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Modal Logic

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CLARENDON PRESS • OXFORD

1997

Oxford University Press, Great Clarendon Street, Oxford OX2 6DP
Oxford New York
Athens Auckland Bangkok Bogota Bombay
Buenos Aires Calcutta Cape Town Dar es Salaam
Delhi Florence Hong Kong Istanbul Karachi
Kuala Lumpur Madras Madrid Melbourne
Mexico City Nairobi Paris Singapore
Taipei Tokyo Toronto
and associated companies in
Berlin Ibadan

Oxford is a trade mark of Oxford University Press

Published in the United States
by Oxford University Press, Inc., New York

A catalogue record for this book is available from the British Library

Library of Congress Cataloging in Publication Data
Data available

ISBN 0 19 853779 4

Typeset by the authors
Printed in Great Britain by
Bookcraft (Bath) Ltd
Midsomer Norton, Avon

To our wives
Lilia and Olga

PREFACE

Modal logic is a branch of mathematical logic studying mathematical models of correct reasoning which involves various kinds of necessity-like and possibility-like operators.

The first modal systems were created in the 1910s and later by Lewis (cf. Lewis and Langford, 1932) who used the operators "it is necessary" and "it is possible" for analyzing other logical connectives, in particular implication. Orlov (1928) and Gödel (1933a) constructed modal systems with the operator "it is provable" and exploited them to interpret Heyting's intuitionistic logic. More recently numerous modal systems have originated from different sources. They include:

- Philosophy, which studies the categories of necessity, contingency, causality, etc., and gives rise to logics with alethic ("it is necessary" and "it is possible"), deontic ("it is obligatory" and "it is permitted"), epistemic ("it is known" and "it does not contradict to what is known"), tense ("at all future times" and "eventually"), and some other modal operators;
- Foundations of mathematics, in which intuitionistic logic and provability logic (with the modal operators "it is provable in a given formal theory, say Peano arithmetic" and "it is consistent with the theory") were created;
- Computer science, which developed dynamic logic (with operators like "after every execution of the program" and "after some execution of the program") and temporal logic (with "henceforth", "sometimes" and other temporal operators) for describing the behavior of computer programs;
- Cognitive science, in which nonmonotonic modal logics, default and autoepistemic logics (with the operators "it is believed" and "it is consistent with the current knowledge base") were designed;
- Linguistics studying modalities in natural languages.

(This list is by no means complete; modal logics may have rather unexpected sources, for instance, quantum mechanics.) Although created in different fields and for different purposes, all these systems (their fragments with the corresponding necessity-like and possibility-like operators, to be more exact) have so much in common that can be definitely attributed to the same family of logics. This family turns out to be very extensive, and not only because there are many kinds of modal operators. Each particular operator may be explicated in different ways, which gives rise to subfamilies of deontic logics, epistemic logics, etc. For example, one application may require a temporal logic of discrete linear time, while another a temporal logic of branching continuous time.

Modal logic is not just a collection of systems of that sort: in fact they are subjects of more special disciplines. Modern modal logic—at least as it is understood in this book—abstracts from those particular systems and considers a general notion (or notions) of modal logic as a set of formulas in a certain language containing certain axioms and closed under certain inference rules. In other words, it deals with a class of extensions of a certain minimal modal system and its main concern is to develop general methods for investigating properties of logics in the class. It is this step of abstraction, made in the 1950s and 1960s, that distinguished modal logic as a separate discipline within mathematical logic and clearly formulated its object of studies.

There are several degrees of freedom in the choice of the minimal modal system. We can choose between a propositional language and a predicate one, between a language with a single basic modal operator and a polymodal one. We should decide which non-modal basis—classical, intuitionistic, or some other—is preferable. And of course there is a wide choice of modal axioms and inference rules. (For a detailed classification of modal logics consult Segerberg (1982).)

In this book our minimal system is the well known propositional unimodal classical logic **K**, and we consider the class of its quasi-normal (i.e., closed under modus ponens and substitution) extensions. This choice is motivated by two reasons. First, almost all important modal systems belong to this class or are reducible in one sense or another to its logics, or can be handled by a similar technique. It is this class that has mostly attracted modal logicians' attention, and for which sufficiently general methods have been developed. And second, modal operators behave, in a sense, like quantifiers and so even the propositional modal language turns out to be very rich and expressive. The class under consideration contains logics with any conceivable combination of properties and clearly demonstrates principal difficulties and problems in modal logic.

Another important family of propositional logics considered in this book is the class of superintuitionistic (or intermediate) logics which are extensions of Heyting's intuitionistic logic **Int**. From the technical and even philosophical point of view superintuitionistic logics are closely related to modal ones, and we use this opportunity to present a theory of such logics, at least in the background.

The purpose of the book is to give a systematic treatment of the most important methods and results concerning these two kinds of logics.

There exist three general ways of manipulating logics: syntactical, semantic and algebraic. The syntactical way, which uses various kinds of proof systems, like Gentzen-style calculi, natural deduction, semantic tableaux, etc., is hardly suitable for our aims. Although such systems have been constructed for a few particular modal and superintuitionistic logics, they are too special to be extended to big classes. The most widely used semantic way, exploiting "geometrical" features of Kripke frames, comes across the effect of Kripke incompleteness. We will go along this way as far as possible and then combine it with the universal algebraic way (which lacks geometrical insight) by adding to Kripke frames the algebraic component and considering general frames. Since the end of the 1970s, when duality theory started by Jónsson and Tarski (1951) was finally developed, this approach to modal (and other non-classical) logics has become dominating, having reconciled thereby "Kripkeans" and "algebraists" and laid a solid mathematical base under the edifice of modal logic.

The existing textbooks on modal logic reflect the state of the discipline as it was in the mid-1970s. From the technical point of view, they practically do not go further than applying the methods of canonical models and filtration to a few particular systems. The modern algebraic semantics (varieties of modal algebras and matrices), duality theory, general completeness results, investigations into metalogical properties of logics, algorithmic and complexity problems remain still scattered over numerous journals and proceedings of conferences. (Partially this situation is mitigated by books in boundary fields, for instance, correspondence theory, logic of time, provability logic, and the handbook series.)

We believe this book will make understandable these important methods, tools and results of modal logic to students specialized in mathematics or computer science as well as in philosophy or linguistics. It should be useful for both novices without any previous knowledge of modal logic and specialists in the subject. We start with the very basic definitions and gradually advance to the front line of the current researches. Each chapter ends with a brief commentary and exercises, often supplemented with open problems.

Modal logic is too extensive a field to be covered comprehensively only by one book. Besides, it can be looked at from different points of view. For instance, from the algebraic standpoint modal logics can be considered as equational theories of Boolean algebras with operators. Also, one can look at modal formulas as a language for describing classes of relational structures and compare it with other languages, say, the classical first order language. In this book our main object of studies are modal logics *per se*; algebras and relational structures provide us with the relevant technical tools. Facing the problem of selecting material, we gave priority to ideas and methods rather than facts concerning individual systems. A number of interesting results are presented as exercises. On the other hand, sometimes it was very difficult to resist the temptation to include in the text quite new theorems, especially if we felt that otherwise the picture would be incomplete. We understand the danger of mixing genres and yet hope that we have managed to find a reasonable compromise between a textbook and a monograph.

Now a few words about the content of the book. Part I introduces in full detail the syntax as well as the semantics of basic superintuitionistic and modal systems and studies their properties. In fact it illustrates in miniature what kinds of problems are to be considered later for big classes of logics. Technically one of the central points here is the construction of Kripke countermodels for a given formula, which is the first step in understanding the "geometry" of arbitrary (refutation) frames for the formula, and also the truth-preserving operations on frames.

In Part II we first consider the method of canonical models for proving Kripke completeness and various forms of filtration for establishing the finite model property, which is called in this book the finite approximability. And then we present a series of "negative" results giving examples of logics lacking the finite approximability, canonicity, compactness, elementarity and Kripke completeness.

Part III introduces adequate semantics for modal and superintuitionistic logics. We translate the language of logic into the language of algebra and arrive at varieties of modal and pseudo-Boolean algebras. Using the Stone–Jónsson–Tarski representation, we convert these algebras into general frames and study the relationship between the algebraic and generalized Kripke semantics. Then we develop a frame-theoretic language in terms of which one can characterize the constitution of transitive refutation frames for a given modal or intuitionistic formula.

Part IV studies various properties of modal and superintuitionistic logics. Here we deal with different forms of completeness (raising problems like "what is the structure of frames for a given logic?", "what is the simplest class of frames characterizing it?"), and touch upon correspondence theory. We consider also lattice-theoretic and metalogical properties (e.g. Post completeness, interpolation, the disjunction property).

Finally, Part V is devoted to algorithmic and complexity problems. Our concern here is not only the traditional problem of the decidability of logics. We are also interested in the decidability of logics' properties and the decidability of the admissibility and derivability problems for inference rules. In complexity theory we focus our attention mainly on estimating the size of minimal refutation frames for finitely approximable logics.

Acknowledgments. We are indebted to Dov Gabbay for initiating the project of writing this book. We are also grateful to Sergei Artemov, Wim Blok, Johan van Benthem, Silvio Ghilardi, Carsten Grefe, Tsutomu Hosoi, Yurij Janov, Dick de Jongh, Max Kanovich, Marcus Kracht, Larisa Maksimova, Hiroakira Ono, Wolfgang Rautenberg, Mefodij Ratsa, Vladimir Rybakov, Valentin Shehtman, Tatsuya Shimura, Dmitrij Skvortsov, Alexander Tsytkin, Frank Wolter, Vladimir Zakharov for stimulating discussions. Thanks are due to Lilia Chagrova, who used the first version of the book for her course in modal logic at Tver University, and to Ivan Zakharyaschev for drawing numerous diagrams of frames in LaTeX (in fact, that was his first experience in geometry).

In different periods the work on the book was supported by the Russian Fundamental Research Foundation, Soros Foundation and Alexander von Humboldt Foundation.

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I INTRODUCTION

Part I

Introduction

The word "logic" is used in this book in two senses. In the broader sense *logic* or better *mathematical logic* is the discipline studying mathematical models of correct human reasoning. While constructing such models, it is usually assumed that reasoning consists of *propositions*, that is sentences whose content may be evaluated as true or not true. For example, Goldbach's conjecture

Every even number that is greater than 2 can be represented as the sum of two prime numbers,

Gödel's second theorem

If the formula $0 = 1$ is not provable in formal Peano arithmetic PA, then the statement "$0 = 1$ is not provable in PA" is not provable in PA,

and Winnie-the-Pooh's song[1]

If Rabbit	Then Tigger's bad habit
Was bigger	Of bouncing at Rabbit
And fatter	Would matter
And stronger,	No longer,
Or bigger	If Rabbit
Than Tigger,	Was taller
If Tigger was smaller,	

are propositions. Gödel's second theorem and Winnie-the-Pooh's song provide us with examples of *compound propositions*: they can be constructed from simpler propositions such as $0 = 1$, Rabbit is bigger than Tigger, etc., with the help of *logical connectives* which are expressed by the words like "and", "or", "if ... then ...", "not", "provable in PA", "no longer". In this sense Goldbach's conjecture is an *elementary* or *atomic proposition.*

If the intrinsic structure of atomic propositions is of no concern to us then we are in the realm of *propositional logic* which studies schemes of correct reasoning on the base of how propositions are constructed from atoms regardless of their content. "If φ then φ or ψ" is a simple example of a propositional scheme which is valid for all concrete propositions φ and ψ.

Propositional logic deals with formal languages containing *propositional variables* whose values may be arbitrary propositions, *propositional constants* like "truth" and "falsehood" and *formulas* constructed from variables and constants using logical connectives. In this book we will consider only languages with the constant "falsehood" ($\perp$), the connectives "and", "or", "if ... then ...", which are denoted by $\wedge$, $\vee$, $\rightarrow$ and called *conjunction*, *disjunction* and *implication*, respectively, and the modal connective $\Box$ called the *necessity operator* which, depending on the context, is read as "it is necessary" or as "it is obligatory" or as "it is provable" or "it is true now and always will be true", etc.

[1] A.A. Milne. The house at Pooh corner.

In the narrower sense, by a *logic* in a given propositional language we will mean simply the set of all formulas in the language representing propositional schemes which are valid from a certain point of view. Different logics appear not only because of the possibility of varying the language, i.e., on account of the desire to study various logical connectives, but also for the reason that the same connectives may be interpreted in different ways.

In this part we briefly consider a few most important propositional logics which give rise to those big families of logics we shall deal with in the sequel.

1

CLASSICAL LOGIC

Classical propositional logic was created by Boole about 150 years ago (see Boole, 1947). It holds the central position among propositional logics not only due to its venerable age. In fact, it represents the simplest model of reasoning based upon the assumption that every proposition is either true or false. Many other logics are either contained in the classical one or built on its basis by enriching the language with new connectives.

1.1 Syntax and semantics

Fix the *propositional language* $\mathcal{L}$ whose *primitive symbols* (alphabet) are:

- the *propositional variables* $p_0, p_1, \ldots$;
- the *propositional constant* $\bot$ (falsehood);
- the *propositional connectives*: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication);
- the *punctuation marks*: (and),

and the *formulas* of $\mathcal{L}$ (or $\mathcal{L}$-*formulas*, or simply *formulas* if $\mathcal{L}$ is understood) are defined inductively:

- all the variables in $\mathcal{L}$ and the constant $\bot$ are *atomic* $\mathcal{L}$-*formulas* (or simply *atoms*);
- if φ and ψ are $\mathcal{L}$-formulas then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \rightarrow \psi)$ are also $\mathcal{L}$-formulas;
- a sequence of primitive symbols in $\mathcal{L}$ is a formula iff[2] this follows from the two preceding items.

Example 1.1 The following sequence of symbols is a formula:

$$(((p_0 \rightarrow \bot) \vee p_1) \rightarrow ((p_1 \wedge p_2) \wedge p_3)).$$

We will denote propositional variables by the small Roman letters p, q, r, possibly with subscripts or superscripts; the small Greek letters φ, ψ, χ and maybe some others are reserved for formulas, and capital Greek letters like Γ, Δ, Σ are used for denoting sets of formulas.

The set of all variables in $\mathcal{L}$ is denoted by $\mathbf{Var}\mathcal{L}$. Unless otherwise indicated, we will assume $\mathbf{Var}\mathcal{L}$ to be countable. This restriction is of not so great importance, and almost all the results to be obtained below can be generalized (in one way or another) to languages with finitely or uncountably many variables.

[2]Iff is the standard abbreviation for "if and only if".

The set of all $\mathcal{L}$-formulas is denoted by $\mathbf{For}\mathcal{L}$. The formulas used in the construction of a formula φ according to the definition above as well as φ itself are called *subformulas* of φ. $\mathbf{Sub}\varphi$ is the set of all φ's subformulas and $\mathbf{Var}\varphi$ is the set of all variables in $\mathbf{Sub}\varphi$. If $\mathbf{Var}\varphi = \emptyset$ then φ is called a *variable free formula.* We use the notation $\varphi(q_1, \dots, q_n)$ to reflect the fact that $\mathbf{Var}\varphi \subseteq \{q_1, \dots, q_n\}$. We rely upon the reader's common sense and give no exact definition of *occurrence* of a subformula in a formula.

The propositional connectives $\neg$ (*negation*), $\leftrightarrow$ (*equivalence*) and the constant $\top$ (*truth*) can be defined as abbreviations:

$$(\neg\varphi) = (\varphi \to \bot),$$

$$(\varphi \leftrightarrow \psi) = (\varphi \to \psi) \wedge (\psi \to \varphi),$$

$$\top = (\bot \to \bot).$$

If a formula φ is of the form $(\neg\psi)$ or $(\psi \odot \chi)$, for $\odot \in \{\wedge, \vee, \to, \leftrightarrow\}$, then $\neg$ or, respectively, $\odot$ is called the *main connective* of φ. The formula ψ is said to be the *premise* of the implication $(\psi \to \chi)$ and χ its *conclusion.*

We shall use the following standard conventions on representation of formulas: we assume $\neg$ to connect formulas stronger than $\wedge$ and $\vee$, which in turn are stronger than $\to$ and $\leftrightarrow$, and omit those brackets that can be recovered according to this priority of the connectives. We shall also write $\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \dots \vee \varphi_n$ or $\bigvee_{i=1}^n \varphi_i$ instead of $(\dots((\varphi_1 \vee \varphi_2) \vee \varphi_3) \vee \dots \vee \varphi_n)$ and $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \dots \wedge \varphi_n$ or $\bigwedge_{i=1}^n \varphi_i$ instead of $(\dots((\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \wedge \dots \wedge \varphi_n)$; $\bigvee_{i\in\emptyset} \varphi_i$ and $\bigwedge_{i\in\emptyset} \varphi_i$ mean $\bot$ and $\top$, respectively. Each φ_i in a formula of the form $\bigvee_{i=1}^n \varphi_i$ or $\bigwedge_{i=1}^n \varphi_i$ is called a *disjunct* or a *conjunct* of the formula, respectively.

With the help of these abbreviations and conventions, the formula in Example 1.1 can be now written much more briefly:

$$\neg p_0 \vee p_1 \to p_1 \wedge p_2 \wedge p_3.$$

We have introduced the syntax of classical logic and now turn to its semantics, i.e., define the classical interpretation of the language $\mathcal{L}$.

The fundamental semantic assumption characterizing classical logic is as follows:

- each atomic proposition is either true or false (but not simultaneously), with $\bot$ being always false;
- the truth-values of compound propositions are uniquely defined by the following *truth-table*, where T and F stand for "true" and "false", respectively:

ψ	χ	$\psi \wedge \chi$	$\psi \vee \chi$	$\psi \to \chi$	$\neg\psi$	$\psi \leftrightarrow \chi$
F	F	F	F	T	T	T
F	T	F	T	T	T	F
T	F	F	T	F	F	F
T	T	T	T	T	F	T

Thus, according to this truth-table, "false" means just "not true".

Starting from the assumption above, we can give now an exact definition of classical model of the language $\mathcal{L}$.

A *classical model* of $\mathcal{L}$ is any subset $\mathfrak{M}$ of $\mathbf{Var}\mathcal{L}$. Less formally this means that $\mathfrak{M}$ contains those and only those atomic propositions that are regarded to be true. By induction on the construction of a formula φ we define a relation $\mathfrak{M} \models \varphi$ which is read either as "*φ is true in the model $\mathfrak{M}$*" or as "*$\mathfrak{M}$ is a model for φ*":

$$
\begin{array}{ll}
\text{not } \mathfrak{M} \models \bot; & \\
\mathfrak{M} \models p & \text{iff } p \in \mathfrak{M}, \text{ for every } p \in \mathbf{Var}\mathcal{L}; \\
\mathfrak{M} \models \psi \wedge \chi & \text{iff } \mathfrak{M} \models \psi \text{ and } \mathfrak{M} \models \chi; \\
\mathfrak{M} \models \psi \vee \chi & \text{iff } \mathfrak{M} \models \psi \text{ or } \mathfrak{M} \models \chi; \\
\mathfrak{M} \models \psi \to \chi & \text{iff } \mathfrak{M} \models \chi \text{ whenever } \mathfrak{M} \models \psi.
\end{array}
$$

If $\mathfrak{M} \models \varphi$ does not hold then we write $\mathfrak{M} \not\models \varphi$ and say that either *φ is false in $\mathfrak{M}$* or *$\mathfrak{M}$ is a countermodel for φ* or *$\mathfrak{M}$ refutes φ*.

Observe at once that truth or falsity of a formula φ in a model $\mathfrak{M}$ depends only on the truth-values of φ's variables in $\mathfrak{M}$. In other words, the following proposition holds.

Proposition 1.2 *Suppose that models $\mathfrak{M}$ and $\mathfrak{N}$ are such that*

$$\mathfrak{M} \models p \textit{ iff } \mathfrak{N} \models p$$

for all variables p in some set $\mathbf{Var} \subseteq \mathbf{Var}\mathcal{L}$. Then, for every formula φ with $\mathbf{Var}\varphi \subseteq \mathbf{Var}$,

$$\mathfrak{M} \models \varphi \textit{ iff } \mathfrak{N} \models \varphi.$$

Proof An easy induction on the construction of φ. ❑

A model $\mathfrak{M}$ is called a *model for a set Γ of formulas* (notation: $\mathfrak{M} \models \Gamma$) if all formulas in Γ are true in $\mathfrak{M}$.

A formula φ is said to be (*classically*) *valid* if it is true in all models of $\mathcal{L}$; in this case we write $\models \varphi$.

Example 1.3 To show the validity of the formula

$$p \vee (p \to \bot),$$

known as the *law of the excluded middle*, it suffices to construct the *truth-table* for $p \vee (p \to \bot)$, which looks like this

p	$\vee$	(	p	$\to$	$\bot$	)
F	T		F	T	F	
T	T		T	F	F	

and make sure of that the column under the main connective of our formula contains only T.

Example 1.4 The truth-table for $(p \wedge q \to \bot) \to (p \to \bot)$

(p	$\wedge$	q	$\to$	$\bot$)	$\to$	(p	$\to$	$\bot$)
F	F	F	T	F	T	F	T	F
F	F	T	T	F	T	F	T	F
T	F	F	T	F	F	T	F	F
T	T	T	F	F	T	T	F	F

contains F in the column under the main connective, which means that this formula is not valid.

Finally, we define *classical logic in the language* $\mathcal{L}$ as the set $\mathbf{Cl}_{\mathcal{L}}$ of all valid $\mathcal{L}$-formulas or in symbols

$$\mathbf{Cl}_{\mathcal{L}} = \{\varphi \in \mathbf{For}\mathcal{L} : \models \varphi\}.$$

Since $\mathcal{L}$ is always understood, we drop the subscript and write simply **Cl**.

1.2 Semantic tableaux

Having defined classical or some other logic, we naturally face the problem of recognizing, given an arbitrary formula, whether it belongs to the logic or not. If there is an algorithm deciding this problem for a logic then the logic is called *decidable.*

The decidability of classical logic becomes evident as soon as we observe that the truth-value of a formula $\varphi(p_1, \dots, p_n)$ depends only on the truth-values assigned to $p_1, \dots, p_n$. A trivial decision algorithm is as follows: we just write down all 2^n possible assignments of F and T to $p_1, \dots, p_n$ and calculate the truth-value of φ for each of them; φ is in **Cl** iff all calculated values are T.

Yet there are a dozen subtler ways of determining validity. Here we consider one of them, a variant of the *semantic* (or *Beth*) *tableau method.* Roughly the underlying idea is that instead of climbing bottom-up from the truth-values of φ's variables to the truth-values of φ, we can move top-down, purposefully constructing a countermodel for φ. The semantic tableau method not only gives a more convenient tool for handling classical formulas (though in the worst case it works as ineffectively as the truth-table method). It is more important for us that the method can be extended to some other logics with different semantics which does not admit truth-tables.

Let us start with examples.

Example 1.5 Suppose we want to determine whether the formula

$$\varphi = ((p \to q) \to p) \to p,$$

known as *Pierce's law,* is valid or not. To solve this problem let us try to construct a countermodel for φ.

We begin the construction with forming a tableau consisting of two parts: in the left one we put those subformulas of φ which we want to be true, while the right part contains subformulas which are to be made false. Since we want φ to be false, it should be put in the right part. The truth-table for $\rightarrow$ tells us that φ is false iff $(p \rightarrow q) \rightarrow p$ is true and p is false; so we put the former formula in the left and the latter in the right part of the tableau:

$$\begin{array}{r|l}
\hline
 & ((p \rightarrow q) \rightarrow p) \rightarrow p \\
(p \rightarrow q) \rightarrow p & p
\end{array}$$

Now, to make $(p \rightarrow q) \rightarrow p$ true we have two possibilities, namely, either to make p true or to make $p \rightarrow q$ false. So the tableau above can be extended in two ways:

$$\begin{array}{r|l}
\hline
 & ((p \rightarrow q) \rightarrow p) \rightarrow p \\
(p \rightarrow q) \rightarrow p & p \\
p &
\end{array}$$

$$\begin{array}{r|l}
\hline
 & ((p \rightarrow q) \rightarrow p) \rightarrow p \\
(p \rightarrow q) \rightarrow p & p \\
 & p \rightarrow q \\
p & q
\end{array}$$

But then we arrive at a contradiction: both tableaux require p to be simultaneously true and false. This means that there is no countermodel for φ, and hence

$$((p \rightarrow q) \rightarrow p) \rightarrow p \in \mathbf{Cl}.$$

Example 1.6 Now let us use the same technique to construct a countermodel for

$$\varphi = r \wedge (\neg p \vee \neg q) \rightarrow r \wedge (p \vee \neg q).$$

The first four lines in the tableau are clear:

$$\begin{array}{r|l}
\hline
 & r \wedge (\neg p \vee \neg q) \rightarrow r \wedge (p \vee \neg q) \\
r \wedge (\neg p \vee \neg q) & r \wedge (p \vee \neg q) \\
r & \\
\neg p \vee \neg q &
\end{array}$$

But now there are two ways to make $r \wedge (p \vee \neg q)$ false: to put r in the right part or to put $p \vee \neg q$ there. Thus we obtain two extensions of the tableau:

(a)

$$\begin{array}{r|l}
\hline
 & r \wedge (\neg p \vee \neg q) \rightarrow r \wedge (p \vee \neg q) \\
r \wedge (\neg p \vee \neg q) & r \wedge (p \vee \neg q) \\
r & \\
\neg p \vee \neg q & \\
 & r
\end{array}$$

(b)

$$\begin{array}{r|l}
\hline
 & r \wedge (\neg p \vee \neg q) \to r \wedge (p \vee \neg q) \\
r \wedge (\neg p \vee \neg q) & r \wedge (p \vee \neg q) \\
r & \\
\neg p \vee \neg q & \\
 & p \vee \neg q \\
 & p \\
q & \neg q
\end{array}$$

The requirements of the tableau (a) are inconsistent. And (b) again has two extensions: to make $\neg p \vee \neg q$ true, we can put in the left part either $\neg p$ or $\neg q$. The latter alternative leads immediately to a contradiction, while the former one gives us the tableau

$$\begin{array}{r|l}
\hline
 & r \wedge (\neg p \vee \neg q) \to r \wedge (p \vee \neg q) \\
r \wedge (\neg p \vee \neg q) & r \wedge (p \vee \neg q) \\
r & \\
\neg p \vee \neg q & \\
 & p \vee \neg q \\
 & p \\
q & \neg q \\
\neg p &
\end{array}$$

whose requirements can be satisfied by assigning F to p and T to q and r. Hence φ is false in every model $\mathfrak{M}$ such that $\mathfrak{M} \models q$, $\mathfrak{M} \models r$ and $\mathfrak{M} \not\models p$.

Now we present this procedure of juggling with formulas as a formal system and show that applying it to an arbitrary given formula, in a finite number of steps we shall either construct a countermodel for it or establish its validity.

We will represent a tableau as a pair of sets of formulas: one of them contains all the formulas in the left part of the tableau and the other those in the right part. Thus, a *semantic tableau* in the language $\mathcal{L}$ is just a pair $t = (\Gamma, \Delta)$ with $\Gamma, \Delta \subseteq \mathbf{For}\mathcal{L}$.

A tableau (Γ, Δ) is called (*downward*) *saturated* in **Cl** if, for all formulas $\psi, \chi \in \mathbf{For}\mathcal{L}$,

(S1)	$\psi \wedge \chi \in \Gamma$	implies	$\psi \in \Gamma$	and	$\chi \in \Gamma$,
(S2)	$\psi \wedge \chi \in \Delta$	implies	$\psi \in \Delta$	or	$\chi \in \Delta$,
(S3)	$\psi \vee \chi \in \Gamma$	implies	$\psi \in \Gamma$	or	$\chi \in \Gamma$,
(S4)	$\psi \vee \chi \in \Delta$	implies	$\psi \in \Delta$	and	$\chi \in \Delta$,
(S5)	$\psi \to \chi \in \Gamma$	implies	$\psi \in \Delta$	or	$\chi \in \Gamma$,
(S6)	$\psi \to \chi \in \Delta$	implies	$\psi \in \Gamma$	and	$\chi \in \Delta$.

(Γ, Δ) is *disjoint* if $\Gamma \cap \Delta = \emptyset$ and $\bot \notin \Gamma$. Say that a tableau $t' = (\Gamma', \Delta')$ is an *extension* of a tableau $t = (\Gamma, \Delta)$ (or t is a *subtableau* of t') and write $t \subseteq t'$ if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

A tableau $t = (\Gamma, \Delta)$ is called *realizable* if there is a model $\mathfrak{M}$ such that

$$\mathfrak{M} \models \psi \text{ for all } \psi \in \Gamma \text{ and } \mathfrak{M} \not\models \chi \text{ for all } \chi \in \Delta;$$

in this case $\mathfrak{M}$ is said to *realize* t.

Proposition 1.7 *A tableau* $t = (\Gamma, \Delta)$ *is realizable iff it can be extended to a disjoint saturated tableau* $t' = (\Gamma', \Delta')$.

Proof ($\Rightarrow$) Suppose $\mathfrak{M}$ is a model realizing t. Put $\Gamma' = \{\varphi \in \mathbf{For}\mathcal{L} : \mathfrak{M} \models \varphi\}$ and $\Delta' = \{\varphi \in \mathbf{For}\mathcal{L} : \mathfrak{M} \not\models \varphi\}$. It is clear that $\Gamma' \cap \Delta' = \emptyset$, $\perp \notin \Gamma'$, $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Comparing conditions (S1)–(S6) with the definition of the truth-relation $\models$ in Section 1.1, it is easy to see that t' satisfies those conditions and so is saturated.

($\Leftarrow$) It suffices to show that every disjoint saturated tableau (Γ', Δ') is realizable. Define a model $\mathfrak{M}$ by taking $\mathfrak{M} = \Gamma' \cap \mathbf{Var}\mathcal{L}$. By induction on the construction of φ one can readily establish that $\varphi \in \Gamma'$ implies $\mathfrak{M} \models \varphi$ and $\varphi \in \Delta'$ implies $\mathfrak{M} \not\models \varphi$. ❑

Proposition 1.7 provides us in fact with an algorithm for verifying realizability of finite tableaux. Indeed, conditions (S1)–(S6) can be read as the *saturation rules*:

(SR1)	if $\psi \wedge \chi \in \Gamma$ then add ψ and χ to Γ,
(SR2)	if $\psi \wedge \chi \in \Delta$ then add ψ or χ to Δ,
etc.	

And then we obtain

Proposition 1.8 *A finite tableau* t_1 *is realizable iff there is a sequence* $t_1, \dots, t_n$ *such that* t_n *is a disjoint saturated tableau and each* t_{i+1} *is obtained from* t_i *by applying to it one of the saturation rules.*

Proof Exercise. ❑

As another exercise we invite the reader to prove that all formulas in Table 1.1 are in **Cl**. In what follows we will use those formulas without any comments.

1.3 Classical calculus

Classical logic can be represented as a formal axiomatic system, i.e., as a calculus, in several ways. Since in this book we are not going to deal with proof theory, we consider here only a Hilbert-type calculus which is rather convenient for theoretical constructions but not for practical use.

Classical propositional calculus Cl in the language $\mathcal{L}$ contains the following *axioms* and *inference rules*:

Axioms:

(A1) $p_0 \to (p_1 \to p_0)$,
(A2) $(p_0 \to (p_1 \to p_2)) \to ((p_0 \to p_1) \to (p_0 \to p_2))$,
(A3) $p_0 \wedge p_1 \to p_0$,
(A4) $p_0 \wedge p_1 \to p_1$,
(A5) $p_0 \to (p_1 \to p_0 \wedge p_1)$,
(A6) $p_0 \to p_0 \vee p_1$,

Table 1.1 *A list of classically valid formulas*

Formula	Name
$p \wedge p \leftrightarrow p,\ p \vee p \leftrightarrow p$	The laws of idempotency
$p \wedge q \leftrightarrow q \wedge p,\ p \vee q \leftrightarrow q \vee p$	The laws of commutativity
$p \wedge \bot \leftrightarrow \bot,\ p \wedge \top \leftrightarrow p$	
$p \vee \bot \leftrightarrow p,\ p \vee \top \leftrightarrow \top$	
$\bot \to p,\ p \to \top$	
$p \wedge \neg p \to q$	Duns Scotus' law
$\left.\begin{array}{l} p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r \\ p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r \end{array}\right\}$	The laws of associativity
$(p \wedge q) \vee q \leftrightarrow q,\ p \wedge (p \vee q) \leftrightarrow p$	The laws of absorption
$\left.\begin{array}{l} p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r) \end{array}\right\}$	The laws of distributivity
$p \to (q \to p)$	The law of simplification
$(p \to q) \to ((q \to r) \to (p \to r))$	The law of syllologism
$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$	Frege's law
$p \wedge q \to p,\ p \to p \vee q$	
$(p \vee q) \wedge (p \vee \neg q) \leftrightarrow p$	
$p \to (q \to p \wedge q)$	The law of adjunction
$(p \to (q \to r)) \leftrightarrow (p \wedge q \to r)$	The law of importation and exportation
$(p \to q) \to ((p \to r) \to (p \to q \wedge r))$	
$(p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r)$	
$(p \to q) \wedge (p' \to q') \to (p \vee p' \to q \vee q')$	
$(p \to q) \wedge (p' \to q') \to (p \wedge p' \to q \wedge q')$	
$(p \to r) \to ((q \to r) \to (p \vee q \to r))$	
$\left.\begin{array}{l} \neg(p \vee q) \leftrightarrow \neg p \wedge \neg q \\ \neg(p \wedge q) \leftrightarrow \neg p \vee \neg q \end{array}\right\}$	De Morgan's laws
$(p \to q) \leftrightarrow \neg p \vee q$	
$(p \to q) \leftrightarrow \neg(p \wedge \neg q)$	
$((p \to q) \to p) \to p$	Pierce's law
$p \vee \neg p$	The law of the excluded middle
$(p \to q) \leftrightarrow (\neg q \to \neg p)$	The law of contraposition
$p \leftrightarrow \neg\neg p$	The law of double negation
$(p \wedge q) \vee (p \wedge \neg q) \leftrightarrow p$	

(A7) $p_1 \to p_0 \vee p_1$,
(A8) $(p_0 \to p_2) \to ((p_1 \to p_2) \to (p_0 \vee p_1 \to p_2))$,
(A9) $\bot \to p_0$,
(A10) $p_0 \vee (p_0 \to \bot)$;

Inference rules:

Modus ponens (MP): given formulas φ and $\varphi \to \psi$, we obtain ψ,

Substitution (Subst): given a formula φ, we obtain $\varphi\boldsymbol{s}$,

where $\boldsymbol{s}$, a *substitution*, is a map from $\mathbf{Var}\mathcal{L}$ to $\mathbf{For}\mathcal{L}$ and $\varphi\boldsymbol{s}$ is defined by induction on the construction of φ: $p\boldsymbol{s} = \boldsymbol{s}(p)$ for every $p \in \mathbf{Var}\mathcal{L}$, $\bot\boldsymbol{s} = \bot$ and $(\psi \odot \chi)\boldsymbol{s} = \psi\boldsymbol{s} \odot \chi\boldsymbol{s}$, for $\odot \in \{\wedge, \vee, \to\}$.

A substitution $\boldsymbol{s}$ such that $\boldsymbol{s}(p) = \psi, \dots, \boldsymbol{s}(q) = \chi$ and $\boldsymbol{s}(r) = r$, for all variables r different from $p, \dots, q$, will be denoted by $\{\psi/p, \dots, \chi/q\}$. Given substitutions $\boldsymbol{s}'$ and $\boldsymbol{s}''$, we denote by $\boldsymbol{s}'\boldsymbol{s}''$ their *composition*, i.e., the substitution $\boldsymbol{s}$ such that $p\boldsymbol{s} = (p\boldsymbol{s}')\boldsymbol{s}''$ for every variable p.

A formula φ is said to be *derivable* in Cl if there is a *derivation* of φ in Cl, i.e., a sequence $\varphi_1, \dots, \varphi_n$ of formulas such that $\varphi_n = \varphi$ and for every i, $1 \le i \le n$, φ_i is either an axiom or obtained from some of the preceding formulas in the sequence by one of the inference rules; the number n is called the *length* of this derivation. If φ is derivable in Cl then we write $\vdash_{Cl} \varphi$ or simply $\vdash \varphi$ when this does not involve ambiguity.

Example 1.9 The following sequence is a derivation of $\varphi \to \varphi$, for any formula φ:

(1)	$(p_0 \to (p_1 \to p_2)) \to ((p_0 \to p_1) \to (p_0 \to p_2))$	(A2)
(2)	$(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$	(by Subst from (1))
(3)	$p_0 \to (p_1 \to p_0)$	(A1)
(4)	$\varphi \to ((\varphi \to \varphi) \to \varphi)$	(by Subst from (3))
(5)	$(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$	(by MP from (2), (4))
(6)	$\varphi \to (\varphi \to \varphi)$	(by Subst from (3))
(7)	$\varphi \to \varphi$	(by MP from (5), (6)).

Example 1.10 Below is a derivation of $\varphi \vee \psi \to \psi \vee \varphi$, for any formulas φ and ψ:

(1)	$p_1 \to p_0 \vee p_1$	(A7)
(2)	$p_0 \to p_0 \vee p_1$	(A6)
(3)	$\varphi \to \psi \vee \varphi$	(by Subst from (1))
(4)	$\psi \to \psi \vee \varphi$	(by Subst from (2))
(5)	$(p_0 \to p_2) \to ((p_1 \to p_2) \to (p_0 \vee p_1 \to p_2))$	(A8)
(6)	$(\varphi \to \psi \vee \varphi) \to ((\psi \to \psi \vee \varphi) \to (\varphi \vee \psi \to \psi \vee \varphi))$	(by Subst from (5))
(7)	$(\psi \to \psi \vee \varphi) \to (\varphi \vee \psi \to \psi \vee \varphi)$	(by MP from (3), (6))
(8)	$\varphi \vee \psi \to \psi \vee \varphi$	(by MP from (4), (7)).

As an exercise we invite the reader to construct a derivation of an arbitrary formula of the form $(\varphi \vee \psi) \vee \chi \leftrightarrow \varphi \vee (\psi \vee \chi)$.

Observe that in the derivations above the rule Subst was applied only to *axioms*. We call such kind of derivations *substitutionless*.

Proposition 1.11 *Each formula φ derivable in Cl has a substitutionless derivation in Cl.*

Proof The proof proceeds by induction on the length of a derivation of φ. The basis of induction is trivial, since in this case φ is an axiom.

Suppose now that the claim of the proposition holds for all formulas having derivations of length $< n$, for some $n > 1$, and let $\varphi_1, \dots, \varphi_n$ be a derivation of $\varphi = \varphi_n$. If φ_n is the result of applying MP to φ_i and φ_j, for $1 \leq i, j < n$, then we can readily construct a substitutionless derivation of φ from substitutionless derivations of φ_i and φ_j, which exist by the induction hypothesis.

Suppose that $\varphi_n = \varphi_i s$. Let $\psi_1, \dots, \psi_m$ be a substitutionless derivation of $\varphi_i = \psi_m$ and $\chi_1, \dots, \chi_l$ all the axioms occurring in it. Then the sequence $\chi_1, \dots, \chi_l, \psi_1 s, \dots, \psi_m s$ is a substitutionless derivation of φ_n, which follows from the fact that $(\psi \to \chi)s = \psi s \to \chi s$ and $(\psi s')s = \psi(s's)$, for all formulas ψ, χ and every substitution s'. ❑

Proposition 1.11 shows that classical calculus can be defined without using Subst. We can, for instance, replace p_0, p_1, p_2 in axioms (A1)–(A10) with the symbols φ_0, φ_1, φ_2 in our metalanguage and regard the resulting expressions as *axiom schemes* representing in fact the infinite set of substitution instances of (A1)–(A10).

Let Γ be a set of formulas. A sequence $\varphi_1, \dots, \varphi_n$ is called a *derivation of φ from the set of assumptions* Γ if $\varphi_n = \varphi$ and for every i, $1 \leq i \leq n$, φ_i is either an axiom or an assumption in Γ or obtained from some of the preceding formulas by one of the inference rules, with Subst being applied only to axioms. If there is a derivation of φ from Γ, we say that φ is *derivable from* Γ and write $\Gamma \vdash_{Cl} \varphi$ or simply $\Gamma \vdash \varphi$ if understood. By Proposition 1.11, $\vdash \varphi$ iff $\emptyset \vdash \varphi$. For brevity we will write $\Gamma, \psi_1, \dots, \psi_n \vdash \varphi$ instead of $\Gamma \cup \{\psi_1, \dots, \psi_n\} \vdash \varphi$ and $\Gamma, \Delta \vdash \varphi$ instead of $\Gamma \cup \Delta \vdash \varphi$. It follows immediately from the definition that

$$\Gamma \vdash \varphi \text{ and } \Gamma \subseteq \Delta \text{ imply } \Delta \vdash \varphi,$$

$$\Gamma \vdash \varphi \to \psi \text{ and } \Delta \vdash \varphi \text{ imply } \Gamma, \Delta \vdash \psi.$$

Now we prove a theorem which turns out to be very useful for establishing derivability.

Theorem 1.12. (Deduction) *If* $\Gamma, \psi \vdash \varphi$ *then* $\Gamma \vdash \psi \to \varphi$.

Proof Let $\varphi_1, \dots, \varphi_n$ be a derivation of $\varphi = \varphi_n$ from $\Gamma \cup \{\psi\}$. By induction on i we show that $\Gamma \vdash \psi \to \varphi_i$ for every $i \in \{1, \dots, n\}$.

If φ_i is an axiom or a formula in Γ then the sequence

$$\begin{array}{lll} (1) & \varphi_i & \\ (2) & p_0 \to (p_1 \to p_0) & \text{(A1)} \\ (3) & \varphi_i \to (\psi \to \varphi_i) & \text{(by Subst from (2))} \\ (4) & \psi \to \varphi_i & \text{(by MP from (1) and (3))} \end{array}$$

is a derivation of $\psi \to \varphi_i$ from Γ.

If $\varphi_i = \psi$ then, as was shown in Example 1.9, $\vdash \psi \to \varphi_i$, and so $\Gamma \vdash \psi \to \varphi_i$.

If φ_i is obtained from φ_j and $\varphi_k = \varphi_j \to \varphi_i$ by MP then, by the induction hypothesis, $\Gamma \vdash \psi \to (\varphi_j \to \varphi_i)$, $\Gamma \vdash \psi \to \varphi_j$, and using (A2), Subst and twice MP we obtain $\Gamma \vdash \psi \to \varphi_i$.

Finally, if $\varphi_i = \varphi_j s$ then φ_j is an axiom and the derivation of $\psi \to \varphi_i$ from Γ we need is the sequence $\varphi_j, (1), \dots, (4)$. ❑

The following examples show how the deduction theorem can be used for proving derivability.

Example 1.13 For every formulas φ, ψ, χ, we have

$$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)).$$

Indeed, by the deduction theorem it suffices to show that

$$\varphi \to \psi,\ \psi \to \chi,\ \varphi \vdash \chi$$

which can be done simply by applying MP twice.

Example 1.14 Let us prove that

$$\Gamma \vdash \psi \lor \varphi \text{ and } \Gamma, \varphi \vdash \psi \text{ imply } \Gamma \vdash \psi.$$

By the deduction theorem, $\Gamma \vdash \varphi \to \psi$. Besides, as we know, $\vdash \psi \to \psi$. By Subst and (A8), we have

$$\vdash (\psi \to \psi) \to ((\varphi \to \psi) \to (\psi \lor \varphi \to \psi))$$

from which, using MP thrice, we obtain $\Gamma \vdash \psi$.

Example 1.15 Now we show that, for every ψ and χ,

$$\vdash \psi \vee (\psi \to \chi).$$

We have:

$\vdash \psi \to \psi \vee (\psi \to \chi)$	(by Subst from (A6))
$\psi \to \bot,\ \psi \vdash \bot$	(by MP)
$\vdash \bot \to \chi$	(by Subst from (A9))
$\psi \to \bot \vdash \psi \to \chi$	(by MP and deduction theorem)
$\psi \to \bot \vdash \psi \vee (\psi \to \chi)$	(by (A7) and MP)
$\vdash (\psi \to \bot) \to \psi \vee (\psi \to \chi)$	(by deduction theorem)
$\vdash (\psi \to \psi \vee (\psi \to \chi)) \to (((\psi \to \bot) \to \psi \vee (\psi \to \chi)) \to (\psi \vee (\psi \to \bot) \to \psi \vee (\psi \to \chi)))$	(by Subst from (A8))

whence using (A10), Subst and MP thrice we obtain $\vdash \psi \vee (\psi \to \chi)$.

Calculus Cl is said to be *sound* if $\vdash \varphi$ implies $\models \varphi$, for all $\varphi \in \mathbf{For}\mathcal{L}$, and *complete* if the converse implication holds. Thus, the soundness and completeness of Cl means that the set of derivable formulas coincides with the set of valid formulas.

Theorem 1.16. (Soundness and completeness of Cl) *For each formula φ, $\vdash \varphi$ iff $\models \varphi$.*

Proof $(\Rightarrow)$ To prove the soundness it suffices to verify that all axioms of Cl are valid and the inference rules preserve the validity. We leave this to the reader.

$(\Leftarrow)$ Suppose $\nvdash \varphi$ and show that the tableau $t_0 = (\emptyset, \{\varphi\})$ is realizable, which means that $\not\models \varphi$.

Say that a tableau (Γ, Δ) is *consistent* in Cl if $\Gamma \vdash_{Cl} \psi_1 \vee \ldots \vee \psi_m$ holds for no $\psi_1, \ldots, \psi_m \in \Delta$, $m \geq 0$. We remind the reader that the disjunction of the empty set of formulas is $\bot$, and so the consistency of (Γ, Δ) means in particular that $\Gamma \nvdash \bot$. Since $\nvdash \varphi$ and $\nvdash \bot$, the tableau t_0 is consistent.

Let $\varphi_1, \ldots, \varphi_n$ be a list of all formulas in $\mathbf{Sub}\varphi$. Define a sequence of tableaux $t_0 = (\Gamma_0, \Delta_0), \ldots, t_n = (\Gamma_n, \Delta_n)$ by taking

$$t_{i+1} = \begin{cases} (\Gamma_i, \Delta_i \cup \{\varphi_{i+1}\}) \text{ if } (\Gamma_i, \Delta_i \cup \{\varphi_{i+1}\}) \text{ is consistent} \\ (\Gamma_i \cup \{\varphi_{i+1}\}, \Delta_i) \text{ otherwise.} \end{cases}$$

Notice that $\Gamma_n \cup \Delta_n = \mathbf{Sub}\varphi$. Let us show that t_{i+1} is consistent whenever t_i is consistent. Indeed, otherwise using Example 1.10 and axioms (A6)–(A8) we could find formulas $\psi_1, \ldots, \psi_m \in \Delta_i$ such that

$$\Gamma_i \vdash \psi_1 \vee \ldots \vee \psi_m \vee \varphi_{i+1},$$

$$\Gamma_i, \varphi_{i+1} \vdash \psi_1 \vee \ldots \vee \psi_m.$$

But then, by Example 1.14, $\Gamma_i \vdash \psi_1 \vee \ldots \vee \psi_m$, contrary to the consistency of t_i. Thus t_n is consistent.

Now we show that the tableau t_n is disjoint and saturated. By Proposition 1.7, it will follow that t_0 is realizable. Since t_n is consistent, $\vdash \psi \to \psi$ and $\vdash \bot \to \psi$ for every formula ψ, t_n is disjoint.

To verify condition (S1), suppose that $\psi \wedge \chi \in \Gamma_n$ and $\psi \in \Delta_n$. However by (A3), $\psi \wedge \chi \vdash \psi$, which is a contradiction, since t_n is consistent. Conditions (S2)–(S5) are checked analogously with the help of axioms (A5)–(A7) and Example 1.9.

As for (S6), suppose that $\psi \to \chi \in \Delta_n$, but either $\psi \notin \Gamma_n$ or $\chi \notin \Delta_n$. Then either $\chi \in \Gamma_n$ or $\psi \in \Delta_n$. Both these cases contradict the consistency of t_n, since, by (A1), $\chi \vdash \psi \to \chi$ and, as was shown in Example 1.15, $\vdash \psi \vee (\psi \to \chi)$.

Observe by the way that axiom (A10) was used only in the proof of (S6). ❑

Corollary 1.17 $\mathbf{Cl} = \{\varphi \in \mathbf{For}\mathcal{L} : \vdash_{Cl} \varphi\}$.

Thus, validity is a semantic counterpart of derivability in *Cl*. The following generalization of Theorem 1.16 provides a semantic counterpart for the notion of derivability from assumptions.

Theorem 1.18. (Strong completeness of *Cl*) *Every tableau* (Γ, Δ) *consistent in Cl is realizable. In particular, for every* Γ *and every* φ, $\Gamma \vdash \varphi$ *iff* $\mathfrak{M} \models \Gamma$ *implies* $\mathfrak{M} \models \varphi$, *for every model* $\mathfrak{M}$.

Proof The proof proceeds by the same scheme as the proof of ($\Leftarrow$) in Theorem 1.16. The only difference is that now the process of saturating (Γ, Δ) may be infinite (cf. the proof of Lindenbaum's lemma in Section 5.1). ❑

The same technique yields

Theorem 1.19. (Compactness) *A tableau* (Γ, Δ) *is realizable iff every tableau* (Γ', Δ') *with finite* $\Gamma' \subseteq \Gamma$ *and* $\Delta' \subseteq \Delta$ *is realizable. In particular, a set of formulas has a model iff its every finite subset has a model.*

Proof ($\Rightarrow$) is trivial and to prove ($\Leftarrow$) it is enough to observe that every derivation involves only finitely many formulas and use Theorem 1.18. Details are left to the reader. ❑

1.4 Basic properties of Cl

In this section we formulate a number of important syntactical properties of logics in the language $\mathcal{L}$ and prove or disprove them for classical logic **Cl**.

By a *logic* in the language $\mathcal{L}$ we mean here an arbitrary set $L \subseteq \mathbf{For}\mathcal{L}$ which is closed under the inference rules modus ponens and substitution. Derivations in L are defined in the same way as in **Cl** with the exception that axioms now are not those of **Cl** but all formulas in L. If L_1, L_2 are logics and $L_1 \subseteq L_2$ then L_2 is called an *extension* of L_1 and L_1 a *sublogic* of L_2.

CONSISTENCY. A logic L is called *consistent* if $L \neq \mathbf{For}\mathcal{L}$. If L contains formula (A9) then it is consistent iff $\bot \notin L$. And if L accepts the law of Duns Scotus (see Table 1.1) then L is consistent iff $\varphi \in L$ and $\neg\varphi \in L$ for no formula φ. Since $\not\models \bot$, we have

Theorem 1.20 **Cl** *is consistent.*

DECIDABILITY. As was already observed in Section 1.2, we have

Theorem 1.21 **Cl** *is decidable.*

POST COMPLETENESS. A logic is said to be *Post complete* if it is consistent and has no proper consistent extension.

Theorem 1.22 **Cl** *is Post complete.*

Proof Suppose L is a logic such that $\mathbf{Cl} \subset L$ and $\varphi \in L - \mathbf{Cl}$ for some $\varphi \in \mathbf{For}\mathcal{L}$. Let $\mathfrak{M}$ be a model refuting φ. Define a substitution $\boldsymbol{s}$ by taking

$$p_i\boldsymbol{s} = \begin{cases} \top \text{ if } \mathfrak{M} \models p_i \\ \bot \text{ otherwise.} \end{cases}$$

Then $\varphi\boldsymbol{s}$ is obviously false in every model. Therefore, $\varphi\boldsymbol{s} \to \bot \in \mathbf{Cl}$ and, since $\varphi\boldsymbol{s} \in L$, we obtain by MP that $\bot \in L$. But this means that L is inconsistent. ❑

We say a logic L is *0-reducible* if, for every formula $\varphi \notin L$, there is a variable free substitution instance $\varphi\boldsymbol{s} \notin L$. As a consequence of the proof of Theorem 1.22 we immediately obtain

Theorem 1.23 **Cl** *is 0-reducible.*

INDEPENDENT AXIOMATIZABILITY. A logic L in the language $\mathcal{L}$ is *independently axiomatizable* by a set (of *independent axioms*) $\Gamma \subseteq \mathbf{For}\mathcal{L}$ if the closure of Γ under MP and Subst is L but no proper subset of Γ possesses this property.

Theorem 1.24 **Cl** *is independently axiomatizable.*

Proof Follows from Theorem 1.16 according to which **Cl** is the closure under MP and Subst of a *finite* set of formulas. In fact one can show that (A1)–(A10) is a set of independent axioms for **Cl**. ❑

STRUCTURAL COMPLETENESS. Let $\varphi_1, \dots, \varphi_n, \varphi$ be some formulas. We will understand the figure

$$\frac{\varphi_1, \dots, \varphi_n}{\varphi} \tag{1.1}$$

as the inference rule which, for every substitution $\boldsymbol{s}$, derives $\varphi\boldsymbol{s}$ from the formulas $\varphi_1\boldsymbol{s}, \dots, \varphi_n\boldsymbol{s}$. Rule (1.1) is called *admissible* in a logic L if, for every substitution $\boldsymbol{s}$, $\varphi\boldsymbol{s} \in L$ whenever $\varphi_1\boldsymbol{s}, \dots, \varphi_n\boldsymbol{s} \in L$. By definition, the rule $p, p \to q/q$ (i.e., modus ponens) is admissible in any logic. We say also that rule (1.1) is *derivable* in L if there is a derivation of φ in L from the set of assumptions $\{\varphi_1, \ldots, \varphi_n\}$. It should be clear that every derivable rule in L is also admissible in L.

By the deduction theorem and the law of importation and exportation, (1.1) is derivable in **Cl** iff $\varphi_1 \wedge \ldots \wedge \varphi_n \to \varphi \in \mathbf{Cl}$. A logic L is called *structurally complete* if every admissible rule in L is derivable in L.

Theorem 1.25 **Cl** *is structurally complete.*

Proof Suppose rule (1.1) is admissible in **Cl** but not derivable, i.e.,

$$\varphi_1 \wedge \ldots \wedge \varphi_n \to \varphi \notin \mathbf{Cl}.$$

By Theorem 1.23, there is a variable free formula $\varphi_1\boldsymbol{s} \wedge \ldots \wedge \varphi_n\boldsymbol{s} \to \varphi\boldsymbol{s}$ which is false in every model. This means that the formulas $\varphi_1\boldsymbol{s}, \ldots, \varphi_n\boldsymbol{s}$ are valid, while $\varphi\boldsymbol{s}$ is not. Therefore, $\varphi_1\boldsymbol{s}, \ldots, \varphi_n\boldsymbol{s} \in \mathbf{Cl}$ but $\varphi\boldsymbol{s} \notin \mathbf{Cl}$, which is a contradiction. ❑

It follows from Theorem 1.25 and the decidability of **Cl** that there is an algorithm which can recognize whether an arbitrary given rule is admissible in **Cl**. In other words we obtain

Corollary 1.26 *The admissibility problem for inference rules in* **Cl** *is decidable.*

As examples of admissible inference rules in **Cl** we present here the following *congruence rules*:

$$\frac{p \leftrightarrow q}{p \wedge r \leftrightarrow q \wedge r} \qquad \frac{p \leftrightarrow q}{r \wedge p \leftrightarrow r \wedge q}$$

$$\frac{p \leftrightarrow q}{p \vee r \leftrightarrow q \vee r} \qquad \frac{p \leftrightarrow q}{r \vee p \leftrightarrow r \vee q}$$

$$\frac{p \leftrightarrow q}{(p \to r) \leftrightarrow (q \to r)} \qquad \frac{p \leftrightarrow q}{(r \to p) \leftrightarrow (r \to q)}$$

Taken together these rules yield the following theorem which is useful for the equivalent transformation of formulas.

Theorem 1.27. (Equivalent replacement) *Let $\varphi(\psi)$ be a formula containing an occurrence of a formula ψ and $\varphi(\chi)$ obtained from $\varphi(\psi)$ by replacing this occurrence with an occurrence of a formula χ. Then, for every logic L in which the congruence rules are admissible, $\psi \leftrightarrow \chi \in L$ implies $\varphi(\psi) \leftrightarrow \varphi(\chi) \in L$.*

Proof An easy induction on the construction of φ using the admissibility of the congruence rules above is left to the reader as an exercise. ❑

CRAIG INTERPOLATION PROPERTY. Say that a logic L has the *Craig interpolation property* if, for every formula $\varphi \to \psi \in L$, there is a formula χ, whose variables, if any, occur both in φ and ψ, such that $\varphi \to \chi \in L$ and $\chi \to \psi \in L$; the formula χ is called then an *interpolant* for φ and ψ in L.

Theorem 1.28. (Craig interpolation) Cl *has the Craig interpolation property.*

Proof Suppose formulas φ and ψ have no interpolant. Our aim is to show that in this case the tableau $t_0 = (\{\varphi\}, \{\psi\})$ is realizable, and so $\varphi \to \psi \notin$ **Cl**. The proof below resembles the saturation technique used in the proof of Theorem 1.16, though it is based on somewhat different principles.

Say that a tableau (Γ, Δ) is *separable* (relative to φ and ψ) if there is a formula χ such that $\mathbf{Var}\chi \subseteq \mathbf{Var}\varphi \cap \mathbf{Var}\psi$ and both tableaux $(\Gamma, \{\chi\})$ and $(\{\chi\}, \Delta)$ are not realizable (= inconsistent). According to our assumption, t_0 is not separable.

Call (Γ, Δ) *complete* (relative to φ and ψ) if, for every $\varphi' \in \mathbf{Sub}\varphi$, $\psi' \in \mathbf{Sub}\psi$, one of the formulas φ' or $\neg\varphi'$ is in Γ and one of the formulas ψ' or $\neg\psi'$ is in Δ. Starting from t_0 we will construct a complete inseparable extension of t_0 and then show that it is realizable.

Let $\varphi_1, \dots, \varphi_k$ and $\psi_1, \dots, \psi_m$ be lists of all φ's and ψ's proper subformulas, respectively. Define a sequence $t_0 = (\Gamma_0, \Delta_0), \dots, t_n = (\Gamma_n, \Delta_n)$, where $n = k + m$, by taking, for $i < k$ and $j < m$,

$$t_{i+1} = \begin{cases} (\Gamma_i \cup \{\varphi_{i+1}\}, \Delta_0) & \text{if } (\Gamma_i \cup \{\varphi_{i+1}\}, \Delta_0) \text{ is inseparable} \\ (\Gamma_i \cup \{\neg\varphi_{i+1}\}, \Delta_0) & \text{otherwise,} \end{cases}$$

$$t_{k+j+1} = \begin{cases} (\Gamma_k, \Delta_{k+j} \cup \{\psi_{j+1}\}) & \text{if } (\Gamma_k, \Delta_{k+j} \cup \{\psi_{j+1}\}) \text{ is inseparable} \\ (\Gamma_k, \Delta_{k+j} \cup \{\neg\psi_{j+1}\}) & \text{otherwise.} \end{cases}$$

Clearly, t_n is complete. We show that, for $i < k$, t_{i+1} is inseparable whenever t_i is inseparable. Indeed, otherwise we would have two formulas χ_1 and χ_2, whose variables are in $\mathbf{Var}\varphi \cap \mathbf{Var}\psi$, such that the tableaux $(\Gamma_i \cup \{\varphi_{i+1}\}, \{\chi_1\})$, $(\{\chi_1\}, \Delta_0)$, $(\Gamma_i \cup \{\neg\varphi_{i+1}\}, \{\chi_2\})$ and $(\{\chi_2\}, \Delta_0)$ are not realizable. But then the tableaux $(\Gamma_i, \{\chi_1 \vee \chi_2\})$ and $(\{\chi_1 \vee \chi_2\}, \Delta_0)$ are not realizable either, contrary to t_i being inseparable. In a similar way one can show that, for $j < m$, t_{k+j+1} is inseparable if t_{k+j} is so.

Thus, t_n is complete and inseparable. Define a model $\mathfrak{M}$ by taking, for every $p \in \mathbf{Var}\mathcal{L}$,

$$p \in \mathfrak{M} \text{ iff } p \in \Gamma_n \text{ or } \neg p \in \Delta_n.$$

We show that $\mathfrak{M}$ realizes t_n and so t_0 as well. Namely, by induction on the construction of χ we prove that

$$\chi \in \Gamma_n \text{ iff } \mathfrak{M} \models \chi, \ \text{ for } \chi \in \mathbf{Sub}\varphi,$$

$$\chi \in \Delta_n \text{ iff } \mathfrak{M} \not\models \chi, \ \text{ for } \chi \in \mathbf{Sub}\psi.$$

The basis of induction is obvious. Suppose $\chi = \chi_1 \to \chi_2$, $\chi \in \mathbf{Sub}\varphi$, $\chi \in \Gamma_n$ and $\mathfrak{M} \not\models \chi$. Then $\mathfrak{M} \models \chi_1$, $\mathfrak{M} \not\models \chi_2$ and so, by the induction hypothesis, $\chi_1 \in \Gamma_n$ and $\neg\chi_2 \in \Gamma_n$, contrary to the inseparability of t_n, since in that case both $(\Gamma_n, \{\bot\})$ and $(\{\bot\}, \Delta_n)$ are not realizable. Thus $\chi \in \Gamma_n$ implies $\mathfrak{M} \models \chi$. To prove the converse suppose $\mathfrak{M} \models \chi$ and $\chi \notin \Gamma_n$. Then $\neg(\chi_1 \to \chi_2) \in \Gamma_n$, from which $\chi_1 \in \Gamma_n$ and $\neg\chi_2 \in \Gamma_n$, for otherwise $(\Gamma_n, \{\bot\})$ would be not realizable, contrary to the inseparability of t_n. So, by the induction hypothesis, $\mathfrak{M} \models \chi_1$ and $\mathfrak{M} \not\models \chi_2$, whence $\mathfrak{M} \not\models \chi$, which is a contradiction.

The other cases are considered analogously. We leave them to the reader. ❑

LOCAL TABULARITY. Formulas φ and ψ are said to be *equivalent* in a logic L if $\varphi \leftrightarrow \psi \in L$. A logic L is called *locally tabular* (or *locally finite*) if, for every natural $n \geq 0$, L contains only a finite number of pairwise nonequivalent formulas built from variables $q_1, \ldots, q_n$.

Theorem 1.29 **Cl** *is locally tabular.*

Proof With every formula $\varphi(q_1, \ldots, q_n)$ we associate the n-ary *Boolean function* F_φ which maps n-tuples of T and F to the set {T, F} in accordance with the truth-table for $\varphi(q_1, \ldots, q_n)$. It is clear that the formulas $\varphi(q_1, \ldots, q_n)$ and $\psi(q_1, \ldots, q_n)$ are equivalent in **Cl** iff $F_\varphi = F_\psi$. And since there are exactly 2^{2^n} distinct n-ary Boolean functions, the number of pairwise nonequivalent formulas of the variables $q_1, \ldots, q_n$ is also 2^{2^n}. ❑

HALLDÉN COMPLETENESS. A logic L is said to be *Halldén complete* if, for every formulas φ and ψ containing no common variables, $\varphi \vee \psi \in L$ iff $\varphi \in L$ or $\psi \in L$.

Theorem 1.30 **Cl** *is Halldén complete.*

Proof Suppose φ and ψ have no variables in common, $\varphi \notin \mathbf{Cl}$ and $\psi \notin \mathbf{Cl}$. Then there are models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ refuting φ and ψ, respectively. Define a model $\mathfrak{M}$ by taking, for each variable p, $p \in \mathfrak{M}$ iff either $p \in \mathbf{Sub}\varphi$ and $p \in \mathfrak{M}_1$ or $p \in \mathbf{Sub}\psi$ and $p \in \mathfrak{M}_2$. By Proposition 1.2, we then have $\mathfrak{M} \not\models \varphi$, $\mathfrak{M} \not\models \psi$, whence $\mathfrak{M} \not\models \varphi \vee \psi$ and $\varphi \vee \psi \notin \mathbf{Cl}$.

The converse implication is trivial. ❑

DISJUNCTION PROPERTY. A logic L is said to have the *disjunction property* if, for every formulas φ and ψ, $\varphi \vee \psi \in L$ iff $\varphi \in L$ or $\psi \in L$. Since classical logic accepts the law of the excluded middle $p \vee \neg p$, we obviously have

Theorem 1.31 **Cl** *does not have the disjunction property.*

1.5 Exercises

Exercise 1.1 A formula φ is said to be in *disjunctive* (*conjunctive*) *normal form* if $\varphi = \psi_1 \vee \ldots \vee \psi_n$ (respectively, $\varphi = \psi_1 \wedge \ldots \wedge \psi_n$) where $n \geq 1$ and each ψ_i is a conjunction (disjunction) of atoms or negations of atoms. Show that every formula can be effectively transformed to an equivalent (in **Cl**) formula which is in disjunctive (conjunctive) normal form. (Hint: use the equivalence $(p \to q) \leftrightarrow \neg p \vee q$, de Morgan's laws, the law of double negation, the laws of distributivity and the equivalent replacement theorem.)

Exercise 1.2 A formula $\varphi(q_1, \dots, q_n)$ is in *full disjunctive* (*conjunctive*) *normal form* if it is either $\bot$ ($\top$) or a disjunctive (conjunctive) normal form whose every disjunct (conjunct) contains exactly one occurrence of each of the variables $q_1, \dots, q_n$. Show that every formula can be effectively transformed to an equivalent (in **Cl**) formula which is in full disjunctive (conjunctive) normal form. (Hint: with each line in the truth-table for $\varphi(q_1, \dots, q_n)$, in which it has value T, associate the conjunction $\chi_1 \wedge \dots \wedge \chi_n$, where $\chi_i = q_i$ if q_i is true in the line and $\chi_i = \neg q_i$ otherwise, and take the disjunction of all these conjunctions.)

Exercise 1.3 Show that each of the following sets $\{\wedge, \neg\}$, $\{\vee, \neg\}$, $\{\to, \bot\}$ is *truth-functionally complete* in the sense that every Boolean function (i.e., a function from $\{\mathrm{F}, \mathrm{T}\}^n$ to $\{\mathrm{F}, \mathrm{T}\}$) can be represented as F_φ, for some formula φ containing only connectives and constants in the set; in particular, every $\mathcal{L}$-formula is equivalent in **Cl** to such a formula.

Exercise 1.4 (Principle of duality) Let φ be a formula whose connectives are only $\wedge$, $\vee$ and $\neg$. The *dual* of φ is the formula φ^* which is obtained by replacing simultaneously every $\wedge$, $\vee$, $\bot$, $\top$ in φ with $\vee$, $\wedge$, $\top$, $\bot$, respectively. Show that for all formulas φ and ψ, $\varphi(p_1, \dots, p_n) \leftrightarrow \neg\varphi^*(\neg p_1, \dots, \neg p_n) \in \mathbf{Cl}$ and that $\varphi \leftrightarrow \psi \in \mathbf{Cl}$ iff $\varphi^* \leftrightarrow \psi^* \in \mathbf{Cl}$. In particular, $\varphi \in \mathbf{Cl}$ iff $\neg\varphi^* \in \mathbf{Cl}$.

Exercise 1.5 Let $\overline{a} = (a_1, \dots, a_n)$ and $\overline{b} = (b_1, \dots, b_n)$ be n-tuples of F and T and let $a_i \leq b_i$ iff $a_i = \mathrm{F}$ or $b_i = \mathrm{T}$. Put $\overline{a} \leq \overline{b}$ iff $a_i \leq b_i$ for every $i \in \{1, \dots, n\}$. A formula $\varphi(p_1, \dots, p_n)$ is called *monotone* if $F_\varphi \overline{a} \leq F_\varphi \overline{b}$ whenever $\overline{a} \leq \overline{b}$. Show that every formula containing only the connectives $\wedge$, $\vee$ and the constants $\bot$, $\top$ is monotone.

Exercise 1.6 Show that every monotone formula is equivalent in **Cl** to a formula in the language with the connectives $\wedge$, $\vee$ and the constants $\bot$ and $\top$.

Exercise 1.7 Say that a formula $\varphi(p_1, \dots, p_n)$ is *monotone relative to* p_i if

$$(q \to r) \to (\varphi(\dots, p_{i-1}, q, p_{i+1}, \dots) \to \varphi(\dots, p_{i-1}, r, p_{i+1}, \dots)) \in \mathbf{Cl}$$

and *antimonotone relative to* p_i if

$$(q \to r) \to (\varphi(\dots, p_{i-1}, r, p_{i+1}, \dots) \to \varphi(\dots, p_{i-1}, q, p_{i+1}, \dots)) \in \mathbf{Cl}.$$

Prove that (i) a formula is monotone iff it is monotone relative to its every variable; (ii) $p \to q$ is monotone relative to q and antimonotone relative to p; (iii) every formula φ is monotone or antimonotone relative to each variable occurring at most once in φ.

Exercise 1.8 A *matrix* for $\mathcal{L}$ is a structure $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, D\rangle$, where A is a non-empty set, D its non-empty subset, $\wedge$, $\vee$, $\to$ are binary operations on A and $\bot \in A$. A *valuation* in $\mathfrak{A}$ is a map $\mathfrak{V}$ from $\mathbf{Var}\mathcal{L}$ to A. Considering the connectives as the corresponding operations on A, we can extend inductively $\mathfrak{V}$ to a map from $\mathbf{For}\mathcal{L}$ to A. The pair $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{V}\rangle$ is an *n-universal model for* a logic L if

$\varphi \in L$ iff $\mathfrak{V}(\varphi) \in D$, for every formula $\varphi(p_1, \dots, p_n)$. For each $n < \omega$, construct a finite n-universal model for **Cl**. (Hint: take $\mathfrak{A} = \langle A, \wedge, \vee, \to, \|\bot\|, \{\|\top\|\}\rangle$, where A consists of the sets

$$\|\varphi(p_1, \dots, p_n)\| = \{\psi(p_1, \dots, p_n) : \ \varphi \leftrightarrow \psi \in \mathbf{Cl}\},$$

$$\|\varphi\| \odot \|\psi\| = \|\varphi \odot \psi\|, \text{ for } \odot \in \{\wedge, \vee, \to\},$$

$$\mathfrak{V}(p_i) = \begin{cases} \|p_i\| \text{ if } 1 \le i \le n \\ \|\bot\| \text{ otherwise.)} \end{cases}$$

Exercise 1.9 Prove that, for every $n \geq 1$, one can axiomatize **Cl** in the language with the connectives $\to$ and $\neg$ using n independent axioms and the rules Subst and MP. (Hint: for $n = 1$, take the axiom

$$\beta = ((((p_0 \to p_1) \to (\neg p_2 \to \neg p_3)) \to p_2) \to p_4) \to ((p_4 \to p_0) \to (p_3 \to p_0))$$

and for $n > 1$, use the axioms

$$\alpha_1 = \neg\neg(p \to p),$$

$$\alpha_i = \neg^{2i}(p \to p), \ 1 \le i \le n-1,$$

$$\alpha_n = \neg\neg(p \to p) \to (\dots \to (\neg^{2(n-1)}(p \to p) \to \beta) \dots),$$

where $\neg^n$ is the string of n negations.) Is it possible to extend this result to the language used in this book?

1.6 Notes

This chapter contains only those basic facts concerning classical logic that will be used in the sequel. We did not touch upon, for instance, Gentzen-style systems or Post's theory of Boolean functions. A more comprehensive exposition of classical propositional logic can be found in other textbooks on mathematical logic, say in Church (1956), Kleene (1967), Mendelson (1984) or Takeuti (1975).

There are several ways of proving the completeness theorem for *Cl*. We took that one which can be easily extended to other logics to be considered in the book. In fact, it goes back to Beth (1959), though the notion of semantic tableau we use here is somewhat different from the standard one, say that in Fitting (1983). Usually a semantic tableau is defined as a sort of derivation from a given pair $t = (\Gamma, \Delta)$ using inference rules like (SR1)–(SR6). This yields an alternative proof system for **Cl**. We apply essentially the same method but for constructing countermodels. All we need is just one disjoint saturated pair obtained from t with the help of those rules. Since we do not require tableaux to be finite, our completeness proof can be easily extended to the standard Henkin construction used for establishing completeness; cf. e.g. Chang and Keisler (1990).

Cl is the simplest logic among those to be considered in this book. Some of its properties (e.g. Halldén completeness) are trivial and were presented only for comparison with properties of non-classical logics. Although everything seemed to be known about **Cl** in the 1940s, from time to time new results continue to appear. Hodges (1983) claims that Craig's (1957) interpolation theorem was the last important achievement. That **Cl** is structurally complete was also observed not so long ago; see Belnap *et al.* (1963). Anisov (1982) showed that for any $n > 1$, **Cl** can be axiomatized by n independent axioms, with Subst and MP being the inference rules (see Exercise 1.9 the formula β in which was found by Meredith (1953)). Note also that if we do not use the rule of substitution (even in axioms) then there is a little hope to get an independent axiomatization, see Dale (1983). In this connection one more result deserves mentioning. Diamond and McKinsey (1947) constructed an algebra which is not Boolean itself but its all subalgebras generated by two elements are. It follows in particular that one cannot axiomatize **Cl** by axioms containing < 3 variables.

2

INTUITIONISTIC LOGIC

From the set-theoretic point of view intuitionistic propositional logic is a subset of the classical one: it can be defined by the calculus which is obtained from Cl by discarding the law of the excluded middle (A10). It is Brouwer's (1907, 1908) criticism of this law that intuitionistic logic stems from. However, the philosophical and mathematical justifications of these two logics are fundamentally different.

2.1 Motivation

The law of the excluded middle allows proof of disjunctions $\varphi \vee \psi$ such that neither φ nor ψ is provable. It is equivalent in **Cl** to the formula $\neg\neg p \rightarrow p$ justifying proofs by *reductio ad absurdum*, which make it possible to prove the existence of an object (having some given properties) without showing a way of constructing it. Proofs of that sort are known as *non-constructive*. The aim of intuitionistic logic is to single out and describe the laws of "constructive" reasoning.

The main principle of intuitionism asserts that the truth of a mathematical statement can be established only by producing a constructive proof of the statement. So the intended meaning of the intuitionistic logical connectives is defined in terms of *proofs* and *constructions*. The notions "proof" and "construction" themselves are regarded as primary, and it is assumed that we understand what a proof of an atomic proposition is.

- A proof of a proposition $\varphi \wedge \psi$ consists of a proof of φ and a proof of ψ.
- A proof of $\varphi \vee \psi$ is given by presenting either a proof of φ or a proof of ψ.
- A proof of $\varphi \rightarrow \psi$ is a construction which, given a proof of φ, returns a proof of ψ.
- $\bot$ has no proof and a proof of $\neg\varphi$ is a construction which, given a proof of φ, would return a proof of $\bot$.

This interpretation, given by Brouwer, Kolmogorov[3] (1932) and Heyting (1956), can hardly be reckoned as a precise semantic definition and used for constructing intuitionistic logic, as it was done for **Cl**. Nevertheless, it is not difficult to see that the first nine axioms of classical calculus Cl are entirely acceptable from the intuitionistic point of view, while the law of the excluded middle must be

[3]Kolmogorov treated formulas as schemes of solving (or posing) problems; for example, $\varphi \rightarrow \psi$ means the problem: given any solution to the problem φ, find a solution to the problem ψ.

rejected (indeed, we cannot present now a proof of Goldbach's conjecture or that $\boldsymbol{P} = \boldsymbol{NP}$, etc., nor are we able to show that these statements do not hold).

Intuitionistic logic was first constructed in the form of calculus by Heyting (1930). This calculus (an equivalent one, to be more exact) is obtained from Cl by discarding axiom (A10).

As to the interpretation above, it can be made more precise in various ways. Two of them—Kleene's realizability interpretation and Medvedev's finite problem interpretation—will be briefly discussed in Section 2.9. Another way, connected with the explicit introduction of a new provability operator, will be considered in Section 3.9 of Chapter 3 dealing with modal logic.

More suitable for the practical use strict and philosophically significant definitions of semantics for intuitionistic logic were given by Beth (1956) and Kripke (1965a) (see also Grzegorczyk, 1964). Their semantics does not exploit the notions of proof and construction; instead, it explicitly expresses an epistemic feature of intuitionistic logic. We will give now some informal motivation of the Kripke semantics; the corresponding formal definitions will be introduced in the next section.

By accepting the fundamental semantic assumption of classical logic—each proposition is either true or false—we completely abstract from the fact that actually it may be *a priori* unknown whether this or that proposition is true or false. We do not know now, for instance, if Goldbach's conjecture is true, if the equality $\boldsymbol{P} = \boldsymbol{NP}$ holds, whether there are rational beings in the Archer constellation, and so forth. But it is quite possible that we can know about this in the future, acquiring new information on mathematics and the world around us.

It is this epistemic aspect of the notion of truth that intuitionistic logic, as opposed to the classical one, takes into account.

Let us imagine that our knowledge is developing discretely, nondeterministically passing from one state to another. When at some state of knowledge (or information) x, we can say which facts are known at x and which are not established yet. Besides, we know what states of information y are possible in the future. Of course, this does not mean that we shall necessarily reach all these possible states (for instance, we can imagine now not only a course of events under which Goldbach's conjecture will be proved, but also such a situation when it will remain unproved or will be refuted). It is reasonable also to assume that while passing to a new state y all the facts known at x will be preserved, and some new facts will possibly be established.

It is natural to regard an atomic proposition, established at a state x, to be true at x; it will remain true at all further possible states. A proposition which is not true at x cannot be in general regarded as false, for it may become true at one of the subsequent states.

The truth of compound propositions can be defined now as follows.

- $\varphi \wedge \psi$ is true at a state x if both φ and ψ are true at x.
- $\varphi \vee \psi$ is true at x if either φ or ψ is true at x.

- $\varphi \to \psi$ is true at a state x if, for every subsequent possible state y, in particular x itself, φ is true at y only if ψ is true at y.
- $\bot$ is true nowhere.

It follows from this definition that the negation $\neg\varphi = \varphi \to \bot$ is true at x if φ is true at no subsequent possible state. A proposition φ may be regarded to be false at x if $\neg\varphi$ is true at x.

All axioms (A1)–(A9) (under every substitution of concrete propositions instead of variables) turn out to be true at all conceivable states, which cannot be said about (A10), i.e., $p_0 \vee (p_0 \to \bot)$. Indeed, if a proposition φ is not true at a state x, but becomes true at a subsequent state y, then $\neg\varphi$ is not true at x and so neither is $\varphi \vee \neg\varphi$.

2.2 Kripke frames and models

As in Section 1.1, let us fix the propositional language $\mathcal{L}$ with the connectives $\wedge$, $\vee$, $\to$ and the constant $\bot$. Starting from the informal interpretation above, we give now a precise definition of an intuitionistic model for $\mathcal{L}$.

An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ consisting of a nonempty set W and a partial order R on W, i.e., $\mathfrak{F}$ is just a partially ordered set. We remind the reader that a binary relation R on W is called a *partial order* if the following three conditions[4] are satisfied for all $x, y, z \in W$:

$$\begin{array}{ll} xRx & \text{(reflexivity)},\\ xRy \wedge yRz \to xRz & \text{(transitivity)},\\ xRy \wedge yRx \to x = y & \text{(antisymmetry)}.\end{array}$$

The elements of W are called the *points* of the frame $\mathfrak{F}$ and xRy is read as "y is accessible from x" or "x sees y".

A *valuation* of $\mathcal{L}$ in an intuitionistic frame $\mathfrak{F} = \langle W, R\rangle$ is a map $\mathfrak{V}$ associating with each variable $p \in \mathbf{Var}\mathcal{L}$ some (possibly empty) subset $\mathfrak{V}(p) \subseteq W$ such that, for every $x \in \mathfrak{V}(p)$ and $y \in W$, xRy implies $y \in \mathfrak{V}(p)$. Subsets of W satisfying this condition are called *upward closed*. The set of all upward closed subsets of W will be denoted by $\mathrm{Up}W$. Thus, a valuation in $\mathfrak{F}$ is a map $\mathfrak{V}$ from $\mathbf{Var}\mathcal{L}$ into $\mathrm{Up}W$.

An *intuitionistic Kripke model* of the language $\mathcal{L}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ where $\mathfrak{F}$ is an intuitionistic frame and $\mathfrak{V}$ a valuation in $\mathfrak{F}$.

In the terminology of the preceding section points in a frame $\mathfrak{F} = \langle W, R\rangle$ of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ represent states of information; if we are now at a state x then in the sequel we may reach a state y such that xRy. An atomic proposition p is regarded to be true at x if $x \in \mathfrak{V}(p)$. Since $\mathfrak{V}(p)$ is upward closed, all atomic propositions that are true at x remain true at all subsequent possible states.

[4]Here and below, to represent various properties of frames we use the language of classical predicate logic with the predicates R and $=$.

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be an intuitionistic Kripke model and x a point in the frame $\mathfrak{F} = \langle W, R \rangle$. By induction on the construction of a formula φ we define a relation $(\mathfrak{M}, x) \models \varphi$, which is read as "$\varphi$ is true at x in $\mathfrak{M}$":

$$\begin{array}{lll}
(\mathfrak{M}, x) \models p & \text{iff} & x \in \mathfrak{V}(p); \\
(\mathfrak{M}, x) \models \psi \wedge \chi & \text{iff} & (\mathfrak{M}, x) \models \psi \text{ and } (\mathfrak{M}, x) \models \chi; \\
(\mathfrak{M}, x) \models \psi \vee \chi & \text{iff} & (\mathfrak{M}, x) \models \psi \text{ or } (\mathfrak{M}, x) \models \chi; \\
(\mathfrak{M}, x) \models \psi \to \chi & \text{iff} & \text{for all } y \in W \text{ such that } xRy, \\
 & & \quad (\mathfrak{M}, y) \models \psi \text{ implies } (\mathfrak{M}, y) \models \chi; \\
(\mathfrak{M}, x) \not\models \bot. & &
\end{array}$$

It follows from this definition that

$$(\mathfrak{M}, x) \models \neg\psi \quad \text{iff} \quad \text{for all } y \in W \text{ such that } xRy,\ (\mathfrak{M}, y) \not\models \psi.$$

If $\mathfrak{M}$ is understood we write $x \models \varphi$ instead of $(\mathfrak{M}, x) \models \varphi$. The truth-set of φ in $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, i.e., the set $\{x : x \models \varphi\}$, will be denoted by $\mathfrak{V}(\varphi)$.

Notice that an intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ on the frame $\mathfrak{F}$ containing only a single point, say x, is in essence the same as the classical model

$$\mathfrak{N} = \{p \in \mathbf{Var}\mathcal{L} : \ x \in \mathfrak{V}(p)\},$$

because $(\mathfrak{M}, x) \models \varphi$ iff $\mathfrak{N} \models \varphi$, for every formula φ.

Proposition 2.1 *For every intuitionistic Kripke model on a frame $\mathfrak{F} = \langle W, R \rangle$, every formula φ and all points $x, y \in W$, if $x \models \varphi$ and xRy then $y \models \varphi$.*

Proof An easy induction on the construction of φ is left to the reader as an exercise. ❑

In other words, Proposition 2.1 states that the set of points where φ is true is upward closed. On the contrary, the set of points at which φ is not true may be called *downward closed*, since $x \not\models \varphi$ and yRx imply $y \not\models \varphi$.

We say a formula φ is *satisfied* in a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ if $x \models \varphi$ for some point x in $\mathfrak{F}$. φ is *true* in $\mathfrak{M}$ if $x \models \varphi$ for every x in $\mathfrak{F}$; in this case we write $\mathfrak{M} \models \varphi$. If φ is not true in $\mathfrak{M}$ then we say that φ is *refuted* in $\mathfrak{M}$ or $\mathfrak{M}$ is a *countermodel* for φ, and write $\mathfrak{M} \not\models \varphi$.

A formula φ is *satisfied* in a frame $\mathfrak{F}$ if φ is satisfied in some model based on $\mathfrak{F}$. φ is *true at a point* x in $\mathfrak{F}$ (notation: $(\mathfrak{F}, x) \models \varphi$) if φ is true at x in every model based on $\mathfrak{F}$. φ is called *valid* in a frame $\mathfrak{F}$, $\mathfrak{F} \models \varphi$ in symbols, if φ is true in all models based on $\mathfrak{F}$. Otherwise we say that φ is *refuted* in $\mathfrak{F}$ and write $\mathfrak{F} \not\models \varphi$.

If every formula in a set Γ is true at a point x in a model $\mathfrak{M}$, we write $(\mathfrak{M}, x) \models \Gamma$ or simply $x \models \Gamma$. $\mathfrak{M} \models \Gamma$ and $\mathfrak{F} \models \Gamma$ mean that all formulas in Γ are true in $\mathfrak{M}$ and are valid in $\mathfrak{F}$, respectively.

Frames $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$ are said to be *isomorphic* if there is a 1–1 map f from W onto V such that xRy iff $f(x)Sf(y)$, for all $x, y \in W$. The map f is called then an *isomorphism* of $\mathfrak{F}$ onto $\mathfrak{G}$. Models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$

are *isomorphic* if there is an isomorphism f of $\mathfrak{F}$ onto $\mathfrak{G}$ such that, for every $p \in \mathbf{Var}\mathcal{L}$, $\mathfrak{U}(p) = f(\mathfrak{V}(p))$, i.e., for every $x \in W$,

$$(\mathfrak{M}, x) \models p \text{ iff } (\mathfrak{N}, f(x)) \models p.$$

In this case we say that f is an *isomorphism* of $\mathfrak{M}$ onto $\mathfrak{N}$.

The following two propositions are direct consequences of the given definitions.

Proposition 2.2 *If f is an isomorphism of a model $\mathfrak{M}$ onto a model $\mathfrak{N}$ then, for every point x in $\mathfrak{M}$ and every formula φ,*

$$(\mathfrak{M}, x) \models \varphi \textit{ iff } (\mathfrak{N}, f(x)) \models \varphi.$$

This gives us the ground not to distinguish between isomorphic models as well as isomorphic frames.

Proposition 2.3 *Suppose $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{N} = \langle \mathfrak{F}, \mathfrak{U} \rangle$ are models on a frame $\mathfrak{F}$ such that the valuations $\mathfrak{V}$ and $\mathfrak{U}$ coincide on the variables in some set* $\mathbf{Var} \subseteq \mathbf{Var}\mathcal{L}$. *Then for every point x in $\mathfrak{F}$ and every formula φ with* $\mathbf{Var}\varphi \subseteq \mathbf{Var}$,

$$(\mathfrak{M}, x) \models \varphi \textit{ iff } (\mathfrak{N}, x) \models \varphi.$$

Thus, if we want to construct a countermodel for a formula φ on a frame $\mathfrak{F}$, it suffices to define a valuation $\mathfrak{V}$, refuting φ, only on the variables in φ; the values of $\mathfrak{V}$ on other variables have no effect on the truth of φ at points in $\mathfrak{F}$.

We shall often represent intuitionistic frames in the form of diagrams by depicting points as circles $\circ$ and drawing an arrow from x to y if xRy. To avoid awkwardness, we will not draw those arrows that can be uniquely reconstructed by the properties of reflexivity and transitivity. For technical reasons it is sometimes impossible to connect x and y with an arrow; we then connect them with a (broken) line, and the fact that xRy is reflected by placing y higher than x. When representing models, we shall sometimes write some formulas near points: on the left side of a point x we write those formulas that are true at x and those that are not true are written on the right.

Example 2.4 Suppose $\mathfrak{F} = \langle W, R \rangle$ is the frame in which $W = \{a, b\}$, $R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle\}$ and let $\mathfrak{V}(p) = \{b\}$ and $\mathfrak{V}(q) = \{a, b\}$ for all $q \in \mathbf{Var}\mathcal{L}$ different from p. Then the formula $p \vee (p \to \bot)$ is true at b and not true at a in the model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. This situation is represented graphically in Fig. 2.1. Thus, $p \vee (p \to \bot)$ is satisfied as well as refuted in $\mathfrak{F}$. The formula $((p \to \bot) \to \bot) \to p$ is also refuted in $\mathfrak{M}$, since $a \models (p \to \bot) \to \bot$ and $a \not\models p$.

Example 2.5 The formula $p \to ((p \to \bot) \to \bot)$ is valid in all intuitionistic frames. Indeed, suppose otherwise. Then there is a model on a frame $\mathfrak{F} = \langle W, R \rangle$ such that $x \models p$ and $x \not\models (p \to \bot) \to \bot$ for some $x \in W$, and so there is $y \in W$ for which xRy and $y \models p \to \bot$. By the definition of valuation, we must have $y \models p$, whence $y \not\models p \to \bot$, which is a contradiction.

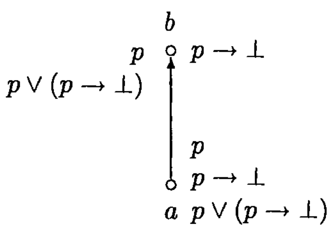


FIG. 2.1.

We define *intuitionistic propositional logic* $\mathbf{Int}_{\mathcal{L}}$ in the language $\mathcal{L}$ as the set of all $\mathcal{L}$-formulas that are valid in all intuitionistic frames, i.e.,

$$\mathbf{Int}_{\mathcal{L}} = \{\varphi \in \mathbf{For}\mathcal{L} : \ \mathfrak{F} \models \varphi \text{ for all frames } \mathfrak{F}\}.$$

Usually we will drop the subscript $\mathcal{L}$ and write simply **Int**.

Since the classical validity is nothing else but the validity in the single-point intuitionistic frame, we obtain the inclusion

$$\mathbf{Int} \subseteq \mathbf{Cl}.$$

And since $p \vee \neg p$ is in **Cl** but does not belong to **Int**, this inclusion is proper.

2.3 Truth-preserving operations

In comparison with classical models intuitionistic ones are much more complex structures. So before proceeding to the study of **Int** let us develop some notions and technical means for handling them. In this section we introduce three very important operations on intuitionistic models and frames which preserve truth and validity.

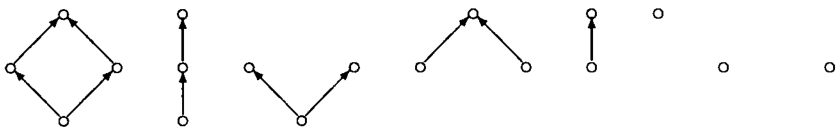
A frame $\mathfrak{G} = \langle V, S \rangle$ is called a *subframe* of a frame $\mathfrak{F} = \langle W, R \rangle$ (notation: $\mathfrak{G} \subseteq \mathfrak{F}$) if $V \subseteq W$ and S is the restriction of R to V ($S = R \restriction V$, in symbols), i.e., $S = R \cap V^2$. The subframe $\mathfrak{G}$ is a *generated subframe* of $\mathfrak{F}$ (notation: $\mathfrak{G} \mathrel{\underset{\rightarrow}{\subset}} \mathfrak{F}$) if V is an upward closed subset of W.

Example 2.6 Let $\mathfrak{F}$ be the frame depicted in Fig. 2.2 (a). Then the frames shown in Fig. 2.2 (a)–(g) are (isomorphic to) subframes of $\mathfrak{F}$, with (a), (d), (e) and (f) being the only pairwise non-isomorphic generated subframes.

If $\mathfrak{G} = \langle V, S \rangle$ is a generated subframe of $\mathfrak{F} = \langle W, R \rangle$ and V is the upward closure of some set $X \subseteq W$, i.e., V is the minimal upward closed subset of W to contain X, then we say that V and $\mathfrak{G}$ are *generated by the set* X. Notice that since R is reflexive and transitive,

$$V = \{x \in W : \ \exists y \in X \ yRx\}.$$

If $\mathfrak{F}$ is generated by a singleton $\{x\}$ then $\mathfrak{F}$ is called *rooted* and x is called the *root* (or the *least point*) of $\mathfrak{F}$. All frames in Fig. 2.2, except (d) and (g), are rooted.



(a) (b) (c) (d) (e) (f) (g)

FIG. 2.2.

We introduce special notations for the operations of upward and downward closure. Namely, if $\mathfrak{F} = \langle W, R\rangle$ is a frame and $X \subseteq W$ then we let

$$X{\uparrow}R = \{x \in W : \exists y \in X\ yRx\},$$

$$X{\downarrow}R = \{x \in W : \exists y \in X\ xRy\}.$$

If $\mathfrak{F}$ is understood then we drop R and write simply $X{\uparrow}$ and $X{\downarrow}$; we also write $x{\uparrow}$ and $x{\downarrow}$ instead of $\{x\}{\uparrow}$ and $\{x\}{\downarrow}$, respectively. All the points in $x{\uparrow}$ ($x{\downarrow}$) are called *successors* (*predecessors*) of x; a successor (predecessor) y of x is *proper* if $x \neq y$. A proper successor (predecessor) y of x is an *immediate successor* (respectively, *immediate predecessor*) of x if $xRzRy$ ($yRzRx$) implies $z = x$ or $z = y$, for every $z \in W$. A point x is a *final* (or *maximal*) *point* in $\mathfrak{F}$ if $x{\uparrow} = \{x\}$; x is the *last* (or *greatest*) *point* in $\mathfrak{F}$ if $x{\downarrow} = W$. More generally, a point $x \in X \subseteq W$ is called *final* (or *maximal*) in X if no proper successor of x is in X.

Thus, $\mathfrak{G} = \langle V, S\rangle$ is a subframe of $\mathfrak{F} = \langle W, R\rangle$ generated by a set X if $V = X{\uparrow}R$ and $S = R \cap V^2$; x is the root of $\mathfrak{G}$ if $V = x{\uparrow}S$. Using arrows, instead of xRy we can write now either $y \in x{\uparrow}$ or $x \in y{\downarrow}$.

A model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ is a *submodel* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ (notation: $\mathfrak{N} \subseteq \mathfrak{M}$) if $\mathfrak{G} = \langle V, S\rangle$ is a subframe of $\mathfrak{F} = \langle W, R\rangle$ and, for every $p \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{U}(p) = \mathfrak{V}(p) \cap V.$$

In the case when $\mathfrak{G} \subsetneq \mathfrak{F}$ the model $\mathfrak{N}$ is called a *generated submodel* of $\mathfrak{M}$ (notation: $\mathfrak{N} \subsetneq \mathfrak{M}$).

The formation of generated submodels is the first truth-preserving operation of the three mentioned above.

Theorem 2.7. (Generation) *Suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ is a generated submodel of $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$. Then for every formula φ and every point x in $\mathfrak{G}$,*

$$(\mathfrak{N}, x) \models \varphi \text{ iff } (\mathfrak{M}, x) \models \varphi.$$

Proof The proof proceeds by induction on the construction of φ. The basis of induction is obvious. Let $\varphi = \psi \to \chi$, $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$. Then we have:

$$\begin{aligned}(\mathfrak{N}, x) \models \varphi &\text{ iff } \forall y \in x{\uparrow}S\ ((\mathfrak{N}, y) \models \psi \to (\mathfrak{N}, y) \models \chi)\\ &\text{ iff } \forall y \in x{\uparrow}R\ ((\mathfrak{M}, y) \models \psi \to (\mathfrak{M}, y) \models \chi)\\ &\text{ iff } (\mathfrak{M}, x) \models \varphi.\end{aligned}$$

Here the second equivalence is justified by the induction hypothesis and the fact that $x{\uparrow}S = x{\uparrow}R$, for every point $x \in V$.

The cases $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ are trivial. ❑

The generation theorem means that the truth-values of formulas at a point x are completely determined by the truth-values of their variables at the points in $x{\uparrow}$ and do not depend on other points in the model.

Corollary 2.8 *If $\mathfrak{G} \subsetneq \mathfrak{F}$ then, for every formula φ,*
(i) $(\mathfrak{G}, x) \models \varphi$ *iff* $(\mathfrak{F}, x) \models \varphi$, *for all points x in $\mathfrak{G}$;*
(ii) $\mathfrak{F} \models \varphi$ *implies* $\mathfrak{G} \models \varphi$.

Proof (i) Suppose $(\mathfrak{G}, x) \not\models \varphi$. Then $(\mathfrak{N}, x) \not\models \varphi$ for some model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$. Define a valuation $\mathfrak{V}$ on $\mathfrak{F}$ by taking

$$\mathfrak{V}(p) = \mathfrak{U}(p) \text{ for all } p \in \mathbf{Var}\mathcal{L}.$$

Then $\mathfrak{N} \subsetneq \mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and so, by the generation theorem, $(\mathfrak{M}, x) \not\models \varphi$. Therefore, $(\mathfrak{F}, x) \models \varphi$ implies $(\mathfrak{G}, x) \models \varphi$. The converse implication is a direct consequence of the generation theorem.

(ii) follows from (i). ❑

We draw two more simple consequences of the generation theorem.

Corollary 2.9 *For every frame $\mathfrak{F}$ and every formula φ, the following conditions are equivalent:*
(i) $\mathfrak{F} \models \varphi$;
(ii) $\mathfrak{G} \models \varphi$, *for every* $\mathfrak{G} \subsetneq \mathfrak{F}$;
(iii) $\mathfrak{G} \models \varphi$, *for every rooted* $\mathfrak{G} \subsetneq \mathfrak{F}$.

Corollary 2.10 $\mathbf{Int}_{\mathcal{L}} = \{\varphi \in \mathbf{For}\mathcal{L} :\ \mathfrak{F} \models \varphi$ *for all rooted frames* $\mathfrak{F}\}$.

Our second truth-preserving operation is defined in a slightly more complicated way.

Suppose we have two frames $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$. A map f from W *onto* V is called a *reduction of $\mathfrak{F}$ to $\mathfrak{G}$* if the following conditions hold for every $x, y \in W$:

$$\text{(R1)} \quad xRy \text{ implies } f(x)Sf(y);$$
$$\text{(R2)} \quad f(x)Sf(y) \text{ implies } \exists z \in W\ (xRz \wedge f(z) = f(y)).$$

In this case we say also that f *reduces $\mathfrak{F}$ to $\mathfrak{G}$* or $\mathfrak{G}$ is an *f-reduct* (or simply a *reduct*) of $\mathfrak{F}$ or $\mathfrak{F}$ is *f-reducible* (or simply *reducible*) to $\mathfrak{G}$. Such a map f is often called a *pseudo-epimorphism* or just a *p-morphism* as well.

Proposition 2.11 *A one-to-one reduction of $\mathfrak{F}$ to $\mathfrak{G}$ is an isomorphism between $\mathfrak{F}$ and $\mathfrak{G}$.*

Proof Exercise. ❑

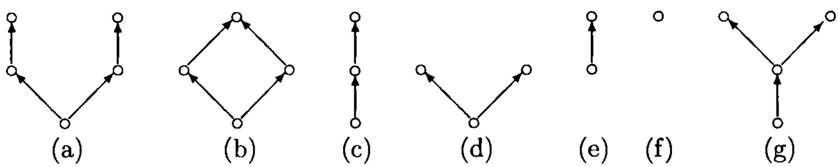


FIG. 2.3.

Example 2.12 The frame in Fig. 2.3 (a) is reducible to all frames (a)–(f), but not to (g).

Proposition 2.13 *Let f be a reduction of $\mathfrak{F} = \langle W, R\rangle$ to $\mathfrak{G} = \langle V, S\rangle$, $X \in \mathrm{Up}W$ and $Y \in \mathrm{Up}V$. Then $f(X) \in \mathrm{Up}V$ and $f^{-1}(Y) \in \mathrm{Up}W$.*

Proof Suppose that $f(x)Sy$ for some $x \in X$ and $y \in V$. Then, by (R2), there is $z \in x\uparrow$ such that $f(z) = y$. Since X is upward closed, $z \in X$ and so $y \in f(X)$. Hence $f(X) \in \mathrm{Up}V$.

Now let xRy, for some $x \in f^{-1}(Y)$ and $y \in W$. Then, by (R1), $f(x)Sf(y)$, whence $f(y) \in Y$ and $y \in f^{-1}(Y)$. So $f^{-1}(Y) \in \mathrm{Up}W$. ❑

Proposition 2.14 *If f is a reduction of $\mathfrak{F}$ to $\mathfrak{G}$ and g a reduction of $\mathfrak{G}$ to $\mathfrak{H}$ then the composition gf is a reduction of $\mathfrak{F}$ to $\mathfrak{H}$.*

Proof Exercise. ❑

A reduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called a *reduction* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ to a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ if, for every $p \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{V}(p) = f^{-1}(\mathfrak{U}(p)),$$

i.e., if for every point x in $\mathfrak{F}$,

$$(\mathfrak{M}, x) \models p \text{ iff } (\mathfrak{N}, f(x)) \models p.$$

Theorem 2.15. (Reduction) *If f is a reduction of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ to a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ then, for every point x in $\mathfrak{F}$ and every formula φ,*

$$(\mathfrak{M}, x) \models \varphi \textit{ iff } (\mathfrak{N}, f(x)) \models \varphi.$$

Proof We conduct the proof by induction on the construction of φ. The basis of induction is trivial. Let $\varphi = \psi \to \chi$.

If $(\mathfrak{M}, x) \not\models \varphi$ then there is a point $y \in x\uparrow$ such that $(\mathfrak{M}, y) \models \psi$ and $(\mathfrak{M}, y) \not\models \chi$. By the induction hypothesis, $(\mathfrak{N}, f(y)) \models \psi$ and $(\mathfrak{N}, f(y)) \not\models \chi$, and by (R1), $f(x)Sf(y)$. Therefore, $(\mathfrak{N}, f(x)) \not\models \varphi$.

Conversely, suppose $(\mathfrak{N}, f(x)) \not\models \varphi$, i.e., there is a point $u \in f(x)\uparrow$ such that $(\mathfrak{N}, u) \models \psi$ and $(\mathfrak{N}, u) \not\models \chi$. Since f is a map "onto", there is $y \in f^{-1}(u)$. Then $f(x)Sf(y)$. By (R2), there is $z \in x\uparrow$ such that $f(z) = f(y) = u$. By the induction hypothesis, $(\mathfrak{M}, z) \models \psi$ and $(\mathfrak{M}, z) \not\models \chi$, whence $(\mathfrak{M}, x) \not\models \varphi$.

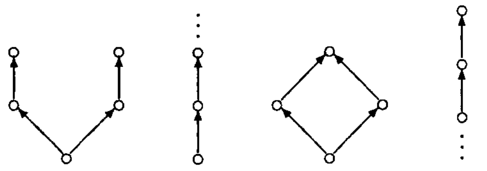


FIG. 2.4.

The cases $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ present no difficulties. ❑

Corollary 2.16 *Let f be a reduction of $\mathfrak{F}$ to $\mathfrak{G}$. Then, for every formula φ and every point x in $\mathfrak{F}$,*

$$(\mathfrak{F}, x) \models \varphi \textit{ implies } (\mathfrak{G}, f(x)) \models \varphi.$$

Proof Assuming otherwise, we have a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ in which $f(x) \not\models \varphi$. Construct a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ by taking, for all $p \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{V}(p) = f^{-1}(\mathfrak{U}(p)).$$

By the reduction theorem, we must then have $(\mathfrak{M}, x) \not\models \varphi$, which is a contradiction. ❑

Corollary 2.17 *If $\mathfrak{F}$ is reducible to $\mathfrak{G}$ then, for every formula φ,*

$$\mathfrak{F} \models \varphi \textit{ implies } \mathfrak{G} \models \varphi.$$

As an example of the use of the reduction theorem we will show that every formula $\varphi \notin \mathbf{Int}$ is refuted in a frame having the tree form.

Say that a frame $\mathfrak{F} = \langle W, R \rangle$ is a *tree* if

- $\mathfrak{F}$ is rooted and
- for every point $x \in W$, the set $x{\downarrow}$ is finite and linearly ordered by R.

We remind the reader that a set X of points in a frame $\mathfrak{F} = \langle W, R \rangle$ is *linearly ordered* by R if xRy or yRx, for every distinct $x, y \in X$. In such a case X is also called a *chain* in $\mathfrak{F}$. In particular, a sequence $x_1, x_2, \ldots$ of (distinct) points in $\mathfrak{F}$ is a (*strictly*) *ascending chain* if $x_1 R x_2 R \ldots$ and a (*strictly*) *descending chain* if $\ldots R x_2 R x_1$.

Example 2.18 The frames shown in Fig. 2.4 (a), (b) are trees (the latter one is an infinite ascending chain), while those in Fig. 2.4 (c), (d) are not trees (the latter one is an infinite descending chain).

Theorem 2.19 *Every rooted frame $\mathfrak{G} = \langle V, S \rangle$ is a reduct of some tree $\mathfrak{F} = \langle W, R \rangle$, which is finite if $\mathfrak{G}$ is finite.*

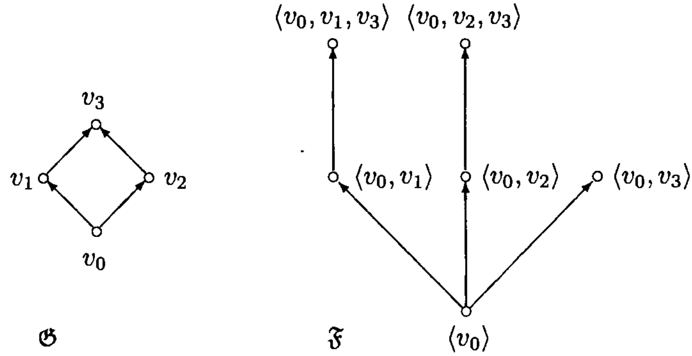


FIG. 2.5.

Proof Suppose v_0 is the root of $\mathfrak{G}$. Define W as the set of all finite strictly ascending chains of the form $\langle v_0, v_1, \dots, v_{n-1}, v_n \rangle$ in $\mathfrak{G}$ and put

$$\langle v_0, \dots, v_n \rangle \, R \, \langle u_0, \dots, u_m \rangle \text{ iff } n \le m \text{ and } v_i = u_i \text{ for } i = 0, \dots, n.$$

Clearly R is a partial order on W. (See Fig. 2.5.)

We show first that $\mathfrak{F} = \langle W, R \rangle$ is a tree. Indeed, $\langle v_0 \rangle$ is the root of $\mathfrak{F}$ and the set $\langle v_0, v_1, \dots, v_{n-1}, v_n \rangle\!\downarrow$ is the finite chain

$$\langle v_0 \rangle \, R \, \langle v_0, v_1 \rangle \, R \dots R \, \langle v_0, v_1, \dots, v_{n-1} \rangle \, R \, \langle v_0, v_1, \dots, v_{n-1}, v_n \rangle .$$

Now we define a map f from W onto V by taking $f(\langle v_0, \dots, v_n \rangle) = v_n$. Let $x, y \in W$ and xRy. Then, by the definition of R, $x = \langle v_0, \dots, v_n \rangle$, $y = \langle v_0, \dots, v_n, v_{n+1}, \dots, v_m \rangle$ and so $v_n S v_{n+1} S \dots S v_{m-1} S v_m$, whence, by the transitivity of S, $v_n S v_m$, i.e., $f(x) S f(y)$. Therefore, f satisfies (R1).

Let $f(x) = v_n$ (i.e., $x = \langle v_0, \dots, v_n \rangle$), $f(y) = v_m$ and $v_n S v_m$. If $v_n = v_m$ then obviously xRx and $f(x) = f(y)$. Otherwise, for $z = \langle v_0, \dots, v_n, v_m \rangle$, we have xRz and $f(z) = v_m$. Thus f satisfies (R2) and so is a reduction of the tree $\mathfrak{F}$ to $\mathfrak{G}$. ❑

Corollary 2.20 **Int** $= \{\varphi \in \mathbf{For}\mathcal{L} : \ \mathfrak{F} \models \varphi$ *for every tree* $\mathfrak{F}\}$.

A tree $\mathfrak{F}$ is said to be *n-ary*, for $n \ge 1$, if every non-final point in $\mathfrak{F}$ has exactly n immediate successors. If, for some $m < \omega$, every strictly ascending chain in a finite n-ary tree $\mathfrak{F}$ can be extended to a strictly ascending chain of length m then we say $\mathfrak{F}$ is the *full n-ary tree of depth m*. And if an n-ary tree has no final points at all then it is called the *full n-ary tree*. It is clear that, for each $n \ge 1$, there is only one full n-ary tree (modulo isomorphism, of course); we denote it by $\mathfrak{T}_n$. Every rooted generated subtree of $\mathfrak{T}_n$ is isomorphic to $\mathfrak{T}_n$, i.e., is again the full n-ary tree.

Theorem 2.21 *Every finite tree* $\mathfrak{F} = \langle W, R \rangle$ *is a reduct of* $\mathfrak{T}_n$, *for each* $n \ge 2$.

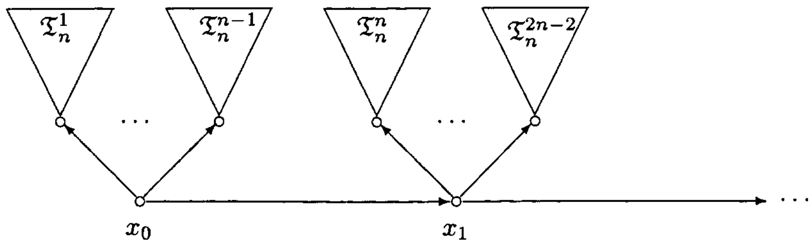


FIG. 2.6.

Proof We proceed by induction on the number of points in $\mathfrak{F}$. If $\mathfrak{F}$ is a singleton then the map of $\mathfrak{T}_n$ to $\mathfrak{F}$ is clearly a reduction.

Suppose now that $\mathfrak{F}$ contains $k+1$ points, v is the root of $\mathfrak{F}$ and $v_0, \dots, v_m$ are all its distinct immediate successors. Denote by $\mathfrak{F}_i = \langle W_i, R_i \rangle$ the subtree of $\mathfrak{F}$ generated by v_i, for $i = 0, \dots, m$.

Let us represent $\mathfrak{T}_n$ as is shown in Fig. 2.6. Here $\mathfrak{T}_n^i$, $i = 1, 2, \dots$, are disjoint isomorphic copies of $\mathfrak{T}_n$. By the induction hypothesis, for each $i \geq 1$, there is a reduction f_i of $\mathfrak{T}_n^i$ to $\mathfrak{F}_{\mathrm{mod}_{m+1}(i)}$. Define a map f from $\mathfrak{T}_n$ onto $\mathfrak{F}$ by taking, for every point x in $\mathfrak{T}_n$,

$$f(x) = \begin{cases} v & \text{if } x = x_i, \text{ for } i \geq 0 \\ f_i(x) & \text{if } x \text{ is a point in } \mathfrak{T}_n^i. \end{cases}$$

It should be clear that f is a reduction of $\mathfrak{T}_n$ to $\mathfrak{F}$. ❑

Corollary 2.22 *Every finite rooted frame is a reduct of $\mathfrak{T}_n$, for each $n \geq 2$.*

Proof Follows from Proposition 2.14 and Theorems 2.19 and 2.21. ❑

Our third truth-preserving operation is the disjoint union of frames.

Let $\{\mathfrak{F}_i = \langle W_i, R_i \rangle : i \in I\}$ be a family of frames such that $W_i \cap W_j = \emptyset$, for all $i \neq j$. The *disjoint union of the family* $\{\mathfrak{F}_i : i \in I\}$ is the frame $\sum_{i \in I} \mathfrak{F}_i = \langle \bigcup_{i \in I} W_i, \bigcup_{i \in I} R_i \rangle$. If the set I is finite, say $I = \{1, \dots, n\}$, then along with $\sum_{i \in I} \mathfrak{F}_i$ we write also $\mathfrak{F}_1 + \ldots + \mathfrak{F}_n$. We obtain a diagram of $\sum_{i \in I} \mathfrak{F}_i$ by drawing side by side the diagrams of all frames $\mathfrak{F}_i$, for $i \in I$, and regarding them as one big diagram. It is clear that every $\mathfrak{F}_i$ is a generated subframe of $\sum_{i \in I} \mathfrak{F}_i$.

The *disjoint union of the family of models* $\{\mathfrak{M}_i = \langle \mathfrak{F}_i, \mathfrak{V}_i \rangle : i \in I\}$ with pairwise disjoint frames is the model $\sum_{i \in I} \mathfrak{M}_i = \langle \sum_{i \in I} \mathfrak{F}_i, \sum_{i \in I} \mathfrak{V}_i \rangle$ where $(\sum_{i \in I} \mathfrak{V}_i)(p) = \bigcup_{i \in I} \mathfrak{V}_i(p)$, for every $p \in \mathbf{Var}\mathcal{L}$. Obviously, each model $\mathfrak{M}_i$ is a generated submodel of $\sum_{i \in I} \mathfrak{M}_i$.

Theorem 2.23. (Disjoint union) *Let $\sum_{i \in I} \mathfrak{M}_i$ be the disjoint union of a family $\{\mathfrak{M}_i : i \in I\}$. Then for every $i \in I$, every point x in $\mathfrak{M}_i$ and every formula φ,*

$$(\sum_{i \in I} \mathfrak{M}_i, x) \models \varphi \textit{ iff } (\mathfrak{M}_i, x) \models \varphi.$$

Proof Follows from the generation theorem. ❑

Corollary 2.24 *Let* $\sum_{i\in I}\mathfrak{F}_i$ *be the disjoint union of a family* $\{\mathfrak{F}_i : i \in I\}$. *Then, for every formula* φ, $\sum_{i\in I}\mathfrak{F}_i \models \varphi$ *iff* $\mathfrak{F}_i \models \varphi$ *for all* $i \in I$.

The following proposition is left to the reader as an exercise.

Proposition 2.25 *Every frame is a reduct of the disjoint union of some family of rooted frames.*

We use the reduction and disjoint union theorems to show that, as in **Cl**, there are only two non-equivalent variable free formulas in **Int**.

Proposition 2.26 *For every variable free formula* φ, *either* $\varphi \leftrightarrow \top \in \mathbf{Int}$ *or* $\varphi \leftrightarrow \bot \in \mathbf{Int}$.

Proof If $\varphi \in \mathbf{Int}$ then clearly $\varphi \leftrightarrow \top \in \mathbf{Int}$. We show that if $\varphi \notin \mathbf{Int}$ then $\varphi \leftrightarrow \bot \in \mathbf{Int}$. Since $\bot \to \varphi \in \mathbf{Int}$, it suffices to prove that $\neg\varphi$ is in **Int**. Suppose otherwise. Then we have two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ refuting φ and $\neg\varphi$, respectively. Since φ is variable free, by Proposition 2.3 we may assume that no variable is true at any point in $\mathfrak{M}_1$ or $\mathfrak{M}_2$. The single-point model $\mathfrak{N}$ refuting all variables is then a reduct of $\mathfrak{M}_1 + \mathfrak{M}_2$, and therefore $\mathfrak{N} \not\models \varphi$ and $\mathfrak{N} \not\models \neg\varphi$, which is a contradiction. ❑

Corollary 2.27 *For every variable free formula* φ, $\varphi \in \mathbf{Int}$ *iff* $\varphi \in \mathbf{Cl}$.

Proof ($\Rightarrow$) is trivial. Suppose $\varphi \in \mathbf{Cl}$. By Proposition 2.26, either $\varphi \leftrightarrow \top \in \mathbf{Int}$ or $\varphi \leftrightarrow \bot \in \mathbf{Int}$. In the former case $\varphi \in \mathbf{Int}$. And the latter means that $\neg\varphi \in \mathbf{Int}$ and so $\neg\varphi \in \mathbf{Cl}$, contrary to the consistency of **Cl**. ❑

Corollary 2.28 **Int** *is not 0-reducible.*

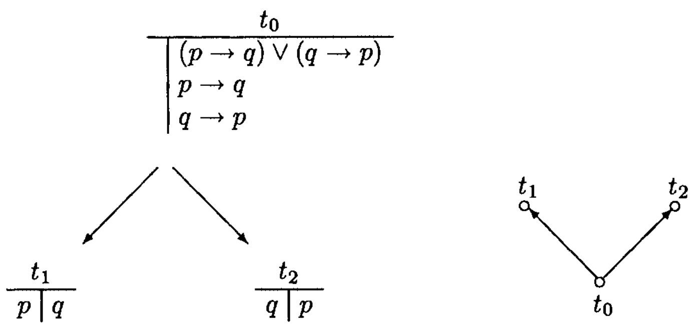
Proof Take any formula $\varphi \in \mathbf{Cl} - \mathbf{Int}$. Then every variable free substitution instance of φ is in **Cl** and so in **Int**. ❑

2.4 Hintikka systems

We have defined both classical and intuitionistic logics as sets of formulas which are valid in some frames. The fundamental difference between these two definitions is, however, that **Cl** is the set of formulas which are valid in a *single finite* frame, while **Int** contains formulas that are valid in *all* frames, including *infinite* ones. In other words, to answer the question "$\varphi \in \mathbf{Cl}$?", it suffices to fulfill a finite number of computations, whereas for a positive solution to the problem "$\varphi \in \mathbf{Int}$?" we must produce a *proof* of the validity of φ in all frames.

In this section we will develop an apparatus of semantic tableaux for intuitionistic logic and show that for every formula $\varphi \notin \mathbf{Int}$ one can construct a countermodel containing at most $2^{|\mathbf{Sub}\varphi|}$ points. (Here and below $|X|$ denotes the *cardinality* of the set X.) Thus, the validity of φ in all frames is completely determined by its validity in the frames of cardinality $\leq 2^{|\mathbf{Sub}\varphi|}$.

Let us again begin with examples.



(a) (b)

FIG. 2.7.

Example 2.29 Suppose that we want to determine whether the formula

$$\boldsymbol{da} = (p \to q) \vee (q \to p),$$

known as the *Dummett formula* (or *axiom*), is in **Int**. To this end let us try to construct a countermodel for it using the same idea as was exploited in Section 1.2 for finding countermodels in **Cl**.

First we form a tableau t_0 by putting $\boldsymbol{da}$ in its right part, indicating thereby that we wish this formula to be not true at the point t_0 in the model to be constructed. Since a disjunction is not true at a point x iff both of its disjuncts are not true at x, we must put in the right part of t_0 two more formulas: $p \to q$ and $q \to p$. An implication is not true at x iff there is a point y accessible from x, where the premise of the implication is true and the conclusion is not (in particular y may coincide with x). So we form two new tableaux t_1 and t_2 accessible from t_0: t_1 contains p in the left part, q in the right and t_2, conversely, p in the right part and q in the left. (See Fig. 2.7 (a).)

Now we construct a frame $\mathfrak{F} = \langle W, R\rangle$ and a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ on it in accordance with our system of tableaux, i.e., by taking

$$W = \{t_0, t_1, t_2\},$$

$$R = \{\langle t_0, t_1\rangle, \langle t_0, t_2\rangle, \langle t_i, t_i\rangle : \; i = 0, 1, 2\},$$
$$\mathfrak{V}(p) = \{t_1\}, \; \mathfrak{V}(q) = \{t_2\}.$$

(The diagram of $\mathfrak{F}$ is depicted in Fig. 2.7 (b).) Then we shall have: $t_1 \models p$, $t_1 \not\models q$ and so $t_0 \not\models p \to q$; $t_2 \models q$, $t_2 \not\models p$ and so $t_0 \not\models q \to p$. Hence $(\mathfrak{M}, t_0) \not\models \boldsymbol{da}$.

Example 2.30 Let us consider now the formula

$$p_2 \vee (p_2 \to p_1 \vee \neg p_1).$$

As before, we form a tableau t_0 by putting this formula in its right column. Then both p_2 and $p_2 \to p_1 \vee \neg p_1$ must also be put in the same column. To make the

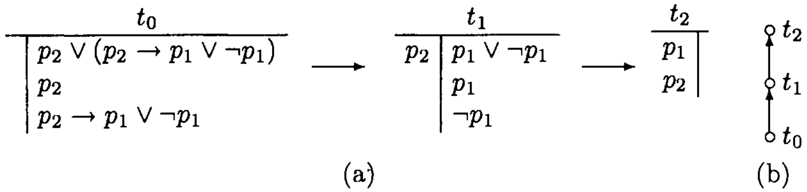


FIG. 2.8.

latter formula not true at t_0, we form a new tableau t_1 accessible from t_0, which contains p_2 in the left part and $p_1 \vee \neg p_1$, and hence p_1 and $\neg p_1$ in the right. Now to ensure that $\neg p_1$ is not true at t_1, we again form a new tableau t_2 accessible from t_1 where p_1 is true, i.e., stands in the left column. We should not forget either that all the formulas which are true at t_1 must be true at t_2 as well; so we put p_2 in the left part of t_2. (See Fig. 2.8 (a).)

Now we construct a frame $\mathfrak{F} = \langle W, R \rangle$ and a valuation $\mathfrak{V}$ in it by taking

$$W = \{t_0, t_1, t_2\},$$

$$R = \{\langle t_i, t_j \rangle : \ i, j = 0, 1, 2 \text{ and } i \leq j\},$$

$$\mathfrak{V}(p_1) = \{t_2\}, \ \mathfrak{V}(p_2) = \{t_1, t_2\}.$$

(The diagram of $\mathfrak{F}$ is shown in Fig. 2.8 (b).) The reader can readily check that all formulas in the left part of the tableau t_i are true at the point t_i in the model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, while those in the right part are not true. Therefore, $(\mathfrak{M}, t_0) \not\models p_2 \vee (p_2 \to p_1 \vee \neg p_1)$.

Our next aim is to show that the refutation procedure described above always succeeds: after a finite number of steps we shall either construct a countermodel for a given formula φ or establish its irrefutability, i.e., that $\varphi \in \mathbf{Int}$.

As before, a *tableau* is a pair $t = (\Gamma, \Delta)$ with $\Gamma, \Delta \subseteq \mathbf{For}\mathcal{L}$. A tableau t is called *saturated* in $\mathbf{Int}$ if it satisfies the conditions (S1)–(S5) in Section 1.2. Thus, every tableau which is saturated in $\mathbf{Cl}$ is saturated in $\mathbf{Int}$ as well; the converse does not hold, as follows from the examples above. A saturated tableau (Γ, Δ) is *disjoint* if $\Gamma \cap \Delta = \emptyset$ and $\bot \notin \Gamma$.

A *Hintikka system* in $\mathbf{Int}$ is a pair $\mathfrak{H} = \langle T, S \rangle$ consisting of a non-empty set T of disjoint saturated tableaux and a partial order S on T and satisfying the following conditions:

(HS_I1) if $t = (\Gamma, \Delta)$, $t' = (\Gamma', \Delta')$ are in T and tSt' then $\Gamma \subseteq \Gamma'$;

(HS_I2) if $t = (\Gamma, \Delta)$ is in T and $\psi \to \chi \in \Delta$ then there is $t' = (\Gamma', \Delta')$ in T such that tSt', $\psi \in \Gamma'$ and $\chi \in \Delta'$.

We say $\mathfrak{H} = \langle T, S \rangle$ is a *Hintikka system for a tableau* t if $t \subseteq t'$ for some $t' \in T$. A tableau $t = (\Gamma, \Delta)$ is called *realizable* in $\mathbf{Int}$ if there are an intuitionistic model $\mathfrak{M}$ and a point x in $\mathfrak{M}$ such that

$$(\mathfrak{M}, x) \models \psi \text{ for every } \psi \in \Gamma \text{ and } (\mathfrak{M}, x) \not\models \chi \text{ for every } \chi \in \Delta.$$

Proposition 2.31 *A tableau t is realizable in* **Int** *iff there is a Hintikka system $\mathfrak{H}$ for t.*

Proof ($\Rightarrow$) Suppose that t is realizable in a model $\mathfrak{M}$ based on a frame $\mathfrak{F} = \langle W, R\rangle$. With each $x \in W$ we associate the tableau $t_x = (\Gamma_x, \Delta_x)$, where

$$\Gamma_x = \{\varphi \in \mathbf{For}\mathcal{L} : \; x \models \varphi\}, \;\; \Delta_x = \{\varphi \in \mathbf{For}\mathcal{L} : \; x \not\models \varphi\},$$

and define a partial order S on the set $T = \{t_x : x \in W\}$ by taking

$$t_x S t_y \text{ iff } xRy.$$

It follows immediately from the definition of intuitionistic model and Proposition 2.1 that $\mathfrak{H} = \langle T, S\rangle$ is a Hintikka system. Besides, $t \subseteq t_x$ for some $x \in W$.

($\Leftarrow$) Let $\mathfrak{H} = \langle T, S\rangle$ be a Hintikka system for t. We will regard $\mathfrak{H}$ as an intuitionistic frame. Define a model $\mathfrak{M} = \langle \mathfrak{H}, \mathfrak{V}\rangle$ on it by taking, for every variable p,

$$\mathfrak{V}(p) = \{u = (\Gamma, \Delta) : \; u \in T \text{ and } p \in \Gamma\}.$$

(HS_I1) ensures that $\mathfrak{V}(p) \in \mathrm{Up}T$. By induction on the construction of φ we show that for any tableau $u = (\Gamma, \Delta)$ in T

$$\varphi \in \Gamma \text{ implies } (\mathfrak{M}, u) \models \varphi,$$

$$\varphi \in \Delta \text{ implies } (\mathfrak{M}, u) \not\models \varphi.$$

The basis of induction is obvious and the formulas $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ are considered in the same way as in **Cl**. So let $\varphi = \psi \to \chi$.

Suppose $\varphi \in \Gamma$ but $u \not\models \varphi$. Then there is a point $v = (\Pi, \Sigma)$ in T such that uSv, $v \models \psi$ and $v \not\models \chi$. By (HS_I1), $\varphi \in \Pi$ and by (S5), either $\chi \in \Pi$ or $\psi \in \Sigma$. Then, by the induction hypothesis, we must have either $v \models \chi$ or $v \not\models \psi$, which is a contradiction. Hence $u \models \varphi$.

Now suppose that $\varphi \in \Delta$. Then, by (HS_I2), there is a tableau $v = (\Pi, \Sigma)$ such that uSv, $\psi \in \Pi$ and $\chi \in \Sigma$. Using the induction hypothesis, we obtain $v \models \psi$ and $v \not\models \chi$, whence $u \not\models \psi \to \chi$. ❑

As follows from Proposition 2.31, $\varphi \notin$ **Int** iff there exists an (infinite, in general) Hintikka system for the tableau $t = (\emptyset, \{\varphi\})$. However, in fact we can obtain a much stronger result if observe that when constructing in the proof of Proposition 2.31 a Hintikka system for t, we may deal only with subformulas of φ. The number of distinct tableaux, corresponding to points in $\mathfrak{F}$, will then be finite, will not exceed $2^{|\mathbf{Sub}\varphi|}$ to be more exact, and an accessibility relation on the tableaux can always be defined in such a way that the conditions (HS_I1) and (HS_I2) are satisfied.

More generally, we have the following

Theorem 2.32 *A tableau t is realizable in* **Int** *iff there is a Hintikka system $\mathfrak{H} = \langle T, S\rangle$ for t such that $|T| \le 2^{|\Sigma|}$, where Σ is the set of all subformulas of the formulas in t.*

Proof $(\Rightarrow)$ We will modify the "only if" part of the proof of Proposition 2.31 according to the idea above. This time we associate with every point $x \in W$ the tableau $t_x = (\Gamma_x, \Delta_x)$ in which

$$\Gamma_x = \{\varphi \in \Sigma : \ x \models \varphi\}, \ \Delta_x = \{\varphi \in \Sigma : \ x \not\models \varphi\}.$$

Putting $T = \{t_x : x \in W\}$, we clearly have $|T| \le 2^{|\Sigma|}$. Define a relation S on T by taking, for tableaux $t_x = (\Gamma_x, \Delta_x)$ and $t_y = (\Gamma_y, \Delta_y)$,

$$t_x S t_y \text{ iff } \Gamma_x \subseteq \Gamma_y.$$

To show that $\mathfrak{H} = \langle T, S\rangle$ is a Hintikka system for t, it suffices to verify only (HS$_I$2). Let $t_x = (\Gamma_x, \Delta_x)$ be a tableau in T and $\psi \to \chi \in \Delta_x$. Then $x \not\models \psi \to \chi$ and so there is a point y such that xRy, $\psi \in \Gamma_y$ and $\chi \in \Delta_y$. By Proposition 2.1, $\Gamma_x \subseteq \Gamma_y$, and hence $t_x S t_y$.

The proof of $(\Leftarrow)$ remains the same as in Proposition 2.31. ❑

Corollary 2.33 (i) *For every formula $\varphi \notin$* **Int** *there is a rooted frame refuting φ and containing at most $2^{|\mathbf{Sub}\varphi|}$ points.*

(ii) *For every $\varphi \notin$* **Int** *there is a finite tree refuting φ.*

(iii) *For every $n \ge 2$,* **Int** $= \{\varphi \in \mathbf{For}\mathcal{L} : \ \mathfrak{T}_n \models \varphi\}$.

Proof Follows from Theorems 2.32, 2.19 and Corollaries 2.9, 2.17, 2.22. ❑

Example 2.34 We show that

$$\neg p \leftrightarrow \neg\neg\neg p \in \mathbf{Int}.$$

Suppose otherwise. Then, in view of $\neg p \to \neg\neg\neg p \in \mathbf{Int}$ (see Example 2.5), there is a finite model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ and a point x in $\mathfrak{F}$ such that $x \models \neg\neg\neg p$ and $x \not\models \neg p$. Take a point $y \in x\uparrow$ where $y \models \neg\neg\neg p$ and $y \models p$. Since $\mathfrak{F}$ is finite, there is a final point $z \in y\uparrow$. Clearly, $z \models \neg\neg\neg p$ and $z \models p$. But then $z \not\models \neg\neg p$ and, since z is final, $z \models \neg p$, whence $z \not\models p$, which is a contradiction.

Theorem 2.32 means in particular that starting with the tableau $(\emptyset, \{\varphi\})$ and using saturation rules (SR1)–(SR5) in Section 1.2 and

(SR$_I$6) if $t = (\Gamma, \Delta)$ and $\psi \to \chi \in \Delta$ then either add ψ to Γ' and χ to Δ' in some $t' = (\Gamma', \Delta')$ accessible from t or construct a new tableau $t' = (\Gamma', \Delta')$ accessible from t by taking $\Gamma' = \Gamma \cup \{\psi\}$, $\Delta' = \{\chi\}$,

in a finite number of steps we shall either construct a Hintikka system for $(\emptyset, \{\varphi\})$, and so a countermodel for φ, or show that there is no Hintikka system for $(\emptyset, \{\varphi\})$ with $\le 2^{|\mathbf{Sub}\varphi|}$ tableaux, and so no Hintikka system for the tableau at all, i.e., $\varphi \in \mathbf{Int}$.

We will not formulate here a procedure of constructing countermodels for intuitionistic formulas in full details. It will be more useful for the reader who is not experienced in intuitionistic logic to have a good informative example.

Example 2.35 Let us try to find a countermodel for the formula

$$\boldsymbol{sa} = ((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p,$$

which is known as the *Scott formula* (or *axiom*).

The attempt of constructing a Hintikka system for $(\emptyset, \{\boldsymbol{sa}\})$ shown in Fig. 2.9 (a) failed. However, applying (SR5) to $(\neg\neg p \to p) \to p \vee \neg p$ in the left column of t_0, we may not only put $p \vee \neg p$ on the left, but also $\neg\neg p \to p$ on the right. And this alternative way succeeds, as is shown in Fig. 2.9 (b).

Taking now the frame $\mathfrak{F} = \langle W, R \rangle$ depicted in Fig. 2.9 (c) and defining a valuation $\mathfrak{V}$ in it by $\mathfrak{V}(p) = t_2$, we, according to Proposition 2.31, obtain the countermodel $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ for the Scott formula.

As an easy exercise we invite the reader to show that all the formulas in the upper part of Table 1.1 including the first de Morgan's law are in **Int**, while all those below this law do not belong to **Int**.

2.5 Intuitionistic frames and formulas

In the preceding section we used the method of semantic tableaux for constructing a countermodel for a given formula φ or proving that such a countermodel does not exist, i.e., $\varphi \in$ **Int**. Now we touch on a more general problem: given a formula φ, to characterize in some non-trivial way the class of all frames validating φ. This problem turns out to be rather complicated. It will play an important role in the sequel. But here we consider it only for a few concrete formulas just to gain more experience in handling Kripke models.

For the beginning let us take again the Dummett formula $\boldsymbol{da}$. It follows from Example 2.29 that in every Hintikka system for $(\emptyset, \{\boldsymbol{da}\})$ there must be (extensions of) three tableaux t_0, t_1 and t_2 shown in Fig. 2.7. It is clear that in this situation t_1 is not accessible from t_2, for otherwise q must belong to the left part of t_1. Likewise, t_1 does not see t_2.

This observation gives us a necessary criterion for $\boldsymbol{da}$ to be refuted in a frame $\mathfrak{F} = \langle W, R \rangle$, which can be represented as the following classical first-order condition on R:

$$\exists x, y, z\ (xRy \wedge xRz \wedge \neg yRz \wedge \neg zRy).$$

So, by the law of contraposition, the Dummett formula is valid in $\mathfrak{F}$ if the following condition holds:

$$\forall x, y, z\ (xRy \wedge xRz \to yRz \vee zRy).$$

A frame $\mathfrak{F}$ satisfying this condition is called *strongly connected*. Notice that every rooted strongly connected frame is a chain.

t_0	
	$((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$
$(\neg\neg p \to p) \to p \vee \neg p$	$\neg p \vee \neg\neg p$
	$\neg p$
	$\neg\neg p\ (= \neg p \to \bot)$
$p \vee \neg p$	
p	

$\downarrow$

t_1	
$\neg p$	$\bot,\ p$
$(\neg\neg p \to p) \to p \vee \neg p$	
$p \vee \neg p$	
p	

(a)

t_0	
	$((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$
$(\neg\neg p \to p) \to p \vee \neg p$	$\neg p \vee \neg\neg p$
	$\neg p$
	$\neg\neg p\ (= \neg p \to \bot)$
	$\neg\neg p \to p$

$\swarrow \quad \searrow$

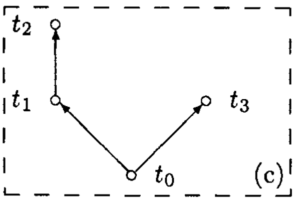
t_1	
$\neg\neg p$	p
$(\neg\neg p \to p) \to p \vee \neg p$	
	$\neg p$
	$\neg\neg p \to p$

t_3	
$\neg p$	
$(\neg\neg p \to p) \to p \vee \neg p$	
	p
$p \vee \neg p$	

(b)

$\downarrow$

t_2	
p	
$\neg\neg p$	
$(\neg\neg p \to p) \to p \vee \neg p$	
	$\neg p$
$p \vee \neg p$	



(c)

FIG. 2.9.

Proposition 2.36 *A frame $\mathfrak{F}$ validates* $\boldsymbol{da}$ *iff $\mathfrak{F}$ is strongly connected.*

Proof ($\Rightarrow$) Suppose $\mathfrak{F} = \langle W, R\rangle$ validates $(p \to q) \vee (q \to p)$ but is not strongly connected. Then there are points $x, y, z \in W$ such that xRy, xRz, $\neg yRz$ and $\neg zRy$. Define a valuation $\mathfrak{V}$ on $\mathfrak{F}$ by taking

$$\mathfrak{V}(p) = y\uparrow \text{ and } \mathfrak{V}(q) = z\uparrow.$$

Then $y \not\models p \to q$, $z \not\models q \to p$ and so $x \not\models \boldsymbol{da}$, which is a contradiction.

($\Leftarrow$) has been already established above. ❑

Now let us consider the formula

$$\boldsymbol{wem} = \neg p \vee \neg\neg p,$$

which is known as the *weak law of the excluded middle*, and again try to find first a necessary condition for its refutability, and thereby a sufficient condition of its validity.

Suppose $\neg p \vee \neg\neg p$ is not true at a point x in a frame $\mathfrak{F} = \langle W, R\rangle$ under some valuation. Then $x \not\models \neg p$ and $x \not\models \neg\neg p$. Hence there are points $y, z \in x\uparrow$ such that $y \models p$ and $z \models \neg p$. It should be clear that y and z do not see each other. The necessary refutability condition thus obtained does not differ from that for the Dummett formula. However, now it is too weak to be a sufficient one. For the frame in Fig. 2.2 (a) satisfies the condition and validates $\neg p \vee \neg\neg p$. The problem is that the points y and z not only do not see each other but have no common successors at all. Indeed, if yRu and zRu then, by Proposition 2.1, $u \models p$, $u \models \neg p$ and so $u \not\models p$, which is impossible.

Thus, as a sufficient condition for the validity of $\neg p \vee \neg\neg p$ in a frame $\mathfrak{F} = \langle W, R\rangle$ we can take the following one:

$$\forall x, y, z\ (xRy \wedge xRz \to \exists u\ (yRu \wedge zRu)).$$

A frame $\mathfrak{F}$ satisfying it is called *strongly directed* or *convergent*. A rooted frame is strongly directed iff every two points in it have a common successor.

Proposition 2.37 *A frame $\mathfrak{F}$ validates* $\boldsymbol{wem}$ *iff $\mathfrak{F}$ is strongly directed.*

Proof Again only the ($\Rightarrow$) part needs a proof. If $\mathfrak{F} = \langle W, R\rangle$ is not strongly convergent then there are points $x, y, z \in W$ such that xRy, xRz and there is no point u accessible from both y and z. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = y\uparrow$. Then $z \models \neg p$, for otherwise there is $u \in z\uparrow$ such that $u \models p$, whence $u \in y\uparrow$, which is a contradiction. Therefore, $z \not\models \neg\neg p$. Besides, $y \not\models \neg p$ and so $x \not\models \neg p \vee \neg\neg p$. ❑

We define now inductively a sequence of formulas $\boldsymbol{bd}_n$:

$$\boldsymbol{bd}_1 = p_1 \vee \neg p_1,$$

$$\boldsymbol{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \boldsymbol{bd}_n).$$

The formulas $\boldsymbol{bd}_1$ and $\boldsymbol{bd}_2$ were already considered in Examples 2.4 and 2.30, from which it follows that to refute $\boldsymbol{bd}_1$ a frame must contain a chain of two points and to refute $\boldsymbol{bd}_2$ a three-point chain is required. In general, by induction on n one can readily show that $\mathfrak{F} \not\models \boldsymbol{bd}_n$ only if there is a chain of $n+1$ points in $\mathfrak{F}$.

We say a frame $\mathfrak{F}$ is of *depth* $n < \omega$, $d(\mathfrak{F}) = n$ in symbols, if there is a chain of n points in $\mathfrak{F}$ and no chain of more than n points. If for every $n < \omega$, $\mathfrak{F}$ contains an n-point chain then $\mathfrak{F}$ is said to be of *infinite depth* ∞.

Proposition 2.38 *A frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{bd}_n$ iff $d(\mathfrak{F}) \leq n$, i.e., iff $\mathfrak{F}$ satisfies the following condition*

$$\forall x_0, \ldots, x_n \ (\bigwedge_{i=0}^{n-1} x_i R x_{i+1} \to \bigvee_{i \neq j} x_i = x_j).$$

Proof Exercise. ❑

After depth let us introduce a notion of width of frames. A set of points $X \subseteq W$ is called an *antichain* in a frame $\mathfrak{F} = \langle W, R\rangle$ if, for every $x, y \in X$, xRy implies $x = y$. In other words X is an antichain if distinct points in X do not see each other. We say a frame $\mathfrak{F}$ is of *width* n if it contains an antichain of n points and there is no antichain of greater cardinality.

Are there any intuitionistic formulas which bound the width of a frame as $\boldsymbol{bd}_n$s bound the depth? The frame $\langle\{0, 1, 2, \ldots\}, =\rangle$ shows that such formulas do not exist, since it is the disjoint union of ω single-point frames and so validates all formulas in **Cl**. However, we can bound the width of *rooted frames* by taking, for instance, the following formulas

$$\boldsymbol{bw}_n = \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j), \quad n \geq 1.$$

Notice that $\boldsymbol{bw}_1$ is the Dummett formula (modulo renaming the variables). We invite the reader to investigate the structure of refutation frames for $\boldsymbol{bw}_n$ and prove

Proposition 2.39 *A frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{bw}_n$ iff every rooted subframe of $\mathfrak{F}$ is of width $\leq n$, i.e., iff*

$$\forall x, x_0, \ldots, x_n \ (\bigwedge_{i=0}^{n} x R x_i \to \bigvee_{i \neq j} x_i R x_j).$$

The following formulas bound the cardinality of rooted frames:

$$\boldsymbol{bc}_n = p_0 \vee (p_0 \to p_1) \vee \ldots \vee (p_0 \wedge \ldots \wedge p_{n-1} \to p_n), \ n \geq 1.$$

Proposition 2.40 *A frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{bc}_n$ iff each rooted subframe of $\mathfrak{F}$ contains $\leq n$ points, i.e.,*

$$\forall x_0, x_1, \ldots, x_n \ (\bigwedge_{i=1}^{n} x_0 R x_i \to \bigvee_{i\neq j} x_i = x_j).$$

Proof ($\Rightarrow$) Suppose $\mathfrak{F}$ contains $n+1$ distinct points $x_0, x_1, \ldots, x_n$ such that $\{x_1, \ldots, x_n\} \subseteq x_0\!\uparrow$. Without loss of generality we may assume that these points are indexed in such a way that $x_i R x_j$ implies $i \leq j$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking, for $i = 0, \ldots, n$,

$$\mathfrak{V}(p_i) = \{x \in W : \ \neg x R x_i\} = W - x_i\!\downarrow.$$

Then we shall have $x_0 \not\models p_0$ and, for $i > 0$, $x_i \models p_0 \wedge \ldots \wedge p_{i-1}$ and $x_i \not\models p_i$. Indeed, otherwise either $x_i \models p_i$, contrary to $x_i R x_i$, or $x_i \not\models p_j$ for some $j < i$, whence $x_i R x_j$, contrary to our indexing of points. Therefore, since x_0 sees all points $x_1, \ldots, x_n$, we obtain $x_0 \not\models \boldsymbol{bc}_n$.

($\Leftarrow$) Suppose $\boldsymbol{bc}_n$ is false at a point x_0 in $\mathfrak{F}$ under some valuation. Then $x_0 \not\models p_0$ and, for every i, $0 < i \leq n$, there is a point $x_i \in x_0\!\uparrow$ such that $x_i \models p_0 \wedge \ldots \wedge p_{i-1}$, $x_i \not\models p_i$. Clearly, the points $x_0, \ldots, x_n$ are distinct and so the subframe of $\mathfrak{F}$ generated by x_0 contains $\geq n+1$ points. ❑

To conclude this section we consider one more interesting family of formulas, namely,

$$\boldsymbol{bb}_n = \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i, \ n \geq 1.$$

It turns out that their arbitrary validating frames cannot be characterized by first order conditions on the accessibility relation (see Chapter 6). However, their finite frames are quite manageable.

Say that a finite frame $\mathfrak{F}$ is of *branching* $\leq n$ if every point in $\mathfrak{F}$ has at most n distinct immediate successors.

Proposition 2.41 *A finite frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{bb}_n$ iff $\mathfrak{F}$ is of branching $\leq n$.*

Proof ($\Rightarrow$) Suppose otherwise. Then there is a point x in $\mathfrak{F}$ having at least $n+1$ distinct immediate successors, say, $x_0, \ldots, x_n$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking

$$\mathfrak{V}(p_i) = W - \bigcup_{i\neq j} x_j\!\downarrow$$

and show that $\boldsymbol{bb}_n$ is not true at x under $\mathfrak{V}$. Indeed, we have $x_i \not\models p_j$ for all $j \neq i$, and so $x \not\models \bigvee_{i=0}^{n} p_i$. Suppose now that the premise of $\boldsymbol{bb}_n$ is not true at x. Then there are $y \in x\!\uparrow$ and $i \in \{0, \ldots, n\}$ such that $y \models p_i \to \bigvee_{i\neq j} p_j$ and $y \not\models \bigvee_{i\neq j} p_j$, from which we obtain $y \not\models p_i$. By the definition of $\mathfrak{V}$, this means that y sees at least two distinct points among $x_0, \dots, x_n$, which is possible only if $y = x$. But then we have $x_i \models p_i$, $x_i \not\models \bigvee_{i \neq j} p_j$ and so $y \not\models p_i \to \bigvee_{i \neq j} p_j$, which is a contradiction. Thus, $x \not\models \boldsymbol{bb}_n$.

($\Leftarrow$) Suppose $\mathfrak{F}$ is a finite frame of branching $\leq n$, but $\mathfrak{F} \not\models \boldsymbol{bb}_n$ under some valuation. Let x be a maximal point in $\mathfrak{F}$ where $\boldsymbol{bb}_n$ is not true. Then we have $x \models \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i \neq j} p_j) \to \bigvee_{i \neq j} p_j)$ and $x \not\models \bigvee_{i=0}^{n} p_i$. Therefore, $x \not\models p_i \to \bigvee_{i \neq j} p_j$, for all $i = 0, \dots, n$, and so there are $x_i \in x\uparrow$ such that $x_i \models p_i$ and $x_i \not\models \bigvee_{i \neq j} p_j$. It follows that x_i and x_j do not see each other if $i \neq j$. Since $\mathfrak{F}$ is of branching $\leq n$, x has a proper successor y seeing at least two distinct points x_i and x_j. But then $y \not\models \bigvee_{i=0}^{n} p_i$ and, since $y \models \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i \neq j} p_j) \to \bigvee_{i \neq j} p_j)$, we have $y \not\models \boldsymbol{bb}_n$, contrary to x being a maximal point in $\mathfrak{F}$ refuting $\boldsymbol{bb}_n$. ❑

Remark By Corollary 2.22, Proposition 2.41 cannot be generalized to infinite frames.

The reader can find more examples among the exercises at the end of this chapter. The general problem of characterizing frames validating (or refuting) an arbitrary given formula will be considered in Chapter 9.

2.6 Intuitionistic calculus

The Hilbert-type *intuitionistic propositional calculus Int* in the language $\mathfrak{L}$ is defined by axioms (A1)–(A9) and the inference rules MP and Subst of Section 1.3. The notions of derivation and derivation from assumptions are defined in exactly the same way as for classical calculus Cl. The fact of derivability of a formula φ in Int is denoted by $\vdash_{Int} \varphi$, and $\Gamma \vdash_{Int} \varphi$ means that φ is derivable in Int from a set of assumptions Γ. If there is no danger of confusion, we write simply $\vdash \varphi$ and $\Gamma \vdash \varphi$.

In this section we show that Int is *sound* and *complete* with respect to the Kripke semantics introduced above. First we observe that when proving the deduction theorem for Cl, we used only axioms (A1) and (A2), and so this theorem holds for Int as well.

Theorem 2.42. (Deduction) *If* $\Gamma, \psi \vdash_{Int} \varphi$ *then* $\Gamma \vdash_{Int} \psi \to \varphi$.

The soundness and completeness of Int is proved by the same scheme as Theorem 1.16.

Theorem 2.43. (Soundness and completeness of Int) *For any formula* φ, $\vdash_{Int} \varphi$ *iff* $\mathfrak{F} \models \varphi$ *for every frame* $\mathfrak{F}$.

Proof ($\Rightarrow$) It suffices to verify that (i) axioms (A1)–(A9) are valid in all intuitionistic frames and (ii) the inference rules MP and Subst preserve the validity. Using the apparatus of semantic tableaux, the reader will easily establish (i). The fact that MP preserves the validity follows immediately from the definition of the truth-relation $\models$.

Let us consider Subst. Suppose that $\mathfrak{F} \models \varphi$ but $\mathfrak{F} \not\models \varphi s$ for some substitution s. Then there is a countermodel $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ for φs. Define a new valuation $\mathfrak{U}$ in

$\mathfrak{F}$ by taking $\mathfrak{U}(p) = \mathfrak{V}(ps)$, for all $p \in \mathbf{Var}\mathcal{L}$, and put $\mathfrak{N} = \langle \mathfrak{F}, \mathfrak{U} \rangle$. Then clearly we have $(\mathfrak{N}, x) \models \varphi$ iff $(\mathfrak{M}, x) \models \varphi s$, for all x in $\mathfrak{F}$. Therefore, $\mathfrak{N}$ is a countermodel for φ, contrary to our assumption.

($\Leftarrow$) Suppose $\not\vdash_{Int} \varphi$. We show then that there is a Hintikka system $\mathfrak{H} = \langle T, S \rangle$ for the tableau $(\emptyset, \{\varphi\})$, and so $\mathfrak{H} \not\models \varphi$.

Call a tableau (Γ, Δ) *consistent* in Int if $\Gamma \vdash_{Int} \psi_1 \vee \ldots \vee \psi_n$ holds for no formulas $\psi_1, \ldots, \psi_n \in \Delta$. Thus, the tableau $(\emptyset, \{\varphi\})$ is consistent.

Let $t_0 = (\Gamma_0, \Delta_0)$ be a consistent tableau such that $\Gamma_0, \Delta_0 \subseteq \mathbf{Sub}\varphi$. In exactly the same way as in the proof of Theorem 1.16 we show that t_0 can be extended to some disjoint saturated (in **Int**) and consistent (in Int) tableau $t_n = (\Gamma_n, \Delta_n)$ such that $\Gamma_n \cup \Delta_n = \mathbf{Sub}\varphi$. But this time t_n does not in general satisfy condition (S6).

Denote by T the set of all disjoint saturated consistent tableaux (Γ, Δ) such that $\Gamma \cup \Delta = \mathbf{Sub}\varphi$. T is clearly non-empty. Define a partial order S on T by taking, for any $t = (\Gamma, \Delta)$ and $t' = (\Gamma', \Delta')$,

$$tSt' \text{ iff } \Gamma \subseteq \Gamma' \text{ iff } \Delta \supseteq \Delta'.$$

We show now that $\mathfrak{H} = \langle T, S \rangle$ is a Hintikka system. It is clear that only ($\mathrm{HS}_I 2$) requires verification. Suppose $t = (\Gamma, \Delta) \in T$ and $\psi \to \chi \in \Delta$. Consider the tableau $t_0 = (\Gamma \cup \{\psi\}, \{\chi\})$. It is consistent, for otherwise we would have $\Gamma, \psi \vdash \chi$ and so, by the deduction theorem, $\Gamma \vdash \psi \to \chi$, contrary to the consistency of t. Therefore, t_0 can be extended to a disjoint saturated consistent tableau $t' = (\Gamma', \Delta')$ which belongs to T. Since $\Gamma \subseteq \Gamma'$, we have tSt'. And by the definition, $\psi \in \Gamma'$ and $\chi \in \Delta'$.

