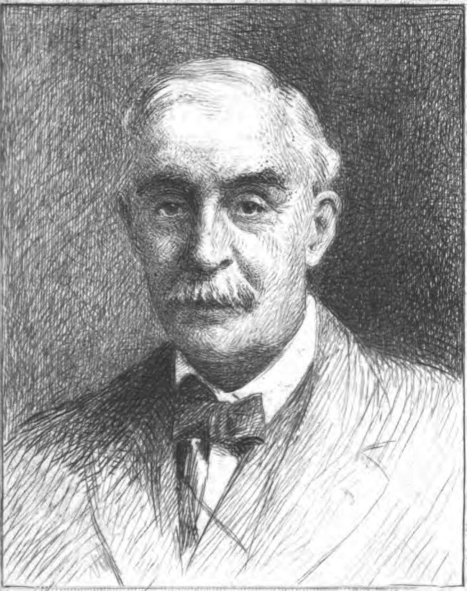
THE FOUNDATIONS OF MATHEMATICS

CARUS



WHW Bicknell del. et sc.

THE FOUNDATIONS OF MATHEMATICS

A CONTRIBUTION TO

THE PHILOSOPHY OF GEOMETRY

BY

DR. PAUL CARUS

ὁ θεὸς ἀεὶ γεωμετρεῖ.—PLATO.

CHICAGO
THE OPEN COURT PUBLISHING CO.
LONDON AGENTS
KEGAN PAUL, TRENCH, TRÜBNER & CO., LTD.
1908

11-22-38

TABLE OF CONTENTS.

THE SEARCH FOR THE FOUNDATIONS OF GEOMETRY: HISTORICAL SKETCH.

	PAGE
Axioms and the Axiom of Parallels.	1
Metageometry.	5
Precursors.	7
Gauss.	11
Riemann.	15
Lobatchevsky.	20
Bolyai.	22
Later Geometricians.	24
Grassmann.	27
Euclid Still Unimpaired.	31

THE PHILOSOPHICAL BASIS OF MATHEMATICS.

The Philosophical Problem.	35
Transcendentalism and Empiricism.	38
The A Priori and the Purely Formal.	40
Anyness and its Universality.	46
Apriority of Different Degrees.	49
Space as a Spread of Motion.	56
Uniqueness of Pure Space.	61
Mathematical Space and Physiological Space.	63
Homogeneity of Space Due to Abstraction.	66
Even Boundaries as Standards of Measurement.	69
The Straight Line Indispensable.	72
The Superreal.	76
Discrete Units and the Continuum.	78

MATHEMATICS AND METAGEOMETRY.

Different Geometrical Systems.	82
Tridimensionality.	84
Three a Concept of Boundary.	88

PAGE
Space of Four Dimensions. 90
The Apparent Arbitrariness of the A Priori. 96
Definiteness of Construction. 99
One Space, But Various Systems of Space Measurement... 104
Fictitious Spaces and the Apriority of All Space Measurement. 109
Infinitude. 116
Geometry Remains A Priori. 119
Sense-Experience and Space. 122
The Teaching of Mathematics. 127

EPILOGUE. 132

INDEX. 139

THE SEARCH FOR THE FOUNDATIONS OF GEOMETRY: HISTORICAL SKETCH.

AXIOMS AND THE AXIOM OF PARALLELS.

MATHEMATICS as commonly taught in our schools is based upon axioms. These axioms so called are a few simple formulas which the beginner must take on trust.

Axioms are defined to be self-evident propositions, and are claimed to be neither demonstrable nor in need of demonstration. They are statements which are said to command the assent of every one who comprehends their meaning.

The word axiom[1] means "honor, reputation, high rank, authority," and is used by Aristotle, almost in the modern sense of the term, as "a self-evident highest principle," or "a truth so obvious as to be in no need of proof." It is derived from the verb **ἀξιοῦν**, "to deem worthy, to think fit, to maintain," and is cognate with **ἄξιος**, "worth" or "worthy."

Euclid does not use the term "axiom." He starts with Definitions,[2] which describe the meanings of point, line, surface, plane, angle, etc. He

[1] *ἀξίωμα*. [2] *ὅροι*.

then proposes Postulates[3] in which he takes for granted that we can draw straight lines from any point to any other point, and that we can prolong any straight line in a straight direction. Finally, he adds what he calls Common Notions[4] which embody some general principles of logic (of pure reason) specially needed in geometry, such as that things which are equal to the same thing are equal to one another; that if equals be added to equals, the wholes are equal, etc.

I need not mention here perhaps, since it is a fact of no consequence, that the readings of the several manuscripts vary, and that some propositions (e. g., that all right angles are equal to one another) are now missing, now counted among the postulates, and now adduced as common notions.

The commentators of Euclid who did not understand the difference between Postulates and Common Notions, spoke of both as axioms, and even to-day the term Common Notion is mostly so translated.

In our modern editions of Euclid we find a statement concerning parallel lines added to either the Postulates or Common Notions. Originally it appeared in Proposition 29 where it is needed to prop up the argument that would prove the equality of alternate angles in case a third straight line falls upon parallel straight lines. It is there enunciated as follows:

"But those straight lines which, with another straight

[3] αἰτήματα. [4] κοιναὶ ἔννοιαι.

line falling upon them, make the interior angles on the same side less than two right angles, do meet if continually produced."

Now this is exactly a point that calls for proof. Proof was then, as ever since it has remained, altogether lacking. So the proposition was formulated dogmatically thus:

"If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles."

And this proposition has been transferred by the editors of Euclid to the introductory portion of the book where it now appears either as the fifth Postulate or the eleventh, twelfth, or thirteenth Common Notion. The latter is obviously the less appropriate place, for the idea of parallelism is assuredly not a Common Notion; it is not a rule of pure reason such as would be an essential condition of all thinking, reasoning, or logical argument. And if we do not give it a place of its own, it should either be classed among the postulates, or recast so as to become a pure definition. It is usually referred to as "the axiom of parallels."

It seems to me that no one can read the axiom of parallels as it stands in Euclid without receiving the impression that the statement was affixed by a later redactor. Even in Proposition 29, the original place of its insertion, it comes in as an afterthought; and if Euclid himself had considered the difficulty of the parallel axiom, so called, he would have placed it among the postulates in the first edition of his book, or formulated it as a definition.[5]

Though the axiom of parallels must be an interpolation, it is of classical origin, for it was known even to Proclus (410-485 A. D.), the oldest commentator of Euclid.

By an irony of fate, the doctrine of the parallel axiom has become more closely associated with Euclid's name than anything he has actually written, and when we now speak of Euclidean geometry we mean a system based upon that determination of parallelism.

We may state here at once that all the attempts made to derive the axiom of parallels from pure reason were necessarily futile, for no one can prove the absolute straightness of lines, or the evenness of space, by logical argument. Therefore these concepts, including the theory concerning parallels, cannot be derived from pure reason; they are not Common Notions and possess a character of their own. But the statement seemed thus to hang in the air, and there appeared the possibility of a geometry, and even of several geometries, in whose domains the parallel axiom would not hold good. This large field has been called metageometry, hyper-

[5] For Professor Halsted's ingenious interpretation of the origin of the parallel theorem see *The Monist*, Vol. IV, No. 4, p. 487. He believes that Euclid anticipated metageometry, but it is not probable that the man who wrote the argument in Proposition 29 had the fifth Postulate before him. He would have referred to it or stated it at least approximately in the same words. But the argument in Proposition 29 differs considerably from the parallel axiom itself.

geometry, or pangeometry, and may be regarded as due to a generalization of the space-conception involving what might be called a metaphysics of mathematics.

METAGEOMETRY.

Mathematics is a most conservative science. Its system is so rigid and all the details of geometrical demonstration are so complete, that the science was commonly regarded as a model of perfection. Thus the philosophy of mathematics remained undeveloped almost two thousand years. Not that there were not great mathematicians, giants of thought, men like the Bernoullis. Leibnitz and Newton, Euler, and others, worthy to be named in one breath with Archimedes, Pythagoras and Euclid, but they abstained from entering into philosophical speculations, and the very idea of a pangeometry remained foreign to them. They may privately have reflected on the subject, but they did not give utterance to their thoughts, at least they left no records of them to posterity.

It would be wrong, however, to assume that the mathematicians of former ages were not conscious of the difficulty. They always felt that there was a flaw in the Euclidean foundation of geometry, but they were satisfied to supply any need of basic principles in the shape of axioms, and it has become quite customary (I might almost say orthodox) to say that mathematics is based upon axioms. In fact, people enjoyed the idea that mathematics, the most lucid of all the sciences, was at bottom as mysterious as the most mystical dogmas of religious faith.

Metageometry has occupied a peculiar position among mathematicians as well as with the public at large. The mystic hailed the idea of "*n*-dimensional spaces," of "space curvature" and of other conceptions of which we can form expressions in abstract terms but which elude all our attempts to render them concretely present to our intelligence. He relished the idea that by such conceptions mathematics gave promise to justify all his speculations and to give ample room for a multitude of notions that otherwise would be doomed to irrationality. In a word, metageometry has always proved attractive to erratic minds. Among the professional mathematicians, however, those who were averse to philosophical speculation looked upon it with deep distrust, and therefore either avoided it altogether or rewarded its labors with bitter sarcasm. Prominent mathematicians did not care to risk their reputation, and consequently many valuable thoughts remained unpublished. Even Gauss did not care to speak out boldly, but communicated his thoughts to his most intimate friends under the seal of secrecy, not unlike a religious teacher who fears the odor of heresy. He did not mean to suppress his thoughts, but he did not want to bring them before the public unless in mature shape. A letter to Taurinus concludes with the remark:

"Of a man who has proved himself a thinking mathematician, I fear not that he will misunderstand what I say, but under all circumstances you have to regard it merely as a private communication of which in no wise public use, or one that may lead to it, is to be made. Perhaps I shall publish them myself in the future if I should gain more leisure than my circumstances at present permit.

"C. F. Gauss.

"Goettingen, 8. November, 1824."

But Gauss never did publish anything upon this topic although the seeds of his thought thereupon fell upon fertile ground and bore rich fruit in the works of his disciples, foremost in those of Riemann.

PRECURSORS.

The first attempt at improvement in the matter of parallelism was made by Nasir Eddin (1201-1274) whose work on Euclid was printed in Arabic in 1594 in Rome. His labors were noticed by John Wallis who in 1651 in a Latin translation communicated Nasir Eddin's exposition of the fifth Postulate to the mathematicians of the University of Oxford, and then propounded his own views in a lecture delivered on July 11, 1663. Nasir Eddin takes his stand upon the postulate that two straight lines which cut a third straight line, the one at right angles, the other at some other angle, will converge on the side where the angle is acute and diverge where it is obtuse. Wallis, in his endeavor to prove this postulate, starts with the auxiliary theorem:

"If a limited straight line which lies upon an unlimited straight line be prolonged in a straight direction, its prolongation will fall upon the unlimited straight line."

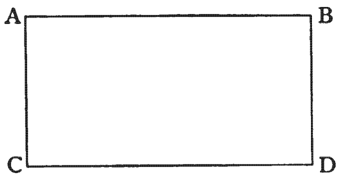
There is no need of entering into the details of his proof of this auxiliary theorem. We may call his theorem the proposition of the straight line and may grant to him that he proves the straightness of the straight line. In his further argument Wallis shows the close connection of the problem of parallels with the notion of similitude.

Girolamo Saccheri, a learned Jesuit of the seventeenth century, attacked the problem in a new way. Saccheri was born September 5, 1667, at San Remo. Having received a good education, he became a member of the Jesuit order March 24, 1685, and served as a teacher of grammar at the Jesuit College di Brera, in Milan, his mathematical colleague being Tommaso Ceva (a brother of the more famous Giovanni Ceva). Later on he became Professor of Philosophy and Polemic Theology at Turin and in 1697 at Pavia. He died in the College di Brera October 25, 1733.

Saccheri saw the close connection of parallelism with the right angle, and in his work on Euclid[6] he examines three possibilities. Taking a quadrilateral ABCD with the angles at A and B right angles and the sides AC and BD equal, the angles at C and D are without difficulty shown to be equal each to the other. They are moreover right angles or else they are either obtuse or acute. He undertakes to

[6] *Euclides ab omni naevo vindicatus; sive conatus geometricus quo stabiliuntur prima ipsa universae geometriae principia.* Auctore Hieronymo Saccherio Societatis Jesu. Mediolani, 1773.

prove the absurdity of these two latter suppositions so as to leave as the only solution the sole possibility left, viz., that they must be right angles. But he finds difficulty in pointing out the contradiction to which these assumptions may lead and thus he opens a path on which Lobatchevsky (1793-1856) and Bolyai (1802-1860) followed, reaching a new view which makes three geometries possible, viz., the geometries of (1) the acute angle, (2) the obtuse angle, and (3) the right angle, the latter being the Euclidean geometry, in which the theorem of parallels holds.



While Saccheri seeks the solution of the problem through the notion of the right angle, the German mathematician Lambert starts from the notion of the angle-sum of the triangle.

Johann Heinrich Lambert was born August 26, 1728, in Mühlhausen, a city which at that time was a part of Switzerland. He died in 1777. His *Theory of the Parallel Lines*, written in 1766, was not published till 1786, nine years after his death, by Bernoulli and Hindenburg in the *Magazin für die reine und angewandte Mathematik.*

Lambert points out that there are three possibilities: the sum of the angles of a triangle may be exactly equal to, more than, or less than 180 degrees. The first will make the triangle a figure in a plane, the second renders it spherical, and the third produces a geometry on the surface of an imaginary sphere. As to the last hypothesis Lambert said not without humor:[7]

"This result[8] possesses something attractive which easily suggests the wish that the third hypothesis might be true."

He then adds:[9]

"But I do not wish it in spite of these advantages, because there would be innumerable other inconveniences. The trigonometrical tables would become infinitely more complicated, and the similitude as well as proportionality of figures would cease altogether. No figure could be represented except in its own absolute size; and astronomy would be in a bad plight, etc."

Lobatchevsky's geometry is an elaboration of Lambert's third hypothesis, and it has been called "imaginary geometry" because its trigonometric formulas are those of the spherical triangle if its sides are imaginary, or, as Wolfgang Bolyai has shown, if the radius of the sphere is assumed to be imaginary $=(\sqrt{-1})r$.

France has contributed least to the literature on the subject. Augustus De Morgan records the following story concerning the efforts of her greatest mathematician to solve the Euclidean problem. La-

[7] P. 351, last line in the *Magazin für die reine und angewandte Mathematik*, 1786.

[8] Lambert refers to the proposition that the mooted angle might be less than 90 degrees.

[9] *Ibid.*, p. 352.

grange, he says, composed at the close of his life a discourse on parallel lines. He began to read it in the Academy but suddenly stopped short and said: "Il faut que j'y songe encore." With these words he pocketed his papers and never recurred to the subject.

Legendre's treatment of the subject appears in the third edition of his elements of Euclid, but he omitted it from later editions as too difficult for beginners. Like Lambert he takes his stand upon the notion of the sum of the angles of a triangle, and like Wallis he relies upon the idea of similitude, saying that "the length of the units of measurement is indifferent for proving the theorems in question."[10]

GAUSS.

A new epoch begins with Gauss, or rather with his ingenious disciple Riemann. While Gauss was rather timid about speaking openly on the subject, he did not wish his ideas to be lost to posterity. In a letter to Schumacher dated May 17, 1831, he said:

> "I have begun to jot down something of my own meditations, which are partly older than forty years, but which I have never written out, being obliged therefore to excogitate many things three or four times over. I do not wish them to pass away with me."

The notes to which Gauss here refers have not been found among his posthumous papers, and it

[10] *Mémoires de l'Académie des Sciences de l'Institut de France.* Vol. XII, 1833.

therefore seems probable that they are lost, and our knowledge of his thoughts remains limited to the comments that are scattered through his correspondence with mathematical friends.

Gauss wrote to Bessel (1784-1846) January 27, 1829:

"I have also in my leisure hours frequently reflected upon another problem, now of nearly forty years' standing. I refer to the foundations of geometry. I do not know whether I have ever mentioned to you my views on this matter. My meditations here also have taken more definite shape, and my conviction that we cannot thoroughly demonstrate geometry *a priori* is, if possible, more strongly confirmed than ever. But it will take a long time for me to bring myself to the point of working out and making public my *very extensive* investigations on this subject, and possibly this will not be done during my life, inasmuch as I stand in dread of the clamors of the Bœotians, which would be certain to arise, if I should ever give *full* expression to my views. It is curious that *in addition to* the celebrated flaw in Euclid's Geometry, which mathematicians have hitherto endeavored in vain to patch and never will succeed, there is still another blotch in its fabric to which, so far as I know, attention has never yet been called and which it will by no means be easy, if at all possible, to remove. This is the definition of a plane as a surface in which a straight line joining *any two* points lies *wholly* in that plane. This definition contains *more* than is requisite to the determination of a surface, and tacitly involves a theorem which is in need of prior proof."

Bessel in his answer to Gauss makes a distinction between Euclidean geometry as practical and metageometry (the one that does not depend upon the theorem of parallel lines) as true geometry. He writes under the date of February 10, 1829:

"I should regard it as a great misfortune if you were to allow yourself to be deterred by the 'clamors of the Bœotians' from explaining your views of geometry. From what Lambert has said and Schweikart orally communicated, it has become clear to me that our geometry is incomplete and stands in need of a correction which is hypothetical and which vanishes when the sum of the angles of a plane triangle is equal to 180°. This would be the *true* geometry and the Euclidean the *practical*, at least for figures on the earth."

In another letter to Bessel, April 9, 1830, Gauss sums up his views as follows:

"The ease with which you have assimilated my notions of geometry has been a source of genuine delight to me, especially as so few possess a natural bent for them. I am profoundly convinced that the theory of space occupies an entirely different position with regard to our knowledge *a priori* from that of the theory of numbers (*Grössenlehre*); that perfect conviction of the necessity and therefore the absolute truth which is characteristic of the latter is totally wanting to our knowledge of the former. We must confess in all humility that a number is *solely* a product of our mind. Space, on the other hand, possesses also a reality outside of our mind, the laws of which we cannot fully prescribe *a priori*."

Another letter of Gauss may be quoted here in full. It is a reply to Taurinus and contains an appreciation of his essay on the Parallel Lines. Gauss writes from Göttingen, Nov. 8, 1824:

"Your esteemed communication of October 30th, with the accompanying little essay, I have read with considerable pleasure, the more so as I usually find no trace whatever of real geometrical talent in the majority of the people who offer new contributions to the so-called theory of parallel lines.

"With regard to your effort, I have nothing (or not much) more to say, except that it is incomplete. Your presentation of the demonstration that the sum of the three angles of a plane triangle cannot be greater than 180°, does indeed leave something to be desired in point of geometrical precision. But this could be supplied, and there is no doubt that the impossibility in question admits of the most rigorous demonstration. But the case is quite different with the second part, viz., that the sum of the angles cannot be smaller than 180°; this is the real difficulty, the rock on which all endeavors are wrecked. I surmise that you have not employed yourself long with this subject. I have pondered it for more than thirty years, and I do not believe that any one could have concerned himself more exhaustively with this second part than I, although I have not published anything on this subject. The assumption that the sum of the three angles is smaller than 180° leads to a new geometry entirely different from our Euclidean,—a geometry which is throughout consistent with itself, and which I have elaborated in a manner entirely satisfactory to myself, so that I can solve every problem in it with the exception of the determining of a constant, which is not *a priori* obtainable. The larger this constant is taken, the nearer we approach the Euclidean geometry, and an infinitely large value will make the two coincident. The propositions of this geometry appear partly paradoxical and absurd to the uninitiated, but on closer and calmer consideration it will be found that they contain in them absolutely nothing that is impossible. Thus, the three angles of a triangle, for example, can be made as small as we will, provided the sides can be taken large enough; whilst the area of a triangle, however great the sides may be taken, can never exceed a definite limit, nay, can never once reach it. All my endeavors to discover contradictions or inconsistencies in this non-Euclidean geometry have been in vain, and the only thing in it that conflicts with our reason is the fact that if it were true there would necessarily exist in space a linear magnitude quite *determinate in itself*, yet unknown to us. But I opine that, despite the empty word-wisdom of the metaphysicians, in reality we know little or nothing of the true nature of space, so much so that we are not at liberty to characterize as *absolutely impossible* things that strike us as unnatural. If the non-Euclidean geometry were the true geometry, and the constant in a certain ratio to such magnitudes as lie within the reach of our measurements on the earth and in the heavens, it could be determined *a posteriori*. I have, therefore, in jest frequently expressed the desire that the Euclidean geometry should not be the true geometry, because in that event we should have an absolute measure *a priori*."

Schweikart, a contemporary of Gauss, may incidentally be mentioned as having worked out a geometry that would be independent of the Euclidean axiom. He called it astral geometry.[11]

RIEMANN.

Gauss's ideas fell upon good soil in his disciple Riemann (1826-1866) whose Habilitation Lecture on "The Hypotheses which Constitute the Bases of Geometry" inaugurates a new epoch in the history of the philosophy of mathematics.

Riemann states the situation as follows. I quote

[11] *Die Theorie der Parallellinien, nebst dem Vorschlag ihrer Verbannung aus der Geometrie.* Leipsic and Jena, 1807.

from Clifford's almost too literal translation (first published in *Nature*, 1873):

"It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible.

"From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it."

Riemann arrives at a conclusion which is negative. He says:

"The propositions of geometry cannot be derived from general notions of magnitude, but the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience."

