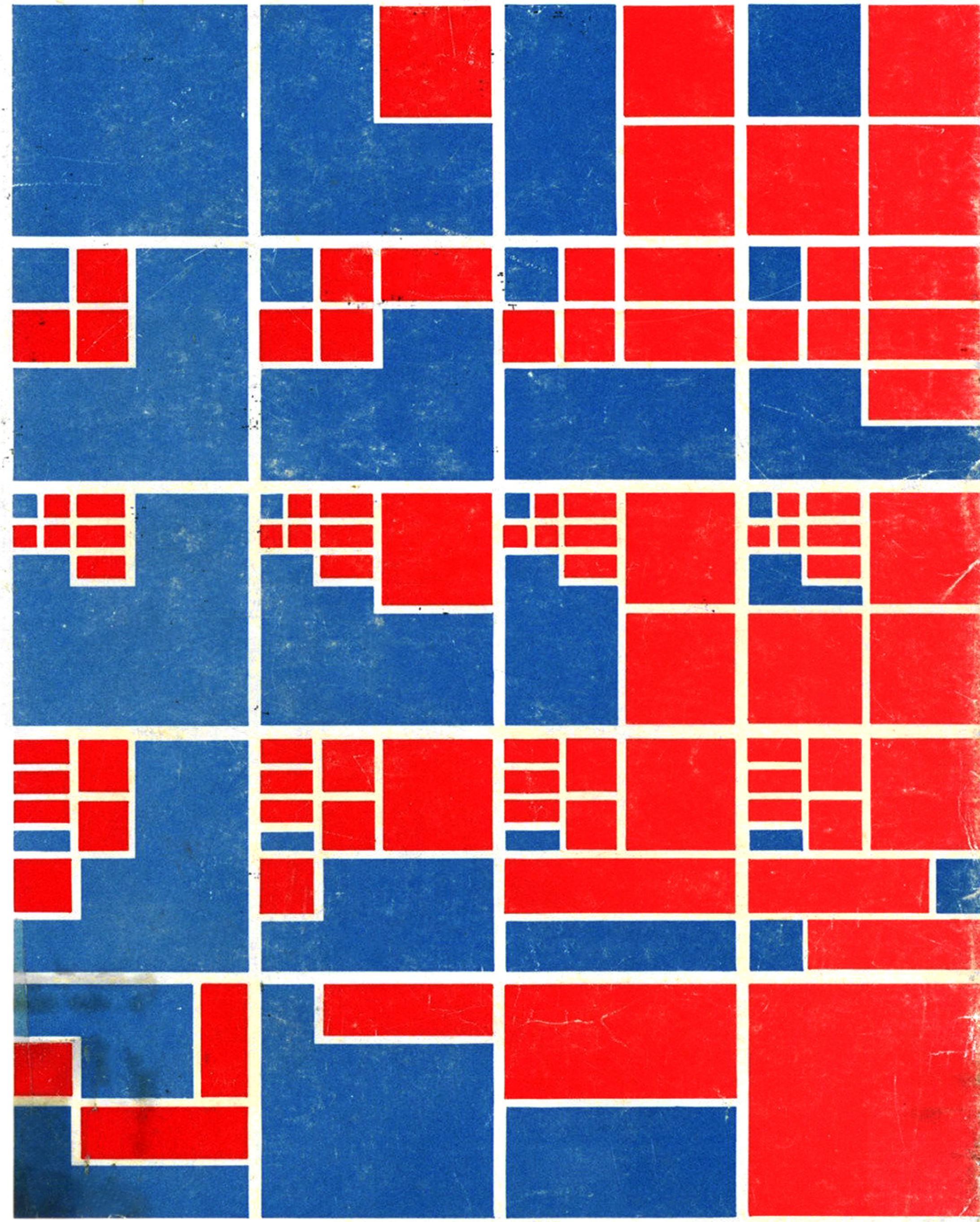
Architectural Morphology

J P Steadman



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Architectural Morphology

J P Steadman

An introduction to the geometry of building plans

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Preface and acknowledgements

"If something like a theory of architecture will ever be developed, then one of its first chapters will deal with the theory of cell configurations ..."

H Rittel (1970)

This book provides an introduction to an area of architectural research which has been emerging over the last ten years, and which has gone under the name, variously, of 'architectural morphology' or 'configurational studies' in architecture. It is concerned centrally with the limits which geometry places on the possible forms and shapes which buildings and their plans may take. The use of the term 'morphology' alludes then to Goethe's original notion, of a general *science of possible forms*, covering not just forms in nature, but forms in art, and especially the forms of architecture.

This research work has been published up to now in scattered papers, many of them so technical as to be inaccessible to the general reader. The material is here brought together for the first time, put into order, and, together with the necessary mathematical foundations, set out in a self-contained treatment for the nonspecialist student of architecture. The last three chapters of the book explore in a more speculative way the broader implications of the work for design, for building science, and for architectural history.

The hope is that the book may be of interest to practising designers and architectural students. But it is directed most specifically to theoreticians: to building scientists, understood in the most general sense of that term; to historians of architecture who have interests in the description and explanation of the basic underlying forms of buildings, and their methods of composition; and to those working on computer aids for the representation and manipulation of building form in design.

Much of the work described here is that of my own immediate colleagues and students in three university departments: the Martin Centre at the Cambridge School of Architecture, the School of Architecture and Urban Planning at the University of California, Los Angeles, and the Centre for Configurational Studies at the Open University. Of these colleagues I should like to name specifically, at Cambridge, Leonardo Combes and Cecil Bloch; at UCLA, Bill Mitchell, Robin Liggett, and George Stiny; and at the Open University, Lionel March, Chris Earl, Ray Matela, and Ramesh Krishnamurti. I owe a great deal to the many discussions which I have had with these individuals and others in these three places; and I hope I have done justice to their work where I have presented it, in necessarily highly summarised form, here.

A glance at the references will show how a very large proportion of the entries refer to a single source—the journal *Environment and Planning B*. Quite apart from his own protean contributions, Lionel March in his editorial capacity has promoted this field of research to an international audience with such enthusiasm and vigour that everyone working on these subjects must be greatly in his debt. If this book can introduce the field to readers unfamiliar with *Environment and Planning B*, and lead them on into the fuller and more technical discussions of the journal itself, then it will have served much of its purpose.

Outside these immediate circles, I must mention the intellectual stimulus and challenge continuously provided by the morphological work of Bill Hillier and of his colleagues and students at University College London. Some of the arguments recounted here have been raised and debated in an interuniversity seminar series held jointly between the Open University, Cambridge, and University College groups.

I would like to thank Chuck Eastman, as editor of this series, and also Chris Earl, for reading the manuscript, for making many helpful suggestions, and for saving me from at least some of my errors. I am especially grateful to Cecil Bloch for his kindness in allowing me to reproduce part of his catalogue of rectangular dissections as an appendix. The manuscript was typed by Sue Ayers and the diagrams drawn by Sarah Couch.

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For Leonardo:
another exhibit for Dubuffet's museum?

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1

Introduction

> "It seems very unaccountable that the generality of our late architects dwell so much upon [the] ornamental, and so slightly pass over the geometrical, which is the most essential part of architecture."
>
> Sir Christopher Wren (1750)

In 1975 William Mitchell, Robin Liggett, and I developed a computer program which generated architectural plans of a certain type automatically (Mitchell et al, 1976). These were plans consisting of rectangular rooms, set together to form arrangements with a rectangular shape overall—the sorts of plans typical of many small houses and flats. Constraints could be specified on the topology of the plan, requiring that certain rooms be adjacent, or not be adjacent to each other, and on the dimensions of the rooms, limiting their areas, lengths, widths, or proportions. The method is explained in more detail here in chapter 9. The special and novel character of the program was its capacity to produce exhaustively *all possible* plans in which the given constraints were satisfied.

This work provoked some strange reactions. Mitchell outlined the system to an architect acquaintance in Los Angeles, and was told flatly "That's impossible". Later, Mitchell and I submitted a paper describing the work to the British *Architects' Journal.* The article was refused by the then editor in a letter of scarcely concealed hysteria: "This work is strictly non-architectural, ... it has nothing to do with architecture".

I tell these anecdotes not out of any sense of grievance (our feelings at the time were more ones of surprise and amusement), but in recognition of the fact that any book which treats architectural subjects from a mathematical point of view, and even more so one which mentions computers, is bound immediately in the present climate of ideas to come up against preconceptions in the minds of many readers.

There is a widespread reaction today, and for good reasons, against the architectural functionalism of the modern movement. That functionalism, to speak very generally, took two forms, one more benign and less dangerous than the other[(1)]. The functionalism of Sullivan, or of Lethaby,

[(1)] Two of the best and most succinct discussions of these distinct meanings of 'functionalism' are to be found in Summerson (1949, page 149), and in Goodman and Goodman (1947, pages 8-9). As the Goodmans say, 'Form follows function' "... in the original statement of Louis Sullivan ... meant that the form is not given by the function but is appropriate to the function; in his words 'a store must look like a store, a bank must look like a bank'. This is an aesthetic principle; for it includes certain ideas, the genres of buildings, whose unity is formal over and above the utility; they are given by the sensibility of the culture ... But in the more radical interpretation of the Bauhaus the formula means that the form is given by the function: there is no addition to the arrangement of the utility, but it is presented just as it works. As such, this is not an aesthetic principle at all ...". (See also Steadman, 1979, chapter 13.)

involved an aesthetic belief that buildings should 'explain themselves', should present in their design a rational argument about their function and means of construction. By contrast the functionalism of, say, Hannes Meyer, or more recently of some adherents of the 'design methods' movement, tended towards a much more radical doctrine: that functional considerations could, if subjected to sufficiently precise analysis, be made to define the form of a building in a necessary and automatic way—what has been called 'functional determinism'.

It is popularly assumed that any architectural research of a mathematical nature must have functionalist aims in this second sense: that it seeks to devise ways in which the design of a building can be formulated as a mathematical 'problem', and mathematically 'solved'.

This book aims to provide an introduction to an area of research which, though mathematical in nature, and although computers are certainly used, has quite different aims and is based on a very different conception of the nature of architectural design. It takes the view that design is, always has been, and always will be concerned at its central core with the manipulation of *form*, with *composition*, understood as the putting together of two-dimensional and three-dimensional components, either spaces or material elements, in arrangements or configurations.

The architect's choice of forms is made according to his artistic purpose, and is directed towards the satisfaction of (though not by any means uniquely determined by) his client's tastes, desires, and utilitarian requirements, as well as being limited by technical and structural possibility. But *whatever* is expressed or signified by an architectural work, *whatever* practical functions it might serve, and *however* it is constructed, this choice of form in design is constrained above all by limits on what is geometrically and topologically possible. It is the purpose of this whole book to demonstrate the detailed nature of some of these limits. As an example, and speaking generally, if it is required that a number of rooms be laid out on a single floor level such that specified pairs of rooms are or are not adjacent (without consideration of their shape or size) then there exists only a finite number of possible such arrangements (perhaps none at all). Such limitations are of a *topological* nature.

Again there exist choices for the overall *geometrical discipline* according to which plans may be laid out, for instance with their walls aligned on a rectangular or some other form of grid; for the *geometrical elements* from which the plan is to be made up—for example, rectangles or other shapes corresponding perhaps to rooms; and for conditions on the assembly of those elements—for example, that they should pack closely without gaps, that they should not overlap, and so on. Limits on the variety of possible arrangements of such elements under such conditions can then be expressed in terms of well-defined rules for their composition. The introduction of *dimensional constraints* will reduce this variety of arrangement still further.

None of these limitations force the designer's hand. Rather they serve to determine the extent of the field of possibilities within which his choice must be exercised. As he restricts *himself* to a geometrical discipline, to a set of formal elements, to some dimensional constraints, so his choice is further narrowed.

Other authors have adopted the terminology of linguistics in this context, and have referred to the *syntax* (Frank Lloyd Wright spoke of 'grammars') of architectural form. We might pursue this analogy with language a little farther. The academic study of architecture can be seen as being divisible into three areas or disciplines. There is the professional training of designers, in architectural schools. The analogy here is with learning to speak a language. There is the study of architectural history, from a critical and aesthetic point of view. This must correspond to the study of literary history and literary criticism. And, third, there is what I would suggest is the architectural counterpart to the discipline of linguistics: that is, an architectural science, devoted to a general investigation of the cultural and technological systems within which all architects work, and all buildings are produced.

This 'architectural linguistics' is itself divisible into two parts. There is that part which deals with the syntax of possible architectural forms and arrangements. And there is that part which deals with the semantics, the systems of meaning, which the syntactic forms and structures come to support. (For example, see Norberg-Schulz, 1974; Bonta, 1979; Broadbent et al, 1980.) The present book is confined entirely to the subject of architectural syntax.

It is in many ways a sequel to *The Geometry of Environment* which Lionel March and I published just twelve years ago (March and Steadman, 1971). (In fact the material presented in the two books overlaps to an extent, especially that on symmetry, and graph theory.) One of our slightly hidden motives in writing *The Geometry of Environment* was a belief that learning to understand geometrical limitations and geometrical possibility formed a valuable, and at that time neglected, part of the general education of the designer(2).

Not that this knowledge would be applied necessarily in specific mathematical *techniques* or *methods*, but that it could constitute part of the broad intellectual makeup of the designer, part of the mental apparatus which he brings to bear in design. There was admittedly a short presentation in *The Geometry of Environment* of some computerised 'design methods'

(2) It is perhaps worth recalling that we felt it necessary, in the introduction to *The Geometry of Environment*, to apologise for our "total disregard in this book for any but the most simple functional requirements". Thus we could have been more justifiably accused on that count of formalist rather than functionalist tendencies. For coverage of functional aspects we referred readers, among other works, to Alexander's (1964) *Notes on the Synthesis of Form*, which as George Stiny has said, should more properly have been called *Notes on the Analysis of Function*.

intended for generating supposedly 'optimal' architectural layouts on the basis of a circulation-minimising criterion. Such methods had been formulated very much in a 'functional determinist' spirit, and were criticised in the book on those grounds.

Some similar methods are described briefly here at the beginning of chapter 9; but the purpose is most emphatically not to endorse the view of architectural design which they embody. They are introduced because they make use of some of the mathematical ideas and techniques which are described here; and more importantly because they led the way towards the development of methods for enumerating all permutations of plan arrangement within given constraints, and hence of exploring the 'outer limits' of geometrical possibility in design, as already described.

Some designers are strangely unwilling to acknowledge the fact of the existence of such geometrical limitations at all—witness the reaction of Mitchell's acquaintance—and adopt evasive tactics in discussion. When one makes the argument that, say the number of rectangular plans with so many rooms is (in a certain sense, defined in chapter 2 in terms of topological equivalence) strictly finite, then the response tends to be "Ah well, but you haven't included triangular plans ..." or "What about circular plans? ... or 'free-form' plans? ...". By moving the ground of the discussion, by trying to step outside the system, the architect attempts vainly to escape this geometrical prison, as he sees it, in an effort to reassert his creative freedom.

But this supposed freedom is illusory; and anyone who denies the existence of geometrical limitations in design is only doomed to stumble against them blindly. The fact is that for many good geometrical and practical reasons, as will be clear from later chapters, architects *do* choose to confine themselves to a rectangular discipline in design. And *where they do so*, then certain limitations on geometrical possibility must necessarily apply. They have perfect freedom, certainly, to reject the discipline of the right-angle if they so wish, and adopt say a triangular or a curvilinear geometry. But of course these disciplines in their turn impose their own limits on possibility—in some respects *more* severe, as we shall see, than those of rectangular geometry.

It might be objected that architects have been managing to compose forms in design very satisfactorily for a couple of thousand years or so, without any conscious knowledge of such limitations. In terms of our earlier analogy, a person can certainly achieve fluency in a language—become a poet indeed—without any formal knowledge of the science of linguistics. Perhaps there is no call for the practical training of designers to include any formal geometrical education in these subjects.

On the other hand it is possible to point to some undeniably creative practitioners of other arts, who have argued that a mastery of the formal 'rules of composition' serves precisely to liberate, rather than to enslave, the artist; that it is exactly the dizzying (and deceptive) notion of an 'infinitude of possibilities' which paralyses the imagination. Thus Stravinsky (1970, page 85) says:

"What delivers me from the anguish into which an unrestricted freedom plunges me is the fact that I am always able to turn immediately to the concrete things that are here in question. [He is referring to such acoustic and physiological facts as the nature of the octave, its chromatic intervals, and their perception, etc.] Let me have something finite, definite—matter that can lend itself to my operation only insofar as it is commensurate with my possibilities. And such matter presents itself to me together with its limitations. I must in turn impose mine on it. So here we are, whether we like it or not, in the realm of necessity. And yet which of us has ever heard talk of art as other than a realm of freedom? This sort of heresy is uniformly widespread because it is imagined that art is outside the bounds of ordinary activity."

In architecture too there are individuals whose creativity might be credited at least in part to their knowledge of geometry as a subject; either explicit, as in the case of Wren, or of a more intuitive and informal kind, such as that of Wright, who is well known to have made much of his education from an early age in the constructive appreciation of abstract pattern and form.

A final word should be said about the mathematical treatment. For many people, I know, the mere sight of a mathematical symbol is sufficient to make the eyes glaze over and the brain go numb. This, however, is an irrational phobia, and like all phobias it is treated by urging the sufferer to face and handle the feared object, to discover for himself that it is by no means as frightening as he had always believed.

If you are a mathematophobe, you should be reassured that the prior knowledge of mathematics required is no more than basic arithmetic and a few simple algebraic concepts. The mathematics is very informally presented, most equations are spelled out in words, and frequent examples are given, illustrated with explanatory diagrams. Do not be put off by algebraic formulations (of which there are few). You should expect to read mathematics slowly in any case; and more is learned by *doing* rather than just reading—so exercises are included at the end of several of the chapters. All the treatment here is self-contained; but in some of the topics covered, background textbooks have been cited for those who want to prepare more widely around the subject.

For the mathematically equipped reader, on the other hand, I should apologise for the absence of formal definitions and proofs, and for any possible tedium which the repetition of explanations or illustrations might cause. References are given to the specialised literature in which formal treatment of the mathematical issues may be found.

The 'dimensionless' representation of rectangular plans

> "The splitting into something discrete and something continuous seems to me a basic issue in all morphology, and the morphology of ornaments and crystals establishes a paragon by the clearcut way in which this distinction is carried out."
>
> Hermann Weyl (1952)

There are some famous drawings by Albrecht Dürer (1528), in which he uses a particular geometrical method to describe the proportions of the human head and face. The method consists in drawing a square or rectangle to enclose the head, and dividing this rectangle up into a grid of lines which mark the positions of the various features—brow, eyes, nose, chin, and so on. Dürer shows how a series of different faces may be produced by altering the relative spacing of the lines of the grid (figure 2.1).

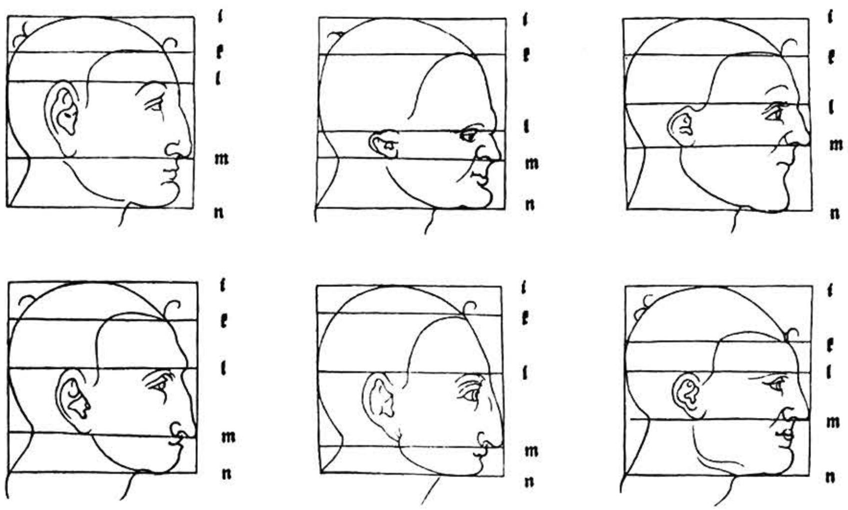


Figure 2.1. Different faces produced by assigning varying dimensions to the intervals of the superimposed grid (from Dürer, 1528).

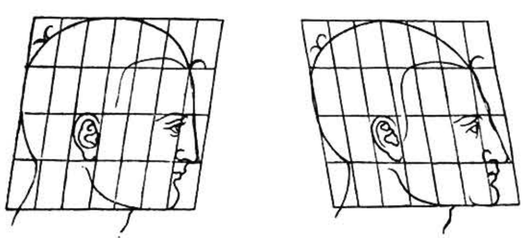


Figure 2.2. Different faces produced by changing the angle between the coordinates of the superimposed grid, in a *shear* transformation (from Dürer, 1528).

All the faces are made from essentially the same drawing, from the same set of curved lines joined in an unvarying arrangement; but squeezed or stretched, so to speak, as the grid dimensions are altered. It is as though the drawing were made on a sheet of rubber.

In other illustrations, Dürer encloses the head in an oblique grid of equal parallelograms (figure 2.2). The grid is again transformed, this time by altering the angle between the coordinates—so subjecting the drawing to a *shear* transformation—to generate a different kind of series of related profiles.

The same idea is developed by D'Arcy Wentworth Thompson (1961) in one of the best-known chapters "On the theory of transformations, or the comparison of related forms", in his book on biological morphology *On Growth and Form*. Thompson's purpose here is to compare the shapes of animals which belong to the same zoological class. To do this he makes

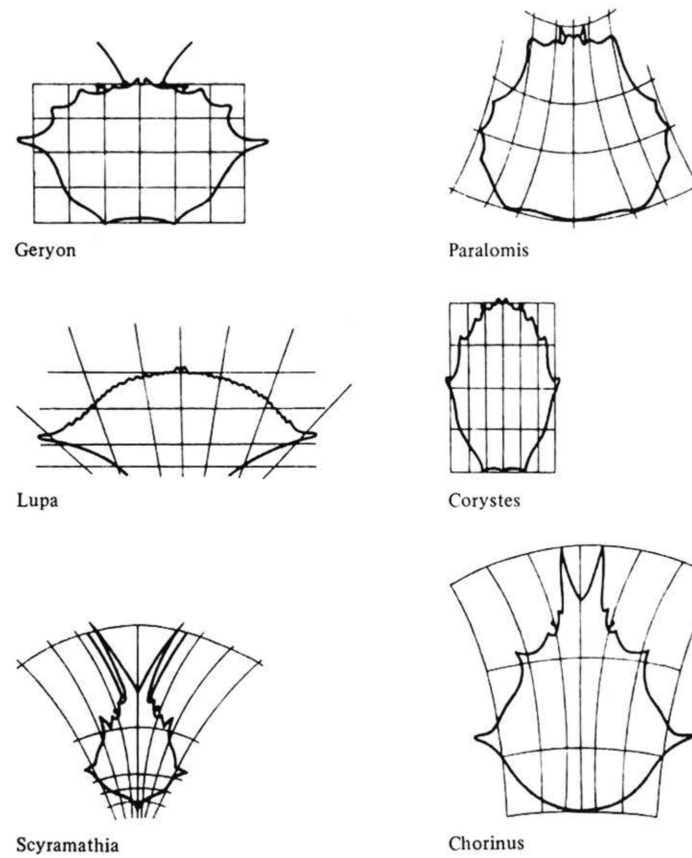


Figure 2.3. Carapaces of various crabs, related together in shape by D'Arcy Thompson's 'method of coordinates' (from Thompson, 1961, figure 142).

use of several types of 'deformation' of systems of rectangular coordinates: by simple enlargement or reduction, by stretching along one or other of the axes, by shearing, by various logarithmic transformations of the dimensions of the grid, and through different forms of curvature of the grid about one or more centres (figure 2.3).

By these means, Thompson illustrates how species which are closely related in evolutionary terms can often be shown to possess forms which can be produced one from another by simple transformations (figure 2.4). By use of similar techniques, that is, through transformations which can be referred to mechanical or geometrical principles of growth, he demonstrates alternatively how the *same* animal or plant changes in shape as it develops. Thompson reproduces in the chapter some of the drawings by Dürer already mentioned. The same method of coordinates can be extended to three dimensions, and Thompson discusses how the solid forms of the bodies of fishes can be related together by means of geometric transformations in which say a 'round' fish is 'rolled out' into a flat fish, 'as a baker rolls a piece of dough'.

In every case the related shapes can be said to show a 'topological similitude'. As Thompson (1961, page 321) expresses it, "There is something, an essential and indispensable something, which is common to them all, something which is the subject of all our transformations, and remains *invariant* (as the mathematicians say) under them all. In these transformations of ours every point may change its place, every line its

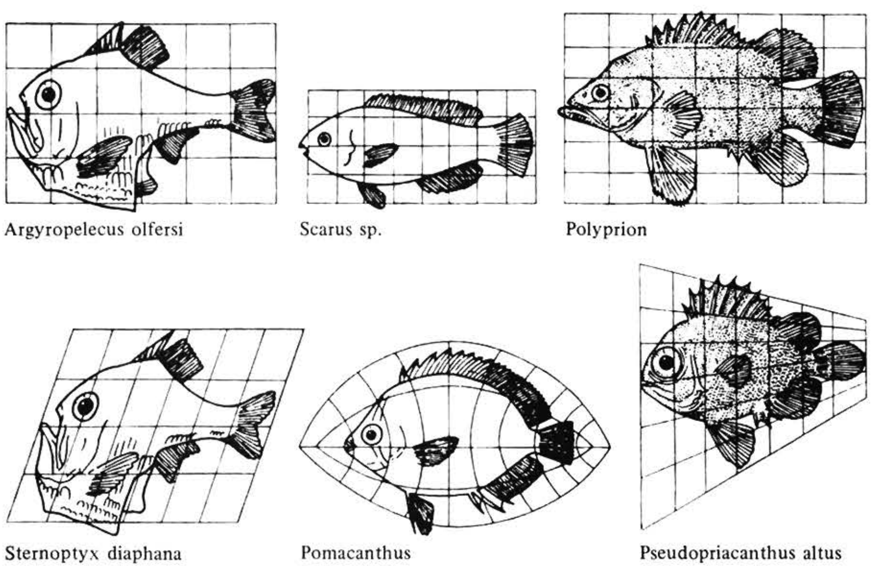
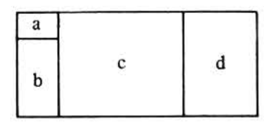


Figure 2.4. Shapes of the bodies of fishes, related together by D'Arcy Thompson's 'method of coordinates' (from Thompson, 1961, figures 146 to 151). Each fish in the top row should be compared with the one immediately below it.

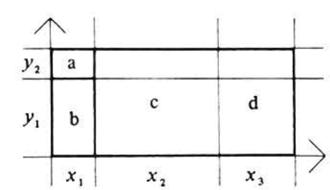
curvature, every area its magnitude; but on the other hand every point and every line continues to exist, and keeps its relative order and position throughout all distortions and transformations". Speaking rather loosely, we might say that D'Arcy Thompson's, and Dürer's, method separates out the essential configuration or *Gestalt* of the shapes of faces, bodies, or other figures, from the particular relative *sizes* which the separate parts of the figure may assume.

Let us see how a similar approach might be taken to describing the shapes of architectural plans (compare Eastman, 1970; March, 1972). We will start with plans organised according to a rectangular geometry, for the sake of simplicity—although the coordinate method is by no means confined to the description of rectangular forms, as we have seen from these biological applications—and we can extend the argument to take in other geometries in due course.

The following figure illustrates in very diagrammatic form a small plan of four rooms:

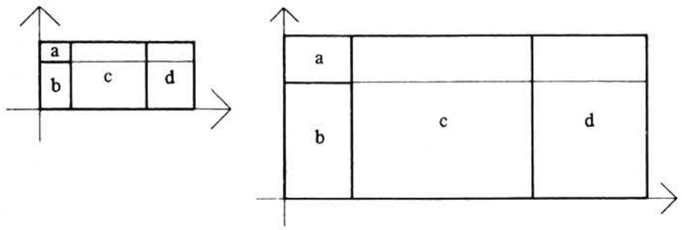


Let us assume at this stage that the thicknesses of walls can be ignored, so that the walls are represented by single lines. Also we ignore any door or window openings. We impose on this plan a coordinate system or grid. There is a sufficient number of grid lines in either direction to mark the positions of all walls, and no more (that is, there are no empty grid lines). Thus we have three grid lines in the 'east-west' direction (as the plan is oriented north to the top on the page) and four lines in the 'north-south' direction. These lines are of course unequally spaced; we can show the dimensions of their spacing along two orthogonal (x and y) axes as in:

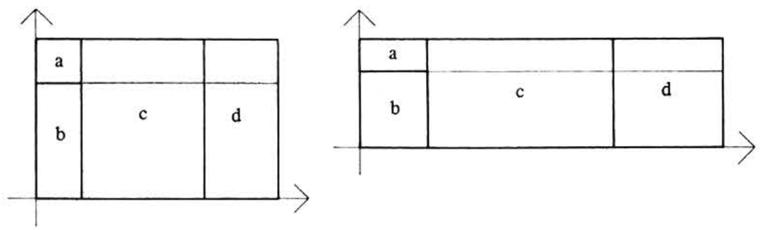


From this point we will refer to an orthogonal grid bounded within a rectangular frame of this kind, as a *grating* (Newman, 1964).

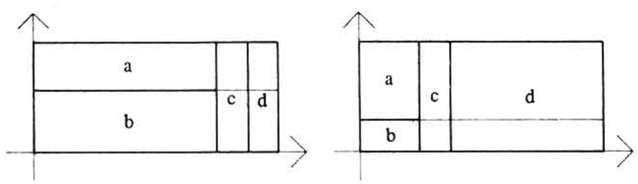
Now imagine the grating, and the plan, subjected to some of D'Arcy Thompson's transformations. We will not consider any shear or curvilinear transformations, but confine ourselves to those transformations which preserve rectangularity. The grating might be simply enlarged or reduced:



it could be stretched in either direction:

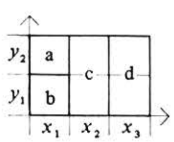


or the relative spacing of the grating lines might be changed—in the same way as Dürer's first series of portrait heads—to any dimensions we choose:



It will be seen that in this way an infinite number of particular plans might be produced, all of which would, however, share the same essential four-room 'shape'. The actual *sizes* of all the rooms might change, but not their overall *disposition* in relation to each other. See how this follows from the fact that the *walls* cannot change their disposition: a wall which is to the 'north' of another, or to the 'east' of another, must always remain so. As a consequence, in the example, room a is adjacent to rooms b and c in the original plan, and will remain so under *all* transformations of the kind we are considering. The same will be true of all other such 'topological' relations of adjacency between the rooms—since these are dependent in turn on the relative positions of the walls.

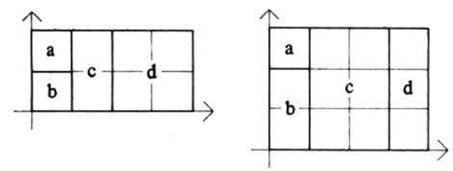
Now take that specific transformation in which the grating intervals are all made equal in both directions:



This representation of the shape is unique—given a unit dimension for the grating—and we can choose for convenience to regard it as the standard or canonical version. We have removed any relative differences in dimension between the grating intervals, and so it may be termed the 'dimensionless representation' of the plan.

It is possible to get back to the original dimensioned plan from the 'dimensionless' version, clearly, by means of the inverse transformation, in which the grid intervals are reassigned their correct sizes. What is more, *any other* differently dimensioned version of the plan may be completely described by means of two types of information: the 'dimensionless representation' to describe the basic shape, together with an appropriate set of dimensions, giving the required spacing for the grating in the x and y directions.

The following figure illustrates some examples of dimensionless representations of the same plan, which are not minimal in the sense described earlier:



That is to say, there exists one or more empty grating lines: these can be removed without losing any part of the shape represented. It is important to avoid this condition in the standard versions of shapes, since otherwise they will not be unique; there are many such nonminimal versions for every shape.

This method of representing rectangular forms has effected a distinction, then, between the description of configuration, and the description of the relative sizes of the parts. It is the same kind of separation, between 'something discrete' and 'something continuous', to which Hermann Weyl (1952) refers in the quotation which heads this chapter. This splitting he says is "a basic issue in all morphology". In the study of crystal forms and the patterns of repetitive ornament with which Weyl is specifically concerned, the reference is to the distinction between the underlying symmetry *lattice* or grid by which the pattern is organised (the 'something discrete') and its metric properties, the particular sizes which the grid units and the angles between the grid lines take (the 'something continuous').

The distinction is a most important one, as Weyl emphasises, because although the metric properties may vary continuously—so giving rise to an infinity of possibilities of scale or size—it can be shown that there are only a strictly limited number of distinct *types* or configurations of symmetry lattice. For regular two-dimensional patterns which fill the plane, the so-called wallpaper groups, this number is 17. And for the equivalent three-dimensional case, the space-filling symmetric lattices which correspond to the structures of crystals, the number is 230. (On symmetry in the arts and sciences generally, see Weyl, 1952; Shubnikov and Koptsik, 1974; Rosen, 1975. Plane-filling or space-filling symmetry of infinite extent is nevertheless perhaps of limited interest for architecture.)

It is in this sense that it is legitimate to speak of there being only a certain restricted range of possibilities in, for example, the design of repetitive wallpaper, mosaic, or fabric patterns. The limitation is on the possible *types* of symmetry; whereas the particular details of the repeated motifs, their size, colour, and so on are capable of indefinite variation.

We shall see in the next chapter how our method for describing architectural plans, and indeed any comparable kinds of rectangular designs, allows us to make some equivalent statements about the range of possibilities for such plans. There are only certain limited numbers of arrangements for the 'dimensionless representations', depending on how, precisely, we distinguish those arrangements. And so we can say that the range of 'possible designs' in this sense is finite, and all possibilities can be listed. None of this alters the fact, of course, that *each one* of these standard or canonical designs can be transformed into an infinite range of particular dimensioned plans, since those dimensions are capable in principle of continuous variation.

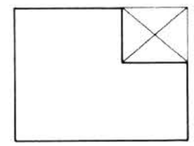
Meanwhile, let us look briefly at some other applications of the method of coordinates, to show how different kinds of representation of buildings are possible with its use. Figure 2.5(a) shows the ground-floor plan of an English vernacular house typical of the Eastern counties, and taken to New England by colonists in the seventeenth century. [It is of the type classified by R W Brunskill (1971) as the 'central fireplaces family'.] As a matter of fact this is the same plan as in the previous examples; but now it is shown in greater detail, with walls drawn at their true thickness and window and door openings and the central fireplaces shown.

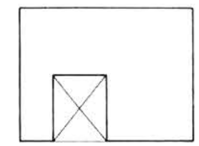
The principle of the method of description is exactly the same, only in this case we require additional lines in the grating to capture the extra detail. The grating lines correspond in effect to all those positions to which a draughtsman would have to move the edges of his T-square and set-square when drawing the plan. Figure 2.5(b) shows the dimensionless version of the plan, together with two lists of numbers (or *vectors*) to indicate the dimensions of the grating intervals. Since walls are now given thickness, it is necessary to have some convention for distinguishing those cells in the grating which represent solid material from those which represent voids (here shown as shaded and unshaded cells, respectively).

It might be objected that as a means of representing architectural plans the rectangular grating is rather limited; not only are we restricted to a rectangular geometric discipline overall, but the perimeter of the building represented must itself be a simple rectangle. However, this is not strictly

so, since it requires only a slight extension of the idea to describe plans with nonrectangular boundaries.

If, for example, one of the component rectangles at the corner of the arrangement is taken to be, not a room, but simply a part of the exterior space surrounding the plan, then in this way L-shaped plans can be represented as in the left-hand figure below and U-shaped plans by means of a similar 'dummy room' on one of the edges as in the right-hand figure:





Ring-shaped or courtyard plans can be treated by the expedient of taking one of the rectangles in the centre of the arrangement to be an open space

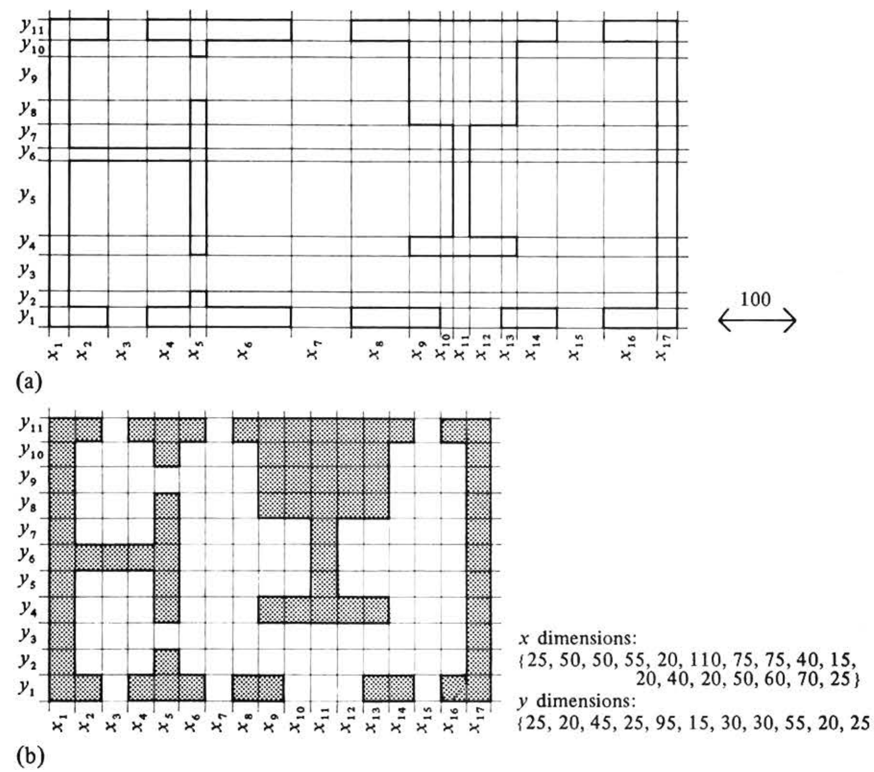
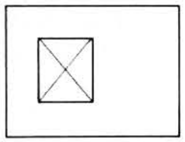
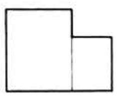


Figure 2.5. (a) Ground-floor plan of an English seventeenth-century house of the 'central fireplaces' type (after Brunskill, 1971, page 103) with superimposed grating to mark the positions of details. (b) The same plan in dimensionless form. Shaded cells represent solids, and unshaded cells voids. The two *dimensioning vectors* give the dimensions of the grating intervals in x and y directions, in arbitrary units.

or light-well instead of a room:

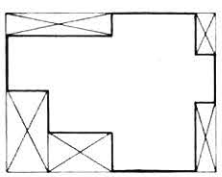


Similarly we are not obliged to regard the rectangles in the interior as separate rooms. They might be zones or areas of any kind, not necessarily separated by walls; for example an L-shaped room could be represented by a pair of adjacent rectangles:



In principle, very complex room or plan shapes might be depicted, given that they could be broken down into rectangular pieces.

In the case of the plan we must imagine a bounding rectangle set around the outside, touching it on all four sides, with the interior component rectangles distinguished as to whether they represent exterior open spaces, or rooms (or parts of rooms) in the interior:



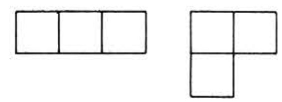
The penalty, for a plan with a highly indented shape, is that large numbers of rectangles may be required in the representation, which have no real physical meaning in the plan itself. Also there may be many different ways in which the same plan can be represented.

We might notice, by the way, that many of the computer methods devised in recent years for assembling floor plans automatically have made use of square grid forms of representation which might seem superficially to be similar [figure 2.6(a) shows the 'central fireplaces' house approximated by such a grid representation]. Observe the key difference, however: those grids have some *fixed* modular dimension for the grid interval. Thus dimensional and shape properties are *not* separated in the way in which the dimensioned grating allows.

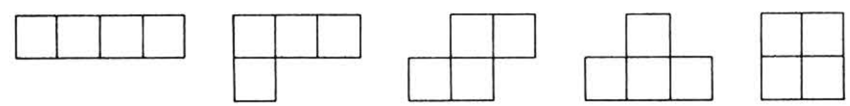
If a large grid dimension is chosen for a modular grid of this kind, and all walls must lie on the grid, then true dimensions are only coarsely approximated [figure 2.6(b)]. On the other hand if a small interval is used so as to give closer dimensional approximation, there will tend to be

large numbers of grid lines and grid cells unoccupied, and the representation becomes highly uneconomical [figure 2.6(c)]. (This can be an important consideration from the computing point of view.) The dimensioned grating by contrast allows precise dimensional accuracy in the representation, while using the minimum of lines and cells in the descriptive grid. These points are discussed more fully in relation to the question of computer representation in Eastman (1970) and Mitchell (1977, chapter 6).

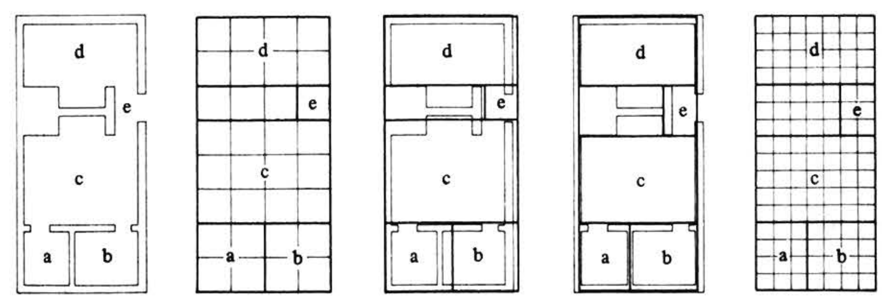
Several authors, of whom the first was Frew (1973), have suggested that a different class of rectangular designs might be used to represent architectural plans (see also Frew et al, 1972; Mitchell and Dillon, 1972; Matela, 1974). These are the so-called 'animals' or 'polyominoes', the latter being the name given by Golomb (1966) who was largely responsible for creating popular interest in their study. A polyomino is analogous to a domino, but may possess more than two faces. It consists of a number of square cells joined together along their edges. There are only two types of 'tromino' with three cells, a straight one and a bent one:



With four cells there are several possibilities:



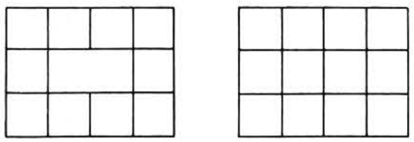
and so on. We will come back to polyominoes in later chapters. As representations of floor plans they look promising, since they can take



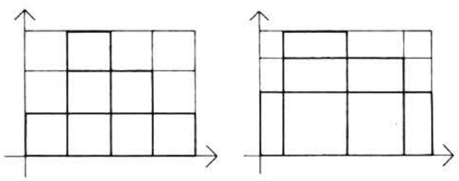
(a) (b) (c)

Figure 2.6. Square grid representation of plans, with *fixed* modular grid dimension, illustrated in relation to the 'central fireplaces' type house (a). With a coarse grid of large modular dimension (b) the plan is only crudely approximated. With a fine grid of small modular dimension the plan is more accurately represented (c), but many grid lines are now unoccupied and so redundant.

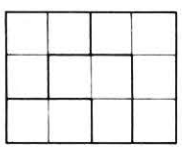
many shapes. Some can have holes in the middle, whereas others are rectangular overall:



Whether rectangular or not, any polyomino can be set on a grating within a bounding rectangle, and can then be dimensioned in any desired way, exactly as before:



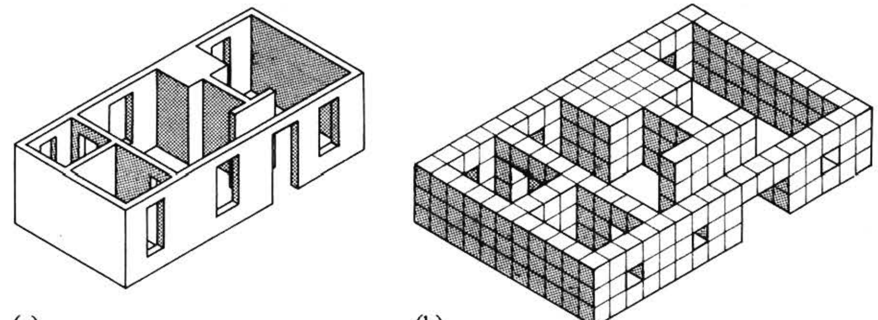
Notice, however, one respect in which they are severely limited as a form of representation of architectural arrangement. If they are taken to represent plans, and each separate cell is a room, then these can only be rooms which coincide exactly along the whole of one wall—since this is how the polyomino is defined. There is no possibility of representing the 'overlapping' arrangement of rooms which is characteristic of almost all real plans. On the other hand we might represent each *room* by a polyomino, in which case this objection is overcome, so that the plan as a whole is a packing together of several polyominoes into a mosaic:



These are possibilities to which we will return.

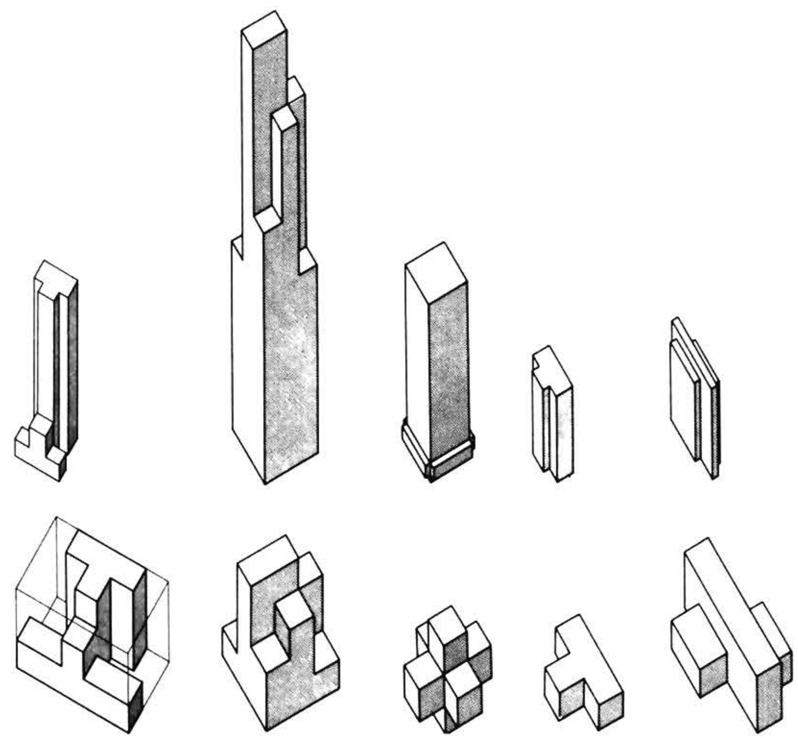
There is no great conceptual difficulty in carrying equivalent techniques of representation into the third dimension. The three-dimensional analogue of the polyomino is the 'polycube'. Some studies exist of the properties of these objects and how they may be packed together in space. Meanwhile, the dimensionless grating form of representation now consists of a solid lattice of cubes, and three dimensioning vectors are required, for the x, y, and z axes (see March, 1972; Mitchell, 1977, chapter 6; Krishnamurti, 1979).

Different levels of detail might be incorporated into the representation: perhaps full details of walls, windows, doors, and so on, as illustrated in



(a) (b)

Figure 2.7. Solid form of the ground-floor arrangement of the 'central fireplaces' type house (a); and in dimensionless representation (b) within a lattice of cubes.



(a) (b) (c) (d) (e)

Figure 2.8. Solid forms of high-rise office buildings in dimensional form (top row), and in dimensionless representation as *polycubes* (bottom row). From left to right: (a) Seagram Building, New York (Ludwig Mies van der Rohe); (b) Sears Tower, Chicago (Skidmore, Owings, and Merrill); (c) Place Victoria, Montreal (Luigi Moretti and Pier Luigi Nervi); (d) One Charles Center, Baltimore (Ludwig Mies van der Rohe); (e) Thyssen-Rohrenwerke Office, Düsseldorf (Hentrich and Perschnigg). (Source: Mitchell, 1977, figures 6.26 and 6.27.)

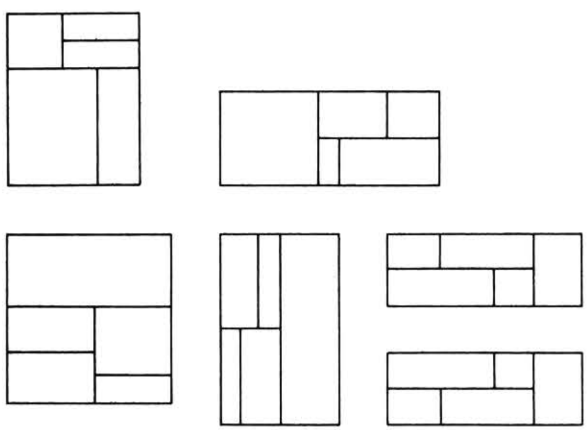
the version of the ground floor of the central fireplaces house type in figure 2.7. Alternatively it would be possible to treat the form of the whole building 'sculpturally', as though it were a solid mass, without any account taken of internal organisation. Some examples of high-rise office buildings treated in this way are illustrated in figure 2.8. The whole building is approximated in each case as a single polycube.

Exercises

2.1 Draw dimensionless representations for some of the plans given in figure 11.5. Treat the walls as having no thickness, ignore all openings in the walls and other minor details, and approximate the rooms as simple rectangles or combinations of rectangles. (You will discover that in some cases this involves some interpretation, and that there is no single definitive way of representing the plan.)

2.2 Take one of the plans given in figure 11.5, and draw a dimensionless representation to *include* wall thicknesses, openings, etc. List the dimensions of the plan in the two dimensioning vectors for the x and y axes. (Use the scale given in the figure; or else make some arbitrary assumptions as to dimensions.)

2.3 Which of the plans illustrated share the same dimensionless representation? (Ignore the particular orientation of each plan on the page.)



2.4 There is an analogous class of design to the polyominoes, called the 'polyiamonds', in which the cells are not squares but equilateral triangles, joined along their edges. Find the three distinct polyiamonds with four cells, and the four distinct polyiamonds with five cells.

2.5 How would you go about extending the method of 'dimensionless representation' to the description of plans organised on a 60° geometry,

that is, composed of equilateral triangles, regular hexagons, and other shapes made from the combination of these elements? The plans of some of the later houses of Frank Lloyd Wright provide good examples.

(This question is not as straightforward as it perhaps appears. You might immediately consider the idea of *three* dimensioning vectors, giving dimensions in the directions of three sets of grating lines at 60° to each other. However, the position of any point can be fixed by any two such coordinates; and indeed the size of an equilateral triangle or of a regular hexagon is fixed by giving the dimension of just one side. The point is that these shapes cannot be squeezed and stretched in the same way as can rectangles, without their internal angles changing. Alternatively, if the angles are kept fixed, such shapes may be transformed in size only by an overall reduction or enlargement. This fact places severe constraints on the possible ways of packing together such shapes, and must be recognised as one of the reasons for plans with triangular and hexagonal geometry being rare in architecture, compared with those of rectangular geometry.)

2.6 As a more open-ended exercise, you might like to look through some books illustrating a variety of architectural plans, and see to what extent these can be represented by the methods indicated in this chapter. You might look for plans which can be approximated as polyominoes, or packings of polyominoes; or for buildings whose whole forms could be represented as polycubes. You could notice the frequency of occurrence of plans with different types of geometry, and try to see what problems are presented by their dimensionless representation. (Circular plans are not so infrequent, but often consist in effect of just one principal room. It is not unusual to find circular, elliptical, or octagonal *rooms* in plans whose geometry is largely rectangular, but which are formed by 'carving' these shapes out of the thickness of the walls of what are basically rectangular units.)

Of the books listed in the references, Durand (1801), Fletcher and Fletcher (1896), and Pevsner (1976) might be suitable for this exercise.

3

The symmetries of rectangular plans

> "The basic concepts of ... symmetry are those of rotations, such as the symmetry of a flower; of inversion, as in the difference between the right hand and the left hand; and the combination of these with each other and with direct movements in space."
>
> J D Bernal (1937)

We have seen in the last chapter how the treatment of shape and dimension can be separated in the description of rectangular designs such as plans. Let us now set all consideration of dimensional properties on one side, and concentrate for the moment wholly on the question of shape. Let us further narrow our focus, and examine those designs which, like the first diagrammatic plan of the 'central fireplaces' house, consist of an arrangement of rectangles (corresponding to rooms) set within a boundary which is itself a simple rectangle. The thicknesses of walls, and all openings in those walls, are ignored.

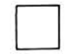
We can imagine that these rectangular arrangements will thus represent, in a somewhat approximate way, a certain particular class of architectural plans. Later on we will consider more critically what degree of approximation this involves: how closely, that is, such representations correspond to the typical plans of real buildings. And we will also examine other classes of arrangement, both rectangular like the polyominoes, and nonrectangular.

Meanwhile, for the sake of simplicity in the discussion which follows, the simple component rectangles of the arrangement will be referred to as 'rooms', and the arrangement of the whole as the 'plan'. But it should be remembered that, as we have already seen, this is not necessarily their sole interpretation in architectural terms, and the arrangements would serve equally to depict L-shaped, U-shaped, and many other shapes of rooms or plans.

Such designs or plans can be thought of either as being made up of sets of rectangular pieces, like tiles or jigsaw pieces, set together by a process of *addition* to make larger rectangular mosaics. Or else they may be imagined as being produced through a process of *division*, by cutting a large rectangle into smaller rectangular pieces. For the latter reason they have been termed 'rectangular dissections', and will be referred to as such here; although, as we shall see, there have been different approaches taken to the problem of enumerating such designs, which have adopted, variously, both dissection, and addition or 'tiling' techniques.

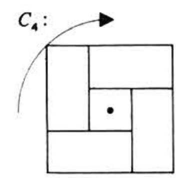
We consider then the dimensionless representations of these 'rectangular dissections', depicted on gratings in which the spacing is equal in both directions. Taking the very simplest case first, a single rectangle, that is, a plan composed of one rectangular room only, the dimensionless

representation is a single square:



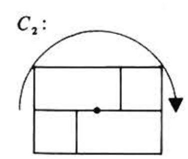
We can now start the process of dissection proper, and divide this original rectangle into two rectangular parts. Before we do so, however, it will be useful to take a slight detour via the subject of symmetry; since symmetry properties are of some importance in distinguishing between, and counting, the possibilities of arrangement with larger numbers of rectangular parts.

We need concern ourselves with two types of planar, that is, two-dimensional, symmetry: *rotational* symmetry and *reflected* symmetry. A figure is said to possess rotational symmetry if, when it is turned in the plane by a certain angle about some point (the *centre of rotation*), it remains unchanged. This is true for *any* planar figure for a turn through 360°, which merely returns it to its original position. If, however, we take a figure such as the square 'pinwheel' made up of five rectangles:



we can appreciate that a rotation of 90° (or a quarter-turn) will leave the arrangement unaltered. The same is true of a rotation through 180° (a half-turn), through 270° (a three-quarter turn), through 360°, or indeed through any number of right angles or multiples of 90°. Such a figure is said to possess *cyclic* symmetry C_4, where the subscript 4 indicates that the figure is symmetrical with respect to (that is, is left unchanged by) a rotation of (360/4) or 90°.

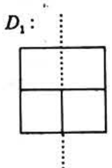
Another example of this type of cyclic symmetry is afforded by the arrangement of four rectangles:



In this case the figure is unchanged by a rotation through 180°, or any multiple of 180°. It thus has the symmetry C_2.

The second type of symmetry we have to consider is symmetry under reflection. We must imagine a mirror set perpendicular to the plane of the paper. The mirror intersects the plane in a *reflection line*. If the figure remains unchanged by reflection in this mirror, then it is said to possess reflection symmetry. Consider the following arrangement:

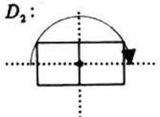
D_1:



It shows mirror symmetry about the central reflection line (shown dotted) as indicated. This is a very rudimentary example of what is perhaps the most commonly found type of symmetry in organic nature, at least in the forms of animals: mirror symmetry about a single central axis, otherwise *bilateral* symmetry, as in the division of the human body into matching left and right halves.

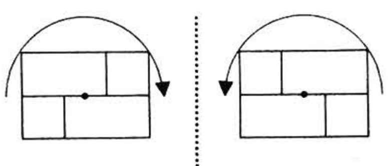
Now it is possible for a figure to show both rotational and reflected symmetry. The arrangement made up from two component rectangles:

D_2:



is an example with rotational symmetry through 180°, as well as mirror symmetry in two perpendicular reflection lines (dotted) as shown. Where the rotated figure has bilateral symmetry about a line passing through the centre of rotation, as here, the rotational symmetry is referred to as *dihedral.* This is a case of D_2 dihedral symmetry, where the subscript 2 indicates the rotational symmetry through 180° as before[3]. The previous example of bilateral symmetry was a case of what is denoted by strict convention as D_1. (It has rotational symmetry only through 360°, that is, by returning the figure to its starting position.)

Notice that, by contrast, our earlier example of symmetry C_2 had no reflected symmetry. There is a distinct 'direction of spin' in its rotational symmetry. We may imagine two otherwise equivalent arrangements, but with opposite directions of spin, clockwise and anticlockwise:

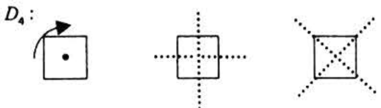


(3) Any rectangle possesses these same symmetry properties: reflected symmetry about two perpendicular axes, and rotational symmetry through a half-turn, together with symmetry under the 'identity transformation' (see below). These four symmetries of the rectangle together form a *group*, in mathematical terms, known as the *Klein four-group* and indicated by the notation K_4.

The one cannot be produced from the other by rotation, only by reflection. Taken together in the positions illustrated, the *pair* of arrangements shows reflected symmetry. They are 'handed' like a pair of gloves or shoes (although as mentioned neither figure has reflected symmetry on its own).

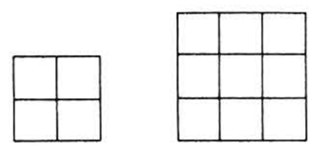
To come back to our dimensionless representation of a one-room plan—the simple square. It might seem pedantic to insist on an enumeration of the symmetry properties of this more or less trivial case. However, by starting at this simplest arrangement we can be clearer about some of the more complicated considerations of symmetry, which will emerge later. The square possesses rotational symmetry of the dihedral type D_4; dihedral, because the square also has reflected symmetry, about the two perpendicular axes:

D_4:



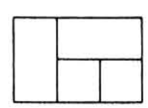
(We may notice that, in addition, it has reflected symmetry about the two diagonal lines passing through opposite corners, though this is of limited interest for our present purposes.)

All these same symmetry properties are displayed by the square arrangement of four squares and also the square arrangement of nine squares in the following figure:



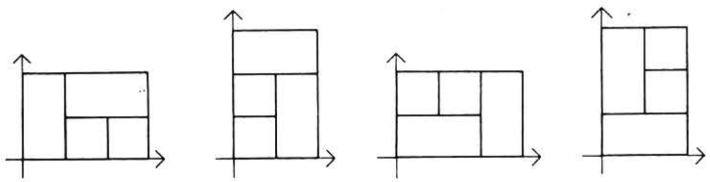
We can say that, of all kinds of dissections of a rectangle into rectangles, these 'square patterns of squares' possess the highest possible degree of symmetry (the number of their symmetries is at a maximum).

At the opposite extreme, there will be arrangements with no symmetries by reflection or rotation at all. One of the simplest of such cases is the dissection composed of four rectangles[4]:

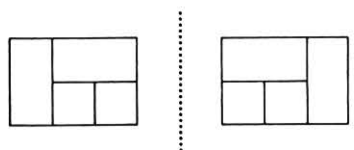


(4) For the sake of completeness in the mathematical treatment of symmetry, particularly in the application of group theory to the subject, a transformation is introduced termed the *identity transformation*, which is in fact no transformation at all. The figure is not altered in any way by this so-called 'transformation'. Thus this figure can, in a rather odd sense, be described as being 'symmetrical under the identity transformation'.

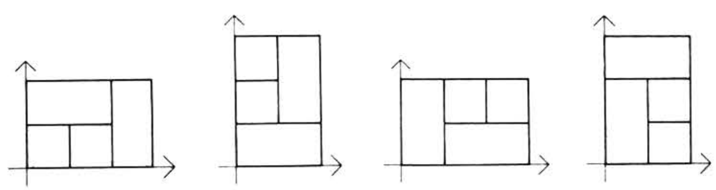
Suppose that the grating on which this figure is represented in dimensionless form is set on a pair of x and y axes, like the examples in chapter 2. How many ways—that is, in how many orientations—might it be set in relation to those axes? We can turn it through 90°, 180°, or 270°, and in each case, since the figure has no symmetry, produce an arrangement which, considered in relation to the axes, is differently oriented:



Or we can reflect the arrangement in a mirror line parallel to one or the other of the sides—it does not matter which:



and again rotate the resulting mirror image through three successive quarter-turns:



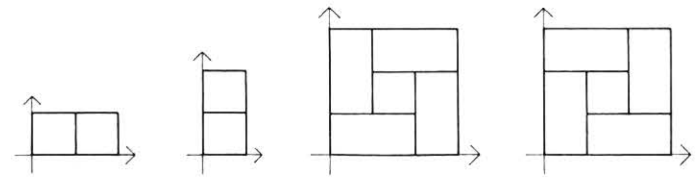
These procedures will generate eight distinctly oriented or handed versions of the original arrangement. Mathematically these are known as *isometries*: transformations which preserve the shape and size of a figure, specifically rotations and reflections[5]. It is possible to carry out the reflections and rotations in different sequences—you might like to try for yourself, if you find it difficult to imagine the operations involved—but there will only ever be eight different end results. These handed and/or rotated versions of a figure are termed *isomorphs*.

Now imagine going through the same process with a square, or with a 'square pattern of squares'. Because of the symmetries of the square, all

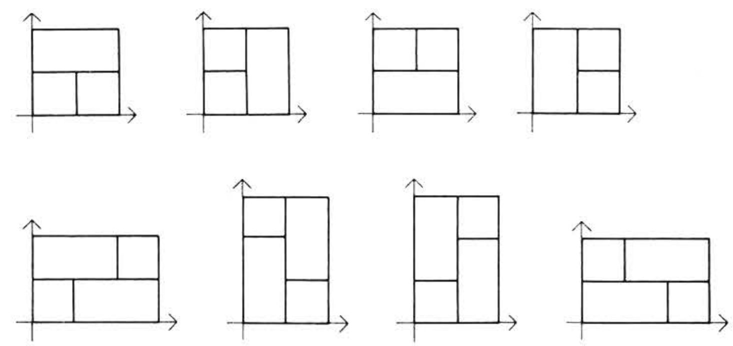
[5] The third type of isometry transformation, that of *translation*, which is essential to the subject of symmetry generally, is not immediately relevant here. A translation is a direct movement of a figure in the plane, in which orientation and handedness are left unchanged.

rotations and reflections of the kind described will leave it unchanged. There is only in effect one possible orientation on the coordinate system.

In between these extremes, of eight possible versions, and only one possible version, there are two intermediate situations which may arise. In the case of dissections with symmetry D_2 or C_4 there are, under reflection and/or rotation, only *two* possible versions in either case:

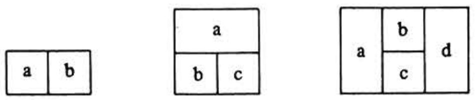


For dissections with bilateral symmetry D_1, or with symmetry C_2 there are only *four* distinct isomorphs:

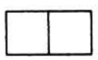


In summary, then, it is possible for any rectangular dissection to be set on the coordinate system in either one, two, four, or eight distinctly oriented or handed positions, depending on its particular symmetries. It is a matter of convenience, or arbitrary convention, as to whether we regard these various isomorphs as the 'same' or as 'different' arrangements. If we are counting or cataloguing all possibilities, then the numbers of arrangements to be listed will be much reduced by treating the various isomorphs as effectively the same dissection. This is the convention we will adopt here. It seems a natural approach in any case, to consider only the basic underlying configuration; from which it will always be possible to generate the various oriented or handed versions by an appropriate series of isometry transformations.

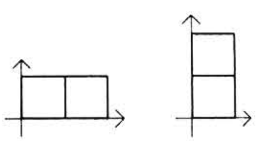
In all the discussion so far it has been assumed that rectangles or rooms in the arrangement were distinguishable only by their relative positions; and the various symmetries of the dissections examined were dependent on this assumption. If, however, distinctions *are* made, by assigning letters let us say to rectangles, or names to the rooms, then the symmetries of many arrangements will be lost. Thus if the two component rectangles in the dissection with symmetry D_2 in the left-hand diagram below, are labelled a and b, the rotational symmetry through 180° is lost, as is one of the symmetries by reflection, so that only symmetry D_1 remains. The labelled dissections in the other two cases lose all their symmetries as a consequence of the labelling:



We are now ready to go back to the process of dissection, and to consider the case where the original single rectangle is cut into two rectangular parts. It will be clear that there is, in the terms of the present discussion, only one way in which this can be done; its dimensionless representation being:

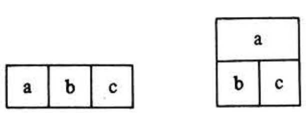


We have just seen how this particular arrangement has dihedral symmetry D_2 and can be set on the coordinate system in two distinct orientations, that is,



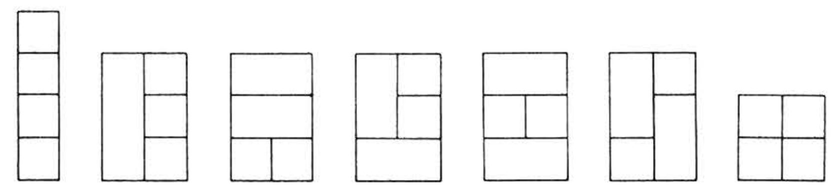
These correspond in effect to the situations in which the dissection, the cut, is made either 'east–west' or 'north–south' (taking north to be at the top of the page). However, the results of the two operations are only rotations of the same arrangement, and so we treat them as the same basic configuration.

When the original rectangle is divided into three rectangular parts, then matters become slightly more interesting. A little experimenting will show us that there are now two quite distinct ways in which this can be done; two different configurations which are not simply isomorphs. There is the arrangement where the rectangle is cut into three parallel slices and also the arrangement when the three parts are so to speak in a 'triangular' configuration:



The first case can be oriented in two ways (if unlabelled, it has symmetry D_2). Notice that the room labelled b in the first arrangement is adjacent both to room a and to room c, but that a and c are separate. The second arrangement can be oriented in four ways (the arrangement—if the labelling is ignored—is bilaterally symmetric in itself, but is only symmetrical under rotation through a full turn of 360°, so that it is strictly a case of D_1 symmetry). In this case any one of the rooms a, b, and c is adjacent to both of the others. Notice also in this dissection that there is an internal T-shaped junction between the walls of the rooms, of a kind which does not occur in the first arrangement.

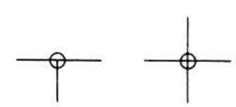
The process of dissection can be carried on, to make divisions of the original rectangle into four parts. At this point we begin to need some systematic procedure for determining all possibilities, other than simple trial and error. We will look at several such procedures in the next chapter. For the moment suffice it to say that there turn out to be seven distinct alternatives, all of which are shown below:



(a) (b) (c) (d) (e) (f) (g)

Several of these we have met before: arrangement (d) is without reflection or rotation symmetries, arrangement (g) has all the symmetries of the square, whereas arrangement (f) is our earlier example of C_2 symmetry. You might care to identify the symmetries of the remaining four dissections, as an exercise.

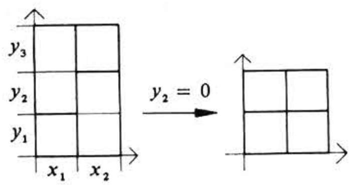
A new feature which emerges, in these dissections into four parts, is exemplified in the single case of arrangement (g). Here the four rooms meet at a single point, giving rise to a cross-shaped or four-way junction between walls. This type, and the three-way or T-junction:



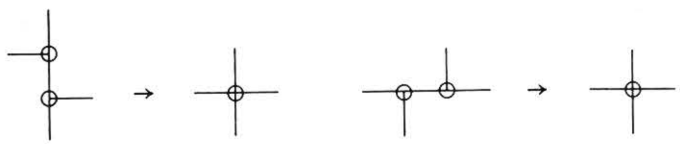
are the only types of junction which can occur in the interior of any dissection (see Biggs, 1969; Combes, 1976).

The four-way type of junction will, however, present us with a problem at a later stage. At this point we can just take note of the special relationship which exists between dissections (f) and (g).

Imagine arrangement (f) set on the coordinate system:



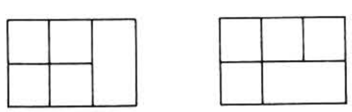
and suppose that the dimensions of the arrangement are described by the dimensioning vectors (x_1, x_2) and (y_1, y_2, y_3). Now suppose that the dimension y_2 is made equal to zero. What will happen? The arrangement will be transformed into the dissection (g); that is, the two three-way junctions will 'coalesce' into a single four-way junction:



If we performed a similar operation on most other dissections—setting one of the dimensions in a dimensioning vector to zero—the effect, as is easily imagined, would be the disappearance of one or more rectangles from the arrangement. In this case, however, no component rectangle is lost. There are four to start with and four remain.

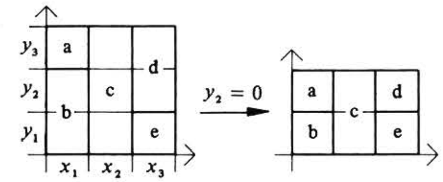
For reasons associated with this fact it has proved convenient in some circumstances when counting or making a catalogue, to regard arrangements related in the way which (f) and (g) are, as one and the same dissection. Arrangement (g) is thought of as a particular dimensioned instance of (f), and, in general, four-way junctions are treated as 'degenerate' pairs of three-way junctions, where the dimension by which they are separated has shrunk to zero. On this view we would therefore count the total number of distinct dissections into four rectangular components as six, not seven. Dissections in which all junctions are three-way have been termed *trivalent*.

With five component rectangles, the number of distinct arrangements *including* those with four-way junctions is twenty-three. There are two with four-way junctions:



The first example of C_4 symmetry, the pinwheel or swastika, appears at this stage.

One particular dissection into five rectangles presents us with a further potential problem in dimensioning. Consider the following two dimensionless representations:



In the case on the right the dissection occupies a grating of size 3 x 2, and there are two separate internal walls which are aligned, the wall between rooms a and b, and the wall between rooms d and e. In the case on the left, which is obviously closely related, the two corresponding walls are not aligned, and the representation occupies a grating of size 3 x 3. If we imagine dimensions assigned to the dissection on the left, and the dimension y_2 set to zero, then the result will be to align the walls in question and thus produce precisely the arrangement on the right. However, the converse is not true: there is no way in which dimensioning of the second arrangement can ever result in the two walls moving out of alignment.

For this reason it seems preferable to adopt the 'nonaligned' case as our basic configuration, and to regard the aligned case as a particular dimensioned instance of the nonaligned representation. Of course, the symmetries of the nonaligned version are different, but all eventualities can be taken care of by appropriate rotations and/or reflections. There are many dissections into six or more rectangles where equivalent problems arise, and which can be taken account of in a similar way by avoiding all such alignments across the representation.

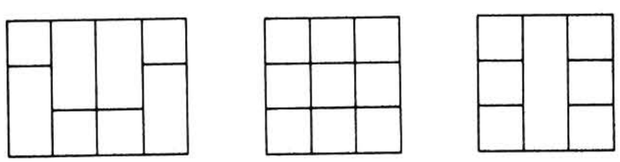
I have been using the word 'wall' in a general and obvious sense up to now. But it will be convenient to introduce some more formal terminology at this stage, to distinguish between different kinds of walls. I will refer to a section of wall which runs between two junctions as a *wall segment*. This can either be an *external wall segment*, forming part of the perimeter of the plan, or an *internal wall segment*, in the interior. In the following arrangement (left) there are six internal wall segments and ten external wall segments [notice that for this purpose the four outer corners of the plan are counted as ('two-way') junctions]:

I shall reserve the word *wall*, plain and simple, to mean the maximum extent of a straight run of wall—which might be made up of one or more continuous aligned wall segments. In this last figure (right), there are two *internal walls* in this sense running 'east–west', across the page, which consist of one wall segment each; and two more internal walls running 'north–south' consisting of two wall segments each. Clearly any plan whose overall shape is rectangular like this one must have just four *external walls*.

Exercises

3.1 Construct polyominoes with the symmetries D_1, D_2, D_4, C_2, and C_4, respectively. Construct a polyomino with no symmetries (that is, 'symmetrical' only under the 'identity transformation'). What are the *smallest* polyominoes (that is, with the least number of cells) displaying each of these various symmetries? (Notice that many polyominoes possess reflected symmetry across *diagonal* axes.)

3.2 The illustration shows rectangular dissections with four-way junctions, and/or alignments of internal walls of the kind discussed in relation to the first figure on page 29. Draw the 'nonaligned' versions of these dissections (this means that there may be only one section of continuous straight wall on each grating line), and replace four-way junctions by pairs of three-way junctions, as discussed in relation to the second figure on page 28.



4

Generating and counting rectangular arrangements: dissection and additive methods

"... we want to cover the field by a systematic research into possibilities. The possibilities of walls and vaults, and of the relations between the walls and the cell, and between one cell and another, want investigating, as Lord Kelvin investigated the geometry of crystalline structures and the 'packing of cells'."

W R Lethaby (1922)

The numbers of distinct possible dissections of a rectangle into successive numbers of rectangular parts are as follows:

component rectangles:	1	2	3	4	5	6
numbers of dissections:	1	1	2	7	23	119

By 'distinct' is meant that the canonical versions of the dissections are distinct. Isomorphs by rotation and/or reflection are not counted. Dissections with four-way junctions are included. Plainly, the number of dissections is growing at an accelerating rate, as is typical of many combinatorial systems of this kind. Any attempt to find and list all possibilities exhaustively must involve devising some organised logical procedure, or *algorithm*, so as to ensure that none are missed and none are counted twice. Once such a procedure is formulated and has been shown to work, it can be implemented as a computer program, and the machine used to do the work of counting, where the numbers are large.

Several such algorithms have been written for generating rectangular dissections; and since their results are known and the dissections can be tabulated, the detailed technical working of these various methods might perhaps be thought to have limited interest for the general reader, or for the reader who is concerned principally with applications to architecture and design. Moreover the invention of a more efficient, that is, faster and more economical method, will for all practical purposes render previous approaches obsolete.

Nevertheless I believe it is worthwhile looking briefly at the broad *principles* of the various methods which have been used, even those which are now effectively superseded; since this will draw attention to various *properties* of dissections, various means whereby they can be *represented*, and various techniques with which they can be *manipulated* and *transformed* one into another. For those readers more deeply interested in the technicalities and in the computing aspects, references are given to the specialised literature.

The first attempt historically to devise an algorithm for generating rectangular dissections was I believe made by myself (Steadman, 1973), and was conceived very much as a 'cutting' or, precisely, a dissection method. In the original single rectangle, it is possible to make a cut in either the 'east–west' or the 'north–south' sense, so as to divide it into two rectangular parts (figure 4.1). The result of both operations, as we have seen, is to give the same basic dissection, but rotated through 90°. Each can be converted to its appropriate dimensionless representation.

The next step is to take the dissection with two rectangles, and further subdivide *one* of its component rectangles, so as to give a dissection into three rectangles. Supposing we start with the two-rectangle dissection in a given orientation, as in figure 4.2, then we can in principle make the one cut in four different ways. We can cut either of the two rectangles; and in each case we can make the cut either 'east-west' or 'north-south'. This exhausts all possibilities. The results are shown at the bottom of the figure. Again they are converted to their dimensionless representation. Of the two distinct possible dissections one is produced twice in the same orientation, and the other in two different orientations. (By applying the process to the two-rectangle dissection in its alternative orientation, we would produce the three-rectangle arrangements in all their other possible orientations.)

So the procedure carries on, taking each dissection with $n-1$ rectangles in turn, taking each of those $n-1$ component rectangles in turn, and cutting

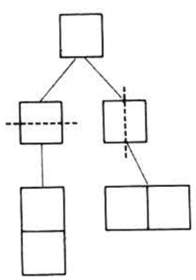


Figure 4.1. Generation of rectangular dissections with a 'cutting' method due to Steadman (1973). The original single rectangle (top) is cut either in the 'east-west' or in the 'north-south' sense (centre), to produce the dissection of two rectangles in its two possible orientations (below).

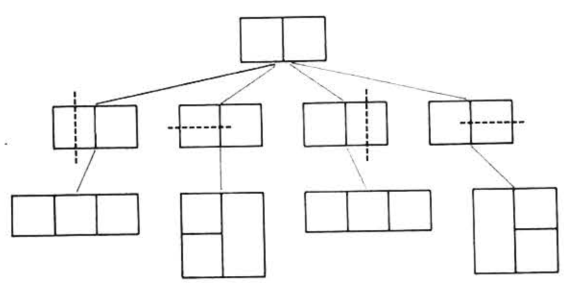


Figure 4.2. One cut is made to the dissection of two rectangles (top). It can be made to either of the rectangles, and in an 'east-west' or a 'north-south' sense in each case (centre), to produce the two dissections of three rectangles (below).

it into two parts by either an 'east-west' or a 'north-south' cut, so as to yield a dissection into n rectangles.

With larger numbers of rectangles in the arrangement than three, there may possibly be several ways of making *each* of these cuts. Take the example of the four-rectangle arrangement illustrated in figure 4.3. We make a 'north-south' cut, as it is oriented, to the rectangle labelled a. There are two possible positions for this cut, resulting in two distinct dissections of five components (which are not just rotations or reflections

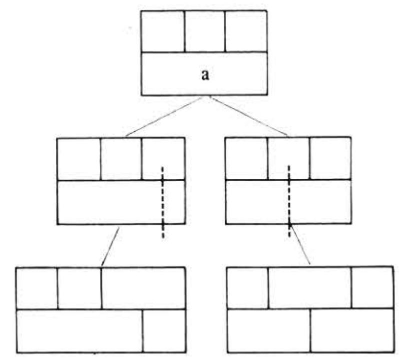


Figure 4.3. The rectangle labelled a in the arrangement of four rectangles (top) can be cut in a 'north-south' sense in two possible positions (centre), resulting in two non-isomorphic dissections of four rectangles (below).

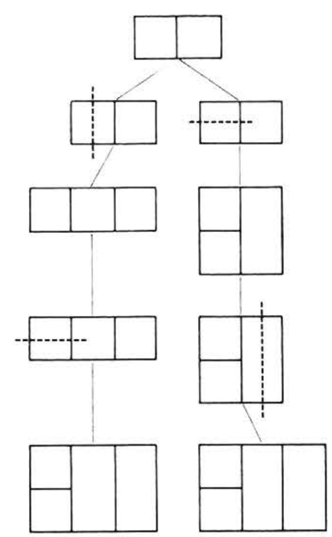


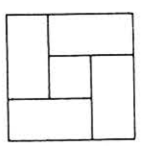
Figure 4.4. Two distinct possible sequences of cuts by which the dissection of two rectangles (top) is converted to the same dissection of four rectangles in the same orientation (below).

of each other). The method must therefore ensure that all such possible cuts are made in turn.

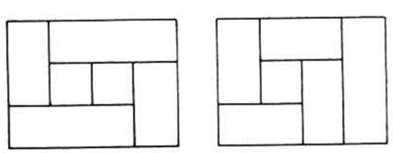
This algorithm is somewhat wasteful, in that it may generate one dissection several times in either the same or differently handed or rotated versions. It is possible that different sequences of cuts can produce the same dissection in the same orientation, several times. We have already seen this in figure 4.2. Another example, at the stage where three rectangles are converted to four, is shown in figure 4.4. After each operation it is thus necessary to compare and reject duplicates, including isomorphs of the same dissection, so as to minimise the work involved in succeeding steps.

The method has never been implemented as a computer program; however, I used it to generate the dissections with six component rectangles, by hand, and succeeded in coming reasonably close to what is now believed to be the complete list of 119 dissections.

Unfortunately it is easy to show that this procedure cannot be exhaustive. The 'pinwheel' dissection of five rectangles with symmetry C_4:



cannot be produced from any arrangement of four rectangles by means of a single cut[6]. This is the first point in the generating process at which such a situation arises. However, there are many comparable arrangements of n rectangles, where $n = 6$ or greater, which cannot be made from arrangements of $n-1$ rectangles by the simple subdivision of one rectangle. Furthermore if the five-rectangle pinwheel and related arrangements are not introduced into the generating process, then certain of their 'descendants', such as those illustrated below:



[6] Compare March and Steadman (1971, page 160). Curiously, and quite incidentally, this C_4 'swastika' is a figure which has cropped up more than once in architectural history, as a way of arranging joists to support a square floor whose span is greater than the length of the available timbers. Serlio first mentioned the problem, and the idea was taken up by J Wallis (1670) in his *Mechanica, sive De Motu*, who illustrates exactly the arrangement of this figure, as well as many other geometric schemes for the structural design of floors. Both Serlio and Wallis are quoted by Wren (1750) in his *Parentalia* in connection with a description of his flat roof design for the Sheldonian Theatre, Oxford.

will not be produced either. (I recognised this problem while working the method by hand, and added in the missing dissections.)