Thus, $\mathfrak{H} = \langle T, S \rangle$ is a Hintikka system for $(\emptyset, \{\varphi\})$. By Proposition 2.31, this means that $\mathfrak{H} \not\models \varphi$. Notice by the way that $|T| \leq 2^{|\mathbf{Sub}\varphi|}$. ❑

Corollary 2.44 $\mathbf{Int} = \{\varphi \in \mathbf{For}\mathcal{L} : \vdash_{Int} \varphi\}$.

The following two theorems are proved by the same argument as Theorem 2.43, although applied to infinite tableaux (for details see Section 5.1).

Theorem 2.45. (Strong completeness) *Each tableau consistent in Int is realizable. In particular, $\Gamma \vdash_{Int} \varphi$ iff $(\mathfrak{M}, x) \models \varphi$ for every model $\mathfrak{M}$ and every point x in $\mathfrak{M}$ such that $(\mathfrak{M}, x) \models \Gamma$.*

Theorem 2.46. (Compactness) *A tableau is realizable in* **Int** *iff its every finite subtableau is realizable in* **Int**.

2.7 Embeddings of Cl into Int

In this section we will consider some connections between classical and intuitionistic propositional logics. As we know, **Int** $\subset$ **Cl**. On the other hand, we can try to embed **Cl** into **Int** in the following sense.

Let L_1 and L_2 be some logics, possibly in distinct languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. An effective function Tr from $\mathbf{For}\mathcal{L}_1$ into $\mathbf{For}\mathcal{L}_2$ is called an *embedding* (or a *translation*) of L_1 into L_2 if, for all $\varphi \in \mathbf{For}\mathcal{L}_1$,

$$\varphi \in L_1 \text{ iff } Tr(\varphi) \in L_2.$$

(In general, this definition is too extensive, for it admits, for instance, such a trivial "embedding" of **Cl** into **Int** as

$$Tr(\varphi) = \begin{cases} \top \text{ if } \varphi \in \mathbf{Cl} \\ \bot \text{ if } \varphi \notin \mathbf{Cl}. \end{cases}$$

However, we are not going to develop here a theory of embeddings and confine ourselves to considering only a number of concrete ones. For more elaborate definitions of translations preserving the structure of formulas see, e.g. Epstein, 1990.)

Embedding operations may be useful at least in two respects. First, sometimes they make it possible to interpret logical connectives in $\mathcal{L}_1$ in terms of those in $\mathcal{L}_2$. And second, embeddings may preserve various properties of logics; for example, if L_1 is embeddable into a decidable logic L_2 then L_1 is also decidable.

A simple embedding of **Cl** into **Int** is provided by the following:

Theorem 2.47. (Glivenko's theorem) *For every φ, $\varphi \in$ **Cl** iff $\neg\neg\varphi \in$ **Int**.*

Proof ($\Rightarrow$) Suppose otherwise, i.e., $\varphi \in \mathbf{Cl}$ and $\neg\neg\varphi \notin \mathbf{Int}$. Then there are a finite model $\mathfrak{M}$ and a point x in $\mathfrak{M}$ such that $x \not\models \neg\neg\varphi$. Hence there is $y \in x\uparrow$ for which $y \models \neg\varphi$. Let z be some *final* point in the set $y\uparrow$. By Proposition 2.1, $z \models \neg\varphi$ and so $z \not\models \neg\neg\varphi$. Let $\mathfrak{M}_1$ be the submodel of $\mathfrak{M}$ generated by z, i.e., $(\mathfrak{M}_1, z) \models p$ iff $(\mathfrak{M}, z) \models p$, for every variable p. According to the generation theorem, $\mathfrak{M}_1$ refutes $\neg\neg\varphi$. But since this model contains only one point, it follows that $\neg\neg\varphi \notin \mathbf{Cl}$, which, by the law of double negation, is a contradiction.

($\Leftarrow$) $\neg\neg\varphi \in \mathbf{Int}$ implies $\neg\neg\varphi \in \mathbf{Cl}$, whence, using that law again, we obtain $\varphi \in \mathbf{Cl}$. ❑

Corollary 2.48 *The map Tr_1 defined by $Tr_1(\varphi) = \neg\neg\varphi$, for every formula φ, is an embedding of* **Cl** *into* **Int**.

Corollary 2.49 *For every formula φ, $\neg\varphi \in$ **Cl** iff $\neg\varphi \in$ **Int**.*

Proof According to Example 2.34, $\neg\varphi \leftrightarrow \neg\neg\neg\varphi \in \mathbf{Int}$. The rest follows from Glivenko's theorem. ❑

Corollary 2.50 *For every formula $\varphi = \psi \to \neg\chi$, $\varphi \in$ **Cl** iff $\varphi \in$ **Int**.*

Proof $\psi \to \neg\chi$ is the abbreviation for the formula $\psi \to (\chi \to \bot)$, which is equivalent in **Int** to $\psi \wedge \chi \to \bot$, i.e., $\neg(\psi \wedge \chi)$. And by Corollary 2.49, $\neg(\psi \wedge \chi) \in \mathbf{Cl}$ iff $\neg(\psi \wedge \chi) \in \mathbf{Int}$. ❑

Corollary 2.51 *For every formula φ containing no connectives different from $\wedge$ and $\neg$, $\varphi \in$ **Cl** iff $\varphi \in$ **Int**.*

Proof If φ contains neither $\to$ nor $\vee$ then it can be represented in the form $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_n$ where each φ_i is either an atom or has the form $\neg\psi_i$ for some

ψ_i. By axioms (A3) and (A4), $\varphi \in \mathbf{Cl}$ implies $\varphi_i \in \mathbf{Cl}$ for all $i = 1, \dots, n$. Since atoms are outside of $\mathbf{Cl}$, $\varphi \in \mathbf{Cl}$ only if $\varphi_i = \neg\psi_i \in \mathbf{Cl}$ for all $i = 1, \dots, n$, and so, by Corollary 2.49, $\varphi_i \in \mathbf{Int}$. Now, applying (A5), we obtain $\varphi \in \mathbf{Int}$. ❑

Corollary 2.51 gives rise to another embedding of $\mathbf{Cl}$ into $\mathbf{Int}$. Indeed, let Tr_2 be a map from $\mathcal{L}$ to $\mathcal{L}$ defined as follows:

$$\begin{array}{ll} Tr_2(\varphi) & = \varphi, \text{ for all atomic } \varphi, \\ Tr_2(\psi \wedge \chi) & = Tr_2(\psi) \wedge Tr_2(\chi), \\ Tr_2(\psi \vee \chi) & = \neg(\neg Tr_2(\psi) \wedge \neg Tr_2(\chi)), \\ Tr_2(\psi \to \chi) & = \neg(Tr_2(\psi) \wedge \neg Tr_2(\chi)). \end{array}$$

Corollary 2.52 *Tr_2 is an embedding of* $\mathbf{Cl}$ *into* $\mathbf{Int}$.

Proof By induction on the construction of φ it is not hard to show that $\varphi \leftrightarrow Tr_2(\varphi) \in \mathbf{Cl}$. It remains to observe that $Tr_2(\varphi)$ contains neither $\vee$ nor $\to$ and use Corollary 2.51. ❑

Theorem 2.53 *If a formula φ contains no $\vee$ and every occurrence of a variable in φ is in the scope of some $\neg$ then $\varphi \leftrightarrow \neg\neg\varphi \in$* $\mathbf{Int}$.

Proof As follows from Example 2.5, $\varphi \to \neg\neg\varphi \in \mathbf{Int}$ for every φ. So we prove only that $\neg\neg\varphi \to \varphi \in \mathbf{Int}$. We will do this by induction on the construction of φ, regarding φ as constructed from formulas of the form $\neg\psi$ and $\bot$ with the help of $\to$ and $\wedge$. The basis of induction follows from Example 2.34.

Suppose that $\varphi = \psi \to \chi$ and $\neg\neg(\psi \to \chi) \to (\psi \to \chi) \notin \mathbf{Int}$. Then there is a finite model $\mathfrak{M}$ such that $x \models \neg\neg(\psi \to \chi)$, $x \models \psi$ and $x \not\models \chi$ for some point x in $\mathfrak{M}$. By the induction hypothesis, $\chi \leftrightarrow \neg\neg\chi \in \mathbf{Int}$ and so $x \not\models \neg\neg\chi$. Hence $y \not\models \chi$, for some final point $y \in x\uparrow$. We also have $y \models \neg\neg(\psi \to \chi)$, whence $y \not\models \neg(\psi \to \chi)$ and so, since y is final, $y \models \psi \to \chi$. And since $y \models \psi$, we get $y \models \chi$, which is a contradiction.

The case of $\varphi = \psi \wedge \chi$ is considered analogously. ❑

We recommend the reader to analyze $\varphi = \neg p \vee \neg q$ to make sure that Theorem 2.53 cannot be extended to formulas containing $\vee$.

Using Theorem 2.53, we can construct one more embedding of $\mathbf{Cl}$ into $\mathbf{Int}$.

Theorem 2.54 *The map Tr_3 defined by the equalities*

$$\begin{array}{ll} Tr_3(\bot) & = \bot, \\ Tr_3(p) & = \neg\neg p, \ \textit{for all } p \in \mathbf{Var}\mathcal{L}, \\ Tr_3(\psi \wedge \chi) & = Tr_3(\psi) \wedge Tr_3(\chi), \\ Tr_3(\psi \vee \chi) & = \neg(\neg Tr_3(\psi) \wedge \neg Tr_3(\chi)), \\ Tr_3(\psi \to \chi) & = Tr_3(\psi) \to Tr_3(\chi) \end{array}$$

is an embedding of $\mathbf{Cl}$ *into* $\mathbf{Int}$.

Proof It is not hard to see that $\varphi \leftrightarrow Tr_3(\varphi) \in \mathbf{Cl}$ for every formula φ. Besides, by Theorem 2.53, we have $Tr_3(\varphi) \leftrightarrow \neg\neg Tr_3(\varphi) \in \mathbf{Int}$. Therefore, by Glivenko's theorem, $\varphi \in \mathbf{Cl}$ iff $Tr_3(\varphi) \in \mathbf{Int}$. ❑

2.8 Basic properties of Int

Now we shall see which of the properties considered in Section 1.4 hold for **Int** and introduce some more.

CONSISTENCY. **Int** is consistent, since $\mathbf{Int} \subset \mathbf{Cl} \subset \mathbf{For}\mathcal{L}$.

DECIDABILITY.

Theorem 2.55 **Int** *is decidable.*

Proof According to Theorem 2.32, $\varphi \notin \mathbf{Int}$ iff there is a Hintikka system $\mathfrak{H} = \langle T, S \rangle$ for $(\emptyset, \{\varphi\})$ with $|T| \leq 2^{|\mathbf{Sub}\varphi|}$. So a decision algorithm for **Int** may be as follows. We form all partially ordered sets containing at most $2^{|\mathbf{Sub}\varphi|}$ tableaux (Γ, Δ) such that $\Gamma, \Delta \subseteq \mathbf{Sub}\varphi$. If at least one of them is a Hintikka system for $(\emptyset, \{\varphi\})$ then $\varphi \notin \mathbf{Int}$; otherwise $\varphi \in \mathbf{Int}$. ❑

The difference between the decision algorithms for **Cl** and **Int** is that in the former case we check if a given formula φ is valid in a single finite frame, while in the latter one we have to check its validity in all frames with $\leq 2^{|\mathbf{Sub}\varphi|}$ points, and so the longer φ is, the more complicated frames must be considered. Is that unavoidable? Couldn't one find a finite intuitionistic frame $\mathfrak{F}$ such that $\mathbf{Int} = \{\varphi \in \mathbf{For}\mathcal{L} : \mathfrak{F} \models \varphi\}$?

TABULARITY. A logic L is called *tabular* if there is a finite frame $\mathfrak{F}$ such that

$$L = \{\varphi \in \mathbf{For}\mathcal{L} : \mathfrak{F} \models \varphi\}.$$

By the definition, classical propositional logic is tabular.

Theorem 2.56 **Int** *is not tabular.*

Proof Suppose otherwise. Then there is a finite frame $\mathfrak{F}$, containing, say, n points, which refutes all formulas that do not belong to **Int**, in particular, $\boldsymbol{bd}_{n+1}$, $\boldsymbol{bw}_{n+1}$ and $\boldsymbol{bc}_{n+1}$ defined in Section 2.5, contrary to Propositions 2.38, 2.39 and 2.40. ❑

Thus, no finite frame is able to characterize **Int**. However, the set of all finite frames or the set of all finite trees can do this.

FINITE APPROXIMABILITY. A logic L is said to be *finitely approximable* (or to have the *finite frame property*) if there is a class $\mathcal{C}$ of finite frames such that

$$L = \{\varphi \in \mathbf{For}\mathcal{L} : \forall \mathfrak{F} \in \mathcal{C}\ \mathfrak{F} \models \varphi\}.$$

Theorem 2.57 **Int** *is finitely approximable.*

Proof Follows from Theorem 2.32. ❑

The property of finite approximability plays a very important role in non-classical logic, since, as we shall see in Section 16.2, by proving the finite approximability of a finitely axiomatizable logic, we thereby establish its decidability as well.

Notice by the way that in fact Theorem 2.32 not only yields the finite approximability of **Int** but also indicates an *upper bound* for the number of points in a minimal refutation frame for $\varphi \notin \mathbf{Int}$. This upper bound determines the complexity of the decision algorithm presented in the proof of Theorem 2.55, and so we are naturally interested in its reduction. A detailed discussion of this and other questions concerning complexity theory can be found in Chapter 18.

POST COMPLETENESS. **Int** is not Post complete, since it has at least one proper consistent extension, namely **Cl**. It is of interest, however, that the following result holds.

Theorem 2.58 **Cl** *is the only Post complete extension of* **Int**.

Proof Suppose L is a Post complete extension of **Int** different from **Cl**. Then there is a formula $\varphi \in L - \mathbf{Cl}$. By Theorem 1.23, we can find a variable free substitution instance ψ of φ which is not in **Cl**. But then $\neg\psi \in \mathbf{Cl}$ and, by Corollary 2.49, $\neg\psi \in \mathbf{Int}$, whence $\neg\psi \in L$, contrary to $\psi \in L$ and L being consistent. ❑

INDEPENDENT AXIOMATIZABILITY.

Theorem 2.59 **Int** *is independently axiomatizable.*

Proof Follows from the fact that **Int** is finitely axiomatizable. A subtler argument shows that axioms (A1)–(A9) are independent. ❑

STRUCTURAL COMPLETENESS. It is not difficult to verify that the congruence rules in Section 1.4 are both admissible and derivable in **Int**, and so the equivalent replacement theorem holds for **Int** as well. However, unlike **Cl**, **Int** is not structurally complete:

Proposition 2.60 *The Scott rule*

$$\frac{(\neg\neg p \to p) \to p \vee \neg p}{\neg p \vee \neg\neg p}$$

is admissible but not derivable in **Int**.

Proof The fact that this rule is not derivable follows from the deduction theorem and Example 2.35, where a countermodel for the Scott formula was constructed. Let us show now that the Scott rule is admissible in **Int**.

Suppose that $\neg\varphi \vee \neg\neg\varphi \notin \mathbf{Int}$ for some formula φ. Then, according to (A6) and (A7), $\neg\varphi \notin \mathbf{Int}$ and $\neg\neg\varphi \notin \mathbf{Int}$. By Corollary 2.49, $\neg\varphi \notin \mathbf{Cl}$ and $\neg\neg\varphi \notin \mathbf{Cl}$. So there are single-point models $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathfrak{F}_2, \mathfrak{V}_2 \rangle$ refuting $\neg\varphi$ and $\neg\neg\varphi$, respectively. Let x_1 be the point in $\mathfrak{F}_1$ and x_2 the point in $\mathfrak{F}_2$. Construct a new frame $\mathfrak{F}$ whose diagram is shown in Fig. 2.10 and define a valuation $\mathfrak{V}$ in it by taking, for every variable p,

$$\mathfrak{V}(p) = \mathfrak{V}_1(p) \cup \mathfrak{V}_2(p).$$

$\mathfrak{M}_1$ and $\mathfrak{M}_2$ are obviously generated submodels of $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, and so $(\mathfrak{M}, x_1) \not\models \neg\varphi$, $(\mathfrak{M}, x_2) \not\models \neg\neg\varphi$, whence $(\mathfrak{M}, x_1) \models \varphi$, $(\mathfrak{M}, x_2) \models \neg\varphi$ and $(\mathfrak{M}, x_2) \not\models \varphi$.

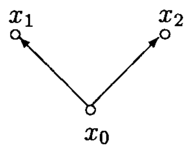


FIG. 2.10.

Then $x_0 \not\models \varphi$, $x_0 \not\models \neg\varphi$, and hence $x_0 \not\models \varphi \vee \neg\varphi$. On the other hand, we have $x_0 \models \neg\neg\varphi \to \varphi$. Indeed, otherwise $x_i \models \neg\neg\varphi$ and $x_i \not\models \varphi$ for some $i \in \{0,1,2\}$. Clearly, $i \neq 1, 2$. And if $i = 0$ then, by Proposition 2.1, $x_2 \models \neg\neg\varphi$, which is a contradiction.

Thus, $\mathfrak{M}$ refutes the formula $(\neg\neg\varphi \to \varphi) \to \varphi \vee \neg\varphi$, and so it does not belong to **Int**. ❑

Theorem 2.61 *If an inference rule is admissible in* **Int** *then it is derivable in* **Cl**.

Proof Suppose on the contrary that a rule $\varphi_1, \dots, \varphi_n/\varphi$ is admissible in **Int** but $\varphi_1 \wedge \dots \wedge \varphi_n \to \varphi \notin$ **Cl**. By Theorem 1.23, a variable free formula of the form $\varphi_1 s \wedge \dots \wedge \varphi_n s \to \varphi s$ is not in **Cl**, from which $\varphi_1 s \wedge \dots \wedge \varphi_n s \in$ **Cl** and $\varphi s \notin$ **Cl**. By Corollary 2.27, we then have $\varphi_1 s \wedge \dots \wedge \varphi_n s \in$ **Int** and $\varphi s \notin$ **Int**, which is a contradiction. ❑

The structural completeness and decidability of **Cl** provide us with an algorithm for recognizing whether a given rule is admissible in **Cl** or not. However, for **Int** the admissibility problem turns out to be much more complicated. We shall consider it in Section 16.7.

INTERPOLATION PROPERTY. **Int** like **Cl** has the interpolation property. The proof of this fact can be obtained by generalizing the construction we used for proving Theorem 1.28. We postpone it till Section 14.1.

LOCAL TABULARITY. It will be shown in Section 7.7 that there exist infinitely many formulas of only one variable which are pairwise non-equivalent in **Int**. Thus, we have

Theorem 2.62 **Int** *is not locally tabular.*

Proof Follows from Example 7.66. ❑

HALLDÉN COMPLETENESS. Clearly, Halldén completeness follows from the disjunction property. So the following theorem is an immediate consequence of Theorem 2.64, which is proved below.

Theorem 2.63 **Int** *is Halldén complete.*

DISJUNCTION PROPERTY.

Theorem 2.64 **Int** *has the disjunction property, i.e., for any formulas* φ *and* ψ, $\varphi \vee \psi \in$ **Int** *iff* $\varphi \in$ **Int** *or* $\psi \in$ **Int**.

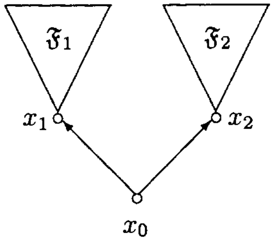


FIG. 2.11.

Proof Suppose that φ and ψ do not belong to **Int** and show that in this case $\varphi \vee \psi \notin$ **Int**.

Let $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathfrak{F}_2, \mathfrak{V}_2 \rangle$ be countermodels for φ and ψ based on disjoint frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$ with roots x_1 and x_2, respectively. Construct a new frame $\mathfrak{F} = \langle W, R \rangle$ by adding root x_0 to $\mathfrak{F}_1 + \mathfrak{F}_2$ (see Fig. 2.11). In other words, $W = \{x_0\} \cup W_1 \cup W_2$ and xRy iff $x = x_0$ or xR_1y or xR_2y, for all $x, y \in W$. Put $\mathfrak{V}(p) = \mathfrak{V}_1(p) \cup \mathfrak{V}_2(p)$, for every $p \in$ **Var**$\mathcal{L}$, and consider the model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. It is clear that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are generated submodels of $\mathfrak{M}$. Then $(\mathfrak{M}, x_0) \not\models \varphi$, $(\mathfrak{M}, x_0) \not\models \psi$ and so $(\mathfrak{M}, x_0) \not\models \varphi \vee \psi$.

The converse implication follows from (A6) and (A7). ❑

2.9 Realizability logic and Medvedev's logic

We conclude the discussion of intuitionistic logic by outlining two ways of refining the proof interpretation.

Kleene (1945) formalized it by treating the intuitionistic connectives algorithmically: for example,

- a proof of $\varphi \vee \psi$ is given by presenting a program establishing φ or a program establishing ψ together with an effective test indicating which disjunct is established;
- a proof of $\varphi \to \psi$ is given by presenting a program which transforms any program establishing φ into a program establishing ψ.

Since programs in a fixed algorithmic language (say, the language of Minsky machines to be introduced in Section 16.1) can be effectively coded by the Gödel numbers (see e.g. Mendelson, 1984), the above definition can be represented in the (first order) language of formal arithmetic. Namely, with every arithmetic sentence φ we associate a formula $x\mathbf{r}\varphi$, which is read as "the number x realizes φ", in the following way:

$$x\mathbf{r}\psi = \psi, \quad \psi \text{ atomic},$$
$$x\mathbf{r}(\psi \wedge \chi) = \exists y, z\ (x = 2^y \cdot 3^z \wedge y\mathbf{r}\psi \wedge z\mathbf{r}\chi),$$

$$
\begin{aligned}
x\mathbf{r}(\psi \vee \chi) &= \exists y, z\ ((x = 2^0 \cdot 3^y \wedge y\mathbf{r}\psi) \vee (x = 2^1 \cdot 3^z \wedge z\mathbf{r}\chi)),\\
x\mathbf{r}(\psi \to \chi) &= \forall y\ (y\mathbf{r}\psi \to f_x(y)\mathbf{r}\chi),\\
x\mathbf{r}\forall y\psi(y) &= \forall y\ (f_x(y)\mathbf{r}\psi(y)),\\
x\mathbf{r}\exists y\psi(y) &= \exists u, z\ (x = 2^u \cdot 3^z \wedge u\mathbf{r}\psi(z)),
\end{aligned}
$$

where f_x is the program with the Gödel number x (for a precise definition consult Mendelson, 1984). And now we call an $\mathcal{L}$-formula φ *realizable* if the first order formula $\exists x(x\mathbf{r}(\varphi s))$ is true for every substitution s of arithmetical sentences instead of the propositional variables in φ.

It is not hard to see that the set of realizable $\mathcal{L}$-formulas is closed under MP and Subst; it is called *realizability logic*. Nelson (1947) proved that it contains **Int**. It turned out, however, that realizability logic is a proper extension of **Int**: Rose (1953) showed that it contains the formula $\boldsymbol{sa}\{\neg q \vee \neg r/p\}$ which does not belong to **Int**. Unfortunately, very little is known about realizability logic. One of a few established facts is that it has the disjunction property; see Varpakhovskij (1965). A class of realizable propositional formulas containing all known formulas of that sort was described by Varpakhovskij (1973).

Another formalization of the proof interpretation (of Kolmogorov's interpretation, to be more precise) was proposed by Medvedev (1962), who treated intuitionistic formulas as finite problems. Formally, a *finite problem* is a pair $\langle X, Y\rangle$ of finite sets such that $Y \subseteq X$ and $X \neq \emptyset$; elements in X are called *possible solutions* and elements in Y *solutions* to the problem. The operations on finite problems, corresponding to the logical connectives, are defined as follows:

$$\langle X_1, Y_1\rangle \wedge \langle X_2, Y_2\rangle = \langle X_1 \times X_2, Y_1 \times Y_2\rangle\,,$$

$$\langle X_1, Y_1\rangle \vee \langle X_2, Y_2\rangle = \langle X_1 \sqcup X_2, Y_1 \sqcup Y_2\rangle\,,$$

$$\langle X_1, Y_1\rangle \to \langle X_2, Y_2\rangle = \left\langle X_2^{X_1}, \{f \in X_2^{X_1} : f(Y_1) \subseteq Y_2\}\right\rangle,$$

$$\bot = \langle X, \emptyset\rangle\,.$$

Here $X \sqcup Y = (X \times \{1\}) \cup (Y \times \{2\})$ (i.e., $X \sqcup Y$ is the ordered union of X and Y) and X^Y is the set of all functions from X into Y. Note that in the definition of $\bot$ the set X is fixed, but arbitrary; for definiteness one can take $X = \{\emptyset\}$.

Now we can interpret formulas by finite problems. Namely, given a formula φ, we replace its variables by arbitrary finite problems and perform the operations corresponding to the connectives in φ. If the result is a problem with a non-empty set of solutions no matter what finite problems are substituted for the variables in φ, then φ is called *finitely valid*. One can show that the set of all finitely valid formulas is closed under MP and Subst and contains **Int**; it is called *Medvedev's logic* and denoted by **ML**.

In fact **ML** can be defined semantically, similarly to how **Int** was introduced. Let W_n be the family of non-empty subsets of a set with $n > 0$ elements and xR_ny mean $y \subseteq x$, for every $x, y \in W_n$. The pair $\mathfrak{B}_n = \langle W_n, R_n\rangle$ is clearly a Kripke frame; we call it a *Medvedev frame*. Medvedev frames have an elegant

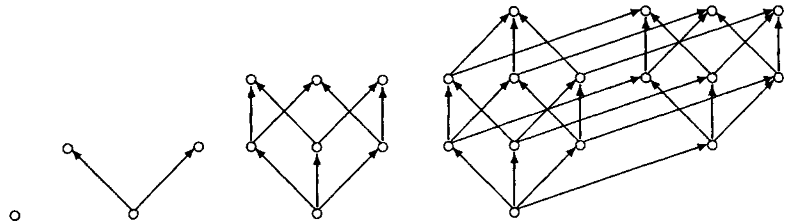


FIG. 2.12.

geometrical form: they look like n-ary Boolean cubes with the top point deleted (for $n = 1, 2, 3, 4$ they are depicted in Fig. 2.12). Medvedev (1966) showed that **ML** coincides with the set of $\mathcal{L}$-formulas that are valid in all Medvedev frames. We offer the reader to check that **ML** contains the formulas ***sa*** and ***kp*** (see Exercise 2.10) which do not belong to **Int**.

2.10 Exercises

Exercise 2.1 Show that, for any family $\{X_i : i \in I\}$ of subsets of W in a frame $\mathfrak{F} = \langle W, R \rangle$,

$$(\bigcup_{i\in I} X_i)\uparrow = \bigcup_{i\in I}(X_i\uparrow), \quad (\bigcup_{i\in I} X_i)\downarrow = \bigcup_{i\in I}(X_i\downarrow),$$

$$(\bigcap_{i\in I} X_i)\uparrow \subseteq \bigcap_{i\in I}(X_i\uparrow), \quad (\bigcap_{i\in I} X_i)\downarrow \subseteq \bigcap_{i\in I}(X_i\downarrow).$$

Is it possible to replace $\subseteq$ here with $=$?

Exercise 2.2 Can the generation theorem be extended to not necessarily generated submodels? Does the operation of forming subframes preserve validity?

Exercise 2.3 Show that an infinite frame contains either an infinite ascending chain or an infinite descending chain or an infinite antichain. (Hint: use König's lemma, according to which every infinite tree of finite branching contains an infinite ascending chain.)

Exercise 2.4 Let $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathfrak{F}_2, \mathfrak{V}_2 \rangle$ be two models based on frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$, respectively. A non-empty binary relation $S \subseteq W_1 \times W_2$ is said to be a *bisimulation* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ if the following conditions are satisfied:

- if $x_1 S x_2$ then $x_1 \models p$ iff $x_2 \models p$, for every variable p;
- if $x_1 S x_2$ and $x_1 R_1 y_1$ then there is $y_2 \in W_2$ such that $y_1 S y_2$ and $x_2 R_2 y_2$;
- if $x_1 S x_2$ and $x_2 R_2 y_2$ then there is $y_1 \in W_1$ such that $y_1 S y_2$ and $x_1 R_1 y_1$.

Prove that if S is a bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and $x_1 S x_2$, then $x_1 \models \varphi$ iff $x_2 \models \varphi$, for every formula φ. Derive from this the generation, reduction and disjoint union theorems.

Exercise 2.5 Show that each finite frame of branching $\leq n$ is a reduct of some finite n-ary tree.

Exercise 2.6 Show that two finite rooted frames are isomorphic iff they validate the same formulas.

Exercise 2.7 Give a purely syntactic proof of Proposition 2.26 (by induction on the construction of φ).

Exercise 2.8 Prove that every formula $\varphi \notin \mathbf{Int}$ is refuted by a tree of depth and branching $\leq |\mathbf{Sub}\varphi|$.

Exercise 2.9 Prove that every disjunction free formula $\varphi \notin \mathbf{Int}$ is refuted by a finite binary tree.

Exercise 2.10 Show that a frame $\mathfrak{F} = \langle W, R \rangle$ validates the *Kreisel–Putnam formula*

$$\boldsymbol{kp} = (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

iff $\mathfrak{F}$ satisfies the following condition

$$\forall x, y, z\ (xRy \wedge xRz \wedge \neg yRz \wedge \neg zRy \to \exists u\ (xRu \wedge uRy \wedge uRz \wedge \\ \forall v\ (uRv \to \exists w\ (vRw \wedge (yRw \vee zRw))))).$$

Exercise 2.11 Show that a frame $\mathfrak{F} = \langle W, R \rangle$ validates the formula

$$\boldsymbol{btw}_n = \bigwedge_{0 \leq i < j \leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n} (\neg p_i \to \bigvee_{j \neq i} \neg p_j)$$

iff $\mathfrak{F}$ satisfies the condition

$$\forall x, x_0, \dots, x_n (\bigwedge_{i=1}^{n} xRx_i \to \exists y \bigvee_{i \neq j} (x_i Ry \wedge x_j Ry)).$$

If $\mathfrak{F}$ is rooted and finite, then this condition means that $\mathfrak{F}$ has $\leq n$ final points, i.e., $\boldsymbol{btw}_n$ bounds the top-width of $\mathfrak{F}$.

Exercise 2.12 Prove that $\mathfrak{F}$ validates $\boldsymbol{sa}$ iff no generated subframe of $\mathfrak{F}$ is reducible to the frame shown in Fig. 2.9 (c).

Exercise 2.13 Show that a frame $\mathfrak{F} = \langle W, R \rangle$ refutes $\boldsymbol{bb}_n$ iff there is a subframe $\mathfrak{G} = \langle V, S \rangle$ of $\mathfrak{F}$ such that $xRyRz$ implies $y \in V$ whenever $x, z \in V$ and $\mathfrak{G}$ is reducible to the $n+1$-ary tree of depth 2.

Exercise 2.14 Prove that $\mathfrak{F} \not\models \bigwedge_{i=0}^{n}(\neg p_i \leftrightarrow \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$ iff there is a generated subframe of $\mathfrak{F}$ reducible to the $n+1$-ary tree of depth 2.

Exercise 2.15 Show that a rooted frame validates the formula

$$\boldsymbol{sm} = (\neg q \to p) \to (((p \to q) \to p) \to p)$$

iff it contains ≤ 2 points.

Exercise 2.16 Show that the *Skvortsov formula*

$$(\neg(p \wedge q) \to \neg(\neg p \wedge q) \vee \neg(p \wedge \neg q)) \to \neg(\neg p \wedge q) \vee \neg(p \wedge \neg q)$$

belongs to $\mathbf{ML} - \mathbf{Int}$.

Exercise 2.17 Define by induction a sequence of finite trees $\mathfrak{J}_n$, known as *Jaśkowski's frames*: $\mathfrak{J}_1$ is the single-point frame and $\mathfrak{J}_{n+1}$ is the result of adding a root to the disjoint union of n copies of $\mathfrak{J}_n$. Prove that

$$\mathbf{Int} = \{\varphi \in \mathbf{For}\mathcal{L} : \ \mathfrak{J}_n \models \varphi \text{ for every } n > 0\}.$$

Exercise 2.18 Say that a connective $\odot$ is *independent* in a logic L if there is no formula φ without occurrences of $\odot$ such that $p \odot q \leftrightarrow \varphi \in L$ (if $\odot = \bot$ then, of course, $\bot \leftrightarrow \varphi \in L$). Prove that $\wedge$, $\vee$, $\to$ and $\bot$ are independent in **Int**. (Hint: to prove that $\wedge$ and $\vee$ are independent use the disjoint union of one- and two-point rooted frames and the three-point rooted frame, respectively.)

Exercise 2.19 Prove that for every set of formulas Γ and every formula φ,

$$\Gamma \vdash_{Cl} \varphi \text{ iff } Tr_i(\Gamma) \vdash_{Int} Tr_i(\varphi),$$

where $Tr_i(\Gamma) = \{Tr_i(\psi) : \ \psi \in \Gamma\}$ and $i = 1, 3$. Does this equivalence hold for $i = 2$?

Exercise 2.20 Define by induction the set $\mathbf{H}$ of *Harrop formulas*: (i) all variables are in $\mathbf{H}$; (ii) if φ and ψ are in $\mathbf{H}$ then $\varphi \wedge \psi$ and $\chi \to \psi$ are also in $\mathbf{H}$, for every formula χ. Prove that for any set Γ of Harrop formulas and all formulas φ and ψ,

$$\Gamma \vdash_{Int} \varphi \vee \psi \text{ implies } \Gamma \vdash_{Int} \varphi \text{ or } \Gamma \vdash_{Int} \psi.$$

Is this true for an arbitrary set of formulas Γ?

2.11 Notes

Intuitionistic logic as a formal explication of Brouwer's (1907, 1908) ideas was constructed in the form of Hilbert-style calculus by Kolmogorov (1925), Orlov (1928) and Glivenko (1929), who considered the propositional language, and then, for the predicate case, by Heyting (1930). For more detailed historical information about intuitionistic logic and its relation to intuitionism and constructivism the reader can consult Troelstra (1969), Dummett (1977), Beeson (1985), van Dalen (1986).

The intended meaning of the intuitionistic connectives was explained first in terms of the proof interpretation due to Brouwer, Kolmogorov and Heyting. It was not clear, however, how to construct a reasonable formal semantics consistent with this informal interpretation, or even just any semantics with respect to which **Int** would be complete (Gödel had just proved his completeness theorem for classical predicate logic). The semantical studies of **Int** were started by

Gödel (1932), who showed that it is not tabular. Jaśkowski (1936) constructed a sequence of finite matrices characterizing **Int** (see Exercise 2.17 giving the frame variant of Jaśkowski's construction). In fact he was the first to prove that **Int** is finitely approximable. Stone (1937) and Tarski (1938) discovered a connection between the derivability in *Int* and topological spaces, which was developed by McKinsey (1941), McKinsey and Tarski (1944, 1946) into the algebraic semantics for **Int** to be considered in full detail in Chapter 7. For a category-theoretical generalization of the topological semantics see Goldblatt (1979).

Note that in the 1940s and 1950s Novikov in his course on intuitionistic logic at Moscow University proposed an informal arithmetic interpretation of **Int** which also led to the topological completeness (much later these lectures were published as book Novikov, 1977). Loosely, the idea of Novikov's interpretation is as follows. Atomic propositions are regarded as statements about comparing weights: $t_1 < t_2$, $t_1 > t_2$, $t_1 = t_2$. To check whether they are true or not, we have at our disposal an unlimited collection of arbitrarily precise (but not absolutely precise!) scales. So by a finite number of weighing we can prove or disprove propositions of the form $t_1 < t_2$, $t_1 > t_2$ but we can never establish that a proposition of the form $t_1 = t_2$ is true, though we may be able to refute it.

Kolmogorov (1932) proposed to consider **Int** as a logic of problems but did not formalize his idea, which was partly fulfilled later by Kleene (1945), Gödel (1958), Medvedev (1962), Artemov (1987b). Gödel (1933a) gave an interpretation of the intuitionistic connectives via the corresponding classical ones by embedding **Int** into Lewis' modal system **S4** (based on classical logic) and treating its necessity operator as "it is provable" (for details see Section 3.9). Before Gödel actually the same results were obtained by Orlov (1928). However, his paper remained unnoticed for a rather long time. This "classical" view on intuitionistic logic was developed further by Novikov (1977). Embeddings of **Cl** into **Int** were constructed by Glivenko (1929), Gödel (1933b), Gentzen (1934–35) and Łukasiewicz (1952).

The relational semantics we considered in this chapter was introduced by Kripke (1965a). In fact it can be traced back to Jónnson and Tarski (1951) who represented algebras for the modal logic **S4**, and hence implicitly for **Int**, in the form of frames, and to Dummett and Lemmon (1959) who did this explicitly for finite algebras. A somewhat different relational semantics was constructed by Beth (1956); a close interpretation of intuitionistic connectives was proposed by Grzegorczyk (1964). Semantics combining in themselves both Kripke and Beth frames are considered in Dragalin (1979). In general, the semantical apparatus for **Int** was developed after the corresponding apparatus for modal logics to be considered in the next chapter. Sometimes, however, new semantical concepts were first introduced for **Int**, witness a sort of p-morphism considered by de Jongh and Troelstra (1966).

Our proof of completeness is similar to that of Fitting (1969), although again, as in the case of **Cl**, we define Hintikka systems as a tool for constructing countermodels rather than for obtaining a proof system for **Int**. There exist other proofs of completeness. For instance, one can extract from Dragalin (1979) a direct proof that *Int* is complete with respect to the full binary tree. This result was first obtained by Smoryński (1973); see also Kirk (1979) who showed that, for each $n \geq 2$, **Int** is characterized by the class of all n-ary trees.

Gentzen (1934–35) represented **Int** as a system of natural deduction and as a calculus of sequents. In a purely syntactic way he proved that **Int** is decidable and has the disjunction property. A syntactic proof of the interpolation property can be found in Schütte (1962).

An interesting syntactical property of **Int** was discovered by Wajsberg (1938) (see also Horn, 1962) who constructed a variant of intuitionistic calculus to derive a formula φ in which it is sufficient to use (A1), (A2) and only those axioms that contain connectives really occurring in φ. Logics which can be represented by calculi with this property are called *separable*. Many extensions of **Int** were proved to be separable, in particular, **Cl** (see Hosoi, 1966c). It is unknown whether one can effectively recognize the separation property, given a finite set of axioms extending **Int**. Although it follows from Khomich (1979) that this can be done in the case of extra axioms in one variable, in general we conjecture that this algorithmic problem has a negative solution. The problem is not trivial even for tabular superintuitionistic logics.

According to Exercise 1.3, the connectives in **Cl** are interconnected and they are enough to express all possible logical connectives which can be represented by truth-tables. This property is known as *truth-functional completeness*. The situation with the connectives in **Int** is much more complicated. First, as was noted by McKinsey (1939), they are independent (see Exercise 2.18). This result was developed then in two directions: expressing intuitionistic formulas in each other and adding new connectives to the language of **Int**.

Since **Int** is not truth-functional, the notion of expressibility in **Int** is defined in the following way. Say that a finite sequence of formulas $\varphi_1, \dots, \varphi_n$ is an *expression* in **Int** via a list of formulas Γ if one of the following four conditions holds for each φ_i:

- φ_i is a variable;
- $\varphi_i \in \Gamma$;
- there are $j, k < i$ and a variable p such that $\varphi_i = \varphi_k\{\varphi_j/p\}$;
- there is $l < i$ such that $\varphi_i \leftrightarrow \varphi_l \in$ **Int**.

A formula φ is said to be *expressible* in **Int** via Γ if φ is a member of some expression via Γ. Expressibility in **Int** turned out to be much more complicated than expressibility in **Cl** and many-valued logics (see Jablonskij, 1979). In any case, no algorithm is known to determine whether a formula is expressible in **Int** via a list of formulas. A significant result in this direction was obtained by Ratsa (1982), who gave a positive solution to the algorithmic problem of recognizing whether a given list of formulas is *functionally complete* in **Int** (and even in any superintuitionistic calculus!) in the sense that every formula is expressible via the list. Here are two examples of functionally complete sets in **Int** found by Kuznetsov (1965):

$$\{p \to (q \wedge \neg r) \wedge (q' \vee r')\},$$

$$\{((p \vee q) \wedge \neg r) \vee (\neg p \wedge (q \leftrightarrow r))\}.$$

The following sequence is an expression via the latter formula containing all intuitionistic connectives (as formulas, of course):

$((p \vee q) \wedge \neg r) \vee (\neg p \wedge (q \leftrightarrow r))$, p, q, r, $((p \vee q) \wedge \neg q) \vee (\neg p \wedge (q \leftrightarrow q))$, $(p \wedge \neg q) \vee \neg p$, $(p \wedge \neg p) \vee \neg p$, $\neg p$, $(p \wedge \neg\neg p) \vee \neg p$, $p \vee \neg p$, $\neg(p \vee \neg p)$, $\bot$, $\neg\bot$, $((p \vee q) \wedge \neg\bot) \vee (\neg p \wedge (q \leftrightarrow \bot))$, $(p \vee q) \vee (\neg p \wedge \neg q)$, $((p \vee q) \wedge \neg\neg\bot) \vee (\neg p \wedge (q \leftrightarrow \neg\bot))$, $\neg p \wedge q$, $\neg q$, $\neg p \wedge \neg q$, $\neg p \wedge r$, $\neg(\neg p \wedge \neg q) \wedge r$, $\neg(\neg p \wedge \neg q) \wedge ((p \vee q) \vee (\neg p \wedge \neg q))$, $p \vee q$, $((\bot \vee q) \wedge \neg r) \vee (\neg\bot \wedge (q \leftrightarrow r))$, $(q \wedge \neg r) \vee (q \leftrightarrow r)$, $(q \wedge \neg p) \vee (q \leftrightarrow p)$, $\neg(\neg p \wedge q) \wedge r$, $\neg(\neg p \wedge q) \wedge ((q \wedge \neg p) \vee (q \leftrightarrow p))$, $p \leftrightarrow q$, $(p \vee q) \leftrightarrow q$, $p \to q$, $p \leftrightarrow (p \to q)$, $p \wedge q$.

Functional incompleteness of the usual systems of connectives in **Int** made it possible to introduce various "new" connectives generalizing the standard ones (infinitary disjunctions, conjunctions, etc.; see for instance, Nadel (1978), Goad (1978), de Jongh (1980), Kalicki (1980), Wojtylak (1983)). On the other hand the language of **Int** was enriched by modal operators; we shall give some references in Section 3.12. An interesting connective $\sqcup$, called the *weak disjunction*, was introduced by Medvedev (1966) for **ML** and then considered by Skvortsov (1983) for **Int** and its extensions. Semantically $\sqcup$ may be defined like this:

$$\begin{aligned}(\mathfrak{M}, x) \models \varphi \sqcup \psi \text{ iff } & \forall y\ (xRy \to ((\mathfrak{M}, y) \models \varphi) \vee ((\mathfrak{M}, y) \models \psi) \vee \\ & \exists u, v\ (yRu \wedge yRv \wedge ((\mathfrak{M}, u) \models \varphi) \wedge ((\mathfrak{M}, v) \models \psi) \wedge \\ & \forall w\ (yRw \to wCu \vee wCv))),\end{aligned}$$

where aCb means $\exists z(aRz \wedge bRz)$. Skvortsov (1983) justifies the given definition by the following considerations. Let us understand points in frames as "reasons" or "arguments" in a controversy. Chains are regarded as possible ways of its development (or possible ways of researches, if we argue with nature). A point x in a model is a reason for φ if $(\mathfrak{M}, x) \models \varphi$. Say that φ is *intuitionistically established* at a point y by reason x if $((\mathfrak{M}, x) \models \varphi) \wedge xRy$; φ is *classically established* at y by reason x if $((\mathfrak{M}, x) \models \varphi) \wedge xCy$.

The intuitionistic disjunction $\varphi \vee \psi$ is true at x if either φ or ψ is intuitionistically established at x. The classical disjunction (we use for it another symbol) $\varphi \dot{\vee} \psi = \neg\neg(\varphi \vee \psi)$ is true at x if, for every initial development of the controversy, either φ or ψ remains classically established (by some reasons) at any point. The relation $(\mathfrak{M}, x) \models \varphi \sqcup \psi$ can be represented in the form $\forall y(xRy \to (\mathfrak{M}, y) \models' \varphi \sqcup \psi)$, where the restricted quantifier $\forall y(xRy \to$ corresponds to an arbitrary initial development of the controversy and $(\mathfrak{M}, y) \models' \varphi \sqcup \psi$ means that either the intuitionistic disjunction $\varphi \vee \psi$ has been already established at y or (above x) there are two reasons for φ and ψ, respectively, which alone establish the classical disjunction at x. Roughly speaking, the weak disjunction $\varphi \sqcup \psi$ reduces to the problem of classical establishing φ or ψ by some unique, concrete reasons for them and not on the ground of the general distribution of truth-values of φ and ψ.

Note also that Medvedev (1979) and Skvortsov (1979) introduced some interesting variants of negation added to **Int**. Vorob'ev (1952a, 1952b, 1972) added to **Int** the so called strong negation; the resulting logic was investigated by Gurevich (1977), Vakarelov (1977), Sendlevski (1984), Goranko (1985).

New intuitionistic connectives can be introduced semantically. Yashin (1985) used some ideas of McCullough (1971) to define an intuitionistic connective as a formula in the language of the elementary theory of Kripke models with one parameter satisfying the following conditions: the monotonicity with respect to accessibility; all quantifiers are of the form $\forall x \geq y$ or $\exists x \geq y$; a proposition with such a connective should not distinguish between two models one of which is obtained from the other using the operations of reduction and the formation of elementary equivalent models. It turns out that intuitionistic connectives in this sense are only the standard intuitionistic propositional formulas. A similar result for the modal case was obtained by Yashin (1986). Yashin (1989) described the connectives that result from relaxing the conditions above.

Novikov (see Smetanich, 1960) and Gabbay (1977) gave syntactical definitions of new intuitionistic connectives. Let λ be an extra unary connective and $Int(\lambda)$ a calculus obtained by adding to Int some new axioms describing λ. According to Novikov, $Int(\lambda)$ defines a new connective if

- $Int(\lambda)$ is conservative over Int, i.e., if $Int(\lambda) \vdash \varphi$ and φ does not contain λ, then $Int \vdash \varphi$;
- $Int(\lambda) \vdash (p \leftrightarrow q) \rightarrow (\lambda(p) \leftrightarrow \lambda(q))$;
- for every λ-free formula φ, $Int(\lambda) + \lambda(p) \leftrightarrow \varphi$ is not conservative over Int.

According to Gabbay, $Int(\lambda)$ defines a new connective if

- $Int(\lambda)$ is conservative over Int;
- no explicit definition of λ is derivable in $Int(\lambda)$;
- $Int(\lambda)$ has the disjunction property;
- some explicit definition of λ is derivable in $Int(\lambda) + \neg\neg p \rightarrow p$;
- the axioms of $Int(\lambda)$ define the meaning of λ uniquely in the sense that $Int(\lambda) + Int(\lambda') \vdash \lambda \rightarrow \lambda'$;
- λ is definable in the second order intuitionistic calculus.

Smetanich (1960) showed that we get a new Novikov connective by adding to Int the axioms

$$\lambda(p) \leftrightarrow \lambda(q),\ \neg\neg\lambda(p),\ \lambda(p) \rightarrow q \vee \neg q.$$

Bessonov (1977) constructed a continuum of similar axiomatic systems defining new connectives and Yashin (1994) showed that the axioms

$$\neg\neg\lambda(p),\ \lambda(p) \rightarrow q \vee \neg q,$$

define a new connective as well.

The result of Exercise 2.9 is due to Segerberg (1974).

3

MODAL LOGICS

When discussing in Section 2.1 the meaning of intuitionistic connectives, we used in our language—a metalanguage with respect to $\mathcal{L}$—the undefined notion "proof". Making the proof interpretation somewhat rougher, we can treat, for example, the intuitionistic formula $p \to q \vee r$ as the proposition

"it is provable (it is provable $p \to$ it is provable $q \vee$ it is provable r)"

with the *classical* connectives $\to$ and $\vee$. "Modalized" propositions of that sort, containing such operators as "it is provable", "it is necessary", "it is obligatory", etc., are the subject of *modal logic*, another branch of mathematical logic.

3.1 Possible world semantics

The expressive capacities of the language $\mathcal{L}$ of classical (or intuitionistic) logic do not allow us to decompose such propositions as

(A) It is possible that water boils at 70°C

or

(B) It is necessary that water boils at 70°C

into a combination of simpler propositions. Like the proposition

(C) Water boils at 70°C,

they can be regarded only as atomic. So we are able to express correctly in $\mathcal{L}$ neither the implications "if (B) then (C)" and "if (C) then (A)", which are naturally considered to be true, nor the implications "if (C) then (B)" and "if (A) then (C)", which are probably recognized to be false.

The *propositional modal language* $\mathcal{ML}$ is obtained by enriching the language $\mathcal{L}$ with the new unary connective $\Box$ and the corresponding formula formation rule

- if φ is an $\mathcal{ML}$-formula then $(\Box\varphi)$ is also an $\mathcal{ML}$-formula,

which is added to the rules in Section 1.1 (with $\mathcal{L}$ being replaced with $\mathcal{ML}$, of course). The set of all $\mathcal{ML}$-formulas is denoted by **For**$\mathcal{ML}$ and the set of all variables in $\mathcal{ML}$ by **Var**$\mathcal{ML}$. In addition to the conventions on formula representation, which were accepted in Section 1.1, we will assume $\Box$ to bind formulas stronger than $\wedge$, $\vee$, $\to$ and $\leftrightarrow$. Thus, $\Box p \to \Box q \vee \Box r$ is an abbreviation for $((\Box p) \to ((\Box q) \vee (\Box r)))$.

We define the connective $\Diamond$ as dual to $\Box$, i.e., by taking

$$\Diamond\varphi = \neg\Box\neg\varphi, \text{ for every } \varphi \in \mathbf{For}\mathcal{ML},$$

and consider it as strong as $\Box$ or $\neg$.

The connectives $\Box$ and $\Diamond$ are usually read as "it is necessary" and "it is possible" and called the *necessity* and *possibility operators*, respectively. So (A) and (B) above can be represented now as $\Diamond$(C) and $\Box$(C). However, the intended meaning of these connectives may vary. Here are only a few possible interpretations of $\Box$ and $\Diamond$.

(i) $\Box$ is understood as logical necessity, i.e., as "it is necessary from the point of view of logical laws", and $\Diamond$ as logical possibility, i.e., "it does not contradict the logical laws".

(ii) $\Box$ may be regarded as epistemic necessity, i.e., as "it is known" (or "it is believed"). This interpretation seems to require some refinement, since at least two questions arise: "whom is this known to?" and "are the logical consequences of known propositions also known; say, is ψ known provided that φ and $\varphi \to \psi$ are known?". We will assume that there is some ideal perceiving person, and the set of propositions which are known to him is closed under the logical consequence. In this case $\Diamond$ may be read as "it does not contradict to anything that is known".

(iii) Another interpretation, closely related to (ii), is to understand $\Box$ as "it is (informally) provable (by an ideal mathematician) in some mathematical theory"; $\Diamond$ means then "it does not contradict to the postulates of the theory".

(iv) $\Box$ may be also regarded as provability in some formal system, for instance, in formal Peano arithmetic PA.

(v) One can understand $\Box$ as deontic necessity, that is as "it is obligatory"; $\Diamond$ is then read as "it is permitted".

(vi) Sometimes $\Box$ is interpreted as tense necessity, that is as "it is true now and always will be true" and $\Diamond$ as "it is true now or will be true afterwards".

Some modal formulas, which are acceptable under one interpretation of $\Box$, may turn out to be unacceptable under another one. For example, an arbitrary proposition of the form $\Box(\Box\varphi \to \varphi)$ may be regarded as true in the cases (i), (ii), (iii) and (vi), but neither in (iv) nor in (v). Indeed, by accepting this principle for the formal provability in PA, we would then have that the formula $\Box(0 = 1) \to 0 = 1$, and so its contraposition $\neg 0 = 1 \to \neg\Box(0 = 1)$, are provable in PA[5]. And since the premise of the latter formula is provable in PA, the conclusion $\neg\Box(0 = 1)$ must also be provable, contrary to Gödel's second theorem, according to which the consistency of PA cannot be proved only by its own means. In the deontic case, $\Box\varphi \to \varphi$ does not hold, since obligations may be not fulfilled. Another example: without stretching a point the principle $\Diamond\varphi \to \Box\Diamond\varphi$ can be accepted only for the logical necessity.

On the other hand, all the interpretations of the operator $\Box$ listed above have many common traits. For instance, for all of them the principles

[5]How $\Box$ is formalized in PA is explained in Section 3.8.

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

and

$$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$$

are acceptable. This makes it possible to consider them, at least to a certain extent, from a common standpoint by treating $\Box$ as some abstract necessity. Moreover, we shall see in the sequel that the differences in the interpretations we have just observed can be provided with a strict mathematical meaning.

The interpretation of the modal language $\mathcal{ML}$ we are going to introduce now, first at the intuitive level and then, in the next section, in the form of precise definitions, is often called the *relational* or *possible world semantics*. Philosophers trace it back to Leibniz who understood necessity as truth in all possible worlds and possibility as truth in at least one possible world.

As in classical logic, we assume that every proposition is either true or false. For example, it is natural to evaluate proposition (C) as false. However, it would be more exact to say that (C) is false under ordinary circumstances, in the ordinary world where we live. For we can imagine some other circumstances, another world in which water really boils at 70°C (in principle, we can even find ourselves in this world having climbed the summit of the Everest). That world where (C) becomes true may be called an *alternative* to our world or a *possible world* relative to it. Using Leibniz's definition, we can say that proposition (A) should be recognized as true in our world, and (B), on the contrary, as false.

In general, by abstracting from concrete details, we can imagine a system of worlds in which each world has some (possibly empty) set of alternatives. The *alternativeness relation* will be denoted by R, so that xRy means that y is an alternative (or possible) world for x. Every world x "lives" under the classical laws: an atomic proposition is either true or false in it and the truth-values of compound non-modal propositions are determined by the usual truth-tables. A modal proposition $\Box\varphi$ is regarded to be true in a world x if φ is true in all the worlds alternative to x; $\Diamond\varphi$ is true in x if φ is true at least in one world y such that xRy.

Concrete properties of the alternativeness relation depend on the type of the modality under consideration. If we deal with the logical necessity then it is natural to regard any two worlds to be alternatives to each other; in other words, the alternativeness relation in this case is *universal*. However, if we consider the tense necessity then possible worlds are states of our world (or some other developing process, e.g. a computer program) at different moments of time. The choice of a suitable alternativeness relation R depends then on our aims and views on the nature of time. For example, we may consider the course of time to be linear, and then R will be a linear ordering of the set of worlds, or we may think that time has a branching nature and take R to be a tree-like ordering of possible worlds.

Alternativeness relations for other interpretations of $\Box$ (say, epistemic, provability or deontic) may be not so clear. To characterize them, we should first describe more precisely the corresponding modalities, by defining them axiomatically, for instance. And after that, given a set of worlds, we can regard a world y as an epistemic (provability, deontic, etc.) alternative to x iff all that is known (respectively, provable, obligatory, etc.) at x is necessarily true at y.

Epistemic, deontic, provability and a number of other modal logics will be introduced in Section 3.8. However, mostly in this chapter we will be considering the logic **K** of some abstract necessity describing those common properties that are characteristic for *all* interpretations of the operator $\Box$ above.

3.2 Modal frames and models

In an intuitionistic frame $\mathfrak{F} = \langle W, R\rangle$, which was used for representing possible states of information, the accessibility relation R between states was a partial order on W. We will represent systems of possible worlds with alternativeness relations between them in the form of frames as well, but for the present no conditions will be imposed on R.

A *modal Kripke frame* $\mathfrak{F} = \langle W, R\rangle$ consists of a non-empty set (of worlds) W and an arbitrary binary (alternativeness) relation R on W. Thus, intuitionistic frames are a special case of modal ones. Elements of W are called *worlds* or, as before, more neutrally, *points*. If xRy, we say that y is an *alternative* to x, or that y is *accessible* from x, or x *sees* y. Other synonyms and notations are: y is a *successor* of x, x is a *predecessor* of y, $y \in x\uparrow$, $x \in y\downarrow$. The notions of *proper* and *immediate successor* or *predecessor* are defined as in the intuitionistic case.

Let us fix some propositional modal language $\mathcal{ML}$. A *valuation* of $\mathcal{ML}$ in a frame $\mathfrak{F} = \langle W, R\rangle$ is a map $\mathfrak{V}$ associating with each variable p in $\mathbf{Var}\mathcal{ML}$ a set $\mathfrak{V}(p)$ of points in W, i.e., $\mathfrak{V}$ is a map from $\mathbf{Var}\mathcal{ML}$ to 2^W. $\mathfrak{V}(p)$ is understood as the set of worlds at which p is true.

A *Kripke model* of $\mathcal{ML}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ where $\mathfrak{F} = \langle W, R\rangle$ is a frame and $\mathfrak{V}$ a valuation in $\mathfrak{F}$. Let x be a point in $\mathfrak{F}$. By induction on the construction of φ we define a *truth-relation* $(\mathfrak{M}, x) \models \varphi$, "$\varphi$ is true at the world x in the model $\mathfrak{M}$", by taking

$$
\begin{array}{ll}
(\mathfrak{M}, x) \models p & \text{iff } x \in \mathfrak{V}(p), \text{ for every } p \in \mathbf{Var}\mathcal{ML};\\
(\mathfrak{M}, x) \models \psi \wedge \chi & \text{iff } (\mathfrak{M}, x) \models \psi \text{ and } (\mathfrak{M}, x) \models \chi;\\
(\mathfrak{M}, x) \models \psi \vee \chi & \text{iff } (\mathfrak{M}, x) \models \psi \text{ or } (\mathfrak{M}, x) \models \chi;\\
(\mathfrak{M}, x) \models \psi \to \chi & \text{iff } (\mathfrak{M}, x) \models \psi \text{ implies } (\mathfrak{M}, x) \models \chi;\\
(\mathfrak{M}, x) \not\models \bot; & \\
(\mathfrak{M}, x) \models \Box\psi & \text{iff } (\mathfrak{M}, y) \models \psi \text{ for all } y \in W \text{ such that } xRy,
\end{array}
$$

and so

$$
\begin{array}{l}
(\mathfrak{M}, x) \models \neg\psi \text{ iff } (\mathfrak{M}, x) \not\models \psi\\
(\mathfrak{M}, x) \models \Diamond\psi \text{ iff } (\mathfrak{M}, y) \models \psi \text{ for some } y \in W \text{ such that } xRy.
\end{array}
$$

If $(\mathfrak{M}, x) \not\models \varphi$ then we say φ is *false* at the world x in $\mathfrak{M}$. Instead of $(\mathfrak{M}, x) \models \varphi$ and $(\mathfrak{M}, x) \not\models \varphi$ we will write simply $x \models \varphi$ and $x \not\models \varphi$, if understood. The truth-set of φ in $\mathfrak{M}$ is denoted by $\mathfrak{V}(\varphi)$.

The definitions of satisfiability, truth, refutability and validity in modal frames and also those of isomorphism between frames and models remain the same as in Section 2.2. Propositions 2.2 and 2.3 can be transferred without any changes to the modal case as well.

Proposition 2.3, stating that the truth-value of a formula depends only on the truth-values of its variables, can be somewhat strengthened by observing that if a formula φ contains at most n nested modalities then the truth-value of φ at a point x depends only on the truth-values of its variables at the points accessible from x by at most n steps. To formulate this more precisely, we require a few definitions.

The *modal degree* $md(\varphi)$ of a formula φ is defined inductively:

$$\begin{aligned} md(\varphi) &= 0, \text{ for every atomic } \varphi; \\ md(\psi \odot \chi) &= \max\{md(\psi), md(\chi)\}, \text{ for } \odot \in \{\wedge, \vee, \rightarrow\}; \\ md(\Box\psi) &= md(\Diamond\psi) = md(\psi) + 1. \end{aligned}$$

We denote by $\Box^n\varphi$ and $\Diamond^n\varphi$ the formulas $\underbrace{\Box \ldots \Box}_{n}\varphi$ and $\underbrace{\Diamond \ldots \Diamond}_{n}\varphi$, respectively; by the definition, both $\Box^0\varphi$ and $\Diamond^0\varphi$ are just φ. Thus, if φ contains no modal operators at all then $md(\Box^n\varphi) = md(\Diamond^n\varphi) = n$.

Let $\mathfrak{F} = \langle W, R\rangle$ be a frame and $x, y \in W$. Say that y is *accessible from x by $n > 0$ steps* and write xR^ny or $y \in x\uparrow^n$ or $x \in y\downarrow_n$ if there exist (not necessarily distinct) points $z_1, \ldots, z_{n-1}$ in W such that $xRz_1Rz_2 \ldots Rz_{n-1}Ry$. We shall also understand xR^0y, $y \in x\uparrow^0$ and $x \in y\downarrow_0$ as $x = y$. If R is transitive then clearly xR^ny implies xRy, for every $n > 0$, and if R is also reflexive then the converse holds as well. A point x is called *reflexive* if xRx; for such an x, xR^nx holds for every $n \geq 0$. A frame is *(ir)reflexive* if all points in it are (ir)reflexive. A frame $\mathfrak{F} = \langle W, R\rangle$ is said to be *intransitive* if $\forall x, y, z\ (xRy \wedge yRz \rightarrow \neg xRz)$. An intransitive frame is clearly irreflexive.

It is not difficult to see that the definition of the truth-relation for the modal operators can be generalized as follows:

Proposition 3.1 *For every $n \geq 0$,*

$$(\mathfrak{M}, x) \models \Box^n\psi \text{ iff } (\mathfrak{M}, y) \models \psi \text{ for all } y \in x\uparrow^n,$$
$$(\mathfrak{M}, x) \models \Diamond^n\psi \text{ iff } (\mathfrak{M}, y) \models \psi \text{ for some } y \in x\uparrow^n.$$

It follows that if xR^ny does not hold for any point y in a frame $\mathfrak{F}$, i.e., $x\uparrow^n = \emptyset$, then $(\mathfrak{F}, x) \models \Box^n\varphi$ and $(\mathfrak{F}, x) \not\models \Diamond^n\varphi$, for every formula φ. In particular, "everything is necessary" and "nothing is possible" at a point without successors. Such a point is called a *dead end.*

The notions of *subframe* and *submodel* are defined as in the intuitionistic case. Each non-empty set X of points in a frame $\mathfrak{F}$ determines in the unique way the

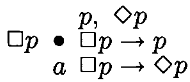
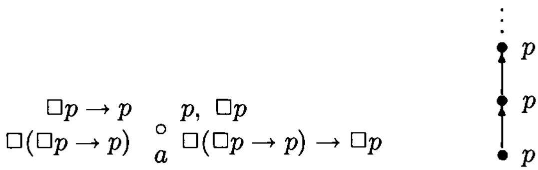


FIG. 3.1.



(a) (b)

FIG. 3.2.

subframe of $\mathfrak{F}$ and the submodel of $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ with the set of worlds X; they are called the *subframe* and the *submodel induced* by X.

Proposition 3.2 *Let $\mathfrak{M}$ be a model, x a point in $\mathfrak{M}$, $n \geq 0$ and $\mathfrak{N}$ the sumbodel of $\mathfrak{M}$ induced by the set $x{\uparrow}^0 \cup \ldots \cup x{\uparrow}^n$. Then, for every formula φ with $md(\varphi) \leq n$,*

$$(\mathfrak{M}, x) \models \varphi \text{ iff } (\mathfrak{N}, x) \models \varphi.$$

Proof By induction on the construction of φ with $md(\varphi) \leq n$. The basis of induction and the cases of $\varphi = \psi \odot \chi$, for $\odot \in \{\wedge, \vee, \to\}$, are trivial.

Suppose that $\varphi = \Box\psi$. Then $(\mathfrak{M}, x) \not\models \Box\psi$ iff there is y in $\mathfrak{M}$ such that $y \in x{\uparrow}$ and $(\mathfrak{M}, y) \not\models \psi$. On the other hand, $(\mathfrak{N}, x) \not\models \Box\psi$ iff there is y in $\mathfrak{N}$ such that $y \in x{\uparrow}$ and $(\mathfrak{N}, y) \not\models \psi$. Construct the submodel $\mathfrak{N}'$ of $\mathfrak{M}$ (or $\mathfrak{N}$) induced by the set $y{\uparrow}^0 \cup \ldots \cup y{\uparrow}^{n-1}$. Since $md(\psi) \leq n-1$, by the induction hypothesis we have $(\mathfrak{M}, y) \not\models \psi$ iff $(\mathfrak{N}', y) \not\models \psi$ iff $(\mathfrak{N}, y) \not\models \psi$. Therefore, $(\mathfrak{M}, x) \not\models \Box\psi$ iff $(\mathfrak{N}, x) \not\models \Box\psi$. ❑

Drawing a frame $\mathfrak{F} = \langle W, R \rangle$ in the form of diagram, we will represent irreflexive points in $\mathfrak{F}$ by bullets $\bullet$ and reflexive ones by circles $\circ$ (in the intuitionistic frames all points were reflexive). We draw an arrow from x to y if $x \neq y$ and xRy. Unless otherwise stated, the frames represented by diagrams are assumed to be transitive. In such cases we do not draw an arrow from x to z if there are arrows from x to y and from y to z. In the diagrams of *nontransitive* frames all arrows are shown explicitly.

When depicting models, alongside their points we shall sometimes write formulas: those that are true at a point are written to the left of it and those that are false to the right.

Example 3.3 Let $\mathfrak{F} = \langle W, R \rangle$ be the frame consisting of a single irreflexive point a, i.e., $W = \{a\}$, $R = \emptyset$, and let $\mathfrak{V}(p) = \emptyset$. Then both $\Box p \to p$ and $\Box p \to \Diamond p$ are false at a under $\mathfrak{V}$, since $a \models \Box p$, $a \not\models p$ and $a \not\models \Diamond p$. This situation is shown graphically in Fig. 3.1.

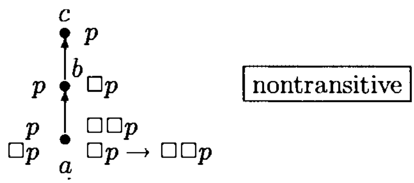


FIG. 3.3.

Example 3.4 Suppose now that $\mathfrak{F} = \langle W, R\rangle$ consists of a single reflexive point a, i.e., $W = \{a\}$, $R = \{\langle a, a\rangle\}$, and let again $\mathfrak{V}(p) = \emptyset$. Then the formulas $\Box p \to p$ and $\Box p \to \Diamond p$ are true at a under $\mathfrak{V}$, while the formula

$$\boldsymbol{la} = \Box(\Box p \to p) \to \Box p,$$

known as the *Löb formula* (or *axiom*), is false (see Fig. 3.2 (a)). $\boldsymbol{la}$ is false also at every point in the (transitive) model shown in Fig. 3.2 (b) and consisting of a strictly ascending chain of irreflexive points. (The definition of ascending and descending chains remains the same as in the intuitionistic case.)

Example 3.5 Now consider the *intransitive* frame $\mathfrak{F} = \langle W, R\rangle$ in Fig. 3.3, i.e., $W = \{a, b, c\}$, $R = \{\langle a, b\rangle, \langle b, c\rangle\}$ (a does not see c!), and put, as shown in Fig. 3.3, $\mathfrak{V}(p) = \{a, b\}$. Then it is easy to see that the formula $\Box p \to \Box\Box p$ is false at a. Notice that by replacing $\bullet$ in Fig. 3.3 with $\circ$, i.e., by taking $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\}$ we again obtain a countermodel for that formula. However, this will not be the case if we take the transitive closure of the depicted accessibilities. For then we shall have aRc from which $a \not\models \Box p$.

An important property of models built upon transitive frames is the following:

Proposition 3.6 *Suppose $\mathfrak{M}$ is a model on a transitive frame. Then for every point x in $\mathfrak{M}$ and every formula φ,*
(i) *$(\mathfrak{M}, x) \models \Box\varphi$ implies $(\mathfrak{M}, y) \models \Box\varphi$, for every $y \in x\uparrow$;*
(ii) *$(\mathfrak{M}, x) \models \Diamond\varphi$ implies $(\mathfrak{M}, y) \models \Diamond\varphi$, for every $y \in x\downarrow$.*

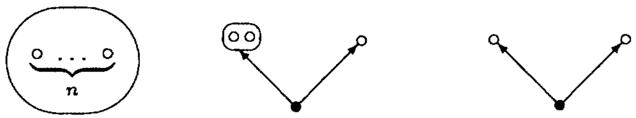
Proof (i) If we assume that $(\mathfrak{M}, y) \not\models \Box\varphi$ for some $y \in x\uparrow$ then there is $z \in y\uparrow$ such that $(\mathfrak{M}, z) \not\models \varphi$. Since $\mathfrak{F}$ is transitive, we must then have $z \in x\uparrow$, contrary to $x \models \Box\varphi$.

(ii) is proved analogously. ❑

It follows that in a transitive model exactly the same formulas of the form $\Box\varphi$ or $\Diamond\varphi$ are true at the points which see each other. We introduce for such points a special terminology.

Let $\mathfrak{F} = \langle W, R\rangle$ be a transitive frame. Define on W an equivalence relation $\sim$ by taking, for every $x, y \in W$,

$$x \sim y \text{ iff either } x = y \text{ or } xRy \text{ and } yRx.$$



(a) (b) (c)

FIG. 3.4.

The equivalence classes with respect to $\sim$ are called *clusters*. The cluster containing a point x will be denoted by $C(x)$. In other words, $C(x)$ contains x and all those points in $\mathfrak{F}$ that are seen from x and see x themselves.

Proposition 3.7 *Suppose x is a point in a model $\mathfrak{M}$ built on a transitive frame and φ an arbitrary formula. Then for every $y \in C(x)$,*

$$(\mathfrak{M}, x) \models \Box\varphi \textit{ iff } (\mathfrak{M}, y) \models \Box\varphi,$$

$$(\mathfrak{M}, x) \models \Diamond\varphi \textit{ iff } (\mathfrak{M}, y) \models \Diamond\varphi.$$

Proof Follows from Proposition 3.6. ❑

The quotient frame of a transitive frame $\mathfrak{F}$ with respect to $\sim$, that is the frame $\langle W/_\sim, R/_\sim \rangle$, where

$$W/_\sim = \{C(x) : \ x \in W\}$$

and

$$C(x)R/_\sim C(y) \text{ iff } xRy,$$

is called the *skeleton* of $\mathfrak{F}$ and denoted by $\rho\mathfrak{F} = \langle \rho W, \rho R \rangle$. It should be clear that the skeleton $\rho\mathfrak{F}$ of every frame $\mathfrak{F}$ is antisymmetric, and if R is reflexive then $\rho\mathfrak{F}$ is partially ordered by ρR. A reflexive and transitive binary relation is called a *quasi-order* (or *preorder*).

We distinguish three types of clusters:

- a *degenerate cluster* consisting of a single irreflexive point,
- a *simple cluster* consisting of a single reflexive point, and
- a *proper cluster* containing at least two (reflexive) points.

We will represent a proper n-point cluster, for $2 \leq n \leq \omega$, as n or as shown in Fig. 3.4 (a). Fig. 3.4 (b) and (c) show a frame with all these types of clusters and its skeleton, respectively.

Example 3.8 Let $\mathfrak{F} = \langle W, R \rangle$ be the frame shown in Fig. 3.5, i.e., $W = \{a_1, a_2\}$, $R = \{\langle a_i, a_j \rangle : \ i, j = 1, 2\}$, and let $\mathfrak{V}(p) = \{a_1\}$. Then the formula

$$\boldsymbol{ma} = \Box\Diamond p \to \Diamond\Box p,$$

known as the *McKinsey formula* (or *axiom*), is false under $\mathfrak{V}$ at both a_1 and a_2.

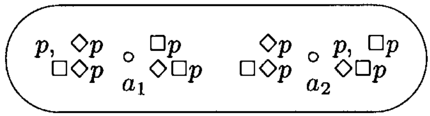


FIG. 3.5.

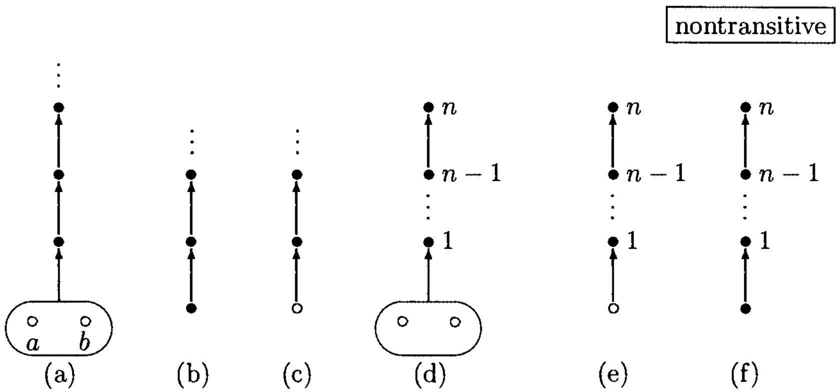


FIG. 3.6.

And finally, we define the *modal logic* $\mathbf{K}_{\mathcal{ML}}$ in the language $\mathcal{ML}$ as the set of all $\mathcal{ML}$-formulas that are valid in all modal Kripke frames, i.e.,

$$\mathbf{K}_{\mathcal{ML}} = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \mathfrak{F} \models \varphi, \ \text{for all frames } \mathfrak{F}\}.$$

As before, we drop the subscript $\mathcal{ML}$ and write, when understood, simply $\mathbf{K}$. It follows from the given definition that

$$\mathbf{Cl}_{\mathcal{L}} \subset \mathbf{K}_{\mathcal{ML}}.$$

Once again we emphasize that the operator $\Box$ in $\mathbf{K}$ should not be understood as some meaningful necessity. From the set-theoretic point of view $\mathbf{K}$ is the minimal logic among all those modal logics that are considered in this book. In Section 3.8 we shall construct modal logics for various meaningful interpretations of $\Box$ by adding new formulas to $\mathbf{K}$ which convey specific traits of these interpretations.

3.3 Truth-preserving operations

The definitions of the truth-preserving operations—generating subframes, reduction and disjoint union—which were introduced in Section 2.3 may be used without any changes in the modal case as well. To refresh them in mind, we just give some examples displaying specific features of modal frames.

Example 3.9 Let us consider once again the intransitive frame in Fig. 3.3. We have: $a{\uparrow}= \{b\}$, although the upward closure of $\{a\}$—the minimal set to contain a and all successors of its points—is $\{a, b, c\}$; $a{\downarrow}= \emptyset$.

This example shows that $\uparrow$ and $\downarrow$ are not upward and downward closure operations in nontransitive or irreflexive frames. So we generalize them as follows. For a frame $\mathfrak{F} = \langle W, R\rangle$ and $X \subseteq W$, we put

$$X{\uparrow}^{\omega}= \bigcup_{n\geq 1} X{\uparrow}^{n},\ X{\downarrow}_{\omega}= \bigcup_{n\geq 1} X{\downarrow}_{n},$$

$$X\underline{\uparrow}^{\xi} = X \cup X{\uparrow}^{\xi},\ X\overline{\downarrow}_{\xi} = X \cup X{\downarrow}_{\xi},\ \text{for } 1 \leq \xi \leq \omega.$$

Using this notation, we can now represent the upward closed set generated by X in $\mathfrak{F}$ as $X\underline{\uparrow}^{\omega}$. A point x is a *root* of $\mathfrak{F}$ if the subframe of $\mathfrak{F}$ generated by x is $\mathfrak{F}$ itself. Notice also that the cluster $C(x)$ generated by a point x is $x\underline{\uparrow} \cap x\overline{\downarrow}$. If $\mathfrak{F}$ is transitive, we say x is a *final point* and $C(x)$ a *final cluster* in X if $x\underline{\uparrow} \cap X = C(x) \cap X$; x is a *last point* and $C(x)$ the *last cluster* in X if $X \subseteq x\overline{\downarrow}$. A set $X \subseteq W$ is called a *cover* for a set $Y \subseteq W$ if $Y \subseteq X\overline{\downarrow}$.

Example 3.10 Let $\mathfrak{F} = \langle W, R\rangle$ be the (nontransitive) frame depicted in Fig. 3.6 (a). $\mathfrak{F}$ is generated by a as well as by b; so $\mathfrak{F}$ is rooted, with both a and b being its roots. All rooted subframes of $\mathfrak{F}$ (modulo isomorphism) are of the form shown in Fig. 3.6 (a)–(f), for $n \geq 0$. The disjoint union of (e) and (f) gives an example of $\mathfrak{F}$'s subframe without a root. The frames (a) and (b) are the only generated subframes of $\mathfrak{F}$.

Theorem 3.11. (Generation) *If $\mathfrak{N}$ is a generated submodel of $\mathfrak{M}$ then, for every point x in $\mathfrak{N}$ and every modal formula φ,*

$$(\mathfrak{N}, x) \models \varphi \textit{ iff } (\mathfrak{M}, x) \models \varphi.$$

Proof We leave the proof, which is similar to that of Theorem 2.7 or Proposition 3.2, to the reader as an exercise. ❑

Corollary 3.12 *If $\mathfrak{G} \subsetneq \mathfrak{F}$ then, for every x in $\mathfrak{G}$ and every formula φ,*
(i) *$(\mathfrak{G}, x) \models \varphi$ iff $(\mathfrak{F}, x) \models \varphi$;*
(ii) *$\mathfrak{F} \models \varphi$ implies $\mathfrak{G} \models \varphi$.*

Corollary 3.13 $\mathbf{K} = \{\varphi \in \mathbf{For}\mathcal{ML} :\ \mathfrak{F} \models \varphi$ *for all rooted frames* $\mathfrak{F}\}$.

Example 3.14 The transitive irreflexive frame shown in Fig. 3.7 (a) is reduced to the frame (b) by the map $f(n) = \mathrm{mod}_2(n)$ and to the frame (c) by the map $g(n) = 0$, for all n. However, if we do not presuppose taking the transitive closure, then (a) is not reducible to (b), but is reducible to (c). It is worth noting also that, by (R1), no reflexive point can be mapped by a reduction to an irreflexive one.

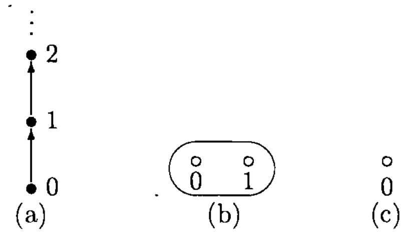


FIG. 3.7.

Theorem 3.15. (Reduction) *If f is a reduction of a model $\mathfrak{M}$ to a model $\mathfrak{N}$ then, for every point x in $\mathfrak{M}$ and every formula φ,*

$$(\mathfrak{M}, x) \models \varphi \textit{ iff } (\mathfrak{N}, f(x)) \models \varphi.$$

Proof Similar to the proof of Theorem 2.15. ❑

Corollary 3.16 *If f is a reduction of $\mathfrak{F}$ to $\mathfrak{G}$ then, for every point x in $\mathfrak{F}$ and every modal formula φ,*
(i) $(\mathfrak{F}, x) \models \varphi$ *implies* $(\mathfrak{G}, f(x)) \models \varphi$*;*
(ii) $\mathfrak{F} \models \varphi$ *implies* $\mathfrak{G} \models \varphi$*.*

Propositions 2.11, 2.13 and 2.14 remain true for the modal case as well.

Proposition 3.17 *If a modal formula φ is valid in some frame then it is also valid either in $\circ$ or in $\bullet$.*

Proof Suppose $\mathfrak{F} \models \varphi$. If $\mathfrak{F}$ contains a dead end, say x, then $\langle \{x\}, \emptyset \rangle$, i.e. $\bullet$, is a generated subframe of $\mathfrak{F}$ which, by Corollary 3.12, validates φ. Otherwise every point in $\mathfrak{F}$ has a successor and so the map f defined by

$$f(x) = \circ, \text{ for all points } x \text{ in } \mathfrak{F},$$

is a reduction of $\mathfrak{F}$ to the frame $\circ$. By Corollary 3.16, this means that $\circ$ validates φ. ❑

A modal frame $\mathfrak{F} = \langle W, R \rangle$ is called a *tree* if the *reflexive and transitive closure* R^* of R (i.e., for every $x, y \in W$, xR^*y iff $y \in x\!\uparrow^{\omega}$) is a tree partial order on W. For instance, the frames in Fig. 3.6 (b), (c), (e) and (f) are trees, while those in (a) and (d) are not, since both of them contain a proper cluster. A transitive frame $\mathfrak{F}$ is a *tree of clusters* (or *quasi-tree*) if its skeleton $\rho\mathfrak{F}$ is a tree.

Slightly modifying the proof of Theorem 2.19 we obtain

Theorem 3.18 *Every rooted modal frame $\mathfrak{G} = \langle V, S \rangle$ is a reduct of some intransitive tree.*

Proof Suppose v_0 is a root of $\mathfrak{G}$. Define a set W and a relation R on W by taking

$$W = \{\langle v_0, v_1, \ldots, v_{n-1}, v_n \rangle : \ v_0 S v_1 S \ldots S v_{n-1} S v_n\},$$

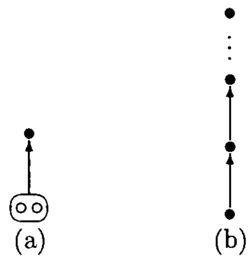


FIG. 3.8.

$$\langle v_0, \dots, v_n \rangle \, R \, \langle u_0, \dots, u_m \rangle \text{ iff } m = n+1 \text{ and } v_i = u_i, \text{ for } i = 0, \dots, n.$$

Clearly $\mathfrak{F} = \langle W, R \rangle$ is an intransitive frame with root $\langle v_0 \rangle$. Moreover, $\mathfrak{F}$ is a tree, since $\langle v_0, v_1, \dots, v_n \rangle \overline{\downarrow}_\omega$ is finite and linearly ordered by the transitive and reflexive closure of R as follows:

$$\langle v_0 \rangle \, R \, \langle v_0, v_1 \rangle \, R \dots R \, \langle v_0, v_1, \dots, v_{n-1} \rangle \, R \, \langle v_0, v_1, \dots, v_{n-1}, v_n \rangle .$$

We leave to the reader to verify that the map f from W onto V defined by $f(\langle v_0, v_1, \dots, v_n \rangle) = v_n$ is a reduction of $\mathfrak{F}$ to $\mathfrak{G}$. ❑

Corollary 3.19 $\mathbf{K} = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \mathfrak{F} \models \varphi$ *for all intransitive trees* $\mathfrak{F}\}$.

Unlike the intuitionistic case, the tree $\mathfrak{F}$ constructed in the proof above may be infinite even if $\mathfrak{G}$ is finite. For example, suppose $\mathfrak{G}$ consists of only one reflexive point; then $\mathfrak{F}$ will have the form shown in Fig. 3.7 (a). However, for finite $\mathfrak{G}$, every point in the corresponding tree $\mathfrak{F}$ has finitely many successors.

The tree $\mathfrak{F}$ constructed in the proofs of Theorems 3.18 and 2.19 is said to be obtained by *unravelling* $\mathfrak{G}$. To "unravel" a transitive frame $\mathfrak{G}$ without loss of transitivity, we should take the transitive closure of R in $\mathfrak{F}$. There is also another technique of "flattening out" clusters in a transitive $\mathfrak{G}$, which is known as *bulldozing* $\mathfrak{G}$: roughly, each non-degenerate cluster in $\mathfrak{G}$ is "bulldozed" into an infinite ascending chain of irreflexive points. Recall that an irreflexive transitive binary relation is called a *strict partial order*. More exactly bulldozing is defined in the proof of the following:

Theorem 3.20. (Bulldozer) (i) *Every transitive frame is a reduct of some strictly partially ordered frame.*

(ii) *Every quasi-ordered frame is a reduct of some partially ordered frame.*

Proof (i) Suppose $\mathfrak{G} = \langle V, S \rangle$ is a transitive frame. With each point $x \in V$ we associate a set x^+ which is $\{\langle x, i \rangle : \ i = 0, 1, \dots\}$ if $C(x)$ is non-degenerate and the singleton $\{\langle x, 0 \rangle\}$ if $C(x)$ is degenerate. Let W be the union of all x^+. Fix some well-ordering $x_0, x_1, \dots, x_\xi, \dots$ of every cluster C in $\mathfrak{G}$ and define a relation R on W by taking

$$\langle x_\xi, i \rangle \, R \, \langle x_\zeta, j \rangle \text{ iff either } i < j \text{ or } \xi < \zeta \text{ and } i = j$$

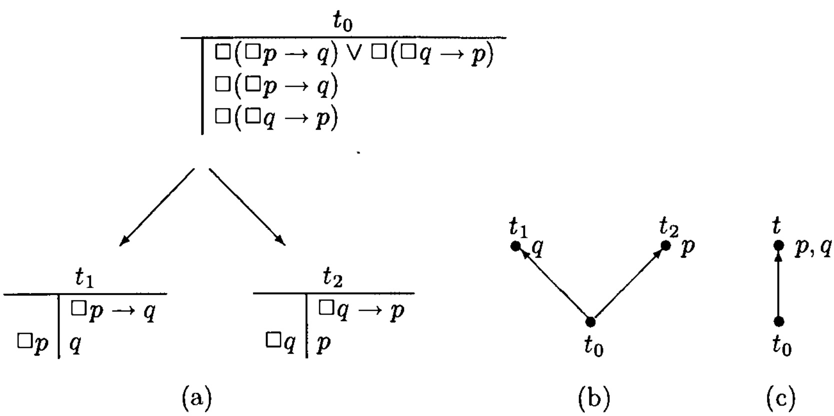


FIG. 3.9.

if $C(x_\xi) = C(x_\zeta)$ and, for distinct $C(x)$ and $C(y)$,

$$\langle x, i\rangle \, R \, \langle y, j\rangle \text{ iff } xSy.$$

It is easy to see that R is transitive and irreflexive, i.e., it is a strict partial order. We show that $\mathfrak{G}$ is a reduct of $\mathfrak{F} = \langle W, R\rangle$.

Define a map f from W onto V by taking $f(\langle x, i\rangle) = x$, for every $\langle x, i\rangle \in W$. By the definition, f is a map "onto" satisfying (R1). Suppose now that xSy. If $C(x) = C(y)$ then $\langle x, i\rangle \, R \, \langle y, i+1\rangle$, for every $i \geq 0$. And if x and y are in distinct clusters then $\langle x, i\rangle \, R \, \langle y, j\rangle$ for all possible i and j. Thus f satisfies (R2) and so is a reduction of $\mathfrak{F}$ to $\mathfrak{G}$.

(ii) The only difference from (i) is that in the definition of R we take $\leq$ instead of $<$. ❑

Observe, however, that the result of bulldozing is not in general a tree. For instance, by bulldozing the frame in Fig. 3.8 (a), we obtain the frame in Fig. 3.8 (b), which is not a tree, since an infinite ascending chain precedes its last point.

The disjoint union of modal frames behaves exactly like the disjoint union of intuitionistic ones.

3.4 Hintikka systems

In this section we extend the semantic tableau method to the modal case. As before, this method will not only provide us with a convenient tool for constructing countermodels but also help us proving the completeness theorem for the calculus K in Section 3.6. Again we begin with a few examples.

Example 3.21 Suppose that we want to construct a countermodel for the formula

$$\boldsymbol{sc} = \Box(\Box p \to q) \vee \Box(\Box q \to p).$$

t_0	
$\Box(\Box(p \to \Box p) \to p)$	$\Box(\Box(p \to \Box p) \to p) \to p$
	p

(a)

t_0
$\bullet\ p$

(b)

FIG. 3.10.

Then we form the tableau $t_0 = (\emptyset, \{\boldsymbol{sc}\})$. Its purpose is, as before, to describe the desirable distribution of the truth-values over (some) subformulas of $\boldsymbol{sc}$ in one world of the model to be constructed. By the saturation rule (SR4), we should add $\Box(\Box p \to q)$ and $\Box(\Box q \to p)$ to the right part of t_0. Recall now that a formula $\Box\psi$ is false at a point x iff there is a point y accessible from x at which ψ is false. So we form two new tableaux t_1 and t_2 and put to their right parts $\Box p \to q$ and $\Box q \to p$, respectively; we regard these tableaux as accessible from t_0. The only thing that is left to do is to apply to t_1 and t_2 rule (SR6), which again is correct, since $\to$ is classical. All steps of this construction are shown in Fig. 3.9 (a).

It is not hard to check that $\boldsymbol{sc}$ is refuted at the point t_0 in the model shown in Fig. 3.9 (b). Nothing prevents us from joining t_1 and t_2 into one tableau, say t, and then we obtain another countermodel for $\boldsymbol{sc}$ which is depicted in Fig. 3.9 (c). Observe that if we need a reflexive countermodel for $\boldsymbol{sc}$ then we must add p and q to the left parts of t_1 and t_2, respectively. However, this does not go through for the countermodel in Fig. 3.9 (c).

Example 3.22 Now let us use this method of constructing countermodels for the formula

$$\boldsymbol{grz} = \Box(\Box(p \to \Box p) \to p) \to p$$

which is known as the *Grzegorczyk formula* (or *axiom*). Only one application of rule (SR6) (see Fig. 3.10 (a)) yields the simplest countermodel for $\boldsymbol{grz}$ built upon the single-point irreflexive frame shown in Fig. 3.10 (b).

Example 3.23 Suppose, however, that we are interested only in *reflexive* countermodels for $\boldsymbol{grz}$. In this case t_0 in Fig. 3.10 must be self-accessible, and so we should put $\Box(p \to \Box p) \to p$ to its left part, which completely changes the matter. Indeed, after that we, in accordance with (SR5), put $\Box(p \to \Box p)$ to the right part of t_0 and then, to make this formula false at t_0, form a new tableau t_1 which is accessible from t_0 and contains $p \to \Box p$ in its right part. So p should be added to the left part of t_1 and $\Box p$ to the right one. But that is not enough: to ensure that $\Box(\Box(p \to \Box p) \to p)$ is true at t_0, we must put $\Box(p \to \Box p) \to p$ to the left part of t_1.

Our next step is to form a tableau t_2 accessible from t_1 and put p in its right part, which guarantees the falsity of $\Box p$ at t_1. Notice that t_0 does not see t_2. All these steps are shown in Fig. 3.11 (a). The reflexive nontransitive countermodel for $\boldsymbol{grz}$ corresponding to this tableau system is depicted in Fig. 3.11 (b).

t_0

$\Box(\Box(p \to \Box p) \to p)$	$\Box(\Box(p \to \Box p) \to p) \to p$
$\Box(p \to \Box p) \to p$	p
	$\Box(p \to \Box p)$

$\downarrow$

t_1

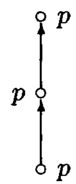
	$p \to \Box p$
p	$\Box p$
$\Box(p \to \Box p) \to p$	

$\downarrow$

t_2

	p

(a)



(b)

FIG. 3.11.

Example 3.24 Now suppose that we need a reflexive and transitive (i.e., quasi-ordered) countermodel for $\boldsymbol{grz}$. Then we should take the transitive closure of the accessibility relation between t_0, t_1 and t_2 in Fig. 3.11 and so, according to Proposition 3.6, copy the left part of t_0 to the left parts of t_1 and t_2. But then, by (SR5), $\Box(p \to \Box p)$ should be written in the right part of t_2, which actually returns us to the same situation as was in t_0. Thus we obtain an infinite sequence of tableaux $t_0 \to t_1 \to t_2 \to \dots$, in which t_{2i} is a copy of t_0 and t_{2i+1} a copy of t_1, for $i = 1, 2, \dots$. The reflexive and transitive countermodel corresponding to this tableau system is depicted in Fig. 3.12 (a).

We can avoid the infinite chain of alternating tableaux if instead of constructing t_2 we just draw an arrow from t_1 to t_0, thus getting a system of two tableaux seeing each other. The corresponding countermodel is shown in Fig. 3.12 (b). Observe that the map $f(t_i) = t_{\mathrm{mod}_2(i)}$ is a reduction of the model (a) to the model (b).

Now we present these considerations in a more formal way. A *tableau* in the language $\mathcal{ML}$ is any pair $t = (\Gamma, \Delta)$ of subsets of $\mathbf{For}\mathcal{ML}$. It is *saturated* if conditions (S1)–(S6) in Section 1.2 are satisfied; t is *disjoint* if $\Gamma \cap \Delta = \emptyset$ and $\bot \notin \Gamma$.

A *Hintikka system* in $\mathbf{K}$ is a pair $\mathfrak{H} = \langle T, S \rangle$, where T is a non-empty set of disjoint saturated tableaux and S a binary relation on T satisfying the following two conditions:

($\mathrm{HS}_M 1$) if $t = (\Gamma, \Delta)$, $t' = (\Gamma', \Delta')$ and tSt' then $\varphi \in \Gamma'$ for every $\Box\varphi \in \Gamma$;

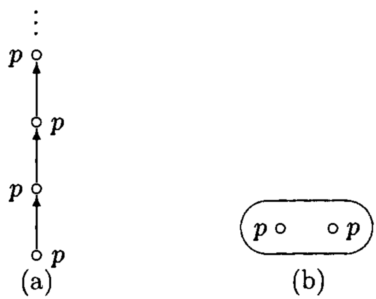


FIG. 3.12.

(HS$_M$2) if $t = (\Gamma, \Delta)$ and $\Box\varphi \in \Delta$ then there is $t' = (\Gamma', \Delta')$ in T such that tSt' and $\varphi \in \Delta'$.

Say that $\mathfrak{H}$ is a *Hintikka system for a tableau* t if $t \subseteq t'$ for some t' in $\mathfrak{H}$.

A tableau (Γ, Δ) is *realized* in (a point x of) a model $\mathfrak{M}$ if $(\mathfrak{M}, x) \models \varphi$, for every $\varphi \in \Gamma$, and $(\mathfrak{M}, x) \not\models \psi$, for every $\psi \in \Delta$. A tableau t is called *realizable* in **K** if it is realized in some model.

In the same way as was done in the proof of Proposition 2.31, given a Hintikka system $\mathfrak{H}$, one can construct a model based on the frame $\mathfrak{H}$ in which every point t realizes the tableau t and conversely, given a model $\mathfrak{M}$ realizing t, one can construct a Hintikka system for t. Thereby we obtain

Proposition 3.25 *A tableau t is realizable in* **K** *iff there is a Hintikka system for t.*

Corollary 3.26 *If $\mathfrak{H}$ is a Hintikka system for $(\emptyset, \{\varphi\})$ then $\mathfrak{H} \not\models \varphi$.*

To construct a Hintikka system for a tableau t, we can deal only with subformulas of formulas contained in t. This observation gives an upper bound for the number of tableaux in a minimal Hintikka system for t.

Theorem 3.27 *A tableau t is realizable in* **K** *iff there is a Hintikka system for t containing at most $2^{|\Sigma|}$ tableaux, where Σ is the set of all subformulas of the formulas in t.*

Proof ($\Rightarrow$) Suppose t is realized in $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. For every point x in the frame $\mathfrak{F} = \langle W, R \rangle$, we form a tableau $t_x = (\Gamma_x, \Delta_x)$ by taking

$$\Gamma_x = \{\varphi \in \Sigma : \ x \models \varphi\}, \ \Delta_x = \{\varphi \in \Sigma : \ x \not\models \varphi\}.$$

Let $\mathfrak{H} = \langle T, S \rangle$, where $T = \{t_x : \ x \in W\}$ and, for every $t_x = (\Gamma_x, \Delta_x)$ and $t_y = (\Gamma_y, \Delta_y)$ in T,

$$t_x S t_y \text{ iff } \Box\varphi \in \Gamma_x \text{ implies } \varphi \in \Gamma_y \text{ for all } \Box\varphi \in \Sigma.$$

This definition guarantees that (HS$_M$1) is satisfied. We show that (HS$_M$2) also holds.

Let $t_x = (\Gamma_x, \Delta_x)$ and $\Box\varphi \in \Delta_x$. Then $(\mathfrak{M}, x) \not\models \Box\varphi$ and so there is a point $y \in W$ such that xRy and $(\mathfrak{M}, y) \not\models \varphi$. By the definition of $\models$, $y \models \psi$ whenever $x \models \Box\psi$. So $t_x S t_y$ and φ belongs to the right part of t_y.

Thus, $\mathfrak{H}$ is a Hintikka system for t. It is also clear that the number of tableaux in T does not exceed the number of subsets in Σ.

($\Leftarrow$) follows from Proposition 3.25. ❑

Corollary 3.28 (i) *For every formula $\varphi \notin$ **K** there is a rooted frame refuting φ and containing at most $2^{|\mathbf{Sub}\varphi|}$ points.*

(ii) *Every $\varphi \notin$ **K** is refuted in some finite intransitive tree.*

Proof (i) follows from Theorem 3.27, since $(\emptyset, \{\varphi\})$ is realizable in **K**.

(ii) Take a finite frame $\mathfrak{H}$ refuting φ and apply to it the unravelling procedure described in the proof of Theorem 3.18, thus obtaining an intransitive tree $\mathfrak{F} = \langle W, R\rangle$ which is reducible to $\mathfrak{H}$ and so, by the reduction theorem, refutes φ. Although $\mathfrak{F}$ may be infinite, every point in it has finitely many successors.

Suppose $md(\varphi) = n$ and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is a model such that $\langle \mathfrak{M}, x\rangle \not\models \varphi$, for some point x. By Proposition 3.2, the submodel $\mathfrak{N}$ of $\mathfrak{M}$, induced by the set $x{\uparrow}^0 \cup \ldots \cup x{\uparrow}^n$, also refutes φ. It remains to notice that $\mathfrak{N}$ is based upon a finite intransitive tree. ❑

Corollary 3.29

$$\mathbf{K} = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \mathfrak{F} \models \varphi \textit{ for all finite intransitive trees } \mathfrak{F}\}.$$

So, by applying saturation rules (SR1)–(SR6) to each individual tableau and also rules

($\mathrm{SR}_M 7$) if $t = (\Gamma, \Delta)$ and $\Box\varphi \in \Gamma$ then add φ to the left part of every t' such that tSt';

($\mathrm{SR}_M 8$) if $t = (\Gamma, \Delta)$ and $\Box\varphi \in \Delta$ then either add φ to the right part of some tableau accessible from t or form a new tableau $t' = (\Gamma', \Delta')$ by taking $\Gamma' = \{\psi : \Box\psi \in \Gamma\}$, $\Delta' = \{\varphi\}$ and put tSt'

we can always construct a finite Hintikka system for each finite realizable tableau.

As an easy exercise we invite the reader to show that all the formulas in Table 3.1 are in **K**.

3.5 Modal frames and formulas

Having got some experience in constructing countermodels, let us try now to find characterizations of frames validating a number of important modal formulas we shall deal with in the sequel.

First we observe that Example 3.3 suggests the following:

Proposition 3.30 *A frame $\mathfrak{F}$ validates $\Box p \to p$ iff $\mathfrak{F}$ is reflexive.*

Proof If $\mathfrak{F} = \langle W, R\rangle$ is not reflexive then $\neg xRx$, for some $x \in W$. So we can put $\mathfrak{V}(p) = W - \{x\}$, which gives us $x \models \Box p$ and $x \not\models p$, whence $x \not\models \Box p \to p$.

Table 3.1 *A list of modal formulas in* **K**

$\Box^m(p_1 \wedge \ldots \wedge p_n) \leftrightarrow \Box^m p_1 \wedge \ldots \wedge \Box^m p_n,$	for $n \geq 0,\ m \geq 0$
$\Diamond^m(p_1 \vee \ldots \vee p_n) \leftrightarrow \Diamond^m p_1 \vee \ldots \vee \Diamond^m p_n,$	for $n \geq 0,\ m \geq 0$
$\Box^m p_1 \vee \ldots \vee \Box^m p_n \to \Box^m(p_1 \vee \ldots \vee p_n),$	for $n \geq 0,\ m \geq 0$
$\Diamond^m(p_1 \wedge \ldots \wedge p_n) \to \Diamond^m p_1 \wedge \ldots \wedge \Diamond^m p_n,$	for $n \geq 0,\ m \geq 0$
$\Box^m(p \to q) \to (\Box^m p \to \Box^m q),$	for $m \geq 0$
$\Box^m(p \to q) \to (\Diamond^m p \to \Diamond^m q),$	for $m \geq 0$
$\Box^m p \wedge \Diamond^m q \to \Diamond^m(p \wedge q),$	for $m \geq 0$
$\Box^n \bot \to \Box^m \bot,$	for $m \geq n$
$\Diamond^m \top \to \Diamond^n \top,$	for $m \geq n$
$\top \leftrightarrow \Box^m \top,$	for $m \geq 0$
$\bot \leftrightarrow \Diamond^m \bot,$	for $m \geq 0$
$(\Box\Diamond p \to \Diamond\Box p) \leftrightarrow \Diamond(\Diamond p \to \Box p)$	

Conversely, if $\mathfrak{F}$ is reflexive then $\mathfrak{F} \models \Box p \to p$, for otherwise there is a model $\mathfrak{M}$ on $\mathfrak{F}$ such that $(\mathfrak{M}, x) \models \Box p$ and $(\mathfrak{M}, x) \not\models p$, for some $x \in W$; but since xRx, we must also have $(\mathfrak{M}, x) \models p$, which is a contradiction. ❑

Likewise, Example 3.5 suggests

Proposition 3.31 *$\mathfrak{F}$ validates $\Box p \to \Box\Box p$ iff $\mathfrak{F}$ is transitive.*

Proof Exercise. ❑

We will denote the formulas $\Box p \to p$ and $\Box p \to \Box\Box p$ by ***re*** and ***tra***, respectively.

Let us consider now the formula $p \to \Box\Diamond p$ and suppose that $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a countermodel for it based on a frame $\mathfrak{F} = \langle W, R \rangle$. Then $x \models p$ and $x \not\models \Box\Diamond p$, for some $x \in W$, and so there is a successor y of x such that $y \not\models \Diamond p$. Observe also that $x \models p$ and $y \not\models \Diamond p$ imply $\neg yRx$.

Thus, a necessary condition for $\mathfrak{F} \not\models p \to \Box\Diamond p$ is $\exists x, y(xRy \wedge \neg yRx)$ and so a sufficient condition for the validity of $p \to \Box\Diamond p$ in $\mathfrak{F}$ is

$$\forall x, y\ (xRy \to yRx).$$

A frame $\mathfrak{F}$ satisfying this condition is called *symmetric*.

Proposition 3.32 *$\mathfrak{F}$ validates **sym** $= p \to \Box\Diamond p$ iff $\mathfrak{F}$ is symmetric.*

Proof Only ($\Rightarrow$) requires a proof. If $\mathfrak{F} = \langle W, R \rangle$ is not symmetric then there are $x, y \in W$ such that xRy and $\neg yRx$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = \{x\}$. Then we have $x \models p$, $y \not\models \Diamond p$, whence $x \not\models \Box\Diamond p$ and $x \not\models p \to \Box\Diamond p$. ❑

Our next example is the formula $\Box p \to \Diamond p$. Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be its countermodel on a frame $\mathfrak{F} = \langle W, R \rangle$. Then $x \models \Box p$ and $x \not\models \Diamond p$ for some $x \in W$. The only conclusion we can derive from this piece of information is that x is a dead end in $\mathfrak{F}$, for if xRy for some $y \in W$ then $y \models p$ and $y \not\models p$, which is a contradiction. Therefore, a necessary condition for $\mathfrak{F} \not\models \Box p \to \Diamond p$ is $\exists x \forall y \neg xRy$. And a sufficient condition for $\mathfrak{F} \models \Box p \to \Diamond p$ is then the *seriality* condition $\forall x \exists y\ xRy$.

Proposition 3.33 *$\mathfrak{F}$ validates* $\boldsymbol{ser} = \Box p \to \Diamond p$ *iff $\mathfrak{F}$ is serial.*

Proof Exercise. ❑

Let us consider now the family of formulas of the form

$$\boldsymbol{ga}_{klmn} = \Diamond^k \Box^l p \to \Box^m \Diamond^n p,$$

where k, l, m, n are arbitrary natural numbers, possibly equal to 0. All formulas we have already dealt with in this section are in this family.

Suppose $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a countermodel for $\Diamond^k \Box^l p \to \Box^m \Diamond^n p$, i.e., $x \models \Diamond^k \Box^l p$ and $x \not\models \Box^m \Diamond^n p$ for some x in $\mathfrak{F} = \langle W, R \rangle$. By Proposition 3.1, there are $y, z \in W$ such that $xR^k y$, $y \models \Box^l p$ and $xR^m z$, $z \not\models \Diamond^n p$. Notice also that $y \models \Box^l p$ and $z \not\models \Diamond^n p$ tell us that there is no point u in $\mathfrak{F}$ which is accessible from y by l steps and from z by n steps. These observations lead to the following

Proposition 3.34 *$\mathfrak{F} = \langle W, R \rangle$ validates $\boldsymbol{ga}_{klmn}$ iff*

$$\forall x, y, z\ (xR^k y \land xR^m z \to \exists u\ (yR^l u \land zR^n u)).$$

Proof Again, only ($\Rightarrow$) requires a proof. Suppose otherwise. Then there are $x, y, z \in W$ such that $xR^k y$, $xR^m z$ and for every $u \in W$, either $\neg yR^l u$ or $\neg zR^n u$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = \{v \in W : yR^l v\}$ and show that $x \models \Diamond^k \Box^l p$ and $x \not\models \Box^m \Diamond^n p$. Indeed, by Proposition 3.1, $y \models \Box^l p$ and, since there is no point $u \in W$ for which $u \models p$ and $zR^n u$, we have $z \not\models \Diamond^n p$, whence $x \models \Diamond^k \Box^l p$ and $x \not\models \Box^m \Diamond^n p$. ❑

Even more extensive families of formulas can be found in Exercise 3.22 and Section 10.3. Propositions 3.30–3.33 are just special cases of Proposition 3.34. Here are a few more useful consequences.

Call a frame $\mathfrak{F} = \langle W, R \rangle$ *n-transitive* if

$$\forall x, y\ (xR^{n+1} y \to xRy \lor xR^2 y \lor \ldots \lor xR^n y),$$

which is read: if it is possible to reach y from x by $n+1$ steps then one can do this by $\leq n$ steps as well. 1-transitivity is nothing else but the standard transitivity.

Corollary 3.35 *$\mathfrak{F}$ validates* $\boldsymbol{tra}_n = \bigwedge_{i=0}^{n} \Box^i p \to \Box^{n+1} p$ *iff $\mathfrak{F}$ is n-transitive.*

A frame $\mathfrak{F} = \langle W, R \rangle$ is said to be *dense* if $\forall x, y\ (xRy \to xR^2 y)$. More generally, $\mathfrak{F}$ is *n-dense* if $\forall x, y\ (xR^n y \to xR^{n+1} y)$.

Corollary 3.36 *$\mathfrak{F}$ validates* $\boldsymbol{den_n} = \Box^{n+1}p \to \Box^n p$ *iff $\mathfrak{F}$ is n-dense.*

A frame $\mathfrak{F}$ is called *Euclidean* if $\forall x, y, z\ (xRy \wedge xRz \to yRz)$.

Corollary 3.37 *$\mathfrak{F}$ validates* $\boldsymbol{euc} = \Diamond\Box p \to \Box p$ *iff $\mathfrak{F}$ is Euclidean.*

Corollary 3.38 $\mathfrak{F} \models \Diamond\Box p \to \Box\Diamond p$ *iff $\mathfrak{F}$ is strongly directed.*

Thus, the formula

$$\boldsymbol{ga} = \Diamond\Box p \to \Box\Diamond p,$$

known as the *Geach formula* (or *axiom*), is similar to the weak law of the excluded middle in **Int**. However, this analogy is not completely perfect. For reflexive frames the condition of strong directedness is equivalent to the *directedness* condition

$$\forall x, y, z\ (xRy \wedge xRz \wedge y \neq z \to \exists u(yRu \wedge zRu)),$$

which in general is weaker. For example, the two-point irreflexive frame is directed but not strongly directed.

A simple modification $\boldsymbol{dir} = \Diamond(\Box p \wedge q) \to \Box(\Diamond p \vee q)$ of $\boldsymbol{ga}$ is valid in directed frames and only in them.

Proposition 3.39 $\mathfrak{F} \models \boldsymbol{dir}$ *iff $\mathfrak{F}$ is directed.*

Proof Exercise. ❑

A similar situation is with the condition of strong connectedness, which in the intuitionistic case corresponds to $\boldsymbol{da}$.

Proposition 3.40 *$\mathfrak{F}$ validates* $\boldsymbol{sc} = \Box(\Box p \to q) \vee \Box(\Box q \to p)$ *iff $\mathfrak{F}$ is strongly connected.*

Proof Exercise. ❑

For reflexive frames the condition of strong connectedness is equivalent to that of *connectedness*

$$\forall x, y, z\ (xRy \wedge xRz \wedge y \neq z \to yRz \vee zRy),$$

which is weaker in general.

Proposition 3.41 *$\mathfrak{F}$ validates* $\boldsymbol{con} = \Box(p \wedge \Box p \to q) \vee \Box(q \wedge \Box q \to p)$ *iff $\mathfrak{F}$ is connected.*

Proof Exercise. ❑

The connectedness of a frame means that no point in it has two distinct successors which do not see each other. Let, for $n \geq 1$,

$$\boldsymbol{bw_n} = \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leq i \neq j \leq n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j)).$$

Proposition 3.42 *$\mathfrak{F}$ validates $\boldsymbol{bw}_n$ iff each point in $\mathfrak{F}$ has at most n successors which do not see each other.*

Proof Exercise. ❑

Using the notion of width, defined in Section 2.5, for transitive frames this result can be formulated similar to Proposition 2.39.

Corollary 3.43 *A transitive frame $\mathfrak{F}$ validates $\boldsymbol{bw}_n$ iff every rooted subframe of $\mathfrak{F}$ is of width $\leq n$.*

The notion of depth loses its initial meaning when arbitrary modal frames are considered, since they may contain circles of the form

$$x_0Rx_1R\dots Rx_nRx_0.$$

However, if we restrict ourselves only to transitive frames then the depth of a frame $\mathfrak{F}$ can be defined as the depth of its skeleton $\rho\mathfrak{F}$ in the sense of Section 2.5.

So we say that a transitive frame $\mathfrak{F} = \langle W, R\rangle$ is of *depth* n, $d(\mathfrak{F}) = n$, if there is a chain $x_1Rx_2R\dots Rx_n$ of points from distinct clusters in $\mathfrak{F}$ (which means that $\neg x_{i+1}Rx_i$) and there is no chain of greater length satisfying this condition. $\mathfrak{F}$ is *of depth* ∞ if, for every $n < \omega$, it contains a chain of n points belonging to distinct clusters.

As modal analogues of the formulas $\boldsymbol{bd}_n$ "restricting" the depth of intuitionistic frames we can take the modal formulas $\boldsymbol{bd}_n$ which are defined as follows:

$$\boldsymbol{bd}_1 = \Diamond\Box p_1 \to p_1,$$

$$\boldsymbol{bd}_{n+1} = \Diamond(\Box p_{n+1} \wedge \neg\boldsymbol{bd}_n) \to p_{n+1}.$$

Proposition 3.44 *A transitive frame $\mathfrak{F}$ validates $\boldsymbol{bd}_n$ iff $d(\mathfrak{F}) \leq n$.*

Proof ($\Rightarrow$) is proved by induction on n. The basis of induction follows from Proposition 3.34. Suppose $\mathfrak{F} = \langle W, R\rangle$ is of depth $n+1$. Then there is a chain $x_0Rx_1R\dots Rx_n$ of points from distinct clusters in $\mathfrak{F}$. Consider the subframe $\mathfrak{G} = \langle V, S\rangle$ of $\mathfrak{F}$ generated by x_1. Then $\mathfrak{G}$ is of depth n and by the induction hypothesis, there is a model $\mathfrak{N} = \langle\mathfrak{G}, \mathfrak{U}\rangle$ such that $x_1 \not\models \boldsymbol{bd}_{n-1}$. Without loss of generality we may assume that $\mathfrak{U}(p_n) = V$. Define a valuation $\mathfrak{V}$ on $\mathfrak{F}$ by taking $\mathfrak{V}(p_n) = W - \{x_0\}$ and $\mathfrak{V}(p_i) = \mathfrak{U}(p_i)$, for $1 \leq i < n$. Let $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$. Then $\mathfrak{N}$ is a generated submodel of $\mathfrak{M}$ and, by the generation theorem, $(\mathfrak{M}, x_1) \not\models \boldsymbol{bd}_{n-1}$, $(\mathfrak{M}, x_1) \models \Box p_n$, whence $(\mathfrak{M}, x_1) \models \Box p_n \wedge \neg\boldsymbol{bd}_{n-1}$ and so $(\mathfrak{M}, x_0) \models \Diamond(\Box p_n \wedge \neg\boldsymbol{bd}_{n-1})$. It remains to recall that $x_0 \not\models p_n$, which gives us $x_0 \not\models \boldsymbol{bd}_n$.

($\Leftarrow$) An easy induction is left to the reader. ❑

The following formulas are similar to the intuitionistic formulas $\boldsymbol{bc}_n$ bounding the cardinality of rooted frames (see Section 2.5):

$$\boldsymbol{alt}_n = \Box p_1 \vee \Box(p_1 \to p_2) \vee \ldots \vee \Box(p_1 \wedge \ldots \wedge p_n \to p_{n+1}), \quad n \geq 0.$$

However in the modal case frames may be nontransitive, and so $\boldsymbol{alt}_n$ bounds only the number of alternatives of every point in a frame validating it.

Proposition 3.45 *A frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{alt}_n$ iff each point in $\mathfrak{F}$ has at most n distinct alternatives, i.e.,*

$$\forall x, x_1, \ldots, x_{n+1} \ (\bigwedge_{i=1}^{n+1} xRx_i \to \bigvee_{i \neq j} x_i = x_j).$$

Proof Exercise. ❑

It is much more difficult to characterize frames for the conversion of the Geach formula, i.e., for the McKinsey formula $\boldsymbol{ma} = \Box\Diamond p \to \Diamond\Box p$. We shall get a characterization of only *transitive* frames validating $\boldsymbol{ma}$.

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ be a countermodel for $\boldsymbol{ma}$ on a transitive frame $\mathfrak{F} = \langle W, R\rangle$, i.e., $x_0 \models \Box\Diamond p$ and $x_0 \not\models \Diamond\Box p$ for some $x_0 \in W$. Then either x_0 is a dead end or there is a point $x_1 \in W$ accessible from x_0 and so $x_1 \models \Diamond p$, $x_1 \not\models \Box p$. Hence there are successors x_2 and x_3 of x_1 such that $x_2 \models p$, $x_3 \not\models p$. Clearly, $x_2 \neq x_3$. Since x_2 and x_3 are accessible from x_0, we can apply to them the same argument as to x_1. As a result we arrive at the following necessary condition for the refutability of $\boldsymbol{ma}$:

$$\exists x \forall y \ (xRy \to \exists u, v \ (yRu \wedge yRv \wedge u \neq v)).$$

(The case when x_0 is a dead end is evidently covered by this condition.) Since R is transitive, the condition can be somewhat simplified:

$$\exists x \forall y \ (xRy \to \exists z \ (yRz \wedge y \neq z)).$$

Thus, a sufficient condition for the validity of $\boldsymbol{ma}$ in a transitive $\mathfrak{F}$ is the following *McKinsey condition:*

$$\forall x \exists y \ (xRy \wedge \forall z \ (yRz \to y = z)),$$

which can be read as: each point in $\mathfrak{F}$ sees a final simple cluster.

Proposition 3.46 *A transitive frame $\mathfrak{F}$ validates $\boldsymbol{ma}$ iff $\mathfrak{F}$ satisfies the McKinsey condition.*

Proof Only ($\Rightarrow$) requires a proof. Suppose the McKinsey condition does not hold in $\mathfrak{F} = \langle W, R\rangle$. Then there is a point $x \in W$ such that either it is a dead end or every successor of x has its own proper successor. In the former case $x \not\models \boldsymbol{ma}$ under any valuation in $\mathfrak{F}$. So let us consider the latter one. Using transfinite induction, we can choose a subset Y of $X = x\!\uparrow$ such that

$$\forall u \in X \exists v \in Y \exists w \in X - Y(uRv \wedge uRw).$$

Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = Y$. By the choice of Y, we must have $u \models \Diamond p$ and $u \not\models \Box p$, for all $u \in X$, whence $x \models \Box\Diamond p$ and $x \not\models \Diamond\Box p$, i.e., $x \not\models \Box\Diamond p \to \Diamond\Box p$. ❑

Now let us consider once again the Löb formula

$$\boldsymbol{la} = \Box(\Box p \to p) \to \Box p.$$

Suppose $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is a countermodel for $\boldsymbol{la}$ based on a frame $\mathfrak{F} = \langle W, R\rangle$, i.e., there is $x \in W$ such that $x \models \Box(\Box p \to p)$ and $x \not\models \Box p$. Then there exists a successor y of x for which $y \not\models p$, $y \models \Box p \to p$, and hence $y \not\models \Box p$. So we must have a successor z of y such that $z \not\models p$. (We emphasize that the points x, y, z are not necessarily distinct.) If x does not see z, then all we can say about $\mathfrak{F}$ is that it is not transitive. But if $\mathfrak{F}$ is transitive then $z \models \Box p \to p$, whence $z \not\models \Box p$, and we can apply to z the same argument as to y.

Thus, a necessary condition for $\mathfrak{F} \not\models \boldsymbol{la}$ is the nontransitivity of $\mathfrak{F}$ or the existence of an infinite ascending chain $x_0Rx_1R\dots$ of not necessarily distinct points in $\mathfrak{F}$. Taking the negation of this proposition, we obtain a sufficient condition for the validity of $\boldsymbol{la}$ in $\mathfrak{F}$: $\mathfrak{F}$ must be transitive, irreflexive and contain no infinite ascending chains. A frame without infinite strictly ascending chains is called *Noetherian.*

Proposition 3.47 *A frame validates* $\boldsymbol{la}$ *iff it is a Noetherian strict partial order.*

Proof Exercise. ❑

It is worth noting that unlike the other properties we met in this section, the absence of infinite ascending chains cannot be expressed by a first order condition on the accessibility relation. For details see Section 6.2.

We recommend the reader to analyze (using Examples 3.22–3.24) the constitution of countermodels for the Grzegorczyk formula and prove the following:

Proposition 3.48 *A frame validates* $\boldsymbol{grz}$ *iff it is a Noetherian partial order, i.e., iff it is reflexive, transitive, antisymmetric and contains no infinite ascending chains of distinct points.*

Proof Exercise. ❑

3.6 Calculus K

The *modal propositional calculus* K in the language $\mathcal{ML}$, which, as will be shown in this section, is sound and complete with respect to the possible world semantics, has the following axioms and inference rules.

Axioms: (A1)–(A10) of Cl (see Section 1.3) and one more proper modal axiom

(A11) $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)$;

Inference Rules: *modus ponens* (MP), substitution (Subst) of modal formulas instead of variables and the rule of

Necessitation (RN): given a formula φ, we infer $\Box\varphi$.

The definition of *derivation* in K is analogous to that in Cl and Int; the only difference is that now we have more axioms and inference rules. The fact of derivability of a formula φ in K is denoted by $\vdash_K \varphi$.

Example 3.49 Let us show that, for every formulas φ and ψ,

$$\vdash_K \varphi \to \psi \text{ implies } \vdash_K \Box\varphi \to \Box\psi.$$

We have:

$$\begin{array}{lr}
\vdash_K \varphi \to \psi & \text{(given)} \\
\vdash_K \Box(\varphi \to \psi) & \text{(by RN)} \\
\vdash_K \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & \text{(from (A11))} \\
\vdash_K \Box\varphi \to \Box\psi & \text{(by MP).}
\end{array}$$

Example 3.50 Now we show that, for any φ and ψ,

$$\vdash_K \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi.$$

Indeed, we have:

$$\begin{array}{lll}
(1) & \vdash_K \varphi \wedge \psi \to \varphi & \text{(from (A3))} \\
(2) & \vdash_K \Box(\varphi \wedge \psi) \to \Box\varphi & \text{(by Example 3.49)} \\
(3) & \vdash_K \varphi \wedge \psi \to \psi & \text{(from (A4))} \\
(4) & \vdash_K \Box(\varphi \wedge \psi) \to \Box\psi & \text{(by Example 3.49)} \\
(5) & \vdash_K \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi & \text{(from (2), (4))} \\
(6) & \vdash_K \varphi \to (\psi \to \varphi \wedge \psi) & \text{(from (A5))} \\
(7) & \vdash_K \Box\varphi \to \Box(\psi \to \varphi \wedge \psi) & \text{(by Example 3.49)} \\
(8) & \vdash_K \Box(\psi \to \varphi \wedge \psi) \to (\Box\psi \to \Box(\varphi \wedge \psi)) & \text{(from (A11))} \\
(9) & \vdash_K \Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi)) & \text{(from (7), (8))} \\
(10) & \vdash_K \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi) & \text{(from (9))} \\
(11) & \vdash_K \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi & \text{(from (5), (10)).}
\end{array}$$

More generally, by induction on n the reader can readily prove that

$$\vdash_K \Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n.$$

Since Proposition 1.11 on substitutionless derivations is obviously extended to K, we can define the notion of *derivation from assumptions* in K in the same manner as in Cl and Int: $\Gamma \vdash_K \varphi$ if there is a sequence $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and each φ_i is either a substitution instance of an axiom of K or an assumption in Γ or obtained by MP or RN from some of the preceding formulas.

However, the deduction theorem, as it was formulated for Cl and Int, should not hold for K if we want K to be sound with respect to the Kripke semantics. For by RN, we have $p \vdash_K \Box p$. But on the other hand, $p \to \Box p$ is false at the point b in the model shown in Fig. 3.3.

To formulate a modal version of the deduction theorem, we require the following definition. Let $\varphi_1, \ldots, \varphi_n$ be a derivation from assumptions. Say that a formula φ_k *depends on a formula* φ_i in this derivation if either $k = i$ or φ_k is obtained by MP or RN from formulas, at least one of which depends on φ_i.

Theorem 3.51. (Deduction theorem for K) *Suppose $\Gamma, \psi \vdash_K \varphi$ and there exists a derivation of φ from the assumptions $\Gamma \cup \{\psi\}$ in which RN is applied to formulas depending on ψ $m \geq 0$ times. Then*

$$\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^m\psi \to \varphi.$$

Proof The proof is conducted by the same scheme as for Cl: we consider a derivation $\varphi_1, \ldots, \varphi_n$ of φ from $\Gamma \cup \{\psi\}$, in which RN is applied to formulas depending on ψ m times, and show by induction on i that

$$\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^l\psi \to \varphi_i, \tag{3.1}$$

where l is the number of applications of RN to formulas depending on ψ in the derivation $\varphi_1, \ldots, \varphi_i$. The cases when φ_i is a substitution instance of an axiom or belongs to $\Gamma \cup \{\psi\}$ are justified in the same way as in the proof of Theorem 1.12: we get $\Gamma \vdash_K \psi \to \varphi_i$ and so (3.1).

Suppose φ_i is obtained from $\varphi_k = \varphi_j \to \varphi_i$ and φ_j by MP, and RN is applied to formulas depending on ψ in $\varphi_1, \ldots, \varphi_k$ and $\varphi_1, \ldots, \varphi_j$ l_1 and l_2 times, respectively. Then, by the induction hypothesis,

$$\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi \to (\varphi_j \to \varphi_i), \quad \Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l_2}\psi \to \varphi_j$$

and we obtain (3.1), since $l_1, l_2 \leq l$.

Thus, it remains to consider only one case: φ_i is obtained from φ_j by RN. If φ_j does not depend on ψ then there is a derivation of $\Box\varphi_j$ from Γ which clearly yields $\Gamma \vdash_K \psi \to \varphi_i$ and so (3.1).

Suppose now that φ_j depends on ψ and RN is applied $l_1 < l$ times to formulas depending on ψ in $\varphi_1, \ldots, \varphi_j$. By the induction hypothesis, we then have

$$\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi \to \varphi_j$$

and so, by Examples 3.49 and 3.50,

$$\Gamma \vdash_K \Box\psi \wedge \ldots \wedge \Box^{l_1+1}\psi \to \Box\varphi_j,$$

which implies (3.1). ❑

Corollary 3.52 *Suppose $\Gamma, \psi \vdash_K \varphi$ and there exists a derivation of φ from the assumptions $\Gamma \cup \{\psi\}$ in which RN is not applied to formulas depending on ψ. Then $\Gamma \vdash_K \psi \to \varphi$.*

In the sequel we will be distinguishing between derivations in which the rule RN is applied exceptionally to formulas that depend only on axioms and derivations without this restriction. For the former we shall use the usual "turnstile"

$\vdash$; the deduction theorem for it is simplified and formulated like that for Cl and Int: $\Gamma, \psi \vdash_K \varphi$ implies $\Gamma \vdash_K \psi \to \varphi$. The latter kind of derivability will be denoted by $\vdash^*$.

Theorem 3.53. (Soundness and completeness of K) *$\vdash_K \varphi$ iff $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F}$.*

Proof ($\Rightarrow$) All the axioms of K are valid in every frame and the inference rules preserve the validity.

($\Leftarrow$) Suppose $\nvdash_K \varphi$. Our aim is to construct a Hintikka system $\mathfrak{H}$ for the tableau $(\emptyset, \{\varphi\})$. Having succeeded in this, we, by virtue of Corollary 3.26, shall establish thereby that $\mathfrak{H} \not\models \varphi$.

Say that a tableau $t = (\Gamma, \Delta)$ is *consistent* in K if $\Gamma \vdash_K \varphi_1 \vee \ldots \vee \varphi_n$ for no $\varphi_1, \ldots, \varphi_n \in \Delta$, $n \geq 0$. Since $\nvdash_K \varphi$, $(\emptyset, \{\varphi\})$ is consistent. The tableau t is called *maximal* (relative to φ) if $\Gamma \cup \Delta = \mathbf{Sub}\varphi$.

By the same argument as in the proof of Theorem 1.16 we can show that every consistent tableau consisting of some subformulas of φ can be extended to a maximal consistent tableau satisfying (S1)–(S6).

Now take the set T of all maximal (relative to φ) consistent tableaux and define a binary relation S on it by putting, for every $t = (\Gamma, \Delta)$ and $t' = (\Gamma', \Delta')$ in T,

$$tSt' \text{ iff } \psi \in \Gamma' \text{ whenever } \Box\psi \in \Gamma.$$

The condition (HS_M1) is satisfied by the definition. So it remains to verify that (HS_M2) also holds.

Let $t = (\Gamma, \Delta) \in T$ and $\Box\psi \in \Delta$. Consider the tableau $t' = (\Gamma', \{\psi\})$ where $\Gamma' = \{\chi : \Box\chi \in \Gamma\}$. We show that it is consistent in K. Indeed, assuming otherwise, we would have $\Gamma' \vdash_K \psi$ and so, by the deduction theorem, $\vdash_K \chi_1 \wedge \ldots \wedge \chi_n \to \psi$, where $\chi_1, \ldots, \chi_n$ are all distinct formulas in Γ'. Then, using Examples 3.49 and 3.50, we obtain $\vdash_K \Box\chi_1 \wedge \ldots \wedge \Box\chi_n \to \Box\psi$ and $\Box\chi_1, \ldots, \Box\chi_n \vdash_K \Box\psi$, contrary to the consistency of t. Thus, t' is consistent and so it is contained in some maximal consistent tableau $t'' = (\Gamma'', \Delta'') \in T$. By the definition of t', we must then have tSt'' and $\psi \in \Delta''$.

Therefore, $\mathfrak{H} = \langle T, S \rangle$ is a Hintikka system for $(\emptyset, \{\varphi\})$, from which $\mathfrak{H} \not\models \varphi$. Notice by the way that $|T| \leq 2^{|\mathbf{Sub}\varphi|}$. ❑

Corollary 3.54 $\mathbf{K} = \{\varphi \in \mathbf{For}\mathcal{ML} : \vdash_K \varphi\}$.

By applying the same kind of argument as in the proof of Theorem 3.53 to infinite consistent tableaux, one can prove the following theorems (for details see Section 5.1).

Theorem 3.55. (Strong completeness) *Every tableau consistent in K is realizable. In particular, $\Gamma \vdash_K \varphi$ iff, for every model $\mathfrak{M}$ and every point x in $\mathfrak{M}$, $(\mathfrak{M}, x) \models \Gamma$ implies $(\mathfrak{M}, x) \models \varphi$.*

Theorem 3.56. (Compactness) *A tableau is realizable in $\mathbf{K}$ iff its every finite subtableau is realizable in $\mathbf{K}$.*

Now we use the completeness theorem for K to obtain an upper bound for the parameter m in the deduction theorem.

Theorem 3.57 *Suppose that* $\psi \vdash_K^* \varphi$. *Then* $\vdash_K \Box^0\psi \wedge \ldots \wedge \Box^m\psi \to \varphi$, *where* $m = 2^{|\mathbf{Sub}\psi \cup \mathbf{Sub}\varphi|}$.

Proof Suppose otherwise. By Theorem 3.53 and Corollary 3.29, we then have a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on a finite intransitive tree $\mathfrak{F} = \langle W, R \rangle$ and refuting $\bigwedge_{i=0}^m \Box^i\psi \to \varphi$ at its root v. With every point x in $\mathfrak{M}$ we associate the tableau $t_x = (\Gamma_x, \Delta_x)$ where

$$\Gamma_x = \{\chi \in \mathbf{Sub}\psi \cup \mathbf{Sub}\varphi : x \models \chi\},$$

$$\Delta_x = \{\chi \in \mathbf{Sub}\psi \cup \mathbf{Sub}\varphi : x \not\models \chi\}.$$

Now construct a new model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ on a frame $\mathfrak{G} = \langle V, S \rangle$ in the following way. V is the set of all points x in $\mathfrak{F}$ such that for no distinct y, z in the chain $x{\overline{\downarrow}}_\omega$ do we have $t_y = t_z$. Let S' be the restriction of R to V. The frame $\langle V, S' \rangle$ is clearly an intransitive tree in which every point is accessible from v by $\leq m$ steps and so $(\mathfrak{M}, x) \models \psi$ for all $x \in V$. If $x \in V$ has a successor $y \in W - V$ then there must be a point $z \in x{\overline{\downarrow}}_\omega$ such that $t_y = t_z$. In this case we draw an arrow from x to z, i.e., add $\langle x, z \rangle$ to S'. The resulting relation is denoted by S. Finally, we define $\mathfrak{U}$ as the restriction of $\mathfrak{V}$ to V.

By induction on the construction of $\chi \in \mathbf{Sub}\psi \cup \mathbf{Sub}\varphi$ we show now that, for every $x \in V$, $(\mathfrak{M}, x) \models \chi$ iff $(\mathfrak{N}, x) \models \chi$. The basis of induction and the cases of non-modal connectives are trivial. Let $\chi = \Box\chi'$. If $(\mathfrak{M}, x) \not\models \Box\chi'$ then $(\mathfrak{M}, y) \not\models \chi'$ for some $y \in W$ such that xRy. By the construction, there is a point $z \in V$ for which $t_z = t_y$ in $\mathfrak{M}$ and xSz. By the induction hypothesis, we then have $(\mathfrak{N}, z) \not\models \chi'$, from which $(\mathfrak{N}, x) \not\models \Box\chi'$. Conversely, assume $(\mathfrak{N}, x) \not\models \Box\chi'$, i.e., $(\mathfrak{N}, y) \not\models \chi'$ for some successor y of x in $\mathfrak{N}$. By the construction of $\mathfrak{G}$, we can find then a point z in $\mathfrak{M}$ such that xRz and $t_y = t_z$. Consequently, $(\mathfrak{M}, z) \not\models \chi'$ and $(\mathfrak{M}, x) \not\models \Box\chi'$.

Thus, $(\mathfrak{N}, v) \not\models \varphi$ and, for every $x \in V$, $(\mathfrak{N}, x) \models \psi$. So we must have

$$(\mathfrak{N}, v) \not\models \Box^0\psi \wedge \ldots \wedge \Box^n\psi \to \varphi$$

for every $n < \omega$, contrary to the deduction theorem and the soundness of K. ❑

3.7 Basic properties of K

In this section we mean by a *logic* any set L of $\mathcal{ML}$-formulas containing **K** and closed under MP, Subst and RN. Derivations in L may use any formulas in L as axioms. The tabularity, finite approximability and other properties are defined for such logics in the same way as for logics in the language $\mathcal{L}$.

CONSISTENCY. **K** is consistent, since the constant $\bot$ is false at every point in every model.

DECIDABILITY. The decidability of **K** is proved analogously to the decidability of **Int** using Theorem 3.27.

Theorem 3.58 **K** *is decidable.*

TABULARITY. Since the formulas $\boldsymbol{bw_n}$ and $\boldsymbol{alt_n}$, defined in Section 3.5, are not in **K** and each frame refuting one of them contains $> n$ points, we have

Theorem 3.59 **K** *is not tabular.*

FINITE APPROXIMABILITY. The fact that **K** is finitely approximable is an immediate consequence of Theorem 3.27.

Theorem 3.60 **K** *is finitely approximable.*

POST COMPLETENESS. As we shall see later, **K** has a continuum of proper consistent extensions. Here we construct only one of them.

Theorem 3.61 **K** *is Post incomplete.*

Proof Let L be the smallest set of formulas containing **K**, the formula $\Box\bot$ and closed under MP, Subst and RN. By the definition, $\Box\bot$ is valid in the frame $\mathfrak{F}$ consisting of a single irreflexive point. And since all formulas in **K** are also valid in $\mathfrak{F}$ and the inference rules preserve validity, we obtain $\mathfrak{F} \models L$, which means that L is consistent. Thus, L is a proper consistent extension of **K**. ❑

Theorem 3.62 **K** *is not* 0*-reducible.*

Proof The formula

$$\Box(\Box\bot \to p) \vee \Box(\Box\bot \to \neg p).$$

does not belong to **K** because it is refuted by the frame in Fig. 3.9 (b). On the other hand, $\Box(\Box\bot \to \varphi) \vee \Box(\Box\bot \to \neg\varphi) \in \mathbf{K}$ for every variable free formula φ. For to refute this substitution instance a frame must contain two dead ends such that at one of them φ is true and at another false, which is impossible. ❑

INDEPENDENT AXIOMATIZABILITY. Since **K** is finitely axiomatizable, we have

Theorem 3.63 **K** *is independently axiomatizable.*

STRUCTURAL COMPLETENESS. The definitions of admissible and derivable rules as well as that of structural completeness remain the same as in Section 1.4. It is to be noted, however, that in the modal case we can use RN and so a rule $\varphi_1, \dots, \varphi_n/\varphi$ is *derivable* in a logic L if $\varphi_1, \dots, \varphi_n \vdash_L^* \varphi$. Since **K** is decidable and in view of Theorem 3.57, we can always recognize whether a given rule is derivable in K. However, it is unknown whether the admissibility problem for inference rules in **K** is decidable.

Theorem 3.64 **K** *is not structurally complete.*

Proof Consider the rule $\Box\bot/\bot$. Since $\Box\bot \notin \mathbf{K}$, it is admissible in **K**. On the other hand, for any $m \geq 1$, the formula $\Box\bot \wedge \ldots \wedge \Box^m\bot \to \bot$ is not in **K** because it is refuted in the frame consisting of a single irreflexive point. ❑

All the congruence rules in Section 1.4 are clearly admissible and even derivable in **K**. Example 3.49 establishes in fact the derivability in **K** of the *regularity rule*

$$\frac{p \to q}{\Box p \to \Box q},$$

and so of the *congruence rule*

$$\frac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q}$$

which gives us the following:

Theorem 3.65. (Equivalent replacement) *Suppose $\varphi(\psi)$ is a modal formula containing an occurrence of a subformula ψ and $\varphi(\chi)$ is obtained from $\varphi(\psi)$ by replacing this occurrence of ψ with an occurrence of a formula χ. Then, for every logic L in the language $\mathcal{ML}$ in which the congruence rules for $\wedge$, $\vee$, $\to$, $\Box$ are admissible, $\psi \leftrightarrow \chi \in L$ implies $\varphi(\psi) \leftrightarrow \varphi(\chi) \in L$.*

Proof An easy induction on the construction of $\varphi(\psi)$ is left to the reader as an exercise. ❑

INTERPOLATION PROPERTY. The following theorem will be proved in Section 14.1.

Theorem 3.66 **K** *has the interpolation property.*

REDUCTIONS OF MODALITIES. A *modality* is a (possibly empty) string of $\Box$, $\Diamond$, $\neg$. Since the formulas of the form $\neg\Box\varphi \leftrightarrow \Diamond\neg\varphi$, $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$ and $\neg\neg\varphi \leftrightarrow \varphi$ are in **K**, we may assume that every modality $\boldsymbol{M}$ contains at most one symbol $\neg$ which is the last one in $\boldsymbol{M}$. A modality $\boldsymbol{M}$ is called *affirmative* (*negative*) if $\neg$ does not occur (occurs) in $\boldsymbol{M}$. By a *modal reduction principle* we mean any formula of the form $\boldsymbol{M}p \to \boldsymbol{N}p$ with distinct affirmative modalities $\boldsymbol{M}$ and $\boldsymbol{N}$.

Theorem 3.67 *No modal reduction principle is in* **K**.

Proof Let $\varphi = \boldsymbol{M}_1 p \to \boldsymbol{M}_2 p$ be a modal reduction principle. Consider two possible cases.

Case 1: $md(\boldsymbol{M}_1 p) = md(\boldsymbol{M}_2 p)$. Since $\boldsymbol{M}_1 \neq \boldsymbol{M}_2$, φ can be represented either as $\boldsymbol{M}\Box\boldsymbol{N}_1 p \to \boldsymbol{M}\Diamond\boldsymbol{N}_2 p$ or as $\boldsymbol{M}\Diamond\boldsymbol{N}_1 p \to \boldsymbol{M}\Box\boldsymbol{N}_2 p$. The former formula is refuted (under every valuation) at the root of the frame shown in Fig. 3.13 (a), where $m \geq 0$ is the length of the string $\boldsymbol{M}$. And the latter one is refuted at the root of the frame shown in Fig. 3.13 (b) under the valuation $\mathfrak{V}(p) = \{a\}$, since $m \models \Diamond\boldsymbol{N}_1 p$ and $m \not\models \Box\boldsymbol{N}_2 p$.

Case 2: $md(\boldsymbol{M}_1 p) \neq md(\boldsymbol{M}_2 p)$. Let $m = \max\{md(\boldsymbol{M}_1 p), md(\boldsymbol{M}_2 p)\}$ and $k = md(\boldsymbol{M}_1 p)$. Then φ is refuted at the root of the frame in Fig. 3.13 (a) under the valuation $\mathfrak{V}(p) = \{k\}$. ❑

This result can be easily extended to

Theorem 3.68 *If* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *are distinct modalities then* $\boldsymbol{M}p \to \boldsymbol{N}p \notin$ **K**.

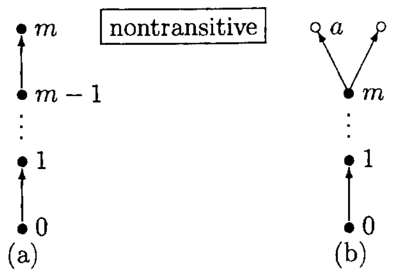


FIG. 3.13.

Proof Exercise. ❑

Modalities $\boldsymbol{M}$ and $\boldsymbol{N}$ are *equivalent* in a logic L if $\boldsymbol{M}p \leftrightarrow \boldsymbol{N}p \in L$. $\boldsymbol{M}$ is said to be *irreducible* in L if it is not equivalent to any modality $\boldsymbol{N}$ with $md(\boldsymbol{N}p) < md(\boldsymbol{M}p)$.

Corollary 3.69 *No distinct modalities are equivalent in* **K**. *All modalities are irreducible in* **K**.

LOCAL TABULARITY.

Theorem 3.70 **K** *is not locally tabular.*

Proof Follows from Corollary 3.69. ❑

HALLDÉN COMPLETENESS.

Theorem 3.71 **K** *is Halldén incomplete.*

Proof Let us consider the formula $\Diamond\top \vee \Box\bot$. Since $\bullet \not\models \Diamond\top$ and $\circ \not\models \Box\bot$, neither of its disjuncts is in **K**. However, $\Diamond\top \vee \Box\bot$ is in **K**, since it is equivalent to the formula $\neg\Box\bot \vee \Box\bot$ which is a substitution instance of (A10) and so belongs to **K**. ❑

DISJUNCTION PROPERTY. **K**, as well as all other modal logics to be considered in this book, contains all the axioms of Cl including $p_0 \vee \neg p_0$ and so does not have the disjunction property. The disjunction property, as it was formulated in Section 1.4, served as some measure of constructivity of the connectives in the language $\mathcal{L}$. In the modal case, especially when $\Box$ is interpreted as "it is provable", a somewhat different formulation is of interest.

We say that a modal logic L has the *modal disjunction property* if, for all formulas $\varphi_1, \dots, \varphi_n$, $\Box\varphi_1 \vee \dots \vee \Box\varphi_n \in L$ iff $\varphi_i \in L$ for some $i \in \{1, \dots, n\}$.

Theorem 3.72 **K** *has the modal disjunction property.*

Proof ($\Leftarrow$) is clear. To prove ($\Rightarrow$) suppose that $\varphi_1, \dots, \varphi_n \notin$ **K**. Then there are models $\mathfrak{M}_i = \langle \mathfrak{F}_i, \mathfrak{V}_i \rangle$, for $i = 1, \dots, n$, based on disjoint rooted frames $\mathfrak{F}_i = \langle W_i, R_i \rangle$ such that φ_i is false at the root x_i of $\mathfrak{F}_i$. Now we form a new frame $\mathfrak{F} = \langle W, R \rangle$ by adding the root x_0 to $\sum_{i=1}^{n} \mathfrak{F}_i$, i.e., put

$$W = \{x_0\} \cup W_1 \cup \dots \cup W_n,$$

$$R = \{\langle x_0, x_i\rangle : \ i = 1, \ldots, n\} \cup R_1 \cup \ldots \cup R_n.$$

Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = \mathfrak{V}_1(p) \cup \ldots \cup \mathfrak{V}_n(p)$, for every variable p, and let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$. It is clear that every $\mathfrak{M}_i$ is a generated submodel of $\mathfrak{M}$. By the generation theorem, we have $(\mathfrak{M}, x_i) \not\models \varphi_i$, for any $i = 1, \ldots, n$, and so $x_0 \not\models \Box\varphi_1 \vee \ldots \vee \Box\varphi_n$. Therefore, $\Box\varphi_1 \vee \ldots \vee \Box\varphi_n \notin \mathbf{K}$. ❑

3.8 A few more modal logics

In this section we define a few more modal logics. They are of different origin. Some of them, like **S4**, **S5**, **GL**, were created to characterize various interpretations of the operator $\Box$, while others, such as **K**, **T**, **K4**, originated for purely technical reasons. We must warn the reader that he should not look for a deep sense or a system in the names of modal logics. Some were given to logics in honor of the logicians whose work led to their creation (for instance, **G**ödel–**L**öb, **K**ripke, **S**olovay), **D** stands for *deontic*; however, in many cases the names are rather arbitrary.

Most of the logics to be presented here will be defined, like **Cl**, **Int** and **K**, semantically, i.e., as the sets of formulas valid in certain frames. But sometimes, as was observed at the end of Section 3.1, preferable is a syntactical definition of a logic in the form of calculus. The following notions are intended to bridge these two methods.

A calculus (axiomatic system) C is said to be *sound* with respect to a class $\mathcal{C}$ of frames if, for every formula φ, $\vdash_C \varphi$ implies $\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in \mathcal{C}$. C is called *complete* with respect to $\mathcal{C}$ (or $\mathcal{C}$-*complete*) if φ is derivable in C whenever it is valid in every frame in $\mathcal{C}$.

The same logic can be defined by different classes of frames. For instance, **K** was defined as the set of formulas which are valid in all frames. On the other hand, as follows from Corollary 3.29, it is determined by the class of all finite trees. Say that a logic L is *characterized* (or *determined*) by a class of frames $\mathcal{C}$ if

$$L = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \forall \mathfrak{F} \in \mathcal{C}\ \mathfrak{F} \models \varphi\}.$$

A frame $\mathfrak{F}$ validating all formulas in L is called a *frame for* L.

Being equipped with these notions, we can proceed now to defining our modal logics.

LOGIC **T**. Semantically the logic **T** in the language $\mathcal{ML}$ is determined by the class of all reflexive frames:

$$\mathbf{T} = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \mathfrak{F} \models \varphi, \text{ for every reflexive frame } \mathfrak{F}\}.$$

Syntactically **T** can be defined by the calculus T which is obtained by adding to K one more modal axiom $\boldsymbol{re} = \Box p \to p$.

Proposition 3.73 *T is sound with respect to the class of reflexive frames.*

Proof Follows from Proposition 3.30. ❑

The completeness of T with respect to the class of reflexive frames will be established in Section 5.2 (but we recommend the reader to try to modify the proof of Theorem 3.53 for T and other calculi to be considered in this section). Thus, $\mathbf{T}$ can be obtained by adding $\boldsymbol{re}$ to $\mathbf{K}$ and taking the closure under MP, Subst and RN. In symbols this fact will be written as

$$\mathbf{T} = \mathbf{K} \oplus \Box p \to p.$$

From the set-theoretic point of view $\mathbf{T}$ is an *extension* of $\mathbf{K}$, i.e., $\mathbf{K} \subseteq \mathbf{T}$. Since $\boldsymbol{re}$ is refuted in any frame containing an irreflexive point, this inclusion is proper.

LOGIC **K4** is characterized by the class of all transitive frames:

$$\mathbf{K4} = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \mathfrak{F} \models \varphi, \ \text{for every transitive frame } \mathfrak{F}\}.$$

The corresponding calculus $K4$ is obtained by adding to K the transitivity axiom $\boldsymbol{tra} = \Box p \to \Box\Box p$.

Proposition 3.74 *$K4$ is sound with respect to the class of transitive frames.*

Proof Follows from Proposition 3.31. ❑

The completeness of $K4$ relative to the same class is proved in Section 5.2. Thus, we have

$$\mathbf{K4} = \mathbf{K} \oplus \Box p \to \Box\Box p.$$

It is clear that $\mathbf{K} \subset \mathbf{K4}$ (see Example 3.5). However, $\mathbf{T}$ and $\mathbf{K4}$ turn out to be incomparable by inclusion, since $\boldsymbol{re} \notin \mathbf{K4}$ and $\boldsymbol{tra} \notin \mathbf{T}$ (why?).

The modal logics above are of rather technical than philosophical interest: they simply correspond to some natural mathematical structures. Our next logic **S4** is also characterized by a class of very natural structures, viz., quasi-ordered sets; however, it can be regarded also as a variant of epistemic logic or a logic of informal provability.[6]

LOGIC **S4** is determined by the class of all quasi-ordered frames, i.e.,

$$\mathbf{S4} = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \mathfrak{F} \models \varphi, \ \text{for every quasi-ordered frame } \mathfrak{F}\}.$$

The calculus $S4$ is K plus two additional axioms $\boldsymbol{re}$ and $\boldsymbol{tra}$. The following proposition is an immediate consequence of the preceding ones.

Proposition 3.75 *$S4$ is sound with respect to the class of quasi-ordered frames.*

In Section 5.2 we shall prove the completeness of $S4$ relative to this class. Therefore,

$$\mathbf{S4} = \mathbf{K} \oplus \boldsymbol{re} \oplus \boldsymbol{tra} = \mathbf{T} \oplus \boldsymbol{tra} = \mathbf{K4} \oplus \boldsymbol{re}.$$

It should be clear that $\mathbf{T} \subset \mathbf{S4}$ and $\mathbf{K4} \subset \mathbf{S4}$.

[6]Recently Artemov (1995) has shown that **S4** coincides with the logic of proofs, for some natural understanding of the concept of proof.

If we impose on a quasi-order R on W the symmetry condition, then R will be an *equivalence relation* on W. When all elements in W are in the same equivalence class, i.e., if $\forall x, y\ xRy$, the relation R is called *universal*. Every frame $\mathfrak{F} = \langle W, R\rangle$ with an equivalence relation R is the disjoint union of some frames $\mathfrak{F}_i = \langle W_i, R_i\rangle$ with universal R_i.

LOGIC **S5**, determined by the class of frames with universal alternativeness relations, can be regarded as the logic of "logical necessity". The corresponding calculus $S5$ is obtained by adding to $S4$ the symmetry axiom $\boldsymbol{sym} = p \to \Box\Diamond p$.

Proposition 3.76 *$S5$ is sound with respect to the class of frames with universal alternativeness relations.*

Proof Follows from Proposition 3.32. ❑

We recommend the reader to show that $\mathbf{S4} \subset \mathbf{S5}$. In Section 5.2 we shall prove that

$$\mathbf{S5} = \mathbf{S4} \oplus p \to \Box\Diamond p.$$

LOGIC **Grz**, the *Grzegorczyk logic*, can be connected as **S4** with the proof interpretation of $\Box$. Semantically **Grz** is determined by the class of Noetherian partial orders, i.e., quasi-ordered frames without proper clusters and infinite ascending chains. Syntactically it may be defined by the calculus Grz which is obtained by adding to K (or $K4$ or $S4$) the Grzegorczyk axiom $\boldsymbol{grz}$. Proposition 3.48 immediately provides us with

Proposition 3.77 *Grz is sound with respect to the class of Noetherian partial orders.*

The completeness of Grz with respect to that class will be established in Section 5.5, so we have

$$\mathbf{Grz} = \mathbf{K} \oplus \boldsymbol{grz} = \mathbf{K4} \oplus \boldsymbol{grz} = \mathbf{S4} \oplus \boldsymbol{grz}.$$

Grz is clearly a proper extension of **S4** incomparable with **S5**.

LOGIC **D**. The deontic logic **D** (the minimal deontic logic, to be more exact) is usually defined by the calculus D obtained by adding to K the seriality axiom $\boldsymbol{ser} = \Box p \to \Diamond p$, which can be read as "what is obligatory is also permitted".

