathcal{L}^i$-left interpolation,*

3. *I' has implications and it is quasi-compact, and*
4. *I' has Craig $(\Phi(\mathcal{L}^i), \Phi(\mathcal{R}^i))$-interpolation.*

Then the institution I has Craig-Robinson $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation.

Proof. The key to the proof of this theorem is that the Grothendieck institution determined by the comorphism (Φ, α, β) has Craig interpolation for pushout squares of the form

$$\begin{tikzcd}
\langle I, \Sigma \rangle \arrow[r, "\langle 1_I, \varphi_1 \rangle"] \arrow[d, "\langle u, \Phi(\varphi_2) \rangle"'] & \langle I, \Sigma_1 \rangle \arrow[d, "\langle u, \Phi(\theta_1) \rangle"] \\
\langle I', \Phi(\Sigma_2) \rangle \arrow[r, "\langle 1_{I'}, \Phi(\theta_2) \rangle"'] & \langle I', \Phi(\Sigma') \rangle
\end{tikzcd}$$

where

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\
\Sigma_2 \arrow[r, "\theta_2"'] & \Sigma'
\end{tikzcd}$$

is a pushout of signature morphisms in I with $\varphi_1 \in \mathcal{L}^i$ and $\varphi_2 \in \mathbb{R}^i$.

For this we first note that according to the construction from the proof of Thm. 12.12 the considered square of Grothendieck signature morphisms is indeed a pushout square. Then we apply the Grothendieck interpolation Thm. 12.17 as follows:

- We take the category of indices to consist of two objects i and i' and one non-identity arrow u, the class of 'left' index arrows (denoted by $\mathcal{L}$ in Thm. 12.17) as $\{1_i, 1_{i'}\}$ and the class of 'right' index arrows (denoted by $\mathcal{R}$ in Thm. 12.17) as $\{1_i, 1_{i'}, u\}$.
- We take $\mathcal{L}^{i'} = \Phi(\mathcal{L}^i)$ and $\mathcal{R}^{i'} = \Phi(\mathcal{R}^i)$.
- I' has Craig $(\mathcal{L}^{i'}, \mathcal{R}^{i'})$-interpolation by hypothesis and I has Craig $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation by the borrowing Prop. 9.30 by using the hypothesis that (Φ, α, β) has Craig $\mathcal{L}^i$-left interpolation.
- The conditions on interpolation squares of institution comorphisms and on the right interpolation property for the comorphism are trivially fulfilled, while the condition on the left interpolation property for the comorphism is directly fulfilled by the hypothesis that (Φ, α, β) has Craig $\mathcal{L}^i$-left interpolation.

By the conclusion of the Grothendieck interpolation Thm. 12.17 we obtain that the desired square of signature morphisms in the Grothendieck institution is indeed a CI square.

Now we proceed with the proof of the Craig-Robinson interpolation property for I. Consider a pushout square of signature morphisms in I as in the second diagram above and $E_1 \subseteq \mathsf{Sen}(\Sigma_1)$ and $E_2, \Gamma_2 \in \mathsf{Sen}(\Sigma_2)$ such that $\theta_1(E_1) \cup \theta_2(\Gamma_2) \models \theta_2(E_2)$. We have to find an interpolant $E \subseteq \mathsf{Sen}(\Sigma)$ such that $E_1 \models \varphi_1(E)$ and $\varphi_2(E) \cup \Gamma_2 \models E_2$. As in the proof of Prop. 9.20 we may assume without loss of generality that E_2 is a singleton, i.e., consists of only one sentence (otherwise we take E to be the union of all interpolants obtained for the individual sentences).

The original problem $\theta_1(E_1) \cup \theta_2(\Gamma_2) \models \theta_2(E_2)$ translates to

$$\alpha_{\Sigma'}(\theta_1(E_1)) \cup \alpha_{\Sigma'}(\theta_2(\Gamma_2)) \models' \alpha_{\Sigma'}(\theta_2(E_2)).$$

By the naturality of α this is the same with

$$\Phi(\theta_1)(\alpha_{\Sigma_1}(E_1)) \cup \Phi(\theta_2)(\alpha_{\Sigma_2}(\Gamma_2)) \models' \Phi(\theta_2)(\alpha_{\Sigma_2}(E_2)).$$

Because of the assumption that $E_2 = \{e\}$ is singleton and by compactness or by the existence of infinite conjunctions we may also assume that E_1 and Γ_2 are finite. Because I' has implications, let $\alpha_{\Sigma_2}(\Gamma_2) \Rightarrow \alpha_{\Sigma_2}(E_2)$ denote the $\Phi(\Sigma_2)$-sentence $\gamma_1 \Rightarrow (\ldots \Rightarrow (\gamma_n \Rightarrow \alpha_{\Sigma_2}(e)))$ where $\alpha_{\Sigma_2}(\Gamma_2) = \{\gamma_1, \ldots, \gamma_n\}$. Then we have that

$$\Phi(\theta_1)(\alpha_{\Sigma_1}(E_1)) \models' \Phi(\theta_2)(\alpha_{\Sigma_2}(\Gamma_2) \Rightarrow \alpha_{\Sigma_2}(E_2))$$

which is a Grothendieck interpolation problem for the above mentioned pushout square of signature morphisms in the Grothendieck institution

$$\langle u, \Phi(\theta_1)\rangle(E_1) \models^\sharp \langle 1_{i'}, \Phi(\theta_2)\rangle(\alpha_{\Sigma_2}(\Gamma_2) \Rightarrow \alpha_{\Sigma_2}(E_2)).$$

Since this pushout square is a CI square, let E be the interpolant. This means we have that

$$E_1 \models^\sharp \langle 1_i, \varphi_1\rangle(E) \text{ and } \langle u, \Phi(\varphi_2)\rangle(E) \models^\sharp \alpha_{\Sigma_2}(\Gamma_2) \Rightarrow \alpha_{\Sigma_2}(E_2).$$

We show that E is also an interpolant for the original Craig-Robinson interpolation problem. Note that $E_1 \models \varphi_1(E)$ is just $E_1 \models^\sharp \langle 1_i, \varphi_1\rangle(E)$. We still have to show that $\varphi_2(E) \cup \Gamma_2 \models E_2$.

Let M_2 be a model for $\varphi_2(E) \cup \Gamma_2$. Because the institution comorphism is conservative, let M_2' be a $\Phi(\Sigma_2)$-model such that $\beta_{\Sigma_2}(M_2') = M_2$. By the satisfaction condition for the institution comorphism we have that $M_2' \models' \alpha_{\Sigma_2}(\varphi_2(E)) \cup \alpha_{\Sigma_2}(\Gamma_2)$. But $\langle u, \Phi(\varphi_2)\rangle(E) \models^\sharp \alpha_{\Sigma_2}(\Gamma_2) \Rightarrow \alpha_{\Sigma_2}(E_2)$ means $\Phi(\varphi_2)(\alpha_{\Sigma_2}(E)) = \alpha_{\Sigma_2}(\varphi_2(E)) \models' \alpha_{\Sigma_2}(\Gamma_2) \Rightarrow \alpha_{\Sigma_2}(E_2)$. From this we deduce that $M_2' \models' \alpha_{\Sigma_2}(E_2)$ and by the satisfaction condition for the institution comorphism we obtain that $M_2 \models E_2$. □

Applications of Thm. 12.18 are related to the applications of the left interpolation Prop. 9.26. For example by considering comorphisms $I \to$ **FOL**, the table of Cor. 9.27 can be also read as a table of Craig-Robinson interpolation properties as in the following.

Corollary 12.19. *The following institutions have Craig-Robinson* $(\mathcal{L}, \mathcal{R})$*-interpolation.*

institution	$\mathcal{L}$	$\mathcal{R}$
EQL	*ie*	$**$
universal **FOL**-*atoms*	*iei*	$***$
HCL	*ie*$*$	$***$
UNIV	*ie*$*$	$***$
$\forall\vee$	*ie*$*$	$***$

One immediate consequence of the new Craig-Robinson interpolation properties given by Cor. 12.19 is that some of the definability results obtained as instances of the definability by axiomatizability Thm. 10.8 and given in Cor. 10.10 can also be obtained as instances of the definability by interpolation Thm. 10.5.

Exercises

12.16. For each pushout of sets (as in the left diagram below) its corresponding square of institution comorphisms (the right diagram below)

$$\begin{tikzcd} S \arrow[r, "u1"] \arrow[d, "u2"'] & S1 \arrow[d, "v1"] \\ S2 \arrow[r, "v2"'] & S' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{FOL}^{S} \arrow[r, "\overline{\mathbf{fol}}^{u1}"] \arrow[d, "\overline{\mathbf{fol}}^{u2}"'] & \mathbf{FOL}^{S1} \arrow[d, "\overline{\mathbf{fol}}^{v1}"] \\ \mathbf{FOL}^{S2} \arrow[r, "\overline{\mathbf{fol}}^{v2}"'] & \mathbf{FOL}^{S'} \end{tikzcd}$$

is a CI square when either $u1$ or $u2$ is injective.

Notes. The theory of (morphism-based) Grothendieck institutions developed by [46] was preceded by ‘extra’ theory morphisms across institution morphisms of [44] with the motivation to provide semantics for heterogeneous multi-logic specification with CafeOBJ [57]. Grothendieck institutions provide a *homogeneous* semantics for heterogeneous multi-logic environments. Comorphism-based Grothendieck institutions were defined in [132] by dualization of the morphism-based Grothendieck institutions and have been extensively used as foundations for heterogeneous specification with CASL extensions [135]. Heterogeneity of the institution mappings involved was also considered in [134] by a “Bi-Grothendieck” construction for an indexed structure of both institution morphisms and comorphisms. The paper [46] shows that Grothendieck institutions are just a special case of the more general concept of Grothendieck construction in an arbitrary 2-category. Cor. 12.8 extends Bénabou’s result [15] to fibred institutions.

‘Globalisation’ results for Grothendieck institutions have been obtained in [46] for theory co-limits, liberality, model amalgamation, and signature inclusions by following the same pattern of lifting each of these properties from the ‘local’ level of the indexed institution to the ‘global’ level of the Grothendieck institution. Although the ‘globalisation’ results can be immediately translated into the language of fibred institutions, the framework of indexed institutions seems to be the most appropriate for applications and for the presentations and development of these results. In the case of theory co-limits and liberality, the sufficient part of the globalisation results was first obtained in [44]. This paper had conjectured an ‘if and only if’ characterization of model amalgamation for extra theory morphisms, and [46] solved it. Later on [132] showed that comorphism-based Grothendieck institutions interact in a simpler way with exactness. The general interpolation problem in Grothendieck institutions was solved in [51].

Chapter 13

Institutions with Proofs

The already familiar semantic consequence relation $E \models E'$ between sets of sentences constitutes the semantic way to establish truth because it involves the models and the satisfaction relation between models and sentences. The syntactic approach to truth consists of establishing consequence relations, called *proofs*, between sets of sentences involving only syntactic entities. Therefore this approach is beyond models and satisfaction relation between models and sentences. The syntactic approach to truth, called 'proof theory', is in many ways complementary to model theory. However the relationship between model theory and proof theory is crucial for any logical system. For example the correctness of a proof theory can be established only in the presence of a model theory.

Proof systems. So what is a proof? It is a one-way move from a set E to a set E' of sentences, called *E proves from E'*, and meaning that E' is established 'true' on the basis of E being established 'true'. And there can be several different ways to prove E' from E. Therefore proofs can be conveniently represented as labeled arrows $E \xrightarrow{p} E'$. Proofs between sets of sentences have two natural compositionality properties:

- an associative *horizontal* one, meaning that proofs $E \xrightarrow{p} E'$ and $E' \xrightarrow{p'} E''$ determine a proof $E \xrightarrow{p;p'} E''$, and
- a *vertical* one meaning that assuming that $E_1 \cap E_2 = \emptyset$, any proofs $E \xrightarrow{p_1} E_1$ and $E \xrightarrow{p_2} E_2$ determine a proof $E \xrightarrow{\langle p_1, p_2 \rangle} E_1 \cup E_2$

$$\begin{tikzcd}
E_1 & E_1 \cup E_2 \arrow[l] \arrow[r] & E_2 \\
& E \arrow[ul, "p_1"] \arrow[u, "\langle p_1, p_2 \rangle"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

such that each p_i can be 'extracted' from $\langle p_1, p_2\rangle$ by horizontal composition with a canonical *monotonicity* proof $E_1 \cup E_2 \longrightarrow E_i$.

Thus horizontal composition gives proofs the structure of a category, whose objects are the sets of sentences of a fixed signature Σ. Let us denote this category by $\mathsf{Pf}(\Sigma)$. The vertical composition just says that $\mathsf{Pf}(\Sigma)$ has finite products of disjoint sets of sentences.

It is also natural to assume that any signature morphism $\varphi : \Sigma \to \Sigma'$ gives a translation from $\mathsf{Pf}(\Sigma)$ to $\mathsf{Pf}(\Sigma')$ which extends the translation of the sentences to proofs in such a way that the horizontal composition is preserved. The latter property means we have a functor $\mathsf{Pf}(\varphi) : \mathsf{Pf}(\Sigma) \to \mathsf{Pf}(\Sigma')$. Moreover, for any signature morphisms $\varphi : \Sigma \to \Sigma'$ and $\varphi' : \Sigma' \to \Sigma''$, Pf should preserve their composition. All these are collected by the following proof theoretic counterpart for the concept of institution.

A *proof system* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ consists of

- a category of 'signatures' $\mathbb{S}ig$,
- a 'sentence functor' $\mathsf{Sen} : \mathbb{S}ig \to \mathbb{S}et$, and
- a 'proof functor' $\mathsf{Pf} : \mathbb{S}ig \to \mathbb{C}at$ (giving for each signature Σ the category of the Σ-proofs)

such that

1. $\mathsf{Sen}; \mathcal{P}; (-)^{\mathrm{op}}$ is a sub-functor of Pf, and
2. the inclusion $\mathcal{P}(\mathsf{Sen}(\Sigma))^{\mathrm{op}} \hookrightarrow \mathsf{Pf}(\Sigma)$ is broad and preserves finite products of disjoint sets (of sentences) for each signature Σ, where $\mathcal{P} : \mathbb{S}et \to \mathbb{C}at$ is the ($\mathbb{C}at$-valued) power-set functor.

$$
\begin{tikzcd}
\mathsf{Pf}(\Sigma) \arrow[r, "\mathsf{Pf}(\varphi)"] & \mathsf{Pf}(\Sigma') \\
\mathcal{P}(\mathsf{Sen}(\Sigma))^{\mathrm{op}} \arrow[u, hook] \arrow[r, "\mathcal{P}(\mathsf{Sen}(\varphi))^{\mathrm{op}}"'] & \mathcal{P}(\mathsf{Sen}(\Sigma'))^{\mathrm{op}} \arrow[u, hook]
\end{tikzcd}
$$

Note that the inclusion $\mathcal{P}(\mathsf{Sen}(\Sigma))^{\mathrm{op}} \hookrightarrow \mathsf{Pf}(\Sigma)$ is broad means that $\mathsf{Pf}(\Sigma)$ has subsets of $\mathsf{Sen}(\Sigma)$ as objects, that preservation of products implies that there are distinguished monotonicity proofs $\supseteq_{\Gamma,E} : \Gamma \to E$ whenever $E \subseteq \Gamma$ which are preserved by signature morphisms, i.e., $\varphi(\supseteq_{\Gamma,E}) = \supseteq_{\varphi(\Gamma),\varphi(E)}$, and that proofs $\Gamma \to E_1 \uplus E_2$ are in one-one natural correspondence with pairs of proofs $\langle \Gamma \to E_1, \Gamma \to E_2\rangle$.

Within the context of proof systems, in order to simplify notation, singleton sets $\{\rho\}$ may be sometimes denoted just by their element ρ.

Infinitary proof systems. When we allow infinite products of disjoint sets, i.e., infinite vertical compositions of proofs, we call the proof system *infinitary*. Unless specified as infinitary, our proof systems are considered by default finitary.

Entailment systems. Thin proof systems, i.e., such that $\mathsf{Pf}(\Sigma)$ are preorders, are called *entailment systems*. The preorder $\mathsf{Pf}(\Sigma)$ is then called an *entailment relation* while its proofs are called *entailments*. Thus entailment systems can only tell that a certain set of sentences E is *provable* from another set of sentences Γ, without the possibility to distinguish between different sets of sentences.

Each proof system can be 'flattened' canonically to an entailment system given by the preorder $\Gamma \vdash E$ on the sets of sentences defined by "there exists at least one proof" from Γ to E, which can also be read as 'Γ entails E'.

An important technical simplification which arises as a consequence of the fact that there exists at most one entailment between any sets of sentences is the fact that the vertical composition of entailments becomes total rather than partial.

Fact 13.1. *In any entailment system, for any signature Σ and sets E, E_1, E_2 of Σ-sentences,*

$$E \vdash E_1 \ \textit{and}\ E \vdash E_2 \ \textit{implies}\ E \vdash E_1 \cup E_2.$$

The semantic entailment system. In any institution, the semantic consequence relation between sets of sentences gives an example of an infinitary entailment system, which is called the *semantic proof system* or the *semantic entailment system* of the institution. This shows that proof systems are more abstract than institutions.

Summary of the chapter. In practice, proof systems are usually presented by 'systems of rules', which means that in fact they are 'freely generated' by these systems of rules. We introduce the concept of system of (proof) rules and develop an adjunction between these and proof systems, which gives the free proof systems mentioned above. We show that if all rules are finitary, then the resulting free proof system is compact.

In another section we approach internal logic concepts such as Boolean connectives and quantifiers from a proof theoretic perspective. We show that quantifiers can be added 'freely' to any proof system, which corresponds to the (meta-)rule of 'Generalization' of conventional concrete logic. This process preserves the compactness.

The next section discusses entailment and presents a general construction of a model theory on top of any proof system. The resulting institution comes equipped with a canonical system of elementary diagrams, yielding a good argument for the naturalness of the concept of elementary diagram.

Institutions and proof systems can be combined into the concept of 'institution with proofs' which constitutes a meta-theory for logical systems capturing both the model and the proof theoretic sides of logics. In this framework we develop a general soundness result for institutions with free proof systems, and a general Birkhoff proof theory together with a corresponding completeness result, which is applicable to general Horn institutions but quite surprisingly also to other types of 'universal' calculi. The general methodology used for Birkhoff completeness involves a 'layered' approach in which the proof calculus and the completeness results are developed according to the layered syntactical structure of the institution. This layered approach will be used in Sect. 14.2 to develop a general completeness result for structured specifications. It can also be used to develop other completeness results at a general institution-independent level.

13.1 Free Proof Systems

Systems of proof rules

The PL example. Readers familiar with conventional logic will recognize the following set of proof rules as the proof system of propositional logic **PL**.

$$
\begin{array}{ll}
(P1) & \emptyset \vdash \rho_1 \Rightarrow (\rho_2 \Rightarrow \rho_1) \\
(P2) & \emptyset \vdash (\rho_1 \Rightarrow (\rho_2 \Rightarrow \rho_3)) \Rightarrow ((\rho_1 \Rightarrow \rho_2) \Rightarrow (\rho_1 \Rightarrow \rho_3)) \\
(P3) & \emptyset \vdash (\neg\rho_1 \Rightarrow \neg\rho_2) \Rightarrow (\rho_2 \Rightarrow \rho_1) \\
(P4) & \emptyset \vdash (\rho_1 \Rightarrow \rho_2) \Rightarrow (\neg\rho_2 \Rightarrow \neg\rho_1) \\
(MP) & \{\rho_1, \rho_1 \Rightarrow \rho_2\} \vdash \rho_2
\end{array}
$$

This is a rather typical case of presentations of proof systems as a set of rules of the form $E \vdash E'$. These rules are the primitive proofs from which all proofs are generated by closure under the horizontal and vertical compositions of proofs. In general, this process can be explained as an adjunction between proof systems and 'systems of rules'.

Systems of rules. The following defines the general concept of system of rules. A *system of (proof) rules* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$ consists of

- a category of 'signatures' $\mathbb{S}ig$,
- a 'sentence functor' $\mathsf{Sen} : \mathbb{S}ig \to \mathbb{S}et$,
- a '(proof) rule functor' $\mathsf{Rl} : \mathbb{S}ig \to \mathbb{S}et$, and
- two natural transformations $h, c : \mathsf{Rl} \Rightarrow \mathsf{Sen}; \mathcal{P}$, where $\mathcal{P} : \mathbb{S}et \to \mathbb{S}et$ is the $\mathbb{S}et$-valued power-set functor.

Therefore, for each signature Σ, $\mathsf{Rl}(\Sigma)$ gives the set of the Σ-proof rules, $h_\Sigma : \mathsf{Rl}(\Sigma) \to \mathcal{P}(\mathsf{Sen}(\Sigma))$ gives the hypotheses of the rules, and $c_\Sigma : \mathsf{Rl}(\Sigma) \to \mathcal{P}(\mathsf{Sen}(\Sigma))$ gives the conclusions. A Σ-rule r can be therefore written as $h_\Sigma(r) \xrightarrow{r} c_\Sigma(r)$. The functoriality of Rl and the naturality of the hypotheses h and of the conclusions c, say that the translation of rules along signature morphisms is coherent with the translation of the sentences.

Sometimes, systems of rules may be defined as signature indexed families $\{rl(\Sigma)\}_{\Sigma \in |\mathbb{S}ig|}$ with $rl(\Sigma) \subseteq \mathcal{P}(\mathsf{Sen}(\Sigma)) \times \mathcal{P}(\mathsf{Sen}(\Sigma))$. Notice that this can be extended canonically to a proper system of rules by adding freely the translations of the rules by the signature morphisms.

Proof-theoretic morphisms and comorphisms

We can easily notice that each proof system can be seen as a system of rules by regarding each proof as a rule (the hypotheses being given by the domain of the proof, and the conclusions by the codomain). Forgetting from proof systems to systems of rules can be understood as a forgetful functor from the category of proof systems to the category of rule systems provided we organize proof systems and systems of rules as categories. For this we need to define mappings between proof systems, and systems of rules, respectively. This can be done naturally in the style we have previously defined morphisms and comorphisms for institutions.

Morphisms and comorphisms of proof systems. Let us consider first the case of comorphisms. A *proof system comorphism* between proof systems $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ and $(\mathbb{S}ig', \mathsf{Sen}', \mathsf{Pf}')$ consists of

- a 'signature' functor $\Phi: \mathbb{S}ig \to \mathbb{S}ig'$,
- a 'sentence translation' natural transformation $\alpha: \mathsf{Sen} \Rightarrow \Phi; \mathsf{Sen}'$, and
- a 'proof translation' natural transformation $\gamma: \mathsf{Pf} \Rightarrow \Phi; \mathsf{Pf}'$ such that translation of sets of sentences is compatible with translation of single sentences:

$$\begin{tikzcd}
\mathsf{Pf}(\Sigma) \arrow[r, "\gamma_\Sigma"] & \mathsf{Pf}'(\Phi(\Sigma)) \\
\mathsf{Sen}(\Sigma) \arrow[u, hook] \arrow[r, "\alpha_\Sigma"'] & \mathsf{Sen}'(\Phi(\Sigma)) \arrow[u, hook]
\end{tikzcd}$$

Proof systems morphisms are defined by analogy with institution morphisms by reversing the direction of the signature mapping (in the definition of the proof system comorphisms). Let $\mathbb{P}f\mathbb{S}ys$ denote the category of proof system morphisms, and $co\mathbb{P}f\mathbb{S}ys$ denote the category of proof system comorphisms.

Morphisms and comorphisms of systems of rules. A *comorphism of systems of (proof) rules* between systems of rules $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$ and $(\mathbb{S}ig', \mathsf{Sen}', \mathsf{Rl}', h', c')$ consists of

- a 'signature' functor $\Phi: \mathbb{S}ig \to \mathbb{S}ig'$,
- a 'sentence translation' natural transformation $\alpha: \mathsf{Sen} \Rightarrow \Phi; \mathsf{Sen}'$,
- a 'rule translation' natural transformation $\gamma: \mathsf{Rl} \Rightarrow \Phi; \mathsf{Rl}'$ which is compatible with the hypotheses and the conclusions, i.e., the diagram below commutes:

$$\begin{tikzcd}
\mathsf{Rl} \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, shift right, "h"'] \arrow[d, Rightarrow, shift left, "c"] & \Phi; \mathsf{Rl}' \arrow[d, Rightarrow, shift right, "h'"'] \arrow[d, Rightarrow, shift left, "c'"] \\
\mathsf{Sen}; \mathcal{P} \arrow[r, Rightarrow, "\alpha"'] & \Phi; \mathsf{Sen}'; \mathcal{P}
\end{tikzcd}$$

Morphisms of systems of rules are defined similarly by reversing the direction of the signature mapping. Let $\mathbb{R}l\mathbb{S}ys$ denote the category of proof rule system morphisms, and $co\mathbb{R}l\mathbb{S}ys$ denote the category of proof rule system comorphisms.

Fact 13.2. *There exist forgetful functors $\mathbb{P}f\mathbb{S}ys \to \mathbb{R}l\mathbb{S}ys$ and $co\mathbb{P}f\mathbb{S}ys \to co\mathbb{R}l\mathbb{S}ys$ mapping each proof system $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ to the system of rules $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, dom, cod)$ (i.e., the hypothesis of a Σ-proof is its domain and the conclusion is its codomain).*

Free proof systems

The free proof system construction is left adjoint to the forgetful functor $co\mathbb{P}f\mathbb{S}ys \to co\mathbb{R}l\mathbb{S}ys$ when working with comorphisms, and right adjoint to $\mathbb{P}f\mathbb{S}ys \to \mathbb{R}l\mathbb{S}ys$ when working with morphisms.

Theorem 13.3. *Each system of proof rules such that its sentence translations are injective generates freely a proof system.*

Proof. Let $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$ be a system of proof rules such that $\mathsf{Sen}(\varphi)$ is injective for each signature morphism $\varphi \in \mathbb{S}ig$. We fix a signature $\Sigma \in |\mathbb{S}ig|$ and define the single-sorted **PA** signature consisting of the following:

– total constants, all sets of sentences $E \subseteq \mathsf{Sen}(\Sigma)$, all sets of sentences inclusions $E \supseteq E'$, and all elements of $\mathsf{Rl}(\Sigma)$,

– unary total operation symbols, h and c, and

– binary partial operation symbols, $\_;\_$ and $\langle \_, \_ \rangle$.

Let $E \xrightarrow{p} \Gamma$ abbreviate $(h(p) \stackrel{e}{=} E) \wedge (c(p) \stackrel{e}{=} \Gamma)$.

We consider the initial partial algebra PT^{Σ} of the following set of quasi-existence equations:

(R^{Σ})	$h_{\Sigma}(r) \xrightarrow{r} c_{\Sigma}(r)$ for all $r \in \mathsf{Rl}(\Sigma)$
(S^{Σ})	$E \xrightarrow{E} E$ for all $E \subseteq \mathsf{Sen}(\Sigma)$
$(M1^{\Sigma})$	$(E \supseteq E) \stackrel{e}{=} E$ for all $E \subseteq \mathsf{Sen}(\Sigma)$
$(M2^{\Sigma})$	$E \xrightarrow{E \supseteq E'} E'$ for all $E' \subseteq E \subseteq \mathsf{Sen}(\Sigma)$
$(M3^{\Sigma})$	$(E \supseteq E');(E' \supseteq E'') \stackrel{e}{=} (E \supseteq E'')$ for all $E'' \subseteq E' \subseteq E \subseteq \mathsf{Sen}(\Sigma)$
$(C1^{\Sigma})$	$(\forall p, p')(E \xrightarrow{p} E') \wedge (E' \xrightarrow{p'} E'') \Rightarrow (E \xrightarrow{p;p'} E'')$ for all $E, E', E'' \subseteq \mathsf{Sen}(\Sigma)$

$(C2^{\Sigma})$ $(\forall p,p',p'')(E \xrightarrow{p} E') \wedge (E' \xrightarrow{p'} E'') \wedge (E'' \xrightarrow{p''} E''') \Rightarrow p;(p';p'') \stackrel{e}{=} (p;p');p''$
for all $E,E',E'',E''' \subseteq \mathsf{Sen}(\Sigma)$

$(C3^{\Sigma})$ $(\forall p)E \xrightarrow{p} E' \Rightarrow (E;p \stackrel{e}{=} p) \wedge (p;E' \stackrel{e}{=} p)$
for all $E,E' \subseteq \mathsf{Sen}(\Sigma)$

$(P1^{\Sigma})$ $(\forall p,p')(E \xrightarrow{p} \Gamma) \wedge (E \xrightarrow{p'} \Gamma') \Rightarrow$
$\Rightarrow (E \xrightarrow{\langle p,p'\rangle} \Gamma \cup \Gamma') \wedge (\langle p,p'\rangle;(\Gamma \cup \Gamma' \supseteq \Gamma) \stackrel{e}{=} p) \wedge (\langle p,p'\rangle;(\Gamma \cup \Gamma' \supseteq \Gamma') \stackrel{e}{=} p')$
for all $E,\Gamma,\Gamma' \subseteq \mathsf{Sen}(\Sigma)$ with $\Gamma \cap \Gamma' = \emptyset$

$(P2^{\Sigma})$ $(\forall p,p')(E \xrightarrow{p} \Gamma \cup \Gamma') \wedge (E \xrightarrow{p'} \Gamma \cup \Gamma') \wedge$
$\wedge (p;(\Gamma \cup \Gamma' \supseteq \Gamma) \stackrel{e}{=} p';(\Gamma \cup \Gamma' \supseteq \Gamma)) \wedge (p;(\Gamma \cup \Gamma' \supseteq \Gamma') \stackrel{e}{=} p';(\Gamma \cup \Gamma' \supseteq \Gamma')) \Rightarrow$
$p \stackrel{e}{=} p'$ for all $E,\Gamma,\Gamma' \subseteq \mathsf{Sen}(\Sigma)$ with $\Gamma \cap \Gamma' = \emptyset$

The category $\mathsf{Pf}(\Sigma)$ of the Σ-proofs is defined by $|\mathsf{Pf}(\Sigma)| = \mathcal{P}(\mathsf{Sen}(\Sigma))$ and $\mathsf{Pf}(\Sigma)(\Gamma,E) = \{p \in PT^{\Sigma} \mid PT^{\Sigma}_{h}(p) = \Gamma, PT^{\Sigma}_{c}(p) = E\}$. The composition of proofs is given by $p;p' = p(PT^{\Sigma}_{\_;\_})p'$ and the monotonicity proofs $\supseteq_{\Gamma,E} :\ \Gamma \to E$ are defined as $PT^{\Sigma}_{\Gamma \supseteq E}$. Notice also that $(PT^{\Sigma})_E = E$. By the last equations above, $\mathcal{P}(\mathsf{Sen}(\Sigma))^{\mathrm{op}} \hookrightarrow \mathsf{Pf}(\Sigma)$ preserves products as each $\Gamma \supseteq E$ gets mapped to $\supseteq_{\Gamma,E}$.

Any signature morphism $\varphi :\ \Sigma \to \Sigma'$ induces a morphism $\overline{\varphi}$ between the theories corresponding to Σ and Σ'. Notice that $\overline{\varphi}$ maps $P1^{\Sigma}$ to $P1^{\Sigma'}$ and $P2^{\Sigma}$ to $P2^{\Sigma'}$ because $\mathsf{Sen}(\varphi)$ is injective. Then we define the functor $\mathsf{Pf}(\varphi)$ as the unique partial algebra homomorphism $PT^{\Sigma} \to PT^{\Sigma'}\restriction_{\overline{\varphi}}$. We have therefore defined a proof system $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$, which we will show that it is the free proof system over $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$.

For each signature Σ, let $\eta_{\Sigma} :\ \mathsf{Rl}(\Sigma) \to \mathsf{Pf}(\Sigma)$ map any Σ-rule to its congruence class. We show that the comorphism

$$(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta) :\ (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, dom, cod)$$

is universal.

$$\begin{tikzcd}
(\mathbb{S}ig,\mathsf{Sen},\mathsf{Rl},h,c) \arrow[rr, "(1_{\mathbb{S}ig},1_{\mathsf{Sen}},\eta)"] \arrow[dr, "(\Phi,\alpha,\gamma)"'] & & (\mathbb{S}ig,\mathsf{Sen},\mathsf{Pf},dom,cod) \arrow[dl, "(\Phi,\alpha,\gamma')"] \\
& (\mathbb{S}ig',\mathsf{Sen}',\mathsf{Pf}',dom,cod) &
\end{tikzcd}$$

For each comorphism $(\Phi,\alpha,\gamma) :\ (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Pf}', dom, cod)$, each signature $\Sigma \in |\mathbb{S}ig|$ determines a partial algebra A of the theory of quasi-existence equations defining PT^{Σ} by letting its carrier be $\mathsf{Pf}'(\Phi(\Sigma))$, $A_E = \alpha_{\Sigma}(E)$ for each set E of Σ-sentences, $A_r = \gamma_{\Sigma}(r)$ for each Σ-rule r, A_h, A_c, and $A_{\_;\_}$ respectively, are the canonical extensions of $dom_{\Phi(\Sigma)}$, $cod_{\Phi(\Sigma)}$, and of the composition in $\mathsf{Pf}'(\Phi(\Sigma))$ respectively. Finally, by the universal property of products, we define $A_{\langle\_,\_\rangle}(p_1,p_2)$ to be the unique proof q such that $q;(cod(p_1) \cup cod(p_2) \supseteq cod(p_i)) = p_i$.

Then $\gamma'_{\Sigma} :\ \mathsf{Pf}(\Sigma) \to \mathsf{Pf}'(\Phi(\Sigma))$ is given by the unique algebra homomorphism $PT^{\Sigma} \to A$. □

For the actual systems of rules, the injectivity of the sentence translations comes as a consequence of the injectivity of the signature morphisms. For example this can be noticed easily in the case of **FOL**. Therefore, we cannot have a proof system for **FOL** freely generated from the rules unless we consider its sub-institution determined by all injective signature morphisms.

Free infinitary proof systems can be obtained by an infinitary version of Thm. 13.3. This requires an extension of partial algebras with infinitary operations for dealing with the infinitary vertical compositions of proofs.

Free entailment systems. Thm. 13.3 and its proof can be downgraded to a theorem on existence of free entailment systems. Of course this requires downgrading also the system of proof rules to a concept of *system of entailment rules* consisting for each signature Σ of a binary relation $\vdash_\Sigma \subseteq \mathcal{P}(\mathsf{Sen}(\Sigma)) \times \mathcal{P}(\mathsf{Sen}(\Sigma))$ between sets of Σ-sentences such that for any signature morphism $\varphi : \Sigma \to \Sigma'$, if $E \vdash_\Sigma E'$ then $\varphi(E) \vdash_{\Sigma'} \varphi(E')$. In that situation the condition of the injectivity of the sentence translation is not needed because this is used only for translations of families of equations $(P1^\Sigma)$ and $(P2^\Sigma)$ which in the simplified setting of entailment systems may be replaced by just one family of equations $(E \vdash \Gamma) \wedge (E \vdash \Gamma') \Rightarrow (E \vdash \Gamma \cup \Gamma')$ for all $E, \Gamma, \Gamma' \subseteq \mathsf{Sen}(\Sigma)$ (thus without the condition $\Gamma \cap \Gamma' = \emptyset$). Hence we can formulate the following:

Corollary 13.4. *Each system of entailment rules generates freely an entailment system.*

Soundness of institutions with free proof systems

Institutions with proofs. Institutions can be enhanced with proof systems as follows. An *institution with proofs* is a tuple $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ such that

- $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is an institution, and
- $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ is a proof system.

The fundamental coherence relationship between the model theory and the proof theory of any institution with proofs is that of *soundness*. An institution with proofs is *sound* when for each proof $E \to E'$ we have that $E \models E'$.

Institutions with (proof) rules. An *institution with proof rules* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Rl}, h, c)$ combines an institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ with a system of rules $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$. Likewise for institutions with proofs, an institution with rules is *sound* when for each rule $r \in \mathsf{Rl}(\Sigma)$, $h_\Sigma(r) \models c_\Sigma(r)$.

The result below shows that the free construction of proof systems from systems of rules preserves soundness and explains the actual practice of establishing soundness of institutions with proofs that consist only of checking the soundness of the rules.

Proposition 13.5. *The institution with proofs* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ *such that the proof system* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ *is freely generated by a system of rules* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$ *is sound whenever the institution with rules* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Rl}, h, c)$ *is sound.*

Proof. Because $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \mathsf{Rl}, h, c)$ is sound we consider the canonical comorphism of systems of proof rules $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ig, \mathsf{Sen}, \models, dom, cod)$ to the institution with semantic proofs that maps any rule $E \xrightarrow{r} E'$ to the semantic proof $E \models E'$.

$$\begin{tikzcd}
(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c) \arrow[rr, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta)"] \arrow[dr, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma)"'] & & (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \arrow[dl, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma')"] \\
& (\mathbb{S}ig, \mathsf{Sen}, \models) &
\end{tikzcd}$$

By the universal property of the free proof system $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$, $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma)$ can be extended to a comorphism of proof systems $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma') : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ig, \mathsf{Sen}, \models)$. But the existence of a comorphism $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma') : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ig, \mathsf{Sen}, \models)$ is equivalent to the soundness of $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$. $\square$

Completeness. This is the opposite property to soundness. Informally, it says that for each semantic deduction there exists at least one (syntactic) proof. Usually it is much more difficult to establish completeness properties than soundness properties.