For the purposes of a computer method for generating dissections of a higher order, it is obviously necessary to overcome this drawback. A second algorithm, devised by myself and Mitchell (Mitchell et al, 1976) and implemented as a computer program by Sauda (1975), included operations designed to generate the C_4 swastika and related arrangements. There are three types of operation involved in this method, which works directly on the dimensionless representation:

(1) The first of these operations is very similar to that employed in the hand-worked method as just described. In this case, however, the operation treats only those component rectangles which are made up, in any given dimensionless representation, from more than one cell in the grating. In the case of rectangles made up from two cells only, the division can be made in one way only, converting each cell into a separate rectangle [figure 4.5(a)]. Where the original rectangle is made from three cells in the grating, then the division may be made in two ways [figure 4.5(b)]. Where the original rectangle is made from four cells, there are five possibilities [figure 4.5(c)]. And so on. This type of subdivision operation is applied to all the relevant component rectangles in the given dissection of $n-1$ rectangles in all possible ways in turn, so as to add a new nth rectangle in each case.

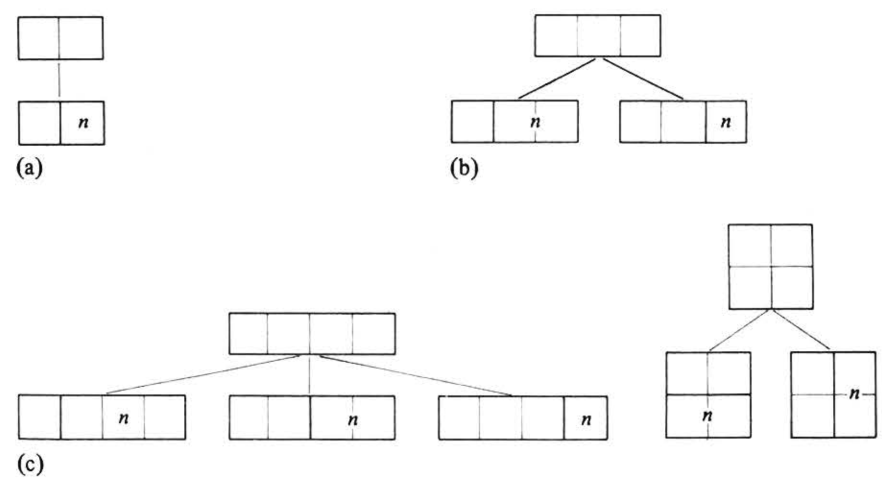


Figure 4.5. Generation of rectangular dissections with operation (1) of a computer algorithm due to Mitchell et al (1976). A rectangle made up from more than one cell in the grating of a dimensionless representation, is divided into two separate rectangles. The figure shows possible subdivisions (a) where the original rectangle consists of two cells only, (b) where it consists of three cells, and (c) where it consists of four cells. Label n indicates the new nth rectangle added in each case to the original dissection of $n-1$ rectangles.

(2) In the second type of operation a new nth rectangle is added along the whole boundary edge of an existing dissection with $n-1$ rectangles (figure 4.6). Clearly there are four potentially distinct ways in which this might be done, depending on the symmetries of the dissection in question, corresponding to its four outside edges. If the size of the original dissection, measured as the number of columns x rows in the grating, is $l \times m$, then this operation will produce a dissection either of size $(l+1) \times m$ or size $l \times (m+1)$.

(3) The third sort of operation is slightly more complex than the previous two, and is intended to take care of the pinwheel type dissections (amongst others). It may be described thus. Take one of the outside boundary edges of a given dissection with $n-1$ rectangles. The operation is applied only where two or more rectangles border that edge. We work along from one end of the edge in question to the other. Let us suppose that this is left to right in the example shown in figure 4.7(a). Take the first rectangle a which borders the edge, and extend this along its length by an increment of one unit in the grating. Fill the remainder of the new row with a new nth rectangle. This operation produces a new dissection with n rectangles as on the left-hand side of figure 4.7(a).

Repeat the operation, by extending both the first rectangle a and the second rectangle b occurring along the edge, again by increments of one grating unit along their lengths. Fill the remainder of the new row with a new nth rectangle (as shown on the right-hand side of figure 4.7(a). Where there are r rectangles bordering the edge of the dissection which is being operated on, then the process is repeated, working 'left to right' along the edge in this fashion, a total of $r-1$ times; that is, up to and including the penultimate rectangle. Since there are three rectangles bordering the edge in the illustration given, the number of possible operations is two, as shown.

A similar process is now carried out along the same edge, but working back along in the opposite direction. This gives, from right to left in the

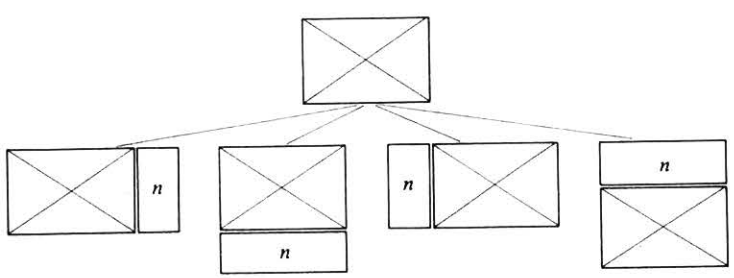


Figure 4.6. Operation (2) of the Mitchell et al (1976) algorithm. A new nth rectangle is added along the whole boundary edge of a dissection of $n-1$ rectangles. This operation can be carried out in four potentially distinct ways, corresponding to the four sides of the original dissection.

example, a further two possibilities [figure 4.7(b)]. The same procedures are repeated in turn (where applicable) to all four boundary edges of the dissection. It will be seen that the operation, in general terms, consists of extending certain component rectangles so as to turn the $n-1$ dissection of size $l \times m$ into an L-shape of size $l \times (m+1)$ or $(l+1) \times m$; and filling the corner of this L with an nth rectangle. Figure 4.8 shows one way in which the pinwheel dissection for $n = 5$ may be produced with this operation from a dissection with $n = 4$.

All three of the operations described are applied separately, in turn, to all dissections with $n-1$ rectangles. The method as a whole, it is clear, comprises a mixture of 'dissection' operations with what are in effect 'additive' procedures. The results will again include large numbers of duplicates; that is, the same basic configuration may be generated several times, as well as in various possible isomorphs.

The method employed in the computer program for removing these duplicates is very straightforward. Each new dissection into n rectangles which is generated by some operation is subjected to three quarter-turns relative to the coordinate system. All four oriented arrangements so produced are then subjected to reflection (figure 4.9). In this way there is produced a total of eight versions which, as we saw in the last chapter, might comprise either one, two, four, or eight distinct isomorphs, depending

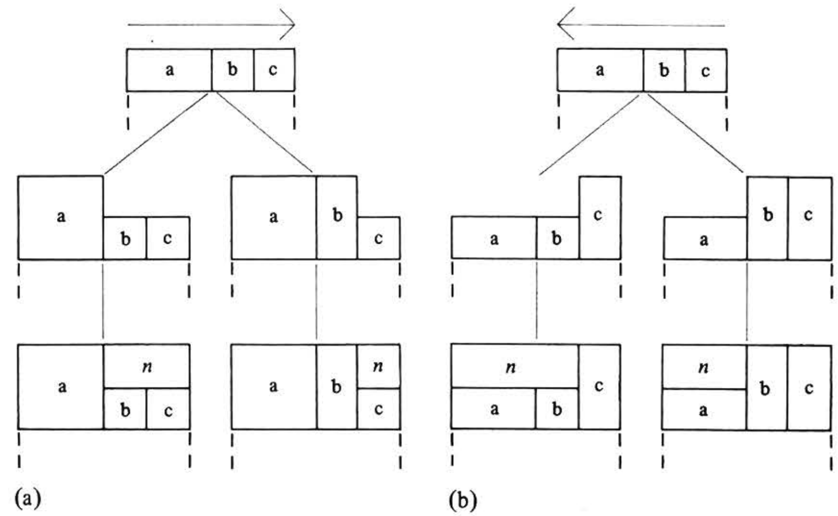


Figure 4.7. Operation (3) of the Mitchell et al (1976) algorithm. (a) The operation proceeds along the edge of a dissection of $n-1$ rectangles, in this example from left to right (top). Rectangles bordering the edge are extended by one unit in the grating—in this case rectangle a, or rectangles a and b together (centre)—and the space remaining within the resulting L-shape is filled by an nth rectangle (below). (b) As in (a), but with the operation now proceeding from right to left along the edge of the dissection.

on the symmetries of the basic configuration in question. By a matching of the eight against each other, any duplicate versions resulting from such symmetries can be detected and removed. The remainder are matched against all other dissections with n rectangles so far produced. If one of the new arrangements duplicates some dissection already found, then they are all rejected; if not, one version is chosen and retained. In this way the final list contains only one version for each dissection.

The computer program embodying these procedures was used to generate dissections for $n = 7$ and $n = 8$. The numbers of dissections produced differ slightly from those found by other authors however, since the

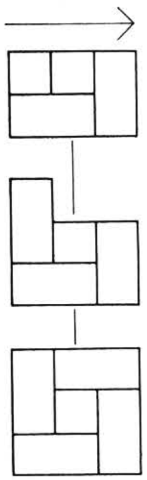


Figure 4.8. One way in which the C_4 'swastika' dissection for $n = 5$ may be produced from a dissection for $n = 4$, with operation 3 of the Mitchell et al (1976) algorithm.

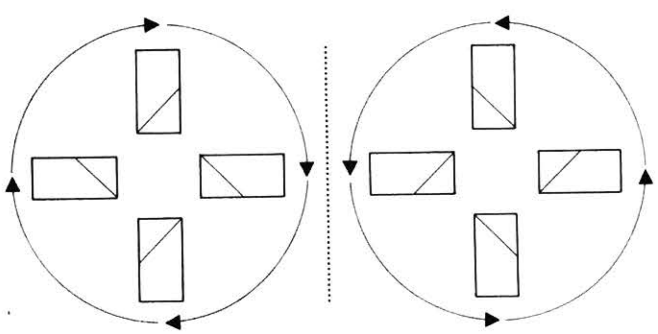
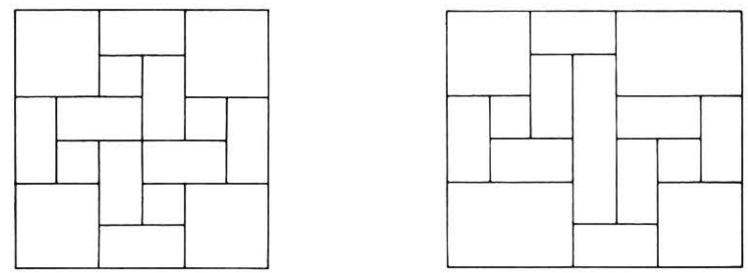


Figure 4.9. Rejection of isomorphs in the Mitchell, Steadman, and Liggett algorithm. Each dissection generated is subjected to three quarter-turns relative to the coordinate system; and all four oriented arrangements so produced are subject to reflection. (The diagonal line marks the dissection here in such a way as to illustrate that potentially it may have eight isomorphs.) All these eight versions are then matched against each other, and any duplicates removed.

program did not distinguish arrangements with wall alignments from their equivalent 'nonaligned' versions (compare the first figure on page 29), and consequently gave totals which are lower than where these are separately counted. (This question is taken up again in chapter 5, where table 5.2 gives counts for the inclusion of and also for the exclusion of arrangements with wall alignments. For nonaligned dissections, and including those with four-way junctions, the numbers are 735 for $n = 7$ and 5527 for $n = 8$.) The program was also relatively slow by computing standards; in particular the process of sorting out duplicates is inefficient, since it involves comparing up to eight isomorphs of each new dissection against all of up to several thousand other dissections (in the case of $n = 8$) in turn.

It has furthermore been shown by Earl (1977) that, quite apart from the problems with alignment, this algorithm is not completely exhaustive



(a) (b)

Figure 4.10. Examples of dissections with sixteen rectangles (a) and fifteen rectangles (b) of the type which Earl (1977) shows cannot be generated with the Mitchell et al (1976) algorithm.

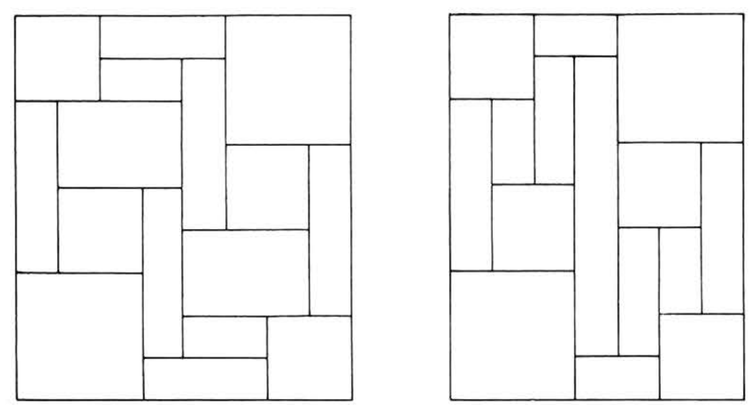


Figure 4.11. The two dissections of figure 4.10 in their nonaligned versions.

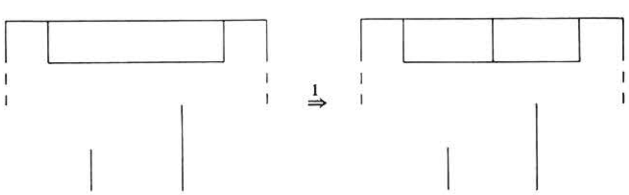
for larger values of n. This he demonstrates with the 'particularly elegant' counterexample of figure 4.10(a), which contains sixteen rectangular components. This dissection would not be produced by any of the three operations described.

Notice that no two adjacent rectangles in the arrangement can be coalesced into a single rectangle by the removal of one wall; so, conversely, no pair of rectangles could have been produced by the dissection of a single existing rectangle, as with operation (1). The arrangement along any of the four outside edges of the dissection is not such as could have been produced by the addition of a single $l \times 1$ or $1 \times m$ rectangle along the side of an existing dissection, as with operation (2). Nor is the edge arrangement such as could have been produced by operation (3), since this increases the dimension of the grating by one unit only; and the corner rectangles here all have dimensions 2×2. Figure 4.10(b) shows a further example of a dissection impossible to produce with the algorithm. In this case the number of components is fifteen. Earl believes this to be the smallest value of n at which the particular situation arises.

A new algorithm, devised by Earl (1977), avoids the difficulty, and is claimed by him to be completely exhaustive for all n. Earl's method is designed, unlike those so far described, to generate only trivalent nonaligned, or *fundamental* dissections. (Aligned arrangements and those with four-way junctions may be generated by appropriate dimensioning operations on nonaligned dissections as explained in chapter 3.) The counterexamples of figure 4.10 would appear in their nonaligned forms as in figure 4.11.

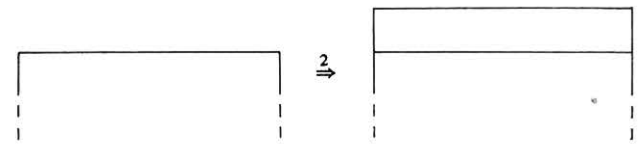
Earl shows that it is possible to dispense with 'dissection' or subdivision operations in the interior of an existing arrangement, of the kind represented by the first operation in the previous algorithm. Instead it is sufficient to carry out four kinds of procedure at the *edge* of any dissection. Imagine that the edge in question is at the 'top' or 'north' of the dissection as it appears on the coordinate system and on the page. Then:

(1) Take a rectangle which lies at the top edge, and divide it into two rectangles with a wall running in the vertical direction, not aligned with any other existing vertical wall in the remainder of the dissection:



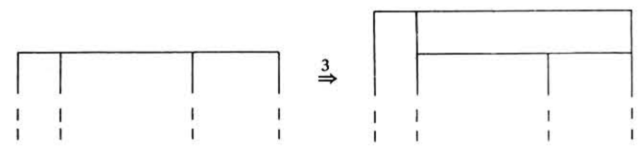
(It follows that this operation involves an increase in the *width* of the grating by one unit.)

(2) Add a new rectangle of depth one unit and filling the whole width of the dissection:



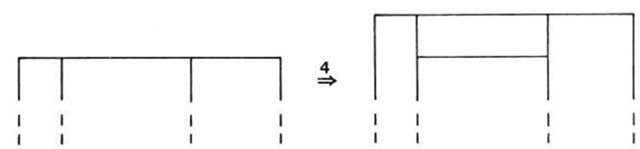
(This corresponds to the second operation in the previous algorithm.)

(3) Extend one or more rectangles at a corner of the dissection by one unit, to form an L-shape, and fill the opposite corner with a new rectangle:



(This corresponds to the third operation in the previous algorithm.)

(4) Extend one or more rectangles at both corners of the dissection by one unit, to form a U-shape, and fill the central gap with a new rectangle:



(It is this operation which is required to generate Earl's counterexamples of figure 4.11.)

All those operations are to be carried out along the edge in question in all possible ways, very much as described for the previous algorithm. It is not, however, necessary as before to apply the rules to all four edges of each dissection in turn. The reason will be clear from figure 4.12.

Suppose we start with the unit rectangle, and work only on its 'north' edge (drawn in thicker line). By applying rules 1 and 2, respectively, we obtain the dissection for $n = 2$ in its two possible orientations. By then applying to these the various rules which are relevant in each case, as indicated on the figure, and still working always at the top edge only, we generate the two dissections for $n = 3$, one in its two isomorphs and the other in its four isomorphs. [Only rules (1) to (3) are employed, since there is no situation where rule (4) applies up to this point.]

Earl's method thus has the great merit that each dissection is produced once and once only in each and all of its possible rotations and reflections. (It is obviously necessary to maintain consistency about which edge is the 'working' edge.) It is not possible for the same isomorph of the same dissection to be generated via two distinct sequences of operations.

Another way of representing and working on dissections which can also be used for an exhaustive enumeration has been devised by Flemming (1978). Flemming makes use of what he calls *wall representations* of rectangular plans. He uses 'wall' here in the formal sense introduced in the last chapter, to mean a maximal continuous straight run of wall. Thus in the dissection of figure 4.13(a) there are three walls oriented 'east-west' and four walls oriented 'north-south'. Flemming confines his attention to trivalent dissections—he calls them 'T-plans', since every junction is T-shaped. Notice how the external walls on the north and south sides of the dissection in figure 4.13(a) are by convention shown extended beyond the outer corners of the plan, so as to turn these into three-way junctions.

Suppose the rooms in figure 4.13(a) are labelled a, b, c, and d as shown and the four regions around the outside of the dissection are labelled N, E, S, and W for the points of the compass. The extensions of the external walls on the north and south sides serve to separate these four exterior regions.

The wall representation now consists of a list of walls, first all east-west oriented walls, and then all north-south walls. Each wall is represented by listing in order the rooms or regions which lie on either side of it. An east-west wall is represented by listing first the rooms to the north of it, in order from west to east; and then the rooms to the south of it, in the same order. Similarly for a north-south wall, first the rooms to the west are listed, in order from north to south; and then the rooms to the east, in the same order. Thus the northern boundary wall of the dissection in figure 4.13(a) is represented as (N), (W, a, b, E), where the brackets ( ) divide the rooms or regions to north and south, respectively. The complete wall representation for this dissection is

{⟨1, (N), (W, a, b, E)⟩, ⟨1, (a, b), (c, d)⟩, ⟨1, (W, c, d, E), (S)⟩,

⟨−1, (W), (a, c)⟩, ⟨−1, (c), (d)⟩, ⟨−1, (a), (b)⟩, ⟨−1, (b, d), (E)⟩}.

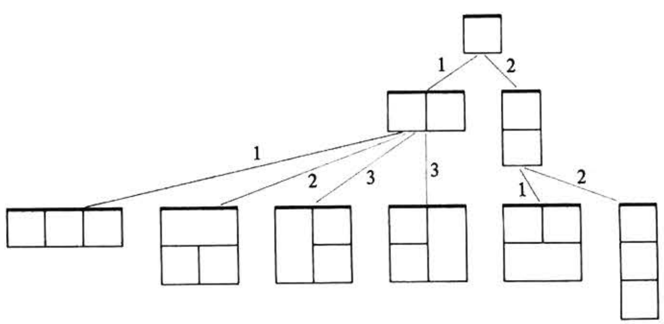


Figure 4.12. Generation of dissections of up to three rectangles, with an algorithm due to Earl (1977). The process starts with the single rectangle (top); and all rules are applied in turn where relevant to the 'north' edge (shown in heavier line). The numbers indicate the rule applied in each case. See how every dissection is produced in all its isomorphs. [There is no situation in which rule (4) applies up to this level.]

The enclosing brackets ⟨⟩ serve to identify each wall, the prefixes 1 and −1 are used to signify east-west and north-south walls, respectively, and the complete wall representation of the plan as a whole is enclosed in braces { }.

A typical operation in Flemming's method for generating dissections is illustrated in figure 4.13(b). A new fifth rectangle, labelled e, is introduced into the dissection with four rectangles, by 'pushing' d over towards the east and inserting a new north-south wall as shown. Besides this new wall, there are three old walls which are affected by this operation: all four are drawn in heavier lines in the figure. The new wall representation is as follows. See how the change is registered in the lines corresponding to the three altered walls and the new wall (in heavy type).

{⟨1, (N), (W, a, b, E)⟩, ⟨**1, (a, b), (c, e, d)**⟩, ⟨**1, (W, c, e, d, E), (S)**⟩,

⟨−1, (W), (a, c)⟩, ⟨**−1, (c), (e)**⟩, ⟨**−1, (e), (d)**⟩, ⟨−1, (a), (b)⟩, ⟨−1, (b, d), (E)⟩}.

Figure 4.13. *Wall representations* of rectangular dissections, from Flemming (1978). A *wall* here is a maximal continuous straight run of wall. Thus there are three walls in the dissection illustrated in (a) oriented east-west, and four walls oriented north-south. The two external east-west walls are extended beyond the corners of the plan, so that all junctions are three-way (the dissection is trivalent). In (b) a new rectangle e is inserted into the dissection of (a), through an operation in which rectangle d is pushed to the east, and a new north-south wall is introduced. Besides this new wall there are three existing walls affected by this operation—they are all four marked in heavier line. The new wall representation is shown beside the dissection. The changes are registered in the four lines corresponding to the three altered walls and the new wall (marked with asterisks).

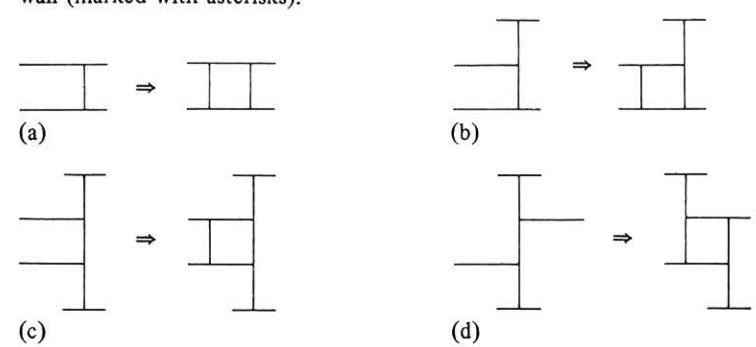
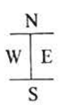


Figure 4.14. Types of operation on wall representations of (trivalent) rectangular dissections, from Flemming (1978). In each case a new rectangle is introduced by pushing aside some existing rectangle towards the east and inserting one new north-south wall. Operation (a) is the one employed in the previous figure. Operation (d) is of the general type required for making the C_4 swastika and related arrangements.

Every operation in the generating process is similar to this one just illustrated, in that a new rectangle is introduced by pushing aside some existing rectangle and inserting one new wall. In some cases this requires that other existing adjacent rectangles are 'pulled' or extended to fill part of the space where the first is pushed aside.

Differences arise out of the ways in which the continuations of the four walls bounding the new rectangle are oriented, north–south or east–west. Four possibilities are illustrated in figure 4.14. In all cases existing rectangles are pushed towards the east and new north–south walls introduced. The operation of figure 4.13(b) appears here as figure 4.14(a). The operation shown in figure 4.14(d) is of the general type needed, as will be appreciated, to make 'pinwheel' and related dissections. Reflections and rotations of these four basic situations give rise to a total of sixteen possibilities as illustrated in figure 4.15.

Flemming shows how each of these geometrical operations can be expressed as a corresponding logical operation on the wall representation, by making substitutions in the sequences of letters or symbols representing rooms. To generate all trivalent dissections using Flemming's approach, it would be necessary to start from the configuration with no rooms, just the external walls, as in



and to apply suitable sequences of operations as described, to yield arrangements for successive values of n. It should be said though that Flemming's approach is a more general and abstract form of representation of a packing of rectangles into a rectangle than is a dimensionless representation on a grating. The wall representation, by its nature, does not always specify the 'overlapping' adjacencies of rooms *across* a wall. Thus the wall representation given above for the dissection of figure 4.13(a)

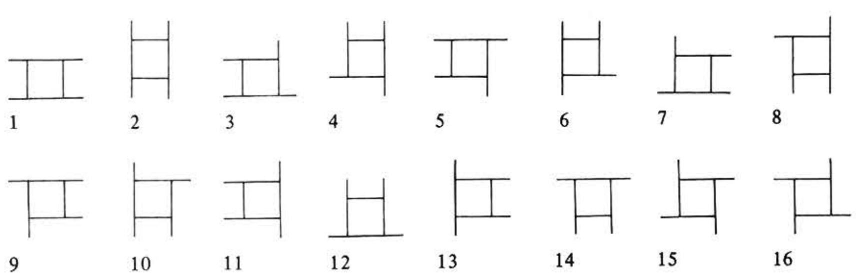
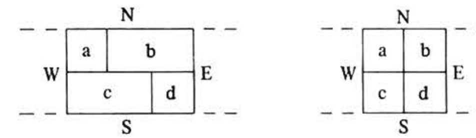


Figure 4.15. Reflections and rotations of the basic operations on wall representations shown in figure 4.14, giving rise to sixteen possibilities in all; from Flemming (1978).

would serve equally to describe the two dissections:



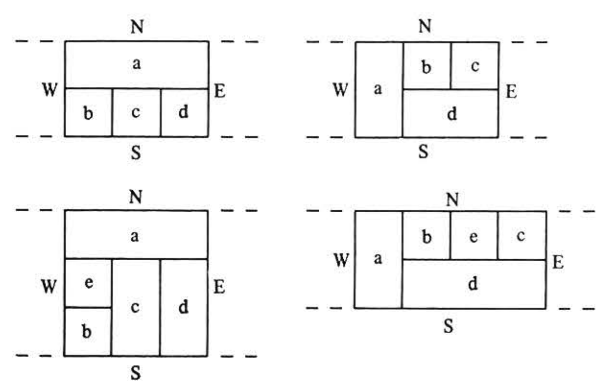
Of these, one would be without its labels a mirrored isomorph of figure 4.13(a), but differs by virtue of the labelling so that, for example, whereas rooms a and d are adjacent in figure 4.13(a), they are not in the above figure. In the other, the two three-way junctions are coalesced into a four-way junction. In general this whole question of overlaps or alignments across walls is taken care of by Flemming at the stage when dimensions are reintroduced into the wall representation, as described in chapter 9. So an enumeration of (undimensioned) wall representations would yield smaller numbers than that for rectangular dissections as previously defined.

Exercises

4.1 Use Earl's procedure to carry the process illustrated in figure 4.12 one stage further, to generate dissections with four component rectangles. The results can be checked by referring to the first figure on page 27 of chapter 3. There should be, in all, twenty-six arrangements counting all isomorphs. [Notice that the one dissection containing a four-way junction, dissection (g) in the first figure on page 27 is *not* produced.] Rule (4) will be applicable in one instance only.

You might wish to go on to enumerate dissections for $n = 5$. There are twenty-one of these, excluding arrangements with four-way junctions (not counting isomorphs). You can check your results against figure 5.1.

4.2 Make wall representations of the dissections shown in the figure. By which of the operations, as illustrated in figure 4.15, can the dissections with four rectangles be made into the dissections with five rectangles?



5

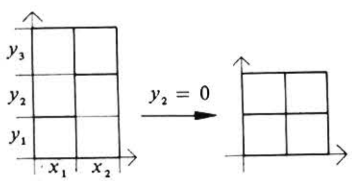
Generating and counting rectangular arrangements: tilings and colourings on grids

"... we need a true science of architecture, a sort of architectural biology which shall investigate the unit cell and all the possibilities of combination."

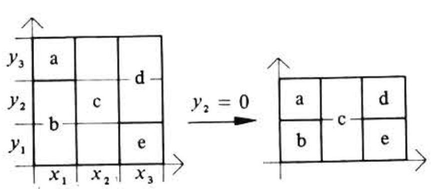
W R Lethaby (1912)

Figure 5.1 shows all rectangular dissections up to $n = 5$ in dimensionless representation, this time including aligned arrangements and those with four-way junctions. Notice the different *sizes of grating* which the various dissections occupy. We denote the size of grating as before by $l \times m$ where m is greater than or equal to l, that is, the smaller dimension is given first.

The single dissection for $n = 2$ occupies a grating 1×2; and the two dissections for $n = 3$ occupy gratings of size 1×3 and 2×2. Dissections for $n = 4$ occupy gratings of size 1×4, 2×2, and 2×3. Notice that the single arrangement of size 2×2 contains a four-way junction and can be produced from a dissection with two three-way junctions of size 2×3, as was shown in chapter 3:



Dissections for $n = 5$ take up gratings of size 1×5, 2×3, 2×4, and 3×3. In this case there are three arrangements of size 2×3: two with four-way junctions and one with an alignment. The two with four-way junctions may be generated from nonaligned dissections of size either 2×4 or 3×3, whereas the remaining aligned arrangement takes size 3×3 when nonaligned as was shown in chapter 3:



These results, together with the numbers of dissections of each size, are set out in table 5.1.

Even considering these relatively few cases, it emerges that there exists, for each value of n, a range of possible grating sizes, from the extremely elongated to the square or 'squarish'. Of these the dissections most nearly square are relatively more numerous, whereas there is for each value of n only one dissection of size $1 \times n$. It is obvious that this latter type is the

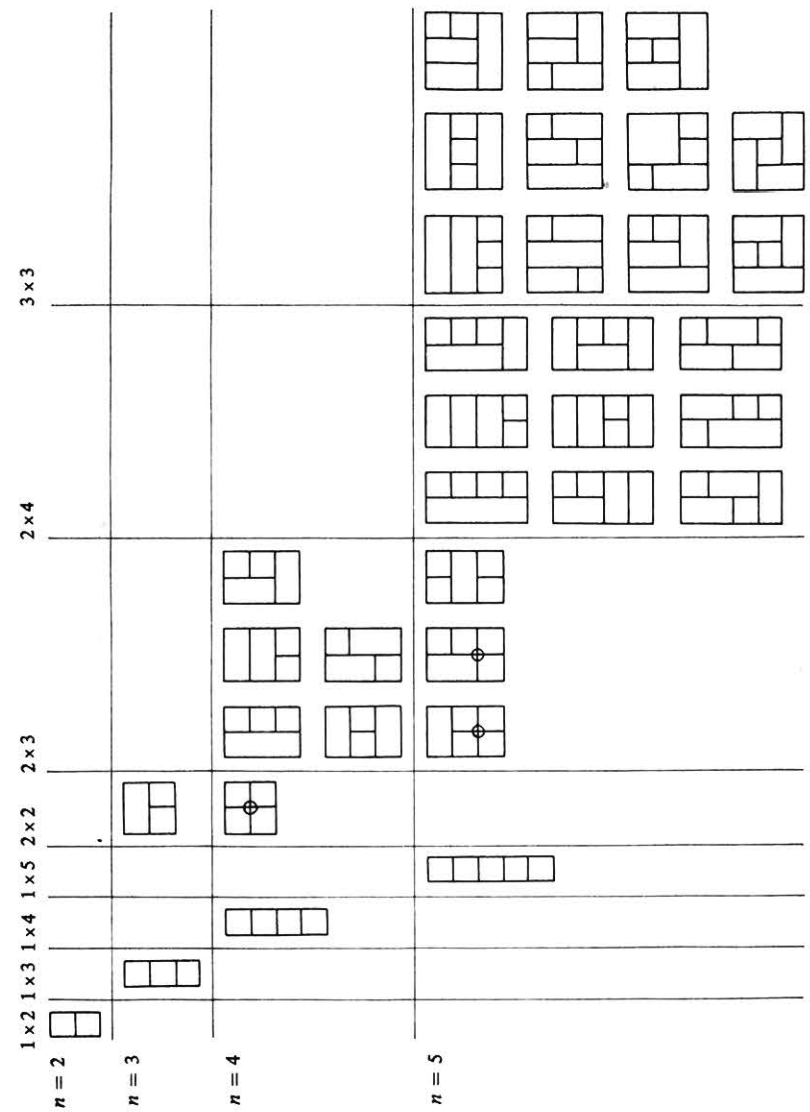


Figure 5.1. All rectangular dissections up to $n = 5$ in dimensionless representation, classified by their grating size $l \times m$. Four-way junctions are shown circled. Arrangements with wall alignments are included.

most elongated shape possible for given n; and equally that there can be only one such dissection for *any* value of n, that is, the arrangement where all n rectangles are strung out in a row.

It is possible to predict theoretically the range of grating sizes required for dissections with n rectangles, without the need to generate and measure all individual cases. The following demonstration is due very largely to Bloch (1976). Consider in the first place fundamental dissections, without four-way junctions or alignments. For this type there is a simple relationship between l, m, and n (Biggs, 1969; Earl, 1977), such that

$$n = l+m-1 \; . \qquad (5.1)$$

You might like to confirm that this relationship does indeed apply for the values of n in table 5.1 (excluding 2 x 2 for $n = 4$, and 2 x 3 for $n = 5$).

The reason is quite straightforward. Consider the smallest possible dissection, of one cell in the grating, in which n, l, and m all have the value 1. The relation obtained here is $1 = 1+1-1$. A new rectangle is introduced by the addition of just one internal wall. (It follows that the number of internal walls—in trivalent dissections—must always equal $n-1$, that is, one less than the number of rectangles.) See how each of Earl's four operations in the last chapter (pages 40 and 41), for example, had the effect of introducing just one new internal wall.

Now since there are to be no alignments of walls in these dissections, every new internal wall must require the creation of a new line in the grating (again as in Earl's algorithm)—that is to say, *either l or m* must be increased by 1. Thus whenever n is increased by 1, $l+m$ must be increased by 1; and the relationship of equation (5.1) follows.

Fundamental dissections are the most 'sparse' kinds of dissection, where n is at a minimum for given values of l and m. An example of size 4 x 5 is:

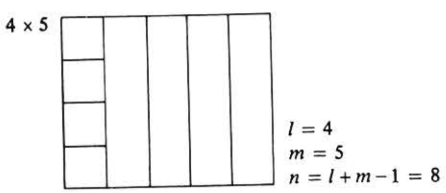
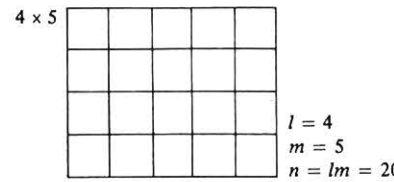


Table 5.1. Grating sizes $l \times m$ for dissections up to $n = 5$.

n	$l \times m$	number	n	$l \times m$	number	n	$l \times m$	number
1	1 x 1	1	4	1 x 4	1	5	1 x 5	1
2	1 x 2	1		2 x 2	1 [a]		2 x 3	3 [b]
				2 x 3	5		2 x 4	9
3	1 x 3	1					3 x 3	11
	2 x 2	1						

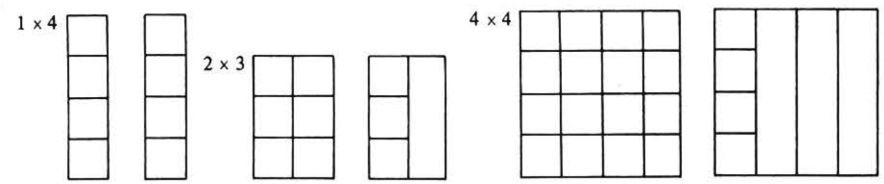
[a] With four-way junction. [b] With four-way junction or alignment.

which is chosen deliberately to emphasise how every grating line is made necessary by the presence of one wall. By contrast the 'densest' kinds of dissection, where n is at a maximum for the given grating dimensions, must be those where every cell of the grating represents a component rectangle, and $n = lm$. The case for grating size 4 x 5 is:



Obviously for given values of l and m there is only ever one dissection of this type.

The following figure gives examples of the two extreme values for n for various grating sizes:



These facts show us that for given l and m, n must lie at or between the two extreme values; it gives us the relationship

$$l+m-1 \leqslant n \leqslant lm \qquad (5.2)$$

(where $\leqslant$ means 'is less than or equal to').

Now to come to the problem in hand: suppose conversely that n is given, what are the limits on the possible values which l and m may take? Suppose we plot a graph of l against m [figure 5.2(a)]. Pairs of values of l and m, that is, sizes of gratings, will appear as points on this graph; and since these values must always be integers (whole numbers) these points must fall on the intersections of lines representing such integral values. A line at 45° through the origin represents pairs of values where $l = m$, that is square gratings. Points placed symmetrically on either side of this diagonal will correspond to arrangements of the same size, but in which the values of l and m are interchanged. (The same shape is rotated through 90°; see, for example, the two points representing 6 x 4 and 4 x 6 in the figure.) It is sufficient therefore to consider only one half of the area of the graph—that is, points on or below the line $l = m$.

In the most dense kinds of dissection $lm = n$. If we draw a corresponding line in the graph it will appear as a curve (a hyperbola), representing the lower limit of the area, for given n, within which allowable pairs of values of l and m will be found. Figure 5.2(a) shows this curve illustrated for $n = lm = 6$. Any points below this curve cannot represent compatible values of l and m for $n = 6$.

All that we need now is to specify the upper limit on combinations of l and m. This we do by plotting the straight line corresponding to $l+m-1 = n$, to represent the most sparse types of dissection—those where the sum of $l+m$ is at its greatest for given n. The overall result is to define a closed, roughly triangular area on the graph [figure 5.2(b)], bounded by the two straight lines $l = m$ and $l+m-1 = n$, and by the curve $lm = n$. All points representing integral values of l and m falling on or inside this boundary represent possible grating sizes for dissections with n component rectangles. Figure 5.2(b) illustrates the situation for $n = 6$.

Figure 5.3 shows the points plotted similarly for successive values of n between 1 and 9. See how in all cases they fall within a triangular area delimited in the way described; and how in all cases there is one grating size, at the extreme right-hand corner, where $l = 1$, corresponding to that unique dissection where the n rectangles are set out in a single row. We can confirm these results for n up to 5, by comparison with the sizes listed previously in table 5.1. Remember that it is possible that to one point, or size $l \times m$, particularly where the shape is squarish, there may correspond a large number of different dissections.

This demonstration of Bloch's allows a new approach to the problem of generating dissections exhaustively. For given n we now know all the

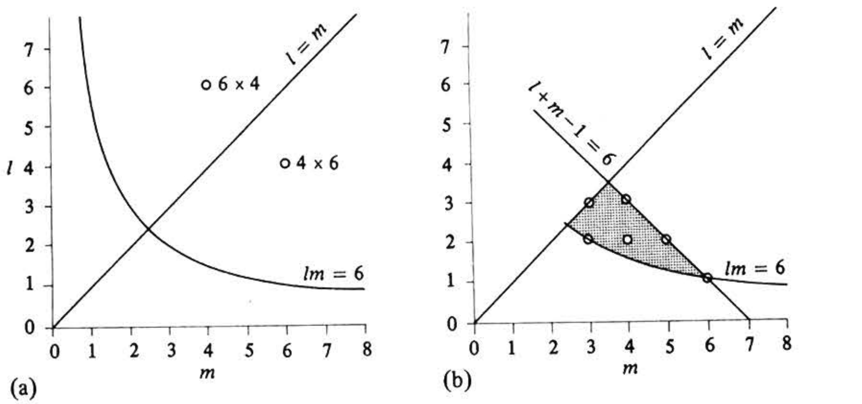


Figure 5.2. (a) Graph of possible grating sizes, corresponding to pairs of integral values for grating dimensions l and m; after Bloch (1976). Square gratings lie on the line $l = m$ passing through the origin at 45° to the coordinates. Points placed symmetrically either side of this diagonal correspond to gratings of the same size, but with the values l and m exchanged; as with the sizes 6 x 4 and 4 x 6 illustrated. The lower limit on pairs of values for l and m for given n, is given by the hyperbola $lm = n$, plotted here for $n = 6$. (b) The upper limit on pairs of values for l and m for given n, is given by the straight line $l+m-1 = n$, plotted here for $n = 6$. All compatible pairs of values for l and m for $n = 6$ lie inside or on the boundaries of the area (shown shaded) defined by $l = m$, $lm = 6$, and $l+m-1 = 6$.

possible sizes of gratings needed. We take each size in turn and divide it up into, or fill it with, n rectangular pieces. Bloch (1978; 1979a) has himself employed just this technique; his algorithm is conceived in terms of placing rectangular 'tiles' in such a way as to fill empty gratings of predetermined size[7].

The first step is to take the total number of cells in the grating, given by $l \times m$, and divide this number into n parts, each of which will correspond to a tile. Thus a grating size 4 x 5 will contain twenty cells, which might be partitioned (in just one way out of many) into eight tiles as

$$\{4, 3, 3, 3, 3, 2, 1, 1\}.$$

The numbers in the partition are arranged for convenience in descending order of size; each represents the area of a single tile; together they sum to the area of the whole grating.

Each value in the partition must be further broken down into a pair of factors to represent the actual dimensions (in grating intervals) of that particular tile. The result is what Bloch calls a 'factored representation' of the partition. For the partition given above, we might have a factored representation

$$\{(2 \times 2), (1 \times 3), (1 \times 3), (1 \times 3), (1 \times 3), (1 \times 2), (1 \times 1), (1 \times 1)\}.$$

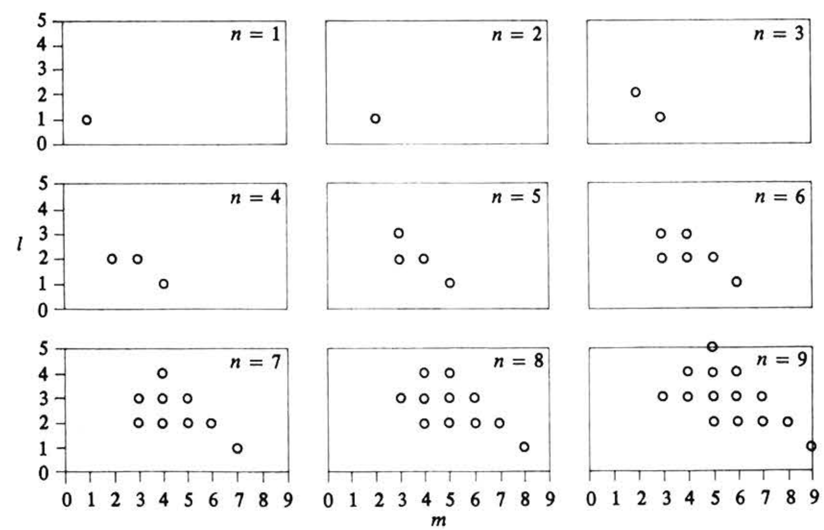
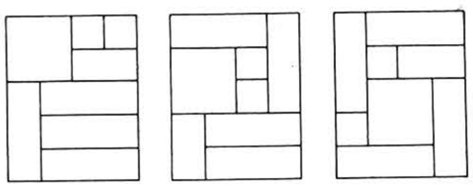


Figure 5.3. Possible grating sizes $l \times m$ for values of n between 1 and 9; after Bloch (1976).

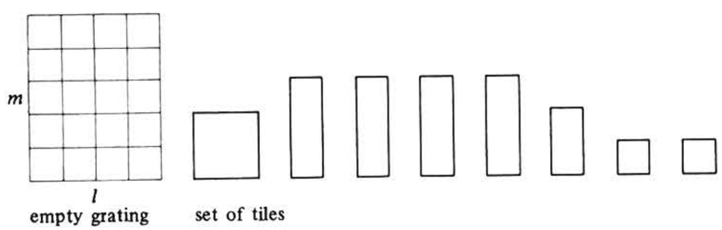
(7) In the mathematical literature, problems of this type are strictly classified as 'packing' problems. The term 'tiling' refers to filling the plane with regular patterns composed of repeated copies of some few component shapes. However, 'tiling' very aptly describes the nature of the process involved here.

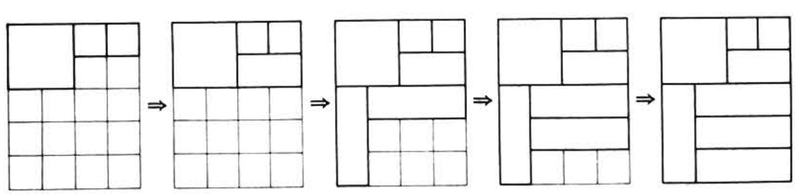
For each specified grating size (that is, each pair of values for l and m) it is necessary to determine every possible partition—of which there may be many—and for each partition every possible factoring—of which there may again be many. The result is a listing of all possible sets of n tiles, with which the grating might be filled. There are various conditions applying to factored representations—restrictions on admissible sets of tiles—which Bloch discusses. These include conditions on the largest size of tile, and on the maximum dimensions or combinations of dimensions for large tiles.

I will not go into the details of Bloch's algorithm for generating dissections, which is rather complex; but in general outline he proceeds as follows. Every grating size is taken in turn, and in each case every factored representation is taken in turn. The set of tiles which the factored representation comprises is fitted into the empty grating, in all the arrangements which are possible. Three of the possibilities for the 4 x 5 grating and factored representation given above are:



The tiles are fitted by working systematically from one end of the grating to the other, in the direction of the longer grating dimension m:





By taking in turn all feasible sequences and permutations in which the tiles can be placed, and so proceeding until the grating is filled (or the process fails), all possible tilings are generated. The process is such that many dissections are finally produced which are nonminimal (compare the figure on page 11) or which are isomorphic despite precautions being taken to avoid some such conditions arising. Such configurations must be

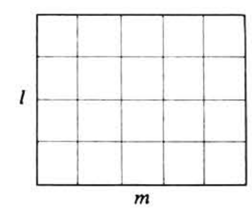
identified and rejected. Those that remain at the end *include* dissections with four-way junctions, as well as dissections with wall alignments.

Using this technique, Bloch has generated all dissections up to and including $n = 9$; and he has used the computer to make a catalogue of drawings of each dissection up to $n = 8$ (see Bloch and Krishnamurti, 1978; Bloch, 1979a; 1979b). This catalogue is reproduced here (page 250) as an appendix, but with results for $n = 8$ omitted for lack of space. The catalogue is organised according to a whole series of classifications which are explained in chapter 8. It seems that Bloch's method is not capable of an enumeration beyond $n = 9$, because of the fact that it produces increasingly larger numbers of the nonminimal and isomorphic arrangements which have to be weeded out.

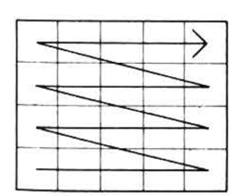
The exact numbers of dissections in the various categories—with and without alignments and four-way junctions—for these higher values of n, are given at the end of this chapter. Meanwhile, we should look at one last method for generating dissections, which exactly confirms Bloch's results, and which is due to Krishnamurti (Krishnamurti and Roe, 1978; Bloch and Krishnamurti, 1978).

Rather than filling the empty grating with tiles, the essential concept here is that of assigning imaginary *colours* to the grating cells. Each rectangle in the dissection is then represented either by a single coloured cell or by a group of adjacent cells all coloured the same. Different rectangles are coloured differently; and therefore as many colours are needed as there are rectangles, n, in the dissection.

Imagine a grating set with its long dimension m 'east–west' on the page:



In Krishnamurti's algorithm, the colours are assigned to the cells in a standard pattern, starting from the bottom left-hand corner of the grating and working along the bottom row to the right; then jumping to the left-hand end of the next row up and working to the right again; and so on until the grating is completely coloured:

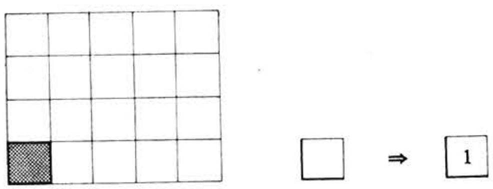


Let us call the cell in this sequence which is to be coloured at each step, the qth cell.

The colours are applied according to a set of rules whose combined effect is to ensure that only groups of adjacent cells which together form rectangular shapes are coloured the same; that exactly n colours are used; and that the resulting dissections are minimal both in 'east–west' and in 'north–south' directions. The rules are of two principal types: those which colour the qth cell the same as one of its predecessors in the sequence (that is, they extend an existing rectangle), and those which apply a new colour (that is, they start a new rectangle). Let us symbolise the new colour by μ.

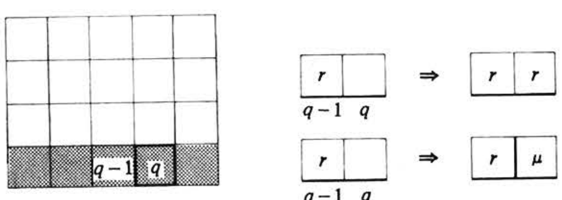
Within this the rules may be further divided into four categories, according to the cells in the grating to which they can apply.

(1) The simplest situation is that applying solely to *the single cell at bottom left* in the grating, which is the first in the sequence, and which therefore must be assigned the first colour:



cell type 1

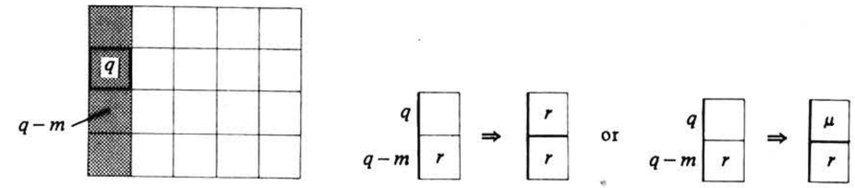
(2) For those cells which are in *the bottom row*, each one can only have one neighbour already coloured, and that is the $(q-1)$th cell on its immediate left. There are two options for the qth cell here: either it is coloured the same as cell $q-1$, or it is coloured differently, with a new colour μ:



cell type 2

Either of these is compatible with the arrangement becoming a rectangular dissection.

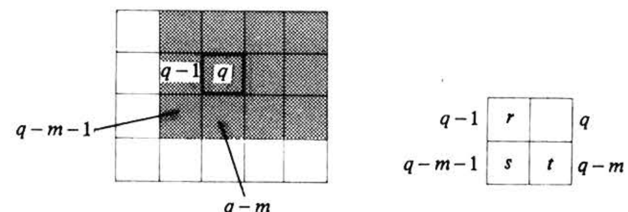
(3) When one or more rows have been completely coloured, a special condition arises with each new cell in *the column at the extreme left*. In this case the only neighbouring cell already coloured is that immediately below:



cell type 3

It is easy to see that this must be the $(q-m)$th cell, counting back in sequence a number of cells equal to the width of the grating. Again the options are to colour the new qth cell the same as the $(q-m)$th, or to assign a new colour μ.

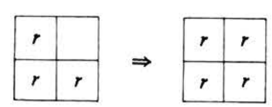
(4) Finally, there are *all the remaining cells* in the grating, for which in every case there will be three neighbouring cells already coloured. These neighbours are the cells numbered $q-1$, $q-m-1$, and $q-m$. Let us suppose that these have been coloured r, s, and t respectively, where r, s, and t might be the same or they might be different colours:



cell type 4

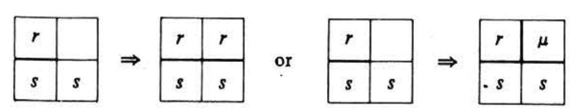
Only certain combinations of colourings of this group of four cells (including the qth cell) are permissible, within a rectangular dissection. Thus it is not allowable for three cells to be of one colour, and the fourth a second colour, since this would create an L-shape in the final arrangement. Four situations *are* permitted for rule 4:

4(a) $r = s = t$. Only one rule applies: the qth cell must be coloured the same as the other three:

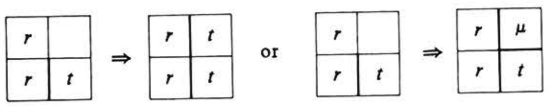


(The four cells belong to the same rectangle.)

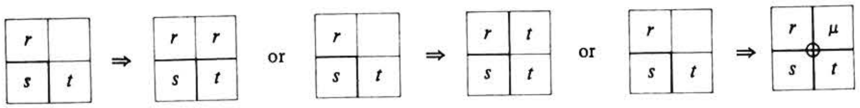
4(b) $s = t$ and r is different. Two rules can apply: the qth cell may be coloured the same as r, or it may be assigned a new colour μ:



(The four cells belong either to two or to three rectangles.)
4(c) $r = s$ and t is different. Two rules can apply: the qth cell may be coloured the same as t, or it may be assigned a new colour μ:



(Again, the four cells belong either to two or to three rectangles.)
4(d) r, s, and t are all different. Three rules can apply: the qth cell may be coloured the same as r, the same as t, or with a new colour μ:

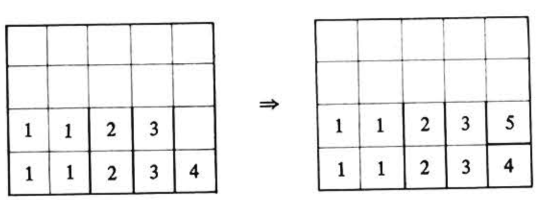


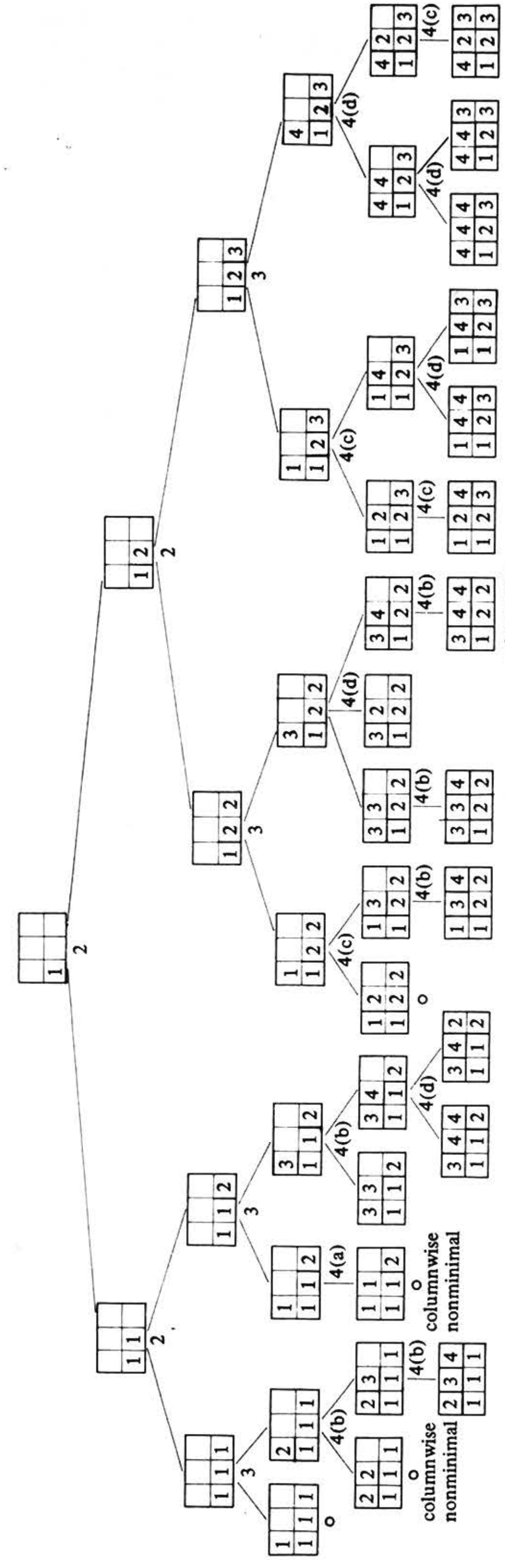
(The four cells belong either to three or to four rectangles.) Notice that of all these operations it is only in this very last case, where the four cells in question are all coloured differently, that a four-way junction is created.

Two general conditions on the applications of these rules ensure that exactly n colours are used for colouring the complete grating. Rules of the first general type, those which extend an existing rectangle, apply only where $lm - q \geqslant n - \mu$. The total number of cells in the grating is given by lm, and since q is the number of the current cell, $lm - q$ must give the number of cells remaining to be coloured. The number of unused colours, that is, the number of new rectangles remaining to be created is given by $n - \mu$. The condition therefore states, in effect, that where there are as many cells remaining as there are rectangles still to be created, each one must be coloured differently.

Meanwhile, only when $n > \mu$, that is, the number of colours used so far is less than n, are the rules of the second general type—those which assign a new colour and thus start a new rectangle—applicable.

It remains to ensure that the dissections are minimal both in the 'north-south' and in the 'east-west' senses. Recall that in a minimal arrangement there must be at least one wall on every line in the grating. The first point at which it is necessary to make a check for minimality (in the 'north-south' sense), is reached at the extreme right-hand cell in the second row up (assuming $l > 2$):





o $lm - q < n - \mu$, that is, number of remaining colours exceeds number of remaining cells

Figure 5.4. Generation of rectangular dissections with an algorithm due to Krishnamurti (Krishnamurti and Roe, 1978). The illustration shows the 'colouring' of a 2 × 3 grating with four 'colours' 1 to 4, to generate dissections for $n = 4$. The numbers below the gratings indicate the colouring rule applied in each case. Colourings for which $lm - q < n - \mu$, that is, the number of remaining colours exceeds the number of remaining cells, are marked with hollow circles. Impermissible colourings yielding nonminimal dissections where there is not at least one wall on each grating line are marked as such. The algorithm produces all five dissections for these values of l, m, and n, in all the isomorphs which the given orientation of the grating allows.

Suppose, as in the example illustrated, that there is no wall so far on the grating line between first and second rows. In these circumstances, the colouring rule applied *must* be of the second general type, which creates an east–west wall on the grating line. Similar tests must be applied to all cells further up the extreme right-hand column of the grating. They must also be carried out when the colouring process reaches the top-most row, starting from the second cell in that row, so as to ensure minimality in the 'east–west' direction.

Krishnamurti is able to generate separately the different classes of dissection—with or without alignments or four-way junctions—by the judicious omission or inclusion of selected rules. There is only one type of rule, as already noted—that of 4(d) above, where four cells are all coloured differently—which creates a four-way junction. Omission of this rule results in only trivalent dissections being produced. If alignments of walls are to be avoided, then rules of the type which create a new rectangle cannot be applied in two circumstances, both of them relating only to cells of the fourth category. These are either when $s = t$ [refer to 4(b)] and there is a wall already on the relevant 'north–south' grid line (with which the 'north–south' wall created by the new rectangle will be aligned but not continuous). Or similarly where $r = s$ [refer to 4(c)], and there is a wall already on the relevant 'east–west' grid line.

Figure 5.4 gives a sample illustration of the whole algorithm in action, for a grating size 2 x 3 and $n = 4$. The 'colours' here are signified by the numbers 1 to 4. Where appropriate the tests for minimality are applied and their results indicated. See how the application of all relevant rules at each step produces the five dissections for these values, in all the isomorphs which this particular orientation of the grating allows.

Krishnamurti and Roe have an economical technique for detecting and removing isomorphs. Consider the following four isomorphs of the dissection with four rectangles, labelled (a) to (d) as they were generated in order by the colouring process:

3	4	2
1	1	2

(a)

1	3	4
1	2	2

(b)

1	4	4
1	2	3

(c)

4	4	3
1	2	3

(d)

Suppose the numbers representing the colours of the cells are read off in the same order in which they were applied. This gives for each arrangement a unique sequence of six digits, which are, respectively,

112342 , 122134 , 123144 , and 123443 .

Indeed the colouring process encodes *every* dissection in each of its isomorphs by a unique 'code number' (containing *lm* digits) in this way.

(Although by no means all numbers of the appropriate length will correspond, obviously, to dissections.)

Krishnamurti and Roe choose to retain as their representative or *canonical* isomorph in each case, that arrangement which possesses the *lowest* number. In the above example, this would therefore be arrangement (a) whose code number is 112342. Consider the stage at which this particular colouring is first generated. It is subjected to rotations and reflections (as in previous methods) to produce its possible isomorphs:

3	4	2
1	1	2

2	4	3
2	1	1

2	1	1
2	4	3

1	1	2
3	4	2

These isomorphs are then 'recoloured' in the standard cell sequence, and their corresponding code numbers derived [as in (b), (c), and (d) above]. Since these new numbers are all *greater* than that of the original colouring (a), if follows that (a) must be the canonical isomorph. When the same process comes to be applied to (b), (c), or (d), and these are reflected, rotated, and recoloured in a similar way, at least one of the resulting code numbers in every case will be *lower* than that of the original colouring—hence none of these can be canonical and they are all in turn rejected.

The ingenious feature of this scheme is that there is no need to store and compare colourings to detect isomorphs. It is possible to inspect a legitimate colouring in isolation, and determine whether or not it is the canonical version and should be retained.

With this algorithm Krishnamurti was able to generate dissections up to and including $n = 10$, the furthest any enumeration has been taken to date. Table 5.2 summarises the results obtained by Bloch and Krishnamurti (1978) for values of n from 5 to 10, with the different classes of dissections counted separately in each case. Krishnamurti quotes a figure of three

Table 5.2. Numbers of dissections of different classes up to $n = 10$.

n	Number of dissections			
	general [a]	nonaligned [b]	trivalent [c]	fundamental [d]
5	24	23	22	21
6	126	119	108	101
7	815	735	668	591
8	6465	5527	5026	4168
9	58072	46204	43005	32754
10	578663	423724	389803	282605

[a] Including those with alignments and/or four-way junctions.
[b] That is, without alignments.
[c] Without four-way junctions.
[d] Without alignments or four-way junctions.

minutes computation time on an IBM 370/165, for generating the 58072 arrangements for $n = 9$.

In Krishnamurti (1979) the enumeration of rectangular dissections is extended to the third dimension, that is, to packings of rectangular blocks into rectangular boxes. Such packings could clearly serve as geometric models for the spatial forms of buildings. Considerations apply which are equivalent to those in plans, concerning the alignment of walls and the possible types of junctions between walls, but relating now to the alignment of *planes* (that is, walls or floors); and junctions between planes (see Earl, 1978). Using a colouring technique similar to that described in this chapter, Krishnamurti has enumerated the possibilities up to $n = 8$, where n is the number of blocks.

Exercises

5.1 List the possible grating sizes for dissections of order 12. (Draw a graph as in figure 5.2(b), plot the relevant equations, and identify the integer points.)

5.2 Use the general principles of Bloch's approach to generate all possible tilings on 2 x 3 gratings. You will need first to partition the six grating cells into n tiles in all possible ways. Notice that tiles of area 5 are not possible, since these could only take the shape 1 x 5 which will not fit on the grating. With the partition {5, 1} omitted, then, you should find that there are ten remaining partitions, including the extreme possibilities {6} and {1, 1, 1, 1, 1, 1}. Each of these has as it turns out only one factored representation (the shapes of tiles are here given uniquely by their areas). You will now need to work by trial and error, fitting each set of tiles into the grating in all possible ways. See how many of the arrangements so produced are nonminimal—indeed the simple presence of a 2 x 2 tile, or of the single 2 x 3 tile ensures that the arrangement *must* be nonminimal. You should end up with nine distinct dissections: five of order 4, three of order 5, and one of order 6.

5.3 Use Krishnamurti's colouring technique to generate all dissections of order 5 on 2 x 3 gratings, following the lines of figure 5.4. The first stages will be identical to those in figure 5.4; but the criterion $lm - q < n - \mu$ (fewer cells remain than colours) will begin to apply sooner. Remember to make the checks for minimality. The results should be as for the dissections of order 5 in exercise 5.2 above—but they will occur in all the different isomorphs which the orientation of the grating allows.

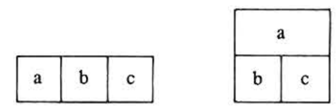
5.4 Of the isomorphs of the three dissections produced in the previous exercise, which is the *canonical* isomorph in Krishnamurti and Roe's terms, in each case? (Derive the number for each isomorph as in the figure on page 58 and identify that with the lowest number.)

6

Graphs of plans and their arrangement

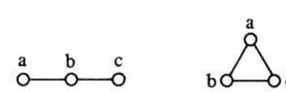
> "In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch ... which Leibniz first mentioned, calling it the *geometry of position*. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them."
>
> L Euler (1736)

We noticed in chapter 3 how, in different rectangular dissections, the relationships of *adjacency* between the component rectangles varied. The examples were cited of the two dissections for $n = 3$:



In the first case the room labelled b is adjacent to both a and c; but a and c are separate. In the second dissection each of the three rooms is adjacent to both the others. Imagine these, or indeed any other dissections, subjected to the kinds of transformation described in chapter 2, where the grating intervals in the x and y directions are assigned particular dimensions. Evidently, the relationships of adjacency between rooms will not be changed under such transformation, no matter how the relative sizes of the rooms may be altered.

Suppose that we represent each room in an arrangement by a point; and that wherever two rooms are adjacent, we draw a line joining the respective points. (For these purposes we define rooms to be adjacent where they share some length of wall in common. So two rooms which touch only at a corner, for example, rooms diagonally opposite at a four-way junction, are not regarded as adjacent.) Such figures constitute what are known mathematically as *graphs*; in this case we can refer to them as *adjacency graphs.* The adjacency graphs of the two dissections for $n = 3$ are:



These adjacency graphs are rather similar to what architects call 'functional' or 'linkage' diagrams, and sometimes make use of in the early stages of layout design. Another term is 'bubble diagram'—where the rooms are the 'bubbles'—although this, as we shall see, is closer to what strictly should be termed a 'plan graph'. In general, the meaning and definition of such diagrams is imprecise, however, and they are used to combine topological with shape and dimensional information in a way which is inconsistent and conceptually confused. Here I shall try to be more explicit in defining precisely which features of a plan the graph does and does not represent.

Relationships of adjacency between rooms in the plan of a building are clearly of the greatest functional significance. Where two rooms share a sufficient length of wall in common then it is possible for them to be made accessible one from the other via a door. The overall patterns of adjacency between corridors, halls, and other rooms determine the ways in which routes can run, along which people can circulate through the building. Again, the adjacencies of rooms to the exterior constrain the possibilities for placing windows in those rooms; and the same is true for doors giving direct access to the outside.

There are other aspects of the planning of buildings which may have to do with the adjacencies of rooms. For example, it may be desirable that two rooms be adjacent because they share common services, for example, plumbing; or that two rooms *not* be adjacent, for reasons of sound insulation perhaps, or privacy. We shall look more closely at some of these functional and planning matters in chapter 10.

The adjacency graph of a plan, then, captures important topological features of the relationship between rooms—topology being that branch of mathematics which deals with the properties of spaces as they form connected pieces and have boundaries, independent of their size and shape. Continuity, connectedness, and adjacency are thus all properties of topological interest.

We can make use of some of the techniques and findings of the theory of graphs in representing and manipulating plan arrangements. In fact there is now a sizeable body of work on the application of graph theory to architecture. It is beyond the scope of this and the next chapter to review all that work in detail. So I will try here to give a general account of the most relevant ideas and results, together with sufficient references to lead the reader into the literature of graph theory in general, and applications to architecture in particular. [For general accounts of architectural applications see March and Steadman (1971, chapters 10 and 11), Steadman (1973), Mitchell (1977, chapter 6), and Earl and March (1979). For introductions to graph theory see Ore (1963) and Preparata and Yeh (1973, chapter 2). For a more advanced and comprehensive text with useful appendices of diagrams see Harary (1969). An entertaining and discursive historical account of the subject is given in Biggs et al (1976).]

For the present purpose we will need some basic definitions and explanations of concepts and terminology. A point in a graph is termed a *vertex*, and a line joining two vertices is an *edge*[8]. It is of no importance

[8] There are many variations in terminology here. Harary (1969), one of the principal authorities, uses the words 'point' and 'line' themselves. Since we are using graphs in some cases here, however, to *represent* geometric arrangements of lines (that is, plans), it seems less confusing to refer to 'vertices' and 'edges' in the corresponding graphs.

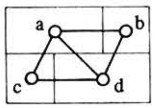
to the theory exactly how the graph is drawn. The vertices may be placed anywhere on the page, and the edges joining them may be straight or curved, even crossing each other. What matters is how many vertices and edges there are, and which edges join which pairs of vertices. So there is a large, indeed an infinite, number of ways of drawing the same graph. The graph, despite its name, is an abstract structure, which is depicted for convenience in graphical form.

In general, graphs are used to represent sets of relationships (denoted by the edges) between sets of entities or elements of some kind (the vertices). In our case these relationships are spatial ones—relationships of adjacency—but in other applications the relationships represented might be completely nongeometric. For example, graphs have been used in sociology to record personal relationships between members of social groups; in anthropology to represent marital relations and kinship structures (family trees); and in the theory of organisations to show hierarchies and the structure of command or responsibility in businesses or institutions.

This said, however, graphs are especially well-suited to the description of spatial configurations, and many of the mathematical puzzles and scientific problems from which the subject of graph theory sprang, have been concerned with the structure of two-dimensional or three-dimensional arrangements in space. The subject is agreed to have begun with a paper by Euler (1736) (from which the quotation at the head of this chapter is drawn), in which he described a problem to do with the plan of the city of Königsberg and its bridges. Since then the connection of graph theory with the study of the general properties of maps has been a continuing one; notably through the infamous and only recently proven 'four-colour conjecture'. (The conjecture concerns the printing of maps, such that adjacent regions are differently coloured. Will a maximum of four colours suffice for *any* map? The answer is yes, it transpires.)

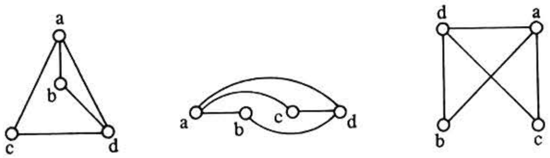
Another early application was to chemistry, in notations for the arrangement of atoms in molecules. It is from this source that the term 'graph' itself—from 'chemicograph'—originates; whereas graph theory borrowed the chemical idea of valency in return. The *valency* of a vertex in a graph (sometimes called its 'degree' or 'order') is the number of edges which meet at that vertex. Yet a further application of graphs has been in the description of networks, for example, traffic networks or electrical circuits. Methods and results from several of these areas—specifically electrical networks and the study of maps—have relevance, as will emerge, to the morphology of architectural plans.

Graphs have been used therefore to represent a variety of spatial structures; but we should still keep it in mind that the graph is not a *picture* of some such structure, in the conventional sense. We use a graph to represent the adjacencies between rooms in a plan; and in this case we could, for clarity in drawing the graph, place the vertices in the centres of the rooms, and draw the edges between them as lines crossing the respective partition walls;

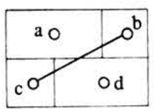


Each edge has the *meaning* 'rooms r_i and r_j are adjacent'.

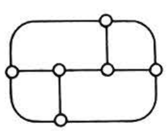
However, we could draw the identical graph in many other ways in which the correspondence with the plan would not be visually so obvious:



And it would be equally possible (though probably less useful) to draw a graph—just to make this point—in which each edge had the *opposite* meaning: that 'rooms r_i and r_j are *not* adjacent'. A 'nonadjacency graph' of this kind, for the same plan, is:



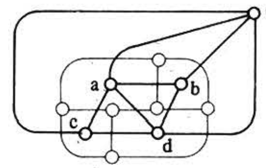
There are yet other possible graphs of buildings, in which the edges (and vertices) have different meanings again. Suppose, for example, we represent the *junctions* between walls by vertices; and for each *wall segment* running between two given junctions, we draw an edge. Keeping in mind the strictly symbolic character of the graph—each edge is not a drawing of a wall segment, but rather has the *meaning* that 'a wall segment runs between junctions v_i and v_j'— nevertheless the result is a kind of diagrammatic version of the plan itself. We can call it the *plan graph*:



(Notice that the four outer corners of the plan are not counted here as junctions.)

A plan graph and its corresponding adjacency graph bear a special relationship to each other. This relationship holds, however, only if we are careful to take account of the exterior region around the plan. We include one more vertex o in the plan graph to represent the whole of this outside space; and we now need edges in the adjacency graph to indicate the

adjacencies to the exterior of all rooms around the periphery of the plan:



The closed regions which are defined by the edges of a graph, as it is drawn in the plane, are called *faces*. In the plan graph each room is therefore a face. And the infinite region around the outside of the plan is the *exterior face*.

Let us now examine the numbers of vertices, edges, and faces in both graphs in the above example. In the plan graph there are six vertices, nine edges, and five faces (including the exterior face). In the adjacency graph there are five vertices, nine edges, and six faces (again counting the exterior face). The number of edges is the same in both cases; we see that this must be so, in the nature of the original definition of the graphs. Each wall segment or edge in the plan graph separates two faces—either two rooms, or a room and the exterior region—and each corresponding edge in the adjacency graph represents the adjacency of those same two faces. As they are drawn in the figure, each edge in one graph crosses the corresponding edge in the other graph.

Furthermore, to every *face* in one graph there corresponds a *vertex* in the other. For faces in the plan graph this property follows directly from the way in which we produced the adjacency graph in the first place, by assigning a vertex to every room (or region) in the plan. But the converse is also true, and there is a face in the adjacency graph for every vertex in the plan graph. Each vertex here represents a junction between wall segments; and the corresponding face in the adjacency graph is bounded by edges representing the adjacencies across all wall segments which meet at that junction. Graphs related in this way are said to be *duals*; and in the architectural context the combination of plan and adjacency graphs has been termed the *dual graph representation.*

In this example we used one vertex to denote the whole of the exterior face in the plan graph. Notice, however, that the dual relationship still holds, if we divide the space outside the plan into several separate regions. For a plan of overall rectangular shape, it may be convenient for example to distinguish four external regions, on the 'north', 'east', 'south', and 'west' sides (as in Flemming's wall representations). In functional terms this makes it possible to represent the orientation of rooms in relation to sunlight or views, say; or to show adjacency relationships to particular areas around the plan such as the street front, a garden, other neighbouring buildings, and so on.

We can represent the exterior regions by adding four more infinite faces in the plan graph (figure 6.1). These faces are separated by four infinite edges attached to vertices at the four outer corners of the plan. We now need appropriate edges in the adjacency graph to show both adjacencies of the rooms to these regions, and the adjacencies between the four regions themselves. This type of adjacency graph for a rectangular plan has been termed its *augmented dual* (Earl and March, 1979).

In some circumstances in the architectural context it is useful to draw an adjacency graph in which only the interior adjacencies between rooms in a plan are included, and adjacencies to the exterior are ignored (as was the case in the figure at the beginning of the chapter). Here then there is a vertex in the adjacency graph for every room as before, an edge for every *internal* wall segment, and a face for every interior junction between those wall segments. However, the full dual relationship between plan graph and adjacency graph is not now preserved. Earl and March (1979) call this type of adjacency graph the *weak dual* of the plan graph.

It is irrelevant to the theory, as mentioned, if graphs are drawn with their edges crossing. It naturally tends to be clearer if they are shown in such a way that edges do not cross; and graphs which it is *possible* to draw without crossings—that is, to *embed* in the plane—are called *planar* graphs. A graph which has been embedded in the plane is called a *plane* graph. It follows that any plan graph of the kind we have been examining must be plane, since the edges represent wall segments, and the vertices, junctions between those wall segments. There is no way in which it would make sense to speak of walls crossing each other, except at a junction.

There do, however, exist *nonplanar* graphs: no matter how their vertices are disposed on the plane of the page, or how circuitously the edges are drawn, there will still necessarily be edges which cross somewhere. Certain traditional puzzles from the literature of recreational mathematics depend

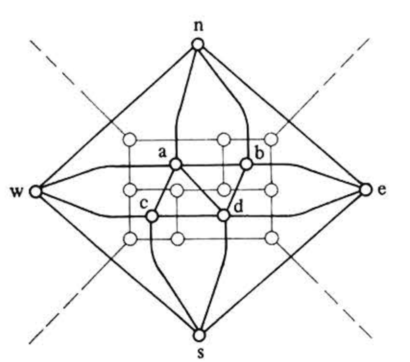
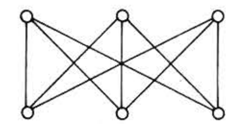


Figure 6.1. A *plan graph* (thin line) and its *augmented dual adjacency graph* (heavy line). Four exterior regions, n, e, s, w are separated by the four infinite edges attached to the vertices at the corners of the plan. Adjacencies of rooms to these regions, and of the regions to each other, are included in the adjacency graph.

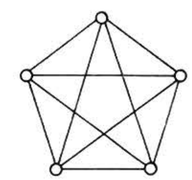
on this fact. They involve setting the reader, in effect, the impossible task of drawing nonplanar graphs without crossings.

One such puzzle is the 'utilities' problem, otherwise the puzzle of the houses and wells. In the latter version nine paths are to be drawn, connecting each of three houses to all of three wells, in such a way that no paths intersect. The corresponding (nonplanar) graph is:

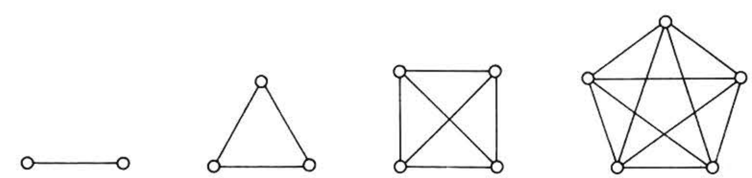


It is known as $K_{3,3}$, where the subscripts signify that three vertices are each connected to all of three other vertices.

A second problem, discussed by Möbius (see Biggs et al, 1976), involves dividing an inherited estate between five brothers such that each holding is adjacent to all four others; that is, a plan or map is required for which the adjacency graph has five vertices, with edges joining all pairs of vertices. This graph, also nonplanar, and referred to as K_5, is as follows:

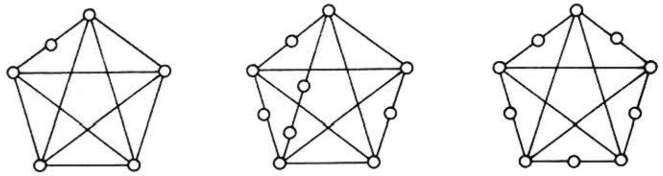


In general, graphs in which edges join all pairs of vertices, like K_5, are called *complete* graphs. All of the complete graphs K_p for values of p up to 5 are:



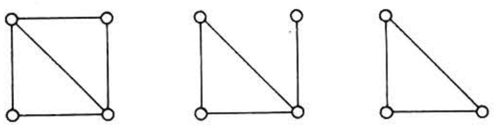
Notice that the complete graphs for p less than 5 are all planar.

The two graphs K_5 and $K_{3,3}$ have a special significance in the characterisation of planarity. To explain this, some further definitions are required. Two graphs are said to be *homeomorphic* ('of a similar form') if one can be obtained from the other through a process of subdivision of edges. This subdivision is effected by a series of progressive substitutions of one edge by a pair of edges plus a vertex:



The graphs illustrated are all homeomorphs of K_5. (It is as though 'extra' vertices were introduced along the existing edges.)

A graph is a *subgraph* of a second graph, if it has all of its vertices and edges in that graph. (The relationship is thus something like that of a subset to a set.) As an example, in the following figure the two graphs on the right are subgraphs of the graph on the left:



We are now equipped to state a theorem due to Kuratowski (1930) which defines the conditions for planarity in graphs: "A graph is planar if and only if it does not contain any subgraph homeomorphic to $K_{3,3}$ or K_5". So the presence of one or other (or both) of these types of subgraph is sufficient to render a graph nonplanar. And conversely, if a test shows that neither is present in a graph, then the graph is planar.

Now every plan graph has a planar dual: and no nonplanar graph can have a dual which is planar. We already know that any plan graph must be plane. It follows that any adjacency graph must in turn be planar, if it is to correspond to a plan. Or, looked at in another way, if a series of adjacencies are specified between rooms or regions, which together form an adjacency graph which is nonplanar (as in the problem of Möbius) then those adjacencies cannot all be realised in any actual plan.