The logic

$$\mathbf{D} = \mathbf{K} \oplus \Box p \to \Diamond p$$

(i.e., the set of all formulas derivable in D) is characterized, as we shall see in Section 5.2, by the class of serial frames. One part of this result is an immediate consequence of Proposition 3.33.

Proposition 3.78 *D is sound with respect to the class of serial frames.*

D is located between **K** and **T**: $\mathbf{K} \subset \mathbf{D} \subsetneq \mathbf{T}$.

Further refinements of the modality "it is obligatory", e.g. obligation in the moral sense or obligation expressed by sentences in the imperative mood, can lead to stronger deontic logics such as $\mathbf{D4} = \mathbf{D} \oplus \boldsymbol{tra}$ and $\mathbf{D5} = \mathbf{D} \oplus \boldsymbol{sym}$, which are called *deontic* **S4** and *deontic* **S5**, respectively.

LOGIC **S4.3**. If we understand $\Box$ as "it is true now and always will be true" and time is considered to be linear, then the logic

$$\mathbf{S4.3} = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \mathfrak{F} \models \varphi, \text{ for every linearly ordered frame } \mathfrak{F}\}$$

can be regarded as the logic of the tense necessity. The corresponding calculus $S4.3$ is obtained by adding to $S4$ the strong connectedness axiom $\boldsymbol{sc}$ (which is equivalent in **S4** to the connectedness axiom $\boldsymbol{con}$).

Proposition 3.79 *$S4.3$ is sound with respect to the class of linearly ordered frames.*

Proof Follows from Theorem 3.40. ❑

The completeness of $S4.3$ with respect to linearly ordered frames is proved in Section 5.2. Thus, we have

$$\mathbf{S4.3} = \mathbf{S4} \oplus \Box(\Box p \to q) \lor \Box(\Box q \to p).$$

It is easy to see that $\mathbf{S4} \subset \mathbf{S4.3} \subset \mathbf{S5}$.

LOGIC **GL**. Now let us consider the necessity operator $\Box$ as provability in formal Peano arithmetic PA. Unlike the previous interpretations of $\Box$, which had after all a more or less vague character, the provability interpretation of modal formulas can be defined in a quite precise manner. To this end we need some facts concerning Gödel's numbering of arithmetic formulas (the reader can find the details in every serious textbook on mathematical logic).

All syntactical constructions of the arithmetic language (terms, formulas, derivations, etc.) can be effectively coded by natural numbers; the code $\ulcorner\phi\urcorner$ of an arithmetic formula ϕ is called the *Gödel number* of ϕ. Gödel constructed a formula $Pr(x)$ with a single free variable x such that, for every natural n,

$$\vdash_{PA} Pr(\overline{n}) \text{ iff } \overline{n} = \ulcorner\phi\urcorner \text{ and } \vdash_{PA} \phi \text{ for some arithmetic formula } \phi.$$

Here $\overline{n}$ is the term representing the number n. In other words, $Pr(\ulcorner\phi\urcorner)$ asserts that the formula ϕ is provable in PA.

By an *arithmetic interpretation* of the language $\mathcal{ML}$ of modal logic we mean any map * from $\mathbf{For}\mathcal{ML}$ to the set of arithmetic sentences such that

- $\bot^*$ is $\overline{0} = \overline{1}$;
- $(\varphi \odot \psi)^* = \varphi^* \odot \psi^*$, for $\odot \in \{\land, \lor, \to\}$;
- $(\Box\varphi)^* = Pr(\ulcorner\varphi^*\urcorner)$.

The main properties of the provability predicate $Pr(x)$ are as follows:

(i) $\vdash_{PA} \phi$ implies $\vdash_{PA} Pr(\ulcorner\phi\urcorner)$;

(ii) $\vdash_{PA} Pr(\ulcorner\phi \to \phi'\urcorner) \to (Pr(\ulcorner\phi\urcorner) \to Pr(\ulcorner\phi'\urcorner))$;

(iii) $\vdash_{PA} Pr(\ulcorner\phi\urcorner) \to Pr(\ulcorner Pr(\ulcorner\phi\urcorner)\urcorner)$;

(iv) $\vdash_{PA} Pr(\ulcorner Pr(\ulcorner\phi\urcorner) \to \phi\urcorner) \to Pr(\ulcorner\phi\urcorner)$.

(The last one is a formalization of Löb's theorem: $\vdash_{PA} Pr(\ulcorner\phi\urcorner) \to \phi$ implies $\vdash_{PA} \phi$.) In any case, these properties are enough to prove Gödel's incompleteness theorems for PA.

The apparent similarity of (i) with the rule RN, (ii) with axiom (A11), (iii) with ***tra*** and (iv) with the Löb axiom ***la*** gives rise to the calculus GL which is obtained by adding ***la*** to $K4$. And it turns out that the modal propositional calculus GL adequately describes the properties of the predicate $Pr(x)$ which are provable in PA. Namely, as was established by Solovay (1976), for every modal formula φ, $\vdash_{GL} \varphi$ iff $\vdash_{PA} \varphi^*$ for all arithmetic interpretations *.

As a consequence of Proposition 3.47, we have

Proposition 3.80 *GL is sound with respect to the class of Noetherian strict partial orders.*

In Section 5.5 we shall show that GL is complete with respect to that class. Thus the provability logic

$$\mathbf{GL} = \mathbf{K4} \oplus \boldsymbol{la} = \mathbf{K} \oplus \boldsymbol{la}$$

is characterized by the class of Noetherian strict partial orders.

It should be clear that $\mathbf{K4} \subset \mathbf{GL}$ and that $\mathbf{GL}$ is incomparable by inclusion with **T**, **S4**, **S5**, **Grz**, **D**, **S4.3**.

The last (but not the least) logic to be considered in this section is

LOGIC **S**. The necessity operator $\Box$ is understood in it as in **GL**, but the purpose of **S** is to describe those properties of the provability predicate that are *true* in the standard model of PA (according to Gödel's first incompleteness theorem, there are sentences which are true in the standard arithmetic model, but not derivable in PA).

Syntactically **S** can be obtained by adding to **GL** the reflexivity axiom ***re*** and then taking the closure under MP and Subst only (so that RN is not applied to ***re***). In symbols this will be written as

$$\mathbf{S} = \mathbf{GL} + \boldsymbol{re} = (\mathbf{K4} \oplus \boldsymbol{la}) + \boldsymbol{re},$$

i.e. $+$, unlike $\oplus$, presupposes taking the closure only under MP and Subst.

By Solovay's (1976) second theorem, for every modal formula φ, $\varphi \in \mathbf{S}$ iff, for all arithmetic interpretations *, φ^* is true in the standard model of PA.

The semantics of **S** is a bit mysterious. Indeed, ***re*** claims that frames for **S** are reflexive, while ***la***, on the contrary, requires the frames to be irreflexive. It follows that there is no Kripke frame validating all formulas in **S**. We will not develop a semantics for **S** here, leaving this question for a more serious consideration in Sections 5.6 and 11.4.

3.9 Embeddings of Int into S4, Grz and GL

As was noticed in the preceding section, the operator $\Box$ in the logic **S4**, characterized by the class of quasi-ordered frames, may be understood as "it is provable". So we can try to formalize the proof interpretation of the intuitionistic connectives in Section 2.1 simply by replacing the words "proof" and "construction" in it with $\Box$. Thus we come to the following translation T of intuitionistic $\mathcal{L}$-formulas into modal $\mathcal{ML}$-formulas: for all $p \in \mathbf{Var}\mathcal{L}$ and all $\varphi, \psi \in \mathbf{For}\mathcal{L}$,

- $\mathsf{T}(p) = \Box p$;
- $\mathsf{T}(\bot) = \Box\bot$;
- $\mathsf{T}(\varphi \wedge \psi) = \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi)$;
- $\mathsf{T}(\varphi \vee \psi) = \mathsf{T}(\varphi) \vee \mathsf{T}(\psi)$;
- $\mathsf{T}(\varphi \to \psi) = \Box(\mathsf{T}(\varphi) \to \mathsf{T}(\psi))$.

The intuitionistic connectives are transformed by T into the corresponding classical ones, but they are understood now in the context of "provability".

We are going to show now that the map $\mathsf{T} : \mathbf{For}\mathcal{L} \to \mathbf{For}\mathcal{ML}$, known as the *Gödel translation*, is an embedding of **Int** into both **S4** and **Grz**.

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a modal model on a quasi-ordered frame $\mathfrak{F}$. Define in the skeleton $\rho\mathfrak{F}$ of $\mathfrak{F}$ (which is partially ordered) an intuitionistic valuation $\rho\mathfrak{V}$ by taking, for every $p \in \mathbf{Var}\mathcal{L}$,

$$\rho\mathfrak{V}(p) = \{C(x) : \ (\mathfrak{M}, x) \models \Box p\}.$$

By Proposition 3.6, this definition does not depend on the choice of x and the set $\rho\mathfrak{V}(p)$ is upward closed in $\rho\mathfrak{F}$. We call the model $\rho\mathfrak{M} = \langle \rho\mathfrak{F}, \rho\mathfrak{V} \rangle$ the *skeleton* of the model $\mathfrak{M}$.

Conversely, if $\mathfrak{N} = \langle \rho\mathfrak{F}, \mathfrak{U} \rangle$ is an intuitionistic model based on the skeleton of a quasi-ordered frame $\mathfrak{F} = \langle W, R \rangle$, then by taking for every $p \in \mathbf{Var}\mathcal{ML}$

$$\mathfrak{V}(p) = \{x \in W : \ (\mathfrak{N}, C(x)) \models p\}$$

we get a modal model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ whose skeleton is (isomorphic to) $\mathfrak{N}$. In particular, if all clusters in $\mathfrak{F}$ are simple and so $\mathfrak{F}$ is isomorphic to $\rho\mathfrak{F}$, the model $\mathfrak{M}$ is also isomorphic to its skeleton $\mathfrak{N}$.

Lemma 3.81. (Skeleton) *For every model $\mathfrak{M}$ of $\mathcal{ML}$ based on a quasi-ordered frame, every point x in $\mathfrak{M}$ and every $\mathcal{L}$-formula φ,*

$$(\rho\mathfrak{M}, C(x)) \models \varphi \textit{ iff } (\mathfrak{M}, x) \models \mathsf{T}(\varphi).$$

Proof By induction on the construction of φ. The basis of induction follows from the definitions of $\rho\mathfrak{M}$ and $\mathsf{T}(\varphi)$. Suppose $\varphi = \psi \to \chi$. Then

$$\begin{aligned}(\rho\mathfrak{M}, C(x)) \not\models \varphi \text{ iff } & \exists y \in x{\uparrow}\ ((\rho\mathfrak{M}, C(y)) \models \psi \text{ and } (\rho\mathfrak{M}, C(y)) \not\models \chi)\\ \text{iff } & \exists y \in x{\uparrow}\ ((\mathfrak{M}, y) \models \mathsf{T}(\psi) \text{ and } (\mathfrak{M}, y) \not\models \mathsf{T}(\chi))\\ \text{iff } & (\mathfrak{M}, x) \not\models \Box(\mathsf{T}(\psi) \to \mathsf{T}(\chi)), \text{ i.e., } (\mathfrak{M}, x) \not\models \mathsf{T}(\varphi).\end{aligned}$$

For $\varphi = \psi \vee \chi$ we have

$$\begin{aligned}(\rho\mathfrak{M}, C(x)) \models \varphi \text{ iff } & (\rho\mathfrak{M}, C(x)) \models \psi \text{ or } (\rho\mathfrak{M}, C(x)) \models \chi \\ \text{iff } & (\mathfrak{M}, x) \models \mathsf{T}(\psi) \text{ or } (\mathfrak{M}, x) \models \mathsf{T}(\chi) \\ \text{iff } & (\mathfrak{M}, x) \models \mathsf{T}(\varphi).\end{aligned}$$

The case $\varphi = \psi \wedge \chi$ is considered in the same way. ❑

Corollary 3.82 *For every quasi-ordered frame $\mathfrak{F}$ and every $\mathcal{L}$-formula φ,*

$$\rho\mathfrak{F} \models \varphi \textit{ iff } \mathfrak{F} \models \mathsf{T}(\varphi).$$

Theorem 3.83 *The Gödel translation* T *is an embedding of* **Int** *into both* **S4** *and* **Grz**.

Proof We must show that, for every $\mathcal{L}$-formula φ,

$$\varphi \in \mathbf{Int} \text{ iff } \mathsf{T}(\varphi) \in \mathbf{S4} \text{ and } \varphi \in \mathbf{Int} \text{ iff } \mathsf{T}(\varphi) \in \mathbf{Grz}.$$

Suppose $\mathsf{T}(\varphi) \notin \mathbf{S4}$ (or $\mathsf{T}(\varphi) \notin \mathbf{Grz}$). Then there is a quasi-ordered frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \mathsf{T}(\varphi)$. According to Corollary 3.82, $\rho\mathfrak{F} \not\models \varphi$ and so $\varphi \notin \mathbf{Int}$.

Conversely, suppose $\varphi \notin \mathbf{Int}$. Then, by Theorem 2.57, there is a finite intuitionistic frame $\mathfrak{F}$ refuting φ. As was observed above, it can be treated as a modal frame isomorphic to its skeleton. Therefore, by Corollary 3.82, $\mathfrak{F} \not\models \mathsf{T}(\varphi)$, from which $\mathsf{T}(\varphi) \notin \mathbf{S4}$ and $\mathsf{T}(\varphi) \notin \mathbf{Grz}$ (since $\mathfrak{F}$ contains neither proper clusters nor infinite ascending chains). ❑

Remark The proof of the skeleton lemma will not change if we replace T by the translation prefixing $\Box$ to *every* subformula of a given intuitionistic formula (see also Exercise 3.25). So this translation embeds **Int** into **S4** and **Grz** too.

The results above not only give a classical interpretation of the intuitionistic connectives but also have purely technical applications.

Corollary 3.84 *Neither* **S4** *nor* **Grz** *is tabular.*

Proof Suppose that **S4** or **Grz** is characterized by a finite frame $\mathfrak{F}$. Then **Int** is characterized by $\rho\mathfrak{F}$. Indeed, if $\varphi \notin \mathbf{Int}$ then, by Theorem 3.83, $\mathsf{T}(\varphi) \notin \mathbf{S4}$ (or $\mathsf{T}(\varphi) \notin \mathbf{Grz}$) and so $\mathfrak{F} \not\models \mathsf{T}(\varphi)$, from which, by Corollary 3.82, $\rho\mathfrak{F} \not\models \varphi$. Thus, **Int** is tabular, contrary to Theorem 2.56. ❑

Corollary 3.85 *Neither* **S4** *nor* **Grz** *is locally tabular.*

Proof Exercise. ❑

For other uses of the Gödel translation T see Section 9.6.

A frame-theoretic counterpart of T is the operator ρ which squeezes proper clusters into reflexive points. Noetherian strictly ordered frames $\mathfrak{F} = \langle W, R \rangle$, which characterize **GL**, can also be easily transformed into partially ordered ones—we should only take the *reflexive closure* R^r of R:

$$xR^r y \text{ iff } x = y \text{ or } xRy.$$

Given a modal frame $\mathfrak{F} = \langle W, R\rangle$ and a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ on it, the frame $\mathfrak{F}^r = \langle W, R^r\rangle$ and the model $\mathfrak{M}^r = \langle \mathfrak{F}^r, \mathfrak{V}\rangle$ are called the *reflexivizations* of $\mathfrak{F}$ and $\mathfrak{M}$, respectively.

A syntactic analog of the reflexivization operator ^{r} is the following translation $^+$ of modal formulas into modal formulas. Let $\Box^+\varphi$ be an abbreviation for the formula $\varphi \wedge \Box\varphi$. Then, for every $\varphi \in \mathbf{For}\mathcal{ML}$, we denote by φ^+ the result of simultaneous replacing all occurrences of $\Box$ in φ with $\Box^+$.

Lemma 3.86. (Reflexivization) *For every model $\mathfrak{M}$ of $\mathcal{ML}$, every point x in $\mathfrak{M}$ and every $\mathcal{ML}$-formula φ,*

$$(\mathfrak{M}, x) \models \varphi^+ \textit{ iff } (\mathfrak{M}^r, x) \models \varphi.$$

Proof By induction on the construction of φ. The basis of induction follows from the fact that $\mathfrak{M}$ and $\mathfrak{M}^r$ share the same valuation. Suppose $\varphi = \Box\psi$. Then

$$\begin{aligned}(\mathfrak{M}, x) \models \varphi^+ &\text{ iff } (\mathfrak{M}, x) \models \psi^+ \wedge \Box\psi^+\\ &\text{ iff } (\mathfrak{M}, x) \models \psi^+ \text{ and } \forall y \in x\!\uparrow\ (\mathfrak{M}, y) \models \psi^+\\ &\text{ iff } \forall y \in x\!\uparrow\ (\mathfrak{M}^r, y) \models \psi\\ &\text{ iff } (\mathfrak{M}^r, x) \models \Box\psi.\end{aligned}$$

The cases $\varphi = \psi \to \chi$, $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ are trivial. ❑

Corollary 3.87 *For every frame $\mathfrak{F}$ and every $\mathcal{ML}$-formula φ,*

$$\mathfrak{F} \models \varphi^+ \textit{ iff } \mathfrak{F}^r \models \varphi.$$

Provided that **GL**, as was claimed in Section 3.8, is characterized by the class of Noetherian strict partial orders, we obtain now

Theorem 3.88 *The translation $^+$ is an embedding of* **Grz** *into* **GL**.

Proof Our aim is to show that, for every modal formula φ,

$$\varphi \in \mathbf{Grz} \text{ iff } \varphi^+ \in \mathbf{GL}.$$

Suppose $\varphi^+ \notin \mathbf{GL}$. Then there is a Noetherian strict partial order $\mathfrak{F}$ refuting φ^+. The reflexivization $\mathfrak{F}^r$ of $\mathfrak{F}$ is clearly a Noetherian partial order which, by Corollary 3.87, refutes φ. So $\varphi \notin \mathbf{Grz}$.

Conversely, if $\varphi \notin \mathbf{Grz}$ then $\mathfrak{F} \not\models \varphi$, for some Noetherian partial order $\mathfrak{F} = \langle W, R\rangle$. Take its "irreflexivization" $\mathfrak{F}^{ir} = \langle W, R^{ir}\rangle$, i.e., put

$$xR^{ir}y \text{ iff } x \neq y \text{ and } xRy.$$

Clearly, $\mathfrak{F}^{ir}$ is a Noetherian strict order and $(\mathfrak{F}^{ir})^r$ is isomorphic to $\mathfrak{F}$. Therefore, by Corollary 3.87, $\mathfrak{F}^{ir} \not\models \varphi^+$, from which $\varphi^+ \notin \mathbf{GL}$. ❑

Putting together Theorems 3.83 and 3.88, we immediately obtain that **Int** is embeddable into **GL**.

Theorem 3.89 *The translation* T^+ *defined by* $\mathsf{T}^+(\varphi) = (\mathsf{T}(\varphi))^+$, *for every* $\mathcal{L}$*-formula* φ, *is an embedding of* **Int** *into* **GL**.

Corollary 3.90 **GL** *is neither tabular nor locally tabular.*

Proof Exercise. ❑

3.10 Other types of modal logics

The modal logics presented in the previous sections by no means exhaust the existing formalizations of various modal operators. Not trying to list all of them here, we just point out some other kinds of modal logics which are in a sense (mainly in the style of their semantic definitions) close to those we considered above.

First of all, it should be emphasized that our choice of **K** as the basic system is explained by its "purity"—in essence it is a usual mathematical practice to abstract from some details in order to clarify the nature of the object under consideration. In principle, there is a wide spectrum of other modal systems that could be chosen as basic ones. From the semantical point of view this would mean to extend our class of frames and models.

For example, sometimes it is useful to consider frames as quadruples $\mathfrak{F} = \langle W, N, R, D\rangle$, where $\langle W, R\rangle$ is a usual Kripke frame, $N \subseteq W$ is a set of so called *normal worlds* and $D \subseteq W$ a set of *distinguished worlds.* A valuation in such a frame is, as before, a function $\mathfrak{V}$ from $\mathbf{Var}\mathcal{ML}$ into 2^W, and the pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is a model. However, the truth-relation for $\Box$ is defined now as follows:

$$(\mathfrak{M}, x) \models \Box\psi \text{ iff } x \in N \text{ and } (\mathfrak{M}, y) \models \psi \text{ for all } y \in W \text{ such that } xRy,$$

and a formula is regarded to be true in $\mathfrak{M}$ if it is true at all points in D. We get usual Kripke frames if $D = N = W$. By imposing various conditions on R, N and D we can define many modal logics known in the literature. For instance,

- the set of formulas that are valid in all reflexive frames with $D \subseteq N$ is known as the logic **S2**;
- the set of formulas that are valid in all quasi-ordered frames such that $D \subseteq N$ is the logic **S3**;
- the set of formulas that are valid in all reflexive frames such that $D \subseteq N$ and $\forall x \in D \exists y \in W - N\ xRy$ is **S6**.

The logic **S1** can be defined analogously but using a somewhat more complicated definition of the truth-relation for $\Box$. The reader can find it in Cresswell (1972).

There are other generalizations of the notion of frame. For example, applications in computer science and linguistics often require more than one operator of the type "it is necessary". We consider then *polymodal logics* with several operators $\Box_i$, for $i = 1, \dots, n$, each of which is interpreted by its own accessibility relation R_i in frames. The set of formulas that are valid in all frames $\langle W, R_1, \dots, R_n\rangle$ (with arbitrary binary relations R_i) is denoted by $\mathbf{K}_n$ and called the *minimal normal n-modal logic.* Of course, these frames can also be enriched by non-normal and distinguished worlds.

Modal operators $\Box_i$ and $\Box_j$ can interact, which is reflected by some connection between R_i and R_j, and by axioms containing both $\Box_i$ and $\Box_j$. For instance, if we want R_2 to be the conversion of R_1 (meaning that a moment x is earlier than a moment y iff y is later than x) then we should accept the formulas

$$p \to \Box_1\Diamond_2 p, \quad p \to \Box_2\Diamond_1 p,$$

where $\Diamond_1$ and $\Diamond_2$ are the dual operators for $\Box_1$ and $\Box_2$, respectively. More precisely, we have

$$\langle W, R_1, R_2\rangle \models (p \to \Box_1\Diamond_2 p) \wedge (p \to \Box_2\Diamond_1 p)$$

iff $\forall x, y \in W\ (xR_1y \leftrightarrow yR_2x)$. We can denote then $\Box_2$ as $\Box_1^{-1}$ (using this notation for the dual operators as well), R_2 as R_1^{-1} and drop the subscripts if $n = 2$. In view of the clear tense character of such an interaction between the modal operators and the corresponding accessibility relations the set of bimodal formulas that are valid in all frames of the form $\mathfrak{F} = \langle W, R, R^{-1}\rangle$ is called the *minimal normal tense logic*, and the symbols $\Box$, $\Box^{-1}$, $\Diamond$, $\Diamond^{-1}$ are replaced by G, H, F(uture), P(ast), respectively.

Other operations on binary relations provide us with other examples of interaction between modal operators. Here are two of them. Consider a frame $\mathfrak{F} = \langle W, R_1, R_2, R_3\rangle$. Then

$$\mathfrak{F} \models \Box_3 p \leftrightarrow \Box_1 p \wedge \Box_2 p \text{ iff } R_3 = R_1 \cup R_2,$$

$$\mathfrak{F} \models \Box_3 p \leftrightarrow \Box_1\Box_2 p \text{ iff } R_3 = R_1 \circ R_2,$$

where $R_1 \circ R_2$ is the composition of R_1 and R_2, i.e., $xR_1 \circ R_2 y$ iff xR_1zR_2y for some $z \in W$.

Models with several accessibility relations appear also in the study of modal logics on the intuitionistic base. In this case models may contain three relations: a partial order for the intuitionistic connectives and two relations for the operators $\Box$ and $\Diamond$, which are not supposed to be dual from the intuitionistic point of view.

Another source of generalizations and even completely different semantical constructions is the problem of formalizing the epistemic necessity. If we deal with modal operators like "it is known that", "an agent A knows that" then some postulates of modal logic, acceptable in other situations, may turn out to be not justified. For example, the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and the inference rule $\varphi/\Box\varphi$ claim that we (or agent A) know(s) all logical consequences of our (his) knowledge—the so called *omniscience paradox.* There are various ways to avoid such kind of danger. We show here only one of them: the neighborhood semantics.

A *neighborhood frame* is a pair $\mathfrak{F} = \langle W, N\rangle$ where W, as before, is a non-empty set (of worlds or points) and N a function associating with every $x \in W$ a family $N(x)$ of subsets of W, called *neighborhoods* of x. A valuation and the truth–relation in such a frame are defined as usual with only one exception:

$$(\mathfrak{M}, x) \models \Box\psi \text{ iff } \{y \in W : y \models \psi\} \in N(x).$$

It is easy to construct a neighborhood model refuting (A11). In fact, one can show that it is valid in a neighborhood frame $\mathfrak{F} = \langle W, N\rangle$ iff the following conditions are satisfied:

- the intersection of two neighborhoods of a point is again its neighborhood;
- if $W \supseteq X' \supseteq X \in N(x)$ then $X' \in N(x)$.

These conditions mean that the set $N(x)$, for every $x \in W$, is a filter in the Boolean algebra of all subsets of W (for the definition of filter consult Chapter 7). We call such frames *normal.*

We shall not continue describing possible generalizations further. To conclude our discussion we would like just to attract the reader's attention to two points:

- modal logic contains much more various systems than one is able to consider in one book;
- the ideas and methods studied in this book can be extended in a natural way to other systems, though possibly with some modifications.

As to the latter, from time to time we shall illustrate it in exercises and commentaries.

3.11 Exercises

Exercise 3.1 Let K' be the calculus whose axioms are those of Cl, two modal axioms $\Box\top$, $\Box p \wedge \Box q \to \Box(p \wedge q)$ and the inference rules are MP, Subst and the regularity rule $\varphi \to \psi / \Box\varphi \to \Box\psi$. Prove that for every formula φ, $\vdash_K \varphi$ iff $\vdash_{K'} \varphi$.

Exercise 3.2 Show that $\mathbf{D} = \mathbf{K} \oplus \Diamond\top$.

Exercise 3.3 Prove syntactically that the different-axiomatizations of **Grz** and **GL** presented in Section 3.8 really define the same logics.

Exercise 3.4 Prove that every formula $\varphi \notin \mathbf{K}$ is refuted by an intransitive tree of branching and depth $\leq |\mathbf{Sub}\varphi|$.

Exercise 3.5 (Deduction theorem for $K4$ and $S4$) Show that

$$\Gamma, \varphi \vdash^*_{K4} \psi \text{ implies } \Gamma \vdash^*_{K4} \Box^+\varphi \to \psi,$$

$$\Gamma, \varphi \vdash^*_{S4} \psi \text{ implies } \Gamma \vdash^*_{S4} \Box\varphi \to \psi.$$

Exercise 3.6 Show that the inference rules in K are independent (i.e., none of them can be deleted without changing the set of derivable formulas).

Exercise 3.7 Show that the rules $p \to q/\Diamond p \to \Diamond q$ and $p \leftrightarrow q/\Diamond p \leftrightarrow \Diamond q$ are derivable in **K**.

Exercise 3.8 Show that the rules $\Box p \to \Box q/p \to q$ and $\Box p \to p/p$ are admissible in **K**. Are they derivable in **K**?

Exercise 3.9 Do Exercise 2.4 for the modal case.

Exercise 3.10 (i) Show that for no set Γ of modal formulas, $\mathfrak{F} \models \Gamma$ iff $\mathfrak{F}$ is irreflexive.

(ii) Show that for no set Γ of modal formulas, $\mathfrak{F} \models \Gamma$ iff $\mathfrak{F}$ is intransitive.

(iii) Show that for no set Γ of modal formulas, $\mathfrak{F} \models \Gamma$ iff $\mathfrak{F}$ is antisymmetric.

(iv) Show that for no set Γ of modal formulas, $\mathfrak{F} \models \Gamma$ iff $\mathfrak{F}$ is a tree.

(v) Prove that the *Gabbay rule* $(\Box p \to p) \vee \varphi/\varphi$, for $p \notin \mathbf{Var}\varphi$, holds in a frame $\mathfrak{F}$ (in the sense that for every formula φ and every variable p not occurring in φ, φ is valid in $\mathfrak{F}$ whenever $(\Box p \to p) \vee \varphi$ is valid in $\mathfrak{F}$) iff $\mathfrak{F}$ is irreflexive. Show also that **K** is closed under this rule.

Exercise 3.11 (i) Prove that **K4** is characterized by the class of strict partial orders and **S4** by the class of partial orders.

(ii) Prove that **K4** and **S4** are not characterized by the classes of finite strict partial orders and finite partial orders, respectively.

Exercise 3.12 Show that every rooted strict partial order $\mathfrak{F}$ is a reduct of some strictly ordered tree, which is finite if $\mathfrak{F}$ is finite.

Exercise 3.13 Show that every formula $M\varphi$, M a modality, is equivalent in **S5** to one of φ, $\neg\varphi$, $\Box\varphi$, $\Diamond\varphi$, $\Box\neg\varphi$, $\Diamond\neg\varphi$. (Hint: the formulas $\Box^2 p \leftrightarrow \Box p$, $\Diamond^2 p \leftrightarrow \Diamond p$, $\Box\Diamond p \leftrightarrow \Diamond p$, $\Diamond\Box p \leftrightarrow \Box p$ are in **S5**.)

Exercise 3.14 Show that every non-empty modality is equivalent in **S4** to one of $\Box$, $\Diamond$, $\Diamond\Box$, $\Box\Diamond$, $\Box\Diamond\Box$, $\Diamond\Box\Diamond$, $\neg$, $\Box\neg$, $\Diamond\neg$, $\Diamond\Box\neg$, $\Box\Diamond\neg$, $\Box\Diamond\Box\neg$, $\Diamond\Box\Diamond\neg$, which are not equivalent to each other.

Exercise 3.15 Show that the equivalences $\Box\Diamond^2 p \leftrightarrow \Box\Diamond p$, $\Diamond\Box^2 p \leftrightarrow \Diamond\Box p$, $\Box\Diamond\Box\Diamond p \leftrightarrow \Box\Diamond p$, $\Diamond\Box\Diamond\Box p \leftrightarrow \Diamond\Box p$ are in **K4**.

Exercise 3.16 Show that every formula is equivalent in **K** to the conjunction of formulas of the form

$$\varphi \vee \Diamond\psi \vee \Box\chi_1 \vee \ldots \vee \Box\chi_n, \tag{3.2}$$

where φ contains neither $\Box$ nor $\Diamond$.

Exercise 3.17 Show that if a formula of the form (3.2) (where φ contains no $\Box$ and $\Diamond$) is in **K** then either $\varphi \in \mathbf{K}$ or $\psi \vee \chi_i \in \mathbf{K}$ for some $i \in \{1, \ldots, n\}$.

Exercise 3.18 (Principle of duality) Let φ be a modal formula whose connectives are only $\bot$, $\top$, $\wedge$, $\vee$, $\Box$, $\Diamond$ and $\neg$. The *dual* of φ is the formula φ^* which is obtained by replacing simultaneously every $\wedge$, $\vee$, $\Box$, $\Diamond$, $\bot$, $\top$ in φ with $\vee$, $\wedge$, $\Diamond$, $\Box$, $\top$, $\bot$, respectively. Show that for all formulas φ and ψ, $\varphi \leftrightarrow \psi \in \mathbf{K}$ iff $\varphi^* \leftrightarrow \psi^* \in \mathbf{K}$. In particular, $\varphi \in \mathbf{K}$ iff $\neg\varphi^* \in \mathbf{K}$.

Exercise 3.19 Show that every variable free formula is equivalent in $\mathbf{D}$ either to $\bot$ or to $\top$.

Exercise 3.20 Let $\varphi(\dots, p_i, \dots)$ be a modal formula containing only (some of) the connectives $\bot$, $\top$, $\wedge$, $\vee$, $\Box$, $\Diamond$. Show that for every frame $\mathfrak{F}$ and all valuations $\mathfrak{V}$ and $\mathfrak{U}$ in $\mathfrak{F}$ such that $\mathfrak{V}(p_j) = \mathfrak{U}(p_j)$ if $i \neq j$ and $\mathfrak{V}(p_i) \subseteq \mathfrak{U}(p_i)$, we have $\mathfrak{V}(\varphi) \subseteq \mathfrak{U}(\varphi)$.

Exercise 3.21 Modal formulas containing only $\bot$, $\top$, $\wedge$, $\vee$, $\Box$, $\Diamond$ are called *positive*. If a formula $\varphi(p_1, \dots, p_n)$ is positive then $\varphi(\neg p_1, \dots, \neg p_n)$ is *negative*. Show that if $\varphi(\neg p_1, \dots, \neg p_n)$ is negative then $\neg\varphi(\neg p_1, \dots, \neg p_n)$ is equivalent in $\mathbf{K}$ to a positive formula, namely, to $\varphi^*(p_1, \dots, p_n)$.

Exercise 3.22 For an affirmative modality $\boldsymbol{M} = \Box^{i_1}\Diamond^{j_1} \dots \Box^{i_k}\Diamond^{j_k}$ and $n \geq 0$, denote by $xR^{M,n}y$ the first order formula

$$\begin{aligned}&\forall z_1\ (yR^{i_1}z \to \exists u_1\ (z_1R^{j_1}u_1 \wedge\\ &\qquad\qquad \forall z_2\ (u_1R^{i_2}z_2 \to \dots \exists u_k\ (z_kR^{j_k}u_k \wedge xR^n u_k)\dots))).\end{aligned}$$

Prove that a frame $\mathfrak{F} = \langle W, R\rangle$ validates the *Hintikka formula*

$$\begin{aligned}\boldsymbol{hin} = {}& \Diamond^{m_1}\Box^{n_1}p_1 \wedge \dots \wedge \Diamond^{m_k}\Box^{n_k}p_k \to\\ & \Box^{s_1}\Diamond^{t_1}(\boldsymbol{M}^1_1 p_1 \wedge \dots \wedge \boldsymbol{M}^1_k p_k) \vee \dots\\ & \dots \vee \Box^{s_l}\Diamond^{t_l}(\boldsymbol{M}^l_1 p_1 \wedge \dots \wedge \boldsymbol{M}^l_k p_k)\end{aligned}$$

with affirmative modalities $\boldsymbol{M}^i_j$, $i = 1, \dots, l$, $j = 1, \dots, k$, iff it satisfies the condition

$$\begin{aligned}&\forall x, y_1, \dots, y_k\ (xR^{m_1}y_1 \wedge \dots \wedge xR^{m_k}y_k \to\\ &\quad \forall z_1\ (xR^{s_1}z_1 \to \exists u_1\ (z_1R^{t_1}u_1 \wedge y_1R^{M^1_1,n_1}u_1 \wedge \dots \wedge y_kR^{M^1_k,n_k}u_1)) \vee \dots\\ &\quad \dots \vee \forall z_l\ (xR^{s_l}z_l \to \exists u_l\ (z_lR^{t_l}u_l \wedge y_1R^{M^l_1,n_1}u_l \wedge \dots \wedge y_kR^{M^l_k,n_k}u_l))).\end{aligned}$$

Exercise 3.23 A finite transitive frame is called a *balloon* if it is a chain of clusters of which only the last one is non-degenerate. Show that a finite transitive frame $\mathfrak{F}$ validates the formula

$$\boldsymbol{z} = \Box(\Box p \to p) \wedge \Diamond\Box p \to \Box p$$

iff $\mathfrak{F}$ is either irreflexive or a balloon.

Exercise 3.24 A finite quasi-order is a *reflexive balloon* if it is a chain of (non-degenerate) clusters of which only the last one is proper. Show that a finite quasi-order $\mathfrak{F}$ validates the formula

$$\boldsymbol{dum} = \Box(\Box(p \to \Box p) \to p) \wedge \Diamond\Box p \to p$$

iff $\mathfrak{F}$ is either a partial order or a reflexive balloon.

Exercise 3.25 Let T_1 and T_2 be the translations of $\mathcal{L}$ into $\mathcal{ML}$ prefixing $\Box$ to every subformula and every proper non-atomic subformula of a given formula, respectively. Prove that both T_1 and T_2 are embeddings of **Int** into **S4** and **Grz**. (Hint: one way of proving is to show that, for every intuitionistic formula φ, $\mathsf{T}(\varphi) \leftrightarrow \mathsf{T}_1(\varphi) \in \mathbf{S4}$ and $\mathsf{T}(\varphi) \leftrightarrow \Box\mathsf{T}_2(\varphi) \in \mathbf{S4}$.)

Exercise 3.26 Let φ be a modal formula. Define by induction the notions of *positive* and *negative occurrences* of subformulas in φ. The occurrence of φ in φ is positive. If $\Box\psi$ or $\psi \odot \chi$, for $\odot \in \{\wedge, \vee\}$, occurs in φ positively (negatively) then the occurrences of ψ and χ in them are also positive (negative) in φ. And if $\psi \to \chi$ occurs in φ positively (negatively) then the occurrence of ψ in it is negative (positive) in φ and that of χ is positive (negative).

Provided that **GL** is characterized by the class of Noetherian strict partial orders, show that the translation † of $\mathcal{ML}$ into $\mathcal{ML}$ replacing each positive occurrence of $\Box\varphi$ in a given formula with $\Box(\Box\varphi \to \varphi^\dagger)$ and leaving other subformulas intact is an embedding of **GL** into **K4**.

Exercise 3.27 Show that a formula all occurrences of variables in which are positive (negative) is equivalent in **K** to a positive (respectively, negative) formula. Is this true for **Int**?

Exercise 3.28 Show that the truth-values of modal formulas at points in a model with non-normal worlds will remain the same if we arbitrarily change the set of points that are accessible from non-normal worlds, in particular, we can always assume that those worlds are dead ends.

Exercise 3.29 Prove that

(i) if $\varphi \in \mathbf{S2}$ then $\mathbf{S2} + \Box\Box\varphi = \mathbf{T}$;

(ii) if $\varphi \in \mathbf{S3}$ then $\mathbf{S2} + \Box\Box\varphi = \mathbf{S4}$.

Exercise 3.30 Prove that **S3** can be represented as the calculus with axioms (A1)–(A10), $\Box(p \to q) \to \Box(\Box p \to \Box q)$ and the inference rules MP, Subst and the rule of necessitation applicable only to the axioms of **S3**.

Exercise 3.31 Prove that the Gödel translation T is an embedding of **Int** into **S3**. Show that $\mathbf{S3} + \{\mathsf{T}_1(\varphi) : \varphi \in \mathbf{Int}\} = \mathbf{S4}$.

Exercise 3.32 Prove that $\mathbf{K} \oplus \{\mathsf{T}(\varphi) : \varphi \in \mathbf{Int}\} = \mathbf{K} \oplus \Box(\Box p \leftrightarrow \Box\Box p)$.

Exercise 3.33 Prove that

(i) $\mathrm{NExt}(\mathbf{K} \oplus \Box(\Box p \leftrightarrow \Box\Box p))$ contains a continuum of maximal (with respect to $\subseteq$) logics into which **Int** is embeddable by T;

(ii) $\mathrm{Ext}(\mathbf{K} \oplus \Box(\Box p \leftrightarrow \Box\Box p))$ contains a continuum of logics L_i, for $i \in I$, such that $L_i + L_j = \mathbf{For}\mathcal{ML}$ if $i \neq j$, and each L_i has a continuum of maximal extensions into which **Int** is embeddable by T.

Exercise 3.34 Show that for every tense frame $\mathfrak{F} = \langle W, R, R^{-1}\rangle$, (i) $\mathfrak{F}$ validates $\Diamond p \to \Diamond^{-1}\Diamond p$ iff R is transitive, (ii) $\mathfrak{F} \models \Diamond p \to \Box(\Diamond^{-1}p \vee p \vee \Diamond p)$ iff $\mathfrak{F} \models \Diamond^{-1}\Diamond p \to (\Diamond^{-1}p \vee p \vee \Diamond^{-1}p)$ iff $\mathfrak{F}$ satisfies the condition of right linearity $\forall x, y, z\ (xRy \wedge xRz \to y = z \vee yRz \vee zRy)$, (iii) $\mathfrak{F}$ validates the formula $(p \wedge \Box^{-1}p) \to \Diamond\Box^{-1}p$ iff $\mathfrak{F}$ satisfies the conditions of right succession and right discreteness $\forall x \exists y\ (xRy \wedge \forall z\ (zRy \to z = x \vee zRx))$, and (iv) the Hamblin axiom $p \wedge Gp \to PGp$ (i.e., $\Box^{+}p \to \Diamond^{-1}\Box p$) is valid in $\mathfrak{F}$ iff $\forall x \exists y\ (yRx \wedge \forall z\ (yRz \to x = z \vee xRz))$.

Exercise 3.35 Prove that there is a tense formula φ such that $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}$ is a disjoint union of finite partially ordered trees.

3.12 Notes

The main object of studies in this book—modal logics resulting from adding to classical logic one modal operator together with axioms and inference rules describing its properties—was originally created for solving problems that were not directly connected to modal logic. We mean here primarily the Lewis systems **S1**–**S5** of Lewis and Langford (1932). The first system in the series, namely **S3**, was formulated by Lewis (1918) as a logic without the so called *paradoxes of material implication*, i.e., formulas like (A1), which asserts that a true proposition follows from any other proposition even if they speak of entirely different things. His idea was to consider the *strict implication* $\Box(\varphi \to \psi)$ instead of the usual material implication $\varphi \to \psi$. However, this solution was not completely satisfactory because all Lewis systems contain other types of paradoxical formulas—*paradoxes of strict implication*—like $\Box(\Box p \to \Box(q \to p))$. (It is worth noting also that there is a converse approach, when one first formulates a non-modal system axiomatizing some implication $\Rightarrow$ without "paradoxes" and then introduces a necessity operator, for instance by taking $\Box\varphi = (\varphi \Rightarrow \varphi) \Rightarrow \varphi$. We shall not consider systems of that sort and refer the reader to Anderson and Belnap (1975).) The history of the development of (philosophical) modal logic before Lewis is discussed in Lemmon and Scott (1977).

We will not present here axioms of the Lewis systems; the reader can find them as well as formulations in the form of Gentzen-style calculi, say in Feys (1965) and Zeman (1973). Note only that the modern way of axiomatizing modal logics is quite different from that in Lewis and Langford (1932). After Gödel (1933a), the majority of existing modal calculi were constructed by adding to a non-modal basic calculus (say *Cl* or *Int*) a number of modal axioms and rules which do not change the non-modal basis.

The semantical approach to defining logics as the sets of formulas that are valid in frames from certain classes is now also generally accepted. And even if some authors prefer axiomatic systems, they try to formulate modal axioms and inference rules in such a way that the desirable properties of the corresponding frames were quite clear, and the first results that are established for such systems are completeness theorems. Sometimes, however, new logics are formulated purely syntactically without any connection to their relational semantics. Such were, for instance, the provability logics **GL** and **S** (Solovay 1976), the provability semantics of which (see Smoryński, 1985) is of much more import than the relational one, or Grzegorczyk's (1967) logic **Grz**.

The relational semantics for modal logics, presented in this chapter, was created by a number of philosophers and mathematicians. Carnap (1942, 1947) constructed a semantics for **S5** which was actually the same as Kripke models with the universal accessibility relation (although Carnap did not mention any relation). Jónsson and Tarski (1951) explicitly introduced (generalized) frames as relational representations of modal algebras (see Chapter 8). However, at that time this very important paper was not noticed. For instance, Dummett and Lemmon (1959) constructed analogous representations of finite algebras for **S4** apparently not knowing about the work of Jónsson and Tarski. Prior (1957) considered frames of the form $\langle \omega, \leq \rangle$ for interpreting tense operators. And then Kanger (1957a, 1957b), Hintikka (1957, 1961, 1963) and Kripke (1959, 1963a, 1963b, 1965b) developed finally the concept of relational model and proved completeness theorems for a few particular systems. The neighborhood semantics (briefly discussed in Section 3.10) was constructed by Montague (1968) and Scott (1970).

The truth-preserving operations on frames were introduced by Segerberg (1968, 1970, 1971). The technique of unravelling was developed by Dummett and Lemmon (1959) and Sahlqvist (1975); the bulldozer theorem is due to Segerberg (1970). The connection between modal formulas and first (and higher) order properties of their frames is the subject of van Benthem (1983, 1984). Although the deduction theorem was known long ago, Theorem 3.57 seems to be quite new; we were informed about it by M. Kracht.

In Section 3.7 we considered examples of properties that are usually investigated for various kinds of logics. One of them—the problem of reducing modalities—is specifically modal; it is connected with the problem of using and understanding iterated modalities in natural languages. In fact, one of the first achievements of mathematical studies in modal logic was the famous result of Parry (1939) according to which **S3** contains precisely 42 irreducible modalities. It follows in particular that all extensions of **S3** have finitely many irreducible modalities. It is to be noted that to find a complete solution to the problem of reducing modalities in a given modal logic, i.e., to find a set of pairwise non-equivalent irreducible modalities such that any other modality is reducible to one of them, may be rather difficult. For example, although the fact that **T** has infinitely many non-equivalent irreducible modalities had been known for a rather long time—the proof of the similar result for **S2** given by McKinsey (1940) goes through for **T** as well—only Mints (1974) proved that distinct modalities are not equivalent in **T** (see Exercise 5.26). It may be of interest to note in this connection that, as was observed by Bellissima (1989), the set of logics in which no distinct modalities are equivalent contains at least two maximal (with respect to $\subseteq$) logics; see Exercise 5.26. One can show in fact that this set, as well as the set of logics with infinitely many irreducible modalities, contains a continuum of maximal logics, and that no algorithm can recognize, given a formula φ, whether $\mathbf{K}\oplus\varphi$ belongs to one these classes. To conclude the discussion of this topic (we will not return to it later), we mention two more results. Bellissima (1985b) presents a test which gives a complete solution to the problem of reducing modalities in a finitely approximable normal extension of **S4**. Of course, this test cannot be effective, since there is a continuum of such extensions. However, it is not hard to construct an algorithm which solves this problem for finitely axiomatizable (not necessarily finitely approximable) normal modal logics containing **S4**. Bellissima and Mirolli (1989) introduce the functions $\mu(P(L)) = |\{L' : P(L') = P(L)\}|$, $\lambda(n) = |\{L : |P(L)| = n\}|$, $\pi(n) = |\{P(L) : |P(L)| = n\}|$, where $P(L)$ is the set of classes of L-equivalent modalities, study their possible behavior for normal extensions L of **K** and leave as an open problem to investigate it in the class of normal extensions of **D**.

We do not discuss here other properties of modal logics; they will be considered in further chapters. In the *Handbook of Philosophical Logic* (Gabbay and Guenthner 1984) the reader can find brief introductions to deontic, epistemic, tense and provability logic with further references to textbooks and monographs.

In the 1970s different people using different methods (details are in Smoryński, 1985) proved the fixed point theorem of the provability logic **GL**: for any modal formula $\varphi(p, q_1, \dots, q_n)$, where p occurs only within the scope of $\Box$, there is a formula $\psi(q_1, \dots, q_n)$ such that

$$\varphi(\psi(q_1, \dots, q_n), q_1, \dots, q_n) \to \psi(q_1, \dots, q_n) \in \mathbf{GL}.$$

A rather simple semantic proof of this theorem was given by Reinhaar-Olson (1990). Its various arithmetic applications (in particular in the proofs of Gödel's theorems) can be found in Smoryński (1985).

Note by the way that the idea of interpreting the necessity operator as provability in Peano arithmetic was proposed also by Kripke (1963b). Buss (1990) realized this idea; the resulting set of modal formulas contains in particular the logic **S4.1**. Kuznetsov and Muravitskij (1980), Kuznetsov (1985) and Muravitskij (1985) developed an approach to describing provability in PA from the standpoint of intuitionistic propositional logic enriched by a modal provability operator, and established a connection of the resulting logic and its extensions with extensions of **GL**.

Artemov (1980, 1985) considered the problem of describing the modal logics having the arithmetic provability interpretation; a complete solution to this problem was found by Beklemishev (1990).

Although the following notion resembles the fixed point theorem above, its true origin is in the concept of the so called Σ-programming (see Goncharov and Sviridenko, 1985). Mardaev (1992, 1993a, 1993b) calls a *positive* (*modal* or *intuitionistic*) *propositional scheme* any set

$$p_1 = \varphi_1(p_1, \dots, p_m, q_1, \dots, q_n), \dots, p_m = \varphi_m(p_1, \dots, p_m, q_1, \dots, q_n),$$

where all $\varphi_i(p_1, \dots, p_m, q_1, \dots, q_n)$ are (modal or intuitionistic) formulas with only positive occurrences of the variables $p_1, \dots, p_m$. Identifying a formula in a model $\mathfrak{M}$ with its truth-set, we say that a tuple $P_1, \dots, P_m$ is a *fixed point* of this scheme under given values Q_i of q_i if the following holds in $\mathfrak{M}$:

$$P_1 = \varphi_1(P_1, \dots, P_m, Q_1, \dots, Q_n), \dots, P_m = \varphi_m(P_1, \dots, P_m, Q_1, \dots, Q_n).$$

In the cited papers Mardaev solves the problem of finding such fixed points in models for **S4**, **Grz**, **GL** and **Int**. For intuitionistic formulas in one variable similar problems were considered by Ruitenburg (1984).

That **Int** can be embedded in **S4** and so can be considered from a "classical" point of view was noticed by Orlov (1928) and Gödel (1933a). (In fact, Orlov (1928) introduced a provability operator, described the axioms of provability, which were the same as Gödel's axioms for **S4**, and treated the intuitionistic validity of a proposition in the context of its provability. Besides, he introduced the first system of relevant logic.) It is of interest that the first Lewis system **S3** turned out to be a "modal companion" of **Int** too, as was shown by Hacking (1963) and strengthened by Chagrov (1981). Kuznetsov and Muravitskij (1977, 1980), Goldblatt (1978) and Boolos (1980) observed independently that **Grz** is embedded by $^+$ into **GL** and T^+ embeds **Int** into **GL**. The embedding † of **GL** into **K4** in Exercise 3.26 is due to Balbiani and Herzig (1994). For more information and references see Chagrov and Zakharyaschev (1992).

The Gödel embedding of **Int** into **S4** can be extended to an embedding of modal logics on the intuitionistic base into classical polymodal logics; see Fischer-Servi (1977), Shehtman (1979) and Wolter and Zakharyaschev (1996, 1997). For further references concerning intuitionistic modal logics the reader can consult Sotirov (1984) or Božić and Došen (1984).

4

FROM LOGICS TO CLASSES OF LOGICS

We have already met with sufficiently many concrete logics to make some generalizations. Instead of proving the same sort of theorems for each logic separately, we can consider big classes of logics and try to develop general methods for investigating their properties *en masse*. In this chapter we introduce rather abstract concepts of superintuitionistic and modal logics and discuss the general settings of problems associated with them to be examined in the rest of the book.

4.1 Superintuitionistic logics

All the logics considered in the first two chapters have the same type of language and from the set-theoretic point of view are extensions of **Int**. Besides, all of them are closed under MP and Subst. This observation motivates the following definition.

A *superintuitionistic logic* (*si-logic*, for short) in the language $\mathcal{L}$ is any set L of $\mathcal{L}$-formulas satisfying the conditions:

- $\mathbf{Int} \subseteq L$;
- L is closed under modus ponens, i.e., $\varphi \in L$ and $\varphi \to \psi \in L$ imply $\psi \in L$, for every $\varphi, \psi \in \mathbf{For}\mathcal{L}$;
- L is closed under uniform substitution, i.e., $\varphi \in L$ implies $\varphi s \in L$, for every $\varphi \in \mathbf{For}\mathcal{L}$ and every substitution s.

According to the given definition, the set $\mathbf{For}\mathcal{L}$ of all $\mathcal{L}$-formulas is a si-logic; we call it the *inconsistent si-logic*. Clearly, $\mathbf{For}\mathcal{L}$ is the greatest si-logic with respect to inclusion and **Int** is the smallest one. Moreover, it follows from the proof of Theorem 2.58 that we have

Theorem 4.1 *For every consistent si-logic L,* $\mathbf{Int} \subseteq L \subseteq \mathbf{Cl}$.

For this reason consistent si-logics are often called *intermediate logics*. (In the propositional case these two notions are practically identical. However, for first order logics and theories on superintuitionistic bases Theorem 2.58 as well as many other results connecting intuitionistic and classical variants (say, Glivenko's theorem) fail and the term "intermediate logic" becomes almost meaningless.)

Theorem 4.2 *For every family $\{L_i : i \in I\}$ of si-logics, the intersection $\bigcap_{i\in I} L_i$ is also a si-logic.*

Proof Follows immediately from the definition of si-logics. ❑

We introduced **Cl**, **Int** and **ML** semantically, as sets of formulas that are valid in certain frames. Many other si-logics can be constructed in a similar way. For we have

Theorem 4.3 *Let $\mathcal{C}$ be an arbitrary class of intuitionistic frames. Then the set of $\mathcal{L}$-formulas that are valid in all frames in $\mathcal{C}$ is a si-logic.*

Proof Exercise. ❑

The si-logic defined in Theorem 4.3 will be called the *logic of the class $\mathcal{C}$* and denoted by Log$\mathcal{C}$. If $\mathcal{C}$ consists of a single frame $\mathfrak{F}$ then instead of Log$\mathcal{C}$ we write Log$\mathfrak{F}$ and call this logic the *logic of* $\mathfrak{F}$. For example, by Corollary 2.33, $\mathbf{Int} = \mathrm{Log}\mathfrak{T}_n$, for each $n \geq 2$. It is to be noted that Theorem 4.3 does not hold if instead of frames we take models (the set of formulas that are true in a model is not necessarily closed under Subst; see Exercise 4.1). Besides, nothing guarantees that every si-logic is the logic of some class of frames (see Section 6.5).

Another way of constructing si-logics follows directly from the definition: we can take any set of formulas Γ, add it to **Int** and then close the result under MP and Subst. The si-logic L thus obtained is denoted by $\mathbf{Int} + \Gamma$; the formulas in Γ are called *additional* or *extra axioms* of L *over* **Int** and L itself the *extension* of **Int** with Γ. If $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ then along with $\mathbf{Int} + \Gamma$ we write also $\mathbf{Int} + \varphi_1 + \ldots + \varphi_n$. For example, $\mathbf{Cl} = \mathbf{Int} + p \vee \neg p$, $\mathbf{For}\mathcal{L} = \mathbf{Int} + p$.

If a si-logic L can be represented as $L = \mathbf{Int} + \Gamma$ with a finite set Γ then L is said to be *finitely axiomatizable.* Notice that, by the soundness and completeness theorem, the first condition in the definition of si-logics can be replaced by the following one:

- L contains the formulas (A1)–(A9).

By the axioms (A3)–(A5) we clearly have

$$\mathbf{Int} + \varphi_1 + \ldots + \varphi_n = \mathbf{Int} + \varphi_1 \wedge \ldots \wedge \varphi_n,$$

i.e., a si-logic is finitely axiomatizable iff it is axiomatizable by a single extra axiom.

Given logics $L_1 = \mathbf{Int} + \Gamma_1$ and $L_2 = \mathbf{Int} + \Gamma_2$, the logic $L = \mathbf{Int} + \Gamma_1 \cup \Gamma_2$ is called the *sum* of L_1 and L_2. If in the definition of $\mathbf{Int} + \Gamma$ we replace **Int** with a si-logic L then the resulting si-logic $L' = L + \Gamma$ is the *extension* of L with Γ; in this case we say that the formulas in Γ are *additional* or *extra axioms* of L' *over* L. L' is *finitely axiomatizable over* L if $L' = L + \Gamma$ for some finite set Γ. The sum of L_1 and L_2 can be represented now as $L_1 + L_2$ or $L_2 + L_1$. The sum of a family of si-logics $\{L_i : i \in I\}$, i.e., the closure of $\bigcup_{i \in I} L_i$ under MP and Subst, is denoted by $\sum_{i \in I} L_i$.

Derivations in a si-logic $L = \mathbf{Int} + \Gamma$ are defined similarly to derivations in **Int**: the only difference is that now together with the axioms of **Int** we can use the extra axioms in Γ. If φ is derivable in L then we write $\vdash_L \varphi$. Clearly, $\varphi \in L$ iff $\vdash_L \varphi$. In the same way as in Section 1.3 we can define a *derivation of φ in L from a set of assumptions* Γ (notation: $\Gamma \vdash_L \varphi$) and prove the following generalization of the deduction theorem for **Int**:

Theorem 4.4 $\Gamma, \psi \vdash_L \varphi$ *iff* $\Gamma \vdash_L \psi \to \varphi$.

It should be clear that $\varphi \in L$ iff $L \vdash_{\mathbf{Int}} \varphi$. Since the congruence rules (for $\wedge$, $\vee$, $\to$) are derivable in **Int**, they are derivable in every si-logic too. So the equivalent replacement theorem of Section 1.4 holds for all si-logics as well.

To axiomatize the sum of si-logics, we can simply join their axioms. It is somewhat more difficult to axiomatize the intersection. Call the formula

$$\varphi(p_1, \dots, p_n) \vee \psi(p_{n+1}, \dots, p_{n+m})$$

the *repeatless disjunction* of the formulas $\varphi(p_1, \dots, p_n)$ and $\psi(p_1, \dots, p_m)$ and denote it by $\varphi \underline{\vee} \psi$.

Theorem 4.5 *Let* $L_1 = \mathbf{Int} + \{\varphi_i : i \in I\}$ *and* $L_2 = \mathbf{Int} + \{\psi_j : j \in J\}$. *Then* $L_1 \cap L_2 = \mathbf{Int} + \{\varphi_i \underline{\vee} \psi_j : i \in I, j \in J\}$.

Proof Suppose $\chi \in L_1 \cap L_2$. By the deduction theorem and the properties of $\wedge$, we have $\bigwedge_{i \in I'} \varphi_i' \to \chi \in \mathbf{Int}$ and $\bigwedge_{j \in J'} \psi_j' \to \chi \in \mathbf{Int}$ for some finite I' and J' such that every φ_i' and ψ_j', for $i \in I'$, $j \in J'$, are substitution instances of some φ_k and ψ_l, for $k \in I$, $l \in J$, respectively. Using the axiom (A8) and the law of distributivity, we obtain then

$$\bigwedge_{i \in I', j \in J'} (\varphi_i' \vee \psi_j') \to \chi \in \mathbf{Int},$$

from which $\chi \in \mathbf{Int} + \{\varphi_i \underline{\vee} \psi_j : i \in I, j \in J\}$ because $\varphi_i' \vee \psi_j'$ is a substitution instance of $\varphi_i \underline{\vee} \psi_j$.

Conversely, assume that $\chi \in \mathbf{Int} + \{\varphi_i \underline{\vee} \psi_j : i \in I, j \in J\}$. Then χ is derivable in **Int** from some finite set of substitution instances $\varphi_i' \vee \psi_j'$ of axioms of this logic. Using (A6) and (A7), we can also derive χ from the set of φ_i' as well as from the set of ψ_i'. Consequently, $\chi \in L_1 \cap L_2$. ❑

Clearly, **Int** in the formulation of Theorem 4.5 can be replaced with any other si-logic.

Although the sum of logics differs in general from the union of them (see Exercise 4.3), they have a few important common properties.

Theorem 4.6 *The sum of si-logics is idempotent, commutative, associative and distributes over the intersection; the intersection of si-logics distributes over the (infinite) sum.*

Proof We show only that

$$L \cap \sum_{i \in I} L_i = \sum_{i \in I} (L \cap L_i)$$

and leave the rest to the reader. Suppose $L = \mathbf{Int} + \Gamma$ and $L_i = \mathbf{Int} + \Delta_i$, for $i \in I$. Then we have

Table 4.1 *A list of standard superintuitionistic logics*

For	=	$\mathbf{Int} + p$
Cl	=	$\mathbf{Int} + p \vee \neg p$
SmL	=	$\mathbf{Int} + (\neg q \to p) \to (((p \to q) \to p) \to p)$
KC	=	$\mathbf{Int} + \neg p \vee \neg\neg p$
LC	=	$\mathbf{Int} + (p \to q) \vee (q \to p)$
SL	=	$\mathbf{Int} + ((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$
KP	=	$\mathbf{Int} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$
WKP	=	$\mathbf{Int} + (\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r)$
$\mathbf{ND}_k$	=	$\mathbf{Int} + (\neg p \to \neg q_1 \vee \ldots \vee \neg q_k) \to$ $(\neg p \to \neg q_1) \vee \ldots \vee (\neg p \to \neg q_k),\ k \geq 2$
$\mathbf{BD}_n$	=	$\mathbf{Int} + \boldsymbol{bd}_n$
$\mathbf{BW}_n$	=	$\mathbf{Int} + \bigvee_{i=0}^{n}(p_i \to \bigvee_{j \neq i} p_j)$
$\mathbf{BTW}_n$	=	$\mathbf{Int} + \bigwedge_{0 \leq i < j \leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{j \neq i} \neg p_j)$
$\mathbf{T}_n$	=	$\mathbf{Int} + \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i \neq j} p_j) \to \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$
$\mathbf{B}_n$	=	$\mathbf{Int} + \bigwedge_{i=0}^{n}(\neg p_i \leftrightarrow \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$
$\mathbf{NL}_n$	=	$\mathbf{Int} + \boldsymbol{nf}_n$, where $\boldsymbol{nf}_0 = \bot$, $\boldsymbol{nf}_1 = p$, $\boldsymbol{nf}_2 = \neg p$, $\boldsymbol{nf}_\omega = \top$ $\boldsymbol{nf}_{2m+3} = \boldsymbol{nf}_{2m+1} \vee \boldsymbol{nf}_{2m+2}$, $\boldsymbol{nf}_{2m+4} = \boldsymbol{nf}_{2m+3} \to \boldsymbol{nf}_{2m+1}$

$$
\begin{aligned}
L \cap \textstyle\sum_{i\in I} L_i &= (\mathbf{Int} + \Gamma) \cap (\mathbf{Int} + \textstyle\bigcup_{i\in I} \Delta_i) \\
&= \mathbf{Int} + \{\varphi \underline{\vee} \psi : \varphi \in \Gamma, \psi \in \textstyle\bigcup_{i\in I} \Delta_i\} \\
&= \mathbf{Int} + \textstyle\bigcup_{i\in I}\{\varphi \underline{\vee} \psi : \varphi \in \Gamma, \psi \in \Delta_i\} \\
&= \textstyle\sum_{i\in I}(\mathbf{Int} + \{\varphi \underline{\vee} \psi : \varphi \in \Gamma, \psi \in \Delta_i\}) \\
&= \textstyle\sum_{i\in I}((\mathbf{Int} + \Gamma) \cap (\mathbf{Int} + \Delta_i)).
\end{aligned}
$$

❑

Note, however, that in general the sum does not distribute over the infinite intersection, i.e., $L + \bigcap_{i\in I} L_i$ may differ from $\bigcap_{i\in I}(L + L_i)$ (see Exercise 6.16).

The family of si-logics together with the operations $\cap$ and $+$ is called the *lattice of si-logics*[7] and denoted by Ext**Int**. More generally, if $L, L' \in$ Ext**Int** and $L \subseteq L'$ then we call L' an *extension* of L, L a *sublogic* of L' and denote the family of L's extensions by Ext$L$.

A list of standard superintuitionistic logics is presented in Table 4.1.

[7] For a definition of lattice see Section 7.3.

4.2 Modal logics

All the modal logics we met with in Chapter 3 (except those in Section 3.10) contain the logic **K** and are closed under MP and Subst. All of them except **S** are also closed under the rule of necessitation RN.

A *quasi-normal modal logic* in the language $\mathcal{ML}$ is any set L of $\mathcal{ML}$-formulas such that

- $\mathbf{K} \subseteq L$;
- L is closed under MP and Subst.

The smallest (with respect to inclusion) quasi-normal modal logic is **K** and the greatest one is the *inconsistent modal logic* $\mathbf{For}\mathcal{ML}$.

A quasi-normal modal logic L is called *normal* if

- L is closed under RN, i.e., $\varphi \in L$ implies $\Box\varphi \in L$, for every formula φ.

Every quasi-normal logic L can be represented in the form

$$L = \mathbf{K} + \Gamma, \tag{4.1}$$

where $\Gamma \subseteq \mathbf{For}\mathcal{ML}$ and $+$ means, as before, the closure (of $\mathbf{K} \cup \Gamma$) under MP and Subst. Every normal logic L is represented as

$$L = \mathbf{K} \oplus \Gamma, \tag{4.2}$$

where $\oplus$ means the closure under MP, Subst and RN.

The formulas in Γ in the representation (4.1) are called the *additional* or *extra axioms of L over* **K**; a quasi-normal logic L is *finitely axiomatizable* if it can be represented in the form (4.1) with a finite Γ. The corresponding notions are defined for normal logics by replacing (4.1) with (4.2) and dropping the prefix "quasi". It is to be noted that a finitely axiomatizable normal logic is not necessarily finitely axiomatizable if we consider it as a quasi-normal one (see Exercise 4.6). Replacing **K** in (4.1) and (4.2) by an arbitrary (normal or quasi-normal) modal logic L, we get the notions of axiomatizability over L.

As in the case of si-logics, the intersection of (quasi-) normal modal logics is again a (quasi-) normal modal logic. The sum can be defined now in two ways:

- $\sum_{i\in I} L_i$ is the closure of $\bigcup_{i\in I} L_i$ under MP and Subst, and
- $\bigoplus_{i\in I} L_i$ is the closure of $\bigcup_{i\in I} L_i$ under MP, Subst and RN.

The reader can easily check that $\mathbf{K} + \varphi_1 + \ldots + \varphi_n = \mathbf{K} + \varphi_1 \wedge \ldots \wedge \varphi_n$, $\mathbf{K} \oplus \varphi_1 \oplus \ldots \oplus \varphi_n = \mathbf{K} \oplus \varphi_1 \wedge \ldots \wedge \varphi_n$.

The family of normal (quasi-normal) modal logics, containing a logic L, together with the operations $\cap$ and $\oplus$ $(+)$ is called the *lattice of normal (quasi-normal) extensions of L* and denoted by $\mathrm{NExt}L$ (respectively, $\mathrm{Ext}L$).

The two kinds of modal logics—the two ways of forming the closure under inference rules, to be more exact—give us two variants of derivations from assumptions: with RN and without it. In the same way as in Section 3.6 one can prove the following generalization of the deduction theorem for **K** (in which $\vdash^*$ means the derivability with RN and $\vdash$ without it).

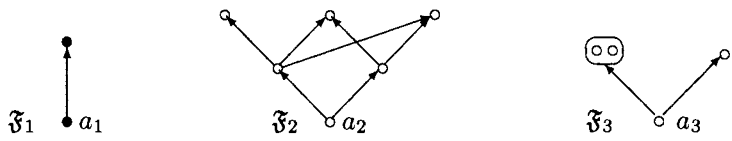


FIG. 4.1.

Theorem 4.7 *For every* $L \in \text{Ext}\mathbf{K}$,
(i) $\Gamma, \psi \vdash_L \varphi$ *iff* $\Gamma \vdash_L \psi \to \varphi$;
(ii) $\Gamma, \psi \vdash_L^* \varphi$ *iff there is* $m \geq 0$ *such that* $\Gamma \vdash_L^* \Box^0\psi \wedge \ldots \wedge \Box^m\psi \to \varphi$.

If $L \in \text{NExt}(\mathbf{K} \oplus \boldsymbol{tra_n})$ then we can clearly take $m = n$. Moreover, Exercise 4.13 gives a sort of conversion of this observation.

The semantical way of constructing modal logics analogous to that in Theorem 4.3 provides us with only (some) logics in NExt**K**.

Theorem 4.8 *Let* $\mathcal{C}$ *be a class of modal frames. Then the set* Log$\mathcal{C}$ *of* $\mathcal{ML}$-*formulas that are valid in all frames in* $\mathcal{C}$ *is a normal modal logic.*

Proof Exercise. ❑

Log$\mathcal{C}$ is called the *logic of the class* $\mathcal{C}$. If $\mathcal{C}$ consists of a single frame $\mathfrak{F}$ then the logic of $\mathcal{C}$ is denoted also by Log$\mathfrak{F}$.

A Kripke semantics for quasi-normal modal logics will be introduced in Section 5.6. Here we only note that for every frame $\mathfrak{F}$ and every point x in it, the set

$$\text{Log}\,\langle \mathfrak{F}, \{x\}\rangle = \{\varphi \in \mathbf{For}\mathcal{ML} : (\mathfrak{F}, x) \models \varphi\}$$

is a quasi-normal but not necessarily normal modal logic.

Example 4.9 Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ be the transitive frames shown in Fig. 4.1. Then the quasi-normal logics $L_i = \text{Log}\,\langle \mathfrak{F}_i, \{a_i\}\rangle$, for $i = 1, 2, 3$, are not normal. Indeed, consider the formulas

$$\varphi_1 = \Diamond\top, \quad \varphi_2 = \bigwedge_{i=1}^{3} \Diamond\psi_i \to \bigvee_{i=1}^{3} \Diamond(\bigwedge_{i \neq j} \Diamond\psi_j \wedge \neg\Diamond\psi_i), \quad \varphi_3 = \Diamond\boldsymbol{grz},$$

where $\psi_1 = \Box(p \wedge q)$, $\psi_2 = \Box(\neg p \wedge q)$, $\psi_3 = \Box(p \wedge \neg q)$. The reader can check that $\varphi_i \in L_i$ but $\Box\varphi_i \notin L_i$, for $i = 1, 2, 3$.

Since the congruence rules for $\wedge$, $\vee$, $\to$ and $\Box$ are derivable in **K**, the equivalent replacement theorem holds for all logics in NExt**K**. However, this is not the case for logics in Ext**K**. For we have

Theorem 4.10 *A quasi-normal logic* L *is normal iff* $p \leftrightarrow q/\Box p \leftrightarrow \Box q$ *is an admissible rule in* L *(or, which is equivalent, iff the equivalent replacement theorem holds for* L*).*

Proof The implication ($\Rightarrow$) is clear. To show ($\Leftarrow$), suppose that $\varphi \in L$. Then $\varphi \leftrightarrow \top \in L$ and so $\Box\varphi \leftrightarrow \Box\top \in L$, from which $\Box\varphi \in L$, since $\Box\top \leftrightarrow \top \in \mathbf{K}$. ❑

Analogously to Theorem 4.5 one can prove the following:

Theorem 4.11 (i) *Let* $L_1 = \mathbf{K} + \{\varphi_i : i \in I\}$ *and* $L_2 = \mathbf{K} + \{\psi_j : j \in J\}$. *Then* $L_1 \cap L_2 = \mathbf{K} + \{\varphi_i \underline{\vee} \psi_j : i \in I, j \in J\}$.

(ii) *Let* $L_1 = \mathbf{K} \oplus \{\varphi_i : i \in I\}$ *and* $L_2 = \mathbf{K} \oplus \{\psi_j : j \in J\}$. *Then* $L_1 \cap L_2 = \mathbf{K} \oplus \{\Box^k\varphi_i \underline{\vee} \Box^l\psi_j : i \in I, j \in J, k, l \geq 0\}$.

Proof Exercise. ❑

The reader can easily check also that Theorem 4.6 holds for both types of sum of modal logics.

A few standard normal modal logics are listed in Table 4.2.

4.3 "The roads we take"

The act of abstraction we made in the two previous sections is aimed mainly to work out a general theory which would provide us with tools for dealing with arbitrary modal and si-logics and methods allowing to solve problems not for each logic individually, but for big classes of them at once. In this section we discuss the most important directions in which this theory will be developed.

Let us begin with methods of constructing logics. We have met with two of them: the *syntactical* or *axiomatic method* which defines a logic by means of indicating its axioms and inference rules, and the *semantical* one which describes a logic as the set of formulas that are "valid" (in one sense or another) in some "model structures" like truth-tables, Kripke frames or models.

Constructing a logic axiomatically, its creator is trying to select a possibly minimal list of axioms and inference rules which reflect his ideas of what principles of reasoning should be included in the logic. **Int**, **S4**, **S5**, **GL** and many other logics were constructed in this way. To aim at minimality or laconicity of axiomatic systems means the desire to present them in the simplest and clearest manner (besides, it is often an interesting mathematical problem).

We can distinguish, for instance, between finitely and infinitely axiomatizable logics. A finitely axiomatizable logic, its finite set of axioms and inference rules, to be more precise, will be called, as before, a *calculus*. Dealing with a calculus, we have at hand only its axioms and inference rules; the logic represented by the calculus is what is deducible in it. The very same logic can be represented by different calculi. This leads to the (algorithmic) problem of deciding whether two given calculi are *equivalent*, i.e., axiomatize the same logic. A closely related problem is to recognize if two given formulas φ and ψ are *deductively equal* in Ext$L$ (NExt$L$) in the sense that $L + \varphi = L + \psi$ (respectively, $L \oplus \varphi = L \oplus \psi$).

As we shall see later, far from all modal and si-logics can be represented by calculi. The following criterion is useful for proving that a given logic is not finitely axiomatizable.

Table 4.2 *A list of standard normal modal logics*

D	=	$\mathbf{K} \oplus \Box p \to \Diamond p$
T	=	$\mathbf{K} \oplus \Box p \to p$
KB	=	$\mathbf{K} \oplus p \to \Box\Diamond p$
K4	=	$\mathbf{K} \oplus \Box p \to \Box\Box p$
K5	=	$\mathbf{K} \oplus \Diamond\Box p \to \Box p$
$\mathbf{Alt}_n$	=	$\mathbf{K} \oplus \Box p_1 \vee \Box(p_1 \to p_2) \vee \ldots \vee \Box(p_1 \wedge \ldots \wedge p_n \to p_{n+1})$
D4	=	$\mathbf{K4} \oplus \Diamond\top$
S4	=	$\mathbf{K4} \oplus \Box p \to p$
GL	=	$\mathbf{K4} \oplus \Box(\Box p \to p) \to \Box p$
For	=	$\mathbf{K4} \oplus p$
Grz	=	$\mathbf{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p$
K4.1	=	$\mathbf{K4} \oplus \Box\Diamond p \to \Diamond\Box p$
K4.2	=	$\mathbf{K4} \oplus \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$
K4.3	=	$\mathbf{K4} \oplus \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$
S4.1	=	$\mathbf{S4} \oplus \Box\Diamond p \to \Diamond\Box p$
S4.2	=	$\mathbf{S4} \oplus \Diamond\Box p \to \Box\Diamond p$
S4.3	=	$\mathbf{S4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$
Triv	=	$\mathbf{K4} \oplus \Box p \leftrightarrow p$
Verum	=	$\mathbf{K4} \oplus \Box p$
S5	=	$\mathbf{S4} \oplus p \to \Box\Diamond p$
K4B	=	$\mathbf{K4} \oplus p \to \Box\Diamond p$
$\mathbf{A}^*$	=	$\mathbf{GL} \oplus \Box\Box p \to \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$
K4Z	=	$\mathbf{K4} \oplus \Box(\Box p \to p) \to (\Diamond\Box p \to \Box p)$
Dum	=	$\mathbf{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p)$
$\mathbf{D4G}_1$	=	$\mathbf{D4} \oplus \Box(\Box^+ p \vee \Box^+ \neg p) \to \Box p \vee \Box\neg p$
K4H	=	$\mathbf{K4} \oplus p \to \Box(\Diamond p \to p)$
$\mathbf{K4Alt}_n$	=	$\mathbf{K4} \oplus \Box p_1 \vee \Box(p_1 \to p_2) \vee \ldots \vee \Box(p_1 \wedge \ldots \wedge p_n \to p_{n+1})$
$\mathbf{K4BW}_n$	=	$\mathbf{K4} \oplus \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \le i \ne j \le n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j))$
$\mathbf{K4BD}_n$	=	$\mathbf{K4} \oplus \boldsymbol{bd}_n$
$\mathbf{K4}_{n,m}$	=	$\mathbf{K4} \oplus \Box^n p \to \Box^m p$, for $1 \le m < n$

Theorem 4.12. (Tarski's criterion) *Let L_0 be a superintuitionistic or quasi-normal modal logic in a countable language. A logic $L \in \mathrm{Ext}L_0$ is not finitely axiomatizable over L_0 iff there exists an infinite sequence of logics $L_1 \subset L_2 \subset L_3 \ldots$ in $\mathrm{Ext}L_0$ such that $L = \sum_{i>0} L_i$. A modal logic $L \in \mathrm{NExt}L_0$ is not finitely axiomatizable over L_0 iff there is an infinite sequence of logics $L_1 \subset L_2 \subset L_3 \ldots$*

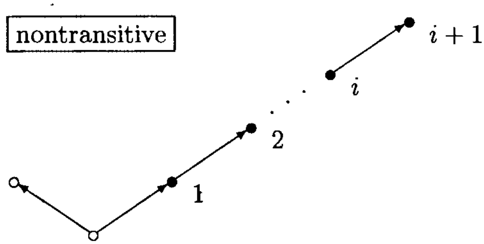


FIG. 4.2.

in $\mathrm{NExt}L_0$ *such that* $L = \bigoplus_{i>0} L_i$.

Proof ($\Rightarrow$) Let $\psi_1, \psi_2, \ldots$ be an enumeration of all formulas in the language of L_0. Define a sequence $\varphi_1, \varphi_2, \ldots$ as follows: φ_1 is the first formula in this enumeration that belongs to $L - L_0$, and for $i \geq 1$, φ_{i+1} is the first formula in the list $\psi_1, \psi_2, \ldots$ that belongs to L but not to $L_i = L_0 + \varphi_1 + \ldots + \varphi_i$. As a result we have $L_i \subset L_{i+1}$ and $L = \sum_{i>0} L_i$. In the case of normal modal logics it suffices to replace $+$ in the proof above by $\oplus$.

($\Leftarrow$) If we assume that L is finitely axiomatizable then there must be i such that L_i contains all axioms of L and so $L_i = L$, which is a contradiction. ❑

We demonstrate the use of this criterion by the following:

Example 4.13 According to Theorems 4.5 and 4.11, the intersection of two finitely axiomatizable quasi-normal or si-logics is finitely axiomatizable too. However, this is not the case for logics in $\mathrm{NExt}\mathbf{K}$. Consider, for instance, the logics $L_1 = \mathbf{K} \oplus \Diamond\top$ and $L_2 = \mathbf{K} \oplus \Box p \vee \Box\neg p$ and show that $L_1 \cap L_2$ is not finitely axiomatizable as a normal logic.

By Theorem 4.11, $L_1 \cap L_2 = \mathbf{K} \oplus \{\Box^k\Diamond\top \vee \Box^l(\Box p \vee \Box\neg p) : k, l \geq 0\}$ and so $L_1 \cap L_2 = \bigcup_{i \geq 0} L^i$, where

$$L^i = \mathbf{K} \oplus \{\Box^k\Diamond\top \vee \Box^l(\Box p \vee \Box\neg p) : 0 \leq k, l \leq i\}.$$

Thus, according to Theorem 4.12, it is enough to show that the formula $\Box^{i+1}\Diamond\top \vee \Box^{i+1}(\Box p \vee \Box\neg p)$ is not in L^i. To this end one can use the frame $\mathfrak{F}$ shown in Fig. 4.2. Indeed, it is easy to see that $\mathfrak{F} \models L^i$ and $\mathfrak{F} \not\models \Box^{i+1}\Diamond\top \vee \Box^{i+1}(\Box p \vee \Box\neg p)$. It should be clear, however, that the intersection of finitely axiomatizable logics in $\mathrm{NExt}\mathbf{K4}$ is finitely axiomatizable as well (see Exercise 4.12).

The next level of complexity in axiomatic representations of logics is the so called *recursive axiomatization*, which means that there is an algorithm recognizing axioms, and the *recursively enumerable axiomatization*, when there is an algorithm generating a sequence of all axioms. In Section 16.2 we shall see that in fact these two notions are equivalent. Besides an effective description of axioms of a logic L a recursive axiomatization provides an algorithm enumerating (generating) precisely all the formulas in L.

However, there are more logics than algorithms: later on we shall meet with various continual families of modal and si-logics, while there are "only" countably many algorithms. Another important characteristic of a "simple" infinite axiomatization is its independence.

To understand the structure of the class (N)ExtL_0 it may be useful to find a set Γ of formulas which is *complete* in the sense that its formulas are able to axiomatize all logics in the class and *independent* in the sense that it contains no complete proper subsets. Such a set (if it exists) may be called an *axiomatic basis* of (N)ExtL_0. Its role is comparable with the role of a basis in a vector space. The existence of an axiomatic basis depends on whether every logic in the class can be represented as the sum of "indecomposable" or prime logics. A logic $L \in$ (N)ExtL_0 is said to be *prime* in (N)ExtL_0 if for any family $\{L_i : i \in I\}$ of logics in (N)ExtL_0, $L = \sum_{i \in I} L_i$ (respectively, $L = \bigoplus_{i \in I} L_i$) implies $L = L_i$ for some $i \in I$. A formula φ is *prime* in (N)ExtL_0 if $L_0 + \varphi$ ($L_0 \oplus \varphi$) is prime.

Proposition 4.14 *Suppose a set of formulas Γ is complete for* (N)ExtL_0 *and contains no distinct deductively equal in* (N)ExtL_0 *formulas. Then Γ is an axiomatic basis for* (N)ExtL_0 *iff every formula in Γ is prime.*

Proof We consider only the class ExtL_0.

($\Rightarrow$) If $\varphi \in \Gamma$ is not prime then $L_0 + \varphi = L_0 + \Delta_1 + \Delta_2$ for some sets $\Delta_1, \Delta_2 \subseteq \Gamma$ such that $L_0 + \Delta_i \subset L_0 + \varphi$, $i = 1, 2$. Consequently, $\varphi \notin \Delta_1 \cup \Delta_2$ and so $\Gamma - \{\varphi\}$ is complete for ExtL_0, which is a contradiction.

($\Leftarrow$) Suppose otherwise. Then for some formula $\varphi \in \Gamma$, the set $\Gamma - \{\varphi\}$ is complete for ExtL_0 and so there is a finite set $\Delta \subset \Gamma$ such that $\varphi \notin \Delta$ and $L_0 + \varphi = L_0 + \Delta$. But then $L_0 + \varphi = L_0 + \psi$, for some $\psi \in \Delta$, which is a contradiction. ❑

Let us turn now to the semantical way of constructing logics. Until now we have operated with two semantical structures: Kripke frames and models. In the sequel we shall consider also logical matrices, algebras and general frames. As before we say that a logic L is *characterized* (or *determined*) by a class $\mathcal{C}$ of such kind of structures if L coincides with the set of formulas (in the language of L) that are valid in all members of $\mathcal{C}$. We can also divide the notion of characterization into the two parts: *soundness* and *completeness*. The soundness means that all structures in $\mathcal{C}$ validate L and the completeness that any formula that is not in L is separated from L by a structure in $\mathcal{C}$.

Dealing with the Kripke semantics, we can try to characterize logics by classes of models or by classes of frames. Neither of these ways is perfect. As we shall see in the next part, all logics under consideration are determined by suitable classes of models. However, Kripke frames fail to do this. On the other hand, not every class $\mathcal{C}$ of models determines a logic: the set of formulas that are true in $\mathcal{C}$ is not necessarily closed under Subst. Such sets of formulas are called *theories*, and models are their semantical counterparts. Frames, as we saw, determine logics. Moreover, being properly generalized, they can determine all of them and so can be regarded as semantical counterparts of logics in Ext**Int** and Ext**K**.

The same logic can be characterized by different classes of structures. For example, **Int** is determined by the class of all Kripke frames or the class of finite frames or that of finite trees. Of course, we are interested in finding the simplest (in one sense or another) classes of structures characterizing a given logic. One possible measure of complexity is the cardinality. For Kripke frames we can define then the following hierarchy of modal and si-logics.

The simplest in this sense are *tabular logics* each of which is characterized by some finite frame. These logics are very nice to deal with: the key problem of recognizing whether a formula φ belongs to a tabular logic L is decided by the routine inspection of all possible valuations of φ's variables in the finite frame characterizing L. Other important properties of tabular logics will be considered in Chapter 12. A good example of a non-trivial class of tabular logics is Ext**S5**: each logic in it except **S5** itself is characterized by a finite cluster. However, the majority of interesting logics are not tabular.

The next in our hierarchy is the class of *finitely approximable logics* which are characterized by (infinite in general) classes of finite Kripke frames. The reason for this name is that every such logic L is the intersection of tabular logics (those determined by the frames in the class characterizing L), i.e., can be "approximated" by a descending sequence of tabular logics. The finite approximable logics are known also as the logics with the *finite frame property*. In Section 8.4 we shall see that the finitely approximable logics are exactly the logics having the *finite model property*, i.e., those that are complete with respect to classes of finite Kripke models.

The class of finitely approximable logics is of great importance. First, it contains almost all standard modal and si-logics. And second, all finitely axiomatizable logics in this class turn out to be decidable, as follows from Harrop's theorem to be proved in Section 16.2 (the reader can easily find a decision algorithm himself). Note, however, that proving the decidability of **Int** and **K** we used not the finite approximability in general but the fact that to separate a formula φ from **Int** or **K** it suffices to consider frames with $\leq 2^{|\mathbf{Sub}\varphi|}$ points. The number of subformulas in φ may be called the *length* of φ; we denote it by $l(\varphi)$. And a logic L such that every $\varphi \notin L$ is separated from L by a frame of cardinality $\leq 2^{l(\varphi)}$ is called *exponentially approximable*. In this connection the questions arise whether it is possible to reduce the exponential approximability to the *polynomial* or even *linear* one, and what kind of lower bounds for the complexity of refutation frames we can expect in general for finitely approximable logics. Complexity problems of that kind will be discussed in Chapter 18. Meanwhile, we just show an example of a linearly approximable (but not tabular) logic.

Example 4.15 Let L be the si-logic determined by the class of finite linearly ordered frames. (In the next chapter we shall see that L coincides with $\mathbf{LC} = \mathbf{Int} + (p \to q) \vee (q \to p)$, known as the *Dummett logic* or the *chain logic*.) If $\varphi \notin L$ then φ is refuted in a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on a finite linear frame $\mathfrak{F}$. Construct a submodel $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ of $\mathfrak{M}$ by putting into it only the final point in $\mathfrak{F}$, the final points in the sets $\{x : x \not\models p\}$ for $p \in \mathbf{Var}\varphi$ and taking the restriction $\mathfrak{U}$ of $\mathfrak{V}$ to $\mathfrak{G}$. One can readily prove by induction on the construction of $\psi \in \mathbf{Sub}\varphi$ that for every point x in $\mathfrak{G}$, $(\mathfrak{N}, x) \models \psi$ iff $(\mathfrak{M}, x) \models \psi$ and that if $(\mathfrak{M}, x) \not\models \psi$ for some x in $\mathfrak{F}$ then there is $y \in x\uparrow$ in $\mathfrak{G}$ such that $(\mathfrak{N}, x) \not\models \psi$. It follows that $\mathfrak{G} \not\models \varphi$ and $|\mathfrak{G}| \leq l(\varphi)$, i.e., L is linearly approximable.

As we shall see in Chapter 6, not all logics are finitely approximable. So the next level in our hierarchy is the family of logics characterized by countable frames. More generally, for an infinite cardinal $\varkappa$, we may say a logic is *$\varkappa$-approximable* if it is determined by frames of cardinality $\leq \varkappa$. That this division makes sense is also shown in Chapter 6.

Finally, we call a logic just *Kripke complete* if it is characterized by some class of Kripke frames. We already know an example of a Kripke incomplete logic: it is Solovay's **S** which has no Kripke frames at all. In Chapter 6 we shall construct incomplete logics in NExt**K4**, NExt**S4**, and Ext**Int**.

In view of these incompleteness results we are facing the problem of finding more powerful semantical instruments than Kripke frames. One way of constructing an adequate semantics for modal and si-logics is to look at them from the algebraic standpoint. As a result of the algebraization, carried out in Chapter 7, with each logic under consideration we associate a variety of nice algebraic structures—Boolean algebras with an additional operator (which is similar to the topological interior operator), and pseudo-Boolean algebras (closely related to the algebras of open sets in topological spaces). Another way of recovering completeness is to impose a restriction on possible valuations in Kripke frames, which leads us to the so called general frames. A remarkable result, discovered by Jónsson and Tarski (1951) (a few years before the creation of Kripke semantics) is that general frames are relational representations of the corresponding algebras, naturally generalizing Stone's representation of Boolean algebras as set fields. We shall consider general frames and duality theory, studying the relationship between algebras and general frames, in Chapter 8.

So far we have considered semantical characterizations of logics, i.e., of the formulas derivable in them. But there is one more fundamental syntactical notion for which we should also find a semantical counterpart, namely, that of derivability from assumptions. Dealing with the Kripke semantics and the relation $\vdash$ (allowing only MP), we say a normal modal or si-logic L is *strongly characterized* (or *determined*) by a class $\mathcal{C}$ of Kripke frames if for any set of formulas Γ and any formula φ (in the language of L), $\Gamma \vdash_L \varphi$ iff for every model $\mathfrak{M}$ based on a frame in $\mathcal{C}$ and every point x in $\mathfrak{M}$, $(\mathfrak{M}, x) \models \Gamma$ implies $(\mathfrak{M}, x) \models \varphi$. A logic that is strongly characterized by some class of Kripke frames is called *strongly Kripke complete*. An equivalent definition of strong completeness is provided by the following:

Proposition 4.16 *A logic $L \in$ NExt**K** or $L \in$ Ext**Int** is strongly Kripke complete iff every L-consistent tableau is realizable in a model based on a Kripke frame for L.*

Proof The implication ($\Leftarrow$) is clear: if $\Gamma \not\vdash_L \varphi$ then there is a model $\mathfrak{M}$ based on a frame for L realizing $(\Gamma, \{\varphi\})$, i.e., there is a point x in $\mathfrak{M}$ such that $(\mathfrak{M}, x) \models \Gamma$ and $(\mathfrak{M}, x) \not\models \varphi$. To prove the converse, suppose a tableau (Γ, Δ) is L-consistent and let p be a variable not occurring in it. We claim that the tableau $(\Gamma \cup \{\varphi \to p : \varphi \in \Delta\}, \{p\})$ is L consistent too. For suppose otherwise. Then

$$\bigwedge_{i=1}^{n} \psi_i \wedge \bigwedge_{i=1}^{m} (\varphi_i \to p) \to p \in L$$

for some $\psi_i \in \Gamma$ and $\varphi_i \in \Delta$. It follows by Subst that

$$\bigwedge_{i=1}^{n} \psi_i \wedge \bigwedge_{i=1}^{m} (\varphi_i \to \bigvee_{j=1}^{m} \varphi_j) \to \bigvee_{j=1}^{m} \varphi_j \in L$$

and so $\bigwedge_{i=1}^{n} \psi_i \to \bigvee_{j=1}^{m} \varphi_j \in L$, contrary to (Γ, Δ) being L-consistent. (Note that in the modal case instead of p one can use $\bot$.) Since L is strongly complete, we have a model $\mathfrak{M}$ based on a frame for L and a point x in it such that $x \models \Gamma \cup \{\varphi \to p : \varphi \in \Delta\}$ and $x \not\models p$, from which $x \not\models \varphi$ for all $\varphi \in \Delta$. ❑

Strongly Kripke complete logics are known also as *compact* ones; however, we shall use the term "compactness" in another sense.

For the consequence relation $\vdash^*$ (allowing both MP and RN) we seem to need a somewhat different semantical counterpart. Say that a logic L in NExt**K** is *globally Kripke complete* if for any finite Γ and φ, $\Gamma \vdash_L^* \varphi$ iff for every model $\mathfrak{M}$ based on a Kripke frame for L, $\mathfrak{M} \models \Gamma$ implies $\mathfrak{M} \models \varphi$. L is *strongly globally complete* if this holds for arbitrary (not necessarily finite) sets of formulas Γ. However, we shall prove in Section 10.1 that actually for logics in infinite languages the notions of strong completeness and strong global completeness are equivalent. Of course, global Kripke completeness can be relativized to finite frames, in which case we talk about *global finite approximability*.

It is worth mentioning here that to formalize the notion "φ logically entails ψ" is one of the central problems in logic. Syntactically one can explicate it as $\varphi \to \psi \in L$, or $\Box(\varphi \to \psi) \in L$, or $\varphi \vdash_L^* \psi$ for some suitable logic L. At the semantical level it is of interest to consider the relation $\varphi \models_{\mathcal{C}} \psi$ which means that ψ is valid in all those frames in the class $\mathcal{C}$ that validate φ, or the local, i.e., point-wise variant of this relation.

Neither the syntactical nor the semantical way of constructing logics is satisfactory if taken alone.

Given a class $\mathcal{C}$ of frames (or other semantical structures), we may wish to find a simple axiomatization of the logic determined by $\mathcal{C}$. A challenge in this direction is to find a recursive axiomatization of the Medvedev logic, determined by the rather transparent class of "topless" Boolean cubes. Or we may need first to elucidate whether $\mathcal{C}$ is *modally* (or *intuitionistically*) *definable* in the sense that it coincides with the class of *all* frames for Log$\mathcal{C}$. (Notice that because of incompleteness there may exist different, non-equivalent axiomatizations of $\mathcal{C}$, and only one of them generates Log$\mathcal{C}$.) For example, the class of reflexive frames is defined by $\Box p \to p$, while that of all irreflexive frames is not modally definable (see Exercise 3.10).

On the other hand, given a formula (an axiom of a logic), we are facing the problem of characterizing the class of frames (or other model structures) validating it. Of course, much depends here on the language in which we want to formulate such a characterization. For example, one can easily describe the class of Kripke models for a formula $\varphi(p_1, \dots, p_n)$ using the first order language with the monadic predicate symbols $P_1, \dots, P_n$ and the binary predicate symbol R. Indeed, define a first order formula $ST(\varphi)$ with one free individual variable x by induction on the construction of φ:

$$ST(p_i) = P_i(x), \ \ ST(\bot) = \bot;$$
$$ST(\psi \odot \chi) = ST(\psi) \odot ST(\chi), \text{ for } \odot \in \{\wedge, \vee, \to\};$$
$$ST(\Box\psi) = \forall y\ (xRy \to ST(\psi)\{y/x\}),$$

where y is an individual variable not occurring in $ST(\psi)$. In the intuitionistic case the definition of $ST(\psi \to \chi)$ should be replaced with

$$ST(\psi \to \chi) = \forall y\ (xRy \to (ST(\psi) \to ST(\chi))\{y/x\}).$$

The first order formula $ST(\varphi)$ is called the *standard translation* of φ.

Example 4.17

$$ST(\Box p \to \Box\Box p) = \forall y\ (xRy \to P(y)) \to \forall y\ (xRy \to \forall z\ (yRz \to P(z))).$$

Every Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on a frame $\mathfrak{F} = \langle W, R \rangle$ can be regarded then as a classical model of this first order language: W is the domain for individual variables, $P_1, \dots, P_n$ are interpreted as $\mathfrak{V}(p_1), \dots, \mathfrak{V}(p_n)$ and R as the accessibility relation on $\mathfrak{F}$.

Proposition 4.18 *For every formula φ, every model $\mathfrak{M}$ and every point a in $\mathfrak{M}$,*

$$(\mathfrak{M}, a) \models \varphi \textit{ iff } \mathfrak{M} \models ST(\varphi)[a],$$
$$\mathfrak{M} \models \varphi \textit{ iff } \mathfrak{M} \models \forall x ST(\varphi).$$

Proof An easy induction on the construction of φ. ❑

$ST(\varphi)$ is a first order equivalent of φ as far as models are concerned. If we deal with frames then P_i are interpreted as arbitrary monadic predicates on the (upward closed, in the intuitionistic case) sets of worlds, and so φ corresponds to the second order formula $\forall P_1 \dots \forall P_n ST(\varphi)$. More exactly, we have the following:

Proposition 4.19 *For every formula $\varphi(p_1, \dots, p_n)$, every Kripke frame $\mathfrak{F}$ and every point a in $\mathfrak{F}$,*

$$(\mathfrak{F}, a) \models \varphi \textit{ iff } \mathfrak{F} \models \forall P_1 \dots \forall P_n ST(\varphi)[a],$$

$$\mathfrak{F} \models \varphi \textit{ iff } \mathfrak{F} \models \forall x \forall P_1 \dots \forall P_n ST(\varphi).$$

This trivial solution to the characterization problem is hardly satisfactory. However, as we saw in Sections 2.5 and 3.5, for many standard modal and intuitionistic formulas the second order equivalents can be improved to nice first order conditions in the language with R and $=$. These observations lead naturally to the general problem of correspondence between modal (intuitionistic) formulas and modally (intuitionistically) definable classes of frames, on the one hand, and formulas of first or higher order predicate logic and classes of frames definable by them, on the other. In this book we shall touch upon only a small fragment of correspondence theory; for a more complete presentation the reader is referred to van Benthem (1983, 1984).

In Chapter 6 we shall see, however, that not all modal and intuitionistic formulas correspond to first order conditions on the accessibility relation. For example, the Löb axiom ***la*** has none. (This means that in a sense propositional modal and intuitionistic formulas can be more expressive then classical first order ones.) Yet, there are other ways to characterize frames for ***la***. It is not hard to see that a transitive frame $\mathfrak{F}$ refutes ***la*** iff there is a (not necessarily generated) subframe of $\mathfrak{F}$ reducible to the single reflexive point. In Chapter 9 we develop a universal frame-theoretic language giving a solution to the characterization problem on transitive (general) frames.

A characterization of model structures for a formula φ serves often as the first step in investigating various properties of the logic axiomatized by φ. Dealing with classes of logics, we are interested naturally in finding sufficiently general methods of establishing the decidability, completeness, finite approximability, etc., and describing (in a syntactical and/or semantical way) families of logics with this or that property. Classical examples here are the method of canonical models for proving Kripke completeness and Bull's theorem claiming that all logics in NExt**S4.3** are finitely approximable. For syntactical properties of logics, such as the disjunction or interpolation property, first we should find their semantical equivalents. Many results of that kind can be found in Parts II and IV.

The problem of recognizing whether a calculus enjoys a given property can be also looked at from the algorithmic point of view. Decidable and undecidable properties of calculi in various classes of logics are considered in Chapter 17.

One more interesting problem, to which we shall turn from time to time in this book, is to clarify the structure of the lattices of extensions of various logics and to connect it with properties of logics.

4.4 Exercises and open problems

Exercise 4.1 Show that Theorems 4.3 and 4.8 do not hold if instead of frames we take models.

Exercise 4.2 Give an example of a model in which the set of true formulas is a si-logic (e.g. **Cl**). Show that every model of that sort is infinite if the language $\mathcal{L}$ is infinite. Give an example of a finite model determining a si-logic in a finite language.

Exercise 4.3 Show that the union of two si-logics (or modal logics) is also a si-logic (respectively, modal logic) iff one of them is contained in another, and only in this case the union and the sum of logics coincide.

Exercise 4.4 Check that $\subseteq$ is a partial order on (N)Ext$L$ and that $\bigcap$ and $\sum$ (or $\bigoplus$) are, respectively, the supremum and infimum in the resulting partially ordered set.

Exercise 4.5 Show that $\oplus$ does not in general distribute over infinite intersections of modal logics. (Hint: consider $\mathbf{D}$ and $\mathbf{K} \oplus \Box^n \bot$, for $1 \leq n < \omega$.)

Exercise 4.6 Prove that it is impossible to represent $\mathbf{K}$ by a calculus with MP and Subst as the only inference rules.

Exercise 4.7 Show that each derivation in a normal logic may be reconstructed in such a way that the rule of necessitation is applied only to axioms.

Exercise 4.8 Show that $\mathbf{K} \oplus \Box\bot = \mathbf{K} + \Box\bot$.

Exercise 4.9 Show that $\mathbf{D}$ is not finitely axiomatizable as a quasi-normal logic. Which of the standard normal modal logics are finitely axiomatizable without the postulated RN? Show that if such a logic contains $\boldsymbol{tra}_n$, for some $n \geq 0$, then it is finitely axiomatizable as a quasi-normal logic.

Exercise 4.10 Show that $\sum_{i \in I} L_i$ ($\bigoplus_{i \in I} L_i$) is the smallest quasi-normal (normal) modal logic containing $\bigcup_{i \in I} L_i$.

Exercise 4.11 Prove that every intuitionistic formula without negative occurrences of $\vee$ (or $\bot$) is deductively equal to some disjunction (respectively, $\bot$-) free formula. Show also that every finitely axiomatizable si-logic can be axiomatized over $\mathbf{Int}$ by a single conjunction free formula. (Hint: use the formulas $(p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r)$, $(p \wedge q \to r) \leftrightarrow (p \to (q \to r))$ that are in $\mathbf{Int}$, and $\varphi_1 \wedge \ldots \wedge \varphi_n$, $(\varphi_1 \wedge \ldots \wedge \varphi_n \to p) \to p$ (in which p does not occur in φ_i) that are deductively equal in $\mathbf{Int}$.)

Exercise 4.12 Let $L_1 = \mathbf{K4} \oplus \{\varphi_i : i \in I\}$ and $L_2 = \mathbf{K4} \oplus \{\psi_j : j \in J\}$. Show that $L_1 \cap L_2 = \mathbf{K4} \oplus \{\Box^+\varphi_i \underline{\vee} \Box^+\psi_j : i \in I, j \in J\}$. Extend this result to logics containing $\boldsymbol{tra}_n$.

Exercise 4.13 Prove that for a modal logic L there exists a formula $\chi(p, q)$ such that, for all φ, ψ and Γ,

$$\Gamma, \psi \vdash_L^* \varphi \text{ iff } \Gamma \vdash_L^* \chi(\psi, \varphi)$$

iff $\boldsymbol{tra}_n \in L$ for some $n < \omega$. (Hint: apply the deduction theorem to $\chi(p, q), p \vdash_L^* q$ and take $q = \Box^{m+1}p$.)

Exercise 4.14 Show that if a logic L is finitely axiomatizable then any set of formulas axiomatizing L contains a finite subset generating L as well.

Exercise 4.15 Give an example of two modal (si-) logics which are not finitely axiomatizable themselves, but whose sum is.

Exercise 4.16 Prove Tarski's criterion without stipulating that the language is countable.

Exercise 4.17 Prove that for no modal or intuitionistic formula φ the first order formula $\forall y\, xRy$ is equivalent to $ST(\varphi)$. (Hint: consider the disjoint union of two reflexive points.)

Exercise 4.18 Show that the notions of global and "local" completeness (finite approximability) coincide for modal logics containing $\boldsymbol{tra}_n$, for some $n < \omega$.

Exercise 4.19 Show that if L_i is characterized by a class $\mathcal{C}_i$ of frames, for $i \in I$, then $\bigcap_{i\in I} L_i$ is characterized by $\bigcup_{i\in I} \mathcal{C}_i$. In particular, Kripke completeness and finite approximability are preserved under intersections of logics.

Exercise 4.20 Show that $\varphi \in L$, L a si- or modal logic, iff there is a derivation of φ in L all variables in which occur in φ.

Exercise 4.21 Prove that **Cl** is the only 0-reducible consistent si-logic.

Exercise 4.22 Show that every logic in NExt**K** with infinitely many pairwise non-equivalent modalities is contained in a maximal normal modal logic with infinitely many non-equivalent modalities.

Exercise 4.23 (i) Prove that for every modal formula φ there is a formula ψ such that $md(\psi) \leq 1$ and $\varphi \leftrightarrow \psi \in \mathbf{S5}$.

(ii) Prove that the sets of formulas of modal degree ≤ 1 in logics from the interval $[\mathbf{T}; \mathbf{S5}]$ coincide and that this is not so for any proper extension of the interval.

Exercise 4.24 Prove that for every logic $L \in$ NExt**K4** and every formula φ there is a formula ψ with $md(\psi) \leq 2$ such that $L \oplus \varphi = L \oplus \psi$. Show that this does not hold for **K**, **T**, $\mathbf{T} \oplus p \to \Box\Diamond p$.

Exercise 4.25 Prove that if L_1 and L_2 are consistent normal modal logics with the necessity operators $\Box_1$ and $\Box_2$, respectively, then the smallest normal bimodal logic L containing $L_1 \cup L_2$ is a conservative extension of both L_1 and L_2 (i.e., for every formula φ in the language with $\Box_i$, $i = 1, 2$, $\varphi \in L$ only if $\varphi \in L_i$).

Problem 4.1 *Is it possible to axiomatize every logic in* Ext**K** *by an independent set of axioms (with the rules MP and Subst)?*

4.5 Notes

Gödel (1932) noticed that there are infinitely many logics between **Int** and **Cl**. Developing this observation, Umezawa (1955, 1959) started considering the whole class of superintuitionistic logics, which he called "intermediate logics". The notion of normal modal logic, as it is understood in this book, was introduced by Lemmon and Scott (1977). The term "normal" is due to McKinsey and Tarski (1948), who showed in particular that there are non-normal extensions of **S4** closed under MP and Subst. Segerberg (1971) called such logics quasi-normal.

There are two main reasons for considering big families of modal logics and developing a general theory for them. First, there exist so many concrete modal systems in the literature that generalizations and various kinds of classifications become inevitable. The second reason (of course connected with the first one) is typical in mathematics and science in general: having analyzed a number of particular objects of the same nature, we turn to studying the "nature" itself, acquiring at this higher level new knowledge of the phenomena we are interested in. Recall, for instance, that the notions of straight line, plane and three dimensional space led to the general concept of vector space, and group theory originated from the study of permutations of n-tuples of natural numbers.

The following example should be closer to the reader of this book. Suppose that we want to prove a theorem according to which all pretabular logics in NExt**S4** are finitely approximable. There are two ways to do this: (i) to analyze each of the five pretabular logics in NExt**S4** individually or (ii) to use Corollary 12.12 which concerns all pretabular logics in NExt**K4**. In case (i) the theorem turns out to be our brilliant observation, while (ii) explains why this observation holds.

That modal logic is not just a collection of individual systems, that a general mathematical theory of modal logics is required was clearly recognized in the late 1960s. Even before that modal logicians from time to time dealt with classes of extensions of some logics. For instance, Scroggs (1951) described the lattice NExt**S5**, Umezawa (1955, 1959) started investigating si-logics, Dummett and Lemmon (1959) considered logics between **S4** and **S5** and embedded si-logics into them, Hosoi (1967) classified si-logics by means of dividing them into slices. However, the mainstream of studies in modal logic was to examine individual systems and construct new ones with given properties. Moreover, during a rather long time there was a hope to find a complete description of the lattices of modal and si-logics. If this hope were realized and the class of, say si-logics, turned out to be countable and describable in a visual way, then the "individualistic" approach would certainly be enough. The turning point was probably the discovery of Jankov (1968b) that there exists a continuum of si-logics and the subsequent constructions of modal and si-logics with various "negative" properties.

It would be difficult now to give a complete list of logicians whose work led to the creation of the general theory of modal and si-logics, but undoubtedly E.J. Lemmon and A.V. Kuznetsov were among the pioneers.

To illustrate how this theory can help to solve "individual" problems, we present here two interesting examples. For many years the problem of independent axiomatizability of modal and si-logics resisted all attempts to solve it. A way to construct a logic without an independent axiomatization was opened by the following observation of Kleyman (1984) which is reformulated here in terms of modal logics (Kleyman's paper deals with varieties of groups).

Proposition 4.20 *Suppose a normal modal logic L_1 has an independent axiomatization. Then, for every finitely axiomatizable normal modal logic $L_2 \subset L_1$, the interval of logics $[L_2, L_1] = \{L \in \mathrm{NExt}\mathbf{K} : L_2 \subseteq L \subseteq L_1\}$ contains an immediate* *predecessor of* L_1.

Proof If L_1 is finitely axiomatizable then the existence of an immediate predecessor of L_1 in $[L_2, L_1]$ follows from Zorn's lemma (see Section 7.4).

Suppose now that L_1 has an infinite independent set of axioms $\{\varphi_i : i < \omega\}$. Since L_2 is a finitely axiomatizable sublogic of L_1, there is $n < \omega$ such that L_2 is contained in the logic with the axioms $\varphi_0, \dots, \varphi_n$. Let L_3 be the logic with the axioms $\varphi_0, \dots, \varphi_n, \varphi_{n+2}, \varphi_{n+3}, \dots$. Since the set of L_1's axioms is independent, $L_2 \subset L_3 \subset L_1$ and $\varphi_{n+1} \notin L_3$. And now again Zorn's lemma provides us with an immediate predecessor of L_1 in the interval $[L_3, L_1]$. ❑

With the help of this proposition Chagrov and Zakharyaschev (1995a) constructed concrete modal and si-logics without independent axiomatizations. The reader can fulfill the construction by himself following the hints in Exercises 8.20–8.22. Note, by the way, that the problem of independent axiomatizability of quasi-normal logics still remains open.

Another example of that kind is connected with attempts to describe big families of logics. A hypothetical way to do this may be illustrated by the following observation. As we shall see later, the class Ext$(\mathbf{Int} + \boldsymbol{bd}_3)$ contains a continuum of logics. But, according to Segerberg's theorem (to be proved in Section 8.6), all of them are finitely approximable and so there is a countable sequence of frames (of depth ≤ 3) such that every logic in the class is characterized by one of its subsequences. What if logics in bigger classes can be determined in a similar way? However, the following proposition holds.

Proposition 4.21 *Suppose a logic L in some class of logics has a continuum of immediate predecessors in the class. Then there is no countable sequence C of semantical structures such that its subsequences characterize all logics in this class.*

Proof Suppose otherwise and let L_i, for $i \in I$, be all distinct immediate predecessors of L in our class. Then, as is easy to see, for every $i \in I$ there is $\mathfrak{S}_i \in C$ such that $\mathfrak{S}_i \models L_i$ and $\mathfrak{S}_i \not\models L_j$ for $j \in I - \{i\}$, contrary to C being countable. ❑

This proposition, which also follows from Kleyman (1984), was applied to modal and si-logics by Chagrov and Tsytkin (1987) and Chagrov (1994a).

Maksimova *et al.* (1979) proved that $\mathbf{ML}$ is not finitely axiomatizable; Shehtman (1990b) extended this result to all normal modal companions of $\mathbf{ML}$ in NExt$\mathbf{S4}$.

Exercise 4.25 is due to Thomason (1980).

Part II

Kripke semantics

Proceeding to the systematic study of superintuitionistic and modal logics, first of all we are interested in finding good semantic instruments. To begin with, let us try to manage with what we already have, namely, Kripke frames and models. So the main questions we address in this part are whether logics in Ext**Int** and NExt**K** are characterized by suitable classes of Kripke frames and whether they are finitely approximable. First we generalize the completeness proofs of Sections 2.6 and 3.6 and show that this approach works for a good many other logics.

5

CANONICAL MODELS AND FILTRATION

In this chapter we consider two best known methods of obtaining completeness results. One of them—the *method of canonical models*—given a consistent logic L in **ExtInt** or NExt**K**, constructs a canonical Kripke model $\mathfrak{M}_L = \langle \mathfrak{F}_L, \mathfrak{V}_L \rangle$ characterizing L. Sometimes the frame $\mathfrak{F}_L$ turns out to be a frame for L, and then we can say at once that L is Kripke complete. Another method, known as *filtration*, is intended for establishing the finite approximability by means of extracting from the canonical models finite refutation frames.

5.1 The Henkin construction

Suppose L is a superintuitionistic or normal modal logic, and we want to find a class of Kripke models characterizing L. The proofs of the completeness theorems for **Int** and **K** above provide us with a method of constructing models refuting formulas outside of L. However formulas in L need not be true in them. So let us try to modify this method in such a way that it would ensure not only completeness but also soundness.

Recall that the worlds in those models are tableaux $t = (\Gamma, \Delta)$ consistent in **Int** and **K**, with the truth-relation in them being chosen so that $t \models \varphi$, for all $\varphi \in \Gamma$, and $t \not\models \psi$, for all $\psi \in \Delta$. The condition guaranteeing in this situation that all formulas in L are true in all worlds is, of course, the inclusion $L \subseteq \Gamma$.

Now we restore this construction in full detail. It is often called the *Henkin construction* in view of its conceptual closeness to the construction used by Henkin for proving the completeness of first order classical calculus.

So, let L be a consistent si-logic in the language $\mathcal{L}$ or a normal modal logic in the language $\mathcal{ML}$. A tableau $t = (\Gamma, \Delta)$ in the language of L is said to be *L-consistent* if for no $\varphi_1, \ldots, \varphi_n$ in Δ do we have $\Gamma \vdash_L \varphi_1 \vee \ldots \vee \varphi_n$. t is called *maximal* if $\Gamma \cup \Delta$ is the set of all formulas in the language of L. It should be clear that every maximal L-consistent tableau is saturated in **Int** or **K** (for details see the proof of Theorem 1.16).

The following two lemmas guarantee that if we succeed in constructing (according to our plan) a model whose points are all maximal L-consistent tableaux then all formulas in L will be true in this model, while all those outside of L will be refuted by it.

Lemma 5.1. (Lindenbaum's lemma) *Every L-consistent tableau $t = (\Gamma, \Delta)$ can be extended to a maximal L-consistent tableau.*

Proof Let $\varphi_1, \varphi_2, \ldots$ be some enumeration of all formulas in the language of L. Define a sequence of tableaux $t_0 = (\Gamma_0, \Delta_0)$, $t_1 = (\Gamma_1, \Delta_1), \ldots$ by taking $t_0 = t$ and, for $i \geq 0$,

$$t_{i+1} = \begin{cases} (\Gamma_i, \Delta_i \cup \{\varphi_i\}) \text{ if } (\Gamma_i, \Delta_i \cup \{\varphi_i\}) \text{ is } L\text{-consistent} \\ (\Gamma_i \cup \{\varphi_i\}, \Delta_i) \text{ otherwise.} \end{cases}$$

In exactly the same way as in the proof of Theorem 1.16 one can show that L-consistency of t_i entails the L-consistency of t_{i+1}. Thus all the constructed tableaux $t_0, t_1, \ldots$ prove to be L-consistent and for every formula φ there is i such that either $\varphi \in \Gamma_i$ or $\varphi \in \Delta_i$.

Let us consider now the tableau $t^* = (\Gamma^*, \Delta^*)$ where

$$\Gamma^* = \bigcup_{i<\omega} \Gamma_i, \quad \Delta^* = \bigcup_{i<\omega} \Delta_i.$$

It is clear that $\Gamma^* \cup \Delta^*$ contains all the formulas in the language of L and so t^* is maximal. To prove that it is L-consistent, suppose otherwise. Then for some $\varphi_1, \ldots, \varphi_n \in \Delta^*$, there is a derivation of $\varphi_1 \vee \ldots \vee \varphi_n$ from the set Γ^* in L. Since this derivation uses only a finite number of assumptions in Γ^*, there exists i such that $\varphi_1, \ldots, \varphi_n \in \Delta_i$ and $\Gamma_i \vdash_L \varphi_1 \vee \ldots \vee \varphi_n$, contrary to t_i being L-consistent. ❑

Lemma 5.2 *Suppose Λ is a set of formulas and φ a formula in the language of L. Then $\Lambda \vdash_L \varphi$ iff, for every maximal L-consistent tableau $t = (\Gamma, \Delta)$, $\varphi \in \Gamma$ whenever $\Lambda \subseteq \Gamma$. In particular, $\varphi \in L$ iff $\varphi \in \Gamma$ for every maximal L-consistent tableau $t = (\Gamma, \Delta)$.*

Proof ($\Rightarrow$) If $\Lambda \subseteq \Gamma$ and $\varphi \notin \Gamma$, for some maximal L-consistent tableau $t = (\Gamma, \Delta)$, then, by the maximality of t, $\varphi \in \Delta$. Since $\Lambda \vdash_L \varphi$, it follows that t is not L-consistent, which is a contradiction.

($\Leftarrow$) Suppose $\Lambda \not\vdash_L \varphi$. Then the tableau $t = (\Lambda, \{\varphi\})$ is L-consistent. By Lindenbaum's lemma, t is contained in a maximal L-consistent tableau, which is a contradiction. ❑

Remarks (1) When proving Lemmas 5.1 and 5.2, we did not use the rule of necessitation. So these lemmas hold for quasi-normal modal logics as well.

(2) It should be clear that with the help of transfinite induction the lemmas above can easily be extended to logics in uncountable languages. And certainly they hold for logics in finite languages.

Now we can construct the model we are looking for. First we form a frame $\mathfrak{F}_L = \langle W_L, R_L \rangle$ by taking W_L to be the set of all maximal L-consistent tableaux and, for any $t_1 = (\Gamma_1, \Delta_1)$ and $t_2 = (\Gamma_2, \Delta_2)$ in W_L,

$$t_1 R_L t_2 \text{ iff } \Gamma_1 \subseteq \Gamma_2 \text{ iff } \Delta_1 \supseteq \Delta_2, \text{ if } L \in \text{Ext}\mathbf{Int}$$

and

$$t_1 R_L t_2 \text{ iff } \{\varphi : \Box\varphi \in \Gamma_1\} \subseteq \Gamma_2, \text{ if } L \in \mathrm{NExt}\mathbf{K}.$$

The frame $\mathfrak{F}_L$ is called the *canonical frame* for L.

Lemma 5.3 *$\mathfrak{F}_L$ is a Hintikka system in* **Int**, *if $L \in \mathrm{Ext}\mathbf{Int}$, and in* **K**, *if $L \in \mathrm{NExt}\mathbf{K}$.*

Proof Follows from the proofs of Theorems 2.43 and 3.53 and Lindenbaum's lemma. ❑

Define a valuation $\mathfrak{V}_L$ in $\mathfrak{F}_L$ by taking, for every variable p,

$$\mathfrak{V}_L(p) = \{(\Gamma, \Delta) \in W_L : p \in \Gamma\}.$$

The resulting model $\mathfrak{M}_L = \langle \mathfrak{F}_L, \mathfrak{V}_L \rangle$ is called the *canonical model* for L.

Theorem 5.4. (Canonical model) *Let L be a consistent superintuitionistic or normal modal logic and $\mathfrak{M}_L = \langle \mathfrak{F}_L, \mathfrak{V}_L \rangle$ its canonical model on the frame $\mathfrak{F}_L = \langle W_L, R_L \rangle$. Then for every formula φ and every tableau $t = (\Gamma, \Delta)$ in W_L,*
(i) *$\varphi \in \Gamma$ implies $(\mathfrak{M}_L, t) \models \varphi$,*
(ii) *$\varphi \in \Delta$ implies $(\mathfrak{M}_L, t) \not\models \varphi$*
and so
(iii) *$(\mathfrak{M}_L, t) \models \varphi$ iff $\varphi \in \Gamma$.*

Proof By Lemma 5.3, $\mathfrak{F}_L$ is a Hintikka system. It remains to observe that the valuation $\mathfrak{V}_L$ was defined in exactly the same way as in the proofs of Propositions 2.31 and 3.25, where the implications (i) and (ii) were established. ❑

Theorem 5.5 *Suppose L is a consistent logic in* Ext**Int** *or* NExt**K**. *Then every L-consistent tableau is realized in $\mathfrak{M}_L$. In particular, $\Lambda \vdash_L \varphi$ iff, for every point x in $\mathfrak{M}_L$, $x \models \Lambda$ implies $x \models \varphi$, and $\varphi \in L$ iff $\mathfrak{M}_L \models \varphi$.*

Proof Follows from Lemmas 5.1, 5.2 and Theorem 5.4. ❑

Of course, it seems unlikely that a completeness result of such generality can be used as a technical tool for deciding whether a given formula is in L. Indeed, the model $\mathfrak{M}_L$ was defined via the derivability in L and so it is nothing more than a model-theoretic reformulation of L. The role of canonical models is different: they give us that starting point from which we can develop a model-theoretic approach to investigating the logics under consideration. In the next sections of this chapter we shall use canonical models to prove the Kripke completeness and finite approximability of many superintuitionistic and modal logics. But before that we observe some important properties of canonical models which will be required in the sequel.

A model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ for **Int** or **K** is said to be *differentiated* if, for any two points x, y in $\mathfrak{F}$, $x = y$ whenever exactly the same formulas are true at x and y. As a direct consequence of the canonical model theorem we obtain

Proposition 5.6 *Every canonical model is differentiated.*

A model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ for **Int** on a frame $\mathfrak{F} = \langle W, R \rangle$ is called *tight* if, for any $x, y \in W$, xRy whenever $x \models \varphi$ implies $y \models \varphi$ for every $\varphi \in \mathbf{For}\mathcal{L}$. A model $\mathfrak{M}$ for **K** is *tight* if, for any $x, y \in W$, xRy whenever $x \models \Box\varphi$ implies $y \models \varphi$ for every $\varphi \in \mathbf{For}\mathcal{ML}$. It follows immediately from the definition of $\mathfrak{M}_L$ that the following proposition holds:

Proposition 5.7 *Every canonical model is tight.*

A model is called *refined* if it is both differentiated and tight. Notice that in the intuitionistic case differentiatedness follows from tightness.

Corollary 5.8 *Every canonical model is refined.*

In the modal case Propositions 5.6 and 5.7 can be generalized as follows:

Proposition 5.9 *Suppose $\mathfrak{M}_L = \langle \mathfrak{F}_L, \mathfrak{V}_L \rangle$ is the canonical model for a normal modal logic L. Then for any $x, y \in W_L$ and any $n \geq 0$, $xR_L^n y$ iff $x \models \Box^n\varphi$ implies $y \models \varphi$ for every modal formula φ.*

Proof $(\Rightarrow)$ Follows from Proposition 3.1.

$(\Leftarrow)$ is proved by induction on n. The case $n = 0$ means nothing else but that $\mathfrak{M}_L$ is differentiated.

Suppose now that our proposition holds for n and let $x \models \Box^{n+1}\varphi$ imply $y \models \varphi$ for every $\varphi \in \mathbf{For}\mathcal{ML}$. We must prove is that there is z such that $xR_L z$ and $zR_L^n y$. Consider the tableau $t = (\Gamma, \Delta)$ where

$$\Gamma = \{\varphi : \ x \models \Box\varphi\}, \quad \Delta = \{\Box^n\psi : \ y \not\models \psi\}$$

and show that it is L-consistent. Suppose otherwise. Then

$$\vdash_L \varphi_1 \wedge \ldots \wedge \varphi_k \to \Box^n\psi_1 \vee \ldots \vee \Box^n\psi_m,$$

for some $\varphi_1, \ldots, \varphi_k \in \Gamma$ and $\Box^n\psi_1, \ldots, \Box^n\psi_m \in \Delta$, whence, by Examples 3.49 and 3.50,

$$\vdash_L \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_k \to \Box(\Box^n\psi_1 \vee \ldots \vee \Box^n\psi_m).$$

Therefore, $x \models \Box(\Box^n\psi_1 \vee \ldots \vee \Box^n\psi_m)$ and so $x \models \Box^{n+1}(\psi_1 \vee \ldots \vee \psi_m)$. But then $y \models \psi_1 \vee \ldots \vee \psi_m$, whence $y \models \psi_i$ for some i, contrary to $\Box^n\psi_i \in \Delta$.

By Lindenbaum's lemma, t is contained in some maximal L-consistent tableau $t^* = (\Gamma^*, \Delta^*)$. By the definition of Γ, we must have $xR_L t^*$. Furthermore, by the definition of Δ, $t^* \models \Box^n\varphi$ implies $y \models \varphi$, for every φ, and so, by the induction hypothesis, $t^* R_L^n y$. ❑

Corollary 5.10 *For all points x and y in the canonical model for a normal modal logic L and every $n \geq 0$, $xR_L^n y$ iff $y \models \varphi$ implies $x \models \Diamond^n\varphi$, for all $\varphi \in \mathbf{For}\mathcal{ML}$.*

A model $\mathfrak{M}$ is called *compact* if a tableau t is realizable in $\mathfrak{M}$ whenever every finite subtableau of t is realizable in $\mathfrak{M}$. For modal models $\mathfrak{M}$ this definition is clearly equivalent to the more familiar one: $\mathfrak{M}$ is compact if a set of formulas Σ is satisfied in $\mathfrak{M}$ whenever every finite subset of Σ is satisfied in $\mathfrak{M}$.

Proposition 5.11 *Every canonical model is compact.*

Proof Exercise. ❑

5.2 Completeness theorems

According to Theorem 5.5, all formulas that do not belong to a superintuitionistic or normal modal logic L are refuted in the canonical frame $\mathfrak{F}_L$. So if we prove that $\mathfrak{F}_L$ is a frame for L, then the Kripke completeness of L will be established. Moreover, in this case L will be even strongly complete.

A logic L with $\mathfrak{F}_L \models L$ is called *canonical*. For example, **Int** and **K** are obviously canonical. Using this notion, the observation above can be formulated as follows:

Theorem 5.12 *Every canonical superintuitionistic or normal modal logic is strongly Kripke complete.*

In fact, sometimes we can derive a much more useful result than simply strong Kripke completeness. Suppose that we have already proved the soundness of L with respect to the class of frames satisfying some property $\mathcal{P}$. If now we succeed in proving that $\mathfrak{F}_L$ satisfies $\mathcal{P}$, then we shall establish not only that L is canonical but also that it is characterized by the class of frames satisfying $\mathcal{P}$.

For instance, Proposition 3.73 asserts that the calculus T or, equivalently, the logic $L = \mathbf{K} \oplus \Box p \to p$, is sound with respect to the class of reflexive frames. To prove the completeness of this logic, it suffices to establish the reflexivity of its canonical frame $\mathfrak{F}_L = \langle W_L, R_L \rangle$. Suppose otherwise. Then there is a tableau $t = (\Gamma, \Delta) \in W_L$ such that not tR_Lt. By the definition of R_L, this is possible only if $\Box\varphi \in \Gamma$ and $\varphi \in \Delta$, for some formula φ. But then, by Lemmas 5.2 and 5.3, $\Box\varphi \to \varphi \notin L$, which is a contradiction.

Note by the way that in this argument we have used only that $\boldsymbol{re}$ belongs to L. So actually we have proved the following:

Theorem 5.13 *Suppose a logic $L \in$ NExt**K** is consistent and contains $\boldsymbol{re}$. Then the canonical frame $\mathfrak{F}_L$ for L is reflexive.*

Corollary 5.14 *The calculus T is complete with respect to the class of reflexive frames.*

If we recall that the logic **T** was defined as the logic characterized by the class of reflexive frames, then as a consequence of Proposition 3.73 and Corollary 5.14 we immediately derive

Corollary 5.15 (i) $\mathbf{T} = \mathbf{K} \oplus \boldsymbol{re}$.
(ii) **T** *is canonical.*

By the same scheme we can prove the canonicity of the logics in Ext**Int** and NExt**K** axiomatizable by the formulas which were supplied by first order equivalents in Chapters 2 and 3.

Theorem 5.16 *Suppose L is a consistent superintuitionistic or normal modal logic and $\varphi \in L$, for some formula φ in the list* **da**, **wem**, $\boldsymbol{bd}_n$, $\boldsymbol{bw}_n$, $\boldsymbol{bc}_n$, $\boldsymbol{btw}_n$, **kp** *or, respectively,* $\boldsymbol{tra}_n$, **sym**, **ser**, $\boldsymbol{ga}_{klmn}$, **euc**, $\boldsymbol{den}_n$, **sc**, **con**, **ga**, **dir**, $\boldsymbol{bw}_n$, $\boldsymbol{bd}_n$, $\boldsymbol{alt}_n$. *Then the canonical frame $\mathfrak{F}_L$ for L satisfies the condition corresponding to φ.*

Proof We will consider only three formulas, leaving the others to the reader as an exercise.

(i) Let $\Diamond^k\Box^l p \to \Box^m\Diamond^n p \in L$, for some $k, l, m, n \geq 0$, and show that the canonical frame $\mathfrak{F}_L = \langle W_L, R_L\rangle$ satisfies the condition

$$\forall x, y, z\ (xR_L^k y \wedge xR_L^m z \to \exists u\ (yR_L^l u \wedge zR_L^n u)).$$

Let $t_i = (\Gamma_i, \Delta_i)$, for $i = 0, 1, 2$, be some tableaux in W_L such that $t_0 R_L^k t_1$ and $t_0 R_L^m t_2$. In order to show that there exists a tableau $t = (\Gamma, \Delta)$ for which $t_1 R_L^l t$ and $t_2 R_L^n t$, we should prove, by Lindenbaum's lemma, Proposition 5.9 and Corollary 5.10, that the tableau

$$t' = (\{\chi :\ \Box^l\chi \in \Gamma_1\}, \{\psi :\ \Diamond^n\psi \in \Delta_2\})$$

is L-consistent. Suppose otherwise. Then $\chi \to \psi \in L$ for some formulas $\Box^l\chi \in \Gamma_1$ and $\Diamond^n\psi \in \Delta_2$. Applying the regularity rule l times, we obtain $\Box^l\chi \to \Box^l\psi \in L$, whence $\Box^l\psi \in \Gamma_1$ and so $\Diamond^k\Box^l\psi \in \Gamma_0$. Since $\Diamond^k\Box^l\psi \to \Box^m\Diamond^n\psi \in \Gamma_0$, we have also $\Box^m\Diamond^n\psi \in \Gamma_0$. But then $\Diamond^n\psi \in \Gamma_2$, contrary to t_2 being L-consistent.

(The reader can find a more general result in Exercise 5.25, while the strongest generalization, known as Sahlqvist's theorem, will be proved in Section 10.3.)

(ii) Suppose now that $\Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p) \in L$ and show that $\mathfrak{F}_L$ is connected, i.e., satisfies the condition

$$\forall x, y, z\ (xR_L y \wedge xR_L z \wedge y \neq z \to yR_L z \vee zR_L y).$$

Suppose otherwise. Then we have three tableaux $t_i = (\Gamma_i, \Delta_i)$ in W_L, for $i = 0, 1, 2$, such that (a) $t_0 R_L t_1$, (b) $t_0 R_L t_2$, (c) $t_1 \neq t_2$, (d) not $t_1 R_L t_2$ and (e) not $t_2 R_L t_1$. By (d), there is $\varphi_1 \in \Delta_2$ such that $\Box\varphi_1 \in \Gamma_1$, while by (c), we have some $\chi_1 \in \Gamma_1 \cap \Delta_2$. Let $\varphi = \varphi_1 \vee \chi_1$. Then $\Box^+\varphi \in \Gamma_1$ and $\varphi \in \Delta_2$. By using (e) and (c) in exactly the same way, we can find $\psi \in \Delta_1$ such that $\Box^+\psi \in \Gamma_2$. Therefore, $\Box(\Box^+\varphi \to \psi) \vee \Box(\Box^+\psi \to \varphi) \in \Delta_0$, which is a contradiction.

(iii) Finally, we consider a si-logic L containing the Kreisel–Putnam formula

$$\boldsymbol{kp} = (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

and prove that $\mathfrak{F}_L$ satisfies the condition

$$\forall x, y, z\ (xR_L y \wedge xR_L z \wedge \neg yR_L z \wedge \neg zR_L y \to \exists u\ (xR_L u \wedge uR_L y \wedge uR_L z \wedge \\ \forall v\ (uR_L v \to \exists w\ (vR_L w \wedge (yR_L w \vee zR_L w))))).$$

Suppose $t_1 = (\Gamma_1, \Delta_1)$, $t_2 = (\Gamma_2, \Delta_2)$, $t_3 = (\Gamma_3, \Delta_3)$ are points in W_L such that $t_1 R_L t_2$, $t_1 R_L t_3$ and $\neg t_2 R_L t_3$, $\neg t_3 R_L t_2$. Form a tableau $t = (\Gamma, \Delta)$ by taking

$$\Gamma = \Gamma_1 \cup \{\neg\varphi : \ \neg\varphi \in \Gamma_2 \cap \Gamma_3\},$$

$$\Delta = \Delta_2 \cup \Delta_3$$

and show that t is L-consistent. Indeed, if this is not the case then (using the first de Morgan law which belongs to **Int**) we would have

$$\Gamma_1 \vdash_L \neg\varphi \to \psi \vee \chi,$$

for some $\neg\varphi \in \Gamma_2 \cap \Gamma_3$, $\psi \in \Delta_2$, $\chi \in \Delta_3$, and so, by $\boldsymbol{kp}$,

$$\Gamma_1 \vdash_L (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi).$$

Therefore, either $\neg\varphi \to \psi \in \Gamma_1$ or $\neg\varphi \to \chi \in \Gamma_1$. In the former case we would then have $\psi \in \Gamma_2$ and in the latter $\chi \in \Gamma_3$, contrary to the L-consistency of t_2 and t_3.

Thus, t is L-consistent and, by Lindenbaum's lemma, it can be extended to a maximal L-consistent tableau, say, $t_4 = (\Gamma_4, \Delta_4)$. By the definition, $t_1 R_L t_4$, $t_4 R_L t_2$ and $t_4 R_L t_3$. It remains to show that every successor of t_4 has a common successor with t_2 or t_3. Suppose otherwise, i.e., some successor $t' = (\Gamma', \Delta')$ of t_4 has no common successors with t_2 and t_3. Then there are formulas $\neg\varphi_2 \in \Gamma_2$ and $\neg\varphi_3 \in \Gamma_3$ such that $\varphi_2, \varphi_3 \in \Gamma'$. Indeed, the tableau $(\Gamma_2 \cup \Gamma', \emptyset)$ is L-inconsistent (for otherwise t_2 and t' would have a common successor) and so $\varphi, \varphi_2 \vdash_L \bot$, for some $\varphi \in \Gamma_2$, $\varphi_2 \in \Gamma'$, from which $\varphi \vdash_L \neg\varphi_2$ and hence $\neg\varphi_2 \in \Gamma_2$.

Therefore, $\neg\varphi_2 \vee \neg\varphi_3$ and so $\neg(\varphi_2 \wedge \varphi_3)$ are in $\Gamma_2 \cap \Gamma_3$. But then, since $t_4 R_L t'$, we have $\neg(\varphi_2 \wedge \varphi_3) \in \Gamma'$. On the other hand $\varphi_2 \wedge \varphi_3 \in \Gamma'$, contrary to the L-consistency of t'. ❑

As a consequence of Theorem 5.16 we immediately obtain

Theorem 5.17 *Every logic L in* Ext**Int** *and* NExt**K** *axiomatizable by some of the formulas mentioned in Theorem 5.16 is canonical, with the canonical frame $\mathfrak{F}_L$ satisfying the first order conditions corresponding to the axioms of L.*

In particular we have the following completeness results:

Corollary 5.18 (i) *The calculus $K4$ is characterized by the class of transitive frames.*

(ii) *$S4$ is characterized by the class of quasi-ordered frames.*

(iii) *$S5$ is characterized by the class of frames with universal alternativeness relations.*

(iv) *D is characterized by the class of serial frames.*

(v) *$S4.3$ is characterized by the class of connected quasi-orders and by the class of linear partial orders.*

Proof (i), (ii), (iv) and the first part of (v) are immediate consequences of Theorem 5.17 and the soundness results in Section 3.8, (iii) follows from these and the generation theorem. As to the completeness of $S4.3$ with respect to linear partial orders, suppose $\not\vdash_{S4.3} \varphi$. Then φ is refuted in a connected quasi-order and so, by the generation theorem, in a frame which is a chain of clusters. By bulldozing this chain (see the proof of Theorem 3.20), we can construct a linear order which is reducible to it and so, by the reduction theorem, also refutes φ. ❑

For **S5** Theorem 5.17 yields an even better result.

Corollary 5.19 **S5** *is locally tabular and characterized by the class of finite frames with universal alternativeness relations.*

Proof For $n > 0$ let $\mathcal{ML}_n$ be a modal language with n variables. By Theorem 5.17, the logic $\mathbf{S5}(n) = \mathbf{S5} \cap \mathbf{For}\mathcal{ML}_n$ is canonical and $\mathfrak{F}_{\mathbf{S5}(n)}$ is the disjoint union of clusters. Since $\mathfrak{M}_{\mathbf{S5}(n)}$ is differentiated and by Proposition 3.7, each of these clusters may contain at most 2^n points and the total number of clusters does not exceed 2^{2^n}. So $\mathfrak{F}_{\mathbf{S5}(n)}$ is finite. Therefore, there are only finitely many pairwise non-equivalent formulas with n variables in **S5** and each of them that is not in **S5** is refuted in $\mathfrak{F}_{\mathbf{S5}(n)}$. ❑

Unfortunately the method of establishing completeness using canonical models is far from being universal: there are normal modal and superintuitionistic logics which are Kripke complete but not canonical, witnesses **GL**, **Grz**, **SL** (see Section 6.2) and the McKinsey logic

$$\mathbf{KM} = \mathbf{K} \oplus \Box\Diamond p \to \Diamond\Box p.$$

It also turns out that the axioms of these logics do not correspond to any first order condition on their Kripke frames.

As to the McKinsey axiom $\boldsymbol{ma}$, we saw in Section 3.5 that in the class of transitive frames it corresponds to the McKinsey condition. Moreover, we will show now that the canonical frame for every normal extension of

$$\mathbf{K4.1} = \mathbf{K4} \oplus \Box\Diamond p \to \Diamond\Box p$$

satisfies it. To this end we require the following:

Lemma 5.20 $\Diamond \bigwedge_{i=1}^{n}(\Diamond\varphi_i \to \Box\varphi_i) \in \mathbf{K4.1}$, *for any formulas* $\varphi_1, \dots, \varphi_n$.

Proof Observe first that since $(\Box\Diamond p \to \Diamond\Box p) \leftrightarrow \Diamond(\Diamond p \to \Box p) \in \mathbf{K}$ (see Table 3.1), $\Diamond(\Diamond\varphi \to \Box\varphi) \in \mathbf{K4.1}$ for any φ. Let $\psi_i = \Diamond\varphi_i \to \Box\varphi_i$, $i = 1, \dots, n$. Then $\Diamond\psi_1, \dots, \Diamond\psi_n \in \mathbf{K4.1}$. By RN, $\Box\Diamond\psi_1, \Box\Diamond\psi_2 \in \mathbf{K4.1}$ and so $\Diamond\Box\psi_2 \in \mathbf{K4.1}$. By using twice $\Box p \wedge \Diamond q \to \Diamond(p \wedge q)$ in Table 3.1, we get $\Diamond\Diamond(\psi_1 \wedge \psi_2) \in \mathbf{K4.1}$ and so $\Diamond(\psi_1 \wedge \psi_2) \in \mathbf{K4.1}$, since $\Diamond\Diamond p \to \Diamond p$ is in **K4**. Now by applying the same argument to $\Diamond(\psi_1 \wedge \psi_2)$ and $\Diamond\psi_3$, we obtain $\Diamond(\psi_1 \wedge \psi_2 \wedge \psi_3) \in \mathbf{K4.1}$ and so forth. Eventually we shall have $\Diamond(\psi_1 \wedge \ldots \wedge \psi_n) \in \mathbf{K4.1}$. ❑

Theorem 5.21 *Suppose $L \in$* NExt**K4** *is consistent and contains* $\boldsymbol{ma}$. *Then $\mathfrak{F}_L = \langle W_L, R_L \rangle$ satisfies the McKinsey condition*

$$\forall x \exists y \ (xR_L y \wedge \forall z \ (yR_L z \rightarrow y = z)).$$

Proof Let $t_0 = (\Gamma_0, \Delta_0)$ be a tableau in W_L. Consider the tableau $t' = (\Gamma', \emptyset)$ with

$$\Gamma' = \{\varphi : \ \Box\varphi \in \Gamma_0\} \cup \{\Diamond\varphi \rightarrow \Box\varphi : \ \varphi \in \mathbf{For}\mathcal{ML}\}$$

and show that it is L-consistent. Suppose otherwise. Then

$$\varphi, \Diamond\varphi_1 \rightarrow \Box\varphi_1, \dots, \Diamond\varphi_n \rightarrow \Box\varphi_n \vdash_L \bot,$$

for some $\Box\varphi \in \Gamma_0$ and $\varphi_1, \dots, \varphi_n \in \mathbf{For}\mathcal{ML}$. By the deduction theorem and the regularity rule, it follows that

$$\vdash_L \Box\varphi \rightarrow \Box\neg \bigwedge_{i=1}^{n} (\Diamond\varphi_i \rightarrow \Box\varphi_i),$$

and so

$$\neg\Diamond \bigwedge_{i=1}^{n} (\Diamond\varphi_i \rightarrow \Box\varphi_i) \in \Gamma_0,$$

contrary to Lemma 5.20.

Now take a maximal L-consistent extension $t_1 = (\Gamma_1, \Delta_1)$ of t'. Clearly $t_0 R_L t_1$. We are going to show that either t_1 itself or any $t_2 \in t_1\!\uparrow$ has no proper successors. Indeed, otherwise we have three tableaux $t_i = (\Gamma_i, \Delta_i)$ in W_L, for $i = 1, 2, 3$, such that $t_2 \neq t_3$ and $t_1 R_L t_2 R_L t_3$. But then there is a formula φ such that $\varphi \in \Gamma_2 \cap \Delta_3$ and so, by the transitivity of R_L, $\Diamond\varphi \in \Gamma_1$ and $\Box\varphi \in \Delta_1$, whence $\Diamond\varphi \rightarrow \Box\varphi \in \Delta_1$, contrary to t_1 being L-consistent. ❑

As a consequence of this theorem and results in Section 3.5 we derive

Corollary 5.22 (i) **K4.1** = **K4** $\oplus \Box\Diamond p \rightarrow \Diamond\Box p$ *is canonical, with $\mathfrak{F}_{\mathbf{K4.1}}$ being transitive and satisfying the McKinsey condition.*

(ii) **S4.1** = **S4** $\oplus \Box\Diamond p \rightarrow \Diamond\Box p$ *is canonical, with $\mathfrak{F}_{\mathbf{S4.1}}$ being a quasi-order satisfying the McKinsey condition.*

5.3 The filtration method

The canonical model for a consistent logic L refutes all the formulas which do not belong to L. It is very big (contains continuum many points, to be more exact) and complicated. On the other hand, the examples of **Int** and **K** show that each formula $\varphi \notin L$ may be separated from L by a finite frame. Provided that L is finitely axiomatizable, this immediately yields the decidability of L (for details consult Section 16.2).

The filtration method is intended to establish such completeness results and sometimes it may succeed even if the method of canonical models fails to prove canonicity.

To establish the finite approximability of a logic L, we need to prove that for every formula φ there is a frame $\mathfrak{F}$ satisfying the following three conditions: (1) $\mathfrak{F} \not\models \varphi$, (2) $\mathfrak{F}$ is finite, (3) $\mathfrak{F} \models L$. By Theorem 5.5, to ensure (1) it suffices to take the canonical frame $\mathfrak{F}_L$ for L. It is somewhat more difficult to satisfy (2), but, as we saw in Sections 2.4 and 3.4, also possible. Since we are interested only in truth-values of φ, all the formulas which are not subformulas of φ may be discarded from the tableaux in W_L. Or better we shall regard tableaux $t_1 = (\Gamma_1, \Delta_1)$ and $t_2 = (\Gamma_2, \Delta_2)$ in W_L as $\mathbf{Sub}\varphi$-*equivalent* if $\Gamma_1 \cap \mathbf{Sub}\varphi = \Gamma_2 \cap \mathbf{Sub}\varphi$. And then we see that modulo the $\mathbf{Sub}\varphi$-equivalence tableaux in W_L mostly duplicate each other. More exactly, there are at most $2^{|\mathbf{Sub}\varphi|}$ pairwise non-$\mathbf{Sub}\varphi$-equivalent tableaux in W_L. Is it possible to construct from them some Hintikka system $\mathfrak{H} = \langle T, S\rangle$? To do this it suffices to define an accessibility relation S so that the conditions (HS_I1) and (HS_I2), if $L \in \mathrm{Ext}\mathbf{Int}$, and ($\mathrm{HS}_M1$) and ($\mathrm{HS}_M2$), if $L \in \mathrm{NExt}\mathbf{K}$, are satisfied. The former of these two conditions can always be satisfied, for instance, by taking it as a necessary condition for S. To meet the latter, we can use the fact that $\mathfrak{F}_L$ satisfies it and simply put $t_1 S t_2$ if $t_1' R_L t_2'$, for some $t_1', t_2' \in W_L$ that are $\mathbf{Sub}\varphi$-equivalent to t_1 and t_2, respectively. The restrictions thus obtained give in general a spectrum of suitable S. And this is very much to the point, since we still need to take care of the condition (3). Whether (3) can be met by a proper choice of S depends on the particular logic L. So let us first consider in more detail the construction sketched above and then apply it to establish the finite approximability of a few superintuitionistic and modal logics.

Suppose we have a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ of the language $\mathcal{L}$ or $\mathcal{ML}$ on a frame $\mathfrak{F} = \langle W, R\rangle$ and let Σ be a set of ($\mathcal{L}$- or $\mathcal{ML}$-) formulas closed under subformulas, i.e., $\mathbf{Sub}\varphi \subseteq \Sigma$ whenever $\varphi \in \Sigma$. We say points $x, y \in W$ are Σ-*equivalent* in $\mathfrak{M}$ and write $x \sim_\Sigma y$ if

$$(\mathfrak{M}, x) \models \varphi \text{ iff } (\mathfrak{M}, y) \models \varphi, \text{ for every } \varphi \in \Sigma.$$

Clearly $\sim_\Sigma$ is an equivalence relation on W. Denote by $[x]_\Sigma$ the equivalence class generated by x, i.e., put $[x]_\Sigma = \{y \in W : x \sim_\Sigma y\}$. As a rule we will drop the subscript Σ and write simply $[x]$ and $x \sim y$ if this does not involve ambiguity.

A *filtration of* $\mathfrak{M}$ *through* Σ is any model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ based on a frame $\mathfrak{G} = \langle V, S\rangle$ such that

(i) $V = \{[x] : x \in W\}$;

(ii) $\mathfrak{U}(p) = \{[x] : x \in \mathfrak{V}(p)\}$, for every variable $p \in \Sigma$;

(iii) xRy implies $[x]S[y]$, for all $x, y \in W$;

(iv) if $[x]S[y]$ then $y \models \varphi$ whenever $x \models \Box\varphi$, for $x, y \in W$ and $\Box\varphi \in \Sigma$,

in the modal case and

(iv′) if $[x]S[y]$ then $y \models \varphi$ whenever $x \models \varphi$, for all $x, y \in W$ and $\varphi \in \Sigma$,

in the intuitionistic one.

Theorem 5.23. (Filtration) *Let $\mathfrak{N}$ be a filtration of a model $\mathfrak{M}$ through a set of formulas Σ. Then for every point x in $\mathfrak{M}$ and every formula $\varphi \in \Sigma$,*

$$(\mathfrak{M}, x) \models \varphi \;\; \textit{iff} \;\; (\mathfrak{N}, [x]) \models \varphi.$$

Proof The proof proceeds by induction on the construction of φ. The basis of induction follows from (ii). Now let $\varphi = \Box\psi \in \Sigma$ and $x \models \varphi$. To prove $[x] \models \varphi$, we need to show that $[y] \models \psi$ for every successor $[y]$ of $[x]$. So suppose $[x]S[y]$. Then, by (iv), $y \models \psi$ and, by the induction hypothesis, $[y] \models \psi$. Conversely, let $[x] \models \Box\psi$. Take any $y \in x\!\uparrow$. Then, by (iii), $[x]S[y]$ and so $[y] \models \psi$, whence, by the induction hypothesis, $y \models \psi$. The induction step for $\varphi = \psi \wedge \chi$, $\varphi = \psi \vee \chi$ and $\varphi = \psi \to \chi$ in the modal case follows immediately from the truth-definition and the induction hypothesis.

The intuitionistic case is considered analogously by using (iv′) instead of (iv). ❑

In general, the conditions (iii) and (iv) (or (iv′)) do not determine S uniquely. Actually, they allow us to choose any relation S in the interval $\underline{S} \subseteq S \subseteq \overline{S}$, where

$$\underline{S} = \{\langle [x], [y] \rangle : \; \exists x', y' \in W \; (x' \sim x \wedge y' \sim y \wedge x'Ry')\},$$

$$\overline{S} = \{\langle [x], [y] \rangle : \; \forall \Box\varphi \in \Sigma \; (x \models \Box\varphi \to y \models \varphi)\}$$

or, in the intuitionistic case,

$$\overline{S} = \{\langle [x], [y] \rangle : \; \forall \varphi \in \Sigma \; (x \models \varphi \to y \models \varphi)\}.$$

Indeed, if $[x]S[y]$ holds then, by (iv) and (iv′), $[x]\overline{S}[y]$. And if $[x]\underline{S}[y]$ then $x'Ry'$, for some $x' \in [x]$, $y' \in [y]$, and so, by (iii), $[x]S[y]$. The fact that $\underline{S}$ satisfies (iv) or (iv′) and $\overline{S}$ satisfies (iii) follows directly from the truth-definition in modal and intuitionistic models.

For this reason the filtration on the frame $\underline{\mathfrak{G}} = \langle V, \underline{S} \rangle$ is called the *finest* or the *least filtration* of $\mathfrak{M}$ through Σ, while the filtration on the frame $\overline{\mathfrak{G}} = \langle V, \overline{S} \rangle$ is called the *coarsest* or the *greatest*.

It is to be noted that a relation S between $\underline{S}$ and $\overline{S}$ may be nontransitive even if the original R is transitive, in particular, not all S in this interval give rise to filtrations of intuitionistic models. To construct a transitive relation we can take the *transitive closure* $\widehat{\underline{S}}$ of $\underline{S}$, i.e., put

$$\widehat{\underline{S}} = \{\langle [x], [y] \rangle : \; \exists n > 0 \; [x]\underline{S}^n[y]\}.$$

Clearly $\widehat{\underline{S}}$ satisfies (iii). To prove (iv), suppose $[x]\widehat{\underline{S}}[y]$ and $x \models \Box\varphi$, for some x, y in $\mathfrak{M}$ and $\Box\varphi$ in Σ. Then there is a finite sequence of points $[u], \ldots, [v]$

such that $[x]\underline{S}[u]\underline{S}\dots\underline{S}[v]\underline{S}[y]$. By the definition of $\underline{S}$, $x'Ru'$ for some x' and u' that are Σ-equivalent to x and u, respectively. Since R is transitive and by Proposition 3.6, we then have $u' \models \Box^+\varphi$ and so $u \models \Box^+\varphi$. Using the same argument for the sequence $u, \dots, v, y$, we shall eventually obtain $y \models \Box^+\varphi$. The intuitionistic models are considered analogously. Observe that in this case $\widehat{\underline{S}}$ as well as $\overline{S}$ are partial orders.