An institution with proofs $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ is *complete* when

$$E \models_\Sigma \Gamma \ \text{ implies } \ E \vdash_\Sigma \Gamma$$

for all sets $E, \Gamma \subseteq \mathsf{Sen}(\Sigma)$ with Γ finite. An institution with proof rules is complete if and only if the corresponding institution with proofs freely generated by the system of proof rules is complete.

Exercises

13.1. Define $\mathbb{P}f\mathbb{S}ys$, $\mathbb{R}l\mathbb{S}ys$, $co\mathbb{P}f\mathbb{S}ys$ and $co\mathbb{R}l\mathbb{S}ys$ as Grothendieck categories in the style of Fact 3.8. Consequently, establish the completeness properties for $co\mathbb{P}f\mathbb{S}ys$ and $co\mathbb{R}l\mathbb{S}ys$.

13.2. The category $\mathbb{P}f\mathbb{I}ns$ of institutions with proofs is the pullback of the category $\mathbb{I}ns$ of institutions and the category $\mathbb{P}f\mathbb{S}ys$ of proof systems over the Grothendieck category $\mathbb{C}at((-)^{\mathrm{op}}, \mathbb{S}et^{\mathrm{op}})^\sharp$ of the functor $\mathbb{C}at((-)^{\mathrm{op}}, \mathbb{S}et^{\mathrm{op}}) : \mathbb{C}at^{\mathrm{op}} \to \mathbb{C}at$.

$$\begin{tikzcd}
\mathbb{I}ns \arrow[r] & \mathbb{C}at((-)^{\mathrm{op}}, \mathbb{S}et^{\mathrm{op}})^\sharp \\
\mathbb{P}f\mathbb{I}ns \arrow[u] \arrow[r] & \mathbb{P}f\mathbb{S}ys \arrow[u]
\end{tikzcd}$$

13.3. The category $\mathbb{P}f\mathbb{I}ns$ of institutions with proofs admits Grothendieck objects. This gives the construction for Grothendieck institutions with proofs; describe them directly.

13.4. The rules $(P1-4)$ and (MP) which generate the proof system of **PL** are sound in any institution with semantic implications and negations.

13.2 Compactness

The proof theoretic concept of compactness is significantly more refined than its model theoretic counterpart.

Finitary proofs. A proof $E \xrightarrow{p} E'$ is *finitary* when both E and E' are finite. Similarly, a (proof) rule r is finitary when both the hypothesis $h_\Sigma(r)$ and the conclusion $c_\Sigma(r)$ are finite for each signature Σ.

Compact proofs. A proof $E \xrightarrow{p} E'$ is *compact* when it can be represented as $p = \langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle$ with q finitary.

$$\begin{tikzcd}
E_1 & E_1 \uplus E'_2 \arrow[l] \arrow[r] & E'_2 \\
 & E \arrow[ul] \arrow[u, "p=\langle \supseteq_{E,E_1}, \supseteq_{E,E_2}; q\rangle"] \arrow[r] & E_2 \arrow[u, "q"']
\end{tikzcd}$$

This means that the conclusion of p can be split as $E_1 \uplus E'_2$ such that $E_1 \subseteq E$ and there exists a finitary proof q of E'_2 from a subset $E_2 \subseteq E$ such that q constitutes the non-trivial part of the proof p.

Proposition 13.6. *Any compact proof $E \to E'$ with E' finite can be written as a composition between a monotonicity proof and a finitary proof.*

Proof. Consider a compact proof $p : E \to E'$ such that E' is finite. If we represent it as $p = \langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle$ with q finitary, then by using the uniqueness aspect of the product property of $E_1 \uplus E'_2$ we have that $p = \supseteq_{E,E_1 \cup E'_2}; p_0$ where p_0 is the unique proof such that $p; \supseteq_{E',E_1} = \supseteq_{E_1 \cup E_2, E_1}$ and $p_0; \supseteq_{E',E'_2} = \supseteq_{E_1 \cup E_2, E_2}; q$.

$$\begin{tikzcd}
E_1 & E_1 \uplus E'_2 \arrow[l] \arrow[r] & E'_2 \\
 & E_1 \cup E_2 \arrow[ul] \arrow[u, "p_0"'] \arrow[r] & E_2 \arrow[u, "q"'] \\
 & E \arrow[uul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Because E' is finite we have that E_1 is finite, hence p_0 is finitary. $\square$

A proof system is *compact* when each of its proofs is compact. Because of the trivial nature of monotonicity proofs, one can see that in any compact proof system any proof of a finite set of sentences *is* finitary in essence, which is stronger than saying that any provable sentence *admits* a finitary proof. In other words, the compactness of the proof system is a stronger property than the compactness of its corresponding entailment system.

The sub-system of compact proofs. Compact proofs have good compositional properties as shown by the result below.

Proposition 13.7. *For any proof system* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$*, the collection of its compact proofs form a (proof) sub-system, denoted by* $(\mathbb{S}ig, \mathsf{Sen}, C(\mathsf{Pf}))$.

Proof. Note that all monotonicity proofs are trivially compact. We therefore have to show that compact proofs form a sub-category of all proofs, that this sub-category creates (binary) products of disjoint sets of sentences, and that translations along signature morphisms preserve compactness.

1. *Subcategory.* Notice that each identity proof is trivially compact. Consider proofs

– $\langle \supseteq_{\Gamma,E_1}, \supseteq_{\Gamma,E_2'}; q\rangle : \ \Gamma \to E_1 \uplus E_2$ with $E_2' \xrightarrow{q} E_2$ finitary and

– $\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma_2'}; r\rangle : \ E \to \Gamma = \Gamma_1 \uplus \Gamma_2$ with $\Gamma_2' \xrightarrow{r} \Gamma_2$ finitary.

We have to show that their composition is compact.

$$\begin{tikzcd}
 & \Gamma_1 & & E_1 \\
E \arrow[ur] \arrow[r, "\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma_2'}; r\rangle"'] \arrow[d] & \Gamma_1 \uplus \Gamma_2 \arrow[u] \arrow[d] \arrow[r, equal] & \Gamma \arrow[ur] \arrow[r, "\langle \supseteq_{\Gamma,E_1}, \supseteq_{\Gamma,E_2'}; q\rangle"'] \arrow[d] & E_1 \uplus E_2 \arrow[u] \arrow[d] \\
\Gamma_2' \arrow[r, "r"'] & \Gamma_2 \arrow[r, dotted] & E_2' \arrow[r, "q"'] & E_2
\end{tikzcd}$$

We know that $E_2' \subseteq \Gamma = \Gamma_1 \uplus \Gamma_2$. Without any loss of generality we may assume that $E_2' \subseteq \Gamma_2$ since if $E_2' \not\subseteq \Gamma_2$ then because

$$\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma_2'}; r\rangle = \langle \supseteq_{E,\Gamma_1 \setminus E_2'}, \supseteq_{E,\Gamma_2' \cup (\Gamma_1 \cap E_2')}; \langle \supseteq_{\Gamma_2' \cup (\Gamma_1 \cap E_2'),\Gamma_2'}; r, \supseteq_{\Gamma_2' \cup (\Gamma_1 \cap E_2'),\Gamma_1 \cap E_2'}\rangle\rangle$$

we may replace in the original problem

– Γ_2 by $E_2' \cup \Gamma_2$,

– Γ_1 by $\Gamma_1 \setminus E_2'$,

– Γ_2' by $\Gamma_2' \cup (\Gamma_1 \cap E_2')$, and

– r by $\langle \supseteq_{\Gamma_2' \cup (\Gamma_1 \cap E_2'),\Gamma_2'}; r, \supseteq_{\Gamma_2' \cup (\Gamma_1 \cap E_2'),\Gamma_1 \cap E_2'}\rangle$.

The above mentioned equality can be established by projecting both its left-hand side and its right-hand side to $\Gamma_1 \setminus E_2'$, $\Gamma_1 \cap E_2'$ and to Γ_2 and by using the uniqueness aspect of the product $\Gamma_1 \setminus E_2' \uplus \Gamma_1 \cap E_2' \uplus \Gamma_2$ in the category of proofs. The diagram below may help to visualize this.

$$\begin{tikzcd}
 & \Gamma_1 \cap E'_2 & & \\
\Gamma_1 \arrow[ur] \arrow[d] & \Gamma_1 \uplus \Gamma_2 = \Gamma = (\Gamma_1 \setminus E'_2) \uplus (E'_2 \cup \Gamma_2) \arrow[l] \arrow[u] \arrow[r] \arrow[dl] & (E'_2 \cup \Gamma_2) = (\Gamma_1 \cap E'_2) \uplus \Gamma_2 \arrow[ul] \arrow[r] & \Gamma_2 \\
(\Gamma_1 \setminus E'_2) & E \arrow[l] \arrow[u, "x"] \arrow[ul] \arrow[r] & (\Gamma_1 \cap E'_2) \cup \Gamma'_2 \arrow[u] \arrow[uul] \arrow[r] & \Gamma'_2 \arrow[u, "r"]
\end{tikzcd}$$

Then, by assuming that $E'_2 \subseteq \Gamma_2$ we have that

$$\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle; \langle \supseteq_{\Gamma,E_1}, \supseteq_{\Gamma,E'_2}; q\rangle = \langle \supseteq_{E,E_1\cap\Gamma_1}, \supseteq_{E,\Gamma'_2}; r; \langle \supseteq_{\Gamma_2,E'_2}; q, \supseteq_{\Gamma_2,E_1\cap\Gamma_2}\rangle\rangle.$$

This equality can be established noting that $E_1 = (E_1 \cap \Gamma_1) \cap E_1 \cap \Gamma_2)$ and then by projecting both its sides to E_2, $E_1 \cap \Gamma_1$ and to $E_1 \cap \Gamma_2$ and by using the uniqueness aspect of the universal property of the product $E_2 \uplus (E_1 \cap \Gamma_1) \uplus (E_1 \cap \Gamma_2)$. The diagram below helps to visualize this process.

$$\begin{tikzcd}
E_1 \arrow[d] & (E_1 \cap \Gamma_1) \uplus E_2 \uplus (E_1 \cap \Gamma_2) \arrow[l] \arrow[rr] \arrow[dl] \arrow[dr] & & E_2 \\
E_1 \cap \Gamma_1 & & E_1 \cap \Gamma_2 & E'_2 \arrow[u, "q"] \\
\Gamma_1 \arrow[u] & E \arrow[l] \arrow[uu, "y"] \arrow[r] & \Gamma'_2 \arrow[r, "r"] & \Gamma_2 \arrow[ul] \arrow[u]
\end{tikzcd}$$

Now all we have to do is to note that the right-hand side of the above equality is a compact proof.

2. *Direct products of disjoint sets.* Assume compact proofs $\langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle$ and $\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle$ such that q and r are finitary and such that $(E_1 \uplus E_2) \cap (\Gamma_1 \uplus \Gamma_2) = \emptyset$. The fact that

$$\langle\langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle, \langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle\rangle : E \to E_1 \uplus E_2 \uplus \Gamma_1 \uplus \Gamma_2$$

is compact too follows immediately from the equality

$$\begin{aligned}&\langle\langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle, \langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle\rangle = \\ &\langle \supseteq_{E,E_1\uplus\Gamma_1}, \supseteq_{E,E'_2\cup\Gamma'_2}; \langle \supseteq_{E'_2\cup\Gamma'_2,E'_2}; q, \supseteq_{E'_2\cup\Gamma'_2,\Gamma'_2}; r\rangle\rangle\end{aligned}$$

which can be established by projecting both sides to the components of the product $E_1 \uplus E_2 \uplus \Gamma_1 \uplus \Gamma_2$ in the category of proofs.

3. *Translations of proofs along signature morphisms preserve compactness.* In other words we have to prove that for any signature morphism $\varphi\colon \Sigma \to \Sigma'$ we have that $\varphi(C(\mathsf{Pf}(\Sigma))) \subseteq C(\mathsf{Pf}(\Sigma'))$. Let $p = \langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle$ be a compact proof such that q is finitary. That $\varphi(p)$ is compact too follows by the equality

$$\varphi(p) = \langle \supseteq_{\varphi(E),\varphi(E_1)\setminus\varphi(E_2)}, \supseteq_{\varphi(E),\varphi(E'_2)}; \varphi(q)\rangle$$

which is established by noting that $\varphi(E_1 \uplus E_2) = \varphi(E_2) \uplus (\varphi(E_1) \setminus \varphi(E_2))$ and by the uniqueness aspect of the universal property of the product $\varphi(E_1 \uplus E_2) = \varphi(E_2) \uplus (\varphi(E_1) \setminus \varphi(E_2))$ in $\mathsf{Pf}(\Sigma')$. This is based on the commutativity of the diagram

$$\begin{tikzcd}
\varphi(E_1) \setminus \varphi(E_2) & \varphi(E_1 \uplus E_2) = \varphi(E_2) \uplus (\varphi(E_1) \setminus \varphi(E_2)) \arrow[l] \arrow[r] & \varphi(E_2) \\
& \varphi(E) \arrow[ul] \arrow[u, "\varphi(p)"'] \arrow[r] & \varphi(E_2') \arrow[u, "\varphi(q)"']
\end{tikzcd}$$

While the square in the right-hand side of the diagram commutes by applying the functor $\mathsf{Pf}(\varphi)$ to the corresponding commutative square of Σ-proofs, the commutativity of the triangle in the left-hand side of the diagram can be established by the following calculations:

$$\begin{aligned}
\varphi(p);\supseteq_{\varphi(E_1 \uplus E_2),\varphi(E_1)\setminus\varphi(E_2)} &= \varphi(p);\supseteq_{\varphi(E_1 \uplus E_2),\varphi(E_1)};\supseteq_{\varphi(E_1),\varphi(E_1)\setminus\varphi(E_2)} \\
&= \varphi(p);\varphi(\supseteq_{E_1 \uplus E_2,E_1});\supseteq_{\varphi(E_1),\varphi(E_1)\setminus\varphi(E_2)} \\
&= \varphi(p;\supseteq_{E_1 \uplus E_2,E_1});\supseteq_{\varphi(E_1),\varphi(E_1)\setminus\varphi(E_2)} \\
&= \varphi(\supseteq_{E,E_1});\supseteq_{\varphi(E_1),\varphi(E_1)\setminus\varphi(E_2)} \\
&= \supseteq_{\varphi(E),\varphi(E_1)\setminus\varphi(E_2)}.
\end{aligned}$$

□

Compactness of free proof systems. The result below is a corollary of Prop. 13.7 and of the universal property of free proof systems.

Corollary 13.8. *The proof system freely generated by a system of finitary rules is compact.*

Proof. Consider a proof system $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ generated freely by a system of finitary proof rules $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$, with $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta)$ universal arrow.

By Prop. 13.7 let $(\mathbb{S}ig, \mathsf{Sen}, C(\mathsf{Pf}))$ be the compact proof (sub-)system of $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$. Because each proof rule of $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c)$ is finitary, it means that $\eta_\Sigma(\mathsf{Rl}(\Sigma)) \subseteq C(\mathsf{Pf})(\Sigma)$ for each signature Σ, hence $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta)$ is a comorphism of systems of proof rules $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ig, \mathsf{Sen}, C(\mathsf{Pf}), dom, cod)$.

By the universal property of $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta) \colon (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, dom, cod)$ there exists a unique comorphism of proof systems $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma) \colon (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ig, \mathsf{Sen}, C(\mathsf{Pf}))$ such that the triangle below commutes:

$$\begin{tikzcd}
(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Rl}, h, c) \arrow[r, "{(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta)}"] \arrow[dr, "{(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta)}"'] & (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, dom, cod) \arrow[d, shift right, "{(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma)}"'] \\
& (\mathbb{S}ig, \mathsf{Sen}, C(\mathsf{Pf}), dom, cod) \arrow[u, shift right, "{(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma')}"']
\end{tikzcd}$$

Let $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma')$ be the sub-system comorphism $(\mathbb{S}ig, \mathsf{Sen}, C(\mathsf{Pf})) \to (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$, which makes the above triangle commute. By the uniqueness part of the universal property for the free proof system, we get that $\gamma;\gamma' = 1$, and because γ' are inclusions, we obtain that $C(\mathsf{Pf}) = \mathsf{Pf}$, which means that each proof of $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ is compact. $\square$

13.3 Proof-theoretic Internal Logic

In Chap. 5 we introduced an institution-independent semantics for Boolean connectives and quantifiers. Here we extend this to proof-theoretic Boolean connectives and quantifiers. These proof-theoretic concepts can be seen as extensions of their semantic counterparts when considering the semantic proof system of the institution which regards the semantic consequence relations between sets of sentences as proofs.

Boolean connectives

Conjunctions. A proof-system has *(proof-theoretic) conjunction*, if each proof category $\mathsf{Pf}(\Sigma)$ has distinguished products of singletons, which are singletons again and which are preserved by the proof translations $\mathsf{Pf}(\varphi)$.

This means that for any Σ-sentences ρ_1, ρ_2, there exists a product sentence $\rho_1 \wedge \rho_2$, and two 'projection' proofs $\rho_1 \wedge \rho_2 \xrightarrow{p_i} \rho_i$, such that for any proofs $E \xrightarrow{q_i} \rho_i$, there exists a unique proof $E \xrightarrow{q} \rho_1 \wedge \rho_2$ such that $q; p_i = q_i$.

$$\begin{tikzcd}
\{\rho_1\} & \{\rho_1 \wedge \rho_2\} \arrow[l, "p_1"'] \arrow[r, "p_2"] & \{\rho_2\} \\
& E \arrow[ul, "q_1"] \arrow[u, "q"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

Fact 13.9. *An institution has semantic conjunctions if and only if its semantic proof system has proof-theoretic conjunctions.*

Disjunctions, true, false. As expected, proof-theoretic disjunctions are dual to the conjunctions, disjunctions being co-products in the category of proofs. This holds for the situation when the proof system is finitary. When it is finitary the co-products are considered in the full subcategory of the *finite* sets of sentences.

The Boolean constants true and false, respectively, are modeled proof-theoretically as distinguished terminal and initial objects, respectively. These are required to be preserved by the sentence translations along signature morphisms. More precisely, for each signature Σ we have that true_Σ and false_Σ, respectively, are terminal and initial, respectively, in $\mathsf{Pf}(\Sigma)$, an for each signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ we have that $\varphi(\mathsf{true}_\Sigma) = \mathsf{true}_{\Sigma'}$ and $\varphi(\mathsf{false}_\Sigma) = \mathsf{false}_{\Sigma'}$.

Negations. Any proof system with false has *proof-theoretic negation*, if each sentence ρ has a distinguished 'negation' $\neg\rho$ which is preserved by the proof translations $\mathsf{Pf}(\varphi)$ and such that $\mathsf{Pf}(\Sigma)(\Gamma \cup \{\rho\}, \mathsf{false})$ is in natural bijective correspondence to $\mathsf{Pf}(\Sigma)(\Gamma, \{\neg\rho\})$.

Fact 13.10. *An institution with* false *has semantic negation if and only if its semantic proof system has proof-theoretic negation.*

Corollary 13.11. *In any proof system with negations, for each sentence* ρ *there exists a canonical proof* $\rho \to \neg\neg\rho$.

Proof. For each sentence ρ the canonical proof $\rho \to \neg\neg\rho$ is the correspondent of the identity proof $1_{\neg\rho}$ by the natural bijections

$$\mathsf{Pf}(\Sigma)(\neg\rho, \neg\rho,) \cong \mathsf{Pf}(\Sigma)(\{\rho, \neg\rho\}, \mathsf{false}) \cong \mathsf{Pf}(\Sigma)(\rho, \neg\neg\rho).$$

□

When each of the canonical proofs $\rho \to \neg\neg\rho$ is an isomorphism, we say that the proof system has $\neg\neg$*-elimination*.

Fact 13.12. *For any institution with semantic negations, its semantic entailment system has* $\neg\neg$*-elimination.*

Implications. The concept of proof-theoretic implication requires some preparation.

For each set Γ of Σ-sentences, there is a canonical homomorphism of graphs $\_\cup\Gamma\colon\ \mathsf{Pf}(\Sigma) \to \mathsf{Pf}(\Sigma)$ as defined by the following commutative diagram of proofs:

$$\begin{tikzcd}
 & \Gamma \setminus E' \\
E \cup \Gamma \arrow[r, "p\cup\Gamma"'] \arrow[ur] \arrow[d] & E' \cup \Gamma = E' \uplus (\Gamma \setminus E') \arrow[u] \arrow[d] \\
E \arrow[r, "p"] & E'
\end{tikzcd}$$

(where the monotonicity proofs are not labeled). The main point here is that in general $- \cup \Gamma$ does not preserve compositions, hence it is a graph homomorphism rather than a functor. However we have the following.

Proposition 13.13. *The union graph homomorphism* $- \cup \Gamma$ *is preserved by all signature morphisms, i.e., for each signature morphism* $\varphi\colon\ \Sigma \to \Sigma'$ *the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{Pf}(\Sigma) \arrow[r, "-\cup\Gamma"] \arrow[d, "\mathsf{Pf}(\varphi)"'] & \mathsf{Pf}(\Sigma) \arrow[d, "\mathsf{Pf}(\varphi)"] \\
\mathsf{Pf}(\Sigma') \arrow[r, "-\cup\varphi(\Gamma)"'] & \mathsf{Pf}(\Sigma')
\end{tikzcd}$$

Proof. This property can be deduced immediately from the commutativity of the diagram:

$$\begin{tikzcd}
 & \varphi(\Gamma)\setminus\varphi(E') \\
 & \varphi(\Gamma\setminus E') \arrow[u] \\
\varphi(E)\cup\varphi(\Gamma) \arrow[r, "\varphi(p\cup\Gamma)=", "\varphi(p)\cup\varphi(\rho)"'] \arrow[d] \arrow[ruu] & \varphi(E')\cup\varphi(\Gamma) = \varphi(E')\uplus(\varphi(\Gamma)\setminus\varphi(E')) \arrow[u] \arrow[d] \\
\varphi(E) \arrow[r, "\varphi(p)"'] & \varphi(E')
\end{tikzcd}$$

by noting that $\varphi(\Gamma)\setminus\varphi(E') \subseteq \varphi(\Gamma\setminus E')$ and by using the uniqueness part of the universal property of the product $\varphi(E')\uplus(\varphi(\Gamma)\setminus\varphi(E'))$ in the category of proofs. $\square$

A proof system has *(proof-theoretic) implication*, if for each Σ-sentence ρ the graph homomorphism $-\cup\{\rho\} \colon \mathsf{Pf}(\Sigma) \to \mathsf{Pf}(\Sigma)$ has a distinguished 'right adjoint' denoted by $\rho \Rightarrow (-)$,

– which maps singletons to singletons,

– $(\rho \Rightarrow E) = \{\rho \Rightarrow e \mid e \in E\}$, and

– such that it commutes with the proof translations.

The 'right adjoint' property means that there exists a bijective correspondence

$$\mathsf{Pf}(\Sigma)(\Gamma\cup\{\rho\}, E) \cong \mathsf{Pf}(\Sigma)(\Gamma, \rho \Rightarrow E)$$

natural in Γ, E and Σ, known in conventional logic as the *Deduction Theorem*. The naturality in Σ means the commutativity of the diagrams below for each signature morphism $\varphi\colon \Sigma \to \Sigma'$:

$$\begin{tikzcd}
\mathsf{Pf}(\Sigma) \arrow[r, "\rho\Rightarrow_"] \arrow[d, "\mathsf{Pf}(\varphi)"] & \mathsf{Pf}(\Sigma) \arrow[d, "\mathsf{Pf}(\varphi)"] \\
\mathsf{Pf}(\Sigma') \arrow[r, "\varphi(\rho)\Rightarrow_"'] & \mathsf{Pf}(\Sigma')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Pf}(\Sigma)(\Gamma\cup\{\rho\}, E) \arrow[r, "\cong"] \arrow[d, "\mathsf{Pf}(\varphi)"'] & \mathsf{Pf}(\Sigma)(\Gamma, \rho \Rightarrow E) \arrow[d, "\mathsf{Pf}(\varphi)"] \\
\mathsf{Pf}(\Sigma')(\varphi(\Gamma)\cup\{\varphi(\rho)\}, \varphi(E)) \arrow[r, "\cong"] & \mathsf{Pf}(\Sigma')(\varphi(\Gamma), \varphi(\rho) \Rightarrow \varphi(E))
\end{tikzcd}$$

Fact 13.14. *An institution has semantic implications if and only if its semantic proof system has (proof-theoretic) implications.*

It is easy to see that proof theoretic negation can be defined from proof theoretic implication and false.

Fact 13.15. *Any proof system with implications and* false *has negations defined by* $\neg\rho = (\rho \Rightarrow \mathsf{false})$ *for each sentence* ρ.

Quantifiers

For any class $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms that is stable under pushouts, the proof system *has proof-theoretic universal (existential) $\mathcal{D}$-quantification*, if for all signature morphisms $\varphi \in \mathcal{D}$, $\mathsf{Pf}(\varphi)$ have distinguished right (left) adjoints, denoted by $(\forall\varphi)-$ ($(\exists\varphi)-$), and which are preserved by proof translations along signature morphisms. This means that there exists a bijective correspondence

$$\mathsf{Pf}(\Sigma)(E,(\forall\varphi)E') \cong \mathsf{Pf}(\Sigma')(\varphi(E),E')$$

natural in E and E', and such that for each signature pushout with $\varphi \in \mathcal{D}$,

$$\begin{tikzcd} \Sigma \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \Sigma_1 \arrow[d, "\varphi_1"] \\ \Sigma' \arrow[r, "\theta'"'] & \Sigma'_1 \end{tikzcd}$$

both squares below commute:

$$\begin{tikzcd} \mathsf{Pf}(\Sigma) \arrow[r, "\mathsf{Pf}(\theta)"] & \mathsf{Pf}(\Sigma_1) \\ \mathsf{Pf}(\Sigma') \arrow[u, "(\forall\varphi)"] \arrow[r, "\mathsf{Pf}(\theta')"'] & \mathsf{Pf}(\Sigma'_1) \arrow[u, "(\forall\varphi_1)"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Pf}(\Sigma)(E,(\forall\varphi)E') \arrow[r, "\cong"] \arrow[d, "\mathsf{Pf}(\theta)"'] & \mathsf{Pf}(\Sigma')(\varphi(E),E') \arrow[d, "\mathsf{Pf}(\theta')"] \\ \mathsf{Pf}(\Sigma_1)(\theta(E),(\forall\varphi_1)\theta'(E')) \arrow[r, "\cong"] & \mathsf{Pf}(\Sigma'_1)(\varphi_1(\theta(E)),\theta'(E')) \end{tikzcd}$$

In categorical terms, the commutativity of these squares just says that the pair $\langle \mathsf{Pf}(\theta), \mathsf{Pf}(\theta') \rangle$ is a morphism of adjunctions.

Fact 13.16. *An institution has semantic $\mathcal{D}$-quantifiers if and only if its semantic proof system has $\mathcal{D}$-quantifiers.*

The property of existence of proof-theoretic universal quantifiers is known in conventional logic as the *Generalization Rule*. However this is rather a property of the proof system than a generating rule, hence calling it 'meta-rule' instead of 'rule' would be more appropriate.

A proof system with universal, respectively existential, quantification may be denoted by $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, \forall)$, respectively $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, \exists)$.

Free proof systems with quantification

Here we show that quantification can be added freely to proof systems provided there exists already a syntax for quantifiers. This condition is captured by the concept of 'pre-quantifier'.

Pre-quantifiers. A *sentence system with pre-quantifiers* $(\mathbb{S}ig, \mathsf{Sen}, \mathcal{D}, Q)$ consists of

- a category of 'signature' $\mathbb{S}ig$,

- a 'sentence' functor $\mathsf{Sen}: \mathbb{S}ig \to \mathbb{S}et$,
- a subcategory $\mathcal{D}$ of signature morphisms which is stable under pushouts, and
- a functor $Q: \mathcal{D} \to \mathbb{S}et^{\mathrm{op}}$ such that for each pushout of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \Sigma_1 \arrow[d, "\varphi_1"] \\ \Sigma' \arrow[r, "\theta'"'] & \Sigma'_1 \end{tikzcd}$$

with $\varphi \in \mathcal{D}$, the square below commutes:

$$\begin{tikzcd} \mathsf{Sen}(\Sigma) \arrow[r, "\mathsf{Sen}(\theta)"] & \mathsf{Sen}(\Sigma_1) \\ \mathsf{Sen}(\Sigma') \arrow[u, "Q(\varphi)"] \arrow[r, "\mathsf{Sen}(\theta')"'] & \mathsf{Sen}(\Sigma'_1) \arrow[u, "Q(\varphi_1)"'] \end{tikzcd}$$

For example, in the case of the universal $\mathcal{D}$-quantifiers the functor Q is defined as $Q(\chi)(\rho') = (\forall\chi)\rho'$ for each signature morphism $\chi: \Sigma \to \Sigma'$ and each Σ'-sentence ρ'.

A *comorphism of sentence systems with pre-quantifiers* $(\Phi,\alpha): (\mathbb{S}ig, \mathsf{Sen}, \mathcal{D}, Q) \to (\mathbb{S}ig', \mathsf{Sen}', \mathcal{D}', Q')$ consists of

- a functor $\Phi: \mathbb{S}ig \to \mathbb{S}ig'$, and
- a natural transformation $\alpha: \mathsf{Sen} \Rightarrow \Phi;\mathsf{Sen}'$ which is also a natural transformation $Q \Rightarrow \Phi;Q'$.

An *institution/proof system has pre-quantifiers* when its underlying sentence system has pre-quantifiers. A *comorphism of institutions/proof systems with pre-quantifiers* is a comorphism of institutions/proof systems which is also a comorphism between the underlying sentence systems with pre-quantifiers.

Adding quantifiers freely. The following result shows the possibility of systematically adding quantification to any proof systems. We treat only the case of the universal quantifiers, the existential quantifiers may be treated similarly.

Theorem 13.17. *The forgetful functor from proof systems with universal quantification to proof systems with pre-quantification has an adjoint.*

Proof. This proof follows the same method as the proof of Thm. 13.3. Let $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q)$ be a proof system with pre-quantifiers. This defines the following single-sorted quasi-existence equational theory of partial algebras by considering:

- theories (Σ, E) and pairs of presentations $(\Sigma, E) \supseteq (\Sigma, E')$ for all $E' \supseteq E$ as total constants,

- h and c as total unary operation symbols,
- all signature morphisms $\varphi \in \mathbb{S}ig$ as partial unary operation symbols, and
- $\_;\_$ and $\langle \_, \_ \rangle$ as binary partial operation symbols,

and in addition to

– (S), $(M1-3)$, $(C1-3)$, $(P1-2)$ as in the proof of Thm. 13.3 but in a version replacing E's by (Σ, E)'s

also the following set of quasi-existence equations:

$$
\begin{array}{ll}
(FS) & (\forall p)(\Sigma,E) \xrightarrow{p} (\Sigma,E') \Rightarrow (\Sigma',\varphi(E)) \xrightarrow{\varphi(p)} (\Sigma',\varphi(E')) \\
(FM) & \varphi((\Sigma,E') \supseteq (\Sigma,E)) \stackrel{e}{=} (\Sigma',\varphi(E')) \supseteq (\Sigma',\varphi(E)) \\
(FC) & (\forall p,p')((\Sigma,E) \xrightarrow{p} (\Sigma,E') \wedge ((\Sigma,E') \xrightarrow{p'} (\Sigma,E'') \Rightarrow \varphi(p;p') \stackrel{e}{=} \varphi(p);\varphi(p') \\
(FF) & (\forall p)\varphi'(\varphi(p)) \stackrel{e}{=} (\varphi;\varphi')(p) \\
(FI) & (\forall p)1_\Sigma(p) \stackrel{e}{=} p
\end{array}
$$

for all signature morphisms $\varphi : \Sigma \to \Sigma'$ and $\varphi' : \Sigma' \to \Sigma''$, all sets of sentences $E, E', E'' \subseteq \mathsf{Sen}(\Sigma)$, and where $(\Sigma,E) \xrightarrow{p} (\Sigma,\Gamma)$ abbreviate $(h(p) \stackrel{e}{=} (\Sigma,E)) \wedge (c(p) \stackrel{e}{=} (\Sigma,\Gamma))$.

The given proof system determines canonically a partial algebra P of this quasi-existence equational theory with its underlying set being the disjoint union of proof categories $\biguplus_{\Sigma \in |\mathbb{S}ig|} \mathsf{Pf}(\Sigma)$, and interpreting the operation symbols in the obvious way.

Now, we extend the above quasi-existence equational theory with

- partial unary operations $[\forall\varphi]$ for all signature morphisms $\varphi \in \mathcal{D}$,
- total constants $\eta^{\varphi}_{(\Sigma,E)}$ and $\varepsilon^{\varphi}_{(\Sigma,E)}$ for each theory (Σ,E) and each signature morphism $\varphi \in \mathcal{D}$,

and with the following sentences:

$$
\begin{array}{ll}
(QS_0) & [\forall\varphi](E') \stackrel{e}{=} Q(E') \\
(QS) & (\forall p)(\Sigma',\Gamma') \xrightarrow{p} (\Sigma',E') \Rightarrow (\Sigma,[\forall\varphi]\Gamma') \xrightarrow{[\forall\varphi]p} (\Sigma,[\forall\varphi]E') \\
(QC) & (\forall p,p')((\Sigma',\Gamma') \xrightarrow{p} (\Sigma',E')) \wedge ((\Sigma',E') \xrightarrow{p'} (\Sigma',E'')) \Rightarrow \\
 & [\forall\varphi](p;p') \stackrel{e}{=} [\forall\varphi]p;[\forall\varphi]p' \\
(QF) & (\forall p)((\Sigma',\Gamma') \xrightarrow{p} (\Sigma',E')) \Rightarrow \theta([\forall\varphi]p) \stackrel{e}{=} [\forall\varphi_1](\theta'(p)) \\
(ET_0) & (\Sigma,E) \xrightarrow{\eta^{\varphi}_{(\Sigma,E)}} (\Sigma,[\forall\varphi]\varphi(E)) \\
(ET_1) & (\forall p)(\Sigma,\Gamma) \xrightarrow{p} (\Sigma,E) \Rightarrow \eta^{\varphi}_{(\Sigma,\Gamma)};[\forall\varphi]\varphi(p) \stackrel{e}{=} p;\eta^{\varphi}_{(\Sigma,E)} \\
(EP_0) & (\forall p)(\Sigma',\varphi([\forall\varphi]E')) \xrightarrow{\varepsilon^{\varphi}_{(\Sigma',E')}} (\Sigma',E') \\
(EP_1) & (\forall p')(\Sigma',\Gamma') \xrightarrow{p'} (\Sigma',E') \Rightarrow \varepsilon^{\varphi}_{(\Sigma',\Gamma')};p' \stackrel{e}{=} \varphi([\forall\varphi]p');\varepsilon^{\varphi}_{(\Sigma',E')} \\
(TF) & \varphi(\eta^{\varphi}_{(\Sigma,E)});\varepsilon^{\varphi}_{(\Sigma',\varphi(E))} \stackrel{e}{=} (\Sigma',\varphi(E)) \\
(TQ) & \eta^{\varphi}_{(\Sigma,[\forall\varphi]E')};[\forall\varphi]\varepsilon^{\varphi}_{(\Sigma',E')} \stackrel{e}{=} (\Sigma,[\forall\varphi]E') \\
(I) & (\theta(\eta^{\varphi}_{(\Sigma,E)}) \stackrel{e}{=} \eta^{\varphi_1}_{(\Sigma_1,\theta(E))}) \wedge (\theta'(\varepsilon^{\varphi}_{(\Sigma',E')}) \stackrel{e}{=} \varepsilon^{\varphi_1}_{(\Sigma'_1,\theta'(E'))})
\end{array}
$$

for all signature morphisms $(\varphi\colon\ \Sigma \to \Sigma') \in \mathcal{D}$, $\Gamma', E', E'' \subseteq \mathsf{Sen}(\Sigma')$, $E \subseteq \mathsf{Sen}(\Sigma)$, and all signature pushouts

$$\begin{tikzcd} \Sigma \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \Sigma_1 \arrow[d, "\varphi_1"] \\ \Sigma' \arrow[r, "\theta'"'] & \Sigma'_1 \end{tikzcd}$$

Let $\overline{P}$ be the free extension of P along the extension δ of quasi-existence equational theories defined above. Let $\zeta\colon\ P \to \overline{P}\restriction_\delta$ denote the universal partial algebra homomorphism. For each signature $\Sigma \in |\mathbb{S}ig|$, by letting $\overline{\mathsf{Pf}}(\Sigma)(E,E') = \{p \in \overline{P} \mid (\Sigma,E) \xrightarrow{p} (\Sigma,E')\}$ we get a proof system with universal quantification $(\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall)$ and a comorphism of proof systems with pre-quantifiers $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \omega)\colon\ (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q) \to (\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall)$ where $\omega_\Sigma(p) = \zeta(p)$.