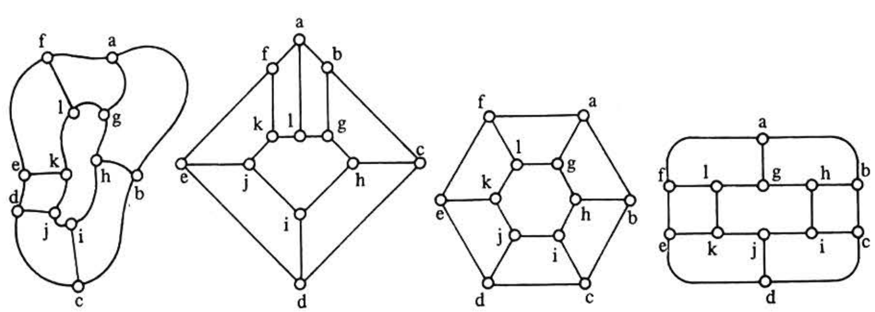


Figure 6.2. Four plans of differing geometry, but with the same (unlabelled) plan graph.

The architectural implication of the fact that K_5 is nonplanar is that in no plan is it possible to have five rooms all mutually adjacent. Four is the greatest number of rooms (whatever their shape, not just rectangular) for which this is possible. This follows since all complete graphs on more than five vertices contain K_5 as a subgraph; hence they too are nonplanar.

So far I have illustrated adjacency and plan graphs exclusively in relation to rectangular arrangements. It should be appreciated, however, that in neither type of graph is the fact of rectangularity of the plan represented as such. Figure 6.2 shows four plans, for example, one of curvilinear form, one with a 45° and 90° geometry, one with 'hexagonal' (60° and 120°) geometry, and one of rectangular form. If the labelling of the vertices is ignored for the moment, all four plan graphs can be said to be the same, in that there is the same number of vertices in all cases, and these vertices are always connected together in pairs in the same arrangement by the same number of edges. The graphs can be matched or 'mapped' one to another in exact correspondence; in technical terms they are said to be *isomorphic* ('of the same form'). The dual adjacency graphs of all four plans are also isomorphic (figure 6.3). (In the example of figure 6.2 the plan graphs can all furthermore be said to be embedded in the plane 'in the same way', in a sense which we will come to examine shortly.)

If, however, the labelling of the vertices *is* taken account of, we see that all four plan graphs are not now precisely the same; we can match edges, vertices, *and* labels only for the first, third and fourth graphs, but not any of these with the second graph. When speaking of isomorphism it is important to be specific therefore as to whether the graphs are taken as labelled or unlabelled.

If an adjacency graph is to correspond to a plan taking the form of a rectangular dissection say, then there are certain limitations which this will impose in turn on the form of the graph itself. Later we will come to look at the precise nature of these limitations.

Let us suppose for the moment, however, that we will allow plans without 'geometrical' constraints of any kind; that is to say, we place no

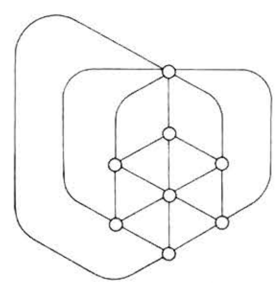
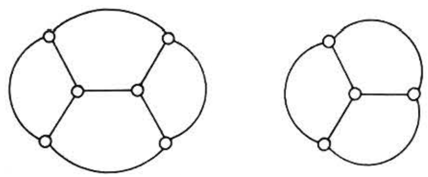


Figure 6.3. The adjacency graph which is the same for all the four plans of figure 6.2 (adjacencies to a single exterior region are included).

limits on the angles between walls in a plan, these walls can be curved or straight, and the rooms, and the exterior perimeter of the plan, may take curvilinear or polygonal shapes of any kind. Such plans can be conveniently drawn as packings of 'bubbles':



[which is not to say that real bubbles do not have very definite geometries (Almgren and Taylor, 1976)], and these bubbles, it must be imagined, can be squeezed and stretched as desired.

It is possible to imagine a process of design in which the architect would start with some set of spaces, together with some requirements as to which of these spaces should ideally be contiguous in a plan. He could represent the spaces by vertices, and the requirements for adjacency between rooms and to the exterior region or regions, by edges joining the relevant vertices. Together these would form what we might call an *adjacency requirement graph.* A hypothetical example for the ground floor of a small house is illustrated in figure 6.4. (It is assumed that this plan is four-sided and that the sides are oriented to points of the compass n, e, s, and w. There are four rooms: a living room l, a dining room d, a kitchen k, and a hall h.) The designer would then want to know what range of possible plan graphs—and ultimately plans—would correspond to, or satisfy, these stated requirements.

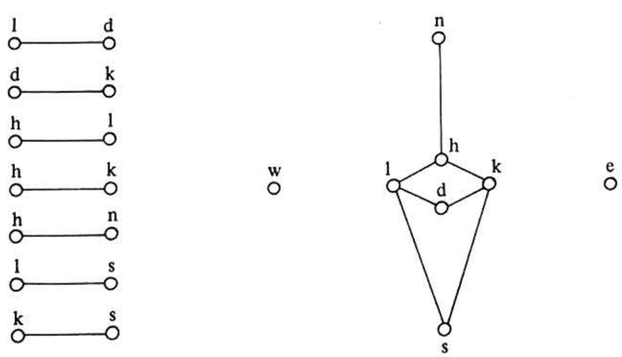


Figure 6.4. Adjacency requirements (left) for a small house plan with four rooms: a living room l, a dining room d, a kitchen k, and a hall h. The plan is assumed to be four-sided: adjacencies are specified to exterior regions at north n and south s. These requirements are then put together into the *adjacency requirement graph* (right).

I do not want to suggest that very many (if any) architects actually *do* design in this way. Most work directly on the geometry of the plan itself. Nor is it true, of course, that adjacency of spaces is the only constraint on the planning of buildings. But for the purposes of the present argument it is convenient to assume this rather simplified or idealised logical sequence of operations. (Amongst other reasons, this will throw light on issues arising in several of the graph-based design methods, to be discussed in chapter 9.)

The first point to notice is that this adjacency requirement graph is rather unlikely to be isomorphic to the adjacency graph of the final plan or plans. On the one hand, it is improbable that the architect will wish to specify the *entire* adjacency graph of the plan in advance. His requirements will generally be confined to those instances where rooms must be contiguous for definite functional reasons; most often so that direct access is possible via a door.

But in any actual plan there will be adjacencies between rooms occurring more or less fortuitously, which are not specifically required on functional or aesthetic grounds, but arise simply out of the packing of spaces into a compact whole. (There are economic reasons for plans to be compact certainly, including the reduction of constructional and heating costs; but these do not necessitate the *particular* adjacencies of specified pairs of rooms.) Here then, for any actual plan in which all the requirements are satisfied, the adjacency requirement graph will be a *subgraph* of the adjacency graph. An example for our house plan of figure 6.4 is illustrated in figure 6.5.

On the other hand, there is the possibility, as we have noted, that an architect might unknowingly stipulate a set of adjacency requirements which together turn out to constitute a nonplanar graph. In a graph-theoretic design method following this line of approach, it would therefore

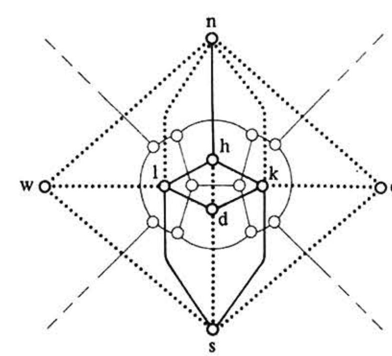


Figure 6.5. A plan (thin line) satisfying the adjacency requirements of figure 6.4. The adjacency requirement graph (heavy solid line) is a *spanning subgraph* of the adjacency graph (solid plus dotted lines). It contains all the vertices, but only some of the edges of the adjacency graph.

be wise to test any requirement graph for planarity before attempting to proceed further. Figure 6.6 shows how the addition of only one more requirement for adjacency to the graph of figure 6.4 serves to make it nonplanar (given that edges ne, es, sw, and wn are included). The requirement graph now contains $K_{3,3}$ as a subgraph.

Several techniques have been formulated as algorithms, and hence as computer programs, by which to determine whether or not a given graph is planar (Shirey, 1969) (although they are mostly rather complicated[(9)]). If the graph failed such a test, then some of the requirements would have to be abandoned, that is, an edge or edges would have to be removed from the graph, before it could correspond to the adjacency graph of any plan.

(There is the further possibility, if additional constraints are placed subsequently on the geometry of the plan or on the shapes and sizes of rooms, that even the requirements in a planar adjacency requirement graph may not be capable of satisfaction simultaneously with these other constraints; in which case the architect must make a choice of which type of requirement to retain and which to relax.)

Let us assume, however, that there exists at least one feasible plan, and that its adjacency graph contains the adjacency requirement graph as a subgraph; this could be one of several different kinds of subgraph. If a subgraph contains all of the vertices, but only some of the edges of a graph, it is said to be a *spanning subgraph*. This is the relationship of the two graphs in figures 6.4 and 6.5. In the architectural interpretation, this

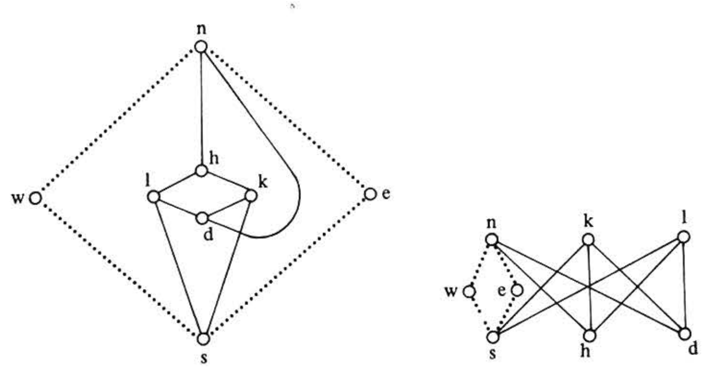


Figure 6.6. One more edge dn added to the adjacency requirement graph of figure 6.4 serves to make it nonplanar (given that the edges ne, es, sw, and wn are included). The graph is redrawn in a way which makes clear that it contains $K_{3,3}$ as a subgraph.

(9) Despite the apparent simplicity of Kuratowski's characterisation, such computer methods do not generally involve a search for K_5 and/or $K_{3,3}$, since this is difficult to program. They mostly employ techniques whereby some planar subgraph is found in the graph, and then other vertices and edges are systematically added back in, in such a way that planarity is preserved, up to the point where either the whole graph is represented in plane form, or else the procedure fails.

would mean that the architect specifies all of the rooms in a plan, but not all of their adjacencies.

Even this would not necessarily be the case. It might for example be that the architect would want to introduce new circulation spaces such as halls, lobbies, or corridors, during the design process. These spaces might not be strictly required in themselves, but are incorporated to meet certain needs, say, for indirect access between other rooms. Thus in a graph-theoretic representation, vertices would be added to the requirement graph and edges also added or rearranged (even removed), to produce the final adjacency graph (as illustrated in figure 6.7).

There are several other possible complications. In some instances the architect might be prepared to allow a series of *alternative* adjacencies: either room a must be next to room b, or else it must be next to room c. This simply means that there will be two, or more, equally acceptable requirement graphs. If it is stipulated that certain rooms must *not* be

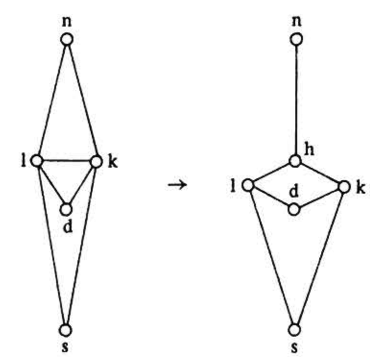


Figure 6.7. Example of a new vertex added to an adjacency requirement graph, to represent a circulation space h, with an accompanying rearrangement of edges.

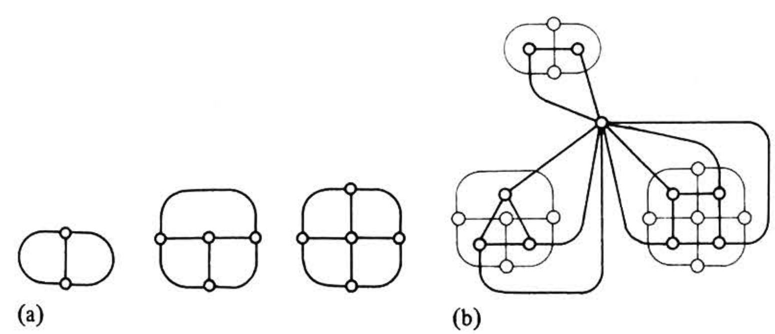


Figure 6.8. (a) A disconnected plan graph, representing the plans of three separate buildings and (b) the corresponding adjacency graph, which becomes connected by virtue of the inclusion of adjacencies to an exterior region.

adjacent, then effectively a second 'nonadjacency requirement graph' is needed.

In general we can say of plan graphs that they must be *connected.* A connected graph is a graph in which all pairs of vertices are joined by continuous sequences of edges and vertices (*paths*); that is, the graph is in one piece and does not fall into two or more separate *components.* A disconnected plan graph could only signify the plans of two or more separate buildings [figure 6.8(a)][10].

The adjacency graph of any plan must also be connected. [In fact even if the plan graph is not connected, its adjacency graph will be, since all the separate 'plans' must be adjacent to the single exterior region, and the corresponding edges will serve to connect the entire graph together—figure 6.8(b).] But this is not necessarily true of an adjacency requirement graph.

Consider, for example, the requirement graph of the ground floor of a block of flats, in which separate access is to be provided to each flat direct from the exterior (not from a common hall). We might suppose that the architect would set a series of requirements for the adjacencies of rooms within each flat, and that he would want to pack all the flats together into the single block; but that he would be largely indifferent as to which rooms of different flats happened to be adjacent across party walls. (Although in this particular example we can imagine that an architect might actually want to specify that certain rooms of different flats should *not* be adjacent across the party walls—for example, that a bedroom in one flat not be next to the living room in another.)

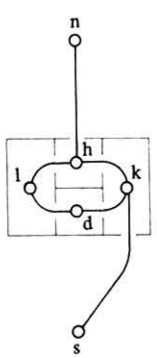


Figure 6.9. The *access graph* of a plan, in which vertices represent rooms or exterior regions, and edges signify the existence of doors or other direct means of access between those rooms or regions. The access graph must be a spanning subgraph of the adjacency graph of a plan.

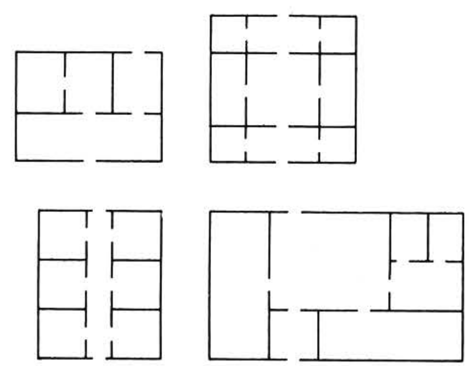
[10] Although, for example, the plan at ground level of a triumphal arch or any comparable bridge-like structure would have a disconnected plan graph.

Several authors have referred to a slightly different type of graph which also has its uses in design methods and in morphological studies (Levin, 1964; Hillier et al, 1978a). This is a graph in which the vertices signify the rooms of a plan, or exterior regions around a plan, as before, but in which each edge signifies the existence of a door or means of access between the two rooms or regions in question. It may be called the *access graph.* Figure 6.9 shows an imagined access graph for the plan illustrated in figure 6.5. Clearly the access graph must always be a spanning subgraph of the adjacency graph of a plan and will generally be a connected spanning subgraph. (Unless some secret rooms are completely walled in, or there is for example an inaccessible light-well.)

We could even define an 'access requirement graph' which would be a subgraph of the more general adjacency requirement graph—there being requirements for adjacency which might be stipulated for reasons other than for access, such as adjacencies of rooms to the exterior to allow for windows.

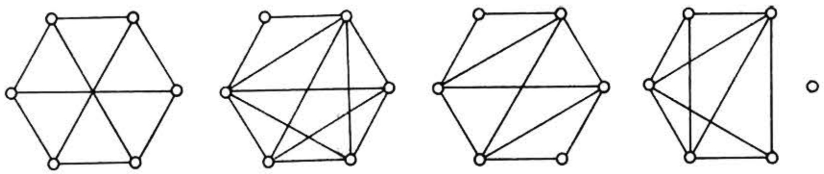
Exercises

6.1 For the plans illustrated (or any other plans with rectangular boundaries that you care to choose), draw in each case
(a) the plan graph,
(b) the weak dual adjacency graph,
(c) the augmented dual adjacency graph, and
(d) the access graph.
Check in each instance that the dual relationship exists between (a) and (c). (You will need to introduce infinite edges at the corners of the plan graph, to divide the exterior into four regions, and show edges in the adjacency graph, for adjacencies between these regions.)

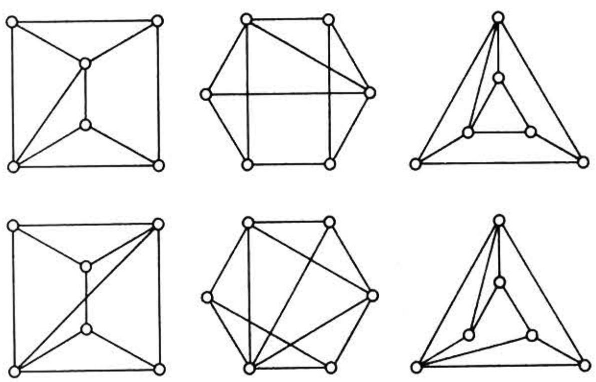


6.2 Draw all graphs on four vertices, including those which are not connected. (There should be eleven of them. You may find it convenient to classify them by their numbers of edges. It is possible to have a graph with *no* edges—just the four isolated vertices. At a maximum there can be six edges, connecting all pairs of vertices, in the *complete* graph.)

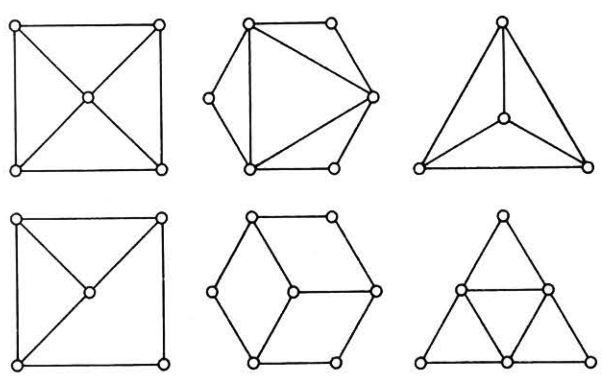
6.3 Which of the graphs on six vertices illustrated is planar, and which nonplanar?



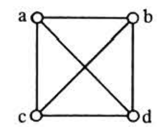
6.4 Which of the graphs illustrated are isomorphic? (A necessary—but not sufficient—condition for two graphs to be isomorphic, is that for every vertex in one graph, there exists a vertex with equal valency in the other graph, to which it can be matched. So a way to start checking for isomorphism is to label vertices with their valencies.)



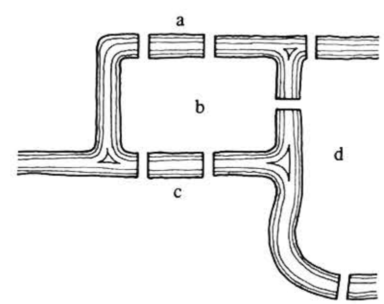
6.5 Which of the graphs illustrated are homeomorphic?



6.6 Draw all distinct possible spanning subgraphs of the graph illustrated. By 'distinct' is meant subgraphs which are not isomorphic. How many possibilities are there if, in determining isomorphism, (a) the labelling of vertices is ignored and (b) the labelling of vertices is taken account of?



6.7 The figure shows the plan of the city of Königsberg (now Kaliningrad), which lies on the river Pregel. There are seven bridges crossing the branches of the river as shown. The problem which Euler (1736) addressed was this: is it possible to take a walk around the city, crossing each bridge once only, and return to one's starting point? Euler showed that the answer is no. Can you show why? Start by drawing a graph in which the four distinct parts of the city are represented by vertices, and the bridges by edges. (This is not strictly a graph proper, since there is more than one edge joining the same vertices in some cases. It is a *multigraph*—see chapter 7.) (Hint: consider the valencies of the vertices.)



6.8 Draw the weak dual adjacency graphs of all rectangular dissections for $n = 4$. Imagine a set of four rooms with distinct functions, labelled α, β, γ, and δ. Choose some set of required adjacencies between those rooms. Draw the adjacency requirement graph. In how many of the dissections can these requirements be satisfied? (That is, in how many cases is the requirement graph a spanning subgraph of the adjacency graph of the dissection?)

For each dissection label the rectangles arbitrarily a, b, c, and d. In every dissection where the same adjacency requirements as before *can* be satisfied, how many distinct possible permutations of position of the room functions are there? (In how many different ways, that is, can α, β, γ, and δ be assigned to a, b, c, and d such that the adjacency requirements are met?)

By this process you will be able to determine all possible plans of rectangular dissection type, which meet the given requirements—without regard, of course, to any possible *dimensional* requirements which might then be imposed.

6.9 Carry out the same exercise as 6.8, but this time draw the augmented dual adjacency graphs of the dissections, and introduce some set of required adjacencies of the four rooms to the four exterior regions on the north, east, south, and west.

You could also elaborate the exercise further by specifying that certain adjacencies, between pairs of rooms or between rooms and certain exterior regions, should *not* occur. In this case it would be convenient to draw both a graph of adjacency requirements, and a graph of adjacencies not required—a 'nonadjacency requirement graph'.

7

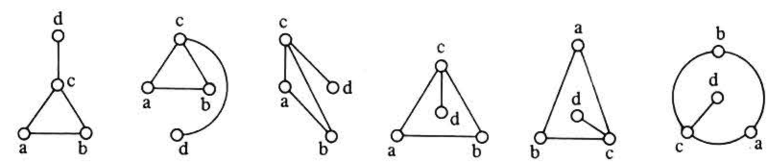
Embedded, coloured, and weighted graphs of plans

"Since much of design, particularly architectural and engineering design, is concerned with objects or arrangements in real Euclidean two-dimensional or three-dimensional space, the representation of space and of things in space will necessarily be a central topic in a science of design."

Herbert Simon (1969)

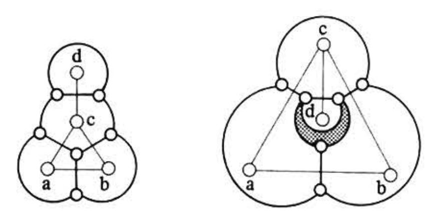
To come back to our imaginary architect of the last chapter, and his graph-theoretic method in design. He is working with an adjacency requirement graph (not just requirements for access). Let us suppose that he has tested this graph for planarity and found that it is planar. The next problem that arises, is that there may be several distinct ways of *embedding* this graph in the plane.

For example, take the following very simple requirement graph on four vertices, which is drawn six times, but in only two different embeddings:



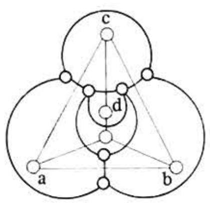
In the first three drawings the vertex d is outside the triangular face abc, and in the other three it is inside. The one class of embedding is distinguished from the other by virtue of this difference; it is as though the edge dc were folded over on the 'joint' represented by the vertex c.

Imagine what this would imply for the plan graph. We will draw the faces (rooms) as curvilinear 'bubbles' as described in the last chapter. We assume that all the four vertices in the requirement graph represent rooms and none of them represents the exterior region. We assume also that no more rooms will be added to the plan beyond these four. There are then two distinct plan graphs, corresponding to the two embeddings of the requirement graph—although exactly the same adjacency requirements are satisfied in each case:

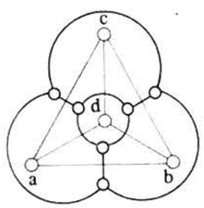


See how the adjacency requirement graph must strictly be interpreted, in the embedding where room d lies inside the triangle of rooms a, b, and c.

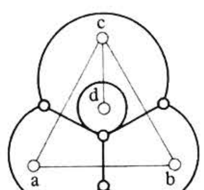
Room d is required only to touch room c, but it is not required to touch a or b. There are various alternatives here. It is possible that there might be a hole—perhaps a courtyard—in the plan (shown shaded in the above figure). If the adjacency graph for this 'courtyard plan' is drawn it ought strictly to include a new vertex to represent the courtyard, however, plus three new edges to represent the adjacencies of a, b, and d to the courtyard:

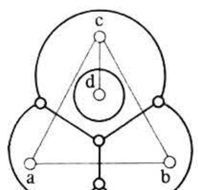


On the other hand the architect might not want such a courtyard, and could extend room d to fill the hole and so make it adjacent to a and b. Here then there would be two new adjacencies ad and bd not expressly required, but introduced because of the constraints of close-packing:



There are two further, slightly weird and unrealistic interpretations which are possible for the embedded requirement graph:

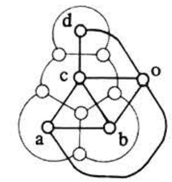


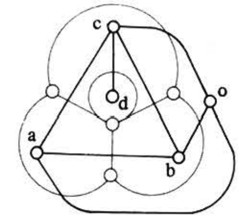


The first is that room d is almost completely engulfed inside room c; thus d touches a and b only at a single point (a five-way junction) which does not count as adjacency. The second possibility is that room c is ring-shaped and completely surrounds d. In either case the plan has no hole, all the required adjacencies are satisfied, and no other adjacencies occur. However, these are rather improbable kinds of plan in architectural terms.

In the full adjacency graphs of the plans corresponding to the two embeddings of the requirement graph, the adjacencies of all rooms to the

exterior region should also be shown:





In the first embedding all four rooms are adjacent to the exterior, whereas in the second only three are, and the room d is enclosed in the interior of the plan.

We can look at this subject of embedding in a more general and systematic way. Let us consider adjacency requirement graphs for the moment which are connected, planar, and in which the n vertices represent rooms in a plan (not the exterior region). We can ask "What are all the possibilities for the embeddings of such graphs, for successive values of n?" That is to say, we can take an exhaustive approach to the enumeration of this class of graphs, rather in the same way that it is possible to take an exhaustive approach to enumerating possible dissections, or possible polyominoes. The general problem of counting planar graphs remains unsolved. (Essentially the difficulty lies not in devising some generating process, but in recognising and removing isomorphs.) The more particular problem of counting planar embeddings *has*, however, been solved. For small n it is possible to enumerate all cases by inspection. Figure 7.1 is adapted from tables published by Korf (1977), and by March and Earl (1977), and illustrates possible embeddings of all connected planar graphs up to $n = 5$. (The two embeddings of the graph of the figure at the beginning of the chapter appear in the fourth row of possibilities for graphs on four vertices.)

The reader is invited to draw the plan graphs corresponding to some of these cases, so as to gain a feeling for the dual relationships involved. These plan graphs are technically speaking *maps* in graph-theoretic terms. A map is a connected plane (that is, embedded planar) graph together with all its faces. Obviously in its interpretation as a real, cartographic map, the faces are countries or regions, and the edges their boundaries; where in architectural terms they are, respectively, rooms and walls.

If it were adjacency *requirement* graphs which were being considered, then it would be quite permissible to add further edges and vertices, to produce adjacency graphs proper. This could be done in each case in very many different ways. Let us for the moment imagine, however, that these graphs in figure 7.1 represent as they stand the weak duals of plan graphs.

Notice some general features. A face in the embedded adjacency graph implies a junction in the plan at which as many wall segments come together as there are edges surrounding the face. Thus in the simplest cases the 'triangular' adjacency graph for $n = 3$ implies that the three rooms in the plan meet at the centre in a three-way junction, the 'quadrangular' graph for $n = 4$ implies a four-way junction, the 'pentagonal'

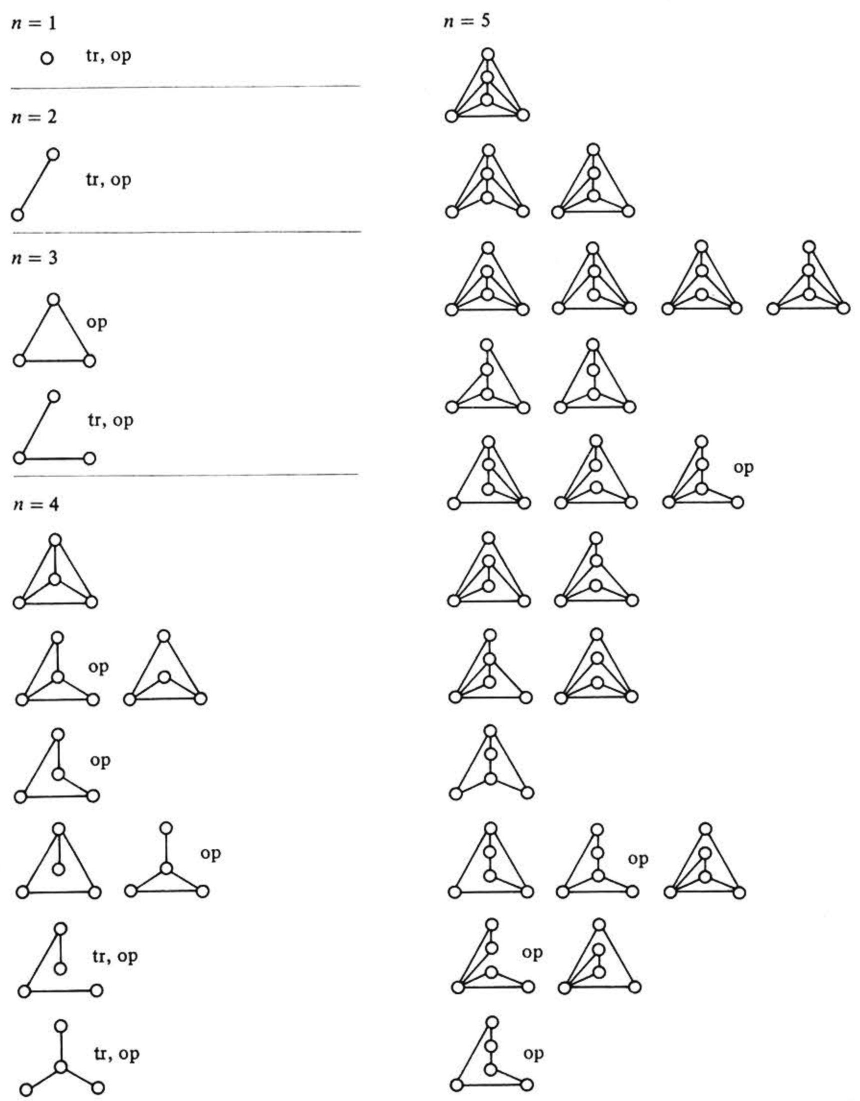
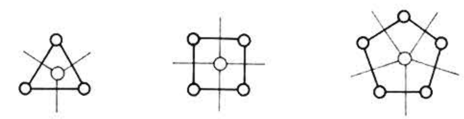
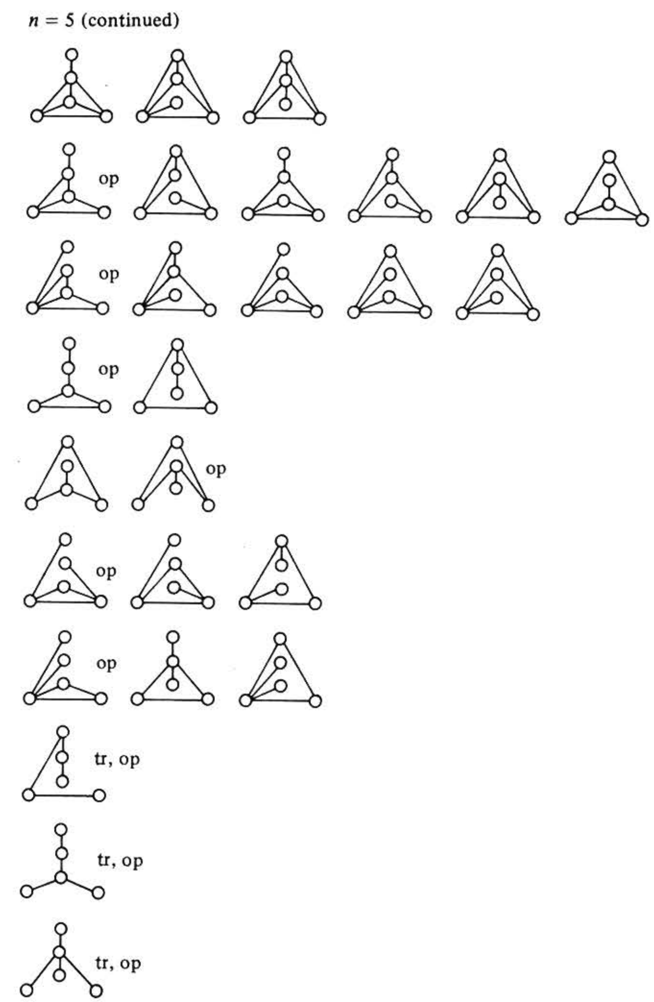


Figure 7.1. Possible embeddings of all connected planar graphs up to $n = 5$ [adapted from Korf (1977) and March and Earl (1977)].

graph for $n = 5$ a five-way junction, and so on:



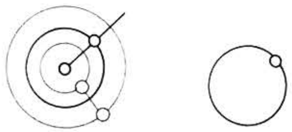


tr indicates a tree
op indicates an outerplanar embedding

Figure 7.1 (continued)

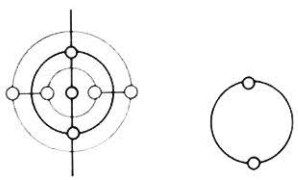
Many of the graphs in figure 7.1 represent plans in which either there are 'holes', or else interior rooms are engulfed inside others, as explained.

It should also be noted that there are certain adjacency relationships which can arise in plans, indeed which occur quite frequently in real buildings, but which are not represented by *any* of these adjacency graphs. It is possible that a room might be *adjacent to itself*, as for example, when it completely surrounds a second room in a ring broken by a partition wall:

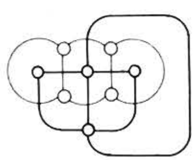


The room is then adjacent to itself across this partition. In the adjacency graph the relationship would have to be represented by a *loop*.

A second possibility is that two rooms might be adjacent across two (or more) distinct sections of common wall. Such a situation is where two rooms surround a third in a ring:



In the adjacency graph this corresponds to the presence of *multiple edges* between the respective vertices. As an aside, any room which touches the boundary of the plan along more than one section of wall, that is, a 'through-room', will have a similar adjacency relationship to the exterior region, which should also be represented by multiple edges:

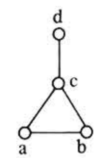


In graph theory, 'graphs' are conventionally defined to exclude loops and multiple edges; but adjacency graphs on this definition are then inadequate to represent such relationships in plans. Graphs in which multiple edges *are* allowed are properly called *multigraphs*, and those in which multiple edges *and* loops are allowed, *pseudographs*. The problem is, for enumerating possible adjacency graphs, that in principle any number of loops or multiple edges may be added to each vertex or pair of already joined vertices, without limit. There is a strong case nevertheless for regarding pseudographs, or at least multigraphs, as a more appropriate

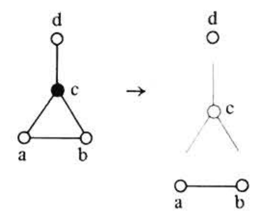
form of representation of possible adjacency relations in architectural plans, than the comparatively limited class of 'graphs' proper.

With this limitation in mind, and coming back to the (true) graphs of figure 7.1: see how the different embeddings are created in each case by the ways in which parts of an adjacency graph can be 'hinged' or 'folded' over to lie in its different faces (including the exterior face). The possibilities in each case here are determined by the *degree of connectedness* of the graph.

With the use of the following example once again; this graph is said to be *1-connected*:

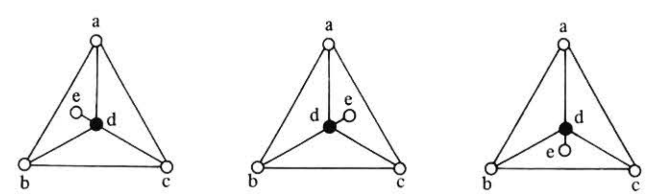


The removal of one vertex (vertex c in this case) together with all edges incident with that vertex, results in the graph falling into two disconnected components (the edge ab and the isolated vertex d):

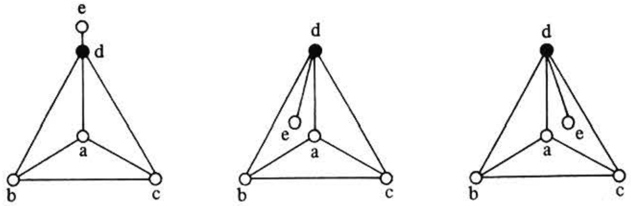


The vertex c is called a *cut vertex*. In the embedding of graphs, any subgraph which would become a disconnected component by the removal of a cut vertex in this way, can be embedded in any of the faces on whose boundary the cut vertex lies.

Thus in this next example:

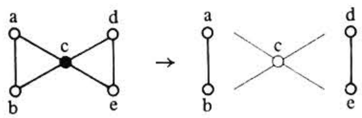


the cut vertex is at d, and the vertex e together with the edge de can be embedded in any of the triangular faces abd, acd, bcd. If the vertices were not labelled these three alternatives would be isomorphs, because of the symmetry of the situation. Suppose the whole of the same graph is reembedded such that b, c, and d lie on the outer boundary, then e can be embedded either in the exterior face, or else in abd or acd (the last two again being isomorphic if the graph is unlabelled):

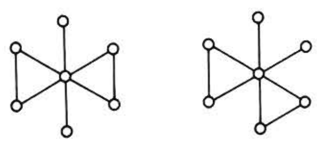


(The graph is that represented in the top row of graphs on five vertices on page 83 in figure 7.1, where all these possible embeddings are accounted for.)

Notice that with 1-connectedness we are not just referring to cases like those so far illustrated where a single vertex is attached to a larger graph by a single edge. For example, in the following figure:

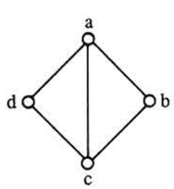


the graph is 1-connected—c is the cut vertex—and the triangular faces abc and cde can be folded at the hinge represented by c, either inside each other, or into the exterior face. The graphs in the next figure illustrate another possibility arising in the embedding of 1-connected graphs, in which a number of 'arms' radiate from a single cut-vertex:



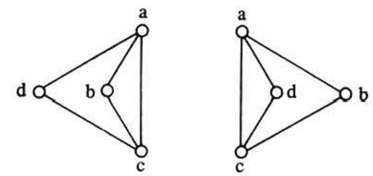
These 'arms' can be arranged around the vertex in different cyclic orders, like the order in which the hands of a clock might occur around the clock-face. Embeddings in which this order is different should be taken as distinct.

A second kind of situation which can give rise to a multiplicity of embeddings is illustrated by the graph



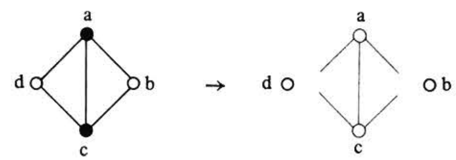
In this case we have the embedding indicated in the figure in which a, b, c, and d all lie on the exterior face. Or we might fold b together with ab and bc over on the pair of hinges a and c, so as to lie inside the face acd.

An exactly symmetrical embedding is achieved by folding over d, with ad and cd, so that they lie inside abc:



If the graph were unlabelled these last two would be the same embedding.

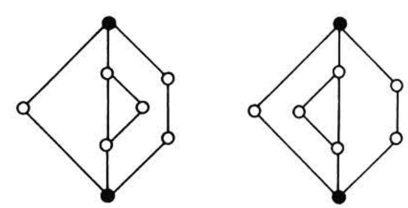
In this case the graph is *2-connected*, as it requires the removal of two vertices (with their incident edges), at a minimum, to disconnect the graph:



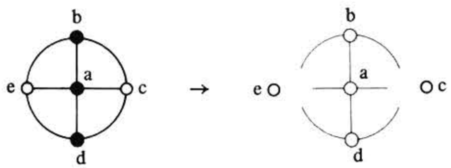
The vertices in question here are a and c, the 'hinges'. If we look at which faces of the graph have these vertices in common, we see that all three do: abc, acd, and abcd. Considering one of the 'folded' subgraphs—let us take b with ab and bc—then there are two possible embeddings of this subgraph (inside or outside acd), that is, one less than the number of faces on which a and c lie.

In general, where a subgraph can be disconnected from a graph by the removal of a pair of vertices, if those vertices are shared by f faces, then the subgraph can be embedded in $f-1$ ways (some of which might be symmetrically equivalent).

There is no necessity for the 'hinge' vertices to be adjacent. A whole section of a graph might be capable of hinging about two widely separated vertices. A 'hinged' subgraph of this kind, embedded in a particular face, could furthermore be turned over or mirrored *within that same face*, so as to give two potentially distinct embeddings:

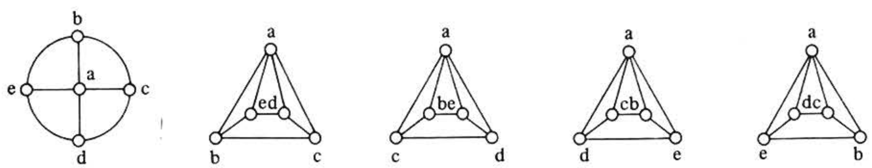


Extending the idea of connectedness one step further: a graph in which three vertices and their incident edges must be removed to disconnect it, will be termed *3-connected*. One of the very simplest of 3-connected graphs is



For reasons which are obvious from the way it is drawn here, it is called a *wheel.* This is the wheel on five vertices. See how the removal of any two vertices leaves the graph still connected. Only the removal of three vertices, with their incident edges, is sufficient to break it into two parts: two isolated vertices.

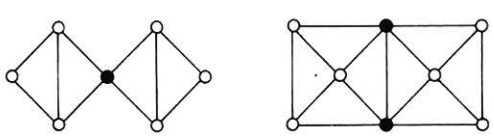
With this, as with all other 3-connected graphs, no possibility exists for 'folding over' or 'pivoting' parts of the graph in the embedding, as was the case with the 1-connected and 2-connected examples. The only alternatives available for distinct embeddings are those presented by the choice of different faces to form the exterior face. Given that the wheel of the last figure is labelled, there are five embeddings, in which either the single quadrilateral face, or else each of the triangular faces in turn, becomes the exterior face:



Once again these last four embeddings are distinguishable only by virtue of the labelling.

If we imagine the graph to be made of a net of elastic strings, it is as though we take each loop of string surrounding a single hole in that net (a face), stretch it out, and lay it down to become the outer edge so that the remainder of the net falls within that loop.

It can then be seen that the general problem of determining the possible embeddings of any given planar connected graph is liable to be a complicated business (compare Korf, 1977). A graph as a whole might be 1-connected; but it could contain subgraphs which in themselves are 2-connected or 3-connected. An example of a graph which is 1-connected, but where the cut vertex is common to two subgraphs which are themselves 2-connected, is the first of

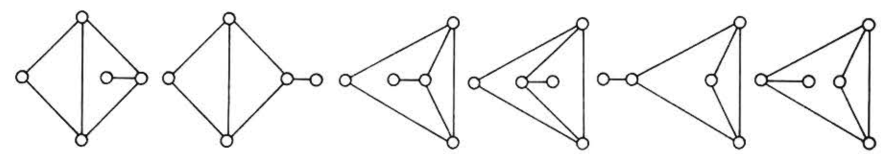


The second shows a 2-connected graph, in which the two 'hinge' vertices are common to two 3-connected subgraphs (both of them wheels on five vertices).

A maximal n-connected subgraph of a graph is called an n-component. The first graph in the above figure contains two 2-components, and the second graph two 3-components. A graph might well consist of some combination of 1-component, 2-components, and 3-components.

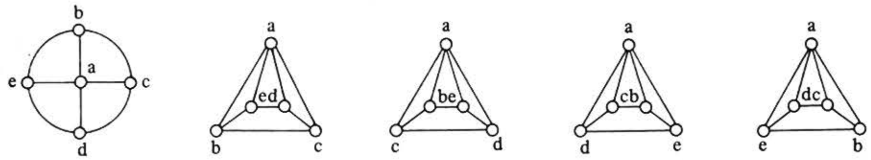
To find all embeddings of a given planar graph involves then essentially isolating each 3-component (if any exist) and taking each of its faces in turn as the exterior face (that is to say, exterior to that component considered in isolation). It is further possible that one entire 3-component might be capable of being, or might be required to be embedded inside one face, or in one of several alternative faces of another embedded 3-component—the possibility depending on the patterns of connectivity between these components. All permutations of possibility here must be accounted for. Then any 2-components must be folded about their 'hinge' vertices into different faces, and (where applicable) mirrored in all those faces, in all distinct possible ways. Finally, every 1-component must be embedded in each of the faces on which its 'hinge' or cut vertex lies.

In many cases this might be a matter of 'hinges on hinges', or 'folds within folds', so to speak. Thus in the following (relatively simple) example:



a 1-component hinges on a 2-component; and we must consider all permutations of embedding of the one with the other. Which of the resulting embeddings in the general case should be regarded as distinct depends, first, on whether the vertices are labelled or not, and, second, if the graph is unlabelled, on the symmetries of the situation. Considering embeddings of entire graphs; there may exist pairs of embeddings where one is a symmetrical reflection of the other, and which differ only by virtue of that reflected symmetry. Whether these are regarded as distinct or not is a matter of convention.

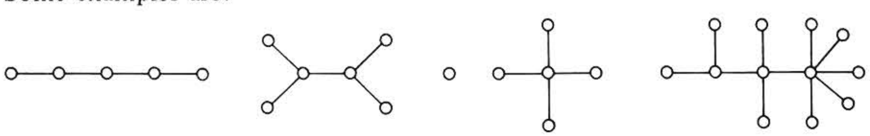
In general, there must be at least as many distinct planar embeddings of a labelled graph, as there are faces in it. In practical architectural terms the choice of a face in a (weak dual) adjacency graph to become the exterior face implies that all those rooms represented by the vertices belonging to that face—and only those rooms—will lie on the outside perimeter of the plan. They will form a continuous 'ring' around all other rooms, which will thus be completely internal and cannot have side-windows or direct access from the outside. With the graph of the figure



which first appeared on page 88, the five embeddings illustrated correspond to plans in which either one or two rooms lie in the centre of the plan, and all others on the exterior, as indicated.

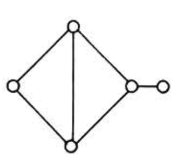
However, if we complete the adjacency graph by including a vertex (or several vertices) explicitly labelled to represent the exterior region (or regions) around the plan, then this has clear implications for possible embeddings. Obviously only those embeddings in which the relevant vertex or vertices lie on the exterior face will be candidates for consideration; and the total number of allowable possibilities will be substantially reduced. Thus in the graph of the above figure, if e is to represent the exterior region, then only three of the five embeddings are acceptable.

One special kind of graph which we will have occasion to refer to in later chapters is the *tree*. A tree is a (connected) graph with no *cycles*. Some examples are:

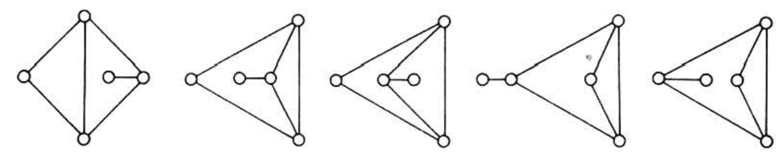


A cycle in a graph is a continuous sequence of vertices and edges, containing at least three vertices, which does not double back on or intersect itself, and which returns to the original vertex. A tree is thus necessarily 1-connected, and when embedded has only one face, the exterior face. All of its vertices lie on this face. Notice that it is nevertheless possible for a tree to have several distinct embeddings, since it may be that its various 'branches' are capable of being arranged in different order around a common vertex, in a manner similar to that illustrated in the third figure on page 86. All trees on n vertices up to $n = 5$ appear in their appropriate positions in the enumeration of embeddings in figure 7.1, and are labelled there as such.

Besides trees, there are other graphs which can also be embedded such that all vertices again lie on the exterior face. All of these (including trees) are known as *outerplanar* graphs. An example of an outerplanar graph containing cycles is



It is illustrated here in an outerplanar embedding—although other embeddings with vertices lying in the interior may be possible—compare the following figure:



It is possible that there might be several distinct outerplanar embeddings of the *same* outerplanar graph, by virtue of there being different cyclic orderings around cut-vertices of arm-like or branch-like parts (as with trees). Architecturally, an outerplanar embedding will correspond to a plan without any internal rooms. If an adjacency requirement graph is *not* outerplanar, then this means that one or several rooms *must* be internal—although it is not necessarily determined which particular rooms these must be. All outerplanar graphs on n vertices up to $n = 5$, and their embeddings, are also marked in figure 7.1.

In general though, as will have become plain, the process of taking some adjacency requirement graph, embedding it in all possible ways, and then adding edges (and possibly new vertices) to produce an adjacency graph from which the dual plan graph can be derived, can become a very laborious and elaborate one. This complexity is reflected in several of the design methods which are described in chapter 9.

Primary plans and their adjacency graphs

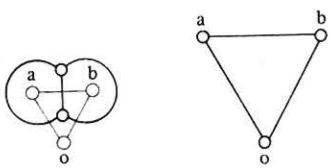
However, there is a converse way of looking at the whole subject, which avoids much of the messiness created by the problem of alternative embeddings. Imagine an adjacency graph to which no more edges can be added in any position without rendering the graph nonplanar. Such a graph is said to be *maximal planar*.

It is the dual of a plan graph in which the adjacencies between rooms (including adjacencies to the exterior region) are at a maximum. It is only possible to *reduce* the total number of adjacencies from this level, through operations which effectively involve the removal or repositioning of edges in the adjacency graph.

March and Earl (1977) have suggested that the set of plans—by which they mean distinct embeddings of plan graphs—in which rooms are maximally adjacent in this way, be considered as *primary*. *All* other plans can then be obtained from the primary plans by processes of what they term *ornamentation*. Through ornamentation, either adjacencies between rooms or regions are lost, or else a room which is adjacent to another (or the exterior) across one wall segment may become adjacent to the same room or region across a second wall segment, that is, multiple edges are introduced into the adjacency graph.

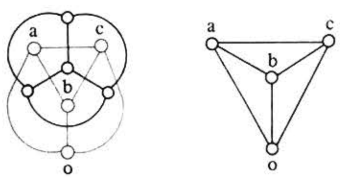
In a maximal plane adjacency graph (that is, an embedded maximal planar adjacency graph) every face is a triangle. This means that in the corresponding primary plan every junction between wall segments is three-way. Junctions in which more than three wall segments meet together imply the loss of adjacencies, by comparison with the primary plan. Such junctions can be created by 'ornamentation', as we shall see.

The simplest primary plan of any interest is that with two rooms:

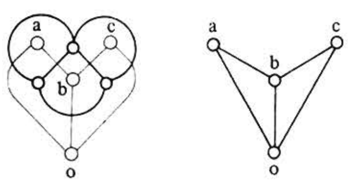


The adjacency graph includes a third vertex for the exterior region, and together the adjacencies make a triangle. There is no possibility of a graph on three vertices becoming nonplanar of course. But this triangle—the complete graph on three vertices—represents the maximal number of adjacencies possible. (It simultaneously represents the minimum number of adjacencies in this case, since the adjacencies of the rooms to the exterior region must occur in the nature of the situation, and if the adjacency between the two rooms were not to occur, then the plan would be disconnected—which is not allowed.)

The (only possible) primary plan for three rooms is:

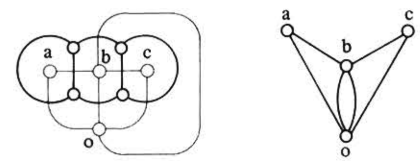


The rooms are all adjacent to each other and to the exterior region, and the adjacency graph is the complete graph on four vertices this time, which when embedded has four triangular faces (including the exterior face). The following shows one possibility for ornamentation of this primary plan, with a consequent reduction in the number of adjacencies:



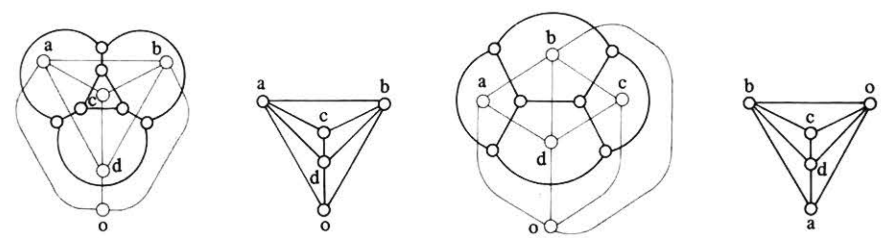
If one interior edge (or wall segment) in the plan graph is removed, as indicated in the figure, the adjacency is lost between two rooms (a and c as they are labelled), and two three-way junctions are 'coalesced' into a single four-way junction.

This four-way junction in the plan graph may then be 'expanded' again into two three-way junctions by the reintroduction of a new wall which connects the plan in a different way:

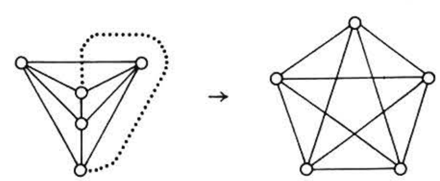


The centre room (labelled b) is now adjacent to the exterior across two walls, and this implies two edges joining the relevant pair of vertices in the adjacency graph. The two successive ornamentation operations described account for all possible plans (besides the primary plan) with three rooms (excluding, that is, plans in which multiple adjacencies occur *between rooms*, or in which rooms are adjacent to themselves—all of which could be created by further ornamentation).

When we come to four rooms, there are two primary plans:



The interesting thing is that these share the *same* embedding of the *same* adjacency graph. (It only has this one embedding.) The difference in the plans arises out of the choice of vertex to represent the exterior region. It may be a vertex with valency three or with valency four, in which case either three rooms or four rooms are on the exterior of the plan as shown. See how the addition of one more edge to this adjacency graph (in the only available position, joining the two vertices of valency three) would turn it into K_5; thus the graph must be maximal planar:



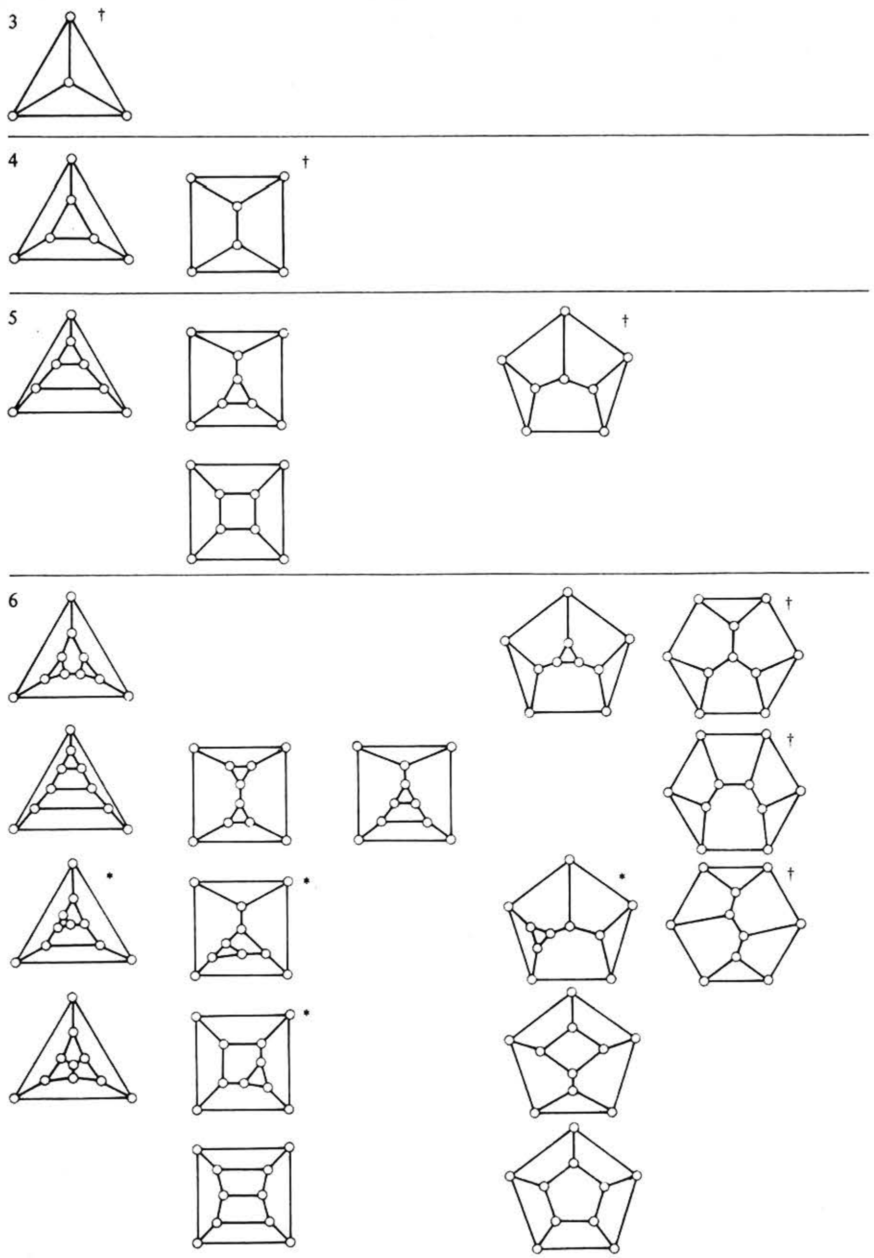


Figure 7.2. *Primary plans* with up to six rooms (left) and their adjacency graphs (right) (from March and Earl, 1977). Each adjacency graph is shown in all its possible embeddings. Notice that the correspondence between plans and graphs is not one-to-one, since the choice of vertex in the adjacency graph to represent the exterior region

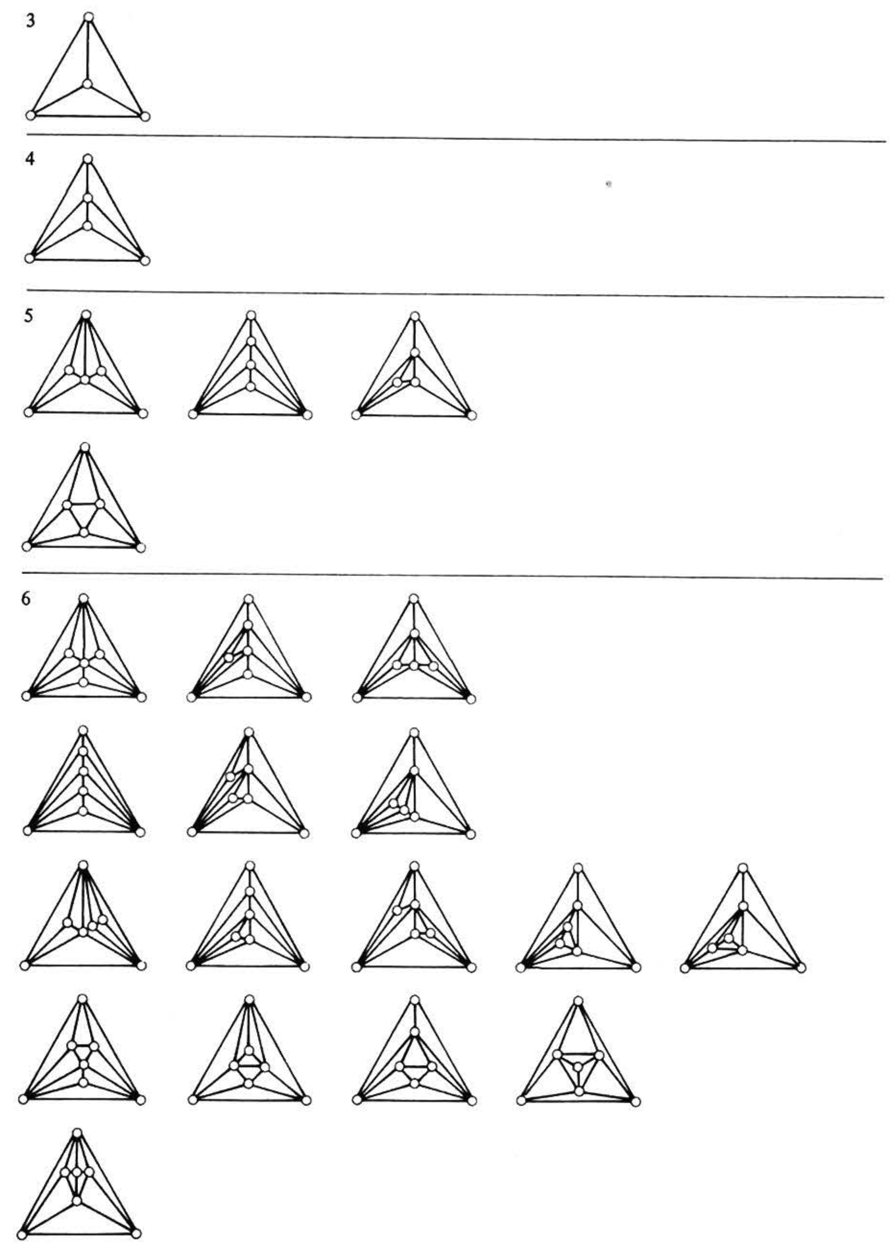
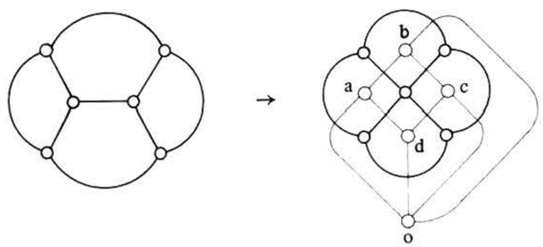


Figure 7.2 (continued)
may give rise to different plans; and conversely the same plan may be produced by suitable choices of vertex for the exterior region in different adjacency graphs. Plans marked * have distinct isomorphs by reflection. Plans marked † have no internal rooms.

The next figure shows the primary plan with four rooms adjacent to the exterior:



Ornamentation by the removal of one edge as shown has the result of making the four rooms meet at a single four-way junction. The adjacency graph correspondingly possesses one less edge than the adjacency graph of the primary plans, and has become the wheel on five vertices.

Figure 7.2 is taken from March and Earl (1977, figures 2 and 3) and shows all primary plans together with their adjacency graphs, up to six rooms. The figure is arranged to show on each row the various possible embeddings of each adjacency graph, and opposite these the corresponding possibilities for plans. The correspondence is not one-to-one, since the choice of vertices in the adjacency graphs to represent the exterior region *may* give rise to different plans; and conversely the same plan *may* be given by suitable choices of vertices for the exterior region in different adjacency graphs.

What is the significance of considering only maximal planar adjacency graphs in this context? First, there are rather few of them, compared with planar graphs generally. Furthermore it can be shown that every maximal planar graph with four or more vertices is 3-connected (Harary, 1969). This means that the task of enumerating possible embeddings of such graphs is a relatively straightforward one. Every such graph has at a maximum only as many distinct embeddings as it has faces—each of these faces in turn being taken as the exterior face, as explained—and if unlabelled it may very well have fewer embeddings, as a result of symmetries. These facts are illustrated in figure 7.2, where the numbers of distinct primary plans for $n = 3, 4, 5$, and 6, and their corresponding embedded adjacency graphs are evidently few in number, and growing with increasing n at a rather gentle rate. Compare, for example, the single maximal planar adjacency graph on five vertices in this table, with the fifty embedded planar connected adjacency graphs (all of them spanning subgraphs of the maximal planar graph) illustrated in figure 7.1[11].

[11] The graphs in figure 7.2 include adjacencies to the exterior region; whereas we were taking those in figure 7.1 to be the weak duals, without the exterior represented. Nevertheless we can make this direct comparison if we regard one vertex in each of the graphs of figure 7.1 to represent the exterior, instead of a room.

Again, since these adjacency graphs are maximal planar, they correspond as described to plans with the highest degree of adjacency, the greatest potential connectivity between rooms which is topologically feasible. There is no point in an architect or his client idly setting 'requirements' for adjacency in excess of these limits, since they can never be satisfied in any actual plan. Seppänen and Moore (1970) remark on the fact that as larger primary plans are considered, the average face (that is, room) approaches to being six-sided. (It is adjacent to six others.) This does not mean, of course, that every face must be hexagonal; rather that if the number of edges in a face is averaged over all faces, then the resulting value tends to six. This is a fact which will have significance for some empirical studies of actual building plans—and other spatial patterns—which we will come to in later chapters.

As a matter of fact even the primary plans can, in turn, be related to a still smaller number of yet more fundamental 'objects', as March and Earl demonstrate. So far we have considered the embedding of graphs in the plane, since it is in this form that they can be interpreted as architectural plans. It is quite possible, however, to embed graphs in other kinds of surfaces, and specifically of interest here, to embed them on the *sphere*. We can envisage this as being like tightening the graph, like a string bag, over the spherical surface.

It has been shown by Whitney (1932) that every 3-connected planar graph has a *unique* embedding on the sphere. The relation of the embedding of such a graph on the sphere, to its possible embeddings in the plane, can be imagined in the following way. The sphere is placed above the plane, and then the vertices and edges are *projected* down, from the spherical surface onto the plane surface (figure 7.3). Perhaps the easiest way to think of this, is to imagine the sphere to be transparent, and that a point source of light is placed above the sphere, shining down

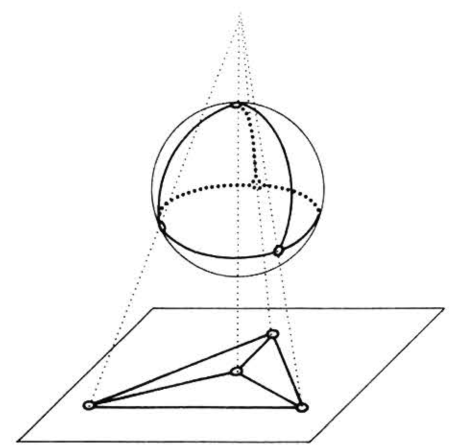


Figure 7.3. A graph embedded on a sphere, and its projection onto the plane.

on the graph as it lies on the sphere and casting its shadow onto the plane. Less fancifully, it is the kind of stereographic projection which is employed in map-making (from the curved surface of the earth onto the flat sheet of the map), or which underlies perspective drawing.

The graph as it is embedded on the sphere, may be projected down in different ways *through each of its faces* in turn. Thus the various possible embeddings of a 3-connected graph in the plane represented by taking each face in turn as the exterior face, correspond to the possible ways of projecting the single embedding on the sphere, through each of its faces.

Similarly, the dual of an adjacency graph, that is the corresponding plan graph itself, can be conceived of as a graph embedded on the sphere, and projected down through each of *its* faces (rooms or regions) so as to give all the possible primary plans.

The embedding of the graph of the plan on the sphere can in its turn be imagined as the skeleton of edges of a solid polyhedral figure. (It is as though each curved face of the graph on the sphere were pressed flat.) Every vertex of this solid must be trivalent, that is, it is the meeting point of three faces. In this way March and Earl establish a correspondence between a class of solid forms—trivalent polyhedra—and primary plans. Because to each polyhedron there corresponds, in general, several primary plans, the number of polyhedra is smaller than the number of plans. The total number of such polyhedra with between four and seven faces, is

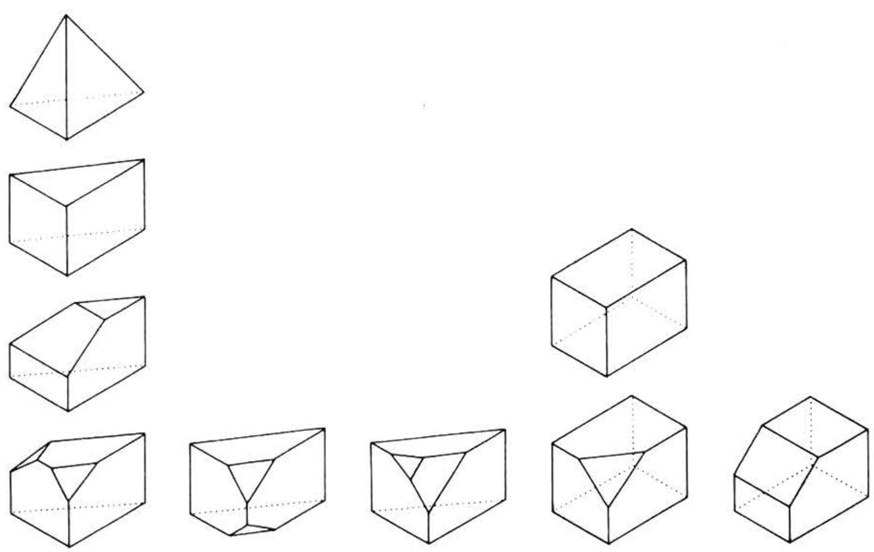
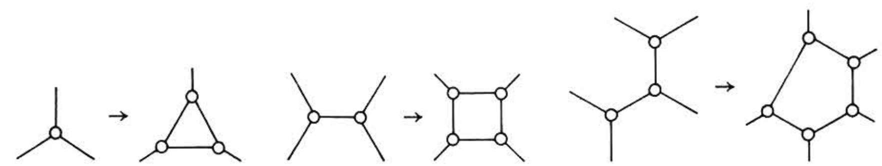


Figure 7.4. All trivalent polyhedra with up to seven faces (from March and Earl, 1977). These can be put into one-to-one correspondence with what March and Earl call *fundamental* architectural plans.

only nine, all of them illustrated in figure 7.4. March and Earl refer to them in this architectural interpretation, as *fundamental* plans.

March and Earl themselves mention how many writers on architecture and the arts, from Dürer to Buckminster Fuller, have been fascinated by polyhedral forms and their properties, and have held them out as perfect models of order and spatial organisation in design. Any connection with the actual design of buildings has often seemed more of a metaphorical one, however, until this work of March and Earl in which a mathematical mapping, albeit a many-stage and in some cases complex one, is established between the class of trivalent polyhedra, and all possible architectural plans.

It is known that, starting from the smallest trivalent polyhedron, the tetrahedron, all others with larger numbers of faces can be generated by combinations of three and only three operations (Eberhard, 1891). These operations can be likened to a process of shaving off parts of the solid with a carpenter's plane: planing off one vertex to produce a new triangular face, planing off two vertices and an edge to produce a new quadrilateral face, or planing off two edges and three vertices to give a new pentagonal face:



A systematic examination of the possible ways of making these cuts provides a means of enumerating trivalent polyhedra, and this has been done up to twelve faces. A general analytical solution to the enumeration problem has not yet, however, been found.

By limiting their attention to maximal planar adjacency graphs then, March and Earl manage to reduce the number of distinct possibilities under consideration to a much smaller number than that of *all* planar connected graphs on the same number of vertices. And because the maximal planar adjacency graphs are 3-connected, the question of enumerating possible embeddings is also much simplified and tidied up.

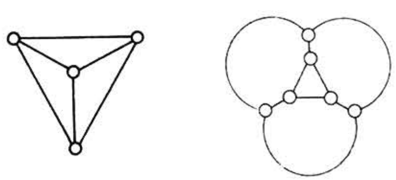
From the point of view of a general theory of possible plans, this work effects a considerable clarification and conceptual organisation of the subject as a whole. From a practical or design standpoint nevertheless, where the purpose is to arrive at some particular plan conforming to some particular set of adjacency constraints, it should be appreciated that the complications which we looked at earlier, in the possibilities for adding edges to and embedding an adjacency graph, have by no means disappeared here, but have simply been transferred into the very extensive and potentially complex possibilities for 'ornamentation' of the primary plans.

Coloured graphs of plans

Up to this point we have been looking at plans considered as graphs, without any constraint placed on the shapes and sizes of rooms. In a graph proper—either an adjacency graph or a plan graph—it is only relations of a topological nature which are expressed. However, in a graph-theoretic approach to design, our imaginary architect would want to go on from a plan graph, to a geometric layout with specific dimensions. There are ways in which graphs can be *coloured*, and values or *weights* added to their edges and vertices, so as to impose on the basic topological model some further representation of these geometrical and dimensional aspects.

A first question is "What limitations are placed on plan graphs and their embeddings, if any, by the requirement that they be capable of transformation into plans with rectangular geometry?" In particular, what class of graphs may correspond, for example, to plans of rectangular geometry? March and Earl state that every primary architectural plan can be represented such that every room is a polyomino and the walls are all set orthogonally. There is no set of adjacency relationships in a plan which *cannot* be realised in an orthogonal geometry and which *necessarily* require any curvilinear or 'free-form' geometry. On the other hand, in a packing of polyomino-shaped rooms, the junctions between walls must be either three-way or four-way. So any ornamentation of a primary plan which produced junctions of a higher order could not be realised in an orthogonal arrangement.

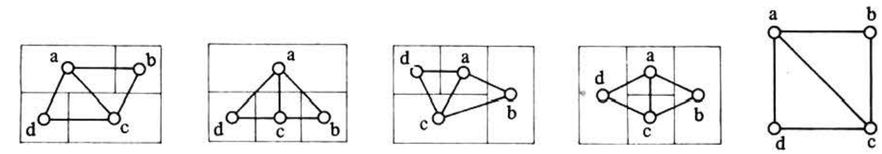
Similarly, if a plan graph is to be realised as a rectangular dissection, the junctions between walls must also either be three-way or four-way. It follows that the corresponding faces in the adjacency graph must be triangles or quadrilaterals. Furthermore, any rectangular room in the interior of a plan of this type must be surrounded by a minimum of four other rooms. Thus any embedded adjacency graph with a subgraph as shown here in which all the vertices represent rooms (not the exterior region) is inadmissible:



The valency of every vertex which represents an interior room must be at least four.

Let us look at the relationships between some rectangular dissections and their adjacency graphs. Taking first just the weak dual graph in which only interior relationships of adjacency between rooms are represented, we find that in general many dissections may share the same graph.

The next figure illustrates four dissections for $n = 4$, all of which have the same weak dual:



In this case we can see that the relationship of the rooms to the exterior regions on the four sides of the plan are all different. If appropriate vertices and edges are added to complete *augmented* dual graphs, then these differences are made manifest, and the four graphs become distinct (figure 7.5). (It is often difficult to determine by inspection whether two graphs are isomorphic. One necessary—but not sufficient—condition is that the number of vertices with a given valency be the same in both groups. In figure 7.5 all vertices are labelled with their valencies. This labelling helps to show that the graphs are indeed different.)

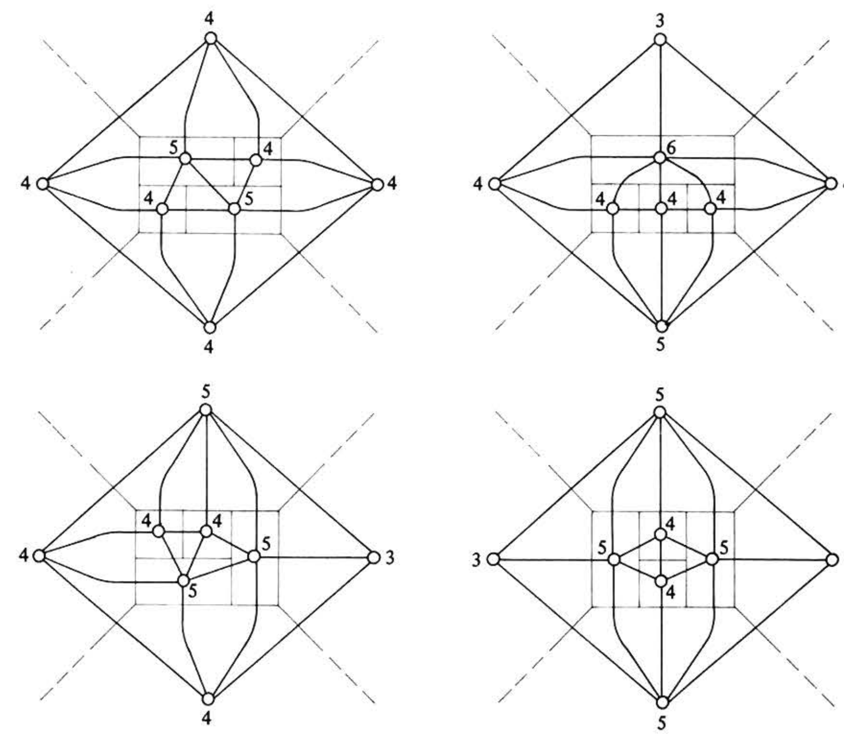


Figure 7.5. Four plans of rectangular dissection form, with their augmented dual adjacency graphs, showing different patterns of adjacency of rooms to the four exterior regions in each case. (Vertices are marked with their valencies to help illustrate these differences.)

However, it should not be imagined, even when these exterior adjacencies on the four sides are taken account of, that the augmented dual graphs of different dissections will themselves necessarily be distinct. Figure 7.6(a) shows two dissections for $n = 6$ in which not only are the adjacencies across the exterior walls of a similar pattern, but the two augmented duals as a whole are absolutely identical.

Notice that this means that the plan graphs themselves are also isomorphic. In what way do the dissections differ then? The difference lies in the fact that where some wall segments in one dissection run in an 'east–west' direction, the corresponding wall segments in the other dissection run in a 'north–south' direction. Suppose that the pattern of wall segments in one of these dissections (its embedded plan graph) was made from elastic strings, knotted together at the junctions. It would be possible to peg out the other dissection from the same set of strings, without altering any of the knots. Figure 7.6(b) shows the first dissection transformed into the second, in this kind of way.

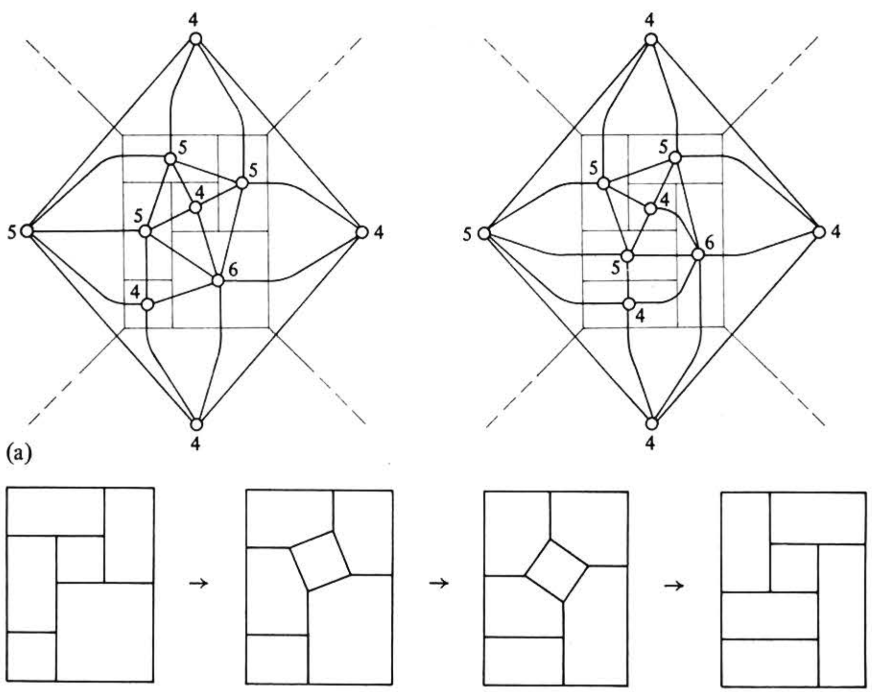
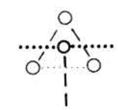


Figure 7.6. (a) Two distinct plans of rectangular dissection form, but whose augmented dual adjacency graphs are identical (vertices are marked with their valencies to help illustrate this identity). (b) A sequence of small shifts of wall segments (without changes to any junctions) transforms the first dissection into the second; the two differ only by virtue of the orientation of those wall segments, in the east–west or north–south directions.

To convert a plan graph into a particular rectangular dissection it is thus necessary to specify in which of the two perpendicular directions at least some of the wall segments run. The orientation of certain wall segments will then determine logically the directions in which others must lie. In graph-theoretic terms this can be formulated as a problem of *colouring* the edges of the plan graph, in either of two 'colours', to signify the orientation of the corresponding wall segments either 'east–west' or 'north–south'.

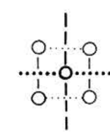
This colouring must obey definite rules, if the resulting plan is to take the form of a rectangular dissection (Grason, 1968; 1970a; 1970b; 1970c; Earl and March, 1979). The 'colours' in the figures which follow are denoted by broken and dotted lines. There are three rules required to ensure that the rooms themselves will be rectangular:

(1) At any vertex of valency three, the three incident edges must not be coloured the same:



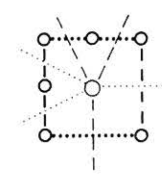
(In any three-way junction two wall segments must run in one orientation and the third wall segment in the perpendicular orientation.)