Alternatively we can define a transitive filtration $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ of a transitive modal model $\mathfrak{M}$ through Σ by taking, for any x and y in $\mathfrak{M}$,

$$[x]S[y] \text{ iff } y \models \Box^+\varphi \text{ whenever } x \models \Box\varphi, \text{ for all } \Box\varphi \in \Sigma.$$

It should be clear that the frame $\mathfrak{G} = \langle V, S \rangle$ is transitive and that $\mathfrak{N}$ is a filtration of $\mathfrak{M}$. It is called the *Lemmon filtration of* $\mathfrak{M}$ *through* Σ.

A very important property of filtrations is that they are finite whenever the "filter" Σ is finite. Moreover, a filtration may be finite even if the filter Σ is infinite. Say that a set Σ is *finitely based over a model* $\mathfrak{M}$ if there is a finite set of formulas Δ, a *finite base of* Σ *over* $\mathfrak{M}$, such that

$$\forall \psi \in \Sigma\ \exists \chi \in \Delta\ \mathfrak{M} \models \psi \leftrightarrow \chi.$$

For example, since **Cl** is locally tabular (see Theorem 1.29), the Boolean closure, i.e., the closure under $\wedge$, $\vee$, $\to$, $\bot$ of every finite set of modal formulas is finitely based over any model. (However, this is not so in the intuitionistic case.)

Proposition 5.24 *Suppose* $\mathfrak{N}$ *is a filtration of a model* $\mathfrak{M}$ *through a set* Σ *which is finitely based over* $\mathfrak{M}$ *and* Δ *is a finite base of* Σ. *Then* $\mathfrak{N}$ *contains at most* $2^{|\Delta|}$ *points.*

Proof Clearly, two points are Σ-equivalent in $\mathfrak{M}$ iff they are Δ-equivalent. So the number of pairwise non-Σ-equivalent points in $\mathfrak{M}$ is not greater than the number of subsets in Δ. ❑

As a consequence of Theorem 5.23 and Proposition 5.24 we obtain

Corollary 5.25 *Suppose* $\mathfrak{M}$ *is a countermodel for* φ *and* Σ *a finitely based over* $\mathfrak{M}$ *set of formulas closed under subformulas. Then every filtration of* $\mathfrak{M}$ *through* Σ *is a finite countermodel for* φ.

Thus, to prove the finite approximability (the finite model property, to be more precise) of a logic L, it suffices to show that for every formula $\varphi \notin L$ there is a filtration $\mathfrak{N}$ of some countermodel $\mathfrak{M}$ for φ through some finitely based (over $\mathfrak{M}$) filter Σ containing φ such that $\mathfrak{N} \models L$. If this is really the case then we say that L *admits filtration.*

Corollary 5.26 *If a logic* L *admits filtration then* L *is finitely approximable.*

The following two remarks are relevant here. First, if a logic L is sound with respect to the class of frames satisfying a property $\mathcal{P}$, then to prove that L admits filtration it is sufficient to show that for every $\varphi \notin L$ there is a filtration

$\mathfrak{N} = \langle\mathfrak{G},\mathfrak{U}\rangle$ of $\mathfrak{M}_L$ (or some other countermodel for φ) through some finitely based Σ containing φ such that $\mathfrak{G}$ satisfies $\mathcal{P}$. Second, it turns out that when filtrating the canonical model $\mathfrak{M}_L$, we have no real choice for S. For the following proposition holds:

Proposition 5.27 *Suppose $\mathfrak{N} = \langle\mathfrak{G},\mathfrak{U}\rangle$ is a filtration of the canonical model $\mathfrak{M}_L = \langle\mathfrak{F}_L,\mathfrak{V}_L\rangle$ through Σ such that $\mathfrak{N} \models L$. Then $\mathfrak{N}$ is the finest filtration of $\mathfrak{M}_L$ through Σ.*

Proof Let $\mathfrak{G} = \langle V,S\rangle$. We need to show that $S \subseteq \underline{S}$, i.e., for any $[x],[y] \in V$ such that $[x]S[y]$, there are $x',y' \in W_L$ for which $x' \sim x$, $y' \sim y$ and $x'R_Ly'$. Consider the tableaux $t_1 = (\Gamma_1,\Delta_1)$ and $t_2 = (\Gamma_2,\Delta_2)$, where

$$\Gamma_1 = \{\varphi : \ (\mathfrak{N},[x]) \models \varphi\}, \quad \Delta_1 = \{\varphi : \ (\mathfrak{N},[x]) \not\models \varphi\},$$

$$\Gamma_2 = \{\varphi : \ (\mathfrak{N},[y]) \models \varphi\}, \quad \Delta_2 = \{\varphi : \ (\mathfrak{N},[y]) \not\models \varphi\}.$$

Since $\mathfrak{N} \models L$, both t_1 and t_2 are L-consistent and so belong to W_L. And since $[x]S[y]$, we must have $t_1R_Lt_2$. ❑

We are in a position now to prove the finite approximability of a few modal and superintuitionistic logics using the filtration method.

First of all, since our basic logics **Int** and **K** are characterized by the class of all frames, they trivially admit filtration.

Next, let us observe that, by (iii), every filtration of a reflexive or serial model is reflexive or serial too. More generally, if the underlying frame of a model $\mathfrak{M}$ satisfies some condition expressed by first order formulas (with R and $=$ as their only predicates), containing no occurrences of $\to$ and $\bot$—such formulas are called *positive*—then (iii) guarantees that the underlying frame of every filtration of $\mathfrak{M}$ also satisfies this condition, which can readily be proved by induction on the construction of the first order positive formulas. In fact, this is a consequence of the result in classical model theory according to which positive formulas are stable under homomorphisms (see Chang and Keisler, 1990, Theorem 3.2.4). Thus we have

Theorem 5.28 *If a normal modal or superintuitionistic logic L is characterized by the class of frames satisfying some first order positive formulas in R and $=$ then L admits filtration and so is finitely approximable.*

Proof The detailed proof is left to the reader as an exercise. ❑

Corollary 5.29 *The logics* **D**, **T** *and* **S5** *are finitely approximable and decidable.*

Proposition 5.30 *The finest filtration of every symmetrical model is also symmetrical.*

Proof Suppose $\mathfrak{N} = \langle\mathfrak{G},\mathfrak{U}\rangle$ is the finest filtration of a model $\mathfrak{M}$ based on a symmetrical frame $\mathfrak{F} = \langle W,R\rangle$ and $[x]S[y]$, for some points $[x]$, $[y]$ in $\mathfrak{G}$. Then by the definition of S, there are $x' \in [x]$ and $y' \in [y]$ such that $x'Ry'$, from which $y'Rx'$ and so, by (iii), $[y]S[x]$. ❑

As a consequence of Proposition 5.30 and Theorem 5.17, according to which **KB** is characterized by symmetrical frames, we obtain

Corollary 5.31 **KB** *admits filtration and so is finitely approximable and decidable.*

Using the transitive closure of the finest filtration or the Lemmon filtration and the fact that **K4**, **D4** and **S4** are characterized by the classes of transitive frames, serial transitive frames and quasi-orders, respectively, we immediately obtain

Corollary 5.32 *The logics* **K4**, **D4**, **S4** *admit filtration and so are finitely approximable and decidable.*

Also we have

Theorem 5.33 *The logics* **K4.2**, **K4.3**, **S4.2**, **S4.3**, **KC**, **LC** *admit filtration and so are finitely approximable and decidable.*

Proof We show how to establish this result only for **K4.2**. The other logics are considered analogously.

By Theorem 5.17 and the generation theorem, **K4.2** is characterized by the class of rooted transitive directed frames. So it suffices to show that, for every model $\mathfrak{M}$ based on such a frame $\mathfrak{F} = \langle W, R \rangle$ and a finite filter Σ, there is a filtration of $\mathfrak{M}$ through Σ which is also based on a transitive directed frame.

Take the transitive closure $\mathfrak{N}$ of the finest filtration of $\mathfrak{M}$ through Σ. Let S be the accessibility relation in $\mathfrak{N}$ and let $[x]S[y]$ and $[x]S[z]$, for some points $[x]$, $[y]$, $[z]$ in $\mathfrak{N}$ such that $[y] \neq [z]$. Then uRy', vRz', for some $y' \sim y$, $z' \sim z$, u and v. Clearly, $y' \neq z'$. Since $\mathfrak{F}$ is rooted and transitive, both y' and z' are seen from the root of $\mathfrak{F}$ and so, by the directedness condition, there is w such that $y'Rw$ and $z'Rw$, from which $[y]S[w]$ and $[z]S[w]$. ❑

Remark It is worth noting that although **S4.3** is characterized by the class of linear partial orders, it is not characterized by the class of finite linear partial orders. For example, the Grzegorczyk formula is refuted by a proper cluster or an infinite ascending chain and so does not belong to **S4.3**. On the other hand, it is valid in every finite partial order. It follows in particular that by filtrating linear orders we may obtain chains with proper clusters.

Our next two results are a bit more complicated. They demonstrate situations when we have to filtrate models through sets which are bigger than the set of subformulas of the refuted formula.

Theorem 5.34 *The logics* **K4.1** *and* **S4.1** *admit filtration and so are finitely approximable and decidable.*

Proof We consider only **K4.1**, leaving **S4.1** to the reader as an exercise. According to Corollary 5.22, **K4.1** is characterized by the class of transitive frames satisfying the McKinsey condition. So, given a countermodel $\mathfrak{M}$ for φ on such a frame $\mathfrak{F}$, we must construct a transitive filtration $\mathfrak{N}$ of $\mathfrak{M}$ through some finite set $\Sigma \supseteq \mathbf{Sub}\varphi$ such that every final cluster in $\mathfrak{N}$ is simple. Observe at once that, by (iii) and the McKinsey condition, no filtration of $\mathfrak{M}$ contains dead ends. Thus, our only problem is to avoid final proper clusters in $\mathfrak{N}$. We recommend the reader first to try filtrating $\mathfrak{M}$ through $\mathbf{Sub}\varphi$ to understand that under such a filtration two final simple clusters in $\mathfrak{M}$ may be put into one proper cluster in $\mathfrak{N}$. To prevent this, we should take a smaller accessibility relation in our filtration which can be done by choosing a bigger filter Σ.

Define Σ as the closure under subformulas of the set

$$\{\Box\Diamond\psi,\ \Diamond\Box\psi :\ \psi \in \mathbf{Sub}\varphi\}$$

and let $\mathfrak{N}$ be the transitive closure of the finest filtration of $\mathfrak{M}$ through Σ. Suppose $[x]$ and $[y]$ belong to a final cluster in $\mathfrak{N}$ and show that $[x] = [y]$. According to the filtration theorem, it suffices to establish that $[x] \sim [y]$.

Take a formula $\psi \in \Sigma$. If $\psi = \Box\chi$ or $\psi = \Diamond\chi$ then, by Proposition 3.6, $[x] \models \psi$ iff $[y] \models \psi$. So the only remaining case is $\psi \in \mathbf{Sub}\varphi$. Suppose $[x] \models \psi$. Then $\Diamond\psi$ is true in $\mathfrak{N}$ at every point in the cluster containing $[x]$; so $[x] \models \Box\Diamond\psi$ and $x \models \Box\Diamond\psi$. Since $\mathfrak{M}$ is a model for **K4.1**, we must then have $x \models \Diamond\Box\psi$ and hence $[x] \models \Diamond\Box\psi$. Therefore, there is a point $[z]$ in the cluster under consideration such that $[z] \models \Box\psi$ and so $[y] \models \psi$. ❑

Theorem 5.35 *The logic* **K5** *admits filtration and so is finitely approximable and decidable.*

Proof By Theorem 5.17, $\mathbf{K5} = \mathbf{K} \oplus \Diamond\Box p \to \Box p$ is characterized by the class of Euclidean frames. Let $\mathfrak{M}$ be a countermodel for a formula φ based on a Euclidean frame. Again, a filtration of $\mathfrak{M}$ through $\mathbf{Sub}\varphi$ need not be Euclidean. So let us try a bigger filter, say,

$$\Sigma = \mathbf{Sub}\varphi \cup \{\Diamond\Box\psi :\ \Box\psi \in \mathbf{Sub}\varphi\}.$$

Let $\mathfrak{N}$ be the coarsest filtration of $\mathfrak{M}$ through Σ. We show that its underlying frame $\mathfrak{G} = \langle V, S\rangle$ is Euclidean.

Suppose $[x]S[y]$ and $[x]S[z]$, for some $[x],[y],[z] \in V$, and prove that $[y]S[z]$. By the definition of S, we need to show that $[y] \models \Box\psi$ implies $[z] \models \psi$, for every $\Box\psi \in \Sigma$. So let $\Box\psi \in \Sigma$ and $[y] \models \Box\psi$. Then $[x] \models \Diamond\Box\psi$ and, by the filtration theorem, $x \models \Diamond\Box\psi$, from which $x \models \Box\psi$, since $\mathfrak{M}$ is a model for **K5**. Therefore, $[x] \models \Box\psi$ and $[z] \models \psi$. ❑

Remark Since **K5** has finitely many distinct modalities (see Exercise 5.10), the modal closure, i.e., the closure under prefixing $\Box$ and $\Diamond$, of every finite set of formulas is finitely based over any model for **K5**. So instead of Σ in the proof above we might use the modal closure of $\mathbf{Sub}\varphi$.

Theorem 5.36 *For every variable free formula ψ, the logic $\mathbf{K} \oplus \psi$ admits filtration and so is finitely approximable and decidable.*

Proof Since ψ contains no variables, every flirtation of a model for $\mathbf{K} \oplus \psi$ refuting φ through $\mathbf{Sub}\varphi \cup \mathbf{Sub}\psi$ is also a model for $\mathbf{K} \oplus \psi$ in which φ is refuted. ❑

It should be clear that instead of $\mathbf{K}$ in Theorem 5.36 we can take any other logic considered in this section.

5.4 Diego's theorem

The bigger the filter, the more properties of the initial model will be inherited by its filtration and the more chances that the filtration will be a model for the logic under consideration. In this section we show that the closure of every finite set of intuitionistic formulas under $\wedge$, $\rightarrow$ and $\bot$ (or $\neg$) is finitely based over any intuitionistic model and so can be exploited as a filter for establishing the finite approximability of superintuitionistic logics. This very useful result is an immediate consequence of the following:

Theorem 5.37. (Diego's theorem) *For every $n \geq 0$, the set Ξ_n of formulas, constructed from the variables $p_1, \dots, p_n$ using $\wedge$, $\rightarrow$ and $\bot$, contains only finitely many pairwise non-equivalent in* **Int** *formulas.*

Proof The proof proceeds via a number of lemmas and requires some auxiliary definitions.

To begin with, we form the coarsest filtration $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ of the canonical model for **Int** through Ξ_n. We will regard points t in $\mathfrak{F} = \langle W, R \rangle$ as tableaux $t = (\Gamma, \Delta)$ such that

$$\Gamma = \{\varphi \in \Xi_n : (\mathfrak{M}, t) \models \varphi\}, \quad \Delta = \{\varphi \in \Xi_n : (\mathfrak{M}, t) \not\models \varphi\}.$$

For an atomic $p \in \Xi_n$, call such a tableau $t = (\Gamma, \Delta)$ *p-prime* (relative to Ξ_n) if $p \in \Delta$ and, for every $\varphi \in \Xi_n$, either $\varphi \in \Gamma$ or $\varphi \rightarrow p \in \Gamma$.

Lemma 5.38 *For any atomic $p \in \Xi_n$, any $t = (\Gamma, \Delta) \in W$ and any $\varphi \in \Xi_n$, if $\varphi \rightarrow p \in \Delta$ then there is a p-prime successor $t^* = (\Gamma^*, \Delta^*)$ of t in $\mathfrak{F}$ such that $\varphi \in \Gamma^*$.*

Proof Since $\varphi \rightarrow p \in \Delta$, there must be a point $t_1 = (\Gamma_1, \Delta_1)$ accessible from t in $\mathfrak{F}$ for which $\varphi \in \Gamma_1$, $p \in \Delta_1$. Let X be a maximal chain of points in $\mathfrak{F}$ refuting p and such that $t_1 \in X$. Put $\Gamma^* = \bigcup_{(\Gamma', \Delta') \in X} \Gamma'$, $\Delta^* = \Xi_n - \Gamma^*$ and $t^* = (\Gamma^*, \Delta^*)$. The tableau t^* is **Int**-consistent, for otherwise we would have $\varphi_1 \wedge \dots \wedge \varphi_k \rightarrow \psi_1 \vee \dots \vee \psi_l \in \mathbf{Int}$, for some $\varphi_1, \dots, \varphi_k \in \Gamma^*$, $\psi_1, \dots, \psi_l \in \Delta^*$. But then, since X is linearly ordered and by ($\mathrm{HS}_I 1$), there exists $t' = (\Gamma', \Delta') \in X$ such that $\varphi_1, \dots, \varphi_k \in \Gamma'$, $\psi_1, \dots, \psi_l \in \Delta'$, contrary to the **Int**-consistency of t'. Therefore t^* is a point in $\mathfrak{F}$. In fact, it is the final point in X. Besides, we clearly have $\varphi \in \Gamma^*$ and tRt^*. It remains to observe that t^* is p-prime. Indeed, by the definition, $p \in \Delta^*$ and if ψ and $\psi \rightarrow p$ are in Δ^* then there is a successor t' of t^* such that $t' \models \psi$, $t' \not\models p$, from which $t' = t^*$, for otherwise we can extend the chain X by adding t' to it, contrary to its maximality. ❑

Let V be the set of all p-prime tableaux in W, for all atomic $p \in \Xi_n$, S the restriction of R to V and $\mathfrak{G} = \langle V, S \rangle$.

Lemma 5.39 *For any $t = (\Gamma, \Delta) \in W$ and any $\varphi \in \Delta$, there is a tableau $t^* = (\Gamma^*, \Delta^*)$ in V such that tRt^* and $\varphi \in \Delta^*$.*

Proof Observe first that, by the intuitionistic equivalences

$$p \leftrightarrow ((\bot \to \bot) \to p), \quad (p \to (q \to r)) \leftrightarrow (p \wedge q \to r)$$

and

$$(p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r),$$

φ is equivalent in **Int** to a formula of the form $\bigwedge_i (\psi_i \to p_i)$, for some atomic $p_i \in \Xi_n$ and $\psi_i \in \Xi_n$. Therefore, $\psi_i \to p_i \in \Delta$, for some i, and so, by Lemma 5.38, there is a p_i-prime tableau $t^* = (\Gamma^*, \Delta^*)$ accessible from t and such that $\psi_i \in \Gamma^*$. It follows immediately that $\varphi \in \Delta^*$. ❑

As a consequence we readily derive that $\mathfrak{G} = \langle V, S \rangle$ is a Hintikka system characterizing Ξ_n in the sense that, for every $\varphi \in \Xi_n$, φ is in **Int** iff $\varphi \in \Gamma$, for all (Γ, Δ) in V.

Our goal now is to show that $\mathfrak{G}$ is finite.

Lemma 5.40 *If $t = (\Gamma, \Delta)$ is a p-prime tableau and $t' = (\Gamma', \Delta')$ a proper successor of t in $\mathfrak{G}$ then $p \in \Gamma'$.*

Proof Since $t \neq t'$ and tSt', there must be some $\varphi \in \Gamma' - \Gamma$. And since t is p-prime, $\varphi \to p \in \Gamma$. Therefore, $\varphi \to p \in \Gamma'$ and so $p \in \Gamma'$. ❑

Suppose $t = (\Gamma, \Delta)$ is a p-prime tableau in $\mathfrak{G}$, for some $p \neq p_n$, and $p_n \in \Gamma$. Form a tableau $t' = (\Gamma', \Delta')$ by taking

$$\Gamma' = \{\varphi \in \Gamma : \ p_n \notin \mathbf{Sub}\varphi\}, \quad \Delta' = \{\varphi \in \Delta : \ p_n \notin \mathbf{Sub}\varphi\}.$$

Clearly t' is a p-prime tableau relative to Ξ_{n-1}. It turns out that t is uniquely determined by t' and p_n in the following sense.

Lemma 5.41 $\Gamma = \{\varphi \in \Xi_n : \ \Gamma', p_n \vdash_{\mathbf{Int}} \varphi\}$.

Proof It suffices to show that $\Gamma', p_n \vdash_{\mathbf{Int}} \varphi$, for every $\varphi \in \Gamma$. So let φ be an arbitrary formula in Γ and $\varphi' = \varphi\{\top / p_n\}$. By the strong completeness theorem for **Int**, we have $p_n \vdash_{\mathbf{Int}} \varphi' \leftrightarrow \varphi$. It follows that $\varphi' \in \Gamma$. Hence $\varphi' \in \Gamma'$ and so $\Gamma', p_n \vdash_{\mathbf{Int}} \varphi$. ❑

We are in a position now to prove the crucial

Lemma 5.42 *$\mathfrak{G}$ is finite.*

Proof The proof proceeds by induction on n. If $n = 0$ then, according to Corollary 2.27, $\mathfrak{G}$ contains only one point.

Suppose now that $n > 0$. By the induction hypothesis and Lemma 5.41, there are finitely many tableaux (Γ, Δ) in $\mathfrak{G}$ such that $p \in \Gamma$, for some atomic $p \in \Xi_n$.

(The variables may be renamed to use Lemma 5.41.) So it is sufficient to show that there is a finite number of tableaux $t = (\Gamma, \Delta)$ in $\mathfrak{G}$ containing all $p_1, \dots, p_n$ in Δ. By Lemma 5.40, every such point t has no predecessors in $\mathfrak{G}$. And by the generation theorem and the fact that all p_i are in Δ, t is uniquely determined by the set of its proper successors in $\mathfrak{G}$. Since, as we have already established, only a finite number of such sets exists, there are only finitely many distinct $t = (\Gamma, \Delta)$ with $p_1, \dots, p_n \in \Delta$. ❑

It is not difficult now to complete the proof of Diego's theorem. Since $\mathfrak{G}$ characterizes Ξ_n, for any $\varphi, \psi \in \Xi_n$ we have $\varphi \leftrightarrow \psi \in \mathbf{Int}$ iff for every (Γ, Δ) in $\mathfrak{G}$, φ and ψ simultaneously belong either to Γ or to Δ. Ergo the number of pairwise non-equivalent in **Int** disjunction free formulas built from $\bot, p_1, \dots, p_n$ is not greater than the number of subsets in V, that is $2^{|V|}$. ❑

As a direct consequence of Diego's theorem we obtain

Corollary 5.43 *Suppose Σ is a finite set of intuitionistic formulas. Then the closure of Σ under $\wedge$, $\rightarrow$ and $\bot$ contains finitely many pairwise non-equivalent in* **Int** *formulas and so is finitely based over any intuitionistic model.*

We take advantage of this result to establish the finite approximability of the Kreisel–Putnam logic **KP**. In Section 7.3 we shall use it to prove the finite approximability of an infinite family of si-logics.

Theorem 5.44 *The Kreisel–Putnam logic*

$$\mathbf{KP} = \mathbf{Int} + (\neg p \rightarrow q \vee r) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$$

admits filtration and so is finitely approximable and decidable.

Proof Suppose $\varphi \notin \mathbf{KP}$ and $\mathfrak{M}$ is a model for **KP** refuting φ. Let Σ be the closure of $\mathbf{Sub}\varphi$ under $\rightarrow$, $\wedge$ and $\bot$, and Δ a finite base of Σ over $\mathfrak{M}$. Construct the coarsest filtration $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ of $\mathfrak{M}$ through Σ. By Proposition 5.24, Corollary 5.43 and the filtration theorem, $\mathfrak{N}$ is a finite countermodel for φ.

To prove that $\mathfrak{G} = \langle V, S \rangle$ is a frame for **KP**, we show that it satisfies the first order condition for ***kp*** given in Exercise 2.10. Suppose otherwise. Since $\mathfrak{G}$ is finite, we then have points $[x], [y], [z] \in V$ such that $[x]S[y]$, $[x]S[z]$, $[y]$ and $[z]$ do not see each other and every successor $[u]$ of $[x]$, seeing both $[y]$ and $[z]$ (in particular $[x]$ itself), sees a final point $[w]$ in $\mathfrak{G}$, which is not accessible from $[y]$ and $[z]$. Let $[w_1], \dots, [w_n]$ be all the final points in $\mathfrak{G}$ that are seen from $[x]$ and are not seen from $[y]$ and $[z]$. According to our assumption, $n > 0$.

For a point $[v] \in V$, denote by γ_v the conjunction of all the formulas in Δ that are true at $[v]$ and by δ_v the disjunction of those formulas in Δ that are false at $[v]$ in $\mathfrak{N}$. Put $\gamma = \bigvee_{i=1}^{n} \gamma_{w_i}$ and consider the following substitution instance of ***kp***:

$$\kappa = (\neg\gamma \rightarrow \delta_y \vee \delta_z) \rightarrow (\neg\gamma \rightarrow \delta_y) \vee (\neg\gamma \rightarrow \delta_z).$$

Since $\mathfrak{M} \models \mathbf{KP}$, we have $(\mathfrak{M}, x) \models \kappa$. Also we must have $x \not\models \neg\gamma \rightarrow \delta_y$ and $x \not\models \neg\gamma \rightarrow \delta_z$. Indeed, if for instance $x \models \neg\gamma \rightarrow \bigvee_i \psi_i$, where ψ_i are the formulas in Δ that are false at $[y]$, then by using ***kp*** a sufficient number of times we obtain that $x \models \bigvee_i(\neg\gamma \to \psi_i)$ and so $x \models \neg\gamma \to \psi_i$, for some i. And since $\neg\gamma$ is equivalent in **Int** to $\bigwedge_{i=1}^{n} \neg\gamma_{w_i} \in \Sigma$, we conclude by the filtration theorem that $[x] \models \neg\gamma \to \psi_i$. On the other hand, we have $[y] \models \neg\gamma$, for otherwise there is a point $[v]$ accessible from $[y]$ and such that $[v] \models \gamma_{w_i}$, for some i, and so $[w_i]S[v]$, which is possible only when $[w_i] = [v]$, since $[w_i]$ is final in $\mathfrak{G}$. It follows that $[y] \models \psi_i$, which is a contradiction.

Therefore, $x \not\models \neg\gamma \to \delta_y \vee \delta_z$ and so there is $u \in x\uparrow$ such that $u \models \neg\gamma$ and $u \not\models \delta_y \vee \delta_z$. Then $[x]S[u]$, $[u] \models \neg\gamma$ and $[u] \not\models \delta_y \vee \delta_z$, from which $[u]S[y]$ and $[u]S[z]$, since $\mathfrak{N}$ is the coarsest filtration of $\mathfrak{M}$. Take a point $[w_i] \in [u]\uparrow$. Clearly $[w_i] \models \gamma_{w_i}$ and so $[u] \not\models \neg\gamma_{w_i}$, contrary to $[u] \models \bigwedge_{i=1}^{n} \neg\gamma_{w_i}$. ❑

Remark In Section 18.2 we shall show that there are formulas $\varphi \notin \mathbf{KP}$ whose smallest refutation frames validating **KP** contain at least $2^{2^{|\mathbf{Sub}\varphi|}}$ points. This means that by filtrating $\mathfrak{M}$ through **Sub**φ we could not establish the finite approximability of **KP**.

5.5 Selective filtration

If we want to use the filtration method for establishing the finite approximability of a logic L without knowing any non-trivial completeness results for it, we have no other choice but to filtrate the canonical model $\mathfrak{M}_L$ through some set of formulas Σ. However, this may yield no result no matter what Σ we choose, even if L is really finitely approximable. For example, as we shall see below, **GL** is characterized by the class of finite strict orders, but the canonical frame $\mathfrak{F}_{\mathbf{GL}}$ contains a reflexive point, and so by (iii) in Section 5.3, every filtration of $\mathfrak{M}_{\mathbf{GL}}$ has a reflexive point as well.

When filtrating a model or better a Hintikka system $\mathfrak{H}$ through Σ, we divide the tableaux in $\mathfrak{H}$ into Σ-equivalence classes, identify the tableaux in each class and try to project the accessibility relation in $\mathfrak{H}$ to the resulting finite set of tableaux so that we again could obtain a Hintikka system. Yet there is another way of constructing finite Hintikka systems starting from $\mathfrak{H}$: instead of factorizing $\mathfrak{H}$ we may try to extract a finite subsystem of $\mathfrak{H}$ by *selecting* some suitable points in the Σ-equivalence classes in accordance with the rules for constructing Hintikka systems. This method is known as *selective filtration*. We use it here to establish the finite approximability of **GL**, **Grz** and $\mathbf{T}_n$. (By the way, none of these logics, except $\mathbf{T}_1$, is canonical.)

A general scheme of selective filtration, which will be enough for our purposes, may be described as follows. Suppose L is a modal or superintuitionistic logic and $\varphi \notin L$. Then there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ separating φ from L, i.e., $\mathfrak{M} \not\models \varphi$ and $\mathfrak{M} \models L$. Suppose also that a set of formulas Σ is finitely based over $\mathfrak{M}$, closed under subformulas and contains φ. We may think of $\mathfrak{F}$ as the Hintikka system $\mathfrak{F} = \langle W, R \rangle$, with points $t \in W$ being the tableaux $t = (\Gamma, \Delta)$, where

$$\Gamma = \{\psi \in \Sigma : (\mathfrak{M}, t) \models \psi\}, \quad \Delta = \{\psi \in \Sigma : (\mathfrak{M}, t) \not\models \psi\}.$$

We start our selective filtration of $\mathfrak{M}$ through Σ by selecting a tableau $t = (\Gamma, \Delta)$ in W such that $\varphi \in \Delta$. The tableau t will be the root of the finite Hintikka system we are going to extract from $\mathfrak{M}$. It may turn out that the pair $\mathfrak{H} = \langle T, S\rangle$, where $T = \{t\}$ and S is the restriction of R to T, is already a Hintikka system. In this situation we are done. Otherwise there are formulas $\Box\psi \in \Delta$ ($\psi \to \chi \in \Delta$, in the intuitionistic case) such that either $\psi \notin \Delta$ or not tSt (respectively, $\psi \notin \Gamma$). Denote by Θ_t the set of all formulas of that sort. Now, at the second step for each $\Box\psi \in \Theta_t$ (respectively, $\psi \to \chi \in \Theta_t$), we select a tableau $t' = (\Gamma', \Delta')$ in W such that tRt' and $\psi \in \Delta'$ (respectively, $\psi \in \Gamma'$ and $\chi \in \Delta'$). Denote by T_t the set of all selected successors of t. Then we add T_t to T, thus obtaining a set T', take the restriction S' of R to T' and check whether $\mathfrak{H}' = \langle T', S'\rangle$ is a Hintikka system. If this is not the case then, for each $t' \in T_t$, we consider formulas in $\Theta_{t'}$, select a set $T_{t'}$ of suitable successors of t', add it to T', and so on till we reach tableaux t^* with $\Theta_{t^*} = \emptyset$. If we succeed then the resulting Hintikka system $\mathfrak{H}^* = \langle T^*, S^*\rangle$ will certainly refute φ.

Two points are essential in this construction. First, we must ensure somehow that the process will eventually terminate. For example, we may try to select successors t' of each tableau t in such a way that $\Theta_{t'}$ contains less formulas than Θ_t. And second, to separate φ from L, $\mathfrak{H}^*$ must be a frame for L. In that respect the definition of the accessibility relation in $\mathfrak{H}^*$ as the restriction of R to T^* may be too severe. For in fact, to obtain a Hintikka system, it is sufficient to define on T^* any relation S in the interval $S_* \subseteq S \subseteq S^*$, where tS_*t' iff either $t = t'$ and tRt' or $t' \in T_t$ (of course, in the intuitionistic case S must be a partial order).

We now apply this scheme to prove the finite approximability of

$$\mathbf{GL} = \mathbf{K4} \oplus \Box(\Box p \to p) \to \Box p.$$

Using the selective filtration, we will extract from the canonical model $\mathfrak{M}_{\mathbf{GL}}$ a finite submodel that refutes $\varphi \notin \mathbf{GL}$ and contains only irreflexive points, which, by Proposition 3.47, is enough to ensure that the model validates the Löb axiom ***la***.

The following observation is the key to the filtration.

Lemma 5.45 *Suppose $x \not\models \Box\psi$ for some point x in a model $\mathfrak{M}$ for* **GL**. *Then there is an (irreflexive) point $y \in x{\uparrow}$ such that $y \not\models \psi$ and $y \models \Box\psi$.*

Proof Since every substitution instance of the Löb axiom is true in $\mathfrak{M}$, we have $x \models \Box(\Box\psi \to \psi) \to \Box\psi$. Therefore, $x \not\models \Box(\Box\psi \to \psi)$ and so there is $y \in x{\uparrow}$ such that $y \models \Box\psi$ and $y \not\models \psi$. Clearly, y is irreflexive. ❑

Theorem 5.46 **GL** *is characterized by the class of finite strict partial orders.*

Proof It suffices to show that every formula $\varphi \notin \mathbf{GL}$ is refuted by some finite strict partial order $\mathfrak{G} = \langle V, S\rangle$. We construct it according to the scheme above by filtrating $\mathfrak{M}_{\mathbf{GL}}$ through $\Sigma = \mathbf{Sub}\varphi$.

Observe first that there is an irreflexive point x_0 in $\mathfrak{M}_{\mathbf{GL}}$ such that $x_0 \not\models \varphi$. For there must be x in $\mathfrak{M}_{\mathbf{GL}}$ refuting φ, and if x is reflexive then $x \not\models \Box\varphi$ and we can use Lemma 5.45.

We define $\mathfrak{G}$ by induction. Put $V_0 = \{x_0\}$ and $\Theta_{x_0} = \{\Box\psi \in \Sigma : x_0 \not\models \Box\psi\}$. Suppose now that $V_n = \{x_1, \dots, x_m\}$ has been already constructed. If $\Theta_{x_i} = \emptyset$ for all $i = 1, \dots, m$, then let $V = \bigcup_{j=0}^{n} V_j$ and S be the restriction of $R_{\mathbf{GL}}$ to V. Otherwise, for each x_i with $\Theta_{x_i} \neq \emptyset$ and each $\Box\psi \in \Theta_{x_i}$, we select according to Lemma 5.45 an irreflexive point $y \in x_i\uparrow$ such that $y \not\models \psi$ and $y \models \Box\psi$. Let V_{n+1} be the set of the selected points y.

Since $|\Theta_y| < |\Theta_{x_i}|$ (because $\mathfrak{F}_{\mathbf{GL}}$ is transitive) and Σ is finite, we must eventually reach a set V_k whose points validate all the boxed formulas in Σ, i.e., $\Theta_x = \emptyset$ for every $x \in V_k$. By the construction, the resulting frame $\mathfrak{G}$ is a strict partial order and $\mathfrak{G} \not\models \varphi$. ❑

Corollary 5.47 (i) **GL** *is characterized by the class of Noetherian strict orders.*
(ii) **GL** *is characterized by the class of finite strictly ordered trees.*

Proof Follows from Theorem 5.46, Proposition 3.47, Exercise 3.12 and the reduction theorem. ❑

It is somewhat more difficult to prove the finite approximability of the Grzegorczyk logic

$$\mathbf{Grz} = \mathbf{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p.$$

First we observe that the canonical frame for **Grz** satisfies two good properties:

Proposition 5.48 $\mathfrak{F}_{\mathbf{Grz}}$ *is reflexive and transitive.*

Proof Suppose there is x in $\mathfrak{F}_{\mathbf{Grz}}$ such that $x \notin x\uparrow$. By the definition of canonical model, this means that $x \models \Box\varphi$ and $x \not\models \varphi$, for some formula φ. But then $x \not\models \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$, contrary to $\mathfrak{M}_{\mathbf{Grz}} \models \mathbf{Grz}$. Thus, $\mathfrak{M}_{\mathbf{Grz}}$ is reflexive.

Now let us prove that $\Box p \to \Box\Box p \in \mathbf{Grz}$. By Theorem 5.16, it will follow that $\mathfrak{F}_{\mathbf{Grz}}$ is transitive. Suppose otherwise. Then $x \not\models \Box p \to \Box\Box p$ and so $x \not\models \varphi$, where $\varphi = (p \wedge \neg\Box p) \vee \Box\Box p$, for some point x in $\mathfrak{M}_{\mathbf{Grz}}$. We will show that in this case $x \models \Box(\Box(\varphi \to \Box\varphi) \to \varphi)$.

Suppose otherwise. Then there exists $y \in x\uparrow$ such that $y \models \Box(\varphi \to \Box\varphi)$, $y \models \Box p$ and $y \not\models \Box\Box p$ (for if $y \not\models p$ then $x \not\models \Box p$, which is a contradiction). Besides, there is $z \in y\uparrow$ for which $z \models \varphi \to \Box\varphi$, $z \models p$ and $z \not\models \Box p$. Therefore, $z \models \varphi$ and $z \models \Box\varphi$. Now we have $u \in z\uparrow$ such that $u \models (p \wedge \neg\Box p) \vee \Box\Box p$ and $u \not\models p$. By the reflexivity, $u \not\models \Box\Box p$ and hence $u \models p \wedge \neg\Box p$, which is a contradiction.

Thus $x \not\models \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$, contrary to $\mathfrak{M}_{\mathbf{Grz}}$ being a model for **Grz**. ❑

Corollary 5.49 $\mathbf{Grz} = \mathbf{K4} \oplus \boldsymbol{grz} = \mathbf{S4} \oplus \boldsymbol{grz}$.

According to Proposition 3.48, to establish the finite approximability of **Grz**, given a formula $\varphi \notin \mathbf{Grz}$, we need to extract from $\mathfrak{M}_{\mathbf{Grz}}$ a finite partially ordered countermodel for φ. The following lemma shows in particular that when filtrating $\mathfrak{M}_{\mathbf{Grz}}$ through $\Sigma = \mathbf{Sub}\varphi$ we may choose successors of a point x in clusters different from $C(x)$. And if two or more successors appear in the same non-degenerate cluster, we simply shall not take into account the accessibility relation between them.

Lemma 5.50 *Suppose* $\Box\psi \in \Sigma$, $x \models \psi$ *and* $x \not\models \Box\psi$, *for some point* x *in* $\mathfrak{M}_{\mathbf{Grz}}$. *Then there is a point* $y \in x\!\uparrow$ *such that* $y \not\models \psi$ *and* $z \sim_\Sigma x$ *for no* $z \in y\!\uparrow$.

Proof Suppose otherwise. Then for every $y \in x\!\uparrow$ such that $y \not\models \psi$, there is a point $z \in y\!\uparrow$ which is Σ-equivalent to x and so $z \models \psi$ and $z \not\models \Box\psi$. It follows that $x \models \Box(\Box(\psi \to \Box\psi) \to \psi)$.

Since $x \not\models \Box\psi$, there is $y \in x\!\uparrow$ such that $y \not\models \psi$, and since $\mathfrak{F}_{\mathbf{Grz}}$ is transitive, $y \models \Box(\Box(\psi \to \Box\psi) \to \psi)$, contrary to $\mathfrak{M}_{\mathbf{Grz}} \models \mathbf{Grz}$. ❑

We are in a position now to prove

Theorem 5.51 **Grz** *is determined by the class of finite partial orders.*

Proof Given a formula $\varphi \notin \mathbf{Grz}$, take $\Sigma = \mathbf{Sub}\varphi$ and use the selective filtration through Σ to extract from $\mathfrak{M}_{\mathbf{Grz}}$ a finite partially ordered frame $\mathfrak{G} = \langle V, S\rangle$ refuting φ. We construct $\mathfrak{G}$ by induction.

To begin with, we take some point x in $\mathfrak{M}_{\mathbf{Grz}}$ such that $x \not\models \varphi$ and put $\mathfrak{G}_0 = \langle V_0, S_0\rangle$, where $V_0 = \{x\}$, $S_0 = \{\langle x, x\rangle\}$, and $\Theta_x = \{\Box\psi \in \Sigma : x \not\models \Box\psi \text{ and } x \models \psi\}$. Suppose now that we have already constructed a partially ordered frame $\mathfrak{G}_n = \langle V_n, S_n\rangle$ with $V_n \subseteq W_{\mathbf{Grz}}$, $S_n \subseteq R_{\mathbf{Grz}}$. Let X_n be the set of final points x in $\mathfrak{G}_n$ such that $\Theta_x \neq \emptyset$. If $X_n = \emptyset$ then put $\mathfrak{G} = \mathfrak{G}_n$. Otherwise for each $x \in X_n$ and each $\Box\psi \in \Theta_x$, fix a point $y(x, \Box\psi) \in x\!\uparrow$ such that $y(x, \Box\psi) \not\models \psi$ and $\neg\exists z \in y(x, \Box\psi)\!\uparrow\; z \sim_\Sigma x$ (that such a point exists is guaranteed by Lemma 5.50). Put

$$V_{n+1} = V_n \cup \{y(x, \Box\psi) : \; x \in X_n \text{ and } \Box\psi \in \Theta_x\},$$

define S_{n+1} to be the reflexive and transitive closure of the relation

$$S_n \cup \{\langle x, y(x, \Box\psi)\rangle : \; x \in X_n \text{ and } \Box\psi \in \Theta_n\}$$

and let $\mathfrak{G}_{n+1} = \langle V_{n+1}, S_{n+1}\rangle$. It should be clear that $S_{n+1} \subseteq R_{\mathbf{Grz}}$ (but S_{n+1} is not in general the restriction of $R_{\mathbf{Grz}}$ to V_{n+1}).

Notice that $\mathfrak{G}_{n+1}$ is a partial order. Indeed, otherwise we would have a cluster in $\mathfrak{G}_{n+1}$ containing both x and $y(x, \Box\psi)$, for some $x \in X_n$ and $\Box\psi \in \Theta_x$. But then $y(x, \Box\psi) R_{\mathbf{Grz}} x$, contrary to our choice of $y(x, \Box\psi)$.

Since no chain in $\mathfrak{G}_{n+1}$ contains distinct Σ-equivalent points and since Σ is finite, at some step m we shall have $X_m = \emptyset$, and so our selection process will terminate. If we regard points x in $\mathfrak{G}$ as the tableaux $t_x = (\Gamma_x, \Delta_x)$ with $\Gamma_x = \{\psi \in \Sigma : \; (\mathfrak{M}_{\mathbf{Grz}}, x) \models \psi\}$ and $\Delta_x = \{\psi \in \Sigma : \; (\mathfrak{M}_{\mathbf{Grz}}, x) \not\models \psi\}$, then $\mathfrak{G}$ will clearly be a Hintikka system. Therefore, $\mathfrak{G} \not\models \varphi$. ❑

Corollary 5.52 (i) **Grz** *is characterized by the class of Noetherian partial orders.*

(ii) **Grz** *is characterized by the class of finite partially ordered trees.*

Proof Follows from Theorems 5.51, 2.19 and Proposition 3.48. ❑

Let us consider now the si-logics

$$\mathbf{T}_n = \mathbf{Int} + \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i, \quad \text{for } n \geq 1,$$

and prove that all of them are finitely approximable. By Proposition 2.41, $\mathbf{T}_n$ is sound with respect to the class of finite frames of branching $\leq n$. We shall use the selective filtration to show that $\mathbf{T}_n$ is also complete with respect to this class.

Theorem 5.53 *$\mathbf{T}_n$ is characterized by the class of finite frames of branching $\leq n$.*

Proof Suppose $\varphi \notin \mathbf{T}_n$ and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a model for $\mathbf{T}_n$ refuting φ. By Theorem 2.19 and the reduction theorem, without loss of generality we may assume that $\mathfrak{F} = \langle W, R \rangle$ is a tree. Let $\Sigma = \mathbf{Sub}\varphi$ and $\Gamma_x = \{\psi \in \Sigma : x \models \psi\}$, for every point x in $\mathfrak{F}$.

Given x in $\mathfrak{F}$, put $rg(x) = \{[y] : y \in x\uparrow\}$ and say that x is of *minimal range* if $rg(x) = rg(y)$ for every $y \in [x] \cap x\uparrow$. Since there are only finitely many distinct Σ-equivalence classes in $\mathfrak{M}$, every $y \in [x]$ sees a point $z \in [x]$ of minimal range.

We are in a position now to extract from $\mathfrak{M}$ a finite refutation frame $\mathfrak{G} = \langle V, S \rangle$ for φ of branching $\leq n$. To begin with, we select some point x of minimal range at which φ is refuted and put $V_0 = \{x\}$.

Suppose now that V_k has already been defined. If $|rg(x)| = 1$ for every $x \in V_k$, then we put $\mathfrak{G} = \langle V, S \rangle$ where $V = \bigcup_{i=0}^{k} V_k$ and S is the restriction of R to V. Otherwise, for each $x \in V_k$ with $|rg(x)| > 1$ and each $[y] \in rg(x)$ different from $[x]$ and such that $\Gamma_z \subset \Gamma_y$ for no $[z] \in rg(x) - \{[x]\}$, we select a point $u \in [y] \cap x\uparrow$ of minimal range. Let U_x be the set of all the selected points for x and $V_{k+1} = \bigcup_x U_x$. It should be clear that $\Gamma_x \subset \Gamma_u$ (and $rg(x) \supset rg(u)$), for every $u \in U_x$, and so the inductive process must terminate. Using the standard tableau argument one can readily show also that $\mathfrak{G} \not\models \varphi$.

It remains to establish that $\mathfrak{G} \models \mathbf{T}_n$, i.e., $\mathfrak{G}$ is of branching $\leq n$. Suppose otherwise. Then there is a point x in $\mathfrak{G}$ with $\geq n+1$ immediate successors $x_0, \dots, x_m$, which are evidently in U_x because $\mathfrak{F}$ is a tree. We are going to construct a substitution instance of $\mathbf{T}_n$'s axiom $\boldsymbol{bb}_n$ which is refuted at x in $\mathfrak{M}$.

Denote by δ_i the conjunction of the formulas in Γ_{x_i}. Since all of them are true at x_i in $\mathfrak{M}$, we have $x_i \models \delta_i$; and since $\Gamma_i \subseteq \Gamma_j$ for no distinct i and j, we have $x_j \not\models \delta_i$ if $i \neq j$. Put $\chi_i = \delta_i$, for $0 \leq i < n$, $\chi_n = \delta_n \vee \ldots \vee \delta_m$ and consider the truth-value of the formula $\psi = \boldsymbol{bb}_n\{\chi_0/p_0, \dots, \chi_n/p_n\}$ at x in $\mathfrak{M}$.

Since xRx_i for every $i = 0, \dots, m$, we have $x \not\models \bigvee_{i=0}^{n} \chi_i$. Suppose, however, that $x \not\models \bigwedge_{i=0}^{n}((\chi_i \to \bigvee_{i\neq j} \chi_j) \to \bigvee_{i\neq j} \chi_j)$. Then $y \models \chi_i \to \bigvee_{i\neq j} \chi_j$ and $y \not\models \bigvee_{i\neq j} \chi_j$, for some $y \in x\uparrow$ and some $i \in \{0, \dots, n\}$, and hence $y \not\models \chi_i$. Since $x_i \models \chi_i$ and $x_i \not\models \bigvee_{i\neq j} \chi_j$, y sees no points in $[x_i]$ and so $y \not\sim_\Sigma x$ (for otherwise

x would not be of minimal range). Therefore, $\Gamma_{x_j} \subseteq \Gamma_y$ for some $j \in \{0, \ldots, m\}$, and then $y \models \chi_j$ if $j < n$ and $y \models \chi_n$ if $j \geq n$, which is a contradiction.

It follows that $x \models \bigwedge_{i=0}^{n}((\chi_i \to \bigvee_{i \neq j} \chi_j) \to \bigvee_{i \neq j} \chi_j)$, from which $x \not\models \psi$, contrary to $\mathfrak{M}$ being a model for $\boldsymbol{bb}_n$. ❑

As a consequence of Theorem 5.53 we obtain the following completeness result justifying, by the way, the name $\mathbf{T}_n$ of the logics under consideration.

Corollary 5.54 $\mathbf{T}_n$ *is characterized by the class of finite n-ary trees.*

Proof Exercise (use the reduction theorem and Exercise 2.5). ❑

5.6 Kripke semantics for quasi-normal logics

The Kripke semantics for modal logics we have dealt with so far is suitable only for *normal* extensions of $\mathbf{K}$. Now we use the concept of canonical model to introduce in a rather natural way a Kripke semantics for all logics in Ext$\mathbf{K}$, including quasi-normal ones.

Suppose L is a consistent quasi-normal logic. Then the set of formulas

$$M = \{\varphi \in \mathbf{For}\mathcal{ML} : \ \forall n \geq 0 \ \Box^n \varphi \in L\}$$

is clearly a normal logic, the greatest one among all normal logics contained in L, to be more exact. We call M the *kernel* of L and denote it by $\ker L$.

Let $\mathfrak{M}_M = \langle \mathfrak{F}_M, \mathfrak{V}_M \rangle$ be the canonical model for M. Each maximal L-consistent tableau t is also a maximal M-consistent tableau, and so t is a point in $\mathfrak{F}_M$. Denote by D_L the set of all maximal L-consistent tableaux. Then by Lemma 5.2 which, as we observed, holds for quasi-normal logics as well, we have

$$\Lambda \vdash_L \varphi \text{ iff for every } (\Gamma, \Delta) \in D_L,\ \Lambda \subseteq \Gamma \text{ implies } \varphi \in \Gamma.$$

Therefore, by Theorem 5.4, for any Λ and φ,

$$\Lambda \vdash_L \varphi \text{ iff for every } t \in D_L,\ (\mathfrak{M}_M, t) \models \Lambda \text{ implies } (\mathfrak{M}_M, t) \models \varphi.$$

Of course, instead of M we can take any other normal logic contained in L.

This result can be interpreted as follows. We distinguish in $\mathfrak{M}_M$ a set of points, namely D_L, and regard them as the only "actual worlds" in $\mathfrak{M}_M$. A formula φ is then assumed to be true in $\mathfrak{M}_M$ if it is true at all the actual worlds.

Thus we arrive at the following Kripke semantics for quasi-normal logics.

A *Kripke frame with distinguished points* is a pair $\langle \mathfrak{F}, D \rangle$ where $\mathfrak{F} = \langle W, R \rangle$ is a Kripke frame and $D \subseteq W$. The points in D are called the *distinguished points* or the *actual worlds* in $\mathfrak{F}$. A *model with distinguished points* (based on $\langle \mathfrak{F}, D \rangle$) is a pair $\langle \mathfrak{M}, D \rangle$ where $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is an ordinary Kripke model based on $\mathfrak{F}$. A formula φ is said to be *true* in $\langle \mathfrak{M}, D \rangle$ (notation: $\langle \mathfrak{M}, D \rangle \models \varphi$) if $(\mathfrak{M}, x) \models \varphi$ for all $x \in D$. φ is *valid* in $\langle \mathfrak{F}, D \rangle$ (notation: $\langle \mathfrak{F}, D \rangle \models \varphi$) if φ is true in all models based on $\langle \mathfrak{F}, D \rangle$.

Clearly, $\mathfrak{F} \models \varphi$ iff $\langle \mathfrak{F}, W \rangle \models \varphi$, so a frame $\mathfrak{F}$ may be identified with $\langle \mathfrak{F}, W \rangle$. As to the other extreme case, it follows from the definition that all formulas, even $\bot$, are valid in $\langle \mathfrak{F}, \emptyset \rangle$.

The model $\langle \mathfrak{M}_{\ker L}, D_L \rangle$ and the frame $\langle \mathfrak{F}_{\ker L}, D_L \rangle$, constructed at the beginning of this section, are called the *canonical model* and the *canonical frame* (*with distinguished points*) *for* L, respectively.

What we have established so far can be summarized as the following:

Theorem 5.55 *Each consistent quasi-normal logic* L *is strongly characterized by its canonical model* $\langle \mathfrak{M}_{\ker L}, D_L \rangle$, *i.e., for every* Λ *and* φ,

$$\Lambda \vdash_L \varphi \text{ iff } \forall x \in D_L \ (x \models \Lambda \to x \models \varphi),$$

in particular,

$$\varphi \in L \text{ iff } \langle \mathfrak{M}_{\ker L}, D_L \rangle \models \varphi.$$

It is worth noting that a formula φ is true in $\langle \mathfrak{M}, D \rangle$ iff φ is true in every model in the class $\{\langle \mathfrak{M}, \{d\} \rangle : d \in D\}$. So we obtain

Theorem 5.56 *Every consistent quasi-normal logic is strongly characterized by a class of models having a single distinguished point.*

Given a class $\mathcal{C}$ of frames with distinguished points, denote by Log$\mathcal{C}$ the set of modal formulas that are valid in all frames in $\mathcal{C}$; if $\mathcal{C} = \{\langle \mathfrak{F}, D \rangle\}$ then we write simply Log$\langle \mathfrak{F}, D \rangle$. As an easy exercise we invite the reader to prove the following:

Proposition 5.57 *For every class* $\mathcal{C}$ *of frames with distinguished points, Log*$\mathcal{C}$ *is a quasi-normal logic.*

To illustrate the introduced semantics for quasi-normal logics we give some examples.

Example 5.58 The first known quasi-normal, but not normal extension of **S4** was

$$\mathbf{S4.1'} = \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p.$$

To understand why $\mathbf{S4.1'}$ is not normal, let us consider the frame $\mathfrak{F}$ shown in Fig. 5.1 (a) with actual world 0. Since $\mathfrak{F}$ does not satisfy the McKinsey condition, it refutes $\boldsymbol{ma}$ and so $(\mathfrak{F}, 0) \not\models \Box\boldsymbol{ma}$. However, $\langle \mathfrak{F}, \{0\} \rangle \models \boldsymbol{ma}$, for otherwise we would have (under some valuation) $1 \not\models \boldsymbol{ma}$, which is impossible. Therefore, $\mathbf{S4.1'} \subseteq \text{Log}\,\langle \mathfrak{F}, \{0\} \rangle$. On the other hand, $\Box\boldsymbol{ma} \notin \text{Log}\,\langle \mathfrak{F}, \{0\} \rangle$, which means that $\mathbf{S4.1'}$ is not closed under necessitation.

Theorem 5.59. (Scroggs' theorem) *All logics in* Ext**S5** *are normal.*

Proof It is enough to show that every quasi-normal extension L of $\mathbf{S5}(n)$ in the language with $n < \omega$ variables is normal. According to Theorem 5.55, L is characterized by $\langle \mathfrak{M}_{\mathbf{S5}(n)}, D_L \rangle$, which in view of Corollary 5.19 is finite. Using the differentiatedness and finiteness of $\mathfrak{F}_{\mathbf{S5}(n)}$ it is readily shown (see Exercise 5.3) that L is characterized by the frame $\langle \mathfrak{F}_{\mathbf{S5}(n)}, D_L \rangle$. Let $\mathfrak{F}$ be the subframe of $\mathfrak{F}_{\mathbf{S5}(n)}$

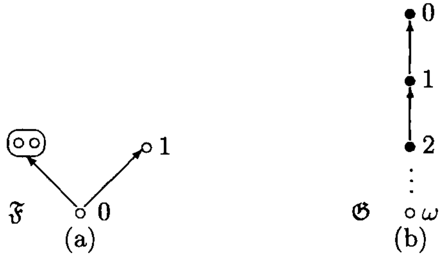


FIG. 5.1.

generated by D_L. Then $L = \mathrm{Log}\,\langle \mathfrak{F}, D_L\rangle$ and so, as is easy to see, $L = \mathrm{Log}\mathfrak{F}$.

❑

Example 5.60 As we observed in Section 3.8, there is no Kripke frame validating all formulas in Solovay's logic $\mathbf{S} = \mathbf{GL} + \Box p \to p$. It follows in particular that $\mathbf{S}$ has no consistent normal extensions. For the same reason no Kripke frame with distinguished points can validate $\mathbf{S}$. Logics with this property may be called *Kripke inconsistent*. All consistent extensions of $\mathbf{S}$, if any, are clearly Kripke inconsistent.

Moreover, there is no (normal) Kripke model for $\mathbf{S}$. For by Lemma 5.45, every model $\mathfrak{M}$ for $\mathbf{GL}$ contains a final irreflexive point x. (Indeed, if y is not a dead end in $\mathfrak{M}$ then $y \not\models \Box\bot$ and so $x \models \Box\bot$ for some $x \in y\uparrow$.) But then $x \not\models \Box\bot \to \bot$.

We construct now a model with a distinguished point for $\mathbf{S}$, which shows by the way that $\mathbf{S}$ is consistent. Let $\mathfrak{G} = \langle V, S\rangle$ be the (transitive) frame depicted in Fig. 5.1 (b), or formally

$$\mathfrak{G} = \langle \{i :\ i \leq \omega\}, \{\langle \omega,\omega\rangle, \langle j,i\rangle :\ 0 \leq i < j \leq \omega\}\rangle .$$

Define a valuation $\mathfrak{U}$ in $\mathfrak{G}$ by taking $\mathfrak{U}(p) = V$, for every variable p. Observe first that all substitution instances of the Löb axiom are true in the (normal) model $\mathfrak{M} = \langle \mathfrak{G}, \mathfrak{U}\rangle$. Indeed, all of them are clearly true at all irreflexive points in $\mathfrak{M}$. As to ω, one can readily prove by induction on the construction of φ that if $\omega \models \varphi$ (or $\omega \not\models \varphi$) then there is some $n < \omega$ such that $m \models \varphi$ (respectively, $m \not\models \varphi$), for all $m \in \{n, n+1, \dots, \omega\}$. So if a substitution instance of the Löb axiom is false at ω then it is also false at some irreflexive point n which, as we know, is impossible.

Thus, $\mathfrak{M} \models \mathbf{GL}$. Now, let us observe that $(\mathfrak{M}, \omega) \models \Box\varphi \to \varphi$, for every formula φ, simply because ω is reflexive. So if we distinguish ω as the only actual world in $\mathfrak{M}$ then we obtain that $\langle \mathfrak{M}, \{\omega\}\rangle \models \mathbf{S}$.

We use this observation to prove the following:

Theorem 5.61 *For every modal formula φ,*

$$\varphi \in \mathbf{S} \textit{ iff } \bigwedge_{\Box\psi \in \mathbf{Sub}\varphi} (\Box\psi \to \psi) \to \varphi \in \mathbf{GL}.$$

Proof The implication ($\Leftarrow$) is evident. To prove ($\Rightarrow$), suppose

$$\bigwedge_{\Box\psi\in\mathbf{Sub}\varphi} (\Box\psi \to \psi) \to \varphi \notin \mathbf{GL}.$$

Since **GL** is finitely approximable, this formula is refuted at the root x of some finite frame $\mathfrak{F} = \langle W, R\rangle$ for **GL** under some valuation. Construct a new frame $\mathfrak{G}$ by adding to $\mathfrak{F}$ the infinite chain depicted in Fig. 5.1 (b) so that it could see all points in $\mathfrak{F}$ and define a valuation in $\mathfrak{G}$ in such a way that the truth-value of each variable remains the same at all points in $\mathfrak{F}$ and at points in the added chain it coincides with that at x. By induction on the construction of $\chi \in \mathbf{Sub}\varphi$ and using the fact that $x \models \bigwedge_{\Box\psi\in\mathbf{Sub}\varphi}(\Box\psi \to \psi)$ one can show that $y \models \chi$ iff $x \models \chi$, for every $y \in x\downarrow$. Since the root of $\mathfrak{G}$ is reflexive, it follows that $\bigwedge_{\Box\psi\in\mathbf{Sub}\varphi}(\Box\psi \to \psi) \to \varphi$ is false at it. And that every substitution instance of $\boldsymbol{la}$ is true there is checked in the same way as in Example 5.60. ❑

5.7 Exercises

Exercise 5.1 Show that $\mathbf{K} \oplus \boldsymbol{ma} = \mathbf{D} \oplus \boldsymbol{ma}$.

Exercise 5.2 Construct modal and intuitionistic models that are not differentiated (tight, compact).

Exercise 5.3 Show that if $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is a differentiated finite model for a logic L then $\mathfrak{F}$ is a frame for L. Use this to prove that a logic is finitely approximable iff it has the finite model property.

Exercise 5.4 Show that $\mathbf{GL.3} = \mathbf{GL} \oplus \boldsymbol{con}$ is characterized by the class of finite strict linear orders and by the frame $\langle \omega, >\rangle$.

Exercise 5.5 Show that $\mathbf{K4Z} = \mathbf{K4} \oplus \boldsymbol{z}$ is characterized by the class consisting of finite irreflexive frames and balloons.

Exercise 5.6 Show that $\mathbf{D4Z.3} = \mathbf{D4} \oplus \boldsymbol{z} \oplus \boldsymbol{con}$ is characterized by the frame $\langle \omega, <\rangle$.

Exercise 5.7 Show that $\mathbf{Dum} = \mathbf{S4} \oplus \boldsymbol{dum}$ is characterized by the class consisting of finite partial orders and reflexive balloons.

Exercise 5.8 Show that $\mathbf{Grz.3} = \mathbf{Grz} \oplus \boldsymbol{sc}$ is characterized by the frame $\langle \omega, \geq\rangle$.

Exercise 5.9 Show that $\mathbf{Dum.3} = \mathbf{Dum} \oplus \boldsymbol{sc}$ is characterized by the frame $\langle \omega, \leq\rangle$.

Exercise 5.10 (i) Show that frames for **K5** are 3-transitive.

(ii) Prove that there are finitely many pairwise non-equivalent modalities in **K5**.

(iii) Prove that all logics in NExt**K5** are locally tabular and finitely axiomatizable.

Exercise 5.11 Show that $\mathbf{D4G_1}$ is finitely approximable.

Exercise 5.12 (i) Prove that extensions of **S4** may have only 14, 10, 8, 6, 2 or 1 pairwise non-equivalent modalities.

(ii) Show that both **S4.1** and **S4.2** have exactly 10 pairwise non-equivalent modalities, and $\mathbf{S4.1} \oplus \mathbf{S4.2}$ has only 8 of them.

Exercise 5.13 Show that $\mathbf{Int} = \bigcap_{i \geq 1} \mathbf{T}_n$ and that

$$\mathbf{Int} \subset \ldots \subset \mathbf{T}_n \subset \ldots \subset \mathbf{T}_2 \subset \mathbf{T}_1.$$

Exercise 5.14 Prove that each $\mathbf{T}_n$, for $n \geq 2$, has the disjunction property.

Exercise 5.15 Prove that all logics $\mathbf{Alt}_n$, for $n < \omega$, are finitely approximable.

Exercise 5.16 Prove that all logics in $\mathrm{NExt}\mathbf{Alt}_1$ are finitely approximable.

Exercise 5.17 Say that a logic L *strongly admits filtration* if for every generated submodel $\mathfrak{M}$ of $\mathfrak{M}_L$ and every finite set of formulas Σ closed under subformulas, there is a filtration of $\mathfrak{M}$ through Σ based on a frame for L. Prove that if L strongly admits filtration then L is globally finitely approximable. Use this to show that the logics **K**, **D**, **T**, **KB** are globally finitely approximable.

Exercise 5.18 Show that $\mathrm{Log}\langle \mathfrak{F}, D\rangle = \bigcap_{x \in D} \mathrm{Log}\,\langle \mathfrak{F}, \{x\}\rangle$.

Exercise 5.19 Show that $\mathbf{K4} = \bigcap_{n \geq 1} \mathbf{K4BD}_n = \bigcap_{n \geq 1} \mathbf{K4BW}_n$.

Exercise 5.20 Prove that $\mathbf{Alt}_3 \oplus \boldsymbol{re} \oplus \boldsymbol{sym}$ has infinitely many non-equivalent modalities. (Segerberg (1971) conjectures that no proper normal extension of $\mathbf{Alt}_3 \oplus \boldsymbol{re} \oplus \boldsymbol{sym}$ has this property.)

Exercise 5.21 Show that $\mathbf{K4H} = \mathbf{K4} \oplus p \to \Box(\Diamond p \to p)$ is canonical, with its canonical frame satisfying the condition

$$xRy \wedge yRz \to x = y \vee y = z.$$

Prove that every $L \in \mathrm{Next}\mathbf{K4H}$ is finitely approximable.

Exercise 5.22 Show that $\mathbf{S4} \oplus \Box\Diamond p \to (p \to \Box p)$ is characterized by the class of quasi-orders satisfying the condition

$$x \neq z \wedge xRz \wedge xRy \to yRz.$$

Exercise 5.23 Show that $\mathbf{S4} \oplus \Box(\Box p \to q) \vee (\Diamond\Box q \to p)$ is characterized by the class of quasi-orders satisfying the condition

$$xRz \wedge \neg zRx \wedge xRy \to yRz.$$

Exercise 5.24 Show that $\Box(p \to q) \to (\Box p \to \Diamond q) \in \mathbf{D}$.

Exercise 5.25 Prove that if a normal modal logic L contains the formula ***hin*** of Exercise 3.22 then $\mathfrak{F}_L$ satisfies the first order condition given in that exercise.

Exercise 5.26 Show that no distinct modalities are equivalent in the logics $\mathbf{T}$ and $\mathbf{K} \oplus \Diamond p \to \Box p$. Derive from this that there are at least two maximal logics in NExt$\mathbf{K}$ in which no distinct modalities are equivalent.

Exercise 5.27 Show that the logic $\mathbf{S4} + \Diamond \boldsymbol{grz}$ is not normal.

Exercise 5.28 Prove that $\mathbf{S4}+\{\Box\varphi_i : i \in I\} = \mathbf{S4}\oplus\{\Box\varphi_i : i \in I\}$. Is it possible to replace $\mathbf{S4}$ in this equality by $\mathbf{K4}$?

Exercise 5.29 Prove that (i) NExt$\mathbf{S4.3}$ = Ext$\mathbf{S4.3}$,
(ii) NExt$(\mathbf{S4.2} \oplus \boldsymbol{bd_3})$ = Ext$(\mathbf{S4.2} \oplus \boldsymbol{bd}_3)$ and
(iii) NExt$(\mathbf{S4} \oplus \boldsymbol{bd_2}) \neq$ Ext$(\mathbf{S4} \oplus \boldsymbol{bd}_2)$.

Exercise 5.30 Show that the reflexive point in the frame considered in Example 5.60 can be replaced by an irreflexive one.

Exercise 5.31 Show that $\ker L$ in Theorem 5.55 can be replaced with any normal logic $L' \subseteq L$ (for instance, $\mathbf{K}$).

Exercise 5.32 Show that $\mathbf{KP} \subseteq \mathbf{ML}$.

Exercise 5.33 (M. Abashidze) Let φ^n be the result of replacing every $\Box$ in φ by $\Box^n$. Prove that for every $n > 0$ and every φ,

$$\varphi \in \mathbf{GL} \text{ iff } \varphi^n \in \mathbf{GL}.$$

5.8 Notes

The construction of the canonical models is conceptually close to that used in the Henkin-style completeness proofs for classical first and second order calculi (see (Church 1956) and (Chang and Keisler 1990)) and to the Tarski–Lindenbaum algebras (see Chapter 7). The method of canonical models was introduced by Lemmon and Scott (1977)[8] and Makinson (1966); cf. also Cresswell (1967) and Schütte (1968). The canonical model of Lemmon and Scott (1977) seems to have its roots in the relational representation of Boolean algebras with modal operators (in particular, the Tarski–Lindenbaum algebras for modal logics), studied by Lemmon (1966a, 1966b). The canonical model of Makinson (1966) is an "improvement" of the tableau construction of Kripke (1963a). Perhaps these different sources explain why Lemmon and Scott (1977) introduce the filtration method, while the construction of the canonical model in Makinson (1966) is combined with selecting (using a sort of selective filtration) from it a countable submodel, which gives an analog of the Löwenheim–Skolem's theorem (see Theorem 6.29). Approximately at the same time the canonical model for intuitionistic (predicate) logic was constructed by Aczel (1968), Fitting (1969) and Thomason (1969).

The method of canonical models turned out to be a powerful tool in nonclassical logic. It was applied systematically to prove completeness theorems for a good many normal modal logics by Lemmon and Scott (1977), who obtained

[8] This book was written in 1966.

in particular Corollary 5.22 and the result of Exercise 5.25, and by Segerberg (1971). Routley (1970) extended the ideas of Makinson (1966) to wide classes of weak modal systems. Rennie (1970) noticed that the method of canonical models works for polymodal logics too. Smoryński (1973) used it for si-logics. Later the method was applied to a great many other types of logics; see for instance Goldblatt (1982) and Segerberg (1994).

The filtration method was introduced simultaneously with the canonical models by Lemmon and Scott (1977); the filtration theorem is due to Segerberg (1968). However, the algebraic variant of filtration goes back to McKinsey (1941) and Lemmon (1966a, 1966b). In modal logic various forms of filtration were used by Bull (1967), Segerberg (1968, 1971), Gabbay (1970b, 1972b, 1976) who developed selective filtrations, Nagle and Thomason (1985), Shehtman (1990a) and many others. Smoryński (1973), Gabbay (1970a), Ono (1972), Gabbay and de Jongh (1974), Ferrari and Miglioli (1993) and others applied it to si-logics. The results and proofs concerning the finite approximability, presented in this chapter, were taken from the cited papers and books. The observation of Exercise 5.17 is due to Goranko and Passy (1992). Diego's theorem was first proved by Diego (1966); the proof above is due to Urquhart (1974). Sobolev (1977b) somewhat generalized Diego's theorem and used it to establish the finite approximability of a wide class of si-logics. Unfortunately, even the formulation of this result is too complicated to be presented here. A consequence of Sobolev's theorem—that all si-logics with extra axioms in one variable are finitely approximable—will be proved by another method in Section 11.6. An interesting result was obtained by Drugush (1984): using a variant of selective filtration of Gabbay and de Jongh (1974) he proved that every si-logic characterized by a class of trees is finitely approximable. Note also that according to Drugush (1982) the union of si-logics characterized by finite trees is determined by finite trees too, i.e., the family of such logics is a sublattice of **ExtInt**. The completeness results of the preceding section concerning logics with linear frames are due to Segerberg (1970).

The semantics for quasi-normal modal logics was developed by Segerberg (1971). Example 5.58 and Exercise 5.28 are due to McKinsey and Tarski (1948). Scroggs's Theorem appeared in Scroggs (1951) and Theorem 5.61 was first proved by Solovay (1976).

Quite recently Shehtman has proved that every Kripke complete si-logic and every logic in **NExtS4** characterized by partially ordered Kripke frames is strongly complete with respect to the neighborhood semantics.

6

INCOMPLETENESS

In the preceding chapter we saw that many standard superintuitionistic and normal modal logics are Kripke complete, even finitely approximable and so decidable. Many of them turned out to be canonical and hence strongly Kripke complete, with their canonical frames satisfying good first order properties. Now we are facing the natural question: isn't it possible to extend these completeness results to all logics in **ExtInt** and **NExtK**? To present examples of incomplete (in one sense or another) logics in these classes and elucidate to some extent the origin of the incompleteness is the main aim of this chapter.

6.1 Logics that are not finitely approximable

As follows from the hierarchy in Section 4.3, the incompleteness with respect to the classes of finite frames accompanies some other incompleteness results, say Kripke incompleteness. So in a sense this section is redundant. However, the stronger the incompleteness result, the more complex logic is involved. Here we construct rather simple normal modal and si-logics that are not finitely approximable (in particular, Kripke complete), so that the origin of this phenomenon will be quite clear.

We begin with modal logics. Let us consider once again the frame $\mathfrak{G}$ shown in Fig. 5.1 (b). The root ω is clearly the only point in $\mathfrak{G}$ capable of refuting the Löb axiom $\boldsymbol{la} = \Box(\Box p \to p) \to \Box p$. Another characteristic property of ω is that it sees infinitely many points. More precisely, ω is the only point in $\mathfrak{G}$ such that if i is accessible from it, for some $i < \omega$, then $i+1$ is also accessible. This may be expressed by modal formulas in the following way. Since i is obviously the only point in $\mathfrak{G}$ at which the formula $\alpha_i = \Box^{i+1}\bot \wedge \Diamond^i\top$ is true, ω is the unique world where $\Diamond\alpha_0$ and all the formulas $\Diamond\alpha_i \to \Diamond\alpha_{i+1}$, for $i < \omega$, are true.

Thus, $\mathfrak{G} \models \neg\boldsymbol{la} \wedge \Diamond\alpha_i \to \neg\boldsymbol{la} \wedge \Diamond\alpha_{i+1}$, for all $i < \omega$, and $\mathfrak{G} \not\models \boldsymbol{la} \vee \neg\Diamond\alpha_0$. And if we notice also that all $\Diamond\alpha_i$ may be simultaneously satisfied only in an infinite frame then we can immediately conclude that $\mathrm{Log}\mathfrak{G}$ is not finitely approximable. Moreover, this observation can be developed into a much stronger result.

Put

$$L_1 = \mathbf{K4} + \{\neg\boldsymbol{la} \wedge \Diamond\alpha_i \to \neg\boldsymbol{la} \wedge \Diamond\alpha_{i+1} : \ i < \omega\}, \quad L_2 = \mathrm{Log}\mathfrak{G}.$$

Since $\mathfrak{G} \models L_1$, we have $L_1 \subseteq L_2$.

Theorem 6.1 (i) *No logic in the interval*

$$[L_1, L_2] = \{L \in \mathrm{Ext}\mathbf{K} : \ L_1 \subseteq L \subseteq L_2\}$$

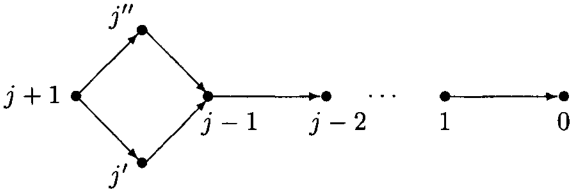


FIG. 6.1.

is finitely approximable.

(ii) *There is a continuum of normal logics in* $[L_1, L_2]$.

(iii) *There are infinitely many finitely axiomatizable normal logics in* $[L_1, L_2]$.

Proof (i) It is sufficient to show that $\boldsymbol{la} \vee \neg\Diamond\alpha_0$ is not in L_2 and cannot be separated from L_1 by a finite model. The former is clear, since $\mathfrak{G} \not\models \boldsymbol{la} \vee \neg\Diamond\alpha_0$.

Suppose $\mathfrak{M}$ is a model with actual world w such that $\langle\mathfrak{M}, \{w\}\rangle \models L_1$ and $\langle\mathfrak{M}, \{w\}\rangle \not\models \boldsymbol{la} \vee \neg\Diamond\alpha_0$. By the definition of L_1, we then have $w \models \Diamond\alpha_i$, for every $i < \omega$, and so there are points x_i in $\mathfrak{M}$ such that $x_i \models \alpha_i$. We show that $x_i \neq x_j$ whenever $i \neq j$.

Suppose otherwise, that is $x_i = x_j$ for some $i > j$. Then we have $x_j \models \Box^{j+1}\bot$ and since $\Box^{j+1}\bot \to \Box^i\bot \in L_1$ (because $\mathbf{K4} \subseteq L_1$), $x_j \models \Box^i\bot$, contrary to $x_i = x_j$ and $x_i \models \Diamond^i\top$ or, equivalently, $x_i \not\models \Box^i\bot$.

(ii) Let us consider the logics

$$L_I = L_1 \oplus \{\varphi_i :\ i \in I\}$$

where $I \subseteq \omega$ and $\varphi_i = \Box(\alpha_i \to p) \vee \Box(\alpha_i \to \neg p)$. Since α_i is true in $\mathfrak{G}$ only at one point, $\mathfrak{G} \models \varphi_i$ (to refute φ_i we need two points at which α_i is true: at one p is true while at the other p is false). So L_I is a normal logic in the interval $[L_1, L_2]$, for every $I \subseteq \omega$.

If $j \notin I$ then the frame $\mathfrak{H}$ in Fig. 6.1 validates φ_i, for every $i \in I$, and all the axioms of L_1 as well, because $\mathfrak{H} \models \boldsymbol{la}$. On the other hand, $\mathfrak{H}$ clearly refutes φ_j under every valuation such that $j' \models p$ and $j'' \not\models p$. Therefore, $\varphi_j \notin L_I$ and so $L_I \neq L_J$ if $I \neq J$. It follows that the cardinality of the set $\{L_I :\ I \subseteq \omega\}$ is that of continuum.

(iii) Let us consider the logic $L_3 = \mathbf{K4} \oplus \psi$, where

$$\psi = \neg\boldsymbol{la} \wedge \Diamond(\neg q \wedge \Box q) \to \neg\boldsymbol{la} \wedge \Diamond(\neg\Box q \wedge \Box\Box q),$$

and show that it belongs to the interval $[L_1, L_2]$.

If (under some valuation) $\neg\boldsymbol{la} \wedge \Diamond(\neg q \wedge \Box q)$ is true at a point x in $\mathfrak{G}$ then clearly $x = \omega$. Besides, $\omega \models \Diamond(\neg q \wedge \Box q)$ means that there is $y \in \omega\!\uparrow$ such that $y \models \neg q \wedge \Box q$. Therefore, y is irreflexive and so $y = i$ for some $i < \omega$. Since $i \not\models q$ and $j \models q$ for every $j < i$, we have $i + 1 \not\models \Box q$ and $i + 1 \models \Box\Box q$, from which $\omega \models \neg\boldsymbol{la} \wedge \Diamond(\neg\Box q \wedge \Box\Box q)$. Thus, $\mathfrak{G} \models \psi$ and $L_3 \subseteq L_2$.

To prove the inclusion $L_1 \subseteq L_3$ it is sufficient to observe that

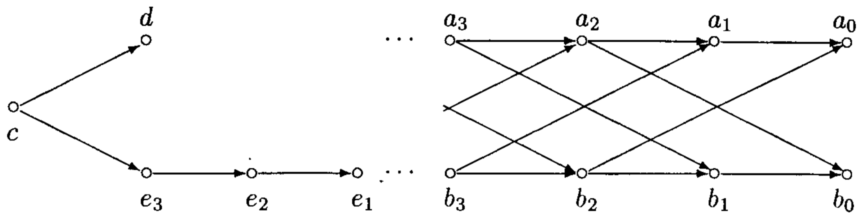


FIG. 6.2.

$$\psi\{\Box^i\bot/q\} \leftrightarrow (\neg \boldsymbol{la} \wedge \Diamond\alpha_i \to \neg \boldsymbol{la} \wedge \Diamond\alpha_{i+1}) \in \mathbf{K}.$$

Infinitely many other examples of finitely axiomatizable logics in the interval $[L_1, L_2]$ can be constructed from L_3 by adding to it formulas φ_i from the proof of (ii). ❑

It is worth noting that the frame $\mathfrak{G}$ in Fig. 5.1 (b) is of width 1, and so $L_1 \subseteq L_3 \oplus \boldsymbol{bw}_1 \subseteq L_2$. Thus, as a consequence of Theorem 6.1 we obtain

Theorem 6.2 *There is a finitely axiomatizable normal modal logic of width* 1 *(i.e., an extension of* **K4.3***) that is not finitely approximable.*

Intuitionistic frames are homogeneous in the sense that all their points are reflexive. So we cannot directly use the construction above to define si-logics that are not finitely approximable. (As we shall see in Section 11.6, all si-logics of width 1 are finitely approximable.) Yet the general idea may be realized in the intuitionistic case as well.

Instead of $\mathfrak{G}$ we use the intuitionistic frame $\mathfrak{F}$ shown in Fig. 6.2. It consists of two parts: the first one, containing the points a_i and b_i, for $i < \omega$, simulates the irreflexive part of $\mathfrak{G}$ and the remaining points c, d, e_1, e_2, e_3, seeing all points in the first part, simulate ω. Formally, $\mathfrak{F} = \langle W, R\rangle$ is defined as follows:

$$\begin{aligned} W &= \{a_i, b_i, c, d, e_j : \ i < \omega, \ j = 1, 2, 3\}, \\ R &= \{\langle x, x\rangle, \langle c, x\rangle, \langle d, a_k\rangle, \langle d, b_k\rangle, \langle e_l, a_k\rangle, \\ &\quad \langle e_l, b_k\rangle, \langle e_l, e_m\rangle, \langle a_{k+i}, a_k\rangle, \langle b_{k+i}, b_k\rangle, \langle a_{k+i+2}, b_k\rangle, \\ &\quad \langle b_{k+i+2}, a_k\rangle : \ x \in W, \ k, i < \omega, \ 1 \le l \le m \le 3\}. \end{aligned}$$

Instead of the Löb axiom and α_i above we take the intuitionistic formulas

$$\alpha = (p \to q) \vee (q \to p_1 \vee (p_1 \to p_2 \vee (p_2 \to p)))$$

and

$$\alpha_i \vee \beta_i,$$

where, for $i < \omega$,

$$\alpha_1 = r \to r' \vee \neg r', \ \beta_1 = \neg r \to r' \vee \neg r',$$

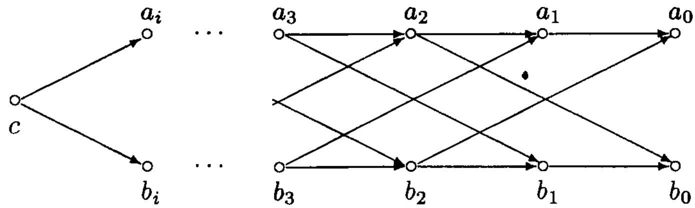


FIG. 6.3.

$$\alpha_2 = \beta_1 \to \alpha_1 \vee \neg\neg r,\ \beta_2 = \alpha_1 \to \beta_1 \vee \neg r,$$

$$\alpha_{i+3} = \beta_{i+2} \to \alpha_{i+2} \vee \beta_{i+1},\ \beta_{i+3} = \alpha_{i+2} \to \beta_{i+2} \vee \alpha_{i+1}.$$

Put

$$L_1 = \mathbf{Int} + \{\alpha \vee \alpha_{i+1} \vee \beta_{i+1} \to \alpha \vee \alpha_i \vee \beta_i :\ i \geq 2\},\quad L_2 = \mathrm{Log}\mathfrak{F}.$$

Theorem 6.3 (i) *No si-logic in the interval*

$$[L_1, L_2] = \{L \in \mathrm{ExtInt} :\ L_1 \subseteq L \subseteq L_2\}$$

is finitely approximable.

(ii) *There are a continuum of si-logics in* $[L_1, L_2]$.

(iii) *There are infinitely many finitely axiomatizable logics in* $[L_1, L_2]$.

Proof We give here only a sketch of the proof and invite the reader to fill the gaps.

(i) The formula $\alpha \vee \alpha_2 \vee \beta_2$ is not in L_2 and cannot be separated from L_1 by a finite model.

(ii) For $i \geq 2$, we put

$$\varphi_i = \alpha_{i+1} \wedge \beta_{i+1} \to \alpha_i \vee \beta_i.$$

It is not hard to verify that $\mathfrak{F} \models \varphi_i$, for every $i \geq 2$. Using the subframe of $\mathfrak{F}$ depicted in Fig. 6.3, one can show also that $\varphi_i \notin L_1 + \{\varphi_j :\ j \geq 2,\ i \neq j\}$.

(iii) As an example of a finitely axiomatizable logic in $[L_1, L_2]$ one can take the logic

$$L_3 = \mathbf{Int} + \alpha \vee \gamma_2 \to \alpha \vee \gamma_1,$$

where

$$\gamma_1 = ((p_2 \to p_1 \vee q_2) \to p_1 \vee (p_1 \to p_2 \vee q_1)) \vee ((p_1 \to p_2 \vee q_1) \to p_2 \vee (p_2 \to p_1 \vee q_2)),$$

$$\gamma_2 = (((p_1 \to p_2 \vee q_1) \to p_2 \vee (p_2 \to p_1 \vee q_2)) \to ((p_2 \to p_1 \vee q_2) \to p_1 \vee (p_1 \to p_2 \vee q_1)) \vee (p_2 \to p_1 \vee q_2)) \vee$$

$$(((p_2 \to p_1 \lor q_2) \to p_1 \lor (p_1 \to p_2 \lor q_1)) \to ((p_1 \to p_2 \lor q_1) \to p_2 \lor (p_2 \to p_1 \lor q_2)) \lor (p_1 \to p_2 \lor q_2)).$$

The inclusion $L_1 \subseteq L_3$ follows from the equalities

$$\gamma_1\{\beta_{i-3}/q_1, \alpha_{i-3}/q_2, \beta_{i-2}/p_1, \alpha_{i-2}/p_2\} = \alpha_i \lor \beta_i,$$

$$\gamma_2\{\beta_{i-3}/q_1, \alpha_{i-3}/q_2, \beta_{i-2}/p_1, \alpha_{i-2}/p_2\} = \alpha_{i+1} \lor \beta_{i+2}.$$

And to prove $L_3 \subseteq L_2$ it suffices to verify that $\mathfrak{F} \models \alpha \lor \gamma_2 \to \alpha \lor \gamma_1$.

Infinitely many other finitely axiomatizable logics in $[L_1, L_2]$ can be constructed by adding formulas φ_i to L_3. ❑

Since $\mathfrak{F}$ is of width 2, we have $\boldsymbol{bw}_2 \in \mathrm{Log}\mathfrak{F}$. So, as a consequence of Theorem 6.3 we obtain

Theorem 6.4 *There is a finitely axiomatizable superintuitionistic logic of width 2 that is not finitely approximable.*

6.2 Logics that are not canonical and elementary

Most of the logics considered in Chapter 5 proved to be Kripke complete simply because they are characterized by their canonical frames. However, canonicity is only a sufficient condition for Kripke completeness. Although the canonical frame $\mathfrak{F}_L$ for a logic L refutes all the formulas that are not in L, it may also refute some of L's axioms. In other words, L is not necessarily sound with respect to $\mathfrak{F}_L$, witness the following simple example.

Theorem 6.5 **GL** *is not canonical.*

Proof Recall that a frame validates the Löb axiom iff it is a Noetherian strict order, in particular it contains no reflexive points. We are going to show that there are reflexive worlds in $\mathfrak{F}_{\mathbf{GL}}$.

As was established in Example 5.60, Solovay's logic $\mathbf{S} = \mathbf{GL} + \Box p \to p$ is consistent. Therefore, by Lindenbaum's lemma, the tableau

$$(\mathbf{GL} \cup \{\Box\varphi \to \varphi : \varphi \in \mathbf{For}\mathcal{ML}\}, \emptyset)$$

can be extended to a maximal **GL**-consistent tableau (Γ, Δ), which is the reflexive point in $\mathfrak{F}_{\mathbf{GL}}$ we need, since $\Box\varphi \in \Gamma$ implies $\varphi \in \Gamma$. ❑

In fact we have even a stronger result.

Theorem 6.6 **GL** *is not strongly Kripke complete.*

Proof Let $\alpha_i = \Box(p_i \to \Diamond p_{i+1} \land \neg\Diamond p_i)$ and $\Gamma = \{\Diamond p_1, \alpha_i : 1 \leq i < \omega\}$. We show that the tableau $(\Gamma, \emptyset)$ is **GL**-consistent but not realizable in any model based upon a frame for **GL**. To prove the former it is enough to observe that the formula $\Diamond p_1 \land \alpha_1 \land \ldots \land \alpha_n$ is true at the root 0 in the model $\langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F} = \langle \{0, \ldots, n+1\}, < \rangle$ and $\mathfrak{V}(p_i) = \{i\}$, which is clearly a model for **GL**. And the latter claim follows from the fact that to make Γ true at a point we need an infinite ascending chain starting from it. ❑

Say that a class $\mathcal{C}$ of Kripke frames is *elementary* if there is a set Φ of first order sentences in R and $=$ such that, for every Kripke frame $\mathfrak{F}$, $\mathfrak{F} \in \mathcal{C}$ iff $\mathfrak{F}$ is a (classical) model for Φ. A logic L is *elementary* if the class of all Kripke frames for L is elementary.

To prove that **GL** is not elementary we use the compactness theorem from classical model theory (see Chang and Keisler, 1990, Theorem 1.3.22).

Theorem 6.7 **GL** *is not characterized by an elementary class of frames. In particular,* **GL** *is not elementary.*

Proof Suppose **GL** is characterized by a class $\mathcal{C}$ of Kripke frames (by Proposition 3.47, all of them are Noetherian strict partial orders) and show that $\mathcal{C}$ is not elementary.

Assume otherwise. Then $\mathcal{C}$ consists of all classical models for some set Φ of first order formulas with R and $=$ as their only predicates. By Theorem 5.46, **GL** is characterized by the class of finite strict orders. Since the formulas $\boldsymbol{bd_n}$ are refuted by transitive frames of depth $> n$ (see Proposition 3.44), none of them is in **GL**. Therefore, for every $n < \omega$, $\mathcal{C}$ contains a frame of depth $> n$.

Let us consider now the first order formulas

$$\phi_n = \bigwedge_{1 \leq i < j \leq n} (a_i R a_j \wedge \neg a_j R a_i)$$

(here a_i are individual constants of the first order language). Clearly, a strict partial order $\mathfrak{F}$ satisfies ϕ_n iff $\mathfrak{F}$ is of depth $\geq n$. So every finite subset of the set $\Phi \cup \{\phi_n : 1 \leq n < \omega\}$ has a model, for instance, a frame in $\mathcal{C}$ of depth $> m$ where m is the maximal subscript of ϕ_ns in the subset. By the compactness theorem, the whole set $\Phi \cup \{\phi_n : 1 \leq n < \omega\}$ has a model as well, say, a strict order $\mathfrak{F}$, which is in $\mathcal{C}$ because it satisfies Φ. But to satisfy all ϕ_n, $\mathfrak{F}$ must contain an infinite ascending chain $a_1 R a_2 R a_3 R \dots$ of distinct points, which is a contradiction, since $\mathfrak{F} \models \mathbf{GL}$ and so $\mathfrak{F}$ is Noetherian. ❑

In exactly the same way one can prove

Theorem 6.8 **Grz** *is not strongly complete and it is not characterized by an elementary class of frames. In particular,* **Grz** *is neither canonical nor elementary.*

In fact the notions of canonicity and elementarity turn out to be closely related: in Section 10.2 we shall prove that every logic in NExt**K** and Ext**Int** is canonical whenever it is characterized by an elementary class of frames. So by proving that a Kripke complete logic is not strongly complete we establish also that it is not elementary.

Theorem 6.9 $\mathbf{T}_2 = \mathbf{Int} + \boldsymbol{bb}_2$ *is not strongly complete. Moreover, no si-logic in the interval* $[\mathbf{Int}, \mathbf{T}_2]$*, save* **Int***, is strongly complete.*

Proof Let $\mathfrak{T}_2$ be the full binary tree. Say that a point a in $\mathfrak{T}_2$ is of *codepth* n ($cd(a) = n$, in symbols) if the chain $a{\downarrow}$ contains $n + 1$ points. With every point a in $\mathfrak{T}_2$ and every $i > 0$ we associate the variables p_a and q_i, respectively.

By the *type* of the root a_0 in $\mathfrak{T}_2$ we mean the tableau $t_{a_0} = (\emptyset, \{p_{a_0}\})$. And if the type of a point a in $\mathfrak{T}_2$ is $(\Theta, \{p_a\})$, and b, c are the immediate successors of a with $cd(b) = cd(c) = n$ then the *types* of b and c are $t_b = (\Theta \cup \{p_a, q_n\}, \{p_b\})$ and $t_c = (\Theta \cup \{p_a, \neg q_n\}, \{p_c\})$, respectively.

Now let us consider the tableau $t = (\Gamma, \{p_{a_0}\})$ in which Γ consists of all formulas of the form

$$\alpha = (\bigwedge \Theta \to p_b) \to p_a$$

such that b is a proper successor of a and $t_b = (\Theta, \{p_b\})$,

$$\beta = (\bigwedge \Theta \to p_c) \to p_a \vee p_b$$

such that $a{\uparrow} \cap b{\uparrow} = \emptyset$, $c{\downarrow} = a{\downarrow} \cap b{\downarrow}$ and $t_c = (\Theta, \{p_c\})$, and

$$\gamma = (\bigwedge \Sigma \to p_b) \to \varphi \vee (\bigwedge \Theta \to p_a)$$

such that $cd(a) > 0$, $t_a = (\Theta, \{p_a\})$, φ is the conjunction of all formulas of the form q_i and $\neg q_i$ in Θ and b is the immediate predecessor of a with the type $t_b = (\Sigma, \{p_b\})$.

It is a matter of routine to check that every finite subtableau of t is realizable in a model based on a sufficiently deep finite binary tree, which is a frame for $\mathbf{T}_2$ (it suffices to put $a \models p_b$ iff $b \notin a{\uparrow}$ and $a \models q_i$ iff q_i belongs to the left part of t_a). Thus, t is L-consistent, for any $L \in [\mathbf{Int}, \mathbf{T}_2]$.

We are going to show now that if t is realized in a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ then a generated subframe of $\mathfrak{F} = \langle W, R \rangle$ is reducible to any finite tree and so, by Corollary 2.33 and the reduction theorem, refutes all the formulas that are not in $\mathbf{Int}$, i.e., $\mathfrak{F} \not\models L$ for any proper extension L of $\mathbf{Int}$.

Without loss of generality we may assume that $\mathfrak{F}$ is rooted and t is realized at its root. For every a in $\mathfrak{T}_2$, put

$$Y_a = \{x \in W : \ x \text{ realizes } t_a\}.$$

Notice that if a point x sees some Y_a but does not belong to any Y_a itself, then the set $Z = \{a : \ x \in Y_a{\downarrow}\}$ has a root (with respect to the partial order in $\mathfrak{T}_2$). Indeed, otherwise there are two distinct minimal points $a, b \in Z$. Let $c{\downarrow} = a{\downarrow} \cap b{\downarrow}$ and $t_c = (\Theta, \{p_c\})$. Since $x \models (\bigwedge \Theta \to p_c) \to p_a \vee p_b$ and $x \in Y_a{\downarrow} \cap Y_b{\downarrow}$, we have $x \not\models p_a \vee p_b$ and so $x \not\models \bigwedge \Theta \to p_c$. It follows that Y_c is accessible from x, which is a contradiction. Denote the root of Z by a_x and put $X_a = Y_a \cup \{x \in W : \ a = a_x\}$. Thus,

$$\bigcup_{a \in \mathfrak{T}_2} X_a{\downarrow} = \bigcup_{a \in \mathfrak{T}_2} X_a = W'.$$

Observe also that if $a{\uparrow} \cap b{\uparrow} = \emptyset$ in $\mathfrak{T}_2$ then $X_a{\uparrow} \cap X_b{\uparrow} = \emptyset$ in $\mathfrak{F}$. To show this suppose φ and ψ are the conjunctions of all the q_i and $\neg q_i$ in the left parts of t_a and t_b, respectively. By the definition, φ is true at all points in Y_a. And if φ is not true at a point $x \in X_a - Y_a$ then, since all formulas of the form γ are true at x, x must see the set Y_c corresponding to the immediate predecessor c of a, which is a contradiction. Therefore, φ is true everywhere in X_a. By the same reason ψ is true everywhere in X_b. It remains to notice that φ and ψ cannot be true at a point simultaneously.

Using this observation and the formulas of the form α it is not hard to check that the map g defined by $g(x) = a$ iff $x \in X_a$ is a reduction of the subframe $\mathfrak{F}' = \langle W', R{\restriction}W' \rangle$ of $\mathfrak{F}$ to $\mathfrak{T}_2$ (we leave this to the reader; some details can be found in the proof of Theorem 9.39).

Let $\mathfrak{G}$ be an arbitrary finite tree. By Theorem 2.21, there is a reduction h of $\mathfrak{T}_2$ to $\mathfrak{G}$. The composition $f' = hg$ is then a reduction of $\mathfrak{F}'$ to $\mathfrak{G}$. So our aim now is to extend it to a reduction f of $\mathfrak{F}$ to $\mathfrak{G}$. For every $x \in W - W'$, the set $f'(x{\downarrow})$ is a chain in $\mathfrak{G}$ (for otherwise $X_a{\uparrow} \cap X_b{\uparrow} \neq \emptyset$ for some a and b in $\mathfrak{T}_2$ without common successors). Let u be a final point in $\mathfrak{G}$ accessible from the last point in this chain. Then we put $f(y) = u$ for all $y \in W - W'$ such that $f'(x{\downarrow}) = f'(y{\downarrow})$. And for $x \in W'$ let $f(x) = f'(x)$. It should be clear from the construction that f reduces $\mathfrak{F}$ to $\mathfrak{G}$, which proves our theorem. ❑

6.3 Logics that are not compact and complete

The compactness theorem from classical model theory, used in Section 6.2, may be formulated as follows: if every finite subset of a set of formulas Σ has a model refuting a formula φ then the whole set Σ also has a model refuting φ.

We say a modal or si-logic L is *compact* (relative to Kripke frames) if each formula $\varphi \notin L$ is separated from L by a Kripke frame whenever φ is separated by a Kripke frame from every finitely axiomatizable sublogic $L' \subseteq L$. Clearly, Kripke completeness implies compactness.

Let us consider the logic

$$L_1 = \mathbf{K4} \oplus \{\gamma_i \to \Diamond\gamma_{i+1} : \ i < \omega\} \oplus \delta,$$

where

$$\gamma_0 = \Diamond\beta_0 \wedge \Diamond\alpha_1, \quad \gamma_{i+1} = \Diamond\beta_{i+1} \wedge \Diamond\alpha_{i+2} \wedge \neg\Diamond^+\gamma_i,$$

$$\beta_i = \Diamond\Diamond\alpha_i \wedge \neg\Diamond\alpha_{i+1}, \quad \alpha_0 = \Box\bot, \quad \alpha_{i+1} = \Diamond\alpha_i \wedge \neg\Diamond\Diamond\alpha_i \quad (i < \omega),$$

$$\delta = \neg(p \wedge \neg q \wedge \Box^+(p \wedge \neg q \to \Diamond(p \wedge q)) \wedge \Box^+(p \wedge q \to \Diamond(\neg p \wedge q)) \wedge$$
$$\wedge \Box^+(\neg p \wedge q \to \Diamond(p \wedge \neg q))).$$

(We remind the reader that $\Box^+\varphi = \varphi \wedge \Box\varphi$, $\Diamond^+\varphi = \varphi \vee \Diamond\varphi$.)

Theorem 6.10 L_1 *is not compact.*

Proof Let us first clarify the semantic meaning of L_1's axioms. To understand the (variable free) axioms $\gamma_i \to \Diamond\gamma_{i+1}$ it is useful to take a look at the frame depicted in Fig. 6.4. The only point in this frame, at which α_0 is true, is clearly a_0. Then by induction on i one can readily show that a_i is the only point at which α_i

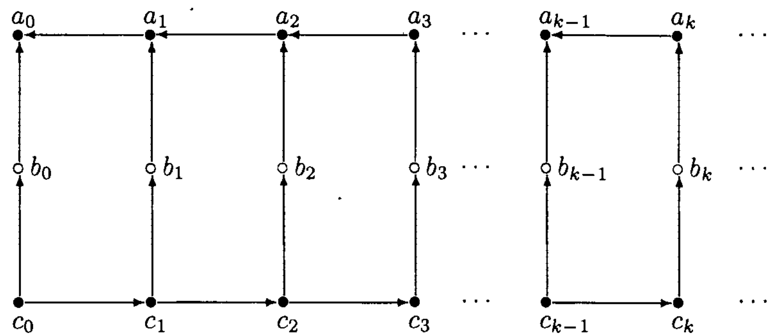


FIG. 6.4.

is true. It follows immediately that $\{x : x \models \beta_i\} = \{b_i\}$ and $\{x : x \models \gamma_i\} = \{c_i\}$, for $i < \omega$.

Thus, $\gamma_i \to \Diamond\gamma_{i+1}$ may be understood as "in the frame under consideration c_i sees c_{i+1}".

The meaning of δ can be expressed more precisely.

Lemma 6.11 *A transitive frame $\mathfrak{F}$ validates δ iff $\mathfrak{F}$ contains neither an infinite ascending chain of distinct points nor a cluster with ≥ 3 points.*

Proof Exercise. ❑

We are in a position now to prove Theorem 6.10. Namely, we are going to show that, for every finitely axiomatizable logic $L \subseteq L_1$, (a) there exists a frame $\mathfrak{F}$ such that $\mathfrak{F} \models L$ and $\mathfrak{F} \not\models \neg\gamma_0$, but (b) $\neg\gamma_0$ cannot be separated from L_1 by any Kripke frame.

Suppose L is a finitely axiomatizable sublogic of L_1. Since derivations of L's axioms involve only a finite number of L_1's axioms, there is $k < \omega$ such that

$$L \subseteq \mathbf{K4} \oplus \{\gamma_i \to \Diamond\gamma_{i+1} : 0 \leq i \leq k-1\} \oplus \delta.$$

So, to prove (a) it suffices to show that, for every $k < \omega$, there is a frame $\mathfrak{F}_k$ such that

$$\mathfrak{F}_k \models \mathbf{K4} \oplus \{\gamma_i \to \Diamond\gamma_{i+1} : 0 \leq i \leq k-1\} \oplus \delta, \tag{6.1}$$

$$\mathfrak{F}_k \not\models \neg\gamma_0. \tag{6.2}$$

Define $\mathfrak{F}_k = \langle W_k, R_k \rangle$ by taking

$$W_k = \{a_i, b_i, c_i : 0 \leq i \leq k\},$$

$$R_k = \{\langle a_i, a_j \rangle, \langle b_l, b_l \rangle, \langle b_l, a_l \rangle, \langle b_i, a_j \rangle, \langle c_l, b_l \rangle, \\ \langle c_j, b_i \rangle, \langle c_j, c_i \rangle, \langle c_l, a_j \rangle, \langle c_l, a_k \rangle : 0 \leq j < i \leq k, \ 0 \leq l \leq k\}.$$

In other words, $\mathfrak{F}_k$ is the subframe of the frame in Fig. 6.4 containing the first k rectangles. (6.1) and (6.2) are now direct consequences of the properties of γ_i discussed above and Lemma 6.11.

To establish (b), suppose $\mathfrak{F}$ is a frame for L_1 refuting $\neg\gamma_0$. Then $y_0 \models \gamma_0$ for some y_0 in $\mathfrak{F}$. Since $\mathfrak{F} \models \gamma_0 \to \Diamond\gamma_1$, there exists $y_1 \in y_0\uparrow$ such that $y_1 \models \gamma_1$. By the definition of γ_{i+1}, it follows in particular that $y_1 \models \neg\Diamond\gamma_0$, and so $y_0 \notin y_1\uparrow$. With the help of the axiom $\gamma_1 \to \Diamond\gamma_2$ in exactly the same way we show that there is $y_2 \in y_1\uparrow$ such that $y_1 \notin y_2\uparrow$, etc. As a result we construct an infinite ascending chain of distinct points in $\mathfrak{F}$, contrary to Lemma 6.11 and $\mathfrak{F} \models \delta$.

❑

It is worth noting that the proof above shows incidentally that the logic $\mathbf{K4} \oplus \{\gamma_i \to \Diamond\gamma_{i+1} : i < \omega\}$ is not finitely approximable. Thus we have got

Theorem 6.12 *There is a normal extension of* **K4** *with variable free additional axioms that is not finitely approximable.*

This result does not hold for Ext**S4** and Ext**Int** because in both **S4** and **Int** every variable free formula is equivalent to $\top$ or $\bot$ (see Proposition 2.26 and Exercise 3.19). As we shall see in Section 8.7, each variable free formula is deductively equal in NExt**GL** to one of the formulas $\top$, $\Box^i\bot$ ($i < \omega$). Since $\Box^i\bot \to \Box^j\bot \in \mathbf{K4} \subseteq \mathbf{GL}$, for $i < j$, all normal extensions of **GL** with variable free formulas are finitely axiomatizable and so finitely approximable, as follows from

Theorem 6.13 *Suppose φ is a variable free modal formula and $L \in$ NExt**K** is globally Kripke complete (globally finitely approximable). Then $L \oplus \varphi$ is globally complete (globally finitely approximable) as well. If $L \in$ NExt**K4** and L is decidable then $L \oplus \varphi$ is also decidable.*

Proof Let $M = L \oplus \varphi$ and $\Gamma \not\vdash^*_M \psi$ for some finite Γ. Then $\Gamma, \varphi \not\vdash^*_L \psi$ and so there is a Kripke (finite) frame $\mathfrak{F}$ for L such that under some valuation $\Gamma \cup \{\varphi\}$ is true in $\mathfrak{F}$ and ψ is refuted. Since φ is variable free, $\mathfrak{F} \models M$. ❑

Theorem 6.14 $\mathbf{GL} + \{\Diamond^i\top : i < \omega\}$ *is not finitely approximable.*

Proof This logic is consistent because it is contained in **S**, but does not have finite models at all. ❑

On the other hand we clearly have

Theorem 6.15 *Every (normal) extension of a canonical logic with variable free additional axioms is also canonical.*

6.4 A calculus that is not Kripke complete

The incomplete logic L_1, constructed in the preceding section, is not finitely axiomatizable (why?), and all the axioms $\gamma_i \to \Diamond\gamma_{i+1}$ were used essentially in the incompleteness proof. Here we show that a single additional axiom is enough to get a Kripke incomplete logic. The idea of replacing the infinite set of axioms with a single formula is similar to that in the proof of Theorem 6.1 (iii).

We continue using the notations introduced in Section 6.3. Define a logic L_2 as follows:

$$L_2 = \mathbf{K4} \oplus \epsilon \oplus \delta,$$

where

$$\epsilon = \lambda_0 \to \Diamond(\lambda_1 \wedge \neg\Diamond^+\lambda_0), \quad \lambda_i = \Diamond\mu_i \wedge \Diamond\nu_{i+1}, \quad \mu_i = \Diamond\Diamond\nu_i \wedge \neg\Diamond\nu_{i+1},$$

$$\nu_0 = p \wedge \neg\Diamond p, \quad \nu_i = \nu_0\{\Diamond^i p/p\}.$$

Theorem 6.16 *The calculus L_2 is not Kripke complete.*

Proof We are going to show that $\neg\gamma_0 \notin L_2$, but $\neg\gamma_0$ is valid in every frame for L_2. The proof is similar to the proof that $\neg\gamma_0$ cannot be separated from L_1 by a Kripke frame. In that proof we used the triple of formulas α_i, β_i, γ_i characterizing in the frame depicted in Fig. 6.4 the triple of points a_i, b_i, c_i (with the same subscripts). The triple ν_i, μ_i, λ_i is also intended for determining in this frame a triple of points a_j, b_j, c_j, possibly with $i \neq j$; and if this is the case then it turns out that the triple ν_{i+k}, μ_{i+k}, λ_{i+k} determines the triple a_{j+k}, b_{j+k}, c_{j+k}, for $k < \omega$. That is in essence the single formula ϵ will play the role of the infinite set $\{\gamma_i \to \Diamond\gamma_{i+1} : i < \omega\}$.

Lemma 6.17 *For every frame $\mathfrak{F}$, if $\mathfrak{F} \models L_2$ then $\mathfrak{F} \models \neg\gamma_0$.*

Proof Observe first that

$$\alpha_i = \nu_i\{\top/p\}, \quad \beta_i = \mu_i\{\top/p\},$$

$$\gamma_0 = \lambda_0\{\top/p\}, \quad \gamma_{i+1} = (\lambda_{i+1} \wedge \neg\Diamond^+\lambda_i)\{\top/p\},$$

$$\lambda_i \to \Diamond(\lambda_{i+1} \wedge \neg\Diamond^+\lambda_i) = \epsilon\{\Diamond^i p/p\},$$

$$\gamma_i \to \Diamond\gamma_{i+1} = (\lambda_i \to \Diamond(\lambda_{i+1} \wedge \neg\Diamond^+\lambda_i))\{\top/p\} = \epsilon\{\Diamond^i\top/p\}.$$

It follows that $\gamma_i \to \Diamond\gamma_{i+1} \in L_2$, for all $i < \omega$, and so $L_1 \subseteq L_2$. It remains to use the proof of Theorem 6.10. ❑

Thus, to complete the proof of our theorem it suffices to establish

Lemma 6.18 $\neg\gamma_0 \notin L_2$.

Proof We need the transitive frame $\mathfrak{F} = \langle W, R\rangle$ shown in Fig. 6.4; here is its formal definition:

$$W = \{a_i, b_i, c_i : i < \omega\},$$

$$R = \{\langle a_i, a_j\rangle, \langle b_j, b_j\rangle, \langle b_i, a_j\rangle, \langle b_j, a_j\rangle, \langle c_j, b_i\rangle,$$
$$\langle c_j, b_j\rangle, \langle c_j, a_k\rangle, \langle c_j, c_i\rangle : 0 \le j < i, \ k < \omega\}.$$

Define a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ by taking $\mathfrak{V}(p) = \emptyset$, for every variable p. Since $\mathfrak{F}$ contains an infinite ascending chain, by Lemma 6.11 we have $\mathfrak{F} \not\models \delta$. However, $\mathfrak{M}$ does not "feel" the chain. More exactly, the following holds.

Lemma 6.19 *$\mathfrak{M} \models \delta^*$ for every substitution instance δ^* of δ.*

Proof Observe first that we have

Lemma 6.20 *For every formula φ, the set $\mathfrak{V}(\varphi)$ is either finite or cofinite.*

Proof The proof proceeds by induction on the construction of φ. The basis of induction and the cases when φ's main connective is not modal are trivial. So let us consider $\varphi = \Box\psi$.

Suppose $\mathfrak{V}(\psi)$ is finite. Then there is a_i such that $a_i \not\models \psi$. This means that

$$\{a_j, b_j, c_k : \ i < j, \ k < \omega\} \subseteq W - \mathfrak{V}(\Box\psi),$$

i.e., $W - \mathfrak{V}(\Box\psi)$ is cofinite and so $\mathfrak{V}(\varphi)$ is finite.

Suppose now that $\mathfrak{V}(\psi)$ is cofinite. Two cases are possible: (a) there is a_i such that $a_i \not\models \psi$ and (b) $a_i \models \psi$ for every $i < \omega$. In Case (a), as before, $\mathfrak{V}(\Box\psi)$ is finite. So let us consider Case (b).

Since the set $W - \mathfrak{V}(\psi)$ is finite, there is $k \geq 0$ such that

$$\{a_i, b_j, c_j : \ i < \omega, \ j \geq k\} \subseteq \mathfrak{V}(\psi).$$

Then

$$\{a_i, b_j, c_j : \ i < \omega, \ j \geq k\} \subseteq \mathfrak{V}(\Box\psi),$$

and hence $\mathfrak{V}(\Box\psi)$ is cofinite. ❑

We are in a position now to prove Lemma 6.19 by *reductio ad absurdum*. Suppose $\delta^* = \delta\{\varphi/p, \psi/q\}$ and $y_0 \not\models \delta^*$, for some point y_0 in $\mathfrak{M}$. Then we have

$$y_0 \models \varphi \wedge \neg\psi,$$

$$y_0 \models \Box^+(\varphi \wedge \neg\psi \to \Diamond(\varphi \wedge \psi)),$$
$$y_0 \models \Box^+(\varphi \wedge \psi \to \Diamond(\neg\varphi \wedge \psi)),$$
$$y_0 \models \Box^+(\neg\varphi \wedge \psi \to \Diamond(\varphi \wedge \neg\psi)).$$

Using the same argument as in the proof of Lemma 6.11, we can construct an infinite ascending chain $y_0 R y_1 R y_2 R \ldots$ in $\mathfrak{F}$ such that, for every $k < \omega$,

$$y_{3k} \models \varphi \wedge \neg\psi, \quad y_{3k+1} \models \varphi \wedge \psi, \quad y_{3k+2} \models \neg\varphi \wedge \psi.$$

Since $\mathfrak{F}$ contains no proper clusters, it follows that $y_i \neq y_j$ if $i \neq j$. Therefore, the sets

$$\{y_{3k}, y_{3k+1} : \ k < \omega\} \subseteq \mathfrak{V}(\varphi)$$

and

$$\{y_{3k+2} : \ k < \omega\} \subseteq W - \mathfrak{V}(\varphi)$$

are infinite, contrary to Lemma 6.20. ❑

Lemma 6.21 $\mathfrak{F} \models \epsilon$. *In particular,* $\mathfrak{M} \models \epsilon^*$ *for every substitution instance* ϵ^* *of* ϵ.

Proof Suppose otherwise. Then under some valuation in $\mathfrak{F}$, ϵ is false at some point y, i.e.,

$$y \models \lambda_0, \tag{6.3}$$

$$y \not\models \Diamond(\lambda_1 \wedge \neg\Diamond^+\lambda_0). \tag{6.4}$$

It follows from (6.3) that there is $z \in y{\uparrow}$ at which both $\Diamond\Diamond\nu_0$ and $\neg\Diamond\nu_1$ are true. This means that we can reach from z by two steps a point u at which $\nu_0 = p \wedge \neg\Diamond p$ is true. Therefore, u is irreflexive and so $u \in \{a_i, c_i : i < \omega\}$. On the other hand, z does not see an irreflexive point $v \in u{\downarrow}$, which is possible only if $z = b_i$, for some $i \geq 0$. But then $u = a_i$.

It follows also from (6.3) that there is a point $x \in y{\uparrow}$ at which ν_1 is true. Since $a_i \models \nu_0$, the only point where ν_1 may be true is a_{i+1}, whence yRa_{i+1}. Then, according to the construction of $\mathfrak{F}$, $y = c_j$ for some $j \leq i$.

Thus, we have $y = c_j$ and yRc_{i+1}. Besides, as we have already established, $a_i \models \nu_0$. It is not difficult to see now (by induction on k) that, for every $k < \omega$,

$$\{x : x \models \nu_k\} = \{a_{i+k}\}$$

and so

$$\{x : x \models \mu_k\} = \{b_{i+k}\}.$$

It follows that $c_{i+1} \models \lambda_1 \wedge \neg\Diamond^+\lambda_0$, contrary to yRc_{i+1} and (6.4). ❑

By Lemmas 6.19 and 6.21, $\mathfrak{M}$ is a model for L_2. It remains to observe that $c_0 \models \gamma_0$ and so $\neg\gamma_0 \notin L_2$. This completes the proof of Lemma 6.18. ❑

Theorem 6.16 follows immediately. ❑

In fact the proofs of Theorems 6.10 and 6.16 provide us with a big family of incomplete logics. Indeed, denote by L_3 the set of modal formulas that are true in all models $\langle\mathfrak{F}, \mathfrak{U}\rangle$ such that, for every variable p, $\mathfrak{U}(p) = \mathfrak{V}(\varphi)$ for some formula φ. It is not hard to see that $L_3 \in$ NExt**K4**. We then have: $L_1 \subseteq L_2 \subseteq L_3$, $\neg\gamma_0$ is not in L_3 and cannot be separated from L_1 by a Kripke frame. Therefore, all the logics between L_1 and L_3 are Kripke incomplete. Using the formulas φ_i from the proof of Theorem 6.1, we can show that the cardinality of the interval $[L_1, L_3]$ is that of continuum. And by adding φ_i to L_2 we can construct infinitely many finitely axiomatizable logics in this interval.

Let us fix these observations as

Theorem 6.22 (i) *No logic in the interval* $[L_1, L_3]$ *is Kripke complete.*

(ii) *There is a continuum of normal logics in* $[L_1, L_3]$.

(iii) *There are infinitely many finitely axiomatizable normal logics in* $[L_1, L_3]$.

Using the fact that every finite irreflexive chain validates L_3, we can obtain one more interesting result:

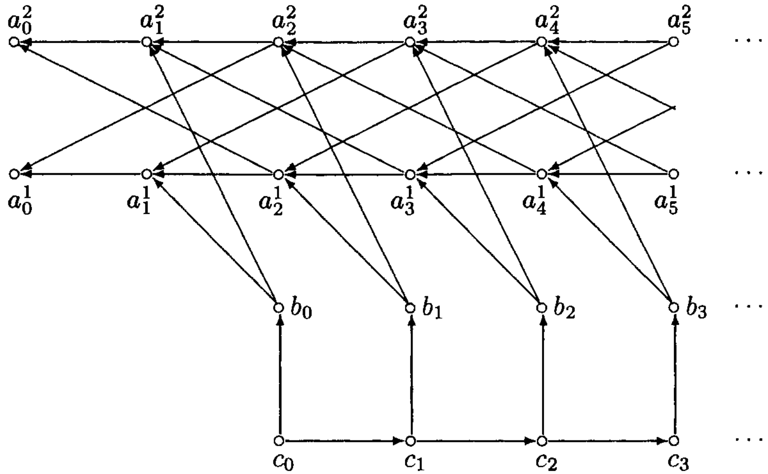


FIG. 6.5.

Theorem 6.23 *No normal logic between* $\mathbf{K4} \oplus \delta$ *and* L_3 *is strongly Kripke complete.*

Proof Similar to the proof of Theorem 6.6. ❑

6.5 More Kripke incomplete calculi

Now we realize the idea of constructing incomplete calculi, developed in the previous section, to find Kripke incomplete extensions of **Grz**, **GL** and **Int**. Since we use the same method, the most part of technical details in the proofs is left to the reader as exercises, sometimes far from being trivial.

For two cases—**Grz** and **Int**—we use the (transitive) frame shown in Fig. 6.5; for **GL** all the reflexive circles in it should be replaced with irreflexive "bullets". Its similarity with the frame in Fig. 6.4 is emphasized by the worlds' names.

We require the following intuitionistic formulas:

$$\alpha_{-1}^1 = p, \quad \alpha_{-1}^2 = q, \quad \alpha_0^1 = q \to p, \quad \alpha_0^2 = p \to q,$$

$$\alpha_{n+1}^1 = \alpha_n^2 \to \alpha_n^1 \vee \alpha_{n-1}^2, \quad \alpha_{n+1}^2 = \alpha_n^1 \to \alpha_n^2 \vee \alpha_{n-1}^1,$$

$$\beta_n = \alpha_{n+1}^1 \wedge \alpha_{n+1}^2 \to \alpha_n^1 \vee \alpha_n^2, \quad (n < \omega)$$

$$\lambda = \beta_0 \to \beta_1 \vee \beta_2, \quad \mu = \beta_0 \vee \beta_1, \quad \epsilon = \lambda \to \mu, \quad \iota = \beta_1 \to \beta_0 \vee \alpha_2^1.$$

Theorem 6.24 (i) $\mathbf{Grz} \oplus \mathsf{T}(\epsilon)$ *is Kripke incomplete.*
(ii) $\mathbf{GL} \oplus \mathsf{T}^+(\epsilon)$ *is Kripke incomplete*[9].

Proof We will establish only (i); (ii) is proved in the same way. The proof consists of two lemmas.

[9]Here T and T^+ are the embeddings of **Int** into **Grz** and **GL** defined in Section 3.9.

Lemma 6.25 *If* $\mathfrak{F}$ *is a frame such that* $\mathfrak{F} \models \mathbf{Grz} \oplus \mathsf{T}(\epsilon)$ *then* $\mathfrak{F} \models \mathsf{T}(\mu)$.

Proof (Sketch) Proving the lemma by *reductio ad absurdum*, we suppose that $\mathfrak{F} \models \mathbf{Grz} \oplus \mathsf{T}(\epsilon)$ and $\mathfrak{F} \not\models \mathsf{T}(\mu)$ and show, as in the proof of Lemma 6.17, that in this case $\mathfrak{F}$ contains an infinite ascending chain of distinct points, contrary to $\mathfrak{F} \models \mathbf{Grz}$.

For $n \geq 1$, let

$$\gamma_0 = \mu, \quad \gamma_n = \mu\{\alpha^1_{n-2} \vee \alpha^2_{n-1}/p, \alpha^1_{n-1} \vee \alpha^2_{n-2}/q\}$$

and suppose that, under some valuation in $\mathfrak{F}$, $\mathsf{T}(\mu)$ is not true at some point y_0, i.e., $y_0 \not\models \mathsf{T}(\gamma_0)$. Since $\mathfrak{F}$ validates $\mathsf{T}(\epsilon)$, we can use the substitution instances $\epsilon\{\alpha^1_{n-2} \vee \alpha^2_{n-1}/p, \alpha^1_{n-1} \vee \alpha^2_{n-2}/q\}$ of ϵ to find first a point $y_1 \in y_0\uparrow$ such that $y_1 \models \mathsf{T}(\gamma_0)$ and $y_1 \not\models \mathsf{T}(\gamma_1)$, then $y_2 \in y_1\uparrow$ such that $y_2 \models \mathsf{T}(\gamma_1)$ and $y_2 \not\models \mathsf{T}(\gamma_2)$, and so on. ❑

Lemma 6.26 $\mathsf{T}(\mu) \notin \mathbf{Grz} \oplus \mathsf{T}(\epsilon)$.

Proof (Sketch) Let $\mathfrak{M}$ be a model based on the frame in Fig. 6.5 and such that $x \models p$ iff $x = a^1_0$ and $x \models q$ iff $x = a^2_0$. In the same manner as in the proofs of Lemmas 6.19 and 6.21, one can show that $\mathfrak{M} \models \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$ and $\mathfrak{M} \models \mathsf{T}(\epsilon)\{\varphi/p, \psi/q\}$, for every formulas φ and ψ. On the other hand, we have $c_0 \not\models \mathsf{T}(\mu)$. ❑

This completes the proof of Theorem 6.24. ❑

Given a si-logic L, we put

$$\mathsf{T}(L) = \{\mathsf{T}(\varphi) : \varphi \in L\}, \quad \mathsf{T}^+(L) = \{\mathsf{T}^+(\varphi) : \varphi \in L\}.$$

Theorem 6.27 *There is a superintuitionistic logic* L *such that*
(i) L *is Kripke complete;*
(ii) $\mathbf{Grz} \oplus \mathsf{T}(L)$ *is not Kripke complete;*
(iii) $\mathbf{GL} \oplus \mathsf{T}^+(L)$ *is not Kripke complete.*

Proof Define L as the si-logic of the frame in Fig. 6.5, so that (i) holds by the definition. Let $\mathfrak{M}$ be the model on this frame introduced in the proof of Lemma 6.26. Then $\mathfrak{M} \not\models \mu$. On the other hand, ϵ is valid in our frame, i.e., $\epsilon \in L$. By Lemma 3.81, we then have $\mathsf{T}(\mu) \notin \mathbf{Grz} \oplus \mathsf{T}(L)$ and $\mathsf{T}(\epsilon) \in \mathbf{Grz} \oplus \mathsf{T}(L)$. The incompleteness of $\mathbf{Grz} \oplus \mathsf{T}(L)$ follows now immediately from Lemma 6.25.

(iii) is proved in the same way. ❑

The proof above may be interpreted as that there are no intuitionistic analogues of the Grzegorczyk and Löb formulas, or δ in Section 6.3, which "feel" the presence of infinite ascending chains. Yet, the proof of Theorem 6.9 shows that for the same purpose one can use the formula $\boldsymbol{bb}_2$.

Theorem 6.28 *The si-logic* $\mathbf{Int} + \epsilon + \iota + \boldsymbol{bb}_2$ *is Kripke incomplete.*

Proof (Sketch) One can show similar to Lemma 6.25 that if a frame validates ϵ and ι and refutes μ then it also refutes $\boldsymbol{bb}_2$, i.e., μ is valid in every frame for the logic $L = \mathbf{Int} + \epsilon + \iota + \boldsymbol{bb}_2$. On the other hand, all substitution instances of ϵ, ι and $\boldsymbol{bb}_2$ are true in the model $\mathfrak{M}$ defined in the proof of Theorem 6.27, from which $\mathfrak{M} \models L$, $\mathfrak{M} \not\models \mu$ and so $\mu \notin L$. ❑

6.6 Complete logics without countable characteristic frames

The Löwenheim–Skolem theorem of classical model theory (see Chang and Keisler, 1990, Corollary 2.1.6) states that if a first order theory (in a countable language) has an infinite model then it has also a countable model. Canonical models for modal and superintuitionistic logics contain, by the definition, a continuum of points. However, most of these points may be safely removed, as is shown by the following

Theorem 6.29 *Every consistent logic in* Ext**K** *and* Ext**Int** *has a countable characteristic Kripke model.*

Proof We construct a countable characteristic model for a consistent logic L in NExt**K**. Quasi-normal modal logics and si-logics are considered analogously.

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be an arbitrary characteristic model for L (for instance, the canonical model) and $\mathfrak{F} = \langle W, R \rangle$. A countable characteristic model we need can be extracted from $\mathfrak{M}$ in a manner similar to the selective filtration method. Let $\Sigma = \mathbf{For}\mathcal{ML} - L$.

Step 0. For every $\varphi \in \Sigma$, fix a point in $\mathfrak{M}$ at which φ is false. Let W_0 be the set of all the fixed points. Clearly, W_0 is countable.

Step $n+1$. Suppose we have already constructed a countable set $W_n \subseteq W$. Now, for every $x \in W_n$ and every $\varphi \in \Sigma$ we fix a point $y \in x\!\uparrow$ in $\mathfrak{M}$, if any, at which φ is false. Let W_{n+1} be the union of W_n and the set of all new fixed points. Again W_{n+1} is countable.

Finally, define a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ on a countable frame $\mathfrak{G} = \langle V, S \rangle$ by taking

$$V = \bigcup_{i<\omega} W_i, \quad S = R \cap V^2,$$

$$\mathfrak{U}(p) = \mathfrak{V}(p) \cap V, \quad \text{for every } p \in \mathbf{Var}\mathcal{ML}.$$

By induction on the construction of a formula φ one can readily show that, for every $x \in V$,

$$(\mathfrak{M}, x) \models \varphi \text{ iff } (\mathfrak{N}, x) \models \varphi.$$

It follows that $\mathfrak{N}$ characterizes L. ❑

Needless to say that this result does not hold for Kripke frames (for there are logics without characteristic Kripke frames at all). Moreover, even if a logic is Kripke complete it may have no countable characteristic frame as is demonstrated by the following theorem, in which cardinal numbers $\beth_\xi$ are defined by transfinite induction on ordinals ξ:

$$\beth_\xi = \begin{cases} \aleph_0 & \text{if } \xi = 0 \\ 2^{\beth_\zeta} & \text{if } \xi = \zeta + 1 \\ \bigcup_{\zeta<\xi} \beth_\zeta & \text{if } \xi \text{ is a limit ordinal.} \end{cases}$$

Theorem 6.30 *There is a logic $L \in$ NExt**K4** which is characterized by a Kripke frame of cardinality $\beth_\omega$, but is not approximable by frames of smaller cardinality.*

Proof Define L as the logic of the transitive frame $\mathfrak{F} = \langle W, R\rangle$ shown in Fig. 6.6. A formal definition of $\mathfrak{F}$ may look like this:

$$W = \{a, b, c, d, a_{-1}, b_{-1}, e', e\} \cup \{a_i, b_i, a'_i, b'_i : \ i < \omega\} \cup$$
$$\{h_i^0, h_i^{0,1}, h_i^{0,2} : \ i < \omega\} \cup \{h_X^i, h_X^{i,1}, h_X^{i,2} : \ 1 \le i < \omega, \ X \in \mathcal{P}^i\omega\}$$

(where $\mathcal{P}Y$ is the power set of Y, $\mathcal{P}^0 Y = Y$ and $\mathcal{P}^{i+1} = \mathcal{P}(\mathcal{P}^i Y)$) and R is the transitive closure of the following binary relation S on W: for every $x, y \in W$,

$$\begin{array}{l}
xSy \text{ iff} \\
\quad (x = e \wedge y \neq e) \vee (x = b \wedge y = b) \vee \\
\quad (x = c \wedge y = a) \vee (x = d \wedge (y = a \vee y = b)) \vee \\
\quad (x = a_{-1} \wedge y = c) \vee (x = b_{-1} \wedge y = d) \vee (x = e' \wedge (y = c \vee y = d)) \vee \\
\quad (x = a'_0 \wedge (y = a_{-1} \vee y = d)) \vee (x = b'_0 \wedge (y = b_{-1} \vee y = c)) \vee \\
\quad \exists i \ \exists X, Y \ ((x = a_i \wedge y = a_{i-1}) \vee (x = b_i \wedge y = b_{i-1}) \vee \\
\quad (x = a'_{i+1} \wedge y = a'_i) \vee (x = b'_{i+1} \wedge y = b'_i) \vee \\
\quad (x = h_i^0 \wedge (y = h_i^{0,1} \vee y = h_i^{0,2})) \vee \\
\quad (x = h_i^{0,1} \wedge (y = a_i \vee y = a'_0)) \vee \\
\quad (x = h_i^{0,2} \wedge (y = b_i \vee y = b'_0)) \vee \\
\quad (x = h_X^i \wedge (y = h_X^{i,1} \vee y = h_X^{i,2})) \vee \\
\quad (x = h_X^{1,1} \wedge (y = a'_1 \vee (y = h_i^{0,1} \wedge i \in X) \vee (y = h_i^{0,2} \wedge i \notin X))) \vee \\
\quad (x = h_X^{1,2} \wedge (y = b'_1 \vee (y = h_i^{0,1} \wedge i \notin X) \vee (y = h_i^{0,2} \wedge i \in X))) \vee \\
\quad (x = h_X^{i+1,1} \wedge (y = a'_{i+1} \vee (y = h_Y^{i,1} \wedge Y \in X) \vee (y = h_Y^{i,2} \wedge Y \notin X))) \vee \\
\quad (x = h_X^{i+1,2} \wedge (y = b'_{i+1} \vee (y = h_Y^{i,1} \wedge Y \notin X) \vee (y = h_Y^{i,2} \wedge Y \in X)))).
\end{array}$$

By the given definition, $|\mathfrak{F}| = \beth_\omega$. So we must show that $L = \text{Log}\mathfrak{F}$ is not approximable by frames of cardinality $< \beth_\omega$.

Although the frame $\mathfrak{F}$ looks rather cumbersome (which is justified by our purpose, of course), its constitution can be made quite clear. Our next aim is to describe points in $\mathfrak{F}$ by means of modal formulas.

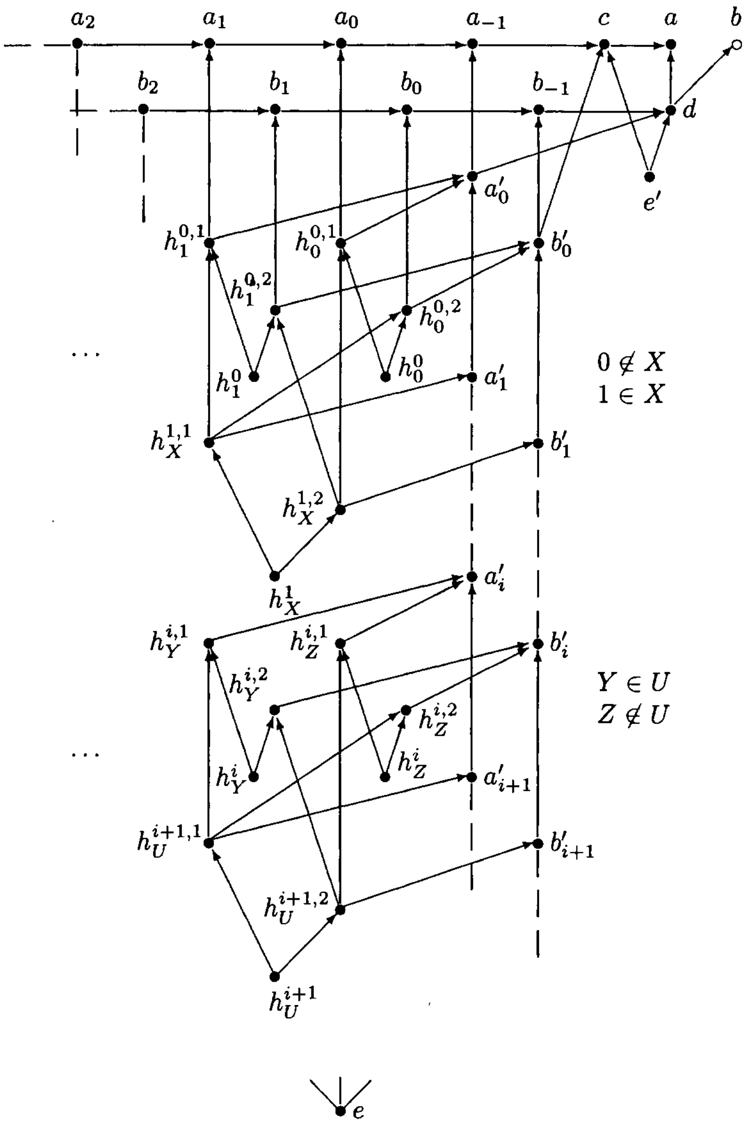


FIG. 6.6.

Let us begin with $\mathfrak{F}$'s points that are characterized by variable free formulas. The reader can easily verify that the formulas

$$\alpha = \Box\bot, \quad \beta = \Diamond\top \wedge \Box\Diamond\top, \quad \gamma = \Box\Box\bot \wedge \Diamond\top,$$

$$\delta = \Diamond\alpha \wedge \Diamond\beta \wedge \neg\Diamond\Diamond\alpha, \quad \epsilon' = \Diamond\gamma \wedge \Diamond\delta \wedge \neg\Diamond\Diamond\gamma \wedge \neg\Diamond\Diamond\delta, \quad \epsilon = \Diamond\epsilon',$$

$$\alpha_{-1} = \Diamond\gamma \wedge \neg\Diamond\Diamond\gamma \wedge \neg\Diamond\beta, \quad \beta_{-1} = \Diamond\delta \wedge \neg\Diamond\Diamond\delta \wedge \neg\Diamond\gamma,$$
$$\alpha'_0 = \Diamond\alpha_{-1} \wedge \Diamond\delta \wedge \neg\Diamond\Diamond\alpha_{-1} \wedge \neg\Diamond\Diamond\delta, \quad \beta'_0 = \Diamond\beta_{-1} \wedge \Diamond\gamma \wedge \neg\Diamond\Diamond\beta_{-1} \wedge \neg\Diamond\Diamond\gamma,$$
$$\alpha_i = \Diamond\alpha_{i-1} \wedge \neg\Diamond\Diamond\alpha_{i-1} \wedge \neg\Diamond\delta, \quad \beta_i = \Diamond\beta_{i-1} \wedge \neg\Diamond\Diamond\beta_{i-1} \wedge \neg\Diamond\gamma,$$
$$\alpha'_{i+1} = \Diamond\alpha'_i \wedge \neg\Diamond\Diamond'\alpha_i \wedge \neg\Diamond\beta_{-1}, \quad \beta'_{i+1} = \Diamond\beta'_i \wedge \neg\Diamond\Diamond\beta'_i \wedge \neg\Diamond\alpha_{-1},$$
$$\chi_i^{0,1} = \Diamond\alpha_i \wedge \neg\Diamond\Diamond\alpha_i \wedge \Diamond\alpha'_0, \quad \chi_i^{0,2} = \Diamond\beta_i \wedge \neg\Diamond\Diamond\beta_i \wedge \Diamond\beta'_0,$$
$$\chi_i^0 = \Diamond\chi_i^{0,1} \wedge \Diamond\chi_i^{0,2} \wedge \neg\Diamond\Diamond\chi_i^{0,1} \wedge \neg\Diamond\Diamond\chi_i^{0,2},$$

for $i \geq 0$, are such that α is true in $\mathfrak{F}$ only at a, β at b, γ at c, δ at d, ϵ' at e', ϵ at e and the formulas denoted by α, β, χ with subscripts and superscripts are true only at the points in $\mathfrak{F}$ denoted by a, b, h, respectively, with the corresponding indices.

Before we continue characterizing $\mathfrak{F}$'s points by modal formulas, let us observe that the following holds.

Lemma 6.31 (i) $\neg\epsilon \notin L$; *more exactly*, $\{x : x \models \epsilon\} = \{e\}$.
(ii) $\epsilon \to \Diamond\chi_i^0 \in L$, *for every* $i < \omega$.
(iii) $\chi_i^0 \to \neg\Diamond\chi_j^{0,1} \wedge \neg\Diamond\chi_j^{0,2} \in L$, *for* $i, j < \omega$, $i \neq j$.

Now define three more sequences of formulas, for $i \geq 1$:

$$\chi^{i,1} = \Diamond\alpha'_i \wedge \neg\Diamond\Diamond\alpha'_i \wedge (\Diamond\alpha_0 \vee \Diamond\beta_0),$$

$$\chi^{i,2} = \Diamond\beta'_i \wedge \neg\Diamond\Diamond\beta'_i \wedge (\Diamond\alpha_0 \vee \Diamond\beta_0),$$
$$\chi^i = \Diamond\chi^{i,1} \wedge \Diamond\chi^{i,2} \wedge \neg\Diamond\Diamond\chi^{i,1} \wedge \neg\Diamond\Diamond\chi^{i,2}.$$

These variable free formulas characterize in $\mathfrak{F}$ not single points but sets of points, namely,

$$\{x : x \models \chi^{i,1}\} = \{h_X^{i,1} : X \in \mathcal{P}^i\omega\},$$
$$\{x : x \models \chi^{i,2}\} = \{h_X^{i,2} : X \in \mathcal{P}^i\omega\},$$
$$\{x : x \models \chi^{i}\} = \{h_X^{i} : X \in \mathcal{P}^i\omega\}.$$

To characterize the relation between the points in $\mathfrak{F}$ involved in representing sets in $\mathcal{P}^{i+1}\omega$, for $i \geq 1$, we require a few more formulas:

$$\pi_i(p) = \chi^i \to (\Diamond(\chi^{i,1} \wedge p) \vee \Diamond(\chi^{i,2} \wedge p)) \wedge \neg(\Diamond(\chi^{i,1} \wedge p) \wedge \Diamond(\chi^{i,2} \wedge p)),$$

$$\sigma_i(p) = \epsilon \wedge \Box\pi_i(p) \to \Diamond\sigma'_i(p),$$

$$\sigma'_i(p) = \chi^{i+1} \wedge \Diamond(\chi^{i+1,1} \wedge \Box(\chi^{i,1} \vee \chi^{i,2} \to p)) \wedge$$
$$\Diamond(\chi^{i+1,2} \wedge \Box(\chi^{i,1} \vee \chi^{i,2} \to \neg p)),$$

$$\rho_i(p,q,r) = \rho_i^-(p,q,r) \to \rho_i^+(p,q,r),$$

$$\rho_i^-(p,q,r) = \epsilon \wedge \Box\pi_i(p) \wedge \Diamond(q \wedge p \wedge (\chi^{i,1} \vee \chi^{i,2})) \wedge$$

$$\Diamond(r \wedge \neg p \wedge (\chi^{i,1} \vee \chi^{i,2})) \wedge$$
$$\Box((\chi^{i,1} \vee \chi^{i,2}) \wedge \neg p \to \neg q) \wedge \Box((\chi^{i,1} \vee \chi^{i,2}) \wedge p \to \neg r),$$

$$\begin{aligned}\rho_i^+(p,q,r) = \Box(\sigma_i'(p) \to\ & \\ & \Box(\chi^{i+1,1} \to \Diamond(q \wedge (\chi^{i,1} \vee \chi^{i,2})) \wedge \neg\Diamond(r \wedge (\chi^{i,1} \vee \chi^{i,2}))) \wedge \\ & \Box(\chi^{i+1,2} \to \Diamond(r \wedge (\chi^{i,1} \vee \chi^{i,2})) \wedge \neg\Diamond(q \wedge (\chi^{i,1} \vee \chi^{i,2})))).\end{aligned}$$

Lemma 6.32 (i) *$\sigma_i(p) \in L$ for every $i < \omega$.*
(ii) *$\rho_i(p,q,r) \in L$ for every $i < \omega$.*

Proof (i) We need to show that if, under some valuation in the frame $\mathfrak{F}$, we have $x \models \epsilon \wedge \Box\pi_i(p)$ then $x \models \Diamond\sigma_i'(p)$. So suppose $x \models \epsilon \wedge \Box\pi_i(p)$. Then by Lemma 6.31, $x = e$ and so $e \models \Box\pi_i(p)$. This means that, for every $X \in \mathcal{P}^i\omega$, either

$$h_X^{i,1} \models p, \quad h_X^{i,2} \not\models p \quad \text{or} \quad h_X^{i,1} \not\models p, \quad h_X^{i,2} \models p.$$

Let $Y = \{X : \ h_X^{i,1} \models p\}$. Then clearly we have $Y = \{X : \ h_X^{i,2} \not\models p\}$. By the construction of $\mathfrak{F}$, there are points h_Y^{i+1}, $h_Y^{i+1,1}$, $h_Y^{i+1,2}$ such that

$$h_Y^{i+1,1} R h_X^{i,j} \ \text{ iff } \ (X \in Y \wedge j = 1) \vee (X \notin Y \wedge j = 2),$$

$$h_Y^{i+1,2} R h_X^{i,j} \ \text{ iff } \ (X \in Y \wedge j = 2) \vee (X \notin Y \wedge j = 1).$$

Then we have

$$h_Y^{i+1,1} \models \chi^{i+1,1} \wedge \Box(\chi^{i,1} \vee \chi^{i,2} \to p),$$
$$h_Y^{i+1,2} \models \chi^{i+1,2} \wedge \Box(\chi^{i,1} \vee \chi^{i,2} \to \neg p),$$

which together with $h_Y^{i+1} R h_Y^{i+1,1}$, $h_Y^{i+1} R h_Y^{i+1,2}$, $h_Y^{i+1} \models \chi^{i+1}$ yields $e \models \Diamond\sigma_i'(p)$.

(ii) Suppose now that, under some valuation in $\mathfrak{F}$, $x \models \rho_i^-(p,q,r)$ and show that $x \models \rho_i^+(p,q,r)$. Since $x \models \epsilon \wedge \Box\pi_i(p)$, we may assume that we are in the same situation as in the proof of (i), in particular, $x = e$ and $h_Y^{i+1} \models \sigma_i'(p)$, with h_Y^{i+1} being that only point at which, under the given valuation, $\sigma_i'(p)$ is true. It follows also from the first assumption that

$$e \models \Diamond(q \wedge p \wedge (\chi^{i,1} \vee \chi^{i,2})), \tag{6.5}$$

$$e \models \Diamond(r \wedge \neg p \wedge (\chi^{i,1} \vee \chi^{i,2})), \tag{6.6}$$

$$e \models \Box((\chi^{i,1} \vee \chi^{i,2}) \wedge \neg p \to \neg q), \tag{6.7}$$

$$e \models \Box((\chi^{i,1} \vee \chi^{i,2}) \wedge p \to \neg r). \tag{6.8}$$

Suppose $e \not\models \rho_i^+(p,q,r)$. Then one of the following holds:

$$h_Y^{i+1,1} \not\models \Diamond(q \wedge (\chi^{i,1} \vee \chi^{i,2})), \tag{6.9}$$

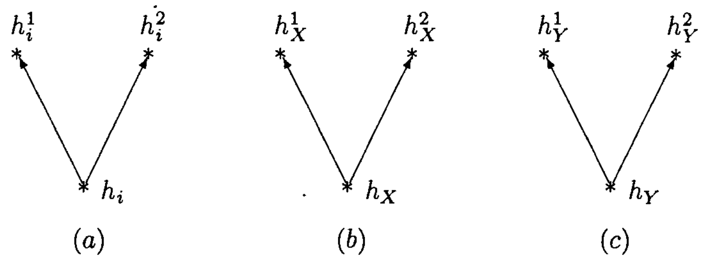


FIG. 6.7.

$$h_Y^{i+1,1} \models \Diamond(r \wedge (\chi^{i,1} \vee \chi^{i,2})), \tag{6.10}$$

$$h_Y^{i+1,2} \not\models \Diamond(r \wedge (\chi^{i,1} \vee \chi^{i,2})), \tag{6.11}$$

$$h_Y^{i+1,2} \models \Diamond(q \wedge (\chi^{i,1} \vee \chi^{i,2})). \tag{6.12}$$

If (6.9) holds then q is false at all points $h_X^{i,j}$ accessible from $h_Y^{i+1,1}$. But according to the definition of Y (see the proof of (i)), $h_Y^{i+1,1}$ sees only those $h_X^{i,j}$ at which p is true. By (6.5), this set must contain a point where q is true, which is a contradiction. Therefore, (6.9) does not hold.

Assume now that (6.10) holds. This means that among the points accessible from $h_Y^{i+1,1}$ there is a point $h_X^{i,j}$ at which r is true. Then by (6.8), $h_X^{i,j} \not\models p$. On the other hand, $h_Y^{i+1,1}$ sees only those points $h_X^{i,j}$ where p is true, which is again a contradiction.

In the same way (6.11) and (6.12) combined with (6.7) and (6.8) lead to a contradiction. ❑

Lemma 6.33 *Every frame $\mathfrak{G} = \langle V, S\rangle$ for L refuting $\neg\epsilon$ contains at least $\beth_\omega$ points.*

Proof Suppose ϵ is true at some point e in $\mathfrak{G}$ validating L. By Lemma 6.31 (ii), for every $i < \omega$, there are points h_i, h_i^1, h_i^2 in $\mathfrak{G}$ forming the diagram shown in Fig. 6.7 (a) and such that $h_i \models \chi_i^0$, $h_i^1 \models \chi_i^{0,1}$, $h_i^2 \models \chi_i^{0,2}$. Using Lemma 6.31 (iii), one can readily prove that the points h_i, h_i^1, h_i^2 are not accessible from h_j, h_j^1, h_j^2, for $i \neq j$.

Given $X \in \mathcal{P}\omega$, define a valuation in $\mathfrak{G}$ in the following way. Suppose h_i, h_i^1, h_i^2 is a triple found above. Put

$$h_i^1 \models p, \quad h_i^2 \not\models p \ \text{ if } \ i \in X,$$

$$h_i^1 \not\models p, \quad h_i^2 \models p \ \text{ if } \ i \notin X.$$

Under this valuation $e \models \Box\pi_0(p)$ and, by Lemma 6.31 (i), $e \models \Diamond\sigma_0'(p)$. Therefore, there are points h_X, h_X^1, h_X^2 in $\mathfrak{G}$ forming the diagram as in Fig. 6.7 (b) and such that

$$\left.\begin{array}{l} h_X \models \chi^1 \\ h_X^1 \models \chi^{1,1} \wedge \Box(\chi^{0,1} \vee \chi^{0,2} \to p) \\ h_X^2 \models \chi^{1,2} \wedge \Box(\chi^{0,1} \vee \chi^{0,2} \to \neg p) \end{array}\right\} \tag{6.13}$$

We show that points in distinct triples of the form h_X, h_X^1, h_X^2 do not see each other. Indeed, suppose $X_1, X_2 \in \mathcal{P}\omega$ and $X_1 \neq X_2$. This means that there is $i < \omega$ such that either $i \in X_1$, $i \notin X_2$ or $i \notin X_1$, $i \in X_2$. Assume for definiteness that $i \in X_1$ and $i \notin X_2$.

Take the triple h_i, h_i^1, h_i^2 determined above and define a valuation in $\mathfrak{G}$ by putting

$$x \models q \ \text{ iff } \ x = h_i^1,$$

$$x \models r \ \text{ iff } \ x = h_i^2.$$

It is not difficult to verify that, under this valuation, $\rho_i^-(p_1, q, r)$ is true at e and so, by Lemma 6.32 (ii), $e \models \rho_i^+(p_1, q, r)$, where p_1 is the variable used for finding the triple h_{X_1}, $h_{X_1}^1$, $h_{X_1}^2$. Using (6.13) with p_1 instead of p, we obtain:

$$h_{X_1} \models \Box(\chi^{1,1} \to \Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2}))), \tag{6.14}$$

$$h_{X_1} \models \Box(\chi^{1,2} \to \Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2}))), \tag{6.15}$$

$$h_{X_1}^1 \models \Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2})), \tag{6.16}$$

$$h_{X_1}^2 \models \Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2})). \tag{6.17}$$

In exactly the same way, using $\rho_i(p_2, q, r)$ instead of $\rho_i(p_1, q, r)$, where p_2 is the variable involved in finding the triple h_{X_2}, $h_{X_2}^1$, $h_{X_2}^2$, we obtain:

$$h_{X_2} \models \Box(\chi^{1,1} \to \Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2}))), \tag{6.18}$$

$$h_{X_2} \models \Box(\chi^{1,2} \to \Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2}))), \tag{6.19}$$

$$h_{X_2}^1 \models \Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2})), \tag{6.20}$$

$$h_{X_2}^2 \models \Diamond(q \wedge (\chi^{0,1} \vee \chi^{0,2})) \wedge \neg\Diamond(r \wedge (\chi^{0,1} \vee \chi^{0,2})). \tag{6.21}$$

Suppose that a point in the triple h_{X_1}, $h_{X_1}^1$, $h_{X_1}^2$ sees a point in the triple h_{X_2}, $h_{X_2}^1$, $h_{X_2}^2$. Then by the transitivity, $h_{X_1} S h_{X_2}^1$ or $h_{X_1} S h_{X_2}^2$. In the former case we arrive at a contradiction between (6.20) and (6.14), and in the latter one between (6.21) and (6.15). Other possibilities are considered analogously.

It follows that there are $\beth_1$ distinct points of the form h_X, h_X^1, h_X^2, for $X \in \mathcal{P}\omega$.

Suppose now that we have already proved that, for every $X \in \mathcal{P}^i\omega$, there exist points h_X, h_X^1, h_X^2 forming the diagram as in Fig. 6.7 (b) and such that $h_X \models \chi^i$, $h_X^1 \models \chi^{i,1}$, $h_X^2 \models \chi^{i,2}$. Suppose also that points in distinct triples of that form do not see each other. Using Lemma 6.32, in the same way as before we obtain that, for every $Y \in \mathcal{P}^{i+1}\omega$, there are points h_Y, h_Y^1, h_Y^2 forming the diagram as in Fig. 6.7 (c) and such that $h_Y \models \chi^{i+1}$, $h_Y^1 \models \chi^{i+1,1}$, $h_Y^2 \models \chi^{i+1,2}$.

Besides, points in distinct triples of the form h_Y, h_Y^1, h_Y^2 do not see each other. Therefore, there are $\beth_{i+1}$ points of this sort. Thus, for each $i < \omega$, the cardinality of $\mathfrak{G}$ is greater than $\beth_i$ and so $|\mathfrak{G}| \geq \beth_\omega$. ❑

This completes the proof of Theorem 6.30. ❑

Slightly modifying the argument above, we can prove

Theorem 6.34 *There is a Kripke complete quasi-normal extension L of* **K4** *such that every frame for L contains at least $\beth_\omega$ points.*

Proof It suffices to take $L = \mathrm{Log}\langle\mathfrak{F}, \{e\}\rangle$, where $\mathfrak{F}$ is the frame in Fig. 6.6. ❑

Of course this result does not hold for Ext**S4** and Ext**Int** (why?). However we still have

Theorem 6.35 *There are logics in* NExt**S4** *and* Ext**Int** *that are characterized by Kripke frames of cardinality $\beth_\omega$ but are not approximable by frames of smaller cardinality.*

The idea of the proof is similar to that of Theorem 6.30 but technically it is somewhat more complicated.