In the attempt at discovering the simplest matters of fact from which the measure-relations of space may be determined, Riemann declares that—

"Like all matters of fact, they are not necessary, but only of empirical certainty; they are hypotheses."

Being a mathematician, Riemann is naturally bent on deductive reasoning, and in trying to find a foothold in the emptiness of pure abstraction he starts with general notions. He argues that position must be determined by measuring quantities, and this necessitates the assumption that length of lines is independent of position. Then he starts with the notion of manifoldness, which he undertakes to specialize. This specialization, however, may be done in various ways. It may be continuous, as is geometrical space, or consist of discrete units, as do arithmetical numbers. We may construct manifoldnesses of one, two, three, or n dimensions, and the elements of which a system is constructed may be functions which undergo an infinitesimal displacement expressible by dx. Thus spaces become possible in which the directest linear functions (analogous to the straight lines of Euclid) cease to be straight and suffer a continuous deflection which may be positive or negative, increasing or decreasing.

Riemann argues that the simplest case will be, if the differential line-element ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx in which the coefficients are continuous functions of the quantities x, viz., $ds = \sqrt{\Sigma dx^2}$, but it is one instance only of a whole class of possibilities. He says:

> "Manifoldnesses in which, as in the plane and in space, the line-element may be reduced to the form $\sqrt{\Sigma dx^2}$, are therefore only a particular case of the manifoldnesses to be here investigated; they require a special name, and therefore these manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call *flat*."

The Euclidean plane is the best-known instance of flat space being a manifold of a zero curvature.

Flat or even space has also been called by the new-fangled word *homaloidal*,[12] which recommends itself as a technical term in distinction from the popular meaning of even and flat.

In applying his determination of the general notion of a manifold to actual space, Riemann expresses its properties thus:

"In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*; the former belongs to the extent-relations, the latter to the measure relations. That space is an unbounded threefold manifoldness, is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is forever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means follows from this; on the other hand, if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite, provided this curvature has ever so small a positive value. If we prolong all the geodetics starting in a given surface-element, we should obtain an unbounded surface of constant curvature, i. e., a surface which in a *flat* manifoldness of three dimensions would take the form of a sphere, and consequently be finite."

It is obvious from these quotations that Riemann is a disciple of Kant. He is inspired by his teacher Gauss and by Herbart. But while he starts a transcendentalist, employing mainly the method of deductive reasoning, he arrives at results which

[12] From the Greek ὁμαλός, level.

would stamp him an empiricist of the school of Mill. He concludes that the nature of real space, which is only one instance among many possibilities, must be determined *a posteriori.* The problem of tri-dimensionality and homaloidality are questions which must be decided by experience, and while upon the whole he seems inclined to grant that Euclidean geometry is the most practical for a solution of the coarsest investigations, he is inclined to believe that real space is non-Euclidean. Though the deviation from the Euclidean standard can only be slight, there is a possibility of determining it by exact measurement and observation.

Riemann has succeeded in impressing his view upon meta-geometricians down to the present day. They have built higher and introduced new ideas, yet the cornerstone of metageometry remained the same. It will therefore be found recommendable in a discussion of the problem to begin with a criticism of his Habilitation Lecture.

It is regrettable that Riemann was not allowed to work out his philosophy of mathematics. He died at the premature age of forty, but the work which he pursued with so much success had already been taken up by two others, Lobatchevsky and Bolyai, who, each in his own way, actually contrived a geometry independent of the theorem of parallels.

It is perhaps no accident that the two independent and almost simultaneous inventors of a non-Euclidean geometry are original, not to say wayward, characters living on the outskirts of European civilization, the one a Russian, the other a Magyar.

LOBATCHEVSKY.

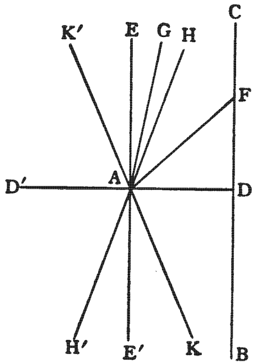
Nicolai Ivanovich Lobatchevsky[13] was born October 22 (Nov. 2 of our calendar), 1793, in the town of Makariev, about 40 miles above Nijni Novgorod on the Volga. His father was an architect who died in 1797, leaving behind a widow and two small sons in poverty. At the gymnasium Lobatchevsky was noted for obstinacy and disobedience, and he escaped expulsion only through the protection of his mathematical teacher, Professor Bartels, who even then recognized the extraordinary talents of the boy. Lobatchevsky graduated with distinction and became in his further career professor of mathematics and in 1827 Rector of the University of Kasan. Two books of his offered for official publication were rejected by the paternal government of Russia, and the manuscripts may be considered as lost for good. Of his several essays on the theories of parallel lines we mention only the one which made him famous throughout the whole mathematical world, *Geometrical Researches on the Theory of Parallels,* published by the University of Kasan in 1835.[14]

Lobatchevsky divides all lines, which in a plane go out from a point A with reference to a given

[13] The name is spelled differently according to the different methods of transcribing Russian characters.

[14] For further details see Prof. G. B. Halsted's article "Lobachévski" in *The Open Court,* 1898, pp. 411 ff.

straight line BC in the same plane, into two classes —cutting and not cutting. In progressing from the not-cutting lines, such as EA and GA, to the cutting lines, such as FA, we must come upon one HA that is the boundary between the two classes; and it is this which he calls the parallel line. He designates the parallel angle on the perpendicular (p= AD, dropped from A upon BC) by Π. If $\Pi(p) < \frac{1}{2}\pi$ (viz., 90 degrees) we shall have on the other side of p another angle DAK=$\Pi(p)$ parallel to DB, so that on this assumption we must make a distinction of *sides in parallelism,* and we must allow two parallels, one on the one and one on the other side. If $\Pi(p) = \frac{1}{2}\pi$ we have only intersecting lines and one parallel; but if $\Pi(p) < \frac{1}{2}\pi$ we have two parallel lines as boundaries between the intersecting and non-intersecting lines.



We need not further develop Lobatchevsky's idea. Among other things, he proves that "if in any rectilinear triangle the sum of the three angles is equal to two right angles, so is this also the case for every other triangle," that is to say, each instance is a sample of the whole, and if one case is established, the nature of the whole system to which it belongs is determined.

The importance of Lobatchevsky's discovery consists in the fact that the assumption of a geometry from which the parallel axiom is rejected, does not lead to self-contradictions but to the conception of a general geometry of which the Euclidean is one possibility. This general geometry was later on most appropriately called by Lobatchevsky "Pangeometry."

BOLYAI.

John (or, as the Hungarians say, János) Bolyai imbibed the love of mathematics in his father's house. He was the son of Wolfgang (or Farkas) Bolyai, a fellow student of Gauss at Göttingen when the latter was nineteen years old. Farkas was professor of mathematics at Maros Vásárhely and wrote a two-volume book on the elements of mathematics[15] and in it he incidentally mentions his vain attempts at proving the axiom of parallels. His book was only partly completed when his son János wrote him of his discovery of a mathematics of pure space. He said:

"As soon as I have put it into order, I intend to write and if possible to publish a work on parallels. At this moment, it is not yet finished, but the way which I have followed promises me with certainty the attainment of my aim, if it is at all attainable. It is not yet attained, but I have discovered such magnificent things that I am myself astonished at the result. It would forever be a pity, if they were lost. When you see them, my father, you yourself will con-

[15] *Tentamen juventutem studiosam in elementa matheos* etc. *introducendi,* printed in Maros Vásárhely. By Farkas Bolyai. Part I. Maros Vásárhely, 1832. It contains the essay by János Bolyai as an Appendix.

cede it. Now I cannot say more, only so much that *from nothing I have created another wholly new world.* All that I have hitherto sent you compares to it as a house of cards to a castle."[16]

János being convinced of the futility of proving Euclid's axiom, constructed a geometry of absolute space which would be independent of the axiom of parallels. And he succeeded. He called it the *Science Absolute of Space*,[17] an essay of twenty-four pages which Bolyai's father incorporated in the first volume of his *Tentamen* as an appendix.

Bolyai was a thorough Magyar. He was wont to dress in high boots, short wide Hungarian trousers, and a white jacket. He loved the violin and was a good shot. While serving as an officer in the Austrian army, János was known for his hot temper, which finally forced him to resign his commission as a captain, and we learn from Professor Halsted that for some provocation he was challenged by thirteen cavalry officers at once. János calmly accepted and proposed to fight them all, one after the other, on condition that he be permitted after each duel to play a piece on his violin. We know not the nature of these duels nor the construction of the pistols, but the fact remains assured that he came out unhurt. As for the rest of the report that "he came out victor from the thirteen duels,

[16] See Halsted's introduction to the English translation of Bolyai's *Science Absolute of Space*, p. xxvii.

[17] *Appendix scientiam spatii absolute veram exhibens: a veritate aut falsitate axiomatis XI. Euclidei (a priori haud unquam decidenda) independentem; Adjecta ad casum falsitatis quadratura circuli geometrica.*

leaving his thirteen adversaries on the square," we may be permitted to express a mild but deep-seated doubt.

János Bolyai starts with straight lines in the same plane, which may or may not cut each other. Now there are two possibilities: there may be a system in which straight lines can be drawn which do not cut one another, and another in which they all cut one another. The former, the Euclidean he calls Σ, the latter S. "All theorems," he says, "which are not expressly asserted for Σ or for S are enunciated absolutely, that is, they are true whether Σ or S is reality."[18] The system S can be established without axioms and is actualized in spherical trigonometry, (*ibid.* p. 21). Now S can be changed to , viz., plane geometry, by reducing the constant i to its limit (where the sect $y = 0$) which is practically the same as the construction of a circle with $r = \infty$, thus changing its periphery into a straight line.

LATER GEOMETRICIANS.

The labors of Lobatchevsky and Bolyai are significant in so far as they prove beyond the shadow of a doubt that a construction of geometries other than Euclidean is possible and that it involves us in no absurdities or contradictions. This upset the traditional trust in Euclidean geometry as absolute truth, and it opened at the same time a vista of new

[18] See Halsted's translation, p. 14.

problems, foremost among which was the question as to the mutual relation of these three different geometries.

It was Cayley who proposed an answer which was further elaborated by Felix Klein. These two ingenious mathematicians succeeded in deriving by projection all three systems from one common aboriginal form called by Klein *Grundgebild* or the Absolute. In addition to the three geometries hitherto known to mathematicians, Klein added a fourth one which he calls *elliptic*.[19]

Thus we may now regard all the different geometries as three species of one and the same genus and we have at least the satisfaction of knowing that there is *terra firma* at the bottom of our mathematics, though it lies deeper than was formerly supposed.

Prof. Simon Newcomb of Johns Hopkins University, although not familiar with Klein's essays, worked along the same line and arrived at similar results in his article on "Elementary Theorems Relating to the Geometry of a Space of Three Dimensions and of Uniform Positive Curvature in the Fourth Dimension.[20]

In the meantime the problem of geometry became interesting to outsiders also, for the theorem of parallel lines is a problem of space. A most

[19] "Ueber die sogenannte nicht-euklidische Geometrie" in *Math. Annalen*, 4, 6 (1871-1872). *Vorlesungen über nicht-euklidische Geometrie*, Göttingen, 1893.

[20] Crelle's *Journal für die reine und angewandte Mathematik*. 1877.

excellent treatment of the subject came from the pen of the great naturalist Helmholtz who wrote two essays that are interesting even to outsiders because written in a most popular style.[21]

A collection of all the materials from Euclid to Gauss, compiled by Paul Stäckel and Friedrich Engel under the title *Die Theorie der Parallellinien von Euklid bis auf Gauss, eine Urkundensammlung zur Vorgeschichte der nicht-euklidischen Geometrie,* is perhaps the most useful and important publication in this line of thought, a book which has become indispensable to the student of metageometry and its history.

A store of information may be derived from Bertrand A. W. Russell's essay on the *Foundations of Geometry*. He divides the history of metageometry into three periods: The synthetic, consisting of suggestions made by Legendre and Gauss; the metrical, inaugurated by Riemann and characterized by Lobatchevsky and Bolyai; and the projective, represented by Cayley and Klein, who reduce metrical properties to projection and thus show that Euclidean and non-Euclidean systems may result from "the absolute."

Among American writers no one has contributed more to the interests of metageometry than the indefatigable Dr. George Bruce Halsted.[22] He has

[21] "Ueber die thatsächlichen Grundlagen der Geometrie," in *Wissenschaftl. Abh.*., 1866, Vol. II., p. 610 ff., and "Ueber die Thatsachen, die der Geometrie zu Grunde liegen," *ibid.*, 1868, p. 618 ff.

[22] From among his various publications we mention only his translations: *Geometrical Researches on the Theory of Parallels by Nicholaus Lobatchewsky*. Translated from the Original. And *The*

not only translated Bolyai and Lobatchevsky, but in numerous articles and lectures advanced his own theories toward the solution of the problem.

Prof. B. J. Delbœuf and Prof. H. Poincaré have expressed their conceptions as to the nature of the bases of mathematics, in articles contributed to *The Monist*.[23] The latter treats the subject from a purely mathematical standpoint, while Dr. Ernst Mach in his little book *Space and Geometry*,[24] in the chapter "On Physiological, as Distinguished from Geometrical, Space," attacks the problem in a very original manner and takes into consideration mainly the natural growth of space conception. His exposition might be called "the physics of geometry."

GRASSMANN.

I cannot conclude this short sketch of the history of metageometry without paying a tribute to the memory of Hermann Grassmann of Stettin, a mathematician of first degree whose highly important results in this line of work have only of late found

Science Absolute of Space, Independent of the Truth or Falsity of Euclid's Axiom XI. (which can never be decided a priori). By John Bolyai. Translated from the Latin, both published in Austin, Texas, the translator's former place of residence. Further, we refer the reader to Halsted's bibliography of the literature on hyperspace and non-Euclidean geometry in the *American Journal of Mathematics*, Vol. I., pp. 261-276, 384, 385, and Vol. II., pp. 65-70.

[23] They are as follows: "Are the Dimensions of the Physical World Absolute?" by Prof. B. J. Delbœuf, *The Monist*, January, 1894; "On the Foundations of Geometry," by Prof. H. Poincaré, *The Monist*, October, 1898; also "Relations Between Experimental and Mathematical Physics," *The Monist*, July, 1902.

[24] Chicago: Open Court Pub. Co., 1906. This chapter also appeared in *The Monist* for April, 1901.

the recognition which they so fully deserve. I do not hesitate to say that Hermann Grassmann's *Lineare Ausdehnungslehre* is the best work on the philosophical foundation of mathematics from the standpoint of a mathematician.

Grassmann establishes first the idea of mathematics as the science of pure form. He shows that the mathematician starts from definitions and then proceeds to show how the product of thought may originate either by the single act of creation, or by the double act of positing and combining. The former is the continuous form, or magnitude, in the narrower sense of the term, the latter the discrete form or the method of combination. He distinguishes between intensive and extensive magnitude and chooses as the best example of the latter the sect[25] or limited straight line laid down in some definite direction. Hence the name of the new science, "theory of linear extension."

Grassmann constructs linear formations of which systems of one, two, three, and n degrees are possible. The Euclidean plane is a system of second degree, and space a system of third degree. He thus generalizes the idea of mathematics, and having created a science of pure form, points out that geometry is one of its applications which originates under definite conditions.

Grassmann made the straight line the basis of

[25] Grassmann's term is *Strecke,* a word connected with the Anglo-Saxon "Stretch," being that portion of a line that stretches between two points. The translation "sect," was suggested by Prof. G. B. Halsted.

his geometrical definitions. He defines the plane as the totality of parallels which cut a straight line and space as the totality of parallels which cut the plane. Here is the limit to geometrical construction, but abstract thought knows of no bounds. Having generalized our mathematical notions as systems of first, second, and third degree, we can continue in the numeral series and construct systems of four, five, and still higher degrees. Further, we can determine any plane by any three points, given in the figures x_1, x_2, x_3, not lying in a straight line. If the equation between these three figures be homogeneous, the totality of all points that correspond to it will be a system of second degree. If this homogeneous equation is of the first grade, this system of second degree will be simple, viz., of a straight line; but if the equation be of a higher grade, we shall have curves for which not all the laws of plane geometry hold good. The same considerations lead to a distinction between homaloidal space and non-Euclidean systems.[26]

Being professor at a German gymnasium and not a university, Grassmann's book remained neglected and the newness of his methods prevented superficial readers from appreciating the sweeping significance of his propositions. Since there was no call whatever for the book, the publishers returned the whole edition to the paper mill, and the complimentary copies which the author had sent

[26] See Grassmann's *Ausdehnungslehre,* 1844, Anhang I, pp. 273-274.

out to his friends are perhaps the sole portion that was saved from the general doom.

Grassmann, disappointed in his mathematical labors, had in the meantime turned to other studies and gained the honorary doctorate of the University of Tübingen in recognition of his meritorious work on the St. Petersburg Sankrit Dictionary, when Victor Schlegel called attention to the similarity of Hamilton's theory of vectors to Grassmann's concept of *Strecke,* both being limited straight lines of definite direction. Suddenly a demand for Grassmann's book was created in the market; but alas! no copy could be had, and the publishers deemed it advisable to reprint the destroyed edition of 1844. The satisfaction of this late recognition was the last joy that brightened the eve of Grassmann's life. He wrote the introduction and an appendix to the second edition of his *Lineare Ausdehnungslehre,* but died while the forms of his book were on the press.

At the present day the literature on metageometrical subjects has grown to such an extent that we do not venture to enter into further details. We will only mention the appearance of Professor Schoute's work on more-dimensional geometry[27] which promises to be the elaboration of the pangeometrical ideal.

[27] *Mehrdimensionale Geometrie* von Dr. P. H. Schoute, Professor der Math. an d. Reichs-Universität zu Groningen, Holland. Leipsic, Göschen. So far only the first volume, which treats of linear space, has appeared.

EUCLID STILL UNIMPAIRED.

Having briefly examined the chief innovations of modern times in the field of elementary geometry, it ought to be pointed out that in spite of the well-deserved fame of the metageometricians from Wallis to Halsted, Euclid's claim to classicism remains unshaken. The metageometrical movement is not a revolution against Euclid's authority but an attempt at widening our mathematical horizon. Let us hear what Halsted, one of the boldest and most iconoclastic among the champions of metageometry of the present day, has to say of Euclid. Halsted begins the Introduction to his English translation of *Bolyai's Science Absolute of Space* with a terse description of the history of Euclid's great book *The Elements of Geometry*, the rediscovery of which is not the least factor that initiated a new epoch in the development of Europe which may be called the era of inventions, of discoveries, and of the appreciation as well as growth of science. Halsted says:

"The immortal *Elements* of Euclid was already in dim antiquity a classic, regarded as absolutely perfect, valid without restriction.

"Elementary geometry was for two thousand years as stationary, as fixed, as peculiarly Greek as the Parthenon. On this foundation pure science rose in Archimedes, in Apollonius, in Pappus; struggled in Theon, in Hypatia; declined in Proclus; fell into the long decadence of the Dark Ages.

"The book that monkish Europe could no longer understand was then taught in Arabic by Saracen and Moor in the Universities of Bagdad and Cordova.

"To bring the light, after weary, stupid centuries, to Western Christendom, an Englishman, Adelhard of Bath, journeys, to learn Arabic, through Asia Minor, through Egypt, back to Spain. Disguised as a Mohammedan student, he got into Cordova about 1120, obtained a Moorish copy of Euclid's *Elements,* and made a translation from the Arabic into Latin.

"The first printed edition of Euclid, published in Venice in 1482, was a Latin version from the Arabic. The translation into Latin from the Greek, made by Zamberti from a manuscript of Theon's revision, was first published at Venice in 1505.

"Twenty-eight years later appeared the *editio princeps* in Greek, published at Basle in 1533 by John Hervagius, edited by Simon Grynaeus. This was for a century and three-quarters the only printed Greek text of all the books, and from it the first English translation (1570) was made by 'Henricus Billingsley,' afterward Sir Henry Billingsley, Lord Mayor of London in 1591.

"And even to-day, 1895, in the vast system of examinations carried out by the British Government, by Oxford, and by Cambridge, no proof of a theorem in geometry will be accepted which infringes Euclid's sequence of propositions.

"Nor is the work unworthy of this extraordinary immortality.

"Says Clifford: 'This book has been for nearly twenty-two centuries the encouragement and guide of that scientific thought which is one thing with the progress of man from a worse to a better state.

" 'The encouragement; for it contained a body of knowledge that was really known and could be relied on.

" 'The guide; for the aim of every student of every subject was to bring his knowledge of that subject into a form as perfect as that which geometry had attained.' "

Euclid's *Elements of Geometry* is not counted among the books of divine revelation, but truly it deserves to be held in religious veneration. There is a real sanctity in mathematical truth which is not sufficiently appreciated, and certainly if truth, helpfulness, and directness and simplicity of presentation, give a title to rank as divinely inspired literature, Euclid's great work should be counted among the canonical books of mankind.

* * *

Is there any need of warning our readers that the foregoing sketch of the history of metageometry is both brief and popular? We have purposely avoided the discussion of technical details, limiting our exposition to the most essential points and trying to show them in a light that will render them interesting even to the non-mathematical reader. It is meant to serve as an introduction to the real matter in hand, viz., an examination of the foundations upon which geometrical truth is to be rationally justified.