(2) At any vertex of valency four, opposite edges must be coloured the same, and adjacent edges must be coloured differently:



(At a four-way junction each pair of opposite wall segments must be aligned, and the two pairs must run in the two perpendicular orientations.)

(3) The edges surrounding each face (not counting the exterior face) must be coloured in such a way as to be divisible into four groups. Within each group all edges must be coloured the same, and adjacent groups must be coloured differently:



(The face is a room, of which the edges making up the four walls must be capable of being oriented so as to form a rectangle.)

Last, if the plan as a whole is to have a rectangular boundary, rule (3) must apply in addition to the edges surrounding the exterior face.

(Without this condition on the exterior face, the application of the previous colouring rules would be sufficient to create plans composed of rectangular rooms but not necessarily with simple rectangular external boundaries.)

Instead of colouring the plan graph according to these rules, it would be possible to colour the embedded adjacency graph according to a complementary set of rules. The two procedures are effectively equivalent.

Colouring the plan graph or the adjacency graph in all possible ways such that the above rules are satisfied, will produce all plans of rectangular dissection form to which the graph corresponds. Figure 7.7 shows the two alternative colourings for the plan graph of figure 7.6(b), giving the two different dissections as illustrated. (Notice that the colouring of the plan boundary, and hence of some of the interior edges, is determined here by the way in which the four corner vertices are positioned. If the positions of the corners were not fixed, then other colourings, giving other dissections, would also be possible.) Some of the design methods to be described in chapter 9 incorporate colouring procedures which work along these lines.

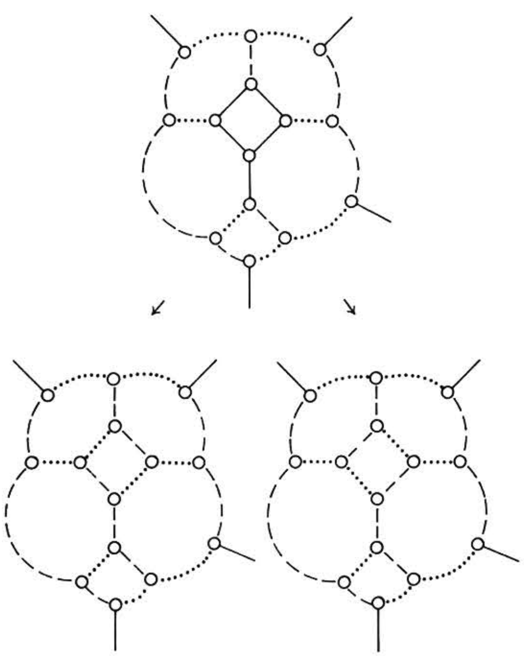
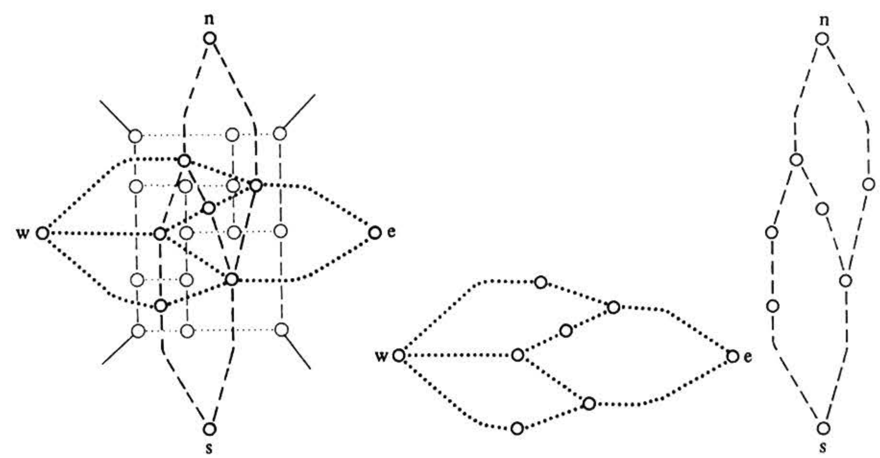


Figure 7.7. A plan graph (top) corresponding to either of the rectangular dissection plans illustrated in figure 7.6. The edges (wall segments) are *coloured* (dotted or broken lines) to correspond to their orientation (east-west or north-south, respectively) Two alternative legitimate colourings exist (below) for the five edges in the centre of the graph, yielding the two distinct dissections of figure 7.6.

Weighted graphs of plans

Finally, there are ways of labelling either an adjacency graph, or else a new type of 'electrical' graph, so as to represent the *dimensions* of a rectangular plan (Grason, 1968; 1970a; 1970b; 1970c; March and Steadman, 1971; Earl and March, 1979). Consider a valid colouring of an augmented dual graph, such that it represents a plan of rectangular dissection form, as in figure 7.8(a). The colouring divides the edges of the graph into two groups, according to whether they correspond to 'north-south' or 'east-west' walls in the plan. The graph can be split into two subgraphs or 'half-graphs', each of which contains the vertices representing all the rooms, two vertices representing opposite exterior regions, and all edges of one colour [figure 7.8(b)]. (The edges representing the adjacencies of the exterior regions to each other are omitted.)

Take the 'half-graph' which joins the vertices e and w representing the exterior regions on east and west, as in figure 7.9. Each edge in the 'half-graph' corresponds to a wall segment aligned north-south in the plan. Suppose that values or *weights* corresponding to the lengths of these wall segments are attached to the edges in the 'half-graph'. Suppose furthermore that each edge is assigned a direction (indicated by an arrowhead) as it crosses the corresponding wall segment from west to east. (This direction is arbitrary: it could equally be east to west—so long as all edges are directed in the same way.)



(a) (b)

Figure 7.8. (a) A plan of rectangular dissection form, with its augmented dual adjacency graph. The edges of the graph are coloured according to whether they represent adjacencies across east-west or north-south wall segments in the plan. The graph may be divided into two 'half-graphs' (b) corresponding respectively to adjacencies across east-west wall segments (broken lines) and adjacencies across north-south wall segments (dotted lines).

A graph with *directed* edges, to which weights are attached in this way, is a *network*[12]. Consider each vertex in this network. The total weight of the edges leaving vertex w, must equal the overall dimension of the plan from north to south. The total weight of edges entering e must be the same. As for each vertex representing a room; the sum of weights on edges *entering* the vertex must equal the sum of weights *leaving* that vertex, and both sums must equal the dimension of the room in question in the north–south direction.

Exactly the same must apply for the second half-graph, running from vertex s to vertex n, in which the weights relate to the dimensions of the plan in the east–west direction.

Looked at the other way round: given an augmented dual adjacency graph, coloured in such a way as to correspond to a rectangular dissection type of plan, then assignments of weights representing dimensions to the coloured edges of this graph must obey this rule governing the sum of weights on edges of one colour at each vertex, for the dimensioning of the plan as a whole to be consistent and feasible.

This condition on the sum of weights at the vertices of a network corresponds to one of two laws stated by Kirchhoff, governing the flow of

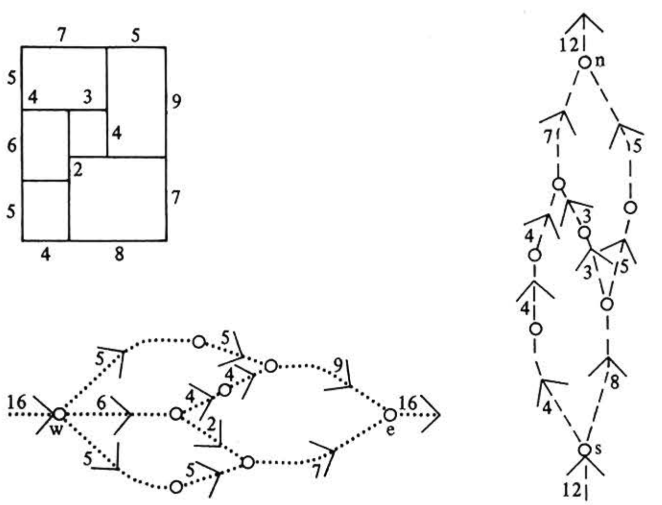


Figure 7.9. A dimensioned rectangular plan and its representation by two *networks*. Each network corresponds to adjacencies across wall segments with a common orientation. Take the network representing adjacencies across north–south wall segments (edges shown in dotted lines): each edge is *directed*, as it crosses the wall segment from west to east. The values (weights) on the edges give the lengths of the wall segments. The sum of weights on edges entering a vertex must equal the sum of weights on edges leaving that vertex. The same applies in the second network representing adjacencies across east–west wall segments (edges shown in broken lines).

(12) A graph with directed edges (but no weights) is called a *digraph*. So a network is a weighted digraph. Edges in networks are sometimes referred to as *arcs*.

current in electrical networks. With an electrical network, clearly, wires are represented by edges, and junctions between wires by vertices. Kirchhoff's law states that the sum of currents entering a junction or vertex must equal the sum of currents leaving that junction.

A different kind of network representation of a rectangular plan has properties which parallel not just this one but both of Kirchhoff's laws for electrical circuits (Kirchhoff, 1845). Vertices and edges here represent quite different entities and relationships from all the graphs of plans considered up to now. Take the plan illustrated in figure 7.10(a). Consider walls running north–south. We take not each separate wall segment this time, but each maximal continuous straight run of wall—each *wall* in the formal sense used in previous chapters—and assign to it a *vertex*. Thus in the example of figure 7.10(a) we take the whole extent of each of the external walls on the east and west sides, and represent them by vertices E and W. We add two more vertices, to represent the wall P between rooms c and f and d and e, and the wall Q which separates rooms a and d from room b.

An *edge* in this network represents one of the *rooms* a to f and the fact that it lies between two of these north–south walls. Thus we place an edge connecting vertex W with vertex Q, to express the fact that room a lies between the corresponding walls; and so on. There are therefore six edges in the complete network, for the six rooms. As before the edges are all assigned an (arbitrarily chosen but consistent) direction, in this case as they run from west to east.

Weights are assigned to the edges, to represent the dimensions of the rooms in the north–south direction [figure 7.10(b)]. An edge is shown entering W, and another leaving E, with weights equal to the north–south dimension of the whole plan. The sums of weights at each vertex now obey Kirchhoff's first law, as they did previously in the 'half-graph' (despite the fact that this network, and the 'half-graph', are quite distinct, and are defined in quite different ways.) The sum of north–south dimensions of rooms on one side of a wall, equals the sum of north–south dimensions of rooms on the other side of that wall.

However, in addition weights are also attached to the *vertices* of the network [shown in italic in figure 7.10(b)]. These represent in effect the dimensions of rooms in the *east–west* direction, but in a particular way. The weights give the distance of each wall from the eastern boundary wall E. Thus the vertex E itself takes the weight 0, Q takes weight 5, P takes weight 8, and W weight 12 for the east–west dimension of the whole plan. Each edge in the network, with the weights attached both to itself and to the two vertices which it joins, now obeys Kirchhoff's second law. That law relates together the resistance R of a wire in an electrical circuit, the difference in voltage $(V - V')$ between the two ends of the wire, and the resulting current A flowing from the larger voltage to the smaller, that is,

$$A = \frac{V - V'}{R} .$$

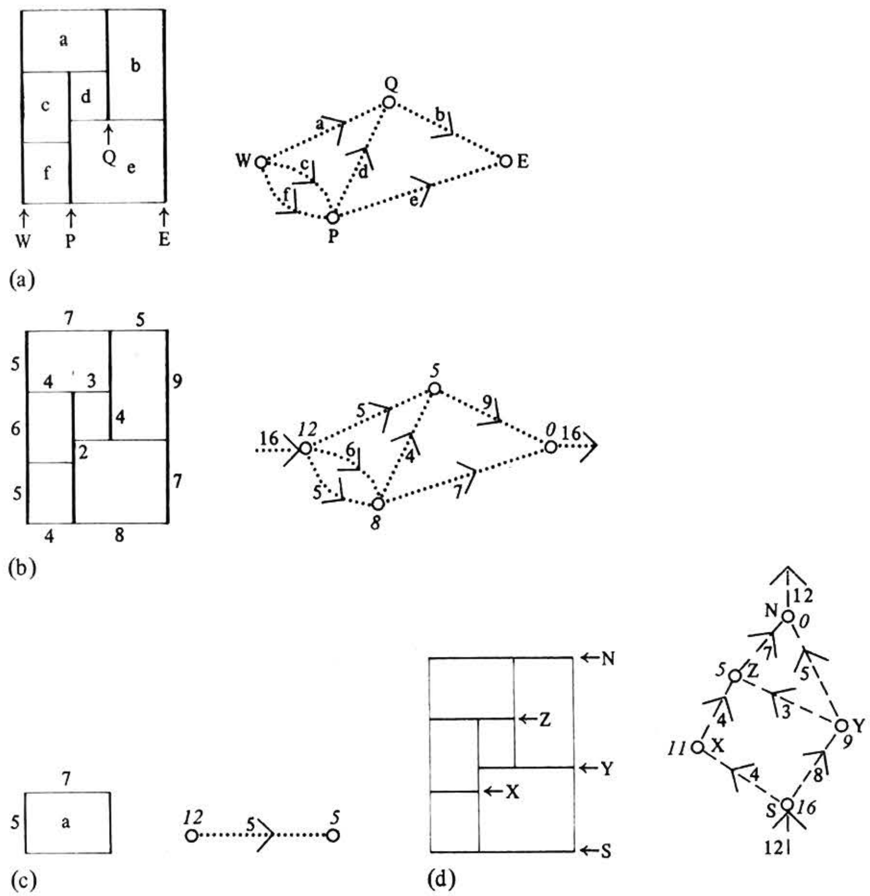


Figure 7.10. (a) A plan of rectangular dissection form with its north–south walls (marked in heavier line) labelled W, P, Q, and E. In the corresponding network, the vertices represent these walls, and the edges (dotted lines) represent rooms lying between the walls. The edges are directed as they run from west to east. (b) The same plan dimensioned. Weights on the edges in the network now give the dimensions of rooms in the north–south direction. Weights on the vertices give (east–west) distances of the corresponding north–south walls from the eastern boundary wall. The sum of weights on edges entering a vertex equals the sum of weights on edges leaving that vertex (the first Kirchhoff law). (c) The single room a with its corresponding edge in the network. The *proportions* of the room (7:5) are given by the ratio of the difference between the weights on the vertices (12 − 5 = 7) to the weight on the edge (5) (the second Kirchhoff law). (d) A *conjugate* network can be constructed for the same plan in its other orientation, in which the vertices represent east–west walls (S, X, Y, Z, N).

In the electrical network, the weight attached to an edge is the current, and the weights attached to the vertices, the voltage levels. To what do these correspond, by analogy, in a rectangular plan? The current, in the network of figure 7.10(b), corresponds to the dimension of a room in the north–south direction; whereas the voltage difference corresponds to its dimension in the east–west direction. It follows that we must interpret the electrical resistance R as the ratio of these dimensions, that is, the *proportion* or shape of the room.

For example, the edge corresponding to room a in figure 7.10(a) has a weight 5 (the 'current'), the vertices W and Q which it joins have weights 12 and 5, respectively (the 'voltages'), and Kirchhoff's law gives

$$5 = \frac{12 - 5}{R} , \qquad \text{or} \qquad R = \frac{7}{5} .$$

These relationships are illustrated in figure 7.10(c).

The same Kirchhoff's laws would apply to a complementary or *conjugate* network (Earl and March, 1979) constructed for the same plan in the other orientation, and representing by its vertices the walls running east–west [figure 7.10(d)]. The dimensions of the whole plan are given by either one of the two networks.

This analogy between electrical networks and rectangular arrangements was first demonstrated in 1937 by the four Cambridge mathematicians Brooks, Smith, Stone, and Tutte who called themselves the 'Important Members', and was used by them to solve the old puzzle of 'squaring the square' (Brooks et al, 1940). The puzzle consists of finding some set of squares, all of different sizes, which will pack together without holes to form a larger square (that is, a dissection of a square into unequal squares). Since then several authors have suggested applications to the dimensioning of architectural plans (see March and Steadman, 1971), a subject which will be taken up again in chapter 9.

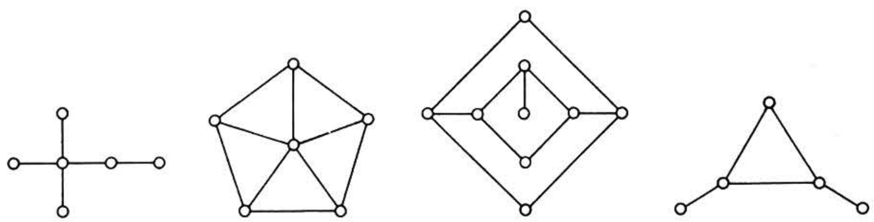
As a final word on architectural applications of graph theory, it should be mentioned that graph representations can be extended, without any great theoretical difficulties arising, to treat 'three-dimensional' adjacency relationships in buildings. That is to say, an edge in a graph might as well represent the fact that two rooms on different floors are adjacent, in the sense that the ceiling of one forms the floor of the other, as represent their adjacency on the same floor across a common wall. On the other hand there are no very significant functional reasons for requiring such adjacencies, and they tend to arise out of the constraints of gravity and close-packing (rooms on one floor must be supported on those of the floor below). Nor is there any need for an adjacency requirement graph of this type to be planar, of course. There are nevertheless limits on the possible adjacencies (across all surfaces) of rooms, whether box-like or of other three-dimensional forms, packed together in space, just as there are limits on adjacencies in the two-dimensional packing of rooms in plans (see Earl, 1978; Krishnamurti, 1979).

The principle of the 'electrical network' can also be extended to represent the dimensional relationships occurring in the packing of cuboids into a cuboid. Teague (1970) suggested the relevance of this representation to building geometry, and proposed its application in computer-aided design.

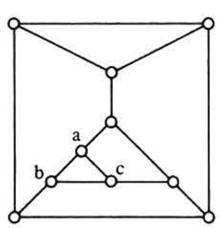
Exercises

7.1 (This is if you did not already take up the invitation in the text.) Draw the plan graphs corresponding to some of the embedded adjacency graphs in figure 7.1 (assume all vertices represent rooms in each case, and the exterior region is not represented). Some of the plans will contain holes, or interior rooms engulfed inside others, as in the second figure on page 79 and the third figure on page 80.

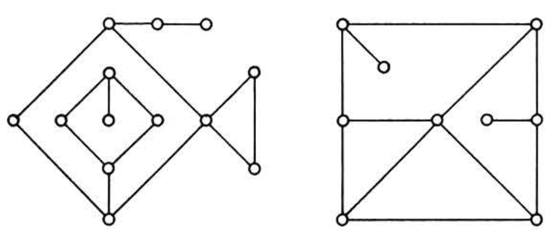
7.2 Draw all distinct planar embeddings of the graphs illustrated. How are the numbers of embeddings altered, by the fact of whether or not the vertices are labelled?



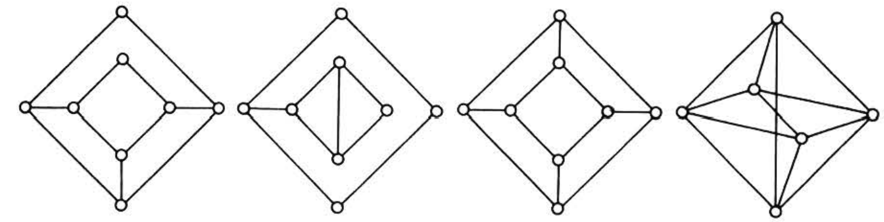
7.3 Redraw the graph illustrated in a planar embedding such that the face abc becomes the exterior face.



7.4 Identify all cut vertices in the graphs illustrated.



7.5 What is the degree of connectedness of the graphs illustrated?



7.6 Draw all trees with eight vertices. (There are twenty-three of them.)

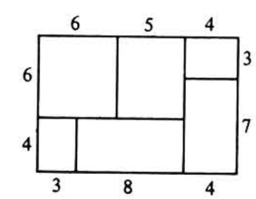
7.7 Draw the projections onto the plane of the tetrahedron (as in figure 7.3), the cube, and the octahedron. Draw their duals. Of which polyhedra are these duals projections onto the plane?

7.8 To which of the embedded adjacency graphs illustrated in figure 7.2 do the primary plans also shown in the same figure correspond? (Remember that this correspondence is not one-to-one, and is dependent on the choice of vertices in the adjacency graph to represent the exterior region.)

7.9 Draw the distinct possible planar embeddings of the trivalent polyhedra (the 'fundamental plans') illustrated in figure 7.4 by projecting them through each of their faces in turn. (Some of the results will be duplicates.) These embeddings should give you the 'primary plans' illustrated in figure 7.2.

7.10 For the dimensioned rectangular dissection illustrated, draw the types of electrical network, representing the adjacencies and dimensions of the rooms in the 'north–south' and 'east–west' directions, whose principles are illustrated in figure 7.9 (where the vertices represent rooms and regions).

For the same dissection, draw the types of electrical network illustrated in figures 7.10(b) and (d) (where the vertices represent walls).



Properties of rectangular arrangements, and their classification

> "Art makes us realize that there are *fixed laws which govern and point to the use of the constructive elements of the composition and of the inherent interrelationships between them.*"
>
> Piet Mondrian (1937)

In chapters 4 and 5 we looked at a number of procedures, or algorithms, for enumerating rectangular arrangements exhaustively. We saw how, as the number n of rectangles in the dissection or polyomino (its *order*) increases, the number of distinct arrangements grows at an accelerating rate. Thus for $n = 10$ there are already 4655 distinct polyominoes, and over 400000 dissections (the exact number depending on whether dissections with alignments and four-way junctions are counted or not).

This 'combinatorial explosion', typical of many systems of similar kinds, raises a series of issues. It suggests that there are effective practical upper limits to which such enumeration, even by the use of computers, can be carried. Many real architectural plans, clearly, have more than nine or ten rooms on a single floor. Does this mean that the approach outlined here is only applicable to small buildings? Even within the practical limits of enumeration, the further question arises whether, for larger values of n, a simple listing of dissections would not be so long and cumbersome as to be of limited practical use.

Certainly where a catalogue of dissections is intended to be referred to directly, it will need to be organised into some systematic order, and properly cross-referenced, if arrangements with specified properties are to be picked out at all easily. And even if a listing of dissections is to be searched by computer, it will still make this search much more efficient and economical if it too can be organised according to some classificatory hierarchy (so that not every entry in the list has to be examined on every run).

We have already looked at a number of properties of dissections in earlier chapters: their symmetries, their graphs, their grating sizes. Indeed in several instances some knowledge of these attributes has been exploited deliberately in the design of algorithms by which the dissections have been enumerated in the first place. Once the enumeration is complete, however, we can turn the question around, and ask how are such attributes distributed over the whole population of possible arrangements? How many dissections possess symmetries of a certain type, how many occupy a certain size of grating, how many possess a certain graph, and so on?

Certainly the first and most obvious basis for classification must be the order of the dissection, n. Beyond this the choice of properties of interest is wide open.

One of the first attempts at a systematic categorisation was that made by Combes (1976), who directed his attention to the number of wall segments in an arrangement. He counted the number of external wall segments w in the perimeter of a dissection and compared this with the number of internal wall segments p [13]. He plotted a graph in which w and p are shown on the vertical and horizontal axes, respectively (figure 8.1).

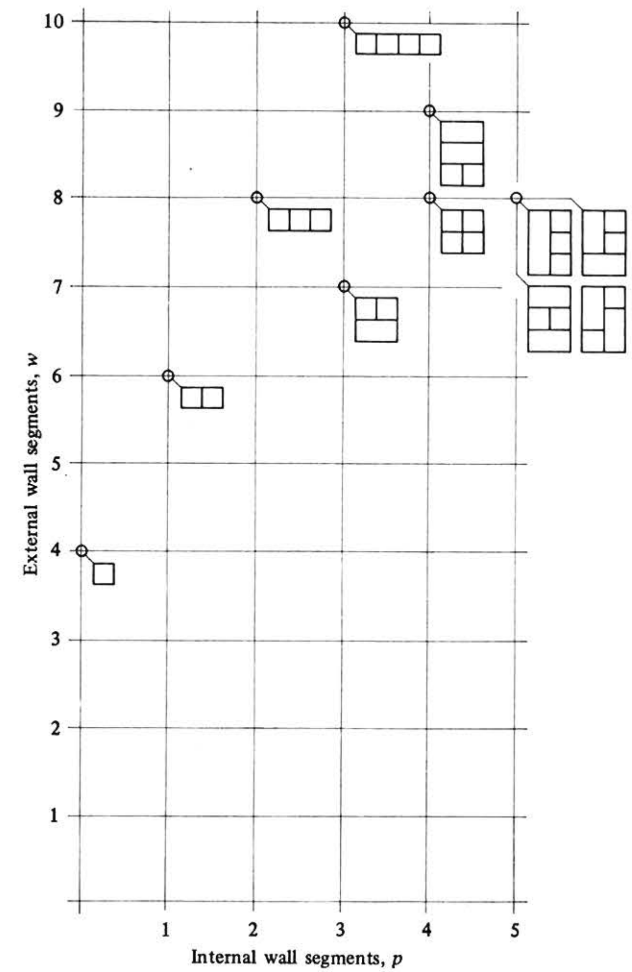


Figure 8.1. Graph of numbers of external wall segments w against numbers of internal wall segments p in rectangular dissections (after Combes, 1976). All individual dissections are illustrated, attached to the appropriate integer points in the graph, for values of $p \leqslant 5$, $w \leqslant 10$.

[13] Combes himself refers to an external wall segment as a 'wall', and an internal wall segment as a 'partition'.

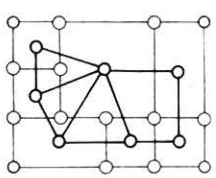
Thus a dissection with given numbers of internal and external wall segments would appear on the graph at the appropriate integer point. The dissection for $n = 1$ has four external and no internal wall segments; the dissection for $n = 2$, six external and one internal wall segments; the two dissections for $n = 3$, eight external and two internal, and seven external and three internal wall segments, respectively; and so on.

Combes was able to demonstrate a number of general relationships existing between w, p, n, and the numbers of junctions, three-way and four-way, occurring in the interior of a dissection. Combes himself arrived at these formulae empirically. They are, however, derivable from the *polyhedral formula* of Euler (Gutiérrez, 1979; see also Biggs et al, 1976, chapter 5). Euler's expression relates together the numbers of vertices v, edges e, and faces f of any polyhedron:

$$v - e + f = 2 .$$

We have seen in the last chapter how any polyhedral form can be 'projected' down through one of its faces onto the plane, to become a plane map. You might like to confirm that Euler's formula applies to any dissection, or indeed any plane map, you care to choose. (You can choose to count the four outer corners of a dissection as vertices of valency 2, or not—the relation applies in both cases. It will be convenient to *include* these vertices here, however, since edges will then correspond one for one to wall segments. Remember to count the exterior face.)

Now the embedded adjacency graph of a rectangular dissection can be considered as a plane map. Suppose we draw the graph of the 'internal' adjacencies between rooms only, and omit the exterior region around the plan and all adjacencies to that region (it is the weak dual):



Each face in this map, with the sole exception of its exterior face, corresponds to a junction in the interior of the dissection. These junctions as we know can only be of two types, three-way or four-way. Let us denote the numbers of each type of junction in the interior by i_3 and i_4, respectively[14]. Meanwhile, the number of edges in the map must, because of the dual relationship, be equal to the number of internal wall segments p.

[14] Readers wishing to refer to Combes's papers should be warned that his notation differs here. He indicates the number of four-way junctions by i, and the *total* number of junctions in the interior of the dissection by j. Thus i is a subset of j. With this difference in mind, equations (8.3), (8.4), and (8.5) below should be compared with equations (7), (8), and (9) in Combes (1976).

From the application of Euler's formula then to this adjacency graph considered as a map: the number of vertices is n, the number of edges is p, and the number of faces is $(i_3 + i_4 + 1)$ (the 1 is for the exterior face). So

$$n - p + i_3 + i_4 + 1 = 2 . \qquad (8.1)$$

In the example illustrated above, $n = 7$, $p = 10$, $i_3 = 3$, $i_4 = 1$, which gives

$$7 - 10 + 3 + 1 + 1 = 2 .$$

Now take the dissection itself, considered as a plane map, and the relation between the total number of wall segments which it contains and the numbers and types of *all* the various junctions (not just those in the interior). There are i_4 four-way junctions, each of which has four internal wall segments incident with it, that is, it has valency 4. Similarly there are i_3 three-way junctions in the interior, each with valency 3. We have not yet counted those three-way junctions which lie on the perimeter of the arrangement. Their number is related to the number of external wall segments w. It is easy to see that the number must be $w-4$. It is four less than w because of the four 'two-way junctions' at the outer corners. (Thus in the above figure, $w = 11$ and the number of three-way junctions in the perimeter is 7.) These four corner junctions themselves have valency 2. Since every wall segment joins two junctions, it follows that by adding all (junctions x valencies) we will count each segment twice. Thus

$$2(p+w) = 4i_4 + 3i_3 + 3(w-4) + (4 \times 2) . \qquad (8.2)$$

For the above example:

$$2(10+11) = 4 + (3 \times 3) + 3(11-4) + 8 = 42 .$$

From equation (8.1) we have a value for p:

$$p = n + i_3 + i_4 - 1 , \qquad (8.3)$$

and substituting into equation (8.2) gives us a value for w:

$$w = 2n - i_3 - 2i_4 + 2 . \qquad (8.4)$$

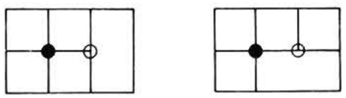
The *total* number of wall segments, which Combes denotes by t, is therefore given by

$$t = p + w = 3n - i_4 + 1 . \qquad (8.5)$$

Notice the implications of this: that in trivalent dissections, that is where $i_4 = 0$, there is a constant relation between the total number of wall segments t and the number of component rectangles n. The introduction of each four-way junction reduces the quantity of wall segments by one. [This fact relates to our observation in chapter 3 (compare the second figure on page 28) that two adjacent three-way junctions may be 'coalesced' into a single four-way junction. In this process one (internal) wall segment disappears.]

These relationships are demonstrated by the example of dissections for $n = 5$, shown in figure 8.2. These are all trivalent, and so the value t $(= p+w)$, is always sixteen, that is, $(3 \times 5)+1$.

Notice how the dissections are arranged in columns, according to the pairs of values w and p, from $w = 12$, $p = 4$ at one extreme, to $w = 8$, $p = 8$ at the other. It is as though, scanning from left to right in the figure, wall segments were progressively removed from the perimeter and transferred into the interior. In the two remaining dissections for $n = 5$ which possess one four-way junction each, t equals 15 in both cases:



To return to Combes's graph on which dissections are plotted, it follows that trivalent dissections will all lie on lines at 45° to the axes, corresponding to the graphs of $t = 3n+1$ for successive values of n (figure 8.3). Any dissection containing four-way junctions will drop down

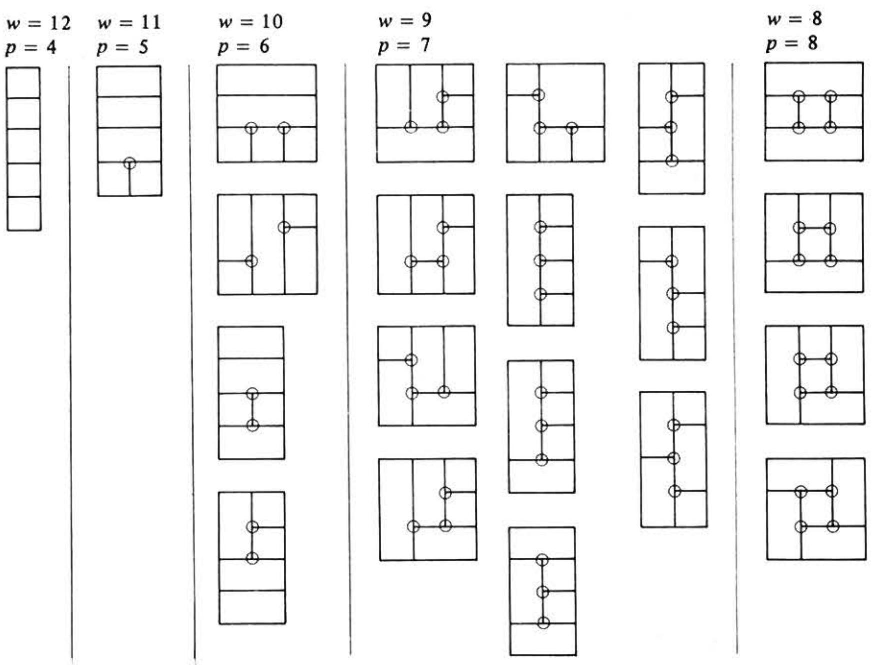


Figure 8.2. Trivalent dissections for $n = 5$, classified according to their values of p and w, from $w = 12$, $p = 4$ at the left, to $w = 8$, $p = 8$ at the right. Notice that the sum $(p+w)$ has a constant value of 16. It is as though wall segments were transferred, one at a time, from the perimeter to the interior of the arrangement.

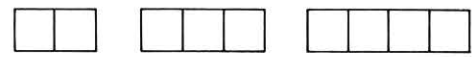
below the relevant line, by a distance in units equal to that number of junctions.

In those dissections without interior junctions of *any* kind, that is, where both i_3 and i_4 are zero, equations (8.3) and (8.4) reduce to

$$p = n-1 \,, \tag{8.6}$$

$$w = 2n+2 \,. \tag{8.7}$$

In fact for each value of n there is only *one* dissection of this type, and with these values for p and w. It is that arrangement where the n rectangles are all arranged in a simple straight line:



It is clear by inspection why the number of internal wall segments must be one less than the number of rectangles, and why the number of external wall segments must equal $2n+2$. These are the dissections in which the ratio of w to p is at its greatest. They have the greatest possible number of sides of their component rectangles lying on the perimeter; they are the most 'outward-looking' arrangements. The line corresponding to the equation $w = 2n+2$ on which they lie, marks an upper boundary to the area in Combes's graph within which all dissections must fall.

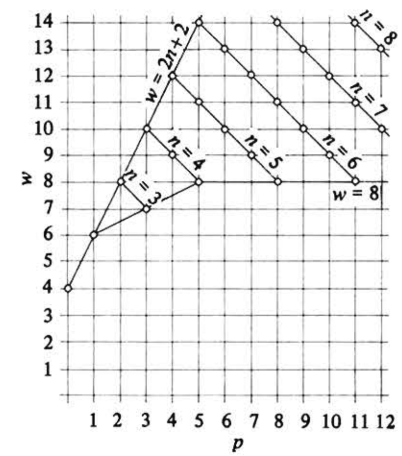
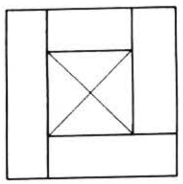


Figure 8.3. The Combes (1976) graph of external wall segments w against internal wall segments p in rectangular dissections. Trivalent dissections for successive values of n lie on the lines $p+w = 3n+1$ at 45° to the axes, as indicated. The space within which all dissections lie is bounded at the upper edge by $w = 2n+2$, corresponding to the most open, 'outward-looking' dissections where the ratio $w{:}p$ is at a maximum. It is bounded at the lower edge by $w = 8$ (for $n \geqslant 4$), corresponding to the most enclosed or 'inward-looking' dissections where the ratio $w{:}p$ is at a minimum.

At the other, lower extreme, the boundary is set, for values of p greater than 4, by the horizontal line at $w = 8$. This is a consequence of the fact that dissections in which the number of external wall segments is least are those where four rectangles form a 'frame' around the outside, and all the remaining rectangles are contained in the interior:



Only where n is less than four, is it possible for there to be fewer external wall segments (compare figure 8.1).

On Combes's diagram then it is possible to delimit a wedge-shaped area within which all dissections fall. Moving from the origin towards the upper right of the graph, corresponds to a progressive increase in the value of n. For given n, all trivalent dissections, as we have seen, lie along the straight line $t = 3n+1$. Where those with four-way junctions are included, then all dissections for given n are contained within a roughly triangular area, of which the upper edge is marked by $t = 3n+1$. For example, the area within which all dissections for $n = 9$ lie is shown in figure 8.4.

Much of Combes's work has been devoted to determining the exact shape of the lower boundary to this triangle in the general case. The demonstration is highly complicated, and need not concern us here. It will be sufficient to observe that a move downwards within the triangle represents an increase in the number of four-way junctions, from 0 to some maximum quantity reached at the lower apex. In the example of $n = 9$ in figure 8.4 there is only one dissection at this extreme lower point. It is the square pattern of nine squares in which there are four four-way junctions.

In general, there will be very many different dissections lying on any given point in Combes's diagram, especially towards the lower right of the triangle for a given n, which is more densely populated with dissections than is the upper left. (Figure 8.4 gives just one example of a dissection for $n = 9$ located on each point.) This classification does not by any means distinguish individual arrangements therefore.

But it *does* serve to classify dissections along a certain number of 'dimensions': first of all by increasing n; and then by the extent to which dissections are 'inward-looking' or 'outward-looking', that is, how many sides of the component rectangles are in contact with the perimeter, and as a consequence what is the ratio of junctions on the perimeter to those in the interior. (See how the number of interior junctions $i_3 + i_4$ increases from 0 to 12, reading from left to right in figure 8.4.)

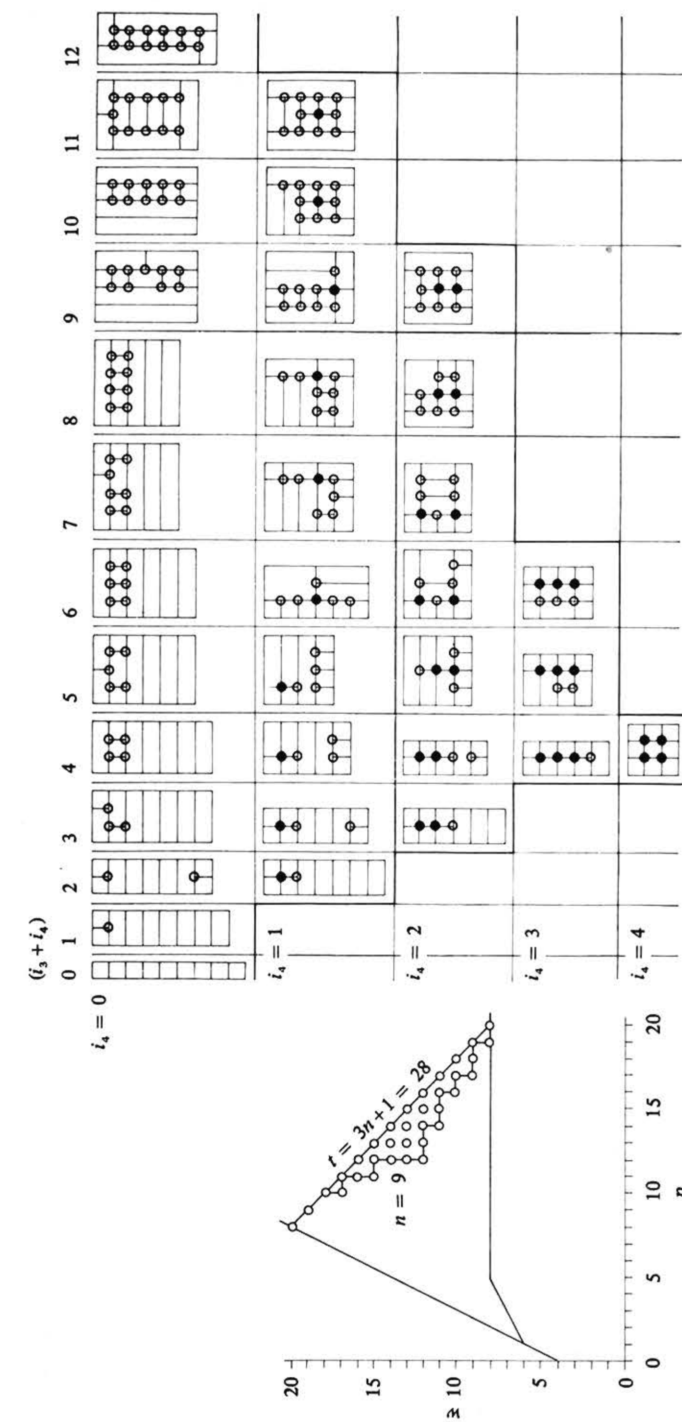


Figure 8.4. (a) The Combes (1976) graph of external wall segments w against internal wall segments p, showing the roughly triangular area in which all dissections for $n = 9$ fall. Trivalent dissections for $n = 9$ lie on the line $p+w = 3n+1 = 28$. All arrangements with four-way junctions fall below this line, by a number of units equal to the number of such junctions. (b) Examples of dissections for $n = 9$ lying on each of the corresponding integer points in the graph. The classification by numbers of wall segments results effectively in a simultaneous classification by number of four-way junctions i_4, and total number of internal junctions, three-way and four-way (i_3+i_4) (three-way junctions are marked by hollow circles, four-way junctions by solid circles).

Finally, there is a classification of the extent to which arrangements contain four-way junctions.

To what degree are these properties of any architectural relevance? Certainly the matter of the contact of the component rectangles with the perimeter of the dissection seems to be important. Rooms on the exterior of a plan can have windows, or they can have doors giving directly onto the outside. Conversely, there are some types of building in whose plans one or a few central spaces are completely enclosed by other rooms clustering around, such as theatres, law courts, or the operating suites of hospitals.

The numbers and types of junctions between walls in a plan might not seem at first sight to be of any architectural interest. Architects do not generally concern themselves over the distinction between three walls or four walls coming together at a point. (Except perhaps in the design of prefabricated walling systems, where different types of connector piece are needed in each case.) But even here there are certain consequences for plan arrangement which are not without practical significance.

Imagine a plan with a given number of rooms, corresponding geometrically to a rectangular dissection. The rooms are all 'overlapped' one with another in their arrangement. Now imagine pairs of adjacent three-way junctions being progressively replaced by four-way junctions. The plan will become increasingly 'squared-up'; and the number of internal wall segments, as we have seen, will decrease as a result. The existence of such a wall segment provides the opportunity to place a door giving access between two rooms. With fewer internal wall segments there is correspondingly less flexibility in the way this placing of doors, and hence the design of the circulation pattern as a whole, can be made. In chapter 11 we will look at some historical evidence (slight though it might be) to suggest changes related to these kinds of properties, going on in the evolution of house planning.

We have been thinking of dissections as representing plans, and the rectangles as rooms. But they might just as well represent architectural

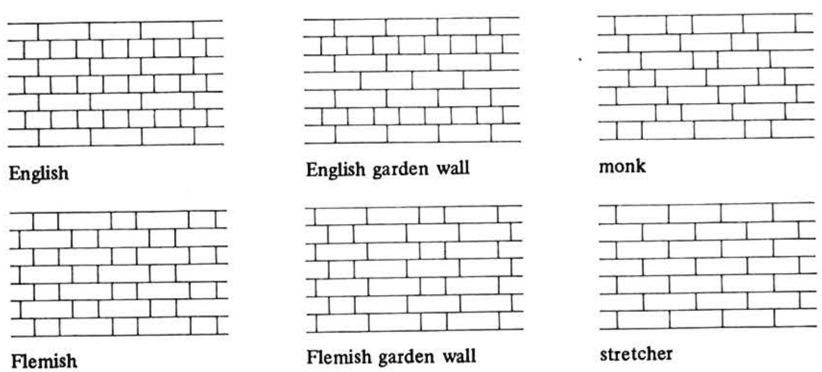


Figure 8.5. Types of traditional pattern, or *bond*, in which bricks are laid.

arrangements at a quite different scale, for example arrangements of constructional components. The various types of 'bond' in which bricks are laid (figure 8.5) are all designed so as to avoid, as far as possible, four-way junctions between bricks (looking at the pattern in the vertical plane). There are sound functional reasons for this, of course: the overlapping is intended to ensure the best possible adhesion between adjacent bricks, and to discourage vertical cracks from opening up. Just as with rooms in plans, trivalent arrangements create more adjacencies between bricks, other things being equal, than do those with four-way junctions.

Bloch's catalogue of dissections

Combes's work provides a classification of dissections by some of their properties, but these properties are limited ones, and in many cases serve to group very large numbers of dissections together. More recently, a comprehensive catalogue of dissections has been produced by Bloch (1979a; 1979b), in which all arrangements up to $n = 8$ are depicted individually, and statistical tabulations are given for $n = 9$. (Part of this catalogue, up to $n = 7$, is reproduced here with his kind permission, as an appendix.) The entries are organised by means of an hierarchical classification. In addition to the characteristics which determine this hierarchy, Bloch has also investigated the frequency of occurrence of certain other properties which are arguably of architectural importance.

To some extent Bloch's catalogue is organised according to the properties by which the dissections were originally generated, by means of his 'tiling' algorithm (compare chapter 5). Dissections are classified first by their order n and then by their grating dimensions. The catalogue contains all dissections including those with alignments and four-way junctions, but with the sole exception of dissections for which $lm = n$, that is, where each rectangle occupies one cell in the grating. Figure 8.6 reproduces a page from the catalogue (Bloch, 1979a). It shows all dissections of order 5 (with the exception of the single arrangement with grating size 1×5). By convention the gratings are set always with the shorter dimension l across the page. The possibilities for grating sizes for $n = 5$ are 2×3, 2×4, and 3×3 (compare figure 5.1 and table 5.1), of which the 2×3 gratings correspond only to arrangements with four-way junctions or alignments.

The next basis for classification is by what Bloch calls *fronts*. A front is a rectangle which is in contact with the perimeter of the dissection (as opposed to one surrounded in the interior). The boundary of the front might contain one, two, or three external wall segments (four only where $n = 1$, obviously). This the term does not distinguish; in all cases the rectangle is a front. As we have seen in Combes's work, for $n \geqslant 4$ the minimum number of fronts is 4; and clearly the maximum number is always n. In figure 8.6 there are only therefore dissections with four or five fronts.

The fourth dimension of Bloch's classification relates to the valencies of vertices in the (weak dual) adjacency graphs of dissections. Bloch refers to the distribution of those valencies between vertices as the *graph partition*. This can be set out in formal notation as a string of digits in which the first digit gives the number of vertices with valency 1, the next gives the number of vertices with valency 2, and so on. For the two dissections which appear at the top left of figure 8.6, for example, the string 0320 indicates that there are no vertices of valency 1, three of valency 2, two

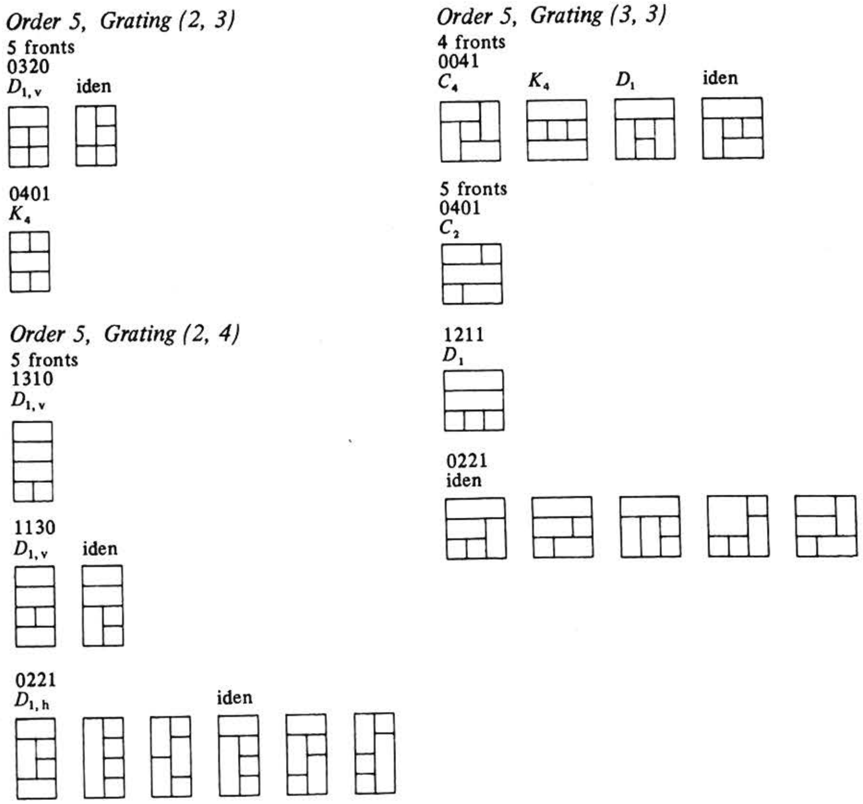
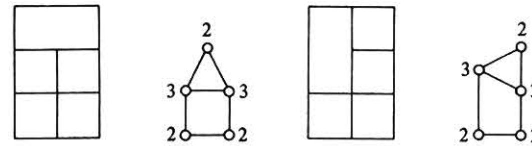


Figure 8.6. Sample page from Bloch's catalogue of rectangular dissections [from Bloch (1979a); see also appendix]. The page shows all dissections for $n = 5$ (with the exception of the single arrangement for grating size 1 x 5). They are classified by grating size; number of *fronts*; *graph partition*; and symmetry properties. A front is a rectangle in contact with the perimeter. The graph partition relates to the valencies of vertices in the weak dual adjacency graph of a dissection. It is set out as a string of digits (four in this instance), where the first digit gives the number of vertices with valency 1, the second digit the number of vertices with valency 2, etc. Dihedral and cyclic symmetries are marked by the usual conventions (D_1, C_2, C_4, etc), and v and h indicate whether the axis of D_1 symmetry is vertical or horizontal. Arrangements marked K_4 have all the symmetries of the rectangle. Those marked iden have no symmetries.

of valency 3, and none of valency 4:



As a final classification, Bloch's catalogue gives the symmetry properties of each dissection. The format is largely self-explanatory in terms of the notation introduced in chapter 3. Those dissections with no symmetries are labelled for 'identity' (they are left unchanged only by the identity transformation). Dissections with all the symmetries of the rectangle are labelled K_4 for the 'Klein four-group [see footnote (3), page 22]. Dissections on nonsquare gratings with bilateral symmetry D_1 are distinguished as to whether the axis of symmetry is vertical ($D_{1,v}$) or horizontal ($D_{1,h}$) (that is, up and down, or across the page). Dissections on square gratings with D_1 symmetry are illustrated always with the axis of symmetry up and down the page. There is the further possibility that square dissections may possess D_1 symmetry about a diagonal axis ($D_{1,dg}$). The cyclic symmetries C_2, C_4 are marked as such.

Statistical analyses of dissections and polyominoes

To come to the statistical analyses which Bloch (1979a) has carried out on these 'populations' of dissections: there is not the space to give his detailed results here, but it is worth commenting on the architectural significance of some of his major findings. *Grating area lm*, which Bloch tabulates for all fundamental dissections, is in itself perhaps not very meaningful in architectural terms. It obviously increases generally with n. And for given n the number of distinct dissections generally (though not universally) increases with increasing grating area, as can be seen from an inspection of the catalogue. We have already noted in chapter 5 how those arrangements which contain four-way junctions or alignments have smaller grating areas than corresponding fundamental dissections, because of the consequent loss of grating lines.

Grating shape l:m on the other hand has some significance, in that for given n the numbers of distinct dissections on long thin gratings are fewer than for gratings whose shape is closer to square. Thus we can imagine that there is more flexibility in the initial planning of architectural arrangements on squarer gratings, and more possibilities for the conversion of such plans after they are built, by rearranging internal walls etc.

The shapes and sizes of gratings are linked with the numbers of *fronts*, since in larger and squarer gratings the proportion of grating cells in the interior to those on the perimeter is higher. There are therefore more opportunities here for arrangements with interior rectangles, and hence with fewer fronts. The ratio of fronts to interior rectangles in a dissection is another measure of how 'outward-looking' or 'enclosed' the arrangement is. The measure is not by any means the same as Combes's ratio of external to internal wall segments; but it points to something of the same qualities.

Bloch's *graph partition* brings out properties which are of the greatest interest architecturally, since the valency of a vertex in the (weak dual) adjacency graph corresponds in the dissection to the number of other rectangles to which one rectangle is adjacent. Some rooms, such as corridors or halls, are typically adjacent to many others (the corresponding vertices have high valencies), indeed this is exactly their purpose; whereas with private rooms such as offices or bedrooms it is perhaps functionally necessary for them to possess only a single adjacency (the corresponding vertex has valency 1): adjacency to the circulation.

Many plan types are characterised by the distribution of this pattern of adjacencies between rooms. There are hierarchical plans in which one or a very few major spaces are adjacent to many other minor ones. A station concourse with its surrounding offices, bars, restaurants, kiosks, etc is a case in point. And there are 'democratic' plans, with a more even distribution of adjacencies throughout. The cellular plans of American Indian pueblos provide an example here.

Since Bloch considers only the weak dual adjacency graph, he does not take account of adjacencies of rooms to exterior regions, whose number must equal the number of external wall segments w. The patterns of distribution of these adjacencies are naturally also of great functional significance in buildings, in characterising different plan types. We shall return to these topics in chapters 10 and 11.

As regards Bloch's results on the occurrence of different types of *symmetry* in dissections, the most important result is that illustrated in figure 8.7. This shows the relative frequency of dissections possessing any symmetry (other than the identity) in the population of all dissections, for values of n between 5 and 9. The percentage of symmetrical dissections drops rapidly, from over 50% for $n = 5$ (compare figure 8.6) to around 2% for $n = 9$. Clearly, for still larger values of n the proportion of all

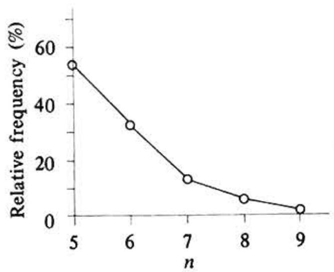


Figure 8.7. Percentage of all rectangular dissections possessing any symmetry (other than the identity), for values of n between 5 and 9 (from Bloch, 1979a).

dissections showing symmetry properties of any kind will become very small indeed. It is a characteristic of many real architectural plans, of course, especially of Renaissance or Classical style buildings such as those planned according to the precepts of the Beaux Arts, even those with very many rooms, that they possess at least bilateral and perhaps other symmetries. Bloch's result suggests that such arrangements are very rare when considered in relation to the whole range of possible rectangular configurations.

The most frequently occurring type of symmetry in dissections generally is D_1, and after this C_2. Symmetries of order 4, that is, D_4 and C_4, are much rarer. Of the D_1 symmetries, the orthogonal (horizontal or vertical) are much more common than the diagonal. The occurrence of symmetries of the different types is related in rather complicated ways to the oddness or evenness in various combinations of l, m, and n, and to the manner in which central 'cores' of one or a few rectangles may be formed, around which others can then be symmetrically disposed.

Bloch derived these statistics by generating all dissections, as explained, and then picking out those with each kind of symmetry. Krishnamurti and Roe (1978) have, however, published a method, using 'colouring' techniques similar to those described in chapter 5, for generating *only* dissections possessing certain specified symmetries. Figure 8.8, for example, illustrates all dissections with C_4 and D_4 symmetries, on square gratings where $l = m = 3$ and $l = m = 4$. (It should be remembered that in all this discussion we are talking of the symmetries of configurations on dimensionless gratings. Such symmetries could well be destroyed with the assignment of dimensions to the grating intervals.)

Besides the classifiers used in the catalogue, Bloch has also compiled statistics for other properties. These include the occurrence of two further

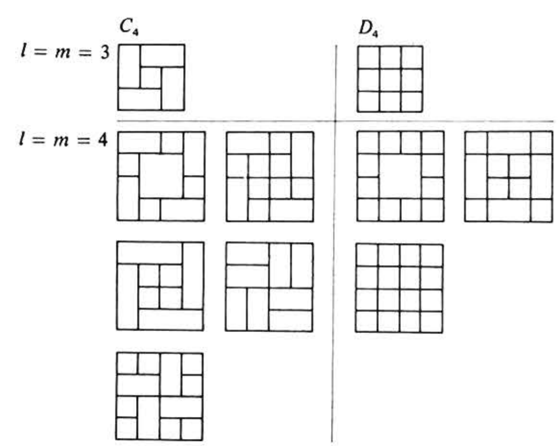
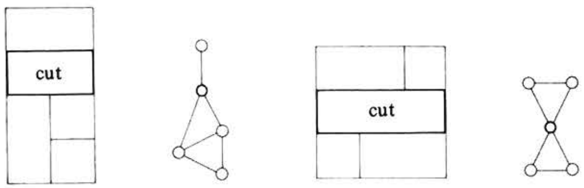


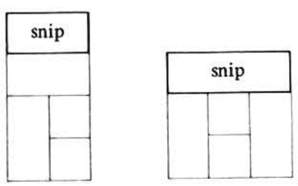
Figure 8.8. All dissections with C_4 and D_4 symmetries on square gratings, $l = m = 3$ and $l = m = 4$ (from Krishnamurti and Roe, 1978).

special types of component rectangle, besides fronts. The first of these is a *cut*: it is a rectangle which extends across the complete width or length of a dissection, and which if removed would leave two separated parts:



(These parts would themselves be rectangular dissections.) In the weak dual adjacency graph of the dissection it corresponds to a cut vertex, hence the term. In architectural parlance a cut is a 'through-room'. Very often a corridor in an office block is a cut in this sense.

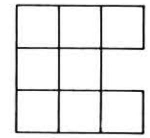
Another type of rectangle is the *snip*. This is a rectangle which lies along the whole length of one of the sides of a dissection, and may be 'snipped' off, leaving a complete dissection of order $n-1$ behind:



It is the type of rectangle which is added in operation (2) of Steadman and Mitchell's generating algorithm, or operation (2) of Earl's algorithm, as described in chapter 4. In architectural terms a snip is something like an end bay. Figure 8.9 illustrates both a snip and a cut in the 'central fireplaces type' house of chapter 2.

We will come back to Bloch's catalogue. In the meantime we should look at some equivalent work which has been done for polyominoes, by March, Matela, and O'Hare. These authors have compiled a number of statistics for various properties of polyominoes, in some cases for $n = 6$, 7, and 8, and in other cases for the whole population up to $n = 9$.

March and Matela (1974) have tabulated the numbers of internal and external wall segments in polyominoes for $n = 6$, 7, and 8, in much the same way Combes did for dissections. Indeed it turns out that, in general, the Combes equations (8.3) and (8.4) governing wall segments apply equally to polyominoes. Of course, there are no internal three-way junctions in polyominoes. But there may be four-way junctions (both internal and on the perimeter); and the total number t of wall segments, for some given n, is dependent on the number i_4 of *internal* four-way junctions, as March and Matela point out. In the following example:



where $n = 8$ and $i_4 = 2$, then

$$p = n + i_4 - 1 = 9, \quad \text{and} \quad w = 2n - 2i_4 + 2 = 14.$$

Our earlier demonstration of the equations for dissections made use of Euler's polyhedral formula, which must still apply. We also relied on the fact of the boundary of the dissection being rectangular to derive equation (8.2), which related wall segments to the numbers and types of all junctions. Clearly for polyominoes we would have to proceed somewhat differently; but it requires only a slight extension of the same approach to show that equations (8.3), (8.4), and (8.5) still hold(15).

Since polyominoes differ from dissections in that their boundaries can take up many shapes, it is possible to apply to polyominoes new measures, which aim to capture some of these shape characteristics. Matela and March compute a simple *shape index* w/p. When this ratio is high the polyomino is long and thin (although it may be branching or bent round on itself); when it is low, it is compact. (The same is true for the measure applied to rectangular dissections.)

A second index devised by March and Matela relates the number of external wall segments to the number of component squares. This is a *perimeter index* and is expressed as $w/(4n)$. In this way its value for any shape is confined to a range between 0 and 1, since for arrangements without internal four-way junctions, $w = 2n+2$, and the index takes the

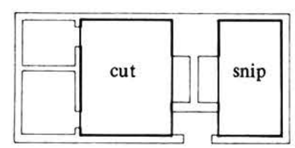


Figure 8.9. Examples of a snip and a cut in the 'central fireplaces' type house of chapter 2. A snip is a room in a plan of rectangular dissection type which lies along the entire edge of the dissection and may be removed (snipped off) to leave a complete dissection of order $n-1$. (It is something like an end bay.) A cut is a room extending across the entire width or length of the dissection, which if removed would leave two separate dissections (it is a through-room).

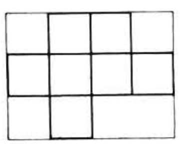
(15) The exception is where the polyomino contains a hole or holes, and these 'courtyards' are treated as exterior regions, and the wall segments which border them as external wall segments. This possibility first arises with $n = 8$, where the eight cells can surround a unit square hole in a complete square ring. (It is possible for a polyomino of order 7 to form a ring, but the two 'ends' of the figure touch only at a corner; and in these circumstances the Combes formulae still apply).

value $(2n+2)/(4n) = (n+1)/(2n)$. When $n = 1$ (and $w = 4$) the index thus reaches its maximum value of 1.

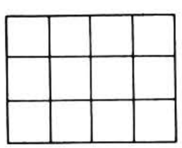
The perimeter index too is a measure of how sprawling or clustered a shape is. It is analogous to a property which is sometimes measured in architectural plans: that is, the ratio of external wall length to floor area. (These are dimensioned quantities, whereas the polyomino is 'dimensionless'). Such a ratio can be related to the rate of heat loss from a building, which is a function of exterior surface area per unit volume. Also it can have implications for building costs, since a lower value means that the same floor area is enclosed within a shorter length of external wall.

Both the shape index and the perimeter index are rather crude measures of shape as such, and although they do vary with shape, they can give the same value for polyominoes which are disposed in very different kinds of patterns. [Several other measures of comparable kinds, and of varying degrees of computational elaboration, have been developed by geographers for describing the shapes of territories or islands (see Bunge, 1966; Blair and Bliss, 1967; Haggett and Chorley, 1969).]

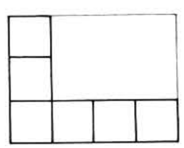
One means of capturing further aspects of shape, as Matela and March show, is to set each polyomino on its grating:



Then the grating size $l \times m$ gives a measure of the 'proportions' of the polyomino as a whole. It is also possible to express a 'density' in terms of the ratio of the order of the polyomino n, to the total number of grid cells lm contained within the complete grating:



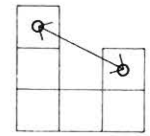
$n = 12$, $lm = 12$, density = 1



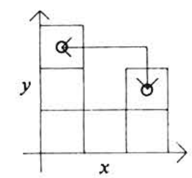
$n = 6$, $lm = 12$, density = 0·5

At a maximum this density is 1 where the polyomino itself has a simple rectangular perimeter and no holes. At lower values the shape is more 'open', with bends, holes, branchings, or convolutions.

A further series of measures applied by March and Matela, and by Matela and O'Hare (1976b) are concerned with the *distances* between pairs of cells in polyominoes. The distances are measured between the centres of cells, and are of three kinds. First there is the direct *straight-line*, or *Euclidean* distance between two centres:

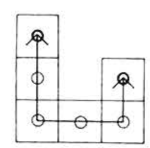


Then there is the *rectangular* or *taxicab* distance which is measured imagining the polyomino to be set orthogonally on a coordinate system. The difference between the x coordinates of the two centres in question, is added to the difference between their y coordinates.



It is the distance traversed moving always in directions parallel with the sides of the figure. It is called taxicab distance in reference to the routes followed by cabs in an American type grid-iron pattern of streets (see Krausse, 1975).

Finally, there is the *graph distance*, which is the shortest distance between two centres along an orthogonal route which passes always inside the figure, that is, a path consisting of successive edges in the adjacency graph, assuming this graph to be drawn with the vertices at the centres of the corresponding cells. Where the polyomino is bent around on itself or contains holes, then this graph distance may differ appreciably from the rectangular distance:



A single aggregate measure may be derived for each cell by adding together the distances (calculated in any one of the three ways) from its centre to the centres of all other cells. And a total figure can be obtained for the whole polyomino by then adding together all these aggregate figures. The measures so obtained describe the relative proximities of cells to all others, and indicate the compactness or elongation of the overall shape of the polyomino, once more. A further possibility is to take the *centroid* of the whole polyomino, defined as the central point of its grating, and measure distances to the centres of all cells from this centroid.

Where the polyominoes are regarded as approximating architectural plans, then it is arguable that the graph distance is likely to be the closest to the actual distance which a person might travel between two rooms.

The taxicab distance is likely to be, for rectangular plans, a second-best approximation, since circulation routes in buildings tend to run orthogonally. Interest in comparing the three distance measures first arose in the architectural context, in connection with computer methods for assembling plan layouts which, as mentioned in chapter 2, tended to make use of regular square grid systems of description. Rooms or zones were then located on this grid so as to minimise circulation distances, supposedly; these were measured as straight-line or Euclidean distances.

As Tabor (1976) pointed out, it would have been much more realistic, in the absence of any predefined circulation routes, to use a taxicab measure of distance. Even this might depart quite substantially from the real distances between rooms, as measured along actual circulation routes, depending on the shape of the plan overall—whether it contained courtyards or was branching in shape, etc—as Tabor demonstrated in a series of analyses. In this light, it is not surprising that by attempting to minimise the sum of Euclidean distances between rooms, such methods tended to produce plans of deep, compact, and clustered form.

There are implications too, in this work of Matela and O'Hare's on distance measures, for the use of regular square grids in representing the land surface of towns or regions in simulation models.

In the present context we might question the significance of the absolute values obtained for distance measures on polyominoes, as such, however; since the polyomino represents an architectural plan in undimensioned form, and when the configuration is dimensioned, so all the distances between cells will change accordingly. The same argument would hold for similar measures applied to 'dimensionless' rectangular dissections. On the other hand the *relative* values of these distance measures would continue to give information about shape and arrangement.

As a final class of properties of polyominoes, Matela and O'Hare (1976a) investigate some of the relations between polyominoes and their (weak dual) adjacency graphs. We have already seen in chapter 6 how the relation between rectangular dissections and their adjacency graphs is not one-to-one; and the same is true for polyominoes. It is, however, possible to distinguish *classes* of polyominoes which share the same adjacency graph. And the properties of the graphs can serve to distinguish some further, characteristic features of these classes of arrangement.

If we take the weak dual graph representing only internal adjacencies between cells, then the number of vertices equals n and the number of edges equals p, as for dissections. Out of all graphs, these must at least be connected and planar. Since the polyomino may have in its interior only four-way junctions, it follows that the (interior) faces in the embedded graph may be only four-sided.

Figure 8.10 is adapted from Matela and O'Hare. It illustrates in successive rows in the diagram, for $n = 4$ in each case, reading from top to bottom: all connected graphs, all graphs which may be the weak dual adjacency

graphs of polyominoes, and the polyominoes themselves. See how, for example, three out of the five polyominoes for $n = 4$ share the same graph. Matela and O'Hare have enumerated all such possible adjacency graphs for polyominoes up to $n = 9$, and illustrate in their paper the possibilities up to $n = 8$.

The numbers of weak dual adjacency graphs for values of n up to $n = 9$, compared with the corresponding numbers of polyominoes is given by:

order	1	2	3	4	5	6	7	8	9
polyominoes	1	1	2	5	12	35	108	369	1285
adjacency graphs	1	1	1	3	4	10	19	49	112

The ratio of graphs to polyominoes decreases with increasing n; that is, for larger values of n there are some graphs which are shared by very many polyominoes. If the number of possible adjacency graphs is compared with the total number of all connected graphs, for increasing values of n, this ratio is also found to decrease, rapidly:

vertices	1	2	3	4	5	6	7	8	9
adjacency graphs	1	1	1	3	4	10	19	49	112
all connected graphs	1	1	2	6	21	112	853	11117	261080

As Matela and O'Hare point out, this implies that out of all the possible patterns of adjacencies between rooms which a designer might wish to create, only a relatively few, for larger values of n, can be accommodated in polyomino-type plans.

It should be said, however, that this result is first of all a consequence of the fact that many connected graphs for large n will be nonplanar, for a start. And after that, it is in large part attributable to the particular geometry of the polyomino itself. The valency of the vertices in permissible

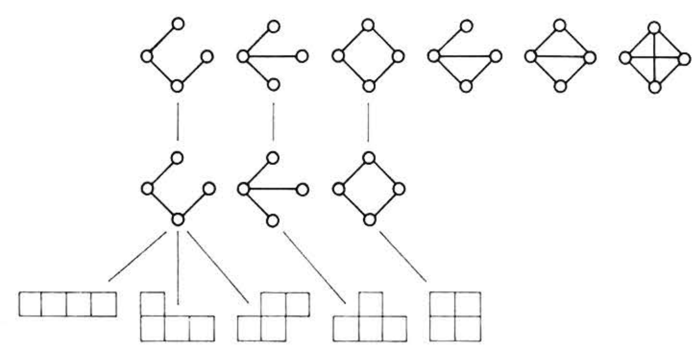
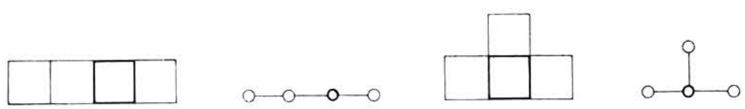


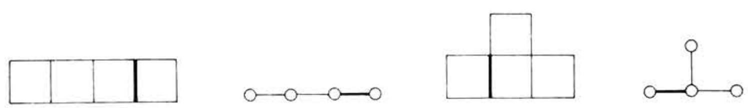
Figure 8.10. Relationship of polyominoes to their adjacency graphs for $n = 4$; after Matela and O'Hare (1976a). The top row shows all connected graphs for $n = 4$. The middle row shows all of those graphs which can be the weak dual adjacency graphs of polyominoes. The bottom row shows the polyominoes themselves. See how three out of the five polyominoes share the same graph.

adjacency graphs must not exceed 4. And faces in the embedded graph must be four-sided. The graphs of rectangular dissections, for example, are not so severely restricted; although here too we would again expect there to be many possible connected graphs, even connected planar graphs, which cannot be mapped into dissections.

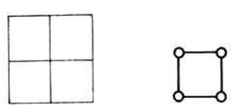
Matela and O'Hare (1976a) have made some studies of the possible *subgraphs* of the adjacency graphs of polyominoes, which are discussed here in chapter 10 in connection with the adaptability of plans. And finally they have investigated the occurrence of *cut vertices* and *bridges* in the adjacency graphs of polyominoes. A cut vertex would correspond to a cell in the polyomino which if removed would leave two or more disconnected parts (themselves smaller polyominoes). It is the exact analogue of Bloch's *cut* in a dissection. Some examples are:



A bridge in the graph, on the other hand, is a single *edge* whose removal disconnects the graph—it represents a single *partition* wall serving to divide the polyomino into two separate parts:



Matela and O'Hare present statistics for the total number of bridges in adjacency graphs for polyominoes, up to $n = 9$. Notice that the graph of the 'square' polyomino for $n = 4$ contains neither a cut point nor a bridge:



Going back to some of the general questions about the classification of dissections which were raised at the beginning of this chapter: once a catalogue, say of dissections, such as Bloch's, or an equivalent catalogue of polyominoes, is completed, to what kinds of use can it be put?

It is possible to imagine that a designer working on a particular planning problem might consult the catalogue to find an arrangement or some few arrangements corresponding to his requirements. The better the catalogue is organised, in terms of a hierarchy of progressively more precise specifications, the easier it will be for him to find the configurations he is seeking. The exact organisation of the classificatory hierarchy would depend, as mentioned, on the particular interests of the users.

But it would be plausible to imagine that it might perhaps be structured on five principal levels (figure 8.11) (compare Korf, 1977)[16].

The first level would be a classification by the simple number of cells or rooms (and hence the number of 'functions' possible to accommodate, one to a room) in the plan. Thus the rooms are considered at this stage as an unordered set. At the second level, adjacency graphs with the appropriate number of vertices (and perhaps with other specified properties, such as particular graph partitions etc) would be considered. At this level, then, topology is introduced. At the third level the adjacency graphs are embedded. At the fourth level shape would be introduced; that is, the classification would move to some particular class of geometrical designs, such as polyominoes or rectangular dissections. At the lowest level,

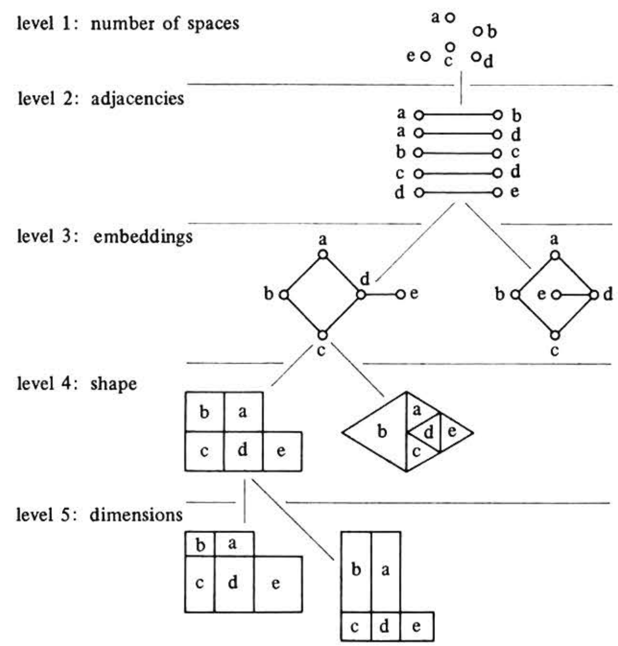


Figure 8.11. Five-level hierarchy for the classification of plans. At the highest level plans are considered as unordered sets of rooms. At the second level, relations of adjacency between rooms are introduced. At the third level the resulting adjacency graphs are embedded. At the fourth level shape is introduced, in the form of some overall geometrical discipline, that is, the plan is organised on a rectangular grid, a triangular grid, it is composed of rectangular rooms, etc. At the fifth level, dimensions are considered.

(16) Guerra (1977) has hinted at some kind of parallel between such a hierarchy of increasingly abstract representations for architectural plans, and the famous classification of geometries by Felix Klein in his 'Erlanger Programme' of 1872 (see Klein, 1939). Compare also March and Steadman (1971, pages 19-25), and March and Steadman (1979).

dimensions would be considered. As we shall see in the next chapter, it is quite possible that a given geometrical arrangement, which meets specified adjacency requirements, may not be capable of being dimensioned to the sizes which are perhaps required for each room.

Bloch (1979a) himself has described (although not implemented) a computer system which would allow the operator to make a search for dissections in essentially this manner. The computer would display on a screen some representation of the 'solution space' for dissections of some given order or orders, such as for example the Combes diagram. The user would then point to and so select, with a light pen or cursor, subsets of arrangements, perhaps specifying further properties, until some single dissection was reached, or some small set defined, which could then be enumerated. (In this case Bloch imagines the specified dissections being generated 'to order', and not being retrieved from some stored catalogue. The effect is the same, although the computing implications, for storage space required, and the speed of response of the system, are very different.)

In the chapter which follows we will look at an actual existing computer program which does something similar. The system in question is perfectly automatic. It is not interactive, nor does it display dissections or groups of dissections graphically in the course of operation. Also it works with a stored and unordered list of all dissections up to $n = 8$, and for this reason is slower than it might be in running. However, it does move in effect through the same five-level classificatory hierarchy as just described, producing at the end all plans, of rectangular dissection form, which conform to specified requirements of adjacency and dimension.

It might with justice be argued, on the other hand, that this suggested form of classification of plans has a distinctly 'functionalist' flavour. It echoes the typical modern movement or 'design methods' approach, starting from a 'programme of requirements' stated as a list of rooms, which are then related together by means of a 'functional diagram' or 'bubble diagram' (the graph), and finally given precise shapes and sizes.

In other approaches to design, or at other periods, the architect might well have started with geometry, with some specific form, perhaps a historically evolved 'type-form', and would have tried to fit the necessary accommodation as best he might into this chosen shape. In the *Beaux Arts* system of composition, the starting point was always a skeleton of what were simultaneously circulation routes and axes of symmetry, around which the rooms were then arranged to form the geometrical *parti*. Even in a functionalist approach, it may be that certain dimensions and shapes are necessarily fixed right from the beginning, as when a building is to be fitted exactly onto a given site. In such a case the shape and size constraints may well have their effects back on possible patterns of adjacency.

All this raises deep and difficult issues about how computer systems intended to aid the designer in producing plans might be organised hierarchically, if they are not to be too constraining on the directions

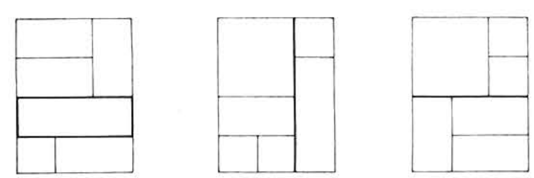
which his search may take. Simon (1975) has even suggested that *style* in design may be connected with the sequence in which constraints are applied, and so to the nature of the route which the search process takes through the space of possibilities considered.

Some of these questions will be taken up again in the next chapter. It is not in such direct applications in design methods, nevertheless, that I myself believe the greatest usefulness of catalogues, and an enumeration approach to plan arrangements in general, lies. Within the limitations imposed by the definition of some class of designs—dissections, polyominoes, or whatever—all permutations of arrangement are exhaustively covered. Hence *general* statements may be made about the range and behaviour of *all possible designs*, within the given definition.

We have been looking—in March, Matela, and O'Hare's work on polyominoes, and Bloch's on dissections—at some theoretical generalisations of just such a statistical kind. These studies are only a beginning, as their authors would admit, and they are by no means complete; but they do begin to suggest how a comprehensive view might be built up of the total extent of plan forms of certain types, of how the behaviour of plans varies systematically with changes in properties of different kinds, and of how rare or frequent plans with given characteristics are in relation to the whole 'universe' of possibilities. (We have noticed, for example, how very scarce are arrangements with any symmetry among all dissections for larger n, however common bilateral symmetry may be among actual large building plans.)

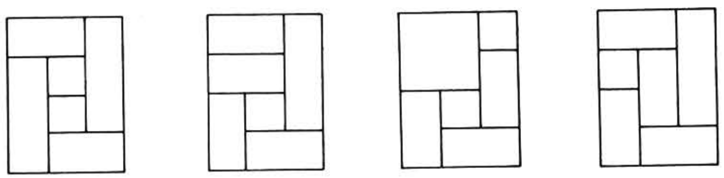
What is more, a *statistical* treatment offers the prospect of extending the scope of this kind of treatment beyond the practical limits imposed on exhaustive enumeration by the accelerating growth in combinatorial possibilities. It may not be feasible to catalogue all dissections, say, for much larger values of n than 10 (although it might be possible to count their total number). But it is quite conceivable that general findings on dissections up to this order could be extrapolated to larger values of n, or that a sampling approach might be used at higher levels.

It is worth pointing out one typical feature of dissections in connection with this problem of how to treat larger configurations. This is that certain large dissections are capable of being separated into two or more smaller dissections, either at *cut* rectangles, or where internal walls run across the whole width of arrangements (they can be cut as though by guillotine):



For larger values of n a great proportion of all dissections are of this character.

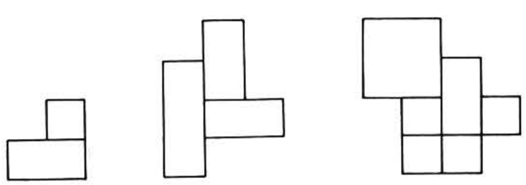
Thus we can identify a much smaller set of 'irreducible' or 'uncuttable' dissections in this sense, from which others can be made up by a process of aggregation (see Bloch, 1979a, chapter 7; Earl, 1980). For example, in the following figure all such dissections are given for $n = 6$, of which there are only four:



What matters then is the pattern of adjacencies which occur across the boundary where the dissections abut. Configurations with very large n might be described by decomposing them by the converse process into a series of smaller dissections. (Every polyomino may be cut into smaller polyominoes; nor need the cuts be 'guillotine' cuts.) Another possibility is the hierarchical nesting of dissections, where a component rectangle of a dissection at one scale, is itself made up from a complete rectangular dissection at a smaller scale.

In chapters 10 and 11 I will propose that the theoretical work described here on plan morphology has many applications in building science and in architectural history. It may be worth noting now some seemingly promising areas for further studies in 'pure morphology'. For example some of the analyses described in this chapter could themselves be extended or completed. In particular, the relationship of rectangular dissections to their adjacency graphs remains to be explored, in the way in which Matela and O'Hare have done for polyominoes.

An obvious and powerful objection to all the work presented so far, is that even within the general limits of rectangular geometry, dissections and polyominoes represent only particular and restricted classes of designs; and there are many real architectural plans which, though all their walls are straight and lie in one or other of two perpendicular directions, do not correspond to either of these types. There are plans, for example, corresponding to packings of rectangles with outside perimeters which are not themselves simple rectangles (that is, they are not rectangular dissections), but which have internal three-way junctions (that is, they are not polyominoes). Some examples are



The rooms in a plan might not be simple rectangles, but could take L, U, T, +, or more complicated shapes. Beyond this, the spaces in a plan might not even be completely enclosed, but only partially delimited with freestanding screen walls—as, for example, in plans characteristic of Mies van der Rohe's early work, or the architectural projects of the de Stijl group (figure 8.12).