6.7 Exercises and open problems

Exercise 6.1 Show that the canonical model for **GL** contains a continuum of reflexive points. (Hint: prove that the sets

$$\mathbf{GL} \cup \{\Box\varphi \to \varphi : \ \varphi \in \mathbf{For}\mathcal{ML}\} \cup$$

$$\{p_i \in \mathbf{Var}\mathcal{ML} : \ i \in I\} \cup \{\neg p_i : p_i \in \mathbf{Var}\mathcal{ML}, \ i \notin I\}$$

are **GL**-consistent for every $I \subseteq \omega$.)

Exercise 6.2 Show that the canonical frames for the logics **GL** and **Grz** are not Noetherian.

Exercise 6.3 Prove that the canonical frame for **Grz** contains a proper cluster. (Hint: show that the tableaux $(\Gamma, \emptyset)$ and $(\Delta, \emptyset)$, where

$$\Gamma = \{p\} \cup \{\neg\Box\varphi : \varphi \notin \mathbf{Grz}\}, \ \Delta = \{\neg p\} \cup \{\neg\Box\varphi : \varphi \notin \mathbf{Grz}\},$$

are **Grz**-consistent and all extensions of them in the canonical model see each other.)

Exercise 6.4 Show that **K4**$\oplus\delta$ is not strongly complete, where δ is the formula defined in Section 6.3.

Exercise 6.5 Show that **GL** in the language with one variable is not strongly complete.

Exercise 6.6 Show that **GL.3** is neither strongly complete nor characterized by an elementary class of frames.

Exercise 6.7 Show that the set of formulas which are true in the model $\mathfrak{M}$ defined in Section 6.4 is not closed under Subst.

Exercise 6.8 Show that **Dum** and **SL** are not strongly complete.

Exercise 6.9 Prove that the logic $\mathbf{T} \oplus \Box(\Box^2 p \to \Box^3 p) \to (\Box p \to \Box^2 p)$ is not finitely approximable. (Hint: show that every intransitive frame for this logic is infinite and that $\boldsymbol{tra}$ does not belong to it; to prove the latter use the frame $\mathfrak{R} = \langle \omega, R\rangle$, where nRm iff $m \geq n-1$, which is known as the *recession frame*.)

Exercise 6.10 Show that there is no finitely approximable logic in the interval $[\mathbf{K} \oplus \Box(\Box^2 p \to \Box^3 p) \to (\Box p \to \Box^2 p), \mathrm{Log}\mathfrak{R}]$.

Exercise 6.11 Show that $\mathbf{T} \oplus \Box p \wedge q \to \Diamond(\Box^2 p \wedge \Diamond q)$ is not finitely approximable.

Exercise 6.12 Prove that $\mathbf{K} \oplus \Diamond\Box p \vee \Box(\Box(\Box q \to q) \to q)$ is incomplete. (Hint: show that the formula $\Diamond\Box p \vee \Box p$ does not belong to this logic and cannot be separated from it by a Kripke frame; to prove the former use the frame $\langle \omega \cup \{\omega, \omega+1\}, R\rangle$ where xRy iff either $x, y \in \omega \cup \{\omega\}$ and $x > y$ or $x = \omega + 1$ and $y = \omega$.)

Exercise 6.13 Show that the formula $\Diamond\Box p \vee \Box(\Box(\Box q \to q) \to q)$ is valid in a frame $\mathfrak{F} = \langle W, R\rangle$ iff $\mathfrak{F}$ satisfies the condition

$$\forall x\ (\neg\exists y\ xRy \vee \exists z\ (xRz \wedge \neg\exists u\ zRu)).$$

Exercise 6.14 Show that the logic $\mathbf{K} \oplus \Diamond\Box p \vee \Box p$ is canonical, with its canonical frame satisfying the condition in the previous exercise.

Exercise 6.15 Prove that $\mathbf{K} \oplus \Box(\Box p \leftrightarrow p) \to \Box p$ is incomplete. (Hint: $\boldsymbol{tra}$ does not belong to this logic.)

Exercise 6.16 Does the equality $L + \bigcap_{i\in I} L_i = \bigcap_{i\in I}(L + L_i)$ hold in ExtInt? (Hint: assuming that it holds, prove that all si-logics are finitely approximable.)

Exercise 6.17 Show that $\mathbf{T} \oplus \Box(\Box(p \to \Box p) \to \Box^3 p) \to p$ is neither complete nor elementary.

Exercise 6.18 Construct a logic in NExt**KB** which is not finitely approximable.

Exercise 6.19 Construct a normal modal logic with arbitrarily large finite rooted frames but without infinite ones.

Exercise 6.20 Construct a complete (finitely approximable) logic $L \in$ NExt**K** and a variable free formula φ such that $L \oplus \varphi$ is not complete (finitely approximable).

Exercise 6.21 Construct a logic in NExt$\mathbf{Alt}_2$ that is not finitely approximable.

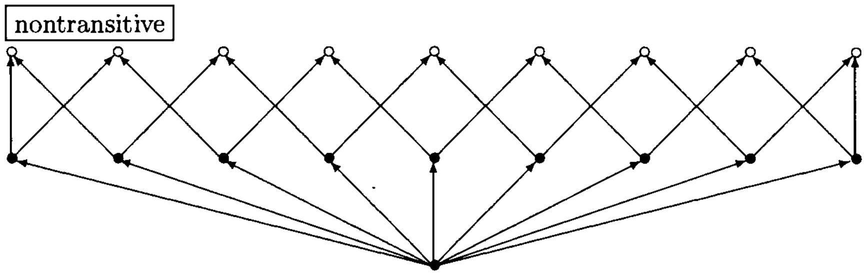


FIG. 6.8.

Exercise 6.22 Prove that the class NExt$\mathbf{Alt}_1$ is countable and NExt$\mathbf{Alt}_2$ is continual.

Exercise 6.23 Prove that the set of tense formulas that are true in the model $\langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F} = \langle \omega, > \rangle$ and $\mathfrak{V}$ is a bijection from the set of variables onto the family of all finite and cofinite subsets of ω, is a consistent tense logic[10] but has no Kripke frames.

Problem 6.1 *Call a logic L locally compact if every fragment of L with $n < \omega$ variables is compact. Are there locally compact logics that are not compact?*

6.8 Notes

The results of investigating modal and si-logics in the first half of the 1960s gave no reason to doubt that all modal and (especially) si-logics can be characterized by Kripke frames. Actually, there were no doubts that these logics are a sort of fragments of classical first order logic. However, in the late 1960s and early 1970s a series of "negative" results appeared, started by Jankov's (1968b) example of a si-logic which is not finitely approximable and modal and si-calculi of that kind constructed by Makinson (1969), Kuznetsov and Gerchiu (1970) and Fine (1972). (The result of Exercise 6.21 is due to Bellissima (1988) and that of Exercise 6.18 to Wolter (1993).)

In fact the "negative" results presented in this chapter show that the languages of modal and si-logics with the frame interpretation have a rather strong expressive power, in some respects stronger than the classical first order language. Moreover, Thomason (1975b) showed that in a sense classical second order logic can be effectively embedded into a propositional modal logic with the frame interpretation. Note, however, that no analogous result has been proved for si-logics, though Thomason's (1975b) idea seems to be enough to justify it.

The first modal formula without a first order equivalent on frames—the McKinsey formula ma—was found by van Benthem (1975) and Goldblatt (1975),

[10] Recall that tense logics are closed under the rules $\varphi/G\varphi$, $\varphi/H\varphi$.

though their proofs were different: the former used countable elementary submodels (i.e., the Löwenheim–Skolem theorem), and the latter ultraproducts. Notice that it is not hard also to prove this result with the help of the compactness theorem in the same manner as in Section 6.2. Later Doets (1987) showed that $\boldsymbol{ma}$ does not have a first order equivalent even on the class of finite frames; see also van Benthem (1989). Indeed, it is easy to see that $\boldsymbol{ma}$ is valid in the frames $\mathfrak{F}_n$ shown in Fig. 6.8, where n is the number of final points, iff n is odd. Now, if $\boldsymbol{ma}$ is first order definable then, according to van Benthem (1976a), it has a single (!) first order formula as its equivalent, and using the technique of Ehrenfeucht (1961) games (see also Exercise 1.3.15 in Chang and Keisler (1990) which does not use the game terminology) one can show that for every first order formula ϕ there is m such that ϕ is valid in all $\mathfrak{F}_n$ for $n > m$ or is refuted in all such frames no matter whether n is even or odd. Goldblatt (1991) proved that $\mathbf{K} \oplus \boldsymbol{ma}$ is not canonical and Wang (1992) showed that it is not strongly Kripke complete. Observe, by the way, that both $\boldsymbol{la}$ and $\boldsymbol{grz}$ are clearly first order definable on finite frames. According to Boolos and Sambin (1991), Fine and Rautenberg were the first to notice that $\mathbf{GL}$ is not strongly complete, and Goldfarb proved this using formulas in one variable. Exercise 6.3 is due to Hughes and Cresswell (1982).

One more interesting example of Doets (1987): the Fine formula

$$\Diamond\Box(p \vee q) \to \Diamond(\Box p \vee \Box q)$$

is equivalent on countable frames to the following first order condition:

$$\forall x, y\ (xRy \to \exists z\ (xRz \wedge \forall u\ (zRu \to yRu) \wedge \forall u, v\ (zRu \wedge zRv \to u = v)))$$

but on the class of all frames it does not have a first order equivalent. The latter is proved with the help of the intransitive frame $\mathfrak{F}$ which consists of a root seeing all points represented by infinite subsets of natural numbers, which in turn see exactly the natural numbers contained in them. It is not hard to check that $\mathfrak{F}$ validates the Fine formula but does not satisfy the first order condition above, which, by the Löwenheim–Skolem theorem, means that the formula is not first order definable. Intuitionistic formulas with similar properties were constructed by Chagrova (1989b). However, the following problem of Doets (1987) is still open: which is the least cardinal $\varkappa$ such that a formula is first order definable whenever it is definable on frames of cardinality $\leq \varkappa$?

First examples of intuitionistic formulas—$\boldsymbol{sa}$ and $\boldsymbol{bb}_n$—without first order equivalents were given by van Benthem (1984) and Rodenburg (1986). In Section 6.2 we established this result for $\boldsymbol{bb}_n$ using Shimura's (1995) theorem (Theorem 6.9) that no logic in the interval $[\mathbf{Int}, \mathbf{T}_2]$ save $\mathbf{Int}$ is strongly Kripke complete and the fact (to be proved in Section 10.2) that Kripke completeness and elementarity imply canonicity. The Scott axiom may also be treated in the same way using another result of Shimura (1995): no si-logic in the interval $[\mathbf{SL}, \mathbf{SL} + \boldsymbol{bd}_3]$ is strongly complete. (Note by the way that $\mathbf{SL}$ in any language

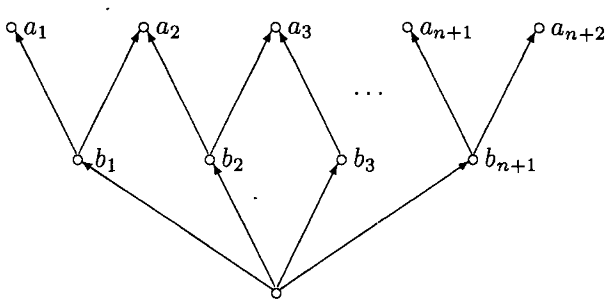


FIG. 6.9.

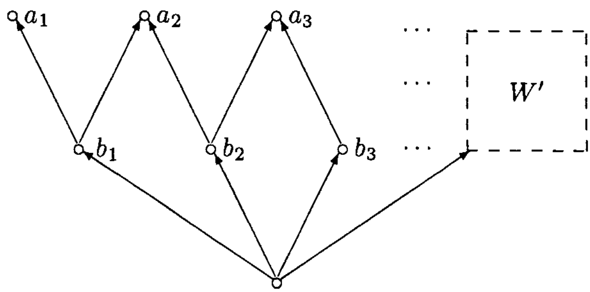


FIG. 6.10.

with finitely many variables is canonical, as has been recently observed by Ghilardi, Meloni and Miglioli.) Here we outline a direct proof due to van Benthem (1984) and Rodenburg (1986), which is based on the compactness theorem.

For the Scott axiom $\boldsymbol{sa}$ we consider frames of the form shown in Fig. 6.9 and describe them by means of first order formulas in the same manner as in the proof of Theorem 6.7. Now, by the compactness theorem, if $\boldsymbol{sa}$ is first order definable then it must be valid in a frame $\mathfrak{F}$ of the form depicted in Fig. 6.10, where points in the "box" W' are incomparable with a_is and b_is. On the other hand, a valuation in $\mathfrak{F}$ such that p is true only at a_i, for all $i < \omega$, refutes $\boldsymbol{sa}$, which is a contradiction.

To prove that $\boldsymbol{bb}_n$s are not first order definable one can use in the same way the frames in Fig. 6.11. In view of the result of Doets (1987) according to which only a finite number of Nishimura formulas are first order definable (and the remaining are not first order definable even on the class of finite frames), it seems that Shimura's (1995) theorem can be extended to almost all si-logics with extra axioms in one variable.

An interesting example was found by Hughes (1990). He showed that the logic $\mathbf{KMT} = \mathbf{K} \oplus \{\Diamond((\Box p_1 \to p_1) \wedge \ldots \wedge (\Box p_n \to p_n)) : n \geq 1\}$ is characterized by the class of frames satisfying the condition $\forall x \exists y\ (xRy \wedge yRy)$, it is finitely approximable and decidable but not finitely axiomatizable and elementary.

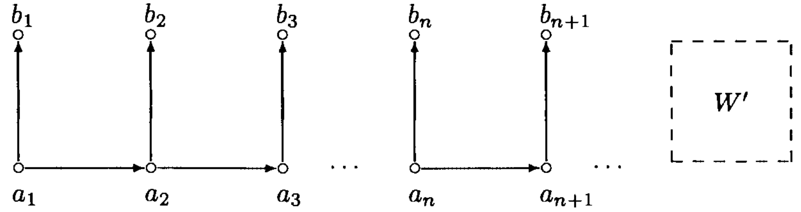


FIG. 6.11.

In general, for modal and intuitionistic formulas with the frame semantics one can refute practically all properties typical for first order formulas. However, there are partial exceptions. For instance, according to Corollary 2.1.5 of Chang and Keisler (1990), if a theory has arbitrarily large finite models, then it has an infinite model. Of course, in our case we should speak about rooted frames. Here is an example of a tense logic with arbitrarily large finite frames but without infinite ones: it suffices to extend the minimal tense logic by the axioms of **GL.3** for both $\Box$ and $\Box^{-1}$. It is easy to see that rooted frames for this logic are of the form $\langle\{1,\dots,n\},<\rangle$. It turns out, however, that for logics in NExt**K4** and Ext**Int** an analog of Corollary 2.1.5 in Chang and Keisler (1990) holds; see Chagrov (1995).

The effect of Kripke incompleteness was first discovered by Thomason (1972b) for tense logics (see Exercise 6.23), and then Thomason (1972a) constructed a non-compact modal logic in NExt**T**. Rybakov (1977, 1978a) and Shehtman (1980) extended the latter result to NExt**Grz** and Ext**Int**. It is worth noting that the non-compact logic of Rybakov (1978a) is decidable and that of Shehtman (1980) is axiomatizable by formulas in two variables. Kripke incomplete normal modal calculi were first constructed by Fine (1974b) and Thomason (1974a), and an incomplete si-calculus by Shehtman (1977). Other examples of that sort can be found in Blok (1978) (see Section 10.5), van Benthem (1978, 1979a), Boolos (1980). Usually incomplete logics in NExt**K** are constructed with the help of various modifications of the so called "recession frame" first used by Makinson (1969); it is defined in Exercise 6.9. Note by the way that the logic of the recession frame was (finitely) axiomatized by Blok (1979). In NExt**K4** and Ext**Int** all known constructions of incomplete logics are based upon modifications of the frame of Fine (1974b); for another application of this frame see Chagrov and Zakharyaschev (1995a).

Every Kripke complete logic is complete with respect to the neighborhood semantics. However, the converse does not hold, as was discovered by Gerson (1975a). Nevertheless it does not guarantee completeness either: Gerson (1975b) constructed the first example of a modal logic that is not complete with respect to the neighborhood semantics and Shehtman (1980) extended this result to the class NExt**Grz**. In Section 6.5, written on the material of Shehtman (1977, 1980), we saw that this does not provide us with si-logics that are not complete with respect to the neighborhood semantics. The question on the existence of such logics, raised by Kuznetsov (1975), is still open. Problem 6.1 is due to Shehtman (1980).

Another variant of the completeness problem is connected with transferring the Löwenheim–Skolem theorem to modal and si-logics. Are countable frames enough to characterize all Kripke complete modal and si-logics? This question was raised by Hosoi and Ono (1973). A negative solution to it for tense logics was obtained by Thomason (1975a) and for modal and si-logics by Shehtman (1983). Theorem 6.30 is due to Chagrov (1986). It is not known, however, what is the minimal cardinality of frames that are enough to characterize all Kripke complete logics. This problem was formulated by Kuznetsov; see Shehtman (1983). Note that all logics of finite width are characterized by countable frames, as will be shown in Section 10.4. In the case of quasi-normal and polymodal logics examples of Kripke complete logics all frames of which contain at least a continuum of points were constructed by Thomason (1975a) and Chagrov (1985b).

Two more open questions concerning the cardinality of frames also deserve mentioning. All the examples above were constructed semantically, and so nothing is known about the cardinality problem for calculi. Besides, we do not know any results of that sort for the neighborhood semantics. Note that these problems are closely related to similar problems for second order logic, which are also far from a complete solution.

Part III

Adequate semantics

As we saw in the previous chapter, not all modal and superintuitionistic logics may be characterized by Kripke frames. There is nothing extraordinary in this unpleasant fact. After all the Kripke semantics was constructed initially just for several particular systems and only after that were we trying it on arbitrary modal and si-logics.

In this part we introduce an adequate semantics for the logics under consideration. First, in Chapter 7 we translate the language of logic into the language of algebra and arrive at the algebraic semantics—modal and pseudo-Boolean algebras. Although this semantics gives no sensible interpretation for logical connectives, it enables us to take advantage of the developed apparatus of universal algebra. Then in Chapter 8, basing on Stone's representation of distributive lattices, we obtain a relational representation of modal and pseudo-Boolean algebras—the so called *general frames*—which combine in themselves the merits of both algebras and Kripke frames.

7

ALGEBRAIC SEMANTICS

Algebraic semantics abstracts from the intended meaning of logical connectives and interprets them just as operations on an arbitrary set A of objects, some of which are regarded as distinguished. Each formula $\varphi(p_1,\ldots,p_n)$ gives rise to a function $f_\varphi(x_1,\ldots,x_n)$ on A, and we may consider $\varphi(p_1,\ldots,p_n)$ to be valid in this "interpretation" if $f_\varphi(a_1,\ldots,a_n)$ is a distinguished object, for every $a_1,\ldots,a_n \in A$. It is not hard to see that all our logics are complete with respect to this highly abstract semantics. But to use it profitably, we should know something about the constitution of algebras corresponding to modal and superintuitionistic logics.

7.1 Algebraic preliminaries

The aim of this section is to introduce the basic algebraic notions and notations to be used in what follows.

Let A be a non-empty set. For $n \geq 1$, by an *n-ary operation* on A we mean any map o from A^n into A; a *0-ary operation* on A is an element in A. For example, the truth-table in Section 1.1 defines $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ as 2-ary or binary operations on the set {F,T}, $\neg$ as a 1-ary or unary operation, and $\bot$ may be regarded as a 0-ary operation on {F,T}, namely F.

A *universal algebra* or simply an *algebra* is a set A, called the *universe* of the algebra, together with some operations $o_1,\ldots,o_n$ on it. We denote the algebra by $\mathfrak{A} = \langle A, o_1,\ldots,o_n\rangle$. For instance, the truth-table for **Cl** determines an algebra of the form $\mathfrak{A} = \langle\{\mathrm{T},\mathrm{F}\},\wedge,\vee,\rightarrow,\leftrightarrow,\neg,\bot\rangle$.

Two algebras $\mathfrak{A} = \langle A, o_1,\ldots,o_n\rangle$ and $\mathfrak{B} = \langle B, o'_1,\ldots,o'_m\rangle$ are said to be *similar* if $n = m$ and, for every $i \in \{1,\ldots,n\}$, the operations o_i and o'_i are of the same arity. As a rule, corresponding operations in similar algebras are denoted by the same symbols, though sometimes different ones may be preferable.

Mostly we shall consider algebras with operations denoted by $\wedge$, $\vee$, $\rightarrow$ (binary), $\bot$ (0-ary) and $\Box$ (unary). It will always be clear from the context whether we deal with algebraic operations or logical connectives. Although on the other hand the set **For**$\mathcal{ML}$ of modal formulas with the formula formation rules may be regarded as an algebra $\langle \mathbf{For}\mathcal{ML},\wedge,\vee,\rightarrow,\bot,\Box\rangle$.

Algebras of the types $\mathfrak{A} = \langle A,\wedge,\vee,\rightarrow,\bot\rangle$ and $\mathfrak{A} = \langle A,\wedge,\vee,\rightarrow,\bot,\Box\rangle$ are called $\mathcal{L}$- and $\mathcal{ML}$-*algebras*, respectively. Each formula $\varphi(p_1,\ldots,p_n)$ in the language $\mathcal{L}$ (or $\mathcal{ML}$) gives rise to an n-ary operation in an $\mathcal{L}$- ($\mathcal{ML}$-) algebra $\mathfrak{A}$ if we interpret φ's connectives as the corresponding operations in $\mathfrak{A}$ and the propositional variables $p_1,\ldots,p_n$ as variables over A. A formula φ considered as the definition of such an operation in $\mathcal{L}$- ($\mathcal{ML}$-) algebras is called a *term* or an $\mathcal{L}$- ($\mathcal{ML}$-) *term*, to be more exact. For $a_1, \dots, a_n \in A$, we denote by $\varphi(a_1, \dots, a_n)$ the result of applying the operation associated with φ in $\mathfrak{A}$ to the arguments $a_1, \dots, a_n$. Given a map $\mathfrak{V}$ from $\mathbf{Var}\mathcal{L}$ in A, called a *valuation* in $\mathfrak{A}$, $\mathfrak{V}(\varphi) = \varphi(\mathfrak{V}(p_1), \dots, \mathfrak{V}(p_n))$ is the *value of* φ *in* $\mathfrak{A}$ *under* $\mathfrak{V}$.

An expression of the form $\varphi = \psi$, φ and ψ terms, is called an *identity*. It is *true* in an algebra $\mathfrak{A}$ if the operations in $\mathfrak{A}$ determined by φ and ψ are the same, i.e., $\mathfrak{V}(\varphi) = \mathfrak{V}(\psi)$ for any valuation $\mathfrak{V}$ in $\mathfrak{A}$. An expression of the form

$$\varphi_1 = \psi_1 \wedge \dots \wedge \varphi_n = \psi_n \to \varphi_0 = \psi_0,$$

in which all φ_i and ψ_i are terms, is called a *quasi-identity*. It is *true* in $\mathfrak{A}$ if for every valuation $\mathfrak{V}$ in $\mathfrak{A}$, $\mathfrak{V}(\varphi_0) = \mathfrak{V}(\psi_0)$ whenever $\mathfrak{V}(\varphi_i) = \mathfrak{V}(\psi_i)$ for all $i = 1, \dots, n$.

For an algebra $\mathfrak{A} = \langle A, o_1, \dots, o_n \rangle$ and a non-empty subset ∇ of A, the pair $\langle \mathfrak{A}, \nabla \rangle$ is called a *matrix* and ∇ its *set of distinguished elements*. If $\mathfrak{A}$ is an $\mathcal{L}$- ($\mathcal{ML}$-) algebra then $\langle \mathfrak{A}, \nabla \rangle$ is an $\mathcal{L}$- ($\mathcal{ML}$-) *matrix*. An $\mathcal{L}$- ($\mathcal{ML}$-) formula φ is said to be *valid* in an $\mathcal{L}$- ($\mathcal{ML}$-) matrix $\langle \mathfrak{A}, \nabla \rangle$ if the value of φ is in ∇ under every valuation in $\mathfrak{A}$. We write $\langle \mathfrak{A}, \nabla \rangle \models \varphi$ to mean that φ is valid in $\langle \mathfrak{A}, \nabla \rangle$. As in the case of the Kripke semantics, we say a logic L is *characterized* by a class $\mathcal{C}$ of matrices (or $\mathcal{C}$ is *characteristic* for L) if L coincides with the set of formulas that are valid in all matrices in $\mathcal{C}$.

We shall often deal with $\mathcal{L}$- and $\mathcal{ML}$-matrices $\langle \mathfrak{A}, \nabla \rangle$ in which ∇ contains only one element $\top = \bot \to \bot$. In this case instead of $\langle \mathfrak{A}, \nabla \rangle \models \varphi$ we write $\mathfrak{A} \models \varphi$ and say that φ is *valid* in $\mathfrak{A}$. Clearly, $\mathfrak{A} \models \varphi$ iff the identity $\varphi = \top$ is true in $\mathfrak{A}$.

An algebra is *finite* if its universe is finite. An algebra whose universe contains only one element is called *degenerate*. A matrix is *degenerate* if its set of distinguished elements coincides with its universe. It should be clear that all identities and quasi-identities are true in every degenerate algebra and all formulas are valid in every degenerate matrix.

Suppose $\mathfrak{A} = \langle A, o_1, \dots, o_n \rangle$ and $\mathfrak{B} = \langle B, o_1, \dots, o_n \rangle$ are similar algebras. A map f from A into B is called a *homomorphism* of $\mathfrak{A}$ in $\mathfrak{B}$ if f preserves the operations in the following sense: for every operation o_i in $\mathfrak{A}$ of arity m and every $a_1, \dots, a_m \in A$,

$$f(o_i(a_1, \dots, a_m)) = o_i(f(a_1), \dots, f(a_m)).$$

A homomorphism f of $\mathfrak{A}$ in $\mathfrak{B}$ is an *isomorphism* or *embedding* of $\mathfrak{A}$ in $\mathfrak{B}$ if f is an *injection*, i.e., $a \neq b$ implies $f(a) \neq f(b)$. And if an isomorphism f of $\mathfrak{A}$ in $\mathfrak{B}$ is also a *surjection*, that is a map "onto", then f is called an *isomorphism* of $\mathfrak{A}$ *onto* $\mathfrak{B}$. In this case $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic*.

Matrices $\langle \mathfrak{A}, \nabla \rangle$ and $\langle \mathfrak{B}, \nabla' \rangle$ are *isomorphic* if there is an isomorphism f of $\mathfrak{A}$ onto $\mathfrak{B}$ such that $f(\nabla) = \nabla'$.

We will not distinguish between isomorphic algebras or isomorphic matrices.

7.2 The Tarski–Lindenbaum construction

It is very easy to find a characteristic matrix for every modal or si-logic. Indeed, suppose $L \in \mathrm{Ext}\mathbf{K}$ (si-logics are treated in exactly the same way simply by omitting $\Box$) and consider the matrix $\langle \mathfrak{A}_{\mathcal{ML}}, L\rangle$ where

$$\mathfrak{A}_{\mathcal{ML}} = \langle \mathbf{For}\mathcal{ML}, \wedge, \vee, \rightarrow, \bot, \Box\rangle$$

is the *algebra of formulas* in which, for $\odot \in \{\wedge, \vee, \rightarrow\}$, $\odot(\varphi, \psi) = \varphi \odot \psi$ and $\Box(\varphi) = \Box\varphi$.

Theorem 7.1 *$\langle \mathfrak{A}_{\mathcal{ML}}, L\rangle$ is a characteristic matrix for L.*

Proof Suppose that $\varphi(p_1, \dots, p_n) \in L$ and $\mathfrak{V}$ is a valuation in $\mathfrak{A}_{\mathcal{ML}}$. Then $\mathfrak{V}(\varphi) = \varphi\{\mathfrak{V}(p_1)/p_1, \dots, \mathfrak{V}(p_n)/p_n\} \in L$, since L is closed under Subst. Thus, $\langle \mathfrak{A}_{\mathcal{ML}}, L\rangle \models \varphi$.

Suppose $\varphi \notin L$. Define a valuation $\mathfrak{V}$ in $\mathfrak{A}_{\mathcal{ML}}$ by taking $\mathfrak{V}(p) = p$ for every variable p. The value of φ under $\mathfrak{V}$ is clearly φ itself and so $\langle \mathfrak{A}_{\mathcal{ML}}, L\rangle \not\models \varphi$. ❑

Of course this theorem conveys nothing else but the fact that L is closed under substitution. Let us now recall another useful fact, namely that equivalent formulas in normal modal and si-logics are interchangeable.

Theorem 7.2 *Every normal modal and si-logic has a characteristic matrix with a single distinguished element.*

Proof We consider only $L \in \mathrm{NExt}\mathbf{K}$. Define an algebra

$$\mathfrak{A}_L = \langle \|\mathbf{For}\mathcal{ML}\|_L, \wedge, \vee, \rightarrow, \bot, \Box\rangle$$

by taking

$$\|\mathbf{For}\mathcal{ML}\|_L = \{\|\varphi\|_L : \varphi \in \mathbf{For}\mathcal{ML}\},$$

$$\|\varphi\|_L = \{\psi \in \mathbf{For}\mathcal{ML} : \varphi \leftrightarrow \psi \in L\},$$

$$\|\varphi\|_L \wedge \|\psi\|_L = \|\varphi \wedge \psi\|_L, \quad \|\varphi\|_L \vee \|\psi\|_L = \|\varphi \vee \psi\|_L,$$

$$\|\varphi\|_L \rightarrow \|\psi\|_L = \|\varphi \rightarrow \psi\|_L, \quad \bot = \|\bot\|_L, \quad \Box\|\varphi\|_L = \|\Box\varphi\|_L.$$

The correctness of this definition is ensured by the equivalent replacement theorem for L according to which the definition of the operations above does not depend on the choice of formulas in the equivalence classes $\|\varphi\|_L$ and $\|\psi\|_L$: for example,

$$\|\varphi\|_L = \|\varphi'\|_L \text{ and } \|\psi\|_L = \|\psi'\|_L \text{ imply } \|\varphi \wedge \psi\|_L = \|\varphi' \wedge \psi'\|_L.$$

As a distinguished element in $\mathfrak{A}_L$ we take $\|\top\|_L = \|\bot \rightarrow \bot\|_L$. Let us prove that the matrix $\langle \mathfrak{A}_L, \{\|\top\|_L\}\rangle$ characterizes L. First, by induction on the construction of $\varphi(p_1, \dots, p_n)$ one can readily show that, for any formulas $\varphi_1, \dots, \varphi_n$,

$$\varphi(\|\varphi_1\|_L, \dots, \|\varphi_n\|_L) = \|\varphi(\varphi_1, \dots, \varphi_n)\|_L.$$

Now, suppose $\varphi(p_1, \dots, p_n) \in L$. Then clearly $\varphi \leftrightarrow \top \in L$. So for all formulas $\varphi_1, \dots, \varphi_n$, we have $\varphi(\varphi_1, \dots, \varphi_n) \leftrightarrow \top \in L$, i.e.,

$$\|\varphi(\varphi_1, \dots, \varphi_n)\|_L = \|\top\|_L.$$

Let $\mathfrak{V}$ be a valuation in $\mathfrak{A}_L$ under which $\mathfrak{V}(p_i) = \|\varphi_i\|_L$ for $1 \le i \le n$. Then

$$\mathfrak{V}(\varphi) = \varphi(\|\varphi_1\|_L, \dots, \|\varphi_n\|_L) = \|\varphi(\varphi_1, \dots, \varphi_n)\|_L = \|\top\|_L,$$

from which $\langle \mathfrak{A}_L, \{\|\top\|_L\}\rangle \models \varphi$.

Suppose that $\varphi(p_1, \dots, p_n) \notin L$. This is equivalent to $\varphi \leftrightarrow \top \notin L$, i.e., $\|\varphi\|_L \neq \|\top\|_L$. Define a valuation $\mathfrak{V}_L$ in $\mathfrak{A}_L$, called the *standard valuation* in $\mathfrak{A}_L$, by taking $\mathfrak{V}_L(p) = \|p\|_L$ for every variable p. Then we have

$$\mathfrak{V}_L(\varphi) = \varphi(\|p_1\|_L, \dots, \|p_n\|_L) = \|\varphi(p_1, \dots, p_n)\|_L \neq \|\top\|_L,$$

from which $\langle \mathfrak{A}_L, \{\|\top\|_L\}\rangle \not\models \varphi$. ❑

Since this proof uses no specific features of normal modal and si-logics except the equivalent replacement theorem, the result above can clearly be extended to other logics for which this theorem holds. However, this cannot be done in the case of quasi-normal modal logics.

Theorem 7.3 *If a logic $L \in$ Ext$\mathbf{K}$ is characterized by a matrix with a single distinguished element then L is normal.*

Proof Let $\langle \mathfrak{A}, \{\top\}\rangle$ be a characteristic matrix for L. We show that L is closed under necessitation.

Suppose $\varphi \in L$. Then the identity $\varphi = \top$ is true in $\mathfrak{A}$ and so $\Box\varphi = \Box\top$ is also true. Since $\Box\top \in \mathbf{K} \subseteq L$, $\Box\top = \top$ in $\mathfrak{A}$, from which $\Box\varphi$ is identically equal to $\top$, i.e., $\langle \mathfrak{A}, \{\top\}\rangle \models \Box\varphi$ and $\Box\varphi \in L$, because $\langle \mathfrak{A}, \{\top\}\rangle$ characterizes L. ❑

Theorem 7.2 can be generalized to quasi-normal logics in the following way.

Theorem 7.4 *Suppose $L \in$ NExt$\mathbf{K}$, $L' \in$ Ext$L$ and $\mathfrak{A}_L$ is the algebra defined in the proof of Theorem 7.2. Then L' is characterized by the matrix $\langle \mathfrak{A}_L, \nabla\rangle$ where $\nabla = \{\|\varphi\|_L : \varphi \in L'\}$.*

Proof Suppose that $\varphi(p_1, \dots, p_n) \in L'$ and $\mathfrak{V}$ is a valuation in $\mathfrak{A}_L$ such that $\mathfrak{V}(p_1) = \|\varphi_1\|_L, \dots, \mathfrak{V}(p_n) = \|\varphi_n\|_L$. Since $\varphi(\varphi_1, \dots, \varphi_n) \in L'$, we then have $\mathfrak{V}(\varphi) = \|\varphi(\varphi_1, \dots, \varphi_n)\|_L \in \nabla$. Therefore, $\langle \mathfrak{A}_L, \nabla\rangle \models \varphi$.

Conversely, suppose $\varphi(p_1, \dots, p_n) \notin L'$. Then

$$\mathfrak{V}_L(\varphi) = \varphi(\|p_1\|_L, \dots, \|p_n\|_L) = \|\varphi(p_1, \dots, p_n)\|_L \notin \nabla,$$

from which $\langle \mathfrak{A}_L, \nabla\rangle \not\models \varphi$. ❑

Corollary 7.5 *Under the conditions of Theorem 7.4 for every formula φ, $\varphi \in L'$ iff $\mathfrak{V}_L(\varphi) \in \nabla$.*

The matrices $\langle \mathfrak{A}_L, \|\top\|_L \rangle$ and $\langle \mathfrak{A}_L, \nabla \rangle$ defined in the proofs of Theorems 7.2 and 7.4 are called the *Tarski–Lindenbaum matrices* for L and L', respectively. If the Tarski–Lindenbaum matrix for L has only one distinguished element then it is called the *Tarski–Lindenbaum algebra* for L. The Tarski–Lindenbaum matrix for a quasi-normal modal logic L' constructed in the proof of Theorem 7.4 depends of course on the choice of L. To define the matrix uniquely we may take as L the maximal normal logic contained in L', that is $\ker L'$.

By the definition, the Tarski–Lindenbaum matrices have countably many elements. So we have

Corollary 7.6 (i) *Every normal modal and si-logic is characterized by a countable algebra.*

(ii) *Every quasi-normal modal logic is characterized by a countable matrix.*

Tarski–Lindenbaum matrices and algebras characterize not only logics themselves but also the inference rules admissible in them.

Theorem 7.7 (i) *A rule $\varphi_1, \dots, \varphi_m/\varphi$ is admissible in a logic $L \in$ NExt**K** or $L \in$ Ext**Int** iff the quasi-identity $\varphi_1 = \top \wedge \dots \wedge \varphi_n = \top \to \varphi = \top$ is true in the Tarski–Lindenbaum algebra $\mathfrak{A}_L$.*

(ii) *Let $L' \in$ Ext**K** and $\langle \mathfrak{A}_L, \nabla \rangle$ be the Tarski–Lindenbaum matrix defined in Theorem 7.4. A rule $\varphi_1, \dots, \varphi_m/\varphi$ is admissible in L' iff for every valuation $\mathfrak{V}$ in $\mathfrak{A}_L$, $\mathfrak{V}(\varphi) \in \nabla$ whenever $\mathfrak{V}(\varphi_1) \in \nabla, \dots, \mathfrak{V}(\varphi_m) \in \nabla$.*

Proof Since (i) is a special case of (ii) (take $L' = L$), we prove only the latter. Let $p_1, \dots, p_n$ be all the variables in $\varphi_1, \dots, \varphi_m, \varphi$.

($\Rightarrow$) Suppose $\mathfrak{V}$ is a valuation in $\mathfrak{A}_L$, $\mathfrak{V}(p_i) = \|\chi_i\|_L$, for $1 \leq i \leq n$, and $\varphi_j(\|\chi_1\|_L, \dots, \|\chi_n\|_L) \in \nabla$, for $1 \leq j \leq m$. Then $\|\varphi_j(\chi_1, \dots, \chi_n)\|_L \in \nabla$, from which $\varphi_j(\chi_1, \dots, \chi_n) \in L'$. Since the rule $\varphi_1, \dots, \varphi_m/\varphi$ is admissible in L', $\varphi(\chi_1, \dots, \chi_n) \in L'$ and so $\mathfrak{V}(\varphi) = \|\varphi(\chi_1, \dots, \chi_n)\|_L \in \nabla$.

($\Leftarrow$) Suppose $\varphi_1, \dots, \varphi_m/\varphi$ is not admissible in L'. This means that there are formulas $\chi_1, \dots, \chi_n$ such that $\varphi_1(\chi_1, \dots, \chi_n) \in L', \dots, \varphi_m(\chi_1, \dots, \chi_n) \in L'$, but $\varphi(\chi_1, \dots, \chi_n) \notin L'$. Then $\mathfrak{V}(\varphi_i) = \varphi_i(\|\chi_1\|_L, \dots, \|\chi_n\|_L) \in \nabla$, $1 \leq i \leq m$, and $\mathfrak{V}(\varphi) = \varphi(\|\chi_1\|_L, \dots, \|\chi_n\|_L) \notin \nabla$, which is a contradiction. ❑

7.3 Pseudo-Boolean algebras

The Tarski–Lindenbaum algebras, being a direct translation of logics into the algebraic language, are too complicated to be a good semantic instrument. Even for **Cl**, which was initially defined as the set of formulas valid in a two-element algebra, the Tarski–Lindenbaum algebra is infinite (if, of course, the language is infinite). However, just as canonical models are only representatives, though very important, of the class of Kripke models, Tarski–Lindenbaum algebras are members of a wider class of algebras validating formulas in **Int** and **K**. In this section we consider the algebras suitable for superintuitionistic logics.

In fact, all we need is $\mathcal{L}$-algebras $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot\rangle$ in which the identity $\varphi = \psi$ is true whenever $\varphi \leftrightarrow \psi \in \mathbf{Int}$. Such algebras are called *pseudo-Boolean algebras* or *Heyting algebras.* A pseudo-Boolean algebra $\mathfrak{A}$ is said to be an *algebra for* a si-logic L if $\mathfrak{A} \models L$. By Theorem 7.2, the Tarski–Lindenbaum algebra $\mathfrak{A}_L$ for every si-logic L is a pseudo-Boolean algebra for L.

Theorem 7.8 *For each si-logic L and each formula φ, $\varphi \in L$ iff φ is valid in every pseudo-Boolean algebra for L.*

Proof ($\Rightarrow$) is trivial and ($\Leftarrow$) is a consequence of the fact that $\mathfrak{A}_L$ is a pseudo-Boolean algebra characterizing L. ❑

Example 7.9 The algebra $\langle\{\mathrm{T},\mathrm{F}\}, \wedge, \vee, \rightarrow, \bot\rangle$ whose operations are defined by the truth-table in Section 1.1 is a pseudo-Boolean algebra because $\mathbf{Int} \subset \mathbf{Cl}$.

The definition above is not convenient for determining if a given $\mathcal{L}$-algebra is a pseudo-Boolean one. The next theorem provides a simpler characterization of pseudo-Boolean algebras.

Given an algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot\rangle$, define a binary relation $\leq$ on A by taking, for every $x, y \in A$,

$$x \leq y \ \text{ iff } \ x \wedge y = x.$$

As will be shown in Theorem 7.13 below, for a pseudo-Boolean $\mathfrak{A}$ the relation $\leq$ turns out to be a partial order on A. So we can use the terminology introduced in Section 2.3 for partial orders, for instance, the greatest element in $\mathfrak{A}$, the least element, etc.

Theorem 7.10 *An algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot\rangle$ is a pseudo-Boolean algebra iff the following conditions hold in $\mathfrak{A}$ for every $x, y \in A$:*

(1) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ *(commutativity of $\wedge$ and $\vee$);*

(2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$ *(associativity of $\wedge$ and $\vee$);*

(3) $(x \wedge y) \vee y = y$, $x \wedge (x \vee y) = x$ *(absorption);*

(4) $z \wedge x \leq y$ *iff* $z \leq x \rightarrow y$ *($x \rightarrow y$ is the greatest element in the set $\{z \in A : z \wedge x \leq y\}$);*

(5) $\bot \leq x$ *($\bot$ is the least element in $\mathfrak{A}$).*

Proof ($\Rightarrow$) Only (4) needs a proof because the other conditions correspond to suitable intuitionistically valid formulas in Table 1.1 ($\bot \leq x$ corresponds to $p \wedge \bot \leftrightarrow \bot$). So suppose $z \wedge x \leq y$, i.e., $z \wedge x \wedge y = z \wedge x$. Since

$$p \leftrightarrow p \wedge (q \rightarrow q \wedge p) \in \mathbf{Int},$$

$$p \wedge (q \rightarrow q \wedge p \wedge r) \leftrightarrow p \wedge (q \rightarrow r) \in \mathbf{Int},$$

we then have

$$z = z \wedge (x \rightarrow x \wedge z) = z \wedge (x \rightarrow x \wedge z \wedge y) = z \wedge (x \rightarrow y),$$

from which $z \leq x \rightarrow y$.

Suppose now that $z \leq x \to y$, i.e., $z \wedge (x \to y) = z$. Since

$$p \wedge q \wedge r \leftrightarrow p \wedge q \wedge (p \to r) \in \mathbf{Int},$$

we then have

$$x \wedge z \wedge y = x \wedge z \wedge (x \to y) = x \wedge z.$$

($\Leftarrow$) The proof in this direction is much harder. First we require

Lemma 7.11 *If (1)–(5) hold in an algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot \rangle$ then the following conditions are also satisfied in $\mathfrak{A}$ for all $x, y \in A$:*

(6) $x \wedge x = x$, $x \vee x = x$ *(idempotency of $\wedge$ and $\vee$);*
(7) $x \to x = y \to y = \bot \to \bot$ $(= \top)$;
(8) $x \wedge (y \to y) = x$ $(= x \wedge \top)$;
(9) $x \wedge (y \to x) = x$;
(10) $x \wedge (x \to y) = x \wedge y$;
(11) $x = y$ *iff* $x \leq y$ *and* $y \leq x$;
(12) $x \leq y$ *iff* $x \to y = \top$;
(13) $x \wedge y = \top$ *iff* $x = y = \top$;
(14) $x = y$ *iff* $x \leftrightarrow y = \top$;
(15) *if* $x = \top$ *and* $x \to y = \top$ *then* $y = \top$;
(16) $x \leq y$ *iff* $x \vee y = y$;
(17) $x \leq z$ *and* $y \leq z$ *iff* $x \vee y \leq z$;
(18) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *(distributivity).*

Proof (6) We use the laws of absorption:

$$x \wedge x = x \wedge (x \vee (x \wedge x)) = x, \quad x \vee x = x \vee (x \wedge (x \vee x)) = x.$$

(7) By (6) we have $(x \to x) \wedge y \wedge y = (x \to x) \wedge y$, i.e., $(x \to x) \wedge y \leq y$, from which by (4), $x \to x \leq y \to y$, i.e., $(x \to x) \wedge (y \to y) = x \to x$. By the same argument we obtain $(x \to x) \wedge (y \to y) = y \to y$. Hence $x \to x = y \to y$ for every $y \in A$, in particular, $y = \bot$.

(8) By (6) we have $x \wedge y \leq y$ and by (4) $x \leq y \to y$, i.e., $x \wedge (y \to y) = x$.

(9) By (6) and (4), $x \wedge y \leq x$ and $x \leq y \to x$, whence $x \wedge (y \to x) = x$.

(10) Again, by (6) and (4) we have $x \to y \leq x \to y$, $(x \to y) \wedge x \leq y$ and so, using (9), $(x \to y) \wedge x = (x \to y) \wedge x \wedge y = x \wedge y$.

(11) If $x = y$ then, by (6), $x \leq y$ and $y \leq x$. Conversely, if $x \leq y$ and $y \leq x$ then, by the definition of $\leq$, we have $x = x \wedge y = y$.

(12) Suppose that $x \to y = \top$. Then using (10) and (8), we obtain $x \wedge y = x \wedge (x \to y) = x \wedge \top = x$, i.e., $x \leq y$. Suppose $x \leq y$. Then, in view of (8), $x \wedge \top \leq y$, from which by (4), $\top \leq x \to y$, i.e., $(x \to y) \wedge \top = \top$. On the other hand, by (8), $(x \to y) \wedge \top = x \to y$, and hence $x \to y = \top$.

(13) If $x = y = \top$ then, by (6), $x \wedge y = \top$. Suppose $x \wedge y = \top$. Since by (6) $x \wedge y \leq x$, we then have $\top \leq x$, i.e., $\top \wedge x = \top$ which together with $\top \wedge x = x$ (by (8)) gives $x = \top$. The equality $y = \top$ is proved analogously.

$$
\begin{array}{llll}
(14) & x = y & \text{iff} \;\; x \le y \text{ and } y \le x & \text{(by (11))} \\
 & & \text{iff} \;\; x \to y = y \to x = \top & \text{(by (12))} \\
 & & \text{iff} \;\; x \leftrightarrow y = \top & \text{(by (13)).}
\end{array}
$$

(15) If $x = x \to y = \top$ then using (6), (8) and (10) we obtain $y = y \wedge x \wedge (x \to y) = x \wedge (x \to y) = \top \wedge \top = \top$.

(16) Suppose $x \le y$, i.e., $x = x \wedge y$. Using the laws of absorption, we then obtain $x \vee y = y \vee (x \wedge y) = y$. If $x \vee y = y$ then, by the same laws, $x \wedge y = x \wedge (x \vee y) = x$, i.e., $x \le y$.

(17) If $x \le z$ and $y \le z$ then, by (16), $x \vee z = z = y \vee z$, whence $x \vee y \vee z = z$, i.e., $x \vee y \le z$. Suppose $x \vee y \le z$, i.e., by (16), $x \vee y \vee z = z$. Using (6), we then obtain $x \vee z = x \vee y \vee z = z$, i.e., $x \le z$. In the same way we get $y \le z$.

(18) According to (11), we need to prove two inequalities:

$$(x \wedge y) \vee (x \wedge z) \le x \wedge (y \vee z)$$

and

$$x \wedge (y \vee z) \le (x \wedge y) \vee (x \wedge z).$$

By (6) and (16), we have $y \le y \vee z$ and so $x \wedge y \le x \wedge (y \vee z)$. By the same argument we obtain $x \wedge z \le x \wedge (y \vee z)$. In view of (17), this establishes the former inequality.

Let us prove the latter. By (16) and (6), we have $x \wedge y \le (x \wedge y) \vee (x \wedge z)$ and $x \wedge z \le (x \wedge y) \vee (x \wedge z)$, from which by (4), $y \le x \to (x \wedge y) \vee (x \wedge z)$ and $z \le x \to (x \wedge y) \vee (x \wedge z)$. Therefore, by (17), $y \vee z \le x \to (x \wedge y) \vee (x \wedge z)$ and so, using (4) once again, we obtain $x \wedge (y \vee z) \le (x \wedge y) \vee (x \wedge z)$. ❑

We can now continue proving Theorem 7.10. We need to show that if an equivalence $\psi \leftrightarrow \chi$ is in **Int** and an algebra $\mathfrak{A}$ satisfies (1)–(5) and so, by Lemma 7.11, (6)–(18) as well, then the identity $\psi = \chi$ is true in $\mathfrak{A}$. In view of (14), it is sufficient to establish that $\varphi \in \mathbf{Int}$ implies $\mathfrak{A} \models \varphi$.

We prove this by induction on the length of a derivation of φ in *Int*. The step of induction is already justified: indeed, it is obvious for Subst and (15) establishes it for MP. So it remains only to check that the axioms of *Int* are valid in $\mathfrak{A}$.

(A1) By (9) we have $x \wedge (y \to x) = x$, i.e., $x \le y \to x$ and so, by (12), $x \to (y \to x) = \top$.

(A2) By applying (10) and (6) several times, we obtain

$$x \wedge (x \to y) \wedge (x \to (y \to z)) \wedge z = x \wedge y \wedge (y \to z) \wedge z =$$

$$x \wedge y \wedge (y \to z) = x \wedge (x \to y) \wedge (x \to (y \to z)),$$

i.e., $x \wedge (x \to y) \wedge (x \to (y \to z)) \le z$, which in view of (4) implies $x \to (y \to z) \le (x \to y) \to (x \to z)$ and, by (12),

$$(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = \top.$$

(A3) By the laws of absorption, we have $x \wedge (x \vee y) = x$, i.e., $x \leq x \vee y$ and by (12), $x \rightarrow x \vee y = \top$.

(A4) follows from $x \rightarrow x \vee y = \top$ by the commutativity of $\vee$.

(A5) By (6), $x \wedge y \leq x \wedge y$ which, by (4), gives $x \leq y \rightarrow x \wedge y$ and so, by (12), $x \rightarrow (y \rightarrow x \wedge y) = \top$.

(A6) By (6) we have $x \wedge y \leq x$ and by (12) $x \wedge y \rightarrow x = \top$.

(A7) is proved in the same way.

(A8) Using (18) and (10), we have

$$\begin{aligned}(x \vee y) \wedge (x \rightarrow z) \wedge (y \rightarrow z) \wedge z &= \\ (x \wedge (x \rightarrow z) \wedge (y \rightarrow z) \wedge z) \vee (y \wedge (x \rightarrow z) \wedge (y \rightarrow z) \wedge z) &= \\ (x \wedge (x \rightarrow z) \wedge (y \rightarrow z)) \vee (y \wedge (x \rightarrow z) \wedge (y \rightarrow z)) &= \\ (x \vee y) \wedge (x \rightarrow z) \wedge (y \rightarrow z),&\end{aligned}$$

from which $(x \vee y) \wedge (x \rightarrow z) \wedge (y \rightarrow z) \leq z$. Now we apply (4), then (12) and obtain $x \rightarrow z \leq (y \rightarrow z) \rightarrow (x \vee y \rightarrow z)$, and hence

$$(x \rightarrow z) \rightarrow ((y \rightarrow z) \rightarrow (x \vee y \rightarrow z)) = \top.$$

(A9) follows from (5) and (12). ❑

As a consequence of Theorem 7.10 we derive an interesting

Corollary 7.12 *Suppose that $\langle A, \wedge, \vee, \rightarrow_1, \bot \rangle$ and $\langle A, \wedge, \vee, \rightarrow_2, \bot \rangle$ are pseudo-Boolean algebras with the same universe and the same operations $\wedge$, $\vee$ and $\bot$. Then $x \rightarrow_1 y = x \rightarrow_2 y$, for every $x, y \in A$.*

Proof According to (4) in Theorem 7.10, the implication in a pseudo-Boolean algebra is completely determined by $\wedge$. ❑

An algebra of the form $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ satisfying the conditions (1)–(3) of Theorem 7.10 is called a *lattice* (we already used this notion in Sections 4.1 and 4.2 when discussing intersections and sums of logics). Pseudo-Boolean algebras may be considered as lattices with two additional operations $\rightarrow$ and $\bot$.

Theorem 7.13 *In every lattice $\langle A, \wedge, \vee \rangle$ the relation $\leq$ defined by*

$$x \leq y \textit{ iff } x \wedge y = x, \textit{ for } x, y \in A,$$

is a partial order on A; besides, for every $x, y \in A$,

$$x \leq y \textit{ iff } x \vee y = y.$$

Proof Notice first that the conditions (6), (11) and (16) in Lemma 7.11 do not depend on (4) and (5) and so hold in every lattice.

The reflexivity of $\leq$ follows from (6). As to the transitivity, if $x \leq y$ and $y \leq z$, i.e., $x = x \wedge y$ and $y = y \wedge z$, then $x = x \wedge y = x \wedge y \wedge z = x \wedge z$, from which $x \leq z$. The antisymmetry follows immediately from (11).

The fact that $\leq$ can be defined via $\vee$ is a consequence of (16). ❑

Given a partial order $\langle A, \leq \rangle$ and a subset $X \subseteq A$, an element $a \in A$ is called the *supremum* or *least upper bound* of X if $X \subseteq a\downarrow$ (i.e., $x \leq a$ for every $x \in X$) and $a \leq b$ whenever $X \subseteq b\downarrow$; a is the *infimum* or *greatest lower bound* of X if $X \subseteq a\uparrow$ (i.e., $x \geq a$ for every $x \in X$) and $a \geq b$ whenever $X \subseteq b\uparrow$. The supremum and infimum of X, if they exist, are denoted by $\bigvee X$ and $\bigwedge X$, respectively. In pseudo-Boolean algebras we clearly have $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$.

Example 7.14 It is not difficult to see that for every lattice $\langle A, \wedge, \vee \rangle$ and every $a_1, \ldots, a_n \in A$ $(n > 0)$,

$$\bigvee\{a_1, \ldots, a_n\} = a_1 \vee \ldots \vee a_n, \quad \bigwedge\{a_1, \ldots, a_n\} = a_1 \wedge \ldots \wedge a_n.$$

It follows that in a finite lattice the supremum and infimum do exist for every set of elements. However, in general this is not so, witness the following:

Example 7.15 Let $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ be the algebra in which

$$A = \{1/n, -1/n : \ n = 1, 2, 3, \ldots\}$$

and $\wedge$ and $\vee$ are defined by

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}, \quad \text{for every } x, y \in A.$$

The reader can readily verify that $\mathfrak{A}$ is a lattice but $\bigvee\{-1/n : n = 1, 2, \ldots\}$ and $\bigwedge\{1/n : n = 1, 2, \ldots\}$ do not exist in $\mathfrak{A}$.

A lattice, in particular a pseudo-Boolean algebra, is *complete* if $\bigwedge X$ and $\bigvee X$ exist in it for every set X.

It is useful to observe that the partial order relation $\leq$ defined in Theorem 7.13 completely determines the lattice operations $\wedge$ and $\vee$.

Theorem 7.16 *Suppose $\langle A, \leq \rangle$ is a partial order such that $\bigwedge\{x, y\}$ and $\bigvee\{x, y\}$ exist for every $x, y \in A$. Then the algebra $\langle A, \wedge, \vee \rangle$, with $\wedge$ and $\vee$ defined by*

$$x \wedge y = \bigwedge\{x, y\} \text{ and } x \vee y = \bigvee\{x, y\},$$

is a lattice and

$$x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y.$$

Proof Exercise. ❑

We use the developed algebraic technique to prove the following remarkable result, which is based upon Diego's theorem from Section 5.4.

Theorem 7.17. (McKay's theorem) *Every si-logic L axiomatizable by disjunction free formulas is finitely approximable and so decidable if the number of its extra axioms is finite.*

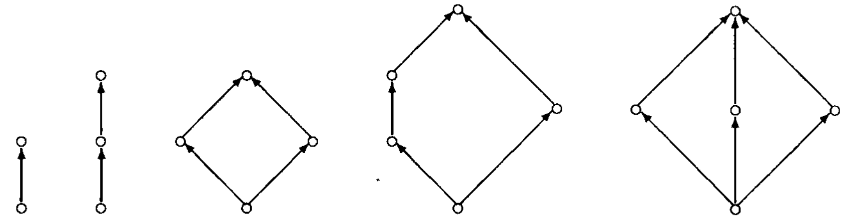


FIG. 7.1.

Proof Suppose $\varphi \notin L$. Take any pseudo-Boolean algebra $\mathfrak{A}$ for L in which $\mathfrak{V}(\varphi) \neq \top$ for some valuation $\mathfrak{V}$ and let B be the closure of the set $\{\mathfrak{V}(\psi) : \psi \in \mathbf{Sub}\varphi\}$ under the operations $\to$, $\wedge$, $\bot$ in $\mathfrak{A}$. By Diego's theorem, there are finitely many pairwise non-equivalent in **Int** disjunction free formulas with $\leq |\mathbf{Sub}\varphi|$ variables. Consequently, B is finite.

Define an operation $\vee^*$ on B by taking, for $x, y \in B$,

$$x \vee^* y = \bigwedge \{z \in B : x, y \leq z\},$$

where $\leq$ is the lattice order in $\mathfrak{A}$. (Since B is finite, $\bigwedge$ in the right-hand part always exists.) Clearly, $x \vee^* y$ is the supremum of x and y in B with respect to $\leq$, $x \wedge y$ is the infimum and so, by Theorems 7.16 and 7.10, $\mathfrak{B} = \langle B, \wedge, \vee^*, \to, \bot \rangle$ is a pseudo-Boolean algebra. In general, $x \vee y \leq x \vee^* y$ ($\mathfrak{B}$ is not necessarily a subalgebra of $\mathfrak{A}$), but if $x \vee y \in B$ then we obviously have $x \vee y = x \vee^* y$. It follows that the value of φ in $\mathfrak{B}$ under $\mathfrak{V}$ coincides with that in $\mathfrak{A}$ and so is different from $\top$. On the other hand, since $\wedge$ and $\to$ in $\mathfrak{B}$ are the restrictions of $\wedge$ and $\to$ in $\mathfrak{A}$, and $\mathfrak{A}$ validates all extra axioms of L (which are disjunction free), $\mathfrak{B}$ must also validate them. Thus, $\mathfrak{B}$ is a finite pseudo-Boolean algebra separating φ from L. Using Theorem 7.30, one can construct a finite frame refuting φ and validating L. ❑

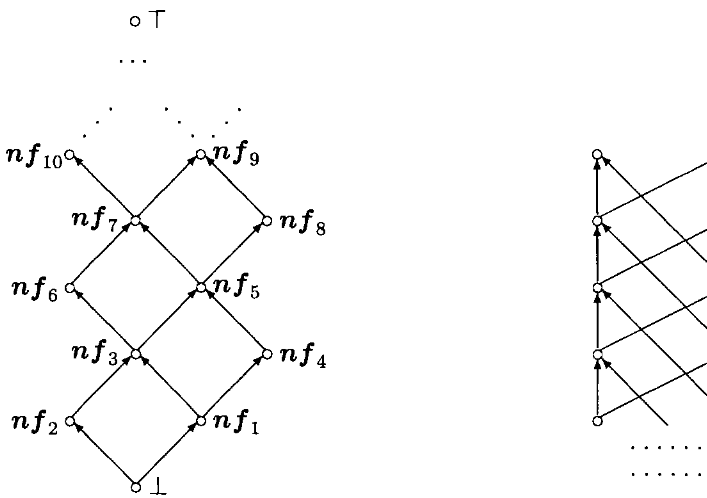
In Sections 4.1 and 4.2 we saw that the set of (normal) extensions of a logic L is a complete lattice with respect to the intersection and sum of logics. The partial order relation $\leq$ in this lattice is the set-theoretic inclusion $\subseteq$, its least element is L and greatest one is the inconsistent logic $L + \bot$. Now we introduce two more operations on Ext$L$ and NExt$L$. For every $L_1, L_2 \in$ Ext$L$, put

$$L_1 \to_1 L_2 = L + \{\varphi : \ \forall\psi \ (\psi \in L_1 \to \varphi \underline{\vee} \psi \in L_2)\},$$

$$L_1 \to_2 L_2 = L \oplus \{\varphi : \ \forall\psi \ (\psi \in L_1 \to \forall i, j \ (\Box^i \varphi \underline{\vee} \Box^j \psi \in L_2))\},$$

where $\underline{\vee}$ is the repeatless disjunction defined in Section 4.1.

Theorem 7.18 (i) *For every modal or si-logic L, $\langle \text{Ext}L, \cap, +, \to_1, L\rangle$ is a complete pseudo-Boolean algebra.*



(a) (b)

FIG. 7.2.

(ii) *For every normal modal logic L, $\langle \mathrm{NExt}L, \cap, \oplus, \rightarrow_2, L\rangle$ is a complete pseudo-Boolean algebra.*

Proof (i) It is sufficient to establish (4) in Theorem 7.10.

Suppose $L_3 \cap L_1 \subseteq L_2$ and $\varphi \in L_3$. Since $L_3 \cap L_1$ is axiomatizable by the formulas of the form $\varphi \underline{\vee} \psi$, for $\varphi \in L_3$, $\psi \in L_1$, we then have $\varphi \in L_1 \rightarrow_1 L_2$ and so $L_3 \subseteq L_1 \rightarrow_1 L_2$.

Now suppose that $L_3 \subseteq L_1 \rightarrow_1 L_2$ and $\varphi \in L_3 \cap L_1$. It follows that $\varphi \underline{\vee} \psi \in L_2$ for every $\psi \in L_1$. In particular, we have $\varphi \underline{\vee} \varphi \in L_2$ and so $\varphi \in L_2$. Therefore, $L_3 \cap L_1 \subseteq L_2$.

(ii) is proved in exactly the same way. ❑

A lattice $\mathfrak{A} = \langle A, \wedge, \vee\rangle$ is called *distributive* if the identities

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \text{ and } p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

are true in $\mathfrak{A}$. Since these identities correspond to the laws of distributivity which are in **Int**, every pseudo-Boolean algebra is a distributive lattice. As a consequence of Theorem 7.18 we obtain

Corollary 7.19 *The lattice of (normal) extensions of every modal or si-logic is distributive.*

Since the lattice operations $\wedge$ and $\vee$ as well as the implication $\rightarrow$ and the least element $\bot$ in pseudo-Boolean algebras are uniquely determined by the partial order $\leq$, we will represent lattices and pseudo-Boolean algebras in pictures as intuitionistic frames $\langle A, \leq\rangle$. For example, the lattices shown in Fig. 7.1 (a), (b), (c) are pseudo-Boolean algebras, whereas those in Fig. 7.1 (d), (e)—the so

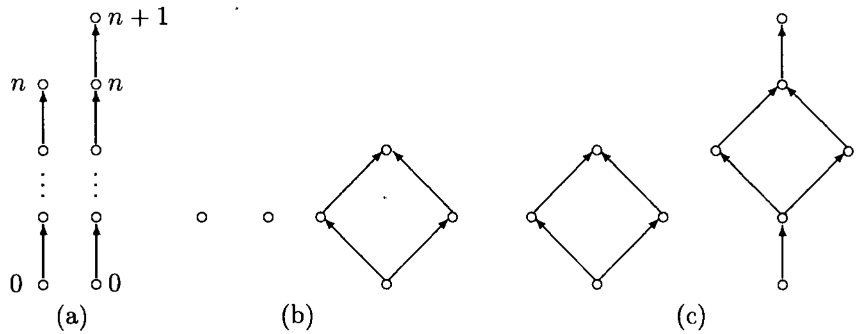


FIG. 7.3.

called *pentagon* and *diamond*—are not, because these lattices are not distributive. By the way, one can prove (see, for instance Grätzer (1978), Theorem 1 in §1, Chapter 2) that a lattice is distributive iff it contains neither the pentagon nor the diamond as its sublattice.

Another example of a lattice, this time infinite, is shown in Fig. 7.2 (a). We recommend the reader to check that this lattice is a pseudo-Boolean algebra.

Now we present an important method of constructing pseudo-Boolean algebras by associating them with intuitionistic frames.

Given an intuitionistic frame $\mathfrak{F} = \langle W, R\rangle$, define an algebra

$$\mathfrak{F}^+ = \langle \mathrm{Up}W, \cap, \cup, \supset, \emptyset\rangle ,$$

where $\mathrm{Up}W$, as before, is the set of upward closed subsets in W, $\cap$ and $\cup$ are the set-theoretic intersection and union and, for every $X, Y \in \mathrm{Up}W$,

$$X \supset Y = \{x \in W : \ \forall y\ (xRy \wedge y \in X \to y \in Y)\}$$

(compare this operations with the definition of the truth-relation in intuitionistic models in Section 2.2). Notice that a valuation in $\mathfrak{F}$ is at the same time a valuation in the algebra $\mathfrak{F}^+$.

Theorem 7.20 (i) *For every intuitionistic frame $\mathfrak{F}$, $\mathfrak{F}^+$ is a pseudo-Boolean algebra.*

(ii) *If $\mathfrak{V}$ is a valuation in $\mathfrak{F}$ (and so in $\mathfrak{F}^+$) and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ then, for every formula φ, the value of φ in $\mathfrak{F}^+$ under $\mathfrak{V}$ is $\{x : \ (\mathfrak{M}, x) \models \varphi\}$. In particular, $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}^+ \models \varphi$.*

Proof Exercise. ❑

The algebra $\mathfrak{F}^+$ defined above is called the *dual* of $\mathfrak{F}$. Fig. 7.3 and Fig. 7.4 show several examples of intuitionistic frames (on the left) and their duals (on the right). As an exercise, we invite the reader to check also that the algebra in Fig. 7.2 (a) is the dual of the frame in Fig. 7.2 (b).

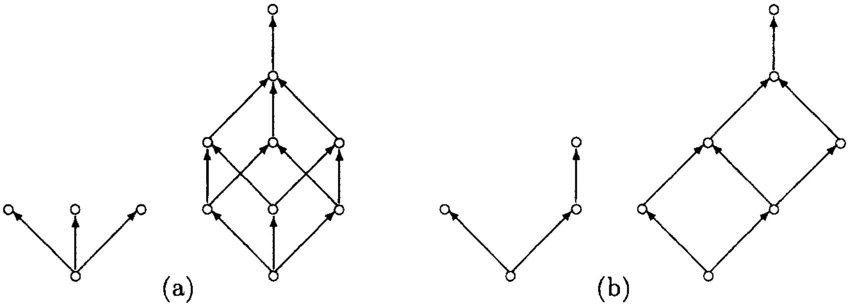


FIG. 7.4.

The completeness results of Chapter 2 together with Theorem 7.20 and the obvious fact that $|\mathfrak{F}^+| \leq 2^{|\mathfrak{F}|}$ yield us

Theorem 7.21 *The following conditions are equivalent for any formula* φ*:*

(i) $\varphi \in \mathbf{Int}$*;*

(ii) φ *is valid in every pseudo-Boolean algebra;*

(iii) φ *is valid in every finite pseudo-Boolean algebra;*

(iv) φ *is valid in every pseudo-Boolean algebra containing* $\leq 2^{2^{|\mathbf{Sub}\varphi|}}$ *elements.*

A pseudo-Boolean algebra is called a *Boolean algebra* if it validates the formula $p \vee (p \to \bot)$ or, equivalently, if the identity $p \vee (p \to \bot) = \top$ is true in it. In other words, Boolean algebras are those pseudo-Boolean algebras that validate all formulas in classical logic **Cl**.

Theorem 7.22 *The following conditions are equivalent for any formula* φ*:*

(i) $\varphi \in \mathbf{Cl}$*;*

(ii) φ *is valid in every Boolean algebra;*

(iii) φ *is valid in some non-degenerate Boolean algebra.*

Proof Exercise. (Hint: show that the two-element Boolean algebra, determined by the truth-table for **Cl**, can be embedded in every non-degenerate Boolean (and even pseudo-Boolean) algebra.) ❑

As follows from Proposition 2.38, all Kripke frames for **Cl** are of depth 1, that is are disjoint unions of single-point frames. Theorem 7.20 provides us then with the following examples of Boolean algebras: $\langle 2^W, \cap, \cup, \supset, \emptyset, \rangle$, where $\supset$ may be defined by $X \supset Y = (X \supset \emptyset) \cup Y = (W - X) \cup Y$. For finite W, these algebras can be represented as n-ary Boolean cubes shown (for $n \leq 4$) in Fig. 7.5. Recall that in Section 2.9 we used these cubes without the top elements as the Kripke frames characterizing Medvedev's logic **ML**.

7.4 Filters in pseudo-Boolean algebras

In this section we consider an algebraic analog of a set of formulas that is closed under *modus ponens*. It will be one of the main links connecting the algebraic and relational semantics.

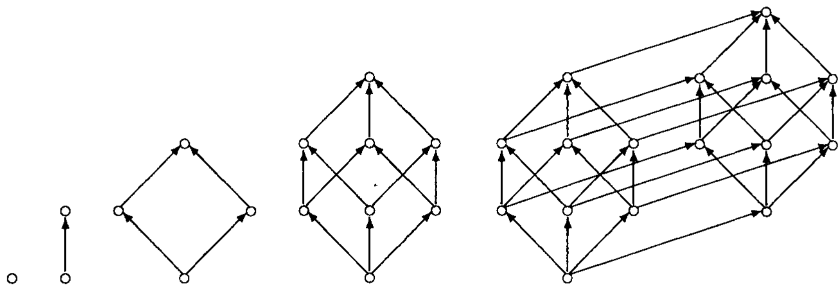


FIG. 7.5.

Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot \rangle$ be a pseudo-Boolean algebra. A set $\nabla \subseteq A$ is called a *filter* in $\mathfrak{A}$ if

- $\top \in \nabla$ and
- for every $x, y \in A$, if $x \in \nabla$ and $x \to y \in \nabla$ then $y \in \nabla$.

Trivial examples of filters in $\mathfrak{A}$ are $\{\top\}$ and A. A filter different from A is called *proper*.

Equivalent definitions of filter, which do not involve $\to$ and $\top$ and so are suitable for arbitrary lattices, are formulated in

Theorem 7.23 *Suppose $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot \rangle$ is a pseudo-Boolean algebra and $\nabla \subseteq A$. Then the following conditions are equivalent:*

(1) *∇ is a filter in $\mathfrak{A}$;*

(2) (2a) *$\nabla \neq \emptyset$ and*
(2b) *$x \in \nabla$ and $y \in \nabla$ iff $x \wedge y \in \nabla$, for every $x, y \in A$;*

(3) (3a) *$\nabla \neq \emptyset$,*
(3b) *if $x \in \nabla$, $y \in \nabla$ then $x \wedge y \in \nabla$,*
(3c) *if $x \in \nabla$, $y \in A$ then $x \vee y \in \nabla$, for every $x, y \in A$;*

(4) (4a) *$\nabla \neq \emptyset$,*
(4b) *if $x \in \nabla$, $y \in \nabla$ then $x \wedge y \in \nabla$,*
(4c) *if $x \in \nabla$, $x \leq y$ then $y \in \nabla$, for every $x, y \in A$.*

Proof We establish the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose that $x, y \in \nabla$. Since $x \to (y \to x \wedge y) = \top \in \nabla$, by the definition of filter we then have $x \wedge y \in \nabla$. The converse follows from $x \wedge y \to x = \top \in \nabla$ and $x \wedge y \to y = \top \in \nabla$.

(2) $\Rightarrow$ (3). Suppose $x \in \nabla$ and $y \in A$. By the law of absorption, we then have $x = x \wedge (x \vee y) \in \nabla$ and so, by (2b), $x \vee y \in \nabla$.

(3) $\Rightarrow$ (4). If $x \in \nabla$ and $x \leq y$ then $y = x \vee y$ and so, by (3c), $y \in \nabla$.

(4) $\Rightarrow$ (1). Let x be an element in ∇. Since $x \leq \top$, (4c) yields us $\top \in \nabla$. Suppose now that $x \in \nabla$ and $x \to y \in \nabla$. By (4b), $x \wedge (x \to y) \in \nabla$ and since

$x \wedge (x \to y) = x \wedge y$, we have $x \wedge y \in \nabla$, from which $y \in \nabla$ because $x \wedge y \leq y$. ❑

A set ∇ of elements in a lattice $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ is a *filter* if it satisfies one of the conditions (2), (3), (4) in Theorem 7.23. The reader can readily show that these conditions are equivalent in every lattice.

The condition (4) shows a way of constructing the smallest filter to contain a given non-empty set X of elements in a lattice $\mathfrak{A} = \langle A, \wedge, \vee \rangle$. That such a filter exists—call it the *filter generated by* X—follows from the evident fact that the intersection of an arbitrary family of filters containing X is again a filter containing X. Put

$$[X) = \{y \in A : \; x_1 \wedge \ldots \wedge x_n \leq y, \text{ for some } x_1, \ldots, x_n \in X\}.$$

Theorem 7.24 *For every $X \neq \emptyset$, $[X)$ is the filter generated by X in $\mathfrak{A}$.*

Proof First we show that $[X)$ satisfies (4). Indeed, clearly $[X) \neq \emptyset$. Suppose $x, y \in [X)$. Then there are $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ such that $x_1 \wedge \ldots \wedge x_n \leq x$ and $y_1 \wedge \ldots \wedge y_m \leq y$. It follows that

$$x_1 \wedge \ldots \wedge x_n \wedge y_1 \wedge \ldots \wedge y_m \leq x \wedge y$$

and so $x \wedge y \in [X)$, which proves (4b). Finally, (4c) holds because $\leq$ is transitive.

Now, by Theorem 7.23, every filter ∇ containing X contains also $[X)$. Therefore, $[X)$ is the smallest filter containing X. ❑

If a lattice $\mathfrak{A}$ has the greatest element $\top$, often called the *unit* of $\mathfrak{A}$, then we may put $[\emptyset) = \{\top\}$. If X is a singleton $\{x\}$ then instead of $[\{x\})$ we write simply $[x)$ and say that this filter is generated by x. A filter generated by a single element is called *principal*. Every filter in a finite lattice is principal, because it is generated by the conjunction of its elements.

In view of the duality between the lattice operations $\wedge$ and $\vee$ we can define a notion dual to the notion of filter. Say that a set Δ of elements in a lattice $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ is an *ideal* if one of the following conditions (2′), (3′), (4′) holds, for every $x, y \in A$:

(2′) (2′a) $\Delta \neq \emptyset$,
(2′b) $x \in \Delta$ and $y \in \Delta$ iff $x \vee y \in \Delta$;

(3′) (3′a) $\Delta \neq \emptyset$,
(3′b) if $x, y \in \Delta$ then $x \vee y \in \Delta$,
(3′c) if $x \in \Delta$ and $y \in A$ then $x \wedge y \in \Delta$;

(4′) (4′a) $\Delta \neq \emptyset$,
(4′b) if $x, y \in \Delta$ then $x \vee y \in \Delta$,
(4′c) if $y \in \Delta$ and $x \leq y$ then $x \in \Delta$.

We leave to the reader proving the fact that these conditions are equivalent. The reader can readily show also that the smallest ideal to contain a non-empty set X—the *ideal generated by* X—is the set

$$(X] = \{y \in A : \ y \leq x_1 \vee \ldots \vee x_n, \text{ for some } x_1, \ldots, x_n \in X\}.$$

If $\mathfrak{A}$ has the least element $\bot$, often called the *zero* of $\mathfrak{A}$, then we put $(\emptyset] = \bot$.

Proposition 7.25 *Suppose $\mathfrak{A}$ is a pseudo-Boolean algebra and ∇ a filter in $\mathfrak{A}$. Then the set of filters in $\mathfrak{A}$ containing ∇ forms a complete distributive lattice with the infimum and supremum defined by*

$$\bigwedge\{\nabla_i : \ i \in I\} = \bigcap_{i \in I} \nabla_i, \quad \bigvee\{\nabla_i : \ i \in I\} = [\bigcup_{i \in I} \nabla_i).$$

Proof Exercise. ❑

The lattice of filters in $\mathfrak{A}$ containing $\nabla = \{\top\}$ is called the *lattice of filters* in $\mathfrak{A}$.

Theorem 7.26 (i) *Suppose L is a normal modal (or si-) logic. Then the lattice $\langle \mathrm{NExt}L, \cap, \oplus, L\rangle$ (respectively, $\langle \mathrm{Ext}L, \cap, +, L\rangle$) is embedded in the lattice of filters in the Tarski–Lindenbaum algebra $\mathfrak{A}_L$ by the map f defined by*

$$f(L') = \{\|\varphi\|_L : \ \varphi \in L'\}.$$

The isomorphism f preserves infimums and supremums in the sense that the equalities

$$f(\bigwedge X) = \bigwedge f(X), \quad f(\bigvee X) = \bigvee f(X)$$

hold for every $X \subseteq \mathrm{NExt}L$ ($X \subseteq \mathrm{Ext}L$).

(ii) *Suppose that L is a quasi-normal modal logic and $\langle \mathfrak{A}_{L_0}, \nabla\rangle$ its Tarski–Lindenbaum matrix for some normal $L_0 \subseteq L$. Then $\langle \mathrm{Ext}L, \cap, +, L\rangle$ is embedded in the lattice of filters in $\mathfrak{A}_{L_0}$ containing ∇ by the map f defined by*

$$f(L') = \{\|\varphi\|_{L_0} : \ \varphi \in L'\}$$

and preserving infimums and supremums.

Proof There is no essential difference between the proofs of (i) and (ii). We confine ourselves to proving (ii).

That f is an injection follows from Theorem 7.4. So it suffices to establish that f preserves $\bigwedge$ and $\bigvee$. Let $X = \{L_i : \ i \in I\} \subseteq \mathrm{Ext}L$.

If $\|\varphi\|_{L_0} \in f(\bigwedge X)$ then $\varphi \in \bigcap_{i \in I} L_i$. It follows that $\|\varphi\|_{L_0} \in f(L_i)$, for every $i \in I$, and so $\|\varphi\|_{L_0} \in \bigwedge f(X) = \bigcap_{i \in I} f(L_i)$. Conversely, if $\|\varphi\|_{L_0} \in \bigwedge f(X)$ then $\|\varphi\|_{L_0} \in f(L_i)$, for every $i \in I$. So we have $\varphi \in L_i$, from which $\varphi \in \bigcap_{i \in I} L_i$ and $\|\varphi\|_{L_0} \in f(\bigcap_{i \in I} L_i) = f(\bigwedge X)$. Thus, $f(\bigwedge X) = \bigwedge f(X)$.

To establish $f(\bigvee X) = \bigvee f(X)$, suppose first that $\|\varphi\|_{L_0} \in f(\bigvee X)$, i.e., $\varphi \in \sum_{i \in I} L_i$. Since every derivation contains only finitely many formulas and by the deduction theorem, there are $J = \{i_1, \dots, i_n\} \subseteq I$ and formulas $\varphi_j \in L_{i_j}$, for $j = 1, \dots, n$, such that $\varphi_1 \wedge \dots \wedge \varphi_n \to \varphi \in L$. But then we have $\|\varphi_1\|_{L_0} \wedge \dots \wedge \|\varphi_n\|_{L_0} \to \|\varphi\|_{L_0} \in \nabla$, $\|\varphi_1\|_{L_0} \in f(L_{i_1}), \dots, \|\varphi_n\|_{L_0} \in f(L_{i_n})$ and so $\|\varphi_1\|_{L_0} \wedge \dots \wedge \|\varphi_n\|_{L_0} \in \bigvee f(X)$, from which $\|\varphi\|_{L_0} \in \bigvee f(X)$.

Now let $\|\varphi\|_{L_0} \in \bigvee f(X)$. Then there are $J = \{i_1, \dots, i_n\} \subseteq I$ and $\|\varphi_j\|_{L_0} \in f(L_{i_j})$, for $j = 1, \dots, n$, such that $\|\varphi_1\|_{L_0} \wedge \dots \wedge \|\varphi_n\|_{L_0} \leq \|\varphi\|_{L_0}$. It follows that $\varphi_1 \wedge \dots \wedge \varphi_n \to \varphi \in L_0$. Since $\varphi_j \in L_{i_j}$, for every $j = 1, \dots, n$, we then have $\varphi \in \sum_{j \in J} L_{i_j}$ and so $\|\varphi\|_{L_0} \in f(\sum_{i \in I} L_i) = f(\bigvee X)$. ❑

Our next aim is to prove the conversion of Theorem 7.20 for finite algebras. In other words, we are going to show that every finite pseudo-Boolean algebra is (isomorphic to) the dual of some intuitionistic frame.

The main role in this representation of pseudo-Boolean algebras is played by prime filters. A filter ∇ in a lattice is said to be *prime* if it is proper and $x \vee y \in \nabla$ implies $x \in \nabla$ or $y \in \nabla$. An ideal Δ is called *prime* if it is proper and with every element of the form $x \wedge y$ it contains also either x or y.

Proposition 7.27 *Suppose ∇ and Δ are disjoint sets in a lattice $\langle A, \wedge, \vee \rangle$ such that $\nabla \cup \Delta = A$. Then ∇ is a prime filter iff Δ is a prime ideal.*

Proof Exercise. ❑

Since all filters in a finite lattice are principal, we associate with every filter in such a lattice the element generating it. Say that an element a in a lattice is *prime* if $a \neq \bot$ and $a = b \vee c$ implies either $a = b$ or $a = c$.

Lemma 7.28 *A principal filter in a distributive lattice is prime iff it is generated by a prime element.*

Proof ($\Rightarrow$) follows directly from the definitions.

($\Leftarrow$) Suppose ∇ is generated by a prime element a and let $b \vee c \in \nabla$. Then $a = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, from which either $a = a \wedge b \in \nabla$ or $a = a \wedge c \in \nabla$ and so, by (2b), either $b \in \nabla$ or $c \in \nabla$. ❑

As an exercise, we recommend the reader to find all prime filters in the pseudo-Boolean algebras shown in Fig. 7.2 and 7.3.

Lemma 7.29 *If a is a prime element in a distributive lattice and $a \leq b \vee c$ then $a \leq b$ or $a \leq c$.*

Proof We have $a = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ from which $a = a \wedge b$ or $a = a \wedge c$, i.e., either $a \leq b$ or $a \leq c$. ❑

Theorem 7.30 *Every finite pseudo-Boolean algebra is isomorphic to the dual of some finite intuitionistic frame.*

Proof Suppose $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot \rangle$ is a finite pseudo-Boolean algebra and W the set of its prime elements. Define a partial order R on W by taking, for every $x, y \in W$,

$$xRy \text{ iff } y \leq x,$$

where $\leq$ is the lattice partial order in $\mathfrak{A}$, and let $\mathfrak{F} = \langle W, R \rangle$. We are going to show that $\mathfrak{A}$ is isomorphic to $\mathfrak{F}^+$.

Notice first that every $a \in A$ is represented as $\bigvee\{b \in W : \ b \leq a\}$, in particular, $\bot = \bigvee \emptyset$. Define a map f from A into UpW by taking, for every $a \in A$,

$$f(a) = \{b \in W : \ b \leq a\} \in \mathrm{Up}W$$

and show that f is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{F}^+$.

Since every element a in $\mathfrak{A}$ is completely determined by the prime elements that are $\leq a$, f is an injection. To show that f is a surjection, take any element $X \in \mathrm{Up}W$ and let $a = \bigvee\{b : \ b \in X\}$. It follows from the definition of f that $X \subseteq f(a)$. The converse inclusion is a consequence of Lemma 7.29.

Let us check now that f preserves the operations. By Corollary 7.12 and Theorem 7.20, it suffices to show that f preserves $\wedge$, $\vee$ and $\bot$.

If $c \in f(a \wedge b)$ then $c \leq a \wedge b$ and so $c \leq a$ and $c \leq b$, from which $c \in f(a)$, $c \in f(b)$ and $c \in f(a) \cap f(b)$. Conversely, if $c \in f(a) \cap f(b)$ then $c \leq a$, $c \leq b$ and hence $c \leq a \wedge b$, i.e., $c \in f(a \wedge b)$. Therefore, $f(a \wedge b) = f(a) \cap f(b)$.

Suppose now that $c \in f(a \vee b)$. Then $c \leq a \vee b$ which, by Lemma 7.29, means that either $c = c \wedge a$ or $c = c \wedge b$, in other words, either $c \leq a$ or $c \leq b$. It follows that $c \in f(a)$ or $c \in f(b)$ and so $c \in f(a) \cup f(b)$. Conversely, if $c \in f(a) \cup f(b)$ then $c \in f(a)$ or $c \in f(b)$. Suppose for definiteness that $c \in f(a)$. Then $c \in W$, $c \leq a$, hence $c \leq a \vee b$ and finally $c \in f(a \vee b)$.

That $f(\bot) = \emptyset$ follows immediately from the definition of f. ❑

The frame $\mathfrak{F}$ constructed in the proof of Theorem 7.30 is called the *dual* of $\mathfrak{A}$; it will be denoted by $\mathfrak{A}_+$.

The following notions will be used mostly for Boolean algebras. A proper filter ∇ in a lattice $\mathfrak{A}$ is called *maximal* if it is not contained in a proper filter in $\mathfrak{A}$ different from ∇. A proper filter ∇ in a pseudo-Boolean algebra $\mathfrak{A}$ is an *ultrafilter* if, for every element a in $\mathfrak{A}$, either $a \in \nabla$ or $\neg a = a \rightarrow \bot \in \nabla$.

Theorem 7.31 *For every filter ∇ in a pseudo-Boolean algebra the following conditions are equivalent:*

(i) ∇ *is a maximal filter;*

(ii) ∇ *is an ultrafilter.*

Proof If ∇ is an ultrafilter then it cannot be extended to another proper filter because for every $a \notin \nabla$, we have $\neg a \in \nabla$ and so $\bot \in [\nabla \cup \{a\})$. The implication (i) $\Rightarrow$ (ii) is a consequence of the following lemma. ❑

Lemma 7.32 *For every proper filter ∇ and every element a in a pseudo-Boolean algebra, at least one of the filters $[\nabla \cup \{a\})$ or $[\nabla \cup \{\neg a\})$ is proper.*

Proof If $[\nabla \cup \{a\})$ is not proper then $\bot \in [\nabla \cup \{a\})$ and so $c \wedge a \leq \bot$, for some $c \in \nabla$. It follows that $c \leq \neg a$, i.e., $[\nabla \cup \{\neg a\}) = \nabla$ is a proper filter. ❑

For Boolean algebras Theorem 7.31 can be generalized to

Theorem 7.33 *For every filter ∇ in a Boolean algebra $\mathfrak{A}$ the following conditions are equivalent:*
(i) ∇ *is a maximal filter;*
(ii) ∇ *is an ultrafilter;*
(iii) ∇ *is a prime filter.*

Proof By Theorem 7.31, it is sufficient to show that (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii).

To prove the former implication, suppose ∇ is a maximal filter. If ∇ is not prime then there are elements a and b in $\mathfrak{A}$ such that $a \vee b \in \nabla$, $a \notin \nabla$ and $b \notin \nabla$. By Theorem 7.31, we then have $\neg a \in \nabla$, $\neg b \in \nabla$ and hence, since $\neg p \to (\neg q \to \neg(p \vee q)) \in \mathbf{Int}$, $\neg(a \vee b) \in \nabla$, contrary to ∇ being a proper filter. The latter implication follows from the fact that in Boolean algebras $a \vee \neg a = \top$, for every element a, and so every prime filter must contain either a or $\neg a$. ❑

As a consequence of Theorems 7.30 and 7.33 we derive

Corollary 7.34 *Every finite Boolean algebra $\mathfrak{A}$ is isomorphic to an algebra of the form $\langle 2^W, \cap, \cup, \supset, \emptyset\rangle$ where $X \supset Y = (W - X) \cup Y$, for every $X, Y \subseteq W$.*

Proof Suppose $\mathfrak{A}_+ = \langle W, R\rangle$, i.e., $\mathfrak{A}$ is isomorphic to $\langle \mathrm{Up}W, \cap, \cup, \supset, \emptyset\rangle$ where $X \supset Y = \{x \in W : \forall y(xRy \wedge y \in X \to y \in Y\}$. We show that the frame $\mathfrak{A}_+$ is of depth 1. Indeed, if xRy, for some $x, y \in W$, then, by the construction of $\mathfrak{A}_+$, $[x) \subseteq [y)$. And since the filters $[x)$ and $[y)$ are prime, they are maximal and so $[x) = [y)$, i.e., $x = y$. Therefore, $\mathrm{Up}W = 2^W$ and $X \supset Y = \{x \in W : x \in X \to x \in Y\} = (W - X) \cup Y$. ❑

It is not difficult to characterize principal ultrafilters in pseudo-Boolean algebras. Say that an element $a \neq \bot$ in such an algebra $\mathfrak{A}$ is an *atom* if, for every x in $\mathfrak{A}$, $x \leq a$ implies $x = \bot$ or $x = a$; in other words, a is a minimal element among those different from the zero.

Theorem 7.35 (i) *An element in a Boolean algebra is prime iff it is an atom.*

(ii) *A principal filter in a pseudo-Boolean algebra is an ultrafilter iff it is generated by an atom.*

Proof Exercise. ❑

However, infinite Boolean algebras contain non-principal ultrafilters.

Example 7.36 Let $\mathfrak{F} = \langle W, =\rangle$ be an infinite frame (of depth 1). Then the set $\nabla \subseteq 2^W$ containing all cofinite subsets of W is clearly a proper filter in the Boolean algebra $\mathfrak{F}^+$. It is non-principal, because the intersection of sets in ∇ is empty, and moreover, according to Theorem 7.35, it cannot be extended to a principal ultrafilter (for otherwise the principal ultrafilter containing ∇ is generated by a point $x \in W$, whereas $W - \{x\} \in \nabla$). On the other hand, as will be shown below, every proper filter is contained in an ultrafilter.

To this end we require the well known

Lemma 7.37. **(Zorn's lemma)** *If the points of every chain in a partial order $\mathfrak{F}$ have a common successor then every point in $\mathfrak{F}$ sees a final point.*

We can apply Zorn's lemma to the partially ordered (by $\subseteq$) set of filters or ideals in an arbitrary lattice. For we clearly have

Lemma 7.38 *The union of any chain of proper filters (or ideals) in a lattice with zero (respectively, unit) element is again a proper filter (ideal).*

Putting these two lemmas together, we obtain

Theorem 7.39 *Every proper filter (ideal) in a lattice with zero (unit) element can be extended to a maximal filter (ideal). In particular, every proper filter in a pseudo-Boolean algebra is contained in an ultrafilter.*

Corollary 7.40 *Every proper filter in a Boolean algebra is the intersection of all ultrafilters containing it.*

Proof Let ∇ be a proper filter in a Boolean algebra $\mathfrak{A}$ and $a \notin \nabla$. Then the filter $[\nabla \cup \{\neg a\})$ is also proper, for otherwise there is $b \in \nabla$ such that $b \wedge \neg a \leq \bot$ and so $b \leq \neg\neg a$, which is a contradiction because $\neg\neg a = a$ and $a \notin \nabla$. By Theorem 7.39, $[\nabla \cup \{\neg a\})$ can be extended to an ultrafilter ∇_a. Therefore, $\nabla = \bigcap_{a \notin \nabla} \nabla_a$. ❑

In pseudo-Boolean algebras every maximal filter is prime, but not the converse (see Fig. 7.3). The following useful result on the existence of prime filters plays in the algebraic semantics the same role as Lindenbaum's lemma plays in the Kripke semantics.

Theorem 7.41 *Suppose ∇ (Δ) is a filter (ideal) in a distributive lattice $\mathfrak{A}$ and $a \notin \nabla$ $(a \notin \Delta)$. Then there is a prime filter ∇' (prime ideal Δ') in $\mathfrak{A}$ such that $\nabla \subseteq \nabla'$ and $a \notin \nabla'$ (respectively, $\Delta \subseteq \Delta'$ and $a \notin \Delta'$).*

Proof By Zorn's lemma and Lemma 7.38, there exists a maximal filter ∇' in $\mathfrak{A}$ which contains ∇ and does not contain a. We shall show that ∇' is prime.

Suppose otherwise. Then there are elements c and d in $\mathfrak{A}$ such that $c \vee d \in \nabla'$, $c \notin \nabla'$ and $d \notin \nabla'$. Let $\nabla_c = [\nabla' \cup \{c\})$, $\nabla_d = [\nabla' \cup \{d\})$. Since ∇_c and ∇_d are different from ∇', we then have $a \in \nabla_c \cap \nabla_d$ and so there are elements $b_1, b_2 \in \nabla'$ such that $b_1 \wedge c \leq a$ and $b_2 \wedge d \leq a$. It follows that $b_1 \wedge b_2 \wedge c \leq a$, $b_1 \wedge b_2 \wedge d \leq a$ and so, by Lemma 7.11 (17), $(b_1 \wedge b_2 \wedge c) \vee (b_1 \wedge b_2 \wedge d) \leq a$. Using distributivity we then obtain $(b_1 \wedge b_2) \wedge (c \vee d) \leq a$. And since $b_1 \wedge b_2 \in \nabla'$, $c \vee d \in \nabla'$, we have $(b_1 \wedge b_2) \wedge (c \vee d) \in \nabla'$, whence $a \in \nabla'$, which is a contradiction.

By duality we obtain the proof for ideals. ❑

Corollary 7.42 *Suppose that a and b are elements in a distributive lattice such that $b \not\leq a$. Then there exists a prime filter ∇' (prime ideal Δ') such that $a \notin \nabla'$ and $b \in \nabla'$ (respectively, $a \in \Delta'$ and $b \notin \Delta'$).*

Proof It is sufficient to take $\nabla = [b)$ and use Theorem 7.41. ❑

7.5 Modal algebras and matrices

In this section we consider algebras and matrices corresponding to normal and quasi-normal modal logics.

An algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot, \Box \rangle$ is called a *modal algebra* if the identity $\varphi = \psi$ is true in $\mathfrak{A}$ for every modal formulas φ and ψ such that $\varphi \leftrightarrow \psi \in \mathbf{K}$. If we replace in this definition $\mathbf{K}$ with a normal modal logic L then $\mathfrak{A}$ is called an *algebra for* L or an L*-algebra*; in particular, all modal algebras are $\mathbf{K}$-algebras. Some modal algebras have their specific names: for instance, $\mathbf{K4}$-algebras are sometimes called *transitive algebras*, $\mathbf{S4}$-algebras *topological Boolean algebras*, $\mathbf{Grz}$-algebras *Grzegorczyk algebras*, $\mathbf{GL}$-algebras *diagonalizable* or *Magarian algebras.*

By Theorem 7.2, the Tarski–Lindenbaum algebra for every normal modal logic L is an L-algebra characterizing L, and so we have

Theorem 7.43 *For each normal modal logic L and each formula φ, $\varphi \in L$ iff φ is valid in every modal algebra for L.*

Theorem 7.44 *An algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot, \Box \rangle$ is modal iff it satisfies the following conditions:*

(i) $\langle A, \wedge, \vee, \rightarrow, \bot \rangle$ *is a Boolean algebra;*

(ii) *for every $x, y \in A$, $\Box(x \wedge y) = \Box x \wedge \Box y$;*

(iii) $\Box\top = \top$.

Proof The implication ($\Rightarrow$) follows from $\mathbf{Cl} \subset \mathbf{K}$, $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q \in \mathbf{K}$ and $\Box\top \leftrightarrow \top \in \mathbf{K}$.

($\Leftarrow$) As in the proof of Theorem 7.10, it suffices to show that $\mathfrak{A} \models \varphi$ for every $\varphi \in \mathbf{K}$, which can be done by induction on the length of a derivation of φ in K. The induction step is clear—Subst and MP were considered in the proof of Theorem 7.10 and the implication $\mathfrak{A} \models \varphi \Rightarrow \mathfrak{A} \models \Box\varphi$ follows from (iii).

So it remains to justify the basis of induction. The axioms of Cl are valid in $\mathfrak{A}$ because it is a Boolean algebra. As to the modal axiom of K, for every $x, y \in A$ we have $(x \rightarrow y) \wedge x \wedge y = (x \rightarrow y) \wedge x$ (since $\mathfrak{A}$ is a Boolean algebra), whence $\Box((x \rightarrow y) \wedge x \wedge y) = \Box((x \rightarrow y) \wedge x)$ and, by (ii), $\Box(x \rightarrow y) \wedge \Box x \wedge \Box y = \Box(x \rightarrow y) \wedge \Box x$. Therefore, $\Box(x \rightarrow y) \wedge \Box x \leq \Box y$, from which we obtain $\Box(x \rightarrow y) \leq \Box x \rightarrow \Box y$ and finally $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) = \top$. ❑

Corollary 7.45 *Suppose $L = \mathbf{K} \oplus \{\varphi_i : i \in I\}$. Then an $\mathcal{ML}$-algebra $\mathfrak{A}$ is an L-algebra iff it satisfies (i)–(iii) in Theorem 7.44 and*

(iv) $\mathfrak{A} \models \varphi_i$, *for every $i \in I$.*

The following construction, connecting modal algebras and frames, provides us with multiple examples of concrete modal algebras.

Given a modal frame $\mathfrak{F} = \langle W, R \rangle$, we define an algebra

$$\mathfrak{F}^+ = \left\langle 2^W, \cap, \cup, \supset, \emptyset, \Box \right\rangle,$$

called the *dual* of $\mathfrak{F}$, by taking, for every $X, Y \subseteq W$,

$$X \supset Y = (W - X) \cup Y,$$

$$\Box X = \{x \in W : \forall y \ (xRy \to y \in X)\}$$

(compare these operations with the definition of the truth-relation in modal models in Section 3.2).

Theorem 7.46 (i) *For every modal frame $\mathfrak{F}$, its dual $\mathfrak{F}^+$ is a modal algebra;*

(ii) *If $\mathfrak{V}$ is a valuation in $\mathfrak{F}$ (and so in $\mathfrak{F}^+$) and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ then, for every formula φ, the value of φ in $\mathfrak{F}^+$ under $\mathfrak{V}$ is $\{x : (\mathfrak{M}, x) \models \varphi\}$. In particular, $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}^+ \models \varphi$.*

Proof Exercise. ❑

This result is a modal counterpart of Theorem 7.20, while Theorem 7.30 is analogous to

Theorem 7.47 *Every finite modal algebra is isomorphic to the dual of some finite modal frame.*

Proof Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box \rangle$ be a finite modal algebra. Since the algebra $\langle A, \wedge, \vee, \to, \bot \rangle$ is Boolean, by Corollary 7.34 it is isomorphic to the algebra $\langle 2^W, \cap, \cup, \supset, \emptyset \rangle$, where W is the set of atoms in $\mathfrak{A}$, an isomorphism being the map f defined by $f(a) = \{b \in W : b \leq a\}$.

Define a binary relation R on W by taking, for every $x, y \in W$,

$$xRy \text{ iff } \forall z \in A \ (x \leq \Box z \to y \leq z)$$

and let $\mathfrak{F} = \langle W, R \rangle$. We prove that f is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{F}^+$. It should be clear from the considerations above that it suffices to show only that f preserves $\Box$.

Suppose $x \in f(\Box a)$. Then $x \in W$ and $x \leq \Box a$. By the definition of $\Box$ in $\mathfrak{F}^+$, we need to show that $\forall y \ (xRy \to y \in f(a))$. So suppose xRy. Then by the definition of R, $y \leq a$ and so $y \in f(a)$.

Conversely, assume $x \in \Box f(a)$ and show that $x \leq \Box a$. The element $\Box v = \bigwedge\{\Box u : x \leq \Box u\} = \Box \bigwedge\{u : x \leq \Box u\}$ is clearly the least "boxed" element in the set $\{\Box u : x \leq \Box u\}$ and so we have $x \leq \Box u$ iff $\Box v \leq \Box u$. By the definitions of R and f, the condition $x \in \Box f(a)$ means that

$$\forall y \in W \ (\forall z \in A \ (x \leq \Box z \to y \leq z) \to y \leq a),$$

which, according to our choice of v, is equivalent to

$$\forall y \in W \ (y \leq v \to y \leq a). \tag{7.1}$$

It follows that $v \leq a$. Indeed, if $v \not\leq a$ then, by Corollary 7.42, $v \wedge \neg a$ belongs to an ultrafilter generated by some $y_0 \in W$ such that $y_0 \leq v$ and $y_0 \not\leq a$, which in view of (7.1) is a contradiction. From $v \leq a$ we obtain $\Box v \leq \Box a$ and so $x \leq \Box a$, i.e., $x \in f(\Box a)$. ❑

The frame $\mathfrak{F}$ defined in the proof above is called the *dual* of the algebra $\mathfrak{A}$ and denoted by $\mathfrak{A}_+$.

Let us now turn to modal logics that are not necessarily closed under the rule of necessitation. If $\mathfrak{A}$ is a modal algebra and ∇ a filter in $\mathfrak{A}$ then the pair $\langle \mathfrak{A}, \nabla \rangle$ is called a *modal matrix*. We say that $\langle \mathfrak{A}, \nabla \rangle$ is a *matrix for* a quasi-normal logic L or simply an *L-matrix* if $\langle \mathfrak{A}, \nabla \rangle \models L$. Since the Tarski–Lindenbaum matrix for L, as defined in Theorem 7.4, characterizes L, we have the following:

Theorem 7.48 *Suppose L is a quasi-normal modal logic and φ a modal formula. Then $\varphi \in L$ iff φ is valid in every modal matrix for L.*

Given a modal frame $\mathfrak{F} = \langle W, R \rangle$ with a set D of distinguished points, define the *dual* $\langle \mathfrak{F}, D \rangle^+$ of $\langle \mathfrak{F}, D \rangle$ as the matrix $\langle \mathfrak{F}^+, D^+ \rangle$ in which

$$D^+ = \{X \subseteq W : \ D \subseteq X\}.$$

Theorem 7.49 (i) *If $\langle \mathfrak{F}, D \rangle$ is a modal frame with distinguished points then $\langle \mathfrak{F}, D \rangle^+$ is a modal matrix.*

(ii) *For every formula φ, $\langle \mathfrak{F}, D \rangle \models \varphi$ iff $\langle \mathfrak{F}, D \rangle^+ \models \varphi$.*

Proof Exercise. ❑

Theorem 7.50 *Every finite modal matrix is isomorphic to the dual of some finite modal frame with distinguished points.*

Proof Let $\langle \mathfrak{A}, \nabla \rangle$ be a modal matrix, f the isomorphism of $\mathfrak{A}$ onto the dual of $\mathfrak{A}_+ = \langle W, R \rangle$ defined in the proof of Theorem 7.47. Suppose also that the filter ∇ is generated by an element a in $\mathfrak{A}$. As a set of distinguished points in $\mathfrak{A}_+$ we take

$$D = f(a) = \{x \in W : \ x \le a\}$$

and show that $x \in \nabla$ iff $f(x) \in D^+$, for every element x in $\mathfrak{A}$.

If $x \in \nabla$, i.e., $a \le x$ then $f(a) \subseteq f(x)$ and so $D \subseteq f(x)$ or, equivalently, $f(x) \in D^+$. Conversely, if $f(x) \in D^+$ then $D = f(a) \subseteq f(x)$, from which $a = \bigvee f(a) \le \bigvee f(x) = x$ and so $x \in [a) = \nabla$. ❑

7.6 Varieties of algebras and matrices

We defined pseudo-Boolean, Boolean and modal algebras as algebras validating some (infinite) collections of identities. In general, the class of all algebras (of the same similarity type), in which all identities in a given set Γ are valid, is called a *variety of algebras* (of this type) and denoted by VarΓ. If Γ is a set of $\mathcal{L}$- or $\mathcal{ML}$-formulas then by VarΓ we mean the variety of $\mathcal{L}$- or, respectively, $\mathcal{ML}$-algebras generated by the identities $\varphi = \top$ such that $\varphi \in \Gamma$.

Conversely, given a class $\mathcal{C}$ of (pseudo-Boolean or modal) algebras, it is natural to consider the set Log$\mathcal{C}$ of formulas that are validated by every algebra in $\mathcal{C}$. The abbreviation Log here is not accidental. For it is quite easy to see that the following is true:

Theorem 7.51 *If C is a non-empty class of pseudo-Boolean or modal algebras, then* Log$C$ *is a superintuitionistic or, respectively, normal modal logic.*

As a consequence of Theorems 7.8 and 7.43 we obtain

Theorem 7.52 *Suppose $\mathcal{V}$ is a variety of pseudo-Boolean or modal algebras and L a superintuitionistic or, respectively, normal modal logic. Then*

$$\mathrm{VarLog}\mathcal{V} = \mathcal{V}, \quad \mathrm{LogVar}L = L.$$

The variety Var$L$ is called the *characteristic variety* for the logic L. Theorem 7.52 states in essence that the relation "logic $\leftrightarrow$ its characteristic variety" is a 1–1 correspondence between the classes of superintuitionistic and normal modal logics and the classes of varieties of pseudo-Boolean and modal algebras, respectively. In fact this correspondence catches much subtler properties of the classes under consideration.

In the classes of varieties of pseudo-Boolean and modal algebras we define lattice operations $\wedge$ and $\vee$ by taking, for any varieties $\mathcal{V}_1$ and $\mathcal{V}_2$,

$$\mathcal{V}_1 \wedge \mathcal{V}_2 = \mathcal{V}_1 \cap \mathcal{V}_2 \text{ and } \mathcal{V}_1 \vee \mathcal{V}_2 = \mathrm{VarLog}(\mathcal{V}_1 \cup \mathcal{V}_2).$$

Theorem 7.53 *Suppose L is a normal modal or si-logic. Then the class of all varieties of L-algebras is a complete lattice with respect to the operations $\wedge$ and $\vee$ defined above.*

Proof Exercise (for details see the proof of Theorem 7.56 below). ❑

Suppose $\mathfrak{A} = \langle A, \wedge, \vee\rangle$ and $\mathfrak{B} = \langle B, \wedge, \vee\rangle$ are lattices. A bijection (i.e., simultaneously injective and surjective map) f from A onto B is said to be a *dual isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$ if it dually preserves the lattice operations in the following sense: for every $x, y \in A$,

$$f(x \wedge y) = f(x) \vee f(y), \quad f(x \vee y) = f(x) \wedge f(y)$$

(or equivalently, if f dually preserves the lattice partial order, i.e., $x \leq y$ iff $f(x) \geq f(y)$). Lattices $\mathfrak{A}$ and $\mathfrak{B}$ are *dually isomorphic* if there is a dual isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. It should be clear that dually isomorphic lattices are complete or incomplete simultaneously and that a dual isomorphism f of a complete lattice $\mathfrak{A}$ onto a lattice $\mathfrak{B}$ dually preserves infimums and supremums, i.e., for every $X \subseteq A$,

$$f(\bigwedge X) = \bigvee f(X), \quad f(\bigvee X) = \bigwedge f(X).$$

Theorem 7.54 *Suppose L is a normal modal or si-logic. Then the map "logic $\rightarrow$ its characteristic variety" is a dual isomorphism of the lattice* NExt$L$ *or, respectively,* Ext$L$ *onto the lattice of all varieties of L-algebras.*

Proof Exercise (for details see the proof of Theorem 7.56 below). ❑

Using this theorem, problems concerning normal modal and si-logics may be reformulated in terms of varieties of corresponding algebras, which is sometimes very helpful because we can take advantage of the developed apparatus of universal algebra.

Now we extend the notion of variety from algebras to modal matrices. Since it is always clear from the context whether we deal with algebras or matrices, we will use for varieties of matrices the same notations as for varieties of algebras.

A *variety of modal matrices* is the class of all modal matrices validating the formulas in a given set Γ; as before it is denoted by $\mathrm{Var}\Gamma$. The set of formulas validated by all matrices in a class $\mathcal{C}$ is also denoted by $\mathrm{Log}\mathcal{C}$. Instead of $\mathrm{VarLog}\mathcal{C}$ we will write $\mathrm{Var}\mathcal{C}$ and say that the variety $\mathrm{Var}\mathcal{C}$ is *generated* by the class of matrices $\mathcal{C}$. The same concerns varieties of algebras as well.

The following theorems are matrix counterparts of Theorems 7.51–7.54.

Theorem 7.55 (i) *If $\mathcal{C}$ is a class of modal matrices then* $\mathrm{Log}\mathcal{C}$ *is a quasi-normal modal logic.*

(ii) *If $\mathcal{V}$ is a variety of modal matrices and L a quasi-normal logic then*

$$\mathrm{VarLog}\mathcal{V} = \mathcal{V}, \quad \mathrm{LogVar}L = L.$$

The variety $\mathrm{Var}L$ of matrices is called the *characteristic variety of matrices* for the quasi-normal logic L. The lattice operations $\wedge$ and $\vee$ on varieties of matrices are defined in exactly the same way as on varieties of algebras.

Theorem 7.56 *Suppose L is a quasi-normal modal logic. Then*

(i) *the class of varieties of modal matrices for L is a complete lattice with respect to $\wedge$ and $\vee$;*

(ii) *the map "quasi-normal logic $\rightarrow$ its characteristic variety" is a dual isomorphism of the lattice* $\mathrm{Ext}L$ *onto the lattice of varieties of L-matrices.*

Proof Since $\langle \mathrm{Ext}L, \cap, + \rangle$ is a complete lattice, it is sufficient to show that the map f defined by $f(L') = \mathrm{Var}L'$, for $L' \in \mathrm{Ext}L$, is a dual isomorphism.

That f is a bijection follows from Theorem 7.55. Let us prove that it preserves the lattice operations, i.e., for $L_1, L_2 \in \mathrm{Ext}L$,

$$\mathrm{Var}(L_1 \cap L_2) = \mathrm{Var}L_1 \vee \mathrm{Var}L_2 = \mathrm{Var}(\mathrm{Var}L_1 \cup \mathrm{Var}L_2),$$

$$\mathrm{Var}(L_1 + L_2) = \mathrm{Var}L_1 \wedge \mathrm{Var}L_2 = \mathrm{Var}L_1 \cap \mathrm{Var}L_2.$$

Suppose first that $\langle \mathfrak{A}, \nabla \rangle \in \mathrm{Var}(L_1 \cap L_2)$, i.e., $\langle \mathfrak{A}, \nabla \rangle \models L_1 \cap L_2$, but $\langle \mathfrak{A}, \nabla \rangle \notin \mathrm{Var}L_1 \vee \mathrm{Var}L_2$. The latter assumption means that there is a formula $\varphi \in L_1 \cap L_2$ such that $\langle \mathfrak{A}, \nabla \rangle \not\models \varphi$, from which $\langle \mathfrak{A}, \nabla \rangle \not\models L_1 \cap L_2$, contrary to the former assumption. Therefore, $\mathrm{Var}(L_1 \cap L_2) \subseteq \mathrm{Var}L_1 \vee \mathrm{Var}L_2$. To establish the converse inclusion, suppose $\langle \mathfrak{A}, \nabla \rangle \in \mathrm{Var}L_1 \vee \mathrm{Var}L_2$. This means that $\langle \mathfrak{A}, \nabla \rangle \models \varphi$, for every $\varphi \in L_1 \cap L_2$, and so $\langle \mathfrak{A}, \nabla \rangle \in \mathrm{Var}(L_1 \cap L_2)$.

Suppose now that $\langle \mathfrak{A}, \nabla \rangle \in \mathrm{Var}(L_1 + L_2)$ but $\langle \mathfrak{A}, \nabla \rangle \notin \mathrm{Var}L_1 \cap \mathrm{Var}L_2$. Then there is a formula $\varphi_i \in L_i$, for some $i \in \{1, 2\}$, such that $\langle \mathfrak{A}, \nabla \rangle \not\models \varphi_i$. However,

$L_i \subseteq L_1 + L_2$ and so we must have $\langle \mathfrak{A}, \nabla \rangle \models \varphi_i$, which is a contradiction. Thus, $\mathrm{Var}(L_1 + L_2) \subseteq \mathrm{Var}L_1 \wedge \mathrm{Var}L_2$.

Finally, let $\langle \mathfrak{A}, \nabla \rangle \in \mathrm{Var}L_1 \wedge \mathrm{Var}L_2$. To establish $\langle \mathfrak{A}, \nabla \rangle \in \mathrm{Var}(L_1 + L_2)$ we must show that $\langle \mathfrak{A}, \nabla \rangle \models \varphi$ for every formula $\varphi \in L_1 + L_2$. So assume $\varphi \in L_1 + L_2$. Then there are formulas $\varphi_1, \dots, \varphi_n \in L_1 \cup L_2$ such that $\varphi_1 \wedge \dots \wedge \varphi_n \to \varphi \in L$. Since $\langle \mathfrak{A}, \nabla \rangle$ validates all formulas in L_1 and L_2, we have $\langle \mathfrak{A}, \nabla \rangle \models \varphi_1 \wedge \dots \wedge \varphi_n$. And since $\langle \mathfrak{A}, \nabla \rangle$ is an L-matrix, we obtain $\langle \mathfrak{A}, \nabla \rangle \models \varphi_1 \wedge \dots \wedge \varphi_n \to \varphi$ and hence $\langle \mathfrak{A}, \nabla \rangle \models \varphi$. ❑

7.7 Operations on algebras and matrices

In this section we consider three fundamental algebra and matrix constructing operations, namely, forming subalgebras and submatrices, direct products of algebras and matrices and their homomorphic images. If we regard pseudo-Boolean and modal algebras (at least some of them) as duals of Kripke frames then it turns out that these operations are algebraic analogues of the frame-theoretic operations of forming reducts, disjoint unions and generated subframes. In full detail this correspondence will be studied in the next chapter as a part of general duality theory. Here it is our main source of examples.

Suppose $\mathfrak{A} = \langle A, o_1, \dots, o_n \rangle$ is an algebra and B a subset of A closed under the operations in $\mathfrak{A}$ (i.e., the result of applying any o_i to elements in B is again an element in B; in particular, if $\mathfrak{A}$ is a pseudo-Boolean algebra then $\bot \in B$). In this case the algebra $\mathfrak{B} = \langle B, o_1, \dots, o_n \rangle$, whose operations are the restrictions of $\mathfrak{A}$'s operations to B, is said to be a *subalgebra* of $\mathfrak{A}$. For $X \subseteq A$, the subalgebra of $\mathfrak{A}$ with the smallest universe containing X is called the *subalgebra generated by* X. (That such a subalgebra exists follows from the obvious fact that the intersection of subalgebras is again a subalgebra.) In particular, if this subalgebra is $\mathfrak{A}$ itself then we say that $\mathfrak{A}$ is *generated by* X. In the case when the algebra $\mathfrak{A}$ is generated by $\emptyset$ it is called *0-generated* (if it is a pseudo-Boolean algebra then A is the closure of $\{\bot\}$ under $\wedge$, $\vee$ and $\to$). More generally, the subalgebra of $\mathfrak{A}$ (in particular, $\mathfrak{A}$ itself) generated by a set of cardinality $\varkappa$ is said to be *$\varkappa$-generated.*

Example 7.57 Let $\mathfrak{F}^+ = \langle 2^W, \cap, \cup, \supset, \emptyset, \Box \rangle$ be the dual of a modal frame $\mathfrak{F}$ and $\mathfrak{V}$ a valuation in $\mathfrak{F}$. Then the set $P \subseteq 2^W$ defined by

$$P = \{\mathfrak{V}(\varphi) : \ \varphi \in \mathbf{For}\mathcal{ML}\}$$

is closed under the operations in $\mathfrak{F}^+$ because, for all formulas φ and ψ, $\emptyset = \mathfrak{V}(\bot)$, $\mathfrak{V}(\varphi) \cap \mathfrak{V}(\psi) = \mathfrak{V}(\varphi \wedge \psi)$, $\mathfrak{V}(\varphi) \cup \mathfrak{V}(\psi) = \mathfrak{V}(\varphi \vee \psi)$, $\mathfrak{V}(\varphi) \supset \mathfrak{V}(\psi) = \mathfrak{V}(\varphi \to \psi)$, $\Box\mathfrak{V}(\varphi) = \mathfrak{V}(\Box\varphi)$. The same, of course, concerns intuitionistic frames. Thus, every valuation in a frame determines a subalgebra of its dual.

Example 7.58 Suppose f is a reduction of a modal frame $\mathfrak{F} = \langle W, R \rangle$ to a frame $\mathfrak{G} = \langle V, S \rangle$. Then it is not hard to check (for details see Section 8.5) that the set $A = \{f^{-1}(X) : \ X \subseteq V\}$ is closed under the operations in $\mathfrak{F}^+$. Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{F}^+$ with the universe A. The map g from 2^V into 2^W defined by $g(X) = f^{-1}(X)$ is an embedding of $\mathfrak{G}^+$ in $\mathfrak{F}^+$ and an isomorphism of $\mathfrak{G}^+$ onto $\mathfrak{A}$.

As to submatrices, we say that a modal matrix $\langle \mathfrak{A}_2, \nabla_2 \rangle$ is a *submatrix* of a modal matrix $\langle \mathfrak{A}_1, \nabla_1 \rangle$ if $\mathfrak{A}_2$ is a subalgebra of $\mathfrak{A}_1$ and ∇_2 is the intersection of ∇_1 with $\mathfrak{A}_2$'s universe (see Exercise 7.11).

Let $\mathcal{C}$ be a class of algebras (or matrices). Denote by $\mathsf{S}\mathcal{C}$ the class of all subalgebras (respectively, submatrices) of algebras (matrices) in $\mathcal{C}$. The following proposition is a direct consequence of the definitions above:

Proposition 7.59 (i) *If a formula φ is valid in an algebra $\mathfrak{A}$ (matrix $\langle \mathfrak{A}, \nabla \rangle$) then φ is valid in every subalgebra (submatrix) of $\mathfrak{A}$ (respectively, $\langle \mathfrak{A}, \nabla \rangle$).*

(ii) $\mathsf{S}\mathcal{C} \subseteq \mathrm{Var}\mathcal{C}$ *for every class $\mathcal{C}$ of algebras or matrices.*

Suppose $\mathfrak{A} = \langle A, o_1, \dots, o_n \rangle$ and $\mathfrak{B} = \langle B, o_1, \dots, o_n \rangle$ are similar algebras. The *direct product* of $\mathfrak{A}$ and $\mathfrak{B}$ is the algebra

$$\mathfrak{A} \times \mathfrak{B} = \langle A \times B, o_1, \dots, o_n \rangle$$

in which the operations $o_1, \dots, o_n$ are defined component-wise, i.e., if o_k is an m-ary operation, $a_1, \dots, a_m \in A$ and $b_1, \dots, b_m \in B$ then

$$o_k(\langle a_1, b_1 \rangle, \dots, \langle a_m, b_m \rangle) = \langle o_k(a_1, \dots, a_m), o_k(b_1, \dots, b_m) \rangle .$$

In particular, if $\mathfrak{A}$ and $\mathfrak{B}$ are pseudo-Boolean algebras then the zero element in $\mathfrak{A} \times \mathfrak{B}$ is $\bot = \langle \bot, \bot \rangle$. It should be clear from the definition that, for every pseudo-Boolean or modal algebras $\mathfrak{A}$ and $\mathfrak{B}$ and every formula φ, $\mathfrak{A} \times \mathfrak{B} \models \varphi$ iff $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \varphi$.

In the same way one can define the direct product $\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n$ of similar algebras $\mathfrak{A}_1, \dots, \mathfrak{A}_n$. More generally, by observing that any n-tuple $\langle x_1, \dots, x_n \rangle \in X_1 \times \ldots \times X_n$ may be interpreted as the function $f : \{1, \dots, n\} \to \bigcup_{i=1}^{n} X_i$ such that $f(i) = x_i \in X_i$, for every $i = 1, \dots, n$, we can extend the definition of direct product to arbitrary families of algebras.

Given a family $\{\mathfrak{A}_i = \langle A_i, o_1, \dots, o_n \rangle : i \in I\}$ of algebras, the *direct product* of $\{\mathfrak{A}_i : i \in I\}$ is the algebra

$$\prod_{i \in I} \mathfrak{A}_i = \left\langle \prod_{i \in I} A_i, o_1, \dots, o_n \right\rangle$$

in which $\prod_{i \in I} A_i$ is the set of all functions f from I into $\bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ and $o_k(f_1, \dots, f_m)$ is a function $g \in \prod_{i \in I} A_i$ defined by

$$g(i) = o_k(f_1(i), \dots, f_m(i)) \in A_i,$$

for every $f_1, \dots, f_m \in \prod_{i \in I} A_i$ and every $i \in I$.

Example 7.60 Let $\{\mathfrak{F}_i : i \in I\}$ be a family of modal (or intuitionistic) frames. We invite the reader to show that $(\sum_{i \in I} \mathfrak{F}_i)^+$ is isomorphic to $\prod_{i \in I} \mathfrak{F}_i^+$ (for details see Section 8.5).

The *direct product* of a family of modal matrices $\{\langle \mathfrak{A}_i, \nabla_i\rangle : i \in I\}$ is the matrix $\prod_{i\in I}\langle \mathfrak{A}_i, \nabla_i\rangle = \langle \mathfrak{A}, \nabla\rangle$ in which $\mathfrak{A} = \prod_{i\in I}\mathfrak{A}_i$ and $\nabla = \prod_{i\in I}\nabla_i$ (the reader should check that ∇ is a filter in $\mathfrak{A}$; see also Exercise 7.12).

For a class C of algebras (or matrices), denote by PC the class of all possible direct products of C's subclasses. Then we clearly have

Proposition 7.61 (i) *If a formula φ is valid in every algebra (or matrix) in some family C then φ is valid in the direct product of C.*

(ii) P$C \subseteq$ Var$C$ *for every class C of algebras or matrices.*

The third operation we need—the formation of homomorphic images—was introduced in Section 7.1. If f is a homomorphism of $\mathfrak{A} = \langle A, o_1, \dots, o_n\rangle$ in $\mathfrak{B} = \langle B, o_1, \dots, o_n\rangle$ then the set $f(A)$ is clearly closed under the operations in $\mathfrak{B}$ and so $\langle f(A), o_1, \dots, o_n\rangle$ is a subalgebra of $\mathfrak{B}$. We call it the *homomorphic image* of $\mathfrak{A}$ (under the homomorphism f) and denote it by $f(\mathfrak{A})$; $\mathfrak{A}$ is called an *inverse homomorphic image* of $f(\mathfrak{A})$. If C is a class of algebras then HC is the class of all homomorphic images of algebras in C. Since every homomorphism preserves the unit element of pseudo-Boolean algebras, we have

Proposition 7.62 (i) *If a formula φ is valid in a pseudo-Boolean or modal algebra $\mathfrak{A}$ then φ is valid in every homomorphic image of $\mathfrak{A}$.*

(ii) H$C \subseteq$ Var$C$ *for every class C of algebras.*

Example 7.63 Suppose $\mathfrak{G} = \langle V, S\rangle$ is a generated subframe of a modal frame $\mathfrak{F} = \langle W, R\rangle$. We invite the reader to prove that the map f from 2^W onto 2^V defined by $f(X) = X \cap V$, for $X \subseteq W$, is a homomorphism of $\mathfrak{F}^+$ onto $\mathfrak{G}^+$ (for details consult Section 8.5).

The following observation provides us with another important example of homomorphism. Let L be a superintuitionistic or normal modal logic and $\mathfrak{A}_L$ its Tarski–Lindenbaum algebra. Suppose also that $\mathfrak{A}$ is an algebra for L generated by a set X such that $|X| \le |\mathbf{Var}\mathcal{L}|$. Then any map f from $\{\|p\|_L : p \in \mathbf{Var}\mathcal{L}\}$ onto X can be extended inductively to a homomorphism of $\mathfrak{A}_L$ onto $\mathfrak{A}$ by taking, for all formulas φ and ψ,

$$f(\|\varphi\|_L \odot \|\psi\|_L) = f(\|\varphi\|_L) \odot f(\|\psi\|_L), \text{ for } \odot \in \{\wedge, \vee, \to\},$$

$$f(\Box\|\varphi\|_L) = \Box f(\|\varphi\|_L), \quad f(\|\bot\|_L) = \bot.$$

Remark The reader may (and should) wonder now, where it was used that $\mathfrak{A}_L$ and $\mathfrak{A}$ belong to the same variety (i.e., validate the same formulas). Why could not we define such a map, say, from $\mathfrak{A}_{\mathbf{Cl}}$ onto $\mathfrak{A}_{\mathbf{Int}}$?

The problem here is that in this case f is not well-defined because it depends now on the choice of φ and ψ in the equivalence classes $\|\varphi\|_L$ and $\|\psi\|_L$. Indeed, we have $\|p \to p\|_{\mathbf{Cl}} = \|p \vee \neg p\|_{\mathbf{Cl}}$, but on the other hand $f(\|p \to p\|_{\mathbf{Cl}}) = \|p \to p\|_{\mathbf{Int}}$ differs from $f(\|p \vee \neg p\|_{\mathbf{Cl}}) = \|p \vee \neg p\|_{\mathbf{Int}}$. The assumption that $\mathfrak{A} \models L$ makes f well-defined. For if $\|\varphi\|_L = \|\psi\|_L$ then $\varphi \leftrightarrow \psi \in L$ and hence $\mathfrak{A} \models \varphi \leftrightarrow \psi$; so if $p_1, \dots, p_n$ are all the variables in φ and ψ and $f(\|p_i\|_L) = a_i$

$(i = 1, \dots, n)$ then $\varphi(a_1, \dots, a_n) = \psi(a_1, \dots, a_n)$ in $\mathfrak{A}$, from which we obtain $f(\|\varphi\|_L) = f(\|\psi\|_L)$.

An algebra $\mathfrak{A}$ in a variety $\mathcal{V}$ is said to be a *free algebra in $\mathcal{V}$ of rank $\varkappa$* if $\mathfrak{A}$ is generated by a set X of cardinality $\varkappa$ and every map of X into an algebra $\mathfrak{B} \in \mathcal{V}$ can be extended to a homomorphism of $\mathfrak{A}$ in $\mathfrak{B}$. The Tarski–Lindenbaum algebra $\mathfrak{A}_L$ for $L = \mathrm{Log}\mathcal{V}$ is a free algebra in $\mathcal{V}$ of rank $|\mathbf{Var}\mathcal{L}|$. Now we generalize the Tarski–Lindenbaum construction to produce a free algebra in $\mathcal{V}$ of an arbitrary given rank.

Let X be an arbitrary set. Define a set $\mathbf{For}X$—the set of "formulas" or *words* over the set of "variables" or *letters* X—as the smallest set to contain X, $\bot$ and such that with every words x and y it contains also the words $x \wedge y$, $x \vee y$, $x \to y$ and $\Box x$. (Of course in the intuitionistic case $\Box$ should be omitted.) The fact that every letter occurring in a word $\varphi \in \mathbf{For}X$ is contained in a set $\{x_1, \dots, x_n\} \subseteq X$ is denoted by $\varphi(x_1, \dots, x_n)$. Two words $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are called *equivalent* in a superintuitionistic or normal modal logic L if $\varphi(p_1, \dots, p_n) \leftrightarrow \psi(p_1, \dots, p_n) \in L$, where $\varphi(p_1, \dots, p_n)$ and $\psi(p_1, \dots, p_n)$ are obtained from $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ by replacing x_i with (real) variables p_i. The class of all words that are equivalent to a word φ in L is denoted by $\|\varphi\|_L$. Now we can define operations $\wedge$, $\vee$, $\to$, $\bot$ and $\Box$ on $\|\mathbf{For}X\|_L = \{\|\varphi\|_L : \varphi \in \mathbf{For}X\}$ in exactly the same way as in the proof of Theorem 7.2. The resulting algebra will be denoted by $\mathfrak{A}_L(X)$ or even $\mathfrak{A}_L(\varkappa)$, where $\varkappa = |X|$, since we consider algebras up to isomorphism.

The apparent similarity between $\mathfrak{A}_L(X)$ and the Tarski–Lindenbaum algebra for L provides us with the following theorems whose proofs are left to the reader as an easy exercise.

Theorem 7.64 (i) *For every normal modal or si-logic L and every formula φ with $\leq \varkappa$ variables,*

$$\varphi \in L \text{ iff } \mathfrak{A}_L(\varkappa) \models \varphi.$$

(ii) *$\mathfrak{A}_L(\varkappa)$ is a free algebra in* $\mathrm{Var}L$ *of rank $\varkappa$.*

Theorem 7.65 *Suppose $L \in$* $\mathrm{NExt}\mathbf{K}$, *$L' \in$* $\mathrm{Ext}L$ *and φ is a formula with $\leq \varkappa$ variables. Then*

$$\varphi \in L \text{ iff } \langle \mathfrak{A}_L(\varkappa), \nabla \rangle \models \varphi$$

where ∇ is the filter in $\mathfrak{A}_L(\varkappa)$ containing all the elements in $\mathfrak{A}_L(\varkappa)$ generated by words that are equivalent to $\top$ in L'.

We shall study the constitution of free algebras of finite rank in varieties of pseudo-Boolean and modal algebras—their relational counterparts, to be more exact—in Section 8.7. Here we consider only one:

Example 7.66 Let us construct the algebra $\mathfrak{A}_{\mathbf{Int}}(1)$. To this end we require the following sequence of formulas $\boldsymbol{nf}_i$, $i \leq \omega$, in one variable:

$$\boldsymbol{nf}_\omega = \top, \;\; \boldsymbol{nf}_0 = \bot, \;\; \boldsymbol{nf}_1 = p, \;\; \boldsymbol{nf}_2 = \neg p,$$

$$\boldsymbol{nf}_{2n+3} = \boldsymbol{nf}_{2n+1} \vee \boldsymbol{nf}_{2n+2}, \quad \boldsymbol{nf}_{2n+4} = \boldsymbol{nf}_{2n+3} \to \boldsymbol{nf}_{2n+1}.$$

These formulas, called the *Nishimura formulas*, are ascribed to the elements of the pseudo-Boolean algebra in Fig. 7.2 (a), known as the *Rieger–Nishimura lattice*. And this is no accident. If we define a valuation $\mathfrak{V}$ in the algebra so that $\mathfrak{V}(p)$ be the element marked by $\boldsymbol{nf}_1 = p$, then clearly $\mathfrak{V}(\boldsymbol{nf}_i)$ is the element marked by $\boldsymbol{nf}_i$, for every $i \leq \omega$. Moreover, the reader can readily check that for any $\odot \in \{\to, \wedge, \vee\}$ and any $i, j, k \leq \omega$,

$$\mathfrak{V}(\boldsymbol{nf}_i \odot \boldsymbol{nf}_j) = \mathfrak{V}(\boldsymbol{nf}_k) \text{ iff } (\boldsymbol{nf}_i \odot \boldsymbol{nf}_j) \leftrightarrow \boldsymbol{nf}_k \in \mathbf{Int}.$$

For instance, we have

$$\mathfrak{V}(\boldsymbol{nf}_{2n+2} \to \boldsymbol{nf}_{2n+1}) = \mathfrak{V}(\boldsymbol{nf}_{2n+4})$$

and

$$(\boldsymbol{nf}_{2n+2} \to \boldsymbol{nf}_{2n+1}) \leftrightarrow \boldsymbol{nf}_{2n+4} \in \mathbf{Int},$$

$$\mathfrak{V}(\boldsymbol{nf}_{2n+3} \wedge \boldsymbol{nf}_{2n+4}) = \mathfrak{V}(\boldsymbol{nf}_{2n+1})$$

and

$$(\boldsymbol{nf}_{2n+3} \wedge \boldsymbol{nf}_{2n+4}) \leftrightarrow \boldsymbol{nf}_{2n+1} \in \mathbf{Int}.$$

A conclusion from this observation may be formulated as

Theorem 7.67 (i) *Every formula in one variable is equivalent in* **Int** *to one of the Nishimura formulas.*

(ii) *If $i \neq j$ then $\boldsymbol{nf}_i \leftrightarrow \boldsymbol{nf}_j \notin$* **Int**.

(iii) *$\mathfrak{A}_{\mathbf{Int}}(1)$ is isomorphic to the pseudo-Boolean algebra in Fig. 7.2 (a).*

(iv) *There are countably many si-logics axiomatizable by formulas in one variable.*

Before we extend the notion of homomorphism to modal matrices, let us consider a connection between homomorphisms of pseudo-Boolean and modal algebras and their filters.

Theorem 7.68 (i) *Let f be a homomorphism of a pseudo-Boolean algebra $\mathfrak{A}$ in $\mathfrak{B}$ and ∇ a filter in $\mathfrak{B}$. Then the set $f^{-1}(\nabla) = \{x : f(x) \in \nabla\}$ is a filter in $\mathfrak{A}$. If in addition ∇ is prime then $f^{-1}(\nabla)$ is also prime.*

(ii) *Suppose f is a homomorphism of a pseudo-Boolean or modal algebra $\mathfrak{A}$ onto $\mathfrak{B}$. Then for every formula φ,*

$$\mathfrak{B} \models \varphi \textit{ iff } \langle \mathfrak{A}, f^{-1}(\top) \rangle \models \varphi.$$

Proof Exercise. ❑

Say that a filter ∇ in a modal algebra $\mathfrak{A}$ is *normal* if $x \in \nabla$ implies $\Box x \in \nabla$, for every element x in $\mathfrak{A}$.

Proposition 7.69 *If f is a homomorphism of a modal algebra $\mathfrak{A}$ in $\mathfrak{B}$ then $f^{-1}(\top)$ is a normal filter in $\mathfrak{A}$.*

Proof Suppose $x \in f^{-1}(\top)$, i.e., $f(x) = \top$. Then $f(\Box x) = \Box f(x) = \Box\top = \top$ and so $\Box x \in f^{-1}(\top)$. ❑

Let ∇ be a filter (normal filter) in a pseudo-Boolean (respectively, modal) algebra $\mathfrak{A}$. Define a relation $\equiv_\nabla$ in $\mathfrak{A}$ by taking

$$x \equiv_\nabla y \text{ iff } x \leftrightarrow y \in \nabla.$$

It is not hard to see that $\equiv_\nabla$ is an equivalence relation in $\mathfrak{A}$. Besides, the relation $\equiv_\nabla$ possesses one more important property.

An equivalence relation $\sim$ in an algebra $\mathfrak{A} = \langle A, o_1, \dots, o_m \rangle$ is said to be a *congruence* if, for every n-ary operation o_i and every $x_1, \dots, x_n, y_1, \dots, y_n$ in A,

$$x_1 \sim y_1, \dots, x_n \sim y_n \text{ imply } o_i(x_1, \dots, x_n) \sim o_i(y_1, \dots, y_n).$$

Theorem 7.70 *Suppose that ∇ is a filter (normal filter) in a pseudo-Boolean (modal) algebra $\mathfrak{A}$. Then the relation $\equiv_\nabla$ is a congruence in $\mathfrak{A}$.*

Proof Let $\odot \in \{\wedge, \vee, \to\}$, $x_1 \equiv_\nabla y_1$ and $x_2 \equiv_\nabla y_2$. Since the identities

$$(x \leftrightarrow y) \to (z \odot x \leftrightarrow z \odot y) = \top, \ (x \leftrightarrow y) \to (x \odot z \leftrightarrow y \odot z) = \top$$

hold in every pseudo-Boolean algebra, by the definition of filter we then obtain $x_1 \odot x_2 \equiv_\nabla y_1 \odot x_2 \equiv_\nabla y_1 \odot y_2$ and so, by the transitivity of $\equiv_\nabla$, $x_1 \odot x_2 \equiv_\nabla y_1 \odot y_2$.

If $\mathfrak{A}$ is modal then the identity $\Box(x \leftrightarrow y) \to (\Box x \leftrightarrow \Box y) = \top$ is true in it. And since ∇ is normal, we then have $\Box x \equiv_\nabla \Box y$ whenever $x \equiv_\nabla y$. ❑

Theorem 7.70 is an algebraic counterpart of the equivalent replacement theorem, which was used essentially in the proof of Theorem 7.2. We give now an algebraic analog of that proof.

Let $\mathfrak{A}$ be a pseudo-Boolean (modal) algebra with a universe A and ∇ a (normal) filter in $\mathfrak{A}$. Denote by $\|x\|_\nabla$ the equivalence class (with respect to $\equiv_\nabla$) generated by an element x in $\mathfrak{A}$, i.e., $\|x\|_\nabla = \{y \in A : x \equiv_\nabla y\}$, and define on the set $\|A\|_\nabla = \{\|x\|_\nabla : x \in A\}$ of these classes operations $\wedge$, $\vee$, $\to$, $\bot$, $\Box$ by taking, for every $x, y \in A$,

$$\|x\|_\nabla \odot \|y\|_\nabla = \|x \odot y\|_\nabla, \text{ for } \odot \in \{\wedge, \vee, \to\},$$

$$\bot = \|\bot\|_\nabla, \ \Box\|x\|_\nabla = \|\Box x\|_\nabla.$$

Since $\equiv_\nabla$ is a congruence, this definition does not depend on the choice of representatives x and y in the classes $\|x\|_\nabla$ and $\|y\|_\nabla$. The resulting algebra $\langle \|A\|_\nabla, \wedge, \vee, \to, \bot \rangle$ (respectively, $\langle \|A\|_\nabla, \wedge, \vee, \to, \bot, \Box \rangle$) is called the *quotient algebra* of $\mathfrak{A}$ with respect to the filter ∇ and denoted by $\mathfrak{A}/\nabla$.

Theorem 7.71 (i) *Suppose f is a homomorphism of a pseudo-Boolean or modal algebra $\mathfrak{A}$ onto $\mathfrak{B}$ and $\nabla = f^{-1}(\top)$. Then the map g defined by*

$$g(f(x)) = \|x\|_\nabla$$

is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}/\nabla$.

(ii) *Suppose ∇ is a (normal) filter in a pseudo-Boolean (modal) algebra $\mathfrak{A}$. Then the map f defined by*

$$f(x) = \|x\|_\nabla$$

is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\nabla$ with $f^{-1}(\top) = \nabla$.

Proof Exercise. ❑

Corollary 7.72 *There are countably many 1-generated algebras in the variety of pseudo-Boolean algebras.*

Proof Follows from Theorems 7.71, 7.67 and the fact that all the filters in $\mathfrak{A}_{\mathbf{Int}}(1)$ are principal. ❑

We use the developed technique to characterize algebraically the consequence relations in modal and si-logics.

Theorem 7.73 (i) *Let $L \in$ NExt**K**. Then $\Gamma \vdash_L \varphi$ iff for any $\mathfrak{A} \in$ Var$L$, any ultrafilter ∇ and any valuation $\mathfrak{V}$ in $\mathfrak{A}$, $\mathfrak{V}(\varphi) \in \nabla$ whenever $\mathfrak{V}(\psi) \in \nabla$ for all $\psi \in \Gamma$.*

(ii) *Let $L \in$ Ext**K**. Then $\Gamma \vdash_L \varphi$ iff for any $\langle \mathfrak{A}, \nabla_0 \rangle \in$ Var$L$, any ultrafilter $\nabla \supseteq \nabla_0$ and any valuation $\mathfrak{V}$ in $\mathfrak{A}$, $\mathfrak{V}(\varphi) \in \nabla$ whenever $\mathfrak{V}(\psi) \in \nabla$ for all $\psi \in \Gamma$.*

(iii) *Let $L \in$ NExt**K**. Then $\Gamma \vdash_L^* \varphi$ (both MP and RN are allowed) iff for any $\mathfrak{A} \in$ Var$L$ and any valuation $\mathfrak{V}$ in $\mathfrak{A}$, $\mathfrak{V}(\varphi) = \top$ whenever $\mathfrak{V}(\psi) = \top$ for all $\psi \in \Gamma$.*

(iv) *Let $L \in$ Ext**Int**. Then $\Gamma \vdash_L \varphi$ iff for any $\mathfrak{A} \in$ Var$L$ and any valuation $\mathfrak{V}$ in $\mathfrak{A}$, $\mathfrak{V}(\varphi) = \top$ whenever $\mathfrak{V}(\psi) = \top$ for all $\psi \in \Gamma$.*

Proof (i) The implication ($\Rightarrow$) follows from the definition of filter. To prove the converse, suppose $\Gamma \not\vdash_L \varphi$ and consider the Tarski–Lindenbaum algebra $\mathfrak{A}_L$ with the standard valuation $\mathfrak{V}_L$. Let ∇' be the filter generated by $\{\mathfrak{V}_L(\psi) : \psi \in \Gamma\}$. Clearly, $\mathfrak{V}_L(\varphi) \notin \nabla'$. By Theorem 7.41, there is a prime filter (= ultrafilter) ∇ in $\mathfrak{A}_L$ such that $\nabla' \subseteq \nabla$ and $\mathfrak{V}_L(\varphi) \notin \nabla$, which is a contradiction.

(ii) is proved analogously to (i).

(iii) Again ($\Rightarrow$) is clear. To prove ($\Leftarrow$), assume $\Gamma \not\vdash_L^* \varphi$ and consider once more $\mathfrak{A}_L$ with $\mathfrak{V}_L$. Let ∇ be the smallest normal filter containing $\{\mathfrak{V}_L(\psi) : \psi \in \Gamma\}$. Then $\mathfrak{V}_L(\varphi) \notin \nabla$, for otherwise we would have $\Gamma \vdash_L^* \varphi$. Now take the algebra $\mathfrak{A} = \mathfrak{A}_L/\nabla$. By Theorem 7.71 and Proposition 7.62, we then have a valuation $\mathfrak{V}$ in $\mathfrak{A} \in$ Var$L$ such that $\mathfrak{V}(\psi) = \top$ for all $\psi \in \Gamma$ and $\mathfrak{V}(\varphi) \neq \top$, which is a contradiction.

(iv) is proved in the same way as (iii). ❑

A map f is called a *homomorphism* of a modal matrix $\langle \mathfrak{A}_1, \nabla_1 \rangle$ in a matrix $\langle \mathfrak{A}_2, \nabla_2 \rangle$ if f is a homomorphism of $\mathfrak{A}_1$ in $\mathfrak{A}_2$ and $\nabla_1 = f^{-1}(\nabla_2)$. If f is a surjection, $\langle \mathfrak{A}_2, \nabla_2 \rangle$ is said to be a *homomorphic image* of $\langle \mathfrak{A}_1, \nabla_1 \rangle$. For a class $\mathcal{C}$ of matrices, denote by $\mathrm{H}\mathcal{C}$ and $\mathrm{H}^{-1}\mathcal{C}$ the classes of all homomorphic images and inverse homomorphic images of matrices in $\mathcal{C}$, respectively. As an easy exercise we invite the reader to prove the following:

Proposition 7.74 (i) *If $\langle \mathfrak{A}_2, \nabla_2 \rangle$ is a homomorphic image of $\langle \mathfrak{A}_1, \nabla_1 \rangle$ then, for every formula φ,*

$$\langle \mathfrak{A}_1, \nabla_1 \rangle \models \varphi \textit{ iff } \langle \mathfrak{A}_2, \nabla_2 \rangle \models \varphi$$

and so

$$\mathrm{Log}\, \langle \mathfrak{A}_1, \nabla_1 \rangle = \mathrm{Log}\, \langle \mathfrak{A}_2, \nabla_2 \rangle .$$

(ii) $\mathrm{H}\mathcal{C} \subseteq \mathrm{Var}\mathcal{C}$, $\mathrm{H}^{-1}\mathcal{C} \subseteq \mathrm{Var}\mathcal{C}$ *for every class $\mathcal{C}$ of matrices.*

As follows from this theorem, the main difference between homomorphisms of matrices and algebras is that forming a homomorphic image of a matrix does not change the set of formulas valid in it, whereas for algebras this is not so.

If f is a homomorphism of a matrix $\langle \mathfrak{A}, \nabla \rangle$ then $\nabla' = f^{-1}(\top)$ is a normal filter contained in ∇, and the homomorphic image of $\langle \mathfrak{A}, \nabla \rangle$ under f can be represented as

$$\langle \mathfrak{A}, \nabla \rangle / \nabla' = \langle \mathfrak{A}/\nabla', \nabla/\nabla' \rangle$$

where $\nabla/\nabla' = \{\|x\|_{\nabla'} : \ x \in \nabla\}$. A matrix of this form is called the *quotient matrix* of $\langle \mathfrak{A}, \nabla \rangle$ with respect to the normal filter ∇'. A matrix $\langle \mathfrak{A}, \nabla \rangle$ is said to be *reduced* if $\{\top\}$ is the only normal filter contained in ∇.

Theorem 7.75 *Every matrix is an inverse homomorphic image of some reduced matrix.*

Proof We require two lemmas.

Lemma 7.76 *The set of normal filters in a modal algebra $\mathfrak{A}$ is a complete sublattice of the lattice of filters in $\mathfrak{A}$.*

Proof The intersection of any family $\{\nabla_i : \ i \in I\}$ of normal filters is clearly a normal filter. We show that $\bigvee_{i \in I}\{\nabla_i : \ i \in I\}$ is also a normal filter.

Let $a \in \bigvee_{i \in I}\{\nabla_i : \ i \in I\}$. Then there are $a_1 \in \nabla_{i_1}, \dots, a_n \in \nabla_{i_n}$, for some $\{i_1, \dots, i_n\} \subseteq I$, such that $a_1 \wedge \dots \wedge a_n \leq a$ and so $\Box a_1 \wedge \dots \wedge \Box a_n \leq \Box a$. Since all filters ∇_i are normal, we have $\Box a_1 \in \nabla_{i_1}, \dots, \Box a_n \in \nabla_{i_n}$, from which $\Box a \in \bigvee_{i \in I}\{\nabla_i : \ i \in I\}$. ❑

Lemma 7.77 *An (inverse) homomorphic image of a normal filter is also a normal filter.*

Proof Let f be a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, ∇ a normal filter in $\mathfrak{B}$ and show that $f^{-1}(\nabla)$ is a normal filter in $\mathfrak{A}$. If $a \in f^{-1}(\nabla)$ then $f(a) \in \nabla$ and so $\Box f(a) = f(\Box a) \in \nabla$, from which $\Box a \in f^{-1}(\nabla)$.

Suppose now that ∇ is a normal filter in $\mathfrak{A}$ and $b \in f(\nabla)$, i.e., $b = f(a)$ for some $a \in \nabla$. Since $\Box a \in \nabla$, we then have $f(\Box a) = \Box f(a) = \Box b \in f(\nabla)$. ❑

We are in a position now to prove Theorem 7.75. Let $\langle \mathfrak{A}, \nabla \rangle$ be a modal matrix and ∇' a maximal (with respect to $\subseteq$) normal filter contained in ∇, which exists by Zorn's Lemma and Lemma 7.76. By Theorem 7.71 (ii), $\langle \mathfrak{A}/\nabla', \nabla/\nabla' \rangle$ is a homomorphic image of $\langle \mathfrak{A}, \nabla \rangle$ and by Lemma 7.77, the latter is reduced. ❑

Thus, homomorphisms of matrices as compared with homomorphisms of algebras are in a sense deficient. For example, every homomorphism between reduced matrices is an embedding. To compensate this deficiency, we introduce one more matrix operation.

Say that a matrix $\langle \mathfrak{A}, \nabla' \rangle$ is an *extension* of a matrix $\langle \mathfrak{A}, \nabla \rangle$ if $\nabla \subseteq \nabla'$. Denote by $\mathrm{E}\mathcal{C}$ the class of all extensions of matrices in a class $\mathcal{C}$. Immediately from this definition we obtain

Proposition 7.78 $\mathrm{E}\mathcal{C} \subseteq \mathrm{Var}\mathcal{C}$ *for every class* $\mathcal{C}$ *of matrices.*

7.8 Internal characterization of varieties

By an internal characterization of a variety $\mathcal{V}$ of algebras or matrices we mean such a representation of $\mathcal{V}$ which does not involve identities, characterizing varieties externally, but uses only purely algebraic tools such as various kinds of operations on algebras and matrices.

The following two results are well known in universal algebra under the names "Birkhoff's theorem" and "Tarski's theorem"; their proofs can be found in any good textbook on universal algebra, say (Grätzer 1979). Although we will not prove them here, the reader can easily reconstruct the proofs by himself, consulting the proofs of similar theorems for varieties of matrices.

Theorem 7.79. (Birkhoff's theorem) *A non-empty class* $\mathcal{C}$ *of algebras is a variety iff* $\mathrm{S}\mathcal{C} \subseteq \mathcal{C}$, $\mathrm{P}\mathcal{C} \subseteq \mathcal{C}$, $\mathrm{H}\mathcal{C} \subseteq \mathcal{C}$.

Theorem 7.80. (Tarski's theorem) *For every non-empty class* $\mathcal{C}$ *of algebras,* $\mathrm{Var}\mathcal{C} = \mathrm{HSP}\mathcal{C}$.

The next result may also me called Birkhoff's Theorem for varieties of matrices.

Theorem 7.81 *A non-empty class* $\mathcal{C}$ *of modal matrices is a variety iff* $\mathrm{S}\mathcal{C} \subseteq \mathcal{C}$, $\mathrm{P}\mathcal{C} \subseteq \mathcal{C}$, $\mathrm{H}\mathcal{C} \subseteq \mathcal{C}$, $\mathrm{H}^{-1}\mathcal{C} \subseteq \mathcal{C}$, $\mathrm{E}\mathcal{C} \subseteq \mathcal{C}$.

Proof ($\Rightarrow$) follows from Propositions 7.59 (ii), 7.61 (ii), 7.74 (ii) and 7.78, because in this case $\mathrm{Var}\mathcal{C} = \mathcal{C}$.

($\Leftarrow$) We need to show that if $\langle \mathfrak{A}, \nabla \rangle \in \mathrm{Var}\mathcal{C}$ then $\langle \mathfrak{A}, \nabla \rangle \in \mathcal{C}$. So suppose a matrix $\langle \mathfrak{A}, \nabla \rangle$ is in $\mathrm{Var}\mathcal{C}$.

Take any set I such that $|I| = \max\{|\mathfrak{A}|, \aleph_0\}$. Let $\mathcal{X}$ be the class of all matrices in $\mathcal{C}$ of cardinality $\leq |I|$. Since $\mathcal{C}$ is non-empty and closed under the formation of submatrices, it contains a 0-generated matrix, which clearly is countable. Therefore, $\mathcal{X} \neq \emptyset$.