Any comorphism of proof systems with pre-quantifiers

$$(\Phi, \alpha, \gamma)\colon\ (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q) \to (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Pf}', \mathcal{D}', \forall)$$

to a proof system with universal quantifiers determines canonically a partial algebra homomorphism $\gamma'\colon\ P \to P'\restriction_\delta$ mapping each Σ-proof $p\colon\ \Gamma \to E$ to the $\Phi(\Sigma)$-proof $\gamma_\Sigma(p)\colon\ \alpha_\Sigma(\Gamma) \to \alpha_\Sigma(E)$, and where P' is the partial algebra of the extended quasi-equational theory with carrier $\biguplus_{\Sigma' \in |\mathbb{S}ig'|} \mathsf{Pf}'(\Sigma')$.

Then the unique partial algebra homomorphism $\overline{\gamma'}\colon\ \overline{P} \to P'$ such that $\zeta;\overline{\gamma'}\restriction_\delta = \gamma'$ determines back a comorphism of proof systems with universal quantifiers $(\Phi, \alpha, \overline{\gamma})$ such that

$$\begin{tikzcd} (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q) \arrow[rr, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \omega)"] \arrow[dr, "(\Phi,\alpha,\gamma)"'] & & (\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall) \arrow[dl, "(\Phi,\alpha,\overline{\gamma})"] \\ & (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Pf}', \mathcal{D}', \forall) & \end{tikzcd}$$

$\square$

A variant of Theorem 13.17 may generate the universally quantified sentences freely, thus eliminating the need for a pre-quantifier structure. The reader is invited to explore the details of this idea. Here we have preferred the approach based on pre-quantifiers mainly because our intention is to add proof system structures to institutions without having to extend their sentences.

Free quantifiers preserve compactness of proofs. This is a direct consequence of the universal property of free proof systems with quantifiers.

Corollary 13.18. *The proof system with universal quantification freely generated by a compact proof system with pre-quantifiers is also compact.*

Proof. This follows an argument very similar to that used for the proof of Cor. 13.8 by taking the compact proof (sub-)system $(\mathbb{S}ig, \mathsf{Sen}, C(\overline{\mathsf{Pf}}))$ (which exists cf. Prop. 13.7) of the free proof system with universal quantification $(\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}})$, and by noting that proof system comorphisms preserve compact proofs (this fact being similar to the fact that signature morphisms preserve compact proofs; see item 3 of the proof of Prop. 13.7). This means that if we assume $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ is compact, then the universal comorphism $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}})$ goes in fact to $(\mathbb{S}ig, \mathsf{Sen}, C(\overline{\mathsf{Pf}}))$. □

Free quantifiers preserve soundness. The result below shows that by adding the meta rule of 'Generalization' to a proof system of an institution with proofs, the soundness property is preserved.

Proposition 13.19. *Let* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ *be any sound institution with proofs and with semantic universal* $\mathcal{D}$*-quantifiers. Then the institution with proofs* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \overline{\mathsf{Pf}})$*, where* $(\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}})$ *is the proof system with universal* $\mathcal{D}$*-quantifiers freely generated by* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ *(with the pre-quantifiers given by the semantic universal quantifiers) is also sound.*

Proof. Note that the semantic proof system $(\mathbb{S}ig, \mathsf{Sen}, \models, \mathcal{D}, \forall)$ is a proof system with universal quantification. By the soundness of $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ we get a comorphism $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, \forall) \to (\mathbb{S}ig, \mathsf{Sen}, \models, \mathcal{D}, \forall)$ of proof systems with pre-quantifiers.

$$\begin{tikzcd}
(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, \forall) \arrow[rr, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \omega)"] \arrow[dr, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \gamma)"'] & & (\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall) \arrow[dl, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \overline{\gamma})"] \\
& (\mathbb{S}ig, \mathsf{Sen}, \models, \mathcal{D}, \forall) &
\end{tikzcd}$$

By the universal property of $(\mathbb{S}ig, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall)$ we get a comorphism $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \overline{\gamma})$ of proof systems with universal quantification. This comorphism gives the soundness of $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \overline{\mathsf{Pf}})$. □

Exercises

13.5. In any proof system, for any Σ-proofs $p : E \to E'$ and $p' : E' \to E''$, for each set of sentences Γ, if $(\Gamma \setminus E'') \subseteq (\Gamma \setminus E')$ then $(p \cup \Gamma); (p' \cup \Gamma) = (p; p') \cup \Gamma$.

13.6. In any entailment system the equivalence relation on sentences $\dashv\vdash$ defined by $\rho_1 \dashv\vdash \rho_2$ iff $\rho_1 \vdash \rho_2$ and $\rho_2 \vdash \rho_1$ determines for each signature Σ a quotient of the preorder $(\mathsf{Sen}(\Sigma), \vdash)$ to a partial order $(\mathsf{Sen}(\Sigma)/_{\dashv\vdash}, \leq)$.

If the entailment system has conjunctions, disjunctions, $\mathtt{true}$, $\mathtt{false}$, and implications, then $(\mathsf{Sen}(\Sigma)/_{\dashv\vdash}, \leq)$ is a Heyting algebra.

13.7. Any entailment system with conjunctions, negations, and $\neg\neg$-elimination has disjunctions and implications. (*Hint:* Define $\rho_1 \vee \rho_2$ as $\neg(\neg\rho_1 \wedge \neg\rho_2))$ and $\rho_1 \Rightarrow \rho_2$ as $\neg\rho_1 \vee \rho_2$. Use the fact that in any entailment system with negations and $\neg\neg$-elimination we have that $\rho \vdash \rho'$ is equivalent to $\neg\rho' \vdash \neg\rho$.) Does this result generalize to proof systems?

13.8. Collapsing theorem

In any proof system with implication, false, and $\neg\neg$-elimination, there exists at most one proof between any two finite sets of sentences. (*Hint:* By using implications and the initiality of $false$, we have that for each finite set of sentences E there exists at most one proof $E\cup\{\mathsf{false}\} \to E\cup\{\mathsf{false}\}$. Use this for showing that the existence of a proof $E \to \mathsf{false}$ implies $E \cong \mathsf{false}$. The conclusion follows by $\neg\neg$-elimination which gives that proofs $E \to \rho'$ are in natural bijective corrrespondence to proofs $E\cup\{\neg\rho'\} \to \mathsf{false}$.)

13.9. Maximally consistent sets, proof theoretically

Consider an entailment system $(\mathbb{S}ig, \mathsf{Sen}, \vdash)$ with negations. A set of Σ-sentences Γ is *consistent* when $\Gamma \not\vdash \mathsf{false}$. It is maximally consistent when it is consistent and it is maximal with respect to this property, i.e., for any other consistent set Γ' such that $\Gamma \subseteq \Gamma'$ we have that $\Gamma = \Gamma'$.

For each signature Σ we let $\mathsf{Mod}(\Sigma) = \{M \subseteq \mathsf{Sen}(\Sigma) \mid M \text{ maximally consistent}\}$ and for each signature morphism $\varphi\colon \Sigma \to \Sigma'$ we let $\mathsf{Mod}(\varphi)\colon \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$ be defined by $\mathsf{Mod}(\varphi)(M') = \varphi^{-1}(M')$. We may define a satisfaction relation $\models_\Sigma \subseteq \mathsf{Mod}(\Sigma) \times \mathsf{Sen}(\Sigma)$ by $M \models \rho$ if and only if $\rho \in M$.

1. $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \vdash)$ is an institution with proofs that is sound and has semantic negations.
2. If in addition we assume that $(\mathbb{S}ig, \mathsf{Sen}, \vdash)$ is compact, then $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \vdash)$ is complete if and only if $(\mathbb{S}ig, \mathsf{Sen}, \vdash)$ has $\neg\neg$-elimination. (*Hint:* Prove and use the generalization of Lindenbaum's Theorem of Ex. 6.17 to entailment systems.)

13.10. Craig interpolation, proof theoretically

The proof-theoretic concept of interpolation refines the semantic concept of interpolation by considering the interpolant to be a set of sentences together with two corresponding proofs. In any proof system, a square of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

is a *Craig Interpolation square* if and only if for each set E_1 of Σ_1-sentences and *finite* set E_2 of Σ_2-sentences and each proof $p\colon\ \theta_1(E_1) \to \theta_2(E_2)$ there exists a set E of Σ-sentences and proofs $p_1\colon\ E_1 \to \varphi_1(E)$ and $p_2\colon\ \varphi_2(E) \to E_2$ such that $p = \theta_1(p_1);\theta_2(p_2)$.

1. Proof-theoretic Craig interpolation squares are closed under both the 'vertical' and the 'horizontal' compositions (in the sense of Ex. 9.1).
2. Formulate a 'single sentence' version for proof-theoretic interpolation and prove that this is a consequence of the 'multiple sentence' version when the proof system is compact and has conjunctions.

13.11. Craig-Robinson interpolation, proof theoretically

Craig interpolation (abbreviated *CI*) for proof systems (see Ex. 13.10 above) can be refined to Craig-Robinson interpolation (abbreviated *CRI*) by generalizing the concept of (model-theoretic) CRI of Sect. 9.4.

1. Generalize the result of Prop. 9.20 (which shows the equivalence between CRI and CI) from semantic entailment systems to arbitrary entailment systems.

2. For the situation when the set of sentences E_2 is a singleton (i.e., contains only one sentence) extend further the result of Prop. 9.20 to proof systems that are *quasi-compact* (i.e., have infinite conjunctions or are compact), have implications, and such that for each signature the sub-sentence relation is well founded. The *sub-sentence relation* $\prec_\Sigma$ is defined as the transitive closure of the binary relation $\{(\rho, \rho \Rightarrow \rho'), (\rho', \rho \Rightarrow \rho') \mid \rho, \rho' \in \mathsf{Sen}(\Sigma)\}$. That $\prec_\Sigma$ is well founded means that there are no infinite 'downwards' chains $\ldots \rho_i \prec \rho_{i-1} \prec \cdots \prec \rho_0$. (*Hint:* The proof needs the result of Ex. 13.5 which can be applied because $\prec_\Sigma$ is well founded.)

13.12. [85] **Proof-theoretic implicit definability**
We say that a signature morphism $\varphi\colon \Sigma \to \Sigma'$ in a proof system is *defined implicitly* (proof theoretically) by $E' \subseteq \mathsf{Sen}(\Sigma')$ if for each signature morphism $\theta\colon \Sigma \to \Sigma_1$ and each Σ'_1-sentence ρ,

$$(\theta';u)(E') \cup (\theta';v)(E') \cup \{u(\rho)\} \vdash v(\rho) \text{ and } (\theta';u)(E') \cup (\theta';v)(E') \cup \{v(\rho)\} \vdash u(\rho)$$

for all pushouts of the form

$$\begin{tikzcd}
 & \Sigma' \arrow[r, "\theta'"] & \Sigma'_1 \arrow[dr, "u"] & \\
\Sigma \arrow[ur, "\varphi"] \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & \Sigma_1 \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_1"'] & & \Sigma'' \\
 & \Sigma' \arrow[r, "\theta'"'] & \Sigma'_1 \arrow[ur, "v"'] &
\end{tikzcd}$$

In any institution with model amalgamation a signature morphism φ

1. is defined implicitly proof theoretically by E' for the semantic entailment system if it is defined implicitly model theoretically by E' (in the sense of Chap. 10), and
2. is defined implicitly model theoretically by E' if it is defined implicitly proof theoretically by E' when it is ι-tight for a system ι of elementary diagrams of the institutions.

13.13. [85] **Definability by interpolation, proof theoretically**
We say that φ is *defined explicitly* by E' when for each $\rho \in \mathsf{Sen}(\Sigma'_1)$, there exists a set of sentences $E_\rho \subseteq \mathsf{Sen}(\Sigma_1)$ such that

$$\theta'(E') \cup \{\rho\} \vdash \varphi_1(E_\rho) \text{ and } \theta'(E') \cup \varphi_1(E_\rho) \vdash \rho.$$

The following constitute proof theoretic variants of Prop. 10.3 and Thm. 10.5, respectively.

1. Any signature is defined implicitly by a theory if it is defined explicitly by that theory.
2. If the entailment system has Craig-Robinson $(\mathcal{L}, \mathcal{R})$-interpolation (see Ex. 13.11) for classes $\mathcal{L}$ and $\mathcal{R}$ of signature morphisms that are stable under pushouts, any signature morphism in $\mathcal{L} \cap \mathcal{R}$ is defined explicitly if it is defined implicitly.

13.4 The Entailment Institution

The semantic proof systems determined by institutions show that proof systems are more general than institutions. Proof systems are more abstract than institutions because they lack the model theory component. In this section we give a general way to enhance proof systems with models and a satisfaction relation between models and sentences.

Proof-theoretic presentations. For any set E of Σ-sentences for a signature Σ, we say that (Σ, E) is a *presentation*. A *presentation morphism* $(\varphi, p) : (\Sigma, E) \to (\Sigma', E')$ consists of a signature morphism $\varphi : \Sigma \to \Sigma'$ and a proof $p : E' \to \varphi(E)$.

Fact 13.20. *Under the obvious composition the proof-theoretic presentation morphisms form a category. Moreover, the category of the presentations of a proof system* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ *is the opposite of the Grothendieck category determined by the* $(\mathbb{S}ig^{\mathrm{op}})$*-indexed functor* Pf*, i.e.,* $\mathbb{P}res = (\mathsf{Pf}^{\sharp})^{\mathrm{op}}$.

The proof-theoretic concept of presentation is significantly more refined than its semantic counterpart in institutions because it takes into account *how* $\varphi(E)$ is a consequence of E' not only that it is a consequence.

As in the case of institutions, the category of presentations of a proof system is denoted by $\mathbb{P}res$.

Theories in proof systems. For any signature Σ of a proof system, a set of sentences (Σ, Γ) is a *theory* when $\Gamma \vdash E$ implies $E \subseteq \Gamma$. The least theory containing a set E of Σ-sentences is denoted by $E^{\bullet}$. It is easy to see that $E^{\bullet} = \{\rho \mid E \vdash \rho\}$.

Given theories (Σ, E) and (Σ', E'), a *(proof theoretic) theory morphism* $\varphi : (\Sigma, E) \to (\Sigma', E')$ is just a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'$. Note that this concept is weaker than that of (proof theoretic) presentation morphism between theories.

The entailment institution of a proof system

Proposition 13.21 (Entailment institution). *Each proof system* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ *determines an institution* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ *called the* entailment institution *of the proof system, where for each signature* $\Sigma \in |\mathbb{S}ig|$,

- *the 'entailment'* Σ*-models are pairs* (ψ, E')*, where* $\psi : \Sigma \to \Sigma'$ *is a signature morphism and* E' *is a* Σ'*-theory,*
- *a* Σ*-model homomorphism* $\varphi : (\psi : \Sigma \to (\Sigma', E')) \to (\psi' : \Sigma \to (\Sigma'', E''))$ *is just a theory morphism* $\varphi : (\Sigma', E') \to (\Sigma'', E'')$ *such that* $\psi;\varphi = \psi'$,
- *a* Σ*-model* (ψ, E') *satisfies a* Σ*-sentence* ρ *if and only if* $\psi(\rho) \in E'$,
- *model reducts are obtained just by composition to the left.*

Proof. The satisfaction condition for the entailment institution can be checked as follows. For any signature morphism $\varphi : \Sigma \to \Sigma'$, any entailment Σ'-model $(\psi : \Sigma' \to \Sigma'', E'')$, and any Σ-sentence ρ,

$$(\psi, E'')\restriction_{\varphi} \models \rho \ \text{ iff } \ (\varphi;\psi, E'') \models \rho \ \text{ iff } \ \psi(\varphi(\rho)) \in E'' \ \text{ iff } \ (\psi, E'') \models \varphi(\rho).$$

$\square$

Entailment institutions have the following important property.

Proposition 13.22. *Any entailment institution is sound and complete.*

Proof. For the completeness property let us assume $E \models \rho$. We consider the Σ-model $1_\Sigma : \Sigma \to (\Sigma, E^\bullet)$. Then $1_\Sigma \models E$ which implies $1_\Sigma \models \rho$. This means $\rho \in E^\bullet$ which means $E \vdash \rho$.

For the soundness property let us assume that $E \vdash \rho$ and consider any Σ-model $\psi : \Sigma \to (\Sigma', E')$ such that $\psi \models E$. Therefore we have that $\psi(E) \subseteq E'$ which means $E \subseteq \psi^{-1}(E')$. Since $\psi^{-1}(E')$ is a theory (it is easy to prove that the inverse image of any theory through a signature morphism is still a theory) we have that $E^\bullet \subseteq \psi^{-1}(E')$. This implies $\rho \in \psi^{-1}(E')$, which means $\psi(\rho) \in E$. This shows $\psi \models \rho$. $\square$

The following corollary may be used as a source of examples that fall between compactness and m-compactness.

Corollary 13.23 (Compactness). *The entailment institution is compact if and only if the entailment system determined by the proof system is compact. However, in the entailment institution each set of sentences is consistent; consequently the entailment institution is trivially model compact.*

An adjunction between proof systems and elementary institutions. Recall that an elementary institution is any institution with elementary diagrams such that for each model M the theory of its elementary diagram is just the theory of the initial model of the elementary diagram, i.e., $M_M^* = E_M^{**}$. Otherwise said, an institution with elementary diagrams is elementary if and only if each model homomorphism is elementary.

Theorem 13.24. *The entailment institution construction is an adjoint to the forgetful functor from the elementary institutions to proof systems.*

Proof. Let $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf})$ be any proof system. The entailment institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is elementary with diagrams ι as follows:

> For each Σ-model (ψ, E'), let the elementary extension $\iota_\Sigma(\psi, E')$ be just ψ and the elementary diagram $E_{(\psi,E')}$ be just E'.

We can notice easily that the isomorphisms $i_{\Sigma,(\psi,E')}$ are identities, and that $E_M^{**} = M_M^*$ for each model M.

For each signature Σ and each Σ-proof $p : E \to \Gamma$, let $\eta_\Sigma(p)$ be the semantic deduction $E \models \Gamma$. This gives a comorphism of proof systems $(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ig, \mathsf{Sen}, \models)$. Let $(\Phi, \alpha, \gamma) : (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ig', \mathsf{Sen}', \models')$ be any comorphism of proof systems, where $(\mathbb{S}ig', \mathsf{Sen}', \models')$ is the semantic entailment system determined by an elementary institution $(\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', \models', \iota')$.

$$\begin{tikzcd}
(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Pf}) \arrow[rr, "(1_{\mathbb{S}ig}, 1_{\mathsf{Sen}}, \eta)"] \arrow[dr, "(\Phi,\alpha,\gamma)"'] & & (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \iota) \arrow[dl, "(\Phi,\alpha,\beta)"] \\
& (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', \models', \iota') &
\end{tikzcd}$$

(Φ,α,γ) can be extended uniquely to a comorphism of institutions with elementary diagrams (Φ,α,β), with β uniquely determined by the comorphism condition on elementary diagrams:

$$\beta_{\Sigma'}(M') = (\Phi(\iota'_{\Sigma'}(M')), \alpha^{-1}_{\Sigma'_{M'}}(E^{**}_{M'}))$$

for each signature $\Sigma' \in |\mathbb{S}ig'|$ and each Σ'-model M'. The fact that $\alpha^{-1}_{\Sigma'_{M'}}(E^{**}_{M'})$ is a theory follows from the fact that (Φ,α,γ) is a comorphism of proof systems.

If $h\colon M' \to N'$ is a Σ'-model homomorphism, then $\beta_{\Sigma'}(h) = \Phi(\iota_{\Sigma'}(h))$. Notice that because $\iota_{\Sigma'}(h)$ is a presentation morphism $(\Sigma'_{M'}, E_{M'}) \to (\Sigma'_{N'}, E_{N'})$, then $\Phi(\iota_{\Sigma'}(h))$ is a theory morphism $(\Phi(\iota'_{\Sigma'}(M')), \alpha^{-1}_{\Sigma'_{M'}}(E^{**}_{M'})) \to (\Phi(\iota'_{\Sigma'}(N')), \alpha^{-1}_{\Sigma'_{N'}}(E^{**}_{N'}))$.

The Satisfaction Condition for the institution comorphism (Φ,α,β) can be obtained as follows:

$$\begin{array}{lll}
\beta_{\Sigma'}(M') \models \rho & & \\
\quad \text{iff} & \Phi(\iota'_{\Sigma'}(M'))(\rho) \in \alpha^{-1}_{\Sigma'_{M'}}(E^{**}_{M'}) & \\
\quad \text{iff} & \alpha_{\Sigma'_{M'}}(\Phi(\iota'_{\Sigma'}(M'))(\rho)) \in E^{**}_{M'} & \\
\quad \text{iff} & \iota'_{\Sigma'}(M')(\alpha_{\Sigma'}(\rho)) & \text{(by the naturality of } \alpha\text{)} \\
\quad \text{iff} & M'_{M'} \models \iota'_{\Sigma'}(M')(\alpha_{\Sigma'}(\rho)) & \text{(since the institution is elementary)} \\
\quad \text{iff} & M' \models \alpha_{\Sigma'}(\rho) & \text{(by the satisfaction condition for } \iota'_{\Sigma'}(M')\text{).}
\end{array}$$

□

Exercises

13.14. A presentation morphism $(\varphi, p)\colon (\Sigma, E) \to (\Sigma', E')$ is

- *closed* when for each proof $p'_1\colon E' \to \varphi(E_1)$ there exists a proof $p_1\colon E \to E_1$ such that $p'_1 = p;\varphi(p_1)$, (In more category theoretic terms this means that p is a 'weak universal arrow' from E' to $\mathsf{Pf}(\varphi)$.)
- *strong* when for each proof $p'_1\colon E' \to E'_1$ there exists a proof $p_1\colon E \to E_1$ such that $p'_1 = p;\varphi(p_1)$.

The closed, respectively strong, presentation morphisms are closed under composition.

13.15. Modularization squares
A commutative square of signature morphisms

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\
\Sigma_2 \arrow[r, "\theta_2"'] & \Sigma'
\end{tikzcd}$$

is said to be a *modularization square* when for any closed presentation morphism $(\varphi_1, p_1)\colon (\Sigma, E) \to (\Sigma_1, E_1)$ and any presentation morphism $(\varphi_2, p_2)\colon (\Sigma, E) \to (\Sigma_2, E_2)$ such that the diagram of presentation morphisms

$$\begin{tikzcd}
(\Sigma, E) \arrow[r, "(\varphi_1, p_1)"] \arrow[d, "(\varphi_2, p_2)"'] & (\Sigma_1, E_1) \arrow[d, "(\theta_1, \supseteq_{\theta_1(E_1)\cup\theta_2(E_2),\theta_1(E_1)})"] \\
(\Sigma_2, E_2) \arrow[r, "(\theta_2, \supseteq_{\theta_1(E_1)\cup\theta_2(E_2),\theta_2(E_2)})"'] & (\Sigma', \theta_1(E_1) \cup \theta_2(E_2))
\end{tikzcd}$$

commutes, $(\theta_2, \supseteq_{\theta_1(E_1)\cup\theta_2(E_2),\theta_2(E_2)})$ is a closed presentation morphism.

1. In any proof system any Craig-Robinson interpolation square is a modularization square.
2. In any entailment system, any commuting square of signature morphisms is a Craig-Robinson interpolation square if and only if it is a modularization square.

13.16. Consider an entailment system $\mathcal{E}$ and let $I(\mathcal{E})$ be its entailment institution.

1. If $\mathcal{E}$ has (proof theoretic) conjunctions, then $I(\mathcal{E})$ has (semantic) conjunctions.
2. Even when $\mathcal{E}$ has (proof theoretic) disjunctions, implications, negations, or false, $I(\mathcal{E})$ does not necessarily have the corresponding connectives.

13.17. Any entailment system $(\mathbb{S}ig, \mathsf{Sen}, \vdash)$ determines an *entailment system of presentations* $(\mathbb{P}res, \mathsf{Sen}^{\text{pres}}, \vdash^{\text{pres}})$ where $\mathbb{P}res$ is the category of presentations, $\mathsf{Sen}^{\text{pres}}(\Sigma, E) = \mathsf{Sen}(\Sigma)$, and

$$\Gamma \vdash^{\text{pres}}_{(\Sigma,E)} \Gamma' \text{ if and only if } \Gamma \cup E \vdash_{\Sigma} \Gamma'.$$

Then

1. $(\mathbb{P}res, \mathsf{Sen}^{\text{pres}}, \vdash^{\text{pres}})$ has conjunctions, false, negations, and implications, respectively, when $(\mathbb{S}ig, \mathsf{Sen}, \vdash)$ has conjunctions, false, negations, and implications, respectively.
2. $(\mathbb{P}res, \mathsf{Sen}^{\text{pres}}, \vdash^{\text{pres}})$ has disjunctions if $(\mathbb{S}ig, \mathsf{Sen}, \vdash)$ has disjunctions and implications.
3. If $(\mathbb{S}ig, \mathsf{Sen}, \vdash)$ has universal $\mathcal{D}$-quantification, then $(\mathbb{P}res, \mathsf{Sen}^{\text{pres}}, \vdash^{\text{pres}})$ has universal $\mathcal{D}^{\text{pres}}$-quantification where $\mathcal{D}^{\text{pres}} = \{\chi : (\Sigma, E) \to (\Sigma', E') \text{ presentation morphism } \mid \chi \in \mathcal{D} \text{ and } \varphi(E) \vdash E'\}$.

13.18. Co-limits of proof theoretic presentations
If the category $\mathbb{S}ig$ of the signatures has J-co-limits and if each category $\mathsf{Pf}(\Sigma)$ of Σ-proofs has J-limits that are preserved by the proof translation functors $\mathsf{Pf}(\varphi)$, then the category $\mathbb{P}res$ of the proof system presentations has J-co-limits.

13.19. For each proof system

1. if it has pushouts for signatures, then the corresponding entailment institution is semi-exact and liberal,
2. each sentence is basic in the corresponding entailment institution,
3. each signature morphism is representable in the corresponding entailment institution,
4. if for a category J its category of signatures has J-limits, respectively co-limits, then for each signature Σ the category of Σ-entailment models has J-limits, respectively co-limits, and
5. any inclusion system for the category of signatures determines 'closed' and 'strong' inclusion systems for the categories of entailment models of signatures (see also Sect. 4.5).

13.5 Birkhoff Completeness

In this section we develop a generic sound and complete proof system for 'universal institutions' which we instantiate to 'Horn institutions'.

Universal institutions. Let $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ be an institution and

- let Sen_1 be a sub-functor of Sen (i.e., $\mathsf{Sen}_1 : \mathbb{S}ig \to \mathbb{S}et$ such that $\mathsf{Sen}_1(\Sigma) \subseteq \mathsf{Sen}(\Sigma)$ and $\varphi(\mathsf{Sen}_1(\Sigma)) \subseteq \mathsf{Sen}_1(\Sigma')$ for each signature morphism $\varphi : \Sigma \to \Sigma'$), and
- let $\mathcal{D} \subseteq \mathbb{S}ig$ be a sub-category of signature morphisms such that $\mathcal{D}$ is stable under pushouts.

We say that I is a *$\mathcal{D}$-universal institution over* I_1, where $I_1 = (\mathbb{S}ig, \mathsf{Sen}_1, \mathsf{Mod}, \models)$, when

- I admits all sentences of the form $(\forall\chi)\rho$ where $\chi : \Sigma \to \Sigma'$ is any signature morphism in $\mathcal{D}$, and ρ is any $\mathsf{Sen}_1(\Sigma')$ sentence, and
- any sentence of I is semantically equivalent to a sentence of the form $(\forall\chi)\rho$ as in the item above.

For example, **UNIV** is a $\mathcal{D}$-universal institution over the restriction of **FOL** to the quantifier-free sentences (i.e., sentences without quantifiers), where $\mathcal{D}$ is the class of all signature extensions with a finite number of constants. Another example is **HCL** which is a $\mathcal{D}$-universal institution over the sentences of the form $H \Rightarrow C$ where H is a finite conjunction of atoms and C is an atom. Similarly, the infinitary versions $\mathbf{UNIV}_\infty$ and $\mathbf{HCL}_\infty$ are also examples of $\mathcal{D}$-universal institutions, but in this case $\mathcal{D}$ is the class of *all* signature extensions with constants (i.e., $\mathcal{D}$ might contain infinitary extensions).

Horn institutions. An institution $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ is *(finitary) $\mathcal{D}$-Horn institution* over $I_0 = (\mathbb{S}ig, \mathsf{Sen}_0, \mathsf{Mod}, \models)$ when I is a $\mathcal{D}$-universal institution over $I_1 = (\mathbb{S}ig, \mathsf{Sen}_1, \mathsf{Mod}, \models)$ and Sen_0 is a sub-functor of Sen_1 such that

- for each signature Σ, the institution I_1 admits all sentences of the form $H \Rightarrow C$ where H is any (finite) set of $\mathsf{Sen}_0(\Sigma)$ sentences, and C is any $\mathsf{Sen}_0(\Sigma)$ sentence, and
- any sentence of I_1 is semantically equivalent to a sentence of the form $H \Rightarrow C$ as in the item above.

For example, **HCL** is a finitary $\mathcal{D}$-Horn institution over **AFOL**, the atomic sub-institution of **FOL** (i.e., the sentences of **AFOL** are the atoms of **FOL**), where $\mathcal{D}$ is the class of all signature extensions with a finite number of constants. Similarly, $\mathbf{HCL}_\infty$ is an (infinitary) $\mathcal{D}$-Horn institution over **AFOL**, but in this case $\mathcal{D}$ is the class of *all* signature extensions with constants (i.e., $\mathcal{D}$ might contain infinitary extensions).

The generic proof system for Horn institutions developed in this section consists of three layers:

1. The 'atomic' layer is that of the proof system of I_0, which in the abstract setting is assumed but which is to be developed in the concrete examples.

2. The layer of the proof system for I_1 which is obtained by assuming the so-called 'Modus Ponens' meta-rule in a restricted form involving the sentences of I_0.
3. The upmost layer is that of the proof system for I, which is obtained by adding the so-called 'Substitutivity' rule and the 'Generalization' meta-rule to the proof system of I_1.

The soundness and the completeness at each layer is obtained relatively to the soundness and the completeness of the layer immediately below.

Such layered decomposition of the proof system of I leads also to sound and complete proof systems for 'universal institutions' which are not necessarily Horn institutions. For this it is enough to start with a sound and complete proof system for I_1. For example, a sound and complete proof system for quantifier free sentences in **FOL** (i.e., sentences formed from atoms by iterative applications of the Boolean connectives) determines automatically a sound and complete proof system for **UNIV**.

The generic universal proof system

Let us assume a $\mathcal{D}$-universal institution $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ over $I_1 = (\mathbb{S}ig, \mathsf{Sen}_1, \mathsf{Mod}, \models)$ with Sen_1 a sub-functor of Sen. We also assume a sub-functor Sen_0 of Sen_1, and denote the corresponding institution by I_0, such that

> For each (finite) set of sentences $B \subseteq \mathsf{Sen}_0(\Sigma)$ and any sentence $e \in \mathsf{Sen}_1(\Sigma)$ there exists a sentence in $\mathsf{Sen}_1(\Sigma)$ which is semantically equivalent to $\wedge B \Rightarrow e$.

Note that this condition is significantly more general than if we assumed that I is a Horn institution over I_0. Indeed, if I were a Horn institution over I_0, for any I_1-sentence $e = (H \Rightarrow C)$ and any (finite) set of I_0-sentences B we would have that $\wedge B \Rightarrow e$ is semantically equivalent to the I_1-sentence $(H \wedge B) \Rightarrow C$. However the above condition holds also for non-Horn settings such as when $I = \mathbf{UNIV}$, the institution of the **FOL** universal sentences, Sen_1 is the (sub-)functor of the quantifier-free sentences, and Sen_0 is the (sub-)functor of the atomic sentences (see Ex. 13.22).

Note also that the above condition comes in two variants: when B is required to be finite, and when B is allowed to be infinite. The infinitary variant is applicable only to the infinitary variants of institutions, such as $\mathbf{HCL}_\infty$ or $\mathbf{UNIV}_\infty$.

We also assume another rather mild technical condition, namely that

> For each $\mathcal{D}$-substitution $\theta : (\varphi : \Sigma \to (\Sigma_1, E_1)) \to (\chi : \Sigma \to (\Sigma_2, E_2))$ in I_0^{pres} (the institution of I_0 presentations), there exists a $\mathcal{D}$-substitution $\varphi \to \chi$ in I_1^{pres} which 'extends' θ. Since there is no danger of confusion we denote this latter substitution by θ too. This means that $\mathsf{Sen}_0(\theta)$ extends to a function $\mathsf{Sen}_1(\theta) : \mathsf{Sen}_1(\Sigma_1) \to \mathsf{Sen}_1(\Sigma_2)$ such that the pair $(\mathsf{Sen}_1(\theta), \mathsf{Mod}^{\mathrm{pres}}(\theta))$ constitute a substitution in I_1^{pres}.

In all examples mentioned above this condition is fulfilled rather easily since the I_1-sentences are Boolean expressions formed from I_0-sentences. For example, if I is a $\mathcal{D}$-Horn institution over I_0, then $\mathsf{Sen}_0(\theta)$ extends canonically to $\mathsf{Sen}_1(\theta)$ by defining $\mathsf{Sen}_1(\theta)(H \Rightarrow C)$ by $\mathsf{Sen}_0(H) \Rightarrow \mathsf{Sen}_0(C)$.

The rules of Substitutivity. For all $\mathcal{D}$-universal sentences $(\forall\varphi)\rho$ and all $\mathcal{D}$-substitutions $\theta : (\Sigma \xrightarrow{\varphi} (\Sigma_1, \emptyset)) \to (\Sigma \xrightarrow{\chi} (\Sigma_2, B))$ in I_0^{pres}

$$\begin{tikzcd}
(\Sigma_1,\emptyset) & & (\Sigma_2,B) \\
& \Sigma \arrow[ul, "\varphi"] \arrow[ur, "\chi"'] &
\end{tikzcd}$$

we consider the following *rule of $\mathcal{D}$-Substitutivity*

$$(\forall\varphi)\rho \vdash (\forall\chi)(\wedge B \Rightarrow \theta(\rho))$$

where $\theta(\rho)$ denotes $\mathsf{Sen}_1(\theta)(\rho)$.

Note that the rule of Substitutivity may also be considered in either a finitary or in an infinitary variant. The above formulation corresponds to the infinitary variant since B may be infinite. If we consider only those I_0-presentations (Σ_2, B) for which B is finite we get the rule of *finitary $\mathcal{D}$-Substitutivity*.

Below we will see that under some technical conditions, which are often met in the applications, the rule of Substitutivity may be rephrased as

$$(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$$

for $\mathcal{D}$-substitutions θ in I_0 rather than I_0^{pres}.

Proposition 13.25. *The rule of $\mathcal{D}$-Substitutivity is sound.*

Proof. Let M be a Σ-model such that $M \models (\forall\varphi)\rho$ and let M_2 be any χ-expansion of M such that $M_2 \models B$. Because $\mathsf{Mod}(\theta)(M_2)$ is a φ-expansion of M (since $\mathsf{Mod}(\theta)(M_2)\restriction_\varphi = M_2\restriction_\chi$) which by the hypothesis satisfies $(\forall\varphi)\rho$, we have that $\mathsf{Mod}(\theta)(M_2) \models \rho$. By the satisfaction condition for substitutions, we obtain that $M_2 \models \theta(\rho)$. Since M_2 was an *arbitrary* expansion of M, we have thus proved that $M \models (\forall\chi)(\wedge B \Rightarrow \theta(\rho))$. $\square$

Another aspect to be considered is whether the translations of the Substitutivity rule along signature morphisms is again a Substitutivity rule in the target signature. Although this property is not strictly necessary at the general level, because in the case of a negative answer to this question we may add such translations to the system of rules, in fact under some mild conditions this property holds (see Ex. 13.20 below).