There are no reasons in principle why such classes of plan might not be enumerated by techniques similar to those described for generating dissections—although the numbers of possibilities for given n (or, say, for given numbers of walls) would naturally, for the more general classes of arrangement, be very much larger. Such enumerations remain to be made.

Earl (1980) though has proposed a classification which embraces all these and other types of rectangular configuration, and which is based in the first place on the nature of the *endpoints of walls* in an arrangement. Such an endpoint may be one of three types:

(a) not coincident with any other wall in the shape,

(b) coincident with another wall at its endpoint (that is, at a 'two-way junction'),

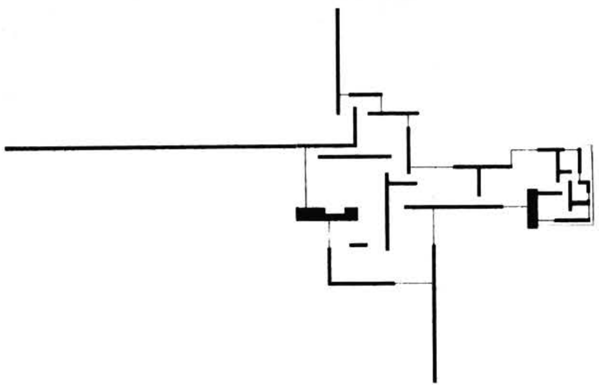
(c) coincident with another wall not at its endpoint (that is, at a three-way junction).

(Notice that at a four-way junction, neither of the walls involved can have its endpoint).

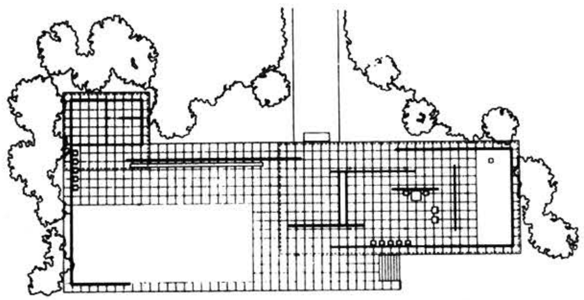
Six types of shape can be defined in these terms, depending on whether walls in those shapes have endpoints of type (a) only; of type (b) only [it is impossible to have shapes with only endpoints of type (c)]; of types (a) and (b), (a) and (c), or (b) and (c); or of all three types (a), (b), and (c) together.

Shapes with wall endpoints only of type (a) are close to the Mies van der Rohe and de Stijl plans [although those illustrated in figure 8.12 also show some wall endpoints of types (b) and (c)]. A rectangular dissection has wall endpoints all of type (c), with the exception of the four walls which form the outside perimeter and meet in points of type (b). Shapes with type (b) wall endpoints only, are like the outside boundaries of polyominoes. A further subclassification is possible in terms of the types of *intersections* of walls.

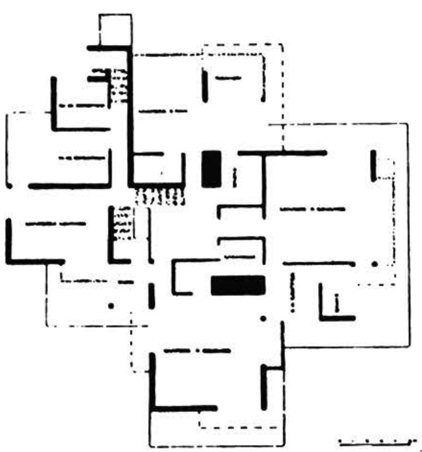
In the same paper Earl goes on to discuss the relations between some of the different graph-theoretic representations of rectangular arrangements, which I have presented here as a rather heterogeneous range of possible approaches, but which Earl is able to show can be connected together by considering their underlying structures at a more general theoretical level. So Earl relates together the forms of representation used variously by Grason (1970c), Mitchell et al (1976), Flemming (1977), the 'electrical network' representation of Brooks et al (1940) and as generalised to three dimensions by Teague (1970), and his own work with March (March and Earl, 1977).

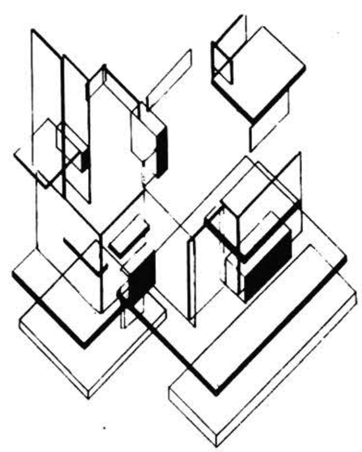


(a)



(b)





(c) (d)

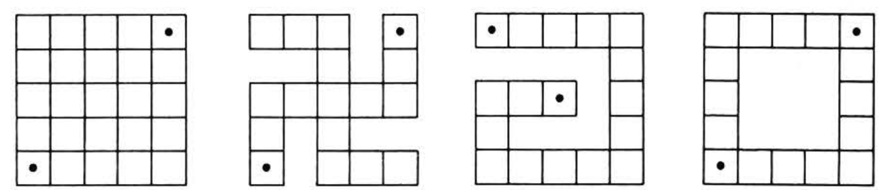
Figure 8.12. Plans in which spaces are only partially delimited, by means of free-standing screen walls: (a) Mies van der Rohe, brick villa project 1923; (b) Mies van der Rohe, German Pavilion, International Exhibition, Barcelona 1929; (c) Theo van Doesburg and C van Eesteren, private house project 1923, first floor plan; (d) Theo van Doesburg, 'Counter-construction'—axonometric of private house project, Paris 1924.

Finally, there are the many classes of regular *nonrectangular* designs, such as, for example, triangular and triangular/hexagonal dissections; the various so-called 'plane tessellations' in which tiles of one or more different shapes are packed together in periodic patterns to fill the plane (see Krishnamurti and Roe, 1979); or the triangular and hexagonal analogues of the polyomino (Golomb, 1966; Lunnon, 1972). Empirical studies of actual buildings, as we shall see, however, suggest that architectural plans corresponding to these types of pattern are rather rare in practice.

Exercises

8.1 Prove that Combes's formulae relating to numbers of external and internal wall segments in an arrangement, apply to polyominoes (see page 127). (You can follow the general lines of the demonstration for dissections on pages 114–115, but you will need to take account of the numbers and types of junctions which can occur in the perimeter of a polyomino.)

8.2 For the polyominoes illustrated, measure the value of March and Matela's 'shape index', the value of their 'perimeter index', and the distances between the pairs of cell centres indicated, as straight-line, taxi-cab and graph distances in each case.



8.3 Figure 8.10 shows the relationship of polyominoes for $n = 4$ to their weak dual adjacency graphs. Explore the relation of rectangular dissections to *their* weak dual adjacency graphs in a similar way, up to $n = 6$, using the catalogue given in the appendix. You should be able to compile a table showing the number of polyominoes and number of weak dual adjacency graphs for each order, equivalent to that for polyominoes given on page 131.

8.4 Using the catalogue of dissections in the appendix, find all 'irreducible' or 'uncuttable' dissections (see page 135) for $n = 7$.

Floor plan morphology in design

"This work is strictly non-architectural, ... it has nothing to do with architecture."
The Editor *The Architects' Journal*

Numerous efforts have been made over the last fifteen years to devise mathematical or computerised methods for generating architectural plans automatically. Some of this work has involved the kinds of graph-theoretical techniques and geometrical ideas which have been described in previous chapters. Despite the strong criticisms which I expressed in chapter 1 of the theoretical basis of much of this work, and in spite of reservations indeed as to its practical usefulness, I nevertheless feel that it will be helpful to describe some of it, if briefly, here. The reasons are that certain of the methods introduce new geometrical techniques for describing plans; or else they throw light on how various morphological issues of a general kind might be investigated.

There is a key distinction to be made right at the start. A method might be intended to generate just *one, or a few plans*, in which certain stated requirements of adjacency between rooms, and perhaps also constraints on the dimensions and shapes of rooms, are satisfied. Such methods are typically intended to mimic the human designer in the processes which he might be assumed to go through in producing a layout. In computing language they are *heuristic* methods, in which strategies are employed which are believed to lead towards some desired result, but are by no means guaranteed to do so.

On the other hand methods might be devised which would produce *all possible plans* (in the sense which will have become clear from previous chapters) conforming to the given requirements. These can be termed *exhaustive* methods.

The distinction is crucial, since in the first case the method or the computer program is itself effectively exercising choice, beyond the explicit specifications laid down by the user. The criteria for this choice are built into the way the method is structured, either deliberately or else more or less fortuitously, as, for example, when the first layout which is found, conforming to the given specifications, is presented as 'the solution'. These criteria may be hidden from the architect who is supposed to make use of the technique; and in any case will very likely not correspond to the criteria which he himself would apply, which would come out of a much broader set of considerations—to do with aesthetics, structure, heating, lighting, or whatever—quite outside the scope of the specific method.

In an exhaustive approach the situation is very different, since there is no choice exercised by the computer program or the technique itself, beyond the constraints which are fixed initially. The architect (or theoretician) is presented with the entire set of feasible alternatives under the specified

definition, and can then apply further criteria of his own for selection within this range. Or else he can, as a result of seeing the possibilities, go back and change the initial constraints so as to generate some different set.

The earliest 'design methods' for plan layout were all of the first, heuristic type. Many of these were concerned with producing plans in which the amount of pedestrian traffic generated would be a minimum. Thus for every pair of rooms or 'zones' in the building a figure would be worked out—perhaps derived from surveys—for the typical frequency of journeys made between those rooms per day or per week. In any actual layout, this figure could be multiplied by the distance separating the rooms in question. Then the total of all such products (trip frequency times distance) could be summed for all pairs of rooms. The design methods were intended to find arrangements in which this sum was minimised.

The typical form of representation used for plans, as mentioned in chapter 2, was a regular square grid of fixed grid dimension. Thus dimension was not separated from shape description, and all rooms had to be approximated both in shape and in area as a number of unit grid cells. Difficulties arose in the measurement of distances between cells or rooms, either as Euclidean, taxicab, or possibly graph distances, as described in the last chapter. The question of fixing values for the frequency of trips between rooms was also problematic, since it is quite plausible to suppose that this frequency might itself vary to some extent with the distance separating the rooms. The algorithms for generating layouts consisted in effect of techniques for packing together polyominoes (the rooms), or of rearranging those packings in such a way as to reduce the value of the circulation measure.

Such methods have been discussed and criticised very widely elsewhere, and so I do not propose to embark on any more detailed account. Comprehensive reviews of the subject have been made by Tabor (1970), by Eastman (1975), and by Mitchell (1977, chapter 13). The largest objection to these methods centres on the fact that they attempted to constrain the process of 'composition' of a plan by means of one single evaluative criterion of functional performance—that of circulation—and to optimise the arrangement (in a mathematical sense) on that basis alone.

Meanwhile, other workers had come to realise that at a smaller scale the absolute distances between rooms are not of such very great functional significance compared with the patterns of adjacency between rooms, and that these adjacencies could be represented by graphs. The topology of the plan might be manipulated by means of graphs, more or less independently of its shape and dimensions.

The first published application of graph theory to small architectural plans was made by Levin (1964). Levin confines his attention to the *access graphs* of plans, where the vertices represent rooms and the edges the existence of doors between those rooms. He discusses the possibility of enumerating all those planar connected graphs which could represent the access graphs of plans. In the first place he considers only outerplanar graphs. An outerplanar graph (see chapter 7, pages 90 and 91) is a graph which possesses at least one and possibly several embeddings such that every vertex lies on the exterior face. Such embeddings therefore represent plans with no interior rooms. There is a finite number of ways, as Levin goes on to show, in which n labels, signifying different functions for the rooms, might be assigned to the n vertices of each outerplanar graph. He enumerates all such graphs and their possible labellings up to $n = 4$. [His enumeration thus corresponds to that of Korf (1977), and of March and Earl (1977) (compare figure 7.1), but for outerplanar graphs only. Also the interpretation of the graphs is different—in Levin's case as access graphs, in the other authors' case as adjacency graphs.]

Levin mentions the problem that planar graphs in general may have more than one embedding. He also introduces the subject of nonplanarity, and shows that certain access graphs (specifically K_5) are not capable of being realised in plans, and that in such cases one or more of the access requirements must be dropped.

Levin in this early paper thus adumbrates an exhaustive approach to the generation of small plans, or at least their graphs; although it should be said that he himself expresses doubt as to whether an enumeration of access graphs beyond $n = 5$ would serve any practical purpose. Also he deals only with graphs, and does not make any systematic approach to the geometry of plans.

In the second half of his paper, however, Levin goes on to describe a design method based on circulation criteria which is much closer to the heuristic techniques mentioned earlier. The method consists in building up an access graph edge by edge. Those edges are placed first which correspond to pairs of rooms between which the frequency of trips is greatest. Edges are added in descending order of their trip-frequency values, and planarity is preserved at the later stages by judiciously omitting edges with low values.

The rationale for this procedure is, clearly, that rooms between which trips are frequent, should be made directly accessible one from another, but that this is less important where the traffic is infrequent. The assumption has difficulties, since if a corridor or other circulation space were introduced rooms could still perhaps be made very close though not adjacent; and many of the values for trip frequencies between pairs or rooms would alter if the new corridor were included as a 'room'.

It is possible that the final resulting access graph might have several planar embeddings. Levin argues that the designer can choose between these on the basis of other considerations. For example, he may want to put rooms which require natural lighting on the perimeter of the plan. Again, the relative sizes of rooms may mean that certain embeddings of the graph are impossible, such as when a cycle of vertices representing small rooms encloses a vertex or vertices representing very large rooms.

This method of Levin's, since it aims to optimise the plan on the basis of a measure of pedestrian movement, is open to all the criticisms made of similar circulation-minimising techniques. Nevertheless it points the way to some exhaustive enumeration methods which do not have the same limitations. And in general a great deal of subsequent research in applications of graph theory can be seen as extensions to or mechanisations of this original work of Levin's.

Two authors following Levin's lead were Cousin, and Friedman. Cousin (1970) describes some of the general ideas of the graph-theoretic treatment of plans, but he does not suggest the idea of an enumeration of possible graphs or plans, nor does he propose any explicit design method.

Friedman (1975) introduces a somewhat eccentric and erratic series of graph-theoretic ideas about the planning of flats into a much larger work on 'scientific' architecture and city planning. His is an exhaustive approach: exhaustive with a vengeance. Like Levin he considers the access graphs of plans, with the vertices labelled to signify the functions of rooms. In addition Friedman allows the possibility that each room might (independently of all others) take a series of geometric shapes, might be oriented to face different directions, and might contain different types of equipment such as plumbing fixtures or furniture. He envisages a computer system—the 'flatwriter'—which would allow the user—not an architect, but the prospective inhabitant of the flat—to permute together a set of such options in *all* combinations, to give what Friedman enthusiastically recognises would be a truly astronomic number of alternative 'designs' (figure 9.1).

Of course, the procedure completely ignores the fact that many, probably the great majority, of these combinations would be incompatible, because the arbitrarily chosen shapes and sizes of the rooms would not pack together geometrically, the required orientations would be impossible to achieve together with the adjacency requirements, and so on. The only constraint which Friedman allows is that embodied by the access graph, and the recognition that it might be nonplanar.

The user of the flatwriter would in theory be helped in his choice from this vast range, by being provided with a figure for each plan, for what is essentially its circulation 'cost' (Friedman calls it an 'effort value'). A set of trip frequencies is specified, not just between adjacent rooms, but between all pairs of rooms. For a given access graph the shortest path between each pair of rooms is calculated as a number of edges, and this number is multiplied by the corresponding trip frequency. All such products are then summed to give the total 'cost' of the plan.

Quite apart from any doubts as to whether this particular measure provides any realistically useful basis for evaluating plans, the main criticism to be made of Friedman's suggested approach concerns the way in which huge numbers of combinations of plan elements are generated blindly with very little account taken, beyond the access graph, of the internal relations between those elements[17]. The flatwriter is a kind of architectural counterpart to the proverbial monkeys typing Shakespeare, or the Lullian combinatorial machines for literary composition which Gulliver found at the Academy of Lagado. Other workers, as we shall see, have given more consideration to the *structure* of the geometrical problem, and in this way have placed severe restrictions on the generation of combinations, so as to bring an exhaustive enumeration of plans within manageable bounds.

Several authors have come to the study of spatial layout problems out of a management science or operations research, rather than a properly architectural interest. They have been concerned with the arrangement of machines, equipment, or industrial activities in factories. Seppänen and Moore (1970) were among the first to propose the use of graphs in this

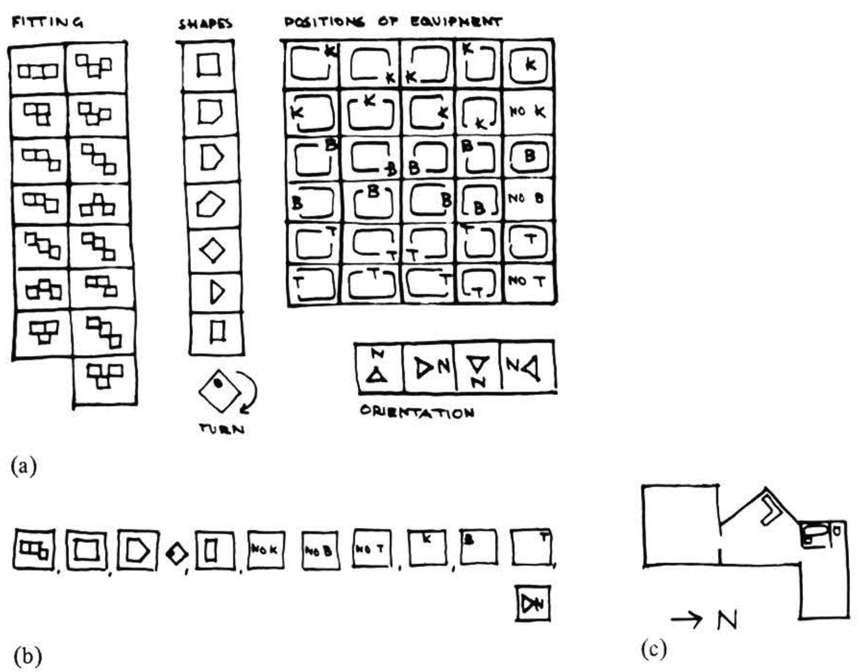


Figure 9.1. The 'flatwriter' computer system for generating floor plans, proposed by Friedman (1975). It is imagined that the user would select out of a variety of options (a) for room arrangement, room shape, positions of equipment, orientation, etc some set of particular specifications (b) for his desired layout. The program would then generate a plan (c) or plans, by permuting these elements together.

(17) Readers should also be warned that Friedman's arithmetic is unreliable. His calculation of the number of possible combinations of a hypothetical set of options offered by the flatwriter (Friedman, 1975, page 36) is wrong by a factor of six. And his example of the frequency with which 'Mr X' moves between the rooms of his house (page 41) implies that the average number of times the man enters a room per day is not equal to the number of times he leaves that same room.

particular context. They consider adjacency graphs and their dual relationship with plan graphs. Their main attention is devoted, however, to the possibility that a requirement graph might not be planar, that is, that the problem is overconstrained and adjacency graphs are to be sought as subgraphs of the requirement graph.

They propose first making a test for planarity[18]. If the requirement graph fails this test, they then go through a process of identifying some minimum 'resolving' set of edges, whose omission will make the graph planar. The method which they use here applies only to graphs containing one or more *Hamiltonian cycles*. (To make a formal test for the presence of Hamiltonian cycles as such—which this method does not attempt—would be difficult.)

Remember that a cycle is a continuous sequence of vertices and edges which does not double back on or intersect itself, and which returns to the original vertex. A Hamiltonian cycle is a cycle in a connected graph which contains every vertex of that graph. It takes its name from the nineteenth century Irish mathematician Hamilton, who made a puzzle consisting of a wooden dodecahedron whose twenty vertices were marked with the names of cities (see Biggs et al, 1976, chapter 2). The problem was to draw a route along the edges of the solid, passing through each city once only and returning to the original city (figure 9.2).

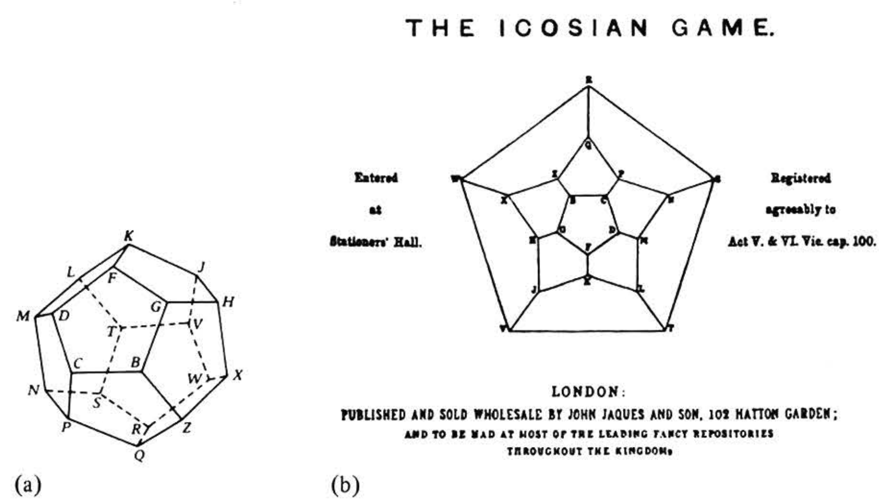


Figure 9.2. The 'icosian game' devised by the Irish mathematician Sir William Hamilton. The problem is to find a sequence of edges in the graph illustrated, which passes through every vertex once, and returns to the original vertex (a *Hamiltonian cycle*). The game was made in two versions: in the form (a) of a wooden dodecahedron, and (b) a projection of the same solid onto the plane.

(18) Actually this is strictly unnecessary to their purposes, since the procedure which follows, itself acts effectively as a planarity test.

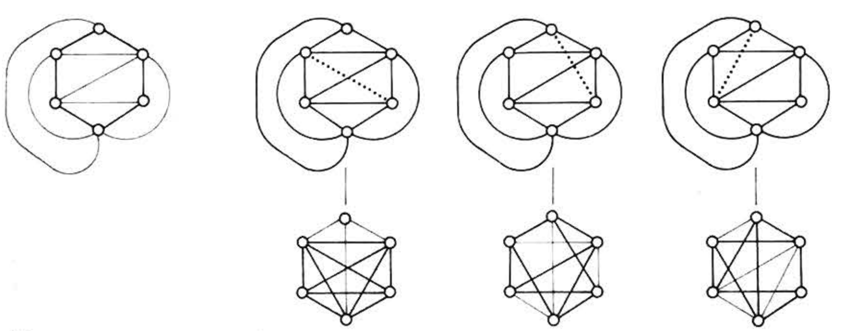
(Unlike some of the rather disappointing graph theory puzzles mentioned in chapter 6, this one is actually soluble.)

It is possible for a graph both to be nonplanar, and not to contain a Hamiltonian cycle, in which case the method described by Seppänen and Moore cannot be used. If the assumption is made, however, that a Hamiltonian cycle *does* exist, then it can be embedded so as to divide the plane into an interior and an exterior region. The remaining edges of the graph can then be reintroduced, and embedded either inside or outside the Hamiltonian cycle. Seppänen and Moore describe an algorithm for identifying incompatibilities between these embeddings of edges, and, further, for identifying sets of edges whose removal will make the graph planar.

Figure 9.3 illustrates the basic idea with a simple example of a graph on six vertices, which possesses a Hamiltonian cycle drawn here in regular hexagonal form. The interior of this hexagon can be 'triangulated' with additional edges, and the exterior face can also be 'triangulated', up to the point where the graph becomes maximal planar. Any edges which still remain to be placed must be incompatible with those already introduced. (In this case there remain only three positions in which edges might be added. An edge in one of these positions gives K_5, and an edge in either of the other two positions gives $K_{3,3}$.)

In a graph with more than one Hamiltonian cycle, Seppänen and Moore's method may find different sets of incompatible edges for different cycles. So all cases would have to be investigated to find the overall minimum 'resolving' set.

Seppänen and Moore recognise that there are further steps, from a planar adjacency graph to a finished plan layout, but make no proposals as to how these might be carried out.



(a) (b)

Figure 9.3. (a) A graph on six vertices containing a hexagonal Hamiltonian cycle (marked in heavier line). The interior and the exterior of the hexagon can be triangulated with additional edges, up to the point where the graph becomes maximal planar. Only three positions then remain (b) for the addition of edges (dotted lines), yielding in one case K_5 (left) and in the other two cases $K_{3,3}$.

A second pair of authors, Lépine and Nakajima (1973), took over this work of Seppänen and Moore, to apply it to architectural design, and suggested some elaboration of the edge-elimination algorithm, by giving weights to all the edges in the requirement graph to express their relative functional importance. They posed the problem then of finding that 'resolving' set of edges whose omission would make the graph planar, and meanwhile whose total value was a minimum.

Another worker who used similar techniques to attack problems of industrial plant layout was Krejčiřík (1969a; 1969b). His was a heuristic method designed to produce one optimal layout satisfying both adjacency *and* dimensional criteria.

In Krejčiřík's method the edges of the requirement graph are again weighted to represent their relative importance. It is assumed that the final plan will consist of rectangular spaces packed into a rectangular shape overall. So four exterior regions are defined on the 'north', 'east', 'south', and 'west' sides, and requirements for adjacencies of spaces to these regions can be introduced.

Krejčiřík's next step is to look for Hamiltonian cycles in the requirement graph. (If no such cycle exists the method is presumably inapplicable.) That cycle is chosen, for which the total weight of the edges which it contains, is greater than in any other Hamiltonian cycle. Meanwhile, the order in which the rooms occur around the cycle must be compatible with the requirements for their adjacency to the exterior regions.

The idea here, speaking rather loosely, is to find a way in which all the rooms can be placed in a 'ring' (the Hamiltonian cycle), which can then be set around the perimeter of the plan so as to satisfy the requirements for adjacencies of rooms to the exterior. Meanwhile, the order of the rooms on the 'ring' is such that adjacency requirements of great functional importance between them are satisfied. Where rooms are not required to be on the perimeter, the 'ring' can be folded back on itself, so that the rooms in question are placed in the interior of the plan. Figure 9.4 gives a notional illustration.

Now remaining edges representing further requirements of adjacency are added back in where possible, in descending order of value. (The most important requirements are satisfied first.) These edges can be added back, up to the point, in principle, where the region inside the embedded Hamiltonian cycle is completely triangulated; and to the point where the region outside the cycle is also as completely triangulated as is compatible with the adjacencies to the north, east, south, and west points. Any edges remaining at this stage must be omitted from the graph.

A test is applied to the resulting embedding of the requirement graph, to detect the presence of subgraphs which cannot be realised in plans made up of rectangular rooms (compare the figure on page 100). Where such subgraphs occur, appropriate adjustments are made, presumably by a suitable rearrangement of edges.

Krejčiřík's method then proceeds to take account of dimensional requirements, and finally produces a complete dimensioned layout. We will come back to this part of his work shortly. Meanwhile it should be said that Krejčiřík's approach can once again be criticised on the same grounds as all such 'optimising' methods. (Nor does his program necessarily produce the optimum arrangement, even in its own terms, although it is claimed to come near this.) What is more, it is not directly applicable to certain kinds of adjacency requirement graphs (those without Hamiltonian cycles).

A more serious and general criticism of many of these authors—Levin, Cousin, Seppänen and Moore, Krejčiřík—is that they really fail to make the distinction, which I emphasised in chapter 6, between an adjacency requirement graph, and the final adjacency graph of some plan in which those requirements are met. Or at least, they imply that typically the designer will specify a large number of adjacency requirements, sufficient to make up a triangulated or maximal planar graph; and that the plan graph can then be derived directly from an embedding of this adjacency requirement graph, through the dual relationship.

Thus they overlook the strong possibility that the requirement graph may only be a spanning subgraph of the final adjacency graph, and that additional edges might be introduced into the embedded requirement graph, simply to represent adjacencies arising through close-packing. Their concern is on the contrary devoted to the possibility that the requirement graph will contain *too many* edges, that is, it is nonplanar and the problem is overconstrained, so that certain edges must be eliminated.

One might question whether the subject of the planarity of requirement graphs really deserves all this attention, however. If one considers typical, realistic sets of adjacency requirements for architectural plans, it is arguably rather rare that they form graphs which are nonplanar. It takes some ingenuity, for example, to concoct plausible illustrations of sets of

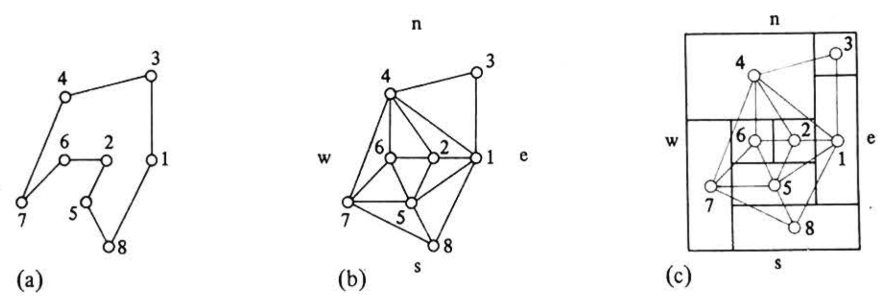


Figure 9.4. Notional illustration of the operation of a computer program for industrial plant layout due to Krejčiřík (1969a; 1969b). A Hamiltonian cycle is found (a) in the adjacency requirement graph. Further required adjacencies are added in (b), in descending order of importance, up to the point where the interior of the cycle is completely triangulated and the exterior of the cycle is also triangulated so far as is compatible with required adjacencies to exterior regions. A plan is derived (c) as the dual of this adjacency graph.

adjacency requirements for small house plans which turn out to comprise K_5 or $K_{3,3}$ (as in the example of figure 6.6). Much more often the process of moving from requirements to an adjacency graph of a feasible plan will involve the addition of edges rather than their subtraction; although obviously these new edges must be added in ways such that planarity is always preserved.

This point was well recognised by Grason (1968; 1970a; 1970b; 1970c), whose computer method for producing small plans conforming to adjacency and dimensional requirements is among the most sophisticated of those devised and implemented to date. Grason confined his attention to rectangular plans made up of rectangular rooms, that is, rectangular dissections. His was the first method to enumerate exhaustively all solutions to such problems.

The representation which he uses throughout to manipulate arrangements, is an augmented dual adjacency graph in which the edges are coloured and directed to represent the orientation of the corresponding walls in the plan, either 'north-south' or 'east-west', as described in chapter 7 (compare figure 7.8). Weights are then placed on the edges to represent the lengths of the wall segments (compare figure 7.9). A complete dimensioned plan would thus be represented in Grason's system by an adjacency graph in which the edges are coloured, directed, and weighted as in figure 9.5.

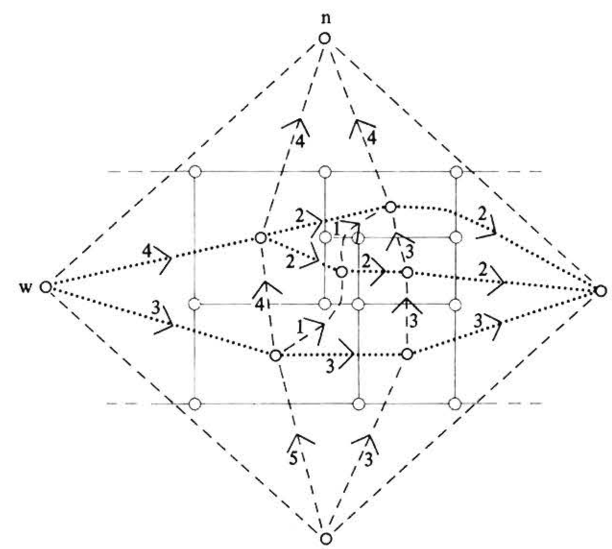
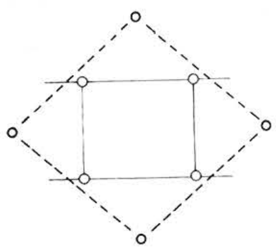


Figure 9.5. Representation of a plan of rectangular dissection form in the computer system for plan generation due to Grason (1970c). The edges of the augmented dual adjacency graph are coloured (dotted or broken lines) according as to whether they represent adjacencies across north-south or east-west walls, respectively. Edges are directed as they cross corresponding wall segments from south to north or from west to east, and weighted to represent the lengths of these wall segments.

Notice that the external walls of the plan running east–west are extended by convention beyond the four corners, and the adjacencies of the four exterior regions at north, south, east, and west are shown across these extended walls:



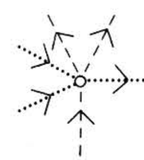
Grason has a special 'graph grammar' which is designed for carrying out operations on graphs. The significance of this grammar is that it represents the graph in an hierarchical manner, corresponding to the connectedness of its components. So the representation distinguishes in effect the 1-connected, 2-connected, and 3-connected components of the graph, and identifies the cut vertices or pairs of 'hinge' vertices (Grason calls them 'pivot pairs') by which these components are connected together (compare chapter 7, pages 86–87). (Grason allows the possibility that a requirement graph as a whole might not actually be connected.)

The first application of the grammar is in simply putting together the set of edges corresponding to the required adjacencies between rooms, to form the adjacency requirement graph itself. This is done edge by edge; and because of the hierarchical nature of the representation in terms of connectedness, it is possible to test without undue complication, at the insertion of each successive edge, whether or not planarity is preserved.

With the assumption that, with all edges inserted, the resulting requirement graph is planar, Grason's next step is to produce, again with the aid of the graph grammar, all possible embeddings of this graph. If the graph as a whole is disconnected, or if it is only 1-connected, it is first modified, by 'anchoring' and 'bracing' the disconnected or 1-connected components, in such a way that the graph is made 2-connected. The graph grammar then acts to produce systematically all embeddings, by choosing different faces for the exterior face, and by folding and twisting the 2-connected components about their 'hinge' vertices into different faces, in all permutations, as described here in chapter 7 (pages 89–91).

Notice that the 'frame' consisting of the adjacencies between the four exterior regions in the figure shown above is fixed for all embeddings, and that the remainder of the graph must obviously be embedded inside this frame. If then there are adjacencies specified between rooms and the exterior regions, the possibilities for embedding are correspondingly restricted by the fact that the remainder of the graph is 'tied' by these edges to the frame.

Each of these embeddings is tested for what Grason has called its 'well-formedness'. If an embedding is 'well-formed', this means it is a subgraph of an adjacency graph or graphs which in their turn are capable of having colours and directions assigned to their edges in such a way as to represent a rectangular packing of rectangular rooms. The tests for well-formedness apply both to the vertices and to the faces of the embedded graph. A vertex in an adjacency graph which represents a (rectangular) room, has (at least four) edges incident with that vertex, which correspond to the segments of wall making up the four sides of that room. We saw in chapter 7 (compare the third figure on page 103) how this implies that the edges in question must be capable of being grouped into four groups, such that in each group all edges are coloured the same, and adjacent groups are coloured differently. Suppose that, as before, the two 'colours' are indicated by dotted and broken lines. Dotted edges in the adjacency graph correspond to walls running north–south in the plan, and broken edges to walls running west–east. The edges are directed, as they cross the walls from west to east, and from south to north. It follows that the edges incident with any room vertex must be arranged in their four groups in order, starting from the west side, and reading clockwise: dotted, inwards; broken, outwards; dotted, outwards; broken, inwards:



Going back now to the embedded *requirement* graph: any 'well-formed' room vertex must either fulfill the above conditions, or else it must be such that the conditions can still be satisfied when further edges are added, in completing the adjacency graph.

The conditions for well-formedness of faces are of an equivalent kind. In the completed adjacency graph the faces can only be of two kinds: triangles (corresponding to three-way junctions in the plan) or quadrangles (corresponding to four-way junctions). Again there are conditions governing permissible colourings and directions for the edges of these faces, as indicated in chapter 7 (first and second figures on page 103). Any face in an embedded requirement graph must either be of one of these two types, or else it must be capable of being 'filled' with such faces.

Grason shows how these conditions can be satisfied by setting conditions on permissible sequences of edges in the boundary of any face. If one imagines oneself taking a walk around the cycle of edges around a face, then each vertex will potentially represent a 'turn' in that walk, depending on how the edges are coloured and directed. The only circumstance in which there is no 'turn' is when two successive edges share the same colour

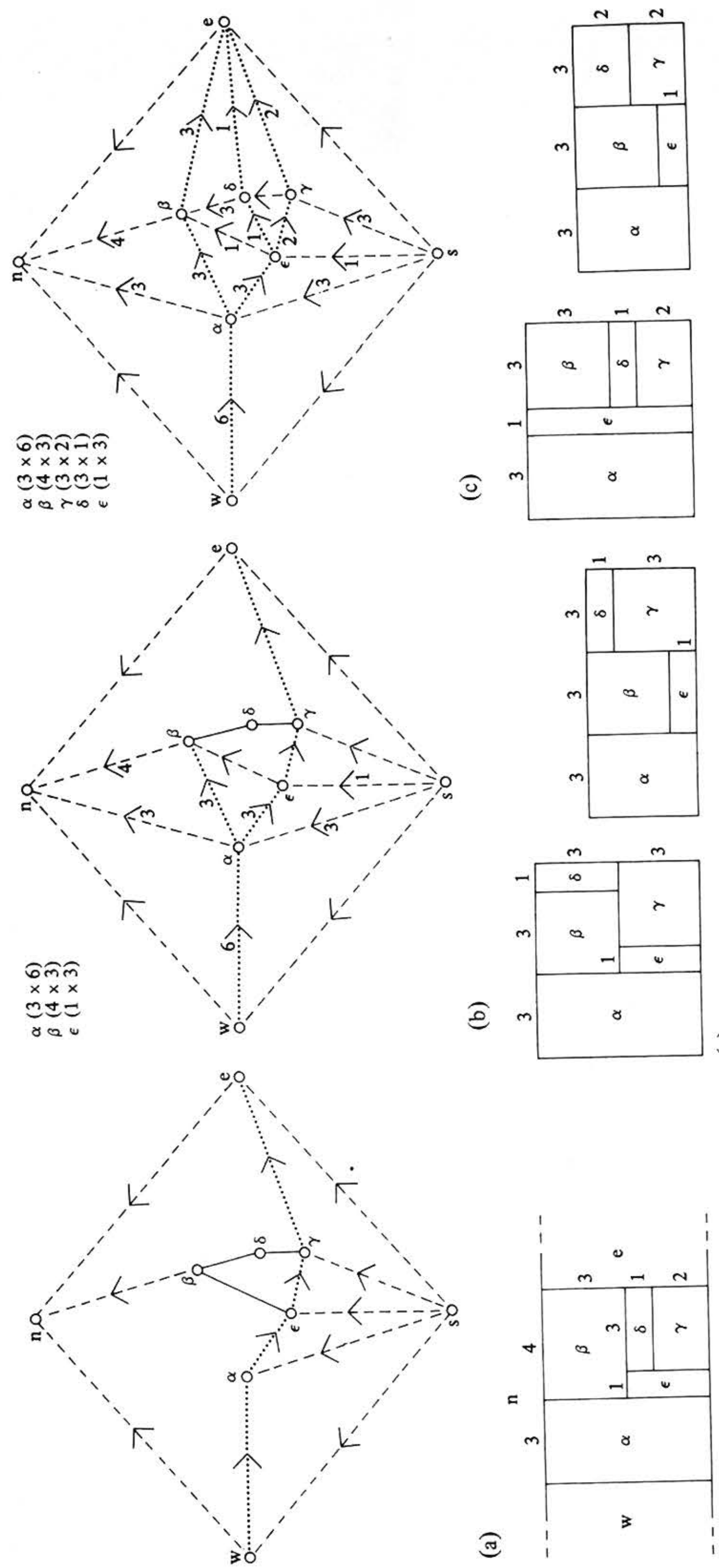
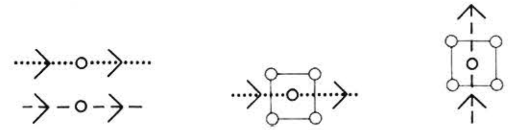
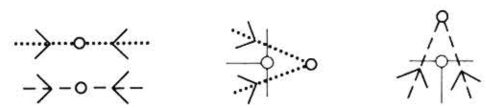


Figure 9.6. Worked example of the operation of Grason's (1970c) computer system, for a five-room plan with the required adjacencies and dimensional constraints on rooms set out in table 9.1. (a) An embedding of the adjacency requirement graph, with the colours of some edges (dotted and broken lines) and their directions necessarily fixed by the adjacencies to exterior regions; and other edges (solid lines) remaining uncoloured and undirected. (b) An intermediate stage in the completion of the embedded adjacency graph, with dimensions (weights on the edges) introduced. (c) The complete adjacency graph, fully directed, coloured, and weighted. (d) The corresponding plan. (e) The four other possible solutions for the same set of constraints.

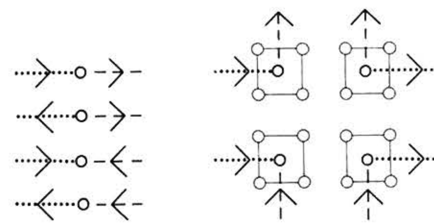
and the same direction:



But two successive edges of the same colour and differently directed, must correspond to a 180° turn in the plan:



And two successive edges of different colours must represent a 'turn' through 90°, which depends on the direction and colours of the edges:



Grason defines certain sequences of 'turns' which are inadmissible in the cycle of edges around a face, if it is to be capable of being realised in a rectangular plan.

These conditions for well-formedness are applied as the embedded requirement graph is built up; and all embeddings which meet the tests are retained for the next part of the process, which completes possible adjacency graphs, and simultaneously assigns dimensions. Figures 9.6 and 9.7 illustrate a worked example given by Grason, showing a set of adjacency requirements between five rooms (table 9.1), and an embedding of the resulting requirement graph [figure 9.6(a)]. See how the colours (dotted or broken) and directions of edges representing adjacencies to the

Table 9.1. Adjacency and dimensional requirements for Grason's worked example.

Room	Required adjacencies	Allowed sizes
α	to w, s, ϵ	6 x 3, 4 x 3
β	to n, δ, ϵ	4 x 3, 3 x 3
γ	to e, s, δ, ϵ	3 x 3, 3 x 2
δ	to β, γ	3 x 2, 3 x 1
ϵ	to α, β, γ, s	6 x 1, 3 x 1

exterior regions are necessarily fixed at this stage, and that this in turn implies the colour and direction of other 'internal' adjacencies; but that certain edges (solid lines) still remain uncoloured and undirected.

Dimensional constraints in Grason's system are specified in terms of a number of precise alternative sizes which each room may take. Thus in the worked example illustrated, room α can take either the size 6 x 3 or the size 4 x 3 (in metres). Possible sizes for other rooms in the example are listed in table 9.1. It is as though the rooms were jigsaw pieces of fixed shape and area. (The orientation of nonsquare rooms is not specified, and they may be placed with the long dimension either north–south or east–west.) In other design methods, as we shall see, dimensional constraints are expressed rather as maximum and minimum values for lengths, widths, or areas of rooms.

There is now a further series of tests for well-formedness applied to the embedded requirement graph, relating to the consistency of the dimensioning process, in which weights are attached to existing edges and new weighted edges are introduced to complete the adjacency graph. A first type of test applies to the edges incident with each vertex, and requires that the 'Kirchhoff law' is obeyed for the sums of weights on those edges, as explained in chapter 7 (compare figure 7.9). Thus the total value of all inward-directed 'dotted' edges at a vertex must equal the total value of all outward-directed 'dotted' edges; and similarly for 'broken' edges. A second type of test relates to the plan dimensions overall, and checks that, for example, the sum of all areas assigned to rooms is compatible with the area of the plan as a whole, and that the sum of dimensions assigned to rooms across the plan in one direction is compatible with the total width of the plan in that direction.

Edges are added one at a time to the embedded requirement graph, so as to fill its faces up to the point where all faces are simple triangles or quadrangles; and tests for well-formedness as described are carried out at each step. This is done in all possible ways, and the corresponding dimensioned plans are then derived from these completed adjacency graphs. Figure 9.6(b) gives a 'snapshot' view of an intermediate stage in the completion of one adjacency graph for Grason's worked example, and figure 9.6(c) gives the completed graph. The corresponding plan is

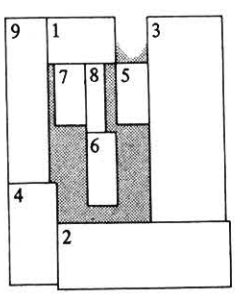


Figure 9.7. An example of a dimensioned plan produced by the computer program of Krejčiřík (1969a; 1969b), in which adjacency requirements are satisfied, and room and plan dimensions lie within specified maxima and minima, but the rooms are not close-packed and the plan contains 'holes' (shaded). It must be adjusted by hand to give the final arrangement.

illustrated in figure 9.6(d). There are only four other possible solutions to this particular problem, and they are illustrated in figure 9.6(e).

Grason's program deserves recognition as the first successful method for generating this type of plan automatically and exhaustively. Without denigrating that achievement, there is a serious criticism to be made, however; indeed it is a drawback which Grason himself was quick to point out. This is that by computing standards, the program is very slow. For instance the worked example shown here required 23 minutes of computing time on an IBM 360/67, to produce the five possible plans with five rooms as illustrated. In other tests, the program took still larger amounts of time—100 minutes for a six room plan, 210 minutes for eight rooms—without in either of these cases completing an exhaustive search. Grason suggested that the program might be rewritten to be more efficient. However, it seems that much of the reason for its slowness is intrinsic, to do with the complex alternating sequences of constructive steps, and tests, which are needed to exhaust all possibilities.

Another minor shortcoming is the fact that the method accepts dimensional constraints, as we have seen, only as fixed sizes on rooms, where more realistically it might be imagined that architects would be prepared to allow any dimension between certain limits. For example in Grason's illustration, room α must be either 6 x 3 or 4 x 3; where perhaps feasible plans exist, besides those produced, in which this room takes size 5 x 3 or even 4 x 4 which lies between the same effective area limits.

Grason's own reaction to the slowness of his program was an inclination to turn from an exhaustive approach, which he imagined had only a research and not any practical interest, to heuristic techniques. However, methods have since been devised which are still exhaustive and are capable of producing similar plans with at least seven or eight rooms, in radically reduced amounts of computing time. Before describing these systems, I should mention several other authors whose work is comparable with Grason's in various respects.

The approach taken by Krejčiřík (1969a; 1969b) to the dimensioning of rectangular plans was to use, like Grason, a representation consisting of an adjacency graph with coloured, directed, and weighted edges. Dimensional constraints in Krejčiřík's method are stated as maxima and minima on the lengths and widths of rooms, and dimensions are allowed to take any modular value between these limits. Krejčiřík's method involves testing the adjacency graph for 'well-formedness' in terms of the colouring and direction of the edges incident with each vertex.

However, there are no tests equivalent to Grason's for 'well-formedness' in terms of the weights on edges (that is, the Kirchhoff law). Instead the program merely assigns weights to edges in such a way as to represent rooms which are adjacent in the required relationships, and have acceptable sizes, but which are not necessarily close-packed (figure 9.7). Within the given dimensional constraints on rooms the program derives a plan whose overall east–west dimension does not exceed some stated maximum, and whose north–south dimension is minimised; but this may be a plan which contains several 'holes'. The idea is that this arrangement can then be adjusted by hand to give the final layout.

A group of workers led by Pereira (LNEC, 1972; Pereira, 1974) developed a program which is in certain respects very close to Grason's. The representation used is the same type of coloured, directed, weighted adjacency graph. A requirement graph is tested for planarity, its different embeddings are produced, and each of these is coloured according to criteria similar to those expressed in Grason's tests for 'well-formedness', to yield all possible corresponding plans of rectangular dissection type.

However, it is not clear from the published accounts exactly how embeddings are generated for 2-connected requirement graphs (this was one of the principal functions of Grason's 'graph grammar'); and details are described only of the embedding of 3-connected graphs, by the choice of different faces for the exterior face.

As for the dimensioning part of the system of Pereira and his colleagues, the dimensional constraints are expressed as maxima and minima on the lengths and widths of rooms; and they discuss the general theory of assigning colours and weights to the adjacency graph of a rectangular dissection, under conditions for 'well-formedness'. However, it seems that their program as actually implemented was again not designed to produce close-packed plans; and that the operator of the program was to be shown partial solutions—with overlapping rooms, and holes—and allowed to interact with the program to modify the dimensional constraints, so as to produce finished plans. Thus the system cannot be claimed to provide exhaustive solutions under stated sets of adjacency *and* dimensional constraints, in the same way as Grason's.

Another author who developed a computer program for generating small plans of rectangular dissection type was Gilleard (1978; 1980). Gilleard's method for completing a set of adjacency requirements so as to give a 'well-formed' embedded adjacency graph differs somewhat from Grason's. He considers in the first place *all* ways in which additional edges can be added in, up to the point at which the graph becomes a triangulation of the quadrilateral represented by the adjacencies of the four exterior regions. Those triangulations which cannot correspond to packings of rectangles, because of subgraphs which are not 'well-formed', are then rejected; and quadrilateral faces—corresponding to four-way junctions in the plan—are produced by the omission of suitable edges from the triangulations, again in all permissible 'well-formed' ways. The resulting embedded adjacency graphs are converted to all corresponding rectangular plans by deriving their duals (the plan graphs) and colouring these plan graphs directly. (The process is perfectly equivalent to Grason's colouring of the adjacency graph.)

Gilleard's approach to dimensioning is to set a range of integer values for the lengths and widths of each room, and then to have the program

cycle repeatedly through possible combinations of these values, according to essentially the same criteria for dimensional 'well-formedness' as Grason's (the Kirchhoff law), until *all* dimensionally feasible solutions are exhaustively generated. As with Pereira's method, 'dimensionless' plans of dissection form are derived to meet the adjacency requirements, and are then dimensioned.

The distinctive feature of Grason's approach by contrast is that adjacency requirements are put together to specify only a *partially completed* adjacency graph, with the colours and directions of some edges left undecided; and then additional edges are inserted to complete the graph, *simultaneously* with the attempt to satisfy the dimensional constraints.

Exhaustive enumeration of dimensioned plans

So far the design methods described have all been of what might be called a 'constructive' character. Graphs, or plans, are built up piece by piece so as to satisfy the given requirements of adjacency and dimension. A process which is in some ways the opposite of this—a 'selective' approach—was suggested by myself (Steadman, 1973), and implemented as a computer system with Mitchell and Liggett (Mitchell et al, 1976).

The essential idea behind our method is that if it is decided that a plan be of rectangular dissection form, and is to contain up to say seven or eight rooms, then a technique for enumerating dissections exhaustively (in our case the technique described in chapter 4) will in effect provide all potential 'dimensionless' solutions in advance. The dissections may simply be stored in a computer file, and a set of adjacency and dimensional requirements may then be tested against all of these possibilities in turn. It is a 'brute force' method, and admittedly lacks a certain intellectual elegance on this count, but it has the virtue of being highly effective.

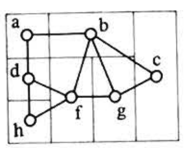
The problem of satisfying a set of adjacency requirements now becomes a matter of taking the adjacency graph of each dissection in turn, and testing to find all the possible ways (if any) in which the requirement graph can form a *labelled spanning subgraph* of the adjacency graph.

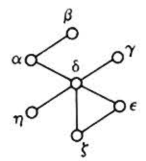
Suppose the rectangles in a specific dissection (the rooms) are labelled a, b, c, ... and the vertices of the requirement graph (the functions or room names) are labelled $\alpha, \beta, \gamma, \ldots$. We want to know in what permutations $\alpha, \beta, \gamma \ldots$ can be matched with, or assigned to a, b, c, ... In the absence of *any* adjacency requirements (that is, if the 'requirement graph' consisted just of n isolated vertices) we would need to consider *all* permutations of assignment of $\alpha, \beta, \gamma \ldots$ to a, b, c, ... The number of these permutations would be factorial n, conventionally symbolised as $n!$, a figure which grows grows rapidly with increasing n [19].

[19] Factorial n is given by $n \times (n-1) \times \ldots \times 2 \times 1$. Values of $n!$ for $n = 1, \ldots, 8$ are:

$n =$	1	2	3	4	5	6	7	8
$n! =$	1	2	6	24	120	720	5040	40320

However, this theoretical maximum number of assignments becomes progressively reduced, as adjacency requirements are introduced, and the requirement graph acquires structure. For example, a vertex in the requirement graph can only be assigned to a vertex in the adjacency graph of a particular dissection which is of equal or higher valency. This condition forms the basis for a test which is applied in our assignment procedure. The valencies of vertices in the adjacency graph of the dissection are listed in descending order of magnitude, and similarly for those in the requirement graph. The two lists are then compared, and if any value in the second list exceeds the corresponding value in the first list, this is sufficient to establish that no assignment can exist. An example is

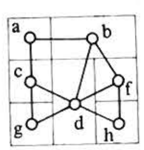


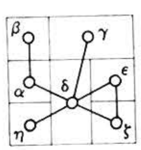


dissection	b	f	d	g	a	c	h
valencies	④	4	3	3	2	2	2
requirements	δ	α	ε	ζ	β	γ	η
valencies	⑤	2	2	2	1	1	1

○ indicates values being compared

If this test is passed, the program proceeds to make assignments of the requirement graph into the dissection. No vertex in the requirement graph is assigned to a vertex in the adjacency graph of the dissection with lower valency than itself. Within this limitation, the requirement vertices $\alpha, \beta, \gamma, \ldots$ are assigned to the room vertices a, b, c, ... in all permutations; and assignments in which edges of the requirement graph match those of the adjacency graph, are retained. The following example illustrates a successful assignment for the requirement graph given above:





We will see shortly how, with realistic sets of adjacency requirements, this process cuts down radically the numbers of alternatives under consideration.

It is possible to set requirements that rooms *not* be adjacent. Assignments in which these prohibited adjacencies occur are all rejected. Since adjacencies may be stipulated to the exterior regions on the north, east, south, and west, it is necessary to rotate and reflect every dissection, and to make assignments to all isomorphs in each case. The process, applied to all dissections with n rectangles, gives all plans in 'undimensioned' form which meet the adjacency (and 'nonadjacency') requirements[20]. Should it happen that the requirement graph is not planar, naturally no successful assignments will result.

[20] One particular failing in the method should be mentioned, which means that it is not strictly exhaustive—although it is nearly so. The list of dissections used in the first stage was not complete (compare page 39), and in particular did not include all

The computing time here is a function both of the number of rooms n, and of the strictness of the adjacency requirements. (With fewer requirements set, the process takes longer.) With typical and realistic sets of requirements, assignments to six rooms can be made in a few seconds with an IBM 360/91. For seven rooms, less than a minute is generally required, whereas the eight-room case often needs several minutes.

Figure 9.8 illustrates a worked example, which uses the system to design a small apartment of seven rooms. Figure 9.8(a) shows the graph of required adjacencies (edges drawn as solid lines) including adjacencies to the exterior regions; and a second graph of prohibited adjacencies (edges drawn as broken lines). Successful assignments of room names to rooms in dissections in this case resulted in 504 different 'dimensionless' plans, of which a small sample is illustrated in figure 9.8(b). Notice that it is assumed that the apartment is entered from the north side, and is flanked by neighbouring apartments on east and west.

It is interesting to compare this number of 504 with the maximum number of potential assignments in the absence of any adjacency constraints. To get that number we must take 7! (= 5040) assignments for each isomorph of each dissection; multiply by eight for the number of isomorphs (this is an overestimate, since some dissections will have symmetries; but as indicated in figure 8.7, the percentage of dissections with *any* symmetries is only 12·5 for $n = 7$); and then multiply again by the number of dissections with seven rectangles, which is 735. This gives a grand total of 29 635 200. Thus successful assignments in this (fairly typical) illustration amount to only some 0·0017% of the theoretical maximum.

In the second part of our system, dimensional constraints are satisfied. These are stated as maximum and minimum values on the dimensions and areas of rooms. Also constraints can be placed on the proportions which rooms may take. The process involves finding sets of values for the dimensioning vectors in the x and y (east–west and north–south) directions, so as to give actual sizes to the intervals of the grating on which the dissection is represented, as described in chapter 2.

It is very possible that certain sets of dimensional requirements may not be capable of being met at all in a given 'dimensionless' dissection, despite the fact that the adjacency requirements are satisfied. If they *can* be satisfied, however, and if dimensions are allowed to vary continuously between certain fixed limits, then in principle there will be infinitely many 'possible solutions', varying only by minute increments of size. The notion of exhausting all possibilities must clearly be interpreted rather differently in this context.

[20] (continued)
nonaligned dissections, only the equivalent aligned versions. Thus certain possibilities for the dimensioning of these dissections were potentially excluded, as explained in chapter 3 (compare the first figure on page 29). Clearly this is a shortcoming which could easily be remedied, by making suitable amendments to the file of dissections.

Grason's approach and Gilleard's too, in effect, was to consider only a small number of fixed modular sizes for each room, and to exhaust all possible arrangements of those specific sizes. This might be reasonable in the design of buildings using a modular system of prefabrication say, with a rather large unit module. In general, however, it seems more plausible that architects would want to allow any variations of size or proportion within acceptable upper and lower bounds. Our own approach has been to devise a method whereby certain dimensional properties of the plan overall can be 'optimised' in a mathematical sense; meanwhile dimensional limits set on individual rooms are all met.

To do this we make use of various techniques of mathematical *programming*, which originated in the subject of operations research (see Churchman et al, 1957). I do not propose to explain these methods

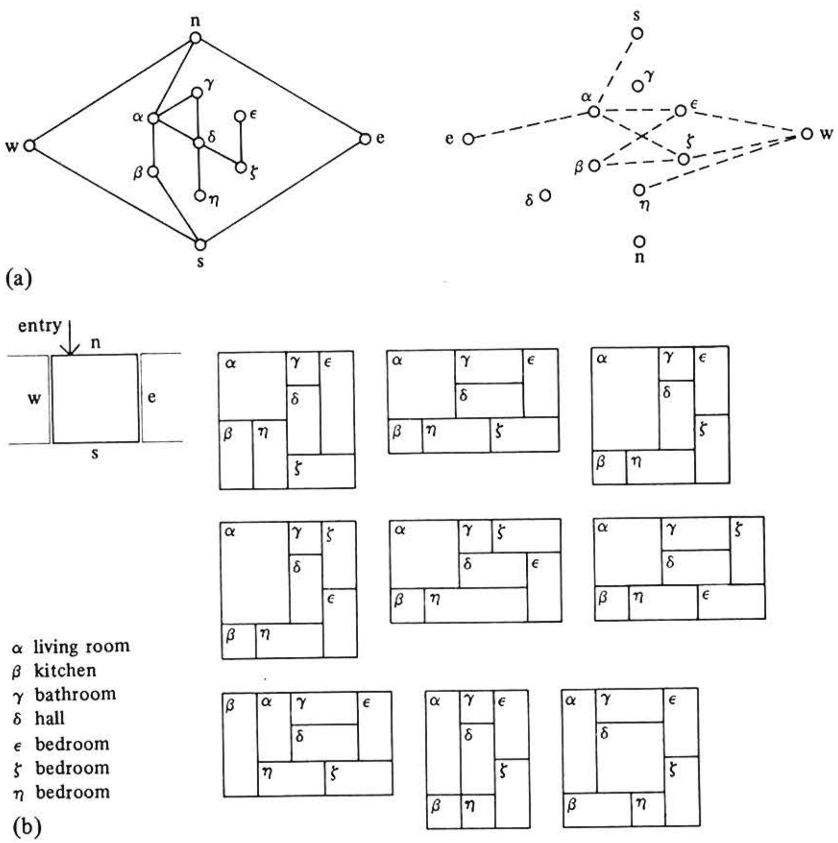


Figure 9.8. Worked example of the operation of a computer system for plan generation due to Mitchell et al (1976). Plans of rectangular dissection type are produced for an apartment with seven rooms, α to η. (a) The graph of required adjacencies (solid lines) and the graph of prohibited adjacencies (broken lines). (b) A sample of the 504 successful assignments of room names to rectangles in dissections for $n = 7$.

in any detail. A basic introduction to the most general and widely used technique, that of *linear programming*, is given in Grawoig (1967). Some applications to architectural problems are described in Mitchell (1977).

In all cases some *objective* capable of being expressed as a simple function of a set of variable quantities is either minimised or maximised. Thus, for example, a monetary cost might be minimised, or a profit maximised, subject to a series of *constraints* which place limits on how much of the objective can be achieved. The constraints are expressed as a series of *inequalities* (or perhaps also equalities), for example, 'x must be greater than or equal to some value', 'y must be less than or equal to some value'.

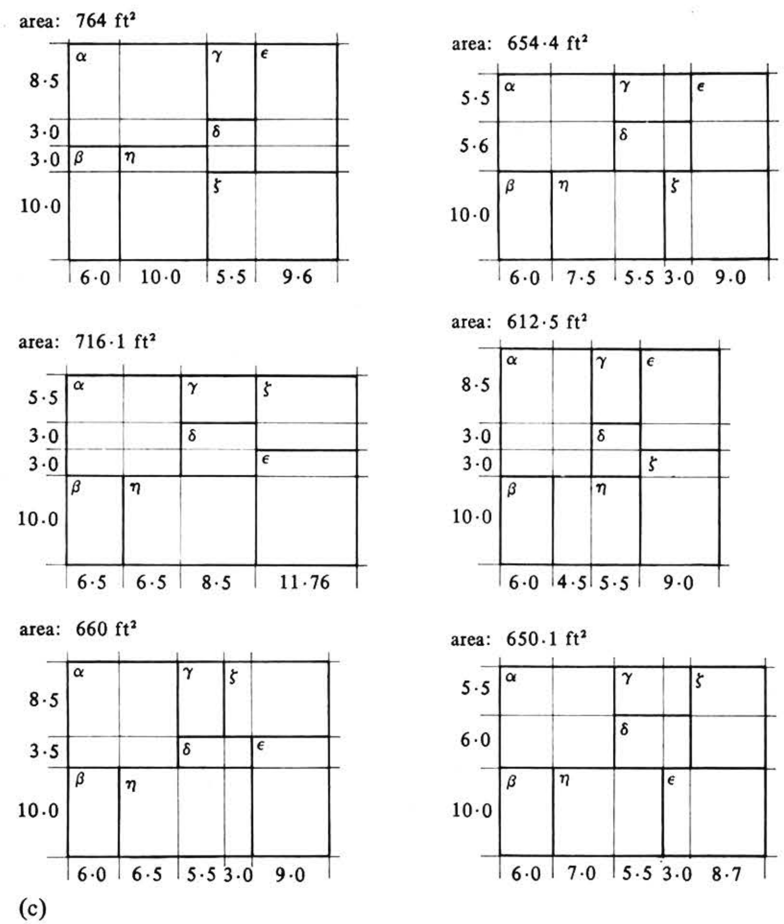


Figure 9.8 (continued)
(c) Results of applying the dimensional constraints on rooms set out in table 9.2, by means of a nonlinear programming technique in which the total areas of the plans are minimised. Only these six arrangements, out of the 504 'dimensionless' plans, are capable of satisfying the specified constraints.

Linear programming may only be used where the constraints and the objective can both be expressed in linear form, and is inapplicable to equations and inequalities involving powers, for example, x^2 or y^3. Objectives which fulfil this condition in the case of a (rectangular) plan would be, for example, the total length of the perimeter of the plan, its proportion, or its overall dimension of width or depth. In each case the objective might be either minimised or maximised. The constraints could be expressed as limits on the widths, lengths, or proportions of individual rooms—for example, that the length of room r should not be greater than l feet, its width not less than m feet, or its shape not more elongated than $m:l$. All these constraints and objectives involve only numerical ratios or linear dimensions.

In Mitchell et al (1976) we demonstrated linear programming applied to the dimensioning of a floor plan, by taking the example of a mobile trailer home whose overall width was assumed to be fixed at the maximum dimension allowable on the roads. It was further assumed that the dimensions of width of the rooms were all fixed. The total length of the trailer was then minimised, for a given plan arrangement of rectangular dissection form, and subject to constraints on the lengths of separate rooms.

This is obviously a slightly artificial example, though, and in general one would want to place limits on the *areas* of rooms, and perhaps to minimise or maximise the area of the plan overall. However, this means that constraints and objective cannot then be expressed linearly (areas being measured in ft^2 or m^2), and so it becomes necessary to use *nonlinear* programming. (The same would apply in any attempt to optimise the three-dimensional form of a building, where measurements of *volume*, ft^3 or m^3, enter into objective or constraints.)

Table 9.2 shows the dimensioning constraints on the various rooms, set for our worked example of the apartment plan as illustrated earlier. In this case we used a nonlinear programming algorithm due to Clasen et al (1974) and implemented for the present problem by McGovern (1976). Besides the constraints listed, it was stipulated that where two rooms were required to be adjacent, their overlap should not be less than three feet—assuming the adjacency being to allow for a door. In the worked example the total areas of plans were *minimised* under the given constraints[21].

Notice nevertheless that this does *not* imply some value judgement to the effect that a plan of minimum area is necessarily the 'best' in a functional or economic sense; since it would be possible to derive both the minimum area, *and* the maximum area for the plan, under the given constraints. Then it is known that *all* possible dimensional solutions—be they in theory infinitely numerous—must lie between these extremes.

(21) In Mitchell et al (1976) it is incorrectly stated that the objective optimised in this same worked example is a notional cost of construction, measured as a function of floor area, in which the cost per square foot for kitchen and bathroom differs from that for other rooms. In fact the objective is simple total floor area, as here.

It transpires that out of 504 'dimensionless' plans, only six are capable of being dimensioned as specified: all six are shown in figure 9.8(c). The fact is that the patterns of 'overlap' or alignment in a dimensionless dissection, without fixing the absolute sizes of rooms, have strong implications for their *relative* dimensions. They imply that certain rooms must have equal lengths or widths, that some must be smaller or larger than others, and that the sums of widths of certain groups of rooms must equal the sums of widths of other groups. (These are the kinds of relations expressed in effect by weights attached to the edges of a coloured adjacency graph.) It is for these reasons that typically, as in the worked example, permissible topologies are severely reduced in number by realistic dimensional requirements.

It is interesting to see how some of the plans with smaller total areas in the example, are ones where the hall space is considerably larger than in plans of greater area. (This hall appears in the centre of the plan in each case, since it is—perhaps unrealistically—not constrained to be adjacent to an exterior region. Access to the apartment from the outside at the north is assumed to be through the living room.) The total computing time taken for both stages of this seven-room worked example, the assignment of the requirement graph and the dimensioning process, was some three minutes on an IBM 360/91[22].

The dimensioning procedures used in Mitchell et al (1976) have been further developed by Gero (1977), who used for illustration the same worked examples of trailer home and apartment plan. Gero, however, used *dynamic programming* (see Bellman, 1957), a technique which in this context has several advantages.

Inside the optimal solution to a dimensioning problem of this kind there is liable to exist a certain *flexibility*, in that say the dimension of

Table 9.2. Dimensional requirements for Mitchell et al's worked example.

Room	Length (ft)		Width (ft)		Area (ft²)		Maximum proportion ratio
	min	max	min	max	min	max	
α living room	8·0	20·0	8·0	20·0	150·0	300·0	1·5:1
β kitchen	6·0	18·0	6·0	18·0	50·0	120·0	–
γ bathroom	5·5	5·5	8·5	8·5	–	–	–
δ hall	0	15·0	3·5	6·0	0	72·0	–
ε bedroom 1	9·0	20·0	9·0	20·0	100·0	180·0	1·5:1
ζ bedroom 2	8·0	18·0	8·0	18·0	100·0	180·0	1·5:1
η bedroom 3	10·0	17·0	10·0	17·0	100·0	180·0	1·5:1

(22) Any comparison, however, with the times quoted above (page 155) for the solution of comparable problems via Grason's method should take account of the fact that the machine used by Grason was slower, perhaps as much as one hundred times slower, than the IBM 360/91. And Grason's program was executed interpretively, not in a compiled language.

one room might be increased and the dimension of an adjacent room correspondingly reduced, up to certain limits, without the overall length, area, or whatever other feature of the overall plan is optimised, being affected. It is obviously of interest to the designer or building scientist to know precisely the extent of this flexibility; and this the use of dynamic programming allows. Figure 9.9(a) shows for example one of the dimensioned solutions to the apartment problem from the top right diagram of figure 9.8(c), with exactly the same total plan area, but with a number of changes made to the dimensions of individual rooms.

Gero also shows how in this same case, if the limits on the dimensions of the bathroom γ are relaxed, and its area is allowed to increase above the effective maximum specified in table 9.2, then a dimensioned solution of *lower* total plan area (627 ft^2) is possible. In this arrangement the long dimension of the bathroom is oriented north–south instead of east–west [figure 9.9(b)].

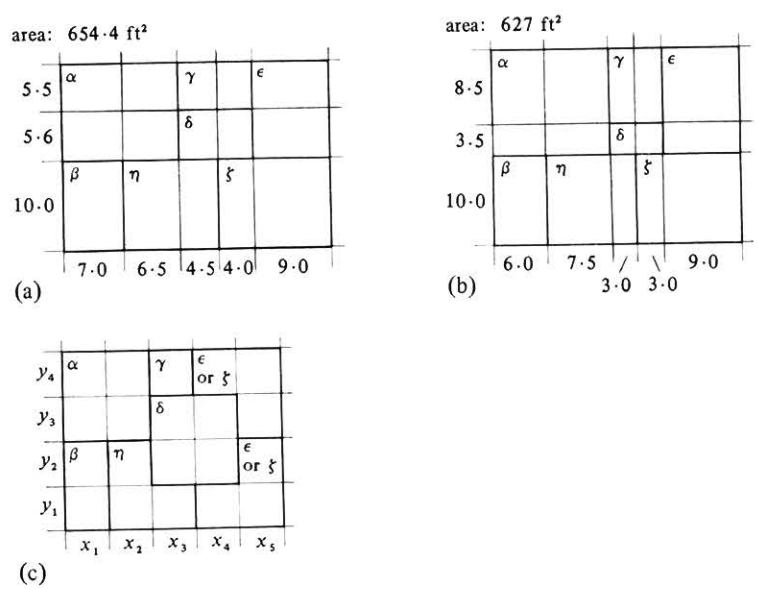


Figure 9.9. Flexibility of dimensioning of individual rooms within an overall set of dimensional constraints. (a) One of the dimensioned plans from figure 9.8(c) (top right), with the same total area (654·4 ft^2), but with changes made to some of the room dimensions within the given constraints. (b) Demonstration by Gero (1977), using dynamic programming, that if the constraints set on the dimensions of the bathroom γ in the same example (see table 9.2) are relaxed, a solution of lower total area (627 ft^2) is possible. (c) Dimensionless plan corresponding in its adjacency relations to all six plans shown in figure 9.8(c) (from Gero, 1977). Each of the six can be produced from this arrangement through appropriate assignments of dimensions, such that one of the x-dimensions or one of the y-dimensions of the grating is set to zero, some wall alignments are offset, and the L-shaped rooms become rectangles.

A further possibility with dynamic programming is that whole classes of dimensioning problems may be solved simultaneously through a process known as 'invariant embedding'. Figure 9.9(c) illustrates a dimensionless plan containing L-shaped rooms, and alternative positions for two bedrooms, which corresponds in its adjacency relations to all six plans shown in figure 9.8(c). Those plans may each of them be derived from the arrangement of figure 9.9(c) by appropriate assignments of dimensions, such that either one of the x or one of the y dimensions of the grating is set to zero, certain alignments of walls are offset, and the rooms all become simple rectangles.

Finally, in this account of design methods for rectangular plans subject to adjacency and dimensional constraints, there is the method devised by Flemming (1977; 1978) and based on his 'wall representations' as described here in chapter 4. Flemming has developed a computer system, the DIS program, with which it is possible to enumerate exhaustively solutions to problems of the same kind addressed by Grason and by Mitchell, Steadman, and Liggett.

Flemming shows how trivalent rectangular dissection-type plans ('T-plans') may be derived, by means of the DIS program, through successive operations of the types depicted in figure 4.15. [Notice that at this stage the plans do not have dimensions, nor is the pattern of overlap of rooms across walls represented—compare figure 4.13(a) and the first figure on page 45.] These geometric operations on the plan are carried out in the program through equivalent logical operations on the wall representations themselves. Figure 9.10 illustrates an example for four rooms in which five required relations of adjacency are specified. The program adds rooms one at a time, such that at every step two general conditions are met. The first is that no required adjacency which has already been satisfied at an earlier stage is disrupted by the introduction of the new room. The second is that all adjacencies required between the new room and those already placed, or with the exterior regions at n, e, s, and w, are satisfied. Flemming (1977) is able to prove that there exists a unique sequence of operations by which each 'topologically feasible solution' may be reached. Figure 9.10 shows only part of the whole tree of alternative sequences by which all such solutions meeting the stated adjacency constraints may be produced. (Other solutions would be possible via the unfinished branches on the right of the figure.)

For the dimensioning of these topological solutions Flemming makes use of linear programming. He argues that although it will often be useful to specify constraints involving areas, these can nevertheless be approximated linearly. It will be clear that constraints on the lengths and widths of rooms or on dimensions of the plan as a whole, can be expressed directly in relation to wall representations, since each of these lists rooms in order along either side of every wall. Constraints on the 'overlapping' adjacency of two rooms *across* a wall can also be expressed, such that, for example, a sufficient dimension is allowed for a door.

Figures 9.11 and 9.12 illustrate a worked example which uses the DIS program, in which an eight-room apartment is planned within an L-shaped area. A ninth dummy space is used to fill out the corner of the L. Figure 9.11 shows the adjacency requirement graph, and table 9.3 lays out

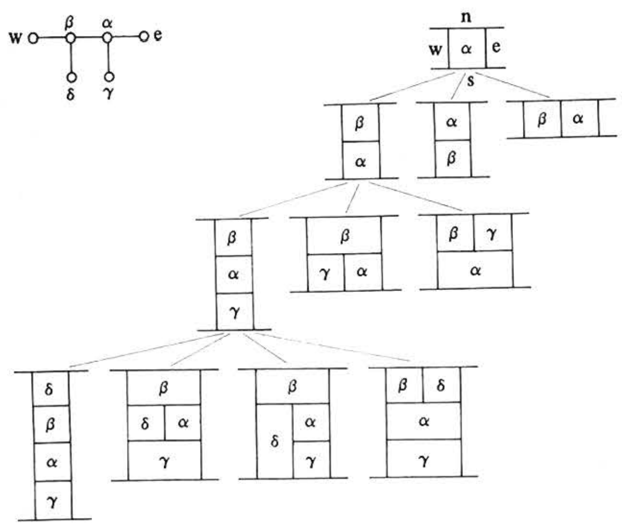


Figure 9.10. Generation of trivalent rectangular dissection-type plans with the DIS program of Flemming (1977). Five adjacencies are required between four rooms α, β, γ, δ and the exterior regions w and e, as shown in the graph. The program adds rooms one at a time through operations on wall representations (see chapter 4). No required adjacency, already satisfied at an earlier stage, is disrupted by the introduction of the new room. Meanwhile the new room is positioned such that its required adjacencies to those already placed, and to the exterior regions, are satisfied. The figure shows only part of the tree of sequences of operations leading towards feasible solutions.

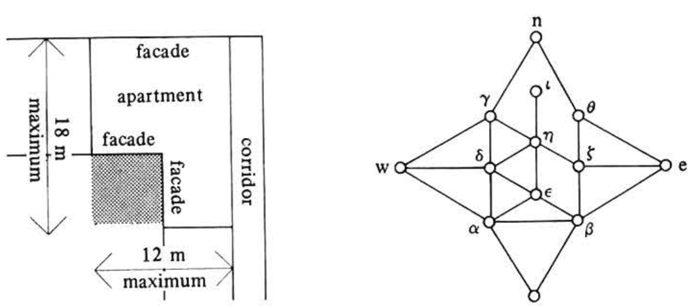


Figure 9.11. A worked example of the use of Flemming's (1977) DIS program, for the planning of an eight-room apartment whose overall perimeter is L-shaped. The general arrangement is shown of the apartment in relation to the access corridor and building facades, with upper limits given on overall dimensions. The L-shape of the apartment is filled out to form a rectangle by the addition of a ninth dummy exterior space α (shaded). Required adjacencies are shown in the graph.

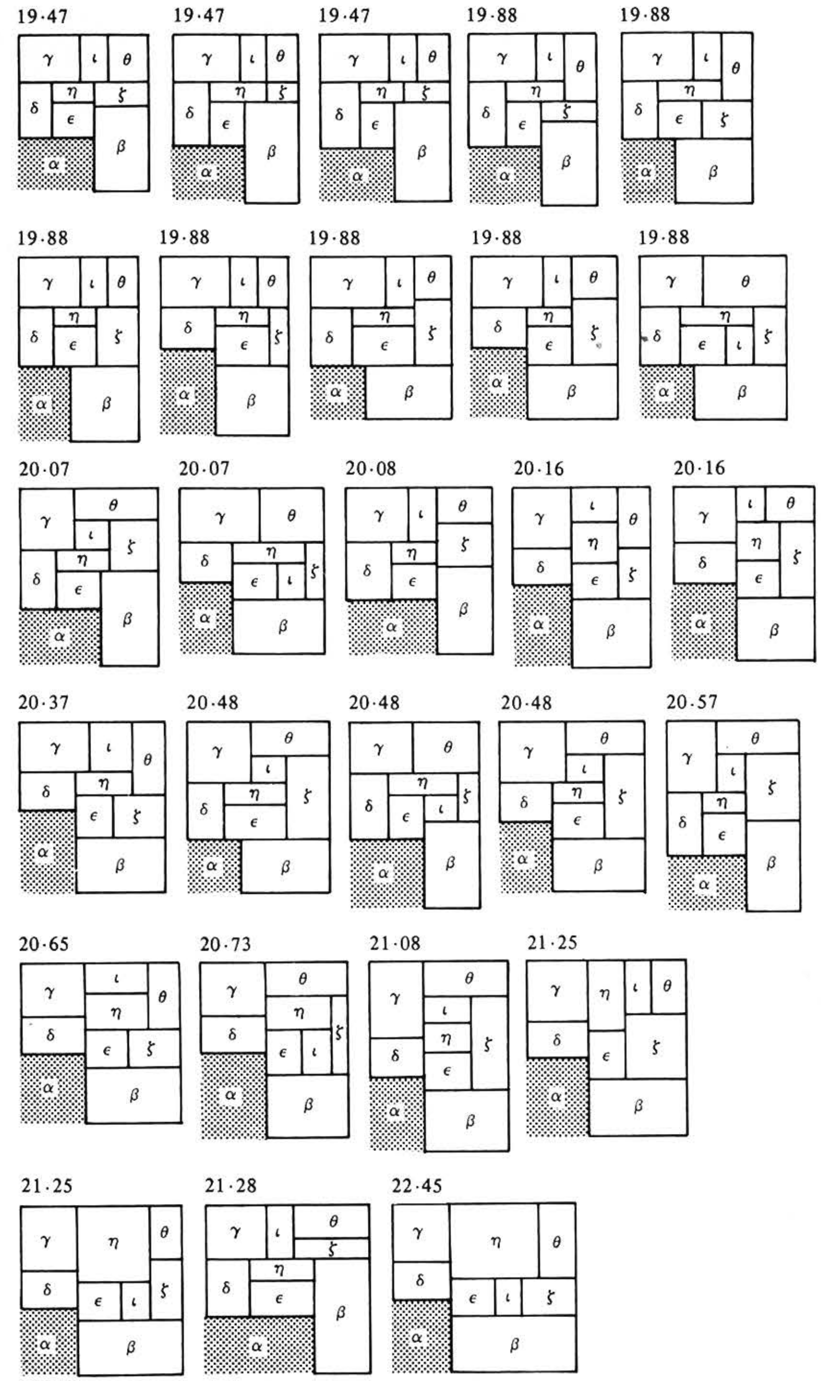


Figure 9.12. The twenty-seven feasible plans which meet both adjacency requirements (figure 9.11) and the dimensional constraints set out in table 9.3. The sum of the overall dimensions of the plan (in effect, half the perimeter length) is minimised in each case by means of linear programming. The figure over each plan gives this value. Notice that two adjacent halls ζ and η are specified, which together form a variety of shapes of circulation area.

the dimensional constraints. These correspond to current standards for public housing schemes in the Federal Republic of Germany. Notice that two rectangular halls are specified, and are required to be adjacent, so as to allow for the possibility of either a T-shaped or an L-shaped circulation area. The adjacency requirements of figure 9.11 represent a selection from a number of alternative possibilities (for adjacencies between rooms, and to the exterior regions) which would be equally acceptable within similar standards.