Let $\left\langle \widetilde{\mathfrak{A}}, \widetilde{\nabla} \right\rangle$ be a matrix in $\mathcal{X}$ and f a map from I into $\widetilde{\mathfrak{A}}$. Denote by $\langle \mathfrak{A}_f, \nabla_f \rangle$ the submatrix of $\left\langle \widetilde{\mathfrak{A}}, \widetilde{\nabla} \right\rangle$ generated by $f(I)$. Since $\mathcal{X}$ is closed under the formation of submatrices, $\langle \mathfrak{A}_f, \nabla_f \rangle \in \mathcal{X}$ and $f(I)$ is a set of $\mathfrak{A}_f$'s generators. Suppose now that $\mathcal{F}$ is the set of all maps as defined above for all $\left\langle \widetilde{\mathfrak{A}}, \widetilde{\nabla} \right\rangle \in \mathcal{X}$ and consider the direct product

$$\left\langle \widehat{\mathfrak{A}}, \widehat{\nabla} \right\rangle = \prod_{f \in \mathcal{F}} \langle \mathfrak{A}_f, \nabla_f \rangle \in \mathrm{PS}\mathcal{X} \subseteq \mathrm{PS}\mathcal{C}. \tag{7.2}$$

For every $i \in I$, $a_i = \{f(i) : f \in \mathcal{F}\}$ is an element in $\widehat{\mathfrak{A}}$ and the set of all these elements characterizes Log$\mathcal{C}$ in the sense that if $\varphi(p_1, \dots, p_n)$ is a formula and $\varphi(a_{i_1}, \dots, a_{i_n}) \in \widehat{\nabla}$, for some set $\{i_1, \dots, i_n\} \subseteq I$, then $\varphi \in$ Log$\mathcal{C}$. Indeed, suppose otherwise, i.e., there is a matrix $\langle \mathfrak{A}', \nabla' \rangle \in \mathcal{C}$ such that $\varphi(b_1, \dots, b_n) \notin \nabla'$, for some elements $b_1, \dots, b_n$ in $\mathfrak{A}'$. Take the submatrix $\langle \mathfrak{A}'', \nabla'' \rangle$ of $\langle \mathfrak{A}', \nabla' \rangle$ generated by $b_1, \dots, b_n$. Since this submatrix is finitely generated, it is countable and so belongs to $\mathcal{X}$; besides, we have $\langle \mathfrak{A}'', \nabla'' \rangle \not\models \varphi$. Let g be a map in $\mathcal{F}$ such that $g(i_1) = b_1, \dots, g(i_n) = b_n$. Then $\langle \mathfrak{A}'', \nabla'' \rangle = \langle \mathfrak{A}_g, \nabla_g \rangle$ is a factor in the product $\left\langle \widehat{\mathfrak{A}}, \widehat{\nabla} \right\rangle$. By the definition of direct product, $\varphi(a_{i_1}, \dots, a_{i_n}) = \{\varphi(f(i_1), \dots, f(i_n)) : f \in \mathcal{F}\}$ and since $\varphi(g(i_1), \dots, g(i_n)) = \varphi(b_1, \dots, b_n) \notin \nabla_g$, we have $\varphi(a_{i_1}, \dots, a_{i_n}) \notin \widehat{\nabla}$, which is a contradiction.

Let us consider the algebra $\mathfrak{A}_{\mathbf{K}}(I)$. By Theorem 7.64 (ii), $\mathfrak{A}$ is a homomorphic image of $\mathfrak{A}_{\mathbf{K}}(I)$ under some homomorphism h and so the matrix $\langle \mathfrak{A}, \nabla \rangle$ is a homomorphic image of $\langle \mathfrak{A}_{\mathbf{K}}(I), h^{-1}(\nabla) \rangle$, i.e.,

$$\langle \mathfrak{A}, \nabla \rangle \in \mathrm{H}\{\langle \mathfrak{A}_{\mathbf{K}}(I), h^{-1}(\nabla) \rangle\}. \tag{7.3}$$

We show now that $\langle \mathfrak{A}_{\mathbf{K}}(I), h^{-1}(\nabla) \rangle \in \mathrm{EH}^{-1}\mathrm{S}\{\left\langle \widehat{\mathfrak{A}}, \widehat{\nabla} \right\rangle\}$.

Put $g(i) = a_i$, for $i \in I$, and extend g to a homomorphism of $\mathfrak{A}_{\mathbf{K}}(I)$ onto a subalgebra $\mathfrak{A}'$ of $\widehat{\mathfrak{A}}$ generated by the set $\{a_i : i \in I\}$, i.e., for every x and y in $\mathfrak{A}_{\mathbf{K}}(I)$, we put

$$g(x \odot y) = g(x) \odot g(y), \text{ for } \odot \in \{\wedge, \vee, \rightarrow\},$$

$$g(\bot) = \bot, \; g(\Box x) = \Box g(x).$$

Denote by ∇' the intersection of $\widehat{\nabla}$ with the universe of $\mathfrak{A}'$ and show that $g^{-1}(\nabla') \subseteq h^{-1}(\nabla)$.

Let $a \in g^{-1}(\nabla')$. Then there are $i_1, \dots, i_n \in I$ and a formula $\varphi(p_1, \dots, p_n)$ such that $a = \varphi(i_1, \dots, i_n)$ and

$$g(a) = g(\varphi(i_1, \dots, i_n)) = \varphi(g(i_1), \dots, g(i_n)) = \varphi(a_{i_1}, \dots, a_{i_n}) \in \nabla' \subseteq \widehat{\nabla}.$$

As was proved above, this means that $\varphi \in$ Log$\mathcal{C}$. Since $\langle \mathfrak{A}, \nabla \rangle \in$ Var$\mathcal{C}$, we have $\langle \mathfrak{A}, \nabla \rangle \models \varphi$ and so $h(a) = \varphi(h(i_1), \dots, h(i_n)) \in \nabla$, i.e., $a \in h^{-1}(\nabla)$.

Thus, we have shown that the matrix $\langle \mathfrak{A}_{\mathbf{K}}(I), h^{-1}(\nabla) \rangle$ is an extension of $\langle \mathfrak{A}_{\mathbf{K}}(I), g^{-1}(\nabla') \rangle$. It follows that

$$\begin{aligned} \langle \mathfrak{A}_{\mathbf{K}}(I), h^{-1}(\nabla) \rangle &\in \mathrm{E}\{\langle \mathfrak{A}_{\mathbf{K}}(I), g^{-1}(\nabla') \rangle\} \subseteq \\ &\subseteq \mathrm{EH}^{-1}\{\langle \mathfrak{A}', \nabla' \rangle\} \subseteq \mathrm{EH}^{-1}\mathrm{S}\{\langle \widehat{\mathfrak{A}}, \widehat{\nabla} \rangle\}. \end{aligned} \tag{7.4}$$

Now, putting together (7.2), (7.3) and (7.4), we finally obtain

$$\langle \mathfrak{A}, \nabla \rangle \in \mathrm{HEH}^{-1}\mathrm{SPS}\mathcal{C} \subseteq \mathcal{C}.$$

❑

Thus, given a non-empty class $\mathcal{C}$ of matrices, we can construct the variety Var$\mathcal{C}$ by taking the closure of $\mathcal{C}$ under the operators S, P, H, H^{-1}, E. Moreover, as a consequence of the proof above we obtain

Corollary 7.82 *For every non-empty class $\mathcal{C}$ of modal matrices,*

$$\mathrm{Var}\mathcal{C} = \mathrm{HEH}^{-1}\mathrm{SPS}\mathcal{C}.$$

We can even improve the latter equality by observing that $\mathrm{PS}\mathcal{C} \subseteq \mathrm{SP}\mathcal{C}$, for every class $\mathcal{C}$ of matrices (prove the inclusion by yourself). The following result may be called Tarski's Theorem for varieties of matrices.

Theorem 7.83 *For every non-empty class $\mathcal{C}$ of matrices,*

$$\mathrm{Var}\mathcal{C} = \mathrm{HEH}^{-1}\mathrm{SP}\mathcal{C}.$$

Proof By Corollary 7.82, it suffices to establish the equality $\mathrm{HEH}^{-1}\mathrm{SP}\mathcal{C} = \mathrm{HEH}^{-1}\mathrm{SPS}\mathcal{C}$. The inclusion $\subseteq$ is trivial because $\mathcal{C} \subseteq \mathrm{S}\mathcal{C}$. And the inclusion $\mathrm{PS}\mathcal{C} \subseteq \mathrm{SP}\mathcal{C}$ gives us $\mathrm{HEH}^{-1}\mathrm{SPS}\mathcal{C} \subseteq \mathrm{HEH}^{-1}\mathrm{SSP}\mathcal{C} = \mathrm{HEH}^{-1}\mathrm{SP}\mathcal{C}$. ❑

7.9 Exercises

Exercise 7.1 Prove that all non-degenerate matrices for **S** have infinite sets of distinguished elements. (Hint: show that in such matrices $\top, \Diamond\top, \dots, \Diamond^n\top, \dots$ belong to the set of distinguished elements.)

Exercise 7.2 Show that the quasi-identity

$$\varphi_1 = \psi_1 \wedge \ldots \wedge \varphi_m = \psi_m \to \varphi = \psi$$

is true in the Tarski–Lindenbaum algebra $\mathfrak{A}_L$ iff the rule

$$\varphi_1 \leftrightarrow \psi_1, \ldots, \varphi_m \leftrightarrow \psi_m / \varphi \leftrightarrow \psi$$

is admissible in L.

Exercise 7.3 Show that a lattice is distributive iff one of the laws of distributivity is true in it.

Exercise 7.4 A filter ∇ is called *critical* in a pseudo-Boolean algebra $\mathfrak{A}$ if there is an element a in $\mathfrak{A}$ such that $a \notin \nabla$, but $a \in \nabla'$ for every filter $\nabla' \supset \nabla$. Show that a principal filter is prime iff it is critical.

Exercise 7.5 Prove that every finite pseudo-Boolean algebra $\mathfrak{A}$ is isomorphic to its bidual $(\mathfrak{A}_+)^+$.

Exercise 7.6 Prove that every finite intuitionistic frame $\mathfrak{F}$ is isomorphic to its bidual $(\mathfrak{F}^+)_+$.

Exercise 7.7 Show that the prime elements in $\mathfrak{F}^+$ are exactly the rooted generated subframes of $\mathfrak{F}$.

Exercise 7.8 Show that an intuitionistic Kripke frame $\mathfrak{F}$ is rooted iff $\mathfrak{F}^+$ contains a *second greatest element*, i.e., an element $a \neq \top$ such that $b \leq a$, for every b in $\mathfrak{F}^+$ different from $\top$.

Exercise 7.9 Show that a modal Kripke frame $\mathfrak{F}$ is rooted iff there is an element $a \neq \top$ in $\mathfrak{F}^+$ (called an *opremum*) such that, for every b in $\mathfrak{F}^+$ different from $\top$, there exists $n < \omega$ for which $\Box^n b \leq a$.

Exercise 7.10 For a class C of algebras (or matrices), prove that Var$C$ is the smallest variety to contain C and that it is the intersection of all varieties containing C.

Exercise 7.11 Let $\langle \mathfrak{A}, \nabla \rangle$ be a modal matrix and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Prove that the intersection of ∇ with the universe of $\mathfrak{B}$ is a filter in $\mathfrak{B}$.

Exercise 7.12 Show that if ∇_1 and ∇_2 are prime filters in pseudo-Boolean algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, then $\nabla_1 \times \nabla_2$ is a prime filter in $\mathfrak{A}_1 \times \mathfrak{A}_2$.

Exercise 7.13 Prove the converse of Proposition 7.61 (i).

Exercise 7.14 Prove that the variety Var$L$ of every normal logic L such that $\mathbf{S4} \subseteq L \subseteq \mathbf{Grz} \oplus \boldsymbol{bw}_3$ has a continuum of 1-generated algebras.

Exercise 7.15 Show that if a modal logic L does not contain $\boldsymbol{tra}_n$, for any $n < \omega$, then $\mathfrak{A}_L(1)$ is infinite.

Exercise 7.16 Show that a modal algebra $\mathfrak{A}$ is an **S4**-algebra iff for every element x in $\mathfrak{A}$, $\Box x \leq x$ and $\Box\Box x = \Box x$.

Exercise 7.17 Show that a filter ∇ in a Boolean algebra $\mathfrak{A}$ is an ultrafilter iff $\mathfrak{A}/\nabla$ is the two-element Boolean algebra.

Exercise 7.18 Suppose that $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ is a distributive lattice and B, C are non-empty subsets of A such that $b_1 \wedge \ldots \wedge b_n \not\leq c$, for any $b_1, \ldots, b_n \in B$, $c \in C$, and for all $c_1, c_2 \in C$ there is $c \in C$ for which $c_1 \vee c_2 \leq c$. Prove that there exists a prime filter ∇ in $\mathfrak{A}$ such that $B \subseteq \nabla$ and $C \cap \nabla = \emptyset$.

Exercise 7.19 Give an example of a distributive lattice that is not a pseudo-Boolean algebra.

Exercise 7.20 Show that a filter (ideal) in a pseudo-Boolean algebra is generated by a finite set iff it is principal.

Exercise 7.21 Show that condition (4) in the formulation of Theorem 7.10 can be replaced by a finite list of identities.

Exercise 7.22 Say that an identity is derivable from a set of identities if it can be obtained starting from those identities and $p = p$ (as axioms) using the following inference rules: substitution of a term instead of a variable and

$$\frac{\varphi = \psi}{\psi = \varphi}, \quad \frac{\varphi = \psi, \psi = \chi}{\varphi = \chi}, \quad \frac{\varphi = \psi}{\chi = \chi'},$$

where χ' is the result of replacing (some occurrences of) φ in χ by ψ. Let Γ be any set of identities. Show that $\varphi = \psi$ is derivable from Γ iff $\varphi = \psi$ is true in every algebra in which all identities in Γ are true.

Exercise 7.23 Call a modal or pseudo-Boolean algebra $\mathfrak{A}$ *subdirectly irreducible* if it contains an element $a \neq \top$ such that $h(a) = \top (= h(\top))$, for every non-trivial (i.e., different from an isomorphism) homomorphism h from $\mathfrak{A}$. Prove that

(i) $\mathfrak{A}$ is subdirectly irreducible iff among its non-trivial (i.e., different from $\{\top\}$) (normal in the modal case) filters there exists a smallest one;

(ii) a finite algebra $\mathfrak{F}^+$ is subdirectly irreducible iff the frame $\mathfrak{F}$ is rooted but

(iii) for infinite algebras this does not hold; generalize (ii) to infinite algebras (see Exercise 7.9).

Exercise 7.24 (Birkhoff's theorem on subdirect irreducibles) Say that an algebra $\mathfrak{B}$ is a subdirect product of algebras $\mathfrak{A}_i$, $i \in I$, if there is a homomorphic embedding f of $\mathfrak{B}$ into $\prod_{i \in I} \mathfrak{A}_i$ such that if π_j, $j \in I$, are the natural projections of $\prod_{i \in I} \mathfrak{A}_i$ onto $\mathfrak{A}_j$ then $\pi_j \circ f$ is a surjection. Prove that

(i) every algebra can be represented as a subdirect product of some subdirect irreducible algebras;

(ii) every variety of modal algebras is generated by its subdirectly irreducibles.

Exercise 7.25 (Łoś' theorem for algebras) Let $\mathfrak{A}_i$, $i \in I$, be a family of modal algebras and ∇ an ultrafilter in the Boolean algebra $\langle 2^I, \cap, \cup, \supset, \emptyset, \rangle$ (an ultrafilter over I for short). Form an algebra $\mathfrak{B} = \langle B, \wedge, \vee, \rightarrow, \bot, \Box \rangle$ by taking $B = \{\|x\| : x \in \prod_{i \in I} A_i\}$, where $\|x\| = \{y \in \prod_{i \in I} A_i : \{i : x(i) = y(i)\} \in \nabla\}$, and

$$\|x\| \odot \|y\| = \|x \odot y\| \text{ for } \odot \in \{\wedge, \vee, \rightarrow\}, \ \bot = \|\bot\|, \ \Box\|x\| = \|\Box x\|.$$

$\mathfrak{B}$ is called the *ultraproduct* of the family $\{\mathfrak{A}_i : i \in I\}$ over the ultrafilter ∇ and denoted by $\prod_{i \in I} \mathfrak{A}_i / \nabla$. Prove that for every first order sentence ϕ in the language with the functional symbols $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\Box$ and the predicate $=$,

$$\prod_{i\in I}\mathfrak{A}_i/\nabla \models \phi \text{ iff } \{i : \mathfrak{A}_i \models \phi\} \in \nabla.$$

Exercise 7.26 (Łoś' theorem for frames) Let $\mathfrak{F}_i = \langle W_i, R_i\rangle$, $i \in I$, be a family of frames and ∇ an ultrafilter over I. Form a frame $\mathfrak{F} = \langle W, R\rangle$ by taking $W = \{\|x\| : x \in \prod_{i\in I} W_i\}$, where $\|x\| = \{y \in \prod_{i\in I} W_i : \{i : x(i) = y(i)\} \in \nabla\}$, and

$$\|x\|R\|y\| \text{ iff } \{i : x(i)R_iy(i)\} \in \nabla.$$

$\mathfrak{F}$ is called the *ultraproduct* of the family $\{\mathfrak{F}_i : i \in I\}$ over ∇ and denoted by $\prod_{i\in I}\mathfrak{F}_i/\nabla$. Prove that for every first order sentence ϕ in the language with the predicates R and $=$,

$$\prod_{i\in I}\mathfrak{F}_i/\nabla \models \phi \text{ iff } \{i : \mathfrak{F}_i \models \phi\} \in \nabla.$$

Exercise 7.27 (Jónsson's (1967) lemma) Prove that if $\mathfrak{A}$ is a subdirectly irreducible modal algebra in Var$\mathcal{C}$ then $\mathfrak{A} \in \mathrm{HSP_U}\mathcal{C}$, where $\mathrm{P_U}\mathcal{C}$ is the class of ultraproducts of algebras in $\mathcal{C}$.

Exercise 7.28 (Blok's (1980a) lemma) Let $\{\mathfrak{A}_i : i \in I\}$ be a family of modal algebras and, for $i \in I$, $\mathfrak{F}_i = \langle W_i, R_i\rangle$ a frame such that $\mathfrak{A}_i \in \mathrm{S}(\mathfrak{F}_i^+)$. Prove that for any $\mathfrak{A} \in \mathrm{P_U}(\{\mathfrak{A}_i : i \in I\})$ there is $\mathfrak{F} = \langle W, R\rangle \in \mathrm{P_U}(\{\mathfrak{F}_i : i \in I\})$ and $\mathfrak{A}' \in \mathrm{S}(\mathfrak{F}^+)$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}'$. Furthermore, if for every $i \in I$ and $w \in W_i$, $\{w\} \in \mathfrak{A}_i$ then for every $w \in W$, $\{w\} \in \mathfrak{A}'$.

7.10 Notes

Chronologically, the first semantics for non-classical logics was the algebraic one. Attempts to generalize the truth-functional semantics for **Cl** led naturally to many-valued tables in which both "truth" and "non-truth" are not necessarily unique. Although these tables (whose exterior form resembled of a usual matrix of numbers, which probably was the reason to call them *logical matrices*) were first constructed in a rather *ad hoc* manner mainly to distinguish between, say modal systems as in Lewis and Langford (1932) or to define modal logics as in Łukasiewicz (1920), shortly they became one of the most important tools for studying logics.

The algebraic semantics for si-logics and extensions of **S4** was constructed and systematically used by McKinsey (1941), McKinsey and Tarski (1944, 1946, 1948), Dummett and Lemmon (1959). Lemmon (1966a, 1966b) introduced modal algebras for many other modal systems.

In this chapter we presented only that minimum of results on the algebraic semantics which will be required in the sequel. The field of studies in (pseudo-) Boolean algebras and Boolean algebras with operators itself is so extensive that it is practically impossible to indicate a reasonably short list of references covering it comprehensively. The book where the reader can find a good many results on pseudo-Boolean and topological Boolean algebras together with references to their sources is Rasiowa and Sikorski (1963). The methods of this book were extended by Rasiowa (1974) to other types of algebras and the corresponding logics. Methodologically, these books reflect the algebraic approach to non-classical logics of the mid-1960s: the central problem (from the point of view of studying logics rather than an intrinsic algebraic problem) was to establish the finite approximability together with an upper bound for the number of elements in the minimal refutation algebra or matrix. This approach culminated in Lemmon (1966a, 1966b). Its essential component were representation theorems for finite algebras (like Theorems 7.30 and 7.47 above) which made it possible to prove first the finite approximability of a logic in algebraic terms and them transfer it to frames. In particular, we obtain that the properties of approximability by finite algebras, finite frames and finite models are equivalent. (That is why we prefer the term "finite approximability" rather than the connected with models and frames well known notions of the finite model and finite frame property.)

Our definitions of pseudo-Boolean and modal algebras are somewhat different from the standard ones: usually pseudo-Boolean algebras are defined by a small set of conditions, for instance those in Theorem 7.10. Our approach here is similar to that in Part I where we began with a set of acceptable (for some reasons) formulas and then showed that one can select in it a short list of formulas (axioms) from which the rest are derived by certain inference rules. A little defect (in this sense) of the conditions in Theorem 7.10 is that condition (4) is not an identity. However, one can easily replace it with a finite number of identities; see Exercise 7.21. This similarity is not only an external one. In fact, starting from the finite list of identities mentioned above and $p = p$ all other identities that are true in all pseudo-Boolean algebras are derivable using the inference rules given in Exercise 7.22; cf. Birkhoff (1935). Note that pseudo-Boolean algebras and **Int** were taken here only as an example. The same holds for L-algebras, where L is any si- or normal modal logic. Thus, those logics and the corresponding algebras can be considered as part of the so called *equational logic* and/or its model-theoretic counterpart—the theory of varieties of algebras—along with groups, rings, lattices and other conventional algebraic objects (see the survey Taylor, 1979). Many problems concerning our logics (say axiomatizability, approximability, decidability, etc.) turn out to be of interest for other algebraic equational theories, and as a result a considerable algebraic apparatus for solving them has been developed.

Theorem 7.17 on the finite approximability of si-logics with disjunction free extra axioms was proved by McKay (1968). Theorem 7.67 describing the construction of $\mathfrak{A}_{\mathbf{Int}}(1)$ is due to Rieger (1949) and Nishimura (1960).

The theory of varieties is connected primarily with universal algebras; varieties of matrices are not standard objects in it. The problem here is that when considering matrices we deal with not the condition of identical equality to a distinguished element but the predicate of belonging to a set of distinguished elements. Although the notion of variety is easily extended to algebraic systems in which we regard as identities not only expressions of the form $\varphi = \psi$ but also $P(\varphi_1, \ldots, \varphi_m)$, where P is a predicate of the language under consideration (cf. Mal'cev, 1973), this yields no immediate effect because the set of distinguished elements in a matrix (as it was defined in this book) is a filter, i.e., we are interested in algebraic systems of the form $\langle A, \wedge, \vee, \rightarrow, \bot, \Box, \nabla \rangle$ in which not only standard identities (saying that $\langle A, \wedge, \vee, \rightarrow, \bot, \Box \rangle$ is an L-algebra, for some modal logic L) and conditions of belonging to ∇ hold, but also *quasi-identities* of the form

$$\varphi \in \nabla \wedge (\varphi \rightarrow \psi) \in \nabla \Rightarrow \psi \in \nabla$$

(guaranteeing that ∇ is a filter) must be true. Of course, one could deal with quasi-varieties instead of varieties but this does not agree with the fact that we do not change the postulated inference rules, and so the lattices of logics under consideration are not in general dually isomorphic to lattices of quasi-varieties of matrices.

That it is not hard to modify the algebraic semantics by introducing a rather natural concept of variety of matrices was observed independently in several papers; cf. for instance Blok and Köhler (1983), Chagrov (1985b), Shum (1985).

One of the most powerful algebraic tools for investigating nonclassical logics is Jónsson's (1967) lemma (see also Grätzer, 1979), which makes it possible to establish in a rather easy way some facts about lattices of logics and the constitution of logics and the corresponding varieties of algebras as well. As examples we mention here two results which can be obtained as immediate consequences of Jónsson's lemma:

- every tabular logic has a finite number of extensions, and they are also tabular;
- if two finite subdirectly irreducible algebras determine the same logic then they are isomorphic.

Numerous examples of applications of Jónsson's lemma to modal logics can be found in Blok (1980b). Analogues of Jónsson's lemma for varieties of matrices and algebraic systems were proved by Blok and Köhler (1983) and Shum (1985). It is to be noted that in this book we give purely semantical proofs for a number of results that were originally proved with the help of Jónsson's lemma (see for instance the proof of Blok's theorem in Section 10.5).

8

RELATIONAL SEMANTICS

Having solved the completeness problem, the algebraic semantics, introduced in the previous chapter, deprives us, however, of that transparent interpretation of logical connectives which made it possible to construct models for formulas by analyzing step by step their subformulas and adding new points, if necessary. In other words, we have lost that thread which connected the structure of formulas with the "geometry" of their models. Fortunately, this is not that case when "gaining in force we lose in distance". In this chapter we define a more general concept of frame, combining in itself the merits of both algebras and Kripke frames.

8.1 General frames

There are two ways leading to the general frames. One of them originates from Theorem 5.5 according to which every superintuitionistic and normal modal logic L is characterized by some (for instance, canonical) model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. If L is Kripke incomplete then $\mathfrak{F} \not\models L$, i.e., there is another model $\mathfrak{N} = \langle \mathfrak{F}, \mathfrak{U} \rangle$ refuting some $\varphi \in L$. So, if we do not want to give up the idea of the Kripke semantics entirely and yet have completeness, we should impose some restriction on possible valuations in $\mathfrak{F}$ which would allow us to construct $\mathfrak{V}$ and $\mathfrak{M}$, but forbid $\mathfrak{U}$ and $\mathfrak{N}$. Let us denote by P the family of all formula truth-sets in $\mathfrak{F}$ under $\mathfrak{V}$, i.e., put

$$P = \{\mathfrak{V}(\psi) : \ \psi \in \mathbf{For}\mathcal{L} \text{ (or } \psi \in \mathbf{For}\mathcal{ML})\},$$

and call P a *set of possible values* in $\mathfrak{F}$. Then $\mathfrak{U}(p_i) \notin P$ for some variable $p_i \in \mathbf{Sub}\varphi$. For otherwise with every $p_i \in \mathbf{Sub}\varphi$, $i = 1, \dots, n$, we can associate a formula ψ_i such that $\mathfrak{U}(p_i) = \mathfrak{V}(\psi_i)$, and then $\mathfrak{U}(\varphi) = \mathfrak{V}(\varphi s)$ where $s = \{\psi_1/p_1, \dots, \psi_n/p_n\}$, whence $\mathfrak{M} \not\models \varphi s$, contrary to $\varphi s \in L$. Thus, if we require that the valuation $\mathfrak{U}$ in $\mathfrak{F}$ should satisfy the condition

$$\mathfrak{U}(p) \in P, \text{ for every variable } p,$$

then $\mathfrak{N} = \langle \mathfrak{F}, \mathfrak{U} \rangle$ will necessarily be a model for L.

In fact sets P of possible values in $\mathfrak{F}$ can be defined without any connection with models on $\mathfrak{F}$. Indeed, let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a model of $\mathcal{ML}$. Then, as was shown in Example 7.57, the algebra $\langle P, \cap, \cup, \supset, \emptyset, \Box \rangle$, where $\cap$ and $\cup$ are the set-theoretic intersection and union, and $\supset$ and $\Box$ are defined as in Section 7.5, is a modal algebra, a subalgebra of $\mathfrak{F}^+ = \langle 2^W, \cap, \cup, \supset, \emptyset, \Box \rangle$ to be more exact. If $\mathfrak{M}$ is an intuitionistic model then $\langle P, \cap, \cup, \supset, \emptyset \rangle$, with $\supset$ defined as in Section 7.3, is a pseudo-Boolean algebra which is a subalgebra of $\mathfrak{F}^+ = \langle \mathrm{Up}W, \cap, \cup, \supset, \emptyset \rangle$. So we can define a set of possible values in $\mathfrak{F}$ simply by taking as P the universe of some (modal or pseudo-Boolean) subalgebra of $\mathfrak{F}^+$, in particular the universe of $\mathfrak{F}^+$ itself. Thus, we arrive at the following definitions.

A *modal general frame* is a triple $\mathfrak{F} = \langle W, R, P \rangle$ in which $\langle W, R \rangle$ is an ordinary Kripke frame and P, a *set of possible values* in $\mathfrak{F}$, is a subset of 2^W containing $\emptyset$ and closed under $\cap$, $\cup$ and the operations $\supset$ and $\Box$ which are defined as follows: for every $X, Y \subseteq W$,

$$X \supset Y = (W - X) \cup Y,$$

$$\Box X = \{x \in W : \ \forall y \in W \ (xRy \to y \in X)\}.$$

It follows from the duality of $\Box$ and $\Diamond$ that the closure under $\Box$ can be replaced with the closure under the operation $\Diamond$:

$$\Diamond X = X{\downarrow} = \{y \in W : \ \exists x \in X \ yRx\}.$$

And since $X \supset \emptyset = W - X$, the set P is also closed under complementation in the space W.

We denote by $\mathfrak{F}^+$ the algebra $\langle P, \cap, \cup, \supset, \emptyset, \Box \rangle$ and call it the *dual* of $\mathfrak{F}$.

Proposition 8.1 *The dual of every modal general frame is a modal algebra.*

Proof Exercise. ❑

An *intuitionistic general frame* is a triple $\mathfrak{F} = \langle W, R, P \rangle$ where $\langle W, R \rangle$ is an intuitionistic Kripke frame and P, a *set of possible values* in $\mathfrak{F}$, is a subset of $\mathrm{Up}W$ containing $\emptyset$ and closed under $\cap$, $\cup$ and the following operation $\supset$: for every $X, Y \subseteq W$,

$$\begin{aligned} X \supset Y &= \{x \in W : \ \forall y \in W \ (xRy \wedge y \in X \to y \in Y)\} \\ &= \Box((W - X) \cup Y). \end{aligned}$$

$\mathfrak{F}^+$, the *dual* of $\mathfrak{F}$, is the algebra $\langle P, \cap, \cup, \supset, \emptyset \rangle$.

Proposition 8.2 *The dual of each intuitionistic general frame is a pseudo-Boolean algebra.*

Proof Exercise. ❑

General frames $\mathfrak{F} = \langle W, R, P \rangle$ and $\mathfrak{G} = \langle V, S, Q \rangle$ are *isomorphic* ($\mathfrak{F} \cong \mathfrak{G}$ in symbols) if there is an isomorphism f of $\langle W, R \rangle$ onto $\langle V, S \rangle$ such that $X \in P$ iff $f(X) \in Q$, for every $X \subseteq W$. As before, we do not distinguish between isomorphic frames.

Let $\mathfrak{F} = \langle W, R, P \rangle$ be an intuitionistic (or modal) general frame. A *model of the language* $\mathcal{L}$ (respectively, $\mathcal{ML}$) on $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ where $\mathfrak{V}$, a *valuation* in $\mathfrak{F}$, is a map from $\mathbf{Var}\mathcal{L}$ in P, i.e., $\mathfrak{V}(p) \in P$ for every variable p. The truth-relation $\models$ in $\mathfrak{M}$ is defined in exactly the same way as in Sections 2.2

and 3.2 for ordinary Kripke models; as before $\mathfrak{V}(\varphi) = \{x \in W : x \models \varphi\}$. Note that $\mathfrak{V}(\bot) = \emptyset$ and $\mathfrak{V}(\top) = W$.

The definitions of truth, validity, countermodel, etc., and the corresponding notations, given in Sections 2.2 and 3.2, can be extended to general frames without changes. In particular, we say a logic L is *characterized* (or *determined*) by a class $\mathcal{C}$ of general frames if L coincides with the set of formulas that are valid in all frames in $\mathcal{C}$.

It should be clear that semantically there is no big difference between $\mathfrak{F}$ and $\mathfrak{F}^+$: every valuation $\mathfrak{V}$ in $\mathfrak{F}$ is also a valuation in $\mathfrak{F}^+$ and vice versa; the truth-set of a formula φ in $\mathfrak{F}$ under $\mathfrak{V}$ coincides with the value of φ in $\mathfrak{F}^+$ under $\mathfrak{V}$, in particular, $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}^+ \models \varphi$.

Given a modal or intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on a Kripke frame $\mathfrak{F} = \langle W, R \rangle$, the general frame $\mathfrak{G} = \langle W, R, P \rangle$ with

$$P = \{\mathfrak{V}(\varphi) : \varphi \in \mathbf{For}\mathcal{ML} \text{ (or } \varphi \in \mathbf{For}\mathcal{L})\}$$

is called the *general frame associated with* $\mathfrak{M}$.

The general frame, associated with the canonical model $\mathfrak{M}_L = \langle \mathfrak{F}_L, \mathfrak{V}_L \rangle$ for a logic L, is denoted by $\gamma\mathfrak{F}_L = \langle W_L, R_L, P_L \rangle$. We will call $\gamma\mathfrak{F}_L$ the *universal* (*general*) *frame for* L. The canonical (Kripke) frame $\mathfrak{F}_L$ for L is obtained from the universal one by omitting P_L.

Theorem 8.3 *For every superintuitionistic or normal modal logic L, the Tarski–Lindenbaum algebra $\mathfrak{A}_L$ for L is isomorphic to the dual $\gamma\mathfrak{F}_L^+$ of the universal frame $\gamma\mathfrak{F}_L$ for L, an isomorphism being the map f defined by $f(\|\varphi\|_L) = \mathfrak{V}_L(\varphi)$, for every formula φ.*

Proof Clearly f is a surjection. Suppose that $\|\varphi\|_L \neq \|\psi\|_L$, i.e., $\varphi \leftrightarrow \psi \notin L$. Then $\mathfrak{V}_L(\varphi) \neq \mathfrak{V}_L(\psi)$, for otherwise $\mathfrak{M}_L \models \varphi \leftrightarrow \psi$, from which $\varphi \leftrightarrow \psi \in L$. Therefore, f is an injection.

Now we must show that f preserves the operations in $\mathfrak{A}_L$. The following equalities as well as the similar ones for $\vee$ and $\to$ are straightforward consequences of the definitions of $\mathfrak{A}_L$ and $\mathfrak{M}_L$ and need no comments:

$$f(\|\bot\|_L) = \mathfrak{V}_L(\bot) = \emptyset,$$

$$f(\|\varphi\|_L \wedge \|\psi\|_L) = f(\|\varphi \wedge \psi\|_L) = \mathfrak{V}_L(\varphi \wedge \psi) =$$

$$= \mathfrak{V}_L(\varphi) \cap \mathfrak{V}_L(\psi) = f(\|\varphi\|_L) \cap f(\|\psi\|_L),$$

$$f(\Box\|\varphi\|_L) = f(\|\Box\varphi\|_L) = \mathfrak{V}_L(\Box\varphi) = \Box\mathfrak{V}_L(\varphi) = \Box f(\|\varphi\|_L).$$

❑

Remark The only property of $\gamma\mathfrak{F}_L$ used in the proof above is that $\mathfrak{M}_L$ characterizes L. So $\mathfrak{A}_L$ is isomorphic to the dual of the general frame associated with any model characterizing L.

Since $\mathfrak{A}_L$ is a characteristic algebra for L, we immediately obtain the following completeness theorem which, of course, is also a direct consequence of Theorem 5.5.

Theorem 8.4 *Every consistent normal modal or si-logic L is characterized by some class of general frames, for instance, by the single universal general frame $\gamma\mathfrak{F}_L$ for L or by the class of all general frames for L.*

The set of formulas which are valid in all general frames in some class $\mathcal{C}$ is obviously a logic. We will denote it, as before, by Log$\mathcal{C}$ or Log$\mathfrak{F}$ if $\mathcal{C} = \{\mathfrak{F}\}$. The set of all general frames for L is denoted by FrL.

Corollary 8.5 (i) *For every superintuitionistic or normal modal logic L,*

$$L = \mathrm{Log}\gamma\mathfrak{F}_L \text{ and } L = \mathrm{LogFr}L.$$

(ii) *For every intuitionistic or modal general frame $\mathfrak{F}$,*

$$\mathrm{Log}\mathfrak{F} = \mathrm{Log}\mathfrak{F}^+.$$

Thus, each general frame is an interlacing of the two structures: an ordinary Kripke frame and a modal or pseudo-Boolean algebra of subsets of this frame. A reasonable compromise is achieved: we retain the Kripke interpretation of logical connectives and, due to the introduction of the algebraic component, acquire the completeness.

The following examples will help the reader to develop some intuition in dealing with general frames.

Example 8.6 The simplest modal general frames are the frames of the form $\mathfrak{F} = \langle W, R, 2^W \rangle$ in which the set of possible values contains all subsets of W, i.e., there are no restrictions on valuations in $\mathfrak{F}$. From the semantic point of view, such a general frame $\mathfrak{F}$ does not differ from the Kripke frame $\kappa\mathfrak{F} = \langle W, R\rangle$ in the sense that the same valuations can be defined in them and the same formulas are true in them at each point $x \in W$. All this of course concerns intuitionistic general frames $\mathfrak{F} = \langle W, R, \mathrm{Up}W\rangle$ and their underlying Kripke frames $\kappa\mathfrak{F} = \langle W, R\rangle$.

From now on we shall deal with only general frames which henceforth will be called simply *frames*. The frames of the form $\mathfrak{F} = \langle W, R, 2^W \rangle$ and $\mathfrak{F} = \langle W, R, \mathrm{Up}W\rangle$ will be called *Kripke frames* and denoted as before by $\mathfrak{F} = \langle W, R\rangle$.

If $\mathfrak{F} = \langle W, R, P\rangle$ is a frame then we denote by $\kappa\mathfrak{F}$ its underlying Kripke frame, i.e., $\kappa\mathfrak{F} = \langle W, R\rangle$. It should be clear that $\mathfrak{F}^+$ is a subalgebra of $\kappa\mathfrak{F}^+$.

Example 8.7 Another boundary case of intuitionistic frames are frames of the form $\mathfrak{F} = \langle W, R, P\rangle$ where the set P contains only the two sets: $\emptyset$ and W. Since under every valuation in $\mathfrak{F}$ each variable is either true at all points in $\mathfrak{F}$ or true nowhere, this frame is semantically equivalent to the single-point frame $\circ$, i.e., Log$\mathfrak{F}$ = **Cl**. Observe that $\mathfrak{F}$ and $\circ$ have isomorphic duals, viz., the two-element Boolean algebra.

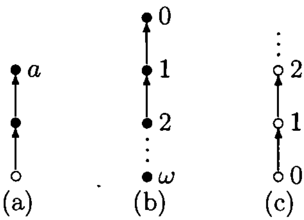


FIG. 8.1.

The same (except the last equality, of course) is true for *reflexive* modal frames. However if $\langle W, R\rangle$ contains a final irreflexive point, e.g., is of the form depicted in Fig. 8.1 (a), then the triple $\langle W, R, \{\emptyset, W\}\rangle$ is not a modal frame, because the set $\{\emptyset, W\}$ is not closed under $\Box$: $\Box\emptyset = \{a\}$. We invite the reader to prove that there is only one set of possible values in this frame, namely 2^W.

Example 8.8 Let us consider the frame $\mathfrak{F} = \langle W, R, P\rangle$ whose underlying (transitive) Kripke frame $\kappa\mathfrak{F}$ is depicted in Fig. 8.1 (b) and P consists of $\emptyset$, W, all finite sets of natural numbers and complements to them in the space W. Or, in other words, P is the union of two sets $\mathcal{X}$ and $\mathcal{Y}$: the elements of $\mathcal{X}$ are all the finite sets of natural numbers, while each element in $\mathcal{Y}$ is the union of a set in $\mathcal{X}$ and the infinite set $\{n, n+1, \dots, \omega\}$, for some $n < \omega$. The fact that P is closed under the Boolean operations is evident and $\Box X \in \mathcal{X}$, for every $X \in P$ different from W, since if $n \notin X$ then $m \notin \Box X$, for all m such that $n < m \leq \omega$.

Note by the way that $(\mathfrak{F}, \omega) \models \Box p \to p$ in spite of the fact that ω is irreflexive. Indeed, if $\omega \models \Box p$ and $\omega \not\models p$ under some valuation $\mathfrak{V}$ in $\mathfrak{F}$ then from the former relation we obtain $\mathfrak{V}(p) = W$, contrary to the latter one. Recall however that $(\kappa\mathfrak{F}, \omega) \not\models \Box p \to p$, since we may put $\mathfrak{V}(p) = W - \{\omega\} \notin P$. (We recommend the reader to compare this example with Example 5.60.)

Example 8.9 Let $\mathfrak{F} = \langle W, R, P\rangle$ be the modal frame such that $\kappa\mathfrak{F}$ has the form as in Fig. 8.1 (c) and P consists of all finite and cofinite (i.e., having finite complements) subsets of W. P is clearly closed under $\cap$, $\cup$ and $\supset$. As to $\Box$, it is not hard to see that $\Box X$ is either empty or cofinite, for every $X \subseteq W$.

Now we show that although $\mathfrak{F}$ contains an infinite ascending chain, the Grzegorczyk formula $\boldsymbol{grz}$ is valid in $\mathfrak{F}$. Suppose otherwise. Then, by Example 3.24, there is an infinite chain $x_0Ry_0Rx_1Ry_1\dots$ in $\mathfrak{F}$ such that, for some valuation $\mathfrak{V}$, $\{y_0, y_1, \dots\} \subseteq \mathfrak{V}(p)$ and $\{x_0, x_1, \dots\} \subseteq W - \mathfrak{V}(p)$. But this is impossible, because $\mathfrak{V}(p)$ is infinite, has an infinite complement in W and so does not belong to P. Let us recall however that, by Proposition 3.48, $\kappa\mathfrak{F} \not\models \boldsymbol{grz}$.

Having constructed the adequate relational semantics for normal modal logics, we can extend it to quasi-normal ones simply by adding to general frames sets of actual worlds. Indeed, as we know from Section 5.6, every logic $L \in \mathrm{Ext}\mathbf{K}$ is characterized by its canonical model $\langle \mathfrak{M}_{\mathrm{ker}\,L}, D_L\rangle$ with distinguished points in the sense that for any $\varphi \in \mathbf{For}\mathcal{ML}$,

$$\varphi \in L \text{ iff } D_L \subseteq \mathfrak{V}_{\ker L}(\varphi).$$

Then, by the definition of the universal general frame $\gamma\mathfrak{F}_{\ker L}$ for $\ker L$, we have $\varphi \in L$ iff $D_L \subseteq \mathfrak{V}(\varphi)$ for every valuation $\mathfrak{V}$ in $\gamma\mathfrak{F}_{\ker L}$. So if we regard the points in D_L as the only actual worlds in $\gamma\mathfrak{F}_{\ker L}$ and consider φ to be true in $\gamma\mathfrak{F}_{\ker L}$ if it is true at all actual worlds then the pair $\langle \gamma\mathfrak{F}_{\ker L}, D_L\rangle$ will characterize L. This observation motivates the following definitions.

A *frame with distinguished points* is a pair $\langle \mathfrak{F}, D\rangle$ such that $\mathfrak{F} = \langle W, R, P\rangle$ is a (general) frame and D a subset of W whose elements are called *distinguished points* or *actual worlds* in $\mathfrak{F}$. A *model with distinguished points based on* $\langle \mathfrak{F}, D\rangle$ is a pair $\langle \mathfrak{M}, D\rangle$, where $\mathfrak{M}$ is a model based on $\mathfrak{F}$. $\langle \mathfrak{M}, D\rangle \models \varphi$ means, as in Section 5.6, that $(\mathfrak{M}, x) \models \varphi$ for all $x \in D$, and $\langle \mathfrak{F}, D\rangle \models \varphi$ means that $\langle \mathfrak{M}, D\rangle \models \varphi$ for all $\langle \mathfrak{M}, D\rangle$ based on $\langle \mathfrak{F}, D\rangle$.

The frame $\langle \gamma\mathfrak{F}_{\ker L}, D_L\rangle$ is called the *universal frame* (with distinguished points) for L.

Every frame $\langle \mathfrak{F}, D\rangle$ with distinguished points gives rise to the modal matrix $\langle \mathfrak{F}^+, D^+\rangle$ where $\mathfrak{F}^+$ is the dual of $\mathfrak{F} = \langle W, R, P\rangle$ and

$$D^+ = \{X \in P : \; D \subseteq X\}$$

is a filter in $\mathfrak{F}^+$. We call $\langle \mathfrak{F}^+, D^+\rangle$ the *dual* of $\langle \mathfrak{F}, D\rangle$. It is clear that $\langle \mathfrak{F}, D\rangle$ is semantically equivalent to $\langle \mathfrak{F}^+, D^+\rangle$.

Theorem 8.10 *The Tarski–Lindenbaum matrix $\langle \mathfrak{A}_{\ker L}, \nabla_L\rangle$ for a quasi-normal logic L is isomorphic to $\langle \gamma\mathfrak{F}^+_{\ker L}, D^+_L\rangle$, an isomorphism being the map f defined by $f(\|\varphi\|_{\ker L}) = \mathfrak{V}_{\ker L}(\varphi)$ for every formula φ.*

Proof By Theorem 8.3, f is an isomorphism of $\mathfrak{A}_{\ker L}$ onto $\gamma\mathfrak{F}^+_{\ker L}$. So we must show that $f(\nabla_L) = D^+_L$. Suppose $\|\varphi\|_{\ker L} \in \nabla_L$. Then $\varphi \in L$, from which $D_L \subseteq \mathfrak{V}_{\ker L}(\varphi)$ and so $\mathfrak{V}_{\ker L}(\varphi) \in D^+_L$. Conversely, if $X \in D^+_L$ then $X = \mathfrak{V}_{\ker L}(\varphi)$ for some formula φ, $D_L \subseteq \mathfrak{V}_{\ker L}(\varphi)$ and so $\varphi \in L$, $\|\varphi\|_{\ker L} \in \nabla_L$ and $f(\|\varphi\|_{\ker L}) = X$. ❑

As a consequence of Theorems 7.4 and 8.10 we obtain the following:

Theorem 8.11 *Every consistent logic $L \in$ Ext**K** is characterized by some class of frames with distinguished points, for instance, by the single universal frame $\langle \gamma\mathfrak{F}_{\ker L}, D_L\rangle$ for L.*

Since $\langle \mathfrak{F}, D\rangle \models \varphi$ iff $\langle \mathfrak{F}, \{d\}\rangle \models \varphi$ for every $d \in D$, we derive one more completeness result.

Theorem 8.12 *Every consistent logic $L \in$ Ext**K** is characterized by some class of frames having a single distinguished point.*

Example 8.13 Let $\mathfrak{F}$ be the frame constructed in Example 8.8. We show that by choosing ω as the single actual world in $\mathfrak{F}$, we obtain a frame for $\mathbf{S} = \mathbf{GL} + \Box p \to p$. Since $\mathfrak{F}$ is transitive, irreflexive and Noetherian, $\kappa\mathfrak{F} \models \mathbf{GL}$, and hence $\mathfrak{F} \models \mathbf{GL}$, in particular, $\langle \mathfrak{F}, \omega\rangle \models \mathbf{GL}$. It remains to recall that, as was shown in Example 8.8, $(\mathfrak{F}, \omega) \models \Box p \to p$. So $\langle \mathfrak{F}, \omega\rangle \models \Box p \to p$ and $\langle \mathfrak{F}, \omega\rangle \models \mathbf{S}$.

8.2 The Stone and Jónsson–Tarski theorems

Another way, using which we also come to the general frames, has its starting point in the realm of algebra. We have already taken one step along this way, having represented (in Sections 7.4 and 7.5) every finite pseudo-Boolean and modal algebra as the dual $\mathfrak{F}^+$ of some finite Kripke frame $\mathfrak{F}$. It is impossible to extend this result to infinite algebras, witness the following cardinality argument: the dual $\mathfrak{F}^+$ of an infinite modal frame $\mathfrak{F}$ contains at least a continuum of elements, although, as we saw in Section 7.2, the Tarski–Lindenbaum algebra for a logic in a denumerable language has only countably many elements.

This section shows however that every pseudo-Boolean and modal algebra $\mathfrak{A}$ is isomorphic to the dual $\mathfrak{F}^+$ of some general frame $\mathfrak{F}$, i.e., to a *subalgebra* of $\kappa\mathfrak{F}$'s dual.

As was shown in the previous section, the Tarski–Lindenbaum algebra $\mathfrak{A}_L$ for a (superintuitionistic or normal modal) logic L is isomorphic to the dual $\gamma\mathfrak{F}_L^+$ of the universal frame $\gamma\mathfrak{F}_L$ for L. So to understand how, given an arbitrary algebra $\mathfrak{A}$, to construct its relational representation it may be useful to see in more detail what the relation between $\mathfrak{A}_L$ and $\gamma\mathfrak{F}_L$ is. Recall first that the elements in $\mathfrak{A}_L$ are the classes $\|\varphi\|_L = \{\psi : \varphi \leftrightarrow \psi \in L\}$, while the points in $\gamma\mathfrak{F}_L$ are the maximal L-consistent tableaux (Γ, Δ). The set Γ in such a tableau has the following properties:

- $\top \in \Gamma$;
- Γ is closed under *modus ponens*, in particular, together with every φ it contains the whole class $\|\varphi\|_L$;
- $\varphi \vee \psi \in \Gamma$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$.

This means that the set $\{\|\varphi\|_L : \varphi \in \Gamma\}$ is a prime filter in $\mathfrak{A}_L$. (Note by the way that $\{\|\varphi\|_L : \varphi \in \Delta\}$ is a prime ideal in $\mathfrak{A}_L$.) And conversely, each prime filter ∇ in $\mathfrak{A}_L$ induces a maximal L-consistent tableau, namely (Γ, Δ) with $\Gamma = \{\varphi \in \mathbf{For} : \|\varphi\|_L \in \nabla\}$, $\Delta = \mathbf{For} - \Gamma$. (Here **For** is the set of all formulas in the language of L.) Thus we can consider points in $\gamma\mathfrak{F}_L$ as prime filters in $\mathfrak{A}_L$.

Recall also that for a superintuitionistic L we defined R_L in $\gamma\mathfrak{F}_L$ by taking $t_1 R_L t_2$ iff $\Gamma_1 \subseteq \Gamma_2$, for any $t_1 = (\Gamma_1, \Delta_1)$ and $t_2 = (\Gamma_2, \Delta_2)$ in W_L. If ∇_1, ∇_2 are the prime filters in $\mathfrak{A}_L$ corresponding to t_1 and t_2, respectively, then

$$t_1 R_L t_2 \text{ iff } \nabla_1 \subseteq \nabla_2.$$

For modal L we defined R_L by taking $t_1 R_L t_2$ iff $\{\varphi : \Box\varphi \in \Gamma_1\} \subseteq \Gamma_2$, which means that

$$t_1 R_L t_2 \text{ iff for every } \|\varphi\|_L \text{ in } \mathfrak{A}_L,\ \Box\|\varphi\|_L \in \nabla_1 \text{ implies } \|\varphi\|_L \in \nabla_2.$$

And finally, $P_L = \{\mathfrak{V}_L(\varphi) : \varphi \in \mathbf{For}\}$. By Theorem 5.4, $\mathfrak{V}_L(\varphi)$ is the set $\{(\Gamma, \Delta) \in W_L : \varphi \in \Gamma\}$ or, algebraically, $\mathfrak{V}_L(\varphi)$ is the set of all prime filters ∇ in $\mathfrak{A}_L$ containing $\|\varphi\|_L$.

Thus we have a method which, given the Tarski–Lindenbaum algebra $\mathfrak{A}_L$ for L, constructs the universal frame $\gamma\mathfrak{F}_L$ for L whose dual $\gamma\mathfrak{F}_L^+$ is isomorphic to $\mathfrak{A}_L$. What if we apply it to an arbitrary pseudo-Boolean or modal algebra?

In fact this method is a generalization, discovered by Jónsson and Tarski (1951), of Stone's (1937) set-theoretic representation of distributive lattices and Boolean algebras which is well-known in lattice theory. We present first Stone's construction for distributive lattices and then extend it to pseudo-Boolean and modal algebras.

Suppose $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ is a distributive lattice. We shall use the following notation and terminology. $W_{\mathfrak{A}}$, the *Stone space* for $\mathfrak{A}$, is the set of all prime filters in $\mathfrak{A}$; $f_{\mathfrak{A}}$, the *Stone isomorphism*, is the map from A in $2^{W_{\mathfrak{A}}}$ defined by

$$f_{\mathfrak{A}}(a) = \{\nabla \in W_{\mathfrak{A}} :\ a \in \nabla\};$$

and $P_{\mathfrak{A}}$, the *Stone lattice* for $\mathfrak{A}$, is the range of $f_{\mathfrak{A}}$, i.e.,

$$P_{\mathfrak{A}} = \{f_{\mathfrak{A}}(a) :\ a \in A\}.$$

Theorem 8.14. (Stone's representation) *Every distributive lattice $\mathfrak{A}$ is isomorphic to $\langle P_{\mathfrak{A}}, \cap, \cup \rangle$, a set ring of the Stone space $W_{\mathfrak{A}}$, with $f_{\mathfrak{A}}$ being an isomorphism.*

Proof By the definition, $f_{\mathfrak{A}}$ is a surjection. Let us show that it is also an injection, i.e., $f_{\mathfrak{A}}(a) = f_{\mathfrak{A}}(b)$ only if $a = b$. Suppose $a \neq b$. Then either $a \not\leq b$ or $b \not\leq a$. In the former case, by Corollary 7.42, there is a prime filter ∇ in $\mathfrak{A}$ such that $a \in \nabla$ and $b \notin \nabla$, whence $f_{\mathfrak{A}}(a) \neq f_{\mathfrak{A}}(b)$. The latter one is considered analogously. Thus, $f_{\mathfrak{A}}$ is a bijection from A onto $P_{\mathfrak{A}}$, and it remains to show that $f_{\mathfrak{A}}$ preserves the lattice operations.

Suppose that $\nabla \in f_{\mathfrak{A}}(a \vee b)$, i.e., $a \vee b \in \nabla$. Since ∇ is prime, we then have either $a \in \nabla$ or $b \in \nabla$ and so $\nabla \in f_{\mathfrak{A}}(a) \cup f_{\mathfrak{A}}(b)$. Conversely, if $\nabla \in f_{\mathfrak{A}}(a) \cup f_{\mathfrak{A}}(b)$ then either $a \in \nabla$ or $b \in \nabla$ and so, by Theorem 7.23, $a \vee b \in \nabla$, i.e., $\nabla \in f_{\mathfrak{A}}(a \vee b)$. Hence $f_{\mathfrak{A}}(a \vee b) = f_{\mathfrak{A}}(a) \cup f_{\mathfrak{A}}(b)$.

Now suppose that $\nabla \in f_{\mathfrak{A}}(a \wedge b)$, i.e., $a \wedge b \in \nabla$. Then $a \in \nabla$, $b \in \nabla$ and so $\nabla \in f_{\mathfrak{A}}(a) \cap f_{\mathfrak{A}}(b)$. On the other hand, if $\nabla \in f_{\mathfrak{A}}(a) \cap f_{\mathfrak{A}}(b)$ then $a \in \nabla$, $b \in \nabla$ and $a \wedge b \in \nabla$, i.e., $\nabla \in f_{\mathfrak{A}}(a \wedge b)$. Thus, $f_{\mathfrak{A}}(a \wedge b) = f_{\mathfrak{A}}(a) \cap f_{\mathfrak{A}}(b)$. ❑

It should be clear that the lattice order $\leq$ in $\langle P_{\mathfrak{A}}, \cap, \cup \rangle$ is the ordinary set-theoretic inclusion $\subseteq$.

A pseudo-Boolean algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot \rangle$ is a distributive lattice and as such it is isomorphic to the Stone lattice $\langle P_{\mathfrak{A}}, \cap, \cup \rangle$. Since $\bot$ does not belong to any prime filter in $\mathfrak{A}$, $f_{\mathfrak{A}}(\bot) = \emptyset$ and so $\emptyset$ is the zero element in the Stone lattice for $\mathfrak{A}$. By Corollary 7.12, the operation $\to$ is uniquely determined by the lattice order $\subseteq$. Now recall that our goal is to represent $\mathfrak{A}$ as a subalgebra of $\mathfrak{F}^+$ for some intuitionistic Kripke frame $\mathfrak{F}$. So what we need is to define a partial order $R_{\mathfrak{A}}$ on $W_{\mathfrak{A}}$ such that (i) all sets in $P_{\mathfrak{A}}$ would be upward closed with respect to $R_{\mathfrak{A}}$ and (ii) $P_{\mathfrak{A}}$ would be closed under the standard operation $\supset$ on $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}} \rangle$.

Then $\mathfrak{F}_{\mathfrak{A}} = \langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, P_{\mathfrak{A}} \rangle$ would be an intuitionistic general frame whose dual $\mathfrak{F}_{\mathfrak{A}}^{+} = \langle P_{\mathfrak{A}}, \cap, \cup, \supset, \emptyset \rangle$ is isomorphic to $\mathfrak{A}$.

Let us define $R_{\mathfrak{A}}$ as is prescribed by our method:

$$\nabla_1 R_{\mathfrak{A}} \nabla_2 \text{ iff } \nabla_1 \subseteq \nabla_2, \text{ for all } \nabla_1, \nabla_2 \in W_{\mathfrak{A}}.$$

Lemma 8.15 *Every set $X \in P_{\mathfrak{A}}$ is upward closed in $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}} \rangle$.*

Proof Suppose $X = f_{\mathfrak{A}}(a)$, for some $a \in A$, $\nabla \in X$ and $\nabla R_{\mathfrak{A}} \nabla'$. Then $a \in \nabla$, $\nabla \subseteq \nabla'$ and so $\nabla' \in f_{\mathfrak{A}}(a)$. ❑

Let $\supset$ be the standard implication in the dual of $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}} \rangle$, i.e., for every $X, Y \subseteq W_{\mathfrak{A}}$

$$X \supset Y = \{\nabla \in W_{\mathfrak{A}} : \forall \nabla' \in W_{\mathfrak{A}} \ (\nabla R_{\mathfrak{A}} \nabla' \wedge \nabla' \in X \to \nabla' \in Y)\}.$$

Lemma 8.16 *$P_{\mathfrak{A}}$ is closed under $\supset$.*

Proof Let $X, Y \in P_{\mathfrak{A}}$, $X = f_{\mathfrak{A}}(a)$ and $Y = f_{\mathfrak{A}}(b)$, for some $a, b \in A$. We show that $X \supset Y = f_{\mathfrak{A}}(a \to b) \in P_{\mathfrak{A}}$.

Suppose $\nabla \in f_{\mathfrak{A}}(a \to b)$, i.e., $a \to b \in \nabla$, $\nabla R_{\mathfrak{A}} \nabla'$ and $\nabla' \in X$, i.e., $a \in \nabla'$. Then $\nabla \subseteq \nabla'$, $a \to b \in \nabla'$ and so, by the definition of filter, $b \in \nabla'$, which means that $\nabla' \in f_{\mathfrak{A}}(b)$. Therefore, $\nabla \in X \supset Y$.

Conversely, let $\nabla \in X \supset Y$ and show that $a \to b \in \nabla$. If $b \in \nabla$ then clearly $a \to b \in \nabla$. So suppose that $b \notin \nabla$. Let ∇_a be the filter in $\mathfrak{A}$ generated by the set $\{a\} \cup \nabla$. It follows from Theorem 7.24 that

$$\nabla_a = \{x \in A : \exists z \in \nabla \ z \wedge a \leq x\}.$$

We are going to show now that $b \in \nabla_a$. Then we shall have $z \wedge a \leq b$, for some $z \in \nabla$, whence, by Theorem 7.10, $z \leq a \to b$ and so $a \to b \in \nabla$.

Suppose $b \notin \nabla_a$. Then, by Theorem 7.41, there is a prime filter ∇' such that $\nabla_a \subseteq \nabla'$ and $b \notin \nabla'$. But this leads to a contradiction, since $\nabla R_{\mathfrak{A}} \nabla'$, $\nabla' \in X$ and so $\nabla' \in Y$, i.e. $b \in \nabla'$. ❑

Thus the triple $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, P_{\mathfrak{A}} \rangle$ is an intuitionistic general frame. We call it the *dual of* $\mathfrak{A}$ and denote it by $\mathfrak{A}_{+}$.

Our observations at the beginning of this section yield

Theorem 8.17 *The dual $(\mathfrak{A}_L)_+$ of the Tarski–Lindenbaum algebra for a si-logic L is isomorphic to the universal frame $\gamma\mathfrak{F}_L$ for L.*

As a consequence of Theorem 8.14, Corollary 7.12 and Lemmas 8.15, 8.16 we obtain

Theorem 8.18. (Stone's representation) *Every pseudo-Boolean algebra $\mathfrak{A}$ is isomorphic to its bidual $(\mathfrak{A}_+)^+$, with $f_{\mathfrak{A}}$ being an isomorphism.*

Corollary 8.19 *Every pseudo-Boolean algebra $\mathfrak{A}$ is (isomorphic to) a subalgebra of $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}} \rangle^+ = (\kappa\mathfrak{A}_+)^+$.*

According to Theorem 8.18, every Boolean algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot\rangle$ is isomorphic to the dual of the frame $\mathfrak{A}_+ = \langle W_\mathfrak{A}, R_\mathfrak{A}, P_\mathfrak{A}\rangle$. Since prime filters in Boolean algebras are ultrafilters, $R_\mathfrak{A}$ is just the identity relation and so $\supset$ in $\mathfrak{A}_+$ is defined by

$$X \supset Y = \{\nabla \in W_\mathfrak{A} : \ \nabla \in X \rightarrow \nabla \in Y\} = (W_\mathfrak{A} - X) \cup Y.$$

Thus we obtain the following:

Theorem 8.20 *Every Boolean algebra $\mathfrak{A}$ is isomorphic to $\langle P_\mathfrak{A}, \cap, \cup, \supset, \emptyset\rangle$, a set field of the Stone space $W_\mathfrak{A}$, with $f_\mathfrak{A}$ being an isomorphism.*

Corollary 8.21 *Every Boolean algebra $\mathfrak{A}$ is isomorphic to a subalgebra of the field $\langle 2^{W_\mathfrak{A}}, \cap, \cup, \supset, \emptyset\rangle$ of all subsets of the Stone space $W_\mathfrak{A}$.*

Now let us turn to modal algebras. Given such an algebra

$$\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot, \Box\rangle,$$

define a relation $R_\mathfrak{A}$ on $W_\mathfrak{A}$ as was described at the beginning of the section, i.e., for $\nabla_1, \nabla_2 \in W_\mathfrak{A}$, we put

$$\nabla_1 R_\mathfrak{A} \nabla_2 \ \text{ iff } \ \forall x \in A \ (\Box x \in \nabla_1 \rightarrow x \in \nabla_2).$$

Lemma 8.22 *For every $a \in A$, $f_\mathfrak{A}(\Box a) = \Box f_\mathfrak{A}(a)$ where $\Box$ in the right-hand part is the standard necessity operation in the frame $\langle W_\mathfrak{A}, R_\mathfrak{A}\rangle$.*

Proof Let $\nabla \in f_\mathfrak{A}(\Box a)$, i.e., $\Box a \in \nabla$, and let $\nabla R_\mathfrak{A} \nabla'$. Then by the definition of $R_\mathfrak{A}$, $a \in \nabla'$, i.e., $\nabla' \in f_\mathfrak{A}(a)$. Therefore, $\nabla \in \Box f_\mathfrak{A}(a)$. Conversely, let $\nabla \in \Box f_\mathfrak{A}(a)$, i.e., for every $\nabla' \in W_\mathfrak{A}$, $\nabla R_\mathfrak{A} \nabla'$ implies $\nabla' \in f_\mathfrak{A}(a)$. Suppose that $\nabla \notin f_\mathfrak{A}(\Box a)$ and consider the set

$$X = \{x \in A : \ \Box x \in \nabla\}.$$

Since $\Box\top = \top \in \nabla$, X is non-empty. Let $[X)$ be the filter generated by X. Then $a \notin [X)$. For otherwise, by Theorem 7.35, $x_1 \wedge \ldots \wedge x_n \leq a$ for some $x_1, \ldots, x_n \in X$, whence $\Box x_1 \wedge \ldots \wedge \Box x_n \leq \Box a$ and so $\Box a \in \nabla$, contrary to our assumption. By Theorem 7.41, there is an ultrafilter ∇' such that $[X) \subseteq \nabla'$ and $a \notin \nabla'$. But then $\nabla R_\mathfrak{A} \nabla'$ and so $\nabla' \in f_\mathfrak{A}(a)$, i.e., $a \in \nabla'$, which is a contradiction. ❑

It follows immediately from this lemma that $P_\mathfrak{A} = \{f_\mathfrak{A}(a) : \ a \in A\}$ is closed under $\Box$ in the frame $\langle W_\mathfrak{A}, R_\mathfrak{A}\rangle$. So $\langle W_\mathfrak{A}, R_\mathfrak{A}, P_\mathfrak{A}\rangle$ is a modal general frame. We call it the *dual of* $\mathfrak{A}$ and denote it by $\mathfrak{A}_+$. Clearly, we have

Theorem 8.23 *The dual $(\mathfrak{A}_L)_+$ of the Tarski–Lindenbaum algebra for a normal modal logic L is isomorphic to the universal frame $\gamma\mathfrak{F}_L$ for L.*

Combining together Theorem 8.20 and Lemma 8.22 we obtain

Theorem 8.24. (The Jónsson–Tarski representation) *Every modal algebra $\mathfrak{A}$ is isomorphic to its bidual $(\mathfrak{A}_+)^+$, with $f_{\mathfrak{A}}$ being an isomorphism.*

Corollary 8.25 *Every modal algebra $\mathfrak{A}$ is (isomorphic to) a subalgebra of the algebra $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}}\rangle^+ = (\kappa\mathfrak{A}_+)^+$.*

And one more algebraic structure needs a relational representation: we mean modal matrices. Suppose that $\langle\mathfrak{A}, \nabla\rangle$ is such a matrix. By Corollary 7.40, every proper filter in $\mathfrak{A}$ is the intersection of all ultrafilters in $\mathfrak{A}$ containing it. Define a set $\nabla_+ \subseteq W_{\mathfrak{A}}$ by taking

$$\nabla_+ = \{\nabla' \in W_{\mathfrak{A}} : \nabla \subseteq \nabla'\}.$$

The general frame $\langle\mathfrak{A}_+, \nabla_+\rangle$ with distinguished points will be called the *dual of* $\langle\mathfrak{A}, \nabla\rangle$.

Theorem 8.26 *Every modal matrix $\langle\mathfrak{A}, \nabla\rangle$ is isomorphic to $\langle(\mathfrak{A}_+)^+, (\nabla_+)^+\rangle$, with $f_{\mathfrak{A}}$ being an isomorphism.*

Proof In view of Theorem 8.24, it suffices to show that $f_{\mathfrak{A}}(\nabla) = (\nabla_+)^+$. This is clear if $\nabla = A$. So suppose that ∇ is a proper filter in $\mathfrak{A}$. If $a \in \nabla$ then

$$f_{\mathfrak{A}}(a) = \{\nabla' \in W_{\mathfrak{A}} : a \in \nabla'\} \supseteq \{\nabla' \in W_{\mathfrak{A}} : \nabla \subseteq \nabla'\}$$

and so $f_{\mathfrak{A}}(a) \in (\nabla_+)^+$. Conversely, if $X \in (\nabla_+)^+$ then there is $a \in A$ such that $X = f_{\mathfrak{A}}(a)$ and $\nabla_+ \subseteq f_{\mathfrak{A}}(a)$, i.e.,

$$\{\nabla' \in W_{\mathfrak{A}} : a \in \nabla'\} \supseteq \{\nabla' \in W_{\mathfrak{A}} : \nabla \subseteq \nabla'\}.$$

So $a \in \nabla$, since $\nabla = \bigcap\{\nabla' \in W_{\mathfrak{A}} : \nabla \subseteq \nabla'\}$, and $f_{\mathfrak{A}}(a) = X$. ❑

Our last result in this section provides a relational characterization of the consequence relations in modal and si-logics. It follows immediately from Theorem 7.73 and the representation theorems proved above.

Theorem 8.27 (i) *For $L \in$ NExt**K**, $\Gamma \vdash_L \varphi$ iff for any model $\mathfrak{M}$ based on a frame for L and any point x in $\mathfrak{M}$, $x \models \Gamma$ implies $x \models \varphi$.*

(ii) *For $L \in$ Ext**K**, $\Gamma \vdash_L \varphi$ iff for any model $\langle\mathfrak{M}, D\rangle$ based on a frame $\langle\mathfrak{F}, D\rangle$ for L and any point $x \in D$, $x \models \Gamma$ implies $x \models \varphi$.*

(iii) *For $L \in$ NExt**K**, $\Gamma \vdash^*_L \varphi$ iff for any model $\mathfrak{M}$ based on a frame for L, $\mathfrak{M} \models \Gamma$ implies $\mathfrak{M} \models \varphi$.*

(iv) *For $L \in$ Ext**Int**, $\Gamma \vdash_L \varphi$ iff for any model $\mathfrak{M}$ based on a frame for L, $\mathfrak{M} \models \Gamma$ implies $\mathfrak{M} \models \varphi$.*

8.3 From modal to intuitionistic frames and back

So far we have considered modal and intuitionistic frames separately. However, as was shown in Section 3.9, every modal Kripke model $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$ based on a quasi-ordered frame $\mathfrak{F}$ induces in a natural way an intuitionistic model, namely the skeleton $\rho\mathfrak{M} = \langle \rho\mathfrak{F}, \rho\mathfrak{V}\rangle$ of $\mathfrak{M}$, such that for any intuitionistic formula φ and any point x in $\mathfrak{F}$,

$$(\rho\mathfrak{M}, C(x)) \models \varphi \text{ iff } (\mathfrak{M}, x) \models \mathsf{T}(\varphi),$$

where T, the Gödel translation, prefixes $\Box$ to every subformula of φ.

The operator ρ can be easily extended to general frames. Given a modal quasi-ordered frame $\mathfrak{F} = \langle W, R, P\rangle$, we define an intuitionistic frame $\rho\mathfrak{F} = \langle \rho W, \rho R, \rho P\rangle$, called the *skeleton* of $\mathfrak{F}$, by taking

$$\rho P = \{\rho X : X \in P \wedge X = \Box X\} = \{\rho X : X \in P \wedge X = X{\uparrow}\},$$

where $\rho X = \{C(x) : x \in X\}$. To verify that $\rho\mathfrak{F}$ is really an intuitionistic frame, it suffices to observe that, for any upward closed $X, Y \in P$,

$$\begin{aligned}\rho(X) \cap \rho(Y) &= \rho(X \cap Y),\\ \rho(X) \cup \rho(Y) &= \rho(X \cup Y),\\ \rho(X) \supset \rho(Y) &= \rho(\Box(X \supset Y)).\end{aligned}$$

(Here $\supset$ in the left-hand side is intuitionistic and that in the right-hand one is Boolean.)

If $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is a model on $\mathfrak{F}$ then $\rho\mathfrak{M} = \langle \rho\mathfrak{F}, \rho\mathfrak{V}\rangle$, the *skeleton* of $\mathfrak{M}$, is the intuitionistic model with $\rho\mathfrak{V}$ defined, as before, by

$$\rho\mathfrak{V}(p) = \rho(\mathfrak{V}(\Box p)), \quad \text{for every variable } p.$$

Using the equations above, one can readily prove the following obvious generalization of Lemma 3.81.

Lemma 8.28. (Skeleton) *For every model $\mathfrak{M}$ based on a quasi-ordered frame $\mathfrak{F}$, every intuitionistic formula φ and every point x in $\mathfrak{F}$,*

$$(\rho\mathfrak{M}, C(x)) \models \varphi \quad \textit{iff} \quad (\mathfrak{M}, x) \models \mathsf{T}(\varphi)$$

and so

$$\rho\mathfrak{F} \models \varphi \quad \textit{iff} \quad \mathfrak{F} \models \mathsf{T}(\varphi).$$

Let us now clarify the algebraic meaning of the operator ρ. Observe first that the operation $\Box$ in a quasi-ordered modal frame $\mathfrak{F} = \langle W, R, P\rangle$ has the following properties: for every $X, Y \subseteq W$,

(I1) $\quad \Box(X \cap Y) = \Box X \cap \Box Y$;

(I2) $\quad \Box X \subseteq X$;

(I3) $\quad \Box\Box X = \Box X$;

(I4) $\quad \Box W = W$.

A set W with an operation $\Box$ on 2^W satisfying these four properties is usually called a *topological space* and $\Box$ an *interior operation* in this space. The dual operation $\Diamond$, defined by $\Diamond X = -\Box - X$ (where $-X = W - X$), is called the *closure operation* in the topological space. That is why the modal algebras $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box \rangle$ whose box $\Box$ satisfies (I1)–(I4) (in which X, Y, $\subseteq$ and W should be replaced by a, b, $\leq$ and $\top$, respectively) are known as *topological Boolean* (or *interior*, or *closure*) *algebras*. According to Exercise 7.16, a modal algebra $\mathfrak{A}$ is a topological Boolean algebra iff $\mathfrak{A} \models$ **S4**. Borrowing the topology terminology, we say that an element a in such an algebra is *open* if $a = \Box a$.

Proposition 8.29 *For every topological Boolean algebra and all open elements a and b in it,*

$$a \vee b = \Box(a \vee b).$$

Proof Since $a \leq a \vee b$, we have $a = \Box a \leq \Box(a \vee b)$. Likewise $b \leq \Box(a \vee b)$ and so $a \vee b \leq \Box(a \vee b)$. The desired equality follows then from (I2). ❑

By the definition, the dual $\mathfrak{F}^+$ of a quasi-ordered frame $\mathfrak{F}$ is a topological Boolean algebra. And conversely we have

Proposition 8.30 *The dual $\mathfrak{A}_+$ of a topological Boolean algebra $\mathfrak{A}$ is a quasi-ordered frame.*

Proof We must show that the accessibility relation $R_{\mathfrak{A}}$ in $\mathfrak{A}_+$ is reflexive and transitive. Let $\nabla \in W_{\mathfrak{A}}$ and $\Box a \in \nabla$. By (I2), $\Box a \leq a$ and so $a \in \nabla$. Therefore, $\nabla R_{\mathfrak{A}} \nabla$. Suppose now that $\nabla_1 R_{\mathfrak{A}} \nabla_2 R_{\mathfrak{A}} \nabla_3$ and $\Box a \in \nabla_1$. By (I3), $\Box\Box a \in \nabla_1$, whence $\Box a \in \nabla_2$ and $a \in \nabla_3$, which means that $\nabla_1 R_{\mathfrak{A}} \nabla_3$. ❑

Given a topological Boolean algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box \rangle$, we define an algebra $\boldsymbol{\rho}\mathfrak{A} = \langle \boldsymbol{\rho}A, \wedge, \vee, \to_\Box, \bot \rangle$ by taking $\boldsymbol{\rho}A = \{a \in A : a = \Box a\}$ and $a \to_\Box b = \Box(a \to b)$, for any $a, b \in \boldsymbol{\rho}A$. (By (I1) and Proposition 8.29, $\boldsymbol{\rho}A$ is closed under $\wedge$ and $\vee$.) $\boldsymbol{\rho}\mathfrak{A}$ is called the *algebra of open elements* of $\mathfrak{A}$.

Proposition 8.31 *For every quasi-ordered modal frame $\mathfrak{F} = \langle W, R, P \rangle$, $(\boldsymbol{\rho}\mathfrak{F})^+$ is isomorphic to $\boldsymbol{\rho}(\mathfrak{F}^+)$. So the algebra $\boldsymbol{\rho}\mathfrak{A}$ of open elements of any topological Boolean algebra $\mathfrak{A}$ is a pseudo-Boolean one; more exactly, $\boldsymbol{\rho}\mathfrak{A} \cong (\boldsymbol{\rho}(\mathfrak{A}_+))^+$.*

Proof It easy to verify that the function mapping $\boldsymbol{\rho}X$ to X, for every upward closed $X \in P$, is an isomorphism of $(\boldsymbol{\rho}\mathfrak{F})^+$ onto $\boldsymbol{\rho}(\mathfrak{F}^+)$. The dual $\mathfrak{A}_+$ of a topological Boolean algebra $\mathfrak{A}$ is a quasi-ordered frame whose dual, by Theorem 8.24, is isomorphic to $\mathfrak{A}$. So $\boldsymbol{\rho}\mathfrak{A} \cong \boldsymbol{\rho}((\mathfrak{A}_+)^+) \cong (\boldsymbol{\rho}(\mathfrak{A}_+))^+$. ❑

What is more important, the converse statement, i.e., that each pseudo-Boolean algebra (or intuitionistic frame) is an algebra of open elements (respectively, a skeleton) of some topological Boolean algebra (quasi-ordered modal frame), also holds. We will prove it first for general frames and then transfer, by duality, to algebras.

Given an intuitionistic frame $\mathfrak{F} = \langle W, R, P \rangle$, the simplest way of constructing a modal frame from it is to take the closure $\boldsymbol{\sigma}P$ of P under the Boolean operations $\cap$, $\cup$ and $\to$.

Lemma 8.32 *For every $X \subseteq W$, X is in σP iff*

$$X = (-X_1 \cup Y_1) \cap \ldots \cap (-X_n \cup Y_n)$$

for some $X_1, Y_1, \ldots, X_n, Y_n \in P$ and $n \geq 1$.

Proof $(\Rightarrow)$ By Exercise 1.1, we can represent each $X \in \sigma P$ as

$$\bigcap_{i=1}^{n}(-U_1^i \cup \ldots \cup -U_{k_i}^i \cup V_1^i \cup \ldots \cup V_{l_i}^i)$$

for some $U_j^i, V_j^i \in P$. Then, taking

$$X_i = \begin{cases} U_1^i \cap \ldots \cap U_{k_i}^i & \text{if } k_i > 0 \\ W & \text{if } k_i = 0 \end{cases}$$

and

$$Y_i = \begin{cases} V_1^i \cup \ldots \cup V_{k_i}^i & \text{if } k_i > 0 \\ \emptyset & \text{if } k_i = 0 \end{cases}$$

we obtain the representation we need.

$(\Leftarrow)$ is trivial. ❑

Now we observe that σP is closed under $\Box$ in $\langle W, R\rangle$ and that P coincides with the set of open (= upward closed) sets in σP. More exactly, the following lemma holds.

Lemma 8.33 *Suppose $X \in \sigma P$ is represented as in Lemma 8.32. Then*

$$\Box X = (X_1 \supset Y_1) \cap \ldots \cap (X_n \supset Y_n) \in P \subseteq \sigma P,$$

where the operations in the right-hand part of = are intuitionistic.

Proof By (I1), it suffices to verify that for every $X, Y \in P$,

$$\Box(-X \cup Y) = X \supset Y.$$

We leave this to the reader as an exercise. ❑

Thus, $\langle W, R, \sigma P\rangle$ is a partially ordered modal frame; we shall denote it by $\sigma\mathfrak{F}$.

Theorem 8.34 *Every intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$ is the skeleton of some quasi-ordered modal frame. For instance, $\mathfrak{F} \cong \rho\sigma\mathfrak{F}$.*

Proof We must show that $\rho\sigma P = P$. Suppose $X \in \rho\sigma P$. By Lemma 8.33, we then have $X = \Box X \in P$. Thus $\rho\sigma P \subseteq P$. The converse inclusion follows immediately from the definitions of ρ and σ. ❑

Notice that if $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is an intuitionistic model then $\boldsymbol{\sigma}\mathfrak{M} = \langle \boldsymbol{\sigma}\mathfrak{F}, \mathfrak{V} \rangle$ is a modal model having $\mathfrak{M}$ as its skeleton. So by the skeleton lemma, we have

$$(\mathfrak{M}, x) \models \varphi \;\text{ iff }\; (\boldsymbol{\sigma}\mathfrak{M}, x) \models \mathsf{T}(\varphi),$$

for every intuitionistic formula φ and every point x in $\mathfrak{F}$.

Given a pseudo-Boolean algebra $\mathfrak{A}$, we denote by $\boldsymbol{\sigma}\mathfrak{A}$ the topological Boolean algebra $(\boldsymbol{\sigma}(\mathfrak{A}_+))^+$.

Corollary 8.35 *Every pseudo-Boolean algebra $\mathfrak{A}$ is isomorphic to the algebra of open elements of some topological Boolean algebra. For instance, $\mathfrak{A} \cong \boldsymbol{\rho\sigma}\mathfrak{A}$.*

Proof By Theorem 8.34, $\mathfrak{A}_+ \cong \boldsymbol{\rho\sigma}(\mathfrak{A}_+)$ and so, by Theorem 8.18 and Proposition 8.31, $\mathfrak{A} \cong (\mathfrak{A}_+)^+ \cong (\boldsymbol{\rho\sigma}(\mathfrak{A}_+))^+ \cong \boldsymbol{\rho}((\boldsymbol{\sigma}(\mathfrak{A}_+))^+)$. ❑

It is worth noting that if $\mathfrak{F} = \langle W, R, \mathrm{Up}W \rangle$ is a finite intuitionistic Kripke frame then $\boldsymbol{\sigma}\mathfrak{F}$ is also a Kripke frame, i.e., $\boldsymbol{\sigma}\mathfrak{F} = \langle W, R, 2^W \rangle$. Indeed, by the finiteness of $\mathfrak{F}$, it suffices to show that $\{x\} \in \boldsymbol{\sigma}\mathrm{Up}W$ for every $x \in W$. But this is evident, since $\{x\} = x\!\uparrow \cap\, x\!\downarrow$, and $x\!\downarrow$ is the complementation of an upward closed set. The latter equality holds of course for infinite frames as well.

Let us say a point x in a modal frame $\mathfrak{F} = \langle W, R, P \rangle$ is an *atom* if $\{x\} \in P$. If $\mathfrak{F}$ is intuitionistic then we call $x \in W$ an *atom* if

$$W - x\!\downarrow\, \in P \;\text{ and }\; \{x\} \cup (W - x\!\downarrow) \in P.$$

A (modal or intuitionistic) frame $\mathfrak{F}$ is *atomic* if every point in $\mathfrak{F}$ is an atom. It should be clear that any finite atomic frame is a Kripke frame.

The observation above means that if $\mathfrak{F}$ is an intuitionistic Kripke frame then $\boldsymbol{\sigma}\mathfrak{F}$ is atomic. However, for an infinite $\mathfrak{F}$, $\boldsymbol{\sigma}\mathfrak{F}$ is not in general a Kripke frame. To see this, consider the intuitionistic frame $\mathfrak{G}$ shown in Fig. 8.1 (c). It is not hard to check that $\boldsymbol{\sigma}\mathfrak{G}$ is exactly the frame defined in Example 8.9, where we observed that $\boldsymbol{\sigma}\mathfrak{G} \models \mathbf{Grz}$, while $\boldsymbol{\kappa\sigma}\mathfrak{G} \not\models \mathbf{Grz}$. On the other hand, it follows from Lemma 8.33 that x is an atom in $\boldsymbol{\sigma}\mathfrak{F}$ only if x is an atom in $\mathfrak{F}$.

The operator $\boldsymbol{\sigma}$ is not the only one which, given an intuitionistic frame $\mathfrak{F}$, returns a modal frame whose skeleton is isomorphic to $\mathfrak{F}$. As an example, we define now an infinite class of such operators.

For Kripke frames $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$, we denote by $\mathfrak{F} \times \mathfrak{G}$ the *direct product* of $\mathfrak{F}$ and $\mathfrak{G}$, i.e., the frame $\langle W \times V, R \times S \rangle$ in which the relation $R \times S$ is defined component-wise:

$$\langle x_1, y_1 \rangle\, (R \times S)\, \langle x_2, y_2 \rangle \;\text{ iff }\; x_1 R x_2 \text{ and } y_1 S y_2.$$

Let $0 < k \leq \omega$. We will regard k as the set $\{0, \dots, k-1\}$ if $k < \omega$ and as $\{0, 1, \dots\}$ if $k = \omega$. Denote by $\boldsymbol{\tau}_k$ an operator which, given an intuitionistic frame $\mathfrak{F} = \langle W, R, P \rangle$, returns a quasi-ordered modal frame $\boldsymbol{\tau}_k\mathfrak{F} = \langle kW, kR, kP \rangle$ such that

FIG. 8.2.

(i) $\langle kW, kR\rangle$ is the direct product of the k-point cluster $\langle k, k^2\rangle$ and $\langle W, R\rangle$ (in other words, $\langle kW, kR\rangle$ is obtained from $\langle W, R\rangle$ by replacing its every point with a k-point cluster; see Fig. 8.2);

(ii) $\rho\tau_k\mathfrak{F} \cong \mathfrak{F}$;

(iii) $I \times X \in kP$, for every $I \subseteq k$ and $X \in \sigma P$.

For instance, we can take as kP the Boolean closure of the set

$$\{I \times X : \ I \subseteq k, \ X \in \sigma P\}.$$

To show that kP is closed under $\Box$ in $\langle kW, kR\rangle$, it suffices, by Lemma 8.32 and the fact that $k \times W - I \times X = (k - I) \times X \cup k \times (W - X)$, to prove the inclusion $\Box \bigcup_{i \in J}(I_i \times X_i) \in kP$ for every finite J and every $I_i \subseteq k$, $X_i \in \sigma P$. And this follows from the equality

$$\Box \bigcup_{i \in J}(I_i \times X_i) = k \times (\Box \bigcup \{\bigcap_{i \in J'} X_i : J' \subseteq J, \ \bigcup_{i \in J'} I_i = k\}),$$

from which, using Lemma 8.33, we can also derive (ii).

For a Kripke frame $\mathfrak{F} = \langle W, R, \mathrm{Up}W\rangle$ we can, of course, take $kP = 2^{kW}$ and then $\tau_k\mathfrak{F} = \langle kW, kR, 2^{kW}\rangle$.

8.4 Descriptive frames

The relationship between general frames and algebras, which was established in Sections 8.1 and 8.2, lacks some symmetry. Indeed, the representation theorems assert that every pseudo-Boolean and modal algebra $\mathfrak{A}$ is isomorphic to its bidual $(\mathfrak{A}_+)^+$, or in symbols

$$\mathfrak{A} \cong (\mathfrak{A}_+)^+. \tag{8.1}$$

But on the other hand, Example 8.7 shows that there are non-isomorphic frames having isomorphic duals. So the relation

$$\mathfrak{F} \cong (\mathfrak{F}^+)_+ \tag{8.2}$$

does not generally hold.

Those frames $\mathfrak{F}$ that satisfy (8.2) are called *descriptive*. Taking into account (8.1), we obtain an equivalent definition: a (modal or intuitionistic) frame is descriptive iff it is isomorphic to the dual of some (modal or, respectively, pseudo-Boolean) algebra.

In this section we give a subtler characterization of descriptive frames. This result is important not only from the aesthetic point of view. For, dealing with logics in **ExtInt** and **ExtK**, we are interested in finding possibly smaller classes of frames which are enough to determine all these logics.

Theorem 8.36 *Every logic in* **ExtInt** *and* **NExtK** *is characterized by a class of descriptive frames.*

Proof Follows from Theorems 8.17, 8.23 and 8.4. ❑

To characterize the constitution of descriptive frames, let us consider once again the universal frames, which are descriptive according to Theorems 8.17 and 8.23. In Section 5.1 we observed that every canonical model is differentiated, tight and compact. Adapting these notions to general frames, we arrive at the following definitions.

A frame $\mathfrak{F} = \langle W, R, P\rangle$ is *differentiated* if for any $x, y \in W$,

$$x = y \text{ iff } \forall X \in P\ (x \in X \leftrightarrow y \in X).$$

An intuitionistic $\mathfrak{F}$ is *tight* if for any $x, y \in W$,

$$xRy \text{ iff } \forall X \in P\ (x \in X \to y \in X).$$

Since in an intuitionistic frame $\mathfrak{F}$ the relation R is antisymmetric, $\mathfrak{F}$ is tight only if $\mathfrak{F}$ is differentiated.

A modal frame $\mathfrak{F}$ is *tight* if for any $x, y \in W$,

$$xRy \text{ iff } \forall X \in P\ (x \in \Box X \to y \in X)$$

or, dually, if

$$xRy \text{ iff } \forall X \in P\ (y \in X \to x \in X{\downarrow}).$$

Those frames that are both differentiated and tight are called *refined*. Finally, a frame $\mathfrak{F}$ is said to be *compact* if, for any families $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \overline{P} = \{W - X : X \in P\}$,

$$\bigcap(\mathcal{X} \cup \mathcal{Y}) = \{x :\ \forall X \in \mathcal{X} \forall Y \in \mathcal{Y}\ (x \in X \wedge x \in Y)\} \neq \emptyset$$

whenever $\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset$ for all finite subfamilies $\mathcal{X}' \subseteq \mathcal{X}$, $\mathcal{Y}' \subseteq \mathcal{Y}$. For modal frames, in which together with any X the set P contains its complement $-X = W - X$, this definition is equivalent to the more familiar one: $\mathfrak{F}$ is compact iff every subset $\mathcal{X}$ of P with the finite intersection property (i.e., with $\bigcap \mathcal{X}' \neq \emptyset$ for any finite subset $\mathcal{X}'$ of $\mathcal{X}$) has non-empty intersection.

Denote by $\mathcal{DF}$, $\mathcal{T}$, $\mathcal{CM}$, $\mathcal{R}$, $\mathcal{D}$ the classes of all differentiated, tight, compact, refined and descriptive frames, respectively. We are going to show that the combination of the first three properties is characteristic for descriptive frames, i.e.,

$$\mathcal{D} = \mathcal{DF} \cap \mathcal{T} \cap \mathcal{CM}.$$

But before that let us take a closer look at those properties.

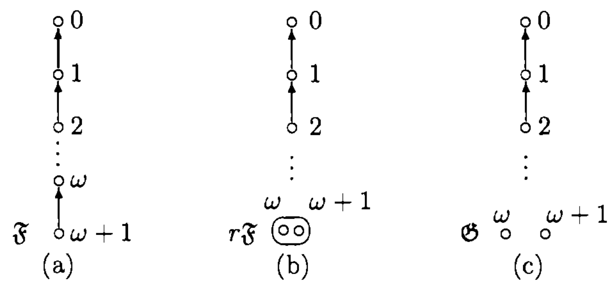


FIG. 8.3.

For a frame $\mathfrak{F} = \langle W, R, P\rangle$ and a point $x \in W$, put

$$Px = \{X \in P : \ x \in X\}, \quad \overline{P}x = \{X \in \overline{P} : \ x \in X\}.$$

If $\mathfrak{F}$ is modal then clearly we have $Px = \overline{P}x$.

Proposition 8.37 *For every frame $\mathfrak{F} = \langle W, R, P\rangle$ and every $x \in W$, Px is a prime filter in $\mathfrak{F}^+$.*

Proof Exercise. ❑

Proposition 8.38 *A frame $\mathfrak{F} = \langle W, R, P\rangle$ is differentiated iff, for every $x \in W$,*

$$\bigcap(Px \cup \overline{P}x) = \{x\}.$$

Proof Exercise. ❑

Proposition 8.39 *If $\mathfrak{F}$ is a differentiated intuitionistic frame then $\boldsymbol{\sigma}\mathfrak{F}$ is also differentiated. However, the operator $\boldsymbol{\rho}$ does not in general preserve differentiatedness.*

Proof The former claim follows from the definition. To prove the latter one, let us consider the modal frame $\mathfrak{F} = \langle W, R, P\rangle$ whose underlying Kripke frame is shown in Fig. 8.3 (a) and $P = \mathcal{X}_f \cup \mathcal{X}_c \cup \mathcal{X}_\omega \cup \mathcal{X}_{\omega+1}$ where

- $\mathcal{X}_f$ contains all finite sets of natural numbers;
- $\mathcal{X}_c$ contains all the complements (in the space W) of the sets in $\mathcal{X}_f$;
- $\mathcal{X}_\omega$ consists of all sets of the form $\{\omega\} \cup \{2n : \ n \geq m\} \cup X$, where $\omega > m \geq 0$ and $X \in \mathcal{X}_f$;
- $\mathcal{X}_{\omega+1} = \{\{\omega+1\} \cup \{2n+1 : \ n \geq m\} \cup X : \ \omega > m \geq 0, \ X \in \mathcal{X}_f\}$.

It is not hard to verify that $\mathfrak{F}$ is a differentiated modal frame and that every upward closed (= open) set in P is either W or consists of all natural numbers in some interval $[0, n]$. Therefore, the points ω and $\omega+1$ cannot be separated by any set in $\boldsymbol{\rho}P = \{X \in P : \ X = X\uparrow\}$ and so $\boldsymbol{\rho}\mathfrak{F}$ is not differentiated. ❑

Every Kripke frame is clearly differentiated. Moreover, for finite frames the converse is also true.

Proposition 8.40 *Every finite differentiated frame $\mathfrak{F} = \langle W, R, P\rangle$ is a Kripke frame.*

Proof In the modal case it suffices to show that $\{x\} \in P$ for any $x \in W$. But this follows from Proposition 8.38 and the finiteness of $\mathfrak{F}$. If $\mathfrak{F}$ is a finite differentiated intuitionistic frame then $\boldsymbol{\sigma}\mathfrak{F}$ is a finite differentiated modal frame. Ergo both $\boldsymbol{\sigma}\mathfrak{F}$ and $\boldsymbol{\rho\sigma}\mathfrak{F} = \mathfrak{F}$ are Kripke frames. ❑

Proposition 8.41 *A frame $\mathfrak{F} = \langle W, R, P\rangle$ is tight iff for every $x \in W$,*

$$x\uparrow = \bigcap\{X \in P : \ x\uparrow \subseteq X\}.$$

Proof ($\Rightarrow$) Suppose $\mathfrak{F}$ is intuitionistic and $y \in \bigcap\{X \in P : \ x\uparrow \subseteq X\}$. Then, since all $X \in P$ are upward closed, $y \in X$ for every $X \in P$ containing x, and so, by the definition of tightness, $y \in x\uparrow$. If $\mathfrak{F}$ is modal then $x\uparrow \subseteq X$ is equivalent to $x \in \Box X$ and so $y \in \bigcap\{X \in P : \ x\uparrow \subseteq X\}$ means that $y \in X$ for every $X \in P$ such that $x \in \Box X$, whence $y \in x\uparrow$.

($\Leftarrow$) Straightforward. ❑

Corollary 8.42 *Both operators $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ preserve tightness and refinedness.*

Example 8.43 The frame $\mathfrak{F}$, constructed in the proof of Proposition 8.39, is not tight, since

$$\bigcap\{X \in P : \ \omega\uparrow \subseteq X\} = \{\omega + 1\}\uparrow.$$

Also not tight is the differentiated intuitionistic frame $\mathfrak{G} = \langle V, S, Q\rangle$ whose underlying Kripke frame is depicted in Fig. 8.3 (c) (ω and $\omega + 1$ see all natural numbers) and $Q = \{V, \emptyset\} \cup \{x\uparrow : \ 0 \leq x \leq \omega\}$.

All Kripke frames are certainly tight. Moreover, by Proposition 8.40, every finite tight intuitionistic frame is a Kripke frame.

Example 8.44 Finite tight modal frames are not in general Kripke frames, as is demonstrated by the frame consisting of the cluster with points 1, 2 and the set of possible values $\{\emptyset, \{1, 2\}\}$. This frame is clearly tight, but not differentiated.

Thus, every Kripke frame is refined and every finite refined frame is a Kripke frame.

Given an arbitrary frame $\mathfrak{F} = \langle W, R, P\rangle$, we can construct a refined frame $r\mathfrak{F} = \langle rW, rR, rP\rangle$, having the same (modulo isomorphism) dual as $\mathfrak{F}$, by identifying some points in W and adding new arrows between them. First, define an equivalence relation $\sim$ on W by taking

$$x \sim y \ \text{ iff } \ \forall X \in P\ (x \in X \leftrightarrow y \in X).$$

Then we let $[x] = \{y \in W : \ x \sim y\}$, for $x \in W$, $rX = \{[x] : \ x \in X\}$, for $X \subseteq W$ and $rP = \{rX : \ X \in P\}$. Notice that $x \in X$ implies $[x] \subseteq X$, for any $X \in P$. Finally, we define a relation rR on rW by taking, for every $[x], [y] \in rW$,

$$[x]rR[y] \text{ iff } \forall X \in P\ (x \in X \to y \in X)$$

in the intuitionistic case and

$$[x]rR[y] \text{ iff } \forall X \in P\ (x \in \Box X \to y \in X)$$

in the modal one. Clearly this definition does not depend on the choice of x in $[x]$.

We denote $\langle rW, rR, rP\rangle$ by $r\mathfrak{F}$ and call it the *refinement* of $\mathfrak{F}$.

Proposition 8.45 *The refinement $r\mathfrak{F}$ of any frame $\mathfrak{F}$ is a refined frame and $\mathfrak{F}^+ \cong r\mathfrak{F}^+$.*

Proof The map r defined by $r(X) = rX$, for $X \in P$, is clearly a bijection from P onto rP. We show that r preserves $\cap$, $\cup$, $\supset$, $\Box$. The first three operations in the modal case present no difficulties. For instance,

$$\begin{aligned}[x] \in rX \cap rY &\text{ iff } [x] \in rX \text{ and } [x] \in rY\\ &\text{ iff } x \in X \text{ and } x \in Y\\ &\text{ iff } x \in X \cap Y\\ &\text{ iff } [x] \in r(X \cap Y).\end{aligned}$$

Suppose now that $\mathfrak{F}$ is intuitionistic, $X, Y \in P$ and $[x] \in rX \supset rY$. Then

$$\forall [y] \in rW\ ([x]rR[y] \wedge [y] \in rX \to [y] \in rY). \tag{8.3}$$

Since xRy implies $[x]rR[y]$, it follows that

$$\forall y \in W\ (xRy \wedge y \in X \to y \in Y),$$

i.e., $x \in X \supset Y$ and so $[x] \in r(X \supset Y)$.

Conversely, let $[x] \in r(X \supset Y)$ and show (8.3). Suppose otherwise, i.e., there is $y \in W$ such that $[x]rR[y]$, $y \in X$ but $y \notin Y$. Then $y \notin X \supset Y$ and so, by the definition of rR, $x \notin X \supset Y$ which is a contradiction.

The modal operation $\Box$ is considered analogously.

Thus, $r\mathfrak{F}$ is really a general frame and r is an isomorphism of $\mathfrak{F}^+$ onto $r\mathfrak{F}^+$. The fact that $r\mathfrak{F}$ is refined follows immediately from the definition. ❑

Example 8.46 The refinement $r\mathfrak{F}$ of the frame $\mathfrak{F}$, considered in the proof of Proposition 8.39, has the underlying Kripke frame as in Fig. 8.3 (b) and $rP = P$. The refinement $r\mathfrak{G}$ of the frame $\mathfrak{G}$ from Example 8.43 has the underlying Kripke frame as in Fig. 8.3 (a) and again $rQ = Q$. Finally, the refinement of the frame, considered in Example 8.44, is $\circ$.

Using the refinement we can show that the notions of finite approximability and finite model property are equivalent.

Theorem 8.47 *A modal or si-logic L is finitely approximable iff it has the finite model property.*

Proof The implication ($\Rightarrow$) is trivial. To show the converse, suppose that $\varphi \notin L$. Then there is a finite model $\mathfrak{M}$ such that $\mathfrak{M} \models L$ and $\mathfrak{M} \not\models \varphi$. Let $\mathfrak{F}$ be the general frame associated with $\mathfrak{M}$. Clearly, $\mathfrak{F}$ validates L and refutes φ. But then $r\mathfrak{F}$ separates φ from L as well. It remains to recall that a finite differentiated frame is a Kripke frame. ❑

Now let us consider compact frames.

Proposition 8.48 *A frame $\mathfrak{F} = \langle W, R, P\rangle$ is compact iff every prime filter ∇ in $\mathfrak{F}^+$ is of the form Px for some $x \in W$.*

Proof ($\Rightarrow$) Let $\Delta = P - \nabla$. Since ∇ is a prime filter, by Proposition 7.27, Δ is a prime ideal. Therefore, ∇ has the finite intersection property and Δ has the finite union property, i.e., $\bigcup \mathcal{Z} \neq W$ for any finite subset $\mathcal{Z}$ of Δ.

Now we take $\mathcal{X} = \nabla$ and $\mathcal{Y} = \{W - X : \ X \in \Delta\}$. Suppose $X_1, \dots, X_n \in \mathcal{X}$, $Y_1, \dots, Y_m \in \mathcal{Y}$ and consider the set

$$Z = X_1 \cap \ldots \cap X_n \cap Y_1 \cap \ldots \cap Y_m.$$

Let $X = X_1 \cap \ldots \cap X_n$ and $Y = Y_1 \cap \ldots \cap Y_m$. Clearly $X \in \mathcal{X}$ and, since $W - Y = (W - Y_1) \cup \ldots \cup (W - Y_m) \in \Delta$, we have also $Y \in \mathcal{Y}$. Suppose $Z = \emptyset$. Then $X \subseteq W - Y \in \Delta$ and so, since $\mathcal{X}$ is a filter, $W - Y \in \mathcal{X}$, which is a contradiction.

Thus, $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property and, by the compactness of $\mathfrak{F}$, there is an $x \in \bigcap(\mathcal{X} \cup \mathcal{Y})$.

We show now that $\nabla = Px$. Clearly $\nabla \subseteq Px$. So suppose $X \in P$ and $x \in X$. By the definition, X is either in ∇ or in Δ. If $X \in \nabla$ then we are done. But in fact this is the only possibility, for if $X \in \Delta$ then $W - X \in \mathcal{Y}$ and so $x \notin \bigcap \mathcal{Y}$, which is a contradiction.

($\Leftarrow$) Suppose $\mathcal{X} \subseteq P$, $\mathcal{Y} \subseteq \overline{P}$ and $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property. We must show that the intersection of all sets in $\mathcal{X} \cup \mathcal{Y}$ is not empty.

Let ∇ be the filter in $\mathfrak{F}^+$ generated by $\mathcal{X}$ and Δ the ideal generated by $\{W - Y : \ Y \in \mathcal{Y}\}$. Then $\nabla \cap \Delta = \emptyset$. For otherwise there are $X = \bigcap \mathcal{X}'$ and $Y = \bigcap \mathcal{Y}'$, for some finite $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$, such that $X \subseteq W - Y$, whence $X \cap Y = \emptyset$, contrary to $\mathcal{X} \cup \mathcal{Y}$ having the finite intersection property. Hence, by Exercise 7.18, there is a prime filter ∇' for which $\nabla \subseteq \nabla'$ and $\nabla' \cap \Delta = \emptyset$.

Let $\nabla' = Px$ for some $x \in W$. Then $x \in Z$ for any $Z \in \nabla$ and $x \notin Z$ for any $Z \in \Delta$, whence $x \in X$ and $x \in Y$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and so $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$. ❑

Proposition 8.49 *The operators $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ preserve compactness.*

Proof That $\boldsymbol{\rho}$ preserves compactness follows immediately from the definition. Indeed, if $\mathfrak{F} = \langle W, R, P\rangle$ is a compact quasi-ordered modal frame, $\mathcal{X} \subseteq \boldsymbol{\rho}P$, $\mathcal{Y} \subseteq \overline{\boldsymbol{\rho}P}$ and $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then

$$\mathcal{X}' \cup \mathcal{Y}' = \{Z \in P : \ (Z = Z{\uparrow} \wedge \boldsymbol{\rho}Z \in \mathcal{X}) \vee (Z = Z{\downarrow} \wedge \boldsymbol{\rho}Z \in \mathcal{Y})\}$$

also possesses this property in $\mathfrak{F}$. Hence $\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset$ and so $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

To prove that $\boldsymbol{\sigma}$ preserves compactness, we use Proposition 8.48. Suppose $\mathfrak{F} = \langle W, R, P \rangle$ is a compact intuitionistic frame, ∇ is a prime filter in $(\boldsymbol{\sigma}\mathfrak{F})^+$ and show that $\nabla \in \boldsymbol{\sigma}Px$, for some $x \in W$. Observe first that

$$\nabla' = \{X \in \nabla : \; X = X\uparrow\}$$

is a prime filter in $\mathfrak{F}^+$. By Proposition 8.48, there is $x \in W$ such that $\nabla' = Px$. We show that $\nabla = \boldsymbol{\sigma}Px$, i.e., $\nabla = \{X \in \boldsymbol{\sigma}P : \; x \in X\}$.

Suppose $X \in \nabla$, but $x \notin X$. As we know, X can be represented in the form

$$X = (-X_1 \cup Y_1) \cap \ldots \cap (-X_n \cup Y_n),$$

for some $X_i, Y_i \in P$. But then there is $i \in \{1, \ldots, n\}$ such that $x \notin -X_i \cup Y_i$, i.e., $x \in X_i$ and $x \notin Y_i$. On the other hand, $-X_i \cup Y_i \in \nabla$ and so either $-X_i \in \nabla$ or $Y_i \in \nabla$, since ∇ is prime. In the former case $X_i \notin \nabla$ and consequently $X_i \notin \nabla'$, which is a contradiction, because $x \in X_i$. And in the latter $Y_i \in \nabla'$, which is again a contradiction, since x then must be in Y_i. Thus, $\nabla \subseteq \{X \in \boldsymbol{\sigma}P : \; x \in X\}$.

To prove the converse inclusion, assume that $x \in X \in \boldsymbol{\sigma}P$, but $X \notin \nabla$. Since ∇ is an ultrafilter, we then have $-X \in \nabla$ and so, as we have just established, $x \in -X$, which is a contradiction. ❑

Proposition 8.50 *No infinite Kripke frame is compact.*

Proof Suppose first that $\mathfrak{F} = \langle W, R, 2^W \rangle$ is an infinite modal Kripke frame. Then the set $\mathcal{X} = \{X \subseteq W : \; W - X \text{ is finite}\}$ has the finite intersection property, but no point x is in $\bigcap \mathcal{X}$, since $W - \{x\} \in \mathcal{X}$.

Now let $\mathfrak{F} = \langle W, R, \mathrm{Up}W \rangle$ be an infinite intuitionistic Kripke frame. Then, according to Exercise 2.3, one of the following three cases holds.

Case 1. $\mathfrak{F}$ contains an infinite descending chain $\ldots Ry_n \ldots Ry_2Ry_1$ of distinct points. Then we take

$$\mathcal{Y} = \{y_i\downarrow : \; i = 1, 2, \ldots\} \subseteq \overline{\mathrm{Up}W}, \quad \mathcal{X} = \{W - \bigcap \mathcal{Y}\} \subseteq \mathrm{Up}W.$$

It is clear that $\mathcal{Y}$ has the finite intersection property and

$$y_i \in y_i\downarrow \cap (W - \bigcap \mathcal{Y}).$$

However, by the definition, $\bigcap(\mathcal{X} \cup \mathcal{Y}) = \emptyset$.

Case 2. $\mathfrak{F}$ contains an infinite ascending chain $x_1Rx_2R\ldots$ of distinct points. In this case we take

$$\mathcal{X} = \{x_i\uparrow : \; i = 1, 2, \ldots\} \subseteq \mathrm{Up}W, \quad \mathcal{Y} = \{W - \bigcap \mathcal{X}\} \subseteq \overline{\mathrm{Up}W}.$$

Again $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, but $\bigcap(\mathcal{X} \cup \mathcal{Y}) = \emptyset$.

Case 3. $\mathfrak{F}$ contains an infinite antichain Z. Consider the sets

$$\mathcal{X} = \{X\uparrow : \; X \subseteq Z \text{ and } Z - X \text{ is finite}\} \subseteq \mathrm{Up}W,$$

$$\mathcal{Y} = \{\dot{Y}{\downarrow}:\ Y \subseteq Z \text{ and } Z - Y \text{ is finite}\} \subseteq \overline{\mathrm{Up}W}.$$

Clearly, $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property. However, $\bigcap(\mathcal{X} \cup \mathcal{Y})$ is empty. ❑

We are now in a position to prove the main result of this section.

Theorem 8.51 *A frame $\mathfrak{F} = \langle W, R, P\rangle$ is descriptive iff it is differentiated, tight and compact.*

Proof $(\Rightarrow)$ It suffices to show that the dual $\mathfrak{A}_+ = \langle W_\mathfrak{A}, R_\mathfrak{A}, P_\mathfrak{A}\rangle$ of every pseudo-Boolean and modal algebra $\mathfrak{A}$ is differentiated, tight and compact.

If ∇_1 and ∇_2 are distinct prime filters in $\mathfrak{A}$ then there is an element a contained in only one of them, say in ∇_1. Then $\nabla_1 \in f_\mathfrak{A}(a) \in P_\mathfrak{A}$ and $\nabla_2 \notin f_\mathfrak{A}(a)$. So $\mathfrak{A}_+$ is differentiated.

The fact that $\mathfrak{A}_+$ is tight follows directly from the definition of $R_\mathfrak{A}$.

To prove that $\mathfrak{A}_+$ is compact, recall that $f_\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ onto $(\mathfrak{A}_+)^+$. So every prime filter in $(\mathfrak{A}_+)^+$ is of the form

$$f_\mathfrak{A}(\nabla) = \{f_\mathfrak{A}(a) :\ \nabla \in f_\mathfrak{A}(a)\} = P_\mathfrak{A}\nabla,$$

for some prime filter ∇ in $\mathfrak{A}$, and we can use Proposition 8.48.

$(\Leftarrow)$ We must show now that $\mathfrak{F} \cong (\mathfrak{F}^+)_+$. By Proposition 8.37, Px is a prime filter in $\mathfrak{F}^+$ and so we can define a map $f_\mathfrak{F}$ from W into $W_{\mathfrak{F}^+}$ by taking $f_\mathfrak{F}(x) = Px$, for any $x \in W$. By Proposition 8.48,

$$W_{\mathfrak{F}^+} = \{Px :\ x \in W\}.$$

So $f_\mathfrak{F}$ is a surjection. Moreover, $f_\mathfrak{F}$ is an injection, since $\mathfrak{F}$ is differentiated.

If $\mathfrak{F}$ is intuitionistic and $x, y \in W$ then, since $\mathfrak{F}$ is tight,

$$xRy \ \text{ iff } \ Px \subseteq Py \ \text{ iff } \ PxR_{\mathfrak{F}^+}Py.$$

If $\mathfrak{F}$ is modal then again, by the tightness of $\mathfrak{F}$, we obtain

$$xRy \ \text{ iff } \ \forall X \in P\ (\Box X \in Px \to X \in Py) \ \text{ iff } \ PxR_{\mathfrak{F}^+}Py.$$

Thus, it remains to show that, for any $X \subseteq W$, $X \in P$ iff $f_\mathfrak{F}(X) \in P_{\mathfrak{F}^+}$. Recall that $P_{\mathfrak{F}^+} = \{f_{\mathfrak{F}^+}(X) :\ X \in P\}$ and $f_{\mathfrak{F}^+}(X) = \{Px :\ x \in X\}$. So if $X \in P$ then $f_\mathfrak{F}(X) = \{Px :\ x \in X\} = f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$. Conversely, if $f_\mathfrak{F}(X) \in P_{\mathfrak{F}^+}$ then $f_\mathfrak{F}(X) = \{Py :\ y \in Y\} = f_\mathfrak{F}(Y)$, for some $Y \in P$, whence $X = Y$, since $f_\mathfrak{F}$ is a bijection. ❑

Example 8.52 Let $\mathfrak{F} = \langle W, R, P\rangle$ be the frame whose underlying Kripke frame is shown in Fig. 8.4 ($\omega + 1$ sees only ω and the subframe generated by ω is transitive) and $P = \{X_1 \cup X_2 \cup X_3 :\ X_i \in \mathcal{X}_i,\ i = 1, 2, 3\}$, where

- $\mathcal{X}_1$ contains all finite sets of natural numbers including $\emptyset$,

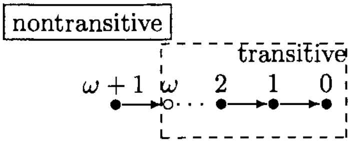


FIG. 8.4.

- $\mathcal{X}_2$ contains $\emptyset$ and all intervals $\{x : \ n \le x \le \omega\}$, for $n = 0, 1, \ldots$,
- $\mathcal{X}_3 = \{\emptyset, \{\omega+1\}\}$.

It is easy to see that P is closed under $\cap$, $-$ and $\downarrow$ (in fact $\mathfrak{F}$ is generated by $\emptyset$). Clearly, $\mathfrak{F}$ is refined. Suppose $\mathcal{X}$ is a subset of P with the finite intersection property. If $\mathcal{X}$ contains a finite set (from $\mathcal{X}_1$ or $\mathcal{X}_3$) then obviously $\bigcap \mathcal{X} \neq \emptyset$. And if $\mathcal{X}$ consists of only infinite intervals from $\mathcal{X}_2$ then $\omega \in \bigcap \mathcal{X}$. Thus, $\mathfrak{F}$ is descriptive.

We invite the reader to check that the frames $\mathfrak{F}$ and $\mathfrak{G}$ considered in Example 8.43 are compact (differentiated but not tight); so their refinements $r\mathfrak{F}$ and $r\mathfrak{G}$ are descriptive.

As a consequence of Theorem 8.51, Proposition 8.49 and Corollary 8.42 we obtain

Theorem 8.53 *The maps ρ and τ preserve descriptiveness.*

It is not hard to extend the results established above a bit further, namely, to modal matrices and frames with distinguished points. A frame with distinguished points $\langle \mathfrak{F}, D\rangle$ is called *descriptive* if $\mathfrak{F} = \langle W, R, P\rangle$ is descriptive and

$$D = \bigcap \{X \in P : \ D \subseteq X\}. \tag{8.4}$$

Theorem 8.54 *Every descriptive frame $\langle \mathfrak{F}, D\rangle$ with distinguished points is isomorphic to its bidual $\langle (\mathfrak{F}^+)_+, (D^+)_+\rangle$.*

Proof By the proof of Theorem 8.51, the map $f_{\mathfrak{F}}$, defined by $f_{\mathfrak{F}}(x) = Px$, is an isomorphism of $\mathfrak{F}$ onto $(\mathfrak{F}^+)_+$. We show that $f_{\mathfrak{F}}(D) = (D^+)_+$. Indeed, $D^+ = \{X \in P : D \subseteq X\}$, $(D^+)_+ = \{Px : D^+ \subseteq Px\} = \{Px : \ x \in \bigcap\{X \in P : D \subseteq X\}\}$ and so, by (8.4), $(D^+)_+ = \{Px : x \in D\} = f_{\mathfrak{F}}(D)$. ❑

Theorem 8.55 *The dual $\langle \mathfrak{A}_+, \nabla_+\rangle$ of every modal matrix $\langle \mathfrak{A}, \nabla\rangle$ is descriptive.*

Proof Exercise. ❑

As a consequence we obtain the following

Theorem 8.56 *Every logic in* **ExtK** *is characterized by a class of descriptive frames with distinguished points.*

8.5 Truth-preserving operations on general frames

To complete the fragment of duality theory suitable for the aims of this book, we will find out what operations on general frames correspond to the three fundamental algebraic operations of forming homomorphic images, subalgebras and direct products. In fact all we need is to extend the notions of generated subframe, reduction and disjoint union from Kripke frames to general ones.

A frame $\mathfrak{G} = \langle V, S, Q\rangle$ is a *generated subframe* of $\mathfrak{F} = \langle W, R, P\rangle$ (notation: $\mathfrak{G} \subseteq \mathfrak{F}$) if $\kappa\mathfrak{G} \subseteq \kappa\mathfrak{F}$ and $Q = \{X \cap V : X \in P\}$.

Theorem 8.57 *If h is an isomorphism of $\mathfrak{G} = \langle V, S, Q\rangle$ onto a generated subframe of $\mathfrak{F} = \langle W, R, P\rangle$ then the map h^+ defined by*

$$h^+(X) = h^{-1}(X) = \{x \in V : h(x) \in X\}, \textit{ for every } X \in P,$$

is a homomorphism of $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.

Proof Without loss of generality we may assume h to be the identity map. Then $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$ and $h^+(X) = X \cap V$.

Clearly, h^+ is a surjection. We show that it preserves $\cup$, $-$ and $\downarrow$, assuming $\mathfrak{G}$ and $\mathfrak{F}$ to be modal frames, and leave the intuitionistic case to the reader. Let $X, Y \in P$. Then we have

$$h^+(X \cup Y) = (X \cup Y) \cap V = (X \cap V) \cup (Y \cap V) = h^+(X) \cup h^+(Y);$$

$$h^+(W - X) = (W - X) \cap V = V - (X \cap V) = V - h^+(X);$$
$$h^+(X{\downarrow}R) = X{\downarrow}R \cap V = (X \cap V){\downarrow}S = h^+(X){\downarrow}S.$$

The only non-trivial passage here is the middle = in the last line where we use the fact that V is upward closed in $\mathfrak{F}$. ❑

Observe that proving this theorem we used only that V is upward closed in $\mathfrak{F}$ and $Q = \{X \cap V : X \in P\}$; the fact that Q is closed under modal or intuitionistic operations was redundant. This means that, given a frame $\mathfrak{F} = \langle W, R, P\rangle$ and a set $Y \subseteq W$, we can take $V = Y{\underline{\uparrow}}^{\omega}R$, $S = R \cap V^2$, $Q = \{X \cap V : X \in P\}$ and then the triple $\mathfrak{G} = \langle V, S, Q\rangle$ will be a general frame which is a generated subframe of $\mathfrak{F}$. We call it the *subframe of $\mathfrak{F}$ generated by Y*.

A model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ on a frame $\mathfrak{G} = \langle V, S, Q\rangle$ is a *generated submodel* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ (notation: $\mathfrak{N} \subseteq \mathfrak{M}$) if $\mathfrak{G} \subseteq \mathfrak{F}$ and $\mathfrak{U}(p) = \mathfrak{V}(p) \cap V$ for every variable p. As a consequence of Theorem 8.57 we immediately obtain that the generation theorems in Sections 2.3 and 3.3 and their corollaries (Theorems 2.7, 3.11 and Corollaries 2.8, 2.9, 3.12) hold for general frames as well. Of course the same results can easily be derived directly from those theorems. Besides, we clearly have

Theorem 8.58 *Every superintuitionistic and normal modal logic is characterized by the class of its rooted general frames.*

Now we prove a theorem which is dual to Theorem 8.57.

Theorem 8.59 *Suppose h is a homomorphism of a modal or pseudo-Boolean algebra $\mathfrak{A}$ onto a modal or, respectively, pseudo-Boolean algebra $\mathfrak{B}$. Then the map*

h_+ defined by $h_+(\nabla) = h^{-1}(\nabla)$, for every prime filter ∇ in $\mathfrak{B}$, is an isomorphism of $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.

Proof By Theorem 7.68, h_+ is a injection from $W_{\mathfrak{B}}$ into $W_{\mathfrak{A}}$. Consider the set

$$W = \{\nabla' \in W_{\mathfrak{A}} : \ h^{-1}(\top) \subseteq \nabla'\}.$$

Clearly W is upward closed in $W_{\mathfrak{A}}$ (in the modal case this follows from the fact that $\Box\top = \top$). We show that h_+ is an isomorphism of $\mathfrak{B}_+$ onto the subframe $\mathfrak{F} = \langle W, R, P\rangle$ of $\mathfrak{A}_+$ generated by W. Notice that $X \in P$ iff, for some element a in $\mathfrak{A}$, $X = \{\nabla' \in W : \ a \in \nabla'\}$.

First we prove that h_+ is a bijection from $W_{\mathfrak{B}}$ onto W. Since every filter contains $\top$, $h_+(\nabla) \in W$ for all $\nabla \in W_{\mathfrak{B}}$. Suppose $\nabla' \in W$ and show that $\nabla' = h^{-1}h(\nabla')$. Indeed, clearly we have $\nabla' \subseteq h^{-1}h(\nabla')$. On the other hand, if $a \in h^{-1}h(\nabla')$ then $h(a) = h(b)$, for some $b \in \nabla'$. And since h is a homomorphism, $h(b \to a) = h(b) \to h(a) = \top$, from which $b \to a \in \nabla'$ and so $a \in \nabla'$. Thus, for every element a in $\mathfrak{A}$ and every $\nabla' \in W$,

$$a \in \nabla' \ \text{ iff } \ h(a) \in h(\nabla'). \tag{8.5}$$

It follows that $h(\nabla')$ is a prime filter in $\mathfrak{B}$, h is a bijection from W onto $W_{\mathfrak{B}}$ and h_+ is a bijection from $W_{\mathfrak{B}}$ onto W. It follows also that, for any $X \subseteq W_{\mathfrak{B}}$, $X \in P_{\mathfrak{B}}$ iff $\mathrm{h}_+(X) \in P$.

It remains to show that $\nabla_1 R_{\mathfrak{B}} \nabla_2$ iff $h_+(\nabla_1) R_{\mathfrak{A}} h_+(\nabla_2)$. This is fairly easy for pseudo-Boolean algebras, since $\nabla_1 R_{\mathfrak{B}} \nabla_2$ means $\nabla_1 \subseteq \nabla_2$. So let us consider the modal case. Suppose that $\nabla_1 R_{\mathfrak{B}} \nabla_2$, i.e., $\Box b \in \nabla_1$ implies $b \in \nabla_2$, for all b in $\mathfrak{B}$, and that $\Box a \in h_+(\nabla_1)$ for some a in $\mathfrak{A}$. Then $h(\Box a) = \Box h(a) \in \nabla_1$, whence $h(a) \in \nabla_2$ and $a \in h_+(\nabla_2)$. Therefore, $h_+(\nabla_1) R_{\mathfrak{A}} h_+(\nabla_2)$. Conversely, suppose $h_+(\nabla_1) R_{\mathfrak{A}} h_+(\nabla_2)$. Then for all a in $\mathfrak{A}$, $\Box a \in h_+(\nabla_1)$ implies $a \in h_+(\nabla_2)$. By (8.5), if $h(a) = b$ and $\Box b \in \nabla_1$ then $\Box a \in h_+(\nabla_1)$, hence $a \in h_+(\nabla_2)$ and so $b \in \nabla_2$. Therefore, $\nabla_1 R_{\mathfrak{B}} \nabla_2$. ❑

For a cardinal $\varkappa$, a frame $\mathfrak{F}$ is said to be *$\varkappa$-generated* if its dual $\mathfrak{F}^+$ is an $\varkappa$-generated algebra. $\mathfrak{F}$ is *finitely generated* if it is n-generated, for some $n < \omega$. Generators of $\mathfrak{F}^+$ will be regarded as *generators* of $\mathfrak{F}$ as well.[11] The dual of the free algebra of rank $\varkappa$ in the variety Var$L$ of a logic L is called the *universal frame of rank $\varkappa$* for L; it will be denoted by $\mathfrak{F}_L(\varkappa)$. Clearly, for every cardinal $\varkappa$, there is only one (up to isomorphism) universal frame $\mathfrak{F}_L(\varkappa)$. So $\mathfrak{F}_L(\aleph_0)$ is the universal frame $\gamma\mathfrak{F}_L$ for L defined in Section 8.1.

Theorem 8.60 *Every descriptive $\varkappa'$-generated frame for a logic L is (isomorphic to) a generated subframe of $\mathfrak{F}_L(\varkappa)$, for any $\varkappa \geq \varkappa'$.*

Proof Follows from Theorems 7.64 and 8.59. ❑

[11]Thus, we have two ways of "generating" frames: relational (i.e., forming generated subframes) and algebraical. It will always be clear from the context which of them is used.

According to Theorem 8.12, every quasi-normal modal logic L is characterized by the class of all frames for L with a single distinguished point. Using the generation theorem, we can somewhat refine this result.

Theorem 8.61 *Every consistent quasi-normal modal logic L is characterized by the class of all frames $\langle \mathfrak{F}, \{d\} \rangle$ for L with root d.*

As to generated subframes of modal frames with distinguished points, let us recall first that h is a homomorphism of a matrix $\langle \mathfrak{A}, \nabla' \rangle$ onto a matrix $\langle \mathfrak{B}, \nabla'' \rangle$ if h is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ and $h^{-1}(\nabla'') = \nabla'$. This means in particular that $h^{-1}(\top) \subseteq \nabla'$ and so the set ∇'_+ of distinguished points in $\mathfrak{A}_+$ (which consists of all ultrafilters in $\mathfrak{A}$ containing ∇') is a subset of W defined in the proof of Theorem 8.59. Moreover, $h_+(\nabla''_+) = \nabla'_+$, i.e., roughly speaking the distinguished points in $\mathfrak{B}_+$ are exactly the same as in $\mathfrak{A}_+$. This observation motivates the following definition.

A modal frame $\langle \mathfrak{G}, E \rangle$ with distinguished points E is a *generated subframe* of a modal frame $\langle \mathfrak{F}, D \rangle$ with distinguished points D (notation: $\langle \mathfrak{G}, E \rangle \subseteq \langle \mathfrak{F}, D \rangle$) if $\mathfrak{G} \subseteq \mathfrak{F}$ and $E = D$.

The next two theorems are left to the reader as an exercise.

Theorem 8.62 *Suppose $\mathfrak{G} = \langle V, S, Q \rangle$ and $\mathfrak{F} = \langle W, R, P \rangle$ are modal frames, E and D are their distinguished points and $\langle \mathfrak{G}, E \rangle \subseteq \langle \mathfrak{F}, D \rangle$. Then the map h^+ defined by $h^+(X) = X \cap V$, for every $X \in P$, is a homomorphism of $\langle \mathfrak{F}^+, D^+ \rangle$ onto $\langle \mathfrak{G}^+, E^+ \rangle$.*

Theorem 8.63 *Suppose that h is a homomorphism of a modal matrix $\langle \mathfrak{A}, \nabla' \rangle$ onto $\langle \mathfrak{B}, \nabla'' \rangle$. Then the map h_+ defined by $h_+(\nabla) = h^{-1}(\nabla)$, for every ultrafilter ∇ in $\mathfrak{B}$, is an isomorphism of $\langle \mathfrak{B}_+, \nabla''_+ \rangle$ onto a generated subframe of $\langle \mathfrak{A}_+, \nabla'_+ \rangle$.*

It is clear that every frame with distinguished points is semantically equivalent to its every generated subframe. The following result which shows the relational meaning of extensions of matrices is also left to the reader.

Theorem 8.64 (i) *If $E \subseteq D$ then $\langle \mathfrak{F}^+, E^+ \rangle$ is an extension of $\langle \mathfrak{F}^+, D^+ \rangle$.*

(ii) *If $\langle \mathfrak{A}, \nabla' \rangle$ is an extension of $\langle \mathfrak{A}, \nabla \rangle$ then $\nabla'_+ \subseteq \nabla_+$.*

The relational counterpart of the notion of subalgebra is that of reduct. Given frames $\mathfrak{F} = \langle W, R, P \rangle$ and $\mathfrak{G} = \langle V, S, Q \rangle$, we say a map f from W onto V is a *reduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if the following three conditions are satisfied, for all $x, y \in W$ and $X \in Q$:

(R1) xRy implies $f(x)Sf(y)$;

(R2) $f(x)Sf(y)$ implies $\exists z \in x\!\uparrow\ f(z) = f(y)$;

(R3) $f^{-1}(X) \in P$.

For Kripke frames this definition is equivalent to the old one given in Section 2.3.

Theorem 8.65 *If f is a reduction of $\mathfrak{F} = \langle W, R, P \rangle$ to $\mathfrak{G} = \langle V, S, Q \rangle$ then the map f^+ defined by $f^+(X) = f^{-1}(X)$, for every $X \in Q$, is an isomorphism of $\mathfrak{G}^+$ in $\mathfrak{F}^+$.*

Proof Clearly f^+ is an injection. So it suffices to show that f^+ preserves all the operations in $\mathfrak{G}^+$. We consider only the modal case and leave the intuitionistic one to the reader. Let $X, Y \in Q$. Then we have

$$f^+(X \cap Y) = f^+(X) \cap f^+(Y);$$

$$f^+(V - X) = W - f^+(X);$$

$$f^+(X{\downarrow}S) = f^+(X){\downarrow}R.$$

Only the last equality needs a justification. Suppose $y \in f^+(X{\downarrow}S)$. Then there is $x \in X$ such that $f(y)Sx$. By (R2), there is $z \in y{\uparrow}$ for which $f(z) = x$. So $y \in f^+(X){\downarrow}R$. Conversely, if $y \in f^+(X){\downarrow}R$ then yRx for some $x \in f^{-1}(X)$, whence by (R1), $f(y)Sf(x)$ which means that $y \in f^+(X{\downarrow}S)$. ❑

A reduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called a *reduction* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ to a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ if $\mathfrak{V}(p) = f^{-1}(\mathfrak{U}(p))$, for every variable p. It follows immediately from Theorem 8.65 that the reduction theorems in Sections 2.3 and 3.3 and their corollaries (Theorems 2.15, 3.15 and Corollaries 2.16, 2.17, 3.16) hold for general frames as well.

Proposition 8.66 *If f_1 is a reduction of a frame $\mathfrak{F}_1$ (or a model $\mathfrak{M}_1$) to $\mathfrak{F}_2$ ($\mathfrak{M}_2$) and f_2 a reduction of $\mathfrak{F}_2$ ($\mathfrak{M}_2$) to $\mathfrak{F}_3$ ($\mathfrak{M}_3$) then the composition $f_2 f_1$ is a reduction of $\mathfrak{F}_1$ ($\mathfrak{M}_1$) to $\mathfrak{F}_3$ ($\mathfrak{M}_3$).*

Proof Exercise. ❑

As a simple example of the use of the reduction and generation theorems we prove the following:

Theorem 8.67. (Makinson's theorem) *Every consistent normal modal logic L is contained either in* **Verum** $= \mathrm{Log}\bullet$ *or in* **Triv** $= \mathrm{Log}\circ$.

Proof We must show that either $\bullet \models L$ or $\circ \models L$. Since L is consistent, there exists a frame $\mathfrak{F}$ for L, which either contains $\bullet$ as a generated subframe or is reducible to $\circ$ (see the proof of Proposition 3.17). Therefore, either $\bullet$ or $\circ$ validates L. ❑

The reductions of frames and models can be defined in somewhat different terms, namely as the quotient frames and models under some congruence relations. Suppose $\mathfrak{F} = \langle W, R, P \rangle$ is a frame and $\sim$ an equivalence relation on W. We denote by $[x]$ the equivalence class under $\sim$ generated by x, i.e., $[x] = \{y \in W : x \sim y\}$, and let $[X] = \{[x] : x \in X\}$ for any $X \subseteq W$. We say $\sim$ is a *congruence* on $\mathfrak{F}$ if xRy implies $[x] \subseteq [y]{\downarrow}$ and $[x] \subseteq X$ for every $X \in P$ and $x \in X$.

Given a congruence relation $\sim$ on $\mathfrak{F}$, define a frame $[\mathfrak{F}] = \langle [W], [R], [P] \rangle$, the *quotient frame* of $\mathfrak{F}$ under $\sim$, by taking

$$[R] = \{\langle [x], [y] \rangle : [x] \subseteq [y]\downarrow\}, \quad [P] = \{[X] : X \in P\}.$$

The fact that $[P]$ is closed under the modal or intuitionistic operations follows from the equalities: $[X \odot Y] = [X] \dot{\odot} [Y]$, for $\odot \in \{\wedge, \vee, \rightarrow\}$, and $[\Box X] = \Box[X]$, which hold for every $X, Y \in P$ (the reader can readily verify them by himself). If $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a model on $\mathfrak{F}$ then by putting $[\mathfrak{V}](p) = [\mathfrak{V}(p)]$, for every variable p, we obtain a model $[\mathfrak{M}] = \langle [\mathfrak{F}], [\mathfrak{V}] \rangle$ which is called the *quotient model* of $\mathfrak{M}$ under $\sim$.

Theorem 8.68 (i) *If $\sim$ is a congruence on $\mathfrak{F}$ then the map f from W onto $[W]$, defined by $f(x) = [x]$, is a reduction of $\mathfrak{F}$ to $[\mathfrak{F}]$ and of $\mathfrak{M}$ to $[\mathfrak{M}]$.*

(ii) *Suppose that f is a reduction of $\mathfrak{F} = \langle W, R, P \rangle$ to $\mathfrak{G} = \langle V, S, Q \rangle$ and $P' = \{f^{-1}(X) : X \in Q\}$. Then the relation $\sim$ on W defined by*

$$x \sim y \quad \textit{iff} \quad f(x) = f(y)$$

is a congruence on $\mathfrak{F}' = \langle W, R, P' \rangle$ and $[\mathfrak{F}']$ is isomorphic to $\mathfrak{G}$, with the map $h([x]) = f(x)$ being an isomorphism.

Proof Exercise. ❑

With the help of Theorem 8.68 we can prove the following:

Theorem 8.69 *If $\mathfrak{F} = \langle W, R, P \rangle$ is a finite (modal or intuitionistic) frame then the refinement map r is a reduction of $\mathfrak{F}$ to $r\mathfrak{F}$. In particular, every finite model is reducible to a refined model.*

Proof Let us consider first the modal case. The relation $\sim$ defined by

$$x \sim y \text{ iff } \forall X \in P\ (x \in X \leftrightarrow y \in X)$$

is a congruence on $\mathfrak{F}$. Indeed, that $[x] \subseteq X$ for all $X \in P$ and $x \in X$ follows immediately from the definition. So suppose that xRy. Since $\mathfrak{F}$ is finite, $[y] = \bigcap\{X \in P : y \in X\} \in P$ and so all the points in $[x]$ must belong to $[y]\downarrow$. Thus, by Theorem 8.68, the map $x \mapsto [x]$ is a reduction of $\mathfrak{F}$ to $[\mathfrak{F}]$. It remains to observe that $[x]rR[y]$ iff $[x][R][y]$.

Now let $\mathfrak{F}$ be intuitionistic. By Propositions 8.45 and 8.40, we then have $r\mathfrak{F} \cong (\mathfrak{F}^+)_+$. The map $f(x) = [x]$ ($[x]$ is clearly the same in both $\mathfrak{F}$ and $\sigma\mathfrak{F}$) is a reduction of $\sigma\mathfrak{F}$ to $((\sigma\mathfrak{F})^+)_+ = \sigma((\mathfrak{F}^+)_+)$ and so of $\mathfrak{F}$ to $(\mathfrak{F}^+)_+$ too. ❑

Example 8.46 shows, however, that Theorem 8.69 does not hold for infinite frames and models.

The notion of congruence enables us to define the limit of an infinite chain of reductions. Suppose that, for every $i < \omega$, we have a reduction f_i of $\mathfrak{F}_i = \langle W_i, R_i, P_i \rangle$ to $\mathfrak{F}_{i+1} = \langle W_{i+1}, R_{i+1}, P_{i+1} \rangle$, or symbolically

$$\mathfrak{F}_0 \xrightarrow{f_0} \mathfrak{F}_1 \xrightarrow{f_1} \mathfrak{F}_2 \xrightarrow{f_2} \dots \tag{8.6}$$

By Proposition 8.66, the composition $g_i = f_{i-1}f_{i-2}\dots f_0$ is a reduction of $\mathfrak{F}_0$ to $\mathfrak{F}_i$. Let $Q_i = \{g_i^{-1}(X) : X \in P_i\}$ and $Q = \bigcap_{i<\omega} Q_i$. Since, by Theorem 8.65, all Q_i are closed under the operations in $\mathfrak{F}_0^+$, Q is also closed under them. Let $\sim_i$ be the congruence on $\langle W_0, R_0, Q_i\rangle$ corresponding to g_i. Clearly, $\sim_i \subseteq \sim_{i+1}$ for every $i < \omega$. It is not hard to verify that $\sim = \bigcup_{i<\omega} \sim_i$ is a congruence relation on $\mathfrak{G} = \langle W_0, R_0, Q\rangle$. And now we can define the *limit* of the chain (8.6) of reductions as the reduction $f(x) = [x]$ of $\mathfrak{G}$, and so of $\mathfrak{F}_0$, to the quotient frame $[\mathfrak{G}]$ of $\mathfrak{G}$ under $\sim$. If we have a sequence

$$\mathfrak{M}_0 = \langle \mathfrak{F}_0, \mathfrak{V}_0\rangle \xrightarrow{f_0} \mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1\rangle \xrightarrow{f_1} \dots$$

of reductions of models then f is also a reduction of $\mathfrak{M}_0$ to the quotient model $[\langle \mathfrak{G}, \mathfrak{V}_0\rangle]$.

To prove a theorem which is dual to Theorem 8.65, we require the following:

Lemma 8.70 *Suppose that $\mathfrak{B} = \langle B, \wedge, \vee\rangle$ is a sublattice of a distributive lattice $\mathfrak{A} = \langle A, \wedge, \vee\rangle$. Then every prime filter ∇ in $\mathfrak{B}$ can be extended to a prime filter ∇' in $\mathfrak{A}$ such that $\nabla = \nabla' \cap B$.*

Proof Let Δ be the prime ideal in $\mathfrak{B}$ dual to ∇, i.e., $\Delta = B - \nabla$. Then by Exercise 7.18, there is a prime filter ∇' in $\mathfrak{A}$ such that $\nabla \subseteq \nabla'$ and $\nabla' \cap \Delta = \emptyset$, whence $\nabla = \nabla' \cap B$. ❑

Theorem 8.71 *If f is an isomorphism of a modal or pseudo-Boolean algebra $\mathfrak{B}$ in $\mathfrak{A}$ then the map f_+ defined by $f_+(\nabla) = f^{-1}(\nabla)$, for every $\nabla \in W_{\mathfrak{A}}$, is a reduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.*

Proof To simplify notation, we assume $\mathfrak{B}$ to be a subalgebra of $\mathfrak{A}$ and so f is the identity map and $f_+(\nabla) = \nabla \cap B$, for every $\nabla \in W_{\mathfrak{A}}$. (Here and below A and B denote the universes of $\mathfrak{A}$ and $\mathfrak{B}$, respectively.) It should be clear that if ∇ is a prime filter in $\mathfrak{A}$ then $f_+(\nabla)$ is a prime filter in $\mathfrak{B}$. So, by Lemma 8.70, f_+ is a map from $W_{\mathfrak{A}}$ onto $W_{\mathfrak{B}}$.

Suppose $\nabla_1 R_{\mathfrak{A}} \nabla_2$, for some $\nabla_1, \nabla_2 \in W_{\mathfrak{A}}$. In the intuitionistic case this means $\nabla_1 \subseteq \nabla_2$, whence $f_+(\nabla_1) \subseteq f_+(\nabla_2)$ and $f_+(\nabla_1) R_{\mathfrak{B}} f_+(\nabla_2)$. In the modal case we have: $\Box a \in \nabla_1$ implies $a \in \nabla_2$, for every a in $\mathfrak{A}$. Since $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, it follows that

$$\Box b \in \nabla_1 \cap B \text{ implies } b \in \nabla_2 \cap B, \text{ for every } b \in B, \tag{8.7}$$

i.e., again $f_+(\nabla_1) R_{\mathfrak{B}} f_+(\nabla_2)$. Thus, f_+ satisfies (R1).

Now suppose that $f_+(\nabla_1) R_{\mathfrak{B}} f_+(\nabla_2)$ which in the modal case is equivalent to (8.7). Let us consider the filter ∇_0 in $\mathfrak{A}$ generated by the set

$$\{a \in A : \Box a \in \nabla_1\} \cup (\nabla_2 \cap B)$$

and the ideal Δ_0 in $\mathfrak{A}$ generated by $B - \nabla_2$. Observe that $\nabla_0 \cap \Delta_0 = \emptyset$. For otherwise there are elements $a \in A$, for which $\Box a \in \nabla_1$, $b \in \nabla_2 \cap B$ and $c \in B - \nabla_2$ such that $a \wedge b \leq c$. Then $a \leq b \to c$, $\Box a \leq \Box(b \to c)$ and hence $\Box(b \to c) \in \nabla_1$. On the other hand, $\Box(b \to c) \in B$ and so, by (8.7), $b \to c \in \nabla_2 \cap B$, whence $c \in \nabla_2 \cap B$, which is a contradiction.

By Exercise 7.18, there is a prime filter ∇' in $\mathfrak{A}$ such that $\nabla_0 \subseteq \nabla'$ and $\Delta_0 \cap \nabla' = \emptyset$. By the definition, $\nabla_1 R_{\mathfrak{A}} \nabla'$ and $\nabla' \cap B = \nabla_2 \cap B$, i.e., $f_+(\nabla') = f_+(\nabla_2)$. Thus, in the modal case f_+ satisfies (R2). The intuitionistic one is considered analogously.

It remains to show that f_+ satisfies (R3). Let $X \in P_{\mathfrak{B}}$, i.e., there is b in $\mathfrak{B}$ such that $X = \{\nabla \in W_{\mathfrak{B}} : \ b \in \nabla\}$. But then $f_+^{-1}(X) = \{\nabla' \in W_{\mathfrak{A}} : \ b \in \nabla'\}$ and so $f_+^{-1}(X) \in P_{\mathfrak{A}}$. ❑

As to general frames with distinguished points, a reduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called a *reduction* of $\langle \mathfrak{F}, D\rangle$ to $\langle \mathfrak{G}, E\rangle$ if $f^{-1}(E) = D$.

We invite the reader to prove the following two theorems as an exercise.

Theorem 8.72 *If f is a reduction of $\langle \mathfrak{F}, D\rangle$ to $\langle \mathfrak{G}, E\rangle$ then f^+ is an isomorphism of $\langle \mathfrak{G}^+, E^+\rangle$ in $\langle \mathfrak{F}^+, D^+\rangle$.*

Theorem 8.73 *If a modal matrix $\langle \mathfrak{B}, \nabla'\rangle$ is a submatrix of $\langle \mathfrak{A}, \nabla''\rangle$ then the map f_+ defined by $f_+(\nabla) = \nabla \cap B$, for every $\nabla \in W_{\mathfrak{A}}$, is a reduction of $\langle \mathfrak{A}_+, \nabla''_+\rangle$ to $\langle \mathfrak{B}_+, \nabla'_+\rangle$.*

It remains to define the relational counterpart of the direct product of modal and pseudo-Boolean algebras.

The *disjoint union* of a family $\{\mathfrak{F}_i = \langle W_i, R_i, P_i\rangle : \ i \in I\}$ of pairwise disjoint frames is the frame $\sum_{i \in I} \mathfrak{F}_i = \langle W, R, P\rangle$ where $W = \bigcup_{i \in I} W_i$, $R = \bigcup_{i \in I} R_i$ and $P = \{\bigcup_{i \in I} X_i : \ X_i \in P_i, \text{ for all } i \in I\}$. The fact that $\sum_{i \in I} \mathfrak{F}_i$ is really a general frame follows from the equations below which hold for every $X_i, Y_i \in P_i$, $i \in I$ (to establish them, only the disjointness of $\mathfrak{F}_i$ is required):

$$\bigcup_{i \in I} X_i \odot \bigcup_{i \in I} Y_i = \bigcup_{i \in I} (X_i \odot Y_i), \text{ for } \odot \in \{\cap, \cup, \supset\};$$

$$\Box \bigcup_{i \in I} X_i = \bigcup_{i \in I} \Box X_i.$$

By the definition, every $\mathfrak{F}_i$ is a generated subframe of $\sum_{i \in I} \mathfrak{F}_i$.

The *disjoint union* $\sum_{i \in I} \mathfrak{M}_i$ of a family of models $\{\mathfrak{M}_i : \ i \in I\}$ is defined in exactly the same way as in Section 2.3. Again $\mathfrak{M}_i$ is a generated submodel of $\sum_{i \in I} \mathfrak{M}_i$ and so, using the generation theorem, we can easily extend Theorem 2.23 and Corollary 2.24 from Kripke frames to general ones.

Theorem 8.74 *Suppose $\{\mathfrak{F}_i = \langle W_i, R_i, P_i\rangle : \ i \in I\}$ is a family of descriptive frames. Then $\sum_{i \in I} \mathfrak{F}_i = \langle W, R, P\rangle$ is descriptive iff I is finite.*

Proof ($\Leftarrow$) It suffices to prove that $\mathfrak{F}_1 + \mathfrak{F}_2$ is compact if both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are compact. Let ∇ be a prime filter in $(\mathfrak{F}_1 + \mathfrak{F}_2)^+$. Then $\nabla_i = \{X \cap W_i : X \in \nabla\}$ is a filter in $\mathfrak{F}_i^+$, for $i = 1, 2$. Moreover, ∇_i is prime if it is proper. Observe now that only one of the filters ∇_1 and ∇_2 is proper. Indeed, let $X_1 \cup X_2 \in \nabla$ for some $X_1 \in P_1$ and $X_2 \in P_2$. Since ∇ is prime, either $X_1 \in \nabla$ or $X_2 \in \nabla$. Suppose for definiteness that $X_1 \in \nabla$. Then, by the definition of filter, $X_1 \cup X \in \nabla$ for every $X \in P_2$ and so $\nabla_2 = W_2$ and $\nabla = \{X \cup Y : X \in \nabla_1, Y \in P_2\}$. By Proposition 8.48, $\nabla_1 = P_1 x$ for some $x \in W_1$, whence $\nabla = Px$. So, by the same proposition, $\mathfrak{F}_1 + \mathfrak{F}_2$ is compact.

($\Rightarrow$) Suppose now that I is infinite and let $\mathcal{Y} = \{W - W_i : i \in I\} \subseteq \overline{P}$. Clearly, $\mathcal{Y}$ has the finite intersection property, but $\bigcap \mathcal{Y} = \emptyset$. ❑

Theorem 8.75 *Let $\{\mathfrak{F}_i = \langle W_i, R_i, P_i\rangle : i \in I\}$ be a family of frames and $\sum_{i\in I} \mathfrak{F}_i = \langle W, R, P\rangle$ their disjoint union. Then the map f defined by $f(X)(i) = X \cap W_i$, for every $X \in P$ and $i \in I$, is an isomorphism of $(\sum_{i\in I} \mathfrak{F}_i)^+$ onto $\prod_{i\in I} \mathfrak{F}_i^+$.*

Proof By the definition, $f(X)$ is an element of $\prod_{i\in I} \mathfrak{F}_i^+$, i.e., a function from I into $\bigcup_{i\in I} P_i$ with $f(X)(i) \in P_i$, for all $i \in I$. It should be clear that f is a bijection. Using the fact that the operations in $\prod_{i\in I} \mathfrak{F}_i^+$ are defined componentwise, one can show that f preserves all the operations in $(\sum_{i\in I} \mathfrak{F}_i)^+$. ❑

According to Theorem 8.74, the dual to Theorem 8.75 does not hold for infinite families of algebras. We have only the following

Theorem 8.76 *Suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are modal or pseudo-Boolean algebras. Then the map f defined by*

$$f(\nabla_1) = \{\langle a_1, a_2\rangle \in A_1 \times A_2 : a_1 \in \nabla_1,\ a_2 \in A_2\},\ \textit{for every } \nabla_1 \in W_{\mathfrak{A}_1},$$

and

$$f(\nabla_2) = \{\langle a_1, a_2\rangle \in A_1 \times A_2 : a_1 \in A_1,\ a_2 \in \nabla_2\},\ \textit{for every } \nabla_2 \in W_{\mathfrak{A}_2},$$

is an isomorphism of $\mathfrak{A}_{1+} + \mathfrak{A}_{2+}$ onto $(\mathfrak{A}_1 \times \mathfrak{A}_2)_+$.

Proof It is easy to see that f is an injection. To show that it is a surjection, suppose ∇ is a prime filter in $\mathfrak{A}_1 \times \mathfrak{A}_2$ and $\nabla_i = \{a_i : \langle a_1, a_2\rangle \in \nabla\}$, for $i = 1, 2$. Then ∇_i either is a prime filter or coincides with the universe of $\mathfrak{A}_i$. And since $\langle a_1, a_2\rangle = \langle a_1, \bot\rangle \vee \langle \bot, a_2\rangle$, only one of ∇_1, ∇_2 may be proper, say ∇_1. But then $\nabla = f(\nabla_1)$.

Suppose that $\nabla', \nabla'' \in W_{\mathfrak{A}_1} \cup W_{\mathfrak{A}_2}$ and $\nabla' R \nabla''$, where R is the accessibility relation in $\mathfrak{A}_{1+} + \mathfrak{A}_{2+}$. Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are generated subframes of $\mathfrak{A}_{1+} + \mathfrak{A}_{2+}$, $\nabla', \nabla'' \in W_{\mathfrak{A}_i}$ for some $i \in \{1, 2\}$. So if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are modal and $\Box \langle a_1, a_2\rangle = \langle \Box a_1, \Box a_2\rangle \in f(\nabla')$ then $\Box a_i \in \nabla'$, whence $a_i \in \nabla''$ and $\langle a_1, a_2\rangle \in f(\nabla'')$. Thus, $f(\nabla') R_{\mathfrak{A}_1 \times \mathfrak{A}_2} f(\nabla'')$.

Conversely, suppose that $f(\nabla') R_{\mathfrak{A}_1 \times \mathfrak{A}_2} f(\nabla'')$ for some $\nabla' \in W_{\mathfrak{A}_1}$ and $\nabla'' \in W_{\mathfrak{A}_1} \cup W_{\mathfrak{A}_2}$. Assume also that $\Box a_1 \in \nabla'$ for some $a_1 \in A_1$. Then $\Box \langle a_1, \bot\rangle =$

$\langle \Box a_1, \Box\bot\rangle \in f(\nabla')$, whence $\langle a_1, \bot\rangle \in f(\nabla'')$ and so $a_1 \in \nabla'' \in W_{\mathfrak{A}_1}$. Thus, $\nabla' R_{\mathfrak{A}_{1+}} \nabla''$ and hence $\nabla' R \nabla''$.

The case of pseudo-Boolean $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is left to the reader.