The author has purposely introduced what might be called a biographical element in these expositions of a subject which is commonly regarded as dry and abstruse, and endeavored to give something of the lives of the men who have struggled and labored in this line of thought and have sacrificed their time and energy on the altar of one of the noblest aspirations of man, the delineation of a philosophy of mathematics. He hopes thereby to relieve the dryness of the subject and to create an interest in the labor of these pioneers of intellectual progress.

THE PHILOSOPHICAL BASIS OF MATHEMATICS.

THE PHILOSOPHICAL PROBLEM.

HAVING thus reviewed the history of non-Euclidean geometry, which, rightly considered, is but a search for the philosophy of mathematics, I now turn to the problem itself and, in the conviction that I can offer some hints which contain its solution, I will formulate my own views in as popular language as would seem compatible with exactness. Not being a mathematician by profession I have only one excuse to offer, which is this: that I have more and more come to the conclusion that the problem is not mathematical but philosophical; and I hope that those who are competent to judge will correct me where I am mistaken.

The problem of the philosophical foundation of mathematics is closely connected with the topics of Kant's *Critique of Pure Reason.* It is the old quarrel between Empiricism and Transcendentalism. Hence our method of dealing with it will naturally be philosophical, not typically mathematical.

The proper solution can be attained only by analysing the fundamental concepts of mathematics and by tracing them to their origin. Thus alone can we know their nature as well as the field of their applicability.

We shall see that the data of mathematics are not without their premises; they are not, as the Germans say, *voraussetzungslos*; and though mathematics is built up from nothing, the mathematician does not start with nothing. He uses mental implements, and it is they that give character to his science.

Obviously the theorem of parallel lines is one instance only of a difficulty that betrays itself everywhere in various forms; it is not the disease of geometry, but a symptom of the disease. The theorem that the sum of the angles in a triangle is equal to 180 degrees; the ideas of the evenness or homaloidality of space, of the rectangularity of the square, and more remotely even the irrationality of π and of e, are all interconnected. It is not the author's intention to show their interconnection, nor to prove their interdependence. That task is the work of the mathematician. The present investigation shall be limited to the philosophical side of the problem for the sake of determining the nature of our notions of evenness, which determines both parallelism and rectangularity.

At the bottom of the difficulty there lurks the old problem of apriority, proposed by Kant and decided by him in a way which promised to give to mathematics a solid foundation in the realm of transcendental thought. And yet the transcendental method finally sent geometry away from home in search of a new domicile in the wide domain of empiricism.

Riemann, a disciple of Kant, is a transcendentalist. He starts with general notions and his arguments are deductive, leading him from the abstract down to concrete instances; but when stepping from the ethereal height of the absolute into the region of definite space-relations, he fails to find the necessary connection that characterizes all *a priori* reasoning; and so he swerves into the domain of the *a posteriori* and declares that the nature of the specific features of space must be determined by experience.

The very idea seems strange to those who have been reared in traditions of the old school. An unsophisticated man, when he speaks of a straight line, means that straightness is implied thereby; and if he is told that space may be such as to render all straightest lines crooked, he will naturally be bewildered. If his metageometrical friend, with much learnedness and in sober earnest, tells him that when he sends out a ray as a straight line in a forward direction it will imperceptibly deviate and finally turn back upon his occiput, he will naturally become suspicious of the mental soundness of his company. Would not many of us dismiss such ideas with a shrug if there were not geniuses of the very first rank who subscribe to the same? So in all modesty we have to defer our judgment until competent study and mature reflection have enabled us to understand the difficulty which they encounter and then judge their solution. One thing is sure, however: if there is anything wrong with metageometry, the fault lies not in its mathematical portion but must be sought for in its philosophical foundation, and it is this problem to which the present treatise is devoted.

While we propose to attack the problem as a philosophical question, we hope that the solution will prove acceptable to mathematicians.

TRANSCENDENTALISM AND EMPIRICISM.

In philosophy we have the old contrast between the empiricist and transcendentalist school. The former derive everything from experience, the latter insist that experience depends upon notions not derived from experience, called transcendental, and these notions are *a priori*. The former found their representative thinkers in Locke, Hume, and John Stuart Mill, the latter was perfected by Kant. Kant establishes the existence of notions of the *a priori* on a solid basis asserting their universality and necessity, but he no longer identified the *a priori* with innate ideas. He granted that much to empiricism, stating that all knowledge begins with experience and that experience rouses in our mind the *a priori* which is characteristic of mind. Mill went so far as to deny altogether necessity and universality, claiming that on some other planet 2×2 might be 5. French positivism, represented by Comte and Littré, follows the lead of Mill and thus they end in agnosticism, and the same result was reached in England on grounds somewhat different by Herbert Spencer.

The way which we propose to take may be characterized as the New Positivism. We take our stand upon the facts of experience and establish upon the systematized formal features of our experience a new conception of the *a priori,* recognizing the universality and necessity of formal laws but rejecting Kant's transcendental idealism. The *a priori* is not deducible from the sensory elements of our sensations, but we trace it in the formal features of experience. It is the result of abstraction and systematization. Thus we establish a method of dealing with experience (commonly called Pure Reason) which is possessed of universal validity, implying logical necessity.

The New Positivism is a further development of philosophic thought which combines the merits of both schools, the Transcendentalists and Empiricists, in a higher unity, discarding at the same time their aberrations. In this way it becomes possible to gain a firm basis upon the secure ground of facts, according to the principle of positivism, and yet to preserve a method established by a study of the purely formal, which will not end in nescience (the ideal of agnosticism) but justify science, and thus establishes the philosophy of science.[1]

[1] We have treated the philosophical problem of the *a priori* at full length in a discussion of Kant's *Prolegomena.* See the author's *Kant's Prolegomena,* edited in English, with an essay on Kant's Philosophy and other Supplementary Material for the study of Kant, pp. 167-240. Cf. *Fundamental Problems,* the chapters "Form and

It is from this standpoint of the philosophy of science that we propose to investigate the problem of the foundation of geometry.

THE A PRIORI AND THE PURELY FORMAL.

The bulk of our knowledge is from experience, i. e., we know things after having become acquainted with them . Our knowledge of things is *a posteriori.* If we want to know whether sugar is sweet, we must taste it. If we had not done so, and if no one had tasted it, we could not know it. However, there is another kind of knowledge which we do not find out by experience, but by reflection. If I want to know how much is 3×3, or $(a+b)^2$ or the angles in a regular polygon, I must compute the answer in my own mind. I need make no experiments but must perform the calculation in my own thoughts. This knowledge which is the result of pure thought is *a priori*; viz., it is generally applicable and holds good even *before* we tried it. When we begin to make experiments, we presuppose that all our *a priori* arguments, logic, arithmetic, and mathematics, will hold good.

Kant declared that the law of causation is of the same nature as arithmetical and logical truths, and that, accordingly, it will have to be regarded as *a priori.* Before we make experiments, we know that every cause has its effects, and wherever there

Formal Thought," pp. 26-60, and "The Old and the New Mathematics," pp. 61-73; and *Primer of Philosophy,* pp. 51-103.

is an effect we look for its cause. Causation is not proved by, but justified through, experience.

The doctrine of the *a priori* has been much misinterpreted, especially in England. Kant calls that which transcends or goes beyond experience in the sense that it is the condition of experience "transcendental," and comes to the conclusion that the *a priori* is transcendental. Our *a priori* notions are not derived from experience but are products of pure reflection and they constitute the conditions of experience. By experience Kant understands sense-impressions, and the sense-impressions of the outer world (which of course are *a posteriori*) are reduced to system by our transcendental notions; and thus knowledge is the product of the *a priori* and the *a posteriori*.

A sense-impression becomes a perception by being regarded as the effect of a cause. The idea of causation is a transcendental notion. Without it experience would be impossible. An astronomer measures angles and determines the distance of the moon and of the sun. Experience furnishes the data, they are *a posteriori*; but his mathematical methods, the number system, and all arithmetical functions are *a priori*. He knows them before he collects the details of his investigation; and in so far as they are the condition without which his sense-impressions could not be transformed into knowledge, they are called transcendental.

Note here Kant's use of the word transcendental which denotes the clearest and most reliable knowledge in our economy of thought, pure logic, arithmetic, geometry, etc. But transcendental is frequently (though erroneously) identified with "transcendent," which denotes that which transcends our knowledge and accordingly means "unknowable." Whatever is transcendental is, in Kantian terminology, never transcendent.

That much will suffice for an explanation of the historical meaning of the word transcendental. We must now explain the nature of the *a priori* and its source.

The *a priori* is identical with the purely formal which originates in our mind by abstraction. When we limit our attention to the purely relational, dropping all other features out of sight, we produce a field of abstraction in which we can construct purely formal combinations, such as numbers, or the ideas of types and species. Thus we create a world of pure thought which has the advantage of being applicable to any purely formal consideration of conditions, and we work out systems of numbers which, when counting, we can use as standards of reference for our experiences in practical life.

But if the sciences of pure form are built upon an abstraction from which all concrete features are omitted, are they not empty and useless verbiage?

Empty they are, that is true enough, but for all that they are of paramount significance, because they introduce us into the *sanctum sanctissimum* of the world, the intrinsic necessity of relations, and thus they become the key to all the riddles of the universe. They are in need of being supplemented by observation, by experience, by experiment; but while the mind of the investigator builds up purely formal systems of reference (such as numbers) and purely formal space-relations (such as geometry), the essential features of facts (of the objective world) are in their turn, too, purely formal, and they make things such as they are. The suchness of the world is purely formal, and its suchness alone is of importance.

In studying the processes of nature we watch transformations, and all we can do is to trace the changes of form. Matter and energy are words which in their abstract significance have little value; they merely denote actuality in general, the one of being, the other of doing. What interests us most are the *forms* of matter and energy, how they change by transformation; and it is obvious that the famous law of the conservation of matter and energy is merely the reverse of the truth that causation is transformation. In its elements which in their totality are called matter and energy, the elements of existence remain the same, but the forms in which they combine change. The sum-total of the mass and the sum-total of the forces of the world can be neither increased nor diminished; they remain the same to-day that they have always been and as they will remain forever.

All *a posteriori* cognition is concrete and particular, while all *a priori* cognition is abstract and general. The concrete is (at least in its relation to the thinking subject) incidental, casual, and individual, but the abstract is universal and can be used as a general rule under which all special cases may be subsumed.

The *a priori* is a mental construction, or, as Kant says, it is ideal, viz., it consists of the stuff that ideas are made of, it is mind-made. While we grant that the purely formal is ideal we insist that it is made in the domain of abstract thought, and its fundamental notions have been abstracted from experience by concentrating our attention upon the purely formal. It is, not directly but indirectly and ultimately, derived from experience. It is not derived from sense-experience but from a consideration of the relational (the purely formal) of experience. Thus it is a subjective reconstruction of certain objective features of experience and this reconstruction is made in such a way as to drop every thing incidental and particular and retain only the general and essential features; and we gain the unspeakable advantage of creating rules or formulas which, though abstract and mind-made, apply to *any* case that can be classified in the same category.

Kant made the mistake of identifying the term "ideal" with "subjective," and thus his transcendental idealism was warped by the conclusion that our purely formal laws were not objective, but were imposed by our mind upon the objective world. Our mind (Kant said) is so constituted as to interpret all facts of experience in terms of form, as appearing in space and time, and as being subject to the law of cause and effect; but what things are in themselves we cannot know. We object to Kant's subjectivizing the purely formal and look upon form as an essential and inalienable feature of objective existence. The thinking subject is to other thinking subjects an object moving about in the objective world . Even when contemplating our own existence we must grant (to speak with Schopenhauer) that our bodily actualization is our own object; i. e., we (each one of us as a real living creature) are as much objects as are all the other objects in the world. It is the objectified part of our self that in its inner experience abstracts from sense-experience the interrelational features of things, such as right and left, top and bottom, shape and figure and structure, succession, connection, etc. The formal adheres to the object and not to the subject, and every object (as soon as it develops in the natural way of evolution first into a feeling creature and then into a thinking being) will be able to build up *a priori* from the abstract notion of form in general the several systems of formal thought: arithmetic, geometry, algebra, logic, and the conceptions of time, space, and causality.

Accordingly, all formal thought, although we grant its ideality, is fashioned from materials abstracted from the objective world, and it is therefore a matter of course that they are applicable to the objective world. They belong to the object and, when we thinking subjects beget them from our own minds, we are able to do so only because we are objects that live and move and have our being in the objective world.

ANYNESS AND ITS UNIVERSALITY.

We know that facts are incidental and haphazard, and appear to be arbitrary; but we must not rest satisfied with single incidents. We must gather enough single cases to make abstractions. Abstractions are products of the mind; they are subjective; but they have been derived from experience, and they are built up of elements that have objective significance.

The most important abstractions ever made by man are those that are purely relational. Everything from which the sensory element is entirely omitted, where the material is disregarded, is called "pure form," and the relational being a consideration neither of matter nor of force or energy, but of number, of position, of shape, of size, of form, of relation, is called "the purely formal." The notion of the purely formal has been gained by abstraction, viz., by abstracting, i. e., singling out and retaining, the formal, and by thinking away, by cancelling, by omitting, by leaving out, all the features which have anything to do with the concrete sensory element of experience.

And what is the result?

We retain the formal element alone which is void of all concreteness, void of all materiality, void of all particularity. It is a mere nothing and a non-entity. It is emptiness. But one thing is left, —position or relation. Actuality is replaced by mere potentiality, viz., the possible conditions of *any* kind of being that is possessed of form.

The word "any" denotes a simple idea, and yet it contains a good deal of thought. Mathematics builds up its constructions to suit *any* condition. "Any" implies universality, and universality includes necessity in the Kantian sense of the term.

In every concrete instance of an experience the subject-matter is the main thing with which we are concerned; but the purely formal aspect is after all the essential feature, because form determines the character of things, and thus the formal (on account of its anyness) is the key to their comprehension.

The rise of man above the animal is due to his ability to utilize the purely formal, as it revealed itself to him especially in types for classifying things, as genera and species, in tracing transformations which present themselves as effects of causes and reducing them to shapes of measurable relations. The abstraction of the formal is made through the instrumentality of language and the result is reason, —the faculty of abstract thought. Man can see the universal in the particular; in the single experiences he can trace the laws that are generally applicable to cases of the same class; he observes some instances and can describe them in a general formula so as to cover any other instance of the same kind, and thus he becomes master of the situation; he learns to separate in thought the essential from the accidental, and so instead of remaining the prey of circumstance he gains the power to adapt circumstances to himself.

Form pervades all nature as an essential constituent thereof. If form were not an objective feature of the world in which we live, formal thought would never without a miracle, or, at least, not without the mystery of mysticism, have originated according to natural law, and man could never have arisen. But form being an objective feature of all existence, it impresses itself in such a way upon living creatures that rational beings will naturally develop among animals whose organs of speech are perfected as soon as social conditions produce that demand for communication that will result in the creation of language.

The marvelous advantages of reason dawned upon man like a revelation from on high, for he did not invent reason, he discovered it; and the sentiment that its blessings came to him from above, from heaven, from that power which sways the destiny of the whole universe, from the gods or from God, is as natural as it is true. The anthropoid did not seek reason: reason came to him and so he became man. Man became man by the grace of God, by gradually imbibing the Logos that was with God in the beginning; and in the dawn of human evolution we can plainly see the landmark of mathematics, for the first grand step in the development of man as distinguished from the transitional forms of the anthropoid is the ability to count.

Man's distinctive characteristic remains, even to-day, reason, the faculty of purely formal thought; and the characteristic of reason is its general application. All its verdicts are universal and involve apriority or beforehand knowledge so that man can foresee events and adapt means to ends.

APRIORITY OF DIFFERENT DEGREES.

Kant has pointed out the kinship of all purely formal notions. The validity of mathematics and logic assures us of the validity of the categories including the conception of causation; and yet geometry cannot be derived from pure reason alone, but contains an additional element which imparts to its fundamental conceptions an arbitrary appearance if we attempt to treat its deductions as rigidly *a priori*. Why should there be straight lines at all? Why is it possible that by quartering the circle we should have right angles with all their peculiarities? All these and similar notions can not be subsumed under a general formula of pure reason from which we could derive it with logical necessity.

When dealing with lines we observe their extention in one direction, when dealing with planes we have two dimensions, when measuring solids we have three. Why can we not continue and construct bodies that extend in four dimensions? The limit set us by space as it positively presents itself to us seems arbitrary, and while transcendental truths are undeniable and obvious, the fundamental notions of geometry seem as stubborn as the facts of our concrete existence. Space, generally granted to be elbow-room for motion in all directions, after all appears to be a definite magnitude as much as a stone wall which shuts us in like a prison, allowing us to proceed in such a way only as is permissible by those co-ordinates and no more. We can by no resort break through this limitation. Verily we might more easily shatter a rock that impedes our progress than break into the fourth dimension. The boundary line is inexorable in its adamantine rigor.

Considering all these unquestionable statements, is there not a great probability that space is a concrete fact as positive as the existence of material things, and not a mere form, not a mere potentiality of a general nature? Certainly Euclidean geometry contains some such arbitrary elements as we should expect to meet in the realm of the *a posteriori*. No wonder that Gauss expressed "the desire that the Euclidean geometry should not be the true geometry," because "in that event we should have an absolute measure *a priori*."

Are we thus driven to the conclusion that our space-conception is not *a priori*; and if, indeed, it is not *a priori*, it must be *a posteriori*! What else can it be? *Tertium non datur.*

If we enter more deeply into the nature of the *a priori*, we shall learn that there are different kinds of apriority, and there is a difference between the logical *a priori* and the geometrical *a priori*.

Kant never investigated the source of the *a priori*. He discovered it in the mind and seemed satisfied with the notion that it is the nature of the mind to be possessed of time and space and the categories. He went no further. He never asked, how did mind originate?

Had Kant inquired into the origin of mind, he would have found that the *a priori* is woven into the texture of mind by the uniformities of experience. The uniformities of experience teach us the laws of form, and the purely formal applies not to one case only but to any case of the same kind, and so it involves "anyness," that is to say, it is *a priori*.

Mind is the product of memory, and we may briefly describe its origin as follows:

Contact with the outer world produces impressions in sentient substance. The traces of these impressions are preserved (a condition which is called "memory") and they can be revived (which state is called "recollection"). Sense-impressions are different in kind and leave different traces, but those which are the same in kind, or similar, leave traces the forms of which are the same or similar; and sense-impressions of the same kind are registered in the traces having the same form. As a note of a definite pitch makes chords of the same pitch vibrate while it passes all others by; so new sense-impressions revive those traces only into which they fit, and thereby announce themselves as being the same in kind. Thus all sense-impressions are systematized according to their forms, and the result is an orderly arrangement of memories which is called "mind."[2]

Thus mind develops through uniformities in sensation according to the laws of form. Whenever a new sense-perception registers itself mechanically and automatically in the trace to which it belongs, the event is tantamount to a logical judgment which declares that the object represented by the sense-impression belongs to the same class of objects which produced the memory traces with which it is registered.

If we abstract the interrelation of all memory-traces, omitting their contents, we have a pure system of genera and species, or the *a priori* idea of "classes and subclasses."

The *a priori*, though mind-made, is constructed of chips taken from the objective world, but our several *a priori* notions are by no means of one and the same nature and rigidity. On the contrary, there are different degrees of apriority. The emptiest forms of pure thought are the categories, and the most rigid truths are the logical theorems, which can be represented diagrammatically so as to be a *demonstratio ad oculos*.

If all *b*s are B and if β is a *b*, then β is a B. If all dogs are quadrupeds and if all terriers are dogs, then terriers are quadrupeds. It is the most rigid

[2] For a more detailed exposition see the author's *Soul of Man*; also his *Whence and Whither*.

kind of argument, and its statements are practically tautologies.

The case is different with causation. The class of abstract notions of which causation is an instance is much more complicated. No one doubts that every effect must have had its cause, but one of the keenest thinkers was in deep earnest when he doubted the possibility of proving this obvious statement. And Kant, seeing its kinship with geometry and algebra, accepted it as *a priori* and treated it as being on equal terms with mathematical axioms. Yet there is an additional element in the formula of causation which somehow disguises its *a priori* origin, and the reason is that it is not as rigidly *a priori* as are the norms of pure logic.

What is this additional element that somehow savors of the *a posteriori*?

If we contemplate the interrelation of genera and species and subspecies, we find that the categories with which we operate are at rest. They stand before us like a well arranged cabinet with several divisions and drawers, and these drawers have subdivisions and in these subdivisions we keep boxes. The cabinet is our *a priori* system of classification and we store in it our *a posteriori* impressions. If a thing is in box β, we seek for it in drawer *b* which is a subdivision of the department B.

How different is causation! While in logic everything is at rest, causation is not conceivable without motion. The norms of pure reason are static, the law of cause and effect is dynamic; and thus we have in the conception of cause and effect an additional element which is mobility.

Causation is the law of transformation. We have a definite system of interrelated items in which we observe a change of place. The original situation and all detailed circumstances are the conditions; the motion that produces the change is the cause; the result or new arrangement of the parts of the whole system is the effect. Thus it appears that causation is only another version of the law of the conservation of matter and energy. The concrete items of the whole remain, in their constitutional elements, the same. No energy is lost; no particle of matter is annihilated; and the change that takes place is mere transformation.[3]

The law of causation is otherwise in the same predicament as the norms of logic. It can never be satisfactorily proved by experience. Experience justifies the *a priori* and verifies its tenets in single instances which prove true, but single instances can never demonstrate the universal and necessary validity of any *a priori* statement.