In this case the objective which is minimised by the linear program is the sum of the north–south and east–west overall dimensions of the plan. (In effect the perimeter length is minimised.) Figure 9.12 shows all twenty-seven plans capable of meeting adjacency and dimensional constraints. Flemming calls these 'principal options'. The figure over each plan gives the sum of its overall dimensions in metres (equals half the perimeter length). Of course, as already discussed, there would be in principle infinitely many solutions differing only slightly in their dimensions, between the extremes determined by say minimising and maximising perimeter length; as well as, most probably, a certain flexibility in room dimensioning even at the extremes. Flemming speaks of the particular dimensioning solutions produced by linear programming as 'representatives' of these potential ranges of dimensional possibility. See how several plans in figure 9.12, with the same perimeter length, differ widely in their internal disposition and room dimensions.

Table 9.3. Dimensional requirements for Flemming's worked example.

Room	Dimension (m)		Area (m^2)	
	min	max	min	max
α [dummy]	3·60			
β living room	3·60		22·00	
γ master bedroom	3·30	5·40	14·00	
δ bedroom	2·40	4·20	7·20	9·99
ϵ bedroom	2·40	4·20	7·20	9·99
ζ hall	1·20	6·00		
η hall	1·20	6·00		
θ kitchen	2·10	5·40	7·20	
ι bathroom	1·80	4·20	4·20	

Total area has a maximum dimension east–west of 12·00 m, and a maximum dimension north–south of 18·00 m.

Minimum lengths of common wall to be shared by rooms or spaces which are required to be adjacent, are:

α to w	3·60 m	α to ϵ	0·90 m	δ to η	0·90 m	ζ to θ	0·90 m
α to s	3·60 m	β to ζ	0·90 m	ϵ to η	0·90 m	η to ι	0·90 m
α to β	2·40 m	γ to η	0·90 m	ζ to e	1·20 m	θ to n	0·90 m
α to δ	0·90 m	γ to n	2·40 m	ζ to η	1·20 m		

A second type of objective function used by Flemming for dimensioning has the effect of keeping certain rooms (for example, circulation) to the minimum areas allowable within a configuration, with any available 'slack' area concentrated into other rooms (for example, living rooms).

Flemming's system is capable of exhausting all solutions where constraints as here are reasonably tight, and the number of spaces not too large. But it can still be useful as a design tool, or for exploratory exercises, where the numbers of potential solutions are so great as to make enumeration impractical. The DIS program has the virtue of displaying to the operator the products of each geometric operation, as in the successive levels of figure 9.10. Thus he can decide not to explore all branches of the solution tree which would be generated in theory by his initial set of constraints. Instead he can pick what seem to be promising partial solutions, on selected branches, and develop only those. Such an approach is only possible with procedures, such as Flemming's, which are of a 'constructive' character.

In general all these methods—Grason's, that of Mitchell, Steadman, and Liggett, Flemming's—come up against limits on the number of rooms which may be treated in the exhaustive enumeration of plans, because of the intrinsically explosive nature of the combinatorial problem. Any technique along these lines which attempted to deal with many more rooms, would be obliged to subdivide the problem hierarchically in some way, and deal only with a subset of rooms at one time. On a single floor level, large plans might be subdivided into discrete zones or bays. Perhaps special conditions could be imposed on the adjacencies of rooms across the boundaries of adjacent zones.

Multistorey plans with large numbers of rooms might be treated, one floor at a time, if the number of rooms on each floor was limited. Special precautions would have to be taken to ensure that vertically interpenetrating elements such as stairs, lifts, or double-height spaces were properly aligned, and to ensure that each floor as a whole had the correct dimensions and was oriented correctly over the lower floors. These possibilities are discussed in Mitchell et al (1976).

A special difficulty posed by all these techniques concerns circulation spaces. When an architect makes a plan in the normal way, he often introduces circulation areas as he proceeds, because, as he thinks, the required relations of adjacency and access cannot be met without them. The present methods, by contrast, require that *all* spaces, including circulation spaces (whether 'necessary' or not) be specified, together with their adjacencies and dimensions, at the outset. Solutions which might exist with fewer circulation spaces, or with more, are not found. It is difficult to see how this objection can be overcome within the terms of the methods, other than by exhausting plan solutions with successive numbers of circulation spaces.

Baybars and Eastman (1980) have outlined a graph-theoretic design method, which consists essentially of a technique for generating maximal planar graphs exhaustively, and for picking out those which possess spanning subgraphs corresponding to a specified set of adjacency requirements. The selected maximal planar adjacency graphs are then embedded, and their duals taken to produce different plans. Baybars and Eastman argue that by subsequently deleting edges corresponding to 'optional' rather than to required adjacencies from the maximal planar graph, it is possible to enumerate systematically all possible positions for extra internal spaces in the plan (in addition to those represented by the vertices of the adjacency graph). The extra spaces might be interpreted either as 'courtyards' or as circulation spaces. However, at the time of writing Baybars and Eastman have not extended this method, as they intend to, to incorporate a dimensioning stage.

No comparable methods exist for geometries other than rectangular, although there are no reasons in principle why they should not be possible. Their applications would be rather limited. Korf (1977) has criticised the restriction of existing methods to rectangular arrangements, and points out (as we have seen in chapter 7) that certain combinations of adjacency requirement cannot be met in plans whose rooms are all simple rectangles. His proposal involves drawing the duals of embedded adjacency graphs as 'bubble diagrams' which he says are 'entirely independent of shape'. However, any actual drawing of the dual must assign shapes and sizes to the 'bubbles', and any actual architectural plan would obviously have to be realised with definite room shapes and dimensions; so that this suggestion of Korf's really just ducks many of the issues which we have been examining in this chapter.

10

Plan morphology and building science

"... our own study of organic form, which we call by Goethe's name of Morphology, is but a portion of that wider Science of Form which deals with the forms assumed by matter under all aspects and conditions, and, in a still wider sense, with forms which are theoretically imaginable."

D'Arcy W Thompson (1961)

This chapter presents some applications of the ideas about floor plan morphology which have been explored so far in the book, to certain topics which are traditionally thought of as falling within the scope of 'building science'—questions of natural lighting, circulation, the adaptability and flexibility of plans. There is more of promise to report here, than solid progress. Nevertheless this promise is very great, much greater indeed, as I have previously argued, than the kinds of 'design method' which formed much of the subject of chapter 9. The equivalence in length of that chapter and this one should be taken therefore as a reflection of how work has been divided so far, rather than as a token of the importance which I, at least, would assign to the two directions of research.

I say questions which are traditionally thought of as those of building science, since part of the philosophy of this morphological or configurational work is that the scope of a 'science of building' should encompass not just the study of building materials, structures, and buildings as environmental enclosures, but that it should include, and indeed be founded on, a study of the forms and arrangements of buildings—their geometry and their topology. This is surely Rittel's conception when he speaks of one of the first chapters of a 'theory of architecture' dealing with 'the theory of cell configurations'. As Hawkes (1980, page 14) has expressed it:
"... the role of scientific research in architecture, as opposed to building science, is primarily to work upon what have been called the 'animals of architecture' (March and Matela, 1974): first to establish a morphology of architectural form and then to use the tools of building science and other methods of analysis to make explanatory statements about the relationship between form and performance. As a practical programme, this has the attraction of offering a more structured basis for the application of traditional analytical research."

At the beginning of chapter 8 I raised some issues posed by the 'combinatorial explosion' in numbers of possibilities for rectangular arrangements with increasing numbers of rectangles n. It appears that for values of n much greater than 10, the extent of combinatorial variety—in rectangular dissections, in polyominoes—becomes so great that a complete enumeration is of little practical purpose; and that indeed for values of n not much larger than this, enumeration itself becomes a practical impossibility.

Meanwhile there are many real buildings, obviously, which contain more than ten or a dozen rooms on a single floor.

I suggested in chapter 8 that it might be feasible to treat this very great theoretical range of possibility, at these higher levels, by *statistical* methods. This problem of combinatorial variety can, however, be viewed in quite a different way. This involves looking rather more closely at what we mean by 'possible' and 'possibility' in this building science context.

The reason for examining the properties of say rectangular dissections as a class of geometrical 'objects' in the first place, was that arguably they serve as fair geometric models, at a certain level of abstraction, for the kinds of wall and room arrangement which are typically to be found in the plans of actual buildings. Rectangular dissections are chosen for study, that is, because they are realistically 'plan-like'. Equally polyominoes might be chosen—although I have suggested (page 16) one respect in which polyominoes are *not* like the plans of many buildings.

In the end then these are empirical matters. The practical relevance of examining the properties of rectangular dissections turns on how many buildings presently existing, or which have existed, in fact do possess a geometry of rectangular dissection form. This perhaps would give some guide to the number of such buildings likely to be designed and constructed in the future. If many buildings were made up of hexagonal rooms, then that would provide practical justification for studying the geometry of the packing of hexagons. Should an architect take it into his head to design a building whose rooms were all irregular pentagons, then of course it would not be included in these enumerations. (Although *all* subdivisions of the plane into regions, that is, all plans treated at the topological level, *are* exhaustively covered by the enumeration of plane maps.) Such a design would not be 'impossible' in any final or absolute way. It would just be rather improbable.

When we speak of 'possible designs', or 'possible plans', it is thus the actual world of real buildings, with all the practical and functional and technological constraints applying to their forms, which serves to define the (theoretical) realms in which these 'possibilities' lie. What we are asking, when we talk of 'possibility' in this context, is "How many other theoretical designs are there, similar in some sense to those which have been actually made?" The problem is that systematic analyses of the geometric forms and arrangements of the mass of actual buildings are rather rare. There does not exist, as there does for natural species, a descriptive anatomy or 'natural history' of architecture(23).

(23) Perhaps the closest approximations in a single volume to a 'natural history' of architecture, are provided by Durand's *Recueil et Parallèle des Édifices* (1801), which as I have suggested elsewhere (Steadman, 1979) may well have had a conscious model in biological classification; and Banister Fletcher's *History of Architecture* (1896) on 'the comparative method'—this a 'comparative anatomy' perhaps? See also my comments on Pevsner (1976) in chapter 11.

It is difficult to find evidence, for example, about even such a simple matter as what proportion of existing buildings possess a rectangular geometry. The only studies known to me are one made by Bemis (1936) of American houses, and one by Krüger (1977; 1979a) of buildings in the city of Reading, Berkshire.

Bemis was an early and influential promoter of industrialisation and prefabrication in the building industry, and he was concerned to know the extent to which systems of prefabrication with a rectangular geometry would restrict variety of design in houses, or alternatively what proportion of 'special' nonrectangular components would be needed to supplement such systems. Thus he did not categorise houses as a whole as to whether their geometry was rectangular or nonrectangular. Instead he took one house at a time and measured the 'proportion of rectangularity' of its form.

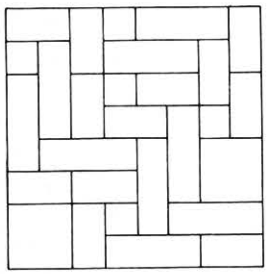
He calculated two figures, one for the rectangularity of the whole house by volume (thus the volumes of pitched roofs, curved bay windows, etc were deducted from the total volume of otherwise rectangular houses), and the second, since he was interested in construction, for that proportion of the physical fabric of the house which was rectangular (again by volume). His basic survey was made in Boston, but the results were extrapolated to the entire United States. He found that 88·5% of all American housing was rectangular by total volume, and 98·3% rectangular in its constructional components.

Krüger worked at a much less detailed level, analysing from large-scale maps the perimeter outlines of the plans of all buildings in the entire city of Reading (see figure 11.14). He found that 98% of those plan shapes were of rectangular geometry.

Equivalent surveys of room shapes and plan shapes, and extended to other types of building, would show the extent of application, for building science, of rectangular dissections, polyominoes, or other classes of arrangements as geometric models for plans.

To come back to the problem of the very large numbers of these kinds of geometric arrangement for large n: I suggest that, despite the merits of rectangular dissections as models of *smaller* plans, there is an increasing proportion of 'theoretical possibilities' for larger dissections which nevertheless become rather *unlike* the plans of buildings, and hence begin to lose their practical interest.

Such dissections consist, certainly, of rectangular components corresponding to rooms, packed together in different configurations. But these configurations are not at all probable architecturally, in ways which are hard to pinpoint precisely, but are no less real for that. It is something to do with such facts as that real buildings tend to have a limited depth, because of the needs of daylighting and natural ventilation, so that when large they become organised into regular patterns of wings and courts. Or that rooms are set along relatively simple and coherent circulation systems consisting of a few branching corridors which extend along the building's whole length. There are many large dissections which are made up, by contrast, of a deep maze-like agglomeration of overlapping rectangles, many of them completely internal, and through which any linking pattern of circulation routes would be circuitous and confusing. An example is



If we could capture properties like these in explicit geometrical measures, then we might be able to limit the study of dissections, for example, to a much reduced class of arrangements which would all be 'building-like' in some such well-defined sense. The identification of these properties can only come from a programme of empirical work, which takes the plans of existing and historical buildings and devises measures whereby these aspects of their characteristic geometry and topology can be described.

What follows is largely speculation then: but one would expect practical limitations on 'possibility' at this larger architectural scale to be of two general kinds, the one configurational or topological, the other dimensional (although dimensional and shape or adjacency constraints are always closely interrelated, as we have seen in the previous chapter).

Dimensional limitations on building form

Let us look first at some *dimensional* limitations which are likely to apply equally to almost all kinds of building. So far in the book the emphasis on possibility in design has been placed very largely on possibility of arrangement in 'undimensioned' terms, with the understanding that the variety of possibilities for assigning dimensions to these arrangements is effectively infinite. However, in real architectural or engineering terms, these dimensional possibilities, although they may be infinitely numerous, are certainly not unbounded.

To take an example from the science of engineering structure: it would be possible in principle to tabulate the numbers of distinct ways in which columns and beams may be put together into structural frameworks—frames with different numbers of those elements, or frames with different numbers of bays and floors—very much as we have done for plan arrangements. Within this, certain subclasses of framework with given structural characteristics might be identified, as, for example, in the work of Bolker and Crapo (1977) on all those structures which are completely braced against lateral loading. This enumeration, like that of rectangular

dissections, is purely configurational, and quite independent of the specific dimensions which the structural members or their relative spacing might take.

Consider, however, the more elementary and conventional way in which the properties of individual beams are studied, by determining the maximum loads above which those beams will deflect beyond acceptable limits, and ultimately fail. This in effect defines for given materials, cross-sectional shapes of beams, etc, the limits of the 'universe of possible beams', *in dimensional terms.*

We can make the same sort of distinction for beams, between their configuration and their dimensions, as was introduced here in chapter 2 for room arrangements (compare Spillers, 1974). Indeed this distinction is manifested in the standard format of handbooks of structural steel tables. Each possible cross-sectional *shape* of steel beam—or at least those which are manufactured—'T', 'L', 'H', square-section, circular section, etc is listed separately.

Then for each shape the structural properties are tabulated for different actual *sizes* of flange, web, etc. The engineer can only choose his designs first of all from among the given configurations. And he is limited a second time, within these shapes, to certain maxima or minima of size—that is, to maximum spans say, or minimum structural depths of beam. These limits, and similar ones applying to columns, will in turn have their implications for the possible dimensioning of different framework configurations as a whole.

There are obviously many comparable ways in which function, technological means (that is, available tools, machines, or manufacturing techniques), and the properties of materials place ultimate dimensional bounds in this sense on the realm of 'possible buildings'. The strengths of materials limit the maximum heights of buildings. The size of the human body limits the minimum dimensions for a door, or the maximum height for a step in a staircase. The necessity or desirability of admitting natural light to buildings limits their maximum depth; or if artificial lighting and ventilation are introduced, then another limit of depth is set perhaps by the need for windows solely to give a view of the outside world. Such limits cannot admittedly be defined with absolute precision. The lines at which they are drawn in practice tend to represent some compromise between considerations of comfort and convenience on the one hand, and limitations of materials, technique, and cost on the other.

Thus it would very likely be possible to construct buildings taller than the highest presently existing, with the expenditure of more money, or with the development of new materials or techniques. Buildings might be made indefinitely deep, if the occupants were prepared to sacrifice the convenience of daylighting, or the pleasure of a view. Limits of these kinds on 'possible buildings' are therefore established largely by judgements of value. In this sense they are not like the absolutely necessary and unavoidable constraints of topology and geometry. This does not, however, alter their force, or the fact that they serve to explain many of the observable restrictions on the variety of building forms which are actually seen in practice.

The kinds of dimensioning technique described in the last chapter might be used for investigating some of these questions experimentally. In the method devised by Mitchell, Liggett, and myself and applied by way of illustration to the example of a small flat, there was a whole series of limits set—in a rather arbitrary way—on the minimum and maximum sizes and proportions for rooms. A nonlinear programming technique was then used to produce plans with overall minimum area, within these constraints on room shape and dimension. It would equally have been possible to minimise or maximise other properties of the plans, such as the lengths of their external perimeter walls, or their linear dimensions of width or depth. [Indeed some studies along precisely these lines are reported in Mitchell (1975a)].

The implication was that the constraints of room shape and size were such as might typically be set by a designer wishing to produce plans on some particular occasion. And that by means of the programming method he would be able to explore the limits of overall dimensional variation (in total area, length of perimeter, overall length or width) allowable *for those specific plans.*

Imagine, however, a similar method being used by a building scientist. He by contrast would be interested in the *absolute extremes* of size and shape allowable for each type of room—insofar as those are determinable. One might imagine, for example, that lower limits of area or dimension could be set on certain room types such as bathrooms, kitchens, or corridors, on the basis of anthropometric standards and the requirement to accommodate certain furniture or fittings. The building scientist would then want to know the complete extent of the range of plan types resulting, with in each case their maximum or minimum overall areas, depths etc. Conversely it might be that limits on these overall dimensions of depth themselves could be established say in terms of maximum allowable structural spans, or some maximum acceptable depth for daylighting—and that these fixed global constraints would in turn have their effect back on individual room shapes and sizes.

The designer looks for a particular plan, or some few plans, with desired properties. The building scientist seeks to map out the whole field within which the designer's search is conducted.

Allometric studies of building form

There is a tradition of research in biology, starting from D'Arcy Thompson's *On Growth and Form*, which has attempted to define limits on organic structure and the 'design' of animals and plants, in a very similar way. Thompson shows for instance how the strengths of woody fibres put a minimum ratio of slenderness on the stems of plants, of which about the extreme limit is represented by bamboo. Equivalent considerations of strength in relation to weight limit the maximum height of trees to approximately 300 feet. Similar reasons applying to bones mean that it is impossible for terrestrial animals to be much larger than the mammoth or the prehistoric dinosaurs.

Other features of the design of organisms besides their 'structural engineering' have analogies with the design of buildings. The ratio of surface area to volume is important in both cases, since it affects the rate of heat loss or gain, to or from the surroundings. In buildings this same ratio of surface to volume is significant, of course, for natural lighting and ventilation. Animals and buildings are both served by different kinds of circulation systems—networks of tubes or passageways, respectively, which must penetrate at a more or less uniform density to every part.

If organisms of differing absolute *size* were to possess exactly the same *shape* (simply scaled up in direct proportion), then they would have very different properties in these respects. This follows from the way in which areas vary with the square of linear dimensions, and volumes vary with the cube. An animal of double the linear dimensions, for example, would have only four times the cross-sectional area of legs with which to support eight times the body weight (assuming weight to be a simple function of volume).

What happens in fact is that these effects are compensated for by *changes in shape.* Such changes may be observed in comparisons of the forms of adults of different but related species which vary in their absolute size. They also occur in the development of individuals of some given species from embryonic to adult form—in which case they are brought about by appropriate adjustments in the differential rates of growth of the parts. Thus the relative shape and proportions of head, trunk, and limbs in human babies are quite different from those in the grown man or woman.

The study of relative growth rates and their effect on shape is termed *allometry.* After D'Arcy Thompson's pioneering work the subject was greatly developed, by Huxley (1932) and others. There has recently been something of a revival of interest in these questions of size and shape in the investigation of biological form (for example, see Alexander, 1971). One study of building forms using allometric techniques borrowed from biology has been made by Bon (1971; 1972a; 1972b; 1973).

Bon took a sample of forty residential buildings—houses, apartments, hotels, including even a houseboat and a railway sleeping car built for a nineteenth-century financier—of various dates and from different parts of the world. Bon's notion here was that, of all functional types, it is dwellings which are the most dependent on natural lighting. Also he was able to find buildings of this general class, of very widely varying sizes. Bon measured for each building its volume V (in cubic feet), and its total surface area of external wall S (in square feet). These values are shown plotted on a graph in figure 10.1. The ratio of surface area to volume decreases generally with increasing absolute size of building, as would be expected. What is interesting here is the comparative rate of growth in each of the variables.

Imagine a building of simple cube shape, with length of side d. The volume V of the building is d^3, and its total surface area of walls S (discounting roof and floor) is $4d^2$. The continuous curve in figure 10.1 represents values of volume and wall surface area of such a cube-shaped building, for increasing d. Notice that the points representing the larger actual buildings in Bon's sample all lie *above* this curve. The implication, clearly, is that in the actual buildings the ratio of surface area to volume has been increased over that of the cube, by some elongation of the shape, or perhaps by some corrugation of the wall surface. This effect is to be explained by the way in which residential buildings—of any size—are so designed as to give a large proportion of the rooms an adequate area of window for the purposes of daylighting, views, and natural ventilation.

It would be expected that if the rates of growth in V and S in the sample of actual buildings were constantly related, then this relation would be of the general allometric form $S = bV^a$, where a and b are constants. With our cube-shaped buildings, for example, $V = d^3$, $S = 4d^2$, which gives $S = 4V^{2/3}$, that is, $S = 4V^{0 \cdot 66}$. The general equation can be restated by means of logarithms, as

$$\log S = \log b + a \log V .$$

Thus if the logarithms of the two variables are plotted, and the values fall on or near a straight line, this shows that the variables are related allometrically. This is indeed the case here, as figure 10.2 demonstrates. The *slope* of the line is given by $a = 0 \cdot 77$. Had the buildings all been of a similar shape (as, for example, cubes or any other fixed form) the slope of the line would have been $0 \cdot 66$. (That line is called the *isometry line.*)

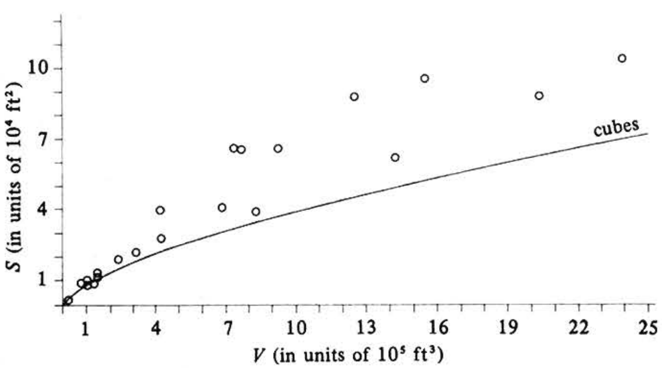


Figure 10.1. Graph of total surface area of external wall S (ft^2) against volume V (ft^3) for a sample of forty residential buildings [from data given in Bon (1972b)]. The continuous curve represents (theoretical) buildings of simple cubic form.

The faster observed rate of growth in wall area, relative to growth in volume, constitutes a case of *positive allometry.*

The consequences can be observed in practice in the way in which the forms of large daylit buildings are flattened in the vertical plane into slabs, and in the horizontal plane become spread out into winged, branching, or courtyard configurations. The actual buildings in Bon's study would not themselves necessarily lie on or near the 'extremes of possibility' which the constraints of daylighting allow. No doubt the standards of lighting in them vary considerably, a supposition which is given weight by the departure of the observed values of S and V somewhat from a smooth curve, and even more so by the fact that the absolute ratio of S to V *does* decrease quite markedly with increasing building size, despite the positive allometric effect. However, the very fact of this positive allometry points to the existence of some kind of dimensional limit on acceptable depths for residential buildings, and an attempt to escape, in the shapes of larger buildings, from the effects of a decreasing ratio of surface area to volume[24].

Dimensional constraints of this kind have their effect back on the properties of the plan and adjacency *graphs* of arrangements. The significance of the depth limit is that in all probability most of the rooms in these buildings are on the perimeter, and only circulation spaces and perhaps bathrooms, storerooms, etc are windowless and in the centre of the plan.

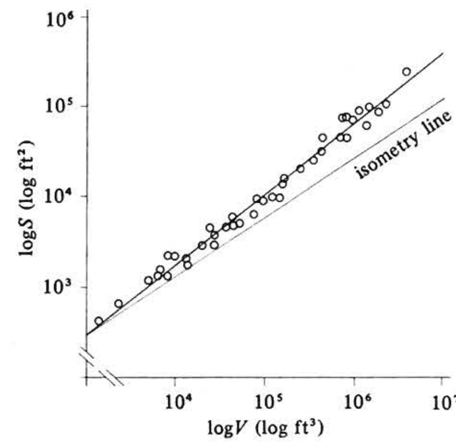


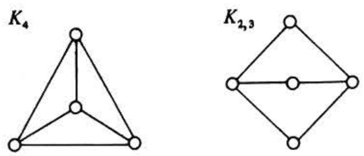
Figure 10.2. The data of figure 10.1 plotted on logarithmic scales, to show the allometric relationship of the two variables S and V (from Bon, 1973). The *isometry line* is marked. The line fitted to the observations has a steeper slope: the faster rate of growth in wall area relative to growth in volume is a case of *positive allometry.*

[24] Bon suggests that similar allometric techniques might be used to investigate structural questions such as the relation between the weight of buildings (as a function of the volume of their materials of construction) and the total cross-sectional area of their structural elements. This could provide a means for determining the height limits imposed by certain types of building material.

This naturally tends to result in the tree-like circulation systems typical of apartment blocks or hotels, with rooms branching off central corridors, or clustered around regularly spaced staircase wells.

It was noted in chapter 7 that there exists a class of graphs which are capable of being embedded in the plane such that all vertices lie on the exterior face. These are *outerplanar* graphs. Considered as adjacency graphs they correspond, in their outerplanar embeddings, to those plans in which *all* rooms lie on the perimeter. All trees, and all graphs with only one cycle, are outerplanar.

Lynes (1977) has commented on the significance of outerplanarity in adjacency graphs for the planning of naturally lit buildings. Harary (1969) has stated the criterion for outerplanarity of a graph: that it must not contain as a subgraph any graph homeomorphic to K_4 or $K_{2,3}$. These subgraphs are:



See how in all embeddings of these graphs, one vertex remains in the interior, and not on the exterior face. The addition of one more vertex connected to all other vertices in each case will produce K_5 and a graph containing $K_{3,3}$, respectively (you might like to check this).

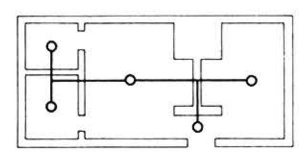
Lynes makes the point that the presence of K_4 or $K_{2,3}$ in an adjacency graph does not mean that some *particular* room has to be internal. There will be several embeddings, in which in each case a different room will fall in the interior. Lynes also remarks on the way in which the placing of windows in a building restricts the ways in which it may be extended in the future. He suggests that this problem too could be investigated by a graph-theoretic approach.

In chapter 8 I listed a whole series of the properties of adjacency graphs which might have practical architectural significance. One of these was the distribution of the valencies of vertices—the number of edges incident with each vertex in the graph. There is plainly *some* relation, if not a necessary or simple one, between this distribution of valencies and the distribution of *room sizes* in a building—since rooms which are relatively large will tend to be adjacent to more other rooms. The two measures together might go some way, together with the structural properties of the access graph, towards the classification of functional building types on a geometric and topological basis.

Some empirical studies exist of the distribution of room sizes in selected types of building, made mostly out of an interest in the potential flexibility and adaptability of plans, or the timetabling of their use. Thus there are surveys of room sizes in hospitals made by Cowan and Nicholson (1965), Weeks (see Llewelyn-Davies et al, 1975), and others; in university buildings (UCERG, 1968; Bullock et al, 1970); and in schools (see Fawcett, 1976b).

There is some tendency for larger buildings to contain larger rooms, especially those which consist of only a few principal spaces, or just a single space, such as auditoria, churches, or markets. One would not, however, expect the range of room sizes in any one *residential* building to be very great, nor would one expect larger residential buildings to contain larger rooms, only a greater number of them. The same would be true of office buildings (made up of individual private offices, that is). In both cases the rooms tend to take their sizes, roughly speaking, from those of the pieces of furniture which they accommodate. In this there is a loose analogy perhaps with the way in which animal bodies of all sizes are made up from cells which do not themselves vary greatly in size, only again in their numbers[25].

Bon (1972a) made a second allometric study of a smaller sample, of twenty buildings, again all houses or dwellings. In this case he measured the total floor area F (in square feet) and what he called the total 'communication network length' L (in feet). This latter he defined as "the minimal orthogonal network connecting all the room centroids in a building". By 'room centroid' he means a point at the notional centre of gravity of the room's shape in plan. Bon forms a circulation network of tree form joining all these points, in which the routes run in one or other of two perpendicular directions, parallel with the walls of the building—that is, the distances along these routes are taxicab distances—and such that the total length of all routes is minimised. The following figure gives an example:



It is thus not a representation of the actual circulation system, but some measure of the theoretical minimum length of circulation needed.

If log F is plotted against log L for Bon's observations, a straight line of slope 1·07 can be satisfactorily fitted (figure 10.3). Had *similar* shapes of building plan and communication network been preserved—which would imply that the *number* of rooms remained the same with increasing building size—the isometry line in this case would have taken a slope of 0·5. This is therefore a very marked case of positive allometry, arising out of the differentiation of the internal plans of the buildings into ever

[25] It was proposed, as early as 1893, by Ryder, that it is the ratio of surface to volume which puts a maximum size limit on the individual organic cell itself. As D'Arcy Thompson (1961) says, "Nature has her materials of predeterminate dimensions, and keeps to the same bricks whether she build a great house or a small".

greater numbers of rooms, all of which must be reached by the 'minimal network' of circulation.

Such a general result is only to be expected. What is perhaps more particularly interesting is that the ratio F/L for Bon's sample varies only slightly, and lies for the great majority of cases between 8 and 12. In a minimal circulation system with the form of a *tree*, the number of edges e must be one less than the number of vertices v, that is, $e = v - 1$. This is a general property of trees. It follows that the number of centroid to centroid distances measured by Bon for each building must be very close to the number of rooms. This suggests that the mean room dimension for these residential buildings is of the order of ten feet, which is again what might be expected.

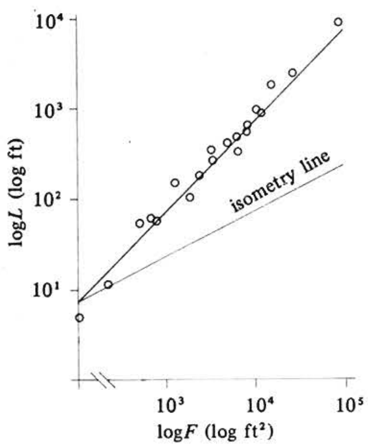


Figure 10.3. Graph of total 'communication network length' L (ft) against total floor area F (ft^2) for a sample of twenty residential buildings, plotted on logarithmic scales to show the allometric relationship of the two variables (from Bon, 1973). The isometry line is marked. The line fitted to the observations has a much steeper slope: the faster rate of growth of the communication network relative to growth in floor area is a strong case of positive allometry.

The adjacency and access graphs of residential buildings

As for the *adjacency graphs* of residential buildings, one would not expect large variations in their vertex valencies precisely because of this fact that the order of room size is the same throughout—assuming the rooms to be of roughly comparable shape, convex for the most part and not excessively elongated. (In a building such as a large hotel, with rooms planned along a series of corridors, these conditions would not apply.) There would be a systematic difference certainly between the valencies in the weak dual graph, of vertices representing, respectively, rooms in the interior and rooms on the perimeter. But this 'boundary effect' is to some extent compensated for by including adjacencies to the exterior region.

And in residential buildings, as we have seen, the majority of the rooms are on the perimeter anyway.

In chapter 7 we saw that in large maximal planar graphs the average valency of vertices approaches the value 6. It would be interesting to know how closely the adjacency graphs of actual buildings approach to this theoretical limit. Two empirical studies have been made whose authors had rather different purposes in mind, but which all the same throw light on this question.

Both studies were of houses. The first was made by myself (Steadman, 1976): my interest was in finding where the plans of typical small modern houses would lie in the whole space of theoretical possibility represented by Combes's diagram (compare the figures at the beginning of chapter 8) for dissections. I took for my sample house plans illustrated in the English National Building Agency's publication *Generic Plans* (NBA, 1965). This

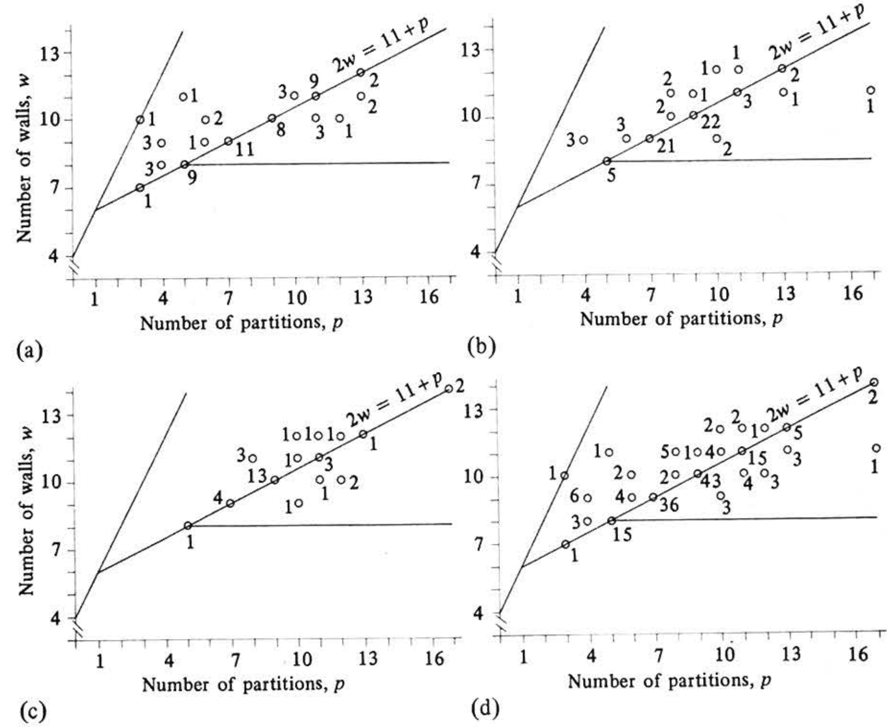
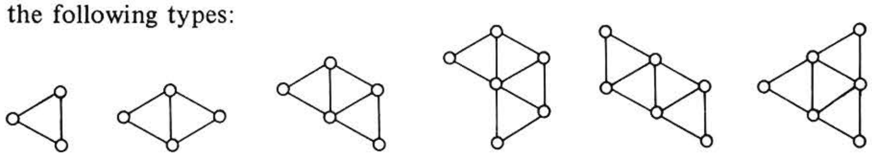


Figure 10.4. Plans of small modern houses of rectangular dissection type, plotted on Combes's graph of numbers of internal and external wall segments (compare figure 8.1) (from Steadman, 1976). The plans are taken from a catalogue, *Generic Plans*, published by the English National Building Agency (NBA, 1965). They are classified as four-person, five-person, and six-person types: these are shown plotted separately in (a), (b), and (c), respectively. The number of plans falling on each point is indicated. Both ground-floor and first-floor plans are included. (d) shows all 164 plans combined in a single diagram. See how of these the great majority (117) fall on the line $2w = 11 + p$.

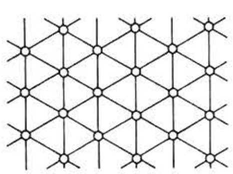
is a catalogue of model plans for the use of architects and house builders. However, they are for the most part derived from a survey of actual plans in use in local authority and privately developed housing schemes.

I took only those in which the shape of the plan overall was rectangular, and approximated the layout in each case as a rectangular dissection. I then plotted the numbers of instances of each dissection on the relevant points in Combes's graph, according to their numbers of internal and external wall segments p and w [figure 10.4(a)–(c)]. *Generic Plans* classifies houses as four-person, five-person, and six-person types. These were plotted separately, and the ground-floor and first-floor plans were both included. The results for all plans together were further summarised in a single diagram [figure 10.4(d)].

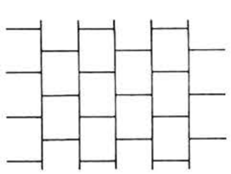
There is a distinctly marked clustering, as will be observed, along a straight line corresponding to the equation $2w = 11+p$. Out of a total of 164 plans, 117 fall precisely on this line. If the weak dual adjacency graphs of dissections on this line are examined, they are found to be of the following types:



(Recall that the number of edges in such a graph equals the number of internal wall segments p in the plan.) These are in fact *triangulations of polygons*—triangle, quadrilateral, pentagon, hexagon, with the interior in each case subdivided into triangles in all distinct possible ways. They are all of them outerplanar graphs (as might be expected for small houses). They can be conceived of as fragments of a larger regular triangulated graph which is maximal planar and in which the mean vertex valency tends towards 6:



The dual of this graph, that is, the corresponding 'plan', is a regular packing of six-sided faces, which may be drawn in rectangular form as a brickwork-type pattern:



Of course, in the small adjacency graphs of the dissections the effect of the lower vertex valencies around the perimeter means that the average figure for these is much less than 6.

Bon investigated the same phenomenon in a sample of seventy house plans drawn from the Central Mortgage and Housing Corporation catalogue *Small House Designs* (CMHC, 1958). He measured vertex valencies in the adjacency graphs of these plans, *including* this time adjacencies to the exterior region, and obtained a mean value of 4·24. He also measured vertex valencies in the *plan graphs* and obtained a mean value of 2·65.

The mean number of rooms in a plan in the sample was 9. Bon compared his survey figures with the same values calculated for a regular plan graph consisting of nine hexagons packed as in the left-hand part of figure 10.5. His purpose here was to allow for equivalent boundary effects in the actual plans and this ideal configuration. The corresponding adjacency graph has a mean vertex valency of 4·45, and the plan graph a mean vertex valency of 2·53. (Notice that vertices of valency 2 are counted at the angles between the external wall segments.) Bon took these results to show a close approximation, in these respects, of his sample of plans to a theoretical 'hexagonal' pattern[26].

The adjacency between two rooms provides the *opportunity* for access between them, but this opportunity is not always exploited. That is to say, the access graph is generally a spanning subgraph of the adjacency graph. Bon asks the interesting question, what proportion of edges in the adjacency graphs of plans are simultaneously edges in the access graphs? In the sample just mentioned the mean vertex valency of the access graphs of the house plans was 1·92, by comparison with 4·24 for the adjacency graphs. This means that for only 45% of adjacencies between rooms, or between rooms and the exterior, were there actually doors connecting the rooms or regions in question.

Bon showed that this percentage, calculated for small house plans, does not seem to vary greatly when calculated for residential buildings of widely differing sizes. He took his sample of forty buildings, already referred to in connection with the allometric studies, and counted in each case the numbers of edges e and vertices v in the access graph. The results are plotted

[26] In point of fact, most of the actual plans would have been rectangular, and it might have been more reasonable for Bon to take for comparison a packing of nine rectangles in 'brickwork' arrangement. This would be no different in its adjacency graph from the hexagon packing. However, there is a slight difference in the mean valency of the vertices of the plan graph due to the counting of vertices with valency 2 on the perimeter. The resulting revised figure for the mean vertex valency of the plan graph is 2·57. Notice also that Bon's method of calculating the mean vertex valency in adjacency graphs is slightly odd, in that for rooms on the perimeter he *includes* adjacencies to the exterior region, but he *excludes* the vertex representing the exterior region itself. Thus for the adjacency graph of the hexagonal plan of figure 10.5, the exterior region, whose vertex has valency 8, is not counted in taking the mean. Where all edges and all vertices *are* counted, the mean vertex valency is given by $2e/v$.

in figure 10.6. See how the ratio of e to v remains remarkably constant with the increase in size of the graph. A line which represents the best fit statistically to the observed values has the equation $e = 1 \cdot 30v - 3 \cdot 04$, as

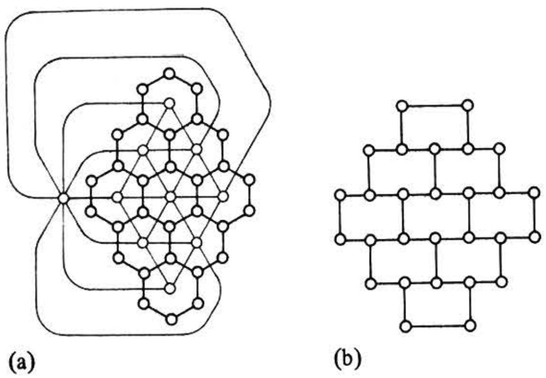


Figure 10.5. (a) Theoretical nine-room plan of hexagonal geometry, for comparison in terms of mean vertex valencies of adjacency graphs and plan graphs, with a sample of seventy actual house plans whose mean number of rooms is nine (from Bon, 1971), and (b) the same plan drawn with a rectangular geometry (see footnote 26).

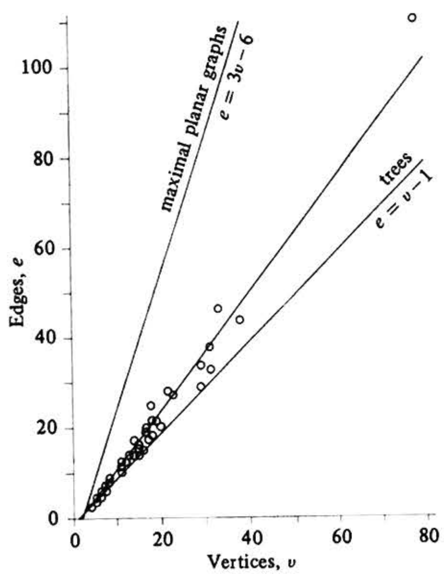


Figure 10.6. Graph of number of edges e against number of vertices v in the access graphs of a sample of the plans of forty residential buildings (from Bon, 1971). (Two observations are omitted here.) A straight line fitted to the observations has the equation $e = 1 \cdot 30v - 3 \cdot 04$. Also shown are straight lines corresponding to *trees*, $e = v - 1$, and *maximal planar graphs*, $e = 3v - 6$. The access graphs of plans, since they must be connected, and planar, must lie between these limits.

shown in figure 10.6. Subsequently Bon (1972b) extended this same analysis to a much larger sample, of 500 buildings, and obtained a similarly consistent ratio e/v although the slope of the allometry line was somewhat shallower (the equation was $e = 1 \cdot 13v - 1 \cdot 34$).

Rather than compare these values of e with the corresponding values for the adjacency graphs of the actual plans, this time, Bon chose rather to make comparisons with *maximal planar* graphs. The real adjacency graphs would not necessarily be maximal planar, of course, although they would probably approach close to that point, as we have just seen.

For maximal planar graphs $e = 3v - 6$. This follows from Euler's formula $v - e + f = 2$, applied to plane triangulations. If every face is a triangle, and since every edge must belong to just two faces, then $f = \frac{2}{3}e$. Substituting in Euler's formula gives $3v - 3e + 2e = 6$, that is, $e = 3v - 6$. See how in the adjacency graph of the hexagonal arrangement of figure 10.5, for example, which is maximal planar, $e = 24$, $v = 10$, and $3v - 6 = 24$.

For each plan, Bon compared the actual number of edges in the access graph with this theoretical maximum for planar graphs on the same number of vertices. This ratio, $e : (3v - 6)$, expressed as a percentage, has been called the 'gamma index' (Garrison and Marble, 1961). It measures the degree of connectedness of a planar graph. The mean value of the gamma index for Bon's sample of forty buildings was 43%. For the seventy small house plans the mean value was 42%.

Bon concludes from this that the 'hexagonality' of the plans, as evidenced by the properties of their adjacency graphs, is more a simple consequence of the close-packing of rooms, rather than a means of maximising access between them. This seems to follow since less than half (42 or 43%) of the theoretical opportunities for providing access are actually made use of. This is a complex matter, however, since there is liable to be a marked variation in vertex valency in the access graphs—which the mean figure conceals—from private rooms such as bedrooms or bathrooms with a valency of 1, to circulation spaces such as halls or landings, with much higher valencies, which give access to several rooms. It is these latter types of space which are placed, of course, in the centre of plans, and the former around the perimeter.

One does not expect there to be doors in very many of the exterior walls of the plan (especially not on floors above the ground!) and it is therefore these walls which would account for at least some of the 'missed opportunities' for access. More would be revealed by counting separately and comparing the percentages of external and internal wall segments which are or are not penetrated by means of access. In the ideal arrangement of nine hexagons in figure 10.5, for example, there are eight adjacencies of rooms to the exterior region, compared with sixteen interior adjacencies between rooms. Thus one-third of the theoretical opportunities for access in this case are in the perimeter. It may be that the 'hexagonality' of plans reflects rather an effort to achieve high levels of *interior* access, between rooms. A measure of general interest for plans would be one showing how adjacencies between rooms and with exterior regions are divided between those made use of for access and those made use of to provide windows.

It is worth making the point that the number of edges in an access graph must have a *lower* bound, if that graph is to be connected, as in practice it would tend to be. (Even the access graphs of pairs of semi-detached or rows of terraced houses are connected, at least at ground level, if access to a common exterior region is included.) The graph which connects v vertices with a minimal number e of edges is a tree, in which case $e = v - 1$ as we saw earlier. The line representing this equation has been plotted on Bon's diagram in figure 10.6. The gamma index for trees must always equal $(v-1):(3v-6)$ which for large graphs tends to the value 33%. This then is the lower bound on the possible value of the index. The distance which the graphs in Bon's sample lie above the line $e = v - 1$ must represent the extent to which they contain cycles, and so are to a degree 'redundant' in providing means of access between rooms.

Gross measures of the properties of adjacency and access graphs

With the exception of certain work on access graphs by Hillier and colleagues, to be described in the next chapter, there are no comparable empirical studies known to me of the properties of the graphs of building types other than residential. However, one can imagine such studies providing an important part of the formal classification of building types, as I suggested earlier.

The adjacency graphs of residential buildings as we see are characterised by no great variations in the valencies of vertices (with the exception of communal access corridors in flats or hotels), and with a high proportion of rooms having adjacency to the exterior. The graphs of other building types might display large variations in vertex valency (the graph partition or distribution of edges between vertices would be much more uneven)—this perhaps reflecting large differences in room sizes, as in, for example, big auditoria with their surrounding lobbies, bars, backstage facilities, and other service spaces. Again the distribution of adjacencies to the exterior region might typify such buildings as recording studios or certain parts of hospitals, in which rooms are deliberately isolated from the outside environment.

As for access graphs, it is certainly possible to imagine larger buildings being classified by the general network properties of their circulation systems. To give some specific examples: the access graphs of blocks of flats consist generally of repeated identical 'knots' of private circulation within the flats, all connected to some larger linear public circulation routes. This public circulation could consist of a series of separate staircases along a block. Or it might be that a block was served by say only two staircases, connected at each level horizontally, in the central corridor or 'balcony access' types of arrangement. High-rise office buildings may be served by a single circulation 'core' from which all rooms on each floor are reached. They may like blocks of flats have a central corridor, in which case the access graph is tree-like; two parallel corridors with a central row of artificially lit service and storage rooms; a continuous cycle of corridor in the 'racetrack' type of plan, and so on (figure 10.7).

Some workers who have applied graph theory to describing networks in subjects other than architecture—in biology (Shimbel, 1953), in transport studies (Garrison and Marble, 1961; Kansky, 1963), in geography (Haggett and Chorley, 1969)—have devised a variety of quantitative measures of graph properties which may be relevant here (compare Tabor, 1976). We have already looked at the gamma index for connectedness. Another property is the *cyclomatic number*, which for a connected graph is given by $e - v + 1$. For trees, where $e = v - 1$, it follows that the cyclomatic number is always zero. The addition of further edges to a tree while keeping the same number of vertices, so that it becomes a graph with cycles, increases the cyclomatic number by 1 for each edge.

Figure 10.8 illustrates the process for a tree on twelve vertices. Notice, however, that the cyclomatic number does *not* count the number of cycles in a graph. In a connected plane graph, the cyclomatic number is equal to the number of *faces* (excluding the exterior face). This follows from Euler's formula. A cycle may pass round any number of faces sharing edges in common. Where a plane graph contains adjacent faces the number of distinct cycles may thus be much greater than the number of faces.

In the access graph of an architectural plan, the presence of cycles means that there exist two or more different routes between certain pairs or groups of rooms. The legal requirement in large modern buildings to provide alternative means of exit in the case of fire means that their access graphs must at least comprise cycles to this extent. The same would apply in any grand house with 'back stairs' or servants' stairs. Any plan with a racetrack arrangement or with continuous circulation around courtyards would equally have an access graph with cycles on each floor considered separately.

The *diameter* of a graph is found by taking the shortest paths between all pairs of vertices (measured as a number of edges in each case) and taking the longest of these—it is the 'longest shortest path'. This is some *very* rough measure, for a graph with a given number of vertices, of how spreading or how compact it is.

Applications of these and similar measures to real plans are discussed further in chapter 11. They give gross indications of the overall structural character of adjacency and access graphs. At a more detailed level it would be possible to describe plan types in terms of classes of specific labelled graphs (the labels signifying room functions). Thus one can imagine small house plans being categorised by distinct patterns of access between living room, dining room, kitchen, bedrooms, etc.

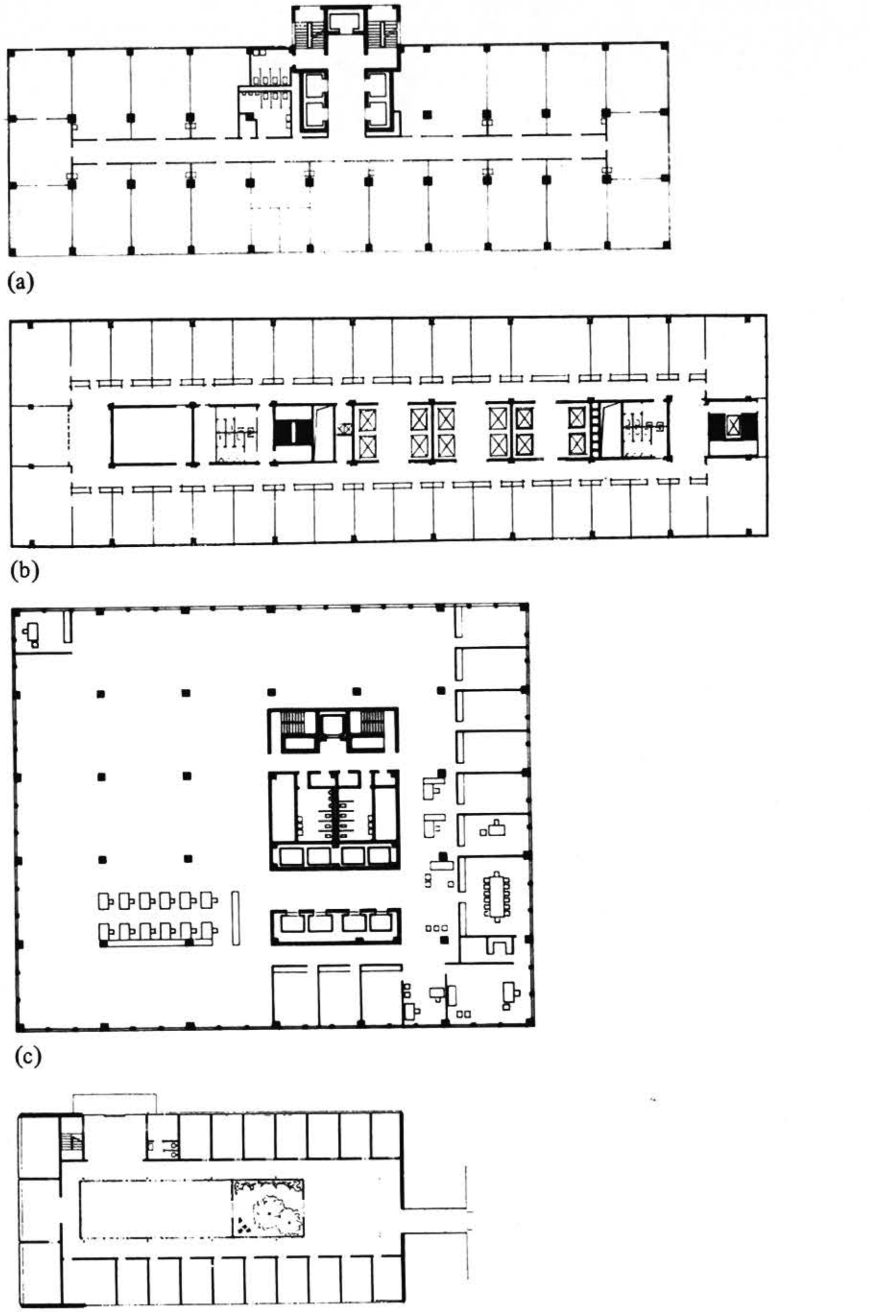
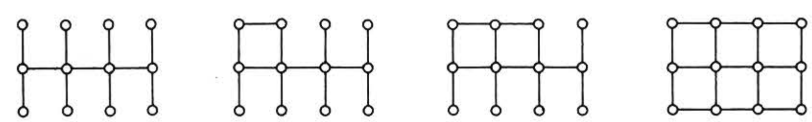


Figure 10.7. Typical examples of plans of office buildings (from Joedicke, 1962, figures 48, 55, 58, and 62): (a) double-zone layout with central utility core—Equitable Savings and Loan Association building, Portland, Oregon (Pietro Belluschi); (b) triple-zone layout with parallel corridors—competition design for Phönix-Rheinrohr AG building, Düsseldorf (Paul Schaeffer-Heyrothsberge); (c) open layout with internal utility core—Mile High Center, Denver (I M Pei); (d) single-storey courtyard layout—Schlumberger building, Ridgefield, Connecticut (Philip Johnson).

Some statistical work along exactly these lines was carried out by Bon (1972b) on his sample of seventy small houses. He categorised their access graphs by numbers of vertices, the distribution of valencies of those vertices, and the numbers of cycles in the graphs (for 27% of the plans the number was zero—the graphs were trees—and in a further 67% there was one cycle only). He measured for the access graphs the values of some of the structural indices described above. He identified characteristic recurrent configurations of labelling of the graphs, that is, typical patterns of access between rooms of different functions. And he measured the mean accessibility of rooms with different functions, that is, the average distance (as a number of edges) from one room to all other rooms—with high values resulting, as would be expected, for halls and corridors.

Besides the functional significance of adjacency between rooms or regions in terms of access or lighting, there are the further factors of privacy and acoustic separation, which could serve to account for the fact of rooms being made deliberately *not* adjacent in certain plan types.

So far in the book we have only considered the functional reasons for the direct binary relation of adjacency or nonadjacency between pairs of rooms or regions. However, there are certain instances, particularly with access relationships, where three or more rooms are made adjacent and accessible one to another in some specific pattern or sequence. This occurs, for instance, where a private office of a manager or a doctor is reached from a public corridor only through a waiting room, or through the intermediate office of a secretary or receptionist. In hospitals, in public baths, or in factories, where people or goods are involved in definite sequences of activities or processes, then the circulation system joining rooms or spaces will naturally follow these sequences. The layouts of museums or exhibitions provide further examples. These topics are also taken up again in the next chapter.



$e = 11$	$e = 12$	$e = 13$	$e = 17$
cyclomatic number: 0	cyclomatic number: 1	cyclomatic number: 2	cyclomatic number: 6

Figure 10.8. The *cyclomatic numbers* of sample graphs on twelve vertices (after Tabor, 1976). For a connected graph the cyclomatic number is given by $e-v+1$: thus for a tree (first graph), where $e = v-1$, the cyclomatic number is zero. The addition of each new edge increases the cyclomatic number by one: in connected plane graphs, as here, it counts the number of interior faces.

Pedestrian circulation

In chapter 9 we examined a number of automatic methods for producing plans supposed to be 'optimal' in terms of their circulation patterns, and I offered several criticisms of the shortcomings of these methods. In each case some regular frequency of daily or weekly pedestrian journeys was assumed between all pairs of 'activities' in an organisation; and the effort was then made to find an arrangement of rooms or positions for those activities such that the total distance travelled in the given time, or else some notional total travel cost, was minimised.

The methods were open to three objections in particular. First, the minimisation of circulation was not in general balanced against other functional (let alone aesthetic) criteria in design such as those of natural lighting, structure, etc. Second, it was difficult to make realistic measurements of distances between pairs of rooms without specifying in advance some general pattern of circulation routes, this implying in turn an approximate overall form for the building. Thus the problem had a certain chicken-and-egg character, since the whole purpose of the automatic layout method was precisely to generate this building form and circulation system in the first place. Third, such methods were implicitly designed to fit the plan of a building as tightly (hence as 'efficiently') as possible to some fixed pattern of activities taking place in and between rooms. Whereas with the design of many building types a certain openness to change, a degree of adaptability and flexibility in the plan, would be more desirable.

In response to these criticisms some authors working on the subject of circulation in buildings, notably Tabor (1976) and Willoughby (1975a; 1975b) have taken a wholly opposite view of the question, and turned from a 'design methods' to a more properly 'building science' approach. They have started instead with specific building plans, with their layout of rooms and circulation routes completely given, and then have made quantitative comparisons between these forms on the basis of their relative 'circulation performance'. The main focus of their work has been on office buildings, hence their chosen forms are several of them similar to those identified above as representative of office types (compare figure 10.7). Tabor compares slab, cross, and court forms [figure 10.9(a)]. Both the slab and the court may have rooms single-banked or double-banked along a corridor. Willoughby in addition examines a 'fishbone' or branching, tree-like plan type [figure 10.9(b)].

It is clear that these type-plans have been chosen to be representative in a diagrammatic way of actual designs of office buildings as found in practice. Room sizes are not widely variable—indeed Tabor and Willoughby assume for the sake of simplicity uniform room areas throughout. And each layout as a whole acknowledges implicitly the constraints of natural lighting to the majority of these rooms, the desirability of a simple and coherent circulation system, and perhaps also the constructional and structural virtues of a regular, modular, rectangular plan.

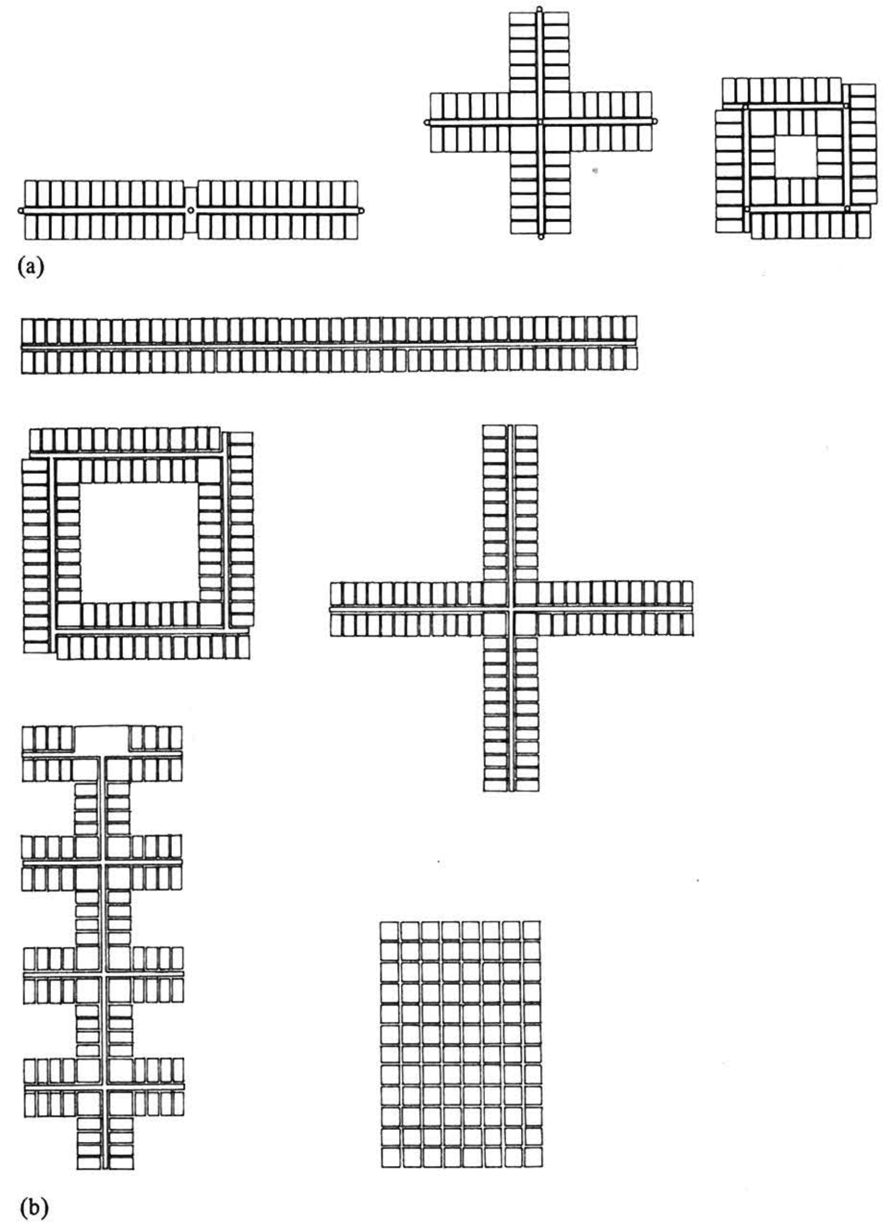


Figure 10.9. Theoretical types of plan for office buildings, each containing the same number of similar-shaped rooms, for comparison in terms of circulation performance. (a) Slab, cross, and court plans (from Tabor, 1976). (b) Slab, court, cross, 'fishbone', and open plan types (from Willoughby, 1975a).

Willoughby comments for instance on the relation of his 'fishbone' type to the arrangement of linked Nissen huts put up by the armed forces in many parts of the United Kingdom during the Second World War, and subsequently converted for use as local government offices, university laboratories, and even hospital buildings. For further comparison Willoughby introduces a continuous deep office plan which would need permanent artificial lighting and ventilation, and would correspond to the familiar *Bürolandschaft* type of the 1960s.

Thus the forms have been chosen to be 'building-like' in the senses vaguely adumbrated earlier[27]. Beyond that the choice is perhaps somewhat arbitrary. Any attempt to take such exercises further would do well in my view to go back to an empirical survey of actual office buildings (or other functional types), to determine from them a realistic range of 'possible' forms and dimensions, rather than to go on ad hoc comparing more hypothetical forms.

It is not my intention to report the results of these circulation 'experiments' in detail, if for no other reason than that they have been fully published elsewhere (refer to Tabor, 1976; Willoughby, 1975a; 1975b). It is, however, well worth looking at their general methodology, to see how this relates to some of the other morphological issues which we have already touched on.

I have suggested in previous chapters that the exact *pattern* of relations of adjacency and of access within some group of rooms only assumes functional significance at a small scale, say that of ten or twelve rooms at the most. In larger plans the properties of access graphs joining hundreds of rooms, one would expect to be hierarchical. The smaller functional groupings of rooms—be they departments or separate tenancies in office buildings, separate flats in a block of flats, or whatever—would be linked to each other and say to the main entrance and any other shared facilities by a system of specialised circulation spaces—corridors, staircases, lifts.

The structure of the access graph at this level of the hierarchy would therefore need to be interpreted both in terms of the overall form of the building, and in terms of the resulting (dimensioned) *distances* separating pairs or groups of rooms along this system. That is to say at this larger scale, the relation of *proximity* between rooms comes to have functional relevance. So the access graph needs to be considered as a *network* in which weights are attached to the edges to represent dimensions of length along corridors etc. (At the scale say of a single house, these distances are effectively negligible since all rooms are 'close' to all others.)

[27] The contrast of inward-looking courtyard buildings with outward-looking slab or 'pavilion' forms is also in part related to a theoretical argument about the general way in which buildings of these generic forms make use of land, put forward by Martin and March (1966; 1972). That argument in its turn was based nevertheless on some considerations of the spacing apart of buildings, having to do at least by implication with lighting, or ventilation, or views.

At the simplest level it would be possible to measure the shortest distances between all pairs of rooms and plot the distribution of these values. This, or the mean of all these distances, would give some indication of the concentrated or dispersed nature of the plan as a whole. The measure of the *diameter* of a graph can be extended to networks, giving the 'longest shortest distance' as a length in feet or metres.

In chapter 8 we saw such calculations made by March and Matela for polyominoes in their undimensioned form (the distances there were expressed as numbers of edges). It would be possible to make similar analyses of the plans of real buildings; or as Tabor has done, of hypothetical 'built forms' with specified dimensions. In March and Matela's work three different kinds of distance measure—straight-line, 'taxicab', and graph distances—were employed. With actual or schematic architectural plans it obviously makes most sense to measure distances as travelled along the access network itself. These will usually be greater than the corresponding straight-line distances. The difference between the two can be expressed as a *diversion factor*, or *detour index* (see Tabor, 1976).

If the access network contains cycles, and hence alternative routes are possible, it seems reasonable to make the behavioural assumption that people will tend to follow the shortest of these routes (although for visitors unfamiliar with a building's layout this would by no means always be true). Several algorithms exist for determining shortest paths in a network [Dreyfus (1969) gives a survey]. A question arises with multi-storey buildings, as to how to compare distances travelled vertically by stairs or lifts, with horizontal distances along corridors. The problem can be circumvented by the measurement of average travel *times* instead of distances, the assumption of certain walking and climbing speeds, etc, although this calculation too has its difficulties, especially in relation to lifts.

Tabor is able to rank the various type-forms which he considers, by taking the mean of the distances between all pairs of rooms in each case. The forms are constructed so as to be directly comparable, in that they each contain the same number of rooms of the same size throughout. The implication is that those forms in which the mean distances are lower have an inbuilt circulation advantage. It would be *possible* to arrange accommodation for an organisation with a given pattern of trip frequencies in these plans such that the total distance travelled was less than in other plans, simply by virtue of their shapes. With multistorey buildings the results can be appreciably affected, for the same form, by the numbers and positions of lifts.

As Tabor allows, however, these demonstrations assume that all distances between pairs of rooms are equally significant functionally. In a well-planned building, as opposed to one in which functions were simply assigned to rooms at random, those activities for which proximity was desirable would naturally be placed closer together. Tabor approaches this problem theoretically by positing a measure of the 'propensity' of the occupants of a building to make shorter journeys. As this value increases towards shorter and shorter trips (the building is better planned), so the ranking of the building forms by mean journey times alters. At the extreme, where all journeys are very short, the effect of the overall building shape is negligible and the forms rank equally. Conversely, it is where journeys are longest that the building form has the greatest relative effect on the resulting mean travel times.

Willoughby tackles the same problem from a slightly different direction. He supposes that a large office building may be divided up into varying numbers of areas or zones, each of which is occupied by a separate department or firm or tenant. He assumes contact between the occupants of all rooms *within* each of these zones, and no contact at all *between* zones. He therefore calculates the total of journey times between all pairs of rooms within the same zone, and sums these values for all zones. For all his experiments, the total number of rooms is again fixed throughout. He has two variables then: the range of building forms on the one hand, and the grouping of rooms into a greater or smaller number of zones on the other.

The results are dependent to quite a large extent on the accidents of the geometric fit of room groupings with building forms. For instance, a division of the rooms into three equal-sized groups does not fit neatly into a four-storey plan, nor into a single-storey building in the shape of a four-armed cross. At least one department or tenant must have accommodation split between wings or floors, and this dislocation increases the distances travelled. Willoughby uses a modified form of one of the automatic layout planning techniques to allocate the room groupings to the building forms in an optimal way, such that the total of travel times is always minimised, so as to make the comparisons on a fair basis.

It would be misleading to list the results of these experiments in detail without giving a more precise account of the working assumptions. However, in general summary, over *all* groupings of the rooms into different numbers and sizes of zone, it is the single-storey open plan which because of its compactness gives the lowest total of travel times, followed by the one-storey and two-storey slab and cross forms. Meanwhile, a decrease in the size and corresponding increase in the number of room groupings results in smaller absolute differences in travel time totals between forms, much as in Tabor's demonstration—although this decline is not a completely regular one.

The purpose of the experiments both by Tabor and by Willoughby, in effect, is to determine how functional characteristics, in this case circulation properties, vary across the range of morphological possibility—to analyse 'the relationship between form and performance' in Hawkes's words. Hawkes himself (1980) has recently proposed, in general terms, an equivalent kind of approach to the question of the relationship of building shape and *energy use*. This work would take account not simply of the insulation properties of the building envelope (compare Martin and March,

1972, chapter 2; Mitchell, 1975b), but other factors such as orientation, area of glazing and hence solar gain, ventilation, and the temporal patterns of occupancy and control of the building's mechanical systems in relation to daily and seasonal climatic fluctuations. It is dynamic simulation models which are the 'tools of building science' needed in this case.

A designer could make use of the knowledge gained from these experiments, when choosing a form to suit some actual project. Those whose interest is rather in describing and explaining the forms and plans of existing buildings could use similar measures or models to predict, or rather 'retrodict', their past and present performance. To what extent buildings are actually planned to be optimal or close to optimal in say circulation or energy conservation terms, or to what extent these are traded off against other considerations, are things which remain to be seen.

To go back to circulation networks, the real functional significance of distances along a route system would clearly be highly dependent on the use of a building and the pattern of movement which it accommodated. In a block of flats, for instance, the distances between separate flats might not be so significant as the distance of each flat from the main entrance. In public buildings accommodating large numbers of people for short periods such as auditoria or the concourses of stations or airports, the distances travelled in these mass movements would naturally be of more importance than the distances separating staff offices or service rooms in those same buildings. A very specific constraint on *maximum* travel distances in many public buildings, a real limit on 'possible' dimensions and one indeed which is widely embodied in legislation, is that imposed by the requirements for means of escape in the case of fire.

So we could imagine the access networks of large buildings being subjected to several levels of analysis: a detailed consideration of the patterns of purely topological access relations within small and relatively isolated groups of rooms within the plan; consideration of the overall graph structure—whether tree-like, containing cycles, etc—of the components of the network corresponding to principal circulation routes; and measurement of distances or travel times along this network, depending on the characteristic movement patterns of the occupants or users of buildings of that functional type.

Before leaving the subject of networks, it is worth mentioning the possibility of using the type of 'electrical network' described in chapter 7, or the extension of the principle by Teague (1970) to three dimensions, for the purposes of certain analyses of building performance. Since these networks represent the systematic relations occurring through the plan between room dimensions, wall and floor areas, and volumes of rooms, they might be used in connection with finite-element methods of structural analysis, as Teague (1970) suggests, or in modelling the flow of heat through the internal structural subdivisions of a building as well as through its external surfaces.

Adaptability and flexibility

I mentioned, but have not yet enlarged on, one criticism of the automatic layout planning methods: their failure to allow for *flexibility* and *adaptability* in plans. There are many features of buildings affecting adaptability, and indeed several senses in which a plan may be said to be adaptable. At a small scale at least, however, the work on plan enumeration already described provides the means for a precise measurement of certain geometrical or topological features of adaptability.

Both terms, 'adaptability' and 'flexibility', are often used loosely. It is best to define several distinct measurable properties of plans vis-à-vis the organisations which they house, which affect the capacity for future changes in use or in form:

(1) There is the capacity of the same fixed plan to accommodate the same organisation, but in different arrangements.

(2) There is the capacity of the same fixed plan to accommodate different organisations.

(3) There is the ease or otherwise with which the interior configuration of some given plan may be physically altered—by removing, moving, or introducing interior walls or wall segments, etc.

(4) There is the ease with which a building may be extended—its capacity for growth.

Suppose that an organisation can be separated into distinct 'activities' or 'functions' each suitable for accommodation in a single room. (There are admittedly difficulties in this assumption, in many cases.) Assume further that a plan consists of a series of separate rooms with different attributes. Then it can be asked of each activity, will it 'fit' into each of the rooms in turn? This might be a matter of simple room size—a class of twenty pupils needs a classroom with space for at least twenty desks. It might be a matter of the fixtures and fittings provided in a room, the services supplied, the shape of the room, whether daylit or not, the strength of the floor, or any of a whole range of other considerations. Nevertheless, for the sake of theoretical simplicity we may imagine that it is possible to give a simple 'yes' or 'no' answer to the question of whether the activity will 'fit'.

The first of our four kinds of adaptability measures then the total number of distinct ways in which some given list or schedule of activities may be assigned to rooms in a given plan, such that each activity has a room in which it fits. If there are no constraints on the adjacency or proximity of activities, then this question may be studied independently of the geometrical organisation of the plan. Such an approach has been taken by Fawcett (1976a, 1978), who counts simply the permutations of arrangement in which a schedule of activities may be matched with a list of rooms, with only such geometric properties of the rooms as their separate sizes and perhaps shapes taken into account (Fawcett terms this measure 'loose-fit adaptability').

Suppose, however, that certain activities in an organisation were required to be adjacent—that an adjacency requirement graph could be specified. Then, clearly the organisation could be accommodated only in those plans the arrangement of which meets the relevant adjacency requirements. We can illustrate the situation with a simplified example which uses rectangular dissections. Suppose that we ignore all considerations of room size, shape, servicing, etc and concentrate solely on the question of 'fit' in relation to room adjacency. Figure 10.10 shows all dissections for $n = 4$, together with their weak dual adjacency graphs.

There are only four different graphs as shown: call them (a), (b), (c), and (d). Notice that (a) contains three edges, (b) and (c) each contain four edges, and (d) contains five edges.

Take one of these graphs (c). Label the vertices α, β, γ, δ as shown. Now let us imagine that the graph of adjacency requirements for the organisation which is to be housed is as in figure 10.11(a), with vertices labelled a, b, c, and d. It is in fact the same graph as (c). However, since the vertices are labelled, they may be mapped onto the vertices of (c) in two distinct ways, as figure 10.11(a) shows. The adaptability of the plan vis-à-vis the organisation is two. Take a second adjacency requirement graph, for a different organisation, as in figure 10.11(b). It may be mapped into the vertices of (c) in six ways. A third requirement graph as in figure 10.11(c) may be mapped into (c) in ten ways. In each case it is a matter of counting permutations of assignment of a, b, c, d to α, β, γ, δ such that the required adjacencies are satisfied.

We can move this study to a more general level, and consider systematically *all possible* adjacency requirement graphs for organisations which might occur. If we continue to take it that there is one vertex or activity to a room, then we need to consider all graphs on four vertices.

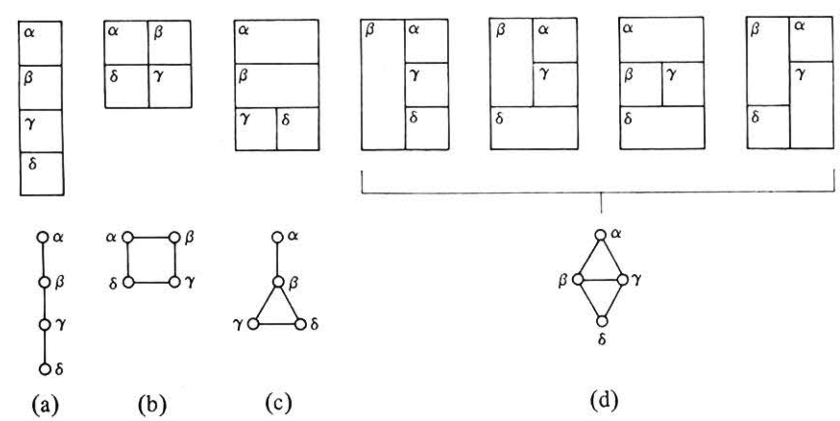


Figure 10.10. All rectangular dissections for $n = 4$, with their weak dual adjacency graphs. There are four distinct graphs, labelled (a) to (d) for reference in table 10.1.

However, there is no point including graphs with more than five edges, in the present context, since as figure 10.10 shows there is no rectangular dissection in which so many requirements could simultaneously be satisfied.

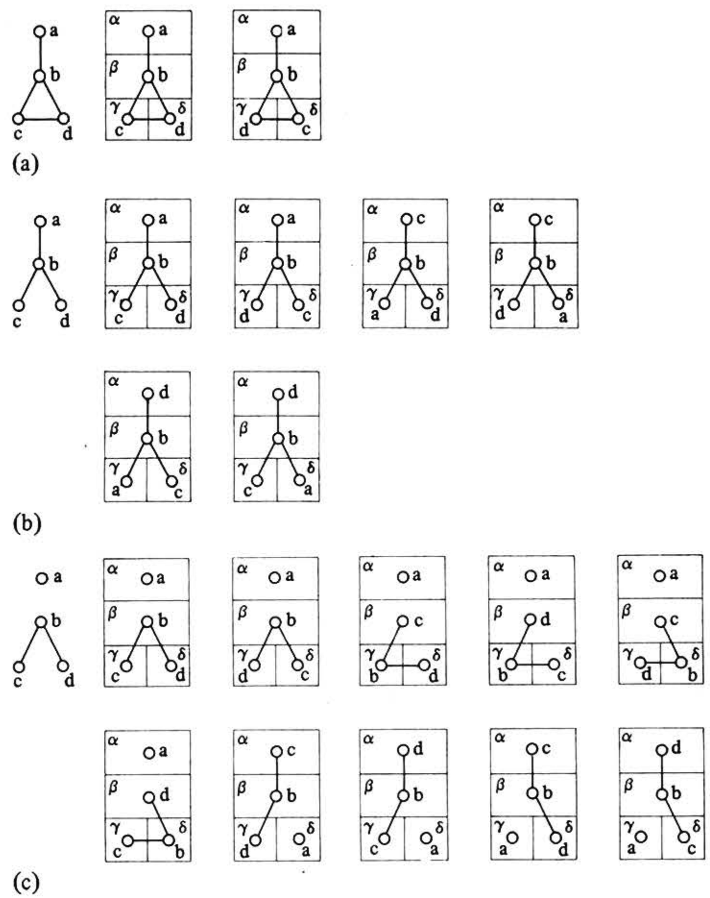


Figure 10.11. Mappings of adjacency requirement graphs into rectangular dissection type plans. (a) A set of required adjacencies between four room functions a, b, c, d, as shown in the graph, may be mapped into the four rooms of the given plan $\alpha, \beta, \gamma, \delta$ in two possible ways: the adaptability of the plan vis-à-vis the organisation is two. (b) A second adjacency requirement graph may be mapped into the plan in six ways. (c) A third graph may be mapped into the plan in ten ways.

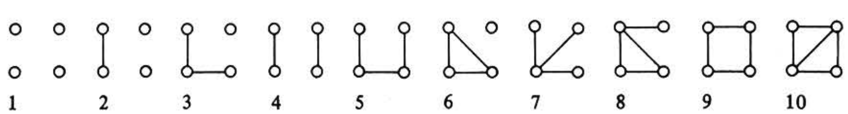


Figure 10.12. All possible adjacency requirement graphs for rectangular dissection type plans, on four vertices. They are numbered 1 to 10 for reference in table 10.1.

This leaves the graphs numbered 1 to 10 depicted in figure 10.12. Half of these are disconnected graphs.

Suppose that the vertices of all these graphs are labelled, and that they are mapped in all allowable ways into the graphs (a), (b), (c), (d) as before. Table 10.1 gives the results. Notice that graphs 1 to 10 are listed in order of increasing numbers of edges, as are graphs (a) to (d). In general, two trends emerge—though they are not completely smooth ones. Looking down the table, we see that the number of fits *decreases* for a given plan as the number of edges in the requirement graph increases. Obviously there comes a point where the number of adjacency requirements exceeds the actual number of adjacencies in the plan, and so where no fit is possible at all (bottom left-hand corner of the table). Meanwhile, looking across the table we find that for a given organisation and set of requirements the number of fits increases as the number of edges in the adjacency graph of the plan increases.

Each entry in the table then represents (in this adjacency respect) adaptability of the first kind, of a particular plan vis-à-vis a particular organisation. Adaptability of the second kind is measured by counting the number of nonzero entries in each column, since this corresponds to the total number of distinct (unlabelled) requirement graphs, that is, the number of possible different organisations, which the plan in question can accommodate. Notice how this value increases in moving across the table from (a) to (d). These are in fact the numbers of (unlabelled) *spanning subgraphs* of the adjacency graph.

Table 10.1. Number of allowable ways graphs 1 to 10 of figure 10.12 can be mapped into plans given by graphs (a) to (d) of figure 10.10. In the lower left-hand corner of the table, the number of adjacency requirements exceeds the number of adjacencies in the plan.

Requirements		Number of allowable ways			
graph number	number of edges	graph (a) 3 edges	graph (b) 4 edges	graph (c) 4 edges	graph (d) 5 edges
1	0	24	24	24	24
2	1	12	16	16	20
3	2	4	8	10	16
4	2	8	16	8	16
5	3	2	8	4	12
6	3	0	0	6	12
7	3	0	0	6	12
8	4	0	0	2	8
9	4	0	8	0	8
10	5	0	0	0	4
column total		50	80	76	132
nonzero entries in column		5	6	8	10

If we count the column totals, these give the numbers of *labelled spanning subgraphs* for each adjacency graph. These figures express some combination of the first two types of adaptability—the number of different organisations which can be housed multiplied by all the ways in which those organisations can be arranged in the plan. In this diagrammatic example it is adjacency graph (d), the graph with the greatest number of edges, which scores the highest on all these adaptability measures.