Suppose now that $X \in P$, where P is the set of possible values in $\mathfrak{A}_{1+} + \mathfrak{A}_{2+}$. Then $X = f_{\mathfrak{A}_1}(a_1) \cup f_{\mathfrak{A}_2}(a_2)$, for some $a_1 \in A_1$, $a_2 \in A_2$, i.e.,

$$X = \{\nabla_1 \in W_{\mathfrak{A}_1} :\ a_1 \in \nabla_1\} \cup \{\nabla_2 \in W_{\mathfrak{A}_2} :\ a_2 \in \nabla_2\}, \tag{8.8}$$

and so, by the definition of f and the property of prime filters in $\mathfrak{A}_1 \times \mathfrak{A}_2$ established above,

$$f(X) = \{\nabla \in W_{\mathfrak{A}_1 \times \mathfrak{A}_2} :\ \langle a_1, a_2\rangle \in \nabla\} \in P_{\mathfrak{A}_1 \times \mathfrak{A}_2}. \tag{8.9}$$

Conversely, if $f(X) \in P_{\mathfrak{A}_1 \times \mathfrak{A}_2}$ then, for some $\langle a_1, a_2\rangle \in A_1 \times A_2$, $f(X)$ is of the form (8.9). Since f is a bijection, X has the form (8.8) and so $X \in P$. ❑

The *disjoint union* of the family $\{\langle \mathfrak{F}_i, D_i\rangle :\ i \in I\}$ of frames with distinguished points is the frame $\langle \sum_{i\in I} \mathfrak{F}_i, \bigcup_{i\in I} D_i\rangle$. The following two theorems are left to the reader as an exercise.

Theorem 8.77 *Let $\{\langle \mathfrak{F}_i, D_i\rangle :\ i \in I\}$ be a family of general frames with distinguished points and $\langle \sum_{i\in I} \mathfrak{F}_i, \bigcup_{i\in I} D_i\rangle$ their disjoint union. Then the map f defined by $f(X)(i) = X \cap W_i$, for every $X \in P$ and $i \in I$, is an isomorphism of $\langle (\sum_{i\in I} \mathfrak{F}_i)^+, (\bigcup_{i\in I} D_i)^+\rangle$ onto $\prod_{i\in I} \langle \mathfrak{F}_i^+, D_i^+\rangle$.*

Theorem 8.78 *Suppose $\langle \mathfrak{A}_1, \nabla'\rangle$ and $\langle \mathfrak{A}_2, \nabla''\rangle$ are modal matrices. Then the map f defined as in Theorem 8.76 is an isomorphism of $\langle \mathfrak{A}_{1+}, \nabla'_+\rangle + \langle \mathfrak{A}_{2+}, \nabla''_+\rangle$ onto $(\langle \mathfrak{A}_1, \nabla'\rangle \times \langle \mathfrak{A}_2, \nabla''\rangle)_+$.*

8.6 Points of finite depth in refined finitely generated frames

Every modal and si-logic L is characterized by the class of its finitely generated descriptive frames. Indeed, by Theorem 8.36, for any formula $\varphi(p_1, \ldots, p_n) \notin L$ there are a descriptive frame $\mathfrak{F}$ for L and a valuation $\mathfrak{V}$ under which φ is refuted in $\mathfrak{F}$. The subalgebra $\mathfrak{A}$ of $\mathfrak{F}^+$ generated by the elements $\mathfrak{V}(p_1), \ldots, \mathfrak{V}(p_n)$ is then an algebra for L refuting φ and so φ is separated from L by the n-generated descriptive frame $\mathfrak{A}_+$.

In this section we study the constitution of an upper part of finitely generated refined transitive frames, namely, the part containing points of finite depth. And in the next section we shall use the results to be obtained here to penetrate into the structure of the universal frames of finite rank for some modal and si-logics.

Say that a point x and the cluster $C(x)$ in a transitive frame $\mathfrak{F}$ are of *depth* d, for $d < \omega$ or $d = \infty$, if the subframe of $\kappa\mathfrak{F}$ generated by x is of depth d. This fact will be denoted by $d(x) = d(C(x)) = d$. We reckon ∞ as being greater than any $d < \omega$. $W^{=d}$ and $W^{\leq d}$ are the sets of all points in $\mathfrak{F} = \langle W, R, P\rangle$ of depth d and $\leq d$, respectively; $W^{<d}$, $W^{>d}$ and $W^{\geq d}$ are defined analogously. The subframe of $\mathfrak{F}$ generated by $W^{\leq d}$ is denoted by $\mathfrak{F}^{\leq d}$.

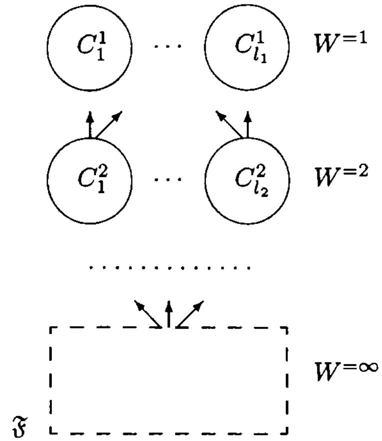


FIG. 8.5.

In general, a transitive frame may contain no points of finite depth at all (see, for instance, Fig. 8.1 (c)). But this is not the case if the frame is finitely generated and refined. In fact, we shall see that every such frame $\mathfrak{F} = \langle W, R, P\rangle$ can be represented as depicted in Fig. 8.5. More exactly, for each natural d such that $0 < d \leq d(\mathfrak{F})$, the set $W^{=d}$ is non-empty and contains a finite number of finite clusters $C_1^d, \ldots, C_{l_d}^d$; all points in $W^{=d}$ turn out to be atoms in $\mathfrak{F}$, and $W^{=d}$ is a cover for the set $W^{\geq d}$, i.e.,

$$\begin{aligned} W &= W^{=1}\overline{\downarrow} \\ &= W^{=1} \cup W^{=2}\overline{\downarrow} \\ &\ldots \\ &= W^{=1} \cup \ldots \cup W^{=m} \cup W^{=m+1}\overline{\downarrow} \\ &\ldots \end{aligned}$$

Frames with such properties are called *top-heavy*. To prove this result, we require some auxiliary notions.

Suppose $\mathfrak{F} = \langle W, R, P\rangle$ is a refined modal or intuitionistic transitive frame generated (as modal or, respectively, pseudo-Boolean algebra) by some sets $G_1, \ldots, G_n \in P$, $0 \leq n < \omega$. Define in $\mathfrak{F}$ a valuation $\mathfrak{V}$ of the language $\mathcal{ML}_n$ or $\mathcal{L}_n$ with the set of variables $\Sigma = \{p_1, \ldots, p_n\}$ by taking $\mathfrak{V}(p_i) = G_i$, for each $i = 1, \ldots, n$. Thus,

$$P = \{\mathfrak{V}(\varphi) : \ \varphi \in \mathbf{For}\mathcal{ML}_n \ (\text{or } \varphi \in \mathbf{For}\mathcal{L}_n)\}$$

and we can work with formulas as well as with sets in P. As in Section 5.3, we regard two points $x, y \in W$ as Σ-*equivalent* in $\mathfrak{F}$ and write $x \sim_\Sigma y$ if the same formulas in Σ are true at them under $\mathfrak{V}$; $[x]_\Sigma$ is the Σ-equivalence class generated by x. Sets $X, Y \subseteq W$ are called Σ-*equivalent* in $\mathfrak{F}$, $X \sim_\Sigma Y$ for short, if every point in X is Σ-equivalent to some point in Y and vice versa.

A non-empty set $X \subseteq W$ is said to be *cyclic* in $\mathfrak{F}$ (relative to $\mathfrak{V}$) if either

$$\forall x, y \in X\ \exists z \in X\ (xRz \wedge z \sim_\Sigma y) \tag{8.10}$$

(which is equivalent to $\forall x \in X\ x{\uparrow} \,\dot{\cap}\, X \sim_\Sigma X$) or

$$\forall x, y \in X\ (x \sim_\Sigma y \wedge \neg xRy). \tag{8.11}$$

These two conditions are mutually exclusive. If the former one is satisfied, X is called a *non-degenerate cyclic set*, while if the latter condition holds we say X is a *degenerate cyclic set*. It should be clear that all cyclic sets in a reflexive (in particular, intuitionistic) frame are non-degenerate and that all clusters are cyclic.

Given d such that $0 \leq d < d(\mathfrak{F})$ and a point $x \in W^{>d}$, we define the *d-span* of x in $\mathfrak{F}$ as the set $sp^d(x) = \{y \in W^{\leq d} : xRy\}$. By the definition, $sp^0(x) = \emptyset$ for every x in $\mathfrak{F}$. A cyclic set X is called *d-cyclic* if

$$X = X\underline{\uparrow} \cap W^{>d} \tag{8.12}$$

and

$$\forall x, y \in X\ (sp^d(x) = sp^d(y)). \tag{8.13}$$

Every non-empty upward closed in $W^{>d}$ subset of a d-cyclic set is also d-cyclic.

Lemma 8.79 *Suppose x and y are Σ-equivalent points in a d-cyclic set X. Then $x \models \varphi$ iff $y \models \varphi$, for every formula φ in* $\mathbf{For}\mathcal{ML}_n$ *or* $\mathbf{For}\mathcal{L}_n$.

Proof We consider only the modal case, leaving the intuitionistic one to the reader. The proof proceeds by induction on the construction of φ.

The basis of induction is ensured by $x \sim_\Sigma y$, and the cases of $\varphi = \psi \wedge \chi$, $\psi \vee \chi$ and $\psi \to \chi$ are trivial. So suppose $x \not\models \Box\psi$. Then $z \not\models \psi$ for some $z \in x{\uparrow}$. If $z \in W^{\leq d}$ then $z \in y{\uparrow}$, since $sp^d(x) = sp^d(y)$, from which $y \not\models \Box\psi$. If $z \in W^{>d}$ then, by (8.12), $z \in X$ and so X is non-degenerate. By (8.10), there is $u \in y{\uparrow}$ such that $u \sim_\Sigma z$, whence by the induction hypothesis, $u \not\models \psi$ and so $y \not\models \Box\psi$. The symmetrical argument shows that $y \not\models \Box\psi$ implies $x \not\models \Box\psi$. ❑

Using the fact that $\mathfrak{F}$ is refined, we obtain a stronger result.

Lemma 8.80 *Suppose $\mathfrak{F}$ is a refined n-generated frame, $0 \leq d < d(\mathfrak{F})$ and X is a d-cyclic set in $\mathfrak{F}$. Then*

(i) *$x = y$, for every Σ-equivalent points $x, y \in X$;*

(ii) *X is a non-degenerate cluster of cardinality $\leq 2^n$, if X is non-degenerate, and*

(iii) *X is an irreflexive singleton, if X is degenerate.*

Proof (i) follows immediately from Lemma 8.79 and the differentiatedness of $\mathfrak{F}$; (iii) is a direct consequence of (i). So let us establish (ii). Observe first that there are at most 2^n pairwise non-Σ-equivalent points in $\mathfrak{F}$, whence by (i), $|X| \le 2^n$. Now suppose $x, y \in X$ and prove that xRy. By the tightness of $\mathfrak{F}$, it suffices to show that (in the modal case—the intuitionistic one is left to the reader) for every $\varphi \in \mathbf{For}\mathcal{ML}_n$, $x \models \Box\varphi$ implies $y \models \varphi$. Assuming otherwise, we must have some φ for which $x \models \Box\varphi$ and $y \not\models \varphi$. By (8.10), there is $z \in X$ such that xRz and $z \sim_\Sigma y$ and so, by Lemma 8.79, $z \not\models \varphi$, which is a contradiction.

Thus, xRy for all $x, y \in X$. It remains to observe that all points in a cluster are of the same depth and that X is upward closed in $W^{>d}$, i.e., X cannot be a proper subset of a cluster. ❑

As a consequence of Lemma 8.80 we obtain the following characterization of clusters of depth $d+1$ in $\mathfrak{F}$.

Lemma 8.81 *Suppose that $\mathfrak{F}$ is a refined finitely generated transitive frame and $d < d(\mathfrak{F})$. Then C is a cluster of depth $d+1$ in $\mathfrak{F}$ iff C is a d-cyclic set in $\mathfrak{F}$.*

Proof ($\Rightarrow$) It is clear that C is cyclic. It is d-cyclic, since all points in C are of the same d-span and besides C has no proper successors in $W^{>d}$, i.e., $C = C\underline{\uparrow} \cap W^{>d}$.

($\Leftarrow$) follows from Lemma 8.80. ❑

It follows, in particular, that a d-cyclic set in a refined finitely generated frame has no proper d-cyclic subsets and so clusters $C(x)$ and $C(y)$ of depth $d+1$ coincide if x and y are of the same d-span and $C(x) \sim_\Sigma C(y)$. Using this observation, we can estimate the number of clusters of depth $d+1$ in $\mathfrak{F}$, if any.

Theorem 8.82 *Suppose $\mathfrak{F}$ is a refined n-generated transitive frame and $d < d(\mathfrak{F})$. Then the number of distinct clusters of depth $d+1$ in $\mathfrak{F}$ is not greater than $c_n(d+1)$ which is defined recursively as follows:*

$$c_n(1) = 2^n + 2^{2^n} - 1;$$

$$c_n(d+1) = c_n(1)2^{c_n(1)+\ldots+c_n(d)}.$$

If $\mathfrak{F}$ is irreflexive or partially ordered then one can take $c_n(1) = 2^n$. Every proper cluster in $\mathfrak{F}$ contains at most 2^n points.

Proof There are at most 2^n pairwise non-Σ-equivalent points and $2^{2^n} - 1$ pairwise non-Σ-equivalent non-empty sets of points in $\mathfrak{F}$. So there are at most 2^n degenerate and $2^{2^n} - 1$ non-degenerate clusters of depth 1 in $\mathfrak{F}$. If $\mathfrak{F}$ is irreflexive or partially ordered then all clusters in $\mathfrak{F}$ are singletons and hence the number of clusters of depth 1 in such a frame is not greater than 2^n.

Distinct clusters of depth $d+1$ may be Σ-equivalent, but then they have distinct d-spans, the total number of which does not exceed the number of all sets of clusters of depth $\le d$. The size of clusters was estimated in Lemma 8.80. ❑

Theorem 8.83 *Suppose $\mathfrak{F}$ is a refined finitely generated transitive frame. Then every point of finite depth in $\mathfrak{F}$ is an atom.*

Proof Observe first that the intuitionistic case reduces to the modal one. Indeed, if $\mathfrak{F} = \langle W, R, P\rangle$ is an intuitionistic n-generated refined frame then $\sigma\mathfrak{F} = \langle W, R, \sigma P\rangle$ is a modal n-generated refined frame. So if x is an atom in $\sigma\mathfrak{F}$, i.e., $\{x\} \in \sigma P$, then $W - x{\downarrow} \in \sigma P$ and $\{x\} \cup (W - x{\downarrow}) \in \sigma P$ and hence the sets $W - x{\downarrow}$ and $\{x\} \cup (W - x{\downarrow})$ are in P, since both of them are upward closed and $P = \rho\sigma P$.

Now we prove our theorem for a modal $\mathfrak{F}$ by induction on depth. Suppose that u is a point in $\mathfrak{F} = \langle W, R, P\rangle$ of depth $d+1$ and that all points of smaller depth, if any, are atoms in $\mathfrak{F}$. It follows from this assumption and the finiteness of $W^{\le d}$ that $W^{>d} \in P$.

For $x \in W^{>d}$, we denote by G_x^d the set

$$(\bigcap\{G_i :\ x \in G_i\} - \bigcup\{G_i :\ x \notin G_i\}) \cap W^{>d} \in P.$$

(We remind the reader that $G_1, \dots, G_n$ generate $\mathfrak{F}$.) It is clear that, for every $y, z \in W^{>d}$, $y \sim_\Sigma z$ iff $G_y^d = G_z^d$. So it suffices to show that $C(u) \in P$, since by Lemma 8.80, $\{u\} = C(u) \cap G_u^d$.

Let us consider the following two cases.

Case 1: The cluster $C(u)$ is non-degenerate. Then we form the set

$$\begin{aligned} X = W^{>d} \cap \\ (\bigcap_{x \in C(u)} G_x^d{\downarrow} - \bigcup_{G_x^d \cap C(u) = \emptyset} \overline{G_x^d{\downarrow}}) \cap \\ (\bigcap_{y \in sp^d(u)} y{\downarrow} - \bigcup_{y \in W^{\le d} - sp^d(u)} y{\downarrow}), \end{aligned} \tag{8.14}$$

which is in P, since there is only a finite number of pairwise distinct sets G_x^d, for $x \in W^{>d}$. By the definition, X consists of all points x of depth $> d$ such that (a) $\underline{x{\uparrow}} \cap W^{>d} \sim_\Sigma C(u)$ and (b) $sp^d(x) = sp^d(u)$. Therefore, $C(u) \subseteq X$. Now, taking the upward closed in $W^{>d}$ part of X, i.e., $\Box(X \cup W^{\le d}) \cap X \in P$, we obtain a d-cyclic set which contains $C(u)$ and so, by Lemma 8.81, must coincide with $C(u)$.

Case 2: The cluster $C(u)$ is degenerate, i.e., u is irreflexive and $C(u) = \{u\}$. By Lemma 8.80, we then have

$$\begin{aligned} C(u) = G_u^d \cap (W^{>d} - W^{>d}{\downarrow}) \cap \\ (\bigcap_{y \in sp^d(u)} y{\downarrow} - \bigcup_{y \in W^{\le d} - sp^d(u)} y{\downarrow}) \end{aligned} \tag{8.15}$$

and so again $C(u) \in P$. ❑

Although we have already learned much about clusters of finite depth in refined finitely generated transitive frames, we do not know still whether they really exist.

$\mathfrak{F}^{\leq 1}_{\mathbf{K4}}(1)$

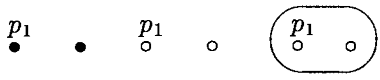


FIG. 8.6.

Theorem 8.84 *Suppose $\mathfrak{F}$ is a refined finitely generated transitive frame and $0 \leq d < d(\mathfrak{F})$. Then for every point $x \in W^{>d}$ there is a cluster C of depth $d+1$ such that $x \in \overline{C}{\downarrow}$. In other words, $W^{=d+1}$ is a (finite) cover for $W^{>d}$.*

Proof If the set $X = x\underline{\uparrow} \cap W^{>d}$ is d-cyclic then, by Lemma 8.81, x is a point of depth $d+1$. Otherwise either (8.10) or (8.13) does not hold for X. So there is a point $y \in x\uparrow \cap W^{>d}$ such that either the number of pairwise non-Σ-equivalent points in $y\underline{\uparrow} \cap W^{>d}$ is smaller than that in X or $sp^d(y) \subset sp^d(x)$. In exactly the same manner we consider now the point y, etc. Since there is only a finite number of pairwise non-Σ-equivalent points in $\mathfrak{F}$ and $W^{\leq d}$ is also finite, we shall eventually find a point $z \in x\uparrow \cap W^{>d}$ for which $z\underline{\uparrow} \cap W^{>d}$ is d-cyclic. Ergo $C(z)$ is a cluster of depth $d+1$ and $x \in C(z){\downarrow}$. ❑

The results obtained above will find many applications later on in the book. Here we show only one immediate consequence.

Say that a logic L in **ExtK4** or **ExtInt** is of *depth* $n < \omega$ if it contains the formula $\boldsymbol{bd}_n$ and does not contain $\boldsymbol{bd}_m$ for any $m < n$; L is of *finite depth* if it is of depth n for some $n < \omega$. This terminology is explained by the following:

Theorem 8.85. (Segerberg's theorem) *Every logic of depth $n < \omega$ is characterized by the class of its finite Kripke frames of depth $\leq n$.*

Proof It suffices to show that every formula $\varphi(p_1, \dots, p_m) \notin L$ is separated from L by a finite Kripke frame of depth $\leq n$. Let $\mathfrak{F}$ be an m-generated refined frame for L refuting φ. By Theorems 8.82 and 8.83, $\mathfrak{F}^{\leq n}$ is a finite Kripke frame. And since $\boldsymbol{bd}_n \in L$ (and in view of Propositions 2.38 and 3.44), $\mathfrak{F}$ contains no points of depth $n+1$ and so, by Theorem 8.84, $\mathfrak{F} = \mathfrak{F}^{\leq n}$. ❑

8.7 Universal frames of finite rank

The most complete information about logics is contained in their universal frames. In this section we give an effective description of the upper part—the points of finite depth—in the universal frames of finite rank for a few standard modal and si-logics and get some general impression of how those frames for other logics may look.

We begin with **K4**. Let $\mathcal{ML}_n$ be the modal language with $n < \omega$ variables, say $p_1, \dots, p_n$ and $\Sigma = \mathbf{Var}\mathcal{ML}_n$. As before, we assume that $G_1, \dots, G_n$ are generators of the universal frame $\mathfrak{F}_{\mathbf{K4}}(n)$ and define a valuation $\mathfrak{V}_{\mathbf{K4}}(n)$ of $\mathcal{ML}_n$ in $\mathfrak{F}_{\mathbf{K4}}(n)$ by taking $\mathfrak{V}_{\mathbf{K4}}(n)(p_i) = G_i$, for $i = 1, \dots, n$ (recall that $\mathfrak{V}_{\mathbf{K4}}(n)$ may be regarded also as a valuation in $\mathfrak{A}_{\mathbf{K4}}(n)$ such that $\mathfrak{V}_{\mathbf{K4}}(n)(p_i) = \|p_i\|_{\mathbf{K4}}$). $\mathfrak{M}_{\mathbf{K4}}(n) = \langle \mathfrak{F}_{\mathbf{K4}}(n), \mathfrak{V}_{\mathbf{K4}}(n) \rangle$ is called the *n-universal model* for **K4**.

Since $\mathfrak{F}_{\mathbf{K4}}(n)$ is a refined n-generated transitive frame, its generated subframe $\mathfrak{F}^{\leq 1}_{\mathbf{K4}}(n)$ of depth 1 consists of clusters with at most 2^n points, with distinct points

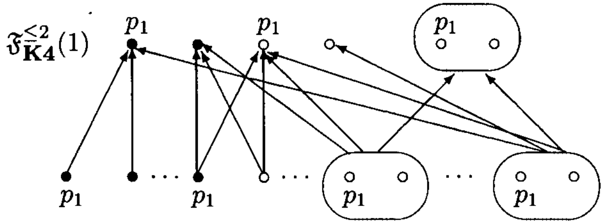


FIG. 8.7.

in each cluster, distinct degenerate clusters as well as distinct non-degenerate ones being pairwise non-Σ-equivalent. The maximal number of such clusters is, as we know, $2^n + 2^{2^n} - 1$. On the other hand, since $\mathfrak{F}_{\mathbf{K4}}(n)$ is the universal frame of rank n and in view of Theorem 8.60, it must contain as a generated subframe any descriptive n-generated frame of depth 1, in particular the frame $\mathfrak{G}^{\le 1}_{\mathbf{K4}}(n)$ associated with the model $\mathfrak{N}^{\le 1}_{\mathbf{K4}}(n)$ of depth 1 containing all possible 2^n pairwise non-Σ-equivalent degenerate clusters and all possible $2^{2^n} - 1$ pairwise non-Σ-equivalent non-degenerate clusters of $\le 2^n$ non-Σ-equivalent points. The model $\mathfrak{N}^{\le 1}_{\mathbf{K4}}(1)$ is shown in Fig. 8.6, where p_1 near a point x means that $x \models p_1$; otherwise $x \not\models p_1$. Since $\mathfrak{N}^{\le 1}_{\mathbf{K4}}(n)$ is finite, to verify that $\mathfrak{G}^{\le 1}_{\mathbf{K4}}(n)$ is descriptive, it suffices to establish its atomicity. We shall do this a bit later, when considering points of arbitrary depth $d < \omega$. Meanwhile, under the assumption that this is the case, we can conclude that $\mathfrak{F}^{\le 1}_{\mathbf{K4}}(n)$ is isomorphic to $\mathfrak{G}^{\le 1}_{\mathbf{K4}}(n)$.

Suppose now that $0 < d < \omega$ and we have already constructed a model $\mathfrak{N}^{\le d}_{\mathbf{K4}}(n)$ of depth d whose associated frame $\mathfrak{G}^{\le d}_{\mathbf{K4}}(n)$ is isomorphic to $\mathfrak{F}^{\le d}_{\mathbf{K4}}(n)$. Define $\mathfrak{N}^{\le d+1}_{\mathbf{K4}}(n)$ by adding to $\mathfrak{N}^{\le d}_{\mathbf{K4}}(n)$ a number of clusters of depth $d + 1$. Namely, for every antichain X of points in $\mathfrak{N}^{\le d}_{\mathbf{K4}}(n)$ containing at least one point of depth d and different from reflexive singletons (i.e., $X \neq \{x\}$, for any reflexive x), we add to $\mathfrak{N}^{\le d}_{\mathbf{K4}}(n)$ copies of all the $2^n + 2^{2^n} - 1$ clusters of depth 1 with the same valuation so that they would be inaccessible from each other and could see only the points in X and their successors. And for every reflexive singleton $X = \{x\}$ of depth d, we add to $\mathfrak{N}^{\le d}_{\mathbf{K4}}(n)$ copies of those clusters of depth 1 which are not Σ-equivalent to any subset of $C(x)$ so that again they would be mutually inaccessible and could see only x and its successors in $\mathfrak{N}^{\le d}_{\mathbf{K4}}(n)$. (It is to be emphasized that we do not distinguish between two antichains whose points generate the same clusters.) The resulting model is denoted by $\mathfrak{N}^{\le d+1}_{\mathbf{K4}}(n)$. A fragment of $\mathfrak{N}^{\le 2}_{\mathbf{K4}}(1)$ is shown in Fig. 8.7.

The frame $\mathfrak{G}^{\le d+1}_{\mathbf{K4}}(n)$ associated with $\mathfrak{N}^{\le d+1}_{\mathbf{K4}}(n)$ is n-generated and descriptive. Indeed, as was observed above, the descriptiveness of a finite frame follows from its atomicity. We prove that $\mathfrak{G}^{\le d+1}_{\mathbf{K4}}(n)$ is atomic by induction on depth. Let $\mathfrak{G}^{\le d+1}_{\mathbf{K4}}(n) = \langle V, S, Q\rangle$ and suppose that all points in $V^{\le m}$ are atoms in $\mathfrak{G}^{\le d+1}_{\mathbf{K4}}(n)$ and u is a point of depth $m+1$. Since $\mathfrak{G}^{\le d+1}_{\mathbf{K4}}(n)$ is finite, it follows in particular that $V^{\le m}$ and $V^{>m}$ are in Q. As in the proof of Theorem 8.83, for $x \in V^{>m}$ we denote by G^m_x the set of points in $V^{>m}$ which are Σ-equivalent to x. Clearly

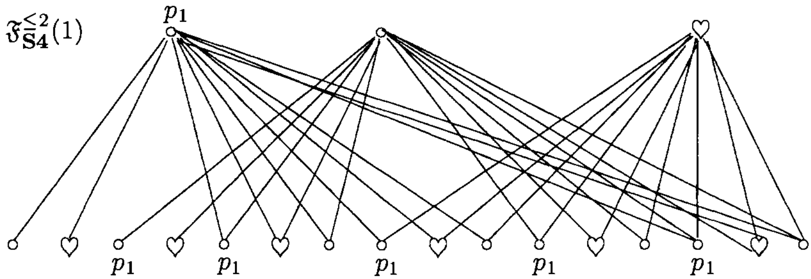
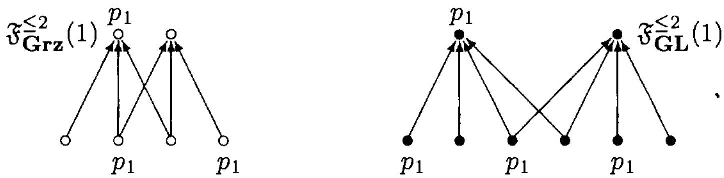


FIG. 8.8.



(a) (b)

FIG. 8.9.

we have $G^m_x \in Q$ and to show that $\{u\} \in Q$ it is sufficient to establish that $C(u) \in Q$ because $\{u\} = C(u) \cap G^m_u$.

If $C(u)$ is a non-degenerate cluster then it is a subset of the set $X \in Q$ defined by (8.14) with d replaced by m. Since for every point $v \in V^{=m+1}$ belonging to a cluster different from $C(u)$, either $C(v)$ is not Σ-equivalent to $C(u)$ or $sp^m(v) \neq sp^m(u)$, we have $X \cap V^{=m+1} = C(u)$. And since there is no point v in $V^{=m+2}$ such that $C(v)$ is Σ-equivalent to a subset of $C(u)$, $C(v){\uparrow} \cap V^{=m+1} = C(u)$ and $sp^m(v) = sp^m(u)$, we have $\Box(X \cup V^{\leq m}) \cap X = C(u)$ and so $C(u) \in Q$.

In the case when $C(u)$ is a degenerate cluster it may be represented in the form (8.15) with d replaced by m. Therefore, again $C(u) \in Q$.

It follows that $\mathfrak{G}^{\leq d+1}_{\mathbf{K4}}(n)$ is a generated subframe of $\mathfrak{F}_{\mathbf{K4}}(n)$. On the other hand, the results of the preceding section show that $\mathfrak{F}_{\mathbf{K4}}(n)$ contains no clusters of depth $d+1$ different from those in $\mathfrak{G}^{\leq d+1}_{\mathbf{K4}}(n)$ and so $\mathfrak{F}^{\leq d+1}_{\mathbf{K4}}(n)$ is isomorphic to $\mathfrak{G}^{\leq d+1}_{\mathbf{K4}}(n)$.

Let $\mathfrak{N}^{<\infty}_{\mathbf{K4}}(n)$ be the union of all models $\mathfrak{N}^{\leq d}_{\mathbf{K4}}(n)$ for $d < \omega$, i.e., its set of worlds, accessibility relation and truth-relation are the unions of those in $\mathfrak{N}^{\leq d}_{\mathbf{K4}}(n)$. We arrive then at the following:

Theorem 8.86 *The frame $\mathfrak{G}^{<\infty}_{\mathbf{K4}}(n)$ associated with $\mathfrak{N}^{<\infty}_{\mathbf{K4}}(n)$ is isomorphic to $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(n)$.*

Since **K4** is finitely approximable, every formula in $\mathbf{For}\mathcal{ML}_n$ that is not in **K4** is refuted by some n-generated descriptive finite frame which must be a generated subframe of $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(n)$. Therefore, both the model $\mathfrak{N}^{<\infty}_{\mathbf{K4}}(n)$ and the frame $\mathfrak{G}^{<\infty}_{\mathbf{K4}}(n)$ characterize $\mathbf{K4} \cap \mathbf{For}\mathcal{ML}$.

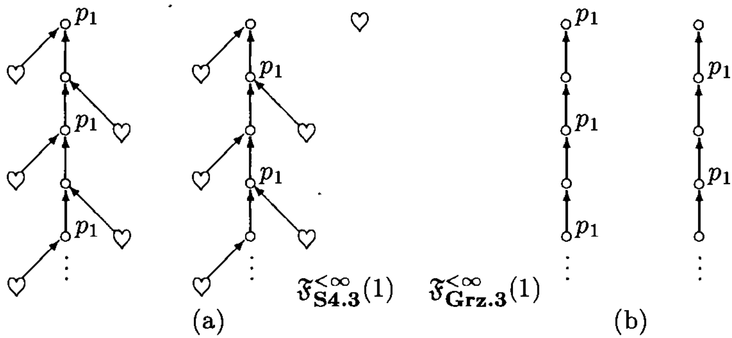


FIG. 8.10.

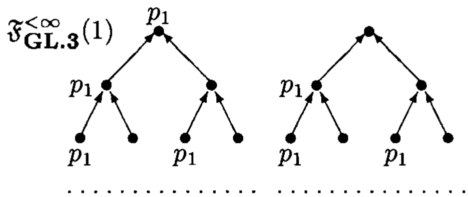


FIG. 8.11.

The universal frame $\mathfrak{F}_L(n)$ for an arbitrary consistent logic L in NExt**K4** is a generated subframe of $\mathfrak{F}_{\mathbf{K4}}(n)$. It can be constructed by removing from $\mathfrak{F}_{\mathbf{K4}}(n)$ those points at which some formulas in L are refuted (under $\mathfrak{V}_{\mathbf{K4}}(n)$). For example, $\mathfrak{F}^{<\infty}_{\mathbf{S4}}(n)$ is obtained by removing from $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(n)$ all the irreflexive points and their predecessors. In other words, $\mathfrak{F}^{<\infty}_{\mathbf{S4}}(n)$ can be constructed in the same way as $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(n)$ but using only non-degenerate clusters. $\mathfrak{F}^{\leq 2}_{\mathbf{S4}}(1)$ (the corresponding model, to be more exact) is shown in Fig. 8.8, where $\heartsuit$ denotes the cluster with two points at one of which p_1 is true. To construct $\mathfrak{F}^{<\infty}_{\mathbf{Grz}}(n)$ and $\mathfrak{F}^{<\infty}_{\mathbf{GL}}(n)$, we take only simple clusters and degenerate clusters, respectively. $\mathfrak{F}^{\leq 2}_{\mathbf{Grz}}(1)$ and $\mathfrak{F}^{\leq 2}_{\mathbf{GL}}(1)$ are depicted in Fig. 8.9 (a), (b). Fig. 8.10 (a), (b) and Fig. 8.11 show the upper parts of the universal frames of rank 1 for the logics **S4.3**, **Grz.3** and **GL.3**, respectively. The universal frames of rank n for logics of finite depth $L = L' \oplus \boldsymbol{bd}_d$ ($L' \in$ NExt**K4**, $d < \omega$) are obtained by removing from $\mathfrak{F}^{<\infty}_{L'}(n)$ all the points of depth $> d$, i.e., $\mathfrak{F}_L(n)$ is isomorphic to the finite frame $\mathfrak{F}^{\leq d}_{L'}(n)$.

$\mathfrak{F}^{<\infty}_{\mathbf{GL}}(0)$ is just an infinite descending chain of irreflexive points. Its points are characterized by the formulas of the form $\varphi_i = \Box^{i+1}\bot \wedge \Diamond^i\top$, for $i \geq 0$. Since **GL** is finitely approximable, every variable free formula $\varphi \notin$ **GL** is refuted in this frame. Let n be the minimal number such that $\mathfrak{F}^{\leq n}_{\mathbf{GL}}(0) \not\models \varphi$. Then clearly $\mathbf{GL} \oplus \varphi = \mathbf{GL} \oplus \Box^{n-1}\bot$. Thus we have

Theorem 8.87 (i) *For every variable free formula φ, there are $i_1, \ldots, i_n$ such that*

$$\varphi \leftrightarrow \bot \vee \varphi_{i_1} \vee \ldots \vee \varphi_{i_n} \in \mathbf{GL} \text{ or } \varphi \leftrightarrow \neg(\bot \vee \varphi_{i_1} \vee \ldots \vee \varphi_{i_n}) \in \mathbf{GL}.$$

(ii) *Every variable free formula is deductively equal in* NExt**GL** *either to* $\top$ *or to* $\Box^n\bot$, *for some* $n \geq 0$.

Proof Exercise. (Hint: (i) is proved by induction on the construction of φ.) ❑

It is worth noting that if a logic L in NExt**K4** or Ext**Int** is finitely approximable then its universal frame $\mathfrak{F}_L(\varkappa)$ of rank $\varkappa$ is completely determined by $\mathfrak{F}_L^{<\infty}(\varkappa)$. For the model $\langle \mathfrak{F}_L^{<\infty}(\varkappa), \mathfrak{V}_L^{<\infty}(\varkappa)\rangle$, where $\mathfrak{V}_L^{<\infty}(\varkappa)$ is the restriction of $\mathfrak{V}_L(\varkappa)$ to $\mathfrak{F}_L^{<\infty}(\varkappa)$, characterizes the logic L (in the language with $\varkappa$ variables) and besides we have the following:

Proposition 8.88 *Suppose L is a normal modal or si-logic in a language with $\varkappa$ variables, $\mathfrak{F}$ an $\varkappa$-generated (but not $\varkappa'$-generated, for any $\varkappa' < \varkappa$) frame, $\mathfrak{V}$ a bijection from the set of variables onto the set of $\mathfrak{F}$'s generators and the model $\langle \mathfrak{F}, \mathfrak{V}\rangle$ characterizes L. Then $\mathfrak{F}^+$ is isomorphic to $\mathfrak{A}_L(\varkappa)$.*

Proof Let f be the bijection from the set of generators in $\mathfrak{A}_L(\varkappa)$ onto the set of generators in $\mathfrak{F}^+$ such that $f(\|p\|_L) = \mathfrak{V}(p)$, for every variable p. Since $\mathfrak{A}_L(\varkappa)$ is a free algebra in Var$L$, f can be extended to a homomorphism h of $\mathfrak{A}_L(\varkappa)$ onto $\mathfrak{F}^+$ such that $h(\|\varphi\|_L) = \mathfrak{V}(\varphi)$. In fact, the map h turns out to be the isomorphism we need. To see this, it suffices to establish that h is an injection. So suppose $\|\varphi\|_L$ and $\|\psi\|_L$ are distinct elements in $\mathfrak{A}_L(\varkappa)$. Then $\varphi \leftrightarrow \psi \notin L$ and hence $\varphi \leftrightarrow \psi$ is refuted in $\langle \mathfrak{F}, \mathfrak{V}\rangle$, from which $h(\|\varphi\|_L) \neq h(\|\psi\|_L)$. ❑

Corollary 8.89 *The universal frame $\mathfrak{F}_L(\varkappa)$ of rank $\varkappa$ for a finitely approximable logic L in* NExt**K4** *or* Ext**Int** *is isomorphic to the bidual of $\mathfrak{F}_L^{<\infty}(\varkappa)$.*

The upper part $\mathfrak{F}_{\mathbf{Int}}^{<\infty}(n)$ of the universal frame $\mathfrak{F}_{\mathbf{Int}}(n)$ for **Int** can be constructed in the same spirit as $\mathfrak{F}_{\mathbf{K4}}^{<\infty}(n)$ but taking into account specific features of intuitionistic frames, namely that they are partially ordered and their sets of possible values consist of upward closed sets of points. First we form a model $\mathfrak{N}_{\mathbf{Int}}^{\leq 1}(n)$ of depth 1 by taking 2^n distinct non-Σ-equivalent reflexive points which do not see each other. (As before, $\Sigma = \{p_1, \ldots, p_n\}$.) Suppose now that we have already constructed a model $\mathfrak{N}_{\mathbf{Int}}^{\leq d}(n)$ of depth $d < \omega$. For every antichain X in $\mathfrak{N}_{\mathbf{Int}}^{\leq d}(n)$ with ≥ 2 points at least one of which is of depth d, we add to $\mathfrak{N}_{\mathbf{Int}}^{\leq d}(n)$ copies of all points y of depth 1 (with the same valuation) such that, for any $x \in X$ and $p \in \Sigma$, $y \models p$ implies $x \models p$. Those copies are arranged so that they would not be accessible from each other and could see only the points in the corresponding antichain and their successors. For a singleton $X = \{x\}$ the added copies of y must satisfy one more condition : $x \not\sim_\Sigma y$. $\mathfrak{N}_{\mathbf{Int}}^{<\infty}(n)$ is defined as the union of $\mathfrak{N}_{\mathbf{Int}}^{\leq d}(n)$ for all $d < \omega$ and $\mathfrak{G}_{\mathbf{Int}}^{<\infty}(n)$ is the n-generated intuitionistic frame associated with $\mathfrak{N}_{\mathbf{Int}}^{<\infty}(n)$.

Theorem 8.90 $\mathfrak{F}_{\mathbf{Int}}^{<\infty}(n) \cong \mathfrak{G}_{\mathbf{Int}}^{<\infty}(n)$.

Proof Exercise. ❑

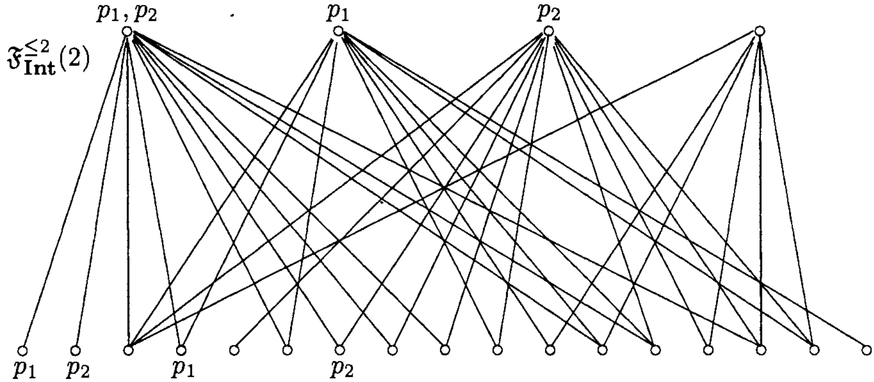


FIG. 8.12.

The model $\mathfrak{N}^{\leq 2}_{\mathbf{Int}}(2)$ is shown in Fig. 8.12 and $\mathfrak{N}^{<\infty}_{\mathbf{Int}}(1)$ in Fig. 8.13. Notice by the way that the following proposition holds.

Proposition 8.91 *For every point k in $\mathfrak{N}^{<\infty}_{\mathbf{Int}}(1)$ and every Nishimura formulas $\boldsymbol{nf}_{2n}$ and $\boldsymbol{nf}_{2n-1}$, $n \geq 1$,*

$$k \not\models \boldsymbol{nf}_{2n} \text{ iff } k \in n\!\downarrow \text{ iff } k = n \text{ or } k \geq n+2;$$

$$k \not\models \boldsymbol{nf}_{2n-1} \text{ iff } k \in \{n+1, n+2\}\!\downarrow \text{ iff } k \geq n+1.$$

Proof This claim can be easily proved either directly by induction or using the observations of Example 7.66 and the fact that the dual of $\mathfrak{G}^{<\infty}_{\mathbf{Int}}(1)$ is isomorphic to the free 1-generated algebra $\mathfrak{A}_{\mathbf{Int}}(1)$ depicted in Fig. 7.2 (a). ❑

Using this proposition, we can obtain a characterization of descriptive frames refuting the Nishimura formulas. Denote by $\mathfrak{H}_n$ the subframe of the frame in Fig. 8.13 generated by n.

Theorem 8.92 *For every descriptive frame $\mathfrak{F}$,*

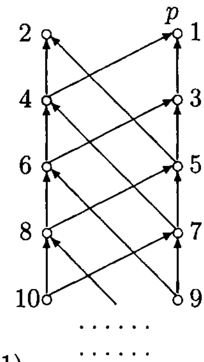
(i) *$\mathfrak{F} \not\models \boldsymbol{nf}_{2n}$ iff there is a generated subframe of $\mathfrak{F}$ reducible to $\mathfrak{H}_n$;*

(ii) *$\mathfrak{F} \not\models \boldsymbol{nf}_{2n-1}$ iff there is a generated subframe of $\mathfrak{F}$ reducible either to $\mathfrak{H}_{n+1}$ or to $\mathfrak{H}_{n+2}$.*

Proof We establish only (i), because (ii) is proved in the same way.

($\Rightarrow$) Suppose $\mathfrak{F}$ refutes $\boldsymbol{nf}_{2n}$ under a valuation $\mathfrak{V}$. Then $\boldsymbol{nf}_{2n}$ is refuted in the subalgebra of $\mathfrak{F}^+$ generated by $\mathfrak{V}(p)$ and so in its dual $\mathfrak{G}$, to which, by Theorem 8.71, $\mathfrak{F}$ is reducible by some map f, under the corresponding valuation $\mathfrak{U}$. Since $\mathfrak{G}$ is a 1-generated descriptive frame, it is (isomorphic to) a generated subframe of $\mathfrak{F}_{\mathbf{Int}}(1)$, $\mathfrak{U}(p) = \{1\}$ and so, by Proposition 8.91, $\mathfrak{G}$ contains $\mathfrak{H}_n$ as a generated subframe. Therefore, $f^{-1}(\mathfrak{H}_n)$ is a generated subframe of $\mathfrak{F}$ reducible to $\mathfrak{H}_n$.

($\Leftarrow$) follows from Proposition 8.91, and the generation and reduction theorems. ❑



$\mathfrak{F}^{<\infty}_{\mathbf{Int}}(1)$

FIG. 8.13.

Figures 8.14 and 8.15 illustrate the frames $\mathfrak{F}^{\le 3}_{\mathbf{KC}}(2)$ and $\mathfrak{F}_{\mathbf{LC}}(3)$, respectively. (For typographical reasons instead of p_1, p_2, p_3 in the latter figure we write their subscripts.) Observe that $\mathfrak{F}^{\le 2}_{\mathbf{KC}}(3)$ is isomorphic to $\mathfrak{F}^{\le 2}_{\mathbf{LC}}(3)$.

Unfortunately, this method of constructing universal frames of finite rank does not go through for logics with nontransitive frames. However, for some particular systems it can be appropriately modified. We show such a modification for **K**.

Again we construct a model $\mathfrak{N}^{\le\infty}_{\mathbf{K}}(n)$ as a "limit" of a sequence of models $\mathfrak{N}^{\le d}_{\mathbf{K}}(n)$, for $d < \omega$. Every point x in this model is characterized by a formula $\chi(x)$. $\mathfrak{N}^{\le 1}_{\mathbf{K}}(n)$ is just the antichain of 2^n non-Σ-equivalent irreflexive points. For these points x we put

$$\chi(x) = \Box\bot \wedge \bigwedge_{x\models p_i} p_i \wedge \bigwedge_{x\not\models p_i} \neg p_i.$$

Suppose now that the model $\mathfrak{N}^{\le d}_{\mathbf{K}}(n)$ and the corresponding formulas $\chi(x)$ have been already constructed. This model is extended to $\mathfrak{N}^{\le d+1}_{\mathbf{K}}(n)$ in the following way. For every set X of points in $\mathfrak{N}^{\le d}_{\mathbf{K}}(n)$ containing at least one point that does not belong to $\mathfrak{N}^{\le d-1}_{\mathbf{K}}(n)$ we add an antichain of 2^n non-Σ-equivalent irreflexive points so that they could see only the points in X and nothing else. The formulas $\chi(x)$ for the new points x look like this:

$$\chi(x) = \Box^d\bot \wedge \bigwedge_{y\in X} \Diamond\chi(y) \wedge \bigwedge_{y\in Y} \neg\Diamond\chi(y),$$

where Y is the complementation of X in $\mathfrak{N}^{\le d}_{\mathbf{K}}(n)$. Finally, let $\mathfrak{N}^{\le\infty}_{\mathbf{K}}(n)$ be the union of all models $\mathfrak{N}^{\le d}_{\mathbf{K}}(n)$ for $d < \omega$.

Using two facts—that every point x in $\mathfrak{N}^{\le\infty}_{\mathbf{K}}(n)$ is characterized by the corresponding formula $\chi(x)$ and that **K** is determined by the class of finite intransitive

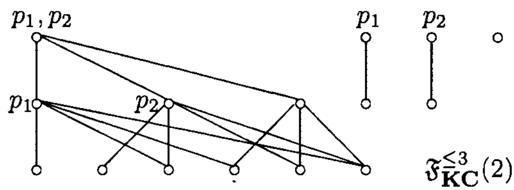


FIG. 8.14.

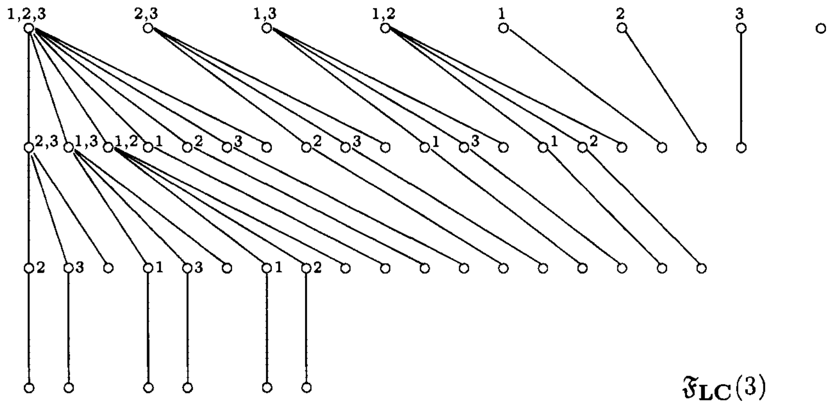


FIG. 8.15.

trees (see Corollary 3.29)—one can prove that $\mathfrak{N}^{<\infty}_{\mathbf{K}}(n)$ characterizes $\mathbf{K}$ and so, by Proposition 8.88, $\mathfrak{A}_{\mathbf{K}}(n)$ is isomorphic to the dual of the frame associated with $\mathfrak{N}^{<\infty}_{\mathbf{K}}(n)$.

8.8 Exercises and open problems

Exercise 8.1 Show that a modal frame $\mathfrak{F}$ is tight iff for every $k \geq 1$ and all $n_1, \dots, n_k \geq 1$,

$$x_1\!\uparrow^{n_1} \cup \ldots \cup x_k\!\uparrow^{n_k} = \bigcap\{X \in P : \; x_1\!\uparrow^{n_1} \cup \ldots \cup x_k\!\uparrow^{n_k} \subseteq X\}.$$

Exercise 8.2 Show that for any family $\mathcal{X}$ of sets in a modal frame

$$\Box \bigcap \mathcal{X} = \bigcap_{X \in \mathcal{X}} \Box X, \quad \Diamond \bigcup \mathcal{X} = \bigcup_{X \in \mathcal{X}} \Diamond X.$$

Is it possible to replace here $\bigcap$ by $\bigcup$ and $\bigcup$ by $\bigcap$?

Exercise 8.3 Show that a modal frame $\mathfrak{F} = \langle W, R, P \rangle$ is compact iff, for any $\mathcal{X} \subseteq P$, $\bigcup \mathcal{X} = W$ only if there is a finite subset $\mathcal{X}'$ of $\mathcal{X}$ such that $\bigcup \mathcal{X}' = W$. Is this true for intuitionistic frames?

Exercise 8.4 Show that the classes $\mathcal{DF}$ and $\mathcal{T}$ are closed under the formation of generated subframes, i.e., every generated subframe of a differentiated or tight frame is differentiated or tight itself. What about $\mathcal{CM}$? (Hint: consider the frame in Example 8.8.)

Exercise 8.5 Show that the class $\mathcal{CM}$ is closed under reductions, while $\mathcal{DF}$ and $\mathcal{T}$ are not.

Exercise 8.6 Show that the classes $\mathcal{DF}$ and $\mathcal{T}$ are closed under disjoint unions.

Exercise 8.7 Show that if $\mathfrak{G}$ and $\mathfrak{F}$ are quasi-ordered modal (intuitionistic) frames and $\mathfrak{G} \subsetneq \mathfrak{F}$ then $\rho\mathfrak{G} \subsetneq \rho\mathfrak{F}$ ($\sigma\mathfrak{G} \subsetneq \sigma\mathfrak{F}$). Prove the analogous results for reductions and disjoint unions.

Exercise 8.8 Prove that the class of finite intransitive trees is closed under finite disjoint unions, reductions and generated subframes but is not modally definable. (Hint: show that $\circ$ validates all formulas validated by all frames in the class.)

Exercise 8.9 Prove that the model $\mathfrak{N}^{<\infty}_{\mathbf{K}}(n)$ constructed at the end of the previous section is n-universal for $\mathbf{K}$.

Exercise 8.10 Show that there is a continuum of 1-generated **Grz**-algebras and a continuum of 1-generated **GL**-algebras.

Exercise 8.11 Show that if h is an isomorphism of a descriptive frame $\mathfrak{G}$ onto a generated subframe of a descriptive frame $\mathfrak{F}$ then $(h^+)_+ f_{\mathfrak{G}} = f_{\mathfrak{F}} h$, and if h is a homomorphism of a modal or pseudo-Boolean algebra $\mathfrak{A}$ onto $\mathfrak{B}$ then $(h_+)^+ f_{\mathfrak{A}} = f_{\mathfrak{B}} h$.

Exercise 8.12 Show that if f is a reduction of a descriptive frame $\mathfrak{F}$ to a descriptive frame $\mathfrak{G}$ then $(f^+)_+ f_{\mathfrak{F}} = f_{\mathfrak{G}} f$, and if f is an isomorphism of $\mathfrak{B}$ in $\mathfrak{A}$ then $(f_+)^+ f_{\mathfrak{B}} = f_{\mathfrak{A}} f$.

Exercise 8.13 Will Theorem 8.87 hold if we replace in it deductive equality by equivalence?

Exercise 8.14 Show that if f reduces $\mathfrak{F}$ to $\mathfrak{G}$ then $d(x) \geq d(f(x))$ for every point x in $\mathfrak{F}$.

Exercise 8.15 For every point x in $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(n)$ construct a formula φ in n variables such that a descriptive frame $\mathfrak{F}$ refutes φ iff there is a generated subframe of $\mathfrak{F}$ reducible to the subframe of $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(n)$ generated by x. Do the same for $\mathfrak{F}^{<\infty}_{\mathbf{Int}}(n)$.

Exercise 8.16 Show that every consistent si-logic either coincides with **Cl** or is contained in **SmL**.

Exercise 8.17 Prove that the dual of the limit of the chain (8.6) is isomorphic to the intersection of all $\mathfrak{F}_i^+$.

Exercise 8.18 Show that for every $\mathfrak{F} = \langle W, R, P\rangle$ the refinement $r\mathfrak{F}$ is isomorphic to the frame $\langle V, S, Q\rangle$ in which $V = \{Px : x \in W\}$, $PxSPy$ iff $Px \subseteq Py$ in the intuitionistic case and $\forall X \in P\ (\Box X \in Px \to X \in Py)$ in the modal one, and $Q = \{\{Px : x \in X\} : X \in P\}$.

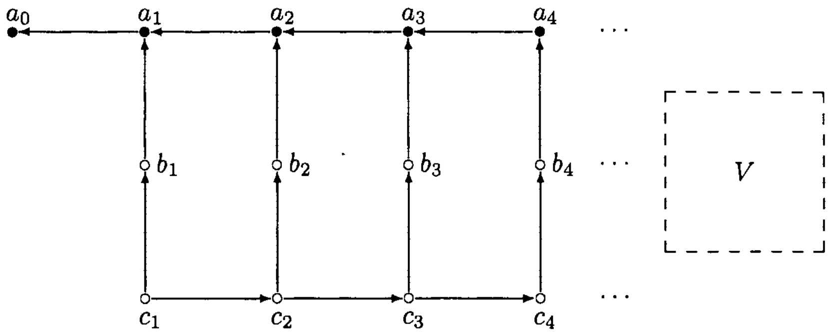


FIG. 8.16.

Exercise 8.19 Show that $\Box(\Box^+p \to q) \vee \Box(\Box^+q \to p)$ is not deductively equal in NExt**K4** to any formula in one variable.

Exercise 8.20 Let $L_2 = \mathbf{K4}\oplus\{ax1, ax2, ax3, ax4, ax5.\psi : \psi \in \{\alpha, \beta, \gamma\}\}$, where

$$ax1 = \alpha_0 \vee \Diamond^+\alpha_1,\ ax2 = \gamma \to \Diamond\gamma,\ ax3 = \gamma \to \Diamond\gamma',$$

$$ax4 = \Diamond\beta' \wedge \Diamond\alpha'' \to \Diamond\gamma,\ ax5.\psi = \Box^+(q \to \neg\psi) \vee \Box^+(\neg q \to \neg\psi),$$
$$\alpha = p \wedge \neg\Diamond p,\ \alpha' = \alpha(\Diamond p/p),\ \alpha'' = \alpha'(\Diamond p/p) = \alpha(\Diamond^2 p/p),$$
$$\alpha_i = \alpha(\Diamond^i\top/p),\ \alpha_{i+1} = \alpha'(\Diamond^i\top/p),\ \alpha_{i+2} = \alpha''(\Diamond^i\top/p),$$
$$\beta = \Diamond\alpha \wedge \neg\Diamond^+\alpha',\ \beta' = \beta(\Diamond p/p),$$
$$\beta_i = \beta(\Diamond^i\top/p) = \Diamond\alpha_i \wedge \neg\Diamond^+\alpha_{i+1},$$
$$\beta_{i+1} = \beta'(\Diamond^i\top/p) = \Diamond\alpha_{i+1} \wedge \neg\Diamond^+\alpha_{i+2},$$
$$\gamma = \Diamond\beta' \wedge \Diamond\alpha'' \wedge \neg\Diamond\beta,\ \gamma' = \gamma(\Diamond p/p),$$
$$\gamma_{i+1} = \gamma(\Diamond^i\top/p) = \Diamond\beta_{i+1} \wedge \Diamond\alpha_{i+2} \wedge \neg\Diamond\beta_i,$$
$$\gamma_{i+2} = \gamma'(\Diamond^i\top/p) = \Diamond\beta_{i+2} \wedge \Diamond\alpha_{i+3} \wedge \neg\Diamond\beta_{i+1} \quad (i \geq 0).$$

Show that if $\mathfrak{F} = \langle W, R, P\rangle$ is a rooted differentiated frame for L_2 then $\langle W, R\rangle$ is isomorphic to a rooted generated subframe of the frame shown in Fig. 8.16, with all $\{a_i\}$, $\{b_j\}$ and $\{c_k\}$ being in P. (Hint: use the following substitution instances of L_2's axioms:

$$ax2.i = \gamma_i \to \Diamond\gamma_i = ax2(\Diamond^i\top/p),$$

$$ax3.i = \gamma_i \to \Diamond\gamma_{i+1} = ax3(\Diamond^i\top/p),$$
$$ax4.i = \Diamond\beta_i \wedge \Diamond\alpha_{i+1} \to \Diamond\gamma_i = ax4(\Diamond^i\top/p) \quad (i \geq 1),$$
$$ax5.\alpha_i = \Box^+(q \to \neg\alpha_i) \vee \Box^+(\neg q \to \neg\alpha_i) = ax5.\alpha(\Diamond^i\top/p),$$
$$ax5.\beta_i = \Box^+(q \to \neg\beta_i) \vee \Box^+(\neg q \to \neg\beta_i) = ax5.\beta(\Diamond^i\top/p),$$
$$ax5.\gamma_{i+1} = \Box^+(q \to \neg\gamma_{i+1}) \vee \Box^+(\neg q \to \neg\gamma_{i+1}) = ax5.\gamma(\Diamond^i\top/p), \quad (i \geq 0).)$$

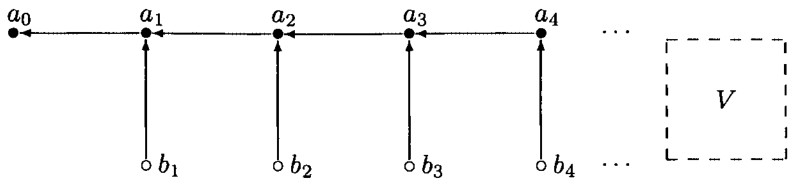


FIG. 8.17.

Exercise 8.21 Let $\mathcal{C}_1$ be the class of all differentiated frames for L_2 whose underlying Kripke frames have the form shown in Fig. 8.17 and $L_1 = \mathrm{Log}\mathcal{C}_1$. Prove that L_1 has no immediate predecessor in the interval $[L_2, L_1]$. (Hint: use the result of the preceding exercise.)

Exercise 8.22 Prove that the logic L_1 in the preceding exercise does not have an independent axiomatization. (Hint: see Section 4.5.)

Exercise 8.23 Show that for every normal logic $L \in [\mathbf{S3}, \mathbf{Grz}]$ and every intuitionistic formula φ, $\mathsf{T}(\Gamma) \vdash_L^* \mathsf{T}(\varphi)$ iff $\Gamma \vdash_{Int} \varphi$.

Problem 8.1 *Are all si-logics complete with respect to topological spaces?*

8.9 Notes

The approach to constructing the adequate semantics for non-classical logics presented in Section 8.1 (it should be clear that it works for, say various kinds of polymodal logics) is similar to Henkin's approach to establishing completeness of higher order classical predicate calculi. The reader can find details of Henkin's method and references in Church (1956). Here we note only that by imposing restrictions on possible valuations in models we in fact introduce interpretations for the unary predicates representing the truth-sets of propositional variables—for that reason general frames are sometimes called *first order frames*. This makes impossible various "negative" effects of Chapter 6 because we are not able any more to change arbitrarily valuations. Moreover, it is not hard to prove the following analog of the Löwenheim–Skolem theorem: for every general frame $\mathfrak{F}$ and a point x in it, one can select a countable general subframe $\mathfrak{G}$ of $\mathfrak{F}$ containing x such that $\mathfrak{G}$ validates the same formulas as $\mathfrak{F}$ and a formula is refutable at x in $\mathfrak{G}$ whenever it is refutable at x in $\mathfrak{F}$.

The approach outlined in Section 8.2 was developed first by Jónsson and Tarski (1951, 1952). In fact, their results were much more general; for example, they added to Boolean algebras collections of arbitrary n-ary operations satisfying some natural properties like conditions (ii) and (iii) in Theorem 7.44. However, chronologically (even in spite of Kripke's (1963a) claim that he had independently obtained the main result of Jónsson and Tarski (1951)) the semantics of general frames for modal logics was explicitly formulated only by Makinson (1970). Thomason (1972b) proved completeness theorems for tense (and so modal) logics with respect to this semantics (which he called *first order*) and introduced the notion of refined frame and the operation of refinement. Goldblatt (1976a, 1976b) contains an extensive and systematical study of the semantics of general frames: first order frames, subframes, homomorphisms, disjoint unions, ultraproducts, compactness and semantical consequence, descriptive frames, the categories of descriptive frames and modal algebras, inverse limits of descriptive frames, modal axiomatic classes, d-persistent formulae, first order definability—these are a few titles of sections in Goldblatt (1976a, 1976b) showing the directions of investigations. Many results in Sections 8.4 and 8.5 were taken from this paper. It is hard to say who was the first to introduce explicitly general intuitionistic frames—in any case it was not too difficult having at hand duality theory for modal logics and the connection between pseudo-Boolean and topological Boolean algebras discovered by McKinsey and Tarski (1946) (we discussed it in Section 8.3). The earliest references we know are Esakia (1974) and Rautenberg (1979).

A topological approach to the Stone–Jónsson–Tarski representation and duality theory was developed by Esakia (1974, 1979b, 1985) and Sambin and Vaccaro (1988). Note also that general frames can be introduced in the case of neighborhood semantics; see Došen (1988).

In view of the duality between algebras and descriptive frames (and the truth-preserving operations on them), Birkhoff's theorem opens a way for solving the problem of characterizing modally and intuitionistically definable classes of (Kripke, general, refined, etc.) frames. Goldblatt and Thomason (1974), van Benthem (1975, 1989), Goldblatt (1976a, 1976b) found various conditions (of closure under certain operations) for a class of frames to be modally definable. For example, as was shown by Goldblatt and Thomason (1974), if a class $\mathcal{C}$ of Kripke frames is closed under elementary equivalence then $\mathcal{C}$ is modally definable iff $\mathcal{C}$ is closed under the formation of generated subframes, disjoint unions and reductions, while its complement is closed under ultrafilter extensions (for the definition see Section 10.2). The case of finite frames is of special interest here. Birkhoff's theorem (for a "finitized" variant of it see Banaschevski, 1983) suggests that as a condition for the modal definability of a class of finite frames one should take the closure of the class under finite disjoint unions, reductions and generated subframes. However, Example 8.8 shows that this is not enough. It is not hard to see that essential in this example is the fact that the frames under consideration are not transitive. Indeed, as was shown by Rodenburg (1986) (for intuitionistic frames) and van Benthem (1989), in the case of transitive frames the conditions above are enough (see Exercise 9.34). In the general case we need also the condition of closure under so called local p-morphic images; see van Benthem (1989). Much less is known about modal definability of classes of frames with actual worlds, although the available variants of Birkhoff's theorem for this case (in particular, Theorem 7.81) give some hope for a progress in this direction too. For definability of frame classes by formulas in richer languages see, for instance, Goranko (1990).

The description of finitely generated universal frames for **K4**, presented in

Sections 8.6 and 8.7, was obtained in essence by Segerberg (1971) and after that was rediscovered in various forms. An important step in understanding the constitution of such frames was made by Shehtman (1978a) who gave a general method of constructing the universal frames of finite rank for finitely approximable logics with transitive frames and illustrated it for **S4**, **Grz** and **Int**. Similar results were obtained by Bellissima (1985a).

Needless to say that if we know the detailed structure of the universal frames for a logic, we have a powerful instrument for studying both the logic itself and the lattice of its extensions. We shall take advantage of it in further chapters. In particular, the solution to the admissibility problem for inference rules, obtained in Section 16.7, would not be possible without this instrument. And the results on m-reducibility in Section 13.1 are based in essence upon considering the form of the upper part of the m-universal frames for the corresponding logics.

However, there are still a lot of open problems concerning universal frames. Actually, the picture is more or less clarified only for extensions of **K4** and **Int**. And even here the behavior of the universal frames for logics that are not finitely approximable may turn out to be rather unexpected; see, for instance, Chagrov (1994b). In the "nontransitive" case, only for very few logics, in particular **K**, universal models have been described. A perspective (though not easy) direction is to consider the constitution of the universal frames for some extensions of $\mathbf{K} \oplus \boldsymbol{tra}_n$, while for extensions of $\mathbf{KTB} = \mathbf{K} \oplus \boldsymbol{re} \oplus \boldsymbol{sym}$ this problem seems to be very hard. It is no accident that so little is known of Ext**KTB**. One of the strongest facts here is that there are infinitely many pretabular logics in NExt**KTB**. It is known, for instance, that the universal frame of rank 2 for $\mathbf{KTB} \oplus \Box^2 p \to \Box^3 p$ is infinite (Byrd 1978), and we have no information about its universal frame of rank 1.

The problem of describing universal frames of finite rank for polymodal and tense logics is much more complicated. Even in the transitive case the situation here resembles that in NExt**K**.

Another interesting problem is to describe atoms (the corresponding formulas, to be more precise) of n-generated free algebras in varieties of modal (tense, etc.) algebras. In accordance with atomicity, atomless of such algebras we call the corresponding logics *n-atomic*, *n-atomless*, etc. Here are some examples:

- **K** is n-atomic, for every n;
- **D** is n-atomless, for every $n > 0$ (there are no 0-atomless modal logics);
- there are normal modal logics which, for any $n > 0$, are neither n-atomic nor n-atomless.

These results were obtained by Bellissima (1984). For finitely approximable logics in NExt**K4**, he proved also that all of them are n-atomic for every n. However, it is not clear whether the finite approximability is essential here. Bellissima (1991) considers similar problems for tense logics. Recently Wolter (1997) has connected atomicity of finitely generated free algebras for polymodal logics with splittings of the corresponding lattices of logics (see Section 10.5). In particular, he proved that if all finitely generated free algebras for L are atomic then L is characterized by the class of frames that split NExt$L$.

Theorems 8.67, 8.85 and 8.92 were proved by Makinson (1971), Segerberg (1971) and Anderson (1972), respectively.

9

CANONICAL FORMULAS

In Sections 2.5 and 3.5 we characterized the geometry of Kripke frames validating some intuitionistic and modal formulas by imposing first order conditions on their accessibility relations. However, as was shown in Section 6.2, there exist formulas which have no first order equivalents. In this chapter we try another, purely frame-theoretic approach to the characterization problem which uses such notions as subframe, reduction, etc. Unfortunately, this approach is not universal either. But its limitation is of a different kind: it characterizes only *transitive* general frames, but for every modal and intuitionistic formula. So all frames in this chapter are assumed to be *transitive*.

The characterization to be obtained below can be roughly described as follows. Given a modal or intuitionistic formula φ, one can effectively construct finite rooted frames $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ such that a general frame $\mathfrak{G}$ refutes φ iff there is a (not necessarily generated) subframe of $\mathfrak{G}$ which is reducible to one of $\mathfrak{F}_i$ and satisfies some other natural conditions. Conversely, with every finite rooted frame $\mathfrak{F}$ we can associate a formula—call it *canonical*—explicitly saying: "I am refuted in a frame iff it contains a subframe reducible to $\mathfrak{F}$ and satisfying those conditions". As a result, we obtain a powerful language of canonical formulas: they axiomatize all logics in Ext**K4** and Ext**Int** and bear explicit information about the constitution of their refutation frames.

9.1 Subreduction

In this section and the next one we give a few examples revealing certain fundamental principles of the constitution of transitive refutation frames for modal and intuitionistic formulas.

Example 9.1 Let us consider once more the Grzegorczyk formula $\boldsymbol{grz}$ (which, as was shown in Section 6.2, is not first order definable). In Examples 3.22 and 3.24 we constructed its two simplest transitive countermodels on the frames $\bullet$ and $\circ\circ$. On the other hand, Proposition 3.48 asserts that a Kripke frame $\mathfrak{F}$ refutes $\boldsymbol{grz}$ iff it contains either an irreflexive point or a proper cluster or an infinite ascending chain of distinct points. Since every infinite ascending chain is reducible to the two point cluster (see Example 3.14), we can reformulate this observation as follows: $\mathfrak{F} \not\models \boldsymbol{grz}$ iff there is a subframe of $\mathfrak{F}$ that is reducible either to $\bullet$ or to $\circ\circ$.

In order to extend this characterization to general frames, we require the following definition.

Given modal frames $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$, a partial (i.e., not completely defined, in general) map f from W onto V is called a *subreduction* (or a *partial p-morphism*) of $\mathfrak{F}$ to $\mathfrak{G}$ if it satisfies the conditions (R1)–(R3) in Section 8.4 for all x and y in the domain of f and all $X \in Q$. In this case we say also that f *subreduces* $\mathfrak{F}$ to $\mathfrak{G}$, $\mathfrak{F}$ is *subreducible* to $\mathfrak{G}$ (by f) and $\mathfrak{G}$ is an (f-)*subreduct* of $\mathfrak{F}$. The domain of f will be denoted by $\mathrm{dom} f$. If $\mathfrak{F}$ and $\mathfrak{G}$ are Kripke frames then the subreducibility of $\mathfrak{F}$ to $\mathfrak{G}$ means that there is a subframe of $\mathfrak{F}$ which is reducible to $\mathfrak{G}$. Note also that if $\mathfrak{G}$ is a finite Kripke frame then (R3) is equivalent to

(R4) $\quad\forall z \in V \; f^{-1}(z) \in P.$

A frame $\mathfrak{G} = \langle V, S, Q\rangle$ is called a *subframe* of $\mathfrak{F} = \langle W, R, P\rangle$ if $V \subseteq W$ and the identity map on V is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, i.e., if S is the restriction of R to V and $Q \subseteq P$. Note that a generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ is not in general a subframe of $\mathfrak{F}$, since V need not be in P; however, if $V \in P$ then $\mathfrak{G}$ is a subframe of $\mathfrak{F}$. More generally, suppose V is a non-empty subset of W in $\mathfrak{F} = \langle W, R, P\rangle$ such that $V \in P$ and S is the restriction of R to V. Define a set of possible values Q in the space V by taking

$$Q = \{X \subseteq V : \; X \in P\}.$$

Q is obviously closed under the Boolean operations and for every $X \in Q$,

$$X{\downarrow}S = V \cap X{\downarrow}R \in Q,$$

so that $\mathfrak{G} = \langle V, S, Q\rangle$ is really a modal frame. Since by the definition, $Q \subseteq P$, the frame $\mathfrak{G}$ is a subframe of $\mathfrak{F}$. We call it the *subframe* of $\mathfrak{F}$ *induced* by V. Thus, an f-subreduct of $\mathfrak{F}$ is a reduct of the $\mathfrak{F}$'s subframe induced by $\mathrm{dom} f$.

Example 9.2 Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G}$ be the Kripke frames shown in Fig. 9.1. Then the map f defined by

$$f(i) = \begin{cases} a & \text{if } i \text{ is even} \\ b & \text{if } i \text{ is odd} \\ \text{undefined} & \text{if } i = \omega \end{cases}$$

is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. Observe that $\mathfrak{F}$ is not reducible to $\mathfrak{G}$. If we define in $\mathfrak{F}$ the set P of possible values consisting of finite sets of natural numbers and complements to them in the space W, then the frame $\mathfrak{F} = \langle W, R, P\rangle$ is not subreducible to $\mathfrak{G}$. For otherwise, when, say, f is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, we would have $f^{-1}(a) \in P$ and $f^{-1}(b) \in P$, which is impossible because $f^{-1}(a)$ and $f^{-1}(b)$ are disjoint and infinite.

Proposition 9.3 *A general frame $\mathfrak{F}$ refutes* $\boldsymbol{grz}$ *iff $\mathfrak{F}$ is subreducible either to $\bullet$ or to* $(\circ\,\circ)$.

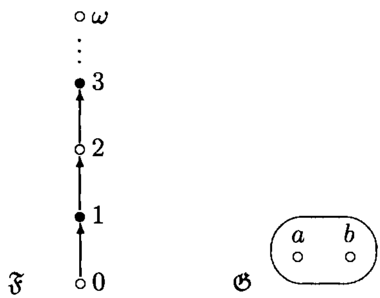


FIG. 9.1.

Proof ($\Rightarrow$) Suppose $\boldsymbol{grz}$ is refuted in $\mathfrak{F} = \langle W, R, P\rangle$ under some valuation. Then the set $X = \{x \in W : x \not\models \boldsymbol{grz}\} \in P$ is non-empty. Let us consider the set $X - X{\downarrow} \in P$ which consists of all final irreflexive points in X, if any.

If $X - X{\downarrow} \neq \emptyset$ then the map f defined by

$$f(x) = \begin{cases} \bullet & \text{if } x \in X - X{\downarrow} \\ \text{undefined} & \text{otherwise} \end{cases}$$

for every $x \in W$, is a subreduction of $\mathfrak{F}$ to $\bullet$. If $X - X{\downarrow} = \emptyset$ then for every $x \in X$, there is $x' \in x{\uparrow} \cap X$. Hence $x' \models \Box(p \to \Box p) \to p$, $x' \not\models p$ and so $x' \not\models \Box(p \to \Box p)$. But then the set

$$Y = \{y \in W : \ y \models \Box(\Box(p \to \Box p) \to p), \ y \models p, \ y \not\models \Box p\} \in P$$

is non-empty, $Y \subseteq X{\downarrow}$ and $X \subseteq Y{\downarrow}$. Therefore, the map f defined by

$$f(x) = \begin{cases} a & \text{if } x \in X \\ b & \text{if } x \in Y \\ \text{undefined} & \text{otherwise} \end{cases}$$

for every $x \in W$, is a subreduction of $\mathfrak{F}$ to the cluster with two points a and b.

($\Leftarrow$) Suppose f subreduces $\mathfrak{F}$ to $\bullet$. Then $\mathrm{dom} f \in P$ is an antichain of irreflexive points. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = W - \mathrm{dom} f$. The reader can easily check that $\boldsymbol{grz}$ is false under $\mathfrak{V}$ at every point $x \in \mathrm{dom} f$.

Suppose now that f is a subreduction of $\mathfrak{F}$ to the cluster with two points a and b. Then we define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking, for instance,

$$\mathfrak{V}(p) = W - f^{-1}(a).$$

We show that $x \not\models \Box(\Box(p \to \Box p) \to p) \to p$, for every $x \in f^{-1}(a)$. By the definition, we have $x \not\models p$. Suppose that $x \not\models \Box(\Box(p \to \Box p) \to p)$. Then there is $y \in x{\uparrow}$ such that $y \models \Box(p \to \Box p)$, $y \not\models p$ and so $y \in f^{-1}(a)$. By (R2), there is $z \in y{\uparrow}$ such that $z \in f^{-1}(b)$. But then $z \models p \to \Box p$, $z \models p$ and so $z \models \Box p$,

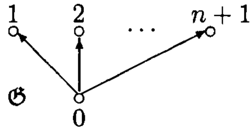
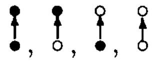


FIG. 9.2.

which is impossible, since by (R2), there is $x' \in z{\uparrow} \cap f^{-1}(a)$ and we must have simultaneously both $x' \models p$ and $x' \not\models p$. ❑

In the same manner one can establish the characterizations presented in Table 9.1, where each $*$ is to be replaced by $\bullet$ and $\circ$ (for instance  represents four frames: ). To be more exact, we have

Proposition 9.4 *A transitive modal frame $\mathfrak{F}$ refutes a formula in the left-hand side of Table 9.1 iff $\mathfrak{F}$ is subreducible to one of the frames in the same line of the right-hand side.*

Proof Exercise. ❑

In the intuitionistic case the definition of subreduction becomes somewhat more complicated. Given intuitionistic frames $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$, a partial map f from W onto V is called a *subreduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if it satisfies (R1) and (R2), for all $x, y \in \mathrm{dom} f$, and also the following condition:

(R3′) $\qquad \forall X \in \overline{Q}\ f^{-1}(X){\downarrow} \in \overline{P}$

where $\overline{Q} = \{V - X : X \in Q\}$ and $\overline{P} = \{W - X : X \in P\}$. For a completely defined f satisfying (R1) and (R2) the condition (R3′) is clearly equivalent to (R3) and so every reduction is also a subreduction. If $\mathfrak{G}$ is a finite Kripke frame then (R3′) is equivalent to

(R4′) $\qquad \forall z \in V\ f^{-1}(z){\downarrow} \in \overline{P}.$

$\mathfrak{G}$ is a *subframe* of $\mathfrak{F}$ if $\kappa\mathfrak{G}$ is a subframe of $\kappa\mathfrak{F}$ and the identity map on V is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$.

Proposition 9.5 *An intuitionistic frame $\mathfrak{F}$ refutes a formula in the left-hand side of Table 9.2 iff $\mathfrak{F}$ is subreducible to one of the frames in the same line of the right-hand side.*

Proof We consider only $\boldsymbol{bw}_n = \bigvee_{i=1}^{n+1}(p_i \to \bigvee_{j\neq i} p_j)$ and leave the other formulas to the reader.

($\Rightarrow$) Suppose $\mathfrak{F} = \langle W, R, P\rangle$ refutes $\boldsymbol{bw}_n$ under some valuation. Define a partial map f from W onto the set of points in the frame $\mathfrak{G}$ in Fig. 9.2 by taking

$$f(x) = \begin{cases} 0 & \text{if } x \not\models \boldsymbol{bw}_n \\ i & \text{if } 0 < i \leq n+1,\ x \models p_i \text{ and } x \not\models \bigvee_{j\neq i} p_j \\ \text{undefined otherwise.} & \end{cases}$$

Table 9.1 *Characterizing refutation frames: subreduction.*

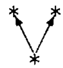
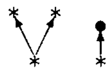
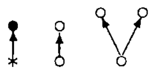
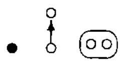
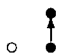
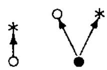
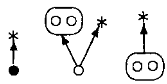
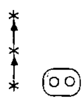
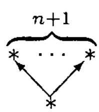
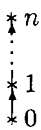
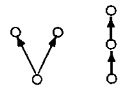
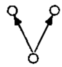
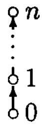
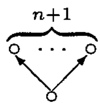
Formula φ	$\mathfrak{F} \not\models \varphi$ iff $\mathfrak{F}$ is subreducible to one of the following frames
$\Box p \to p$	$\bullet$
$\Box(\Box p \to p) \to \Box p$	$\circ$
$p \to \Box\Diamond p$	 (4 frames)
$\Box(\Box(p \to \Box p) \to p) \to p$	
$\Box(\Box^+ p \to q) \lor \Box(\Box^+ q \to p)$	 (6 frames)
$\Box(\Box p \to q) \lor \Box(\Box q \to p)$	 (8 frames)
$\Diamond\Box p \to \Box p$	 (4 frames)
$\Box p \leftrightarrow p$	
$\Box p$	
$\Box(\Box p \to p) \land \Diamond\Box p \to \Box p$	 (4 frames)
$\Box(\Box(p \to \Box p) \to p) \land \Diamond\Box p \to p$	 (6 frames)
$p \to \Box(\Diamond p \to p)$	 (9 frames)
$\boldsymbol{bw}_n$	 ($2n+4$ frames)
$\boldsymbol{bd}_n$	 (2^{n+1} frames)
$\boldsymbol{alt}_n$	rooted frames with $n+1$ distinct points accessible from their roots

Table 9.2 *Characterizing refutation frames: subreduction.*

Formula φ	$\mathfrak{F} \not\models \varphi$ iff $\mathfrak{F}$ is subreducible to one of the following frames
$p \vee \neg p$	
$(\neg q \to p) \to (((p \to q) \to p) \to p)$	
$(p \to q) \vee (q \to p)$	
$\boldsymbol{bd}_n$	
$\boldsymbol{bw}_n$	
$\boldsymbol{bc}_n$	rooted frames with $n+1$ points

Since for all $i \in \{1, \ldots, n+1\}$, if $x \not\models \boldsymbol{bw}_n$ then $x \not\models p_i$ and there exists $y_i \in x\uparrow$ such that $y_i \models p_i$ and $y_i \not\models p_j$ for $j \neq i$, f is a surjection satisfying (R1) and (R2). Besides, we have $f^{-1}(0)\downarrow = \{x : x \not\models \boldsymbol{bw}_n\} \in \overline{P}$ and for every $i \in \{1, \ldots, n+1\}$, $f^{-1}(i)\downarrow = \{x : x \not\models p_i \to \bigvee_{j \neq i} p_j\} \in \overline{P}$. So f satisfies (R4′) as well.

($\Leftarrow$) Suppose f is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking, for every $i \in \{1, \ldots, n+1\}$,

$$\mathfrak{V}(p_i) = W - \bigcup_{j \neq i} f^{-1}(j)\downarrow \in P.$$

Since by (R1), $f^{-1}(i) \cap f^{-1}(j) \downarrow = \emptyset$ for every $i \neq j$, we have $x \models p_i$ and $x \not\models \bigvee_{j \neq i} p_j$ for each $x \in f^{-1}(i)$, whence $x \not\models p_i \to \bigvee_{j \neq i} p_j$. And since by (R2), $f^{-1}(0) \subseteq \bigcap_{i=1}^{n+1} f^{-1}(i)\downarrow$, we have $x \not\models \boldsymbol{bw}_n$ for all $x \in f^{-1}(0)$. ❑

In the intuitionistic case there is a nice algebraic counterpart of the notion of subreduction. Given two pseudo-Boolean algebras $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot \rangle$ and $\mathfrak{B} = \langle B, \wedge, \vee, \to, \bot \rangle$ and a non-empty set $O \subseteq \{\wedge, \vee, \to, \bot\}$, an injection f from B into A is called an *O-isomorphism* of $\mathfrak{B}$ in $\mathfrak{A}$ if f preserves all the operations in O. If $B \subseteq A$ and the identity map on B is an O-isomorphism of $\mathfrak{B}$ in $\mathfrak{A}$ then we call $\mathfrak{B}$ an *O-subalgebra* of $\mathfrak{A}$. In this case the operations from O in $\mathfrak{B}$ are just the restrictions of the corresponding operations in $\mathfrak{A}$ to $\mathfrak{B}$.

The same notions of O-isomorphism and O-subalgebra may be defined of course for modal algebras, but this time $O \subseteq \{\wedge, \vee, \to, \bot, \Box\}$. Denoting the operations $\wedge$, $\vee$, $\to$, $\perp$, $\Box$ by the letters C, D, I, N, B, respectively (N stands for "negation", B for "box"), we shall write "IC-isomorphism" instead of "$\{\to, \wedge\}$-isomorphism", etc.

Example 9.6 Let $\mathfrak{A}$ be the pseudo-Boolean algebra shown in Fig. 9.3. Then the algebra $\mathfrak{B}$ in Fig. 9.3 is an IC-subalgebra of $\mathfrak{A}$, but neither an ICN- nor an ICD-subalgebra, since $\mathfrak{A}$ and $\mathfrak{B}$ have distinct zero elements and distinct $\vee$. Fig. 9.3 shows also that the dual $\mathfrak{A}_+$ of $\mathfrak{A}$ is subreducible (but not reducible) to the dual $\mathfrak{B}_+$ of $\mathfrak{B}$.

Theorem 9.7 *Suppose $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$ are intuitionistic frames and f a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. Then the map f^+ defined by*

$$f^+(X) = W - f^{-1}(V - X)\downarrow$$

for every $X \in Q$, is an IC-isomorphism of $\mathfrak{G}^+$ in $\mathfrak{F}^+$.

Proof Observe first that by (R3$'$), $f^+(X) \in P$ for every $X \in Q$. Notice also that for every $x \in W$ and $X \in Q$,

$$x \in f^+(X) \text{ iff } \forall y \in \mathrm{dom} f\ (xRy \to f(y) \in X). \tag{9.1}$$

It follows from (9.1) and (R1) that for every $X \in Q$ and $y \notin X$, we have $f^{-1}(X) \subseteq f^+(X)$ and $f^{-1}(y) \cap f^+(X) = \emptyset$. Therefore, f^+ is an injection from Q in P.

Let us show now that f^+ preserves $\cap$ and $\supset$, i.e. suppose $X, Y \in Q$ and prove that

$$f^+(X \cap Y) = f^+(X) \cap f^+(Y)$$

and

$$f^+(X \supset Y) = f^+(X) \supset f^+(Y).$$

The former equality follows from the definition of f^+. (Note by the way that f^+ does not preserve $\cup$; in general we have only $f^+(X \cup Y) \supseteq f^+(X) \cup f^+(Y)$. Besides, $f^+(\emptyset)$ may be non-empty.)

Let $x \in f^+(X \supset Y)$. By (9.1), this is equivalent to

$$\forall y \in x\uparrow \cap \mathrm{dom} f\ \forall u \in f(y)\uparrow\ (u \in X \to u \in Y). \tag{9.2}$$

Suppose that xRz, $z \in f^+(X)$ and show that $z \in f^+(Y)$. Indeed, otherwise we must have some $y \in z\uparrow$ such that $f(y)$ is defined, but is not in Y, which by (9.1) and (9.2), is impossible. Therefore, $z \in f^+(Y)$ and so $x \in f^+(X) \supset f^+(Y)$. Conversely, suppose $x \in f^+(X) \supset f^+(Y)$, i.e.,

$$\forall z \in x\uparrow\ (z \in f^+(X) \to z \in f^+(Y)), \tag{9.3}$$

and prove (9.2). Let $y \in x\uparrow \cap \mathrm{dom} f$, $u \in f(y)\uparrow$ and $u \in X$, but $u \notin Y$. Then by (R2), there is $z \in y\uparrow$ such that $f(z) = u$, i.e., $f(z) \in X$ and $f(z) \notin Y$. As we

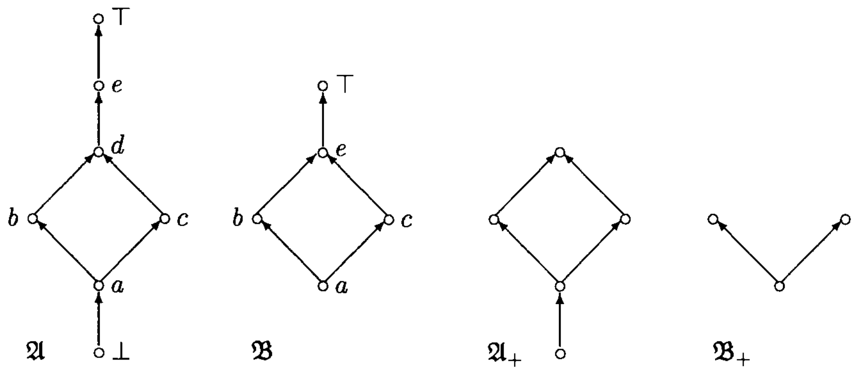


FIG. 9.3.

have already observed, this means that $z \in f^+(X)$ and $z \notin f^+(Y)$, contrary to (9.3). ❑

As a consequence we obtain the following truth-preservation result for intuitionistic IC-formulas, i.e., formulas containing no occurrences of $\vee$ and $\bot$; such formulas are called also *disjunction and negation free formulas.* If a formula has no occurrences of $\vee$ (of $\bot$) then it is called a *disjunction* (respectively, *negation*) *free formula.* It should be emphasized that $\Diamond$ was defined via $\bot$; so negation free modal formulas contain no diamonds.

Corollary 9.8 *Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are intuitionistic frames and $\mathfrak{F}$ is subreducible to $\mathfrak{G}$. Then $\mathfrak{F} \models \varphi$ implies $\mathfrak{G} \models \varphi$ for every disjunction and negation free formula φ.*

Using the method developed for the proof of Theorem 8.71, one can prove a theorem that is dual to Theorem 9.7 (see Exercise 9.2). The algebraic meaning of the notion of subframe in the modal case is explained in Exercise 9.5.

Given intuitionistic or modal frames $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$, a subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called *dense* if $\mathrm{dom}f\underline{\uparrow} \cap \mathrm{dom}f\overline{\downarrow} = \mathrm{dom}f$, i.e., if $\forall x \in W\ \forall y, z \in \mathrm{dom}f\ (yRxRz \to x \in \mathrm{dom}f)$.

Theorem 9.9 *Suppose $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$ are intuitionistic or modal frames, f is a dense subreduction of $\mathfrak{F}$ to $\mathfrak{G}$ and $W = \mathrm{dom}f\underline{\uparrow}$. Then there is an ICD- or, in the modal case, ICDB-isomorphism f^+ of $\mathfrak{G}^+$ in $\mathfrak{F}^+$.*

Proof Let us consider first the modal case. Define a map f^+ from Q into P by taking, for every $X \in Q$, $f^+(X) = W - f^{-1}(V - X)$. It follows from (R3) that $f^+(X) \in P$. Since for every $x \in W$ and every $X \in Q$,

$$x \in f^+(X) \text{ iff } x \notin \mathrm{dom}f \text{ or } f(x) \in X, \tag{9.4}$$

f^+ is an injection. Using (9.4), one can readily check that f^+ preserves $\cap$, $\cup$ and $\supset$. Suppose $X \in Q$ and show that $f^+(\Box X) = \Box f^+(X)$. If $x \in f^+(\Box X)$ then, by

(9.4), either $x \notin \mathrm{dom} f$ or $x \in \mathrm{dom} f$ and $z \in X$ for every $z \in f(x)\uparrow$. Take an arbitrary $y \in x\uparrow$ and show that $y \in f^+(X)$. If $x \notin \mathrm{dom} f$ then $y \notin \mathrm{dom} f$, since $\mathfrak{F}$ is generated by $\mathrm{dom} f$ and f is dense, and so $y \in f^+(X)$. If $x \in \mathrm{dom} f$ and $y \in \mathrm{dom} f$ then, by (R1), we have $f(x)Sf(y)$, whence $f(y) \in X$ and $y \in f^+(X)$.

Conversely, suppose $x \in \Box f^+(X)$, i.e., for every $y \in x\uparrow$, either $y \notin \mathrm{dom} f$ or $f(y) \in X$. It follows by (R2), that either $x \notin \mathrm{dom} f$ or $x \in \mathrm{dom} f$ and $z \in X$ for every $z \in f(x)\uparrow$. Therefore, $x \in f^+(\Box X)$.

As to the intuitionistic case, we define f^+ as in Theorem 9.7. So it suffices to verify that for all $X, Y \in Q$, $f^+(X \cup Y) = f^+(X) \cup f^+(Y)$. The inclusion $f^+(X \cup Y) \supseteq f^+(X) \cup f^+(Y)$ follows directly from (9.1). Suppose $x \in f^+(X \cup Y)$. If $x \notin \mathrm{dom} f$ then, by the density of f, we have $y \notin \mathrm{dom} f$ for every $y \in x\uparrow$, and so x is in $f^+(X)$ as well as in $f^+(Y)$. And if $x \in \mathrm{dom} f$ then, by (9.1), $f(x) \in X$ or $f(x) \in Y$, whence $x \in f^+(X) \cup f^+(Y)$. ❑

As a consequence we obtain one more truth-preservation result.

Corollary 9.10 *If $\mathfrak{F}$ is densely subreducible to $\mathfrak{G}$ then for every negation free formula φ, $\mathfrak{F} \models \varphi$ implies $\mathfrak{G} \models \varphi$.*

9.2 Cofinal subreduction and closed domain condition

Transitive refutation frames for the formulas in Tables 9.1 and 9.2 have a rather simple structure. Roughly, to construct all refutation frames for such a formula, we can first take the frames reducible to one of its refutation patterns in the table and then insert into them new points at any places we want, provided, of course, that the accessibility relation between the old points remains the same. However, there are modal and intuitionistic formulas whose refutation frames are constructed in a more complex way.

Example 9.11 Let us analyze the constitution of transitive refutation frames for the McKinsey formula $\boldsymbol{ma} = \Box\Diamond p \to \Diamond\Box p$. It follows from Proposition 3.46 that the simplest Kripke frames refuting it are again the degenerate cluster $\bullet$ and the two point cluster (∘∘). And again its every refutation frame is subreducible either to $\bullet$ or to (∘∘). Indeed, suppose that $\boldsymbol{ma}$ is false in $\mathfrak{F} = \langle W, R, P\rangle$ under some valuation and let

$$X = \{x \in W : \ x \not\models \boldsymbol{ma}\} \in P.$$

If $X - X\downarrow \neq \emptyset$ then the map f defined by

$$f(x) = \begin{cases} \bullet & \text{if } x \in X - X\downarrow \\ \text{undefined otherwise} & \end{cases}$$

is obviously a subreduction of $\mathfrak{F}$ to $\bullet$. And if $X - X\downarrow = \emptyset$ then we define a map f from $\mathfrak{F}$ onto the frame $\mathfrak{G}$ in Fig. 9.1 by taking

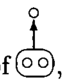
$$f(x) = \begin{cases} a & \text{if } x \not\models \boldsymbol{ma} \text{ and } x \not\models p \\ b & \text{if } x \not\models \boldsymbol{ma} \text{ and } x \models p \\ \text{undefined otherwise.} & \end{cases}$$

It is clear that f satisfies (R1) and (R3), and the fact that it satisfies also (R2) follows from the considerations in Section 3.5.

However, the subreducibility of $\mathfrak{F}$ to $\bullet$ or  is only a necessary condition for $\mathfrak{F} \not\models \boldsymbol{ma}$, but not a sufficient one. For the frame  is subreducible to  but, according to Proposition 3.46, validates $\boldsymbol{ma}$.

Let us take a closer look at the subreductions f defined above. In the former case $\mathrm{dom}f$ contains some final points in $\mathfrak{F}$, dead ends, to be more exact (for if $x \in X - X\downarrow$ is not a dead end then xRy, for some $y \in W$, whence $y \models \Box\Diamond p$, $y \not\models \Diamond\Box p$ and so $y \in X$, which is a contradiction). In the latter one points in $\mathrm{dom}f$ are not necessarily final in $\mathfrak{F}$, but the whole set $\mathrm{dom}f$ behaves itself like a final point in the sense that there is no point in W which is seen from $\mathrm{dom}f$ and does not see $\mathrm{dom}f$ itself.

This observation motivates the following definitions. Given a modal or intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$, a set $X \subseteq W$ is said to be *cofinal* in $\mathfrak{F}$ if $X\uparrow \subseteq X\overline{\downarrow}$. A subframe $\mathfrak{G}$ of $\mathfrak{F}$ is *cofinal* in $\mathfrak{F}$ if its set of worlds is cofinal in $\mathfrak{F}$. A subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called *cofinal* if, for every point x in $\mathfrak{F}$, $x \in \mathrm{dom}f\uparrow$ implies $x \in \mathrm{dom}f\overline{\downarrow}$, i.e., if $\mathrm{dom}f$ is cofinal in $\mathfrak{F}$. If there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{G}$ then we say $\mathfrak{F}$ is *cofinally subreducible* to $\mathfrak{G}$ or $\mathfrak{G}$ is a *cofinal subreduct* of $\mathfrak{F}$.

Example 9.12  is a subframe of , but not cofinal. The frame $\mathfrak{F}$ in Fig. 9.1 is subreducible to , but not cofinally, since $\omega \notin \mathrm{dom}f\overline{\downarrow}$ for any subreduction f of $\mathfrak{F}$ to .

Proposition 9.13 *A frame $\mathfrak{F} = \langle W, R, P\rangle$ refutes the McKinsey formula iff it is cofinally subreducible either to $\bullet$ or to* .

Proof ($\Rightarrow$) was actually established in Example 9.11.

($\Leftarrow$) Suppose f is a cofinal subreduction of $\mathfrak{F}$ to $\bullet$. Then $\mathrm{dom}f$ is a non-empty set of dead ends in $\mathfrak{F}$ and so $\boldsymbol{ma}$ is false at any point in $\mathrm{dom}f$ under any valuation in $\mathfrak{F}$.

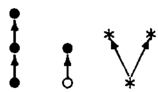
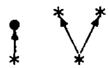
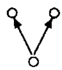
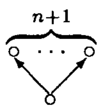
Suppose now that f is a cofinal subreduction of $\mathfrak{F}$ to the cluster with two points a and b. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking

$$\mathfrak{V}(p) = W - f^{-1}(a).$$

Then for each $x \in f^{-1}(a)$, we have $x \models \Box\Diamond p$, $x \not\models \Diamond\Box p$ and so $x \not\models \boldsymbol{ma}$. For otherwise either $x \not\models \Box\Diamond p$ or $x \models \Diamond\Box p$. In the former case $y \not\models \Diamond p$ for some $y \in x\uparrow$, from which $z \not\models p$ for all $z \in y\uparrow$, i.e., $y\uparrow \subseteq f^{-1}(a)$. It follows that $y\uparrow \cap f^{-1}(b) = \emptyset$ and hence, by (R2), $y\overline{\uparrow} \cap \mathrm{dom}f = \emptyset$, contrary to f being cofinal. In the latter case $y \models \Box p$ for some $y \in x\uparrow$ and so $z \models p$ for all $z \in y\uparrow$, i.e., $y\uparrow \cap f^{-1}(a) = \emptyset$, which is again a contradiction. ❑

In the same manner one can prove the following proposition; we leave it to the reader.

Table 9.3 *Characterizing refutation frames: cofinal subreduction.*

Formula φ	$\mathfrak{F} \not\models \varphi$ iff $\mathfrak{F}$ is cofinally subreducible to one of the following frames
$\Box p \to \Diamond p$	
$\Box\Diamond p \to \Diamond\Box p$	
$\Diamond(\Box p \wedge q) \to \Box(\Diamond p \vee q)$	 (8 frames)
$\Diamond\Box p \to \Box\Diamond p$	 (8 frames)
$\neg p \vee \neg\neg p$	
$\boldsymbol{btw}_n$	

Proposition 9.14 *A transitive modal (or intuitionistic) frame $\mathfrak{F}$ refutes a formula in the left-hand side of Table 9.3 iff $\mathfrak{F}$ is cofinally subreducible to one of the frames in the same line of the right-hand side.*

In the intuitionistic case the notion of cofinal subreduction has a clear algebraic meaning.

Theorem 9.15 *Suppose that $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$ are intuitionistic frames and f is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. Then there is an ICN-isomorphism f^+ of $\mathfrak{G}^+$ in a homomorphic image of $\mathfrak{F}^+$.*

Proof Let $\mathfrak{F}_1 = \langle W_1, R_1, P_1\rangle$ be the subframe of $\mathfrak{F}$ generated by dom$f$. It is clear that f is a cofinal subreduction of $\mathfrak{F}_1$ to $\mathfrak{G}$. By Theorem 9.7, the map f^+ defined by $f^+(X) = W_1 - f^{-1}(V - X)\!\downarrow$, for every $X \in Q$, is an IC-isomorphism of $\mathfrak{G}^+$ in $\mathfrak{F}_1^+$. Moreover, f^+ preserves $\emptyset$ because the set dom$f = f^{-1}(V)$ is cofinal in $\mathfrak{F}_1$. So f^+ is an ICN-isomorphism. It remains to recall that, by Theorem 8.57, $\mathfrak{F}_1^+$ is a homomorphic image of $\mathfrak{F}^+$. ❑

Corollary 9.16 *Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are intuitionistic frames and $\mathfrak{F}$ is cofinally subreducible to $\mathfrak{G}$. Then for every disjunction free formula φ, $\mathfrak{F} \models \varphi$ implies $\mathfrak{G} \models \varphi$.*

The constitution of refutation frames for the formulas in Table 9.3 can be roughly described as follows. First we construct the frames $\mathfrak{F}$ that are reducible to one of the refutation patterns for the given formula shown in the table and then insert into $\mathfrak{F}$ new points at any places we want, *but not above* $\mathfrak{F}$.

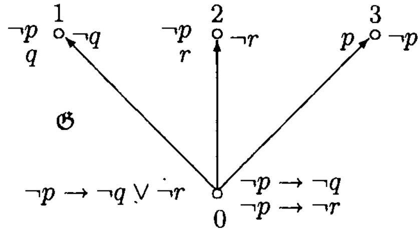


FIG. 9.4.

Our next example shows that some places *inside* frames may also be "closed" for inserting new points.

Example 9.17 Let us try to characterize the class of intuitionistic refutation frames for the *weak Kreisel–Putnam formula*:

$$\boldsymbol{wkp} = (\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r).$$

Using, for instance, the semantic tableau technique, we first construct its simplest countermodel as depicted in Fig. 9.4. Then we observe that every frame $\mathfrak{F}$ refuting $\boldsymbol{wkp}$ is cofinally subreducible to the frame $\mathfrak{G}$ underlying this countermodel by the map f which is defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \models \neg p \to \neg q \vee \neg r,\ x \not\models (\neg p \to \neg q) \vee (\neg p \to \neg r) \\ 1 & \text{if } x \models \neg p \to \neg q \vee \neg r,\ x \models \neg p \text{ and } x \models q \\ 2 & \text{if } x \models \neg p \to \neg q \vee \neg r,\ x \models \neg p \text{ and } x \models r \\ 3 & \text{if } x \models p \text{ or } x \models \neg p \wedge \neg q \wedge \neg r \\ \text{undefined otherwise.} & \end{cases}$$

(The cofinality of f follows from the fact that $f^{-1}(i)$, for $i = 1, 2, 3$, is upward closed and $f^{-1}(0){\uparrow} \subseteq \bigcup_{i=1}^{3} f^{-1}(i){\downarrow}$.)

However, the cofinal subreducibility to $\mathfrak{G}$ turns out to be only a necessary condition for $\mathfrak{F} \not\models \boldsymbol{wkp}$. For the frame shown in Fig. 9.5 is cofinally subreducible to $\mathfrak{G}$, but does not refute $\boldsymbol{wkp}$. Indeed, suppose otherwise. Then there is a valuation in this frame such that $a_0 \models \neg p \to \neg q \vee \neg r$, $a_1 \models \neg p$, $a_1 \not\models \neg q$, $a_2 \models \neg p$ and $a_2 \not\models \neg r$, whence $a \models \neg p \to \neg q \vee \neg r$, $a \not\models \neg q \vee \neg r$ and so $a \not\models \neg p$, i.e., there must be a point $x \in a{\uparrow}$ such that $x \models p$, but such a point does not exist.

This argument shows in fact that the cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ defined above satisfies the condition $\neg\exists x \in \mathrm{dom} f{\uparrow}\ f(x{\uparrow}) = \{1, 2\}$, which turns out to be the sufficient condition we need. For if f is a cofinal subreduction of $\mathfrak{F} = \langle W, R, P\rangle$ to $\mathfrak{G}$ in Fig. 9.4 satisfying it then we define $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = W - f^{-1}(\{1, 2\}){\downarrow}$, $\mathfrak{V}(q) = W - f^{-1}(\{2, 3\}){\downarrow}$, $\mathfrak{V}(r) = W - f^{-1}(\{1, 3\}){\downarrow}$. It is easy to see that under this valuation $x \not\models \neg p \to \neg q$, $y \not\models \neg p \to \neg r$ and $z \models p$, for every $x \in f^{-1}(1)$, $y \in f^{-1}(2)$ and $z \in f^{-1}(3)$. Therefore, $u \not\models (\neg p \to \neg q) \vee (\neg p \to \neg r)$ for every $u \in f^{-1}(0)$. We must also have $u \models \neg p \to \neg q \vee \neg r$, because otherwise

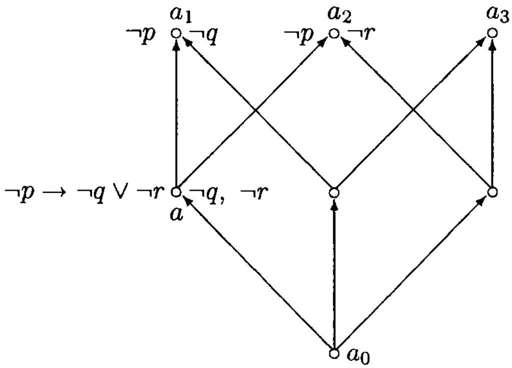


FIG. 9.5.

there is $v \in u\!\uparrow$ such that $v \models \neg p$ and $v \not\models \neg q \vee \neg r$, which is a contradiction, since the former means $v \notin f^{-1}(3)\!\downarrow$ and the latter implies, by the cofinality of f, that $v \in f^{-1}(1)\!\downarrow \cap f^{-1}(2)\!\downarrow$, from which $f(v\!\uparrow) = \{1, 2\}$.

Thus, we can construct all refutation frames for ***wkp*** by taking first the frames $\mathfrak{H}$ that are reducible to $\mathfrak{G}$ by some f and then inserting into them new points anywhere but (i) not above $\mathfrak{H}$ and (ii) not at such places where both $f^{-1}(1)$ and $f^{-1}(2)$ are seen, while $f^{-1}(3)$ is not seen. Figuratively speaking, the place or domain just below 1 and 2 in $\mathfrak{G}$ is closed for inserting new points, while all other domains (e.g. below 1 or below 2 and 3) are open.

Example 9.18 $\mathfrak{F}$ refutes the density axiom $\Box\Box p \to \Box p$ iff there is a subreduction f of $\mathfrak{F}$ to the frame  such that $\neg\exists x \in \mathrm{dom} f\!\uparrow\ f(x\!\uparrow) = \{0\}$. (An equivalent characterization: $\mathfrak{F} \not\models \boldsymbol{den}_1$ iff there is a dense subreduction of $\mathfrak{F}$ to ). This time the domain just below 0 is closed for inserting.

These examples motivate the following definition. Let $\mathfrak{G}$ be a finite frame and $\mathfrak{D}$ a (possibly empty) set of antichains in $\mathfrak{G}$. We say a subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ satisfies the *closed domain condition* for $\mathfrak{D}$ if

$$\text{(CDC)}\ \neg\exists x \in \mathrm{dom} f\!\uparrow - \mathrm{dom} f\ \exists \mathfrak{d} \in \mathfrak{D}\ f(x\!\uparrow) = \mathfrak{d}\underline{\uparrow}$$

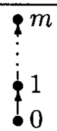
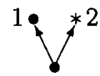
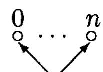
or, which is equivalent, if

$$\text{(CDC)}\ x \in \mathrm{dom} f\!\uparrow \text{ and } f(x\!\uparrow) = \mathfrak{d}\underline{\uparrow} \text{ for some } \mathfrak{d} \in \mathfrak{D} \text{ imply } x \in \mathrm{dom} f.$$

Note that, by the definition, every subreduction satisfies (CDC) for $\mathfrak{D} = \emptyset$. We denote by $\mathfrak{D}^\sharp$ the set of all antichains in $\mathfrak{G}$. It follows also from the definition that a subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ satisfies (CDC) for $\mathfrak{D}^\sharp$ iff f is dense. As an exercise we invite the reader to prove the following two propositions.

Proposition 9.19 *A modal transitive frame $\mathfrak{F}$ refutes a formula in the left side of Table 9.4 iff there is a cofinal subreduction (simply a subreduction for the first*

Table 9.4 *Characterizing refutation frames: closed domain condition.*

Formula φ	$\mathfrak{F} \not\models \varphi$ iff $\mathfrak{F}$ is (cofinally) subreducible to one of the following frames, with (CDC) for $\mathfrak{D}$ being satisfied
$\Box^n p \to \Box^m p$ $(n > m \geq 1)$	 $\mathfrak{D} = \mathfrak{D}^\sharp$
$\Box\Box p \to \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$	 $\mathfrak{D} = \{\{1\}, \{1, 2\}\}$
$\Box\Diamond\top \wedge \Box(\Box^+ p \vee \Box^+\neg p) \to \Box p \vee \Box\neg p$	 $\mathfrak{D} = \mathfrak{D}^\sharp$
$\Box\Diamond\top \wedge \Box \bigvee_{j=0}^{n} \Box^+(p_j \wedge \bigwedge_{i\neq j} \neg p_i) \to \bigvee_{j=0}^{n} \Box\neg(p_j \wedge \bigwedge_{i\neq j} \neg p_i)$	 $\mathfrak{D} = \mathfrak{D}^\sharp$

two formulas) of $\mathfrak{F}$ to one of the frames in the corresponding row of the right side, which satisfies (CDC) for $\mathfrak{D}$ shown near the frame.

Proposition 9.20 *An intuitionistic frame $\mathfrak{F}$ refutes a formula in the left side of Table 9.5 iff there is a cofinal subreduction (a plain subreduction for the first formula) of $\mathfrak{F}$ to one of the frames in the corresponding row of the right side, which satisfies (CDC) for $\mathfrak{D}$ shown near the frame.*

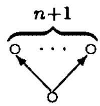
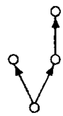
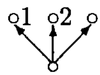
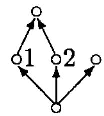
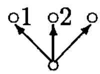
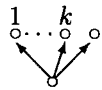
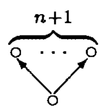
In the next section we will show that in the same manner one can characterize transitive refutation frames for every modal or intuitionistic formula. But before that we obtain some simple general results on subreductions.

Theorem 9.21 *Suppose $\mathfrak{F}_i = \langle W_i, R_i, P_i\rangle$, for $i = 1, 2, 3$, are modal or intuitionistic frames, f_1 is a (cofinal) subreduction of $\mathfrak{F}_1$ to $\mathfrak{F}_2$ and f_2 a (cofinal) subreduction of $\mathfrak{F}_2$ to $\mathfrak{F}_3$. Then the composition $f_3 = f_2 f_1$ is a (cofinal) subreduction of $\mathfrak{F}_1$ to $\mathfrak{F}_3$.*

Proof Since f_1 and f_2 are surjections, their composition is also a surjection. If $x, y \in \mathrm{dom} f_2 f_1$ and xR_1y then, by (R1), $f_1(x)R_2f(y)$ and $f_2f_1(x)R_3f_2f_1(y)$. If $f_2f_1(x)R_3z$ for some $x \in W_1$ and $z \in W_3$ then, by (R2), there are $v \in W_2$ and $y \in W_1$ such that $f_1(x)R_2v$, $f_2(v) = z$ and xR_1y, $f_1(y) = v$, i.e., $f_2f_1(y) = z$. Thus f_2f_1 satisfies (R1) and (R2).

If our frames are modal and $X \in P_3$ then, by (R3), $f_2^{-1}(X) \in P_2$ and $f_1^{-1}(f_2^{-1}(X)) = (f_2f_1)^{-1}(X) \in P_1$. In the intuitionistic case, for $X \in \overline{P}_3$ we have $f_2^{-1}(X){\downarrow} \in \overline{P}_2$ and $f_1^{-1}(f_2^{-1}(X){\downarrow}){\downarrow} \in \overline{P}_1$. And using (R2), one can readily show that $f_1^{-1}(f_2^{-1}(X)){\downarrow} = f_1^{-1}(f_2^{-1}(X){\downarrow}){\downarrow}$. Thus f_2f_1 satisfies also (R3) and so is a subreduction.

Table 9.5 *Characterizing refutation frames: closed domain condition.*

Formula φ	$\mathfrak{F} \not\models \varphi$ iff $\mathfrak{F}$ is (cofinally) subreducible to one of the following frames, with (CDC) for $\mathfrak{D}$ being satisfied
$\boldsymbol{bb_n}$	 $\mathfrak{D} = \mathfrak{D}^\sharp$
$((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$	 $\mathfrak{D} = \mathfrak{D}^\sharp$
$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$	 $\mathfrak{D} = \{\{1,2\}\}$ 
$(\neg p \to \bigvee_{i=1}^{k} \neg q_i) \to \bigvee_{i=1}^{k}(\neg p \to \neg q_i)$	 ...  $\mathfrak{D} = \{\{1,2\}\}$ $\mathfrak{D} = \{\{1,\dots,k\}\}$
$\bigwedge_{i=0}^{n}(\neg p_i \leftrightarrow \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i$	 $\mathfrak{D} = \mathfrak{D}^\sharp$

Now suppose f_1, f_2 are cofinal, $x \in W_1$ and yR_1x for some $y \in \mathrm{dom} f_2 f_1$. Since f_1 is cofinal, we have either $x \in \mathrm{dom} f_1$ or xR_1z for some $z \in \mathrm{dom} f_1$. In the former case $f_1(y)R_2f_1(x)$ and so, by the cofinality of f_2, either $f_1(x) \in \mathrm{dom} f_2$, i.e., $x \in \mathrm{dom} f_2 f_1$, or $f_1(x)R_2v$ for some $v \in \mathrm{dom} f_2$, and then, by (R2), there is $u \in W_1$ such that xR_1u and $f_1(u) = v$, whence $u \in \mathrm{dom} f_2 f_1$. The latter case is considered analogously. ❑

Theorem 9.22 *Suppose $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$ are quasi-ordered modal frames and f is a (cofinal) subreduction of $\mathfrak{F}$ to $\mathfrak{G}$ satisfying (CDC) for a set $\mathfrak{D}$ of antichains in $\mathfrak{G}$. Then there is a (cofinal) subreduction ρf of $\rho\mathfrak{F}$ to $\rho\mathfrak{G}$ satisfying (CDC) for $\rho\mathfrak{D}$, where $\rho\mathfrak{D} = \{\rho\mathfrak{d} : \mathfrak{d} \in \mathfrak{D}\}$ and $\rho\mathfrak{d} = \{C(x) : x \in \mathfrak{d}\}$.*

Proof We define ρf by taking, for any cluster C in $\mathfrak{F}$,

$$\rho f(C) = \begin{cases} C(f(x)) & \text{if } x \in C \text{ and } x \in \mathrm{dom} f \\ \text{undefined if } C \cap \mathrm{dom} f = \emptyset. \end{cases}$$

(This definition does not depend on the choice of $x \in C$, since, by (R1), $C(f(x)) = C(f(y))$ for every $x, y \in C \cap \mathrm{dom} f$.). Clearly, ρf is a partial map from ρW onto ρV satisfying (R1) and (R2) and the cofinality condition as well, provided f is cofinal. Suppose $X \in \rho Q$. Then there is $Y = Y\!\uparrow\, \in Q$ such that

$X = \rho Y$. By (R3), $f^{-1}(V - Y) \in P$, whence $W - f^{-1}(V - Y){\downarrow} \in P$ and so $\rho W - (\rho f)^{-1}(\rho V - \rho Y){\downarrow} = \rho(W - f^{-1}(V - Y){\downarrow}) \in \rho P$. Thus, ρf satisfies (R3′) and it remains to verify that it satisfies (CDC) for $\rho\mathfrak{D}$. Suppose $C \in \mathrm{dom}\rho f{\uparrow}$ and $\rho f(C{\uparrow}) = \rho\mathfrak{d}\underline{\uparrow}$ for some $\rho\mathfrak{d} \in \rho\mathfrak{D}$. Take any $x \in C$. Then $x \in \mathrm{dom}f{\uparrow}$ and $f(x{\uparrow}) = \mathfrak{d}\underline{\uparrow}$, whence by CDC), $x \in \mathrm{dom}f$ and so $C \in \mathrm{dom}\rho f$. ❑

Theorem 9.23 *Suppose that $\mathfrak{F} = \langle W, R, P\rangle$ is a quasi-ordered modal frame, $\mathfrak{G} = \langle V, S\rangle$ a finite intuitionistic frame and f a (cofinal) subreduction of $\rho\mathfrak{F}$ to $\mathfrak{G}$ satisfying (CDC) for a set $\mathfrak{D}$ of antichains in $\mathfrak{G}$. Then there is a (cofinal) subreduction h of a generated subframe of $\mathfrak{F}$ to $\sigma\mathfrak{G}$ satisfying (CDC) for $\mathfrak{D}$.*

Proof Let $\mathfrak{F}' = \langle W', R', P'\rangle$ be the subframe of $\mathfrak{F}$ generated by the set of points $\{x \in W : C(x) \in \mathrm{dom}f\}$. With each $v \in V$ we associate the set

$$X_v = \{y \in W' : f(C(y)) = v\}{\downarrow} - \bigcup_{x \in V - v\uparrow} \{y \in W' : f(C(y)) = x\}{\downarrow}.$$

Since by (R4′), $\rho W' - f^{-1}(x){\downarrow} \in \rho P'$ for every $x \in V$, we have

$$W' - \{y \in W' : \ f(C(y)) = x\}{\downarrow} \in P'$$

and so, by the finiteness of V, $X_v \in P'$.

Now we define a partial map h from W' onto V by taking, for every $x \in W'$,

$$h(x) = \begin{cases} v & \text{if } x \in X_v \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It should be clear that h is a subreduction of $\mathfrak{F}'$ to $\sigma\mathfrak{G}$; moreover, it is cofinal if f is cofinal. Suppose $\mathfrak{d} \in \mathfrak{D}$ and $h(x{\uparrow}) = \mathfrak{d}\underline{\uparrow}$. Then $C(x) \in \mathrm{dom}f{\uparrow}$ and, by the definition of h, $f(C(x)){\uparrow} = \mathfrak{d}\underline{\uparrow}$. Therefore, by (CDC), we have $C(x) \in \mathrm{dom}f$ and so $x \in X_{f(C(x))} \subseteq \mathrm{dom}h$. ❑

Theorem 9.24 *Suppose $\mathfrak{F} = \langle W, R, P\rangle$ is an intuitionistic frame, $\mathfrak{G} = \langle V, S\rangle$ a finite quasi-ordered modal frame, $\mathfrak{D}$ a set of antichains in $\mathfrak{G}$ and f a (cofinal) subreduction of $\mathfrak{F}$ to $\rho\mathfrak{G}$ satisfying (CDC) for $\rho\mathfrak{D}$. Then there is a (cofinal) subreduction h of a generated subframe of $\tau_k\mathfrak{F}$ to $\mathfrak{G}$ satisfying (CDC) for $\mathfrak{D}$, where $k = \max\{|C(x)| : \ x \in V\}$.*

Proof According to the preceding theorem, there are a generated subframe $\mathfrak{F}' = \langle W', R', P'\rangle$ of $\sigma\mathfrak{F}$ and a (cofinal) subreduction g of $\mathfrak{F}'$ to $\sigma\rho\mathfrak{G}$ satisfying (CDC) for $\rho\mathfrak{D}$. Let $\tau_k\mathfrak{F}' = \langle kW', kR', kP'\rangle$ be the subframe of $\tau_k\mathfrak{F}$ generated by $k \times W'$. Define a partial map h from kW' onto V as follows. If $x \in \mathrm{dom}g$, $g(x) = C \in \rho V$ and $C = \{a_1, \ldots, a_n\} \subseteq V$ then we take, for every $i \in k$,

$$h(\langle x, i\rangle) = a_{\mathrm{mod}_n(i)}.$$

(We remind the reader that $n \leq k$.) And if $x \notin \mathrm{dom} g$ then we regard $h(\langle x, i\rangle)$ as undefined for every $i \in k$. It is not difficult to verify now that h is a (cofinal) subreduction of $\tau_k \mathfrak{F}'$ to $\mathfrak{G}$ satisfying (CDC) for $\mathfrak{D}$. ❑

9.3 Characterizing transitive refutation frames

This section shows how, given a modal or intuitionistic formula φ, to construct a finite number of finite rooted frames $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ and to select sets $\mathfrak{D}_1, \dots, \mathfrak{D}_n$ of antichains in them so that an arbitrary transitive frame $\mathfrak{F}$ refutes φ iff there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some $i \in \{1, \dots, n\}$, satisfying the closed domain condition for $\mathfrak{D}_i$.

Let us begin with selecting closed domains. Suppose φ is a modal or intuitionistic formula and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ a model. We say a non-empty antichain $\mathfrak{a}$ in $\mathfrak{G}$ is an *open domain* in $\mathfrak{N}$ relative to φ if there is a disjoint saturated tableau $t_\mathfrak{a} = (\Gamma_\mathfrak{a}, \Delta_\mathfrak{a})$ with $\Gamma_\mathfrak{a} \cup \Delta_\mathfrak{a} = \mathbf{Sub}\varphi$ and such that in the modal case, for every $\Box\psi \in \mathbf{Sub}\varphi$,

(OD$_M$1) $\Box\psi \in \Gamma_\mathfrak{a}$ implies $\psi \in \Gamma_\mathfrak{a}$;

(OD$_M$2) $\Box\psi \in \Gamma_\mathfrak{a}$ iff $a \models \Box^+\psi$ for all $a \in \mathfrak{a}$;

and in the intuitionistic case, for every $\psi \in \mathbf{Sub}\varphi$,

(OD$_I$) $\psi \in \Gamma_\mathfrak{a}$ iff $a \models \psi$ for all $a \in \mathfrak{a}$.

Otherwise $\mathfrak{a}$ is called a *closed domain* in $\mathfrak{N}$ relative to φ.

The motivation behind this definition is as follows. Imagine that we have inserted a new (reflexive or irreflexive) point x just below an antichain $\mathfrak{a}$ in $\mathfrak{G}$, i.e., x sees only the points in $\mathfrak{a}\underline{\uparrow}$ and is accessible from some of those points in $\mathfrak{G}$ that see $\mathfrak{a}$. Is it possible to extend the valuation $\mathfrak{U}$ to x so that the truth-values of φ's subformulas remain the same at the old points in $\mathfrak{G}$ under the extended valuation? The openness of $\mathfrak{a}$ is just a natural sufficient condition for the existence of such an extension no matter what points in $\mathfrak{G}$ see x (cf. Theorem 9.30 below). It is of importance that, given φ and a finite antichain $\mathfrak{a}$, we can always effectively decide whether $\mathfrak{a}$ is open or closed in finite $\mathfrak{N}$ relative to φ.

Example 9.25 The antichain $\{1, 2\}$ is the only closed domain in the counter-model for $\boldsymbol{wkp}$, depicted in Fig. 9.4.

Example 9.26 Let us show that in any intuitionistic model every antichain is open relative to every disjunction free formula. Suppose that $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ is an intuitionistic model, φ a disjunction free formula and $\mathfrak{a}$ a non-empty antichain in $\mathfrak{G}$. Let $(\Gamma_\mathfrak{a}, \Delta_\mathfrak{a})$ be the disjoint tableau defined by (OD$_I$). We show that it is saturated. For $\wedge$ we have

$$\begin{aligned} \psi \wedge \chi \in \Gamma_\mathfrak{a} &\text{ iff } \forall a \in \mathfrak{a}\ a \models \psi \wedge \chi \\ &\text{ iff } \forall a \in \mathfrak{a}\ (a \models \psi \wedge a \models \chi) \\ &\text{ iff } \psi \in \Gamma_\mathfrak{a} \text{ and } \chi \in \Gamma_\mathfrak{a}. \end{aligned}$$

Suppose now that $\psi \to \chi \in \Gamma_\mathfrak{a}$, but $\psi \in \Gamma_\mathfrak{a}$ and $\chi \in \Delta_\mathfrak{a}$. Then $a \models \psi$ for every $a \in \mathfrak{a}$ and $b \not\models \chi$ for some $b \in \mathfrak{a}$, whence $b \not\models \psi \to \chi$, which is a contradiction.

Proposition 9.27 *Suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is a model and $\mathfrak{a}$, $\mathfrak{b}$ are antichains in $\mathfrak{G}$ such that $\mathfrak{a}\underline{\uparrow} = \mathfrak{b}\underline{\uparrow}$. Then for every formula φ, $\mathfrak{a}$ is open in $\mathfrak{N}$ relative to φ iff $\mathfrak{b}$ is open in $\mathfrak{N}$ relative to φ.*

Proof One can take $t_{\mathfrak{a}} = t_{\mathfrak{b}}$. ❑

In view of this proposition we will not distinguish between antichains $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\underline{\uparrow} = \mathfrak{b}\underline{\uparrow}$ (i.e., the points in $\mathfrak{a}$ generate the same clusters as those in $\mathfrak{b}$).

Proposition 9.28 *Every reflexive singleton $\mathfrak{a} = \{x\}$ is an open domain in every model $\mathfrak{N}$ relative to every formula φ.*

Proof The tableau $t_{\mathfrak{a}} = (\{\psi \in \mathbf{Sub}\varphi : x \models \psi\}, \{\psi \in \mathbf{Sub}\varphi : x \not\models \psi\})$ is clearly disjoint, saturated and satisfies (OD_M1) and (OD_M2) in the modal case and (OD_I) in the intuitionistic one. ❑

Proposition 9.29 *Suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is a modal model based on a quasi-ordered frame $\mathfrak{G}$ and $\mathfrak{a}$ is an antichain in $\mathfrak{G}$. Then for every intuitionistic formula φ, $\mathfrak{a}$ is open in $\mathfrak{N}$ relative to $T(\varphi)$ iff $\rho\mathfrak{a}$ is open in $\rho\mathfrak{N} = \langle \rho\mathfrak{G}, \rho\mathfrak{U} \rangle$ relative to φ.*

Proof ($\Leftarrow$) Observe first that every subformula of $T(\varphi)$ is either an atom or has the form $T(\psi)$ for some $\psi \in \mathbf{Sub}\varphi$ or the form $T(\psi) \to T(\chi)$ for some $\psi \to \chi \in \mathbf{Sub}\varphi$. Now, given an intuitionistic tableau $t_{\rho\mathfrak{a}} = (\Gamma_{\rho\mathfrak{a}}, \Delta_{\rho\mathfrak{a}})$ satisfying (OD_I) for φ, we define a modal tableau $t_{\mathfrak{a}} = (\Gamma_{\mathfrak{a}}, \Delta_{\mathfrak{a}})$ as follows. First we put all φ's variables in $\Gamma_{\mathfrak{a}}$, then put $T(\psi)$ in $\Gamma_{\mathfrak{a}}$ if $\psi \in \Gamma_{\rho\mathfrak{a}}$ and in $\Delta_{\mathfrak{a}}$ if $\psi \in \Delta_{\rho\mathfrak{a}}$, and finally we put $T(\psi) \to T(\chi)$ in $\Delta_{\mathfrak{a}}$ if $T(\psi) \in \Gamma_{\mathfrak{a}}$, $T(\chi) \in \Delta_{\mathfrak{a}}$ and put it in $\Gamma_{\mathfrak{a}}$ otherwise. Clearly, $t_{\mathfrak{a}}$ is a disjoint saturated tableau and $\Gamma_{\mathfrak{a}} \cup \Delta_{\mathfrak{a}} = \mathbf{Sub}T(\varphi)$.

Suppose $\Box\psi' \in \Gamma_{\mathfrak{a}}$. Then either ψ' is a variable or $\psi' = T(\psi) \to T(\chi)$. By the definition, in the former case $\psi' \in \Gamma_{\mathfrak{a}}$. As to the latter one, assume $\psi' \in \Delta_{\mathfrak{a}}$, which means that $\psi \in \Gamma_{\rho\mathfrak{a}}$ and $\chi \in \Delta_{\rho\mathfrak{a}}$. Therefore, $a \not\models \psi \to \chi$ for some $a \in \rho\mathfrak{a}$, and so $\psi \to \chi \in \Delta_{\rho\mathfrak{a}}$, whence $T(\psi \to \chi) = \Box\psi' \in \Delta_{\mathfrak{a}}$, which is a contradiction. Thus, $\psi' \in \Gamma_{\mathfrak{a}}$ and $t_{\mathfrak{a}}$ satisfies (OD_M1). To establish (OD_M2), suppose $\Box\psi' = T(\psi)$ for some $\psi \in \mathbf{Sub}\varphi$. Then we have

$$\begin{array}{lll} \Box\psi' \in \Gamma_{\mathfrak{a}} & \text{iff } \psi \in \Gamma_{\rho\mathfrak{a}} & \text{by the definition} \\ & \text{iff } \forall a \in \rho\mathfrak{a}\ a \models \psi & \text{by } (\mathrm{OD}_I) \\ & \text{iff } \forall a \in \mathfrak{a}\ a \models T(\psi) & \text{by Lemma 8.28} \\ & \text{iff } \forall a \in \mathfrak{a}\ a \models \Box^+\psi' & \text{since } \mathfrak{G} \text{ is reflexive.} \end{array}$$

($\Rightarrow$) Now, given a modal tableau $t_{\mathfrak{a}} = (\Gamma_{\mathfrak{a}}, \Delta_{\mathfrak{a}})$, we define an intuitionistic tableau $t_{\rho\mathfrak{a}} = (\Gamma_{\rho\mathfrak{a}}, \Delta_{\rho\mathfrak{a}})$ by taking, for every $\psi \in \mathbf{Sub}\varphi$, $\psi \in \Gamma_{\rho\mathfrak{a}}$ iff $T(\psi) \in \Gamma_{\mathfrak{a}}$ and $\Delta_{\rho\mathfrak{a}} = \mathbf{Sub}\varphi - \Gamma_{\rho\mathfrak{a}}$. One can readily verify that $t_{\rho\mathfrak{a}}$ is saturated. To prove that $t_{\rho\mathfrak{a}}$ satisfies (OD_I), it suffices, by Lemma 8.28, to show that $T(\psi) \in \Gamma_{\mathfrak{a}}$ iff $\forall a \in \mathfrak{a}\ a \models T(\psi)$, which can easily be done by induction on the construction of ψ. ❑

Now we prove a theorem which shows that the notion of closed domain is consistent with the closed domain condition.

Theorem 9.30 *Suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is a finite (modal or intuitionistic) model, φ a (modal or intuitionistic) formula and $\mathfrak{D}$ the set of all closed domains in $\mathfrak{N}$ relative to φ. Then for any (modal or intuitionistic) frame $\mathfrak{F} = \langle W, R, P \rangle$, which is cofinally subreducible to $\mathfrak{G} = \langle V, S \rangle$ by some map f satisfying (CDC) for $\mathfrak{D}$, there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ such that, for any $x \in \mathrm{dom} f$ and any $\psi \in \mathbf{Sub}\varphi$, $(\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{N}, f(x)) \models \psi$.*

Proof First we reduce the intuitionistic case to the modal one. Given an intuitionistic model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$, we construct the modal model $\boldsymbol{\sigma}\mathfrak{N} = \langle \boldsymbol{\sigma}\mathfrak{G}, \mathfrak{U} \rangle$. By Proposition 9.29, the set of closed domains in $\boldsymbol{\sigma}\mathfrak{N}$ relative to $T(\varphi)$ coincides with $\mathfrak{D}$. By Theorem 9.23, there is a cofinal subreduction h of a generated subframe of $\boldsymbol{\sigma}\mathfrak{F}$ to $\boldsymbol{\sigma}\mathfrak{G}$ satisfying (CDC) for $\mathfrak{D}$ and such that $f(x) = h(x)$, for every $x \in \mathrm{dom} f$. So if we prove our theorem for the modal case, we shall have a model $\mathfrak{M}$ based on $\boldsymbol{\sigma}\mathfrak{F}$ such that, for every $x \in \mathrm{dom} f$ and every $\psi \in \mathbf{Sub}\varphi$, $(\mathfrak{M}, x) \models T(\psi)$ iff $(\boldsymbol{\sigma}\mathfrak{N}, f(x)) \models T(\psi)$, and so, by Lemma 8.28, $(\boldsymbol{\rho}\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{N}, f(x)) \models \psi$.

Now, considering the modal case, without loss of generality we may assume that $\mathrm{dom} f{\uparrow} = W$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ as follows. If $x \in \mathrm{dom} f$ then we take $x \models p$ iff $f(x) \models p$, for every $p \in \mathbf{Var}\varphi$. If $x \notin \mathrm{dom} f$ then $f(x{\uparrow}) \neq \emptyset$, since f is cofinal. Let $\mathfrak{a}$ be an antichain in $\mathfrak{G}$ such that $\mathfrak{a}{\uparrow} = f(x{\uparrow})$. By the closed domain condition, $\mathfrak{a}$ is an open domain in $\mathfrak{N}$ and so there is a tableau $t_{\mathfrak{a}} = (\Gamma_{\mathfrak{a}}, \Delta_{\mathfrak{a}})$ satisfying (OD_M1) and (OD_M2). Then we take $y \models p$ iff $p \in \Gamma_{\mathfrak{a}}$, for every $y \notin \mathrm{dom} f$ such that $f(y{\uparrow}) = f(x{\uparrow})$.

Let us first prove that $\mathfrak{V}$ is well-defined, i.e., $\mathfrak{V}(p) = \{x \in W : x \models p\}$ is in P for every variable p. $\mathfrak{V}(p)$ can be represented as the union of the following two sets X and Y:

$$X = \{x \in \mathrm{dom} f : \ x \models p\}, \quad Y = \{x \notin \mathrm{dom} f : \ x \models p\}.$$

According to (R4), we have $X \in P$. By the definition of $\mathfrak{V}$, if $x, y \notin \mathrm{dom} f$ and $f(x{\uparrow}) = f(y{\uparrow})$ then $x \models p$ iff $y \models p$. So, since $\mathfrak{G}$ is finite, there is only a finite number of points $y_1, \dots, y_n \notin \mathrm{dom} f$ such that $Y = Z_1 \cup \ldots \cup Z_n$, where $Z_i = \{z \notin \mathrm{dom} f : \ f(z{\uparrow}) = f(y_i{\uparrow})\}$. Let $A_i = f(y_i{\uparrow})$ and $B_i = V - A_i$. Then we have

$$Z_i = \bigcap_{a \in A_i} f^{-1}(a){\downarrow} \cap - \bigcup_{b \in B_i} f^{-1}(b){\downarrow} \cap - \mathrm{dom} f \in P.$$

Therefore, $Y \in P$ and $\mathfrak{V}(p) = X \cup Y \in P$.

Now by induction on the construction of $\psi \in \mathbf{Sub}\varphi$ we show that

- for $x \in \mathrm{dom} f$, $x \models \psi$ iff $f(x) \models \psi$;
- for $x \notin \mathrm{dom} f$, $x \models \psi$ iff $\psi \in \Gamma_{\mathfrak{a}}$, where $\mathfrak{a}$ is the open domain in $\mathfrak{N}$ associated with x.

The only non-trivial case is $\psi = \Box\chi$. Let $x \in \mathrm{dom} f$. If $x \not\models \Box\chi$ then $y \not\models \chi$ for some $y \in x{\uparrow}$. Suppose $y \in \mathrm{dom} f$. Then, by the induction hypothesis, $f(y) \not\models \chi$ and so $f(x) \not\models \Box\chi$, since $f(x) S f(y)$. Suppose $y \notin \mathrm{dom} f$ and $\mathfrak{b}$ is the open domain in $\mathfrak{N}$ associated with y. By the induction hypothesis, $\chi \in \Delta_{\mathfrak{b}}$ and so, by (OD_M1), $\Box\chi \in \Delta_{\mathfrak{b}}$ and, by (OD_M2), $b \not\models \Box^+\chi$ for some $b \in \mathfrak{b}$. Therefore, $f(x) \not\models \Box\chi$ because $f(x)Sb$. Conversely, if $f(x) \not\models \Box\chi$ then there is $z \in V$ such that $f(x)Sz$ and $z \not\models \chi$. By (R2), there is $y \in x\uparrow$ for which $f(y) = z$; hence $y \not\models \chi$ and so $x \not\models \Box\chi$.

Suppose now that $x \notin \mathrm{dom} f$ and $\mathfrak{a}$ is the open domain in $\mathfrak{N}$ associated with x. If $x \models \Box\chi$ then $y \models \chi$ for all $y \in \dot{x}\uparrow$. Therefore, by the induction hypothesis, $z \models \chi$ for all $z \in \mathfrak{a}\underline{\uparrow}$, and so $a \models \Box^+\chi$ for every $a \in \mathfrak{a}$, i.e., $\Box\chi \in \Gamma_{\mathfrak{a}}$. To prove the converse, suppose $\Box\chi \in \Gamma_{\mathfrak{a}}$ but $x \not\models \Box\chi$. Then there is $y \in x\uparrow$ such that $y \not\models \chi$. If $y \in \mathrm{dom} f$ then $f(y) \not\models \chi$ and so $a \not\models \Box^+\chi$ for some $a \in \mathfrak{a}$, which is a contradiction. If $y \notin \mathrm{dom} f$ then, as we have seen, $z \not\models \Box^+\chi$ for some $z \in f(y\uparrow) \subseteq f(x\uparrow)$, and so again $a \not\models \Box^+\chi$ for some $a \in \mathfrak{a}$, contrary to $\Box\chi \in \Gamma_{\mathfrak{a}}$. ❑

When φ is negation free there is no need to require f to be cofinal.

Theorem 9.31 *Suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is a finite model, φ a negation free formula and $\mathfrak{D}$ the set of all closed domains in $\mathfrak{N}$ relative to φ. Then for any frame $\mathfrak{F} = \langle W, R, P \rangle$, which is subreducible to $\mathfrak{G} = \langle V, S \rangle$ by some f satisfying (CDC) for $\mathfrak{D}$, there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ such that, for any $x \in \mathrm{dom} f$ and any $\psi \in \mathbf{Sub}\varphi$, $(\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{N}, f(x)) \models \psi$.*

Proof The only difference from the proof of Theorem 9.30 concerns the definition of $\mathfrak{V}$, since this time $x \notin \mathrm{dom} f$ does not imply $f(x\uparrow) \neq \emptyset$. If $f(x\uparrow) = \emptyset$ then we put $x \models p$ for all variables p. The definition remains correct:

$$\{x \notin \mathrm{dom} f : \; f(x\uparrow) = \emptyset\} = W - \mathrm{dom} f\overline{\downarrow} \in P.$$

Moreover, since φ is negation free, it should be clear that if $f(x\uparrow) = \emptyset$ then $x \models \psi$ for every $\psi \in \mathbf{Sub}\varphi$. ❑

Thus, if $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is a finite countermodel for a formula φ and $\mathfrak{D}$ the set of all closed domains in $\mathfrak{N}$ relative to φ then $\mathfrak{F} \not\models \varphi$ for every frame $\mathfrak{F}$ which is cofinally subreducible to $\mathfrak{G}$ by some partial map satisfying (CDC) for $\mathfrak{D}$. Now we shall go in the reverse direction and show that, given an arbitrary countermodel $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ for φ, one can construct a finite countermodel $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ for φ such that there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{G}$ satisfying (CDC) for the set $\mathfrak{D}$ of all closed domains in $\mathfrak{N}$. To this end we require two definitions and two propositions.

Let Σ be a non-empty set of formulas closed under subformulas. Given models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$, we say a subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is a Σ-*subreduction* of $\mathfrak{M}$ to $\mathfrak{N}$ if

(i) for each $x \in \mathrm{dom} f$, $(\mathfrak{M}, x) \sim_\Sigma (\mathfrak{N}, f(x))$ and

(ii) for each point x in $\mathfrak{F}$ there is $y \in x\underline{\uparrow} \cap \mathrm{dom} f$ which is Σ-equivalent to x in $\mathfrak{M}$.

It is worth noting that, by (ii), every Σ-subreduction is cofinal.

Proposition 9.32 *Suppose that f_1 is a Σ-subreduction of $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle$ to $\mathfrak{M}_2 = \langle \mathfrak{F}_2, \mathfrak{V}_2 \rangle$ and f_2 a Σ-subreduction of $\mathfrak{M}_2$ to $\mathfrak{M}_3 = \langle \mathfrak{F}_3, \mathfrak{V}_3 \rangle$. Then the composition $f_2 f_1$ is a Σ-subreduction of $\mathfrak{M}_1$ to $\mathfrak{M}_3$.*

Proof By Theorem 9.21, $f_2 f_1$ is a subreduction of $\mathfrak{F}_1$ to $\mathfrak{F}_3$. Clearly, it satisfies (i). To show (ii), suppose x is a point in $\mathfrak{F}_1$. Then, by (ii), there is $y \in x{\uparrow} \cap \mathrm{dom} f_1$ for which $x \sim_\Sigma y$. Using (ii) again, we can find a point $z \in f_1(y){\uparrow} \cap \mathrm{dom} f_2$ such that $f_1(y) \sim_\Sigma z$. By (R2), there is $u \in x{\uparrow}$ such that $f_1(u) = z$. It remains to observe that $u \in \mathrm{dom} f_2 f_1$ and that, by (i), $u \sim_\Sigma x$. ❑

Our second definition is connected with the condition (ii) of the preceding one. Given a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on a frame $\mathfrak{F} = \langle W, R, P \rangle$ and a subset $V \subseteq W$, we say a point $x \in W$ is *Σ-remaindered* in V if $x \sim_\Sigma y$ for some $y \in x{\uparrow} \cap V$. Thus, a subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is a Σ-subreduction of $\mathfrak{M}$ to $\mathfrak{N}$ iff it satisfies (i) and every point in $\mathfrak{F}$ is Σ-remaindered in $\mathrm{dom} f$.

The meaning of this notion is clarified by the following observations. Suppose again that we have a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ on a frame $\mathfrak{F} = \langle W, R, P \rangle$ and $V \subseteq W$. Taking the restrictions S and $\mathfrak{U}$ of, respectively, R and $\mathfrak{V}$ to V, we obtain the model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ on the Kripke frame $\mathfrak{G} = \langle V, S \rangle$, which is called the *Kripke submodel* of $\mathfrak{M}$ *induced* by V. If Q is the set of possible values in $\mathfrak{N}$ and the frame $\mathfrak{G}' = \langle V, S, Q \rangle$ turns out to be a subframe of $\mathfrak{F}$ then $\mathfrak{N}' = \langle \mathfrak{G}', \mathfrak{U} \rangle$ is called a *submodel* of $\mathfrak{M}$ *induced* by V.

Proposition 9.33 *Suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is the Kripke submodel of $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ induced by $V \subseteq W$ and every point in $\mathfrak{F}$ is Σ-remaindered in V. Then, for each $x \in V$, $(\mathfrak{M}, x) \sim_\Sigma (\mathfrak{N}, x)$. So if, in addition, $\mathfrak{N}$ is a submodel of $\mathfrak{M}$ then the identity map on V is a Σ-subreduction of $\mathfrak{M}$ to $\mathfrak{N}$.*

Proof The latter claim is an immediate consequence of the former one, which is proved by induction on the construction of formulas $\varphi \in \Sigma$. We will consider only the modal case, leaving the intuitionistic one to the reader.

The basis of induction and the cases of $\varphi = \psi \wedge \chi$, $\psi \vee \chi$ and $\psi \to \chi$ are trivial. So suppose that $\varphi = \Box\psi$. If $(\mathfrak{M}, x) \not\models \Box\psi$ then there is $y \in x{\uparrow}$ such that $(\mathfrak{M}, y) \not\models \psi$. Since y is Σ-remaindered in V, there must be a $z \in y{\uparrow} \cap V$ for which $(\mathfrak{M}, z) \not\models \psi$. By the induction hypothesis, we then have $(\mathfrak{N}, z) \not\models \psi$ and so $(\mathfrak{N}, x) \not\models \Box\psi$. Conversely, if $(\mathfrak{N}, x) \not\models \Box\psi$ then $(\mathfrak{N}, y) \not\models \psi$ for some point $y \in x{\uparrow} \cap V$, whence, by the induction hypothesis, $(\mathfrak{M}, y) \not\models \psi$ and so $(\mathfrak{M}, x) \not\models \Box\psi$. ❑

We are in a position now to prove the main result of this section.

Theorem 9.34 *Suppose Σ is a finite set of formulas closed under subformulas. Then there is a constant c_Σ such that every model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is Σ-subreducible to some finite model containing at most c_Σ points.*

Proof Note at once that the intuitionistic case reduces to the modal one. For, given a finite set Σ of intuitionistic formulas and an intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, we can first take the closure Π of $\{T(\varphi) : \varphi \in \Sigma\}$ under subformulas and the model $\sigma\mathfrak{M} = \langle\sigma\mathfrak{F}, \mathfrak{V}\rangle$. Then we construct a Π-subreduction f of $\sigma\mathfrak{M}$ to some model $\mathfrak{N} = \langle\mathfrak{G}, \mathfrak{U}\rangle$ containing at most c_Π points and, finally, take the skeleton $\rho\mathfrak{N} = \langle\rho\mathfrak{G}, \rho\mathfrak{U}\rangle$. By Theorem 9.22, ρf is then a subreduction of $\mathfrak{F}$ to $\rho\mathfrak{G}$ satisfying, by Lemma 8.28, the conditions (i) and (ii) for Σ.

Thus, we may consider only the modal case. Suppose $\Theta = \{p_1, \dots, p_n\}$ is the set of variables in Σ. Clearly, without loss of generality we may assume $\mathfrak{F} = \langle W, R, P\rangle$ to be generated by $\mathfrak{V}(p_1), \dots, \mathfrak{V}(p_n)$. The process of Σ-subreducing $\mathfrak{M}$ to a finite model can be described by the slogan "refine and remove". First we take $\mathfrak{M}_0 = \langle\mathfrak{F}_0, \mathfrak{V}_0\rangle = \mathfrak{M}$ and refine only that upper part of $\mathfrak{F}_0$ which gives us the points of depth 1 in the refinement of $\mathfrak{F}_0$. Then we remove from the resulting frame all those points of depth > 1 that have Σ-equivalent successors of depth 1. Thus we obtain a model $\mathfrak{M}_1 = \langle\mathfrak{F}_1, \mathfrak{V}_1\rangle$ which turns out to be a Σ-subreduct of $\mathfrak{M}_0$. After that we refine the part of $\mathfrak{F}_1$ which gives the points of depth 2 and remove all the points of depth > 2 having Σ-equivalent successors of depth 2, thereby obtaining a Σ-subreduct $\mathfrak{M}_2$ of $\mathfrak{M}_1$, and so on. Since there are at most $2^{|\Sigma|}$ pairwise non-Σ-equivalent points, this process of refining and removing must eventually terminate, i.e., we shall construct a Σ-subreduct $\mathfrak{M}_m = \langle\mathfrak{F}_m, \mathfrak{V}_m\rangle$ of $\mathfrak{M}$ whose frame is of depth m. According to Theorem 8.82, the number of points in $\mathfrak{F}_m$ does not exceed $2^n \sum_{i=1}^{2^{|\Sigma|}} c_n(i)$ and so we can take c_Σ to be equal to this constant.

Now we describe this construction in full details. Let $\mathfrak{M}_0 = \mathfrak{M}$ and suppose that we have already constructed a Σ-subreduct $\mathfrak{M}_i = \langle\mathfrak{F}_i, \mathfrak{V}_i\rangle$ of $\mathfrak{M}$ (based upon $\mathfrak{F}_i = \langle W_i, R_i, P_i\rangle$) such that:

- $\mathfrak{F}_i$ is generated by $\mathfrak{V}_i(p_1), \dots, \mathfrak{V}_i(p_n)$;
- for every $d \le i$ ($d \ne 0$), $W_i^{=d}$ is a cover for $W_i^{>d}$;
- every point in $W_i^{\le i}$ is an atom in $\mathfrak{F}_i$ and
- $\left|W_i^{=d}\right| \le 2^n c_n(d)$, for every $d \le i$ ($d \ne 0$).

If $W_i^{>i} = \emptyset$ then $\mathfrak{M}_i = \langle\mathfrak{F}_i, \mathfrak{V}_i\rangle$ is the desirable Σ-subreduct of $\mathfrak{M}$. Otherwise take all distinct maximal i-cyclic sets $X_1, \dots, X_k$ in $\mathfrak{F}_i = \langle W_i, R_i, P_i\rangle$. Unlike Section 8.6, this time $\mathfrak{F}_i$ is not necessarily refined, and so i-cyclic sets are not in general clusters of depth $i+1$. What we are going to show is that they can be *reduced* to clusters of depth $i+1$. It follows from the definition of i-cyclic set that every X_j, for $j = 1, \dots, k$, is uniquely determined by any $x \in X_j$; more precisely,

$$X_j = \{y \in W_i^{>i} : \ y{\uparrow} \cap W_i^{>i} \text{ is non-degenerate } i\text{-cyclic},$$
$$y\underline{\uparrow} \cap W_i^{>i} \sim_\Theta x\underline{\uparrow} \cap W_i^{>i} \text{ and } sp^i(x) = sp^i(y)\}$$

if X_j is non-degenerate and

$$X_j = \{y \in W_i^{>i} : \ y \sim_\Theta x, \ y{\uparrow} \cap W_i^{>i} = \emptyset \text{ and } sp^i(x) = sp^i(y)\}$$

if X_j is degenerate. So all X_j are pairwise disjoint and $k \le c_n(i+1)$. Using the same kind of arguments as in the proofs of Theorems 8.84 and 8.83, we can show that $X_1 \cup \ldots \cup X_k$ is a cover for $W_i^{>i}$ and $X_j \in P_i$ for all $j = 1, \ldots, k$. So, for each $x \in X_j$, $\{y \in X_j : x \sim_\Theta y\} \in P_i$. Recall also that, by Lemma 8.79, the very same formulas (of variables in Θ) are true in $\mathfrak{M}_i$ at Θ-equivalent points in X_j.

Now we define an equivalence relation $\sim$ on W_i by putting

$$x \sim y \text{ iff either } x = y \text{ or } x, y \in X_j, \text{ for some } j \in \{1, \ldots, k\}, \text{ and } x \sim_\Theta y.$$

Let $[x]$ be the equivalence class under $\sim$ generated by x and $[X] = \{[x] : x \in X\}$ for $X \in P_i$. By the definition of i-cyclic set, xR_iy iff $[x] \subseteq [y]{\downarrow}$ for all $x, y \in W_i$. Moreover, since, as we have already observed, the same formulas are true in $\mathfrak{M}_i$ at all points in $[x]$, every $X \in P_i$ is closed under $\sim$ and so $\sim$ is a congruence in $\mathfrak{F}_i$. Therefore, by Theorem 8.68, the quotient model $[\mathfrak{M}_i] = \langle [\mathfrak{F}_i], [\mathfrak{V}_i] \rangle$ under $\sim$ is a reduct (in particular, a Σ-subreduct) of $\mathfrak{M}_i$. Notice also that the reduction $x \mapsto [x]$ of $\mathfrak{F}_i$ to $[\mathfrak{F}_i]$ only "folds" the i-cyclic sets X_j into clusters of depth $i+1$ and leaves other points in $\mathfrak{F}_i$ untouched. Every point of depth $i+1$ is clearly an atom in $[\mathfrak{F}_i]$.