The logical *a priori* is rigidly *a priori*; it is the *a priori* of pure reason. But there is another kind of *a priori* which admits the use of that other abstract notion, mobility, and mobility as much as form is part and parcel of the thinking mind. Our conception of cause and effect is just as ideal as our conception of genera and species. It is just as much

[3] See the author's *Ursache, Grund und Zweck*, Dresden, 1883; also his *Fundamental Problems*.

mind-made as they are, and its intrinsic necessity and universal validity are the same. Its apriority cannot be doubted; but it is not rigidly *a priori*, and we will call it purely *a priori*.

We may classify all *a priori* notions under two headings and both are transcendental (viz., conditions of knowledge in their special fields): one is the *a priori* of being, the other of doing. The rigid *a priori* is passive anyness, the less rigid *a priori* is active anyness. Geometry belongs to the latter. Its fundamental concept of space is a product of active apriority; and thus we cannot derive its laws from pure logic alone.

The main difficulty of the parallel theorem and the straight line consists in our space-conception which is not derived from rationality in general, but results from our contemplation of motion. Our space-conception accordingly is not an idea of pure reason, but the product of pure activity.

Kant felt the difference and distinguished between pure reason and pure intuition or *Anschauung*. He did not expressly say so, but his treatment suggests the idea that we ought to distinguish between two different kinds of *a priori*. Transcendental logic, and with it all common notions of Euclid, are mere applications of the law of consistency; they are "rigidly *a priori*." But our pure space-conception presupposes, in addition to pure reason, our own activity, the potentiality of moving about in any kind of a field, and thus it admits another factor which cannot be derived from pure reason alone. Hence all attempts at proving the theorem on rigidly *a priori* grounds have proved failures.

SPACE AS A SPREAD OF MOTION.

Mathematicians mean to start from nothingness, so they think away everything, but they retain their own mentality. Though even their mind is stripped of all particular notions, they retain their principles of reasoning and the privilege of moving about, and from these two sources geometry can be constructed.

The idea of causation goes one step further: it admits the notions of matter and energy, emptied of all particularity, in their form of pure generalizations. It is still *a priori,* but considerably more complicated than pure reason.

The field in which the geometrician starts is pure nothingness; but we shall learn later on that nothingness is possessed of positive qualifications. We must therefore be on our guard, and we had better inquire into the nature and origin of our nothingness.

The geometrician cancels in thought all positive existence except his own mental activity and starts moving about as a mere nothing. In other words, we establish by abstraction a domain of monotonous sameness, which possesses the advantage of "anyness," i. e., an absence of particularity involving universal validity. In this field of motion we proceed to produce geometrical constructions.

The geometrician's activity is pure motion, which means that it is mere progression; the ideas of a force exerted in moving and also of resistance to be overcome are absolutely excluded.

We start moving, but whither? Before us are infinite possibilities of direction. The inexhaustibility of chances is part of the indifference as to definiteness of determining the mode of motion (be it straight or curved). Let us start at once in all possible directions which are infinite, (a proposition which, in a way, is realized by the light), and having proceeded an infinitesimal way from the starting-point A to the points B, B_1,, B_2, B_3, B_4, ... B_∞; we continue to move in infinite directions at each of these stations, reaching from B the points C, C_1, C_2, C_3, C_4, ... C_∞. From B_1 we would switch off to the points C_1^{B1}, C_2^{B1}, C_3^{B1}, C_4^{B1}, ... C_∞^{B1}, etc. until we reach from B_∞ the points $C_1^{B\infty}$, $C_2^{B\infty}$, $C_3^{B\infty}$, $C_4^{B\infty}$, ... $C_\infty^{B\infty}$, thus exhausting all the points which cluster around every B_1, B_2, B_3, B_4, ... B_∞. Thus, by moving after the fashion of the light, spreading again and again from each new point in all directions, in a medium that offers no resistance whatever, we obtain a uniform spread of light whose intensity in every point is in the inverse square of its distance from its source. Every lighted spot becomes a center of its own from which light travels on in all directions. But among these infinite directions there are rays, A, B, C, ..., A_1, B_1, C_1, ..., A_2, B_2, C_2, ..., etc., that is to say, lines of motion that pursue the original direction and are paths of maximum intensity. Each of these rays, thus ideally constructed, is a representation of the straight line which being the shortest path between the starting-point A and any other point, is the climax of directness: it is the upper limit of effectiveness and its final boundary, a *non plus ultra.* It is a maximum because there is no loss of efficacy. The straight line represents a climax of economy, viz., the greatest intensity on the shortest path that is reached among infinite possibilities of progression by uniformly following up all. In every ray the maximum of intensity is attained by a minimum of progression.

Our construction of motion in all directions after the fashion of light is practically pure space; but to avoid the forestalling of further implications we will call it simply the spread of motion in all directions.

The path of highest intensity in a spread of motion in all directions corresponds to the ray in an ideal conception of a spread of light, and it is equivalent to the straight line in geometry.

We purposely modify our reference to light in our construction of straight lines, for we are well aware of the fact that the notion of a ray of light as a straight line is an ideal which describes the progression of light only as it appears, not as it is. The physicist represents light as rays only when measuring its effects in reflection, etc., but when considering the nature of light, he looks upon rays as transversal oscillations of the ether. The notion of light as rays is at bottom as much an *a priori* construction as is Newton's formula of gravitation.

The construction of space as a spread of motion in all directions after the analogy of light is a summary creation of the scope of motion, and we call it "ideal space." Everything that moves about, if it develops into a thinking subject, when it forms the abstract idea of mobility, will inevitably create out of the data of its own existence the ideal "scope of motion," which is space.

When the geometrician starts to construct his figures, drawing lines and determining the position of points, etc., he tacitly presupposes the existence of a spread of motion, such as we have described. Motility is part of his equipment, and motility presupposes a field of motion, viz., space.

Space is the possibility of motion, and by ideally moving about in all possible directions the number of which is inexhaustible, we construct our notion of pure space. If we speak of space we mean this construction of our mobility. It is an *a priori* construction and is as unique as logic or arithmetic. There is but one space, and all spaces are but portions of this one construction. The problem of tridimensionality will be considered later on. Here we insist only on the objective validity of our *a priori* construction, which is the same as the objective validity of all our *a priori* constructions—of logic and arithmetic and causality, and it rests upon the same foundation. Our mathematical space omits all particularity and serves our purpose of universal application: it is founded on "anyness," and thus, within the limits of its abstraction, it holds good everywhere and under all conditions.

There is no need to find out by experience in the domain of the *a posteriori* whether pure space is curved. Anyness has no particular qualities; we create this anyness by abstraction, and it is a matter of course that in the field of our abstraction, space will be the same throughout, unless by another act of our creative imagination we appropriate particular qualities to different regions of space.

The fabric of which the purely formal is woven is an absence of concreteness. It is (so far as matter is concerned) nothing. Yet this airy nothing is a pretty tough material, just on account of its indifferent "any"-ness. Being void of particularity, it is universal; it is the same throughout, and if we proceed to build our air-castles in the domain of anyness, we shall find that considering the absence of all particularity the same construction will be the same, wherever and whenever it may be conceived.

Professor Clifford says:[4] "We *assume* that two lengths which are equal to the same length are equal to each other." But there is no "assumption" about it. The atmosphere in which our mathematical creations are begotten is sameness. Therefore the same construction is the same wherever and whenever it may be made. We consider form only; we think away all other concrete properties, both

[4] *Loc. cit.*, p. 53.

of matter and energy, mass, weight, intensity, and qualities of any kind.

UNIQUENESS OF PURE SPACE.

Our thought-forms, constructed in the realm of empty abstraction, serve as models or as systems of reference for any of our observations in the real world of sense-experience. The laws of form are as well illustrated in our models as in real things, and can be derived from either; but the models of our thought-forms are always ready at hand while the real things are mostly inaccessible. The anyness of pure form explains the parallelism that obtains between our models and actual experience, which was puzzling to Kant. And truly at first sight it is mystifying that a pure thought-construction can reveal to us some of the most important and deepest secrets of objective nature; but the simple solution of the mystery consists in this, that the actions of nature are determined by the same conditions of possible motions with which pure thought is confronted in its efforts to construct its models. Here as well as there we have consistency, that is to say, a thing done is uniquely determined, and, in pure thought as well as in reality, it is such as it has been made by construction .

Our constructions are made in anyness and apply to all possible instances of the kind; and thus we may as well define space as the potentiality of measuring, which presupposes moving about. Mobility granted, we can construct space as the scope of our motion in anyness. Of course we must bear in mind that our motion is in thought only and we have dropped all notions of particularity so as to leave an utter absence of force and resistance. The motor element, *qua* energy, is not taken into consideration, but we contemplate only the products of progression.

Since in the realm of pure form, thus created by abstraction, we move in a domain void of particularity, it is not an assumption (as Riemann declares in his famous inaugural dissertation), but a matter of course which follows with logical necessity, that lines are independent of position; they are the same anywhere.

In actual space, position is by no means a negligible quantity. A real pyramid consisting of actual material is possessed of different qualities according to position, and the line AB, representing a path from the top of a mountain to the valley is very different from the line BA, which is the path from the valley to the top of the mountain. In Euclidean geometry AB=BA.

Riemann attempts to identify the mathematical space of a triple manifold with actual space and expects a proof from experience, but, properly speaking, they are radically different. In real space position is not a negligible factor, and would necessitate a fourth co-ordinate which has a definite relation to the plumb-line; and this fourth co-ordinate (which we may call a fourth dimension) suffers a constant modification of increase in inverse proportion to the square of the distance from the center of this planet of ours. It is rectilinear, yet all the plumb-lines are converging toward an inaccessible center; accordingly, they are by no means of equal value in their different parts. How different is mathematical space! It is homogeneous throughout. And it is so because we made it so by abstraction.

Pure form is a feature which is by no means a mere nonentity. Having emptied existence of all concrete actuality, and having thought away everything, we are confronted by an absolute vacancy—a zero of existence: but the zero has positive characteristics and there is this peculiarity about the zero that it is the mother of infinitude. The thought is so true in mathematics that it is trite. Let any number be divided by nought, the result is the infinitely great; and let nought be divided by any number, the result is the infinitely small. In thinking away everything concrete we retain with our nothingness potentiality. Potentiality is the empire of purely formal constructions, in the dim background of which lurks the phantom of infinitude.

MATHEMATICAL SPACE AND PHYSIOLOGICAL SPACE.

If we admit to our conception of space the qualities of bodies such as mass, our conception of real space will become more complicated still. What we gain in concrete definiteness we lose from universality, and we can return to the general applicability of *a priori* conditions only by dropping all concrete features and limiting our geometrical constructions to the abstract domain of pure form.

Mathematical space with its straight lines, planes, and right angles is an ideal construction. It exists in our mind only just as much as do logic and arithmetic. In the external world there are no numbers, no mathematical lines, no logarithms, no sines, tangents, nor secants. The same is true of all the formal sciences. There are no genera and species, no syllogisms, neither inductions nor deductions, running about in the world, but only concrete individuals and a concatenation of events. There are no laws that govern the motions of stars or molecules; yet there are things acting in a definite way, and their actions depend on changes in relational conditions which can be expressed in formulas. All the generalized notions of the formal sciences are mental contrivances which comprise relational features in general rules. The formulas as such are purely ideal, but the relational features which they describe are objectively real.

Thus, the space-conception of the mathematician is an ideal construction; but the ideal has objective significance. Ideal and subjective are by no means synonyms. With the help of an ideal space-conception we can acquire knowledge concerning the real space of the objective world. Here the Newtonian law may be cited as a conspicuous example.

How can the thinking subject know *a priori* anything about the object? Simply because the subject is an object moving about among other objects. Mobility is a qualification of the object, and I, the thinking subject, become conscious of the general rules of motion only, because I also am an object endowed with mobility. My "scope of motion" cannot be derived from the abstract idea of myself as a thinking subject, but is the product of a consideration of my mobility, generalized from my activities as an object by omitting all particularities.

Mathematical space is *a priori* in the Kantian sense. However, it is not ready-made in our mind, it is not an innate idea, but the product of much toil and careful thought. Nor will its construction be possible, except at a maturer age after a long development.

Physiological space is the direct and unsophisticated space-conception of our senses. It originates through experience, and is, in its way, a truer picture of actual or physical space than mathematical space. The latter is more general, the former more concrete. In physiological space position is not indifferent, for high and low, right and left, and up and down are of great importance. Geometrically congruent figures produce (as Mach has shown) remarkably different impressions if they present themselves to the eyes in different positions.

In a geometrical plane the figures can be shoved about without suffering a change of form. If they are flopped, their inner relations remain the same, as, *e. g.*, helices of opposite directions are, mathematically considered, congruent, while in actual life they would always remain mere symmetrical counterparts. So the right and the left hands considered as mere mathematical bodies are congruent, while in reality neither can take the place of the other. A glove which we may treat as a two-dimensional thing can be turned inside out, but we would need a fourth dimension to flop the hand, a three-dimensional body, into its inverted counterpart. So long as we have no fourth dimension, the latter being a mere logical fiction, this cannot be done. Yet mathematically considered, the two hands are congruent. Why? Not because they are actually of the same shape, but because in our mathematics the qualifications of position are excluded; the relational alone counts, and the relational is the same in both cases.

Mathematical space being an ideal construction, it is a matter of course that all mathematical problems must be settled by *a priori* operations of pure thought, and cannot be decided by external experiment or by reference to *a posteriori* information.

HOMOGENEITY OF SPACE DUE TO ABSTRACTION.

When moving about, we change our place and pass by different objects. These objects too are moving; and thus our scope of motion tallies so exactly with theirs that one can be used for the computation of the other. All scopes of motion are possessed of the same anyness.

Space as we find it in experience is best defined as the juxtaposition of things. If there is need of distinguishing it from our ideal space-conception which is the scope of our mobility, we may call the former pure objective space, the latter pure subjective space, but, our subjective ideas being rooted in our mobility, which is a constitutional feature of our objective existence, for many practical purposes the two are the same.

But though pure space, whether its conception be established objectively or subjectively, must be accepted as the same, are we not driven to the conclusion that there are after all two different kinds of space: mathematical space, which is ideal, and physiological space, which is real? And if they are different, must we not assume them to be independent of each other? What is their mutual relation?

The two spaces, the ideal construction of mathematical space and the reconstruction in our senses of the juxtaposition of things surrounding us, are different solely because they have been built up upon two different planes of abstraction; physiological space includes, and mathematical space excludes, the sensory data of juxtaposition. Physiological space admits concrete facts,—man's own upright position, gravity, perspective, etc. Mathematical space is purely formal, and to lay its foundation we have dug down to the bed-rock of our formal knowledge, which is "anyness." Mathematical space is *a priori*, albeit the *a priori* of motion.

At present it is sufficient to state that the homogeneity of a mathematical space is its anyness, and its anyness is due to our construction of it in the domain of pure form, involving universality and excluding everything concrete and particular.

The idea of homogeneity in our space-conception is the tacit condition for the theorems of similarity and proportion, and also of free mobility without change, viz., that figures can be shifted about without suffering distortion either by shrinkage or by expanse. The principle of homogeneity being admitted, we can shove figures about on any surface the curvature of which is either constant or zero. This produces either the non-Euclidean geometries of spherical, pseudo-spherical, and elliptic surfaces, or the plane geometry of Euclid—all of them *a priori* constructions made without reference to reality.

Our *a priori* constructions serve an important purpose. We use them as systems of reference. We construct *a priori* a number system, making a simple progression through a series of units which we denominate from the starting-point 0, as 1, 2, 3, 4, 5, 6, etc. These numbers are purely ideal constructions, but with their help we can count and measure and weigh the several objects of reality that confront experience; and in all cases we fall back upon our ideal number system, saying, the table has four legs; it is two and a half feet high, it weighs fifty pounds, etc. We call these modes of determination quantitative.

The element of quantitative measurement is the ideal construction of units, all of which are assumed to be discrete and equivalent. The equivalence of numbers as much as the homogeneity of space, is due to abstraction. In reality equivalent units do not exist any more than different parts of real space may be regarded as homogeneous. Both constructions have been made to create a domain of anyness, for the purpose of standards of reference.

EVEN BOUNDARIES AS STANDARDS OF MEASUREMENT.

Standards of reference are useful only when they are unique, and thus we cannot use any path of our spread of motion in all directions, but must select one that admits of no equivocation. The only line that possesses this quality is the ray, viz., the straight line or the path of greatest intensity.

The straight line is one instance only of a whole class of similar constructions which with one name may be called "even boundaries," and by even I mean congruent with itself. They remain the same in any position and no change originates however they may be turned.

Clifford, starting from objective space, constructs the plane by polishing three surfaces, A, B, and C, until they fit one another, which means until they are congruent.[5] His proposition leads to the same result as ours, but the essential thing is not so much (as Clifford has it) that the three planes are congruent, each to the two others, but that each plane is congruent with its own inversion. Thus,

[5] *Common Sense of the Exact Sciences,* Appleton & Co., p. 66.

under all conditions, each one is congruent with itself. Each plane partitions the whole infinite space into two congruent halves.

Having divided space so as to make the boundary surface congruent with itself (viz., a plane), we now divide the plane (we will call it P) in the same way,—a process best exemplified in the folding of a sheet of paper stretched flat on the table. The crease represents a boundary congruent with itself. In contrast to curved lines, which cannot be flopped or shoved or turned without involving a change in our construction, we speak of a straight line as an even boundary.

A circle can be flopped upon itself, but it is not an even boundary congruent with itself, because the inside contents and the outside surroundings are different.

If we take a plane, represented by a piece of paper that has been evenly divided by the crease AB, and divide it again crosswise, say in the point O, by another crease CD, into two equal parts, we establish in the four angles round O a new kind of even boundary.

The bipartition results in a division of each half plane into two portions which again are congruent the one to the other; and the line in the crease CD, constituting, together with the first crease, AB, two angles, is (like the straight line and the plane) nothing more nor less than an even boundary construction. The right angle originates by the process of halving the straight line conceived as an angle.

Let us now consider the significance of even boundaries.

A point being a mere locus in space, has no extension whatever; it is congruent with itself on account of its want of any discriminating parts. If it rotates in any direction, it makes no difference.

There is no mystery about a point's being congruent with itself in any position. It results from our conception of a point in agreement with the abstraction we have made; but when we are confronted with lines or surfaces that are congruent with themselves we believe ourselves nonplussed; yet the mystery of a straight line is not greater than that of a point.

A line which when flopped or turned in its direction remains congruent with itself is called straight, and a surface which when flopped or turned round on itself remains congruent with itself is called plane or flat.

The straight lines and the flat surfaces are, among all possible boundaries, of special importance, for a similar reason that the abstraction of pure form is so useful. In the domain of pure form we get rid of all particularity and thus establish a norm fit for universal application. In geometry straight lines and plane surfaces are the climax of simplicity; they are void of any particularity that needs further description, or would complicate the situation, and this absence of complications in their construction is their greatest recommendation. The most important point, however, is their quality of being unique by being even. It renders them specially available for purposes of reference.

We can construct *a priori* different surfaces that are homogeneous, yielding as many different systems of geometry. Euclidean geometry is neither more nor less true than spherical or elliptic geometry; all of them are purely formal constructions, they are *a priori,* being each one on its own premises irrefutable by experience; but plane geometry is more practical for general purposes.

The question in geometry is not, as some metageometricians would have it, "Is objective space flat or curved?" but, "Is it possible to make constructions that shall be unique so as to be serviceable as standards of reference?" The former question is due to a misconception of the nature of mathematics; the latter must be answered in the affirmative. All even boundaries are unique and can therefore be used as standards of reference.

THE STRAIGHT LINE INDISPENSABLE.

Straight lines do not exist in reality. How rough are the edges of the straightest rulers, and how rugged are the straightest lines drawn with instruments of precision, if measured by the standard of mathematical straightness! And if we consider the paths of motion, be they of chemical atoms or terrestrial or celestial bodies, we shall always find them to be curves of high complexity. Nevertheless the idea of the straight line is justified by experience in so far as it helps us to analyze the complex curves into their elementary factors, no one of which is truly straight; but each one of which, when we go to the end of our analysis, can be represented as a straight line. Judging from the experience we have of moving bodies, we cannot doubt that if the sun's attraction of the earth (as well as that of all other celestial bodies) could be annihilated, the earth would fly off into space in a straight line. Thus the mud on carriage wheels, when spurting off, and the pebbles that are thrown with a sling, are flying in a tangential direction which would be absolutely straight were it not for the interference of the gravity of the earth, which is constantly asserting itself and modifies the straightest line into a curve.

Our idea of a straight line is suggested to us by experience when we attempt to resolve compound forces into their constituents, but it is not traceable in experience. It is a product of our method of measurement. It is a creation of our own doings, yet it is justified by the success which attends its employment.

The great question in geometry is not, whether straight lines are real but whether their construction is not an indispensable requisite for any possible system of space measurement, and further, what is the nature of straight lines and planes and right angles; how does their conception originate and why are they of paramount importance in geometry.

We can of course posit that space should be filled up with a medium such as would deflect every ray of light so that straight rays would be impossible. For all we know ether may in an extremely slight degree operate in that way. But there would be nothing in that that could dispose of the thinkability of a line absolutely straight in the Euclidean sense with all that the same involves, so that Euclidean geometry would not thereby be invalidated.

Now the fact that the straight line (as a purely mental construction) is possible cannot be denied: we use it and that should be sufficient for all practical purposes. That we can construct curves also does not invalidate the existence of straight lines.