Universal proof systems. Given a proof system $(\mathbb{S}ig, \mathsf{Sen}_1, \mathsf{Pf}_1)$ for I_1 such that its corresponding entailment system is compact, the *$\mathcal{D}$-universal proof system* for I consists of the free proof system

- with universal $\mathcal{D}$-quantification (i.e., the meta-rule of 'Generalization')
- over $(\mathbb{S}ig, \mathsf{Sen}_1, \mathsf{Pf}_1)$ plus the rules of finitary $\mathcal{D}$-Substitutivity,

This is the finitary version of the universal proof system. Its *infinitary* variant is obtained by considering the rules of (infinitary) $\mathcal{D}$-Substitutivity, by dropping off the compactness condition I_0, and by considering the infinitary proof system for I.

Note that the first step of the process of obtaining the $\mathcal{D}$-universal proof system for I consists of the free generation of a proof system over $(\mathbb{S}ig, \mathsf{Sen}_1, \mathsf{Pf}_1)$ which contains the rules of $\mathcal{D}$-Substitutivity. Although this cannot be obtained by applying directly the free proof systems Thm. 13.3, a slightly more general version of Thm. 13.3 does apply. We leave this detail to the reader.

We have the following important consequence of the soundness of the $\mathcal{D}$-Substitutivity rules given by Prop. 13.25.

Corollary 13.26. *If the proof system of I_1 is sound, then the corresponding universal proof system for I is also sound.*

Proof. By a process similar to that of Prop. 13.5 we lift the soundness from the proof system of I_1 to the proof system that adds freely the rules of $\mathcal{D}$-Substitutivity. Then by Prop. 13.19 we lift soundness further to the free proof system with universal $\mathcal{D}$-quantification. □

Universal completeness

Completeness of the universal proof systems is significantly more difficult than the soundness property and therefore requires more conceptual infrastructure. The universal completeness result below comes both in a finite and in an infinite variant, the finite one being obtained by assuming the finitary version for the proof system of I_1 and by adding (to the hypotheses of the infinite one) all the finiteness hypotheses marked in brackets.

Theorem 13.27 (Universal completeness). *The (finitary) $\mathcal{D}$-universal proof system for I determined by the proof system of I_1 as defined above is complete if*

1. *the proof system of I_1 is complete,*
2. *every signature morphism in $\mathcal{D}$ is (finitely) representable,*
3. *every set of sentences in I_0 is epic basic,*
4. *I_0^{pres} has representable $\mathcal{D}$-substitutions, and*
5. *for each set E of sentences $E \subseteq \mathsf{Sen}_1(\Sigma)$ and each sentence $e \in \mathsf{Sen}_1(\Sigma)$ we have that*

$$E \models e \text{ if and only if } M_B \models (\wedge E \Rightarrow e) \text{ for each set of sentences } B \subseteq \mathsf{Sen}_0(\Sigma)$$

(where M_B are the models defining B as basic sets of sentences).

Proof. Assume that $\Gamma \models_\Sigma (\forall\chi)e'$ for a set $\Gamma \subseteq \mathsf{Sen}(\Sigma)$ and $e' \in \mathsf{Sen}_1(\Sigma')$ where $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$. We want to show that $\Gamma \vdash_\Sigma (\forall\chi)e'$. Suppose towards a contradiction that $\Gamma \not\vdash_\Sigma (\forall\chi)e'$.

We define the set of Σ'-sentences

$$\Gamma_1^\chi = \{\rho' \in \mathsf{Sen}_1(\Sigma') \mid \Gamma \vdash (\forall\chi)\rho'\}.$$

Suppose $\Gamma_1^\chi \vdash e'$. For the infinitary case we take $\Gamma' = \Gamma_1^\chi$. For the finitary case, since the entailment system corresponding to the proof system of I_1 is compact there exists a finite $\Gamma' \subseteq \Gamma_1^\chi$ such that $\Gamma' \vdash e'$. Because the universal proof system of I has $\mathcal{D}$-universal quantification we have that $\chi(\Gamma) \vdash \rho'$ for all $\rho' \in \Gamma_1^\chi$, thus $\chi(\Gamma) \vdash \Gamma'$. Therefore $\chi(\Gamma) \vdash e'$ and again by the universal $\mathcal{D}$-quantification property for the proof system of I we obtain $\Gamma \vdash (\forall\chi)e'$. This contradicts our assumption that $\Gamma \not\vdash_\Sigma (\forall\chi)e'$. Thus $\Gamma_1^\chi \not\vdash e'$.

By the completeness of I_1, $\Gamma_1^\chi \not\vdash e'$ implies $\Gamma_1^\chi \not\models e'$. By the hypothesis there exists an epic basic set of sentences $B \subseteq \mathsf{Sen}_0(\Sigma')$ such that $M_B \models \Gamma_1^\chi$ but $M_B \not\models e'$. This implies $M_B\restriction_\chi \not\models (\forall\chi)e'$. If we proved that $M_B\restriction_\chi \models \Gamma$, then we reached a contradiction with $\Gamma \models (\forall\chi)e'$. We will therefore focus on proving that $M_B\restriction_\chi \models \Gamma$.

Let $(\forall\varphi)e_1 \in \Gamma$, where $(\varphi\colon\ \Sigma \to \Sigma_1) \in \mathcal{D}$, and let N be any φ-expansion of $M_B\restriction_\chi$. We have to show that $N \models e_1$. For this we use the following lemma (which we prove later):

Lemma 13.28. *There exists a (finite) subset of sentences $B' \subseteq B$ and a homomorphism $h\colon\ M_\varphi \to M_{B'}\restriction_\chi$ such that the diagram below commutes:*

$$\begin{tikzcd}
M_{B'}\restriction_\chi \arrow[rr, "\mu_{B'}\restriction_\chi"] & & M_B\restriction_\chi = N\restriction_\varphi \\
& M_\varphi \arrow[ul, "h"] \arrow[ur, "i_\varphi(N)"'] &
\end{tikzcd}$$

where $\mu_{B'}$ is the unique homomorphism $M_{B'} \to M_B$ (because B' is epic basic).

Because $\chi\colon\ \Sigma \to \Sigma'$ of Lemma 13.28 is representable it is also quasi-representable. By using the fact that B' are basic it is easy to prove that χ is quasi-representable as a presentation morphism $\chi\colon\ \Sigma \to (\Sigma', B')$ too. Because B' is epic basic the category of models $\mathsf{Mod}(\Sigma', B')$ has the initial model $0_{(\Sigma',B')} = M_{B'}$, hence χ is representable as a presentation morphism.

Because I_0^{pres} has representable $\mathcal{D}$-substitutions, the homomorphism $h\colon\ M_\varphi \to M_{B'}\restriction_\chi$ given by Lemma 13.28 determines a substitution $\theta\colon\ (\varphi\colon\ \Sigma \to \Sigma_1) \to (\chi\colon\ \Sigma \to (\Sigma', B'))$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma', B') \arrow[r, "i_\chi^{B'}", "\cong"'] \arrow[d, "\mathsf{Mod}(\theta)"'] & M_{B'}\restriction_\chi/\mathsf{Mod}(\Sigma) \arrow[d, "h/\mathsf{Mod}(\Sigma)"] \\
\mathsf{Mod}(\Sigma_1) \arrow[r, "\cong", "i_\varphi"'] & M_\varphi/\mathsf{Mod}(\Sigma)
\end{tikzcd}$$

We have that

$$\mathsf{Mod}(\theta)(M_B) = i_\varphi^{-1}(h; i_\chi^{B'}(M_B)) = i_\varphi^{-1}(h; \mu_{B'}\restriction_\chi) = i_\varphi^{-1}(i_\varphi(N)) = N.$$

By (finitary) $\mathcal{D}$-substitutivity we obtain $\Gamma \vdash (\forall\chi)(\wedge B' \Rightarrow \theta(e_1))$. This implies $\wedge B' \Rightarrow \theta(e_1) \in \Gamma_1^\chi$. Since $M_B \models \Gamma_1^\chi$ we obtain $M_B \models \wedge B' \Rightarrow \theta(e_1)$. Because $M_B \models B'$ we get $M_B \models \theta(e_1)$. By the satisfaction condition for substitutions we obtain that $N \models e_1$.

We still owe the following proof:

Proof of Lemma 13.28. The infinitary case is rather simple: we take $B' = B$ and consequently $h = i_\varphi(N)$. For the finitary case, first note by using the fact that each subset of that B is epic basic, we have that $(\mu_{B_f})_{B_f \subseteq B \text{ finite}}$ is the directed co-limit of $(h_{B_f, B'_f})_{B_f \subseteq B'_f \subseteq B \text{ finite}}$

$$\begin{tikzcd}
M_{B_f} \arrow[rr, "h_{B_f,B'_f}"] \arrow[dr, "\mu_{B_f}"'] & & M_{B'_f} \arrow[dl, "\mu_{B'_f}"] \\
& M_B &
\end{tikzcd}$$

where μ_{B_f} and h_{B_f, B'_f} are the unique model homomorphisms given by the fact that each subset of B is epic basic. Because the reduct functors corresponding to representable signature morphisms preserve directed co-limits (cf. Prop. 6.6) we have that $(\mu_{B_f} \upharpoonright_\chi)_{B_f \subseteq B \text{ finite}}$ is also a directed co-limit. Because φ is finitary representable, M_φ is finitely presented. Hence there exists a finite set of sentences $B' \subseteq B$ and a model homomorphism $h\colon\ M_\varphi \to M_{B'} \upharpoonright_\chi$ such that $h; \mu_{B'} \upharpoonright_\chi = i_\varphi(N)$. $\square$

Representable substitutions for presentations. The only condition of the completeness Thm. 13.27 which has a rather technical nature is the existence of representable substitutions for presentations. However in many situations this can be reduced to a simpler form by the following general result.

Proposition 13.29. *In any institution I_0 with a sub-category $\mathcal{D}$ of representable signature morphisms such that*

1. *every set of sentences is epic basic, and*
2. *the representation M_φ of any signature morphism $\varphi \in \mathcal{D}$ is projective with respect to $\mathcal{D}$-reducts of model homomorphisms of the form $0_\Sigma \to M_E$ for all sets E of sentences,*

then the institution of presentations I_0^{pres} has representable $\mathcal{D}$-substitutions whenever I_0 has representable $\mathcal{D}$-substitutions.

Proof. Note that because the institution has only epic basic sets of sentences, each presentation (Σ, E) has an initial model $0_{(\Sigma,E)}$ which is precisely M_E, the model defining E as a basic set of sentences.

Let $\chi_1\colon\ \Sigma \to \Sigma_1$ and $\chi_2\colon\ \Sigma \to \Sigma_2$ and let $h'\colon\ M_{E_1} \upharpoonright_{\chi_1} \to M_{E_2} \upharpoonright_{\chi_2}$ be a Σ-model homomorphism where E_i are sets of Σ_i-sentences. We have to show that h' determines a I_0^{pres}-substitution $\theta\colon\ (\chi_1\colon\ \Sigma \to (\Sigma_1, E_1)) \to (\chi_2\colon\ \Sigma \to (\Sigma_2, E_2))$ such that the diagram below commutes:

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma_2,E_2) \arrow[r, "i^{E_2}_{\chi_2}", "\cong"'] \arrow[d, "\mathsf{Mod}(\theta)"'] & (M_{E_2}\upharpoonright_{\chi_2})/\mathsf{Mod}(\Sigma) \arrow[d, "h'/\mathsf{Mod}(\Sigma)"] \\
\mathsf{Mod}(\Sigma_1,E_1) \arrow[r, "\cong", "i^{E_1}_{\chi_1}"'] & (M_{E_1}\upharpoonright_{\chi_1})/\mathsf{Mod}(\Sigma)
\end{tikzcd}$$

Because M_{χ_1} is projective with respect to $M_{\chi_2} = 0_{\Sigma_2}\upharpoonright_{\chi_2} \to M_{E_2}\upharpoonright_{\chi_2}$ there exists a homomorphism h such that the diagram below commutes:

$$\begin{tikzcd}
M_{\chi_2} = 0_{\Sigma_2}\upharpoonright_{\chi_2} \arrow[r] & M_{E_2}\upharpoonright_{\chi_2} \\
M_{\chi_1} = 0_{\Sigma_1}\upharpoonright_{\chi_1} \arrow[r] \arrow[u, "h"] & M_{E_1}\upharpoonright_{\chi_1} \arrow[u, "h'"']
\end{tikzcd}$$

Because I_0 has representable $\mathcal{D}$-substitutions there exists a $\mathcal{D}$-substitution $\theta :\ \chi_1 \to \chi_2$ in I_0 such that the diagram below commutes:

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma_2) \arrow[r, "i_{\chi_2}", "\cong"'] \arrow[d, "\mathsf{Mod}(\theta)"'] & M_{\chi_2}/\mathsf{Mod}(\Sigma) \arrow[d, "h/\mathsf{Mod}(\Sigma)"] \\
\mathsf{Mod}(\Sigma_1) \arrow[r, "\cong", "i_{\chi_1}"'] & M_{\chi_1}/\mathsf{Mod}(\Sigma)
\end{tikzcd}$$

We show that θ is the desired substitution in I_0^{pres}. For this we first show that for each (Σ_2,E_2)-model M_2, the reduct $\mathsf{Mod}(\theta)(M_2)$ is a (Σ_1,E_1)-model, and second we prove that $i^{E_2}_{\chi_2};(h'/\mathsf{Mod}(\Sigma)) = \mathsf{Mod}(\theta);i^{E_1}_{\chi_1}$ (i.e., the commutativity of the first diagram in this proof).

For showing that $\mathsf{Mod}(\theta)(M_2) \models E_1$ we use that $\mathsf{Mod}(\theta)(M_2) = i^{-1}_{\chi_1}(h;i_{\chi_2}(M_2))$. From the commutativity of the diagram

$$\begin{tikzcd}
M_{\chi_1} = 0_{\Sigma_1}\upharpoonright_{\chi_1} \arrow[r, "h"] \arrow[d] & M_{\chi_2} = 0_{\Sigma_2}\upharpoonright_{\chi_2} \arrow[r, "i_{\chi_2}(M_2)"] \arrow[d] & M_2\upharpoonright_{\chi_2} \\
M_{E_1}\upharpoonright_{\chi_1} \arrow[r, "h'"'] & M_{E_2}\upharpoonright_{\chi_2} \arrow[ur] &
\end{tikzcd}$$

we obtain that

$$i^{-1}_{\chi_1}(h';(M_{E_2} \to M_2)\upharpoonright\chi_2) :\ M_{E_1} \to i^{-1}_{\chi_1}(h;i_{\chi_2}(M_2)) = \mathsf{Mod}(\theta)(M_2)$$

which implies that $\mathsf{Mod}(\theta)(M_2) \models E_1$.

For showing that $i^{E_2}_{\chi_2};(h'/\mathsf{Mod}(\Sigma)) = \mathsf{Mod}(\theta);i^{E_1}_{\chi_1}$ we take any $M_2 \in \mathsf{Mod}(\Sigma_2,E_2)$ and by using again the commutativity of the last diagram above we obtain that $(i^{E_1}_{\chi_1})^{-1}(h';i^{E_2}_{\chi_2}(M_2)) = i^{-1}_{\chi_1}(h;i_{\chi_2}(M_2))$. $\square$

The first condition of Prop. 13.29 is one of the conditions on I_0 required by Birkhoff completeness Thm. 13.37. The second condition of Prop. 13.29 (closely related to the concept of 'projectively representable' of Prop. 6.8) is very easy to establish in institutions in which the model homomorphisms $0_\Sigma \to M_E$ are surjective. One rather typical example is **AFOL**, the atomic sub-institution of **FOL**, with $\mathcal{D}$ being the class of all signature extensions with constants.

Corollary 13.30. $\mathbf{AFOL}^{\mathrm{pres}}$ *has representable $\mathcal{D}$-substitutions.*

Proof. Each set E of atoms is epic basic and the model homomorphism $0_\Sigma \to M_E$ is surjective. The reducts of surjective model homomorphisms are surjective too. For each signature extension with constants χ the model M_χ (which represents χ) is a free model (i.e., term model) hence it is projective with respect to any surjective homomorphism. **AFOL** has representable $\mathcal{D}$-substitutions because each model homomorphism between free models $h\colon\ T_\Sigma(X) \to T_\Sigma(Y)$ determines the Σ-substitution θ defined by $\theta(x) = h(x)$ for each $x \in X$. Thus all conditions of Prop. 13.29 are fulfilled, hence $\mathbf{AFOL}^{\mathrm{pres}}$ has representable $\mathcal{D}$-substitutions. □

This type of argument can be replicated in many institutions, one notable exception being the institution of the existence equations in partial algebra (**PA**). In this example the model homomorphisms $0_\Sigma \to M_E$ are not necessarily surjective.

The Substitutivity rule revisited. The conditions underlying Prop. 13.29 have also another important consequence: they permit a significantly simpler formulation of the Substitutivity rule which uses substitutions in the base institution rather than in the institution of the presentations. As usually, the finitary variant of the result below requires the conditions in the brackets.

Proposition 13.31. *Under the conditions of Prop. 13.29 and if the entailment system corresponding to the proof system of I_1 has* (finitary) Modus Ponens for Sen_0, *meaning that*

$$\Gamma \cup B \vdash_\Sigma e \ \text{ if and only if } \ \Gamma \vdash_\Sigma B \Rightarrow e$$

for any sets of sentences $\Gamma \subseteq \mathsf{Sen}_1(\Sigma)$ and (finite) $B \subseteq \mathsf{Sen}_0(\Sigma)$ and each sentence $e \in \mathsf{Sen}_1(\Sigma)$, then we may use only Substitutivity rules of the form

$$(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$$

where θ is any $\mathcal{D}$-substitution in I_0.

Proof. Let us note that the Substitutivity rules of the form

$$(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$$

for θ any $\mathcal{D}$-substitution in I_0 are just special cases of the full Substitutivity rules by considering $B = \emptyset$. Therefore we have only to show that for any $\mathcal{D}$-substitution $\theta\colon\ (\varphi\colon\ \Sigma \to (\Sigma_1, \emptyset)) \to (\chi\colon\ \Sigma \to (\Sigma_2, B))$ in I_0^{pres} we can have a proof

$$(\forall\varphi)\rho \vdash (\forall\chi)(\wedge B \Rightarrow \theta(\rho))$$

by using the Substitutivity rule in the simpler form proposed above.

The key to obtaining such proof lies in the proof of Prop. 13.29 which shows that each I_0^{pres}-substitution $\theta : (\varphi : \Sigma \to (\Sigma_1, \emptyset)) \to (\chi : \Sigma \to (\Sigma_2, B))$ determines an I_0-substitution $\theta' : (\varphi : \Sigma \to \Sigma_1) \to (\chi : \Sigma \to \Sigma_2)$ such that $\mathsf{Sen}_0(\theta') = \mathsf{Sen}_0(\theta)$. By hypothesis we have that $(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$. Because $\theta(\rho) \cup B \vdash \theta(\rho)$ and because I_1 has *Modus Ponens for* Sen_0 we have that $\theta(\rho) \vdash B \Rightarrow \theta(\rho)$. Because $(\forall\chi)\theta(\rho) \vdash (\forall\chi)\theta(\rho)$ and because the proof system of I has universal $\mathcal{D}$-quantification we have that $\chi((\forall\chi)\theta(\rho)) \vdash \theta(\rho)$. This implies $\chi((\forall\chi)\theta(\rho)) \vdash B \Rightarrow \theta(\rho)$ and again by the universal $\mathcal{D}$-quantification property we obtain $(\forall\chi)\theta(\rho) \vdash (\forall\chi)B \Rightarrow \theta(\rho)$ which leads to $(\forall\varphi)\rho \vdash (\forall\chi)B \Rightarrow \theta(\rho)$. □

General Birkhoff proof systems

Birkhoff proof systems for Horn institutions refine the universal proof systems defined above by assuming a proof system for I_0 and by *defining* a proof system for I_1 rather than assuming a proof system for I_1. Thus

- we assume a proof system $(\mathbb{S}ig, \mathsf{Sen}_0, \mathsf{Pf}_0)$ for I_0 and
- for I_1 we consider the free proof system $(\mathbb{S}ig, \mathsf{Sen}_1, \mathsf{Pf}_1)$ over $(\mathbb{S}ig, \mathsf{Sen}_0, \mathsf{Pf}_0)$ with (finitary) *Modus Ponens for* Sen_0, i.e., there exists a natural isomorphism

  $$\mathsf{Pf}_1(\Sigma)(\Gamma \cup B, e) \cong \mathsf{Pf}_1(\Sigma)(\Gamma, B \Rightarrow e)$$

  for any $\Gamma \subseteq \mathsf{Sen}_1(\Sigma)$, any (finite) $B \subseteq \mathsf{Sen}_0(\Sigma)$ and each $e \in \mathsf{Sen}_1(\Sigma)$.

The Birkhoff proof system is *finitary* if and only if it is finitary as a universal proof system and is generated by using the finitary version of Modus Ponens for Sen_0, otherwise it is infinitary.

Fact 13.32. *The proof system of I_1 is sound if the proof system of I_0 is sound.*

In order to instantiate the general universal completeness Thm. 13.27 to the Birkhoff proof system we need to address the first and the last conditions of Thm. 13.27.

The following result addresses the first condition plus the compactness condition from the definition of the finitary $\mathcal{D}$-universal proof systems. As usual, the result comes in a finitary version (with the information contained within brackets included) and in an infinitary version.

Proposition 13.33. *Let us assume that*

1. *each set of I_0-sentences is basic, and*
2. *the proof system of I_0 is complete (and its entailment system is compact).*

Then the proof system of I_1 is complete (and its entailment system is compact).

Proof. Because the proof system of I_1 has (finitary) *Modus Ponens for* Sen_0 it is enough to prove that

$$\Gamma \models \rho \text{ implies } \Gamma \vdash \rho$$

for each $\Gamma \subseteq \mathsf{Sen}_1(\Sigma)$ and each $\rho \in \mathsf{Sen}_0(\Sigma)$. Let M_{Γ_0} be the model defining the set of sentences $\Gamma_0 = \{e \in \mathsf{Sen}_0(\Gamma) \mid \Gamma \vdash e\}$ as basic. We use the following couple of lemmas.

Lemma 13.34. *$M_{\Gamma_0} \models e$ if and only if $\Gamma \vdash e$ for each sentence $e \in \mathsf{Sen}_0(\Sigma)$.*

Lemma 13.35. *$M_{\Gamma_0} \models \Gamma$.*

If $\Gamma \models \rho$ then by Lemma 13.35 we have that $M_{\Gamma_0} \models \rho$. Now by Lemma 13.34 we obtain $\Gamma \vdash \rho$.

For the compactness of the entailment system of I_1 first let us recall that the sentences of I_1 are of the form $H \Rightarrow C$ where C is an I_0 sentence and H is a *finite* set of I_0 sentences. The compactness conclusion is obtained by applying an argument similar to those of Cor. 13.8 or of Cor.13.18 as follows. We consider the sub-system of the compact entailments of I_1; this is an entailment (sub-)system by Prop. 13.7. It contains the entailment system of I_0 since the latter is compact by the hypotheses. It also has the finitary *Modus Ponens for* Sen_0 because for any finite $B \subseteq \mathsf{Sen}_0(\Sigma)$ the entailment $\Gamma \cup B \vdash e$ is compact if and only if the entailment $\Gamma \vdash B \Rightarrow e$ is compact. Indeed, $\Gamma \cup B \vdash e$ compact means that there exists finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \cup B \vdash e$ which by the *Modus Ponens for* Sen_0 property is equivalent to $\Gamma_0 \vdash B \Rightarrow e$ which means $\Gamma \vdash B \Rightarrow e$ is compact. Now because the entailment system of I_1 is the least one containing the entailment system of I_0 and satisfying the finitary *Modus Ponens for* Sen_0 property we may conclude that this is exactly the compact sub-system of the entailment system of I_1, which just means that the entailment system of I_1 is compact.

We still owe the proof of the Lemmas 13.34 and 13.35.

Proof of Lemma 13.34. The implication from right to left holds by the definition of Γ_0. For the other implication let us consider a sentence e such that $M_{\Gamma_0} \models e$. For any model M such that $M \models \Gamma_0$, because Γ_0 is basic there exists a model homomorphism $M_{\Gamma_0} \to M$. Since $M_{\Gamma_0} \models e$ and e is basic there exists another model homomorphism $M_e \to M_{\Gamma_0}$. These give a model homomorphism $M_e \to M$ which means $M \models e$. We have thus shown that $\Gamma_0 \models e$.

By the completeness of I_0 we obtain that $\Gamma_0 \vdash e$. For the infinitary case let us take $\Gamma_0' = \Gamma_0$. For the finitary case, since an entailment system corresponding to the proof system of I_0 is compact, there exists $\Gamma_0' \subseteq \Gamma_0$ finite such that $\Gamma_0' \vdash e$. By the definition of Γ_0 we obtain that $\Gamma \vdash \Gamma_0'$ hence $\Gamma \vdash e$.

Proof of Lemma 13.35. Let us consider an I_1 sentence $H \Rightarrow C \in \Gamma$ and let us assume that $M_{\Gamma_0} \models H$. By Lemma 13.34 we have that $\Gamma \models H$ and because $H \Rightarrow C \in \Gamma$ and the proof system for I_1 has (finitary) *Modus Ponens for* Sen_0 we obtain that $\Gamma \vdash C$. By Lemma 13.34 again we deduce $M_{\Gamma_0} \models C$. $\Box$

The following shows that the last condition of Thm. 13.27 is also fulfilled.

Proposition 13.36. *Under the conditions of Prop. 13.33, for each set E of sentences $E \subseteq \mathsf{Sen}_1(\Sigma)$ and each sentence $e \in \mathsf{Sen}_1(\Sigma)$ we have that*

$$E \models e \text{ if and only if } M_B \models (\wedge E \Rightarrow e) \text{ for each set of sentences } B \subseteq \mathsf{Sen}_0(\Sigma)$$

(where M_B are the models defining B as basic sets of sentences).

Proof. Let $e = (H \Rightarrow C)$ with $H \subseteq \mathsf{Sen}_0(\Sigma)$ and $C \in \mathsf{Sen}_0(\Sigma)$. Consider the model $M_{(E\cup H)_0}$ defining $(E \cup H)_0 = \{\rho \in \mathsf{Sen}_0(\Sigma) \mid E \cup H \models \rho\}$. By Lemma 13.35 we have that $M_{(E\cup H)_0} \models E \cup H$. By the hypothesis this implies $M_{(E\cup H)_0} \models H \Rightarrow C$. Because $M_{(E\cup H)_0} \models H$ too, it follows that $M_{(E\cup H)_0} \models C$. Since C is basic there exists a homomorphism $M_C \to M_{(E\cup H)_0}$.

Now let M be any models such that $M \models E \cup H$. By Lemma 13.34 we obtain that $M \models (E \cup H)_0$. Because $(E \cup H)_0$ is basic, there exists a homomorphism $M_{(E\cup H)_0} \to M$. Together with the homomorphism $M_C \to M_{(E\cup H)_0}$ this gives a homomorphism $M_C \to M$ which means $M \models C$. $\square$

Propositions 13.33 and 13.36 lead to the following completeness result for Horn institutions obtained as an instance of the general universal completeness Thm. 13.27 (as usual the finitary version is obtained by adding the conditions in the brackets).

Theorem 13.37. *The (finitary) Birkhoff proof system for a (finitary) $\mathcal{D}$-Horn institution is complete if*

1. *the proof system of I_0 is complete (and its entailment system is compact),*
2. *every signature morphism in $\mathcal{D}$ is (finitely) representable,*
3. *every set of sentences in I_0 is epic basic, and*
4. *I_0^{pres} has representable $\mathcal{D}$-substitutions.*

The Birkhoff proof system of HCL. In order to develop a concrete sound and complete Birkhoff proof system for **HCL** we set the parameters of Thm. 13.37 for this example as follows:

- the institution I is **HCL**,
- the institution I_0 is **AFOL** (the atomic sub-institution of **FOL**),
- $\mathcal{D}$ is the class of all signature extensions with a finite number of constants,
- the system of proof rules for **AFOL** is given by the following set of rules for any **FOL** signature (S,F,P):

$$
\begin{array}{ll}
(R) & \emptyset \vdash t = t \;\text{ for each term }\; t \\
(S) & t = t' \vdash t' = t \;\text{ for any terms }\; t \;\text{ and }\; t' \\
(T) & \{t = t', t' = t''\} \vdash t = t'' \;\text{ for any terms }\; t, t' \;\text{ and }\; t'' \\
(F) & \{t_i = t_i' \mid 1 \le i \le n\} \vdash \sigma(t_1,\ldots,t_n) = \sigma(t_1',\ldots,t_n') \;\text{ for any }\; \sigma \in F \\
(P) & \{t_i = t_i' \mid 1 \le i \le n\} \cup \{\pi(t_1,\ldots,t_n)\} \vdash \pi(t_1',\ldots,t_n') \;\text{ for any }\; \pi \in P.
\end{array}
$$

Because of the condition of Thm. 13.3 which requires the sentence translations to be injective, we need to consider the variants of **HCL** and **AFOL** which have only injective signature morphisms. However, by Cor. 13.4, if we work with entailment systems rather than proof systems, this injectivity condition is no longer needed. For the rest of this section we will ignore this issue.

Proposition 13.38. **AFOL** *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by a simple routine check. For proving the completeness, for any set E of atoms for a signature (S,F,P) we define

$$\equiv_E = \{(t,t') \mid E \vdash t = t'\}.$$

By (R), (S), (T) and (P) this is an F-congruence. Then we define a model M_E as follows:

- the (S,F)-algebra part of M_E is defined as the quotient of the initial algebra (term) (S,F)-algebra T_F by $\equiv_E$, and
- for each relation symbol $\pi \in P$, we define $(M_E)_\pi = \{x/_{\equiv_E} \mid E \vdash \pi(x)\}$.

The definition of $(M_E)_\pi$ is correct because of the rule (P). Now we note that for each (S,F,P)-atom ρ,

$$E \vdash \rho \text{ if and only if } M_E \models \rho.$$

Now if $E \models \rho$ then $M_E \models \rho$ which means $E \vdash \rho$. □

We are now able to formulate the following finitary and infinitary instances of the general Birkhoff completeness Thm. 13.37.

Corollary 13.39. *The finitary Birkhoff proof system for* **HCL** *is sound and complete. Moreover, this proof system is obtained as the free proof system*

- *with universal quantification, and*
- *such that for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences there exists a natural isomorphism*

$$\mathsf{Pf}(S,F,P)(\Gamma \cup H, C) \cong \mathsf{Pf}(S,F,P)(\Gamma, H \Rightarrow C)$$

which is generated by the following system of finitary rules for a signature (S,F,P):

$$\begin{array}{rl}
(R) & \emptyset \vdash t = t \ \textit{for each term } t \\
(S) & t = t' \vdash t' = t \ \textit{for any terms } t \textit{ and } t' \\
(T) & \{t = t', t' = t''\} \vdash t = t'' \ \textit{for any terms } t, t' \textit{ and } t'' \\
(F) & \{t_i = t'_i \mid 1 \le i \le n\} \vdash \sigma(t_1,\dots,t_n) = \sigma(t'_1,\dots,t'_n) \ \textit{for any } \sigma \in F \\
(P) & \{t_i = t'_i \mid 1 \le i \le n\} \cup \{\pi(t_1,\dots,t_n)\} \vdash \pi(t'_1,\dots,t'_n) \ \textit{for any } \pi \in P \\
(\mathit{Subst}) & (\forall X)\rho \vdash (\forall Y)\theta(\rho) \ \textit{for any } (S,F,P)\textit{-sentence } \rho \textit{ and} \\
& \ \textit{for any substitution } \theta : X \to T_F(Y)
\end{array}$$

Corollary 13.40. *The infinitary Birkhoff proof system for* $\mathbf{HCL}_\infty$ *is sound and complete for the same rules as those for Cor. 13.39.*

Exercises

13.20. [85] Consider an institution I_0 with a sub-category $\mathcal{D}$ of representable signature morphisms such that

1. I has weak model amalgamation,
2. I has representable $\mathcal{D}$-substitutions,
3. $\mathcal{D}$ is stable under pushouts,
4. for each pushout of signature morphisms such that $\chi \in \mathcal{D}$

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\chi"'] & \Sigma' \arrow[d, "\varphi'"] \\ \Sigma_1 \arrow[r, "\chi_1"'] & \Sigma'_1 \end{tikzcd}$$

for any Σ'-sentence ρ' and Σ_1-sentence ρ_1 if $\varphi'(\rho') = \chi_1(\rho_1)$, then there exists a Σ-sentence ρ such that $\rho' = \varphi(\rho)$ and $\rho_1 = \chi(\rho)$. (Compare this condition with the co-amalgamation property of Ex. 4.19.)

Then the translation of any $\mathcal{D}$-Substitutivity rule along a signature morphism is a $\mathcal{D}$-Substitutivity rule.

13.21. Under the conditions of Thm. 13.27, the entailment system of the finitary $\mathcal{D}$-universal proof system of I is compact.

13.22. Complete proof system for UNIV
Any complete and compact proof system for the sub-institution of **FOL** determined by the quantifier-free sentences (i.e., sentences formed from atoms by Boolean connectives, without any quantifications), by the universal completeness Thm. 13.27 determines a complete universal proof system for **UNIV**(the institution of the universal **FOL** sentences).

13.23. Birkhoff calculus for preordered algebras
HPOA, the institution of Horn preordered algebras, gets a sound and complete proof system obtained as the free proof system

- with universal quantification, and
- such that for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences there exists a natural isomorphism

$$\mathsf{Pf}(S,F)(\Gamma \cup H, C) \cong \mathsf{Pf}(S,F)(\Gamma, H \Rightarrow C)$$

which is generated by the following system of finitary rules for a signature (S,F):

$$\begin{array}{rl}
(R) & \emptyset \vdash t = t \ \text{ for each term } t \\
(RP) & \emptyset \vdash t \leq t \ \text{ for each term } t \\
(S) & t = t' \vdash t' = t \ \text{ for any terms } t \text{ and } t' \\
(T) & \{t = t', t' = t''\} \vdash t = t'' \ \text{ for any terms } t, t' \text{ and } t'' \\
(TP) & \{t \leq t', t' \leq t''\} \vdash t \leq t'' \ \text{ for any terms } t, t' \text{ and } t'' \\
(F) & \{t_i = t'_i \mid 1 \leq i \leq n\} \vdash \sigma(t_1, \ldots, t_n) = \sigma(t'_1, \ldots, t'_n) \ \text{ for any } \sigma \in F \\
(FP) & \{t_i \leq t'_i \mid 1 \leq i \leq n\} \vdash \sigma(t_1, \ldots, t_n) \leq \sigma(t'_1, \ldots, t'_n) \ \text{ for any } \sigma \in F \\
(Comp) & \{t'_1 = t_1, t_1 \leq t_2, t_2 = t'_2\} \vdash t'_1 = t'_2 \ \text{ for all terms } t_1, t'_1, t_2, t'_2 \\
(Subst) & (\forall X)\rho \vdash (\forall Y)\theta(\rho) \text{ for any } (S,F)\text{-sentence } \rho \text{ and for any substitution } \theta : X \to T_F(Y).
\end{array}$$

13.24. [36] Let I_0 be an institution with a sub-category $\mathcal{D}$ of representable signature morphisms such that

1. every set of sentences is epic basic, and
2. for any signature morphism $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$ in $\mathcal{D}$ and any set E_2 of Σ_2-sentences, every homomorphism $h : M_{\chi_1} \to M_{E_2}\restriction_{\chi_2}$ determines an I_0^{pres}-substitution $\theta_h : (\chi_1 : \Sigma \to \Sigma_1) \to (\chi_2 : \Sigma \to (\Sigma_2, E_2))$ such that the diagram below commutes:

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma_2,E_2) \arrow[r, "i_{\chi_2}"] \arrow[d, "\mathsf{Mod}(\theta_h)"'] & (M_{E_2}\restriction_{\chi_2})/\mathsf{Mod}(\Sigma) \arrow[d, "h/\mathsf{Mod}(\Sigma)"] \\
\mathsf{Mod}(\Sigma) \arrow[r, "i_{\chi_1}"'] & \mathsf{Mod}(\Sigma_1)
\end{tikzcd}$$

Then the institution of presentations I_0^{pres} has $\mathcal{D}$-substitutions.