Matela and O'Hare (1976a) discuss this question of the subgraphs of an adjacency graph, in relation to polyominoes, and remark too on the significance for adaptability. They were not able to count all spanning subgraphs of larger polyominoes by computer. However, as a step towards this, they did count numbers of labelled spanning *trees*. For a given number of edges, it is in general those graphs with many cycles which possess the greatest numbers of labelled spanning trees as subgraphs. Figure 10.13 shows all those 'perfectly cyclic' graphs which may be the adjacency graphs of polyominoes up to $n = 9$. In each case all edges lie on cycles.

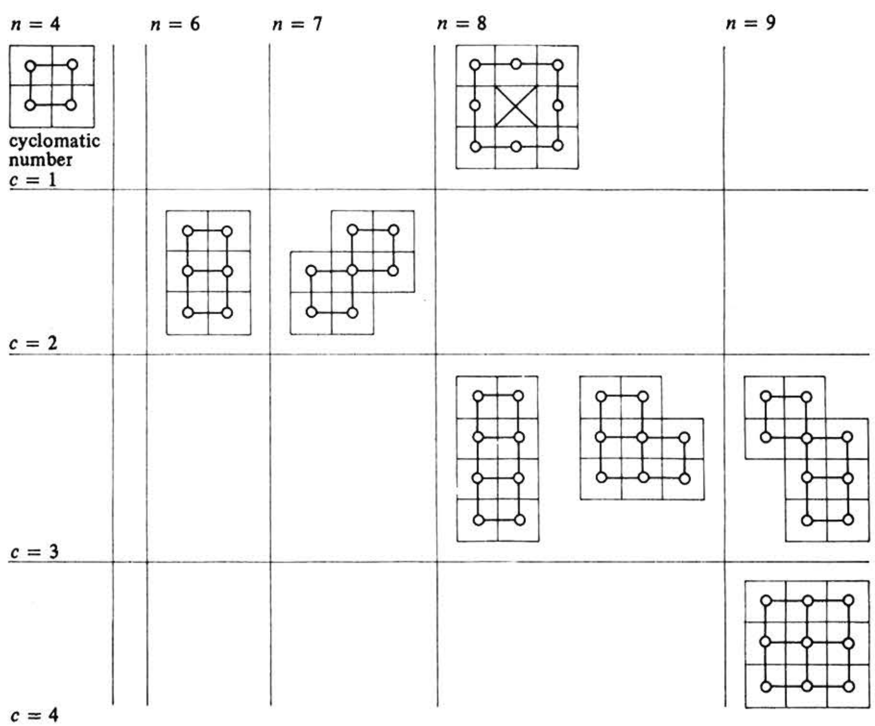
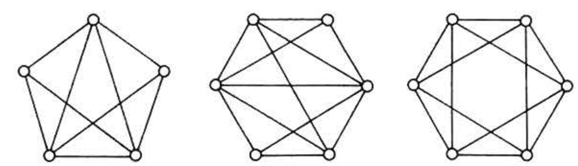


Figure 10.13. All 'perfectly cyclic' graphs which can be the adjacency graphs of polyominoes, up to $n = 9$ (from Matela and O'Hare, 1976a). All edges in these graphs lie on cycles. Such graphs have the maximum number of labelled spanning trees for given n: the highly compact forms of the corresponding polyominoes, considered as plans, offer maximum internal adaptability.

It is these types of graph, or their presence as subgraphs themselves in adjacency graphs for larger n, which determine the maximum possible number of labelled spanning trees. Each of the perfectly cyclic graphs corresponds to a unique polyomino as shown in the figure. Within the limited terms of the exercise, then, it is possible to say that there are only a very few of these highly compact forms in which maximum adaptability of the first two types can be achieved. (On the other hand a compact form has few external walls, and there are thus fewer positions on the boundary at which additional rooms might be attached. So its 'external adaptability', its potential for growth, is, by contrast less than that of more elongated, branching forms.)

Another author who has studied the subgraphs of adjacency graphs in this connection is Marsh (1976). Because, as we have seen, it is plans whose adjacency graphs are most highly connected which offer the greatest (internal) adaptability, Marsh confined his attention to counting the numbers of subgraphs of *maximal planar graphs.* And because he made this enumeration by hand, he considered only such graphs on five and six vertices:



Marsh's results confirm the general trends suggested by our exercise with graphs on four vertices. Marsh also notes the importance for the fit of requirement graph with adjacency graph, of the number of cycles in both, and the lengths of those cycles.

Notice that part of the system of Mitchell, Steadman, and Liggett carries out automatically this exact same process of mapping labelled spanning subgraphs into graphs, and could in principle be used for measuring adaptability.

All this, of course, takes no account of the other factors governing adaptability such as room size, servicing, etc which would need to be introduced into a more realistic exercise (refer to Fawcett, 1976a; 1976b).

The enumeration of rectangular dissections suggests ways for measuring the ease with which plans of this form might be changed internally, or extended on the perimeter. Indeed certain of the original generating algorithms proceed by exactly these operations: an interior room is divided into two parts, or a new room is added along an outside edge of the plan. For every plan with n rooms there is a certain number of its 'descendants' with $n+1$ rooms produced through each of these operations. These numbers could be interpreted as measures of possibilities for physical alteration of that given plan by the subdivision or addition of rooms. The possibilities for *removal* of internal wall segments, thus reducing the number of rooms, are given conversely by the number of 'parent' dissections with $n-1$ rooms from which the dissection in question can be independently generated.

No general investigations along these lines have so far been undertaken. In practice the possibilities for subdivision or extension of building plans are crucially dependent on dimensional as well as on adjacency factors.

One specific study has been made, however, of the adaptability of house plans, which takes account of realistic room size and adjacency constraints together. This is the work of Bailey (1977), who examined rectangular plans with room sizes based on the recommendations of the Department of Environment handbook *Space in the Home* (DoE, 1968) with plan areas and overall plan dimensions corresponding to the 'metric house shells' proposed by the National Building Agency (NBA, 1969). Again because he was working by hand and without the benefit of computer methods, Bailey was obliged to simplify the problem. He considered two-storey, nominally 'five-person' houses, each containing a minimum of three bedrooms. He assumed the ground floor to comprise a

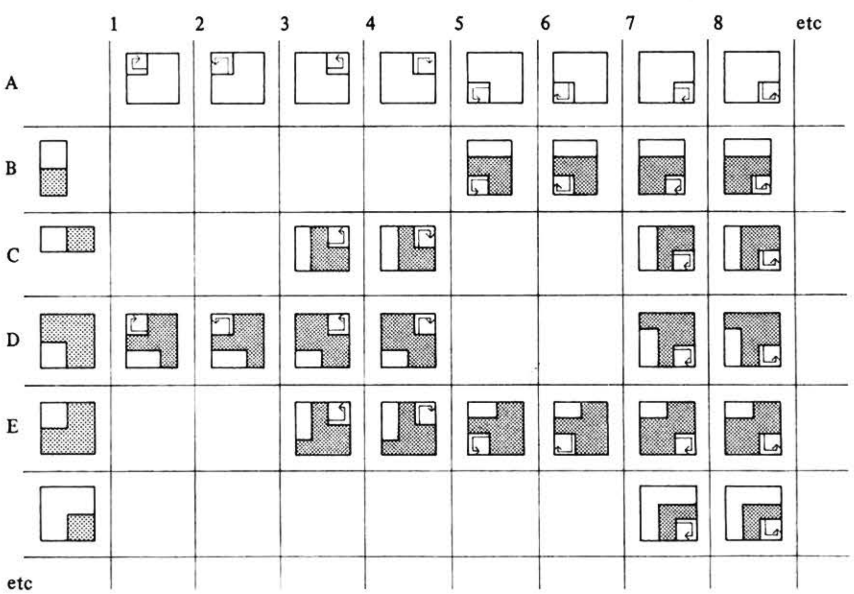


Figure 10.14. Permutations of general arrangement for the ground floor plans of small houses, from a study of adaptability by Bailey (1977). The houses are two-storey, nominally 'five person' with at least three bedrooms. The ground floor comprises in each case a dining room, kitchen, living room, hall, wc, corridor, and staircase. The first three of these are assumed to be planned within a single rectangle or two adjacent rectangles. The figure shows some examples of possible placings A, B, C, ... of these rectangles (unshaded) within the rectangular house 'shell', leaving a circulation and service area (shaded); combined with possible placings 1, 2, 3, ... of staircases—in these instances a dog-leg type.

dining room, kitchen, living room, hall, wc, corridor, and staircase. He assumed that the first three of these could be approximated as two rectangles in plan, either living room plus combined kitchen/dining room, or dining room plus kitchen/living room. (The possibilities that all three rooms might be separate, or all three functions combined in a single room, were also left open.)

Bailey then enumerated all distinct ways of placing these two rectangles in a rectangular shell under certain adjacency constraints, such that a third connected polyomino-shaped area remained for the planning of all the other spaces. These schematic plan outlines were then combined with a systematic permutation of positions and orientations for staircases—either dog-leg or straight flight types. Figure 10.14 shows some examples. Certain of these plan diagrams were selected for dimensioning within the specified standards, and were planned in detail on both floors. In each case Bailey determined whether within the dimensional constraints it was possible to offer certain very specific options for adaptation, such as the ability to offer an extra bedroom, a study, or utility room, or various combinations of dining, kitchen, and living areas. Figure 10.15 gives examples of some of these options for a given 'shell' size, and which correspond to some of the different schematic arrangements of the ground floor which were illustrated in figure 10.14.

Such an exercise could be completed and carried further by means of a computer method such as that of Mitchell, Steadman, and Liggett. Room arrangements on both floors (or indeed on more than two floors) could be exhaustively enumerated under adjacency constraints, and under dimensional constraints both on the rooms and on the plan overall. The resulting variety of plans could be interpreted in two ways.

It could be studied as Bailey does in the context of specific forms of adaptability—how many ways in which functions could be assigned to rooms in some given plan, or how many useful ways in which that plan could be physically altered by moving say one, two, or more internal wall segments, or altering certain services, doors, or windows.

Alternatively the range of such plans could be viewed much more generally as plotting the total extent of possibilities in design within some set dimensional 'discipline'—such as those imposed by the recommendations of *Space in the Home* and the NBA's 'metric shells'. It is clear that such disciplines must limit plan variety—and in practice these limits become partly known with time, in an empirical and anecdotal sort of way, to architects who have to work within them. A systematic approach, however, would measure these effects precisely. Such exercises as the NBA's tabulation of *Generic Plans* (NBA, 1965) could be automated. It should be possible to identify the effect, on increasing or reducing the number of resulting plans, of changes in specific dimensional or adjacency constraints.

Some dimensional disciplines in housing design, notably that put forward by Habraken (1961) in his 'supports' concept for separating the structural

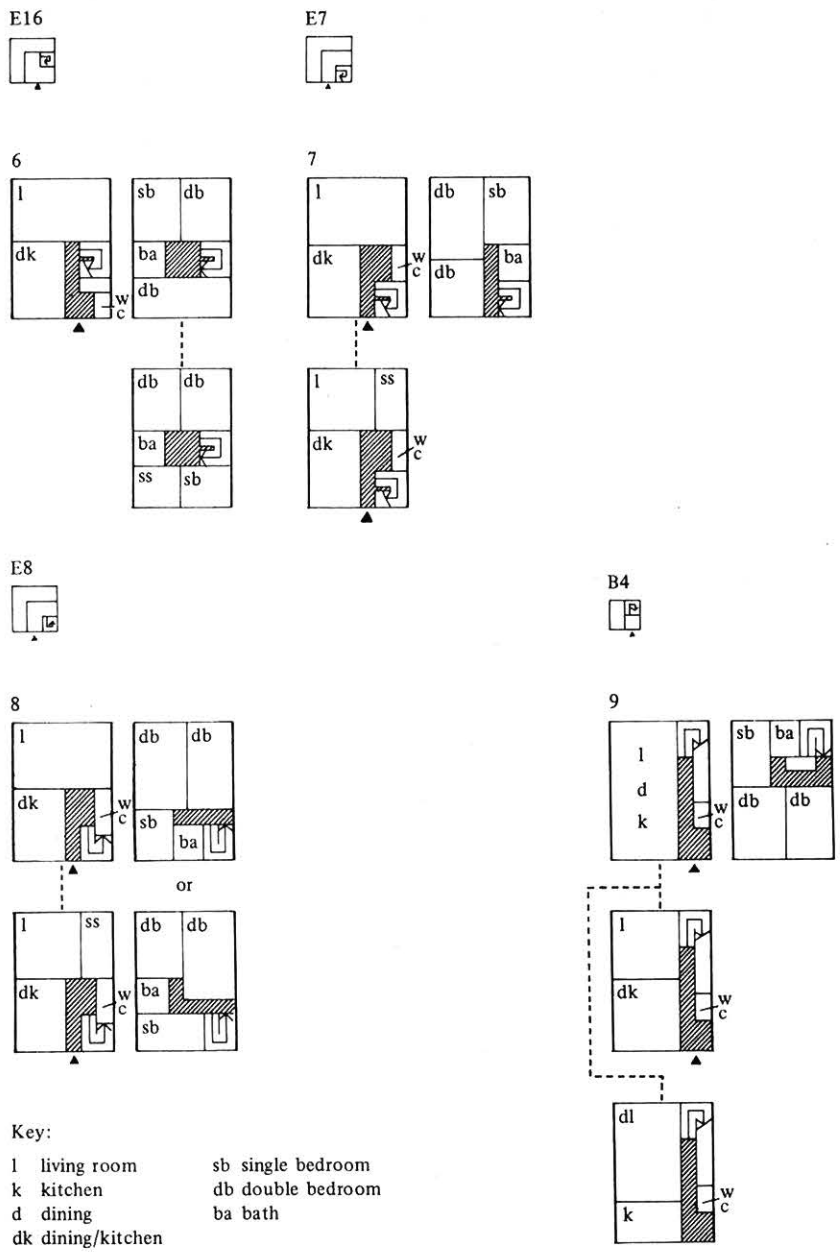


Figure 10.15. Some of the basic dimensionless plan arrangements of figure 10.14, dimensioned according to British governmental recommendations (DoE, 1968; NBA, 1969), from Bailey's (1977) study of adaptability in house plans. Plans on both floors are illustrated, together with various options for alternative room uses or additional rooms. Plan dimensions overall are 5·7 x 8·1 metres.

frame and services in a block of flats from the detailed layout of internal partitions, have been intended to increase the initial choice of plans available to the tenants, and to increase their options for making subsequent alterations to those plans. The Greater London Council's PSSHAK system (Hamdi and Wilkinson, 1971) is another case in point. Again an exhaustive enumeration of those possibilities would give an exact measure of the success of the underlying system of dimensional constraints in increasing the range of plan options.

The actual within the possible

I have tried to present in this chapter a view of building science as a study of *possibility* in buildings, and in particular a study of their possible forms and dimensions. I am not speaking here, therefore, of that kind of functional determinism in building research which attempted to find the unique, necessary, or 'optimal' form to correspond to some fixed set of architectural 'requirements'. The purpose is rather to define the extreme boundaries and within those the broad topography, so to speak, of the fields within which every architect must exercise choice. Within these ranges the architect then identifies further criteria, by which to narrow his selection to some final unique design.

Those limits which are set by the laws of physics and geometry are unchanging over time. However, those bounds which are set by the capacities of building materials, or by available technological means, are ones which may move as technique develops or new materials are discovered—something which events in the history of architecture have repeatedly shown.

This view of building science leads us to see the study of architectural *history* then, insofar as it has to do with building forms, materials, and constructional methods, as a description and analysis of where the *actual* lies, or has lain, *within the possible*. At the same time, it is only through empirical study of actual buildings and their properties, as I have argued, that these theoretical realms of 'possibility', at any period in history, can in the last analysis be defined.

Exercises

Rather than specify formal exercises relating to this chapter (and the next), I suggest that you might like to use the material presented here for the basis of your own research projects.

For instance you could take samples of plans, for example, those of small houses or apartments, and make analyses following the lines of my own work (pages 183–184), plotting the occurrence of dissection type arrangements on Combes's diagram; or following the lines of Bon's work (pages 185–187), drawing and classifying their labelled access graphs, measuring the valencies of vertices, and measuring the numbers of faces or cycles in the graphs.

You could study the relationship for actual plans of access graphs to adjacency graphs (see pages 187-188). What proportion of adjacencies between rooms or with exterior regions are used for access? What proportion of adjacencies with the exterior are used to provide natural light and ventilation?

You might take the access graphs of larger buildings of some given functional type, for example, offices or blocks of flats, and try to classify them, using some of the gross measures of graph structure described here, and making also some measures of distances along the circulation network, as suggested on page 189.

You might explore the questions of adaptability of rectangular dissection type plans by internal rearrangement or by extension on the perimeter (independent of any consideration of dimensions), as suggested on pages 203-204, and making use of the catalogue of dissections in the appendix.

11

Plan morphology and architectural history

> "Archaeological studies and the history of science are concerned with things only as technical products, while art history has been reduced to a discussion of the meanings of things without much attention to their technical and formal organization. The task of the present generation is to construct a history of things that will do justice both to meaning and being, both to the plan and to the fullness of existence, both to the scheme and to the thing."
>
> George Kubler (1962)

With a few outstanding exceptions, applications to architectural history of techniques for morphological analysis are, so far, fewer even than those to the traditional subjects of building science. Therefore this final chapter will be even fuller of speculative suggestions for directions in research, than the last.

Many of the descriptive methods and analytical tools listed in the last chapter may be applied as well to old buildings, of course, as they may to new—although a certain caution needs to be taken, that modern functionalist preoccupations, tacitly embodied in some of these measures, are not carried over to historical and social conditions where they do not apply. For example, the idea of different rooms in a house being allocated to distinct 'activities' was less developed in say the seventeenth century (compare Barley, 1963) than it is today. (Even in modern housing design, functional room uses are often much less tidy in practice than the architect originally supposes.) Again, the typical modern hierarchical classification of roads related to their capacities for vehicular traffic will hardly be applicable to the streets and alleys of the mediaeval city.

I have already made some suggestions for ways of classifying building types on a morphological basis. The kind of multilevel hierarchic taxonomy described in chapter 8, involving a conceptual separation of dimensional, shape, and topological properties, could be applied in architectural history. Indeed something very much along these lines has been proposed by Guerra (1977) in a paper on the rehabilitation of old buildings. Guerra suggests that the access graphs of older public buildings such as churches, convents, or theatres might be compared with the organisational or access requirement graphs of modern cultural or educational institutions, to measure their suitability for accommodating these new uses.

Guerra further imagines a kind of historical/geographical classification of vernacular architectures in which the occurrence of buildings with similar plan layout and construction might be plotted on maps. Contours drawn on these maps could mark the boundaries between distinct local types, and a series of maps made for different time periods could in principle show the movement of these boundaries and types. This is a very grandiose project of course, and one for which the basis in empirical survey work, in the case of many building types and geographical regions, simply does not exist.

Nevertheless there have been collected together a few archives of plans, as, for example, the specialised collections of British vernacular house plans to be found in what is by now an extensive literature, which overlap the fields of architectural history, historical geography, and archaeology [see Brunskill (1971) for a bibliography], as well as in the inventories of the Royal Commission on Historical Monuments for England and the Royal Commission on Ancient Monuments for Wales. It is perhaps unfortunate that the very extensive records of British houses brought together under the original initiative of Cordingley at the University of Manchester, do not always include full details of plan forms. Brunskill and others working from the Manchester materials have prepared maps of exactly the kind envisaged by Guerra, although these largely relate to the materials of construction used in domestic buildings, rather than to their plan types.

For other building types, there are few if any comparable systematic collections of plans. Most of what does exist—for 'polite', Western architecture—has been listed by Pevsner in the encyclopaedic bibliography and notes to his *History of Building Types* (1976); although that book in itself is largely confined to names, dates, and stylistic comments, and strangely, although it includes many illustrations of plans, hardly addresses the question of plan form at an analytic level at all.

It is fair to say that in all this work any classification of plans which is attempted in geometrical terms, tends to be on the basis of rather loose and informally defined 'family resemblances' of shape and layout. Thus Brunskill (1971) distinguishes such types of vernacular house plan as the 'hall', the 'two-unit', the 'inside cross passage', the 'central fireplace', and the 'double-pile' families (figure 11.1) (compare also Faulkner, 1958; Pantin, 1962-1963).

One author who has attempted a greater precision in the geometrical definition of types is Dickens (1977), who examined a sample of seventy-four small Cambridgeshire house plans from the inventory of the Royal Commission (1968). He approximated all of these plans as polyominoes (taking the principal rooms as the cells of the polyomino, and ignoring circulation spaces); and he drew the access graph in each case, labelled by room functions. The houses are all two-storey and the graph describes access on both levels, with the staircase shown as a zig-zag line. The resulting types are illustrated in figure 11.2. They correspond to six different polyominoes with one, two, three, and four cells, and the total number of different labelled graphs is 21. The column of figures accompanying each graph shows the number of occurrences of that type found in each of six successive fifty-year periods, from 1550 to 1850. Notice that all polyominoes are represented up to three cells, but that only two appear out of the five possible polyominoes with four cells.

If the plan shapes had been arrived at by some random process, then for a given number of cells n each polyomino could be said to be equally

'probable', and might be expected to occur with similar frequency. For example, since there are five polyominoes with four cells, so the probability of occurrence of each would be 1/5 or 0·2. If differently handed isomorphs were regarded as distinct, then there would be seven shapes, and the probability of each would become 1/7 or 0·1429. Other ways of measuring the geometrical probability of the occurrence of the same shapes, which give slightly different results, are also possible.

Of course, the actual planning of the houses was by no means random, and the observed frequencies of occurrence of plan shapes in Dickens's sample are very different from these notional probabilities—indeed certain 'theoretically possible' shapes as we have seen do not appear at all. Dickens proposes that these differences might be interpreted as evidence of two factors at work. The actual plans are chosen as being more

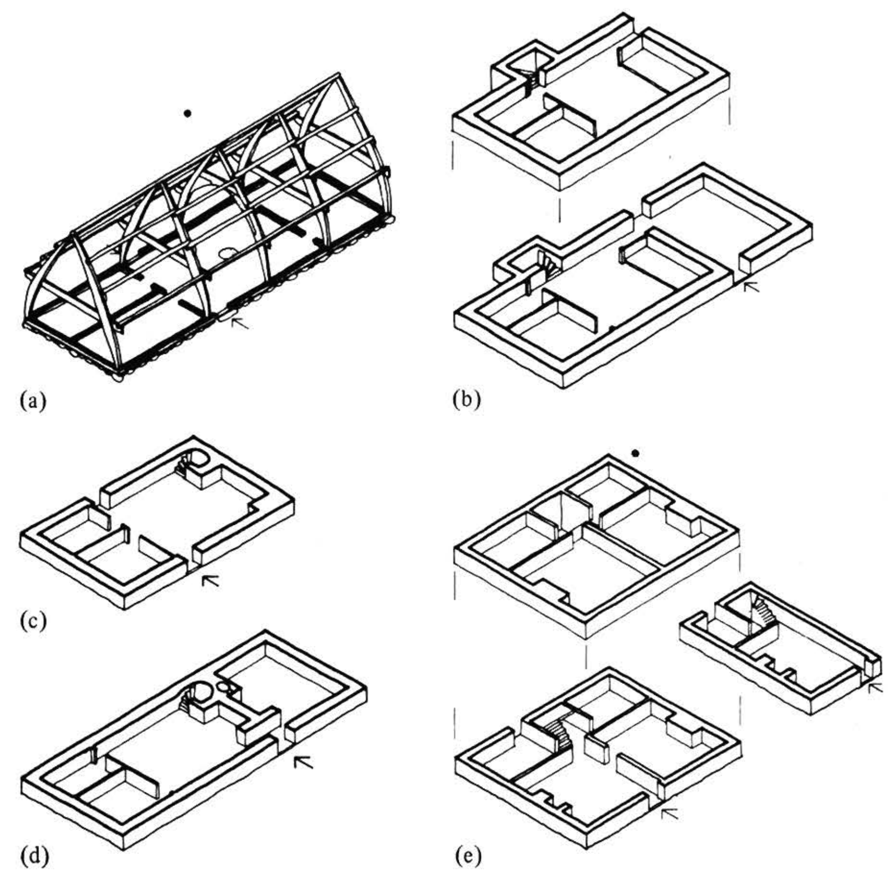


Figure 11.1. Basic types of English vernacular house plan (from Brunskill, 1971): (a) the hall type; (b) the two-unit type; (c) the inside cross-passage type; (d) the central fireplaces type; (e) double-pile plans.

compact, this resulting in a reduction of the ratio of wall surface to plan area (or to volume), and a minimisation of distances within the plan. Or the chosen forms might be those which are easier to construct—specifically, Dickens argues, those in which the plan perimeter has fewer re-entrants, and where as a consequence the roof construction in particular would be simpler.

He uses geometric measures of shape designed to capture these properties, very similar to those used by March and Matela (1974) (see page 127), and ranks the polyominoes on those bases. The square polyomino with four cells scores highly on both counts as would be expected, and it is indeed the most frequently observed four-room plan shape. The ranking of the other four-cell shapes does not, however, correspond with their

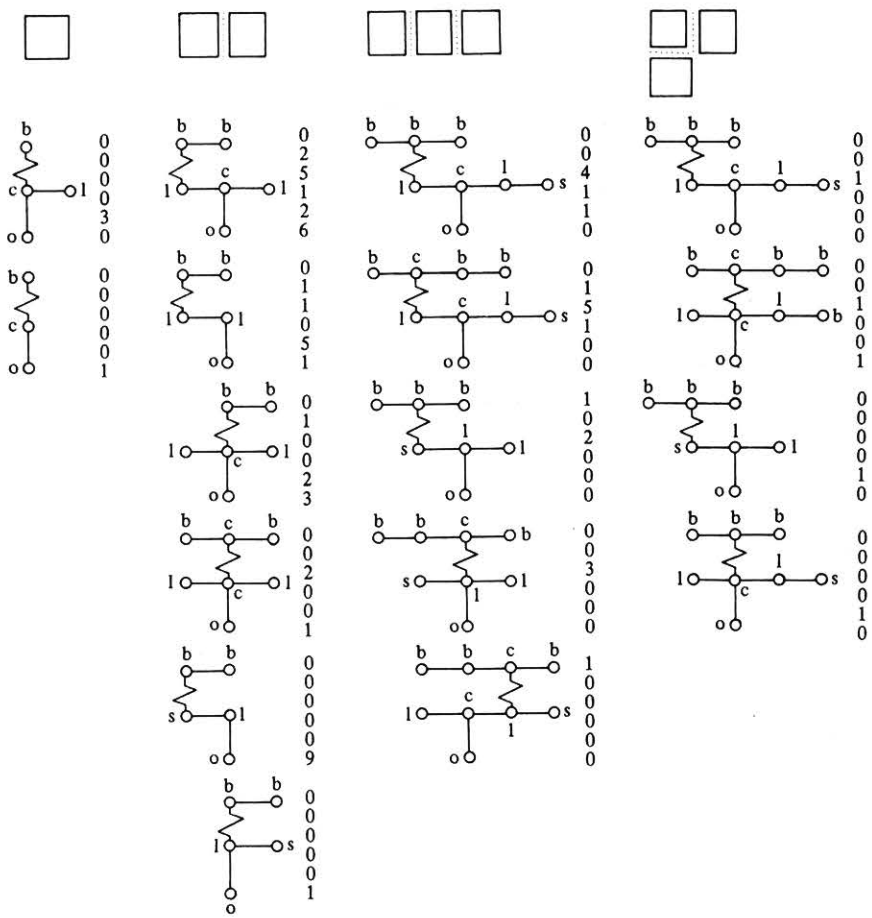


Figure 11.2. Classification of a sample of seventy-four small Cambridgeshire house plans (from Dickens, 1977). The plans are approximated as polyominoes (with the principal rooms taken as the cells of the polyomino and with circulation spaces ignored). The access graphs of all plans are illustrated, with access on both storeys

frequencies of occurrence as plans—although it should be said that the sample is too small for statistically meaningful correlations to be made in any case.

For each overall polyomino shape there are several alternative positions for the entrance to the house and several positions for the staircase. These give rise to a number of theoretically possible combinations, which again may be compared with those actually occurring.

Dickens uses standard statistical tests to determine whether there is any significance in the relative frequencies, in each time period, of the occurrence of plan shapes, plan sizes (numbers of rooms), and types of access graph. These tests reveal certain significant associations of types

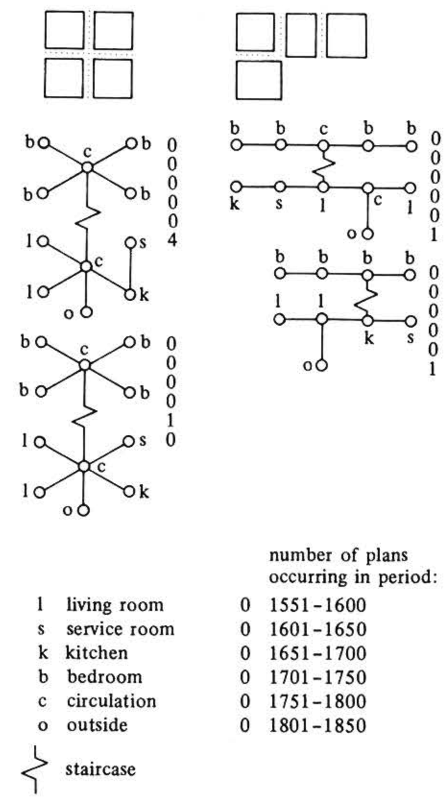
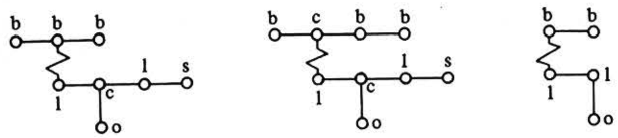


Figure 11.2 (continued)
shown, and with the staircase indicated by a zig-zag line. Against each graph is indicated the number of occurrences of that type in each of six fifty-year periods between 1550 and 1850.

with periods, as, for example, a large number of instances of the straight three-cell plan in the period 1650-1700, and high frequencies of occurrence from 1650 to 1700 for the first two access graphs shown below and from 1750 to 1800 for the third one.



A second rather comparable piece of work on vernacular house plans was carried out by Arbon (reported in Steadman, 1976), and follows exactly my own exercise in locating dissection type plans on Combes's diagram of internal and external wall segments as described in the last chapter. Arbon's sample of thirty-eight plans dating from 1450 to 1690 was taken from Fox and Raglan's (1951-1954) work on the houses of Monmouthshire in Wales. The results are illustrated in figure 11.3 where the houses are classed into three groups: cruck houses built between 1450 and 1520, pre-Renaissance masonry houses of 1540 to 1580, and Renaissance masonry houses of 1600 to 1690.

Recall that modern house plans from the NBA's *Generic Plans* were heavily clustered along the line $2w = 11+p$, marked again in figure 11.3. It will be seen that the earliest, cruck houses lie by contrast predominantly along the upper limit of the diagram at $w = 2p+4$. The dissections on this line are those consisting of straight rows of rectangles, corresponding to the characteristic planning of cruck houses in which each room consists

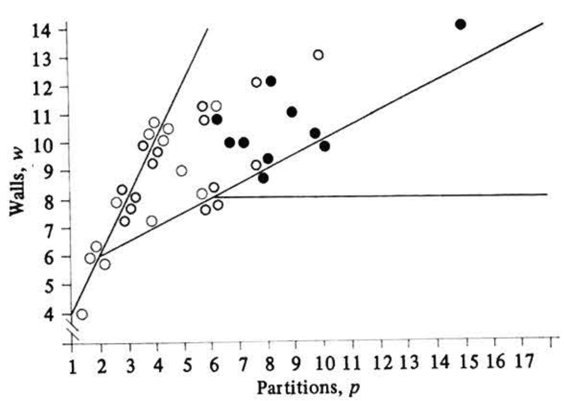


Figure 11.3. Classification by Arbon of thirty-eight Monmouthshire house plans (from Steadman, 1976). The plans are plotted on Combes's graph of numbers of internal and external wall segments (see figures 8.1 and 10.4). The sample is classified into three groups: cruck houses, 1450-1520 (light circles); pre-Renaissance masonry houses, 1540-1580 (heavy circles); Renaissance masonry houses, 1600-1690 (solid circles).

of one or more structural bays, and the plan is extended linearly by the addition of bays at each end.

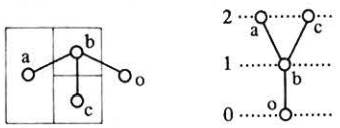
The later houses move across the space of Combes's diagram towards $2w = 11+p$ where modern plans are found. This could be interpreted in terms of several factors. The masonry techniques serve to free the plan from the constraint of a narrow linear shape which was imposed by cruck construction. The increasing differentiation of functions within the house results in a greater complexity of plan form, more rooms, and hence more internal wall segments. And the growing demands for privacy require that rooms should be accessible from common circulation spaces without one having to pass through one room to reach others. In turn this requires plan forms with 'overlapping' adjacencies between rooms, of the kind occurring with the greatest frequency towards the lower boundary of the diagram.

These two studies, Arbon's and Dickens's, would have to be taken much further and applied to larger samples, to produce historically conclusive results. However, they do indicate a general approach, whereby some 'population' of plans of real buildings is compared with the whole range of plans which are 'theoretically possible' on some appropriate geometric definition. In this way those features of their spatial arrangement which are contingent on the necessities of the geometrical discipline are identified and separated out in the analysis. Any further concentration or clustering of the sample within the space of geometrical possibilities, can be attributed to the operation of additional constraints—to the fact of architects or builders limiting their choice (whether consciously or unconsciously), according to more restricted criteria. In the work just described these were inferred to be technical and functional criteria, to do with structure and circulation.

It is in this kind of process, of setting the actual against the possible, as I have described it, that I believe much of the promise of a morphological analysis of historical buildings lies. The mere act of transforming the means by which a plan is represented, from a traditional scale drawing, into 'dimensionless' form or into the form of an adjacency graph say, does not necessarily of itself throw any light on historical issues. In many cases as much can be 'read', and as easily, from a simple inspection of the conventionally represented plan. Nevertheless there are some buildings where a transformation of their plan representation *does* have the effect of illuminating properties which are by no means otherwise apparent.

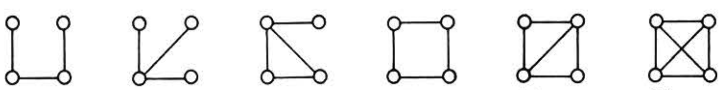
An outstanding example of this is provided by the work of Hillier and colleagues (Hillier et al, 1978a; 1978b; Hanson and Hillier, 1979; Hillier and Hanson, forthcoming) who drew the 'permeability' structures—in our terms the access graphs—of a variety of buildings ancient and modern. The method is most revealing with large and complex buildings. However, it is perhaps best explained first in relation to small plans in order to illustrate the concepts involved.

The figure below on the left shows a rectangular dissection type plan of three rooms labelled a, b, and c as shown:



One possible access graph for this plan is illustrated, with a vertex to represent the exterior region o. The figure on the right shows the same access graph drawn out again but in a specially standardised (Hillier calls it a 'justified') format. The vertex for the exterior o is placed on the lowest level in the diagram, level 0[28]. Those spaces which are directly accessible from the exterior—in this case only room b—are placed on the first level up, level 1. All spaces to which the shortest route from the exterior contains two edges—the remaining rooms a and c—are placed on level 2. For graphs with more edges the process can be continued up to any required number of levels.

The next figure shows *all* possible access graphs for the same arrangement of rooms, the assumption being such graphs are connected (and, of course, planar).



(a) (b) (c) (d) (e) (f)

These are listed (a) to (f) according to increasing numbers of edges. At a maximum there can be six edges, three representing access between rooms, and three representing access from the exterior, at which point access graph and adjacency graph coincide. These six access graphs may each be labelled to represent the rooms they connect, in a number of ways, to give a total of thirty-eight possibilities as illustrated in figure 11.4. Each graph is set out in justified form below the corresponding plan.

Drawing an access graph in this way (especially with larger graphs, as we shall see) brings out a number of features. First of all, since this is the basis of the diagram, it shows immediately the *shortest distance* (as a number of edges) from each room to the exterior. Call this d_{oi} to signify the depth of room i from the outside. Those rooms which are on the highest level in the diagram are the 'deepest' in the building—one must pass from the entrance through the greatest number of other spaces to reach them. Some buildings are *deep* ones in this sense, and others *shallow*. Notice that this property of depth has nothing to do directly with the adjacencies (or dimensions) of rooms, but simply with the ways

(28) In Hillier's terms this exterior space is the 'carrier', and is conceived of as continuous, not divided arbitrarily into exterior regions.

rooms are interconnected by relationships of access, by doors or openings. In the examples of figure 11.4 it is the *same* dissection which provides the basis of the plan throughout, whereas the depth of the plan in this measure of Hillier's varies between 1 and 3.

A *mean depth* figure $\bar{d}_{oi}$ may be derived for the plan as a whole, by summing the depths of all spaces from the exterior, and dividing by their number. These values are given for the plans illustrated in figure 11.4. They may be compared directly since the number of rooms is the same throughout. In general though, values of mean depth are dependent naturally on the size (number of vertices) as well as the structure of an access graph. However, it is possible to obtain a *relative* or normalised measure of mean depth by means of the expression

$$\frac{2(\bar{d}_{oi}-1)}{v-2} .$$

The *maximum* depth for graphs on v vertices occurs with graphs which consist of a simple linear tree, in which the exterior region o is represented by one of the end vertices (as in graphs 1 to 6 of figure 11.4). In such cases the maximum depth must be $v-1$, and the mean depth $\frac{1}{2}v$. The *minimum* depth, and minimum mean depth, which graphs of any size can take is 1. These correspond to plans where all rooms are directly accessible from the exterior (as in graphs 26–28 and 35–38 of figure 11.4). Substituting for $\bar{d}_{oi}$ in the above expression gives

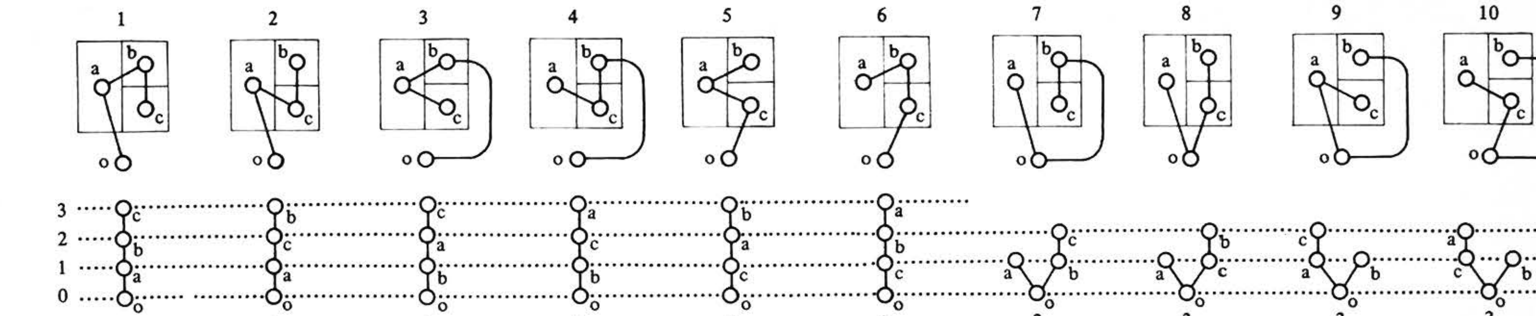
$$\frac{2\left(\frac{v}{2}-1\right)}{v-2} = 1 ,$$

$$\frac{2(1-1)}{v-2} = 0 ,$$

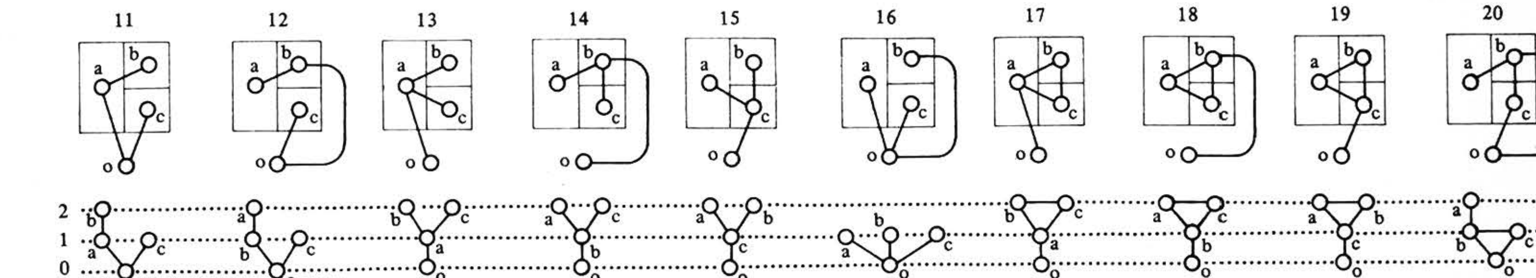
for the deepest and shallowest cases, respectively. Whatever the number of vertices in a graph, this measure can only take values between 0 and 1. It thus allows comparisons to be made of the relative mean depth away from the exterior, of access graphs of widely differing sizes.

It would be possible to draw the justified access graph taking any one of the *rooms*, rather than the exterior region, and setting it on level 0. This would show the shortest distances to all other rooms from the given room. If all such justified versions of an access graph were drawn, for all rooms, then this would reveal all distances d_{ij} from the ith room to the jth room, and the absolute maximum such distance $d_{ij}^{\max}$—the 'longest shortest distance' between any pair of rooms in the plan, that is, the *diameter* of the graph. Diameters for the access graphs in the example are tabulated in figure 11.4.

Another property of the access graph which is important in Hillier's analysis is its number of *faces*. Recall that for a plane graph the *cyclomatic*

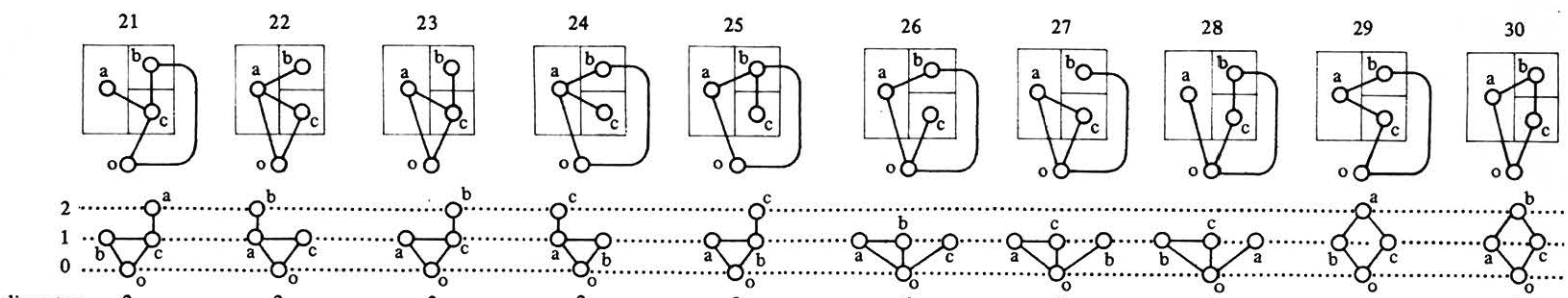


	1	2	3	4	5	6	7	8	9	10
diameter:	3	3	3	3	3	3	3	3	3	3
cyclomatic number:	0	0	0	0	0	0	0	0	0	0
mean depth:	2	2	2	2	2	2	1·33	1·33	1·33	1·33

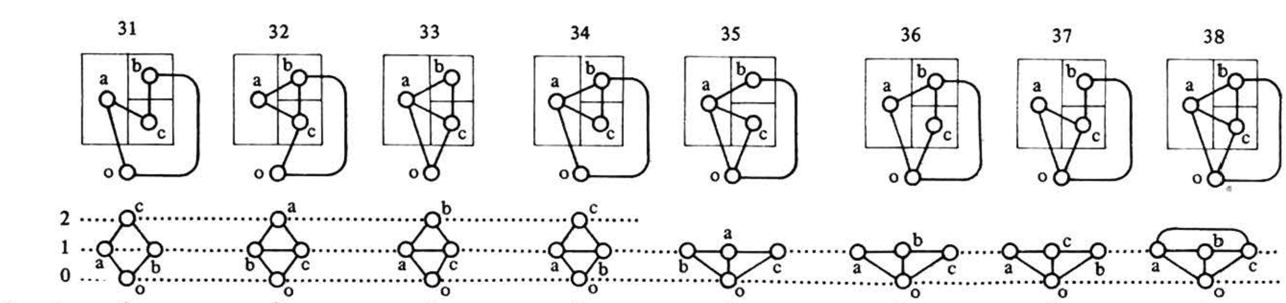


	11	12	13	14	15	16	17	18	19	20
diameter:	3	3	2	2	2	2	2	2	2	2
cyclomatic number:	0	0	0	0	0	0	1	1	1	1
mean depth:	1·33	1·33	1·66	1·66	1·66	1	1·66	1·66	1·66	1·33

Figure 11.4. All thirty-eight possible access graphs for the three-room plan illustrated. The graphs are set out in Hillier's 'justified' format below the plan in each case: the exterior region o is set on level zero, any space directly accessible from the exterior on level 1, any space accessible from the exterior



	21	22	23	24	25	26	27	28	29	30
diameter:	2	2	2	2	2	2	2	2	2	2
cyclomatic number:	1	1	1	1	1	1	1	1	1	1
mean depth:	1·33	1·33	1·33	1·33	1·33	1	1	1	1·33	1·33

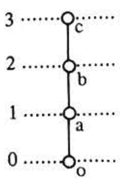


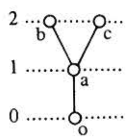
	31	32	33	34	35	36	37	38
diameter:	2	2	2	2	2	2	2	1
cyclomatic number:	1	2	2	2	2	2	2	3
mean depth:	1·33	1·33	1·33	1·33	1	1	1	1

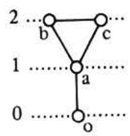
Figure 11.4 (continued)
through an intermediate space on level 2, and so on. In this way the *depth* of each room from the exterior is tabulated. The *diameter* and *cyclomatic number* of each access graph are given, as is the *mean depth* of all rooms from the exterior.

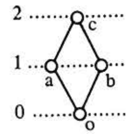
number, $e - v + 1$, is equal to the number of faces (not counting the exterior face). Cyclomatic numbers are given for all the graphs illustrated in figure 11.4. It does not help especially in counting the number of faces in a graph to draw it in the justified format. Consider, however, graphs with no interior faces, that is, trees, where the cyclomatic number is zero. The implication in access terms is that there is a unique route from the exterior to any room, and that shallower rooms lie on this route and hence control the access to deeper rooms. The justified version of the graph shows this very clearly.

For example, in the access graph on the left below, room b controls access to room c, and room a controls access to room b.





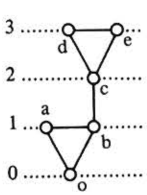




The second graph shows a case in which a controls access simultaneously both to b and to c. Hillier would describe the vertices b and c in this second case as being *symmetric* with respect to a: neither of the spaces b or c controls access to the other from a. By contrast in the first graph, b and c are *asymmetric* with respect to a: b controls access to c from a.

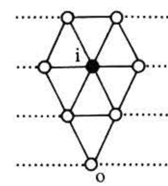
Wherever faces and hence cycles are introduced into the access graph, it follows that there must be alternative routes from the exterior to certain rooms (and alternative routes between pairs of rooms). The third graph gives an example. In many cases control of deeper spaces by shallower ones is lost. In the last graph, c is accessible from the outside either through a or b.

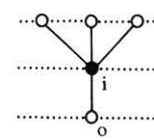
Hillier has a special term to distinguish those graphs which possess cycles passing through the exterior region. He calls these *distributed* graphs. The last graph above is a case in point. Graphs without such cycles (that is, trees, or with cycles which do not pass through the exterior, as in the first three graphs above) are referred to as *nondistributed*. It is clearly possible for distributed graphs to contain subgraphs, each connected at a single vertex, which are in themselves nondistributed. The following graph is distributed in these terms, but the subgraph bcde is a nondistributed subgraph:



Plane graphs on a given number of vertices v may contain any number of interior faces from zero to a maximum given by $2v - 5$. This follows from the fact that for maximal planar graphs $e = 3v - 6$. Euler's formula gives $f = e - v + 1$ if the exterior face is excluded. So for maximal planar graphs $f = 3v - 6 - v + 1 = 2v - 5$. A *relative* measure comparing the number of faces in a plane graph with the maximum possible, is therefore given by the ratio $f/(2v-5)$, which can take values between 0 and 1. Values for this ratio are given for all the access graphs in figure 11.4, which as will be seen, vary between 0 for the first graphs (the trees) and 1 for graph number 38 which is maximal planar and in which the number of interior faces is given by $2v - 5 = 3$. In this way, graphs with differing numbers of vertices can be compared for their degree of connectivity, by a measure which is not unlike Garrison and Marble's gamma index[29].

Individual vertices in the access graph, that is, rooms, may be characterised by the number of faces on which they lie. At a maximum this number can equal the valency of the vertex (the figure on the left below gives an example):

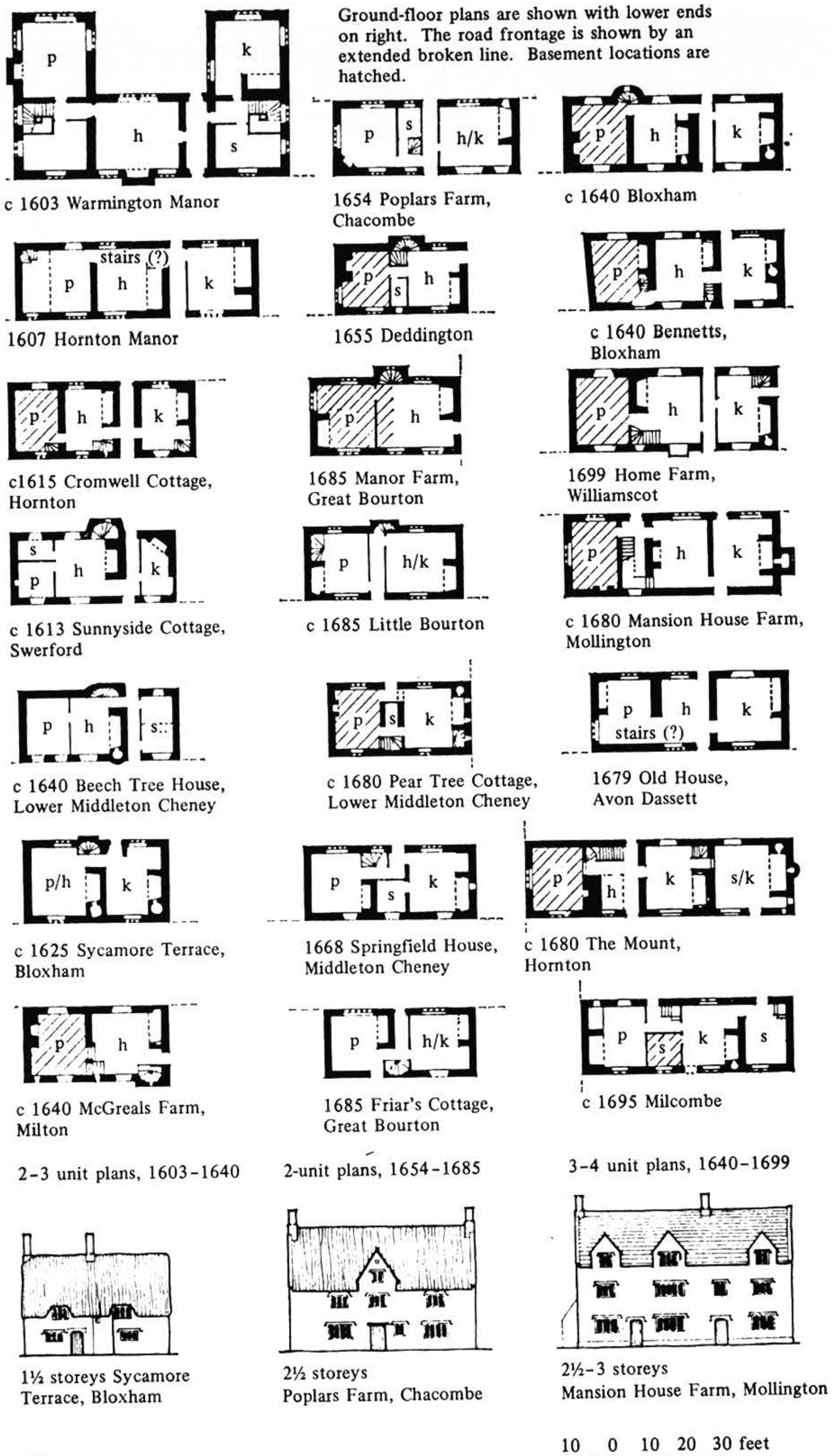




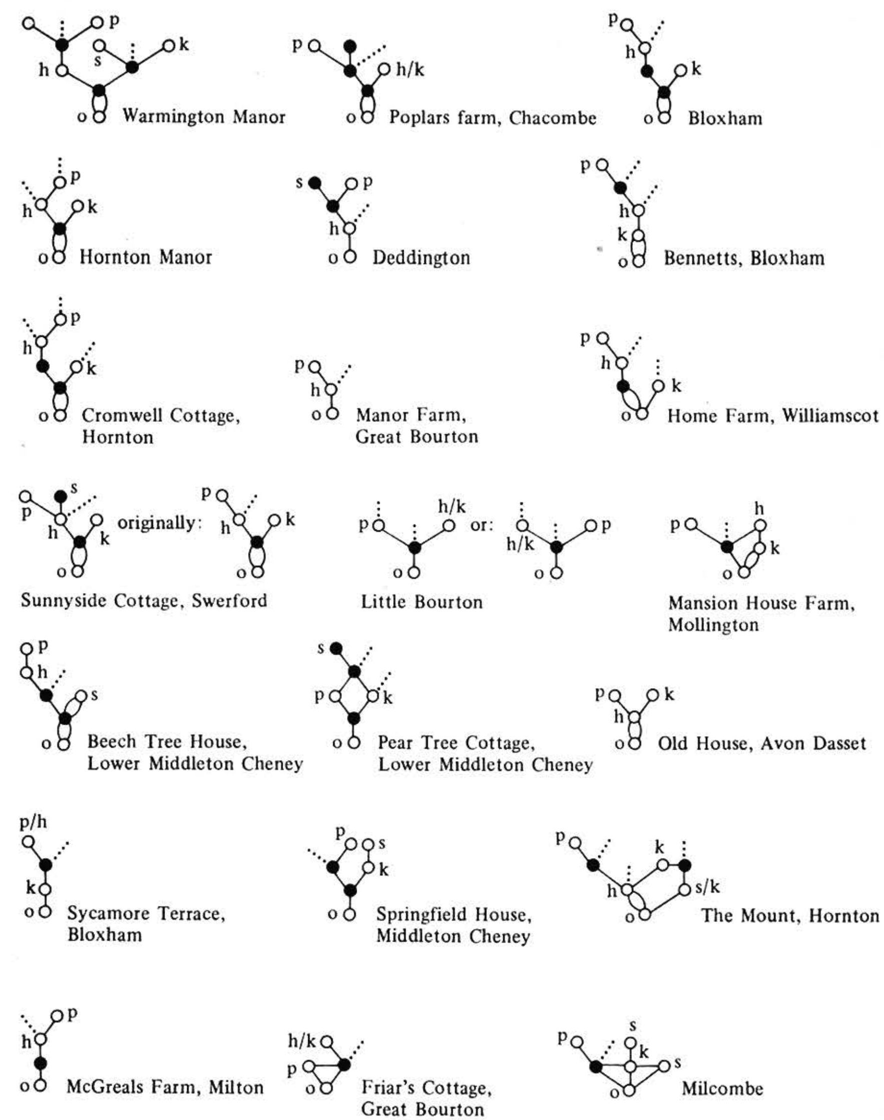
The maximum value which any vertex valency in a graph can take is clearly $v - 1$, where the vertex in question is joined directly to all other vertices (see the figure on the right above).

To turn now to the actual uses made by Hillier and his coworkers of these methods and measures: in one study Hanson and Hillier (1979) have examined the plans of small houses, specifically a sample of seventeenth-century houses drawn from the work of Wood-Jones (1963). Figure 11.5(a) shows the ground-floor plans of twenty-one houses in the area of Banbury, Oxfordshire, dating from 1575 to 1700, and described by Wood-Jones as being variations of a 'through-passage' type. The through-passage, as its name implies, runs from front to back of the house and connects a front entrance from the road directly to a back entrance from a yard or garden. Rooms in the house are set either side of the passage, typically with a kitchen and service rooms on one side, and a hall and parlour on the other. [The through-passage is thus a classic example of a 'cut' room, in Bloch's terminology (compare figure 8.9 on page 127).]

(29) The gamma index by contrast relates the number of *edges* in a graph to the number of edges in a maximal planar graph on the same number of vertices. Hillier refers to the degree of connectivity of a graph measured as described, as its 'relative ringiness'.



(a)

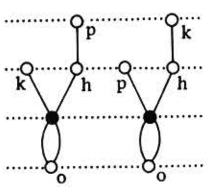


⋮ stairs
● circulation, storeroom
h hall
k kitchen
p parlour
s service

(b)

Figure 11.5. (a) Seventeenth-century regional house plans from the area around Banbury, Oxfordshire, and mostly of the 'through-passage' type (from Wood-Jones, 1963); and (b) their corresponding justified access graphs (from Hanson and Hillier, 1979). Staircases are shown by dotted lines; circulation spaces and storerooms by solid circles. The through-passages appear as pairs of edges incident with the exterior region o.

Hanson and Hillier drew the access graphs of these plans according to the conventions described earlier (see figure 11.5(b). Vertices representing principal rooms are shown as hollow circles and vertices representing circulation spaces and storerooms as solid circles. The through-passage appears immediately, as a pair of edges joining the exterior space o at level 0 to the vertex representing the passage itself at level 1. A cursory look serves to show, as Hanson and Hillier point out, that roughly one-third of Wood-Jones's plans in figure 11.5 do *not* in fact contain a through-passage. Those which do may be characterised as belonging to one of the two following basic arrangements, or minor variants on these:



The structure of the access graph in itself is the same in both these cases, with long and short arms leading off the through-passage vertex. The difference lies in the labelling of the vertices, by the exchange of the positions of kitchen and parlour.

Hanson and Hillier's analysis suggests that these plans are characteristic of the early seventeenth century: some examples are shown on the left of figure 11.6. Notice that the parlour here is generally a deep, hence relatively private, space, and often with only one door (that is, vertex valency 1). Overall, these plans are deep by comparison with later types, nondistributed (tree-like), (the pair of edges representing the through-passage does *not* constitute a cycle), and the relations of the vertices mostly asymmetric.

In the middle of the century there appears to be a transition to what Wood-Jones typifies as a 'lowland' plan form. Figure 11.7 shows ten further plans in this category from various counties of central England. Some of Hanson and Hillier's corresponding graphs are illustrated in the centre of figure 11.6. The new arrangement retains the division of the plan into two unequal length arms, but these are now reached from a single entrance, and the parlour is now generally on the shorter of the two arms.

Towards the end of the century a third type emerges (on the right of figure 11.6) in which the graphs of the plans, from having previously been nondistributed in Hillier's terms, come to contain cycles passing through the exterior on which several of the principal spaces lie. Kitchen and service rooms are often directly accessible from the exterior, whereas the parlour remains a deep nondistributed space with single access and not on the main distributed cycle of rooms.

Hanson and Hillier discuss the possible relations of these evolutionary changes in plan form to economic and social changes, and in particular to

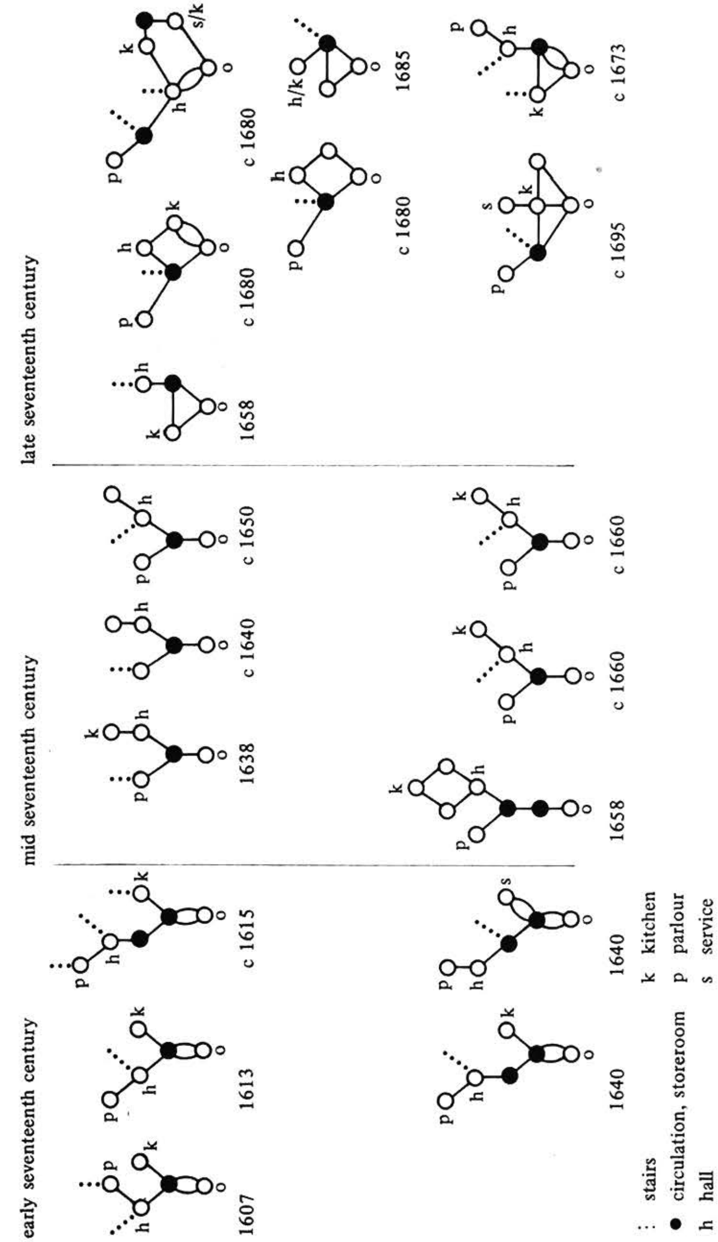


Figure 11.6. Justified access graphs of some of the plans shown in figure 11.5 together with some further plans shown in figure 11.7 (from Hanson and Hillier, 1979). Staircases are shown by dotted lines; circulation spaces and storerooms by solid circles. The graphs are arranged to show types characteristic of the early, middle, and late parts of the seventeenth century.

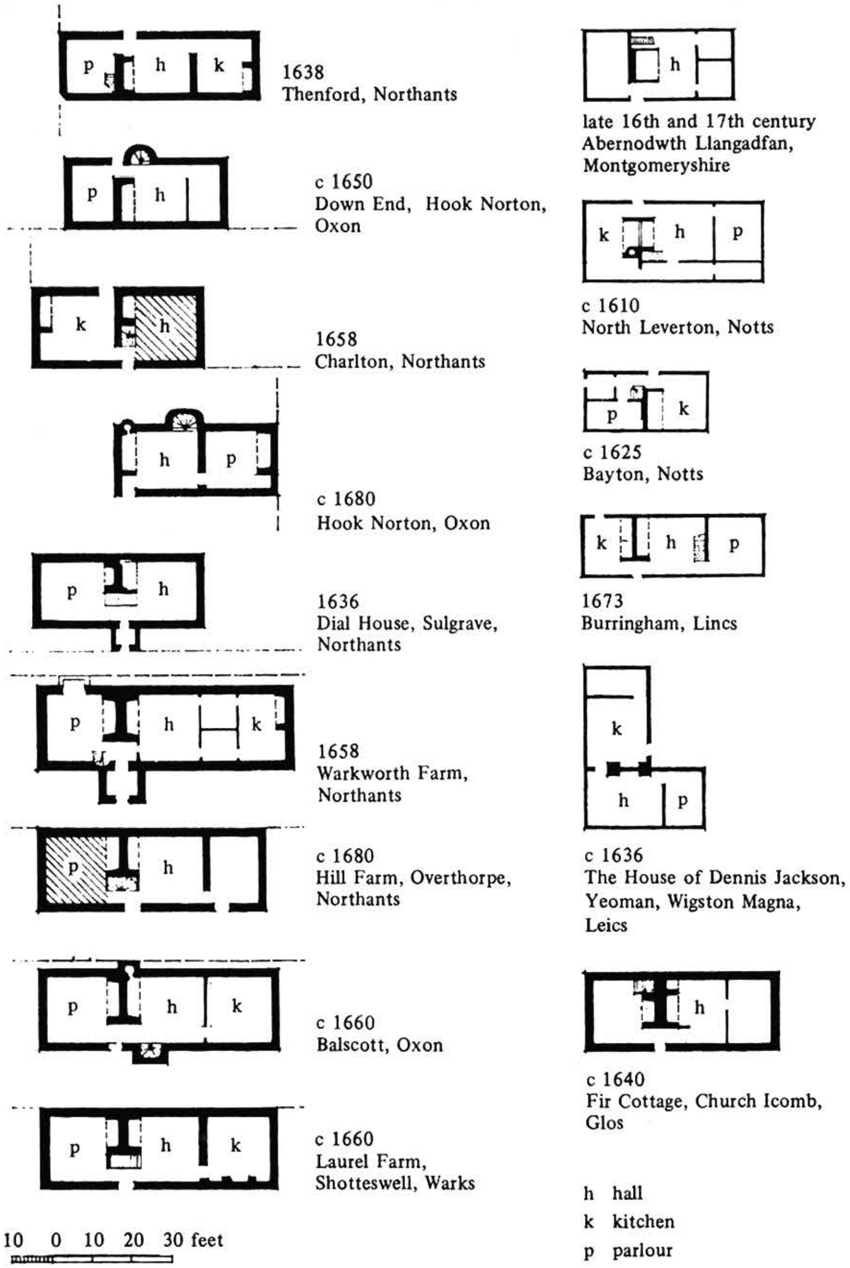


Figure 11.7. Seventeenth-century house plans from various counties of central England, of the 'lowland' type (from Wood-Jones, 1963).

changes in family structure, village life, and the relations between the sexes. In general, and not just in this piece of work on domestic space organisation, the primary interests of Hillier and his colleagues are sociological and anthropological, rather than technological. It is their belief that social relations are not (or not just) expressed or communicated, but that they are actually embodied in or *constituted by* the spatial organisation of buildings and settlement patterns. Thus they are concerned to find formal properties of these access graphs which might coincide with social distinctions, or which might affect the ways in which contact between individuals or social groups is allowed or prevented.

There is not the space here, nor would this be the place, for me to try to do justice to the range and depth of Hillier's sociological thesis. I will try simply to indicate some of the broad lines along which the argument is made.

For many if not the majority of building types it is possible to make a fundamental distinction between 'inhabitants' and 'strangers'. The 'inhabitants' are either the permanent occupants of spaces, such as residents in houses or the occupants of private offices; or else they may be those whose social status or occupation is identified with or 'mapped into' a building or room, such as the priest in a church, the players in an auditorium, or the shopkeeper in a shop. 'Strangers', on the other hand, are obviously the members of the public, who might attend a church service, go shopping, or listen to a concert.

For certain buildings there would be a subset of 'strangers'—they may be called 'visitors'—who are still in a kind of subordinate position vis-à-vis the 'inhabitants', but nevertheless come to occupy the building for extended periods. Besides visitors to private houses or guests in hotels, this category would include pupils in a school, convicts in a prison, or patients in a hospital. The last case, that of a hospital, is especially complicated since there exists a whole gradation of 'strangers' from casual visitors, through out-patients, to long-stay in-patients or permanent inmates.

To return to the distinction between the distributed and nondistributed parts of the access graph: this distinction has the effect of partitioning the vertices of the graph into two sets, as illustrated in figure 11.8, those which lie on cycles passing through the exterior, and those which do not. It is Hillier's proposal that the distributed vertices will in many cases correspond to those parts of a building to which strangers may gain access. Meanwhile, the undistributed vertices, which lie on what amounts to a set of trees or tree-like subgraphs attached at their 'roots' to the distributed system, will tend to be the preserve of the inhabitants. Where this correspondence *does* obtain, then social relations between inhabitants will be mapped into the nondistributed parts of the graph, whose structure differentiates or separates them from strangers, and ensures their privacy.

This is the case with blocks of flats or office buildings (other than open-plan offices), as some of the plans illustrated in the last chapter suggested (compare figure 10.7), and it is also true for many larger

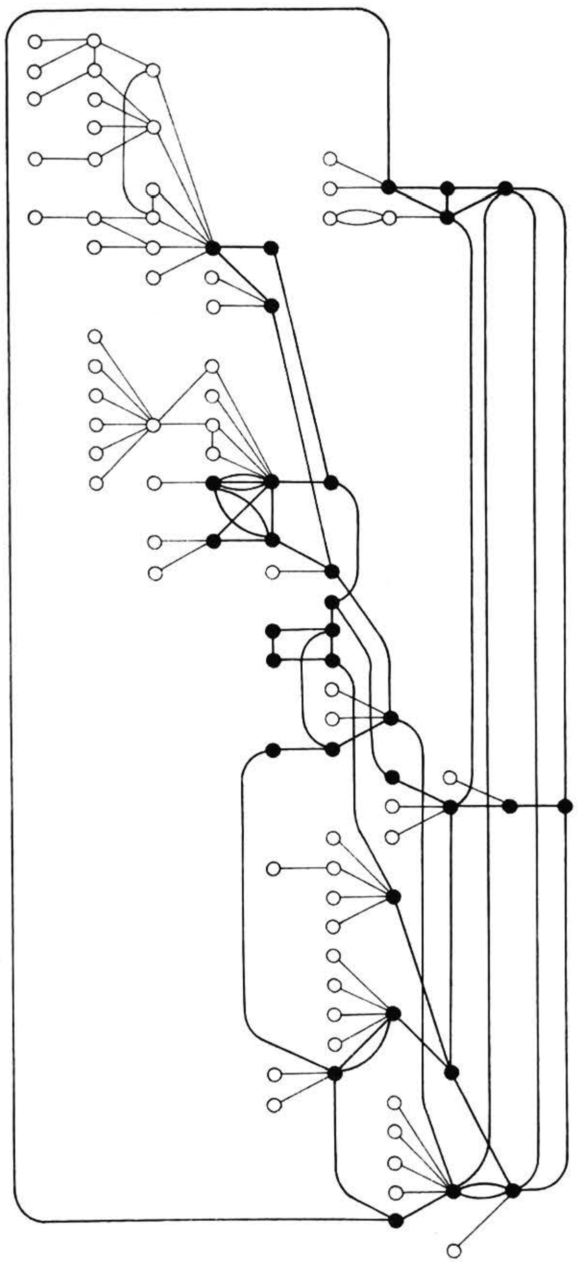


Figure 11.8. Justified access graph of the plan of a complex building, with distributed vertices marked by solid circles, and undistributed vertices by hollow circles (compare figure 11.11). The distributed vertices represent rooms which lie on cycles passing through the exterior (the 'carrier'). The undistributed vertices represent rooms which do not lie on such cycles.

institutional buildings, as examples later in this chapter illustrate (figures 11.9 to 11.11). In these instances the distributed vertices lie on what is clearly distinguished as a 'public' circulation system. However, the same distinction may be carried into plans where the formal division of 'public' from 'private' is not so obvious, and where definite 'circulation spaces' understood in a modern sense do not necessarily exist. Thus in examples of village plans from the anthropological literature, which Hillier and colleagues examine, it is relations between the sexes and between age groups which can often be mapped by reference to distributed and nondistributed vertices.

In 'public buildings', both ancient and modern, there will be *interfaces* where inhabitants and strangers meet, such as halls, meeting rooms, concourses, auditoria. These spaces will tend to correspond to vertices with high valencies lying in the distributed part of the graph. In buildings where casual entry is to be encouraged as much as can be, then these interface spaces will be shallow in the graph. Such is the case with department stores, where the distributed part of the graph is also, as Hillier argues, made maximally connected.

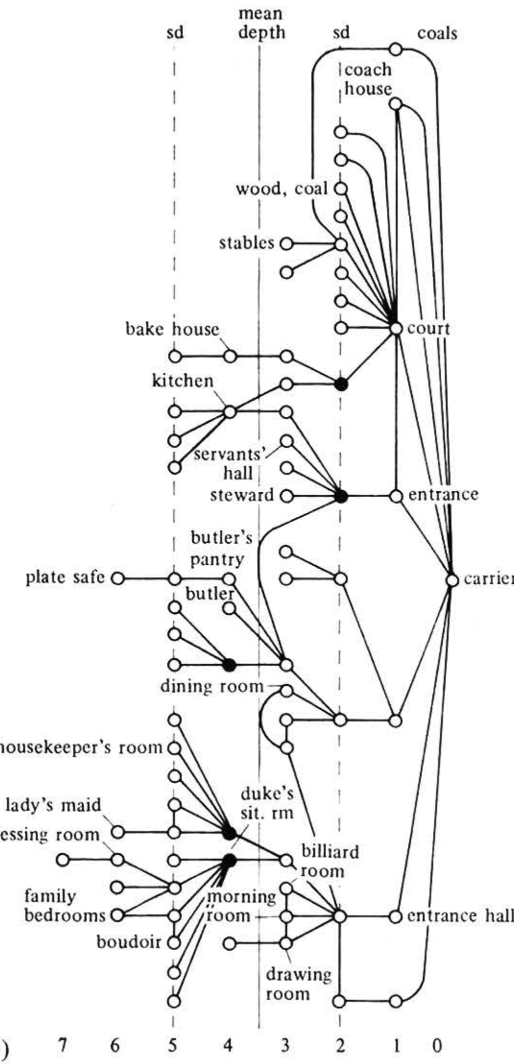
In the case of *bureaucratic* institutions which have dealings with the public such as local government offices, the interface—the waiting room where the jobless are interviewed, the car licenses are issued, the passports are renewed—will be shallow in the building. Behind that interface there will be many layers of organisational hierarchy, up to the levels at which policy is made. Information is fed back up through this system from the interface, and decisions are passed down again as to the standardised procedures to be administered at the interface.

By contrast the contact between strangers and members of the *professions*—patients visiting their doctor, clients seeing their lawyer—must be a direct one, since the professional must exercise his judgement in relation to the case in question. The status of a professional is naturally higher than that of a desk clerk in a bureaucracy, and the level at which he meets strangers is correspondingly deeper in the building.

As I mentioned it is by no means universally true that the distinction of distributed from nondistributed spaces corresponds to a distinction between public circulation and private rooms in a building. The spaces belonging to inhabitants may be distributed ones—indeed in this context Hillier remarks on the way in which the inhabitants of the notorious London rookeries were able to escape the law precisely because they had available to them alternative routes to the exterior through which strangers—the police—were unable to pass. Hillier suggests in general that a distributed or cyclic structure in access graphs corresponds to an integrated and inter-penetrating kind of social organisation, whereas a nondistributed or hierarchical, tree-like structure enforces separation and segregation. In the latter type of structure each inhabitant has few or no neighbours, and meets his fellow inhabitants only at shallower levels in the public circulation hierarchy.

(a)

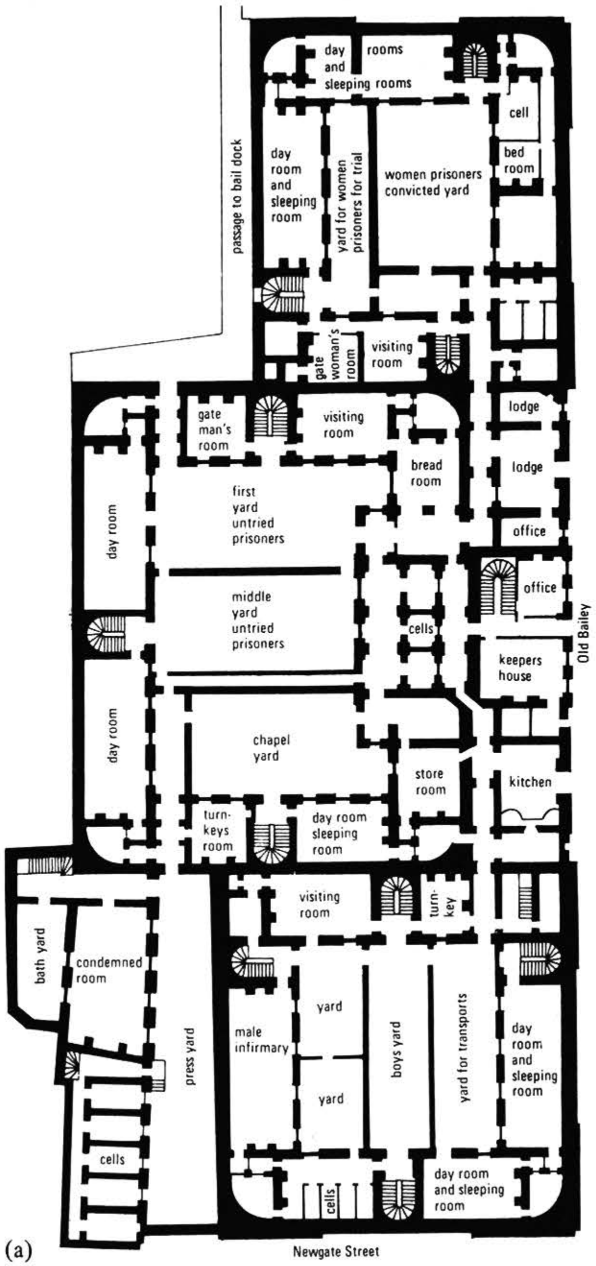
Figure 11.9. (a) Ground-floor plan of Buchanan House, Stirlingshire 1851–1853 (William Burn) (from Girouard, 1971, figure 2, page 23). (b) The corresponding justified access graph (after Hillier et al, 1978b, page 25). At the bottom of the graph are the private quarters of the family; at the top the servants' quarters and service rooms. It is set on the page such that depth increases from right to left. The shallow, distributed vertices with high valencies correspond for the most part to circulation spaces. On the family side these link the rooms in which *visitors* would be entertained—drawing room, morning room, dining room. On the servants' side they are where goods are delivered to the house, and through which the servants penetrate to the family rooms. Deeper in the plan are more tree-like, undistributed parts of the graph, corresponding on the family side to the most private rooms: the Duke's private sitting room, the boudoirs, and bedrooms. Access to these is strictly controlled by shallower spaces. The deepest space on the servants' side is the plate safe. The solid vertical line marks the mean depth of the plan, and the broken lines, the standard deviation of the depth of all rooms from that mean.