So again while a geometry based upon the idea of homaloidal space will remain what it has ever been, the other geometries are not made thereby illegitimate. Euclid disposes as little of Lobachevsky and Bolyai as they do of Euclid.

As to the nature of the straight line and all the other notions connected therewith, we shall always be able to determine them as concepts of boundary, either reaching the utmost limit of a certain function, be it of the highest (such as ∞) or lowest measure (such as 0); or dividing a whole into two congruent parts.

The utility of such boundary concepts becomes apparent when we are in need of standards for measurement. An even boundary being the utmost limit is unique. There are innumerable curves, but there is only one kind of straight line. Accordingly, if we need a standard for measuring curves, we must naturally fall back upon the straight line and determine its curvature by its deviation from the straight line which represents a zero of curvature.

The straight line is the simplest of all boundary concepts. Hence its indispensableness.

If we measure a curvature we resolve the curve into infinitesimal pieces of straight lines, and then determine their change of direction. Thus we use the straight line as a reference in our measurement of curves. The simplest curve is the circle, and its curvature is expressed by the reciprocal of the radius; but the radius is a straight line. It seems that we cannot escape straightness anywhere in geometry; for it is the simplest instrument for measuring distance. We may replace metric geometry by projective geometry, but what could projective geometricians do if they had not straight lines for their projections? Without them they would be in a strait indeed!

But suppose we renounced with Lobatchevsky the conventional method of even boundary conceptions, especially straightness of line, and were satisfied with straightest lines, what would be the result? He does not at the same time, surrender either the principle of consistency or the assumption of the homogeneity of space, and thus he builds up a geometry independent of the theorem of parallel lines, which would be applicable to two systems, the Euclidean of straight lines and the non-Euclidean of curved space. But the latter needs the straight line as much as the former and finds its natural limit in a sphere whose radius is infinite and whose curvature is zero. He can measure no spheric curvature without the radius, and after all he reaches the straight line in the limit of curvature. Yet it is noteworthy that in the Euclidean system the straight line is definite and π irrational, while in the non-Euclidean, π is a definite number according to the measure of curvature and the straight line becomes irrational.

THE SUPERREAL.

We said in a former chapter (p. 48), "man did not invent reason, he discovered it," which means that the nature of reason is definite, unalterable, and therefore valid. The same is true of all anyness of all formal thought, of pure logic, of mathematics, and generally of anything that with truth can be stated *a priori*. Though the norms of anyness are woven of pure nothingness, the flimsiest material imaginable, they are the factors which determine the course of events in the entire sweep of actual existence and in this sense they are real. They are not real in the sense of materiality; they are real only in being efficient and in distinction to the reality of corporeal things we may call them superreal.

On the one hand it is true that mathematics is a mental construction; it is purely ideal, which means it is woven of thought. On the other hand we must grant that the nature of this construction is foredetermined in its minutest detail and in this sense all its theorems must be discovered. We grant that there are no sines, and cosines, no tangents and cotangents, no logarithms, no number π, nor even lines, in nature, but there are relations in nature which correspond to these notions and suggest the invention of symbols for the sake of determining them with exactness. These relations possess a normative value. Stones are real in the sense of offering resistance in a special place, but these norms are superreal because they are efficient factors everywhere.

The reality of mathematics is well set forth in these words of Prof. Cassius Jackson Keyser, of Columbia University:

> "Phrase it as you will, there is a world that is peopled with ideas, ensembles, propositions, relations, and implications, in endless variety and multiplicity, in structure ranging from the very simple to the endlessly intricate and complicate. That world is not the product but the object, not the creature but the quarry of thought, the entities composing it—propositions, for example,—being no more identical with thinking them than wine is identical with the drinking of it. Mind or no mind, that world exists as an extra-personal affair,—pragmatism to the contrary notwithstanding."

While the relational possesses objective significance, the method of describing it is subjective and of course the symbols are arbitrary.

In this connection we wish to call attention to a most important point, which is the necessity of creating fixed units for counting. As there are no logarithms in nature, so there are no numbers; there are only objects or things sufficiently equal which for a certain purpose may be considered equivalent, so that we can ignore these differences, and assuming them to be the same, count them.

DISCRETE UNITS AND THE CONTINUUM.

Nature is a continuum; there are no boundaries among things, and all events that happen proceed in an uninterrupted flow of continuous transformations. For the sake of creating order in this flux which would seem to be a chaos to us, we must distinguish and mark off individual objects with definite boundaries. This method may be seen in all branches of knowledge, and is most in evidence in arithmetic. When counting we start in the domain of nothingness and build up the entire structure of arithmetic with the products of our own making.

We ought to know that whatever we do, we must first of all take a definite stand for ourselves. When we start doing anything, we must have a starting-point, and even though the world may be a constant flux we must for the sake of definiteness regard our starting-point as fixed. It need not be fixed in reality, but if it is to serve as a point of reference we must regard it as fixed and look upon all the rest as movable; otherwise the world would be an indeterminable tangle. Here we have the first rule of mental activity. There may be no rest in the world yet we must create the fiction of a rest as a δός μοι ποῦ στῶ and whenever we take any step we must repeat this fictitious process of laying down definite points.

All the things which are observed around us are compounds of qualities which are only temporarily combined. To call them things as if they were separate beings existing without reference to the rest is a fiction, but it is part of our method of classification, and without this fictitious comprehension of certain groups of qualities under definite names and treating them as units, we could make no headway in this world of constant flux, and all events of life would swim before our mental eye.

Our method in arithmetic is similar. We count as if units existed, yet the idea of a unit is a fiction. We count our fingers or the beads of an abacus or any other set of things as if they were equal. We count the feet which we measure off in a certain line as if each one were equivalent to all the rest. For all we know they may be different, but for our purpose of measuring they possess the same significance. This is neither an hypothesis nor an assumption nor a fiction, but a postulate needed for a definite purpose. For our purpose and according to the method employed they are the same. We postulate their sameness. We have made them the same, we treat them as equal. Their sameness depends upon the conditions from which we start and on the purpose which we have in view.

There are theorems which are true in arithmetic but which do not hold true in practical life. I will only mention the theorem $2+3+4=4+3+2=4+2+3$ etc. In real life the order in which things are pieced together is sometimes very essential, but in pure arithmetic, when we have started in the domain of nothingness and build with the products of our own counting which are ciphers absolutely equivalent to each other, the rule holds good and it will be serviceable for us to know it and to utilize its significance.

The positing of units which appears to be an indispensable step in the construction of arithmetic is also of great importance in actual psychology and becomes most apparent in the mechanism of vision.

Consider the fact that the kinematoscope has become possible only through an artificial separation of the successive pictures which are again fused together into a new continuum. The film which passes before the lens consists of a series of little pictures, and each one is singly presented, halting a moment and being separated from the next by a rotating fan which covers it at the moment when it is exchanged for the succeeding picture. If the moving figures on the screen did not consist of a definite number of pictures fused into one by our eye which is incapable of distinguishing their quick succession the whole sight would be blurred and we could see nothing but an indiscriminate and unanalyzable perpetual flux.

This method of our mind which produces units in a continuum may possess a still deeper significance, for it may mark the very beginning of the real world. For all we know the formation of the chemical atoms in the evolution of stellar nebulas may be nothing but an analogy to this process. The manifestation of life too begins with the creation of individuals—of definite living creatures which develop differently under different conditions and again the soul becomes possible by the definiteness of single sense-impressions which can be distinguished as units from others of a different type.

Thus the contrast between the continuum and the atomic formation appears to be fundamental and gives rise to many problems which have become especially troublesome in mathematics. But if we bear in mind that the method, so to speak, of atomic division is indispensable to change a world of continuous flux into a system that can be computed and determined with at least approximate accuracy, we will be apt to appreciate that the atomic fiction in arithmetic is an indispensable part of the method by which the whole science is created.

MATHEMATICS AND METAGEOMETRY.

DIFFERENT GEOMETRICAL SYSTEMS.

STRAIGHTNESS, flatness, and rectangularity are qualities which cannot (like numbers) be determined in purely quantitative terms; but they are determined nevertheless by the conditions under which our constructions must be made. A right angle is not an arbitrary amount of ninety degrees, but a quarter of a circle, and even the nature of angles and degrees is not derivable either from arithmetic or from pure reason. They are not purely quantitative magnitudes. They contain a qualitative element which cannot be expressed in numbers alone. A plane is not zero, but a zero of curvature in a boundary between two solids; and its qualitative element is determined, as Kant would express it, by *Anschauung,* or as we prefer to say, by pure motility, i. e., it belongs to the domain of the *a priori* of doing. For Kant's term *Anschauung* has the disadvantage of suggesting the passive sense denoted by the word "contemplation," while it is important to bear in mind that the thinking subject by its own activities creates the conditions that determine the qualities above mentioned.

Our method of creating by construction the straight line, the plane, and the right angle, does not exclude the possibility of other methods of space-measurement, the standards of which would not be even boundaries, such as straight lines, but lines possessed of either a positive curvature like the sphere or a negative curvature rendering their surface pseudo-spherical.

Spheres are well known and do not stand in need of description. Their curvature which is positive is determined by the reciprocal of their radius.

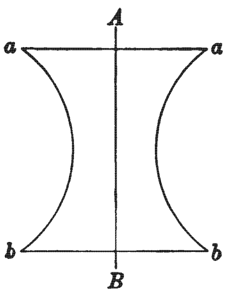


Fig. 1.

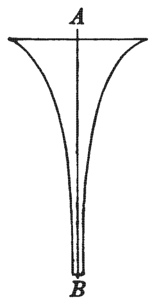


Fig. 2.

Pseudo-spheres are surfaces of negative curvature, and pseudo-spherical surfaces are saddle-shaped. Only limited pieces can be connectedly represented, and we reproduce from Helmholtz,[1] two instances. If arc *ab* in figure 1 revolves round an axis AB, it will describe a concave-convex surface like that of the inside of a wedding-ring; and in the same way, if either of the curves of figure 2 revolve round their axis of symmetry, it will describe one half of a pseudospherical surface resem-

[1] *Loc. cit.*, p. 42.

bling the shape of a morning-glory whose tapering stem is infinitely prolonged. Helmholtz compares the former to an anchor-ring, the latter to a champagne glass of the old style.

The sum of the angles of triangles on spheres always exceeds 180°, and the larger the sphere the more will their triangles resemble the triangle in the plane. On the other hand, the sum of the angles of triangles on the pseudosphere will always be somewhat less than 180°. If we define the right angle as the fourth part of a whole circuit, it will be seen that analogously the right angle in the plane differs from the right angles on the sphere as well as the pseudosphere.

We may add that while in spherical space several shortest lines are possible, in pseudospherical space we can draw one shortest line only. Both surfaces, however, are homogeneous (i. e., figures can be moved in it without suffering a change in dimensions), but the parallel lines which do not meet are impossible in either.

We may further construct surfaces in which changes of place involve either expansion or contraction, but it is obvious that they would be less serviceable as systems of space-measurement the more irregular they grow.

TRIDIMENSIONALITY.

Space is usually regarded as tridimensional, but there are some people who, following Kant, express themselves with reserve, saying that the mind of man may be built up in such a way as to conceive of objects in terms of three dimensions. Others think that the actual and real thing that is called space may be quite different from our tridimensional conception of it and may in point of fact be four, or five, or *n*-dimensional.

Let us ask first what "dimension" means.

Does dimension mean direction? Obviously not, for we have seen that the possibilities of direction in space are infinite.

Dimension is only a popular term for co-ordinate. In space there are no dimensions laid down, but in a space of infinite directions three co-ordinates are needed to determine from a given point of reference the position of any other point.

In a former section on "Even Boundaries as Standards of Measurement," we have halved space and produced a plane P_1 as an even boundary between the two halves; we have halved the plane P_1 by turning the plane so upon itself, that like a crease in a folded sheet of paper the straight line AB was produced on the plane. We then halved the straight line, the even boundary between the two half-planes, by again turning the plane upon itself so that the line AB covered its own prolongation. It is as if our folded sheet of paper were folded a second time upon itself so that the crease would be folded upon itself and one part of the same fall exactly upon and cover the other part. On opening the sheet we have a second crease crossing the first one making the perpendicular CD, in the point O, thus producing right angles on the straight line AB, represented in the cross-creases of the twice folded sheet of paper. Here the method of producing even boundaries by halving comes to a natural end. So far our products are the plane, the straight line, the point and as an incidental but valuable by-product, the right angle.

We may now venture on a synthesis of our materials. We lay two planes, P_2 and P_3, through the two creases at right angles on the original plane P_1, represented by the sheet of paper, and it becomes apparent that the two new planes P_2 and P_3 will intersect at O, producing a line EF common to both planes P_2 and P_3, and they will bear the same relation to each as each one does to the original plane P_1, that is to say: the whole system is congruent with itself. If we make the planes change places, P_1 may as well take the place of P_2 and P_2 of P_3 and P_3 of P_2 or P_1 of P_3, etc., or *vice versa,* and all the internal relations would remain absolutely the same. Accordingly we have here in this system of the three planes at right angles (the result of repeated halving), a composition of even boundaries which, as the simplest and least complicated construction of its kind, recomends itself for a standard of measurement of the whole spread of motility.

The most significant feature of our construction consists in this, that we thereby produce a convenient system of reference for determining every possible point in co-ordinates of straight lines standing at right angles to the three planes.

If we start from the ready conception of objective space (the juxtaposition of things) we can refer the several distances to analogous *loci* in our system of the three planes, mutually perpendicular, each to the others. We cut space in two equal halves by the horizontal plane P_1. We repeat the cutting so as to let the two halves of the first cut in their angular relation to the new cut (in P_2) be congruent with each other, a procedure which is possible only if we make use of the even boundary concept with which we have become acquainted. Accordingly, the second cut should stand at right angles on the first cut. The two planes P_1 and P_2 have one line in common, EF, and any plane placed at right angles to EF (in the point O) will again satisfy the demand of dividing space, including the two planes P_1 and P_2, into two congruent halves. The two new lines, produced by the cut of the third plane P_3 through the two former planes P_1 and P_2, stand both at right angles to EF. Should we continue our method of cutting space at right angles in O on either of these lines, we would produce a plane coincident with P_1, which is to say, that the possibilities of the system are exhausted.

This implies that in any system of pure space *three co-ordinates are sufficient* for the determination of any place from a given reference point.

THREE A CONCEPT OF BOUNDARY.

The number three is a concept of boundary as much as the straight line. Under specially complicated conditions we might need more than three co-ordinates to calculate the place of a point, but in empty space the number three, the lowest number that is really and truly a number, is sufficient. If space is to be empty space from which the notion of all concrete things is excluded, a kind of model constructed for the purpose of determining juxtaposition, three co-ordinates are sufficient, because our system of reference consists of three planes, and we have seen above that there is no possibility of introducing a fourth plane without destroying its character of being congruent with itself, which imparts to it the simplicity and uniqueness that render it available for a standard of measurement.

Three is a peculiar number which is of great significance. It is the first real number, being the simplest multiplex. One and two and also zero are of course numbers if we consider them as members of the number-system in its entirety, but singly regarded they are not yet numbers in the full sense of the word. One is the unit, two is a couple or a pair, but three is the smallest amount of a genuine plurality. Savages who can distinguish only between one and two have not yet evolved the notion of number; and the transition to the next higher stage involving the knowledge of "three" passes through a mental condition in which there exists only the notion one, two, and plurality of any kind. When the idea of three is once definitely recognized, the naming of all other numbers can follow in rapid succession.

In this connection we may incidentally call attention to the significance of the grammatical dual number as seen in the Semitic and Greek languages. It is a surviving relic and token of a period during which the unit, the pair, and the uncounted plurality constituted the entire gamut of human arithmetic. The dual form of grammatical number by the development of the number-system became redundant and cumbersome, being retained only for a while to express the idea of a couple, a pair that naturally belong together.

Certainly, the origin of the notion three has its germ in the nature of abstract anyness. Nor is it an accident that in order to construct the simplest figure which is a real figure, at least three lines are needed. The importance of the triangle, which becomes most prominent in trigonometry, is due to its being the simplest possible figure which accordingly possesses the intrinsic worth of economy.

The number three plays also a significant part in logic, and in the branches of the applied sciences, and thus we need not be astonished at finding the very idea, three, held in religious reverence, for the doctrine of the Trinity has its basis in the constitution of the universe and can be fully justified by the laws of pure form.

SPACE OF FOUR DIMENSIONS.

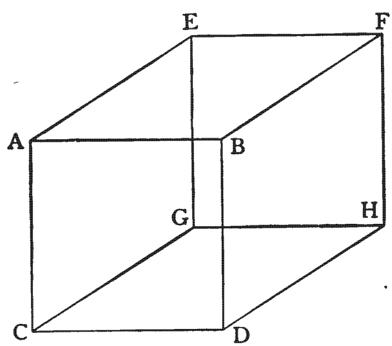
The several conceptions of space of more than three dimensions are of a purely abstract nature, yet they are by no means vague, but definitely determined by the conditions of their construction. Therefore we can determine their properties even in their details with perfect exactness and formulate in abstract thought the laws of four-, five-, six-, and *n*-dimensional space. The difficulty with which we are beset in constructing *n*-dimensional spaces consists in our inability to make them representable to our senses. Here we are confronted with what may be called the limitations of our mental constitution. These limitations, if such they be, are conditioned by the nature of our mode of motion, which, if reduced to a mathematical system, needs three co-ordinates, and this means that our space-conception is tridimensional.

We ourselves are tridimensional; we can measure the space in which we move with three co-ordinates, yet we can definitely say that if space were four-dimensional, a body constructed of two factors, so as to have a four-dimensional solidity, would be expressed in the formula:

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

We can calculate, compute, excogitate, and describe all the characteristics of four-dimensional space, so long as we remain in the realm of abstract thought and do not venture to make use of our motility and execute our plan in an actualized construction of motion. From the standpoint of pure logic, there is nothing irrational about the assumption; but as soon as we make an *a priori* construction of the scope of our motility, we find out the incompatibility of the whole scheme.

In order to make the idea of a space of more than three dimensions plausible or intelligible, we resort to the relation between two-dimensional beings and tridimensional space. The nature of tridimensional space may be indicated yet not fully represented in



two-dimensional space. If we construct a square upon the line AB one inch long, it will be bounded by four lines each an inch in length. In order to construct upon the square ABCD a cube of the same measure, we must raise the square by one inch into the third dimension in a direction at right angles to its surface, the result being a figure bounded by six surfaces, each of which is a one-inch square. If two-dimensional beings who could not rise into the third dimension wished to gain an idea of space of a higher dimensionality and picture in their own two-dimensional mathematics the results of three dimensions, they might push out the square in any direction within their own plane to a distance of one inch, and then connect all the corners of the image of the square in its new position with the corresponding points of the old square. The result would be what is to us tridimensional beings the picture of a cube.

When we count the plane quadrilateral figures produced by this combination we find that there are six, corresponding to the boundaries of a cube. We must bear in mind that only the original and the new square will be real squares, the four intermediary figures which have originated incidentally through our construction of moving the square to a distance, exhibit a slant and to our two-dimensional beings they appear as distortions of a rectangular relation, which faultiness has been caused by the insufficiency of their methods of representation. Moreover all squares count in full and where their surfaces overlap they count double.

Two-dimensional beings having made such a construction must however bear in mind that the field covered by the sides, GEFH and BFDH does not take up any room in their own plane, for it is only a picture of the extension which reaches out either above or below their own plane; and if they venture out of this field covered by their construction, they have to remember that it is as empty and unoccupied as the space beyond the boundaries AC and AB.

Now if we tridimensional beings wish to do the same, how shall we proceed?

We must move a tridimensional body in a rectangular direction into a new (i. e., the fourth) dimension, and being unable to accomplish this we may represent the operation by mirrors. Having three dimensions we need three mirrors standing at right angles. We know by *a priori* considerations according to the principle of our construction that the boundaries of a four-dimensional body must be solids, i. e.,tridimensional bodies, and while the sides of a cube (algebraically represented by a^3) must be six surfaces (i. e., two-dimensional figures, one at each end of the dimensional line) the boundaries of an analogous four-dimensional body built up like the cube and the square on a rectangular plan, must be eight solids, i. e., cubes. If we build up three mirrors at right angles and place any object in the intersecting corner we shall see the object not once, but eight times. The body is reflected below, and the object thus doubled is mirrored not only on both upright sides but in addition in the corner beyond, appearing in either of the upright mirrors coincidingly in the same place. Thus the total multiplication of our tridimensional boundaries of a four-dimensional complex is rendered eightfold.

We must now bear in mind that this representation of a fourth dimension suffers from all the faults of the analogous figure of a cube in two-dimensional space. The several figures are not eight independent bodies but they are mere boundaries and the four dimensional space is conditioned by their interrelation. It is that unrepresentable something which they enclose, or in other words, of which they are assumed to be boundaries. If we were four-dimensional beings we could naturally and easily enter into the mirrored space and transfer tridimensional bodies or parts of them into those other objects reflected here in the mirrors representing the boundaries of the four-dimensional object. While thus on the one hand the mirrored pictures would be as real as the original object, they would not take up the space of our three dimensions, and in this respect our method of representing the fourth dimension by mirrors would be quite analogous to the cube pictured on a plane surface, for the space to which we (being limited by our tridimensional space-conception) would naturally relegate the seven additional mirrored images is unoccupied, and if we should make the trial, we would find it empty.