13.25. [36] **Birkhoff calculus for partial algebras**
$QE^{\omega}(\mathbf{PA})$, the institution of partial algebras with finitary quasi-existence equations as sentences, gets a sound and complete proof system obtained as the free proof system

- with universal $\mathcal{D}$-quantification for $\mathcal{D}$ the class of signature extensions with a finite number of total constants, and
- such that for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences there exists a natural isomorphism

$$\mathsf{Pf}(S,TF,PF)(\Gamma \cup H, C) \cong \mathsf{Pf}(S,TF,PF)(\Gamma, H \Rightarrow C)$$

which is generated by the following system of finitary rules for a signature (S,TF,PF):

$$\begin{array}{rl}
(S) & t \stackrel{e}{=} t' \vdash t' \stackrel{e}{=} t \ \text{ for any terms } t \text{ and } t' \\
(T) & \{t \stackrel{e}{=} t', t' \stackrel{e}{=} t''\} \vdash t \stackrel{e}{=} t'' \ \text{ for any terms } t, t' \text{ and } t'' \\
(F) & \{t_i \stackrel{e}{=} t'_i \mid 1 \le i \le n\} \cup \{\mathsf{def}(\sigma(t_1,\dots,t_n)), \mathsf{def}(\sigma(t'_1,\dots,t'_n))\} \\
 & \vdash \sigma(t_1,\dots,t_n) \stackrel{e}{=} \sigma(t'_1,\dots,t'_n) \ \text{ for any } \sigma \in TF \cup PF \\
(\mathit{Totality}) & \{\mathsf{def}(t_i) \mid 1 \le i \le n\} \vdash \mathsf{def}(\sigma(t_1,\dots,t_n)) \ \text{ for any } \sigma \in TF \\
(\mathit{Subterm}) & \{\mathsf{def}(\sigma(t_1,\dots,t_n))\} \vdash \{\mathsf{def}(t_i) \mid 1 \le i \le n\} \ \text{ for any } \sigma \in TF \cup PF \\
(\mathit{Subst}) & (\forall X)\rho \vdash (\forall Y)(\bigwedge_{x \in X} \mathsf{def}(\theta(x)) \Rightarrow \theta(\rho)) \ \text{ for any } (S,TF,PF)\text{-sentence } \rho \\
 & \ \text{and for any substitution } \theta : X \to T_{TF \cup PF}(Y).
\end{array}$$

(*Hint:* Since the second condition of Prop. 13.29 does not hold, apply the result of Ex. 13.24.)

Extend the Birkhoff calculus for $QE^{\omega}(\mathbf{PA})$ to the institution $QE(\mathbf{PA})$ of the infinitary quasi-existence equations.

Notes. Usually, categorical logic works with categories of sentences, where morphisms are (equivalence classes) of proof terms [109]. However, this captures provability between single sentences only, while logic traditionally studies provability from a set of sentences. Proof systems have been defined in [137] that reconcile both approaches by considering categories of sets of sentences. This also avoids one of the big faults of categorical logic, that the definition of implication depends on (the existence of) conjunctions. Systems of proof rules were introduced in [52] which also developed the free proof system Thm. 13.3 and its compactness Cor. 13.8. Note that our concept of proof rules admits multiple conclusions, which constitute a slight generalization of the usual practice in actual logics which use only single conclusion rules. Lawvere [112] defined quantification as adjoint to substitutions, while [137] defines quantification as adjoint to sentence translation along signature morphisms. The free proof system with (universal) quantification construction (Thm. 13.17) and its compactness property (Cor. 13.18) have been developed in [52].

Entailment systems were probably defined for the first time under the name of π-institutions in [67], and later modified by [123] in order to formalize the notion of syntactic consequence. [90] gives a similar definition but restricted to finite sets of sentences. Meseguer [123] showed how to construct an institution from an entailment system by producing a model theory directly from a comma category construction on theories, and [58] extends this construction by embedding the category of entailment systems into the category of [ordinary] institutions.

The general soundness results given by Prop. 13.5 and 13.19 are due to [52].

Birkhoff calculus and its completeness result have been developed for a single-sorted conditional version of **EQL** in [24]; this has been extended to many sorts in [78], and to arbitrary institutions in [36]. The layered approach to institution-independent completeness was invented by [28] within the framework of specification theory. Later this was extended to Gödel completeness [147] and Birkhoff completeness [36]. The latter work revealed the surprisingly close relationship between the completeness of the institution of universal sentences (**UNIV**) and the general concept of Birkhoff completeness.

Chapter 14

Specification

This chapter is devoted to some applications of institution-independent model theory to specification theory, thus it digresses slightly from the main topic of the book.

Apart from their practical significance, applications to specification theory also have a strong historical significance. The concept of institution and that of institution-independent model theory arose from specification theory due to a high proliferation of logics in the practice of formal software specification. It is now standard practice to base each specification language rigorously on some institution. A 'basic' specification would then appear as a finite set of sentences in some signature of the institution. 'Structured' specifications can be constructed from the basic ones by several specification building operators, which are defined at the institution-independent level. Based on the satisfaction relation of the institution and on the semantics of the specification building operators, we can assign to each structured specification a signature and a class of models. This is the 'denotation' of the specification. In this manner, the structured specifications of an institution can be organized as an institution too, which inherits many of the properties of the base institution.

We show that under some conditions structured specifications can be reduced to normal forms. This result is further used for showing that by assuming model amalgamation and interpolation, any sound and complete proof system for the base institution can be lifted to the structured specifications.

Specification with 'predefined types' extends ordinary specification with predefined entities, such as real numbers, both at the semantic and syntactic level. This extension is required by the reality of actual specification languages. In the final section of this chapter we discuss predefined types from an institution-independent perspective, by constructing an institution of predefined types on top of any base institution that has a system of elementary diagrams.

14.1 Structured Specifications

In this section we define the concept of structured specifications over arbitrary institutions and investigate some fundamental semantic properties of structured specifications. For this we need the following concept.

Intersection of signatures. Recall (from Sect. 4.5) that an institution I is *inclusive* when its category of signatures $\mathbb{S}ig$ comes equipped with an inclusion system such that $\mathsf{Sen}(\Sigma) \subseteq \mathsf{Sen}(\Sigma')$ whenever $\Sigma \hookrightarrow \Sigma'$ is an inclusion of signatures.

Given two signatures Σ_1 and Σ_2 in an inclusive institution, let $\Sigma_1 \cup \Sigma_2$ be their *union*, i.e., their least upper bound in the category of inclusions of signatures. Then their *intersection* $\Sigma_1 \cap \Sigma_2$ is defined as the unique pullback square such that $\Sigma_1 \cap \Sigma_2 \hookrightarrow \Sigma_1$ and $\Sigma_1 \cap \Sigma_2 \hookrightarrow \Sigma_2$ are inclusions.

$$\begin{tikzcd}
\Sigma_1 \cap \Sigma_2 \arrow[r] \arrow[d] & \Sigma_1 \arrow[d] \\
\Sigma_2 \arrow[r] & \Sigma_1 \cup \Sigma_2
\end{tikzcd}$$

The existence and the uniqueness of this pullback square can be shown quite easily (Ex. 4.48). However from this assumption it does not follow that such intersection-union squares are pushouts too. Since this property is rather desirable for applications and it also holds in many concrete situations of interest, let us assume it for this chapter. Hence

The intersection-union squares of signatures are pushouts.

Structured specifications. Given an inclusive institution I, its *structured specifications* (or just *specifications* for short) are defined from the finite presentations by iteration of several specification building operators. The semantics of each specification SP is given by its *signature* $Sig[\mathrm{SP}]$ and its category of *models* $Mod[\mathrm{SP}]$. Below we sometimes define only the class of objects for each $Mod[\mathrm{SP}]$, the category $Mod[\mathrm{SP}]$ being the corresponding full subcategory of $\mathsf{Mod}(Sig(\mathrm{SP}))$.

PRES. Each finite presentation (Σ, E) is a specification such that

- $Sig[(\Sigma, E)] = \Sigma$, and
- $Mod[(\Sigma, E)] = \mathsf{Mod}^I(\Sigma, E)$.

UNION. For any specifications SP_1 and SP_2 we can take their *union* $\mathrm{SP}_1 \cup \mathrm{SP}_2$ with

- $Sig[\mathrm{SP}_1 \cup \mathrm{SP}_2] = Sig[\mathrm{SP}_1] \cup Sig[\mathrm{SP}_2]$, and
- $|Mod[\mathrm{SP}_1 \cup \mathrm{SP}_2]| = \{M \in \mathsf{Mod}^I(Sig[\mathrm{SP}_1 \cup \mathrm{SP}_2]) \mid M\!\upharpoonright_{Sig[\mathrm{SP}_i]} \in Mod[\mathrm{SP}_i]$ for each $i \in \{1,2\}$.

TRANS. For any specification SP and signature morphism $\varphi\colon\ Sig(\mathrm{SP}) \to \Sigma'$ we can take its *translation along* φ denoted by $\mathrm{SP} \star \varphi$ and such that

- $Sig[\mathrm{SP} \star \varphi] = \Sigma'$, and

- $|Mod[\mathrm{SP}\star\varphi]| = \{M' \in \mathsf{Mod}^I(\Sigma') \mid M'\!\upharpoonright_\varphi \in Mod[\mathrm{SP}]\}$.

DERIV. For any specification SP′ and and signature morphism $\varphi : \Sigma \to Sig(\mathrm{SP}')$ we can take its *derivation along* φ denoted by $\varphi \mid \mathrm{SP}'$ and such that

- $Sig[\varphi \mid \mathrm{SP}'] = \Sigma$, and
- $|Mod[\varphi \mid \mathrm{SP}']| = \{M'\!\upharpoonright_\varphi \mid M' \in Mod[\mathrm{SP}']\}$.

FREE. For any specification SP′ and signature morphism $\varphi : \Sigma \to Sig[\mathrm{SP}']$ we can take the *persistently free specification of* SP′ *along* φ denoted SP'^{φ} and such that

- $Sig[\mathrm{SP}'^{\varphi}] = Sig[\mathrm{SP}']$, and
- $|Mod[\mathrm{SP}'^{\varphi}]| = \{M' \in Mod[\mathrm{SP}'] \mid M'$ strongly persistently $\beta_{\mathrm{SP}};\mathsf{Mod}^I(\varphi)$-free$\}$, where β_{SP} is the subcategory inclusion $Mod[\mathrm{SP}] \to \mathsf{Mod}^I(Sig[\mathrm{SP}])$.

Equivalent specifications. Two specifications SP_1 and SP_2 are *equivalent*, denoted $\mathrm{SP}_1 \models\!\!\mid \mathrm{SP}_2$, when $Sig[\mathrm{SP}_1] = Sig[\mathrm{SP}_2]$ and $Mod[\mathrm{SP}_1] = Mod[\mathrm{SP}_2]$. In general it is possible to have different specifications that are equivalent. When we are interested only in the semantics of specifications rather than in the way they are constructed, it does make sense to consider specifications modulo this equivalence relation.

Specification morphisms. A *specification morphism* $\varphi : \mathrm{SP}_1 \to \mathrm{SP}_2$ between specifications SP_1 and SP_2 is a signature morphism $\varphi : Sig[\mathrm{SP}_1] \to Sig[\mathrm{SP}_2]$ such that $M\!\upharpoonright_\varphi \in Mod[\mathrm{SP}_1]$ for each $M \in Mod[\mathrm{SP}_2]$.

Fact 14.1. *For any institution I, the specifications and their morphisms under the obvious composition form a category, denoted $\mathbb{S}pec^I$.*

Models and sentences of specifications. Note that Mod can be therefore regarded as a functor $Mod : \mathbb{S}pec^I \to \mathbb{C}at^{\mathrm{op}}$.

An SP-*sentence* for a specification SP is any $Sig[\mathrm{SP}]$-sentence; this determines a functor $Sen : \mathbb{S}pec^I \to \mathbb{S}et$.

A model $M \in Mod[\mathrm{SP}]$ *satisfies* a SP-sentence ρ if and only if $M \models_{Sig[\mathrm{SP}]} \rho$ in the original institution.

Fact 14.2. *Specifications together with their models and sentences form an institution of (structured) specifications on top of the original institution I. We denote the institutions of the specifications over I by I^{spec}.*

The institution I will be referred to as the 'base institution' while I^{spec} as the 'institution of specifications'.

Proposition 14.3. *For any institution I, I^{spec} is an institution. Moreover, there exists a structural institution adjoint morphism $(Sig, 1, \beta) : I^{\mathrm{spec}} \to I$, where for each specification* SP*, β_{SP} is the subcategory inclusion $Mod^I[\mathrm{SP}] \hookrightarrow \mathsf{Mod}(Sig[\mathrm{SP}])$.*

Special signature morphisms. A specification morphism $\varphi\colon\ \mathrm{SP}_1 \to \mathrm{SP}_2$ is

- *Refinement* when $Sig[\mathrm{SP}_1] = Sig[\mathrm{SP}_2]$ and φ is an identity (as signature morphism).
- *Inclusion* when φ is an abstract inclusion (as signature morphism). Specification inclusions correspond to the software engineering concept of *specification import*.
- *Strong* when $\mathrm{SP}_2 \mathrel{\mid\!\!\equiv\!\!\mid} \mathrm{SP}_1 \star \varphi$.
- *Closed* when $\mathrm{SP}_1 \mathrel{\mid\!\!\equiv\!\!\mid} \varphi \mid \mathrm{SP}_2$.

Inclusions of specifications. Let $\mathbb{S}pec^I/_{\mathrel{\mid\!\!\equiv\!\!\mid}}$ be the 'quotient' of the category of specifications under specification equivalence $\mathrel{\mid\!\!\equiv\!\!\mid}$. The class of objects of $\mathbb{S}pec^I/_{\mathrel{\mid\!\!\equiv\!\!\mid}}$ is $\{\mathrm{SP}/_{\mathrel{\mid\!\!\equiv\!\!\mid}} \mid \mathrm{SP} \in |\mathbb{S}pec^I|\}$ while the set of the arrows $\mathrm{SP}_1/_{\mathrel{\mid\!\!\equiv\!\!\mid}} \to \mathrm{SP}_2/_{\mathrel{\mid\!\!\equiv\!\!\mid}}$ is just the set of the specification morphisms $\varphi\colon\ \mathrm{SP}_1 \to \mathrm{SP}_2$. The correctness of this definition follows by the following simple remark.

Fact 14.4. *If* $\mathrm{SP}_1 \mathrel{\mid\!\!\equiv\!\!\mid} \mathrm{SP}'_1$ *and* $\mathrm{SP}_2 \mathrel{\mid\!\!\equiv\!\!\mid} \mathrm{SP}'_2$ *and* $h\colon\ \mathrm{SP}_1 \mathrel{\mid\!\!\equiv\!\!\mid} \mathrm{SP}_2$ *is a specification morphism, then* $h\colon\ \mathrm{SP}'_1 \mathrel{\mid\!\!\equiv\!\!\mid} \mathrm{SP}'_2$ *is also a specification morphism.*

The following result resembles the lifting of inclusion systems from signatures to theories given by Prop. 4.24.

Proposition 14.5. $\mathbb{S}pec^I/_{\mathrel{\mid\!\!\equiv\!\!\mid}}$ *has two inclusion systems inheriting the inclusion system of the signatures:*

- *a* closed *one, where abstract inclusions of specifications are closed abstract inclusions of signatures and abstract surjections of specifications are abstract surjections of signatures, and*
- *a* strong *one, where abstract inclusions of specifications are abstract inclusions of signatures and abstract surjections of specifications are strong abstract surjections of signatures.*

Moreover, the strong inclusion system for specifications has unions where $(\mathrm{SP}_1/_{\mathrel{\mid\!\!\equiv\!\!\mid}}) \cup (\mathrm{SP}_2/_{\mathrel{\mid\!\!\equiv\!\!\mid}}) = (\mathrm{SP}_1 \cup \mathrm{SP}_2)/_{\mathrel{\mid\!\!\equiv\!\!\mid}}$ *for any specifications* SP_1 *and* SP_2.

Proof. Any specification morphism $\varphi\colon\ \mathrm{SP} \to \mathrm{SP}'$ factors as a composition between a specification surjection and a specification inclusion

$$\begin{tikzcd}
\mathrm{SP} \arrow[rr, "\varphi"] \arrow[dr, "e_\varphi"'] & & \mathrm{SP}' \\
& \varphi(\mathrm{SP}) \arrow[ur, "i_\varphi"'] &
\end{tikzcd}$$

where

- $\varphi(\mathrm{SP}) = i_\varphi \mid \mathrm{SP}'$ in the case of the closed inclusion system, and
- $\varphi(\mathrm{SP}) = \mathrm{SP} \star e_\varphi$ in the case of the strong inclusion system

and $\varphi = e_\varphi ; i_\varphi$ is the factorization of φ as a composition between a surjection of signatures and an inclusion of signatures.

Notice that $\varphi(\mathrm{SP})$ can be replaced by any other equivalent specification, hence the above factorization is unique only modulo equivalence of specifications. $\square$

Co-limits of specifications. Many structuring constructs in actual specification languages rely on co-limits, especially pushouts, of specifications. One of them is the instantiation of parameterized specifications. A parameterized specification can be regarded as a specification morphism $\varphi\colon\ \mathrm{SP} \to \mathrm{SP}'$ where SP is the *parameter* and SP′ is the *body* of the specification. The abstract parameter SP can be interpreted into another concrete specification SP_1 by a specification morphism $\phi\colon\ \mathrm{SP} \to \mathrm{SP}_1$. The result SP'_1 of the instantiation of the parameter is given by a pushout of φ and ϕ.

Co-limits of signatures can be lifted to specifications in a manner similar to that of Prop. 4.2 lifting co-limits from signatures to presentations.

Proposition 14.6. *In any institution I, the signature functor $Sig\colon\ \mathbb{S}pec^I \to \mathbb{S}ig^I$ from the specifications to signatures lifts finite co-limits.*

Proof. We prove this result for the particular case of pushouts, the same argument working as well for any finite co-limit.

Consider any specification morphisms $\phi\colon\ \mathrm{SP} \to \mathrm{SP}'$ and $\varphi\colon\ \mathrm{SP} \to \mathrm{SP}_1$ and take a pushout of signatures

$$\begin{tikzcd}
Sig[\mathrm{SP}] \arrow[r, "\phi"] \arrow[d, "\varphi"'] & Sig[\mathrm{SP}'] \arrow[d, "\varphi'"] \\
Sig[\mathrm{SP}_1] \arrow[r, "\phi_1"'] & \Sigma'_1
\end{tikzcd}$$

We define $\mathrm{SP}'_1 = (\mathrm{SP}' \star \varphi') \cup (\mathrm{SP}_1 \star \phi_1)$. Notice that $\varphi'\colon\ \mathrm{SP}' \to \mathrm{SP}'_1$ and $\phi_1\colon\ \mathrm{SP}_1 \to \mathrm{SP}'_1$ are specification morphisms. We prove that SP'_1 defines a pushout for ϕ and φ.

$$\begin{tikzcd}
\mathrm{SP} \arrow[r, "\phi"] \arrow[d, "\varphi"] & \mathrm{SP}' \arrow[d, "\varphi'"] \arrow[ddr, bend left, "\theta'"] & \\
\mathrm{SP}_1 \arrow[r, "\phi_1"] \arrow[drr, bend right, "\theta_1"'] & \mathrm{SP}'_1 \arrow[dr, "\gamma"] & \\
& & \mathrm{SP}''
\end{tikzcd}$$

Consider $\theta'\colon\ \mathrm{SP}' \to \mathrm{SP}''$ and $\theta_1\colon\ \mathrm{SP}_1 \to \mathrm{SP}''$ specification morphisms. By the pushout property for signatures there exists a unique signature morphism $\gamma\colon\ Sig[\mathrm{SP}'_1] \to Sig[\mathrm{SP}'']$ such that $\varphi';\gamma = \theta'$ and $\phi_1;\gamma = \theta_1$.

We still have to show that γ is a specification morphism. Let $M'' \in Mod[\mathrm{SP}'']$. Then $M''\!\upharpoonright_\gamma\upharpoonright_{\varphi'} = M''\!\upharpoonright_{\theta'} \in Mod[\mathrm{SP}']$ and $M''\!\upharpoonright_\gamma\upharpoonright_{\phi_1} = M''\!\upharpoonright_{\theta_1} \in Mod[\mathrm{SP}_1]$ which shows that $M''\!\upharpoonright_\gamma \in \mathsf{Mod}[\mathrm{SP}'_1]$. $\square$

Model amalgamation for specifications. An immediate but important corollary of the construction of co-limits of specifications given by Prop. 14.6 is that model amalgamation properties also lift from signatures to specifications. Below we formulate this for ordinary model amalgamation, but it can be replicated easily to other forms of model amalgamation such as weak model amalgamation or semi-exactness.

Proposition 14.7. *The institution of specification has model amalgamation whenever the base institution has model amalgamation.*

Proof. For any pushout of specifications

$$\begin{tikzcd} \mathrm{SP} \arrow[r, "\phi"] \arrow[d, "\varphi"'] & \mathrm{SP}' \arrow[d, "\varphi'"] \\ \mathrm{SP}_1 \arrow[r, "\phi_1"'] & \mathrm{SP}'_1 \end{tikzcd}$$

and any SP'-model M' and any SP_1-model M_1 such that $M' \upharpoonright_\phi = M_1 \upharpoonright_\varphi$ we consider the unique amalgamation $M'_1 = M' \otimes M_1$ in $Sig[\mathrm{SP}'_1]$ given by the amalgamation property of the base institution.

By Prop. 14.6 we know that $\mathrm{SP}'_1 = (\mathrm{SP}' \star \varphi') \cup (\mathrm{SP}_1 \star \phi_1)$. This implies that $M'_1 \in Mod[\mathrm{SP}'_1]$. $\square$

Fundamental parameterization theorem

The following result provides foundations for the semantics of parameterized specifications. Although formulated for any institution, in applications this result is in fact interpreted in institutions of specifications for pushout squares of specifications.

Theorem 14.8. *Consider a commuting square of signatures in an arbitrary institution.*

$$\begin{tikzcd} \Sigma \arrow[r, "\phi"] \arrow[d, "\varphi"'] & \Sigma' \arrow[d, "\varphi'"] \\ \Sigma_1 \arrow[r, "\phi_1"'] & \Sigma'_1 \end{tikzcd}$$

1. *If this is a weak model amalgamation square, then ϕ_1 is conservative whenever ϕ is conservative.*
2. *If this is an amalgamation square both for models and model homomorphisms and* $\mathsf{Mod}(\phi)$ *has a retract $(-)^{\phi}$, then* $\mathsf{Mod}(\phi_1)$ *has a retract $(-)^{\phi_1}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathsf{Mod}(\Sigma) \arrow[r, "(-)^{\phi}"] & \mathsf{Mod}(\Sigma') \\ \mathsf{Mod}(\Sigma_1) \arrow[u, "\mathsf{Mod}(\varphi)"] \arrow[r, "(-)^{\phi_1}"'] & \mathsf{Mod}(\Sigma'_1) \arrow[u, "\mathsf{Mod}(\varphi')"'] \end{tikzcd}$$

3. *If this is an amalgamation square both for models and model homomorphisms, then for each Σ'_1-model M'_1, M'_1 is strongly persistently ϕ_1-free whenever $M'_1\restriction_{\varphi'}$ is strongly persistently ϕ-free.*

4. *If in addition ϕ is* strongly persistently liberal *(i.e., the adjunction determined by the reduct model functor is strongly persistent), then ϕ_1 is also strongly persistently liberal.*

   Moreover, for each Σ'_1-model M'_1, M'_1 is strongly persistently ϕ_1-free if and only if $M'_1\restriction_{\varphi'}$ is strongly persistently ϕ-free.

Proof. 1. Any Σ_1-model M_1 admits an ϕ_1-expansion which is the amalgamation between M_1 and any ϕ-expansion M' of $M_1\restriction_{\varphi}$.

2. When $\mathsf{Mod}(\phi)$ has a retract, the argument of 1 can be used to define the retract functor $(-)^{\phi_1}$. The functoriality of $(-)^{\phi_1}$ follows from the amalgamation property in the strong form.

The commutativity of retracts with model reduct functors follows from the fact that for each Σ_1-model M_1, $M_1^{\phi_1}$ is the amalgamation between M_1 and $(M_1\restriction_{\varphi})^{\phi}$.

3. Consider any Σ'_1-model M'_1 such that $M'_1\restriction_{\varphi'}$ is strongly persistently ϕ-free. Let $h_1 : M'_1\restriction_{\phi_1} \to N'_1\restriction_{\phi_1}$ be a Σ_1-model homomorphism for some Σ'_1-model N'_1. Let $h = h_1\restriction_{\varphi} : M'_1\restriction_{\phi_1}\restriction_{\varphi} \to N'_1\restriction_{\phi_1}\restriction_{\varphi}$. Because $M'_1\restriction_{\varphi'}$ is strongly persistently ϕ-free, there exists a unique $h' : M'_1\restriction_{\varphi'} \to N'_1\restriction_{\varphi'}$ such that $h'\restriction_{\phi} = h$. Let $h'_1 : M'_1 \to N'_1$ be the amalgamation of h' and h_1. Then h'_1 is the unique ϕ_1-expansion of h_1.

4. Follows from 2 and 3 $\square$

The following corollary shows the stability of the persistently free specifications under pushouts, which gives foundations for parameterized specification.

Corollary 14.9. *For any pushout of specifications in a semi-exact institution*

$$\begin{array}{ccc} \mathrm{SP} & \xrightarrow{\phi} & \mathrm{SP}' \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\ \mathrm{SP}_1 & \xrightarrow[\phi_1]{} & \mathrm{SP}'_1 \end{array}$$

such that ϕ is a strongly persistent specification morphism, we have that $\mathrm{SP}'^{\phi_1}_1 \models\!\mid \mathrm{SP}'_1 \cup (\mathrm{SP}'^{\phi} \star \varphi')$ and that

$$\begin{array}{ccc} \mathrm{SP} & \xrightarrow{\phi} & \mathrm{SP}'^{\phi} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\ \mathrm{SP}_1 & \xrightarrow[\phi_1]{} & \mathrm{SP}'^{\phi_1}_1 \end{array}$$

is a pushout of specifications.

Normal forms of structured specifications

Flattening structured specifications. Any specification SP can be 'flattened' to a finite presentation $Flat(\mathrm{SP})$ as follows:

- $Flat(\Sigma,E) = (\Sigma,E)$ for any presentation (Σ,E),
- $Flat(\mathrm{SP}\cup\mathrm{SP}') = (\Sigma\cup\Sigma', E\cup E')$ when $Flat(\mathrm{SP}) = (\Sigma,E)$ and $Flat(\mathrm{SP}') = (\Sigma',E')$,
- $Flat(\mathrm{SP}\star\varphi) = (\Sigma', \varphi(E))$ for each signature morphism $\varphi\colon \Sigma\to\Sigma'$ when $Flat(\mathrm{SP}) = (\Sigma,E)$,
- $Flat(\varphi \mid \mathrm{SP}') = (\Sigma, \varphi^{-1}(E'))$ for each signature morphism $\varphi\colon \Sigma\to\Sigma'$ when $Flat(\mathrm{SP}') = (\Sigma',E')$, and
- $Flat(\mathrm{SP}^{\varphi}) = Flat(\mathrm{SP})$ for each signature morphism $\varphi\colon \Sigma\to Sig[\mathrm{SP}]$.

Note that $Flat$ can extended to a functor $\mathbb{S}pec^{I}\to\mathbb{P}res^{I}$.

By simple applications of the satisfaction condition we obtain that:

Fact 14.10. *Any specification* SP *refines its flattening* $Flat(\mathrm{SP})$. *When* SP *is formed only from* PRES, UNION, *and* TRANS, *we have that* $\mathrm{SP} \models\!\!| \, Flat(\mathrm{SP})$.

Normal forms. Although the specifications formed also with DERIV cannot be flattened to semantically equivalent finite presentations, they admit equivalent normal forms of the form $\phi \mid (\Sigma,E)$ where ϕ is a signature morphism and (Σ,E) is a presentation. Normal forms of specifications are not unique, however they are unique modulo isomorphisms.

Let us assume the institution has pushouts for signatures. Below we define the relation *has normal form* between specifications as the least relation defined by

- (Σ,E) has normal form $1_\Sigma \mid (\Sigma,E)$ for each presentation (Σ,E),
- $(\mathrm{SP}_1\cup\mathrm{SP}_2)$ has normal form $\phi \mid (\Sigma, \phi_1'(E_1')\cup\phi_2'(E_2'))$ for each specifications SP_1 and SP_2, where
  - SP_k has normal form $\phi_k \mid (\Sigma_k', E_k')$ for each $k\in\{1,2\}$, with $\phi_k\colon \Sigma_k\to\Sigma_k'$,
  - ϕ_1' and ϕ_2' constitute a pushout co-cone for $i_1;\phi_1$ and $i_2;\phi_2$, where $i_k\colon \Sigma_1\cap\Sigma_2\hookrightarrow\Sigma_k$ are the inclusions of the intersection of the signatures, and

$$\begin{tikzcd}
\Sigma_1\cap\Sigma_2 \arrow[r, "i_1"] \arrow[d, "i_2"'] & \Sigma_1 \arrow[r, "\phi_1"] \arrow[d] & \Sigma_1' \arrow[dd, "\phi_1'"] \\
\Sigma_2 \arrow[r] \arrow[d, "\phi_2"'] & \Sigma_1\cup\Sigma_2 \arrow[dr, "\phi"] & \\
\Sigma_2' \arrow[rr, "\phi_2'"'] & & \Sigma
\end{tikzcd}$$

  - $\phi\colon \Sigma_1\cup\Sigma_2\to\Sigma$ is the unique signature morphism making the diagram above commute (by using the basic assumption of this chapter that the intersection-union of a square of signatures is a pushout).

- $\mathrm{SP} \star \varphi$ has normal form $\phi' \mid (\Sigma'_1, \varphi_1(E_1))$ when SP has normal form $\phi \mid (\Sigma_1, E_1)$ and the square below is a pushout

$$\begin{tikzcd} \Sigma \arrow[r, "\phi"] \arrow[d, "\varphi"'] & \Sigma_1 \arrow[d, "\varphi_1"] \\ \Sigma' \arrow[r, "\phi'"'] & \Sigma'_1 \end{tikzcd}$$

- $\varphi \mid \mathrm{SP}'$ has normal form $(\varphi;\phi) \mid (\Sigma', E')$ when SP' has normal form $\phi \mid (\Sigma', E')$.

The uniqueness of the normal forms.

Theorem 14.11. *Assume the base institution has model amalgamation. For each specification* SP*, if* $\varphi \mid (\Sigma, E)$ *is one of its normal forms then* $\mathrm{SP} \models\!\mid \varphi \mid (\Sigma, E)$.

Moreover, if both $\varphi_1 \mid (\Sigma_1, E_1)$ *and* $\varphi_2 \mid (\Sigma_2, E_2)$ *are normal forms of* SP*, then there exists an isomorphism of presentations* $i\colon\ (\Sigma_1, E_1) \to (\Sigma_2, E_2)$ *such that* $\varphi_1; i = \varphi_2$.

Proof. We prove this result by induction on the structure of the specification SP. The base case is when SP is a presentation; in this case the conclusion of the theorem is obvious.

For the induction step for UNION, consider specifications SP_1 and SP_2 satisfying the conclusion of the theorem and let us prove the conclusion for $\mathrm{SP}_1 \cup \mathrm{SP}_2$. We use the notation from the definition of normal forms. We have that $Sig[\mathrm{SP}_1 \cup \mathrm{SP}_2] = Sig[\phi \mid (\Sigma, \phi'_1(E'_1) \cup \phi'_2(E'_2))]$.

Let $M' \in Mod[\phi \mid (\Sigma, \phi'_1(E'_1) \cup \phi'_2(E'_2))]$. There exists $M \in \mathsf{Mod}(\Sigma, \phi'_1(E'_1) \cup \phi'_2(E'_2))$ such that $M \upharpoonright_\phi = M'$. Let $M'_1 = M \upharpoonright_{\phi'_1}$ and $M'_2 = M \upharpoonright_{\phi'_2}$. We have that $M'_1 \models E'_1$ and $M'_2 \models E'_2$. Also $M' \upharpoonright_{\Sigma_1} = M'_1 \upharpoonright_{\phi_1}$ and $M' \upharpoonright_{\Sigma_2} = M'_2 \upharpoonright_{\phi_2}$, which shows that $M' \upharpoonright_{\Sigma_k} \in Mod[\mathrm{SP}_k]$ since $\mathrm{SP}_k \models\!\mid \phi_k \mid (\Sigma'_k, E'_k)$, hence $M' \in Mod[\mathrm{SP}_1 \cup \mathrm{SP}_2]$.

On the other hand, for each $M' \in Mod[\mathrm{SP}_1 \cup \mathrm{SP}_2]$, $M' \upharpoonright_{\Sigma_k} = M'_k \upharpoonright_{\phi_k}$ with $M'_k \in \mathsf{Mod}(\Sigma'_k, E'_k)$ (because $\mathrm{SP}_k \models\!\mid \phi_k \mid (\Sigma'_k, E'_k)$) for each $k \in \{1,2\}$. Notice that $M'_1 \upharpoonright_{(i_1;\phi_1)} = M' \upharpoonright_{\Sigma_1 \cap \Sigma_2} = M'_2 \upharpoonright_{(i_2;\phi_2)}$, hence, because the institution has model amalgamation, there exists a Σ-model M such that $M \upharpoonright_{\phi'_1} = M'_1$ and $M \upharpoonright_{\phi'_2} = M'_2$. By the satisfaction condition we have that $M \models \phi'_1(E'_1) \cup \phi'_2(E'_2)$. Because $M \upharpoonright_\phi \upharpoonright_{\Sigma_1} = M' \upharpoonright_{\Sigma_1}$ and $M \upharpoonright_\phi \upharpoonright_{\Sigma_2} = M' \upharpoonright_{\Sigma_2}$, by the model amalgamation property of the institution we have that $M \upharpoonright_\phi = M'$. Therefore $M' \in \mathsf{Mod}(\Sigma, \phi'_1(E'_1) \cup \phi'_2(E'_2))$.

For the induction step for TRANS, assume that $\mathrm{SP} \models\!\mid \phi \mid (\Sigma_1, E_1)$ and consider a Σ'-model M'. Then $M' \in Mod[\mathrm{SP} \star \varphi]$ is equivalent to $M' \upharpoonright_\varphi \in Mod[\mathrm{SP}]$ which is equivalent to the existence of $M_1 \in \mathsf{Mod}(\Sigma_1, E_1)$ with $M_1 \upharpoonright_\phi = M' \upharpoonright_\varphi$ which by the model amalgamation property of the institution is further equivalent to the existence of $M'_1 \in \mathsf{Mod}(\Sigma'_1, \varphi_1(E_1))$ with $M'_1 \upharpoonright_{\phi'_1} = M'$.

For DERIV, the first part of the conclusion of the theorem follows by a simple application of the satisfaction condition of the institution.

The second part of the conclusion of the theorem follows by the uniqueness of the pushouts modulo isomorphisms. $\square$

Exercises

14.1. The specification building operator UNION can be replaced by its particular version where SP_1 and SP_2 have the same signature. The general union of any specifications can be obtained from translations and the union over the same signature.

14.2. Algebraic properties of the structuring of specifications

- For any specifications SP, SP$'$ and SP$''$
  - $SP \cup SP' \mathrel{\mid\!\!\models} SP' \cup SP$,
  - $SP \cup SP \mathrel{\mid\!\!\models} SP$,
  - $(SP \cup SP') \cup SP'' \mathrel{\mid\!\!\models} SP \cup (SP' \cup SP'')$.
- For any specification SP and any signature morphisms $\varphi\colon\ Sig[SP] \to \Sigma'$ and $\varphi'\colon\ \Sigma' \to \Sigma''$
$$SP \star (\varphi;\varphi') \mathrel{\mid\!\!\models} (SP \star \varphi) \star \varphi'.$$
- For any specifications SP_1 and SP_2 and any signature morphism $\varphi\colon\ Sig[SP_1 \cup SP_2] \to \Sigma$
$$(SP_1 \cup SP_2) \star \varphi \mathrel{\mid\!\!\models} (SP_1 \star (i_1;\varphi)) \cup SP_2 \star (i_2;\varphi))$$
where i_k is the inclusion $Sig[SP_k] \hookrightarrow Sig[SP_1 \cup SP_2]$ for $k \in \{1,2\}$.
- For any specification SP and any signature morphisms $\varphi'\colon\ \Sigma'' \to \Sigma'$ and $\varphi\colon\ \Sigma' \to Sig[SP]$,
$$(\varphi';\varphi) \mid SP \mathrel{\mid\!\!\models} \varphi' \mid \varphi \mid SP.$$

14.3. Assume the institution is semi-exact and consider any specifications SP_1 and SP_2. Let i be the signature inclusion $Sig[SP_1] \cap Sig[SP_2] \hookrightarrow Sig[SP_1] \cup Sig[SP_2]$, and for $k \in \{1,2\}$ let i_k be the signature inclusion $Sig[SP_1] \cap Sig[SP_2] \hookrightarrow Sig[SP_k]$. Then
$$i \mid (SP_1 \cup SP_2) \mathrel{\mid\!\!\models} (i_1 \mid SP_1) \cup (i_2 \mid SP_2).$$

14.4. (a) Given a specification SP and a signature morphism $\varphi\colon\ Sig[SP] \to \Sigma'$, then $SP \mathrel{\mid\!\!\models} \varphi \mid (SP \star \varphi)$.
(b) On the other hand, given a specification SP$'$ and a signature morphism $\varphi\colon\ \Sigma \to Sig[SP']$, then SP$'$ is a refinement of $(\varphi \mid SP) \star \varphi$.