(b)

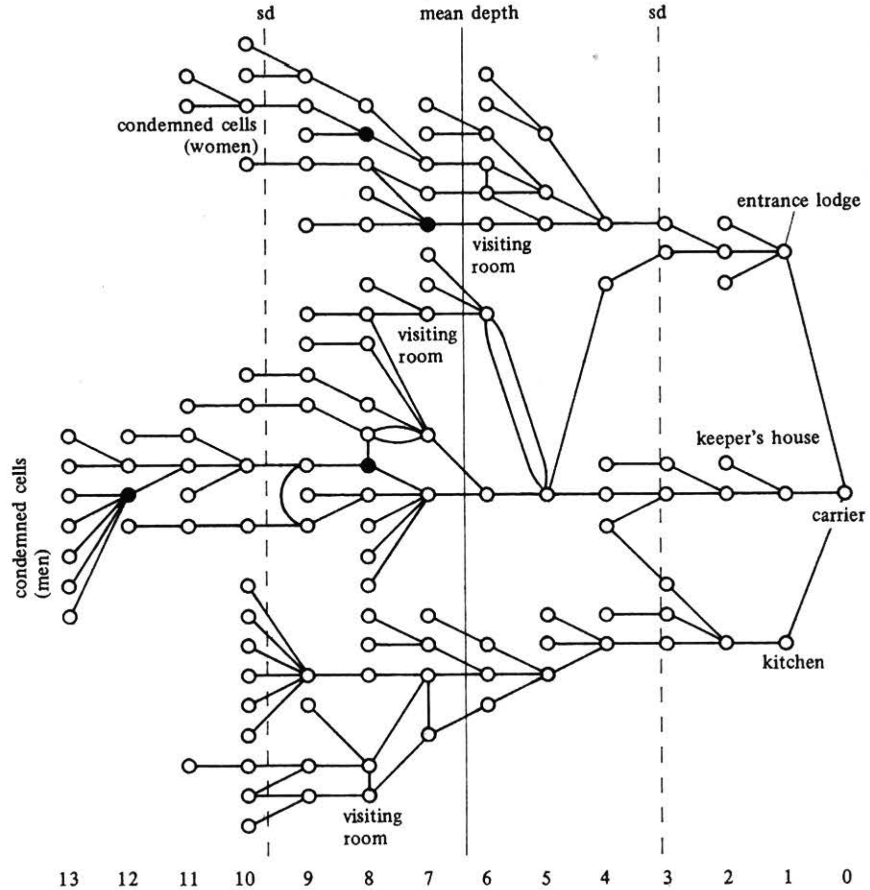
Figure 11.9 (continued)

number of spaces	78	cyclomatic number	14
depth of deepest space	7	measure of relative connectedness	0·093
mean depth of all spaces	3·5	depth of deepest undistributed space	4
relative mean depth	0·066	mean depth of distributed spaces	2·4
maximum vertex valency	12	mean depth of undistributed spaces	1·5
mean vertex valency	2·2		



(a)

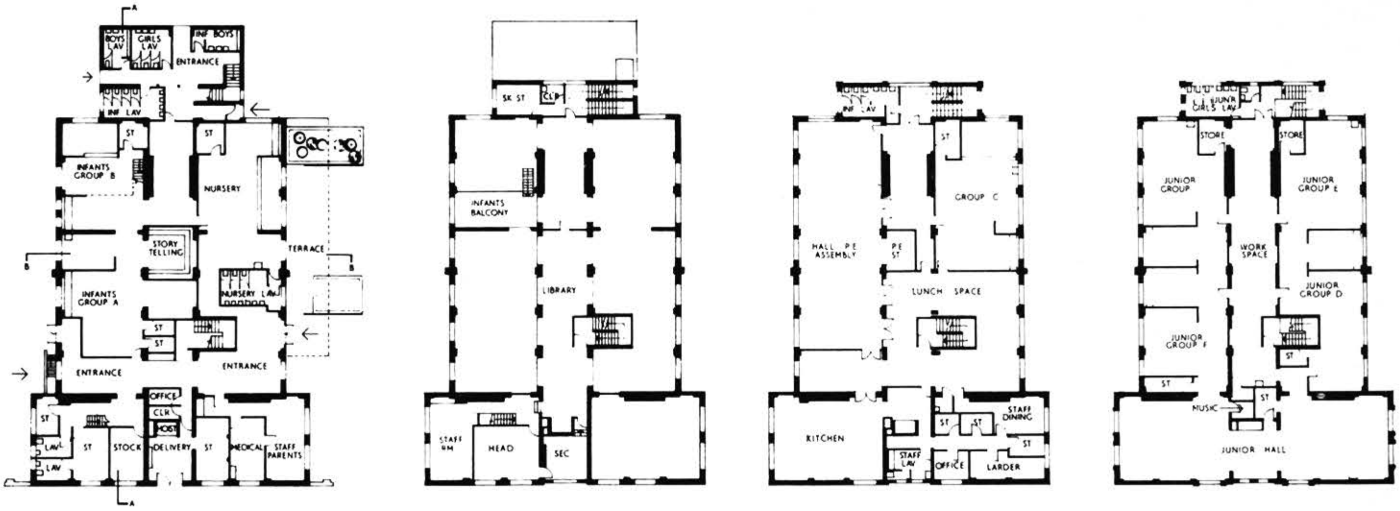
Figure 11.10. (a) Newgate Gaol, London, prior to alterations, circa 1836. (b) The corresponding justified access graph (after Hillier et al, 1978b, page 72). Here, obviously, the control of deep spaces by shallower ones constitutes the whole purpose of the building. The graph as a whole is extremely deep, and highly undistributed. The cells all lie in the tree-like depths of the graph; and those for condemned



(b)

Figure 11.10 (continued)
prisoners are the deepest spaces of all. The shallower, distributed parts are the preserve of the *inhabitants*—the warders. Besides the prisoners who here constitute the *visitors* in Hillier's terms, there are visitors in the more usual sense, who are allowed in under strict supervision, and who meet the prisoners in *interface* spaces—the visiting rooms, situated at about half the depth of the plan.

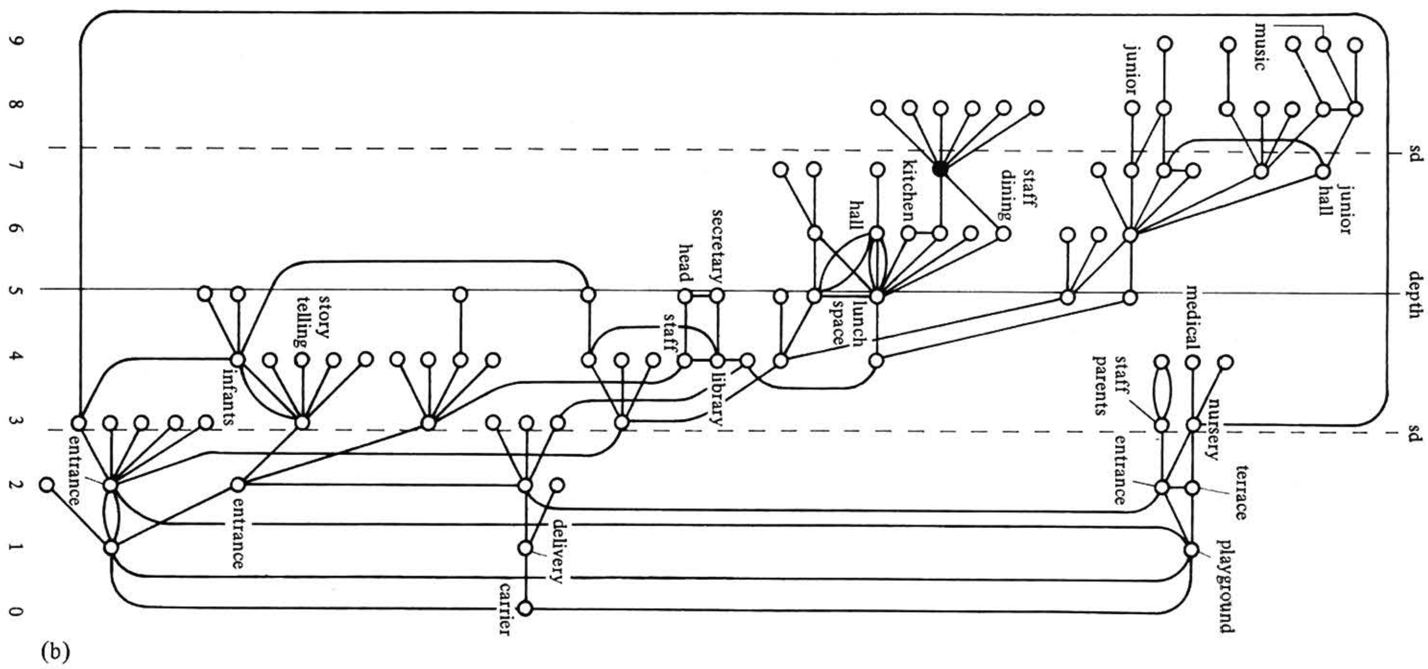
number of spaces	124	cyclomatic number	12
depth of deepest space	13	measure of relative connectedness	0·049
mean depth of all spaces	6·4	depth of deepest undistributed space	9
relative mean depth	0·088	mean depth of distributed spaces	2·7
maximum vertex valency	8	mean depth of undistributed spaces	5·0
mean vertex valency	1·8		



(a)

Figure 11.11. (a) Ground-floor, first-floor, second-floor, and third-floor plans of Compton Primary School, Clerkenwell, London, as remodelled 1970 (from *Architects' Journal*, 1971, pages 708-712). (b) The corresponding justified access graph for all four floors (after Hillier et al, 1978b, page 68). The graph here is much more distributed generally than that for Newgate—there are many more cycles. The *visitors*—in this case the schoolchildren—are much less rigorously supervised than are the prisoners. But it is still true that the *inhabitants*—here the teachers—occupy some of the shallower spaces, whereas the classrooms are deeper in the plan and lie in the undistributed parts of the graph. (Since this particular graph is not planar, the measures of cyclomatic number and relative connectedness are inapplicable.)

number of spaces	91	relative mean depth	0·092	depth of deepest undistributed space	3
depth of deepest space	9	maximum vertex valency	10	mean depth of distributed spaces	3·6
mean depth of all spaces	5·1	mean vertex valency	2·6	mean depth of undistributed spaces	1·5



(b)

Figure 11.11 (continued)

At the very deepest levels in access graphs it seems that the spaces associated with the most elevated social status are to be found. In several examples of villages and palaces which Hillier and colleagues have taken from anthropological and archaeological sources, it is the deepest spaces which are found to be occupied by headmen, chiefs, or the most essential functions of government. The access to these elevated levels is controlled by whole sequences of antechambers, spaces occupied by subsidiary officials, guards, etc.

Spaces containing the most sacred objects seem to have high values for the relative mean depth of all rooms measured from that room—they are remote, that is to say, detached from the remainder of the complex. Among modern buildings, one notable example of a deep (perhaps even sacred?) space of a kind which is at least comparable in its structural position in the access graph, is the operating theatre in a hospital.

I referred to the notion of patients, pupils, or prisoners forming a class of 'visitors' in institutional buildings. Here, as Hillier points out, the spatial relation of visitors in this sense to inhabitants is inverted. It is now the inhabitants—the teachers, hospital staff, warders—who occupy and supervise the distributed spaces, so that they can control entry to and from the nondistributed spaces—the classrooms, the wards, the cells—by the 'visitors'.

Figures 11.9, 11.10, and 11.11 illustrate some examples of plans, together with their justified access graphs, taken from a study of twenty large buildings made by Hillier et al (1978a). The buildings analysed are of very varied kinds, containing anything between 30 and 300 spaces in each. Half of these are palaces, temples, and tribal settlements from the anthropological and archaeological literature. The remainder are modern (nineteenth-century and twentieth-century) buildings including a school, a prison, a hospital, and a range of other types. This is not in any sense therefore a statistically random sample, but rather an attempt to explore the dimensions of variation in their permeability or access structure, of individual buildings of widely different character, age, and function.

Figure 11.9 shows the Buchanan House built in Stirlingshire, Scotland, in the early 1850s. Figure 11.10 shows Newgate Gaol in London as originally built in 1836. And figure 11.11 shows Compton Primary School in Clerkenwell, London (1971). The figures at the bottom of the access graph in each case mark the depths d_{oi} of spaces, as before. The vertical solid line marks the mean depth $\bar{d}_{oi}$, and the two broken lines the standard deviation from this mean. Also tabulated for each graph are the values of the measures explained earlier, specifically:

the number of vertices v, that is, rooms or spaces,
the depth of the deepest space d_{oi}^{max},
the mean depth $\bar{d}_{oi}$,
the relative mean depth $2(\bar{d}_{oi} - 1)/(v - 2)$,
the maximum vertex valency,

the mean valency over all vertices,
the cyclomatic number (that is, the number of interior faces f),
the measure of relative connectedness $f/(2v - 5)$,
the deepest undistributed space, where depth is measured from the nearest distributed space,
the mean depth of distributed spaces, and
the mean depth of undistributed spaces, where depth is again measured from the nearest distributed space.

Hillier and his colleagues have as their long-term aim to work towards a theory of building types based on their spatial organisation, whose classifications will follow the lines of these analyses. They suggest that a primary dimension of classification will be according to variations in the relationships between inhabitants, strangers, and the interfaces at which the two meet. Extensions of these same methods to larger samples of the same functional building type may make it possible to identify repeated structural patterns in access graphs, with characteristic values for at least some of the various measures described. It may be that structural similarities will be found not always at the level of whole buildings, but sometimes for 'subcomplexes' within buildings. It may transpire that the access graphs of actual buildings are constituted out of unique combinations of relatively standardised or regular subcomplexes.

This static analysis of buildings at fixed dates could be extended to the study of historical processes of change. Two types of process can be distinguished here. There is the alteration of a single building by conversions or additions, the sorts of changes which were discussed under the heading of 'adaptability' in the last chapter. Hillier and colleagues address this question in a preliminary way by drawing, in the case of two of their buildings, Newgate Gaol and Compton Primary School, a series of graphs corresponding to the original plans and to succeeding stages in their later remodelling.

The second and generally much more interesting process of change is the evolution of a *design* as it is materialised in successive individual buildings of the same functional or morphological type. Hillier uses a pair of biological terms here, to distinguish the *genotype* from the *phenotype*. The genotype in biology is what is transmitted in heredity by the set of instructions embodied in the genetic code. It is the type or design of the class of organisms—so by analogy the type-design of a class of buildings. The phenotype refers to the actual individual, the organism, or in our case the building, in which that type-design is realised. Phenotypes, actual buildings, may undergo processes of growth or adaptation during their lifetimes. It is only genotypes or designs which can properly be said to 'evolve'.

Empirical studies of the first kind of process, growth and change of individual buildings, remain very few in number. Most of these, since they have been made out of an interest in adaptability, have tended to concentrate on changes in room use or room size without reference to the spatial organisation of the plan as a whole (see Cowan, 1963; Cowan and Nicholson, 1965; Llewellyn-Davies et al, 1973). One exception is a study by Bon (1973) who was able to find data on the stages of development on one building, a palace in Ugarit (Ras Shamra). He counted the numbers of edges and vertices in the plan graph of this building for eleven stages of growth, and in the access graph for nine stages of growth. He found that the ratio e/v remained remarkably constant throughout, that is, a precise allometric relationship applied (figure 11.12).

As for evolutionary studies of types of building characterised in detailed topological and geometric terms, I know of none other than the work on houses already cited in this chapter. Hawkes (1976b), however, has touched in a more discursive way on some of the issues which might arise in this kind of 'geometrical history' of building types. He emphasises the role of what he calls 'stereotypes' in architectural design. A stereotype is "a generally held notion about the nature of a good solution to any recurrent building design problem" (page 465). There is clearly a certain affinity between this idea of a stereotype, and Hillier's genotype—although Hawkes's interests are more those of a building scientist than of a social scientist. (Also he sees the stereotype as something of which designers are consciously, if only perhaps vaguely, aware, which is not necessarily true of the genotype in Hillier's conception.)

In primitive and vernacular architectures it is arguable that designs for buildings are transmitted and reproduced through repeated *copying* of the existing type, or through repetition of a fixed set of inherited constructional procedures, with only small changes to the design being introduced in each 'generation'. The parallel with biological evolution in this respect is

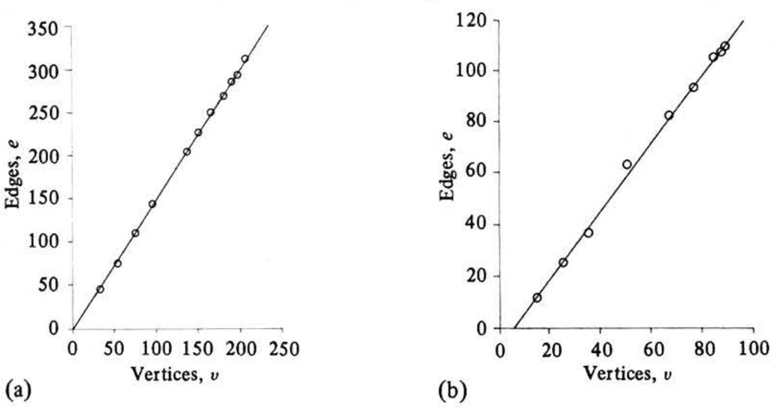


Figure 11.12. Numbers of edges e and vertices v in the plan graph (a) and the access graph (b) of a single building—a palace in Ugarit (Ras Shamra)—at successive stages in its growth (from Bon, 1973). Plan graphs are shown for eleven stages of development, and access graphs for nine stages. In both cases the ratio $e:v$ remains remarkably constant.

therefore quite close (Steadman, 1979). In the modern 'self-conscious' process of design carried on by professional architects, the role of direct copying is obviously much reduced. Nevertheless all architects still work within a tradition, and still rely heavily on precedent and on recourse to tried solutions. This can result in a continuity at the level of spatial organisation and underlying geometric form, while surface features and stylistic treatment are more rapidly transformed.

For example, Hawkes (1980), in a historical review of the designs of auditoria, points to the persistence of a very few basic auditorium forms, from antiquity right up to this century. The Greek and Roman type, as described by Vitruvius, is carried through to the Renaissance. From the late seventeenth to the nineteenth century, a horseshoe-type plan deriving principally from Fontana's Teatro Tor di Nona in Rome (1671) dominates European practice, and remains largely unchanged behind Baroque, Rococo, and Classical revival treatments. During the nineteenth century a new type of rectangular 'shoebox' plan, with one or two balconies, emerges to satisfy the demand for a type of hall specially suited to musical performances.

As Hawkes (1980, page 6) says of the horseshoe form, "There were, clearly, deviations in detail from the precedent in response to local conditions, particularly of size and social organisation, and also to development in the style and scale of productions, particularly to accommodate the needs of opera ...". This is precisely the distinction of phenotypes (the 'deviations in detail') from the genotype (the 'precedent').

Similar themes, of the influence and persistence of stereotypes, are pursued by Hawkes (1976b) in his 'evolutionary tale' of the history of central city office building forms from the beginning of this century to the present.

Hawkes attributes the longevity of some of the earlier forms in both building types to the fact that they were known to be functionally satisfactory, but the basis for this success was not understood at a theoretical level. The horseshoe auditorium form, for example, had been arrived at empirically by a rather slow process of trial and error. Later architects tended to adhere to the stereotype since they had little scientific basis, specifically little acoustical theory, with which to predict the results of radical departures from that form. The same is true, according to Hawkes, for the stereotypical, naturally lit, and naturally ventilated, urban office building of the beginning of this century.

Departures from the stereotype may be occasioned by changes in social structures and functional demands, as, for example, the way in which the classical theatre was 'brought indoors' in the Renaissance, and how a social segregation of the theatre audience was subsequently effected by the separation of boxes from stalls and pit. Later technical innovations, for example, those in artificial lighting and structural materials, provided the *opportunity* for morphological change, by allowing wider and deeper balconies and freeing the stage from the constraints of natural lighting from top or side. The realms of 'technical possibility' were correspondingly extended.

In the case of modern office buildings, Hawkes points to the comparable effects of the constructional innovation of lightweight curtain walling, and as a not unrelated development, the introduction of full air-conditioning. A new situation arises in the twentieth century with the emergence of building science itself as an academic discipline. The findings of building science become part of the mental equipment of architects, and a resulting emphasis is placed in design on those aspects of performance which can be predicted and controlled—first historically, in the case of office buildings, on daylighting and sunlighting, and later on the complete artificial control of temperature and ventilation. As Hawkes (1980, page 471) puts it: "Along with the parallel and frequently related developments in the technologies of construction and equipment of buildings, this growth in our skills of specification and prediction has been a fundamental force in the determination of the nature of the built form".

It ought to go without saying that all these proposals for a morphological history of buildings and building types are made in the frank recognition that such a history would be a partial one, focussing on geometrical, material, and technological constraints, on functional performance, and as in the case of Hillier's work, on the relation of spatial to social organisation. None of this denies the creative capacities of individual architects, the pervasive influence which can be exerted by the example of great buildings, or the status of architecture as an art. The hope is rather that such a programme may to some extent counterbalance or complement the exclusive concentration by some architectural historians and critics on personalities, styles, 'influences' in the narrowest sense, and especially today on questions of semiotics and iconology.

This is not to suggest by any means that matters of architectural *style*, and specifically as expressed in floor plans, are beyond the reach of formal analysis, as the work of Stiny and Mitchell (1978a; 1978b) on the plans of Palladian villas amply demonstrates. This work makes use of the logical apparatus of the *shape grammar*, which has been developed by Stiny (1975) and Gips (1975). A shape grammar can be imagined as analogous to a grammar for an ordinary language. But where the latter specifies the permissible combinations in which words may be assembled in that language into the one-dimensional sequences constituted by grammatical sentences, a shape grammar specifies rather the combinations in which two-dimensional or three-dimensional shapes may be assembled, to form complex compositions in the plane or in space.

It is not my intention to go into the subject of shape grammars here, for the simple reason that they are the subject of the books by Stiny and Gips already cited, as well as being discussed in Stiny and Gips (1978), to which readers are strongly recommended. There is no doubt that the shape grammar is a powerful tool for the investigation of questions of

composition and style in many areas of design and the plastic arts[30]. Stiny and Mitchell in the project just mentioned, for example, have developed a grammar which constructs and enumerates not just the plans of Palladio's (1570) actual projects as illustrated in the *Quattro Libri*, but all those other plans, conforming to the same compositional rules, which Palladio never designed (see figure 11.13).

The subjects of settlement or urban morphology (see Conzen, 1960; Clarke, 1977; Hillier and Hanson, forthcoming; Hillier et al, 1976) are strictly beyond the scope of this book. It is worth making the point, all the same, that no very distinct line of demarcation can be drawn between the architectural and the settlement scales: indeed the size of many a single building can approach that of a village, with an organisation along covered 'streets' or around internal courts. The real distinction is between enclosed space within buildings, and open spaces between buildings.

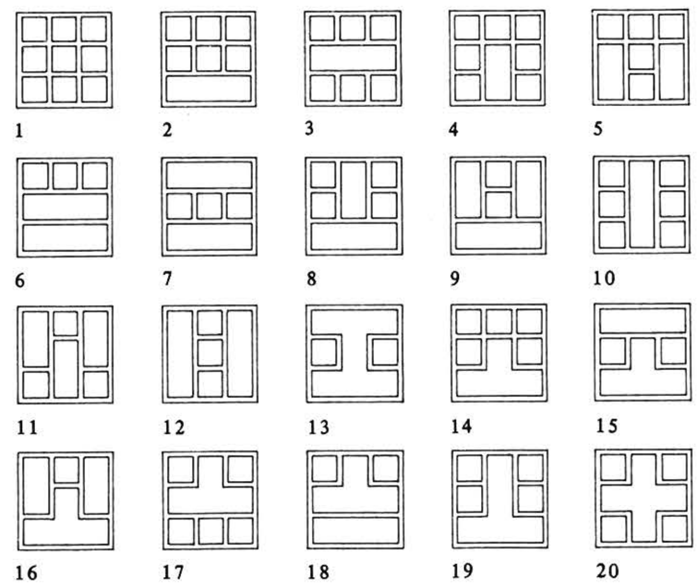


Figure 11.13. Part of a catalogue of possible room layouts (in dimensionless form) for Palladian villas (from Stiny and Mitchell, 1978b). The figure shows all schemes on 3 x 3 grids—including that of Palladio's actual plan for the Villa Angarano (number 4). Stiny and Mitchell also enumerate all schemes on 3 x 5 grids (not shown here).

[30] As a matter of fact shape grammars have already made a covert appearance in earlier chapters, since the method for generating rectangular dissections devised by Earl (chapter 4) was formulated by him precisely as a shape grammar; and Krishnamurti's method of colouring on gratings (chapter 5) also effectively simulates the operation of a shape grammar, with colouring rules taking the place of 'shape rules'.

Nor can the interior layout of a single building be considered without reference to its immediate surroundings, since the plan in general will be constrained by relations to the street or other points of access, by relations to gardens or yards, by the effects of nearby or adjoining buildings on possibilities for the placing of windows, by the orientation of the site and so on. As we have seen already, the relations of adjacency or access of interior rooms to regions on the outside of the plan may be represented using graphs; and this representation can obviously be extended to groups of buildings, up to the scale of the whole settlement. Thus Hillier and colleagues have drawn the access graphs, in 'justified' format, of complete villages, and of large housing estates.

One particularly interesting use of graph theory in this context is that of Krüger (1977; 1979a; 1979b; 1980; 1981a; 1981b), who defines several types of graph to describe an urban area. He has applied his techniques to a study of the city of Reading, Berkshire. In the first type of graph, each vertex represents the interior of the plan of a whole building (without regard to its subdivision into rooms), and an edge expresses the adjacency of one building to another. Thus a pair of semidetached houses would be represented by two vertices joined by an edge:

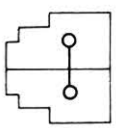
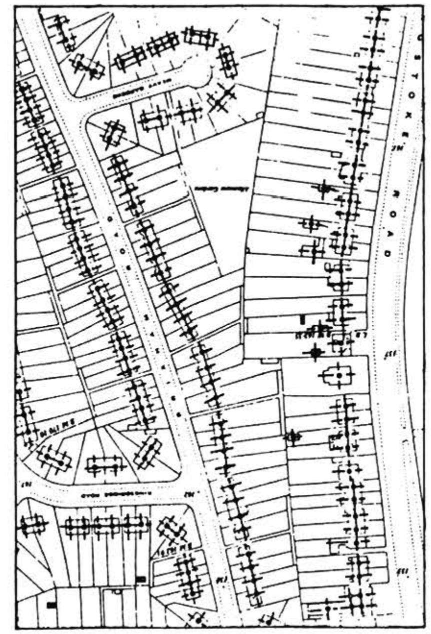


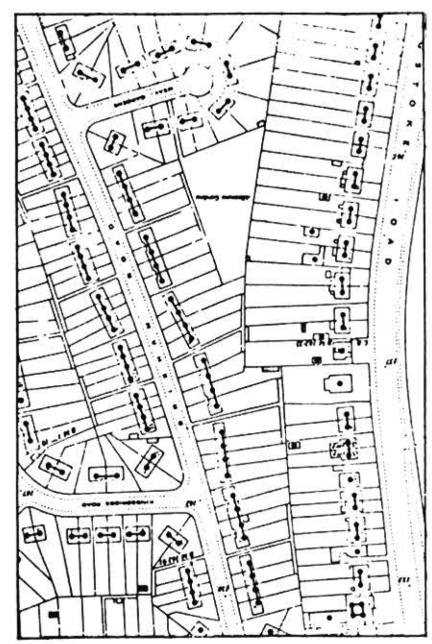
Figure 11.14(a) shows this type of graph for a suburban area of Reading in which detached, semidetached, and terraced houses are all to be found.

The second type of graph represents relations of adjacency of buildings to surrounding open spaces. It is assumed for the sake of simplicity, working at this scale, that minor indentations of the perimeter of a building can be ignored (according to definite rules which Krüger specifies). For instance, a pair of semidetached houses with extensions at the back as shown above is approximated by two simple rectangles. In the graph, the vertices represent buildings again, and also exterior regions; and each edge represents the adjacency of a building to an exterior space across a wall in the building perimeter. Figure 11.14(b) shows graphs of types 1 and 2 combined (in which vertices representing exterior regions have been omitted for clarity).

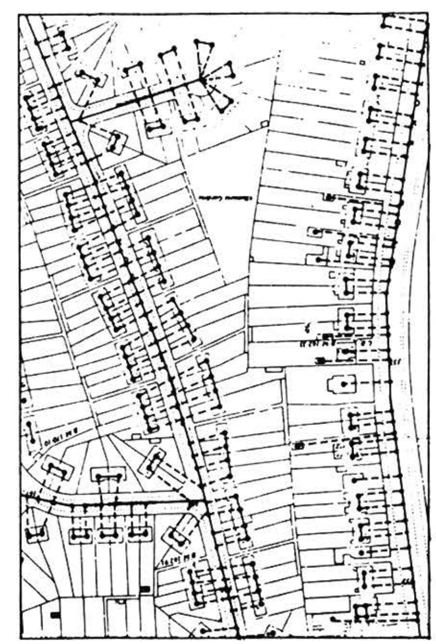
Two more types of graph in Krüger's analysis relate to the road system, and to the means of access to buildings. Here the vertices correspond to buildings once more, and to junctions between roads, driveways, paths, etc (where these give access to buildings); and the edges represent the roads and paths themselves. Figure 11.14(c) illustrates, for the same area of Reading, a combination of these access graphs with the first graph (of adjacency between buildings). The edges shown as broken lines connect each building to its access point on the road network.



(b)



(a)



(c)

Figure 11.14. (a) Map of a suburban part of the city of Reading, Berkshire, with Krüger's (1977) type 1 graph of the adjacencies of buildings superimposed. Each vertex represents a complete building, and an edge the adjacency of two buildings across a party wall. (b) The same map with Krüger's type 1 and type 2 graphs combined. In the type 2 graph, a vertex represents either a complete building or an exterior space, and an edge represents the adjacency of a building to an exterior space. (c) The same map with Krüger's type 1, type 3, and type 4 graphs combined. In the type 3 graph, a vertex represents either a building or a neighbouring point on the street system, and an edge represents a means of access from street to building. In the type 4 graph, a vertex represents a junction between streets, paths, etc, and an edge represents such a route itself.

The road network divides the whole urban area into a series of bounded regions or city blocks (they are the faces of the embedded road access graph). Krüger describes a fifth and final type of graph in which the vertices correspond to these blocks, and the edges represent the adjacencies between blocks. Obviously it is only possible for buildings which are within the same block to be adjacent. Figure 11.15 shows the definitions of all five of Krüger's graphs in diagrammatic form.

Krüger's purpose in all this is to relate the building morphology to the structural characteristics of the road network, and beyond these to other properties of the urban system more traditionally dealt with by urban geographers and planners, such as floor space and population densities, land use, and patterns of travel behaviour. To do this he makes use of measures on all these graphs, relating their numbers of vertices, edges, cycles, and (for disconnected graphs) components, of the kinds mentioned here in previous chapters. For example, he computes their cyclomatic numbers, and the values for various connectivity measures such as the gamma index of Garrison and Marble referred to in chapter 10. He also calculates for the buildings themselves the values for some measures which are rather similar to those defined for polyominoes by March and Matela (1974) and described here in chapter 8.

A set of mutually adjacent buildings, that is, for which Krüger's graph of type 1 is connected, forms what Krüger calls an 'array', in which the party walls between buildings are analogous to the partitions between rooms in a single building plan. Krüger takes the average number of external walls per

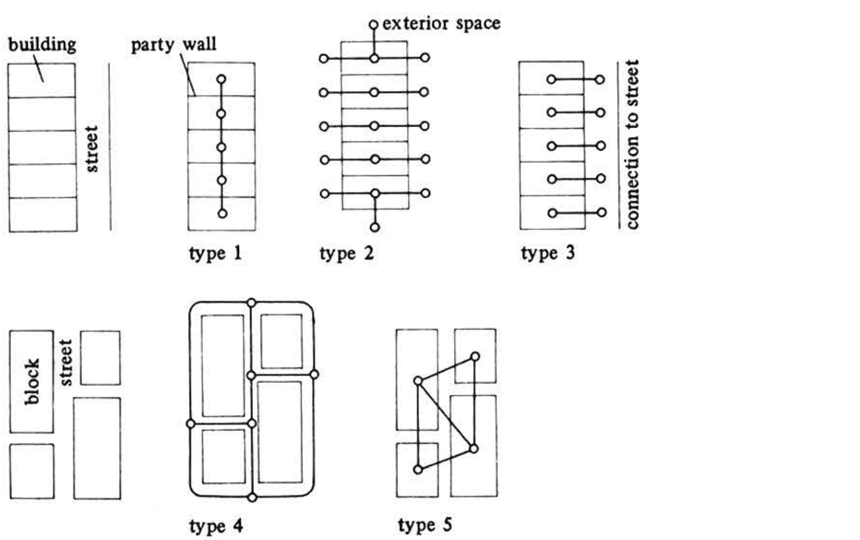


Figure 11.15. Diagrammatic representation of all five of Krüger's (1977) graph types for urban areas. In the type 5 graph a vertex represents a city block, and an edge the adjacency of two blocks across a street.

building (a 'perimeter measure'), the average number of external walls per array (a 'compactness measure'), a ratio of the number of external walls to the number of party-walls in an array (a 'shape measure'), and the number of party-walls per array (a 'connectivity' measure).

With these sorts of measures Krüger examines relationships, for example, between the connectivities of the building arrays and the connectivity of the road system (these relations are shown to be weak); or the effect on the connectivity of the building arrays, of an increase in the number of buildings per unit of land area, that is, an increase in density of ground coverage. In this case, as would be intuitively expected, the connectivity of buildings increases, and the graph of type 1 comes to contain more cycles. Figure 11.16 illustrates such a graph for an area in the centre of Reading, where the housing is mostly terraced, and commercial and industrial buildings are packed closely together. [Over the greater part of the city, by contrast, type 1 graphs are highly disconnected, and without cycles, as in figure 11.14(a).] Krüger points out how in general, in Western industrial cultures at least, the graphs of adjacency of arrays of buildings, up to the point where they are maximally connected, must still preserve outer-planarity—since it is usually required that every separate building have independent access from the road network (graph type 3), and not access solely through another building.

Krüger's empirical work in determining values for and relations between certain of these measures, enables him successfully to simulate, by means of a probabilistic model, the distribution of numbers and types of building arrays (types, that is, distinguished by their connectivities), throughout a series of zones or cells representing the map of the whole city of Reading.

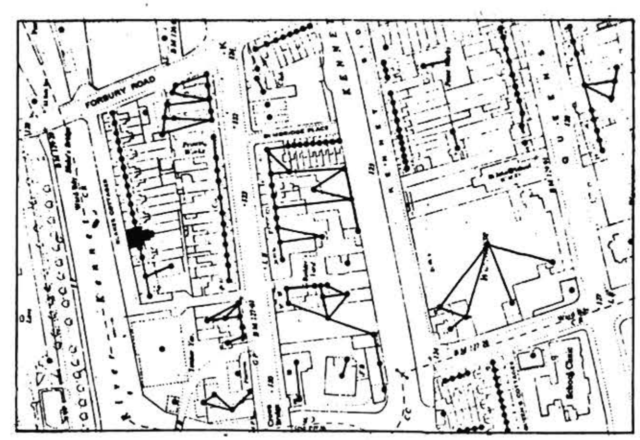


Figure 11.16. Krüger's type 1 graph for a central area of Reading. The density is higher and hence there are more adjacencies between buildings, and more cycles in the graph, than say in a suburban area [compare figure 11.14(a)].

Although Krüger's analysis is made of the contemporary city, and applies to one point in time, it is obviously highly suggestive for similar approaches to the morphology of towns and settlements at earlier dates in history; and indeed Krüger himself mentions possible extensions of his methods to treat the processes of morphological change in cities taking place over time.

Exercises

As in the previous chapter, I suggest that you might like to use the work described here as the starting point for your own investigations of historical plans: analyses of house plans following the approaches taken by Dickens or Arbon; analysis and classification of buildings of different functional types by drawing their access graphs in Hillier's 'justified' format, and by measuring the properties of those graphs; or analyses of the building morphology and road and land-use patterns of villages or parts of towns by means of the approach of Krüger.

12

Afterword: prospects for an architectural morphology

In the introduction I mentioned the relation of the present book to *The Geometry of Environment* (March and Steadman, 1971), and I talked about the purposes which the two books have in common, including that of introducing the architectural designer to some mathematical ways of conceiving and manipulating the geometric forms of buildings.

At that time many workers who were enthusiastic about the new application of computers in architecture saw the promise of the machine as a means for automating the *process* of designing (the computerised layout methods described in the first part of chapter 9 were a product of that philosophy). This was not our view. Rather we believed that the most immediately promising use for the computer was as a tool for the *representation* of designs. The computer would, in this view, replace not the designer, but his pencil and his drawing board. The computer was a powerful tool for building *models* of buildings. If this idea was correct, then mathematical tools would be needed for describing and operating on the two-dimensional or three-dimensional geometry of building form, of a kind which could be expressed in one or another computer language. Developments in computer systems for building description since then have done much to vindicate this belief (for example, see Eastman et al, 1975; Mitchell, 1977.)

Such computer models can serve two kinds of purpose, however. They may be employed directly by the *designer*, not to generate his design for him automatically, but to represent what he designs—to model it and draw it—and to evaluate it, as, for instance, by calculating its cost, its thermal behaviour, or some other aspect of performance.

Alternatively, the same kinds of model may be used by the *building scientist* to make some more general experiments on whole ranges of hypothetical building designs. The product of this experimental work is a body of knowledge—a building science—which is useful to the designer at a more strategic level. It comes to form part of his education, part of his background understanding of the nature of buildings and their performance, out of which his design ideas may be formed and against which they can be criticised and assessed.

It is fair to say, however, that much of building science up to ten years ago, by concentrating attention on the properties of materials and the structural and physical behaviour of the building fabric, seriously neglected the study of building form. (This was perhaps partly a consequence, ironically enough, of the functionalist attitude—form would come out of function, and so it was function which ought to be studied—and partly a result of the fact that many building scientists were physicists or chemists by training, and not very interested in questions of shape or spatial arrangement at an architectural scale.)

The situation had been reached where it was possible to model say the flow of heat, the transmission of structural forces, or the penetration of natural light in some detail; but to do this for only the most elementary shapes of room or building [see the discussion by Hawkes (1976a)]. Alternatively, the results gained from experimental work with such models referred only to a small number of very particular building designs. So it was difficult to *generalise* any findings to other buildings with different geometries, and the work remained at an anecdotal level.

This book has reported a research effort which has been going on over the last ten years to try to lay the foundations of a unified geometrical theory of architectural plan form—a 'theory of cell configurations'. The work has made use of mathematical tools of the kind discussed in *The Geometry of Environment*, much elaborated, as well as other approaches. The computer methods which have been built on the basis of those descriptive techniques are needed, as we have seen, because of the large combinatorial difficulties which arise when all permutations of arrangement or pattern are enumerated. (They have nothing to do directly with 'designing by computer'.)

There are ways in which the geometrical findings have already found immediate applications to certain long-standing problems in building science—for instance the measurement of flexibility and adaptability of plans—as we saw in chapter 10. In the meantime there has been the growth of interest—which we have followed in the last chapter—amongst some architectural historians, and those whose concerns are more sociological and anthropological, in the spatial analysis of building types and settlement patterns. Connections are emerging, as we have seen, between the two strands of work, geometrical and historical.

'Morphology' is the word which Goethe coined to signify a universal science of form and spatial structure. Goethe's method in botany, where his first morphological interests lay, was intended not just to provide abstract representations, and a classification, of the variety of existing plants, but to extrapolate beyond these and to show how recombinations of the basic elements of plant form could create theoretical species unknown to nature. Goethe's morphology remained nevertheless at a metaphysical, transcendental level, and he did not refer organic forms to the mechanics of their anatomical structure, to functional analyses of their physiological working, or to the growth processes by which they were produced.

In the biology of this century it was D'Arcy Thompson who accepted most wholeheartedly Goethe's challenge to develop an organic morphology as part of "that wider Science of Form which deals with the forms assumed by matter under all aspects and conditions, and, in a still wider sense, with forms which are theoretically imaginable" (D'Arcy Thompson, 1961, page 269). D'Arcy Thompson moved from a purely descriptive to an explanatory morphology, seeking causes in physical forces and invoking the aid of mathematical models—most notably the 'method of coordinates'—which Goethe had excluded from his botany (although Thompson too had his blind spots, especially when it came to biochemistry, genetics, and the theory of evolution).

The foregoing chapters record the tentative beginnings, then, on the part of a number of workers, to take up, however belatedly, Goethe's challenge for architecture. There seem to be two large and immediate research tasks. The first is an empirical enterprise: to begin to compile that body of systematic records of the plans of existing and historical buildings, classified somehow both geometrically and by functional type, which would correspond to a 'natural history' of architecture. The second is to integrate the models of the geometric and topological properties of plans, treated in the earlier part of this book, together with representations of physical characteristics such as those relating to structural strengths, flows of heat, light, and sound, or to the movement of people and goods through plans, which are more familiar from traditional building science. Hawkes (1980, page 14) speaks of the priorities being *first* "to establish a morphology of architectural form" and *then* "to use the tools of building science ... to make explanatory statements about the relationship between form and performance". But perhaps, as I have argued here in chapter 10, there cannot be quite such a clear distinction between 'possible geometries' and functional or structural factors—since these latter considerations themselves put dimensional and shape limitations on those forms which are 'technologically and functionally possible'.

Once such a morphology can explore, for buildings, that variety of forms which is 'theoretically imaginable', then the history of actual buildings may be studied as a kind of evolutionary process (in the case of primitive or vernacular architectures at least) through these spaces of hypothetical possibility. (At the same time all this analogy from biology must not be allowed to encourage any of the confusions—the 'biological fallacies' equating organic evolution with cultural evolution—which have bedevilled the architectural theory of the modern movement.)

The study of form in biology itself at the scale of organs or of the whole body, and of the relation of form to organic function—an 'engineering' of animal or plant design—although it flourished in the nineteenth century, has languished somewhat in this century, with the rise in prestige of molecular biology and genetics. Is it too bold to suggest that an architectural morphology might in time have something in turn to offer to biology, in geometrical models of cellular assemblies, under constraints of function and engineering structure, and undergoing long-term processes of evolutionary change?

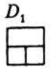
Appendix

Diagrams of rectangular dissections up to $n = 7$

For an explanation of the layout and organisation of the catalogue, see chapter 8, pages 121–123. The catalogue is taken from C J Bloch's PhD thesis *A Formal Catalogue of Small Rectangular Plans: Generation, Enumeration and Classification*, and it includes dissections with alignments and dissections with four-way junctions, but omits dissections for which $lm = n$.

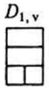
Order 3, Grating (2.2)

3 fronts
03
D_1

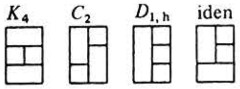


Order 4, Grating (2.3)

4 fronts
121
$D_{1,v}$

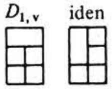


022
K_4 C_2 $D_{1,h}$ iden

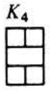


Order 5, Grating (2.3)

5 fronts
0320
$D_{1,v}$ iden

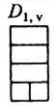


0401
K_4

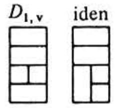


Order 5, Grating (2.4)

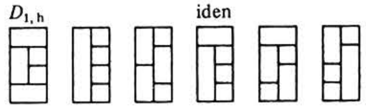
5 fronts
1310
$D_{1,v}$



1130
$D_{1,v}$ iden

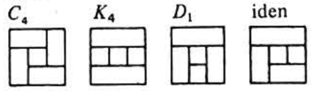


0221
$D_{1,h}$ iden

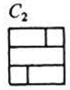


Order 5, Grating (3.3)

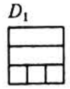
4 fronts
0041
C_4 K_4 D_1 iden



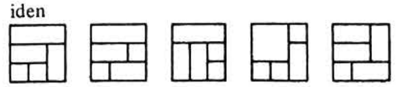
5 fronts
0401
C_2



1211
D_1

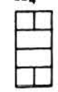


0221
iden

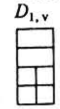


Order 6, Grating (2.4)

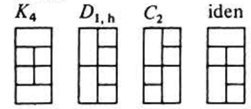
6 fronts
04200
K_4



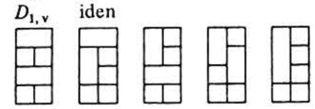
12300
$D_{1,v}$



02400
K_4 $D_{1,h}$ C_2 iden

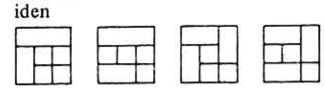


03210
$D_{1,v}$ iden

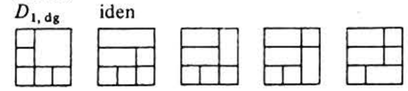


Order 6, Grating (3.3)

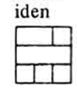
5 fronts
01410
iden



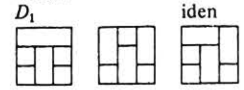
6 fronts
03210
$D_{1,dg}$ iden



04101
iden



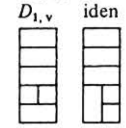
02301
D_1 iden



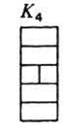
Order 6, Grating (2.5)

6 fronts
14100
$D_{1,v}$

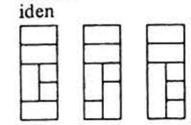
12300
$D_{1,v}$ iden



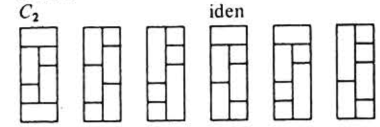
20400
K_4



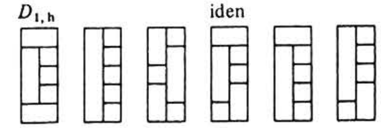
11310
iden



02220
C_2 iden

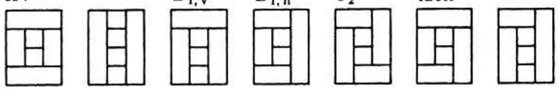


02301
$D_{1,h}$ iden

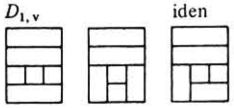


Order 6, Grating (3.4)

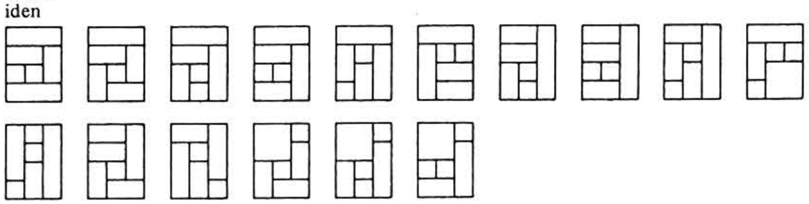
4 fronts
00240
K_4 $D_{1,v}$ $D_{1,h}$ C_2 iden



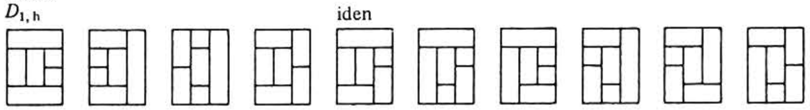
5 fronts
10320
$D_{1,v}$ iden



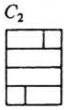
01230
iden



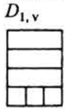
00501
$D_{1,h}$ iden



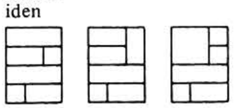
6 fronts
04200
C_2



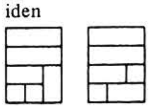
13110
$D_{1,v}$



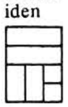
03210
iden



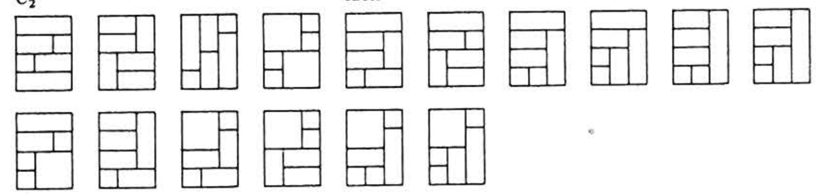
11310
iden



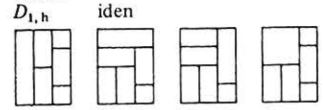
12120
iden



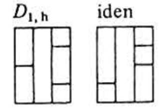
02220
C_2 iden



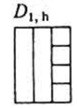
03030
$D_{1,h}$ iden



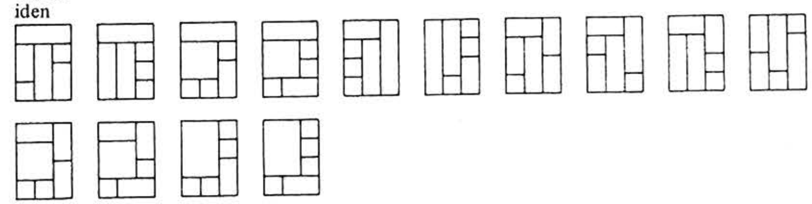
04101
$D_{1,h}$ iden



12201
$D_{1,h}$

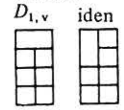


02301
iden

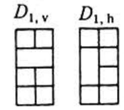


Order 7, Grating (2.4)

7 fronts
034000
$D_{1,v}$ iden

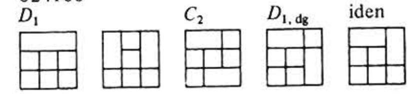


042100
$D_{1,v}$ $D_{1,h}$

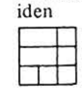


Order 7, Grating (3.3)

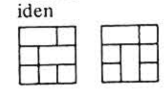
6 fronts
024100
D_1 C_2 $D_{1,dg}$ iden



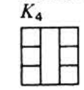
7 fronts
042100
iden



033010
iden

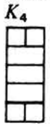


042001
K_4

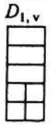


Order 7, Grating (2.5)

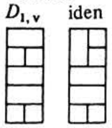
7 fronts
052000
K_4



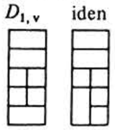
133000
$D_{1,v}$



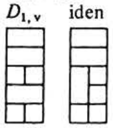
034000
$D_{1,v}$ iden



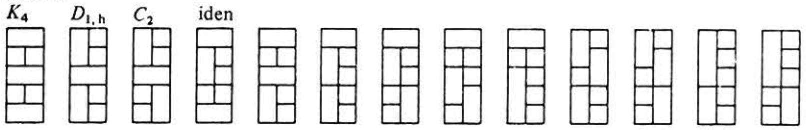
115000
$D_{1,v}$ iden



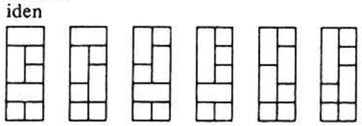
123100
$D_{1,v}$ iden



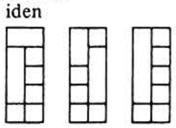
024100
K_4 $D_{1,h}$ C_2 iden



032200
iden

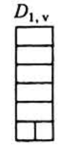


033010
iden

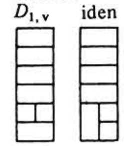


Order 7, Grating (2.6)

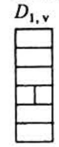
7 fronts
151000
$D_{1,v}$



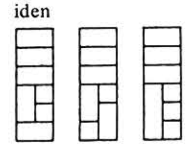
133000
$D_{1,v}$ iden



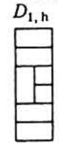
214000
$D_{1,v}$



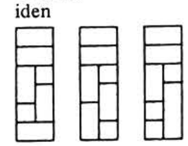
123100
iden



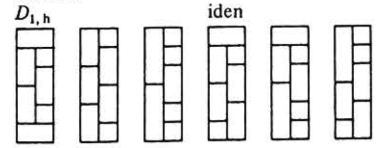
204100
$D_{1,h}$



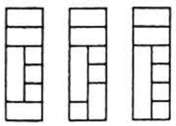
113200
iden



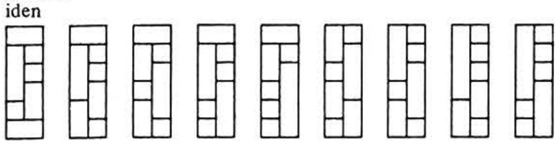
022300
$D_{1,h}$ iden



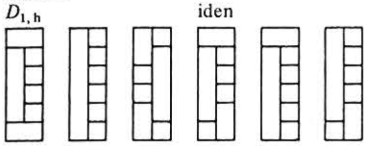
114010
iden



023110
iden

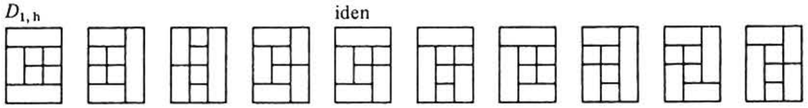


024001
$D_{1,h}$ iden

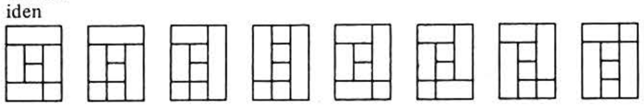


Order 7, Grating (3.4)

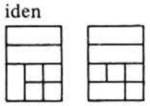
5 fronts
004300
$D_{1,h}$ iden



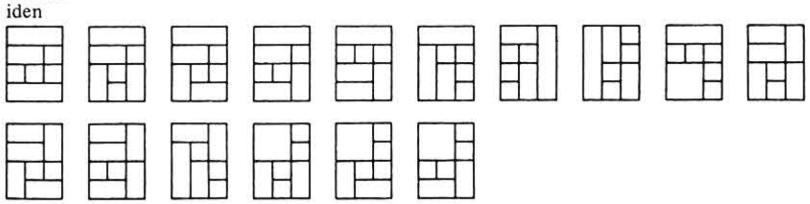
012400
iden



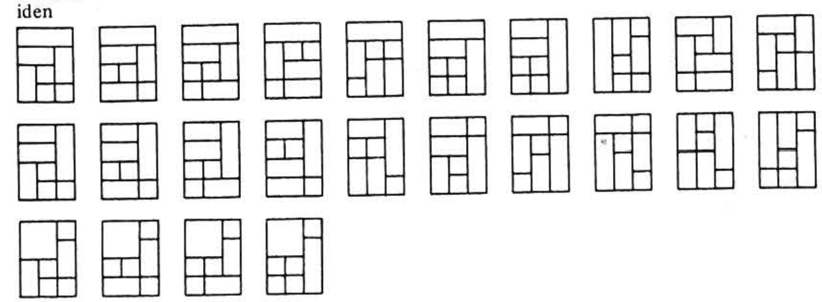
6 fronts
113200
iden



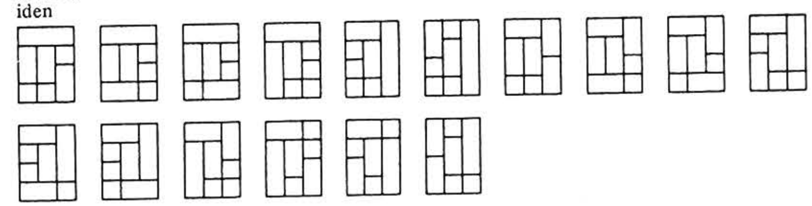
014200
iden



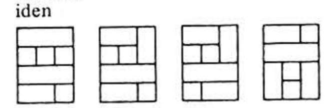
022300
iden



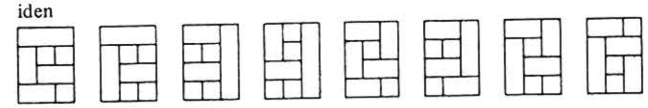
015010
iden



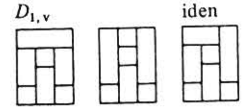
023110
iden



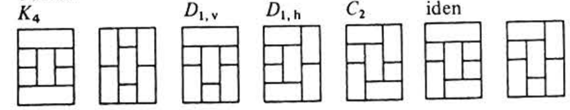
013210
iden



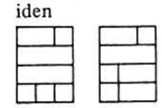
021310
$D_{1,v}$ iden



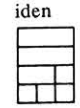
006001
K_4 $D_{1,v}$ $D_{1,h}$ C_2 iden



7 fronts
042100
iden



123100
iden



024100
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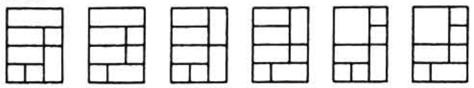
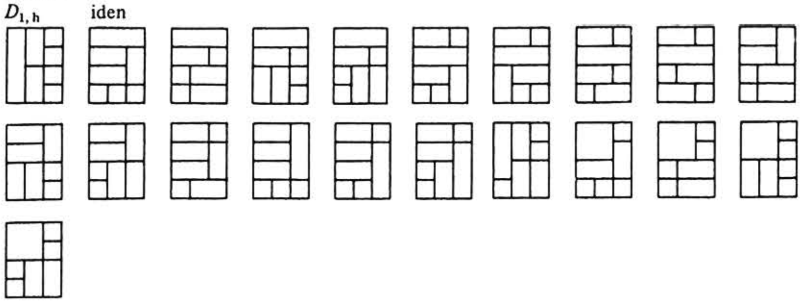
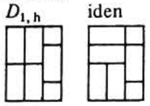


032200
$D_{1,h}$ iden



040300
$D_{1,h}$ iden



022300
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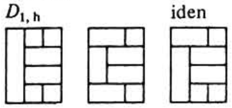
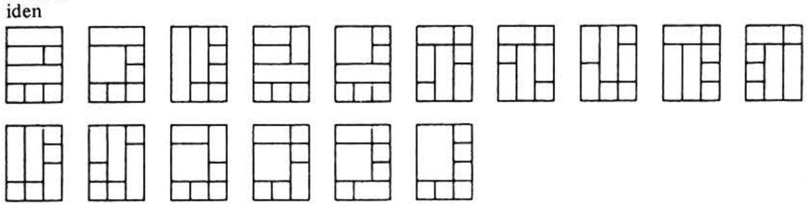
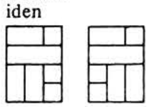


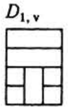
033010
iden



041110
iden



122110
$D_{1,v}$



023110
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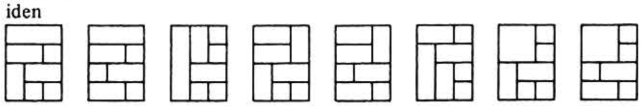
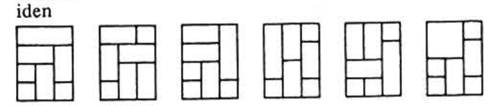
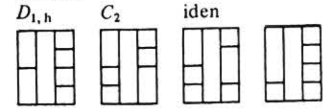


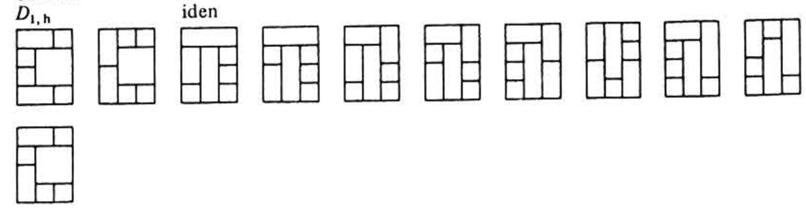
031210
iden



042001
$D_{1,h}$ C_2 iden

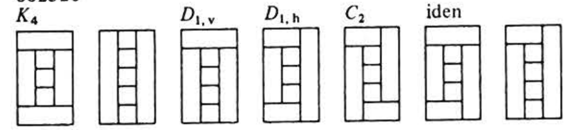


024001
$D_{1,h}$ iden

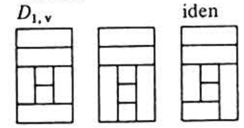


Order 7, Grating (3.5)

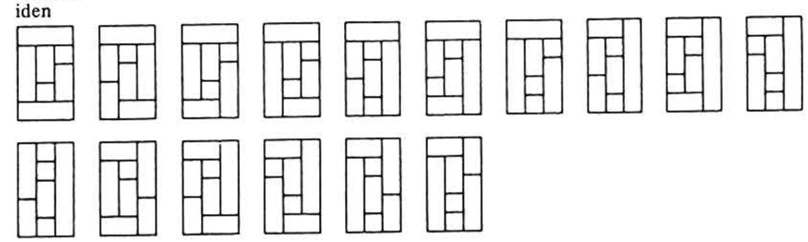
4 fronts
002320
K_4 $D_{1,v}$ $D_{1,h}$ C_2 iden



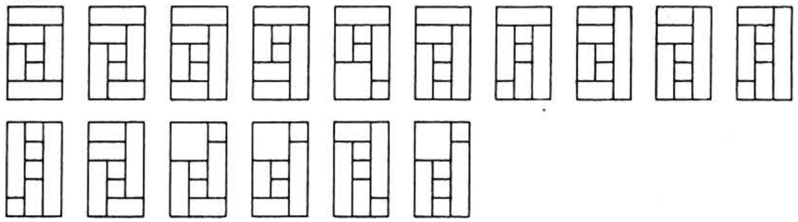
5 fronts
101500
$D_{1,v}$ iden



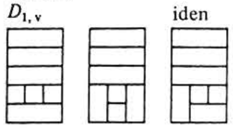
003310
iden



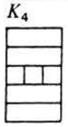
011410
iden



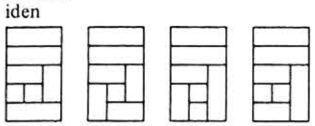
6 fronts
113200
$D_{1,v}$ iden



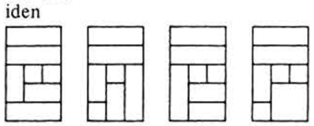
202300
K_4



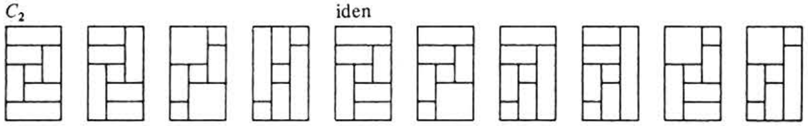
103300
iden



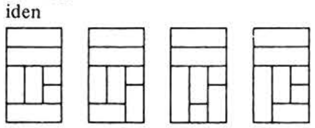
111400
iden



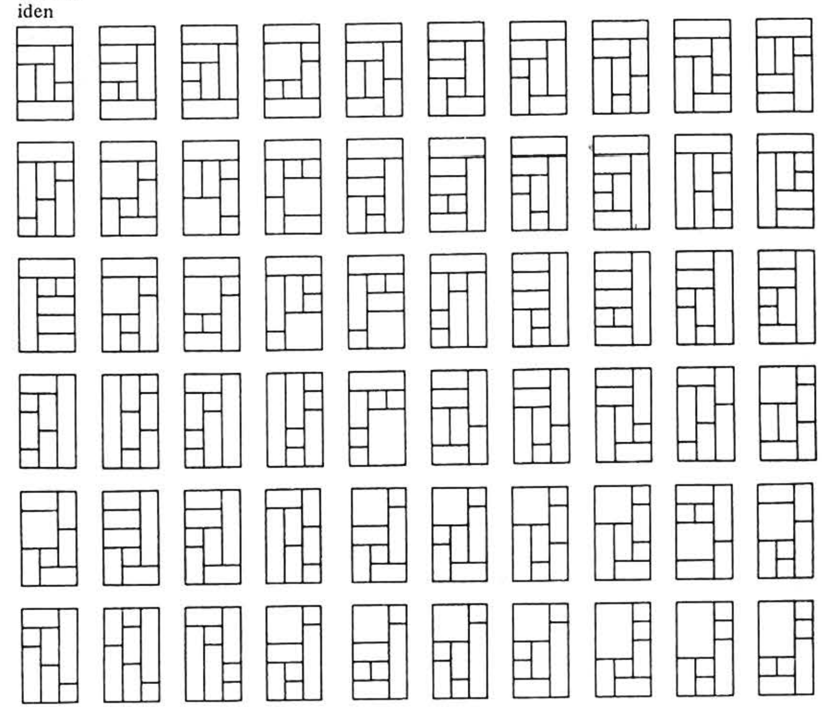
020500
C_2 iden



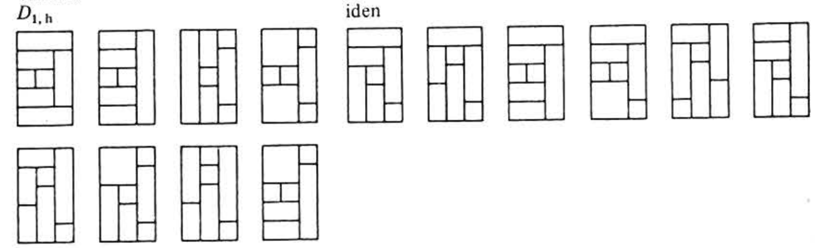
104110
iden



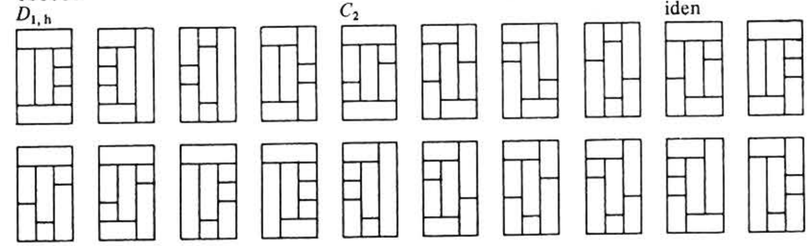
013210
iden



021310
$D_{1,h}$ iden

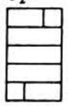


006001
$D_{1,h}$ C_2 iden

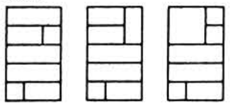


7 fronts

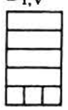
052000
C_2



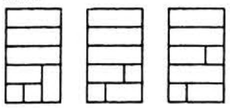
034000
iden



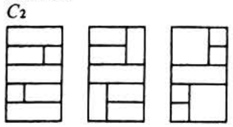
141100
$D_{1,v}$



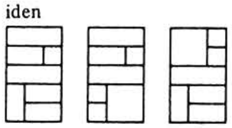
123100
iden



024100
C_2



iden



131200
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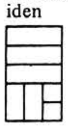
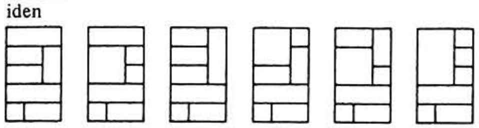
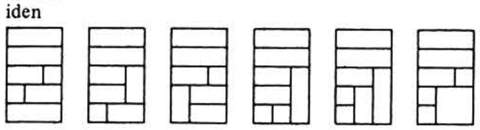


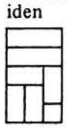
032200
iden



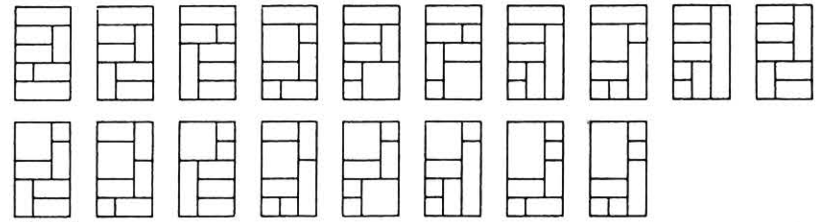
113200
iden



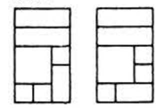
121300
iden



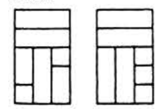
022300
iden



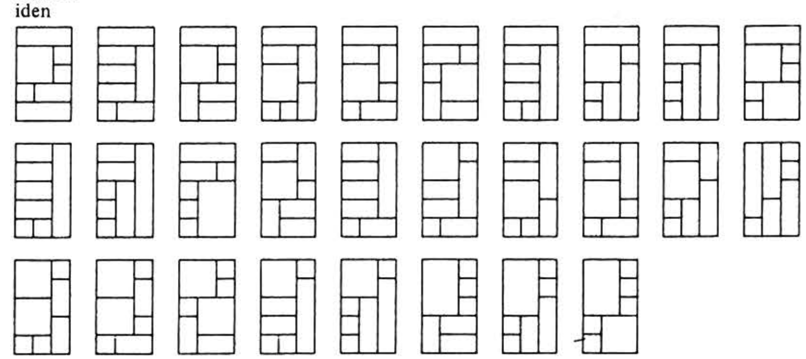
114010
iden



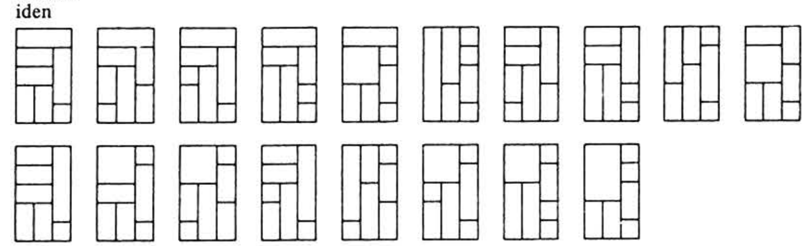
122110
iden



023110
iden

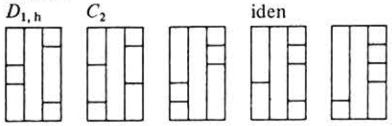


031210
iden



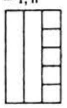
042001

$D_{1,h}$ C_2 iden



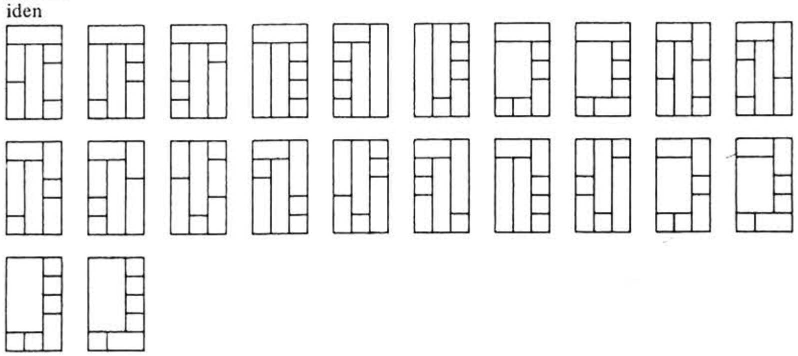
123001

$D_{1,h}$



024001

iden

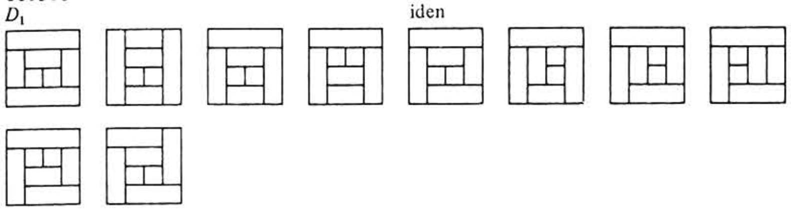


Order 7, Grating (4.4)

4 fronts

001510

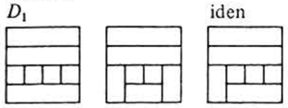
D_1 iden



5 fronts

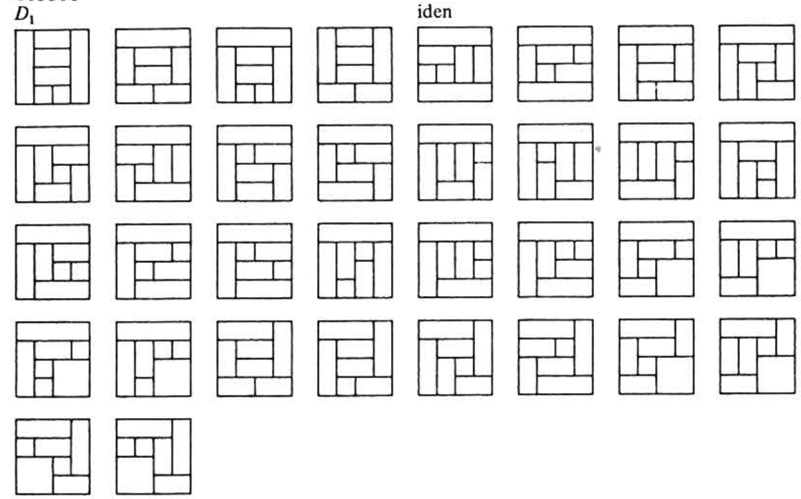
102310

D_1 iden



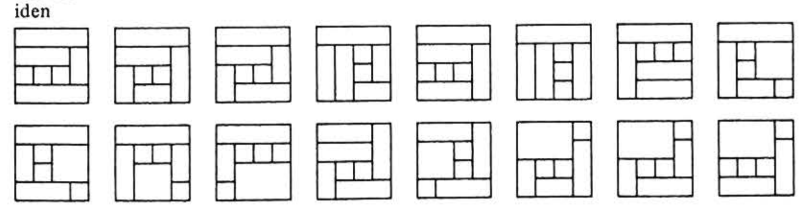
003310

D_1 iden



011410

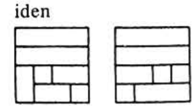
iden



6 fronts

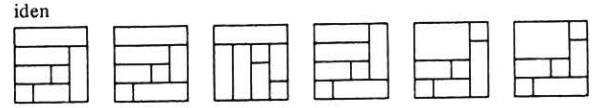
111400

iden



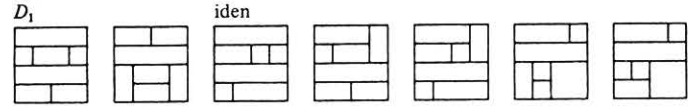
020500

iden



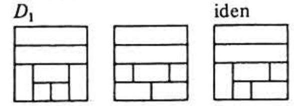
023110

D_1 iden

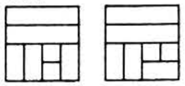


104110

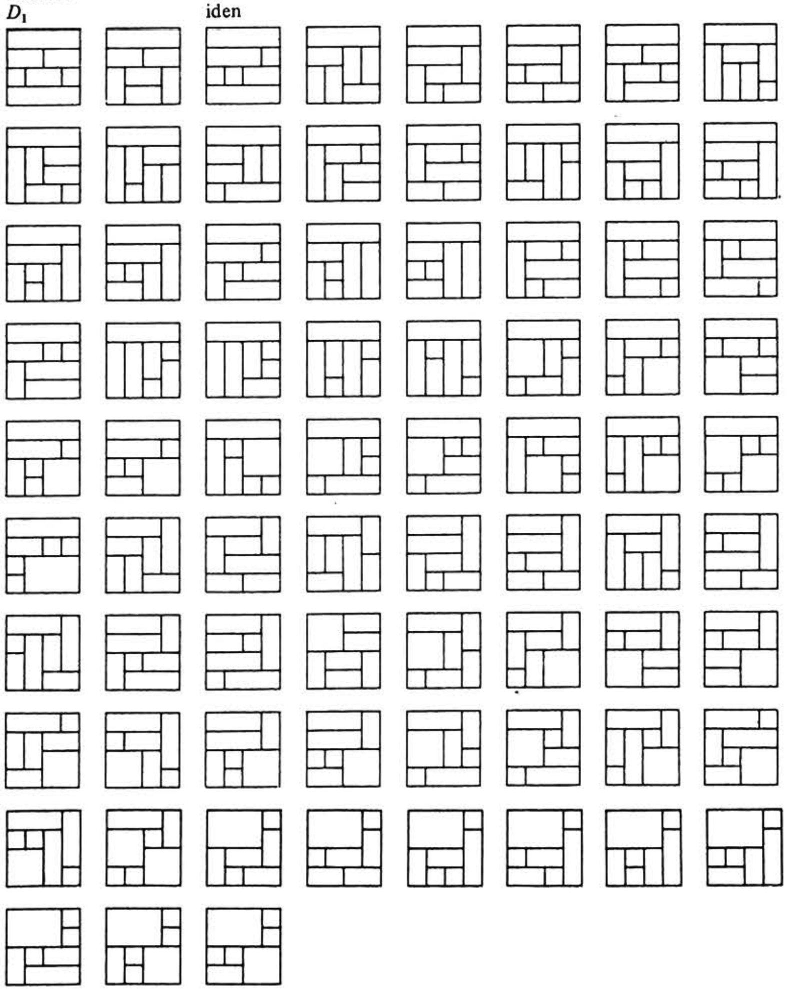
D_1 iden



112210
iden



013210
D_1 iden



021310
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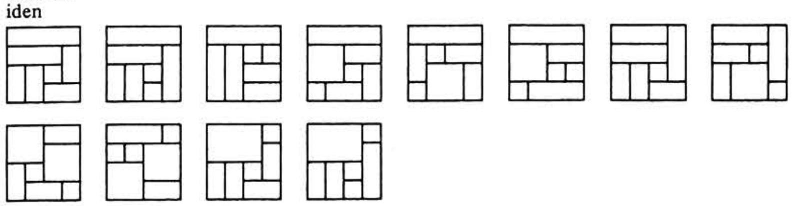

006001
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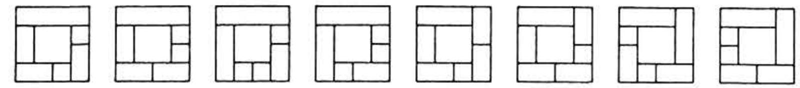
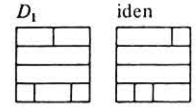


7 fronts
042100
D_1 iden



032200
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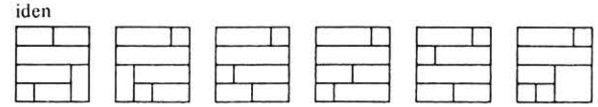
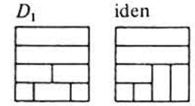


121300
D_1 iden



022300
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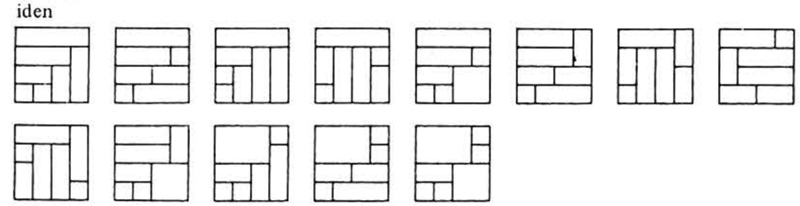
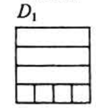
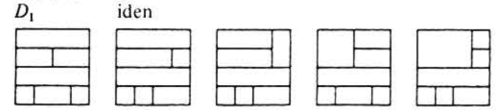


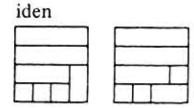
132010
D_1



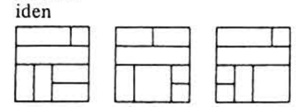
033010
D_1 iden



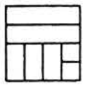
114010
iden



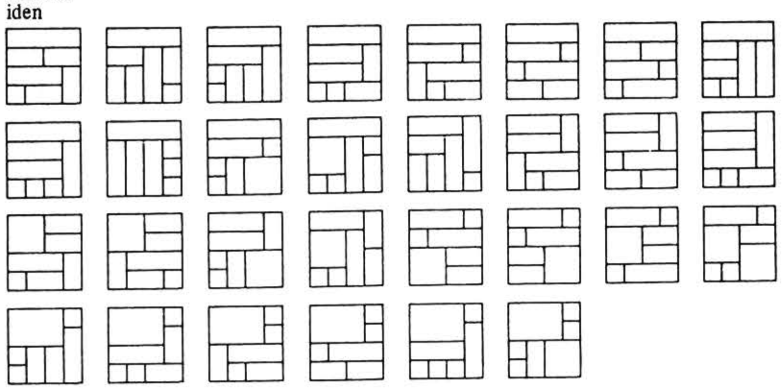
041110
iden



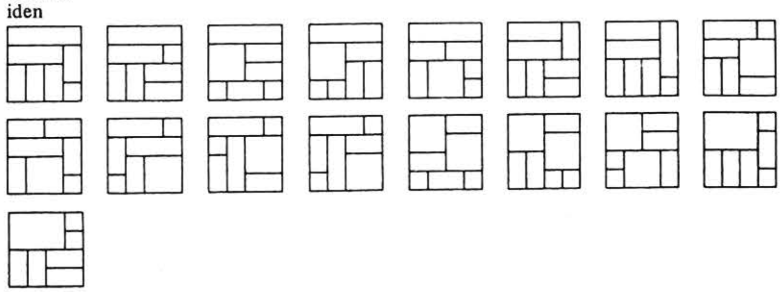
122110
iden



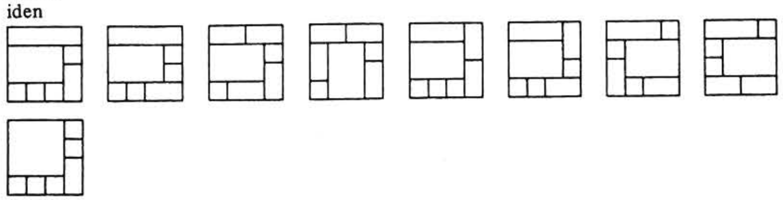
023110
iden



031210
iden



024001
iden



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Architectural Morphology
An introduction to the geometry of building plans

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Just sixty years ago the architect and writer W R Lethaby called for a programme of theoretical work on the geometry of architectural plans, which would as he said "... cover the field by a systematic research into possibilities". "The possibilities of walls and vaults, and of the relations between the walls and the cell, and between one cell and another, want investigating, as Lord Kelvin investigated the geometry of crystalline structures and the 'packing of cells'."

This book is the first introduction to an area of research which has grown up over the last ten years, and which has begun to answer Lethaby's call. The author shows how, given suitable geometrical definitions of certain classes of plans, systematic methods can be devised for enumerating *all possible plans* of each type. Particular attention is devoted to plans consisting of rectangular rooms, set within rectangular boundaries—so-called *rectangular dissections*—since the plans of many actual small buildings, especially houses, approximate to this kind of geometrical arrangement.

Computer methods for generating rectangular dissections are described in some detail and classes of plans with other geometries are also discussed. Mathematical techniques are introduced for the representation of plans and their properties, and, in particular, topological properties of the adjacencies between rooms are represented by using the theory of graphs.

The author goes on to show how these plan-generating methods, and the catalogues of plans which they can produce, may be applied in three areas: in design, in building science, and in the study of architectural history. Design methods are described by which it is possible to enumerate exhaustively all plans for small houses or apartments, conforming to given adjacency and dimensional requirements. In building science, applications of this morphological work are suggested to the study of circulation and environmental performance, to the subject of adaptability and flexibility in plans, and in the formal classification of building types. In architectural history, connections are discussed between this approach to the representation and classification of historical plans, and their interpretation in terms of construction, social function, and artistic style.

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