Further experimenting in this line would render constructions of a more complicated character more and more difficult although not quite impossible. Thus we might represent the formula $(a+b)^4$ by placing a wire model of a cube, representing the proportions $(a+b)^3$, in the corner of our three mirrors, and we would then verify by ocular inspection the truth of the formula

$$a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

However, we must bear in mind that all the solids here seen are merely the boundaries of four-dimensional bodies. All of them with the exception of the ones in the inner corner are scattered around and yet the analogous figures would have to be regarded as being most intimately interconnected, each set of them forming one four-dimensional complex. Their separation is in appearance only, being due to the insufficiency of our method of presentation.

We might obviate this fault by parceling our wire cube and instead of using three large mirrors for reflecting the entire cube at once, we might insert in its dividing planes double mirrors, i. e., mirrors which would reflect on the one side the magnitude *a* and on the other the magnitude *b*. In this way we would come somewhat nearer to a faithful representation of the nature of four-dimensional space, but the model being divided up into a number of mirror-walled rooms, would become extremely complicated and it would be difficult for us to bear always in mind that the mirrored spaces count on both sides at once, although they overlap and (tri-dimensionally considered) seem to fall the one into the other, thus presenting to our eyes a real labyrinth of spaces that exist within each other without interfering with one another. They thus render new depths visible in all three dimensions, and in order to represent the whole scheme of a four-dimensional complex in its full completeness, we ought to have three mirrors at right angles placed at every point in our tridimensional space. The scheme itself is impossible, but the idea will render the nature of four-dimensional space approximately clear. If we were four-dimensional beings we would be possessed of the mirror-eye which in every direction could look straightway round every corner of the third dimension. This seems incredible, but it can not be denied that tridimensional space lies open to an inspection from the domain of the fourth dimension, just as every point of a Euclidean plane is open to inspection from above to tridimensional vision. Of course we may demur (as we actually do) to believing in the reality of a space of four dimensions, but that being granted, the inferences can not be doubted.

THE APPARENT ARBITRARINESS OF THE A PRIORI.

Since Riemann has generalized the conception of space, the tridimensionality of space seems very arbitrary.

Why are three co-ordinates sufficient for pure space determinations? The obvious answer is, Because we have three planes in our construction of space-boundaries. We might as well ask, why do the three planes cut the entire space into 8 equal parts? The simple answer is that we have halved space three times, and $2^3=8$. The reason is practically the same as that for the simpler question, why have we two halves if we divide an apple into two equal parts?

These answers are simple enough, but there is another aspect of the question which here seems in order: Why not continue the method of halving? And there is no other answer than that it is impossible. The two superadded planes P_2 and P_3 both standing at right angles to the original plane, necessarily halve each other, and thus the four right angles of each plane P_2 and P_3 on the center of intersection, form a complete plane for the same reason that four quarters are one whole. We have in each case four quarters, and $^4/_4=1$.

Purely logical arguments (i. e., all modes of reasoning that are rigorously *a priori*, the *a priori* of abstract being) break down and we must resort to the methods of the *a priori* of doing. We cannot understand or grant the argument without admitting the conception of space, previously created by a spread of pure motion. Kant would say that we need here the data of *reine Anschauung*, and Kant's *reine Anschauung* is a product of our motility. As soon as we admit that there is an *a priori* of doing (of free motility) and that our conception of pure space and time (Kant's *reine Anschauung*) is its product, we understand that our mathematical conceptions cannot be derived from pure logic alone but must finally depend upon our motility, viz., the function that begets our notion of space.

If we divide an apple by a vertical cut through its center, we have two halves. If we cut it again by a horizontal cut through its center, we have four quarters. If we cut it again with a cut that is at right angles to both prior cuts, we have eight eighths. It is obviously impossible to insert among these three cuts a fourth cut that would stand at right angles to these others. The fourth cut through the center, if we needs would have to make it, will fall into one of the prior cuts and be a mere repetition of it, producing no new result; or if we made it slanting, it would not cut all eight parts but only four of them; it would not produce sixteen equal parts, but twelve unequal parts, viz., eight sixteenths plus four quarters.

If we do not resort to a contemplation of the scope of motion, if we neglect to represent in our imagination the figure of the three planes and rely on pure reason alone (i. e., the rigid *a priori*), we have no means of refuting the assumption that we ought to be able to continue halving the planes by other planes at right angles. Yet is the proposition as inconsistent as to expect that there should be regular pentagons, or hexagons, or triangles, the angles of which are all ninety degrees.

From the standpoint of pure reason alone we cannot disprove the incompatibility of the idea of a rectangular pentagon. If we insist on constructing by hook or crook a rectangular pentagon, we will succeed, but we must break away from the straight line or the plane. A rectangular pentagon is not absolutely impossible; it is absolutely impossible in the plane; and if we produce one, it will be twisted.

Such was the result of Lobatchevsky's and Bolyai's construction of a system of geometry in which the theorem of parallels does not hold. Their geometries cease to be even; they are no longer Euclidean and render the even boundary conceptions unavailable as standards of measurement.

If by logic we understand consistency, and if anything that is self-contradictory and incompatible with its own nature be called illogical, we would say that it is not the logic of pure reason that renders certain things impossible in our geometric constructions, but the logic of our scope of motion. The latter introduces a factor which determines the nature of geometry, and if this factor is neglected or misunderstood, the fundamental notions of geometry must appear arbitrary.

DEFINITENESS OF CONSTRUCTION.

The problems which puzzle some of the metaphysicians of geometry seem to have one common foundation, which is the definiteness of geometrical construction. Geometry starts from empty nothingness, and we are confronted with rigid conditions which it does not lie in our power to change. We make a construction, and the result is something new, perhaps something which we have not intended, something at which we are surprised. The synthesis is a product of our own making, yet there is an objective element in it over which we have no command, and this objective element is rigid, uncompromising, an irrefragable necessity, a stubborn fact, immovable, inflexible, immutable. What is it?

Our metageometricians overlook the fact that their nothingness is not an absolute nothing, but only an absence of concreteness. If they make definite constructions, they must (if they only remain consistent) expect definite results. This definiteness is the logic that dominates their operations. Sometimes the results seem arbitrary, but they never are; for they are necessary, and all questions *why?* can elicit only answers that turn in a circle and are mere tautologies.

Why, we may ask, do two straight lines, if they intersect, produce four angles? Perhaps we did not mean to construct angles, but here we have them in spite of ourselves.

And why is the sum of these four angles equal to 360°? Why, if two are acute, will the other two be found obtuse? Why, if one angle be a right angle, will all four be right angles? Why will the sum of any two adjacent angles be equal to two right angles? etc. Perhaps we should have preferred three angles only, or four acute angles, but we cannot have them, at least not by this construction.

We have seen that the tridimensionality of space is arbitrary only if we judge of it as a notion of pure reason, without taking into consideration the method of its construction as a scope of mobility. Tridimensionality is only one instance of apparent arbitrariness among many others of the same kind.

We cannot enclose a space in a plane by any figure of two straight lines, and we cannot construct a solid of three even surfaces.

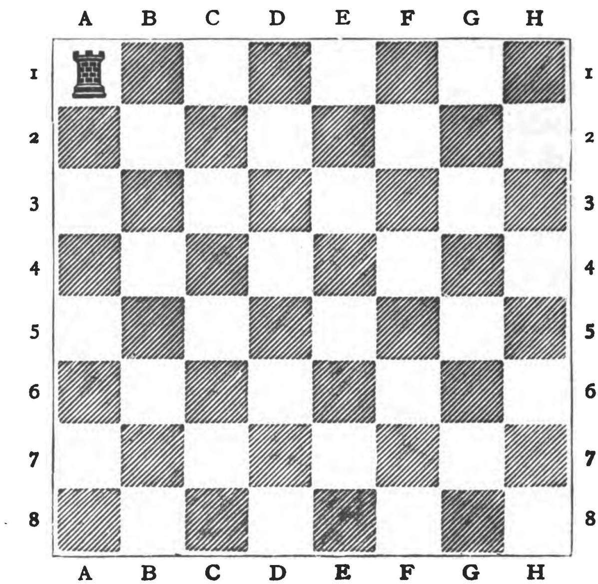
There are only definite forms of polyhedra possible, and the surfaces of every one are definitely determined. To the mind uninitiated into the secrets of mathematics it would seem arbitrary that there are two hexahedra (viz., the cube and the duplicated tetrahedron), while there is no heptahedron. And why can we not have an octahedron with quadrilateral surfaces? We might as well ask, why is the square not round!

Prof. G. B. Halsted says in the Translator's Appendix to his English edition of Lobatchevsky's *Theory of Parallels,* p. 48:

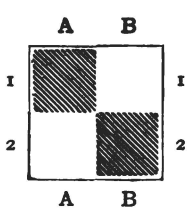
> "But is it not absurd to speak of space as interfering with anything? If you think so, take a knife and a raw potato and try to cut it into a seven-edged solid."

Truly Professor Halsted's contention, that the laws of space interfere with our operations, is true. Yet it is not space that squeezes us, but the laws of construction determine the shape of the figures which we make.

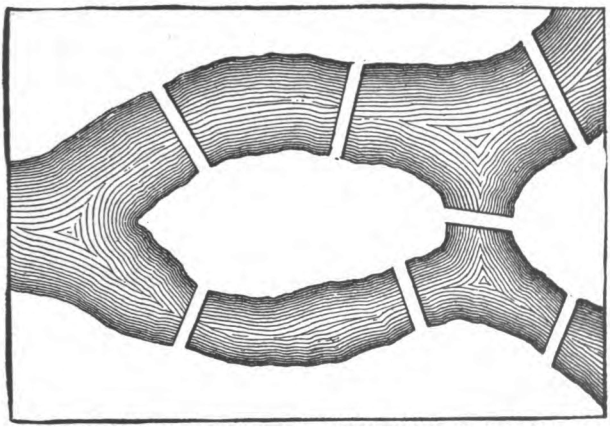
A simple instance that illustrates the way in which space interferes with our plans and movements is the impossible demand on the chessboard to start a rook in one corner (A1) and pass it with the rook motion over all the fields once, but only once, and let it end its journey on the opposite corner (H8). Rightly considered it is not space that interferes with our mode of action, but the law of consistency. The proposition does not contain anything illogical; the words are quite rational and the sentences grammatically correct. Yet is the task impossible, because we cannot turn to the right and left at once, nor can we be in two places at once, neither can we undo an act once done or for the



nonce change the rook into a bishop; but something of that kind would have to be done, if we start from A1 and pass with the rook motion through A2 and B1 over to B2. In other words: Though the demand is not in conflict with the logic of abstract being or the grammar of thinking, it is impossible because it collides with the logic of doing; the logic of moving about, the *a priori* of motility.

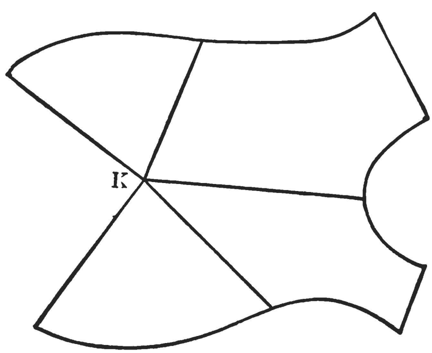


The famous problem of crossing seven bridges leading to the two Königsberg Isles, is of the same kind. Near the mouth of the Pregel River there is an island called Kneiphof, and the situation of the seven bridges is shown as in the adjoined dia-



THE SEVEN BRIDGES OF KÖNIGSBERG.

gram. A discussion arose as to whether it was possible to cross all the bridges in a single promenade



EULER'S DIAGRAM.

without crossing any one a second time. Finally Euler solved the problem in a memoir presented to the Academy of Sciences of St. Petersburg in 1736, pointing out why the task could not be done. He reproduced the situation in a diagram and proved that if the number of lines meeting at the point K (representing the island Kneiphof as K) were even the task was possible, but if the number is odd it can not be accomplished.

The squaring of the circle is similarly an impossibility.

We cannot venture on self-contradictory enterprises without being defeated, and if the relation of the circumference to the diameter is an infinite transcendent *series*, being

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{15} + \tfrac{1}{17} - \tfrac{1}{19} + \tfrac{1}{21} - \tfrac{1}{23} + \cdots$$

we cannot expect to square the circle.

If we compute the series, π becomes 3.14159265 ...., figures which seem as arbitrary as the most whimsical fancy.

It does not seem less strange that $\epsilon = 2.71828$; and yet it is as little arbitrary as the equation $3 \times 4 = 12$.

The definiteness of our mathematical constructions and arithmetical computations is based upon the inexorable law of determinism, and everything is fixed by the mode of its construction.

ONE SPACE, BUT VARIOUS SYSTEMS OF SPACE-MEASUREMENT.

Riemann has generalized the idea of space and would thus justify us in speaking of "spaces." The common notion of space, which agrees best with that of Euclidean geometry, has been degraded into a mere species of space, one possible instance among many other possibilities. And its very legitimacy has been doubted, for it has come to be looked upon in some quarters as only a popular (not to say vulgar and commonplace) notion, a mere working hypothesis, infested with many arbitrary conditions of which the ideal conception of absolute space should be free. How much more interesting and aristocratic are curved space, the dainty two-dimensional space, and above all the four-dimensional space with its magic powers!

The new space-conception seems bewildering. Some of these new spaces are constructions that are not concretely representable, but only abstractly thinkable; yet they allow us to indulge in ingenious dreams. Think only of two-dimensional creatures, and how limited they are! They can have no conception of a third dimension! Then think of four-dimensional beings; how superior they must be to us poor tridimensional bodies! As we can take a figure situated within a circle through the third dimension and put it down again outside the circle without crossing the circumference, so four-dimensional beings could take tridimensional things encased in a tridimensional box from their hiding-place and put them back on some other spot on the outside. They could easily help themselves to all the money in the steal-lined safes of our banks, and they could perform the most difficult obstetrical feats without resorting to the dangerous Cæsarean operation.

Curved space is not less interesting. Just as light may pass through a medium that offers such a resistance as will involve a continuous displacement of the rays, so in curved space the lines of greatest intensity would be subject to a continuous modification. The beings of curved space may be assumed to have no conception of truly straight lines. They must deem it quite natural that if they walk on in the straightest possible manner they will finally but unfailingly come back to the same spot. Their world-space is not as vague and mystical as ours: it is not infinite, hazy at a distance, vague and without end, but definite, well rounded off, and perfect. Presumably their lives have the same advantages moving in boundless circles, while our progression in straight lines hangs between two infinitudes—the past and the future!

All these considerations are very interesting because they open new vistas to imaginative speculators and inventors, and we cannot deny that the generalization of our space-conception has proved helpful by throwing new light upon geometrical problems and widening the horizon of our mathematical knowledge.

Nevertheless after a mature deliberation of Riemann's proposition, I have come to the conclusion that it leads us off in a wrong direction, and in contrast to his conception of space as being one instance among many possibilities, I would insist upon the uniqueness of space. Space is the possibility of motion in all directions, and mathematical space is the ideal construction of our scope of motion in all directions.

The homogeneity of space is due to our abstraction which omits all particularities, and its homaloidality means only that straight lines are possible not in the real world, but in mathematical thought, and will serve us as standards of measurement.

Curved space, so called, is a more complicated construction of space-measurement to which some additional feature of a particular nature has been admitted, and in which we waive the advantages of even boundaries as means of measurement.

Space, the actual scope of motion, remains different from all systems of space-measurement, be they homaloidal or curved, and should not be subsumed with them under one and the same category.

Riemann's several space-conceptions are not spaces in the proper sense of the word, but systems of space-measurement. It is true that space is a tridimensional manifold, and a plane a two-dimensional manifold, and we can think of other systems of *n*-manifoldness; but for that reason all these different manifolds do not become spaces. Man is a mammal having two prehensiles (his hands); the elephant is a mammal with one prehensile (his trunk); tailless monkeys like the pavian have four; and tailed monkeys have five prehensiles. Is there any logic in extending the denomination man to all these animals, and should we define the elephant as a man with one prehensile, the pavian as a man with four prehensiles and tailed monkeys as men with five prehensiles? Our zoologists would at once protest and denounce it as an illogical misuse of names.

Space is a manifold, but not every manifold is a space.

Of course every one has a right to define the terms he uses, and obviously my protest simply rejects Riemann's use of a word, but I claim that his identification of "space" with "manifold" is the source of inextricable confusion.

It is well known that all colors can be reduced to three primary colors, yellow, red, and blue, and thus we can determine any possible tint by three co-ordinates, and color just as much as mathematical space is a threefold, viz., a system in which three co-ordinates are needed for the determination of any thing. But because color is a threefold, no one would assume that color is space.

Riemann's manifolds are systems of measurement, and the system of three co-ordinates on three intersecting planes is an *a priori* or purely formal and ideal construction invented to calculate space. We can invent other more complicated systems of measurement, with curved lines and with more than three or less than three co-ordinates. We can even employ them for space-measurement, although they are rather awkward and unserviceable; but these systems of measurement are not "spaces," and if they are called so, they are spaces by courtesy only. By a metaphorical extension we allow the idea of system of space-measurement to stand for space itself. It is a brilliant idea and quite as ingenious as the invention of animal fables in which our quadruped fellow-beings are endowed with speech and treated as human beings. But such poetical licences, in which facts are stretched and the meaning of terms is slightly modified, is possible only if instead of the old-fashioned straight rules of logic we grant a slight curvature to our syllogisms.

FICTITIOUS SPACES AND THE APRIORITY OF ALL SYSTEMS OF SPACE-MEASUREMENT.

Mathematical space, so called, is strictly speaking no space at all, but the mental construction of a manifold, being a tridimensional system of space-measurement invented for the determination of actual space.

Neither can a manifold of two dimensions be called a space. It is a mere boundary in space, it is no reality, but a concept, a construction of pure thought.

Further, the manifold of four dimensions is a system of measurement applicable to any reality for the determination of which four co-ordinates are needed. It is applicable to real space if there is connected with it in addition to the three planes at right angles another condition of a constant nature, such as gravity.

At any rate, we must deny the applicability of a system of four dimensions to empty space void of any such particularity. The idea of space being four-dimensional is chimerical if the word space is used in the common acceptance of the term as juxtaposition or as the scope of motion. So long as four quarters make one whole, and four right angles make one entire circumference, and so long as the contents of a sphere which covers the entire scope of motion round its center equals $\frac{4}{3}\pi r^3$, there is no sense in entertaining the idea that empty space might be four-dimensional.

But the argument is made and sustained by Helmholtz that as two-dimensional beings perceive two dimensions only and are unable to think how a third dimension is at all possible, so we tridimensional beings cannot represent in thought the possibility of a fourth dimension. Helmholtz, speaking of beings of only two dimensions living on the surface of a solid body, says:

"If such beings worked out a geometry, they would of course assign only two dimensions to their space. They would ascertain that a point in moving describes a line, and that a line in moving describes a surface. But they could as little represent to themselves what further spatial construction would be generated by a surface moving out of itself, as we can represent what should be generated by a solid moving out of the space we know. By the much-abused expression 'to represent' or 'to be able to think how something happens' I understand—and I do not see how anything else can be understood by it without loss of all meaning—the power of imagining the whole series of sensible impressions that would be had in such a case. Now, as no sensible impression is known relating to such an unheard-of event, as the movement to a fourth dimension would be to us, or as a movement to our third dimension would be to the inhabitants of a surface, such a 'representation' is as impossible as the 'representation' of colors would be to one born blind, if a description of them in general terms could be given to him.

"Our surface-beings would also be able to draw shortest lines in their superficial space. These would not necessarily be straight lines in our sense, but what are technically called *geodetic lines* of the surface on which they live; lines such as are described by a *tense* thread laid along the surface, and which can slide upon it freely."....

"Now, if beings of this kind lived on an infinite plane, their geometry would be exactly the same as our planimetry. They would affirm that only one straight line is possible between two points; that through a third point lying without this line only one line can be drawn parallel to it; that the ends of a straight line never meet though it is produced to infinity, and so on."....

"But intelligent beings of the kind supposed might also live on the surface of a sphere. Their shortest or straightest line between two points would then be an arc of the great circle passing through them."....

"Of parallel lines the sphere-dwellers would know nothing. They would maintain that any two straightest lines, sufficiently produced, must finally cut not in one only but in two points. The sum of the angles of a triangle would be always greater than two right angles, increasing as the surface of the triangle grew greater. They could thus have no conception of geometrical similarity between greater and smaller figures of the same kind, for with them a greater triangle must have different angles from a smaller one. Their space would be unlimited, but would be found to be finite or at least represented as such.

"It is clear, then, that such beings must set up a very different system of geometrical axioms from that of the inhabitants of a plane, or from ours with our space of three dimensions, though the logical powers of all were the same."

I deny what Helmholtz implicitly assumes that sensible impressions enter into the fabric of our concepts of purely formal relations. We have the idea of a surface as a boundary between solids, but surfaces do not exist in reality. All real objects are solid, and our idea of surface is a mere fiction of abstract reasoning. Two dimensional things are unreal, we have never seen any, and yet we form the notion of surfaces, and lines, and points, and pure space, etc. There is no straight line in existence, hence it can produce no sense-impression, and yet we have the notion of a straight line. The straight lines on paper are incorrect pictures of the true straight lines which are purely ideal constructions. Our *a priori* constructions are not a product of our sense-impressions, but are independent of sense or anything sensed.