14.5. Interpolation for structured specifications
The following *Craig interpolation property* holds for specifications. For any weak amalgamation square of signatures

$$\begin{tikzcd}
\Sigma \arrow[r, "\phi"] \arrow[d, "\varphi"'] & \Sigma' \arrow[d, "\varphi'"] \\
\Sigma_1 \arrow[r, "\phi_1"'] & \Sigma'_1
\end{tikzcd}$$

any Σ'-specification SP$'$ and any Σ_1-specification SP_1 such that $SP' \star \varphi'$ refines $SP_1 \star \varphi_1$, there exists a Σ-specification SP such that SP$'$ refines $SP \star \phi$ and $SP \star \varphi$ refines SP_1. (*Hint:* define $SP = \phi \mid SP'$.)

14.6. For any specification SP and any signature morphisms $\phi\colon\ \Sigma' \to \Sigma$ and $\varphi\colon\ \Sigma \to Sig[SP]$ such that the model reduct functor $\mathsf{Mod}(\phi)$ is faithful, $SP^{\phi;\varphi}$ refines SP^{φ}.

14.7. For any exact institution that has an initial signature, denoted 0, for each presentation (Σ, E),
$$Mod[(\Sigma,E)^{\varphi_\Sigma}] = \{M \mid M \ \text{ initial } \ (\Sigma,E)-\text{model}\}$$
where $\varphi_\Sigma\colon\ 0 \to \Sigma$ is the unique signature morphism from 0 to Σ.

14.8. Intersection of structured specifications
Define the intersection $\mathrm{SP}_1 \cap \mathrm{SP}_2$ of any specifications SP_1 and SP_2 as a new specification building operation and use it for showing that the signature functor from specifications to signatures $Sig\colon\ \mathbb{S}pec^I \to \mathbb{S}ig^I$ lifts pullbacks.

14.9. Elementary diagrams for structured specifications
Consider an inclusive (base) institution I with elementary diagrams ι such that for each Σ-model M, the elementary extension $\iota_\Sigma(M)\colon\ \Sigma \to \Sigma_M$ is an inclusion. Then I^{spec} has elementary diagrams. (*Hint:* The elementary diagram of a model M of a specification SP is $\mathrm{SP} \to \mathrm{SP} \cup (\Sigma_M, E_M)$ where $\Sigma = Sig[\mathrm{SP}]$ and $\Sigma \to (\Sigma_M, E_M)$ is the elementary diagram of M in I.)

14.10. Each persistently liberal institution comorphism $(\Phi,\alpha,\beta)\colon\ I \to I'$ determines a canonical institution comorphism $(\Phi_1,\alpha_1,\beta_1)\colon\ I^{\mathrm{spec}} \to (I')^{\mathrm{spec}}$ such that the diagram of institution comorphisms below commutes:

$$\begin{tikzcd}
I \arrow[r, "{(\Phi,\alpha,\beta)}"] \arrow[d] & I' \arrow[d] \\
I^{\mathrm{spec}} \arrow[r, "{(\Phi_1,\alpha_1,\beta_1)}"'] & (I')^{\mathrm{spec}}
\end{tikzcd}$$

14.2 Specifications with Proofs

$\langle \mathcal{T}, \mathcal{D}\rangle$-specifications. Often it is necessary and meaningful to restrict the signature morphisms of an institution I used by the TRANS and DERIV specification structuring operators to special classes $\mathcal{T}$ and $\mathcal{D}$, respectively, of signature morphisms. The specifications thus built by PRES, UNION, TRANS by morphisms in $\mathcal{T}$, and DERIV by morphisms in $\mathcal{D}$, are called *$\langle \mathcal{T}, \mathcal{D}\rangle$-specifications*. Note that FREE is not considered here. Let us denote the category of *$\langle \mathcal{T}, \mathcal{D}\rangle$-specifications* by $\mathbb{S}pec_{\mathcal{T},\mathcal{D}}$. The resulting institution of the $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications is denoted $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$.

Extending proof systems to specifications. Given an institution with proofs $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ and classes of signature morphisms $\mathcal{T}, \mathcal{D} \subseteq \mathbb{S}ig$, the institution $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ of $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications can be enhanced with a proof system by taking the initial proof system such that

(*base*) for each specification SP

$$\Gamma \vdash_{\mathrm{SP}} E \ \text{ if } \ \Gamma \vdash_{Sig[\mathrm{SP}]} E$$

(*pres*) for each finite presentation (Σ, E)

$$\emptyset \vdash_{(\Sigma,E)} E$$

(*deriv*) and it satisfies the (meta-)rule

$$d(\Gamma) \vdash_{\mathrm{SP}'} d(E) \ \text{ implies } \ \Gamma \vdash_{d|\mathrm{SP}'} E$$

for each $d \in \mathcal{D}$ with $d\colon\ \Sigma \to Sig[\mathrm{SP}']$.

Proposition 14.12. *There exists the initial proof system* $\mathsf{Pf}_{\mathcal{T},\mathcal{D}}$ *for* $\langle \mathcal{T}, \mathcal{D}\rangle$*-specifications satisfying (*base*), (*pres*) and (*deriv*) defined above.*

Proof. Let us describe rather informally the construction of $\mathsf{Pf}_{\mathcal{T},\mathcal{D}}$. This can be done by the following two steps:

1. for each presentation (Σ, E) we add rules $P_{\Sigma,E} : \emptyset \to E$ to the Σ-proofs (from I) such that $\varphi(P_{\Sigma,E}) = P_{\Sigma',\varphi(E)}$ for each signature morphism $\varphi : \Sigma \to \Sigma'$, and we take the free proof system which preserves the horizontal and vertical composition of the original proofs of I, and
2. while $\Gamma \not\vdash_{d|\mathrm{SP}'} E$ for some specification SP' and $(d : \Sigma \to \mathit{Sig}[\mathrm{SP}']) \in \mathcal{D}$ such that $d(\Gamma) \vdash_{\mathrm{SP}'} d(E)$, we add a $d|\mathrm{SP}'$-proof $\Gamma \to E$ and take again the free proof system which preserves the horizontal and vertical composition of the existing proofs.

The formal construction can be done in the manner of constructions of the free proof systems over systems of rules (Thm. 13.3) and of construction of the free proof systems with quantification (Thm. 13.17). $\square$

Soundness

The soundness of the proof system can be lifted easily from the base institution to the institution of structured specifications.

Proposition 14.13. *For any sound institution with proofs I, the corresponding institution with proofs $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ of structured $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications is also sound.*

Proof. Soundness of $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ means that there exists a comorphism of proof systems

$$(1, 1, \gamma) : (\mathbb{S}\mathit{pec}_{\mathcal{T},\mathcal{D}}, \mathit{Sen}, \mathsf{Pf}_{\mathcal{T},\mathcal{D}}) \to (\mathbb{S}\mathit{pec}_{\mathcal{T},\mathcal{D}}, \mathit{Sen}, \models)$$

where $(\mathbb{S}\mathit{pec}_{\mathcal{T},\mathcal{D}}, \mathit{Sen}, \models)$ is the semantic proof system determined by the institution $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ of $\langle \mathcal{T}, \mathcal{D}\rangle$-specifications.

By Prop. 14.12 it is enough to show that the semantic proof system $(\mathbb{S}\mathit{pec}_{\mathcal{T},\mathcal{D}}, \mathit{Sen}, \models)$ satisfies properties (*base*), (*pres*), and (*deriv*). Indeed these hold as follows:

- Property (*base*) holds because for each specification SP if $\Gamma \vdash_{\mathit{Sig}[\mathrm{SP}]} E$, then by the soundness of the base institution I we have that $\Gamma \models_{\mathit{Sig}[\mathrm{SP}]} E$. This implies $\Gamma \models_{\mathrm{SP}} E$.
- Property (*pres*) means that for each presentation (Σ, E) we have that $\emptyset \models_{(\Sigma,E)} E$, which is immediate.
- Property (*deriv*) means that $d(\Gamma) \models_{\mathrm{SP}'} d(E)$ implies $\Gamma \models_{d|\mathrm{SP}'} E$. Consider a model $M \in \mathit{Mod}[d \mid \mathrm{SP}]$ such that $M \models \Gamma$. There exists a model $M' \in \mathit{Mod}[\mathrm{SP}']$ such that $M = M' \restriction_d$. By the satisfaction condition it follows that $M' \models d(\Gamma)$ which by the hypothesis implies $M' \models d(E)$. By the satisfaction condition (in the reverse direction from its previous use) we have $M \models E$.

$\square$

Completeness

The lifting of completeness from the base institution to the institution of $(\mathcal{T}, \mathcal{D})$-specifications requires more conditions than the lifting of soundness.

Theorem 14.14. *Consider a base inclusive institution with proofs* $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ *and classes of signature morphisms* $\mathcal{T}$ *and* $\mathcal{D}$ *such that*

1. *the classes* $\mathcal{T}$ *and* $\mathcal{D}$ *of signature morphisms satisfy the following properties:*
   (a) $\mathcal{D} \subseteq \mathcal{T}$ *and each signature inclusion belongs to* $\mathcal{T}$,
   (b) *for each* $d_1 : \Sigma \to \Sigma_1$ *and* $d_2 : \Sigma \to \Sigma_2$ *with* $d_1, d_2 \in \mathcal{D}$, *there exists a pushout square*

$$\begin{tikzcd} \Sigma \arrow[r, "d_1"] \arrow[d, "d_2"'] \arrow[dr, "d"] & \Sigma_1 \arrow[d] \\ \Sigma_2 \arrow[r] & \Sigma' \end{tikzcd}$$

   such that $d \in \mathcal{D}$,
   (c) *for each* $t : \Sigma \to \Sigma'$ *in* $\mathcal{T}$ *and* $d : \Sigma \to \Sigma_1$ *in* $\mathcal{D}$ *there exists a pushout square*

$$\begin{tikzcd} \Sigma \arrow[r, "t"] \arrow[d, "d"'] & \Sigma' \arrow[d, "d'"] \\ \Sigma_1 \arrow[r] & \Sigma'_1 \end{tikzcd}$$

   such that $d' \in \mathcal{D}$,
2. *I is complete,*
3. *I has model amalgamation, and*
4. *I has* $(\mathcal{D}, \mathcal{T})$*-Craig-Robinson interpolation.*

Then the proof system $\mathsf{Pf}_{\mathcal{T},\mathcal{D}}$ *for the institution* $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$ *of* $(\mathcal{T}, \mathcal{D})$*-specifications is complete.*

Proof. We prove by induction on the structure of the specification SP that $\Gamma \vdash_{SP} E$ if $\Gamma \models_{\mathrm{SP}} E$. For this proof we will systematically use the existence and uniqueness of normal forms for specifications formed with PRES, UNION, TRANS and DERIV given by Thm.14.11.

PRES. Consider a presentation (Σ, E) and assume $\Gamma \models_{(\Sigma,E)} E_1$. This implies $\Gamma \cup E \models_\Sigma E_1$. By the completeness of the base institution I we have that $\Gamma \cup E \vdash_\Sigma E_1$ which implies $\Gamma \cup E \vdash_{(\Sigma,E)} E_1$.

By (*pres*) we have that $\vdash_{(\Sigma,E)} E$ which gives $\Gamma \vdash_{(\Sigma,E)} \Gamma \cup E$. From this and $\Gamma \cup E \vdash_{(\Sigma,E)} E_1$ we obtain $\Gamma \vdash_{(\Sigma,E)} E_1$.

UNION. Consider a union of specifications $\mathrm{SP}_1 \cup \mathrm{SP}_2$. Because arbitrary union of specifications can be reduced to translations and union of specifications having the same signature, and because all signature inclusions are in $\mathcal{T}$, we may assume that $Sig[\mathrm{SP}_1] = Sig[\mathrm{SP}_2] = \Sigma$. Let us assume $\Gamma \models_{\mathrm{SP}_1 \cup \mathrm{SP}_2} E$. Consider a normal form $d_1 \mid (\Sigma_1, \Gamma_1)$ for

SP_1, a normal form $d_2 \mid (\Sigma_2, \Gamma_2)$ for SP_2, and a normal form $d \mid (\Sigma', d_1'(\Gamma_1) \cup d_2'(\Gamma_2))$ for $\mathrm{SP}_1 \cup \mathrm{SP}_2$ such that the square below is a pushout:

$$\begin{tikzcd} \Sigma \arrow[r, "d_1"] \arrow[d, "d_2"'] \arrow[dr, "d"] & \Sigma_1 \arrow[d, "d_1'"] \\ \Sigma_2 \arrow[r, "d_2'"'] & \Sigma' \end{tikzcd}$$

Let M' be a Σ'-model such that $M' \models d(\Gamma) \cup d_1'(\Gamma_1) \cup d_2'(\Gamma_2)$. This implies that $M' \restriction_d \models \Gamma$ and $M' \restriction_d \in Mod[\mathrm{SP}_k]$ (for each $k \in \{1,2\}$, $M' \restriction_d \in Mod[\mathrm{SP}_k]$ because $M' \restriction_{d_k'} \models \Gamma_k$ and $\mathrm{SP}_k \models\!\!\mid d_k \mid (\Sigma_k, \Gamma_k)$), which implies $M' \restriction_d \models E$, hence $M' \models d(E)$. This shows that $d(\Gamma) \cup d_1'(\Gamma_1) \cup d_2'(\Gamma_2) \models d(E)$.

But $d_2 \in \mathcal{T}$ because $\mathcal{D} \subseteq \mathcal{T}$, hence the pushout above is a CRI square. Therefore, there exists an interpolant $I \subseteq \mathsf{Sen}(\Sigma)$ such that $\Gamma_1 \models d_1(I)$ and $d_2(\Gamma) \cup \Gamma_2 \cup d_2(I) \models d_2(E)$.

Because $\mathrm{SP}_1 \models\!\!\mid d_1 \mid (\Sigma_1, \Gamma_1)$, we have that $\models_{\mathrm{SP}_1} I$. By the induction hypothesis this implies $\vdash_{\mathrm{SP}_1} I$. Translating along the inclusion $\mathrm{SP}_1 \to \mathrm{SP}_1 \cup \mathrm{SP}_2$ we obtain $\vdash_{\mathrm{SP}_1 \cup \mathrm{SP}_2} I$.

Because $\mathrm{SP}_2 \models\!\!\mid d_2 \mid (\Sigma_2, \Gamma_2)$ we have that $\Gamma \cup I \models_{\mathrm{SP}_2} E$. By the induction hypothesis this implies $\Gamma \cup I \vdash_{\mathrm{SP}_2} E$. Translating along the inclusion $\mathrm{SP}_2 \to \mathrm{SP}_1 \cup \mathrm{SP}_2$ we obtain $\Gamma \cup I \models_{\mathrm{SP}_1 \cup \mathrm{SP}_2} E$. From $\vdash_{\mathrm{SP}_1 \cup \mathrm{SP}_2} I$ and $\Gamma \cup I \vdash_{\mathrm{SP}_1 \cup \mathrm{SP}_2} E$ we obtain $\Gamma \vdash_{\mathrm{SP}_1 \cup \mathrm{SP}_2} E$.

TRANS. Consider a translation of specifications $\mathrm{SP} \star t$ with $t \in \mathcal{T}$ such that $t\colon\ Sig[\mathrm{SP}] \to \Sigma'$. Let us assume $\Gamma' \models_{\mathrm{SP}\star t} E'$. Consider a normal form $d' \mid (\Sigma_1', t_1(\Gamma_1))$ for $\mathrm{SP}\star t$ such that $d \mid (\Sigma_1, \Gamma_1)$ is a normal form for SP and the square below is a pushout.

$$\begin{tikzcd} \Sigma \arrow[r, "t"] \arrow[d, "d"'] & \Sigma' \arrow[d, "d'"] \\ \Sigma_1 \arrow[r, "t_1"'] & \Sigma_1' \end{tikzcd}$$

For any Σ_1'-model M_1', when $M_1' \models d'(\Gamma') \cup t_1(\Gamma_1)$ we have that $M_1' \restriction_{d'} \in Mod[\mathrm{SP} \star t]$ (because $\mathrm{SP}\star t \models\!\!\mid d' \mid (\Sigma_1', t_1(\Gamma_1))$) and that $M_1' \restriction_{d'} \models \Gamma'$, which by the hypothesis implies $M_1' \restriction_{d'} \models E'$ which by the satisfaction condition is equivalent to $M_1' \models d'(E')$. We therefore have therefore shown that $d'(\Gamma') \cup t_1(\Gamma_1) \models d'(E')$.

By the CRI property, there exists an interpolant $I \in \mathsf{Sen}(\Sigma)$ such that $\Gamma_1 \models d(I)$ and $\Gamma' \cup t(I) \models E'$.

From $\Gamma_1 \models d(I)$ we deduce $\models_{\mathrm{SP}} I$ which by the induction hypothesis implies $\vdash_{\mathrm{SP}} I$. By translating along the specification morphism $t\colon\ \mathrm{SP} \to \mathrm{SP}\star t$, we obtain that $\vdash_{\mathrm{SP}\star t} I$ which implies $\Gamma' \vdash_{\mathrm{SP}\star t} \Gamma' \cup t(I)$.

On the other hand, from $\Gamma' \cup t(I) \models E'$, by the completeness of the base institution we have $\Gamma' \cup t(I) \vdash_{\Sigma'} E'$, which by (*base*) gives $\Gamma' \cup t(I) \vdash_{\mathrm{SP}\star t} E'$. From $\Gamma' \vdash_{\mathrm{SP}\star t} \Gamma' \cup t(I)$ and $\Gamma' \cup t(I) \vdash_{\mathrm{SP}\star t} E'$ we get $\Gamma' \vdash_{\mathrm{SP}\star t} E'$.

DERIVE. Consider a derived specification $d \mid \mathrm{SP}'$ and assume $\Gamma \models_{d|\mathrm{SP}'} E$. We have $d(\Gamma) \models_{\mathrm{SP}'} d(E)$ (for each model $M' \in Mod[\mathrm{SP}']$ such that $M' \models d(\Gamma)$, then $M' \restriction_d \in$

$Mod[d \mid \mathrm{SP}']$ and $M' \restriction_d \models \Gamma$, which implies $M' \restriction_d \models E$ which means $M' \models d(E)$). Then by the induction hypothesis $d(\Gamma) \vdash_{\mathrm{SP}'} d(E)$, and by (*deriv*) we have $\Gamma \vdash_{d|\mathrm{SP}'} E$. $\square$

Note that the proof of the completeness Thm. 14.14 uses only that the proof system for the structured specifications fulfills properties (*base*), (*pres*), and (*deriv*). The initiality property of $\mathsf{Pf}_{\mathcal{T},\mathcal{D}}$ is necessary only for lifting the soundness of the proof system of the base institution I to the soundness of $\mathsf{Pf}_{\mathcal{T},\mathcal{D}}$ in $I^{\mathrm{spec}}_{\mathcal{T},\mathcal{D}}$.

Fact 14.15. *The condition* 1(b) *of Thm. 14.14 follows from* 1(a) *and* 1(c) *if $\mathcal{D}$ were closed under composition. Consequently conditions* 1(b) *and* 1(c) *can be replaced by the condition that $\mathcal{D}$ is a sub-category stable under pushouts.*

The following is a concrete instance of the soundness Prop. 14.13 and of the completeness Thm. 14.14.

A sound and complete proof system for structured specifications in HCL

Corollary 14.16. *The $((***), (ie*))$-specifications in* **HCL** *admit a sound and complete proof system which is the proof system freely generated by the finitary Birkhoff proof system for* **HCL** *and satisfying (*basic*), (*pres*), and (*deriv*).*

Proof. Cor. 13.39 gives the soundness and the completeness of the finitary Birkhoff calculus for **HCL**. Cor. 12.19 gives the Craig-Robinson $((ie*), (***))$-interpolation for **HCL**. Condition 1 of Thm. 14.14 can be established easily by using Fact 14.15. $\square$

Other kinds of instances of Thm. 14.14 can be obtained in sound and complete institutions with proofs which have implications and Craig $(\mathcal{D}, \mathcal{T})$-interpolation since according to Prop. 9.20 these have Craig-Robinson $(\mathcal{D}, \mathcal{T})$-interpolation. A standard and practically important example for this is **FOL** (cf. Cor. 9.15 having Craig $((i**), (***))$-interpolation).

14.3 Predefined Types

Specification with 'predefined types' extends ordinary specification with predefined entities both at the semantic and syntactic level. The semantics of actual specifications languages, which in general use at least predefined types for the numbers, is based on institutions with predefined types.

In some cases predefined types cannot be avoided at all. For example, specifications using real numbers constitute a non-trivial extension of ordinary **FOL** specifications because the model of the real numbers cannot be specified as an initial model of a finite presentation. This negative fact follows easily from cardinality issues. The following is an example of a specification with predefined types.

The Euclidean plane. When we consider the problem of the specification of the Euclidean plane $\mathbb{R}^2$ as a vector space, its signature $((S,F,P),\mathbb{R}')$ consists of

- an ordinary **FOL** signature (S,F,P), and
- an (S,F,P)-model $\mathbb{R}'$

such that

- $S = \{\mathtt{Real}, \mathtt{Vect}\}$,
- F consists of the usual ring operations for the real numbers plus $F_{\to\mathtt{Vect}} = \{0\}$, $F_{\mathtt{RealReal}\to\mathtt{Vect}} = \{\langle\_,\_\rangle\}$, $F_{\mathtt{VectVect}\to\mathtt{Vect}} = \{\_+\_\}$, $F_{\mathtt{Vect}\to\mathtt{Vect}} = \{-\_\}$, and $F_{\mathtt{RealVect}\to\mathtt{Vect}} = \{\_*\_\}$,
- the set of relation symbols P is empty, and
- $\mathbb{R}'$ is the free (S,F,P)-model over the ring $\mathbb{R}$ of real numbers.

The Euclidean plane is obtained as the quotient of $\mathbb{R}'$ modulo the following set E of equations defining a two-dimensional real vector space:

$$0 = \langle 0, 0\rangle,$$
$$(\forall\{a,b,a',b'\})\langle a, b\rangle + \langle a', b'\rangle = \langle a+a', b+b'\rangle,$$
$$(\forall\{k,a,b\})k * \langle a, b\rangle = \langle k*a, k*b\rangle,$$
$$(\forall\{a,b\}) - \langle a, b\rangle = \langle -a, -b\rangle.$$

Note that there are many models of the 'predefined presentation' $(((S,F,P),\mathbb{R}'),E)$, for example the model $\mathbb{R}+$ interpreting `Vect` as the set of real numbers and $\langle\_,\_\rangle$ as addition of real numbers. The 'intended' model of this presentation, the Euclidean plane $\mathbb{R}^2$, is in fact the initial model of the presentation.

The institution of predefined types

Signatures, sentences, models, and satisfaction with predefined types can be defined on top of any base institution $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$.

Signatures with predefined types. A *signature* (Σ,A) *with predefined type* A consists of a signature Σ in the base institution and a *predefined* Σ-model A.

Models with predefined types. A (Σ,A)-*model* is just a Σ-model M plus an interpretation of A into M in the form of a model homomorphism $h\colon A \to M$.

Note that in the case of the example above a model with predefined types is the same with a model homomorphism from $\mathbb{R}$ to the reduct of an (S,F,P)-model M to the signature of the real numbers.

Model homomorphisms. Homomorphisms of (Σ,A)-models have to 'preserve' A, hence the category of the (Σ,A)-models is the comma category $A/\mathsf{Mod}(\Sigma)$.

Sentences with predefined types. Since sentences with predefined types, such as

$$(\exists\{X,Y\})\, X * \langle 3.14, 5.79\rangle + Y * \langle \pi, e\rangle = \langle 2, -1.65\rangle$$

might involve entities from the predefined model A, we can define them based on the concept of an elementary diagram. Hence, let us further assume that the base institution has elementary diagrams ι.

Then a (Σ,A)-*sentence (with predefined type A)* is just a Σ_A-sentence, where $\iota_\Sigma(A)\colon\ \Sigma \to \Sigma_A$ is the elementary extension of Σ via A.

The satisfaction relation. A (Σ,A)-model $h\colon\ A \to M$ *satisfies* a (Σ,A)-sentence ρ if and only if the (Σ_A,E_A)-model $A_h = i^{-1}_{\Sigma,A}(h)$ satisfies it.

This construction of an institution of predefined types $(\mathbb{S}ig^{\wr},\mathsf{Sen}^{\wr},\mathsf{Mod}^{\wr},\models^{\wr})$ over a base institution with elementary diagrams $(\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models,\iota)$ can be summed up as follows:

- the category of signatures with predefined type $\mathbb{S}ig^{\wr}$ is the Grothendieck category $\mathsf{Mod}^{\sharp}$,
- $\mathsf{Sen}^{\wr}(\Sigma,A) = \mathsf{Sen}(\Sigma_A)$ for each signature (Σ,A), and $\mathsf{Sen}^{\wr}(\varphi,h) = \mathsf{Sen}(\iota_\varphi(h))$ for each signature morphism (φ,h),
- $\mathsf{Mod}^{\wr}(\Sigma,A) = A/\mathsf{Mod}(\Sigma)$ for each signature (Σ,A), and
- for each signature (Σ,A), $(h\colon\ A \to M) \models^{\wr}_{(\Sigma,A)} \rho$ if and only if $i^{-1}_{\Sigma,A}(h) \models_{\Sigma_A} \rho$.

Exercises

14.11. For any base institution $(\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models,\iota)$ with elementary diagrams, there exists a canonical institution morphism $(\mathbb{S}ig^{\wr},\mathsf{Sen}^{\wr},\mathsf{Mod}^{\wr},\models^{\wr}) \to (\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models)$ from the institution of predefined types to the base institution. Moreover this an adjoint institution morphism whenever the categories of models have initial models.

14.12. For any base institution $(\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models,\iota)$ with elementary diagrams, there exists an institution comorphism $(\mathbb{S}ig^{\wr},\mathsf{Sen}^{\wr},\mathsf{Mod}^{\wr},\models^{\wr}) \to (\mathbb{P}res,\mathsf{Sen}^{\mathrm{pres}},\mathsf{Mod}^{\mathrm{pres}},\models^{\mathrm{pres}})$ from the institution of predefined types to the institution of presentations over the base institution. (*Hint:* Each signature of predefined types (Σ,A) gets mapped to the presentation (Σ_A,E_A).)

14.13. Sound/complete proof system for predefined types
Let $(\mathbb{S}ig,\mathsf{Sen},\mathsf{Mod},\models,\vdash,\iota)$ be an institution with entailments and with elementary diagrams ι.

1. The institution with predefined types $(\mathbb{S}ig^{\wr},\mathsf{Sen}^{\wr},\mathsf{Mod}^{\wr},\models^{\wr},\vdash^{\wr})$ admits an entailment system defined by

$$\Gamma \vdash^{\wr}_{(\Sigma,A)} E \ \text{ if and only if } \ \Gamma \cup E_A \vdash_{\Sigma_A} E$$

   where $\iota_\Sigma(A)\colon\ \Sigma \to (\Sigma_A,E_A)$ is the elementary diagram of A.

2. $I^{\wr}$ is sound or complete, respectively, whenever the base institution I is sound or complete, respectively.

3. The entailment system of $I^{\wr}$ is precisely the free entailment system generated by the entailments of I plus the rules $\emptyset \vdash_{\Sigma_A} E_A$ for each Σ-model A.

14.14. For any base institution with elementary diagrams $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \iota)$ such that

- it is semi-exact,
- it is liberal on the signature morphisms,
- its category of signatures $\mathbb{S}ig$ has pushouts, and
- for each signature $\Sigma \in |\mathbb{S}ig|$, the category $\mathsf{Mod}(\Sigma)$ of Σ-models has pushouts,

the corresponding institution of predefined types $(\mathbb{S}ig^{\wr}, \mathsf{Sen}^{\wr}, \mathsf{Mod}^{\wr}, \models^{\wr})$ has pushouts of signatures and is semi-exact.

14.15. Each institution of predefined types has 'empty' elementary diagrams. (*Hint:* For any signature with predefined type (Σ, A) and any (Σ, A)-model with predefined types $h\colon\ A \to M$, the elementary extension $(\Sigma, A) \to (\Sigma, A)_h$ is defined as $(1_\Sigma, h)\colon\ (\Sigma, A) \to (\Sigma, M)$.)

14.16. An institution of predefined types is liberal whenever its base institution is liberal.

14.17. $I^{\wr}$ **as a Grothendieck institution**
For any institution $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models, \iota)$ with elementary diagrams:

- Each signature $\Sigma \in |\mathbb{S}ig|$ determines an institution $I^\Sigma = (\mathbb{S}ig^\Sigma, \mathsf{Sen}^\Sigma, \mathsf{Mod}^\Sigma, \models^\Sigma)$ defined by $\mathbb{S}ig^\Sigma = \mathsf{Mod}(\Sigma)$, $\mathsf{Sen}^\Sigma(A) = \mathsf{Sen}(\Sigma_A)$, $\mathsf{Mod}^\Sigma(A) = A/\mathsf{Mod}(\Sigma)$, and $(h\colon\ A \to B) \models^\Sigma_A \rho$ if and only if $i^{-1}_{\Sigma,A}(h) \models_{\Sigma_A} \rho$.
- Each signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ determines an institution morphism $(\Phi^\varphi, \alpha^\varphi, \beta^\varphi)\colon\ I^{\Sigma'} \to I^\Sigma$ defined by $\Phi^\varphi = \mathsf{Mod}(\varphi)$, $\alpha^\varphi_{A'} = \mathsf{Sen}(\iota_\varphi(1_{A'\restriction_\varphi}))$, and $\beta^\varphi_{A'}(h') = h'\restriction_\varphi$.

The constructions above determine an indexed institution $\mathbb{S}ig^{\mathrm{op}} \to \mathbb{I}ns$ such that its corresponding (morphism-based) Grothendieck institution is precisely $I^{\wr}$.

14.18. Interpolation in $I^{\wr}$
Consider an institution $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ with elementary diagrams and which is liberal on the signature morphisms. Develop an interpolation result for the institution $I^{\wr}$ of pre-defined types over I as an application of the Grothendieck interpolation Thm. 12.17. (*Hint:* We obtain $I^{\wr}$ as a comorphism-based Grothendieck institution by a comorphism-based replica of Ex. 14.17.)

14.19. Basic sentences in $I^{\wr}$
In any institution $I^{\wr}$ of predefined types a (Σ, A)-sentence ρ is (epic) basic when it is (epic) basic as a sentence of the base institution I. (*Hint:* Use the combined results of Exercises 14.11, 5.29 and 5.30.)

Notes. In software system engineering, the class *Mod*[SP] of models of a specification SP of a system is interpreted as the class of all possible implementations of that system.

Inclusive institutions were invented in [58] which presents a software module algebra for theories. The kernel language for structuring specifications presented here has been introduced in [161] but with the union restricted to the situation when the specifications have the same signature. This is a special case of our union when we consider the trivial inclusion system for the signatures with inclusions being the identities. Modern algebraic specification languages provide more sophisticated structuring constructs, however it is possible to translate them to this kernel language (see [131] for CASL). The normal forms for specifications formed only with unions, translations and derivations are well known from [64, 16, 28]. The importance of normal forms is that it allows us to replace any specification by its appropriate normal form, for which some basic properties are more easily available.

The extension of the entailment from a base institution to the institution of its specifications was originally defined in [161]. The idea of $(\mathcal{T}, \mathcal{D})$-specifications was introduced in [28] which under assumptions similar to the conditions of Thm. 14.14 proved the lifting of entailment completeness from the base institution to specifications. However the completeness result of [28] is obtained in a framework assuming implications, conjunctions, and Craig interpolation for the base institution which is significantly narrower in terms of applications than our framework which assumes just Craig-Robinson interpolation. For example important computing logics such as **EQL** or **HCL** are not applications of the completeness results of [28] but they are applications of Thm. 14.14.

The institutions of predefined types were first introduced in [45] under the name of 'constraint institutions' in a slightly different form in the context of the so-called 'category-based equational logic'.

Chapter 15

Logic Programming

The logic programming paradigm in its purely logical form can be defined in arbitrary institutions. This liberation of the logic programming ideal from its conventional framework (based upon the Horn sub-institution of $\mathbf{REL}^1$, the single sorted variant of relational logic) gives the opportunity of developing the logic programming paradigm over various structures. From a computational angle this corresponds to combinations between logic programming and other computing paradigms, such as functional programming, object orientation, concurrency, etc.

A summary of the chapter. We develop the fundamental concepts of logic programming in an institution-independent framework. They include queries, substitutions, solutions, and solution forms. Herbrand theorems play a primary role for the foundations of logic programming. We prove two institution-independent versions of a Herbrand theorem. The first one has a more logical flavor, while the second one has a more operational flavor since it reduces existential satisfaction to universal satisfaction, thus providing foundations for execution in logic programming.

A special section is devoted to unification, which lies at the core of the operational semantics of logic programming. Unification can be regarded as a co-limit construction problem. This categorical viewpoint on unification permits the development of a generic unification algorithm applicable to various institutions.

By abstracting logic programming modules to theories and module imports to theory morphisms, we also discuss the conditions under which the module system of logic programs interacts well with the solutions of queries.

In the final section we extend the logic programming paradigm to 'constraint' logic programming which we regard just as (pure) logic programming over predefined types. Consequent to this view, we obtain general versions of Herbrand theorems for constraint logic programming just as instances of the general institution-independent Herbrand theorems for (pure) logic programming.

15.1 Herbrand Theorems

The logic programming paradigm. The logic programming paradigm in its purely logical form can be described as follows:

> Given a universal Horn finite presentation (Σ, E) (called 'program') and an existentially quantified conjunction of atoms q (called 'query') in $\Sigma \cup Y$ (for Y a new set of 'logical variables'), find a 'solution' ψ for q, i.e., values for the variables Y, such that the corresponding instance $\psi(q)$ of q is satisfied by (Σ, E). This simply means to *prove* that
>
> $$(\Sigma, E) \models (\exists Y)q.$$

In the most conventional form, logic programming is considered over single-sorted first order logic without equality. Less conventional forms of logic programming extends this to multiple sorts, or even considers first order logic with equality as underlying logic, this being considered as a new related paradigm, and known as 'equational logic programming'.

First Herbrand theorem

A careful look at the logic programming paradigm shows it is largely an institution-independent paradigm.

Queries. Given a signature Σ in an arbitrary institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ with a designated class $\mathcal{D}$ of signature morphisms, a *$\mathcal{D}$-query* is any existentially quantified sentence $(\exists\chi)\rho$ such that $\chi \in \mathcal{D}$ is quasi-representable and ρ is a basic sentence.

Herbrand theorems. The Herbrand Theorem reduces the problem of checking the satisfaction of a query by a presentation from all possible models to the initial model only.

Theorem 15.1 (Herbrand theorem I). *In an institution consider a presentation (Σ, E) which has an initial model $0_{\Sigma,E}$. Then for each query $(\exists\chi)\rho$,*

$$E \models (\exists\chi)\rho \quad \textit{if and only if} \quad 0_{\Sigma,E} \models (\exists\chi)\rho.$$

Proof. The implication from left to right is trivial, hence we focus on the other implication. Let $\chi : \Sigma \to \Sigma'$. Assume that $0_{\Sigma,E} \models (\exists\chi)\rho$ and consider a Σ-model M such that $M \models E$. There exists an expansion N' of $0_{\Sigma,E}$ such that $N' \models \rho$. Because χ is quasi-representable let $g' : N' \to M'$ be the χ-expansion of the unique (Σ, E)-model homomorphism $g : 0_{\Sigma,E} \to M$.

Then M' is a χ-expansion of M. Because ρ is basic, there exists a model homomorphism $M_\rho \to N'$, and by composing with g' there exists a model homomorphism $M_\rho \to M'$, which means that $M' \models \rho$. This shows that $M \models (\exists\chi)\rho$. $\square$

In the logic programming culture the initial model $0_{\Sigma,E}$ is known as the *Herbrand model* of (Σ, E).

Solutions for queries. Each χ-expansion N' of $0_{\Sigma,E}$ such that $N' \models \rho$ is called a *solution* for the query $(\exists\chi)\rho$. The importance of Thm. 15.1 is that it reduces the search space for solution of queries from *all* models to only one model. Since this is still not enough computationally, we will develop Thm. 15.1 into a more computation oriented version.

Second Herbrand theorem

The following version of the Herbrand Theorem gives foundations for the execution of logic programming by algorithms such as resolution and paramodulation.

Theorem 15.2. *Consider an institution with representable $\mathcal{D}$-substitutions for a class $\mathcal{D}$ of representable signature morphisms such that*

1. *for each presentation (Σ, E) with initial model, its signature Σ also has an initial model 0_Σ,*
2. *for any presentation (Σ, E) having an initial model $0_{\Sigma,E}$, for each signature morphism $(\chi\colon \Sigma \to \Sigma') \in \mathcal{D}$ its representation M_χ is projective with respect to all 'quotient' homomorphisms $p_{\Sigma,E}\colon 0_\Sigma \to 0_{\Sigma,E}$.*

Then for each presentation (Σ, E) having an initial model, and for any $\mathcal{D}$-query $(\exists\chi_1)\rho$ we have that

> $E \models (\exists\chi_1)\rho$ *if and only if there exists a $\mathcal{D}$-substitution $\psi\colon \chi_1 \to \chi_2$ such that* $E \models (\forall\chi_2)\psi(\rho)$ *and χ_2 is conservative.*

Proof. Assume $E \models (\exists\chi_1)\rho$ and let $\chi_1\colon \Sigma \to \Sigma_1$. We have that $0_{\Sigma,E} \models (\exists\chi_1)\rho$. Let M_1 be the χ_1-expansion of $0_{\Sigma,E}$ such that $M_1 \models \rho$ and let $h = i_{\chi_1}(M_1)\colon M_{\chi_1} \to 0_{\Sigma,E}$. By the projectivity property of M_χ, there exists $h_0\colon M_{\chi_1} \to 0_\Sigma$ such that $h_0; p_{\Sigma,E} = h$.

Because the institution has representable $\mathcal{D}$-substitutions and because 0_Σ represents the identity signature morphism 1_Σ, let $\psi_{h_0}\colon \chi_1 \to 1_\Sigma$ be a substitution determined by h_0. We have the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Mod}(\Sigma) \arrow[r, "i_{1_\Sigma}", "\cong"'] \arrow[d, "\mathsf{Mod}(\psi_{h_0})"'] & 0_\Sigma/\mathsf{Mod}(\Sigma) \arrow[d, "h_0;-"] \\
\mathsf{Mod}(\Sigma_1) \arrow[r, "\cong", "i_{\chi_1}"'] & M_{\chi_1}/\mathsf{Mod}(\Sigma_1)
\end{tikzcd}$$

We show that $E \models \psi_{h_0}(\rho) \models\!\!\!\!\dashv (\forall 1_\Sigma)\psi_{h_0}(\rho)$ (notice also that 1_Σ is trivially conservative).

Let M be a model such that $M \models E$ and let $f\colon 0_{\Sigma,E} \to M$ be the unique (Σ,E)-model homomorphism. Because $i_{\chi_1}^{-1}(h) = M_1 \models \rho$ there exists $g\colon M_\rho \to i_{\chi_1}^{-1}(h) = M_1$ and we have that $i_{\chi_1}(M_\rho); g\restriction_{\chi_1} = h$. Let $u = g\restriction_{\chi_1}; f$. We have $h; f = i_{\chi_1}(M_\rho); u$ hence there exists a model homomorphism $M_\rho \to i_{\chi_1}^{-1}(h; f)$. It therefore follows that $i_{\chi_1}^{-1}(h; f) \models \rho$, which means that $i_{\chi_1}^{-1}(h_0; p_{\Sigma,E}; f) \models \rho$ which means that $i_{\chi_1}^{-1}(h_0; i_{1_\Sigma}(M)) \models \rho$. By the commutativity of the diagram above, $\mathsf{Mod}(\psi_{h_0})(M) \models \rho$. By the satisfaction condition for substitutions we have that $M \models \psi_{h_0}(\rho)$.

$$\begin{tikzcd}
0_\Sigma \arrow[r, "p_{\Sigma,E}"] & 0_{\Sigma,E} \arrow[r, "f"] & M \\
& M_{\chi_1} \arrow[ul, "h_0"] \arrow[u, "h"] \arrow[r, "i_{\chi_1}(M_\rho)"'] & M_\rho\restriction_{\chi_1} \arrow[ul, "g\restriction_{\chi_1}"'] \arrow[u, "u"']
\end{tikzcd}$$

For the converse, we assume there exists a $\mathcal{D}$-substitution $\psi\colon\ \chi_1 \to \chi_2$ such that $E \models (\forall\chi_2)\psi(\rho)$. Because χ_2 is conservative we can find a χ_2-expansion of 0_Σ which means there exists a model homomorphism $u\colon\ M_{\chi_2} \to 0_\Sigma$. We show that $M_1 = i_{\chi_1}^{-1}(M_\psi; u; p_{\Sigma,E})$ is a χ_1-expansion of $0_{\Sigma,E}$ such that $M_1 \models \rho$, where $M_\psi\colon\ M_{\chi_1} \to M_{\chi_2}$ is the model homomorphism determined by ψ.

Let $M_2 = i_{\chi_2}^{-1}(u; p_{\Sigma,E})$. Because $E \models (\forall\chi_2)\psi(\rho)$ and $M_2\restriction_{\chi_2} = 0_{\Sigma,E}$, we have that $M_2 \models \psi(\rho)$. Note that $M_1 = i_{\chi_1}^{-1}(M_\psi; i_{\chi_2}^{-1}(M_2))$. By the satisfaction condition for the substitution ψ this implies that $M_1 \models \rho$. Therefore $0_{\Sigma,E} \models (\exists\chi_1)\rho$. By the first Herbrand Thm. 15.1 we now have that $E \models (\exists\chi_1)\rho$. □

Solution forms. The substitutions ψ of the second Herbrand Thm. 15.2 are called *solution forms*. The proof of the 'only if' part of this theorem shows that each each solution for a query is an instance of a solution form for the query, while the proof of the 'if' part shows that each instance of any solution form for a query to the initial model gives a solution for the query.

The conditions of the second Herbrand Thm. 15.2 are rather mild in actual examples. Below we discuss one of its important instances.

Equational logic programming in HCL. The following instance of Herbrand Thm. 15.2 provides foundations for the operational semantics of conventional equational logic programming.

Corollary 15.3. *Let E be a set of Horn (S,F,P)-sentences for a* **FOL** *signature (S,F,P). Then for any set of new constants X and any $(S,F\cup X,P)$-atom* ρ,

> $E \models (\exists X)\rho$ *if and only if there exists a first order substitution* $\psi\colon\ X \to Y$ *(i.e., a function $X \to T_F(Y)$) such that* $E \models (\forall Y)\psi(\rho)$.

15.2 Unification

Unification algorithms constitute an essential part in the execution of logic programming.

Unification of FOL terms. Let (S,F,P) be a **FOL** signature, X a set of new constants, and t, t' be two $(F\cup X)$-terms of the same sort. A *unifier* for t and t' is any substitution $\theta\colon\ X \to Y$ (i.e., a function $X \to T_F(Y)$) such that $\theta(t) = \theta(t')$.

Let $\overline{t}$ and $\overline{t'}$ be the substitutions $\{*\} \to X$ corresponding to t and t', respectively, defined by $\overline{t}(*) = t$ and $\overline{t'}(*) = t'$, respectively.

Fact 15.4. *A unifier θ for t and t' is precisely a co-cone for the parallel pair of substitutions $\langle \overline{t}, \overline{t'} \rangle$,*

$$\{*\} \overset{\overline{t}}{\underset{\overline{t'}}{\rightrightarrows}} X \xrightarrow{\theta} Y.$$

Unification in institutions. The categorical formulation of unification given by Fact 15.4 allows the following definition of unifiers in abstract institutions.

For any signature Σ in an arbitrary institution with a designated class $\mathcal{D}$ of signature morphisms, a *$\mathcal{D}$-unifier* for any Σ-$\mathcal{D}$-substitutions $\psi_1, \psi_2 : \ \chi \to \chi'$ is any Σ-$\mathcal{D}$-substitution $\theta : \ \chi' \to \chi''$ such that $\psi_1;\theta$ and $\psi_2;\theta$ are equivalent (i.e., $\mathsf{Mod}(\psi_1;\theta) = \mathsf{Mod}(\psi_2;\theta)$).

$$\bullet \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} \bullet \xrightarrow{\theta} \bullet$$

A *most general unifier* is a co-equalizer in the category of Σ-$\mathcal{D}$-substitutions modulo substitution equivalence. Note that in the case of first order substitutions in **FOL**, two substitutions are equivalent if and only if they are equal.

A categorical unification algorithm

An algorithm finding most general unifiers, i.e., co-equalizers in the category of substitutions, consists essentially of reducing the original problem to 'simpler' problems. By 'simpler' we mean 'less symbols'.

In any category, for each parallel pair of arrows u, v let $coeq(u, v)$ denote the class of co-equalizers of u and v.

Reducing the 'operation symbols'. The following general categorical proposition has a rather straightforward proof which we omit here.

Proposition 15.5. *In any category, if $e : \ X' \to X$ is epi and $t, t' : \ X \to Y$, then*

$$coeq(t,t') = coeq(e;t, e;t').$$

The actual meaning of Prop. 15.5 is that it reduces the number of operation symbols of the unification problem for $\langle e;t,\ e;t' \rangle$ to that of the operation symbols of the unification problem $\langle t,\ t' \rangle$. The following **FOL** instance helps to understand this.

Corollary 15.6. *For any operation symbol σ, the pair of terms $\langle \sigma(t_1 \dots t_n), \sigma(t'_1 \dots t'_n) \rangle$ has the same set of most general unifiers as the set of pairs of terms $\{\langle t_1, t'_1 \rangle, \dots, \langle t_n, t'_n \rangle\}$.*

Proof. By interpreting the entities of Prop. 15.5 in the following way:

- $X' = \{*\}$,
- $X = \{x_1, \ldots, x_n\}$,
- $e(*) = \sigma(x_1, \ldots, x_n)$, and
- $t(x_i) = t_i$ and $t'(x_i) = t'_i$.

$\square$

Splitting the problem. Note that the above Cor. 15.6 reduces the total number of symbols by eliminating the symbol σ and by replacing the original problem (which was written for a single pair of terms) with a problem for a set of pairs of terms. The following proposes a method for splitting a unification problem into smaller pieces which can be solved sequentially. As we will see the choice of the splitting is immaterial.

Proposition 15.7. *Let $(\theta : Y \to Z) \in coeq(t_1, t'_1)$ for $t_1, t'_1 : X_1 \to Y$ and $(\gamma : Z \to V) \in coeq(t_2;\theta, t'_2;\theta)$ for $t_2, t'_2 : X_2 \to Y$. Then*

$$\theta;\gamma \in coeq(\langle t_1, t_2\rangle, \langle t'_1, t'_2\rangle)$$

where $X_1 + X_2$ is the co-product of X_1 and X_2, and $\langle t_1, t_2\rangle$ and $\langle t'_1, t'_2\rangle : X_1 + X_2 \to Y$ are the tuplings of t_1 with t_2 and of t'_1 with t'_2, respectively.

$$\begin{tikzcd}
& X_1 \arrow[dl, bend right] \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t'_1"] & & \\
X_1 + X_2 \arrow[r, shift left, "\langle t_1, t_2\rangle"] \arrow[r, shift right, "\langle t'_1, t'_2\rangle"'] & Y \arrow[r, "\theta"] & Z \arrow[r, "\gamma"] & V \\
& X_2 \arrow[ul, bend left] \arrow[u, shift left, "t_2"] \arrow[u, shift right, "t'_2"'] \arrow[ur, shift left, bend right] \arrow[ur, shift right, bend right] & &
\end{tikzcd}$$

The rather straightforward proof of the above Prop. 15.7 is left to the reader. The following is its **FOL** instance.

Corollary 15.8. *The most general unifier for $\{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$, a set of pairs of F-terms with variables Y, when it exists, can be obtained as the substitution θ_n where*

- $\theta_0 = 1_Y$,
- $\theta_k = \theta_{k-1};\gamma_k$ *for* $1 \le k \le n$, *and*
- γ_k *is the most general unifier of* $\langle \theta_{k-1}(t_k), \theta_{k-1}(t'_k)\rangle$ *for* $1 \le k \le n$.

Proof. Follows immediately from Prop. 15.7 by noting that in any **FOL** signature the disjoint union of sets (of variables) is a co-product in the category of first order substitutions. $\square$

The irreducible cases. The unification algorithms consist of alternations of the reduction step given by Prop. 15.5 and of the splitting step given by Prop. 15.7. This process leads eventually to a finite number of irreducible 'atomic' unification problems. In the case of **FOL** unification these can be of the following three kinds:

1. $\sigma(t_1,\dots,t_n) = \sigma'(t'_1,\dots,t'_{n'})$ where σ and σ' are different operation symbols,
2. $x = t$ where x is a variable occurring in t, and
3. $x = t$ where x is a variable *not* occurring in t.

While in the first two cases there are no unifiers, the last case has the substitution $\theta(x) = t$ as most general unifier.

Substitutions that are variables. In order to approach the situation of the last case at a general categorical level, we need to capture categorically the situation of a term consisting only of a variable.

In any institution with a designated class $\mathcal{D}$ of signature morphisms, a $\mathcal{D}$-substitution $v : \chi \to \varphi$ is a *$\mathcal{D}$-variable* when φ is a co-product $\chi + \chi'$ and v is the component of the co-product co-cone corresponding to χ. For example, in the category of **FOL** $\mathcal{D}$-substitutions (for $\mathcal{D}$ being the standard class of signature extensions with a finite number of constants), we may note immediately that the $\mathcal{D}$-variables are just injections between sets (of **FOL** constants).

Occurrence of variables in terms. Next we need to express at the level of abstract substitutions that a variable does not occur in a term. Given a $\mathcal{D}$-variable $v : \chi \to \varphi$ and a $\mathcal{D}$-substitution $t : \chi \to \varphi$ we say that *v does not occur in t* when $t = t';i'$ where

$$\chi \xrightarrow{v} \varphi = \chi + \chi' \xleftarrow{i'} \chi'$$

is a co-product co-cone in the category of $\mathcal{D}$-substitutions and $t' : \chi \to \chi'$.

Of course the above categorical concepts of a variable, and that a variable does not occur in a term, can be defined in any category rather than in the category of $\mathcal{D}$-substitutions. The simple categorical proposition below generalizes the fact that the most general unifier of x and t, where x is a variable that does not occur in t, is given by the substitution mapping x to t.

Proposition 15.9. *For any 'variable' $v : X \to X + X'$ in a category and for any $t' : X \to X'$ we have that $\langle t', 1_{X'}\rangle \in coeq(v, t';i')$.*

$$\begin{tikzcd} X \arrow[r, "t'"'] \arrow[rr, bend left, "v"] & X' \arrow[r, "i'"'] & X + X' \arrow[r, "{\langle t', 1_{X'}\rangle}"] & X'. \end{tikzcd}$$

Termination of unification algorithms. This issue has to be dealt with at the level of particular cases.

For the first order substitution in **FOL**, the algorithm determined by Propositions 15.5, 15.7, and 15.9 terminates by observing that the preorder on sets of pairs of terms defined by the following three criteria (in the order of their priority)

1. the number of variables in the set of pairs of terms,
2. the number of occurrences of operation symbols, and
3. the number of pairs of terms,

is well founded (i.e., does not have infinite strictly decreasing sequences) and that each application of each step given by Propositions 15.5, 15.7, and 15.9 represents a move downwards in this preorder.

The summing up the unification algorithm in **FOL** is given below.

Corollary 15.10. *In* **FOL** *any parallel pair of finitary first order substitutions has a most general unifier if and only if it has a unifier. Moreover the most general unifier can be computed by alternating the reduction steps given by Cor. 15.6 and the splitting step given by Cor. 15.8 and by applying the unification of a variable with a term.*

Exercises

15.1. Prove Propositions 15.5, 15.7, and 15.9.

15.2. [40] **Unification of infinite terms**
In the institution **CA** of contraction algebras, any parallel pair of first order finitary substitutions has a most general unifier if and only if it has a unifier. (*Hint:* By contrast to the case of **FOL** in the case of the infinite terms, the unification of a variable x with a term t in which x occurs has a most general unifier.)

15.3 Modularization

In this section we study modularization in logic programming by abstracting logic programming modules to presentations, and module imports to presentation morphisms.

Translation of queries along signature morphisms. Consider an institution with a class $\mathcal{D}$ of signature morphisms which is stable under pushouts.

Proposition 15.11. *If the institution is semi-exact and all signature morphisms are liberal, then any signature morphism* $\varphi\colon\ \Sigma \to \Sigma'$ *translates any* Σ-$\mathcal{D}$*-query* $(\exists\chi)\rho$ *to the* Σ'-$\mathcal{D}$*-query* $(\exists\chi')\varphi_1(\rho)$ *where*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\chi} & \Sigma_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ \Sigma' & \xrightarrow[\chi']{} & \Sigma'_1 \end{array}$$

is a pushout square of signature morphisms.

Proof. By Prop. 5.12 we know that in semi-exact institutions the quasi-representable signature morphisms are stable under pushouts. This means that $\chi' \in \mathcal{D}$ is quasi-representable.

We still have to show that $\varphi_1(\rho)$ is basic. Let $M_{\varphi_1(\rho)} = M_\rho^{\varphi_1}$ where $M_\rho^{\varphi_1}$ is the free Σ_1-model along φ_1. Then for any Σ'_1-model N'_1 we have that $N'_1 \models \varphi_1(\rho)$ if and only if $N'_1 \upharpoonright_{\varphi_1} \models \rho$ if and only if there exists a homomorphism $M_\rho \to N'_1 \upharpoonright_{\varphi_1}$ if and only if there exists a homomorphism $M_\rho^{\varphi_1} \to N'_1$. $\square$

Translations of solutions along signature morphisms. Consider a pushout of signature morphisms such as in Prop. 15.11 and assume that φ is a presentation morphism $(\Sigma, E) \to (\Sigma', E')$.

Let us recall that a *solution* for the query $(\exists\chi)\rho$ is any χ-expansion M_1 of the initial model $0_{(\Sigma,E)}$ such that $M_1 \models \rho$. The *translation of the solution* M_1 *along* φ is a Σ'_1-model N'_1 defined as follows:

1. Since χ is quasi-representable and $M_1 \upharpoonright_\chi = 0_{\Sigma,E}$, let $h_1 : M_1 \to N_1$ be the unique χ-expansion of the unique model homomorphism $h : 0_{\Sigma,E} \to 0_{\Sigma',E'} \upharpoonright_\varphi$.
2. Let N'_1 be the amalgamation $N_1 \otimes 0_{\Sigma',E'}$.

The following shows that this translation of solutions gives indeed a solution for the translated query (given by Prop. 15.11).

Proposition 15.12. *If M_1 is a solution for the query $(\exists\chi)\rho$, then its translation N'_1 is a solution for the translated query $(\exists\chi')\varphi_1(\rho)$.*

Proof. That M_1 is a solution for $(\exists\chi)\rho$ means that $M_1 \models \rho$. We have to prove that $N'_1 \models \varphi_1(\rho)$. By the satisfaction condition this is equivalent to $N'_1 \upharpoonright_{\varphi_1} \rho$. But $N'_1 \upharpoonright_{\varphi_1} = N_1$. Since $h_1 : M_1 \to N_1$ and $M_1 \models \rho$ we obtain a model homomorphism $M_\rho \to N_1$ which shows that $N_1 \models \rho$. $\square$

The logic programming meaning of Prop. 15.12 is that each module import 'preserve' the solution of queries.

Solutions that are protected. When the module import is a 'protecting' one the translated query does not have any new solutions (i.e., solutions which are not translations of some solution of the original query).

A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is *protecting* when $0_{\Sigma',E'} \upharpoonright_\varphi = 0_{\Sigma,E}$. From the logic programming perspective, a protecting module import just 'protects' the Herbrand model of the imported module.

Proposition 15.13. *If the presentation morphism φ is protecting, then for each solution N for $(\exists\chi')\varphi_1(\rho)$ there exists a solution M_1 for $(\exists\chi)\rho$ such that N is the translation of M_1.*

Proof. We assume that N is a solution for $(\exists\chi')\varphi_1(\rho)$. Let us first show that $N \upharpoonright_{varphi_1}$ is a solution for $(\exists\chi)\rho$. We have that $(N \upharpoonright_{\varphi_1}) \upharpoonright_\chi = N \upharpoonright_{\chi'} \upharpoonright_\varphi = 0_{\Sigma',E'} \upharpoonright_\varphi = 0_{\Sigma,E}$. That $N \upharpoonright_{\varphi_1} \models \rho$ follows by the satisfaction condition from $N \models \varphi_1(\rho)$.

Now we show that the solution N is the translation of $N\upharpoonright_{\varphi_1}$. This follows from the definition of the translation of $N\upharpoonright_{\varphi_1}$ as the amalgamation of $N\upharpoonright_{\varphi_1}$ with $0_{\Sigma',E'}$. $\square$

Exercises

15.3. Consider an institution with representable $\mathcal{D}$-substitutions. For any liberal signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ let $(-)^{\varphi}\colon\ \mathsf{Mod}(\Sigma) \to \mathsf{Mod}(\Sigma')$ be the left adjoint to the model reduct functor $\mathsf{Mod}(\varphi)\colon\ \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$.

For any presentation morphism $\varphi\colon\ (\Sigma,E) \to (\Sigma',E')$ if θ is a solution form for a $\mathcal{D}$-query q in (Σ,E) then any $\mathcal{D}$-substitution ψ determined by M_{θ}^{φ} is a solution form for the translation $\varphi(q)$ in (Σ',E').

15.4 Constraints

Constraint logic programming extends ordinary logic programming to computation with built-in predefined values. In this section we take the viewpoint that constraint logic programming can be regarded as ordinary logic programming over an institution with predefined types. As in Sect. 14.3, for any base institution $I = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ the institution $I^{\wr}$ of predefined types over I is written $(\mathbb{S}ig^{\wr}, \mathsf{Sen}^{\wr}, \mathsf{Mod}^{\wr}, \models^{\wr})$.

Linear equations with real numbers. A typical example is given by systems of linear equations with real numbers. The reader might recall the Euclidean plane example given in Sect. 14.3. The system

$$\begin{cases} 3.14 * x + \pi * y = 2, \\ 5.79 * x + e * y = -1.65 \end{cases}$$

is equivalent to the 'constraint query' in the Euclidean plane $\mathbb{R}^2$,

$$(\exists\{x,y\})\ x * \langle 3.14, 5.79\rangle + y * \langle \pi, e\rangle = \langle 2, -1.65\rangle.$$

Note that this problem cannot be reduced to an ordinary logic programming problem in **FOL** because the Euclidean plane cannot be specified in **FOL** as an initial model of a finite presentation.

General polynomials. Consider a base institution I with elementary diagrams ι that has binary co-products of models. For any signature (Σ,A) with predefined type A, and any representable signature morphism $\chi\colon\ \Sigma \to \Sigma'$, a (Σ,A)-*polynomial with variables* χ is any basic $\mathsf{Sen}(\Sigma_{A+M_\chi})$-sentence of the base institution. The set of all (Σ,A)-polynomial with variables χ is denoted by $A[\chi]$.

For the Euclidean plane example, recall from Sect. 14.3 that the signature Σ is just the **FOL** signature of the specification of a two-dimensional real vector space, while A is $\mathbb{R}'$, the free Σ-model over the ring $\mathbb{R}$ of real numbers. Then, for any set X of (first order) variables, we take the co-product $\mathbb{R}' + T_\Sigma(X)$ between the free Σ-model over the reals $\mathbb{R}$

with the free Σ-model over X. The elements of $\mathbb{R}' + T_\Sigma(X)$ consist of Σ-terms containing both reals and variables from X. Therefore

$$x * \langle 3.14, 5.79\rangle + y * \langle \pi, e\rangle = \langle 2, -1.65\rangle$$

is a $(\Sigma, \mathbb{R}')$-polynomial with variables $\{x, y\}$.

Constraint queries. Given a signature (Σ, A) in the institution $I^{\wr}$ of predefined types, a *constraint* (Σ, A)*-query* $(\exists\chi)\rho$ consists of a representable signature morphism $\chi : \Sigma \to \Sigma'$ in the base institution (giving the 'logical variables') and ρ a (Σ, A)-polynomial with variables χ.

A query $(\exists\chi)\rho$ is *satisfied* by a (Σ, A)-model $h : A \to B$ if and only if there exists a χ-expansion B' of B such that

$$i^{-1}_{\Sigma, A+M_\chi}(\langle h, i_\chi(B')\rangle) \models \rho,$$

$$\begin{tikzcd}
A \arrow[r, "j_A"] \arrow[dr, "h"'] & A + M_\chi \arrow[d, "\langle h, i_\chi(B')\rangle"] & M_\chi \arrow[l, "j_\chi"'] \arrow[dl, "i_\chi(B')"] \\
& B &
\end{tikzcd}$$

By noting that $(1_\Sigma, j_A) : (\Sigma, A) \to (\Sigma, A + M_\chi)$ is representable by j_A in the institution $I^{\wr}$ of predefined types we have

Fact 15.14. *A* (Σ, A)*-constraint query* $(\exists\chi)\rho$ *is just an ordinary query* $(\exists(1_\Sigma, j_A))\rho$ *in the institution of predefined types.*

From now on we will denote $(Q(1_\Sigma, j_A))\rho$ by $(Q\chi)\rho$, for any $Q \in \{\forall, \exists\}$.

First Herbrand theorem for constraint logic programming. By instantiating Herbrand Thm. 15.1 to institutions of predefined types we obtain the following Herbrand theorem for constraint logic programming over arbitrary institutions.

Theorem 15.15 (First Herbrand theorem for constraint logic programming). *In any liberal institution with elementary diagrams and with binary co-products of models, let* $((\Sigma, A), E)$ *be a presentation with predefined types and let* $0_{(\Sigma,A),E} = i_{\Sigma,A}(0_{\Sigma_A, E_A \cup E})$ *be its initial model. Then for each constraint* (Σ, A)*-query* $(\exists\chi)\rho$*,*

$$E \models^{\wr} (\exists\chi)\rho \quad \textit{if and only if} \quad 0_{(\Sigma,A),E} \models^{\wr} (\exists\chi)\rho.$$

Often the sentences E of a logic program do not involve the elements of the built-in model A, which means they are Σ-sentences rather than Σ_A-sentences. In this case the Herbrand model $0_{(\Sigma,A),E}$ is just the 'quotient' of A by E as shown by the following fact.

Fact 15.16. *For any set* E *of* Σ*-sentences, the initial model with predefined types of* $((\Sigma, A), \iota_\Sigma(A)(E))$ *is* $q_E : A \to A_E$*, the universal 'quotient' model homomorphism from* A *to the free* (Σ, E)*-model over* A*.*

Constraint substitutions. Given representable signature morphisms (in the base institution) $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$, for any Σ-model A, a (Σ,A)-*constraint substitution* $\chi_1 \to \chi_2$ is any substitution $\psi_{h_A} : (1_\Sigma, j_A^1) \to (1_\Sigma, j_A^2)$ determined by Σ-model homomorphism $h : M_{\chi_1} \to A + M_{\chi_2}$ by taking the tupling $h_A = \langle j_A^2, h \rangle$,

$$\begin{tikzcd} A \arrow[r, "j_A^1"] \arrow[dr, "j_A^2"'] & A + M_{\chi_1} \arrow[d, "h_A"] & M_{\chi_1} \arrow[l, "j_{\chi_1}"'] \arrow[dl, "h"] \\ & A + M_{\chi_2} & \end{tikzcd}$$

regarded as (Σ,A)-model homomorphism $j_A^1 \to j_A^2$ between the representations of $(1_\Sigma, j_A^1)$ and $(1_\Sigma, j_A^2)$ in the institution $I^\wr$ of predefined types.

Second Herbrand theorem for constraint logic programming. By instantiating Herbrand Thm. 15.2 to the institution of predefined types, we obtain the following constraint Herbrand theorem giving foundations for execution of constraint logic programming.

Theorem 15.17 (Second Herbrand theorem for constraint logic programming). *Consider a liberal institution such that for each representable signature morphism $\chi : \Sigma \to \Sigma'$ the representation M_χ is projective with respect $0_{((\Sigma,A),E)} : A \to A_E$ (the initial model of any presentation $((\Sigma,A),E)$ with predefined types).*

Then for each presentation $((\Sigma,A),E)$ in the institution of predefined types, and for any constraint (Σ,A)-query $(\exists\chi_1)\rho$ we have that

> $E \models^\wr (\exists\chi_1)\rho$ *if and only if there exists a constraint substitution $\psi : \chi_1 \to \chi_2$ such that $E \models^\wr (\forall\chi_2)\mathsf{Sen}^\wr(\psi)(\rho)$ and χ_2 is conservative.*

Similarly to the applications of Thm. 15.2, the projectivity condition of Thm. 15.17 can be established easily in the actual examples, since the initial models $0_{((\Sigma,A),E)} : A \to A_E$ of presentations in the institution of predefined types are 'quotients' of the predefined model A (see Fact 15.16), and hence they are surjective.

Exercises

15.4. Representable signature morphisms in $I^\wr$

1. Any signature morphism $(\chi, f) : (\Sigma,A) \to (\Sigma',A')$ is representable in the institution $I^\wr$ of predefined types if $\chi : \Sigma \to \Sigma'$ is quasi-representable in the base institution I.
2. If the base institution I has binary co-products of models and a signature morphism $\chi : \Sigma \to \Sigma'$ is represented by M_χ (in I), then $(\chi, j_A) : (\Sigma,A) \to (\Sigma', i_\chi^{-1}(j_\chi))$ is also represented by $j_A : A \to A + M_\chi$ (in $I^\wr$).

15.5. In an institution with elementary diagrams the co-product $M_\chi + A$ can be obtained as $0_{\Sigma'(A),\chi'(E_A)}\restriction_\Sigma$ where $\Sigma'(A)$ is the pushout of $\chi : \Sigma \to \Sigma'$ with the elementary extension $\iota_\Sigma(A) : \Sigma \to \Sigma_A$.

$$\begin{tikzcd} \Sigma \arrow[r, "\iota_\Sigma(A)"] \arrow[d, "\chi"'] & \Sigma_A \arrow[d, "\chi'"] \\ \Sigma' \arrow[r, "\iota'"'] & \Sigma'(A) \end{tikzcd}$$

15.6. (a) Let (Σ, E) be a presentation for $\mathbb{R}$-modules, where $\mathbb{R}$ is the ring of the real numbers. Let $\mathbb{R}'$ be the free extension of R to Σ. Show that systems of linear equations are just parallel pairs of constraint $((\Sigma, \mathbb{R}'), E)$-substitutions, and solutions of systems of linear equations are just $((\Sigma, \mathbb{R}'), E)$-unifiers.
(b) By applying Propositions 15.7 and 15.9 show that any system having solutions has a 'most general' solution.

15.7. Birkhoff proof system for $\mathbf{HCL}^{\wr}$
$\mathbf{HCL}^{\wr}$ admits a sound and complete finitary Birkhoff proof system obtained as the free proof system

- with universal $\mathcal{D}$-quantification, where $\mathcal{D}$ is the class of the signature morphisms of the form (χ, j_A) with χ is any signature extension with a finite number of constants $(S,F,P) \to (S, F \uplus X, P)$ and j_A is the canonical injective homomorphism $A \to A + T_{(S,F,P)}(X)$ (for our convenience we will denote $(\forall(\chi, j_A))\rho$ by $(\forall X)\rho$), and
- such that for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences there exists a natural isomorphism

$$\mathsf{Pf}((S,F,P),A)(\Gamma \cup H, C) \cong \mathsf{Pf}((S,F,P),A)(\Gamma, H \Rightarrow C)$$

and which is generated by the following system of finitary rules for a signature $((S,F,P),A)$:

$$\begin{array}{rl}
(A) & \emptyset \vdash E_A \ \text{ where } \ E_A \ \text{ is the elementary diagram of } \ A \\
(R) & \emptyset \vdash t = t \ \text{ for each term } \ t \\
(S) & t = t' \vdash t' = t \ \text{ for any terms } \ t \ \text{ and } \ t' \\
(T) & \{t = t', t' = t''\} \vdash t = t'' \ \text{ for any terms } \ t, t' \ \text{ and } \ t'' \\
(F) & \{t_i = t_i' \mid 1 \le i \le n\} \vdash \sigma(t_1, \dots, t_n) = \sigma(t_1', \dots, t_n') \ \text{ for any } \ \sigma \in F \\
(P) & \{t_i = t_i' \mid 1 \le i \le n\} \cup \{\pi(t_1, \dots, t_n)\} \vdash \pi(t_1', \dots, t_n') \ \text{ for any } \ \pi \in P \\
(\mathit{Subst}) & (\forall X)\rho \vdash (\forall Y)\theta(\rho) \ \text{ for any } ((S,F,P),A)\text{-sentence } \ \rho \\
 & \quad \text{and for any constraint substitution } \theta : \ X \to A + T_{(S,F,P)}(Y).
\end{array}$$

Notes. Logic programming began in the early 1970s as a direct outgrowth of earlier work in automatic theorem proving and artificial intelligence. The theory of clausal-form [first order] logic, and an important theorem by the logician Jacques Herbrand constituted the foundation for most activity in theorem proving in the early 1960s. The discovery of resolution — a major step in the mechanization of clausal-form theorem proving — was due to J. Alan Robinson [153]. In 1972, Robert Kowalski and Alain Colmerauer were led to the crucial idea that *logic could be used as a programming language* [177]. A year later the first Prolog system was implemented. A good reference for foundations of conventional logic programming is [114]. The equational logic programming paradigm unifies logic programming based on Horn clause logic and functional programming based on equational logic. One of the earliest contributions to this field was [143]. Later Goguen and Meseguer provided a definition of equational logic programming as logic programming over classical conventional specification based on (order sorted) equational logic [79, 80], [41] generalized it to logic programming over 'category-based equational logic', and [77] extended it to logic programming over behavioral logic.

The conventional Herbrand Theorem [114] has been extended to many sorted first order logic with equality in [80] and generalized to category-based equational logic in [41, 42]. The latter included as its instance a Herbrand theorem for 'category-based' constraint logic [45]. The Herbrand Theorem relies on the existence of an initial model for the presentation (program), known as the 'Herbrand universe' by the conventional logic programming community. As we have seen in Chap. 8, this initiality requirement determines the restriction of logic programming to universal

Horn theories. This corresponds to the fact that in actual institutions, universal Horn sentences are the most complex sentences supporting algorithms for automatic execution of logic programming.

Our approach to the institution-independent foundations of logic programming based on (quasi-)representable signature morphisms was developed in [49] which also introduced the concept of institution-independent substitution.

The earliest algorithm for computing most general unifiers in first order logic was given by Herbrand [94], and later [153] applied to automated inference. Following the observation of Goguen that most general unifiers are just co-equalizers in categories of substitutions [72], Rydeheard and Burstall developed a generic categorical approach to unification algorithms in [158]. A 2-categorical approach on unification modulo equational theories has been investigated by [159].

Our basic result on modularization for logic programming was first developed in [43] within the context of category-based equational logic programming.