It is of course to be granted that in order to have any conception, we must have first of all sensation, and we can gain an idea of pure form only by abstraction. But having gained a fund of abstract notions, we can generalize them and modify them; we can use them as a child uses its building blocks, we can make constructions of pure thought unrealizable in the concrete world of actuality. Some of such constructions cannot be represented in concrete form, but they are not for that reason unthinkable. Even if we grant that two-dimensional beings were possible, we would have no reason to assume that two-dimensional beings could not construct a tridimensional space-conception.

Two-dimensional beings could not be possessed of a material body, because their absolute flatness substantially reduces their shape to nothingness. But if they existed, they would be limited to movements in two directions and thus must be expected to be incredulous as to the possibility of jumping out of their flat existence and returning into it through a third dimension. Having never moved in a third dimension, they could speak of it as the blind might discuss colors; in their flat minds they could have no true conception of its significance and would be unable to clearly picture it in their imagination; but for all their limitations, they could very well develop the abstract idea of tridimensional space and therefrom derive all particulars of its laws and conditions and possibilities in a similar way as we can acquire the notion of a space of four dimensions.

Helmholtz continues:

"But let us proceed still farther.

"Let us think of reasoning beings existing on the surface of an egg-shaped body. Shortest lines could be drawn between three points of such a surface and a triangle constructed. But if the attempt were made to construct congruent triangles at different parts of the surface, it would be found that two triangles, with three pairs of equal sides, would not have their angles equal."

If there were two-dimensional beings living on an egg-shell, they would most likely have to determine the place of their habitat by experience just as much as we tridimensional beings living on a flattened sphere have to map out our world by measurements made *a posteriori* and based upon *a priori* systems of measurement.

If the several systems of space-measurement were not *a priori* constructions, how could Helmholtz who does not belong to the class of two-dimensional beings tell us what their notions must be like?

I claim that if there were surface beings on a sphere or on an egg-shell, they would have the same *a priori* notions as we have; they would be able to construct straight lines, even though they were constrained to move in curves only; they would be able to define the nature of a space of three dimensions and would probably locate in the third dimension their gods and the abode of spirits. I insist that not sense experience, but *a priori* considerations, teach us the notions of straight lines.

The truth is that we tridimensional beings actually do live on a sphere, and we cannot get away from it. What is the highest flight of an æronaut and the deepest descent into a mine if measured by the radius of the earth? If we made an exact imitation of our planet, a yard in diameter, it would be like a polished ball, and the highest elevations would be less than a grain of sand; they would not be noticeable were it not for a difference in color and material.

When we become conscious of the nature of our habitation, we do not construct *a priori* conceptions accordingly, but feel limited to a narrow surface and behold with wonder the infinitude of space beyond. We can very well construct other *a priori* notions which would be adapted to one, or two, or four-dimensional worlds, or to spaces of positive or of negative curvatures, for all these constructions are ideal; they are mind-made and we select from them the one that would best serve our purpose of space-measurement.

The claim is made that if we were four-dimensional beings, our present three-dimensional world would appear to us as flat and shallow, as the plane is to us in our present tridimensional predicament. That statement is true, because it is conditioned by an "if." And what pretty romances have been built upon it! I remind my reader only of the ingenious story *Flatland, Written by a Square,* and portions of Wilhelm Busch's charming tale *Edward's Dream*[2]; but the worth of conditional truths depends upon the assumption upon which they are made contingent, and the argument is easy enough that if things were different, they would not be what they are. If I had wings, I could fly; if I had gills I could live under water; if I were a magician I could work miracles.

[2] *The Open Court,* Vol. VIII, p. 4266 et seq.

INFINITUDE.

The notion is rife at present that infinitude is self-contradictory and impossible. But that notion originates from the error that space is a thing, an objective and concrete reality, if not actually material, yet consisting of some substance or essence. It is true that infinite things cannot exist, for things are always concrete and limited; but space is pure potentiality of concrete existence. Pure space is materially considered nothing. That this pure space (this apparent nothing) possesses some very definite positive qualities is a truth which at first sight may seem strange, but on closer inspection is quite natural and will be conceded by every one who comprehends the paramount significance of the doctrine of pure form.

Space being pure form of extension, it must be infinite, and infinite means that however far we go, in whatever direction we choose, we can go farther, and will never reach an end. Time is just as infinite as space. Our sun will set and the present day will pass away, but time will not stop. We can go backward to the beginning, and we must ask what was before the beginning. Yet suppose we could fill the blank with some hypothesis or another, mythological or metaphysical, we would not come to an absolute beginning. The same is true as to the end. And if the universe broke to pieces, time would continue, for even the duration in which the world would lie in ruins would be measurable.

Not only is space as a totality infinite, but in every part of space we have infinite directions.

What does it mean that space has infinite directions? If you lay down a direction by drawing a line from a given point, and continue to lay down other directions, there is no way of exhausting your possibilities. Light travels in all directions at once; but "all directions" means that the whole extent of the surroundings of a source of light is agitated, and if we attempt to gather in the whole by picking up every single direction of it, we stand before a task that cannot be finished.

In the same way any line, though it be of definite length, can suffer infinite division, and the fraction $^1/_3$ is quite definite while the same amount if expressed in decimals as 0.333. . . ., can never be completed. Light actually travels in all directions, which is a definite and concrete process, but if we try to lay them down one by one we find that we can as little exhaust their number as we can come to an end in divisibility or as we can reach the boundary of space, or as we can come to an ultimate number in counting. In other words, reality is actual and definite but our mode of measuring it or reducing it to formulas admits of a more or less approximate treatment only, being the function of an infinite progress in some direction or other. There is an objective *raison d'être* for the conception of the infinite, but our formulation of it is subjective, and the puzzling feature of it originates from treating the subjective feature as an objective fact.

These considerations indicate that infinitude does not appertain to the thing, but to our method of viewing the thing. Things are always concrete and definite, but the relational of things admits of a progressive treatment. Space is not a thing, but the relational feature of things. If we say that space is infinite, we mean that a point may move incessantly and will never reach the end where its progress would be stopped.

There is a phrase current that the finite cannot comprehend the infinite. Man is supposed to be finite, and the infinite is identified with God or the Unknowable, or anything that surpasses the comprehension of the average intellect. The saying is based upon the prejudicial conception of the infinite as a realized actuality, while the infinite is not a concrete thing, but a series, a process, an aspect, or the plan of action that is carried on without stopping and shall not, as a matter of principle, be cut short. Accordingly, the infinite (though in its completeness unactualizable) is neither mysterious nor incomprehensible, and though mathematicians be finite, they may very successfully employ the infinite in their calculations.

I do not say that the idea of infinitude presents no difficulties, but I do deny that it is a self-contradictory notion and that if space must be conceived to be infinite, mathematics will sink into mysticism.

GEOMETRY REMAINS A PRIORI.

Those of our readers who have closely followed our arguments will now understand how in one important point we cannot accept Mr. B. A. W. Russell's statement as to the main result of the metageometrical inquisition. He says:

"There is thus a complete divorce between geometry and the study of actual space. Geometry does not give us certain knowledge as to what exists. That peculiar position which geometry formerly appeared to occupy, as an *a priori* science giving knowledge of something actual, now appears to have been erroneous. It points out a whole series of possibilities, each of which contains a whole system of connected propositions; but it throws no more light upon the nature of our space than arithmetic throws upon the population of Great Britain. Thus the plan of attack suggested by non-Euclidean geometry enables us to capture the last stronghold of those who attempt, from logical or *a priori* considerations, to deduce the nature of what exists. The conclusion *suggested* is, that no existential proposition can be deduced from one which is not existential. But to *prove* such a conclusion would demand a treatise upon all branches of philosophy."[3]

It is a matter of course that the single facts as to the population of Great Britain must be supplied by counting, and in the same way the measurements of angles and actual distances must be taken by *a posteriori* transactions; but having ascertained some lines and angles, we can (assuming our data to be correct) calculate other items with absolute

[3] In *the new volumes of the Encyclopædia Britannica,* Vol. XXVIII, of the complete work, *s. v.* Geometry, Non-Euclidean, p. 674.

exactness by purely *a priori* argument. There is no need (as Mr. Russell puts it) "from logical or *a priori* considerations to deduce the nature of what exists,"—which seems to mean, to determine special features of concrete instances. No one ever assumed that the nature of particular cases, the qualities of material things, or sense-affecting properties, could be determined by *a priori* considerations. The real question is, whether or not the theorems of space relations and, generally, purely formal conceptions, such as are developed *a priori* in geometry and kindred formal sciences, will hold good in actual experience. In other words, can we assume that form is an objective quality, which would imply that the constitution of the actual world must be the same as the constitution of our purely *a priori* sciences? We answer this latter question in the affirmative.

We cannot determine by *a priori* reasoning the population of Great Britain. But we can *a posteriori* count the inhabitants of several towns and districts, and determine the total by addition. The rules of addition, of division, and multiplication can be relied upon for the calculation of objective facts.

Or to take a geometrical example. When we measure the distance between two observatories and also the angles at which at either end of the line thus laid down the moon appears in a given moment, we can calculate the moon's distance from the earth; and this is possible only on the assumption that the formal relations of objective space are the same as those of mathematical space. In other words, that our *a priori* mathematical calculations can be made to throw light upon the nature of space,—the real objective space of the world in which we live.

* * *

The result of our investigation is quite conservative. It re-establishes the apriority of mathematical space, yet in doing so it justifies the method of metaphysicians in their constructions of the several non-Euclidean systems. All geometrical systems, Euclidean as well as non-Euclidean, are purely ideal constructions. If we make one of them we then and there for that purpose and for the time being, exclude the other systems, but they are all, each one on its own premises, equally true and the question of preference between them is not one of truth or untruth but of adequacy, of practicability, of usefulness.

The question is not: "Is real space that of Euclid or of Riemann, of Lobatchevsky or Bolyai?" for real space is simply the juxtapositions of things, while our geometries are ideal schemes, mental constructions of models for space measurement. The real question is, "Which system is the most convenient to determine the juxtaposition of things?"

A priori considered, all geometries have equal rights, but for all that Euclidean geometry, which in the parallel theorem takes the bull by the horn, will remain classical forever, for after all the non-

Euclidean systems cannot avoid developing the notion of the straight line or other even boundaries. Any geometry could, within its own premises, be utilized for a determination of objective space; but we will naturally give the preference to plane geometry, not because it is truer, but because it is simpler and will therefore be more serviceable.

How an ideal (and apparently purely subjective) construction can give us any information of the objective constitution of things, at least so far as space-relations are concerned, seems mysterious but the problem is solved if we bear in mind the objective nature of the *a priori*,—a topic which we have elsewhere discussed.[4]

SENSE-EXPERIENCE AND SPACE.

We have learned that sense-experience cannot be used as a source from which we construct our fundamental notions of geometry, yet sense-experience justifies them.

Experience can verify *a priori* constructions as, e. g., tridimensionality is verified in Newton's laws; but experience can never refute them, nor can it change them. We may apply any system if we only remain consistent. It is quite indifferent whether we count after the decimal, the binary or the duodecimal system. The result will be the same. If experience does not tally with our calculations, we have either made a mistake or made a wrong ob-

[4] See also the author's exposition of the problem of the *a priori* in his edition of *Kant's Prolegomena*, pp. 167-240.

servation. For our *a priori* conceptions hold good for any conditions, and their theory can be as little wrong as reality can be inconsistent.

However, some of the most ingenious thinkers and great mathematicians do not conceive of space as mere potentiality of existence, which renders it formal and purely *a priori,* but think of it as a concrete reality, as though it were a big box, presumably round, like an immeasurable sphere. If it were such, space would be (as Riemann says) boundless but not infinite, for we cannot find a boundary on the surface of a sphere, and yet the sphere has a finite surface that can be expressed in definite numbers.

I should like to know what Riemann would call that something which lies outside of his spherical space. Would the name "province of the extra-spatial" perhaps be an appropriate term? I do not know how we can rid ourselves of this enormous portion of unutilized outside room. Strange though it may seem, this space-conception of Riemann counts among its advocates mathematicians of first rank, among whom I will here mention only the name of Sir Robert Ball.

It will be interesting to hear a modern thinker who is strongly affected by metageometrical studies, on the nature of space. Mr. Charles S. Peirce, an uncommonly keen logician and an original thinker of no mean repute, proposes the following three alternatives. He says:

"First, space is, as Euclid teaches, both *unlimited* and *immeasurable,* so that the infinitely distant parts of any plane seen in perspective appear as a straight line, in which case the sum of the three angles amounts to 180°; or,

"Second, space is *immeasurable* but *limited,* so that the infinitely distant parts of any plane seen in perspective appear as in a circle, beyond which all is blackness, and in this case the sum of the three angles of a triangle is less than 180° by an amount proportional to the area of the triangle; or

"Third, space is *unlimited* but *finite* (like the surface of a sphere), so that it has no infinitely distant parts; but a finite journey along any straight line would bring one back to his original position, and looking off with an unobstructed view one would see the back of his own head enormously magnified, in which case the sum of the three angles of a triangle exceeds 180° by an amount proportional to the area.

"Which of these three hypotheses is true we know not. The largest triangles we can measure are such as have the earth's orbit for base, and the distance of a fixed star for altitude. The angular magnitude resulting from subtracting the sum of the two angles at the base of such a triangle from 180° is called the star's *parallax.* The parallaxes of only about forty stars have been measured as yet. Two of them come out negative, that of Arided (α Cycni), a star of magnitude 1½, which is —0."082, according to C. A. F. Peters, and that of a star of magnitude 7¾, known as Piazzi III 422, which is —0."045 according to R. S. Ball. But these negative parallaxes are undoubtedly to be attributed to errors of observation; for the probable error of such a determination is about ±0."075, and it would be strange indeed if we were to be able to see, as it were, more than half way round space, without being able to see stars with larger negative parallaxes. Indeed, the very fact that of all the parallaxes measured only two come out negative would be a strong argument that the smallest parallaxes really amount to +0."1, were it not for the reflexion that the publication of other negative parallaxes may have been suppressed. I think we may feel confident that the parallax of the furthest star lies somewhere between —0."05 and +0."15, and within another century our grandchildren will surely know whether the three angles of a triangle are greater or less than 180°,—that they are *exactly* that amount is what nobody ever can be justified in concluding. It is true that according to the axioms of geometry the sum of the three sides of a triangle is precisely 180°; but these axioms are now exploded, and geometers confess that they, as geometers, know not the slightest reason for supposing them to be precisely true. They are expressions of our inborn conception of space, and as such are entitled to credit, so far as their truth could have influenced the formation of the mind. But that affords not the slightest reason for supposing them exact." (*The Monist*, Vol. I, pp. 173-174.)

Now, let us for argument's sake assume that the measurements of star-parallaxes unequivocally yield results which indicate that the sum of the angles in cosmic triangles is either a trifle more or a trifle less than 180°; would we have to conclude that cosmic space is curved, or would we not have to look for some concrete and special cause for the aberration of the light? If the moon is eclipsed while the sun still appears on the horizon, it proves only that the refraction of the solar rays makes the sun appear higher than it really stands, if its position is determined by a straight line, but it does not refute the straight line conception of geometry. Measurements of star-parallaxes (if they could no longer be accounted for by the personal equation of erroneous observation), may prove that ether can slightly deflect the rays of light, but it will never prove that the straight line of plane geometry is really a cirve. We might as well say that the norms of logic are refuted when we make faulty observations or whenever we are confronted by contradictory statements. No one feels called upon, on account of the many lies that are told, to propose a theory on the probable curvature of logic. Yet, seriously speaking, in the province of pure being the theory of a curved logic has the same right to a respectful hearing as the curvature of space in the province of the scope of pure motility.

Ideal constructions, like the systems of geometry, logic, etc., cannot be refuted by facts. Our observation of facts may call attention to the logical mistakes we have made, but experience cannot overthrow logic itself or the principles of thinking. They bear their standard of correctness in themselves which is based upon the same principle of consistency that pervades any system of actual or purely ideal operations.

But if space is not round, are we not driven to the other horn of the dilemma that space is infinite?

Perhaps we are. What of it? I see nothing amiss in the idea of infinite space.

By the by, if objective space were really curved, would not its twist be dominated in all probability by more than one determinant? Why should it be a curvature in the plane which makes every straight line a circle? Might not the plane in which our straightest line lies be also possessed of a twist so as to give it the shape of a flat screw, which would change every straightest line into a spiral? But the spiral is as infinite as the straight line. Obviously, curved space does not get rid of infinitude; besides the infinitely small, which would not be thereby eliminated, is not less troublesome than the infinitely great.

THE TEACHING OF MATHEMATICS.

As has been pointed out before, Euclid avoided the word axiom, and I believe with Grassmann, that its omission in the *Elements* is not accidental but the result of well-considered intention. The introduction of the term among Euclid's successors is due to a lack of clearness as to the nature of geometry and the conditions through which its fundamental notions originate.

It may be a flaw in the Euclidean *Elements* that the construction of the plane is presupposed, but it does not invalidate the details of his glorious work which will forever remain classical.

The invention of other geometries can only serve to illustrate the truth that all geometries, the plane geometry of Euclid included, are *a priori* constructions, and were not for obvious reasons Euclid's plane geometry preferable, other systems might as well be employed for the purpose of space-determination. Neither homaloidality nor curvature belongs to space; they belong to the several systems of manifolds that can be invented for the determination of the juxtapositions of things, called space.

If I had to rearrange the preliminary expositions of Euclid, I would state first the *Common Notions* which embody those general principles of Pure Reason and are indispensable for geometry. Then I would propose the *Postulates* which set forth our own activity (viz., the faculty of construction) and the conditions under which we intend to carry out our operations, viz., the obliteration of all particularity, characterizable as "anyness of motion." Thirdly, I would describe the instruments to be employed: the ruler and the pair of compasses; the former being the crease in a plane folded upon itself, and the latter to be conceived as a straight line (a stretched string) one end of which is stationary while the other is movable. And finally I would lay down the *Definitions* as the most elementary constructions which are to serve as tools and objects for experiment in the further expositions of geometry. There would be no mention of axioms, nor would we have to regard anything as an assumption or an hypothesis.

Professor Hilbert has methodically arranged the principles that underlie mathematics, and the excellency of his work is universally recognized.[5] It is a pity, however, that he retains the term "axiom," and we would suggest replacing it by some other appropriate word. "Axiom" with Hil-

[5] *The Foundations of Geometry,* The Open Court Pub. Co., Chicago, 1902.

bert does not mean an obvious truth that does not stand in need of proof, but a principle, or rule, viz., a formula describing certain general characteristic conditions.

Mathematical space is an ideal construction, and as such it is *a priori*. But its apriority is not as rigid as is the apriority of logic. It presupposes not only the rules of pure reason but also our own activity (viz., pure motility) both being sufficient to create any and all geometrical figures *a priori*.

Boundaries that are congruent with themselves being limits that are unique recommend themselves as standards of measurement. Hence the significance of the straight line, the plane, and the right angle.

The theorem of parallels is only a side issue of the implications of the straight line.

The postulate that figures of the same relations are congruent in whatever place they may be, and also that figures can be drawn similar to any figure, is due to our abstraction which creates the condition of anyness.

The teaching of mathematics, now utterly neglected in the public schools and not specially favored in the high schools, should begin early, but Euclid's method with his pedantic propositions and proofs should be replaced by construction work. Let children begin geometry by doing, not by reasoning. The reasoning faculties are not yet sufficiently developed in a child. Abstract reasoning is tedious, but if it comes in as an incidental aid to construction, it will be welcome. Action is the main-spring of life and the child will be interested so long as there is something to achieve.[6]

Lines must be divided, perpendiculars dropped, parallel lines drawn, angles measured and transferred, triangles constructed, unknown quantities determined with the help of proportion, the nature of the triangle studied and its internal relations laid down and finally the right-angled triangle computed by the rules of trigonometry, etc. All instruction should consist in giving *tasks to be performed, not theorems to be proved*; and the pupil should find out the theorems merely because he needs them for his construction.

In the triangle as well as in the circle we should accustom ourselves to using the same names for the same parts.[7]

Every triangle is ABC. The angle at A is always α, at B β, at C γ. The side opposite A is a, opposite B b, opposite C c. Altitudes (heights) are h_a, h_b, h_c. The lines that from A, B, and C pass through the center of gravity to the middle of the opposite sides I propose to call gravitals and would designate them g_a, g_b, g_c. The perpendiculars erected upon the middle of the sides meeting in the center of the circumscribed circle are p_a, p_b, p_c. The lines that divide the angles α, β, γ and meet in the center

[6] Cp. the author's article "Anticipate the School" (*Open Court*, 1899, p. 747).

[7] Such was the method of my teacher, Prof. Hermann Grassmann.

of the inscribed circle I propose to call "dichotoms"[8] and would designate them as d_a, d_b, d_c. The radius of the circumscribed circle is r, of the inscribed circle ρ, and the radii of the three ascribed circles are ρ_a, ρ_b, ρ_c. The point where the three heights meet is H; where the three gravitals meet, G; where the three dichotoms meet, O.[9] The stability of designation is very desirable and perhaps indispensable for a clear comprehension of these important interrelated parts.

[8] From διχότομος. I purposely avoid the term *bisector* and also the term *median*, the former because its natural abbreviation b is already appropriated to the side opposite to the point B, and the latter because it has beeen used to denote sometimes the gravitals and sometimes the dichotoms. It is thus reserved for general use in the sense of any middle lines.

[9] The capital of the Greek ρ is objectionable, because it cannot be distinguished from the Roman P.

EPILOGUE.

WHILE matter is eternal and energy is indestructible, forms change; yet there is a feature in the changing of forms of matter and energy that does not change. It is the norm that determines the nature of all formations, commonly called law or uniformity.