Appendix A

Table of Notation

Sets and Categories

$\mathcal{P}(A)$	the set of the subsets of A
$\mathcal{P}_\omega(A)$	the set of the finite subsets of A
$A \setminus B$	the difference between sets A and B, $\{x \in A \mid x \notin B\}$
$card(A)$	the cardinality of the set A
λ^+	the last cardinal strictly greater than the ordinal λ
$F\vert_J$	the reduction of the filter F over I to a subset $J \in F$
$\mathbb{S}et$	the category of sets as objects and functions as arrows
$\mathbb{C}lass$	the hyper-category of classes as objects and functions (between classes) as arrows
$\mathbb{C}at$	the hyper-category of categories as objects and functors as arrows
$\mathbb{G}rp$	the category of groups
$\vert\mathbb{C}\vert$	the class of objects of the category $\mathbb{C}$
$\mathbb{C}(A,B)$	the set of arrows between objects A and B
$dom(f)$	the domain (source) of the arrow f
$cod(f)$	the codomain (target) of the arrow f
$f;g$	the composition of arrows f and g
$\mathbb{C}^{\mathrm{op}}$	the opposite of the category $\mathbb{C}$
$\mathbb{C}^*$	the 2-opposite of a 2-category $\mathbb{C}$
$A \cong B$	the objects A and B are isomorphic
$A \times B$	the (direct) product of objects A and B
$\prod_{i\in I} A_i$	the (direct) product of the family of objects $\{A_i\}_{i\in I}$
$A + B$	the co-product (direct sum) of the objects A and B
$0_{\mathbb{C}}$	the initial object of the category $\mathbb{C}$

$Lim(D)$	vertex of the limiting cone for the diagram D
$Colim(D)$	vertex of the co-limiting co-cone for the diagram D
$A \uplus B$	the disjoint union of sets A and B
$A/\mathcal{U}$	comma category
(f,B)	object of comma category $A/\mathcal{U}$ where $f: A \to \mathcal{U}(B)$
$B^\sharp$	the Grothendieck category determined by the indexed category B
A_F	the directed diagram of products $F \to \mathbb{C}$ for a filter F over I and a family $\{A_i\}_{i\in I}$ of objects in $\mathbb{C}$
$\prod_F A_i$	the filtered product of objects $\{A_i \mid i \in I\}$ by the filter F over an index set I

Institutions

$\mathbb{S}ig^I$	the category of the signatures of institution I
Sen^I	the sentence functor of institution I
Mod^I	the model functor of institution I
$M \models^I_\Sigma \rho$	the Σ-model M satisfies the Σ-sentence ρ in the institution I
$\Gamma \vdash E$	entailment (i.e., there exists a proof) from Γ to E
Pf^I	the proof functor of proof system I
Rl^I	the proof rule functor of system-of-proof rules I
h^I, c^I	the hypotheses and the conclusions for natural transformations of system-of-proof rules I
$(co)\mathbb{I}ns/\mathbb{P}f\mathbb{I}ns$	the category of institution/institution with proofs (co)morphisms
$(co)\mathbb{R}l\mathbb{S}ys/\mathbb{P}f\mathbb{S}ys$	the category of system-of-proof rules/proof system (co)morphisms
$\varphi^{\mathrm{st}}/\varphi^{\mathrm{op}}/\varphi^{\mathrm{rl}}$	the mapping on sort/OPERATION/relation symbols of the morphism φ of **FOL**-signatures
E^*	the class of models satisfying the set of sentences E
$\mathbb{M}^*$	the set of sentences satisfied by the class of models $\mathbb{M}$
$E^\bullet$	the theory generated by the set of sentences E in a proof system
$E \models\!\!\!\dashv E'$	E and E' are semantically equivalent sets of sentences, i.e., $E \models E'$ and $E' \models E$
$E \vdash\!\dashv E'$	E and E' are proof theoretic equivalent sets of sentences, i.e., $E \vdash E'$ and $E' \vdash E$
$\mathbb{M} \equiv \mathbb{M}'$	$\mathbb{M}$ and $\mathbb{M}'$ are elementarily equivalent classes of models

$\lvert M \rvert$	the set of the elements of the carriers of the model M
$(x : s)$	the constant/variable x has sort s
$\overrightarrow{S}$	the set of all (higher-order) types constructed from the sorts S
τ_Σ	the semantic topology on Σ-models
$\mathbb{T}h^I$	the category of theories of institution I
$\mathbb{P}res^I$	the category of presentations of institution I
I^{pres}	the institution of the presentations of the institution I
$0_\Sigma/0_{\Sigma,E}$	the initial $[\Sigma/(\Sigma,E)]$-model
$\mathsf{Mod}^{\text{pres}}$	the model functor $\mathsf{Mod}^{I^{\text{pres}}}$ of I^{pres}
$M_1 \otimes M_2$	the amalgamation of models M_1 and M_2
$\iota_\Sigma(M) : \Sigma \to \Sigma_M$	the elementary extension of the signature Σ via the model M
(Σ_M, E_M)	the elementary diagram of the Σ-model M
$i_{\Sigma,M}$	the natural isomorphism determined by the elementary diagram of the Σ-model M
M_M	the initial model $0_{\Sigma_M,E_M}$ of the elementary diagram of a model M
N_h	$i^{-1}_{\Sigma,M}(h)$ for $h : M \to N$ model homomorphism
$E(I)$	the elementary sub-institution of I
$=_f$	the kernel of homomorphism f
S_w/S_c	class of (plain)/closed injective model homomorphisms in **FOL** and **PA**
S_f	class of full subalgebras in **PA**
H_r	class of surjective model homomorphisms in **FOL** and **PA**
H_s/H_c	class of strong/closed surjective homomorphisms in **FOL**
$\rho_1 \wedge \rho_2$	the conjunction of ρ_1 and ρ_2
$\rho_1 \vee \rho_2$	the disjunction of ρ_1 and ρ_2
$\rho_1 \Rightarrow \rho_2$	the implication of ρ_2 by ρ_1
$\rho_1 \Leftrightarrow \rho_2$	the equivalence between ρ_1 and ρ_2
$t \stackrel{e}{=} t'$	existence equation
$\bigwedge E$	the conjunction of the set of sentences E
$\neg\rho$	the negation of ρ
$\neg E$	$\{\neg\rho \mid \rho \in E\}$
$(\forall\chi)\rho$	universal quantification of Σ'-sentence ρ for $\chi : \Sigma \to \Sigma'$ signature morphism
$(\exists\chi)\rho$	existential quantification of Σ'-sentence ρ for $\chi : \Sigma \to \Sigma'$ signature morphism
T_F	the algebra of F-terms
$T_F(X)$	the F-algebra of F-terms over the set of variables X
M_χ	the model representing the signature morphism χ
i_χ	the natural isomorphism defining a representable signature morphism χ

$\mathsf{Mod}(\psi)/\mathsf{Sen}(\psi)$	the model reduct/sentence translation part of the substitution ψ
M_E	the model defining a basic set of sentences E
$M \models^{\mathrm{inj}} h$	M is with respect to h
$Inj(H)$	the class of objects/models injective with respect to all $h \in H$
$c(\rho_1, \ldots, \rho_n)$	the c-connection of sentences $\rho_1, \ldots, \rho_n$
$Fixed(\mathcal{U})$	the fixed points of the semantic operator $\mathcal{U}$
$\mathrm{Up}(\mathbb{M})$	the class of all ultraproducts of models of $\mathbb{M}$
Ur	the ultraradical relation on models
$\mathsf{Univ}(\Sigma)$	the set of universal Σ-sentences in **FOL**
$\mathsf{Exist}(\Sigma)$	the set of existential Σ-sentences in **FOL**
$M[\mathsf{Sen}^0]N$	$M^* \cap \mathsf{Sen}^0(\Sigma) \subseteq N^* \cap \mathsf{Sen}^0(\Sigma)$
$M \xrightarrow{\mathsf{Sen}^0} N$	there exists a Σ-model homomorphism $h\colon M \to N$ such that $M_M[\mathsf{Sen}^0]N_h$
$\mathsf{K}\text{-}\mathsf{Mod}(\Sigma)$	the category of K-Σ-Kripke models with models from $\mathsf{Mod}(\Sigma)$, where $K \in \{T, S4, S5\}$
$\mathsf{M}\text{-}\mathsf{Sen}(\Sigma)$	the set of 'modal' Σ-sentences over $\mathsf{Sen}(\Sigma)$
$\mathcal{J}^\sharp$	the Grothendieck institution of the indexed (co)institution $\mathcal{J}$
$\mathrm{SP} \models\!\mid \mathrm{SP}'$	equivalence of specifications
$\mathrm{SP} \star \varphi$	renaming of specification SP
$\varphi \mid \mathrm{SP}$	hiding of specification SP
$\mathbb{S}pec^I$	the category of structured specifications of institution I
$\mathbb{S}pec_{\langle \mathcal{T}, \mathcal{D} \rangle}$	the category of $\langle \mathcal{T}, \mathcal{D} \rangle$-specifications
I^{spec}	the institution of structured specifications over I
$I^{\mathrm{spec}}_{\langle \mathcal{T}, \mathcal{D} \rangle}$	the institution of $\langle \mathcal{T}, \mathcal{D} \rangle$-specifications over I

Bibliography

[1] Jirí Adamek, Horst Herrlich, and George Strecker. *Abstract and Concrete Categories*. John Wiley, 1990.

[2] Marc Aiguier and Fabrice Barbier. An institution-independent proof of the beth definability theorem. *Studia Logica*, 85(3):333–359, 2007.

[3] Marc Aiguier and Răzvan Diaconescu. Stratified institutions and elementary homomorphisms. *Information Processing Letters*, 103(1):5–13, 2007.

[4] Eyal Amir and Sheila McIlraith. Improving the efficiency of reasoning through structure-based reformulation. In B.Y. Choueiry and T.Walsh, editors, *Proceedings of the Symposium on Abstraction, Reformulation and Approximation (SARA'2000)*, volume 1864 of *Lecture Notes in Artificial Intelligence*, pages 247–259. Springer-Verlag Berlin Heidelberg, 2000.

[5] Hajnal Andréka and István Németi. Łoś lemma holds in every category. *Studia Scientiarum Mathematicarum Hungarica*, 13:361–376, 1978.

[6] Hajnal Andréka and István Németi. A general axiomatizability theorem formulated in terms of cone-injective subcategories. In B. Csakany, E. Fried, and E.T. Schmidt, editors, *Universal Algebra*, pages 13–35. North-Holland, 1981. Colloquia Mathematics Societas János Bolyai, 29.

[7] Hajnal Andréka and István Németi. Generalization of the concept of variety and quasivariety to partial algebras through category theory. *Dissertationes Mathematicae*, CCIV, 1983.

[8] Peter B. Andrews. *An Introduction to Mathematical Logic and Type Theory*. Kluwer Academic Publishers, Dordrecht, 2nd edition, 2002.

[9] M. Arrais and José L. Fiadeiro. Unifying theories in different institutions. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 81–101. Springer, 1996.

[10] Edigio Astesiano, Michel Bidoit, Hélène Kirchner, Berndt Krieg-Brückner, Peter Mosses, Don Sannella, and Andrzej Tarlecki. CASL: The common algebraic specification language. *Theoretical Computer Science*, 286(2):153–196, 2002.

[11] Bernhard Banaschewski and Horst Herrlich. Subcategories defined by implications. *Houston Journal Mathematics*, 2:149–171, 1976.

[12] Jon Barwise. Axioms for abstract model theory. *Annals of Mathematical Logic*, 7:221–265, 1974.

[13] Jon Barwise and Solomon Feferman. *Model-Theoretic Logics*. Springer, 1985.

[14] John L. Bell and Allan B. Slomson. *Models and Ultraproducts*. North Holland, 1969.

[15] Jean Bénabou. Fibred categories and the foundations of naive category theory. *Journal of Symbolic Logic*, 50:10–37, 1985.

[16] Jan Bergstra, Jan Heering, and Paul Klint. Module algebra. *Journal of the Association for Computing Machinery*, 37(2):335–372, 1990.

[17] Jan Bergstra, Alban Ponse, and Scott A. Smolka. *Process Algebra*. Elsevier, 2001.

[18] Evert Willem Beth. On Padoa's method in the theory of definition. *Indagationes Mathematicæ*, 15:330–339, 1953.

[19] Jean-Yves Béziau. Classical negation can be expressed by one of its halves. *Logic Journal of the IGPL*, 7(2):154–151, 1999.

[20] Jean-Yves Béziau. From consequence operator to universal logic: a survey of general abstract logic. In Jean-Yves Béziau, editor, *Logica Universalis*, pages 3–17. Birkhäuser, 2005.

[21] Jean-Yves Béziau. 13 questions about universal logic. *Bulletin of the Section of Logic*, 35(2/3):133–150, 2006.

[22] Juan Bicarregui, Theodosis Dimitrakos, Dov Gabbay, and Tom Maibaum. Interpolation in practical formal development. *Logic Journal of the IGPL*, 9(2):231–243, 2001.

[23] Michel Bidoit and Rolf Hennicker. On the integration of the observability and reachability concepts. In *Proc. 5th Int. Conf. Foundations of Software Science and Computation Structures (FOSSACS'2002)*, volume 2303 of *Lecture Notes in Computer Science*, pages 21–36, 2002.

[24] Garrett Birkhoff. On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society*, 31:433–454, 1935.

[25] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, 2000.

[26] Francis Borceux. *Handbook of Categorical Algebra*. Cambridge University Press, 1994.

[27] Tomasz Borzyszkowski. Generalized interpolation in CASL. *Information Processing Letters*, 76:19–24, 2001.

[28] Tomasz Borzyszkowski. Logical systems for structured specifications. *Theoretical Computer Science*, 286(2):197–245, 2002.

[29] Peter Burmeister. *A Model Theoretic Oriented Approach to Partial Algebras*. Akademie-Verlag, Berlin, 1986.

[30] Rod Burstall and Joseph Goguen. The semantics of Clear, a specification language. In Dines Bjorner, editor, *Proceedings, 1979 Copenhagen Winter School on Abstract Software Specification*, pages 292–332. Springer, 1980. Lecture Notes in Computer Science, Volume 86; based on unpublished notes handed out at the Symposium on Algebra and Applications, Stefan Banach Center, Warsaw, Poland, 1978.

[31] Carlos Caleiro, Paulo Mateus, Amilcar Sernadas, and Cristina Sernadas. Quantum institutions. In K. Futatsugi, J.-P. Jouannaud, and J. Meseguer, editors, *Algebra, Meaning, and Computation*, volume 4060 of *LNCS*, pages 50–64. Springer, 2006.

[32] Chen-Chung Chang and H. Jerome Keisler. *Model Theory*. North Holland, Amsterdam, 1990.

[33] Alonzo Church. A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5(1):56–68, 1940.

[34] Corina Cîrstea. An institution of modal logics for coalgebras. *Logic and Algebraic Programming*, 67(1-1):87–113, 2006.

[35] Mihai Codescu. The model theory of higher order logic. Master's thesis, Şcoala Normală Superioară Bucureşti, 2007.

[36] Mihai Codescu and Daniel Găină. Birkhoff completeness in institutions. submitted.

[37] Traian Şerbănuţă. Institutional concepts in first-order logic, parameterized specification, and logic programming. Master's thesis, University of Bucharest, 2004.

[38] Virgil Emil Căzănescu and Grigore Roşu. Weak inclusion systems. *Mathematical Structures in Computer Science*, 7(2):195–206, 1997.

[39] Răzvan Diaconescu. Grothendieck inclusion systems. submitted.

[40] Răzvan Diaconescu. Contraction algebras and unification of infinite terms. *Journal of Computer and System Sciences*, 44(1):23–43, 1992.

[41] Răzvan Diaconescu. Category-based semantics for equational and constraint logic programming, 1994. DPhil thesis, University of Oxford.

[42] Răzvan Diaconescu. Completeness of category-based equational deduction. *Mathematical Structures in Computer Science*, 5(1):9–41, 1995.

[43] Răzvan Diaconescu. Category-based modularisation for equational logic programming. *Acta Informatica*, 33(5):477–510, 1996.

[44] Răzvan Diaconescu. Extra theory morphisms for institutions: logical semantics for multi-paradigm languages. *Applied Categorical Structures*, 6(4):427–453, 1998. A preliminary version appeared as JAIST Technical Report IS-RR-97-0032F in 1997.

[45] Răzvan Diaconescu. Category-based constraint logic. *Mathematical Structures in Computer Science*, 10(3):373–407, 2000.

[46] Răzvan Diaconescu. Grothendieck institutions. *Applied Categorical Structures*, 10(4):383–402, 2002. Preliminary version appeared as IMAR Preprint 2-2000, ISSN 250-3638, February 2000.

[47] Răzvan Diaconescu. Institution-independent ultraproducts. *Fundamenta Informaticæ*, 55(3-4):321–348, 2003.

[48] Răzvan Diaconescu. Elementary diagrams in institutions. *Journal of Logic and Computation*, 14(5):651–674, 2004.

[49] Răzvan Diaconescu. Herbrand theorems in arbitrary institutions. *Information Processing Letters*, 90:29–37, 2004.

[50] Răzvan Diaconescu. An institution-independent proof of Craig Interpolation Theorem. *Studia Logica*, 77(1):59–79, 2004.

[51] Răzvan Diaconescu. Interpolation in Grothendieck institutions. *Theoretical Computer Science*, 311:439–461, 2004.

[52] Răzvan Diaconescu. Proof systems for institutional logic. *Journal of Logic and Computation*, 16(3):339–357, 2006.

[53] Răzvan Diaconescu. Borrowing interpolation. Submitted, 2007.

[54] Răzvan Diaconescu. Institutions, Madhyamaka and universal model theory. In Jean-Yves Béziau and Alexandre Costa-Leite, editors, *Perspectives on Universal Logic*, pages 41–65. Polimetrica, 2007.

[55] Răzvan Diaconescu. A categorical study on the finiteness of specifications. *Information Processing Letters*, 2008. to appear.

[56] Răzvan Diaconescu and Kokichi Futatsugi. *CafeOBJ Report: The Language, Proof Techniques, and Methodologies for Object-Oriented Algebraic Specification*, volume 6 of *AMAST Series in Computing*. World Scientific, 1998.

[57] Răzvan Diaconescu and Kokichi Futatsugi. Logical foundations of CafeOBJ. *Theoretical Computer Science*, 285:289–318, 2002.

[58] Răzvan Diaconescu, Joseph Goguen, and Petros Stefaneas. Logical support for modularisation. In Gerard Huet and Gordon Plotkin, editors, *Logical Environments*, pages 83–130. Cambridge, 1993. Proceedings of a Workshop held in Edinburgh, Scotland, May 1991.

[59] Răzvan Diaconescu and Marius Petria. Saturated models in institutions. Unpublished draft.

[60] Răzvan Diaconescu and Petros Stefaneas. Ultraproducts and possible worlds semantics in institutions. *Theoretical Computer Science*, 379(1):210–230, 2007.

[61] Theodosis Dimitrakos. *Formal Support for Specification Design and Implementation*. PhD thesis, Imperial College, 1998.

[62] Theodosis Dimitrakos and Tom Maibaum. On a generalized modularization theorem. *Information Processing Letters*, 74:65–71, 2000.

[63] Charles Ehresmann. Esquisses et types des structures algébriques. *Buletinul Institutului Politehnic Iaşi*, 14(18):1–14, 1968.

[64] Hartmut Ehrig, Eric Wagner, and James Thatcher. Algebraic specifications with generating constraints. In J. Diaz, editor, *Proceedings, 10th Colloquium on Automata, Languages and Programming*, volume 154 of *Lecture Notes in Computer Science*, pages 188–202. Springer, 1983.

[65] E. Allen Emerson. Temporal and modal logic. In van Leeuwen, editor, *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*, pages 995–1072. Elsevier, 1990.

[66] José L. Fiadeiro and José F. Costa. Mirror, mirror in my hand: A duality between specifications and models of process behaviour. *Mathematical Structures in Computer Science*, 6(4):353–373, 1996.

[67] José L. Fiadeiro and Amilcar Sernadas. Structuring theories on consequence. In Donald Sannella and Andrzej Tarlecki, editors, *Recent Trends in Data Type Specification*, volume 332 of *Lecture Notes in Computer Science*, pages 44–72. Springer, 1988.

[68] Thomas Frayne, Anne C. Morel, and Dana S. Scott. Reduced direct products. *Fundamenta Mathematica*, 51:195–228, 1962.

[69] Dov M. Gabbay and Larisa Maksimova. *Interpolation and Definability: modal and intuitionistic logics*. Oxford University Press, 2005.

[70] Valére Glivenko. Sur la logique de M. Brower. *Bulletin de l'Académie Royale de la Belgique, Classe des Sciences, ser. 5*, 14:225–228, 1928.

[71] Joseph Goguen. *Categories of Fuzzy Sets*. PhD thesis, Department of Mathematics, University of California at Berkeley, 1968.

[72] Joseph Goguen. What is unification? A categorical view of substitution, equation and solution. In Maurice Nivat and Hassan Aït-Kaci, editors, *Resolution of Equations in Algebraic Structures, Volume 1: Algebraic Techniques*, pages 217–261. Academic, 1989. Also Report SRI-CSL-88-2R2, SRI International, Computer Science Lab, August 1988.

[73] Joseph Goguen. Data, schema, ontology and logic integration. *Journal of IGPL*, 13(6):685–715, 2006.

[74] Joseph Goguen and Rod Burstall. A study in the foundations of programming methodology: Specifications, institutions, charters and parchments. In David Pitt, Samson Abramsky, Axel Poigné, and David Rydeheard, editors, *Proceedings, Conference on Category Theory and Computer Programming*, volume 240 of *Lecture Notes in Computer Science*, pages 313–333. Springer, 1986.

[75] Joseph Goguen and Rod Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, 1992.

[76] Joseph Goguen and Răzvan Diaconescu. Towards an algebraic semantics for the object paradigm. In Harmut Ehrig and Fernando Orejas, editors, *Recent Trends in Data Type Specification*, volume 785 of *Lecture Notes in Computer Science*, pages 1–34. Springer, 1994.

[77] Joseph Goguen, Grant Malcolm, and Tom Kemp. A hidden Herbrand theorem: Combining the object, logic and functional paradigms. *Journal of Logic and Algebraic Programming*, 51(1):1–41, 2002.

[78] Joseph Goguen and José Meseguer. Completeness of many-sorted equational logic. *Houston Journal of Mathematics*, 11(3):307–334, 1985.

[79] Joseph Goguen and José Meseguer. Eqlog: Equality, types, and generic modules for logic programming. In Douglas DeGroot and Gary Lindstrom, editors, *Logic Programming: Functions, Relations and Equations*, pages 295–363. Prentice-Hall, 1986.

[80] Joseph Goguen and José Meseguer. Models and equality for logical programming. In Hartmut Ehrig, Giorgio Levi, Robert Kowalski, and Ugo Montanari, editors, *Proceedings, TAPSOFT 1987*, volume 250 of *Lecture Notes in Computer Science*, pages 1–22. Springer, 1987.

[81] Joseph Goguen and Grigore Roşu. Institution morphisms. *Formal Aspects of Computing*, 13:274–307, 2002.

[82] Joseph Goguen, Timothy Winkler, José Meseguer, Kokichi Futatsugi, and Jean-Pierre Jouannaud. Introducing OBJ. In Joseph Goguen and Grant Malcolm, editors, *Software Engineering with OBJ: algebraic specification in action*. Kluwer, 2000.

[83] Robert Goldblatt. Mathematical modal logic: a view of its evolution. *Journal of Applied Logic*, 1(6):309–392, 2003.

[84] George Grätzer. *Universal Algebra*. Springer, 1979.

[85] Daniel Găină. Private communications.

[86] Daniel Găină and Andrei Popescu. An institution-independent generalization of Tarski's Elementary Chain Theorem. *Journal of Logic and Computation*, 16(6):713–735, 2006.

[87] Daniel Găină and Andrei Popescu. An institution-independent proof of Robinson consistency theorem. *Studia Logica*, 85(1):41–73, 2007.

[88] René Guitart and Christian Lair. Calcul syntaxique des modèles et calcul des formules internes. *Diagramme*, 4, 1980.

[89] David Harel. Dynamic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Volume II: Extensions of Classical Logic*, pages 497–604. Reidel, 1984.

[90] Robert Harper, Donald Sannella, and Andrzej Tarlecki. Logic representation in LF. In David Pitt, David Rydeheard, Peter Dybjer, Andrew Pitts, and Axel Poigné, editors, *Proceedings, Conference on Category Theory and Computer Science*, volume 389 of *Lecture Notes in Computer Science*, pages 250–272. Springer, 1989.

[91] Felix Hausdorff. Grundzüge der mengenlehre, 1914. Leipzig.

[92] Leon Henkin. Completeness in the theory of types. *Journal of Symbolic Logic*, 15(2):81–91, June 1950.

[93] Matthew Hennessy and Robin Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the Association for Computing Machinery*, 32:137–161, 1985.

[94] Jacques Herbrand. Recherches sur la théorie de la démonstration. *Travaux de la Société des Sciences et des Lettres de Varsovie, Classe III*, 33(128), 1930.

[95] Horst Herrlich and George Strecker. *Category Theory*. Allyn and Bacon, 1973.

[96] Arend Heyting. Die formalen regeln der intuitionistischen logik. *Sitzungberichte der Preussischen Akademie der Wiessenschaften, Physikalisch-mathematische Klasse*, 1930.

[97] Joram Hirschfeld and William H. Wheeler. *Forcing, arithmetic, divison rings*, volume 454 of *Lecture Notes in Mathematics*. Springer, 1975.

[98] Wilfrid Hodges. *Model Theory*. Cambridge University Press, 1993.

[99] Wilfrid Hodges. *A shorter Model Theory*. Cambridge University Press, 1997.

[100] Michale Holz, Karsten Steffens, and E. Weitz. *Introduction to Cardinal Arithmetic*. Birkhäuser, 1999.

[101] Bart Jacobs. *Categorical Type Theory*. PhD thesis, Katholieke Universiteit Nijmegen, 1991.

[102] Ranjit Jhala, Rupak Majumdar, and Ru-Gang Xu. State of the union: Type inference via Craig interpolation. In *Tools and Algorithms for the Construction and Analysis of Systems*, volume 4424 of *Lecture Notes in Computer Science*, pages 553–567. Springer, 2007.

[103] H. Jerome Keisler. Theory of models with generalized atomic formulas. *Journal of Symbolic Logic*, 1960.

[104] H. Jerome Keisler. *Model Theory for Infinitary Logic*. North-Holland, 1971.

[105] Andrei N. Kolmogorov. On the principle of the excluded middle. *Matematicheskii Sbornik*, 32(646–667), 1925. (in Russian).

[106] Dexter Kozen and Jerzy Tiuryn. Logic of programs. In *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*, pages 789–840. Elsevier, 1990.

[107] Saul Kripke. A completeness theorem in modal logic. *Journal of Symbolic Logic*, 24:1–15, 1959.

[108] Oliver Kutz and Till Mossakowski. Modules in transition. conservativity, composition, and colimits, 2007.

[109] Joachim Lambek and Phil Scott. *Introduction to Higher Order Categorical Logic*. Cambridge, 1986. Cambridge Studies in Advanced Mathematics, Volume 7.

[110] Yngve Lamo. *The Institution of Multialgebras – a general framework for algebraic software development*. PhD thesis, University of Bergen, 2003.

[111] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer, second edition, 1998.

[112] William F. Lawvere. Functorial semantics of elementary theories. *Journal of Symbolic Logic*, 31:294–295, 1966.

[113] Steffen Lewitzka. A topological approach to universal logic: model theoretical abstract logics. In Jean-Yves Béziau, editor, *Logica Universalis*, pages 35–63. Birkhäuser, 2005.

[114] John Lloyd. *Foundations of Logic Programming*. Springer, 1984.

[115] Jerzy Łoś. Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres. In *Mathematical Interpretation of Formal Systems*, pages 98–113. North-Holland, Amsterdam, 1955.

[116] Michael Makkai. Ultraproducts and categorical logic. In C.A. DiPrisco, editor, *Methods in Mathematical Logic*, volume 1130 of *Lecture Notes in Mathematics*, pages 222–309. Springer Verlag, 1985.

[117] Michael Makkai. Stone duality for first order logic. *Advances in Mathematics*, 65(2):97–170, 1987.

[118] Michael Makkai and Gonzolo Reyes. *First order categorical logic: Model-theoretical methods in the theory of topoi and related categories*, volume 611 of *Lecture Notes in Mathematics*. Springer, 1977.

[119] Anatoly Malcev. *The Metamathematics of Algebraic Systems*. North-Holland, 1971.

[120] Günter Matthiessen. Regular and strongly finitary structures over strongly algebroidal categories. *Canadian Journal of Mathematics*, 30:250–261, 1978.

[121] Sheila McIlraith and Eyal Amir. Theorem proving with structured theories. In *Proceedings of the 17th Intl. Conf. on Artificial Intelligence (IJCAI-01)*, pages 624 – 631, 2001.

[122] Kenneth McMillan. Applications of Craig interpolants in model checking. In *Proceedings TACAS'2005*, volume 3440 of *Lecture Notes in Computer Science*, pages 1–12. Springer, 2005.

[123] José Meseguer. General logics. In H.-D. Ebbinghaus et al., editors, *Proceedings, Logic Colloquium, 1987*, pages 275–329. North-Holland, 1989.

[124] José Meseguer. Conditional rewriting logic as a unified model of concurrency. *Theoretical Computer Science*, 96(1):73–155, 1992.

[125] José Meseguer. Membership algebra as a logical framework for equational specification. In F. Parisi-Pressice, editor, *Proc. WADT'97*, volume 1376 of *Lecture Notes in Computer Science*, pages 18–61. Springer, 1998.

[126] Bernhard Möller, Andrzej Tarlecki, and Martin Wirsing. Algebraic specifications of reachable higher-order algebras. In Donald Sannella and Andrzej Tarlecki, editors, *Recent Trends in Data Type Specification*, volume 332 of *Lecture Notes in Computer Science*, pages 154–169. Springer Verlag, Berlin, 1987.

[127] J. Donald Monk. *Mathematical Logic*. Springer-Verlag, 1976.

[128] Michael Morley and Robert Vaught. Homogeneous universal models. *Mathematica Scandinavica*, 11:37–57, 1962.

[129] Till Mossakowksi. Charter morphisms. Unpublished note.

[130] Till Mossakowski. Different Types of Arrow Between Logical Frameworks. In F. Meyer auf der Heide and B. Monien, editors, *Proc. of ICALP*, volume 1099 of *Lecture Notes in Computer Science*, pages 158–169. Springer Verlag, 1996.

[131] Till Mossakowski. Specification in an arbitrary institution with symbols. In C. Choppy, D. Bert, and P. Mosses, editors, *Recent Trends in Algebraic Development Techniques, 14th International Workshop, WADT'99, Bonas, France*, volume 1827 of *Lecture Notes in Computer Science*, pages 252–270. Springer-Verlag, 2000.

[132] Till Mossakowski. Comorphism-based Grothendieck logics. In K. Diks and W. Rytter, editors, *Mathematical foundations of computer science*, volume 2420 of *Lecture Notes in Computer Science*, pages 593–604. Springer, 2002.

[133] Till Mossakowski. Relating CASL with other specification languages: the institution level. *Theoretical Computer Science*, 286:367–475, 2002.

[134] Till Mossakowski. Foundations of heterogeneous specification. In R. Hennicker M. Wirsing, D. Pattinson, editor, *Recent Trends in Algebraic Development Techniques*, volume 2755 of *Lecture Notes in Computer Science*, pages 359–375. Springer Verlag, London, 2003.

[135] Till Mossakowski. Heterogeneous specification and the heterogeneous tool set. Habilitation thesis, University of Bremen, 2005.

[136] Till Mossakowski, Răzvan Diaconescu, and Andrzej Tarlecki. What is a logic translation? UNILOG 2007.

[137] Till Mossakowski, Joseph Goguen, Răzvan Diaconescu, and Andrzej Tarlecki. What is a logic? In Jean-Yves Béziau, editor, *Logica Universalis*, pages 113–133. Birkhäuser, 2005.

[138] Till Mossakowski and Markus Roggenbach. Structured CSP – a process algebra as an institution. In J. Fiadeiro, editor, *WADT 2006*, volume 4409 of *Lecture Notes in Computer Science*, pages 92–110. Springer-Verlag Heidelberg, 2007.

[139] Daniele Mundici. Compactness + Craig interpolation = Robinson consistency in any logic, 1979. Manuscript. University of Florence.

[140] Daniele Mundici. Robinson consistency theorem in soft model theory. *Transactions of the AMS*, 263:231–241, 1981.

[141] Greg Nelson and Derek Oppen. Simplication by cooperating decision procedures. *ACM Trans. Program. Lang. Syst.*, 1(2):245–257, 1979.

[142] István Németi and Ildikó Sain. Cone-implicational subcategories and some birkhoff-type theorems. In B. Csakany, E. Fried, and E.T. Schmidt, editors, *Universal Algebra*, pages 535–578. North-Holland, 1981. Colloquia Mathematics Societas János Bolyai, 29.

[143] Michael J. O'Donnell. *Computing in systems described by equation*, volume 58 of *Lecture Notes in Computer Science*. Springer, 1977.

[144] Derek Oppen. Complexity, convexity and combinations of theories. *Theoretical Computer Science*, 12:291–302, 1980.

[145] Wieslaw Pawlowski. Context institutions. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 436–457. Springer, 1996.

[146] Marius Petria. Finitely sized models in institutions. Private communication.

[147] Marius Petria. An institutional version of Gödel Completeness Theorem. In *Algebra and Coalgebra in Computer Science*, volume 4624, pages 409–424. Springer Berlin / Heidelberg, 2007.

[148] Marius Petria and Răzvan Diaconescu. Abstract Beth definability in institutions. *Journal of Symbolic Logic*, 71(3):1002–1028, 2006.

[149] Amir Pnueli. The temporal semantics of concurrent programs. *Theoretical Computer Science*, 13:45–60, 1981.

[150] Andrei Popescu, Traian Şerbănuţă and Grigore Roşu. A semantic approach to interpolation. In *Foundations of software science and computation structures*, volume 3921 of *Lecture Notes in Computer Science*, pages 307–321. Springer Verlag, 2006.

[151] Vaughan R. Pratt. Semantical considerations on Floyd-Hoare logic. In *Proceedings of the 17th Annual IEEE Symposium on Foundations of Computer Science*, pages 109–121, 1976.

[152] Horst Reichel. *Structural Induction on Partial Algebras*. Akademie Verlag Berlin, 1984.

[153] J. Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the Association for Computing Machinery*, 12:23–41, 1965.

[154] Grigore Roşu. Kan extensions of institutions. *Universal Computer Science*, 5(8):482–492, 1999.

[155] Grigore Roşu. Axiomatisability in inclusive equational logic. *Mathematical Structures in Computer Science*, 12(5):541–563, 2002.

[156] Grigore Roşu and Joseph Goguen. On equational Craig interpolation. *Universal Computer Science*, 5(8):482–493, 1999.

[157] Pieter-Hendrik Rodenburg. A simple algebraic proof of the equational interpolation theorem. *Algebra Universalis*, 28:48–51, 1991.

[158] David Rydeheard and Rod Burstall. *Computational Category Theory*. Prentice-Hall, 1988.

[159] David E. Rydeheard and John Stell. Foundations of equational deduction: A categorical treatment of equational proofs and unification algorithms. In D. H. Pitt et al., editor, *Category theory and computer programming, Guilford*, volume 240 of *Lecture Notes in Computer Science*, pages 493–505. Springer, 1985.

[160] Antonio Salibra and Giuspeppe Scollo. Interpolation and compactness in categories of pre-institutions. *Mathematical Structures in Computer Science*, 6:261–286, 1996.

[161] Donald Sannella and Andrzej Tarlecki. Specifications in an arbitrary institution. *Information and Control*, 76:165–210, 1988.

[162] Lutz Schröder, Till Mossakowski, and Christoph Lüth. Type class polymorphism in an institutional framework. In José Fiadeiro, editor, *Recent Trends in Algebraic Development Techniques, 17th Intl. Workshop (WADT 2004)*, volume 3423 of *Lecture Notes in Computer Science*, pages 234–248. Springer, Berlin, 2004.

[163] Lutz Schröder, Till Mossakowski, Andrzej Tarlecki, Piotr Hoffman, and Bartek Klin. Amalgamation in the semantics of CASL. *Theoretical Computer Science*, 331(1):215–247, 2005.

[164] Saharon Shelah. Every two elementary equivalent models have isomorphic ultrapowers. *Israel Journal of Mathematics*, 10:224–233, 1971.

[165] Joseph Shoenfield. *Mathematical Logic*. Addison-Wesley, 1967.

[166] Petros Stefaneas. Chartering some institutions, 1993. Unpublished draft.

[167] Colin Stirling. Modal and temporal logics. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2, pages 477–563. Oxford University Press, 1992.

[168] Andrzej Tarlecki. Bits and pieces of the theory of institution. In David Pitt, Samson Abramsky, Axel Poigné, and David Rydeheard, editors, *Proceedings, Summer Workshop on Category Theory and Computer Programming*, volume 240 of *Lecture Notes in Computer Science*, pages 334–360. Springer, 1986.

[169] Andrzej Tarlecki. On the existence of free models in abstract algebraic institutions. *Theoretical Computer Science*, 37:269–304, 1986.

[170] Andrzej Tarlecki. Quasi-varieties in abstract algebraic institutions. *Journal of Computer and System Sciences*, 33(3):333–360, 1986.

[171] Andrzej Tarlecki. Moving between logical systems. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 478–502. Springer, 1996.

[172] Andrzej Tarlecki. Structural properties of some categories of institutions. Technical report, Institute of Informatics, Warsaw University, 1996.

[173] Andrzej Tarlecki. Towards heterogeneous specifications. In D. Gabbay and M. van Rijke, editors, *Proceedings, International Conference on Frontiers of Combining Systems (FroCoS'98)*, pages 337–360. Research Studies Press, 2000.

[174] Andrzej Tarlecki, Rod Burstall, and Joseph Goguen. Some fundamental algebraic tools for the semantics of computation, part 3: Indexed categories. *Theoretical Computer Science*, 91:239–264, 1991.

[175] Alfred Tarski and Robert Vaught. Arithmetical extensions of relational systems. *Compositio Mathematicæ*, 13:81–102, 1957.

[176] Tenzin Gyatso (the Fourteenth Dalai Lama). *The Universe in a Single Atom*. Wisdom Publications, 2005.

[177] Maartin H. van Emden and Robert Kowalski. The semantics of predicate logic as a programming language. *Journal of the Association for Computing Machinery*, 23(4):733–742, 1976.

[178] Robert Vaught. *Set theory: an introduction*. Birkhäuser, 1985.

[179] Paulo Veloso. On pushout consistency, modularity and interpolation for logical specifications. *Information Processing Letters*, 60(2):59–66, 1996.

[180] Michal Wałicki, Adis Hodzic, and Sigurd Meldal. Compositional homomorphisms of relational structures (modeled as multialgebras). In *Fundamentals of Computing Theory*, volume 2138 of *Lecture Notes in Computer Science*, pages 359–371. Springer, 2001.

[181] Charles F. Wells. Sketches: Outline with references. Unpublished draft.

Index