The term "norm" is preferable to the usual word "law" because the unchanging uniformities of the domain of natural existence that are formulated by naturalists into the so-called "laws of nature," have little analogy with ordinances properly denoted by the term "law." The "laws of nature" are not acts of legislation; they are no ukases of a Czar-God, nor are they any decrees of Fate or of any other anthropomorphic supremacy that sways the universe. They are simply the results of a given situation, the inevitable consequents of some event that takes place under definite conditions. They are due to the consistency that prevails in existence.

There is no compulsion, no tyranny of external oppression. They obtain by the internal necessity of causation. What has been done produces its proper effect, good or evil, intended or not intended, pursuant to a necessity which is not dynamical and from without, but logical and from within, yet, for all that, none the less inevitable. The basis of every so-called "law of nature" is the norm of formal relations, and if we call it a law of form, we must bear in mind that the term "law" is used in the sense of uniformity.

Form (or rather our comprehension of the formal and of all that it implies) is the condition that dominates our thinking and constitutes the norm of all sciences. From the same source we derive the principle of consistency which underlies our ideas of sameness, uniformity, rule, etc. This norm is not a concrete fact of existence but the universal feature that permeates both the anyness of our mathematical constructions and the anyness of objective conditions. Its application produces in the realm of mind the *a priori,* and in the domain of facts the uniformities of events which our scientists reduce to formulas, called laws of nature. On a superficial inspection it is pure nothingness, but in fact it is universality, eternality, and omnipresence; and it is the factor objectively of the world order and subjectively of science, the latter being man's capability of reducing the innumerable sense-impressions of experience to a methodical system of knowledge.

Faust, seeking the ideal of beauty, is advised to search for it in the domain of the eternal types of existence, which is the omnipresent Nowhere, the ever-enduring Never. Mephistopheles calls it the

Naught. The norm of being, the foundation of natural law, the principle of thinking, is non-existent to Mephistopheles, but in that nothing (viz., the absence of any concrete materiality, implying a general anyness) from which we weave the fabric of the purely formal sciences is the realm in which Faust finds "the mothers" in whom Goethe personifies the Platonic ideas. When Mephistopheles calls it "the nothing," Faust replies:

> "In deinem Nichts hoff' ich das All zu finden."
> ['Tis in thy Naught I hope to find the All.]

And here we find it proper to notice the analogy which mathematics bears to religion. In the history of mathematics we have first the rigid presentation of mathematical truth discovered (as it were) by instinct, by a prophetic divination, for practical purposes, in the shape of a dogma as based upon axioms, which is followed by a period of unrest, being the search for a philosophical basis, which finally leads to a higher standpoint from which, though it acknowledges the relativity of the primitive dogmatism, consists in a recognition of the eternal verities on which are based all our thinking, and being, and yearning.

The "Naught" of Mephistopheles may be empty, but it is the rock of ages, it is the divinity of existence, and we might well replace "All" by "God," thus intensifying the meaning of Faust's reply, and say:

> "'Tis in thy naught I hope to find my God."

The norm of Pure Reason, the factor that shapes the world, the eternal Logos, is omnipresent and eternal. It is God. The laws of nature have not been fashioned by a creator, they are part and parcel of the creator himself.

Plutarch quotes Plato as saying that God is always geometrizing.[1] In other words, the purely formal theorems of mathematics and logic are the thoughts of God. Our thoughts are fleeting, but God's thoughts are eternal and omnipresent verities. They are intrinsically necessary, universal, immutable, and the standard of truth and right.

Matter is eternal and energy is indestructible, but there is nothing divine in either matter or energy. That which constitutes the divinity of the world is the eternal principle of the laws of existence. That is the creator of the cosmos, the norm of truth, and the standard of right and wrong. If incarnated in living beings, it produces mind, and it continues to be the source of inspiration for aspiring mankind, a refuge of the struggling and storm-tossed sailors on the ocean of life, and the holy of holies of the religious devotee and worshiper.

The norms of logic and of mathematics are uncreate and uncreatable, they are irrefragable and immutable, and no power on earth or in heaven can change them. We can imagine that the world was made by a great world builder, but we cannot think

[1] Plutarchus *Convivia*, VIII, 2: πῶς Πλάτων ἔλεγε τὸν Θεὸν ἀεὶ γεωμετρεῖν. Having hunted in vain for the famous passage, I am indebted for the reference to Professor Ziwet of Ann Arbor, Mich.

that logic or arithmetic or geometry was ever fashioned by either man, or ghost, or god. Here is the rock on which the old-fashioned theology and all mythological God-conceptions must founder. If God were a being like man, if he had created the world as an artificer makes a tool, or a potter shapes a vessel, we would have to confess that he is a limited being. He might be infinitely greater and more powerful than man, but he would, as much as man, be subject to the same eternal laws, and he would, as much as human inventors and manufacturers, have to mind the multiplication tables, the theorems of mathematics, and the rules of logic.

Happily this conception of the deity may fairly well be regarded as antiquated. We know now that God is not a big individual, like his creatures, but that he is God, creator, law, and ultimate norm of everything. He is not personal but superpersonal. The qualities that characterize God are omnipresence, eternality, intrinsic necessity, etc., and surely wherever we face eternal verities it is a sign that we are in the presence of God,—not of a mythological God, but the God of the cosmic order, the God of mathematics and of science, the God of the human soul and its aspirations, the God of will guided by ideals, the God of ethics and of duty. So long as we can trace law in nature, as there is a norm of truth and untruth, and a standard of right and wrong, we need not turn atheists, even though the traditional conception of God is not free from crudities and mythological adornments. It will be by far preferable to purify our conception of God and replace the traditional notion which during the unscientific age of human development served man as a useful surrogate, by a new conception of God, that should be higher, and nobler, and better, because truer.

INDEX.

Absolute, The, 25.
Anschauung, 82, 97.
Anyness, 46ff., 76; Space founded on, 60.
Apollonius, 31.
A posteriori, 43, 60.
A priori, 38, 64; and the purely formal, 40ff; Apparent arbitrariness of the, 96ff; constructions, 112; constructions, Geometries are, 127; constructions verified by experience, 122; Geometry is, 119ff; is ideal, 44; Source of the, 51; The logical, 54; The purely, 55; The rigidly, 54, 55.
Apriority of different degrees, 49ff; of mathematical space, 121, 129; of space-measurement, 109ff; Problem of, 36.
Archimedes, 31.
As if, 79.
Astral geometry, 15.
Atomic fiction, 81.
Ausdehnungslehre, Grassmann's, 28, 29n., 30.
"Axiom," Euclid avoided, 1, 127; Hilbert's use of, 128.
Axioms, 1ff; not Common Notions, 4.

Ball, Sir Robert, on the nature of space, 123f.
Bernoulli, 9.
Bessel, Letter of Gauss to, 12ff.
Billingsley, Sir H., 82.
Bolyai, János, 22ff, 98; translated, 27.
Boundary concepts, Utility of, 74.
Boundaries, 78, 129; produced by halving, Even, 85, 86.
Bridges of Königsberg, 102f.
Busch, Wilhelm, 115.

Carus, Paul, *Fundamental Problems*, 39n; *Kant's Prolegomena*, 39n, 122n; *Primer of Philosophy*, 40n.
Causation, *a priori*, 53; and transformation, 54; Kant on, 40.
Cayley, 25.
Chessboard, Problem of, 101.
Circle, Squaring of the, 104; the simplest curve, 75.
Classification, 79.
Clifford, 16, 32, 60; Plane constructed by, 69.
Common notions, 2, 4, 128.
Comte, 38.
Concreteness, Purely formal, absence of, 60.
Continuum, 78ff.
Curved space, 106; Helmholtz on, 113.

Definitions of Euclid, 1, 128.
Delboeuf, B. J., 27.
De Morgan, Augustus, 10.
Determinism in mathematics, 104.
Dimension, Definition of, 85.
Dimensions, Space of four, 90ff.
Directions of space, Infinite, 117.
Discrete units, 78ff.
Dual number, 89.

Edward's Dream, 115.
Egg-shaped body, 113f.
Elliptic geometry, 25.
Empiricism, Transcendentalism and, 38ff.
Engel, Friedrich, 26.
Euclid, 1-4, 31ff; avoided "axiom," 1, 127; Expositions of, rearranged, 128; Halsted on, 31f.
Euclidean geometry, classical, 31, 121.
Even boundaries, 122; as standards of measurement, 69ff, 85, 86; produced by halving, 85, 86.
Experience, Physiological space originates through, 65.

Faust, 133.
Fictitious spaces, 109ff.

Flatland, 115.
Form, 60, 133; and reason, 48.
Four-dimensional space and tridimensional beings, 93.
Four dimensions, 109; Space of, 90ff.
Fourth dimension, 25; illustrated by mirrors, 93ff.

Gauss, 11ff; his letter to Bessel, 12ff; his letter to Taurinus, 6f, 13f.
Geometrical construction, Definiteness of, 99ff.
Geometry, *a priori*, 119ff; Astral, 15; Elliptic, 25; Question in, 72, 73, 121.
Geometries, *a priori* constructions, 127.
God, Conception of, 136.
Grassmann, 27ff, 127, 130n.

Halsted, George Bruce, 4n, 20n, 23, 26, 27, 28n, 101; on Euclid, 31f.
Helmholtz, 26, 83; on curved space, 113; on two-dimensional beings, 110f.
Hilbert's use of "axiom," 128.
Homaloidal, 18, 74.
Homogeneity of space, 66ff.
Hypatia, 31.

"Ideal" and "subjective," Kant's identification of, 44f; not synonyms, 64.
Infinite directions of space, 117; division of line, 117; not mysterious, 118; Space is, 116, 126; Time is, 116.
Infinitude, 116ff.

Kant, 35, 40, 61, 84; and the *a priori*, 36, 38; his identification of "ideal" and "subjective," 44f; his term *Anschauung*, 32, 97; his use of "transcendental," 41.
Kant's Prolegomena, 39n.
Keyser, Cassius Jackson, 77.
Kinematoscope, 80.
Klein, Felix, 25.
Königsberg, Seven bridges of, 102f.

Lagrange, 10f.
Lambert, Johann Heinrich, 9f.
Laws of nature, 132.
Legendre, 11.
Line created by construction, 83; independent of position, 62; Infinite division of, 117; Shortest, 84; Straightest, 75, 127.
Littré, 38.
Lobatchevsky, 10, 20ff, 75, 98; translated, 27.
Lobatchevsky's *Theory of Parallels*, 101.
Logic is static, 53.

Mach, Ernst, 27, 65.
Mathematical space, 63ff, 67, 109; *a priori*, 65; Apriority of, 121, 129.
Mathematics, Analogy of, to religion, 134; Determinism in, 104; Reality of, 77; Teaching of, 127ff.
Measurement, Even boundaries as standards of, 69ff, 85, 86; of star parallaxes, 125; Standards for, 74.
Mental activity, First rule of, 79.
Metageometry, 5ff; History of, 26; Mathematics and, 82ff.
Mill, John Stuart, 38.
Mind develops through uniformities, 52; Origin of, 51.
Mirrors, Fourth dimension illustrated by, 93ff.
Monist, 4n, 27, 125.

Names, Same, for parts of figures, 130.
Nasir Eddin, 7.
Nature, 16; a continuum, 78; Laws of, 132.
Newcomb, Simon, 25.

Open Court, 20n, 115n, 130n.
Order in life and arithmetic, 80.

Pangeometry, 22.
Pappus, 31.
Parallel lines in spherical space, 84; theorem, 4n, 25f, 98, 129.
Parallels, Axiom of, 3.
Path of highest intensity a straight line, 58.
Peirce, Charles S., on the nature of space, 123f.
Physiological space, 63ff, 67; originates through experience, 65.
Plane, a zero of curvature, 82; constructed by Clifford, 69; created by construction, 83; Nature of, 73; Significance of, 129.
Plato, 135.
Plutarch, 135.
Poincaré, H., 27.
Point congruent with itself, 71.

Population of Great Britain determined, 119f.
Position, 62.
Postulates, 2, 128.
Potentiality, 63.
Proclus, 4, 31.
Pseudo-spheres, 83.
Pure form, 63; space, Uniqueness of, 61ff.
Purely *a priori*, The, 55; formal, absence of concreteness, 60.

Question in geometry, 72, 73, 121.

Ray a final boundary, 58.
Reason, Form and, 48; Nature of, 76.
Rectangular pentagon, 98.
Religion, Analogy of mathematics to, 134.
Riemann, 15ff, 37, 62, 96, 104, 106ff, 123.
Right angle created by construction, 83; Nature of, 73; Significance of, 129.
Russell, Bertrand A. W., 26; on non-Euclidean geometry, 119.

Saccheri, Girolamo, 8f.
Schlegel, Victor, 30.
Schoute, P. H., 30n.
Schumaker, Letter of Gauss to, 11.
Schweikart, 15.
Sense-experience and space, 122ff.
Shortest line, 84.
Space, a manifold, 108; a spread of motion, 56ff; Apriority of mathematical, 121, 129; curved, 126; founded on "anyness," 60; Helmholtz on curved, 113; Homogeneity of, 66ff; homaloidal, 74; Infinite directions of, 117; Interference of, 101ff; is infinite, 116, 126; Mathematical, 63ff, 67, 109; Mathematical and actual, 62; of four dimensions, 90ff; On the nature of, 123f; Physiological, 63ff, 67; Sense-experience and, 122ff; the juxtaposition of things, 67, 87; the possibility of motion, 59; the potentiality of measuring, 61; Uniqueness of pure, 61ff.
Space-conception, how far *a priori?* 59f; product of pure activity, 55.
Space-measurement, Apriority of, 109ff; Various systems of, 104ff.
Spaces, Fictitious, 109ff.
Squaring of the circle, 104.
Stäckel, Paul, 26.
Standards of measurement, 74; Even boundaries as, 69ff, 85, 86.
Star parallaxes, Measurements of, 125.
Straight line, 69, 71, 112, 122; a path of highest intensity, 59; created by construction, 83; does not exist, 72; indispensable, 72ff; Nature of, 73; One kind of, 75; Significance of, 129; possible, 74.
Straightest line, 75, 127.
Subjective and ideal, Kant's identification of, 44f; not synonyms, 64.
Superreal, The, 76ff.

Taurinus, Letter of Gauss to, 6f, 13f.
Teaching of mathematics, 127ff.
Tentamen, 23.
Theon, 31, 32.
Theory of Parallels, Lobatchevsky's, 101.
Thought-forms, systems of reference, 61.
Three, The number, 88f.
Time is infinite, 116.
"Transcendental," Kant's use of, 41.
Transcendentalism and Empiricism, 35. 38ff.
Transformation, Causation and, 54.
Tridimensional beings, Four-dimensional space and, 93; space, Two-dimensional beings and, 91.
Tridimensionality, 84ff.
Trinity, Doctrine of the, 89.
Two-dimensional beings and tridimensional space, 91.

Uniformities, 132; Mind develops through, 52.
Units, Discrete, 78ff; Positing of, 80.

Wallis, John, 7f.
Why? 100.

Zamberti, 32.
Ziwet, Professor, 135n.

Essays on the Theory of Numbers.

(1) Continuity and Irrational Numbers, (2) The Nature and Meaning of Numbers. By RICHARD DEDEKIND. From the German by W. W. BEMAN. Pages, 115. Cloth, 75 cents net. (3s. 6d. net.)

These essays mark one of the distinct stages in the development of the theory of numbers. They give the foundation upon which the whole science of numbers may be established. The first can be read without any technical, philosophical or mathematical knowledge; the second requires more power of abstraction for its perusal, but power of a logical nature only.

"A model of clear and beautiful reasoning."
—*Journal of Physical Chemistry.*

"The work of Dedekind is very fundamental, and I am glad to have it in this carefully wrought English version. I think the book should be of much service to American mathematicians and teachers."
—*Prof. E. H. Moore, University of Chicago.*

"It is to be hoped that the translation will make the essays better known to English mathematicians; they are of the very first importance, and rank with the work of Weierstrass, Kronecker, and Cantor in the same field."—*Nature.*

Elementary Illustrations of the Differential and Integral Calculus.

By AUGUSTUS DE MORGAN. New reprint edition. With subheadings and bibliography of English and foreign works on the Calculus. Price, cloth, $1.00 net. (4s. 6d net.)

"It aims not at helping students to cram for examinations, but to give a scientific explanation of the rationale of these branches of mathematics. Like all that De Morgan wrote, it is accurate, clear and philosophic."—*Literary World, London.*

On the Study and Difficulties of Mathematics.

By AUGUSTUS DE MORGAN. With portrait of De Morgan, Index, and Bibliographies of Modern Works on Algebra, the Philosophy of Mathematics, Pangeometry, etc. Pages, viii, 288. Cloth, $1.25 net. (5s. net.)

"The point of view is unusual; we are confronted by a genius, who, like his kind, shows little heed for customary conventions. The 'shaking up' which this little work will give to the young teacher, the stimulus and implied criticism it can furnish to the more experienced, make its possession most desirable."—*Michigan Alumnus.*

The Foundations of Geometry.

By DAVID HILBERT, Ph. D., Professor of Mathematics in the University of Göttingen. With many new additions still unpublished in German. Translated by E. J. TOWNSEND, Ph. D., Associate Professor of Mathematics in the University of Illinois. Pages, viii, 132. Cloth, $1.00 net. (4s. 6d net.)

"Professor Hilbert has become so well known to the mathematical world by his writings that the treatment of any topic by him commands the attention of mathematicians everywhere. The teachers of elementary geometry in this country are to be congratulated that it is possible for them to obtain in English such an important discussion of these points by such an authority."—*Journal of Pedagogy.*

Euclid's Parallel Postulate: Its Nature, Validity and Place in Geometrical Systems.

By JOHN WILLIAM WITHERS, Ph. D. Pages vii, 192. Cloth, net $1.25. (4s. 6d. net.)

"This is a philosophical thesis, by a writer who is really familiar with the subject on non-Euclidean geometry, and as such it is well worth reading. The first three chapters are historical; the remaining three deal with the psychological and metaphysical aspects of the problem; finally there is a bibliography of fifteen pages. Mr. Withers's critique, on the whole, is quite sound, although there are a few passages either vague or disputable. Mr. Withers's main contention is that Euclid's parallel postulate is empirical, and this may be admitted in the sense that his argument requires; at any rate, he shows the absurdity of some statements of the *a priori* school."—*Nature.*

Mathematical Essays and Recreations.

By HERMANN SCHUBERT, Professor of Mathematics in Hamburg. Contents: Notion and Definition of Number; Monism in Arithmetic; On the Nature of Mathematical Knowledge; The Magic Square; The Fourth Dimension; The Squaring of the Circle. From the German by T. J. McCormack. Pages, 149. Cuts, 37. Cloth, 75 cents net. (3s. 6d. net.)

"Professor Schubert's essays make delightful as well as instructive reading. They deal, not with the dry side of mathematics, but with the philosophical side of that science on the one hand and its romantic and mystical side on the other. No great amount of mathematical knowledge is necessary in order to thoroughly appreciate and enjoy them. They are admirably lucid and simple and answer questions in which every intelligent man is interested."—*Chicago Evening Post.*

"They should delight the jaded teacher of elementary arithmetic, who is too liable to drop into a mere rule of thumb system and forget the scientific side of his work. Their chief merit is however their intelligibility. Even the lay mind can understand and take a deep interest in what the German professor has to say on the history of magic squares, the fourth dimension and squaring of the circle."

—*Saturday Review.*

A Brief History of Mathematics.

By the late DR. KARL FINK, Tübingen, Germany. Translated by Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in Teachers' College. Columbia University, New York City. With biographical notes and full index. Second revised edition. Pages, xii, 333. Cloth, $1.50 net. (5s. 6d. net.)

"Dr. Fink's work is the most systematic attempt yet made to present a compendious history of mathematics."—*The Outlook.*

"This book is the best that has appeared in English. It should find a place in the library of every teacher of mathematics."
—*The Inland Educator.*

Lectures on Elementary Mathematics.

By JOSEPH LOUIS LAGRANGE. With portrait and biography of Lagrange. Translated from the French by T. J. McCormack. Pages, 172. Cloth, $1.00 net. (4s. 6d. net.)

"Historical and methodological remarks abound, and are so woven together with the mathematical material proper, and the whole is so vivified by the clear and almost chatty style of the author as to give the lectures a charm for the readers not often to be found in mathematical works."—*Bulletin American Mathematical Society.*

A Scrapbook of Elementary Mathematics.

By WM. F. WHITE, State Normal School, New Paltz, N. Y. Cloth. Pages, 248. $1.00 net. (5s. net.)
A collection of Accounts, Essays, Recreations and Notes, selected for their conspicuous interest from the domain of mathematics, and calculated to reveal that domain as a world in which invention and imagination are prodigiously enabled, and in which the practice of generalization is carried to extents undreamed of by the ordinary thinker, who has at his command only the resources of ordinary language. A few of the seventy sections of this attractive book have the following suggestive titles: Familiar Tricks, Algebraic Fallacies, Geometric Puzzles, Linkages, A Few Surprising Facts, Labyrinths, The Nature of Mathematical Reasoning, Alice in the Wonderland of Mathematics. The book is supplied with Bibliographic Notes, Bibliographic Index and a copious General Index.

"The book is interesting, valuable and suggestive. It is a book that really fills a long-felt want. It is a book that should be in the library of every high school and on the desk of every teacher of mathematics."
—*The Educator-Journal.*