For $x \in [W_i]$, let φ_x be the conjunction of all formulas $\psi \in \Sigma$ which are true at x and all formulas $\neg\chi$ such that $\chi \in \Sigma$ and $x \not\models \chi$. Denote by X the set of points of depth $> i+1$ in $[\mathfrak{F}_i]$ which are Σ-remaindered in $[W_i]^{=i+1}$, i.e.,

$$X = \bigcup_{x \in [W_i]^{\leq i+1}} (x{\downarrow} \cap [\mathfrak{V}_i](\varphi_x)) - [W_i]^{\leq i+1}.$$

Let $\mathfrak{F}_{i+1} = \langle W_{i+1}, R_{i+1}, P_{i+1} \rangle$ be the subframe of $[\mathfrak{F}_i]$ induced by $[W_i] - X \in [P_i]$ and $\mathfrak{M}_{i+1} = \langle \mathfrak{F}_{i+1}, \mathfrak{V}_{i+1} \rangle$ the submodel of $[\mathfrak{M}_i]$ based on $\mathfrak{F}_{i+1}$. Every point in $[W_i]$ is Σ-remaindered in W_{i+1} and so, by Proposition 9.33, $\mathfrak{M}_{i+1}$ is a Σ-subreduct of $[\mathfrak{M}_i]$. Finally, using Proposition 9.32, we can conclude that $\mathfrak{M}_{i+1}$ is a Σ-subreduct of $\mathfrak{M}_i$ and hence of $\mathfrak{M}_0 = \mathfrak{M}$ as well. ❑

As a consequence of this result we obtain

Theorem 9.35 *For every formula φ, there is a constant c_φ such that a frame $\mathfrak{F}$ refutes φ only if there are a rooted countermodel $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ for φ with at most c_φ points and a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ satisfying (CDC) for the set $\mathfrak{D}$ of all closed domains in $\mathfrak{N}$ relative to φ.*

Proof Let Σ be the set of φ's subformulas, $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ a countermodel for φ based on $\mathfrak{F}$ and g a Σ-subreduction of $\mathfrak{M}$ to some model $\mathfrak{N}' = \langle \mathfrak{G}', \mathfrak{U}' \rangle$ whose frame $\mathfrak{G}'$ has at most $c_\varphi = c_\Sigma$ points. By the definition of Σ-subreduction, $\mathfrak{N}'$ is a countermodel for φ. If it is not rooted, we take a submodel $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ of $\mathfrak{N}'$ generated by some point in $\mathfrak{N}'$ at which φ is not true. Then the partial map f from $\mathfrak{F}$ onto $\mathfrak{G} = \langle V, S \rangle$ defined by

$$f(x) = \begin{cases} g(x) & \text{if } g(x) \in V \\ \text{undefined} & \text{otherwise} \end{cases}$$

is clearly a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{G}$.

It remains to verify that f satisfies (CDC) for the set $\mathfrak{D}$ of all closed domains in $\mathfrak{N}$. Suppose $x \in \mathrm{dom} f\uparrow$ and $x \notin \mathrm{dom} f$. By (R1), $x \notin \mathrm{dom} g$. Since x is Σ-remaindered in $\mathrm{dom} g$, it is also Σ-remaindered in $\mathrm{dom} f$, i.e., there is a point $y \in x\uparrow \cap \mathrm{dom} f$ such that $x \sim_\Sigma y$. Now, let $\mathfrak{d}$ be an antichain in $\mathfrak{G}$ such that $f(x\uparrow) = \mathfrak{d}\underline{\uparrow}$. We show that $\mathfrak{d}$ is open in $\mathfrak{N}$. Indeed, let $\Gamma_\mathfrak{d} = \{\psi \in \Sigma : x \models \psi\}$, $\Delta_\mathfrak{d} = \{\psi \in \Sigma : x \not\models \psi\}$ and let $t_\mathfrak{d} = (\Gamma_\mathfrak{d}, \Delta_\mathfrak{d})$. Then in the modal case, $t_\mathfrak{d}$ satisfies ($\mathrm{OD}_M 1$), since xRy and $x \sim_\Sigma y$, and the "only if" part of ($\mathrm{OD}_M 2$). To prove the "if" part, suppose that $a \models \Box^+\psi$ for all $a \in \mathfrak{d}$. Then $f(y) \models \Box^+\psi$ as well, since $f(y) \in \mathfrak{d}\underline{\uparrow}$, and so $x \models \Box\psi$. The intuitionistic case is considered analogously.

❑

Now, combining Theorems 9.30, 9.31 and 9.35, we obtain the frame-theoretic characterization of transitive refutation frames for modal and intuitionistic formulas, mentioned at the beginning of the section.

Theorem 9.36 (i) *There is an algorithm which, given a formula φ, returns a finite number of finite rooted frames $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ and sets $\mathfrak{D}_1, \dots, \mathfrak{D}_n$ of antichains in them such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some $1 \le i \le n$, satisfying (CDC) for $\mathfrak{D}_i$. If φ is an intuitionistic disjunction free formula then $\mathfrak{D}_i = \emptyset$ for all $i = 1, \dots, n$.*

(ii) *There is an algorithm which, given a negation free formula φ, returns a finite number of finite rooted frames $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ and sets $\mathfrak{D}_1, \dots, \mathfrak{D}_n$ of antichains in them such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff there is a subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some $1 \le i \le n$, satisfying (CDC) for $\mathfrak{D}_i$. If, in addition, φ is an intuitionistic disjunction free formula then $\mathfrak{D}_i = \emptyset$ for all $i = 1, \dots, n$.*

Proof (i) Let c_φ be the constant mentioned in Theorem 9.35. Construct all possible rooted countermodels $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle, \dots, \mathfrak{M}_n = \langle \mathfrak{F}_n, \mathfrak{V}_n \rangle$ for φ with $\le c_\varphi$ points. Let $\mathfrak{D}_i$ be the set of all closed domains in $\mathfrak{M}_i$ relative to φ. Note that, by Example 9.26, $\mathfrak{D}_i = \emptyset$ if φ is an intuitionistic disjunction free formula. The rest of the proof follows immediately from Theorems 9.30 and 9.35.

(ii) is proved in exactly the same way, but using Theorem 9.31 instead of 9.30. ❑

One more interesting result follows from the proof of Theorem 9.34.

Theorem 9.37 *Suppose $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a model of a modal or intuitionistic language with a finite set of variables. Then $\mathfrak{M}$ is reducible to a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ based upon a top-heavy frame $\mathfrak{G}$. Moreover, if $\mathfrak{F}$ is of finite depth then $\mathfrak{G}$ is finite.*

Proof Construct a sequence of models $\mathfrak{M}_0 = \mathfrak{M}, \mathfrak{M}_1, \dots$ in almost the same way as in the proof of Theorem 9.34. The only difference is that now we do not remove any points from frames, just refining $\mathfrak{F}$ level by level. More exactly, using the terminology of that proof, we define $\mathfrak{M}_{i+1}$ as just $[\mathfrak{M}_i]$. Of course, in general the new construction will not necessarily terminate, unless $\mathfrak{F}$ is of finite depth. But then we have an infinite chain of reductions $\mathfrak{M}_0 \xrightarrow{f_0} \mathfrak{M}_1 \xrightarrow{f_1} \dots$ and can take the limit $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ of this chain. As we know, $\mathfrak{N}$ is a reduct of $\mathfrak{M}$ and clearly, $\mathfrak{G}$ is top-heavy. ❑

9.4 Canonical formulas for K4 and Int

The characterization of refutation frames, found in the preceding section, provides us with a powerful frame-theoretic tool for handling modal and superintuitionistic logics.

Example 9.38 As a simple illustration of its capacity, we show how it can be applied for proving, say, the finite approximability of the Grzegorczyk logic **Grz**. According to Proposition 9.3, a frame refutes ***grz*** iff it is subreducible either to $\bullet$ or to $\circ\circ$. Let φ be an arbitrary modal formula. By Theorem 9.36, we can construct finite frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ and choose sets $\mathfrak{D}_1, \ldots, \mathfrak{D}_n$ of antichains in them so that a frame $\mathfrak{F}$ refutes φ iff there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some $i \in \{1, \ldots, n\}$, satisfying (CDC) for $\mathfrak{D}_i$. Now, if each $\mathfrak{F}_i$ is subreducible to $\bullet$ or $\circ\circ$ (i.e., contains one of them as a subframe, since $\mathfrak{F}_i$ is finite) then every frame refuting φ refutes, by Theorem 9.21, ***grz*** as well, and so $\varphi \in$ **Grz**. And if at least one $\mathfrak{F}_i$ is subreducible neither to $\bullet$ nor to $\circ\circ$ then it is a frame for **Grz** refuting φ. (Note, by the way, that the same argument establishes the finite approximability of all logics in Ext**Int** and NExt**K4** axiomatizable by formulas in Tables 9.1–9.3.)

An important feature of Theorem 9.36 is its invertibility in the sense that with each finite rooted frame $\mathfrak{F}$ and each set $\mathfrak{D}$ of antichains in $\mathfrak{F}$ one can associate a formula which is refuted in a frame $\mathfrak{G}$ iff there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Indeed, let $\mathfrak{F} = \langle W, R\rangle$ be a finite transitive rooted frame, $a_0, \ldots, a_n$ its points, with a_0 being the root. Suppose also that $\mathfrak{D}$ is some (possibly empty) set of antichains in $\mathfrak{F}$ different from reflexive singletons. The *normal modal canonical formula* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ associated with $\mathfrak{F}$ and $\mathfrak{D}$ looks as follows:

$$\alpha(\mathfrak{F}, \mathfrak{D}, \bot) = \bigwedge_{a_i R a_j} \varphi_{ij} \wedge \bigwedge_{i=0}^{n} \varphi_i \wedge \bigwedge_{\mathfrak{d} \in \mathfrak{D}} \varphi_{\mathfrak{d}} \wedge \varphi_\bot \to p_0,$$

where

$$\varphi_{ij} = \Box^+(\Box p_j \to p_i),$$

$$\varphi_i = \Box^+\Big(\big(\bigwedge_{\neg a_i R a_k} \Box p_k \wedge \bigwedge_{j=0, j\neq i}^{n} p_j \to p_i\big) \to p_i,$$

$$\varphi_{\mathfrak{d}} = \Box^+\Big(\bigwedge_{a_i \in W - \mathfrak{d}{\uparrow}} \Box p_j \wedge \bigwedge_{i=0}^{n} p_i \to \bigvee_{a_j \in \mathfrak{d}} \Box p_j\Big),$$

$$\varphi_\bot = \Box^+\Big(\bigwedge_{i=0}^{n} \Box^+ p_i \to \bot\Big).$$

Denote by $\alpha(\mathfrak{F}, \mathfrak{D})$ the result of deleting the conjunct $\varphi_\bot$ from $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$; it is called the *normal modal negation free canonical formula* for $\mathfrak{F}$ and $\mathfrak{D}$.

With intuitionistic $\mathfrak{F}$ and $\mathfrak{D}$ we associate the *intuitionistic canonical formula* $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ and the *intuitionistic negation free canonical formula* $\beta(\mathfrak{F}, \mathfrak{D})$, namely

$$\beta(\mathfrak{F},\mathfrak{D},\bot) = \bigwedge_{a_iRa_j} \psi_{ij} \wedge \bigwedge_{\mathfrak{d}\in\mathfrak{D}} \psi_{\mathfrak{d}} \wedge \psi_\bot \to p_0,$$

where

$$\psi_{ij} = (\bigwedge_{\neg a_jRa_k} p_k \to p_j) \to p_i,$$

$$\psi_{\mathfrak{d}} = \bigwedge_{a_i\in W-\mathfrak{d}\uparrow} (\bigwedge_{\neg a_iRa_k} p_k \to p_i) \to \bigvee_{a_j\in\mathfrak{d}} p_j,$$

$$\psi_\bot = \bigwedge_{i=0}^{n}(\bigwedge_{\neg a_iRa_k} p_k \to p_i) \to \bot,$$

and $\beta(\mathfrak{F},\mathfrak{D})$ is obtained from $\beta(\mathfrak{F},\mathfrak{D},\bot)$ by deleting the conjunct $\psi_\bot$.

The following two results will be referred to as the *refutability criteria for canonical formulas.*

Theorem 9.39 *For any modal transitive frame $\mathfrak{G} = \langle V, S, Q\rangle$,*

(i) *$\mathfrak{G} \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$ iff there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$;*

(ii) *$\mathfrak{G} \not\models \alpha(\mathfrak{F},\mathfrak{D})$ iff there is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.*

Proof Let us first prove (i).

($\Rightarrow$) Suppose $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ is refuted in a model $\mathfrak{N} = \langle\mathfrak{G},\mathfrak{U}\rangle$. Denote by φ the premise of $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ and define a partial map f from V onto W by taking

$$f(x) = \begin{cases} a_i & \text{if } x \not\models \varphi \to p_i \\ \text{undefined} & \text{otherwise.}\end{cases}$$

We show that f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying the closed domain condition for $\mathfrak{D}$. Clearly, f is a function, since $x \models \varphi_i$ and $x \not\models p_i$ imply $x \models p_j$, for all $j \neq i$.

Let xSy, $f(x) = a_i$ and $f(y) = a_j$. Then a_iRa_j, for otherwise $x \models \Box p_j$ (since $\neg a_iRa_j$, $x \models \varphi_i$ and $x \not\models p_i$) and so $y \models p_j$, contrary to $f(y) = a_j$.

Let $f(x) = a_i$ and a_iRa_j. Then $x \models \varphi_{ij}$, $x \not\models p_i$, whence $x \not\models \Box p_j$ and so there is $y \in x\uparrow$ such that $y \not\models p_j$. Since $x \models \varphi$ and xSy, we have $y \models \varphi$. Therefore, $f(y) = a_j$. It follows, in particular, that f is a surjection, since $f^{-1}(a_0) \neq \emptyset$ and a_0Ra_j for all $j \neq 0$.

By the definition, $f^{-1}(a_i) = \{x \in V : \ x \not\models \varphi \to p_i\} \in Q$. Thus, f satisfies (R1)–(R3) and so is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$.

Suppose $x \in \mathrm{dom}f\uparrow$. Then $x \models \varphi$, $x \models \varphi_\bot$ and hence $x \not\models p_i$ or $x \not\models \Box p_i$ for some i. In the former case $x \in \mathrm{dom}f$ and in the latter one there is $z \in x\uparrow$ such that $z \models \varphi$, $z \not\models p_i$ and so $z \in \mathrm{dom}f$. Thus, f is cofinal.

Let $x \in \mathrm{dom}f\uparrow$ and $f(x\uparrow) = \mathfrak{d}\underline{\uparrow}$ for some $\mathfrak{d} \in \mathfrak{D}$. Then $x \models \varphi$, $x \models \varphi_{\mathfrak{d}}$ and $x \not\models \Box p_j$ for all $a_j \in \mathfrak{d}$. Therefore, either $x \not\models \Box p_i$ for some $a_i \in W - \mathfrak{d}\underline{\uparrow}$, or $x \not\models p_i$

for some i. In the former case $a_i \in f(x\uparrow)$, which is a contradiction, whereas the latter means $f(x) = a_i$; that is, $x \in \mathrm{dom} f$. Thus, f satisfies (CDC) for $\mathfrak{D}$.

($\Leftarrow$) Suppose that f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Define a valuation $\mathfrak{U}$ in $\mathfrak{G}$ by taking $\mathfrak{U}(p_i) = V - f^{-1}(a_i)$, for every $i = 0, \dots, n$, and show that in the resultant model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ we have $x \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ for each $x \in f^{-1}(a_0)$.

Let $f(x) = a_0$. Then $x \not\models p_0$, and we must prove that the premise of $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is true at x. Suppose $x \not\models \varphi_{ij}$, for $a_i R a_j$. Then $y \models \Box p_j$ and $y \not\models p_i$ for some $y \in x\uparrow$, whence $f(y) = a_i$. By (R2), there is $z \in y\uparrow$ such that $f(z) = a_j$, and so $z \not\models p_j$, contrary to $y \models \Box p_j$ and ySz.

If $x \not\models \varphi_i$ then there is $y \in x\uparrow$ such that $f(y) = a_i$ and either $y \not\models \Box p_k$ with $\neg a_i R a_k$, or $y \not\models p_j$ for some $j \neq i$. It is clear that neither case can hold, since the former implies $z \not\models p_k$ for some $z \in y\uparrow$, and so $a_i R a_k$, while the latter means $f(y) = a_j$; that is, $j = i$.

In the same way we can show that the assumption $x \not\models \varphi_{\mathfrak{d}}$, for some $\mathfrak{d} \in \mathfrak{D}$, contradicts (CDC) and $x \not\models \varphi_\bot$ is inconsistent with the cofinality condition.

A proof of (ii) can be extracted in the obvious way from that of (i). ❑

In the same manner one can prove

Theorem 9.40 *For any intuitionistic frame $\mathfrak{G}$,*

(i) $\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$ *iff there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$;*

(ii) $\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D})$ *iff there is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.*

In general, with a frame $\mathfrak{F}$ we can associate several canonical formulas by choosing various sets of closed domains: from $\alpha(\mathfrak{F}, \emptyset, \bot)$ to $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$ and from $\alpha(\mathfrak{F}, \emptyset)$ to $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp)$, where $\mathfrak{D}^\sharp$ is the set of all antichains in $\mathfrak{F}$ different from reflexive singletons. These boundary formulas will play a particular role in the sequel, and we give them proper names.

The formulas of the form $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$ and $\beta(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$ are called the (modal and intuitionistic, respectively) *frame formulas* for $\mathfrak{F}$; we denote them by $\alpha^\sharp(\mathfrak{F}, \bot)$ and $\beta^\sharp(\mathfrak{F}, \bot)$. The formulas $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp)$ and $\beta(\mathfrak{F}, \mathfrak{D}^\sharp)$ are called the *negation free frame formulas* for $\mathfrak{F}$ and denoted by $\alpha^\sharp(\mathfrak{F})$ and $\beta^\sharp(\mathfrak{F})$.

Proposition 9.41 (i) $\mathfrak{G} \not\models \alpha^\sharp(\mathfrak{F}, \bot)$ *($\mathfrak{G} \not\models \beta^\sharp(\mathfrak{F}, \bot)$) iff a generated subframe of $\mathfrak{G}$ is reducible to $\mathfrak{F}$.*

(ii) $\mathfrak{G} \not\models \alpha^\sharp(\mathfrak{F})$ *($\mathfrak{G} \not\models \beta^\sharp(\mathfrak{F})$) iff $\mathfrak{G}$ is densely subreducible to $\mathfrak{F}$.*

Proof We consider only the formula $\beta^\sharp(\mathfrak{F})$ and leave the other cases to the reader. Suppose that $\mathfrak{G} \not\models \beta^\sharp(\mathfrak{F})$. Then there is a subreduction g of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}^\sharp$. If g is not dense then there is a point x in the set $\mathrm{dom} g\uparrow \cap \mathrm{dom} g\downarrow - \mathrm{dom} g$. By (CDC), $g(x\uparrow) = a_x\uparrow$ for some point a_x in $\mathfrak{F}$. The dense subreduction f we need can be defined by extending g as follows:

$$f(x) = \begin{cases} g(x) & \text{if } x \in \mathrm{dom} g \\ a_x & \text{if } x \in \mathrm{dom} g{\uparrow} \cap \mathrm{dom} g{\downarrow} - \mathrm{dom} g \\ \text{undefined otherwise.} & \end{cases}$$

The converse implication follows from the refutability criterion. ❑

The formulas $\alpha(\mathfrak{F}, \emptyset)$ and $\beta(\mathfrak{F}, \emptyset)$ are called the *subframe formulas* for $\mathfrak{F}$ and denoted by $\alpha(\mathfrak{F})$ and $\beta(\mathfrak{F})$. Finally, the formulas $\alpha(\mathfrak{F}, \emptyset, \bot)$ and $\beta(\mathfrak{F}, \emptyset, \bot)$ are called the *cofinal subframe formulas* for $\mathfrak{F}$ and denoted by $\alpha(\mathfrak{F}, \bot)$ and $\beta(\mathfrak{F}, \bot)$. Clearly, we have $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$ ($\mathfrak{G} \not\models \beta(\mathfrak{F}, \bot)$) iff $\mathfrak{G}$ is cofinally subreducible to $\mathfrak{F}$ and $\mathfrak{G} \not\models \alpha(\mathfrak{F})$ ($\mathfrak{G} \not\models \beta(\mathfrak{F})$) iff $\mathfrak{G}$ is subreducible to $\mathfrak{F}$.

Proposition 9.42 *For any sets $\mathfrak{D}$ and $\mathfrak{C}$ of antichains in $\mathfrak{F}$ such that $\mathfrak{D} \subseteq \mathfrak{C}$,*

$$\begin{array}{ccccccc} \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot) & \subseteq & \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{C}, \bot) & \subseteq & \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot) & \subseteq & \mathbf{K4} \oplus \alpha(\mathfrak{F}, \bot) \\ \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} \\ \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}) & \subseteq & \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{C}) & \subseteq & \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}) & \subseteq & \mathbf{K4} \oplus \alpha(\mathfrak{F}), \end{array}$$

$$\begin{array}{ccccccc} \mathbf{Int} + \beta^\sharp(\mathfrak{F}, \bot) & \subseteq & \mathbf{Int} + \beta(\mathfrak{F}, \mathfrak{C}, \bot) & \subseteq & \mathbf{Int} + \beta(\mathfrak{F}, \mathfrak{D}, \bot) & \subseteq & \mathbf{Int} + \beta(\mathfrak{F}, \bot) \\ \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} \\ \mathbf{Int} + \beta^\sharp(\mathfrak{F}) & \subseteq & \mathbf{Int} + \beta(\mathfrak{F}, \mathfrak{C}) & \subseteq & \mathbf{Int} + \beta(\mathfrak{F}, \mathfrak{D}) & \subseteq & \mathbf{Int} + \beta(\mathfrak{F}). \end{array}$$

Proof Exercise. ❑

Another important feature of the canonical formulas is that they can axiomatize all logics in NExt**K4** and Ext**Int**. For combining Theorems 9.36, 9.39 and Proposition 9.28, we obtain the following *completeness theorem* for NExt**K4**.

Theorem 9.43 (i) *There is an algorithm which, given a modal formula φ, returns canonical formulas $\alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot), \dots, \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$ such that*

$$\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot) \oplus \ldots \oplus \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

So the set of normal modal canonical formulas is complete for the class NExt**K4**.

(ii) *There is an algorithm which, given a negation free φ, returns negation free canonical formulas $\alpha(\mathfrak{F}_1, \mathfrak{D}_1), \dots, \alpha(\mathfrak{F}_n, \mathfrak{D}_n)$ such that*

$$\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \alpha(\mathfrak{F}_1, \mathfrak{D}_1) \oplus \ldots \oplus \alpha(\mathfrak{F}_n, \mathfrak{D}_n).$$

The combination of Theorems 9.36 and 9.40 yields the *completeness theorem* for Ext**Int**.

Theorem 9.44 (i) *There is an algorithm which, given an intuitionistic φ, returns canonical formulas $\beta(\mathfrak{F}_1, \mathfrak{D}_1, \bot), \dots, \beta(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$ such that*

$$\mathbf{Int} + \varphi = \mathbf{Int} + \beta(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \ldots + \beta(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

So the set of intuitionistic canonical formulas is complete for Ext**Int**.

(ii) *There is an algorithm which, for a negation free* φ, *returns negation free canonical formulas* $\beta(\mathfrak{F}_1, \mathfrak{D}_1), \dots, \beta(\mathfrak{F}_n, \mathfrak{D}_n)$ *such that*

$$\mathbf{Int} + \varphi = \mathbf{Int} + \beta(\mathfrak{F}_1, \mathfrak{D}_1) + \dots + \beta(\mathfrak{F}_n, \mathfrak{D}_n).$$

(iii) *There is an algorithm which, given a disjunction free* φ, *returns cofinal subframe formulas* $\beta(\mathfrak{F}_1, \bot), \dots, \beta(\mathfrak{F}_n, \bot)$ *such that*

$$\mathbf{Int} + \varphi = \mathbf{Int} + \beta(\mathfrak{F}_1, \bot) + \dots + \beta(\mathfrak{F}_n, \bot).$$

(iv) *There is an algorithm which, given a negation and disjunction free* φ, *returns subframe formulas* $\beta(\mathfrak{F}_1), \dots, \beta(\mathfrak{F}_n)$ *such that*

$$\mathbf{Int} + \varphi = \mathbf{Int} + \beta(\mathfrak{F}_1) + \dots + \beta(\mathfrak{F}_n).$$

As an illustration of these completeness theorems, Tables 9.6 and 9.7 show canonical representations of some standard normal modal and si-logics. In fact, these representations can be derived from Propositions 9.4, 9.5, 9.14, 9.19 and 9.20.

Theorem 9.45 *Every si-logic* L *with extra axioms in one variable can be represented either as*

$$L = \mathbf{Int} + \boldsymbol{nf}_{2n} = \mathbf{Int} + \beta^\sharp(\mathfrak{H}_n, \bot)$$

or as

$$L = \mathbf{Int} + \boldsymbol{nf}_{2n-1} = \mathbf{Int} + \beta^\sharp(\mathfrak{H}_{n+1}, \bot) + \beta^\sharp(\mathfrak{H}_{n+2}, \bot),$$

where $\mathfrak{H}_n$, $\mathfrak{H}_{n+1}$, $\mathfrak{H}_{n+2}$ *are the subframes of the frame in Fig. 8.13 generated by the points* n, $n+1$ *and* $n+2$, *respectively.*

Proof By Theorem 7.67, L is axiomatizable by the Nishimura formulas. By Theorem 8.92,

$$\mathbf{Int} + \boldsymbol{nf}_{2n} = \mathbf{Int} + \beta^\sharp(\mathfrak{H}_n, \bot),$$

$$\mathbf{Int} + \boldsymbol{nf}_{2n-1} = \mathbf{Int} + \beta^\sharp(\mathfrak{H}_{n+1}, \bot) + \beta^\sharp(\mathfrak{H}_{n+2}, \bot).$$

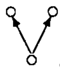
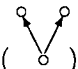
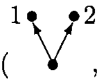
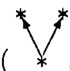
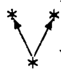
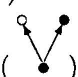
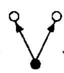
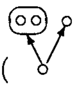
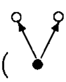
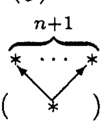
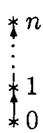
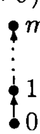
That only two additional axioms of that sort is enough follows from the obvious inclusion $\beta^\sharp(\mathfrak{H}_m, \bot) \in \mathbf{Int} + \beta^\sharp(\mathfrak{H}_n, \bot)$ which holds for every $m \geq n+2$. ❑

It follows from the completeness theorem that as far as such properties of logics as the decidability, completeness or finite approximability are concerned we can deal only with the canonical formulas. Indeed, suppose a logic L and a formula φ are given. By Theorem 9.43, L is axiomatizable by a set of canonical formulas, which is finite if L is finitely axiomatizable. Besides, we can effectively construct canonical formulas $\alpha_1, \dots, \alpha_n$ such that

$$\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \alpha_1 \oplus \dots \oplus \alpha_n.$$

Therefore, we have $\varphi \in L$ iff $\alpha_i \in L$ for every $i \in \{1, \dots, n\}$, and so L is decidable iff there is an algorithm which is capable of deciding, given an arbitrary *canonical*

Table 9.6 *Canonical axioms of standard modal logics*

D4	$=$ **K4** $\oplus\, \alpha(\bullet, \bot)$	**S4** $=$ **K4** $\oplus\, \alpha(\bullet)$	
GL	$=$ **K4** $\oplus\, \alpha(\circ)$	**For** $=$ **K4** $\oplus\, \alpha(*)$ (2 axioms)	
Grz	$=$ **K4** $\oplus\, \alpha(\bullet) \oplus \alpha(\circ\circ)$	**S4.1** $=$ **S4** $\oplus\, \alpha(\circ\circ, \bot)$	
K4.1	$=$ **K4** $\oplus\, \alpha(\bullet, \bot) \oplus \alpha(\circ\circ, \bot)$		
Triv	$=$ **K4** $\oplus\, \alpha(\bullet) \oplus \alpha(\circ\circ) \oplus \alpha($  $)$		
Verum	$=$ **K4** $\oplus\, \alpha(\circ) \oplus \alpha($  $)$		
S5	$=$ **S4** $\oplus\, \alpha($  $)$	**K4B** $=$ **K4** $\oplus\, \alpha($  $)$ (4 axioms)	
S4.2	$=$ **S4** $\oplus\, \alpha($  $, \bot)$	**S4.3** $=$ **S4** $\oplus\, \alpha($  $)$	
A*	$=$ **GL** $\oplus\, \alpha($  $, \{\{1\}, \{1,2\}\})$		
K4.2	$=$ **K4** $\oplus\, \alpha($  $, \bot) \oplus \alpha($  $, \bot) \oplus \alpha($  $, \bot)$ (8 axioms)		
K4.3	$=$ **K4** $\oplus\, \alpha($  $)$ (6 axioms)		
K4Z	$=$ **K4** $\oplus\, \alpha($  $) \oplus \alpha($  $) \oplus \alpha($  $) \oplus \alpha($  $)$		
Dum	$=$ **S4** $\oplus\, \alpha($  $) \oplus \alpha($  $)$		
D4G$_1$	$=$ **D4** $\oplus\, \alpha^\sharp($  $, \bot)$		
K4H	$=$ **K4** $\oplus\, \alpha($  $) \oplus \alpha(\circ\circ)$ (9 axioms)		
K4Alt$_n$	$=$ **K4** $\oplus\, \{\alpha(\mathfrak{F}) : n+1$ points are seen from the root of $\mathfrak{F}\}$		
K4BW$_n$	$=$ **K4** $\oplus\, \alpha($  $)$ $(2n+4$ axioms)		
K4BD$_n$	$=$ **K4** $\oplus\, \alpha($  $)$ $(2^{n+1}$ axioms)		
K4$_{n,m}$	$=$ **K4** $\oplus\, \alpha^\sharp($  $)$		

formula α, whether or not $\alpha \in L$. It follows also from this equality that for every frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff $\mathfrak{F} \not\models \alpha_i$ for some $i \in \{1, \dots, n\}$. Thus, L is complete with respect to a class $\mathcal{C}$ of frames iff for every *canonical* formula $\alpha \notin L$ there is a frame $\mathfrak{F} \in \mathcal{C}$ validating L and refuting α. (The same, of course, concerns logics in Ext**Int**.)

Having proved that the set of canonical formulas is axiomatically complete, it is natural to ask whether it is an axiomatic basis (i.e., contains no proper complete subsets) and if it is not, to find such a basis. It turns out, however, that neither of the classes NExt**K4** and Ext**Int** has an axiomatic basis. By Proposition 4.14, to show this it suffices to find prime modal and intuitionistic formulas and to check whether the set of them is axiomatically complete.

Theorem 9.46 (i) *φ is prime in* NExt**K4** *iff it is deductively equal in* NExt**K4** *to a frame formula $\alpha^\sharp(\mathfrak{F}, \bot)$, i.e.,* $\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot)$.

(ii) *φ is prime in* Ext**Int** *iff* $\mathbf{Int} + \varphi = \mathbf{Int} + \beta^\sharp(\mathfrak{F}, \bot)$*, for some frame formula* $\beta^\sharp(\mathfrak{F}, \bot)$.

Proof We consider only the modal case, since the intuitionistic one is proved in exactly the same way. The proof proceeds via two lemmas.

Lemma 9.47 (i) $\alpha^\sharp(\mathfrak{F}, \bot) \in \mathbf{K4} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ *iff, for some* $i \in I$, $\mathfrak{F} \not\models \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$.

(ii) $\beta^\sharp(\mathfrak{F}, \bot) \in \mathbf{Int} \oplus \{\beta(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ *iff* $\mathfrak{F} \not\models \beta(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$ *for some* $i \in I$.

Proof The implication ($\Rightarrow$) is clear because $\mathfrak{F} \not\models \alpha^\sharp(\mathfrak{F}, \bot)$.

($\Leftarrow$) Suppose $\mathfrak{F} \not\models \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$ for some $i \in I$. Then there is a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{G}_i$ satisfying (CDC) for $\mathfrak{D}_i$. Now, if $\mathfrak{H}$ is an arbitrary frame refuting $\alpha^\sharp(\mathfrak{F}, \bot)$ then a generated subframe $\mathfrak{H}'$ of $\mathfrak{H}$ is reducible to $\mathfrak{F}$ by some g. The composition $h = fg$ is a cofinal subreduction of $\mathfrak{H}'$ to $\mathfrak{G}_i$ which clearly satisfies (CDC) for $\mathfrak{D}_i$. Therefore, $\mathfrak{H} \not\models \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$ and so $\alpha^\sharp(\mathfrak{F}, \bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$. ❑

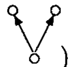
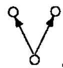
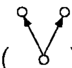
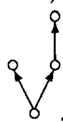
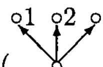
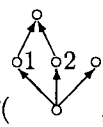
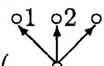
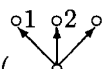
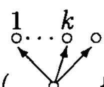
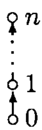
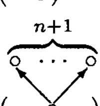
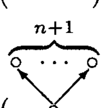
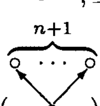
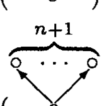
Corollary 9.48 (i) $\alpha^\sharp(\mathfrak{F}, \bot) \in \mathbf{K4} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ *iff, for some* $i \in I$, $\alpha^\sharp(\mathfrak{F}, \bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$.

(ii) $\beta^\sharp(\mathfrak{F}, \bot) \in \mathbf{Int} + \{\beta(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ *iff, for some index* $i \in I$, $\beta^\sharp(\mathfrak{F}, \bot) \in \mathbf{Int} + \beta(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$

It follows from this corollary that each frame formula $\alpha^\sharp(\mathfrak{F}, \bot)$ is prime. Indeed, if $L = \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot) = \mathbf{K4} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ then there is $i \in I$ such that $L = \mathbf{K4} \oplus \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$ and so L cannot be decomposed into a sum of logics different from L.

Now, by the completeness theorem, to prove that each prime formula φ is deductively equal to some frame formula, it suffices to consider only the case of canonical φ. So suppose $\varphi = \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and construct the countermodels $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle, \dots, \mathfrak{M}_n = \langle \mathfrak{F}_n, \mathfrak{V}_n \rangle$ for φ such that $|\mathfrak{F}| < |\mathfrak{F}_i| \leq c_\varphi$, where c_φ is the constant mentioned in Theorem 9.36. Let $\mathfrak{D}_1, \dots, \mathfrak{D}_n$ be the sets of closed domains in $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ relative to φ.

Table 9.7 *Canonical axioms of standard superintuitionistic logics*

For	$=$	$\mathbf{Int} + \beta(\circ)$
Cl	$=$	$\mathbf{Int} + \beta($  $)$
SmL	$=$	$\mathbf{Int} + \beta($  $) + \beta($  $)$
KC	$=$	$\mathbf{Int} + \beta($  $, \bot)$
LC	$=$	$\mathbf{Int} + \beta($  $)$
SL	$=$	$\mathbf{Int} + \beta^\sharp($  $, \bot)$
KP	$=$	$\mathbf{Int} + \beta($  $, \{\{1,2\}\}, \bot) + \beta($  $, \{\{1,2\}\}, \bot)$
WKP	$=$	$\mathbf{Int} + \beta($  $, \{\{1,2\}\}, \bot)$
$\mathbf{ND}_k$	$=$	$\mathbf{Int} + \beta($  $, \{\{1,2\}\}, \bot) + \ldots + \beta($  $, \{\{1,\ldots,k\}\}, \bot)$
$\mathbf{BD}_n$	$=$	$\mathbf{Int} + \beta($  $)$
$\mathbf{BW}_n$	$=$	$\mathbf{Int} + \beta($  $)$
$\mathbf{BTW}_n$	$=$	$\mathbf{Int} + \beta($  $, \bot)$
$\mathbf{T}_n$	$=$	$\mathbf{Int} + \beta^\sharp($  $)$
$\mathbf{B}_n$	$=$	$\mathbf{Int} + \beta^\sharp($  $, \bot)$

Lemma 9.49 (i) *For every modal canonical formula* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$,

$$\mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot) = \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot) \oplus \alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot) \oplus \ldots \oplus \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

(ii) *For every intuitionistic canonical formula* $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$,

$$\mathbf{Int} + \beta(\mathfrak{F}, \mathfrak{D}, \bot) = \mathbf{Int} + \beta^\sharp(\mathfrak{F}, \bot) + \beta(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \ldots + \varphi(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

Proof By Theorem 9.30 and Lemma 9.47, the logic in the right-hand part is contained in that in the left-hand part. To show the converse inclusion, suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is a countermodel for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Let f be the cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ defined in the proof of Theorem 9.39. Two cases are possible.

Case 1: $\mathrm{dom} f = \mathrm{dom} f\!\uparrow$. Then f is a reduction of the subframe of $\mathfrak{G}$ generated by $\mathrm{dom} f$ to $\mathfrak{F}$ and so $\mathfrak{G} \not\models \alpha^\sharp(\mathfrak{F}, \bot)$.

Case 2: $\mathrm{dom} f \subset \mathrm{dom} f\!\uparrow$. Then the number of pairwise non-$\mathbf{Sub}\varphi$-equivalent points in $\mathfrak{N}$ is greater than $|\mathfrak{F}|$ and so, by Theorem 9.34, $\mathfrak{N}$ is $\mathbf{Sub}\varphi$-subreducible to $\mathfrak{M}_i$ for some $i \in \{1, \ldots, n\}$, which, as we know, implies $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$.

Thus, in both cases $\mathfrak{G}$ is not a frame for the logic in the right-hand part and hence $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ belongs to it. ❑

We can now complete the proof of Theorem 9.46. Suppose $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is prime. By Lemma 9.49,

$$\mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot) = \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot) \oplus \alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot) \oplus \ldots \oplus \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$$

with $|\mathfrak{F}_i| > |\mathfrak{F}|$. It follows from these inequalities and the refutability criterion that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \notin \mathbf{K4} \oplus \alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot) \oplus \ldots \oplus \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$. But then we have $\mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot) = \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot)$, since $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is prime. ❑

Thus, we have characterized the sets of prime formulas in NExt**K4** and Ext**Int**. However, they are not complete for these classes. For we have

Proposition 9.50 *Let* $\mathfrak{F}$ *be the frame depicted in Fig. 9.6 (a). Then neither* $\mathbf{K4} \oplus \alpha(\mathfrak{F}, \bot)$ *nor* $\mathbf{Int} + \beta(\mathfrak{F}, \bot)$ *can be axiomatized only by frame formulas.*

Proof Suppose otherwise. Then $\mathbf{K4} \oplus \alpha(\mathfrak{F}, \bot) = \mathbf{K4} \oplus \{\alpha^\sharp(\mathfrak{F}_i, \bot) : i \in I\}$ for some frames $\mathfrak{F}_i$. Let $\mathfrak{G}$ be the Kripke frame shown in Fig. 9.6 (b). Since $\mathfrak{G}$ is cofinally subreducible to $\mathfrak{F}$, it refutes $\alpha(\mathfrak{F}, \bot)$. Then $\mathfrak{G}$ refutes $\alpha^\sharp(\mathfrak{F}_i, \bot)$ for some $i \in I$, and so it is reducible to $\mathfrak{F}_i$ by some reduction f. Clearly, $\mathfrak{F}_i$ is partially ordered and of width ≥ 4. Let $\mathfrak{a} = \{a_1, a_2, a_3, a_4\}$ be an antichain in $\mathfrak{F}_i$ such that, for any antichain $\mathfrak{b}$ of four points in $\mathfrak{F}_i$, $\mathfrak{a} \subseteq \mathfrak{b}\!\uparrow$ implies $\mathfrak{a} = \mathfrak{b}$. Such an antichain certainly exists, since $\mathfrak{F}_i$ is finite. Without loss of generality we may assume that, for some $k < \omega$, $f(c_1^k) = a_1$, $f(c_2^k) = a_2$, $f(c_3^k) = a_3$ and $f(c_4) = a_4$. Suppose $f(c_j^{k+1}) = b_j$ for $j = 1, 2, 3$. By the definition of reduction, b_1, b_2 and b_3 do not see each other in $\mathfrak{F}_i$, are different from a_1, a_2, a_3 and $\mathfrak{a} \subseteq \{b_1, b_2, b_3, a_4\}\!\uparrow$, whence $\{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$, which is a contradiction. ❑

As a consequence of Theorem 9.46 and Propositions 4.14 and 9.50 we derive

Theorem 9.51 NExt**K4** *and* Ext**Int** *have no axiomatic bases.*

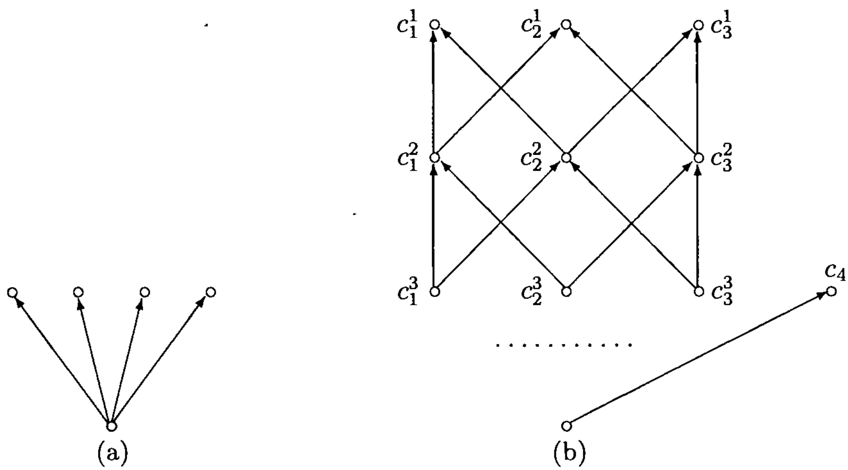


FIG. 9.6.

9.5 Quasi-normal canonical formulas

Theorem 9.35, characterizing the constitution of refutation frames for a given formula by subreducing them to some fixed finite pattern frames, does not take into account at what point in a frame the formula is refuted. As a result, the set of normal modal canonical formulas turns out to be too small to axiomatize all quasi-normal extensions of **K4**, which are not supposed to be closed under necessitation. To see the reason for this, let us recall that logics in Ext**K4** are characterized by frames with distinguished points, with a formula φ being refuted in $\langle \mathfrak{G}, w \rangle$ iff φ is false at w under some valuation in $\mathfrak{G}$. According to the proof of Theorem 9.39, $\langle \mathfrak{G}, w \rangle$ refutes $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ iff there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and the following *actual world condition* as well:

(AWC) $f(w)$ is the root of $\mathfrak{F}$.

Now, consider the frame $\mathfrak{G} = \langle V, S, Q \rangle$, whose underlying Kripke frame is shown in Fig. 8.1 (b) and where Q consists of all finite sets of natural numbers and their complements in the space V. Let ω be the actual world in $\mathfrak{G}$. Since each set $X \in Q$ containing ω is infinite and has a dead end, it is impossible to reduce X to $\circ$ or $\bullet$, and so $\langle \mathfrak{G}, \omega \rangle$ validates all normal canonical formulas. On the other hand, we clearly have $\langle \mathfrak{G}, \omega \rangle \not\models \boldsymbol{bd}_n$ for every $n \geq 1$. It follows in particular that the logics $\mathbf{K4BD}_n$ cannot be axiomatized by normal canonical formulas without the postulated necessitation.

To get over this obstacle and retain the idea of the canonical formulas we are forced to modify the definition of subreduction so that such sets as X above may be "reduced" at least to irreflexive roots of frames. Given a frame $\mathfrak{G} = \langle V, S, Q \rangle$ with an *irreflexive* root u and a frame $\mathfrak{F} = \langle W, R, P \rangle$, we say a partial map f from W onto V is a *quasi-subreduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if it satisfies (R1) for all $x, y \in \mathrm{dom} f$ such that $f(x) \neq u$ or $f(y) \neq u$, (R2) and (R3).

Thus, we may map all points in the frame $\mathfrak{G}$ in Fig. 8.1 (b) to $\bullet$, and this map will be a quasi-subreduction of $\mathfrak{G}$ to $\bullet$ satisfying (AWC). Moreover, every frame is quasi-subreducible to $\bullet$.

Now, given a finite frame $\mathfrak{F}$ with an irreflexive root a_0 and a set $\mathfrak{D}$ of antichains in $\mathfrak{F}$, we define the *quasi-normal canonical formula* $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ as the result of deleting $\Box p_0$ from φ_0 in $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ (which says, in particular, that a_0 is not self-accessible); the *quasi-normal negation free canonical formula* $\alpha^\bullet(\mathfrak{F}, \mathfrak{D})$ is defined in exactly the same way, starting from $\alpha(\mathfrak{F}, \mathfrak{D})$.

Theorem 9.52 *Suppose w is the actual world in a frame $\mathfrak{G}$. Then*

(i) $\langle \mathfrak{G}, w\rangle \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ *iff there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and (AWC);*

(ii) $\langle \mathfrak{G}, w\rangle \not\models \alpha(\mathfrak{F}, \mathfrak{D})$ *iff there is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and (AWC);*

(iii) $\langle \mathfrak{G}, w\rangle \not\models \alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ *iff there is a cofinal quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and (AWC);*

(iv) $\langle \mathfrak{G}, w\rangle \not\models \alpha^\bullet(\mathfrak{F}, \mathfrak{D})$ *iff there is a quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and (AWC).*

Proof Follows from the proof of Theorem 9.39. ❑

Theorem 9.53 (i) *There is an algorithm which, given a modal formula φ, constructs a finite set Δ of normal and quasi-normal canonical formulas such that* $\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \Delta$.

(ii) *There is an algorithm which, given a negation free φ, constructs a finite set Δ of normal and quasi-normal negation-free canonical formulas such that* $\mathbf{K4} + \varphi = \mathbf{K4} + \Delta$.

Proof (i) We put $c = c_\varphi + 1$ and construct all possible rooted models $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1\rangle, \dots, \mathfrak{M}_k = \langle \mathfrak{F}_k, \mathfrak{V}_k\rangle$ refuting φ at their roots $w_1, \dots, w_k$, respectively, and containing $\leq c$ points. Let $\mathfrak{D}_i$ be the set of all closed domains in $\mathfrak{M}_i$. If w_i is irreflexive and, for every $\Box\psi \in \mathbf{Sub}\varphi$, $w_i \models \Box\psi$ only if $w_i \models \psi$, then we associate with $\mathfrak{M}_i$ the quasi-normal canonical formula $\alpha^\bullet(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$. Otherwise we construct the normal canonical formula $\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$. Denote by Δ the set of all resultant canonical formulas and show that $\mathbf{K4} + \varphi = \mathbf{K4} + \Delta$.

Suppose that $\langle \mathfrak{F}, w\rangle \not\models \varphi$, i.e., there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ on the frame $\mathfrak{F} = \langle W, R, P\rangle$ with root w such that $w \not\models \varphi$. Let $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ be the $\mathbf{Sub}\varphi$-subreduct of $\mathfrak{M}$ constructed in the proof of Theorem 9.34 and f the corresponding cofinal subreduction. Two cases are possible.

Case 1: $w \in \mathrm{dom} f$. Then $\mathfrak{G}$ is rooted and contains $< c$ points. So $\mathfrak{N} = \mathfrak{M}_i$ for some $i \in \{1, \dots, k\}$, and f is a cofinal subreduction (in particular, a quasi-subreduction) of $\mathfrak{F}$ to $\mathfrak{F}_i$ satisfying (CDC) for $\mathfrak{D}_i$ and (AWC). Therefore, $\langle \mathfrak{F}, w\rangle$ refutes the canonical formula associated with $\mathfrak{M}_i$.

Case 2: $w \notin \mathrm{dom} f$. Consider the set

$$X = (\bigcap_{y \in V} f^{-1}(y){\downarrow} \cap \{x \in W : x \sim_{\mathbf{Sub}\varphi} w\}) - \mathrm{dom} f$$

consisting of all those points in $W - \mathrm{dom} f$ that are $\mathbf{Sub}\varphi$-equivalent to w and see f-inverse images of all points in $\mathfrak{G}$. Since $\mathfrak{G}$ is finite, $X \in P$. Clearly, $w \in X$ and, for any $\Box\psi \in \mathbf{Sub}\varphi$, $w \models \Box\psi$ implies $w \models \psi$ because w is $\mathbf{Sub}\varphi$-remaindered in $\mathrm{dom} f$ (although w may be irreflexive).

Construct a frame $\mathfrak{G}' = \langle V', S' \rangle$ by adding to $\mathfrak{G}$ the new root u, which is reflexive iff $X \subseteq X\downarrow$, and extend $\mathfrak{U}$ to u so that $u \sim_{\mathbf{Sub}\varphi} w$. Since $|V'| \leq c$, the constructed countermodel for φ coincides with $\mathfrak{M}_i$, for some $i \in \{1, \dots, k\}$. Let f' be a partial map from W onto V' defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom} f \\ u & \text{if } x \in X \\ \text{undefined otherwise.} \end{cases}$$

If $X \subseteq X\downarrow$ or $X = \{w\}$ then f' is clearly a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$ (in particular, this is the case if $\mathfrak{F}$ is reflexive). But if X contains a dead end different from w, f' is only a cofinal quasi-subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$. In both cases f' satisfies (CDC) for $\mathfrak{D}_i$ and (AWC), and so $\langle \mathfrak{F}, w \rangle$ refutes the canonical formula associated with $\mathfrak{M}_i$.

Thus, $\mathbf{K4} + \varphi \subseteq \mathbf{K4} + \Delta$. The converse inclusion and (ii) are established in the same manner as in the proof of Theorem 9.36, taking into account the fact that in the models $\mathfrak{M}_i$, whose associated canonical formulas are quasi-normal, $w_i \models \Box\psi$ implies $w_i \models \psi$, for every $\psi \in \mathbf{Sub}\varphi$ (this is essential for proving an analog of Theorem 9.30). ❑

As an easy exercise, we invite the reader to prove that

$$\mathbf{S} = (\mathbf{K4} \oplus \boldsymbol{la}) + \boldsymbol{re} = \mathbf{K4} + \alpha(\circ) + \alpha(\bullet).$$

Theorem 9.53 and its proof provide us also with the following results:

Theorem 9.54 (i) *There is an algorithm which, given a modal formula φ, constructs a finite set Δ of normal canonical formulas built on reflexive frames such that $\mathbf{S4} + \varphi = \mathbf{S4} + \Delta$.*

(ii) *There is an algorithm which, given a negation free φ, constructs a finite set Δ of normal negation free canonical formulas built on reflexive frames such that $\mathbf{S4} + \varphi = \mathbf{S4} + \Delta$.*

Proof Each quasi-normal logic L containing $\mathbf{S4}$ is characterized by the class $\{\langle \mathfrak{G}, w \rangle : \mathfrak{G} \text{ is reflexive and } \langle \mathfrak{G}, w \rangle \models L\}$. Therefore, all frames in the proof of Theorem 9.53 may be regarded as reflexive and so quasi-normal canonical formulas are redundant. ❑

As a consequence of Theorem 9.54 we obtain

Theorem 9.55 Ext$\mathbf{S4.3}$ = NExt$\mathbf{S4.3}$.

Proof We must show that every logic $L \in$ Ext$\mathbf{S4.3}$ is normal, i.e., $\varphi \in L$ only if $\Box\varphi \in L$, for every formula φ. Suppose otherwise. Then there is φ such that $\varphi \in L$ and $\Box\varphi \notin L$. By Theorem 9.54, φ is deductively equal in Ext$\mathbf{S4}$ to a conjunction of some (normal) canonical formulas, and so there exists $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ such that $\Box\alpha(\mathfrak{F},\mathfrak{D},\bot) \notin L$. Let $\langle\mathfrak{G},w\rangle$ be a frame with root w validating L and refuting $\Box\alpha(\mathfrak{F},\mathfrak{D},\bot)$. Since $\mathfrak{G} \models$ **S4.3**, $\mathfrak{G}$ is a chain of non-degenerate clusters. And since it refutes $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$. It follows, in particular, that $\mathfrak{F}$ is also a chain of non-degenerate clusters and so $\mathfrak{D} = \emptyset$. Let a be the root of $\mathfrak{F}$. Define a map g by taking

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom} f \\ a & \text{if } x \in f^{-1}(a)\downarrow - \mathrm{dom} f \\ \text{undefined otherwise.} \end{cases}$$

It should be clear that g cofinally subreduces $\mathfrak{G}$ to $\mathfrak{F}$ and $g(w) = a$. Consequently, $\langle\mathfrak{G},w\rangle \not\models \alpha(\mathfrak{F},\bot)$, which is a contradiction. ❑

9.6 Modal companions of superintuitionistic logics

As we saw in Section 3.9, the Gödel translation T embeds **Int** into **S4** in the sense that, for any intuitionistic formula φ, $\varphi \in$ **Int** iff $\mathsf{T}(\varphi) \in$ **S4**. Using only this fact and the relationship between intuitionistic and modal frames, established in Section 8.3, one can reduce various problems concerning **Int** (e.g. proving the completeness, finite approximability, disjunction property, etc.) to those for **S4** and *vice versa*. But in fact, it turns out that each logic in Ext**Int** is embedded via T into some logics in NExt**S4**, and for each logic in NExt**S4** there is one in Ext**Int** embeddable in it.

We say a modal logic $M \in$ NExt**S4** is a *modal companion* of a si-logic L if L is embedded in M by T, i.e., if for every intuitionistic formula φ

$$\varphi \in L \text{ iff } \mathsf{T}(\varphi) \in M.$$

If M is a modal companion of L then L is called the *superintuitionistic fragment* of M and denoted by ρM. The reason for denoting the operator "modal logic $\mapsto$ its superintuitionistic fragment" by the same symbol we used for the skeleton operator is explained by the following:

Theorem 9.56 *For every $M \in$ NExt**S4**, $\rho M = \{\varphi \in \mathbf{For}\mathcal{L} : \mathsf{T}(\varphi) \in M\}$. Moreover, if M is characterized by a class $\mathcal{C}$ of modal frames then ρM is characterized by the class $\rho\mathcal{C} = \{\rho\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$ of intuitionistic frames.*

Proof It suffices to show that $\{\varphi \in \mathbf{For}\mathcal{L} : \mathsf{T}(\varphi) \in M\} = \mathrm{Log}\rho\mathcal{C}$. Suppose $\mathsf{T}(\varphi) \in M$. Then $\mathfrak{F} \models \mathsf{T}(\varphi)$ and so, by the skeleton lemma, $\rho\mathfrak{F} \models \varphi$ for every $\mathfrak{F} \in \mathcal{C}$, i.e., $\varphi \in \mathrm{Log}\rho\mathcal{C}$. Conversely, if $\rho\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in \mathcal{C}$ then, by the same lemma, $\mathsf{T}(\varphi)$ is valid in all frames in $\mathcal{C}$ and so $\mathsf{T}(\varphi) \in M$. ❑

Thus, ρ is a map from NExt**S4** into Ext**Int**. The following simple observation shows that actually ρ is a surjection. Given a logic $L \in$ Ext**Int**, we put

$$\tau L = \mathbf{S4} \oplus \{\mathsf{T}(\varphi) : \varphi \in L\}.$$

Theorem 9.57 *For every si-logic L, τL is a modal companion of L, i.e., $L = \rho\tau L$.*

Proof Clearly, $L \subseteq \rho\tau L$. To prove the converse inclusion, suppose $\varphi \notin L$, i.e., there is an intuitionistic frame $\mathfrak{F}$ for L refuting φ. Then, using the skeleton lemma and Theorem 8.34, we obtain $\sigma\mathfrak{F} \models \tau L$ and $\sigma\mathfrak{F} \not\models \mathsf{T}(\varphi)$. Therefore, $\mathsf{T}(\varphi) \notin \tau L$ and so $\varphi \notin \rho\tau L$. ❑

Corollary 9.58 *For every superintuitionistic logic L, τL is the least modal companion of L, i.e., the least (with respect to $\subseteq$) logic in $\rho^{-1}(L)$.*

With the help of the canonical formulas we will obtain now a general characterization of the set $\rho^{-1}(L)$ of all modal companions of a given si-logic L. First let us prove a lemma.

Lemma 9.59 *Suppose $\mathfrak{F}$ is a finite rooted intuitionistic frame, $\mathfrak{D}$ a set of antichains in $\mathfrak{F}$ and $\mathfrak{G}$ a modal quasi-ordered frame. Then*

$$\mathfrak{G} \models \alpha(\mathfrak{F}, \mathfrak{D}, \bot) \text{ iff } \rho\mathfrak{G} \models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$$

and

$$\mathfrak{G} \models \alpha(\mathfrak{F}, \mathfrak{D}) \text{ iff } \rho\mathfrak{G} \models \beta(\mathfrak{F}, \mathfrak{D}).$$

Proof Follows from the refutability criteria for the canonical formulas and Theorems 9.22 and 9.23. ❑

This lemma means that the formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$ behave like the Gödel translations of $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\beta(\mathfrak{F}, \mathfrak{D})$, respectively. More exactly, we have

Corollary 9.60 *For every canonical formulas $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\beta(\mathfrak{F}, \mathfrak{D})$,*

$$\mathbf{S4} \oplus \mathsf{T}(\beta(\mathfrak{F}, \mathfrak{D}, \bot)) = \mathbf{S4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$$

and

$$\mathbf{S4} \oplus \mathsf{T}(\beta(\mathfrak{F}, \mathfrak{D})) = \mathbf{S4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}).$$

Proof Follows from Lemmas 9.59 and 8.28. ❑

Theorem 9.61. (Modal companion) *A logic $M \in$ NExt**S4** is a modal companion of a si-logic*

$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\}$$

iff M can be represented in the form

$$M = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\} \oplus \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j, \bot) : \ j \in J\},$$

where every frame $\mathfrak{F}_j$, for $j \in J$, contains a proper cluster.

Proof ($\Leftarrow$) We must show that for every intuitionistic formula φ, $\varphi \in L$ iff $\mathsf{T}(\varphi) \in M$. Suppose $\mathsf{T}(\varphi) \notin M$. Then for some quasi-ordered frame $\mathfrak{F}$, $\mathfrak{F} \models M$ and $\mathfrak{F} \not\models \mathsf{T}(\varphi)$. By Lemmas 9.59 and 8.28, we then have $\boldsymbol{\rho}\mathfrak{F} \models L$, $\boldsymbol{\rho}\mathfrak{F} \not\models \varphi$ and so $\varphi \notin L$. The converse implication is more complicated. Suppose that $\varphi \notin L$ and $\mathfrak{F} = \langle W, R, P\rangle$ is an intuitionistic frame separating φ from L. We will show that $\boldsymbol{\sigma}\mathfrak{F} = \langle W, R, \boldsymbol{\sigma}P\rangle$ separates $\mathsf{T}(\varphi)$ from M. First, by the skeleton lemma and Theorem 8.34, $\boldsymbol{\sigma}\mathfrak{F} \not\models \mathsf{T}(\varphi)$. Second, by Lemma 9.59, we have $\boldsymbol{\sigma}\mathfrak{F} \models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$ for any $i \in I$. So it remains to show that $\boldsymbol{\sigma}\mathfrak{F} \models \alpha(\mathfrak{F}_j, \mathfrak{D}_j, \bot)$ for every $j \in J$.

Suppose otherwise. Then, for some $j \in J$, we have a subreduction f of $\boldsymbol{\sigma}\mathfrak{F}$ to $\mathfrak{F}_j$. Let a_1 and a_2 be distinct points belonging to the same proper cluster in $\mathfrak{F}_j$. By the definition of subreduction, $f^{-1}(a_1) \subseteq f^{-1}(a_2)\!\downarrow$ and $f^{-1}(a_2) \subseteq f^{-1}(a_1)\!\downarrow$, and so there is an infinite chain $x_1Ry_1Rx_2Ry_2R\dots$ in $\boldsymbol{\sigma}\mathfrak{F}$ such that $\{x_1, x_2, \dots\} \subseteq f^{-1}(a_1)$ and $\{y_1, y_2, \dots\} \subseteq f^{-1}(a_2)$. And since R is a partial order, all the points x_i and y_i are distinct.

The set $f^{-1}(a_1)$ is in $\boldsymbol{\sigma}P$. By Lemma 8.32, we can represent it in the form $f^{-1}(a_1) = (-X_1 \cup Y_1) \cap \dots \cap (-X_n \cup Y_n)$, where $X_i, Y_i \in P$ for any $i = 1, \dots, n$, which means in particular that $X_i = X_i\!\uparrow$ and $Y_i = Y_i\!\uparrow$. Since $f^{-1}(a_1) \cap f^{-1}(a_2) = \emptyset$, for every point y_i there is some number n_i such that $y_i \in X_{n_i}$ and $y_i \notin Y_{n_i}$. But then, for some distinct l and m, the numbers n_l and n_m must coincide, and so if, say, $y_l R y_m$ then $x_m \notin Y_{n_m}$ and $x_m \in X_{n_l}$ (for $y_l R x_m R y_m$). Therefore, $x_m \notin f^{-1}(a_1)$, which is a contradiction.

($\Rightarrow$) Suppose that

$$M = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_k, \mathfrak{D}_k, \bot) : \ k \in K\} \oplus \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j, \bot) : \ j \in J\},$$

where all frames $\mathfrak{F}_k$, for $k \in K$, are partially ordered and all frames $\mathfrak{F}_j$, for $j \in J$, contain proper clusters. By ($\Leftarrow$), we have

$$L = \boldsymbol{\rho}M = \mathbf{Int} + \{\beta(\mathfrak{F}_k, \mathfrak{D}_k, \bot) : \ k \in K\} = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\}$$

and so $\mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\} = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_k, \mathfrak{D}_k, \bot) : \ k \in K\}$, as it follows from Lemma 9.59. ❑

It is worth noting that Theorem 9.61 can be presented in a somewhat more general form. Namely, the very same proof gives us

Theorem 9.62 *$M \in$ NExtS4 is a modal companion of*

$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\} + \{\beta(\mathfrak{F}_j, \mathfrak{D}_j) : \ j \in J\}$$

iff M can be represented in the form

$$M = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\} \oplus \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j) : \ j \in J\} \oplus$$

$$\{\alpha(\mathfrak{F}_k, \mathfrak{D}_k, \bot) : \ k \in K\} \oplus \{\alpha(\mathfrak{F}_n, \mathfrak{D}_n) : \ n \in N\}$$

where all frames $\mathfrak{F}_k$ and $\mathfrak{F}_n$, for $k \in K$ and $n \in N$, contain proper clusters.

Example 9.63 According to Theorem 9.62 and Tables 9.6, 9.7, we have the following equalities:

$$\rho\mathbf{S4} = \rho\mathbf{S4.1} = \rho\mathbf{Dum} = \rho\mathbf{Grz} = \mathbf{Int},$$

$$\rho\mathbf{S4.2} = \rho(\mathbf{S4.2} \oplus \boldsymbol{grz}) = \mathbf{KC},$$
$$\rho\mathbf{S4.3} = \rho(\mathbf{S4.3} \oplus \boldsymbol{grz}) = \mathbf{LC},$$
$$\rho\mathbf{S5} = \rho(\mathbf{S5} \oplus \boldsymbol{grz}) = \mathbf{Cl}.$$

Corollary 9.64 *For every superintuitionistic logic*

$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\} + \{\beta(\mathfrak{F}_j, \mathfrak{D}_j) : \ j \in J\},$$

the set $\rho^{-1}(L)$ *of its modal companions forms the interval in* $\mathrm{NExt}\mathbf{S4}$ *of the form*

$$\rho^{-1}(L) = [\tau L, \tau L \oplus \alpha(\circ\circ)] = \{M \in \mathrm{NExt}\mathbf{S4} : \ \tau L \subseteq M \subseteq \tau L \oplus \mathbf{Grz}\}$$

where $\tau L = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i \in I\} \oplus \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j) : \ j \in J\}$. *If* L *is consistent then this interval contains an infinite descending chain of logics.*

Proof Notice first that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$ are in $\mathbf{Grz}$ iff $\mathfrak{F}$ contains a proper cluster. So $\rho^{-1}(L) \subseteq [\tau L, \ \tau L \oplus \alpha(\circ\circ)]$. On the other hand, the si-fragments of all logics in this interval are the same, namely L. It follows that $\rho^{-1}(L) = [\tau L, \tau L \oplus \alpha(\circ\circ)]$.

Now, if L is consistent then $\beta(\circ) \notin L$ and so we have

$$\tau L \subset \ldots \subset \tau L \oplus \alpha(\mathfrak{C}_n) \subset \ldots \subset \tau L \oplus \alpha(\mathfrak{C}_2) \subset \tau L \oplus \alpha(\mathfrak{C}_1) = \mathbf{For}\mathcal{ML},$$

where $\mathfrak{C}_i$ is the non-degenerate cluster with i points. ❑

Thus, all modal companions of every si-logic L are contained between the least companion τL and the greatest one, viz., $\tau L \oplus \alpha(\circ\circ)$, which will be denoted by σL.

Corollary 9.65 *There is an algorithm which, given a modal formula* φ, *returns an intuitionistic formula* ψ *such that* $\rho(\mathbf{S4} \oplus \varphi) = \mathbf{Int} + \psi$.

Proof Follows from Theorems 9.43 and 9.61. ❑

The following theorem describes lattice-theoretic properties of the maps ρ, τ and σ.

Theorem 9.66 (i) *The map* ρ *is a homomorphism of the lattice* $\mathrm{NExt}\mathbf{S4}$ *onto the lattice* $\mathrm{Ext}\mathbf{Int}$.

(ii) *The map* τ *is an isomorphism of* $\mathrm{Ext}\mathbf{Int}$ *into* $\mathrm{NExt}\mathbf{S4}$.

(iii) **(The Blok–Esakia theorem)** *The map* σ *is an isomorphism of the lattice* $\mathrm{Ext}\mathbf{Int}$ *onto* $\mathrm{NExt}\mathbf{Grz}$.

All these maps preserve infinite unions and intersections of logics.

Proof Exercise. ❑

Now we give frame-theoretic characterizations of the operators $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. First let us note some evident relations between frames for si-logics and their modal companions.

Lemma 9.67 (i) *For every intuitionistic frame $\mathfrak{F}$ and logic $M \in$* **NExtS4**,

$$\mathfrak{F} \models \rho M \textit{ iff } \boldsymbol{\sigma}\mathfrak{F} \models M.$$

(ii) *For every intuitionistic frame $\mathfrak{F}$ and logic $L \in$* **ExtInt**,

$$\mathfrak{F} \models L \textit{ iff } \boldsymbol{\sigma}\mathfrak{F} \models \boldsymbol{\sigma} L.$$

(iii) *For every quasi-ordered frame $\mathfrak{F}$ and superintuitionistic logic L,*

$$\rho\mathfrak{F} \models L \textit{ iff } \mathfrak{F} \models \boldsymbol{\tau} L.$$

(iv) *For every intuitionistic frame $\mathfrak{F}$, si-logic L and every k, $0 < k \leq \omega$,*

$$\mathfrak{F} \models L \textit{ iff } \boldsymbol{\tau}_k\mathfrak{F} \models \boldsymbol{\tau} L.$$

Proof (i) Suppose $\mathfrak{F} \models \rho M$ but $\boldsymbol{\sigma}\mathfrak{F} \not\models M$. In view of our previous results, it should be clear that $\boldsymbol{\sigma}\mathfrak{F} \models \boldsymbol{\tau}\rho M$ and so $\boldsymbol{\sigma}\mathfrak{F}$ refutes some canonical axiom of M built on a frame with a proper cluster, which, as was shown in the proof of Theorem 9.61, is impossible. The converse implication follows from Theorem 8.34 and the skeleton lemma.

(ii) It suffices to put $M = \boldsymbol{\sigma} L$ in (i) and use the fact that $\rho\boldsymbol{\sigma} L = L$.

(iii) and (iv) are left to the reader as an exercise. ❑

Theorem 9.68 *A si-logic L is characterized by a class $\mathcal{C}$ of intuitionistic frames iff $\boldsymbol{\sigma} L$ is characterized by the class $\boldsymbol{\sigma}\mathcal{C} = \{\boldsymbol{\sigma}\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.*

Proof ($\Rightarrow$) According to Lemma 9.67 (ii), we must show that any modal formula $\varphi \notin \boldsymbol{\sigma} L$ is refuted by some frame in $\boldsymbol{\sigma}\mathcal{C}$. And by Theorem 9.43, we may assume φ to be a canonical formula, say, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Besides, we know from the proof of Corollary 9.64 that $\mathfrak{F}$ is partially ordered. Therefore, $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$, i.e., there is $\mathfrak{F} \in \mathcal{C}$ refuting $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ and so, by Lemma 9.59, $\boldsymbol{\sigma}\mathfrak{F} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$.

($\Leftarrow$) is straightforward. ❑

To characterize $\boldsymbol{\tau}$ we require one more lemma.

Lemma 9.69 *For every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ built on a quasi-ordered frame $\mathfrak{F}$, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in$* **S4** $\oplus\, \alpha(\rho\mathfrak{F}, \rho\mathfrak{D}, \bot)$.

Proof Let $\mathfrak{G}$ be a quasi-ordered frame refuting $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Then there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. The map h from $\mathfrak{F}$ onto $\rho\mathfrak{F}$ defined by $h(x) = C(x)$, for every x in $\mathfrak{F}$, is clearly a reduction of $\mathfrak{F}$ to $\rho\mathfrak{F}$. So the composition hf is a cofinal subreduction of $\mathfrak{G}$ to $\rho\mathfrak{F}$, and it is easy to verify that it satisfies (CDC) for $\rho\mathfrak{D}$. ❑

Theorem 9.70 *A si-logic L is characterized by a class $\mathcal{C}$ of frames iff τL is characterized by the class $\bigcup_{0<k<\omega} \tau_k \mathcal{C}$, where $\tau_k \mathcal{C} = \{\tau_k \mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.*

Proof ($\Rightarrow$) By Lemma 9.67 (iv), if $\mathfrak{F}$ is a frame for L then $\tau_k \mathfrak{F}$ is a frame for τL. So suppose that a formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, built on a quasi-ordered frame $\mathfrak{F} = \langle W, R \rangle$, does not belong to τL and show that it is refuted by some frame in $\bigcup_{0<k<\omega} \tau_k \mathcal{C}$. By Lemma 9.69, $\alpha(\rho\mathfrak{F}, \rho\mathfrak{D}, \bot) \notin \tau L$ and so $\beta(\rho\mathfrak{F}, \rho\mathfrak{D}, \bot) \notin L$. Hence there is a frame $\mathfrak{G} = \langle V, S, Q \rangle$ in $\mathcal{C}$ which refutes $\beta(\rho\mathfrak{F}, \rho\mathfrak{D}, \bot)$. But then, by Lemmas 9.67 (ii) and 9.59, $\sigma\mathfrak{G} \models \tau L$ and $\sigma\mathfrak{G} \not\models \alpha(\rho\mathfrak{F}, \rho\mathfrak{D}, \bot)$. Let f be a subreduction of $\sigma\mathfrak{G}$ to $\rho\mathfrak{F}$ satisfying (CDC) for $\rho\mathfrak{D}$ and let $k = \max\{|C(x)| : x \in W\}$. Define a partial map h from $\tau_k\mathfrak{G} = \langle kV, kS, kQ \rangle$ onto $\mathfrak{F}$ as follows: if $x \in V$, $y_0 \in W$, $f(x) = C(y_0)$ and $C(y_0) = \{y_0, \dots, y_n\}$ then we put $h(\langle i, x \rangle) = y_i$, for $i = 0, \dots, n$. By the definition of τ_k (see Section 8.3), for any $i \in \{0, \dots, n\}$ we have

$$h^{-1}(y_i) = \{\langle i, x \rangle : x \in f^{-1}(C(y_0))\} = \{i\} \times f^{-1}(C(y_0)) \in kQ.$$

Now, one can readily prove that h is a cofinal subreduction of $\tau_k\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Therefore, $\tau_k\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$.

($\Leftarrow$) is obvious. ❑

It is worth noting that this proof will not change if we put in it $k = \omega$. So we have

Corollary 9.71 *A logic $L \in$ Ext**Int** is characterized by a class $\mathcal{C}$ of frames iff τL is characterized by the class $\tau_\omega \mathcal{C}$.*

The following theorem gives a deductive characterization of the maps τ and σ.

Theorem 9.72 *For every si-logic L and every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ built on a quasi-ordered frame $\mathfrak{F}$,*

(i) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \tau L$ iff $\beta(\rho\mathfrak{F}, \rho\mathfrak{D}, \bot) \in L$;

(ii) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \sigma L$ iff either $\mathfrak{F}$ is partially ordered and $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ or $\mathfrak{F}$ contains a proper cluster.

Proof (i) The implication ($\Rightarrow$) was actually established in the proof of Theorem 9.70, and the converse one follows from Lemmas 9.69 and 9.59.

(ii) Suppose $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \sigma L$. Then either $\mathfrak{F}$ is partially ordered, and so $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \in L$, or $\mathfrak{F}$ contains a proper cluster. The converse implication follows from (i) and the fact that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in$ **Grz** for every frame $\mathfrak{F}$ with a proper cluster. ❑

The results obtained in this section not only establish some structural correspondences between logics in Ext**Int** and NExt**S4** and their frames, but may be also used for transferring various properties of modal logics to their si-fragments and back. A few results of that sort are collected in Table 9.8; we shall cite them as the *preservation theorem.* The preservation of decidability follows from

Table 9.8 *Preservation theorem*

Property of logics	Preserved under		
	ρ	τ	σ
Decidability	Yes	Yes	Yes
Kripke completeness	Yes	Yes	No
Strong completeness	Yes	Yes	No
Finite approximability	Yes	Yes	Yes
Tabularity	Yes	No	Yes
Pretabularity	Yes	No	Yes
$\mathcal{D}$-persistence	Yes	Yes	No
Local tabularity	Yes	No	No
Disjunction property	Yes	Yes	Yes
Halldén completeness	Yes	No	No
Interpolation property	Yes	No	No
Elementarity	Yes	Yes	No
Independent axiomatizability	No	Yes	Yes

the definition of ρ, Theorem 9.72 and the completeness theorem for the canonical formulas. That ρ preserves Kripke completeness, finite approximability and tabularity is a consequence of Theorem 9.56. The map τ preserves Kripke completeness and finite approximability, since we can define τ_k in Theorem 9.70 so that $\tau_k \langle W, R \rangle = \langle kW, kR \rangle$; however, τ does not in general preserve tabularity, because $\tau\mathbf{Cl} = \mathbf{S5}$ is not tabular. The preservation of finite approximability and tabularity under σ follows from Theorem 9.68; Theorem 6.27 shows on the other hand that σ does not preserve Kripke completeness. The rest of the preservation results in Table 9.8 will be proved later on, when we shall be considering the corresponding properties, or left to the reader as an exercise.

9.7 Exercises and open problems

Exercise 9.1 Show that the classes $\mathcal{DF}$, $\mathcal{T}$ and $\mathcal{CM}$ (and so $\mathcal{R}$ and $\mathcal{D}$) of (not necessarily transitive) modal or intuitionistic frames are closed under the formation of subframes.

Exercise 9.2 Suppose a pseudo-Boolean algebra $\mathfrak{B}$ is an IC-subalgebra of a pseudo-Boolean algebra $\mathfrak{A}$. Prove that the map f_+ defined by

$$f_+(\nabla) = \begin{cases} \nabla \cap B & \text{if } \nabla \cap B \in W_{\mathfrak{B}} \\ \text{undefined otherwise} \end{cases}$$

for every $\nabla \in W_{\mathfrak{A}}$, is a subreduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.

Exercise 9.3 Prove that if a pseudo-Boolean algebra $\mathfrak{B}$ is an ICN-subalgebra of a pseudo-Boolean algebra $\mathfrak{A}$ then $\mathfrak{A}_+$ is cofinally subreducible to $\mathfrak{B}_+$.

Exercise 9.4 (S. Aanderaa) Let $\Box^{\Box\top}\psi = \Box\top \to \Box\psi$, $\Box^{\wedge p}\psi = p \wedge \Box\psi$ and $\varphi^{\Box\top}$ ($\varphi^{\wedge p}$) be the result of replacing every $\Box$ in φ by $\Box^{\Box\top}$ (respectively, $\Box^{\wedge p}$), where $p \notin \mathbf{Var}\varphi$. Denote for convenience $\mathbf{S2}^{\Box\top} = \mathbf{T}$, $\mathbf{S3}^{\Box\top} = \mathbf{S4}$. Prove that for $i \in \{2,3\}$,

$$\varphi \in \mathbf{Si} \text{ iff } p \to \varphi^{\wedge p} \in \mathbf{Si}^{\Box\top}, \quad \varphi \in \mathbf{Si}^{\Box\top} \text{ iff } \varphi^{\Box\top} \in \mathbf{Si}.$$

Exercise 9.5 Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box\rangle$ be a modal algebra and $a \in A$. Define $\mathfrak{A}^a = \langle A^a, \wedge^a, \vee^a, \to^a, \bot, \Box^a\rangle$, the *relativization* of $\mathfrak{A}$ with respect to a, by taking

$$A^a = \{x \cap a : x \in A\},$$

$$(x \cap a) \odot^a (y \cap a) = (x \odot y) \cap a, \text{ for } \odot \in \{\wedge, \vee, \to\},$$
$$\Box^a(x \cap a) = \Box(a \to x) \cap a.$$

Show that $\mathfrak{A}^a$ is a modal algebra and that $\mathfrak{A}^a_+$ is isomorphic to the subframe of $\mathfrak{A}_+$ induced by $f_{\mathfrak{A}}(a)$. Prove also that for every subframe $\mathfrak{G}$ of a modal frame $\mathfrak{F}$ induced by V, $\mathfrak{G}^+ \cong (\mathfrak{F}^+)^V$.

Exercise 9.6 For a formula φ and a variable p not occurring in φ, define φ^p inductively by taking

$$q^p = q \wedge p, \ q \text{ an atom},$$
$$(\psi \odot \chi)^p = \psi^p \odot \chi^p, \quad \text{for } \odot \in \{\wedge, \vee, \to\},$$
$$(\Box\psi)^p = \Box(p \to \psi^p) \wedge p.$$

Show that for every subframe $\mathfrak{G}$ of a modal frame $\mathfrak{F}$ induced by V and valuations $\mathfrak{V}$ in $\mathfrak{F}$ and $\mathfrak{U}$ in $\mathfrak{G}$ such that $\mathfrak{V}(p) = V$ and $\mathfrak{U}(q) = \mathfrak{V}(q) \cap V$, for all q different from p, $\mathfrak{V}(\varphi^p) = \mathfrak{U}(\varphi)$.

Exercise 9.7 Let $\varphi^{sf} = p \to \varphi^p$, where p is a variable having no occurrences in φ. Show that a frame $\mathfrak{F}$ validates φ^{sf} iff all subframes of $\mathfrak{F}$ validate φ.

Exercise 9.8 Show that the si-logic characterized by the frame in Fig. 9.6 (b) is not finitely approximable.

Exercise 9.9 Let $\mathfrak{G}$ be the frame depicted in Fig. 9.6 (a). Show that the logic $\mathbf{K4} \oplus \{\alpha^\sharp(\mathfrak{F}, \bot) : \mathfrak{F}$ is not subreducible to $\mathfrak{G}\}$ is not finitely approximable.

Exercise 9.10 Let a logic $L \in \text{NExt}\mathbf{K4}$ or $L \in \text{Ext}\mathbf{Int}$ be finitely approximable. Show that $\text{NExt}L$ or, respectively, $\text{Ext}L$ has an axiomatic basis iff all logics in the class are finitely approximable.

Exercise 9.11 Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot\rangle$ be a finite pseudo-Boolean algebra with the second greatest element $\top'$. Show that the formula $\gamma(\mathfrak{A})$, called the *characteristic formula* for $\mathfrak{A}$ and defined by

$$\gamma(\mathfrak{A}) = (p_\bot \leftrightarrow \bot) \wedge \bigwedge\{p_a \odot p_b \leftrightarrow p_{a\odot b} : a, b \in A, \ \odot \in \{\wedge, \vee, \to\}\} \to p_{\top'},$$

is deductively equal to $\beta^\sharp(\mathfrak{A}_+, \bot)$ in $\mathbf{ExtInt}$.

Exercise 9.12 Suppose $\mathfrak{F} = \langle W, R\rangle$ is a finite rooted transitive frame, $a_0, \ldots, a_n$ are all its points and a_0 is a root. Show that the conjunction of the formulas p_0, $\Box(p_i \to \neg p_j)$ for $i \neq j$, $\Box(p_i \to \Diamond p_j)$ for $a_i R a_j$, $\Box(p_i \to \neg\Diamond p_j)$ for $\neg a_i R a_j$ is deductively equal in NExt**K4** to $\alpha(\mathfrak{F})$ and that by adding to it the conjunct $\Box(p_0 \vee \ldots \vee p_n)$ we obtain a formula that is deductively equal to $\alpha^\sharp(\mathfrak{F}, \bot)$.

Exercise 9.13 Prove that **GL** cannot be axiomatized by frame formulas over **K4**.

Exercise 9.14 Show that **K4** and **Int** have no immediate successors in NExt**K4** and Ext**Int**, respectively.

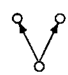
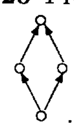
Exercise 9.15 Show that every interval of the form $[\mathbf{K4}, L]$, where L is a proper normal extension of **K4**, contains a continuum of logics. Show the same for extensions of **Int**.

Exercise 9.16 Show that there is a continuum of logics in NExt**K4.3**. (Hint: consider the formulas $\alpha^\sharp(\mathfrak{F}_n, \bot)$, where $\mathfrak{F}_n$ is the chain of n points of which only the root is reflexive, and prove that $\alpha^\sharp(\mathfrak{F}_n, \bot) \in \mathbf{K4.3} \oplus \alpha^\sharp(\mathfrak{F}_m, \bot)$ iff $n = m$.)

Exercise 9.17 Prove that **KC** is the greatest si-logic containing the same negation free formulas as **Int**. (Hint: prove that (a) $\beta(\mathfrak{F}, \mathfrak{D}) \notin \mathbf{KC}$ for any $\mathfrak{F}$ and (b) $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin \mathbf{KC}$ iff $\mathfrak{F}$ contains a last point iff $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ is deductively equal to $\beta(\mathfrak{F}, \mathfrak{D})$.)

Exercise 9.18 Prove that **KC** is the smallest si-logic in which every formula is deductively equal to a negation free formula.

Exercise 9.19 Let $\varphi \in \mathbf{Cl}$. Prove that $\mathbf{Int} + \varphi = \mathbf{Cl}$ iff φ is refuted by the two point chain iff φ is deductively equal to $\beta($  $)$.

Exercise 9.20 Prove that $\mathbf{Int} + \varphi = \mathbf{KC}$ iff φ is refuted by the frame  and validated by  .

Exercise 9.21 Let $\mathfrak{Cl}_n$ be the n-point cluster. Show that

$$\rho^{-1}(\mathbf{Cl}) = \{\mathbf{S5} \oplus \alpha(\mathfrak{Cl}_2), \mathbf{S5} \oplus \alpha(\mathfrak{Cl}_3), \ldots, \mathbf{S5} \oplus \alpha(\mathfrak{Cl}_n), \ldots, \mathbf{S5}\}.$$

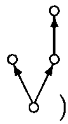
Exercise 9.22 Construct a finitely axiomatizable modal companion of **Int** that is not finitely approximable. (Hint: use the frames in Fig. 9.6 in which c_4 and one of the final points in (a) are replaced by two-point clusters.)

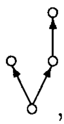
Exercise 9.23 Construct a non-compact modal companion of **Int**.

Exercise 9.24 Show that the lattice Ext$L$ can be embedded into the lattice $\rho^{-1}L$, for every si-logic L.

Exercise 9.25 Prove that if a si-logic L is tabular then all logics in $\rho^{-1}L$ are finitely approximable and finitely axiomatizable.

Exercise 9.26 Show that $\mathbf{A}^*$ is the greatest logic in **NExtGL** into which **Grz** is embeddable by $^+$.

Exercise 9.27 Show that $\mathbf{Grz} \oplus \Box\varphi$ is not embeddable into $\mathbf{A}^* \oplus (\Box\varphi)^+$ by $^+$, where $\varphi = \alpha($  $)$. (Hint: consider the formula $\Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$.)

Exercise 9.28 Show that $\mathbf{Grz} + \alpha^\sharp($  $, \bot)$ is a modal companion of **Int**.

Exercise 9.29 Let $\mathbf{M}^*$ be the quasi-normal modal logic characterized by the Kripke frame $\mathfrak{F}_\omega = \langle W_\omega, R_\omega \rangle$ with actual world o which is defined inductively as follows. Let $\langle W_0, R_0 \rangle$ be the disjoint union of all finite rooted intuitionistic frames, and $\mathfrak{D}_i$, for $i \geq 1$, the set of all antichains in $\langle W_{i-1}, R_{i-1} \rangle$. We then let $W_i = W_{i-1} \cup \{c_\mathfrak{a} : \mathfrak{a} \in \mathfrak{D}_i\}$, R_i be the reflexive and transitive closure of $R_{i-1} \cup \{\langle c_\mathfrak{a}, a \rangle : a \in \mathfrak{a}\}$ and, finally,

$$W_\omega = \bigcup_{i<\omega} W_i \cup \{o\}, \quad R_\omega = \bigcup_{i<\omega} R_i \cup \{\langle o, a \rangle : a \in W_\omega\}.$$

Show that $\mathbf{M}^*$ is the greatest modal companion of **Int** in **ExtS4**.

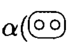
Exercise 9.30 Prove that $L \in \mathbf{ExtS4}$ is a modal companion of **Int** iff it can be represented in the form

$$L = \mathbf{S4} + \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\} + \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j, \bot) : j \in J\},$$

where, for each $i \in I$, there is an antichain $\mathfrak{a} \in \mathfrak{D}_i$ such that $x\uparrow = \{x\} \cup \mathfrak{a}\uparrow$ for no x in $\mathfrak{F}$ and, for each $j \in J$, $\mathfrak{F}_j$ contains a proper cluster.

Exercise 9.31 Show that not all logics in **ExtD4.3** are normal. (Hint: consider the logic characterized by the chain of three points 0, 1, 2 of which only 1 is irreflexive and 0 is the actual world.)

Exercise 9.32 Which of the standard modal and si-logics can be axiomatized by frame formulas?

Exercise 9.33 Show that $\mathbf{S4.1}' = \mathbf{S4} + \alpha($  $, \bot)$.

Exercise 9.34 Prove that the class of finite transitive Kripke frames is modally definable iff it is closed under finite disjoint unions, reductions and generated subframes. (Hint: use the frame formulas.)

Problem 9.1 *What is the structure of modal formulas φ such that $\mathfrak{F} \models \varphi$ implies $\mathfrak{G} \models \varphi$, for every $\mathfrak{F}$ and every (cofinal) subframe $\mathfrak{G}$ of $\mathfrak{F}$?*

Problem 9.2 *Is it true that, for a si-logic L,* Ext$L$ *contains an undecidable logic (an incomplete logic, a logic that is not finitely approximable) iff $\rho^{-1}L$ contains a logic with the same "negative" property?*

Problem 9.3 *Is it true that, for a si-logic L,* Ext$L$ *is continual iff $\rho^{-1}L$ is continual?*

Problem 9.4 *Is* $\mathbf{M}^*$ *finitely axiomatizable?*

Problem 9.5 *Characterize the class of all (not necessarily transitive) refutation frames for a given modal formula and define "canonical" formulas for* Ext$\mathbf{K}$.

9.8 Notes

The first frame-based (algebra-based, to be more precise) formulas were introduced by Jankov (1963b). With every finite pseudo-Boolean algebra $\mathfrak{A}$ having a rooted dual he associated a formula, called the *characteristic formula* for $\mathfrak{A}$, which is deductively equal to the frame formula $\beta^\sharp(\mathfrak{A}_+, \bot)$ (see Exercise 9 11). Jankov (1968b) used the characteristic formulas to construct the first (infinitely axiomatizable) si-logic that is not finitely approximable. He showed also that there is a continuum of logics in Ext$\mathbf{Int}$. Jankov (1969) characterized the prime formulas in $\mathbf{Int}$ (Theorem 9.46 (ii)).

In the modal case formulas deductively equal to $\alpha^\sharp(\mathfrak{F}, \bot)$ were introduced by Fine (1974a), who called them the frame formulas and used for the same purposes as Jankov (1968b) (see Exercise 9.12). The frame formulas are known also as *Jankov* or *Jankov–Fine* or *splitting* formulas (see also de Jongh, 1968).

Blok (1978) noticed in fact that the logics in NExt$\mathbf{K4}$ and Ext$\mathbf{Int}$ axiomatizable by a single frame formula are splittings (see Section 10.5) of these lattices and can be used for studying their structure. An example of such a use is Exercise 9.14. Other applications can be found in Chapter 12. For more references concerning splittings see Section 10.7.

Fine (1985) introduced the subframe formulas (see Exercise 9.12) and studied the logics in NExt$\mathbf{K4}$ axiomatizable by them. For details see Section 11.3.

The apparatus of the canonical formulas was developed in a series of papers (Zakharyaschev 1983, 1984a, 1988, 1989, 1992) first for Ext$\mathbf{Int}$, then for NExt$\mathbf{S4}$ and finally for Ext$\mathbf{K4}$. (Theorem 9.44 (iii) and (iv) was proved in Zakharyaschev (1981).) It is not known whether it can be extended to Ext$\mathbf{K}$. A positive solution to Problem 9.5 would provide us not only with a much deeper understanding of logics in Ext$\mathbf{K}$ but also of polymodal logics, as has been recently shown by Kracht and Wolter (1997). (There are, however, many obstacles to such a solution: frames for $\mathbf{K}$ have no clear upper and bottom parts, not all finite rooted frames give rise to splittings, the notion of subframe reflects the accessibility only "in one step", and so forth; compare also Exercises 8.8 and 9.34.) Theorem 9.55 is due to Segerberg (1975).

The Gödel embedding of logics in **ExtInt** into those in **NExtS4** was first considered by Dummett and Lemmon (1959) who defined the map $\boldsymbol{\tau}$ and showed that $\boldsymbol{\tau}L$ is a modal companion of L. A few years before the Kripke semantics was constructed they conjectured that if a si-logic L is characterized by a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ then $\boldsymbol{\tau}L$ is characterized by $\omega\mathfrak{F} = \langle \omega W, \omega R\rangle$.

A systematic study of the relationship between **ExtInt** and **NExtS4** was started by Maksimova and Rybakov (1974), Blok and Dwinger (1975), Blok (1976) and Esakia (1979a, 1979b). Maksimova and Rybakov introduced the maps $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ (the latter only in an algebraic form), proved Theorems 9.56, 9.66 (i), (ii) and 9.68 and showed that $\boldsymbol{\rho}^{-1}L = [\boldsymbol{\tau}L, \boldsymbol{\sigma}L]$. That $\boldsymbol{\rho}\mathbf{S5} = \mathbf{Cl}$ was first noted by Halldén (1949).

Blok (1976) and Esakia (1979a, 1979b) established that $\boldsymbol{\sigma}L = \boldsymbol{\tau}L \oplus \mathbf{Grz}$ and proved Theorem 9.66 (iii). The semantic characterization of $\boldsymbol{\tau}$, giving in particular a positive solution to the conjecture of Dummett and Lemmon, was obtained by Zakharyaschev (1989, 1991); the modal companion theorem and Theorem 9.72 were also proved there.

The embeddings of logics in **NExtGrz** into those in **NExtGL** via the translation $^+$, and thereby the embeddings of si-logics into normal extensions of **GL** via T^+, were considered by Kuznetsov and Muravitskij (1986). They defined a map $\boldsymbol{\mu}$ from **NExtGL** into **NExtGrz** by taking $\boldsymbol{\mu}L = \{\varphi : \varphi^+ \in L\}$ and proved that $\boldsymbol{\mu}$ is a surjective semilattice $\cap$-homomorphism (but not a lattice homomorphism, since in general we do not have $\boldsymbol{\mu}(L_1 \oplus L_1) = \boldsymbol{\mu}L_1 \oplus \boldsymbol{\mu}L_2$). Muravitskij (1988) observed that $\mathbf{GL}\oplus\Gamma$ is the smallest logic in $\boldsymbol{\mu}^{-1}(\mathbf{Grz}\oplus\Gamma)$. As to the greatest one, Artemov (1987b) discovered that **Grz** is embedded by $^+$ into a proper normal extension of **GL**. Shavrukov (1991) proved that $\mathbf{A}^*$ and $\mathbf{S} + \mathbf{A}^*$ are the greatest logics in **NExtGL** and **ExtS**, respectively, into which **Grz** is embedded by $^+$. (That $^+$ embeds **Grz** into **S** was proved by Boolos (1980).) Both these logics turn out to be decidable, though are not finitely approximable. However, as was observed in Chagrov and Zakharyaschev (1992), the analog of the Blok–Esakia theorem does not hold in this case; for details see Exercise 9.27.

By Shavrukov's result, $\mathbf{A}^*$ is the greatest companion of **Int** in **NExtGL** with respect to T^+. This contrasts to Chagrov's (1990b) result, according to which there is a continuum of maximal T^+-companions of **Int** in **ExtS**.

Chagrov (1985b) observed that there is a modal companion of **Int** in **ExtS4** which contains **Grz** properly (see Exercise 9.28.) Zakharyaschev (1996c) gave a characterization (in terms of canonical formulas) of the set of quasi-normal companions of **Int** and constructed both syntactically and semantically the greatest logic $\mathbf{M}^*$ in this set. Like $\mathbf{A}^*$, $\mathbf{M}^*$ is decidable but not finitely approximable (see Exercises 9.29 and 9.30). For a survey of results concerning embeddings of si-logics into modal logics consult Chagrov and Zakharyaschev (1992).

Exercises 9.5–9.7 were taken from Kracht (1990) and Wolter (1993). The result of Exercise 9.10 was also proved by Wolter (1993). Exercises 9.17–9.20 are due to Jankov (1968a) and Exercises 9.23, 9.24 to Rybakov (1976, 1977). The results of Exercise 9.25 were obtained by Maksimova and Rybakov (1974) and Rybakov (1976).

Part IV

Properties of logics

Having laid in Part III a solid semantic foundation for investigating modal and superintuitionistic logics, we are in a position now to attack more concrete problems. The main question of this part is the following one: given a modal or superintuitionistic logic, how can we recognize whether it has such and such desirable properties? We have already seen how such properties as decidability, Kripke completeness, finite approximability, disjunction property, etc., can be proved for a few particular systems. Now we try to find general methods for proving these properties which cover wide families of logics.

10

KRIPKE COMPLETENESS

Perhaps the most desirable property of a logic is its decidability. However, the main tool for proving it (at least in the realm of modal and superintuitionistic logics) is Harrop's theorem, according to which the decidability of a finitely axiomatizable logic follows from its finite approximability. Likewise, to prove many other properties of a logic, it is desirable first to establish its completeness with respect to some good class of frames, the simpler the better.

That is why we begin our study of logics' properties with general completeness results. We are going to descend the stairs of the hierarchy in Section 4.3, starting from Kripke completeness.

10.1 The method of canonical models revised

The essence of the method of canonical models is to show that the canonical (Kripke) frame $\mathfrak{F}_L$ of a given logic L validates L; if this is the case then L is Kripke complete. Now, being equipped with general frames, we can generalize the method in the following way.

Given a class $\mathcal{C}$ of (general) frames, we say L is *$\mathcal{C}$-persistent* if, for every $\mathfrak{F} \in \mathcal{C}$, $\mathfrak{F} \models L$ implies $\kappa\mathfrak{F} \models L$ (recall that the operator κ gives the underlying Kripke frame of $\mathfrak{F}$). We denote by $\kappa\mathcal{C}$ the class $\{\kappa\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$ and put $\mathcal{C}^* = \mathcal{C} \cup \kappa\mathcal{C}$.

Proposition 10.1 *If a logic L is both $\mathcal{C}$-complete and $\mathcal{C}$-persistent then L is $\kappa\mathcal{C}$-complete and in particular Kripke complete.*

Proof Since L is $\mathcal{C}$-complete, it is characterized by some subclass $\mathcal{C}'$ of $\mathcal{C}$. Since L is $\mathcal{C}$-persistent, it is sound with respect to $\kappa\mathcal{C}'$. And since every formula refuted by a frame $\mathfrak{F}$ is also refuted by $\kappa\mathfrak{F}$, L is characterized by $\kappa\mathcal{C}'$. Hence L is $\kappa\mathcal{C}$-complete. ❑

Since, as we know from Section 8.4, every modal and superintuitionistic logic is $\mathcal{DF}$-, $\mathcal{T}$-, $\mathcal{R}$-, $\mathcal{CM}$- and $\mathcal{D}$-complete, we immediately obtain the following:

Corollary 10.2 *If a logic L is $\mathcal{DF}$-persistent (respectively, $\mathcal{T}$-, $\mathcal{R}$-, $\mathcal{CM}$- or $\mathcal{D}$-persistent) then L is Kripke complete.*

As an illustration let us prove

Theorem 10.3 *Each $L \in$ NExt**Alt**$_n$ is $\mathcal{DF}$-persistent, for any $n < \omega$.*

Proof The key observation in the proof is that for every finite set of points $X = \{x_1, \dots, x_m\}$ in a differentiated modal frame $\mathfrak{F} = \langle W, R, P\rangle$ there are disjoint sets $X_1, \dots, X_m \in P$ such that $X_i \cap X = \{x_i\}$, for $i = 1, \dots, m$.

Suppose $\mathfrak{F}$ is a differentiated frame for L but $\kappa\mathfrak{F} \not\models L$. Notice first that each point x_0 in $\mathfrak{F}$ has $\leq n$ distinct alternatives. For if there are distinct points $x_1, \dots, x_{n+1}$ accessible from x_0 then we can put $X = \{x_1, \dots, x_{n+1}\}$ and define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p_i) = W - X_i$, where $X_i \in P$ are disjoint and $X_i \cap X = \{x_i\}$. It is readily checked that $\boldsymbol{alt_n}$ is false at x_0 under $\mathfrak{V}$, which is a contradiction.

Now, let a formula $\varphi \in L$ be refuted at x in $\kappa\mathfrak{F}$ under some valuation $\mathfrak{V}$. Put $X = x\uparrow^0 \cup \dots \cup x\uparrow^d$, where $d = md(\varphi)$. Since no point in $\mathfrak{F}$ has more than n successors, X is finite, say, $X = \{x_1, \dots, x_m\}$. Take disjoint sets $X_i \in P$ such that $X_i \cap X = \{x_i\}$ and define a valuation $\mathfrak{U}$ in $\mathfrak{F}$ so that for every $y \in X_i$ and every variable p, $y \in \mathfrak{U}(p)$ iff $x_i \in \mathfrak{V}(p)$, $i = 1, \dots, m$. Thus, the valuations $\mathfrak{V}$ and $\mathfrak{U}$ coincide on the points in X and so, by Proposition 3.2, φ is false at x under $\mathfrak{U}$, contrary to $\mathfrak{F} \models L$. ❑

Clearly, $\mathcal{DF}$-, $\mathcal{T}$-, $\mathcal{R}$-, $\mathcal{CM}$- and $\mathcal{D}$-persistence are preserved under sums of logics. Moreover, the operators ρ and τ preserve all these properties save $\mathcal{DF}$-persistence. Indeed, suppose a logic $M \in$ NExt**S4** is $\mathcal{D}$-persistent and $\mathfrak{F}$ is a descriptive frame for ρM. By Theorem 8.53, $\sigma\mathfrak{F}$ is a descriptive modal frame which, by Lemma 9.67, validates M. Since $\kappa\sigma\mathfrak{F} \cong \kappa\mathfrak{F}$, we then have $\kappa\mathfrak{F} \models M$ and so $\kappa\mathfrak{F}$ ($\cong \rho\kappa\mathfrak{F}$) is a frame for ρM. Suppose now that $L \in$ Ext**Int** is $\mathcal{D}$-persistent and let $\mathfrak{F}$ be a descriptive frame for τL. Then $\rho\mathfrak{F}$ is a descriptive frame for L and so $\rho\kappa\mathfrak{F}$ ($\cong \kappa\rho\mathfrak{F}$) validates L. Therefore, by Lemma 9.67, $\kappa\mathfrak{F} \models \tau L$. The same argument shows that ρ and τ preserve $\mathcal{T}$-, $\mathcal{R}$- and $\mathcal{CM}$-persistence. However, the operator σ does not preserve them, witness the couple **Int** and σ**Int** $=$ **Grz**, in which only **Int** is $\mathcal{D}$-persistent.

The following observation may be also of interest.

Proposition 10.4 *A modal or superintuitionistic logic L is $\mathcal{D}$-persistent iff it is $\mathcal{U}_L$-persistent, where $\mathcal{U}_L$ is the class of universal frames for L.*

Proof The implication ($\Rightarrow$) follows from $\mathcal{U}_L \subseteq \mathcal{D}$. To show ($\Leftarrow$) suppose $\mathfrak{F}$ is a descriptive $\varkappa$-generated frame for L. Then by Theorem 8.60, $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}_L(\varkappa)$. And since $\kappa\mathfrak{F}_L(\varkappa) \models L$, we must also have $\kappa\mathfrak{F} \models L$. ❑

Each $\mathcal{D}$-persistent logic is clearly canonical and so strongly Kripke complete. We give now a semantic characterization of strongly complete logics in NExt**K** and Ext**Int**. Say that a normal modal or si-logic L is *$\varkappa$-complex*, $\varkappa$ a cardinal, if every modal (respectively, pseudo-Boolean) algebra for L with $\leq \varkappa$ generators is a subalgebra of $\mathfrak{F}^+$ for some Kripke frame $\mathfrak{F}$ validating L.

Theorem 10.5 *For every logic $L \in$ NExt**K** in an infinite language with $\varkappa$ variables the following conditions are equivalent:*

(i) *L is strongly Kripke complete;*

(ii) *L is $\varkappa$-complex;*

(iii) *L is strongly globally Kripke complete.*

Proof (i) $\Rightarrow$ (ii) Let $\mathfrak{A}$ be a modal algebra for L with $\leq \varkappa$ generators and $\mathfrak{V}$ a valuation in $\mathfrak{A}$ such that the set of all $\mathfrak{V}(p)$, p a variable in the language of L, generates $\mathfrak{A}$. (One can consider $\mathfrak{V}$ as a homomorphism of $\mathfrak{A}_L(\varkappa)$ onto $\mathfrak{A}$.) Let ∇ be a prime filter in $\mathfrak{A}$ and Δ its complement prime ideal. The pair $t = (\nabla', \Delta')$, where

$$\nabla' = \{\varphi : \mathfrak{V}(\varphi) \in \nabla\}, \quad \Delta' = \{\varphi : \mathfrak{V}(\varphi) \in \Delta\},$$

is then a maximal L-consistent tableau (for otherwise we would have $a \in \nabla$ and $b \in \Delta$ such that $a \to b = \top$, which is impossible). Since L is strongly complete, there is a rooted model $\mathfrak{M}_\nabla = \langle \mathfrak{F}_\nabla, \mathfrak{V}_\nabla \rangle$ based upon a Kripke frame $\mathfrak{F}_\nabla$ for L and such that t is realized at its root x_∇. Consider the disjoint union $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{U} \rangle$ of all such $\mathfrak{M}_\nabla$. By the disjoint union theorem, the Kripke frame $\mathfrak{F}$ validates L. Let $\mathfrak{G} = \langle W, R, P \rangle$ be the general frame associated with $\mathfrak{M}$. Clearly $\mathfrak{G}^+$ is a subalgebra of $\mathfrak{F}^+$. We show now that the map $\mathfrak{V}(\varphi) \mapsto \mathfrak{U}(\varphi)$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{G}^+$. It follows immediately from the definition that this map is a surjective homomorphism. So, by Theorem 7.71, it remains to show that $\mathfrak{V}(\varphi) = \top$ iff $\mathfrak{U}(\varphi) = W$, for every formula φ. In the modal case we have

$$\begin{aligned}
\mathfrak{V}(\varphi) = \top &\text{ iff } \forall n < \omega\ \mathfrak{V}(\Box^n \varphi) = \top \\
&\text{ iff } \forall n < \omega \forall \nabla \in W_{\mathfrak{A}}\ \mathfrak{V}(\Box^n \varphi) \in \nabla \\
&\text{ iff } \forall n < \omega \forall \nabla \in W_{\mathfrak{A}}\ \Box^n \varphi \in \nabla' \\
&\text{ iff } \forall n < \omega \forall \nabla \in W_{\mathfrak{A}}\ x_\nabla \models \Box^n \varphi \\
&\text{ iff } \mathfrak{U}(\varphi) = W.
\end{aligned}$$

(ii) $\Rightarrow$ (iii) Suppose $\Gamma \not\vdash_L^* \varphi$. Then by Theorem 7.73, there is an algebra $\mathfrak{A}$ for L and a valuation $\mathfrak{V}$ in it such that $\mathfrak{V}(\psi) = \top$, for all $\psi \in \Gamma$, and $\mathfrak{V}(\varphi) \neq \top$. Without loss of generality we may clearly assume $\mathfrak{A}$ to have $\leq \varkappa$ generators. Since L is $\varkappa$-complex, there is a Kripke frame $\mathfrak{F} = \langle W, R \rangle$ for L such that $\mathfrak{A}$ is (isomorphic to) a subalgebra of $\mathfrak{F}^+$ and we can consider $\mathfrak{V}$ as a valuation in $\mathfrak{F}$. Put $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. But then $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \varphi$.

(iii) $\Rightarrow$ (i) Suppose Γ is an L-consistent set of formulas and p a variable not occurring in Γ (here we use the fact that $\varkappa$ is infinite). Put

$$\Delta = \{p\} \cup \{\Box^n (p \to \varphi) : \varphi \in \Gamma,\ n < \omega\}$$

and show that Δ is L-consistent too. Indeed, let Δ' be a finite subset of Δ. Without loss of generality we may assume that Δ' consists of p and the formulas of the form $\Box^n (p \to \varphi)$, where $n < m$ and $\varphi \in \Gamma'$, for some $m < \omega$ and finite $\Gamma' \subseteq \Gamma$. Since Γ is L-consistent, there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on a frame for L and such that $x \models \Gamma$ for some x in $\mathfrak{F}$. Define a new valuation $\mathfrak{U}$ in $\mathfrak{F}$ by taking $\mathfrak{U}(p) = \mathfrak{V}(\bigwedge \Gamma')$ and $\mathfrak{U}(q) = \mathfrak{V}(q)$ for $q \neq p$. Under $\mathfrak{U}$ we clearly have $x \models \Delta'$. Consequently, Δ is L-consistent.

By the deduction theorem, it follows that $\{p \to \varphi : \varphi \in \Gamma\} \not\vdash_L^* \neg p$. Since L is strongly globally complete, there is a model $\mathfrak{N}$ based on a Kripke frame for L and such that $\mathfrak{N} \models p \to \varphi$, for all $\varphi \in \Gamma$, and $(\mathfrak{N}, x) \models p$, for some x in $\mathfrak{N}$. Therefore, $x \models \Gamma$, which completes the proof. ❑

In the intuitionistic case this theorem reduces to

Corollary 10.6 *A si-logic in a language with $\varkappa$ variables is strongly Kripke complete iff it is $\varkappa$-complex.*

As another consequence we show that the operator $\boldsymbol{\tau}$ preserves strong completeness. Suppose L is a strongly complete si-logic in the infinite language with $\varkappa$ variables and $\mathfrak{F}$ a descriptive frame for $\boldsymbol{\tau} L$ with $\leq \varkappa$ generators. Then $\boldsymbol{\rho}\mathfrak{F}$ is a descriptive frame for L with $\leq \varkappa$ generators. Since L is $\varkappa$-complex, there is a Kripke frame $\mathfrak{G}$ for L such that $\boldsymbol{\rho}\mathfrak{F}^+$ is a subalgebra of $\mathfrak{G}^+$. Let $\varkappa'$ be a cardinal which is bigger than $|\mathfrak{F}|$. Then the Kripke frame $\varkappa'\mathfrak{G}$ (obtained from $\mathfrak{G}$ by replacing its points with $\varkappa'$-point clusters) validates $\boldsymbol{\tau} L$ and it is not hard to check that $\mathfrak{F}^+$ is a subalgebra of $(\varkappa'\mathfrak{G})^+$. Thus, $\boldsymbol{\tau} L$ is $\varkappa$-complex and so strongly complete. That $\boldsymbol{\rho}$ preserves strong completeness can be proved using a syntactical argument; we leave this to the reader.

To establish that the canonical frame $\mathfrak{F}_L$ validates L, we showed in Section 5.2 that $\mathfrak{F}_L$ satisfies some first order sentence ϕ which characterizes the class of Kripke frames for L. Of course, nothing prevents us from trying to characterize classes of general frames by such kind of sentences. Given a class $\mathcal{C}$ of general frames, we say a logic L is *$\mathcal{C}$-elementary* if there is a set Φ of first order sentences (in the language with R and $=$ as its only predicates) such that, for every $\mathfrak{F} \in \mathcal{C}$, $\mathfrak{F}$ is a frame for L iff $\mathfrak{F}$ is a (classical) model for Φ. A first order sentence ϕ in R and $=$ says nothing about sets of possible values, and so a general frame $\mathfrak{F}$ satisfies ϕ iff $\boldsymbol{\kappa}\mathfrak{F}$ satisfies ϕ. Therefore, we have

Proposition 10.7 *If a logic L is $\mathcal{C}^\star$-elementary then it is $\mathcal{C}$-persistent.*

And now, combining Propositions 10.1 and 10.7, we arrive at

Theorem 10.8 *If a logic L is $\mathcal{C}$-complete and $\mathcal{C}^\star$-elementary then L is $\boldsymbol{\kappa}\mathcal{C}$-complete and in particular, Kripke complete.*

Thus, to prove the Kripke completeness of a logic L, it suffices to find a first-order characterization of frames for L in some class $\mathcal{C}^\star$ which is big enough to ensure $\mathcal{C}$-completeness of L. For instance, one can take as $\mathcal{C}$ any of the classes mentioned in Corollary 10.2.

Example 10.9 Suppose $L_0 \in \mathrm{NExt}\mathbf{K}$ is $\mathcal{T}$- (or $\mathcal{R}$- or $\mathcal{D}$-) persistent (and so Kripke complete). With the help of Theorem 10.8 we can prove that the logic $L = L_0 \oplus \Box p \to \Box\Box p$ is also Kripke complete.

Take the class $\mathcal{C}$ of all tight (or, respectively, refined or descriptive) frames for L_0 and establish that L is $\mathcal{C}^\star$-elementary. In fact, we show that, for every $\mathfrak{F} \in \mathcal{C}^\star$, $\mathfrak{F} \models L$ iff $\mathfrak{F}$ is transitive. Indeed, suppose $\mathfrak{F} = \langle W, R, P\rangle$ validates $\Box p \to \Box\Box p$ but there are points x, y, z in $\mathfrak{F}$ such that $xRyRz$ and $\neg xRz$. By the tightness of $\mathfrak{F}$, we then have a set $X \in P$ for which $x \in \Box X$ and $z \notin X$. Define a valuation $\mathfrak{V}$ on $\mathfrak{F}$ by taking $\mathfrak{V}(p) = X$. Then, under this valuation, $x \models \Box p$ but $x \not\models \Box\Box p$, since $z \not\models p$, which is a contradiction. The converse implication follows directly from Proposition 3.31.

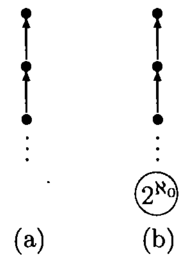


FIG. 10.1.

But how to find a first order equivalent of a given formula, if any, modulo some appropriate class of frames? In Sections 2.5 and 3.5 we used *ad hoc* techniques for obtaining first order equivalents of several particular formulas in the class of Kripke frames. However, it turns out that for an extensive family of modal formulas there is a purely mechanical procedure effectively constructing first order equivalents (in R and $=$) in the class $\mathcal{D}^*$, which immediately gives us $\mathcal{D}$-persistence, canonicity and strong Kripke completeness plus a first order characterization of Kripke frames. This result, known as Sahlqvist's theorem, will be proved in Section 10.3. But before that we establish a deep connection between the notions of elementarity, completeness and $\mathcal{D}$-persistence, which shows that the method of canonical models is applicable to all logics characterized by elementary classes of Kripke frames.

10.2 $\mathcal{D}$-persistence and elementarity

We consider first the modal case and then use the preservation theorem to transfer the main result to superintuitionistic logics.

The difference between ordinary Kripke frames and the underlying Kripke frames of descriptive frames is that the latter may be regarded as the sets of ultrafilters over the world spaces of the former. Given a Kripke frame $\mathfrak{F} = \langle W, R \rangle$, the Kripke frame $\kappa(\mathfrak{F}^+)_+$ is called the *ultrafilter extension* of $\mathfrak{F}$ and denoted by $\widehat{\mathfrak{F}} = \left\langle \widehat{W}, \widehat{R} \right\rangle$. We remind the reader that $\widehat{W}$ is the set of ultrafilters in $\mathfrak{F}^+$ (i.e., in the Boolean algebra with the universe 2^W) and, for all $u_1, u_2 \in \widehat{W}$, $u_1 \widehat{R} u_2$ iff $\forall X \subseteq W\ (\Box X \in u_1 \to X \in u_2)$.

Example 10.10 Let $\mathfrak{F} = \langle W, R \rangle$ be the frame depicted in Fig. 10.1 (a). Then $\widehat{\mathfrak{F}}$ is of the form shown in Fig. 10.1 (b), i.e., $\widehat{\mathfrak{F}}$ can be obtained from $\mathfrak{F}$ by adding to it a continual root cluster. Indeed, the set $\widehat{W}$ consists of two types of ultrafilters: principal and non-principal. Principal ultrafilters are sets of the form $\widehat{a} = \{X \subseteq W :\ a \in X\}$, where $a \in W$. Every non-principal ultrafilter must contain all cofinite subsets in W; such ultrafilters will be denoted by the letters u and v.

Observe first that, for every u, v and $\widehat{a}$, we have $u\widehat{R}\widehat{a}$ and $u\widehat{R}v$. For suppose $\Box X \in u$. Since $\emptyset \notin u$, the set $\Box X$ is not empty. Hence, $\Box X$ must be infinite (because every non-principal ultrafilter contains only infinite sets) and so there is only one possibility: $\Box X = W$, from which $X = W$, $X \in \widehat{a}$ and $X \in v$. It is easily seen that $\widehat{a}\widehat{R}\widehat{b}$ iff aRb. That the root cluster in $\widehat{\mathfrak{F}}$ contains a continuum of points is proved in the next theorem.

Clearly, every finite frame is isomorphic to its ultrafilter extension. However, the ultrafilter extensions of infinite frames are essentially different.

Theorem 10.11 *If a frame $\mathfrak{F}$ is denumerable then $\widehat{\mathfrak{F}}$ is continual.*

Proof It is sufficient to show that over a denumerable set W there is at least a continuum of ultrafilters. Let $W = \{a_0, a_1, a_2, \dots\}$. Construct the sets $X_0 = \{a_0, a_2, a_4, \dots\}$, $X_1 = \{a_1, a_3, a_5, \dots\}$. Notice that they cannot belong to the same ultrafilter, since $X_0 = W - X_1$.

Suppose now that we have already constructed infinite sets

$$X_{i_1}, X_{i_1 i_2}, \dots, X_{i_1 i_2 \dots i_k}$$

such that $X_{i_1} \supset X_{i_1 i_2} \supset \dots \supset X_{i_1 i_2 \dots i_k}$ and

$$X_{i_1 i_2 \dots i_k} = \{a_{j_0}, a_{j_1}, a_{j_2}, \dots\},\ j_0 < j_1 < j_2 < \dots.$$

Consider the sets

$$X_{i_1 i_2 \dots i_k 0} = \{a_{j_0}, a_{j_2}, a_{j_4}, \dots\},\quad X_{i_1 i_2 \dots i_k 1} = \{a_{j_1}, a_{j_3}, a_{j_5}, \dots\}.$$

Since they are disjoint, they cannot belong to the same ultrafilter.

Let $i_1 i_2 i_3 \dots$ and $j_1 j_2 j_3 \dots$ be distinct infinite words in the alphabet $\{0, 1\}$. By the construction, the sets

$$\{X_{i_1}, X_{i_1 i_2}, X_{i_1 i_2 i_3}, \dots\},\ \{X_{j_1}, X_{j_1 j_2}, X_{j_1 j_2 j_3}, \dots\}$$

cannot belong simultaneously to the same ultrafilter. It remains to notice that there are a continuum of sets of that sort, each of them has the finite intersection property and so belong to an ultrafilter over W. ❑

On the other hand, we obviously have

Proposition 10.12 *A frame $\mathfrak{F}$ is isomorphic to a subframe of $\widehat{\mathfrak{F}}$, with the map $x \mapsto \widehat{x}$ being an isomorphism.*

Of course, in general $\mathfrak{F}$ is not a generated subframe of $\widehat{\mathfrak{F}}$. Take, for instance, the frame $\mathfrak{F} = \langle \omega, < \rangle$. Then every point in ω sees a point in $\widehat{W}$. In the ultrafilter extensions of $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{R}, < \rangle$ "old" and "new" points are heavily mixed. However, in some cases we can determine the position of $\mathfrak{F}$ in $\widehat{\mathfrak{F}}$ perfectly well.

Theorem 10.13 *Suppose $\mathfrak{F} = \langle W, R \rangle$ is a transitive frame all points in which are of finite depth and, for every $d < \omega$, $W^{=d}$ is finite. Then $\mathfrak{F}$ is (isomorphic to) a generated subframe of $\widehat{\mathfrak{F}}$, with $W = W^{<\infty} = \widehat{W}^{<\infty}$.*

Proof Let u be a non-principal ultrafilter over W and $\widehat{a}$ a principal one, for $a \in W$. Clearly, $a \in \Box(a\uparrow) \in \widehat{a}$. Since $a\uparrow$ is finite, $a\uparrow \notin u$ and so $\widehat{a}$ does not see u in $\widehat{\mathfrak{F}}$. Therefore, in view of Proposition 10.12, $\mathfrak{F}$ is a generated subframe of $\widehat{\mathfrak{F}}$.

We show that every non-principal ultrafilter u in $\widehat{\mathfrak{F}}$ sees points from $\mathfrak{F}$ of any finite depth. Let us begin with points of depth 1. Suppose $a_1, \dots, a_n$ are all such points and u sees none of them. This means that there are sets $X_1, \dots, X_n \subseteq W$ such that $\Box X_i \in u$ (and so $\Box(X_1 \cap \dots \cap X_n) \in u$), but $X_i \notin \widehat{a}_i$, i.e. $a_i \notin X_i$, for $1 \leq i \leq n$. Let $X = X_1 \cap \dots \cap X_n$. By the definition, $\Box X$ is infinite and $a_i \notin X$. Therefore, there are infinitely many points in $\mathfrak{F}$ which see only points in X. It follows that $X \neq \emptyset$, for otherwise $\mathfrak{F}$ would contain infinitely many dead ends. But if a point x in $\Box X$ sees any point in X then (by transitivity) xRa_i, for some $i \leq n$, and so $a_i \in X$, which is a contradiction.

Let us prove now that u sees points $\widehat{a}$ of an arbitrary finite depth. Suppose otherwise, i.e., u does not see points of depth $> m$, and let $a_1, \dots, a_n$ be all the points in $\mathfrak{F}$ of depth $m+1$. Suppose also that $X_1, \dots, X_n$ are such that $\Box X \in u$ and $a_i \notin X$, for $1 \leq i \leq n$, where again $X = X_1 \cap \dots \cap X_n$. The set X consists of points of depth $\leq m$, for otherwise a point in $\Box X$ would see one of a_i, which means that $a_i \in X$. By the definition, there are finitely many points seeing only points of depth $\leq m$. So the set $\Box X$ is finite, contrary to u being a non-principal ultrafilter. ❑

The requirement of finiteness of $W^{=d}$ in Theorem 10.13 is essential. Without it the result does not hold: the ultrafilter extension of $\langle \omega, \emptyset \rangle$ is just a continual set of mutually inaccessible points. However, we clearly have

Corollary 10.14 *If a Kripke frame $\mathfrak{F}$ is transitive and each of its points has a finite number of successors then $\mathfrak{F} \subseteq\!\!\!\!\rightarrow \widehat{\mathfrak{F}}$.*

Theorem 10.15 *If $\mathfrak{F}$ is a transitive rooted frame then $\widehat{\mathfrak{F}}$ is also rooted.*

Proof Suppose $\mathfrak{F} = \langle W, R \rangle$ and $a\uparrow = W$. We show that $\widehat{a}\widehat{R}x$ for every $x \in \widehat{W}$. If $x = \widehat{b}$, for some $b \in W$, then $\widehat{a}\widehat{R}\widehat{b}$ follows from Proposition 10.12. Let u be a non-principal ultrafilter over W. Take any set $\Box X$ in $\widehat{a}$. Then $a \in \Box X$, from which $X = W$ or $X = W - \{a\}$. In both cases X is in u (as well as in any other non-principal ultrafilter). Therefore, $\widehat{a}\widehat{R}u$. ❑

The following example demonstrates that the requirement of transitivity was essential in Theorem 10.15.

Example 10.16 Let $\mathfrak{F} = \langle \omega, R \rangle$, where $R = \{\langle n, n+1 \rangle : n \in \omega\}$. We show that $\widehat{\mathfrak{F}}$ is not rooted. Observe first that there is no point x in $\widehat{\mathfrak{F}}$ such that $x\widehat{R}\widehat{0}$. Indeed, since $\Box(\omega - \{0\}) = \omega$, we have $\Box(\omega - \{0\}) \in x$ for every x in $\widehat{\mathfrak{F}}$; however, $\omega - \{0\} \notin \widehat{0}$. Thus, if $\widehat{\mathfrak{F}}$ is rooted then $\widehat{0}$ is its root.

Now we show that, for every $n < \omega$, $\widehat{n}\widehat{R}x$ implies $x = \widehat{n+1}$. Since $\Box\{n+1\} = \{n\}$, we have $\Box\{n+1\} \in \widehat{n}$ and then $\widehat{n}\widehat{R}x$ means $\{n+1\} \in x$, i.e., $x = \widehat{n+1}$ (if an ultrafilter contains a singleton then it is generated by the singleton). Hence $\widehat{0}\widehat{R}^n x$ means $x = \widehat{n}$. Ergo $\widehat{0}$ cannot be a root of $\widehat{\mathfrak{F}}$, because the cardinality of $\widehat{\mathfrak{F}}$ is that of the continuum.

After clarifying to some extent the relation between frames and their ultrafilter extensions, let us return to modal logics.

Proposition 10.17 *For every Kripke frame $\mathfrak{F}$ and every logic L, $\widehat{\mathfrak{F}} \models L$ implies $\mathfrak{F} \models L$.*

Proof According to Corollary 8.25, $\mathfrak{F}^+$ is a subalgebra of $(\widehat{\mathfrak{F}})^+$. ❑

It follows immediately from the definition of ultrafilter extension that for a $\mathcal{D}$-persistent logic the converse is also true.

Proposition 10.18 *For every Kripke frame $\mathfrak{F}$ and every $\mathcal{D}$-persistent logic L, $\mathfrak{F} \models L$ implies $\widehat{\mathfrak{F}} \models L$.*

We are in a position now to prove the main result of this section.

Theorem 10.19. (The Fine–van Benthem theorem) *If a logic $L \in$ NExtK is characterized by an elementary class $\mathcal{C}$ of Kripke frames then L is $\mathcal{D}$-persistent.*

Proof We consider here only the case of elementary L, i.e., $\mathcal{C}$ is assumed to be the class of all Kripke frames for L. The general case is left to the reader (for a hint see Exercise 10.11).

Let Φ be a set of first order sentences in the language $\mathcal{L}_1$ with R and $=$ as its only predicate symbols such that, for every Kripke frame $\mathfrak{F}$, $\mathfrak{F} \in \mathcal{C}$ iff $\mathfrak{F} \models \Phi$. Take any $\mathfrak{F} = \langle W, R\rangle$ in $\mathcal{C}$ and enrich the language $\mathcal{L}_1$ with the unary predicate P_X, for every $X \subseteq W$, and the individual constant c_a, for every $a \in W$. We will interpret $P_X(x)$ in $\mathfrak{F}$ as $x \in X$, c_a as a and instead of c_a write simply a if understood. Let Φ' be the set of sentences in the enriched language $\mathcal{L}_1'$ that are true in $\mathfrak{F}$. Clearly, we have $\Phi \subseteq \Phi'$ and for every sentence ϕ in $\mathcal{L}_1'$, either $\phi \in \Phi'$ or $\neg\phi \in \Phi'$ (in particular, if ϕ is a sentence in $\mathcal{L}_1$ and $\phi \notin \Phi$ then $\neg\phi \in \Phi'$).

After that we again extend our first order language in the following way. Let Π be a set of formulas in $\mathcal{L}_1'$ with one free variable x such that, for each finite subset Π' of Π, there is a point a in $\mathfrak{F}$ at which all the formulas $\phi(x) \in \Pi'$ are satisfied, i.e., $\mathfrak{F} \models \phi(a)$. We associate with each such Π a new individual constant c, add it to $\mathcal{L}_1'$, thus obtaining a language $\mathcal{L}_1''$, and add $\phi(c)$ to Φ' for every $\phi(x) \in \Pi$, thus obtaining a new set Φ''.

Note that for each ultrafilter u over W, we have introduced a new constant—denote it by c_u—such that $P_X(c_u) \in \Phi''$ for all $X \in u$. Indeed, since u has the finite intersection property, for every finite subset $\{P_{X_1}(x), \dots, P_{X_n}(x)\}$ of $\{P_X(x) : X \in u\}$, there must be a point $a \in W$ such that $a \in X_1 \cap \dots \cap X_n$, i.e., $\mathfrak{F} \models P_{X_1 \cap \dots \cap X_n}(a)$ or, in other words, $\mathfrak{F} \models P_{X_1}(a), \dots, \mathfrak{F} \models P_{X_n}(a)$.

Since every finite subset of Φ'' has a model (e.g. $\mathfrak{F}$), by the compactness theorem of classical model theory, Φ'' also has a model, say $\mathfrak{F}' = \langle W', R'\rangle$. Clearly, $\mathfrak{F}' \in \mathcal{C}$ and $\mathfrak{F} \subseteq \mathfrak{F}'$.

Define a map f from W' into $\widehat{W}$ by taking, for each $a \in W'$,

$$f(a) = \{X \subseteq W : \mathfrak{F}' \models P_X(x)[a]\}$$

and show that f is a reduction of $\mathfrak{F}'$ to $\widehat{\mathfrak{F}}$, from which it will follow that $\widehat{\mathfrak{F}} \models L$.

Let us check first that $f(a)$ is an ultrafilter over W, i.e., it satisfies the conditions (4a)–(4c) in Theorem 7.23 and, for every $X \subseteq W$, either X or $W - X$ is in $f(a)$. (4a) follows from $\mathfrak{F}' \models P_W(a)$ (since $\forall x\ P_W(x) \in \Phi'$); (4b) is ensured by $\mathfrak{F}' \models P_X(a) \wedge P_Y(a) \leftrightarrow P_{X\cap Y}(a)$. $\mathfrak{F}' \models P_X(a) \to P_Y(a)$, for any $X \subseteq Y$, implies (4c). Finally, suppose $X \notin f(a)$, i.e., $\mathfrak{F}' \models \neg P_X(a)$. Then, using $\mathfrak{F}' \models \neg P_X(a) \leftrightarrow P_{W-X}(a)$, we obtain $\mathfrak{F}' \models P_{W-X}(a)$ and so $W - X \in f(a)$.

Now we show that f is a surjection. Let $u \in \widehat{W}$. If u is a principal ultrafilter, i.e., $u = \widehat{a}$ for some $a \in W$, then clearly $f(a) = \widehat{a}$. And if u is not principal then we have $\mathfrak{F}' \models P_X(c_u)$ iff $X \in u$, and so $u = f(c_u)$.

It remains to verify the reduction conditions (R1) and (R2). Suppose $aR'b$ and $\Box X \in f(a)$, i.e., $\mathfrak{F}' \models P_{\Box X}(a)$. Since the formula

$$\forall x \forall y\ (P_{\Box X}(x) \wedge xRy \to P_X(y))$$

is true in $\mathfrak{F}$, it must be also true in $\mathfrak{F}'$. Therefore, $\mathfrak{F}' \models P_X(b)$, i.e., $X \in f(b)$, and so $f(a)\widehat{R}f(b)$. Thus, (R1) holds. To verify (R2), suppose $f(a)\widehat{R}f(b)$ and show that there is a point c in $\mathfrak{F}'$ such that $aR'c$ and $f(c) = f(b)$. Consider the set of conditions $P_X(z)$, for $X \in f(b)$, and aRz. Every finite subset of this set, say $\{P_{X_1}(z), \dots, P_{X_n}(z), aRz\}$, or equivalently $\{P_X(z), aRz\}$, for $X = X_1 \cap \dots \cap X_n$, is satisfied in $\mathfrak{F}$ at some point z. Indeed, suppose that $\mathfrak{F} \not\models P_X(z) \wedge aRz$ for any $z \in W$, i.e., $\mathfrak{F} \models \forall z\ (aRz \to \neg P_X(z))$. Then $\mathfrak{F} \models \forall z\ (aRz \to P_{W-X}(z))$, whence $\Box(W - X) \in f(a)$ and so, since $f(a)\widehat{R}f(b)$, we have $W - X \in f(b)$, contrary to $X \in f(b)$. By the definition of $\mathfrak{F}'$, there is $c \in W'$ such that $aR'c$ and $\mathfrak{F}' \models P_X(c)$, for all $X \in f(b)$. It remains to establish $f(c) = f(b)$. The inclusion $f(b) \subseteq f(c)$ is evident. Suppose $X \in f(c)$, i.e., $\mathfrak{F}' \models P_X(c)$. If $X \notin f(b)$ then $W - X \in f(b)$ and so $\mathfrak{F}' \models P_{W-X}(c)$, contrary to $\mathfrak{F}' \models P_X(c)$ and $f(c)$ being a proper filter. Therefore, $f(c) \subseteq f(b)$.

Thus, we have showed that $\widehat{\mathfrak{F}} \in \mathcal{C}$ whenever $\mathfrak{F} \in \mathcal{C}$. To complete the proof of the Fine–van Benthem theorem, suppose $\mathfrak{G}$ is a descriptive frame for L and show that $\kappa\mathfrak{G} \models L$.

Algebraically $\mathfrak{G} \models L$ means that $\mathfrak{G}^+ \in \mathrm{Var}L$ and so, by Tarski's theorem (Theorem 7.80), $\mathfrak{G}^+ \in \mathrm{HSP}\mathcal{C}^+$, where $\mathcal{C}^+ = \{\mathfrak{F}^+ :\ \mathfrak{F} \in \mathcal{C}\}$. Clearly, $\mathcal{C}$ is closed under disjoint unions and so, by Theorem 8.75, $\mathcal{C}^+$ is closed under direct products, i.e., $\mathrm{P}\mathcal{C}^+ = \mathcal{C}^+$. Hence, $\mathfrak{G}^+ \in \mathrm{HS}\mathcal{C}^+$. This means that $\mathfrak{G}^+$ is a homomorphic image of an algebra $\mathfrak{A}$ which, in turn, is a subalgebra of some $\mathfrak{B} \in \mathcal{C}^+$. By Theorem 8.59, $\mathfrak{G}$ is a generated subframe of $\mathfrak{A}_+$ and, by Theorem 8.71, $\mathfrak{A}_+$ is a reduct of $\mathfrak{B}_+$. Since $\mathfrak{F} \in \mathcal{C}$ implies $\widehat{\mathfrak{F}} \in \mathcal{C}$, we may assume that $\mathfrak{B}_+$ is isomorphic to a descriptive frame $\mathfrak{H}$ such that $\kappa\mathfrak{H} \models L$. Then $\kappa\mathfrak{A}_+$ is a reduct of $\kappa\mathfrak{H}$, from which $\kappa\mathfrak{A}_+ \models L$ by the reduction theorem, and $\kappa\mathfrak{G}$ is a generated subframe of $\kappa\mathfrak{A}_+$, from which $\kappa\mathfrak{G} \models L$ by the generation theorem. ❑

To transfer the Fine–van Benthem theorem to si-logics we require one more preservation theorem.

Theorem 10.20 *If a logic $L \in$ ExtInt is characterized by an elementary class of Kripke frames then τL is also characterized by an elementary class of Kripke frames.*

Proof Suppose L is characterized by a class $\mathcal{C}$ of intuitionistic Kripke frames and Φ is a set of first order formulas (in $=$ and R) such that, for any Kripke frame $\mathfrak{F}$, $\mathfrak{F} \in \mathcal{C}$ iff $\mathfrak{F} \models \Phi$. Of course, we may assume Φ to contain the axioms of partial order. By Theorem 9.70 (in which we take $\tau_k \langle W, R\rangle = \langle kW, kR\rangle$) and the skeleton lemma, τL is characterized by the class $\mathcal{C}'$ of all quasi-ordered Kripke frames $\mathfrak{G}$ such that $\rho\mathfrak{G} \in \mathcal{C}$. We show that $\mathcal{C}'$ is elementary, namely $\mathcal{C}'$ is the class of models for the set $\Phi' = \{\phi' : \ \phi \in \Phi\}$ of first order formulas, where ϕ' is obtained from ϕ by replacing every subformula of the form $x = y$ with $xRy \wedge yRx$. Indeed, under this transformation the axioms of partial order become the axioms of quasi-order. Besides, by induction on the construction of $\phi(x_1, \dots, x_n)$ it is easy to prove that, for every quasi-order $\mathfrak{G}$ and all points $a_1, \dots, a_n$ in $\mathfrak{G}$, $\mathfrak{G} \models \phi'(a_1, \dots, a_n)$ iff $\rho\mathfrak{G} \models \phi(C(a_1), \dots, C(a_n))$. It follows immediately that $\mathfrak{G} \in \mathcal{C}'$ iff $\rho\mathfrak{G} \in \mathcal{C}$ iff $\rho\mathfrak{G} \models \Phi$ iff $\mathfrak{G} \models \Phi'$. ❑

Corollary 10.21 *If a si-logic is elementary then its smallest modal companion is also elementary.*

With the help of this result and the fact that the operators ρ and τ preserve $\mathcal{D}$-persistence we can easily prove the intuitionistic variant of the Fine–van Benthem theorem.

Theorem 10.22 *If a si-logic L is characterized by an elementary class of Kripke frames then L is $\mathcal{D}$-persistent.*

Proof According to Theorem 10.20, τL is characterized by an elementary class of Kripke frames and so, by the Fine–van Benthem theorem, it is $\mathcal{D}$-persistent. By the preservation theorem, $\rho\tau L = L$ is $\mathcal{D}$-persistent too. ❑

The question as to whether the converse of the Fine–van Benthem theorem holds (in both modal and intuitionistic cases) remains open. Of course, $\mathcal{D}$-persistence implies Kripke completeness; but we need the completeness with respect to an elementary class. There is an example of a logic (see Exercise 10.10) which is $\mathcal{D}$-persistent but not elementary; yet, it is characterized by an elementary subclass of the whole class of its Kripke frames. On the other hand the following remarkable result holds:

Theorem 10.23 *If a (normal modal or superintuitionistic) logic is $\mathcal{R}$-persistent then it is elementary.*

We leave it here without a proof because too much classical model theory is involved in it. As is shown by Exercise 10.4, the converse of Theorem 10.23 does not hold.

10.3 Sahlqvist's theorem

In this section we consider a method which, given a modal formula φ in a rather big family, constructs effectively a first order formula in R and $=$ characterizing descriptive and Kripke frames validating φ.

First we remind the reader that, given a formula $\varphi(p_1, \dots, p_n)$ (whose variables are listed among $p_1, \dots, p_n$), a frame $\mathfrak{F} = \langle W, R, P\rangle$ and sets $X_1, \dots, X_n$ in P, we denote by $\varphi(X_1, \dots, X_n)$ the set of points in $\mathfrak{F}$ at which φ is true under the valuation $\mathfrak{V}$ defined by $\mathfrak{V}(p_i) = X_i$, for $i = 1, \dots, n$, i.e., $\varphi(X_1, \dots, X_n) = \mathfrak{V}(\varphi)$. Using this notation, we can say that

$$(\mathfrak{F}, x) \models \varphi(p_1, \dots, p_n) \text{ iff } \forall X_1, \dots, X_n \in P\ x \in \varphi(X_1, \dots, X_n),$$

$$\mathfrak{F} \models \varphi(p_1, \dots, p_n) \text{ iff } \forall x \in W \forall X_1, \dots, X_n \in P\ x \in \varphi(X_1, \dots, X_n).$$

Example 10.24 Let us imagine that we do not yet know anything about first order equivalents of the formula $\Box p \to p$ in the class of, say, tight frames and let us try to extract such an equivalent directly from the equivalences above and properties of those frames. Then for any tight frame $\mathfrak{F} = \langle W, R, P\rangle$ we shall have:

$$\begin{aligned}(\mathfrak{F}, x) \models \Box p \to p &\text{ iff } \forall X \in P\ x \in (\Box X \to X)\\ &\text{ iff } \forall X \in P\ (x \in \Box X \to x \in X)\\ &\text{ iff } \forall X \in P\ (x{\uparrow} \subseteq X \to x \in X),\end{aligned}$$

since, as we know, for every $n \geq 0$, $x \in \Box^n X$ iff $x{\uparrow}^n \subseteq X$.

We are now at a crucial point. To eliminate the variable X ranging over P, we can use two simple observations. The first one is purely set-theoretic:

$$\forall X \in P\ (Y \subseteq X \to x \in X) \text{ iff } x \in \bigcap\{X \in P :\ Y \subseteq X\}. \tag{10.1}$$

And the second one is the characteristic property of tight frames formulated in Proposition 8.41:

$$\bigcap\{X \in P :\ x{\uparrow} \subseteq X\} = x{\uparrow}. \tag{10.2}$$

With the help of (10.1) and (10.2) we can continue the chain of equivalences above with two more lines:

$$\begin{aligned}(\mathfrak{F}, x) \models \Box p \to p &\text{ iff } \dots\\ &\text{ iff } x \in \bigcap\{X \in P :\ x{\uparrow} \subseteq X\}\\ &\text{ iff } x \in x{\uparrow}.\end{aligned}$$

Therefore, $\mathfrak{F} \models \Box p \to p$ iff $\forall x\ x \in x{\uparrow}$. It remains to notice that the last formula means nothing else but the reflexivity and can be rewritten in the more familiar way as $\forall x\ xRx$.

It would be strange if such a nice technique could not be extended to some other formulas. In fact, it can be considerably generalized.

Recall first that, by Exercise 8.1, we can replace $x\uparrow$ in (10.2) with any term of the form $x_1\uparrow^{n_1} \cup \ldots \cup x_k\uparrow^{n_k}$, thus obtaining the equality

$$\bigcap\{X \in P : x_1\uparrow^{n_1} \cup \ldots \cup x_k\uparrow^{n_k} \subseteq X\} = x_1\uparrow^{n_1} \cup \ldots \cup x_k\uparrow^{n_k} \tag{10.3}$$

which holds for every tight frame $\mathfrak{F} = \langle W, R, P\rangle$, every $x_1, \ldots, x_k \in W$ and every $n_1, \ldots, n_k \geq 0$.

A frame-theoretic term $x_1\uparrow^{n_1} \cup \ldots \cup x_k\uparrow^{n_k}$ with (not necessarily distinct) world variables $x_1, \ldots, x_k$ will be for brevity called an *R-term*. In this section we reserve the letter T for denoting R-terms. Observe that the relation $x \in T$ on $\mathfrak{F} = \langle W, R, P\rangle$ is first order expressible in the predicates R and $=$. Indeed, if $T = x_1\uparrow^{n_1} \cup \ldots \cup x_k\uparrow^{n_k}$, $k > 0$ and $n_1, \ldots, n_k > 0$ then

$$\begin{aligned} x \in T \text{ iff } & \exists y_1^1, \ldots, y_{n_1-1}^1 (x_1 R y_1^1 \wedge y_1^1 R y_2^1 \wedge \ldots \wedge y_{n_1-1}^1 R x) \\ & \vee \ldots \vee \\ & \exists y_1^k, \ldots, y_{n_k-1}^k (x_k R y_1^k \wedge y_1^k R y_2^k \wedge \ldots \wedge y_{n_k-1}^k R x); \end{aligned}$$

if some n_i is 0 then the corresponding disjunct has the form $x = x_i$ and when $k = 0$ we have $x \in T$ iff $x \in \emptyset$ iff $x \neq x$. This observation gives us the following

Lemma 10.25 *Suppose $\varphi(p_1, \ldots, p_n)$ is a modal formula and $T_1, \ldots, T_n$ are R-terms. Then the relation $x \in \varphi(T_1, \ldots, T_n)$ is expressible by a first order formula (in R and $=$) having x as its only free variable.*

Proof By induction on the construction of φ. The basis of induction follows from the observation above, and first order equivalents of compound formulas are constructed in the same way as in the definition of the standard translation ST in Section 4.3. ❑

Syntactically, R-terms with a single world variable correspond to modal formulas of the form $\Box^{m_1}p_1 \wedge \ldots \wedge \Box^{m_k}p_k$ with not necessarily distinct propositional variables $p_1, \ldots, p_k$. Such formulas are called *strongly positive formulas.*

Lemma 10.26 *Suppose $\varphi(p_1, \ldots, p_n)$ is a strongly positive formula containing all the variables $p_1, \ldots, p_n$ and $\mathfrak{F} = \langle W, R, P\rangle$ is a frame. Then one can effectively construct R-terms $T_1, \ldots, T_n$ (of one variable x) such that for any $x \in W$ and any $X_1, \ldots, X_n \in P$,*

$$x \in \varphi(X_1, \ldots, X_n) \text{ iff } T_1 \subseteq X_1 \wedge \ldots \wedge T_n \subseteq X_n.$$

Proof The proof proceeds by induction on the number of conjuncts in φ. If $\varphi(p_1) = \Box^m p_1$ then we have

$$\begin{aligned} x \in \varphi(X_1) &\text{ iff } x \in \Box^m X_1 \\ &\text{ iff } x\uparrow^m \subseteq X_1. \end{aligned}$$

Suppose now that $\varphi(p_1, \dots, p_n) = \psi(p_1, \dots, p_n) \wedge \Box^m p_i$ where $\psi(p_1, \dots, p_n)$ is a strongly positive formula with $\leq k$ conjuncts and $1 \leq i \leq n$. Then we have R-terms $T_1, \dots, T_n$ of one variable x such that

$$\begin{aligned} x \in \varphi(X_1, \dots, X_n) &\text{ iff } T_1 \subseteq X_1 \wedge \ldots \wedge T_n \subseteq X_n \wedge x{\uparrow}^m \subseteq X_i \\ &\text{ iff } T_1 \subseteq X_1 \wedge \ldots \wedge T_i \cup x{\uparrow}^m \subseteq X_i \wedge \ldots \wedge T_n \subseteq X_n. \end{aligned}$$

❑

Now, trying to extend the method of Example 10.24 to a wider class of formulas, we see that it still works if we replace the antecedent $\Box p$ in $\Box p \to p$ with an arbitrary strongly positive formula ψ. As to generalizations of the consequent, let us take first an arbitrary formula χ instead of p and see what properties it should satisfy to be handled by our method.

Thus, for a modal formula $(\psi \to \chi)(p_1, \dots, p_n)$ with strongly positive ψ and a tight frame $\mathfrak{F} = \langle W, R, P \rangle$, we have:

$$\begin{aligned} (\mathfrak{F}, x) \models \psi \to \chi &\text{ iff } \forall X_1, \dots, X_n \in P\ (x \in \psi(X_1, \dots, X_n) \to \\ &\qquad x \in \chi(X_1, \dots, X_n)) \\ \text{(by Lemma 10.26)} &\text{ iff } \forall X_1, \dots, X_n \in P\ (T_1 \subseteq X_1 \wedge \ldots \wedge T_n \subseteq X_n \to \\ &\qquad x \in \chi(X_1, \dots, X_n)) \\ &\text{ iff } \forall X_1, \dots, X_{n-1} \in P\ (T_1 \subseteq X_1 \wedge \ldots \wedge T_{n-1} \subseteq X_{n-1} \to \\ &\qquad \forall X_n \in P\ (T_n \subseteq X_n \to x \in \chi(X_1, \dots, X_n))). \end{aligned}$$

(10.1) does not help us here, but we can readily generalize it to

$$\begin{aligned} &\forall X \in P\ (Y \subseteq X \to x \in \chi(\dots, X, \dots)) \text{ iff} \\ &\qquad x \in \bigcap\{\chi(\dots, X, \dots) :\ Y \subseteq X \in P\}. \end{aligned} \tag{10.4}$$

So

$$\begin{aligned} (\mathfrak{F}, x) \models \psi \to \chi &\text{ iff } \forall X_1, \dots, X_{n-1} \in P\ (T_1 \subseteq X_1 \wedge \ldots \wedge T_{n-1} \subseteq X_{n-1} \to \\ &\qquad x \in \bigcap\{\chi(X_1, \dots, X_n) :\ T_n \subseteq X_n \in P\}). \end{aligned}$$

(Note that if p_n does not occur in ψ, and so the conjunct $T_n \subseteq X_n$ is missing, we can always insert the new conjunct $X_n \subseteq X_n$.) But now (10.2) and (10.3) are useless. In fact, what we need is the equality

$$\bigcap\{\chi(\dots, X, \dots) :\ T \subseteq X \in P\} = \chi(\dots, \bigcap\{X \in P :\ T \subseteq X\}, \dots) \tag{10.5}$$

which, with the help of (10.3), would give us

$$\bigcap\{\chi(\dots, X, \dots) :\ T \subseteq X \in P\} = \chi(\dots, T, \dots). \tag{10.6}$$

Of course, (10.5) is too good to hold for an arbitrary χ, but suppose for a moment that our χ satisfies it. Then we can eliminate step by step all the variables $X_1, \dots, X_n$ like this:

$$(\mathfrak{F}, x) \models \psi \to \chi \text{ iff } \forall X_1, \dots, X_{n-1} \in P\ (T_1 \subseteq X_1 \wedge \dots \wedge T_{n-1} \subseteq X_{n-1} \to$$
$$x \in \chi(X_1, \dots, X_{n-1}, T_n))$$
$$\text{iff} \dots \text{ (by the same argument)}$$
$$\text{iff } x \in \chi(T_1, \dots, T_n).$$

And the last relation can be effectively rewritten in the form of a first order formula $\phi(x)$ in R and $=$ having x as its only free variable. So finally we shall have $\mathfrak{F} \models \psi \to \chi$ iff $\forall x\ \phi(x)$.

Now, to satisfy (10.5) χ should have the property that all its operators could be distributed over intersections. Clearly, $\to$ and $\neg$ are not suitable for this goal. But all the other operators, as it will be shown below, turn out to be good enough at least in descriptive and Kripke frames. So we can take as χ any *positive* modal formula which may contain only $\bot$, $\top$, $\wedge$, $\vee$, $\Box$ and $\Diamond$. The main property of an arbitrary positive formula $\varphi(\dots, p, \dots)$ is its *monotonicity* in every variable p, which means that, for all subsets X, Y of worlds in a frame $\mathfrak{F}$, $X \subseteq Y$ implies $\varphi(\dots, X, \dots) \subseteq \varphi(\dots, Y, \dots)$ (see Exercise 3.20).

To prove that all positive formulas satisfy (10.5) in the class $\mathcal{D}^*$ of descriptive and Kripke frames, we require a lemma. A family $\mathcal{X}$ of non-empty subsets of some space W is called *downward directed* if for every $X, Y \in \mathcal{X}$ there is $Z \in \mathcal{X}$ such that $Z \subseteq X \cap Y$. Note that every downward directed family has the finite intersection property.

Lemma 10.27. (Esakia's lemma) *Suppose $\mathfrak{F} = \langle W, R, P \rangle$ is a descriptive frame. Then for every downward directed family $\mathcal{X} \subseteq P$,*

$$\Big(\bigcap_{X \in \mathcal{X}} X\Big)\downarrow = \bigcap_{X \in \mathcal{X}} (X\downarrow).$$

Proof The inclusion $(\bigcap_{X \in \mathcal{X}} X)\downarrow \subseteq \bigcap_{X \in \mathcal{X}} X\downarrow$ is quite clear. So suppose that $x \in \bigcap_{X \in \mathcal{X}} X\downarrow$, i.e., $x \in X\downarrow$ and so $x\uparrow \cap X \neq \emptyset$ for every $X \in \mathcal{X}$. It follows that the family $\{x\uparrow\} \cup \mathcal{X}$ has the finite intersection property. Since $\mathfrak{F}$ is tight, $x\uparrow = \bigcap\{X \in P : x\uparrow \subseteq X\}$ and so the family $\{X \in P : x\uparrow \subseteq X\} \cup \mathcal{X}$ has the finite intersection property as well. By the compactness of $\mathfrak{F}$, we then have $x\uparrow \cap \bigcap_{X \in \mathcal{X}} X \neq \emptyset$, from which $x \in (\bigcap_{X \in \mathcal{X}} X)\downarrow$. ❑

This lemma means that $\Diamond$ in every tight and compact frame $\mathfrak{F} = \langle W, R, P \rangle$ distributes over the intersection of any downward directed subset of P. And as we already know (see Exercise 8.2), the necessity operator $\Box$ distributes in every frame over the intersection of an arbitrary family $\{X_i : i \in I\}$ of subsets of W, that is $\Box \bigcap_{i \in I} X_i = \bigcap_{i \in I} \Box X_i$.

Lemma 10.28. (Intersection) *Suppose $\varphi(p, \dots, q, \dots, r)$ is a positive formula and $\mathfrak{F} = \langle W, R, P\rangle \in \mathcal{D}^*$. Then for every $Y \subseteq W$ and all $U, \dots, V \in P$,*

$$\bigcap\{\varphi(U, \dots, X, \dots, V) : Y \subseteq X \in P\} = \varphi(U, \dots, \bigcap\{X \in P : Y \subseteq X\}, \dots, V). \tag{10.7}$$

Proof If $\mathfrak{F}$ is a Kripke frame then the variable X ranges over all subsets of W containing Y and so, by the monotonicity of φ, both sides of (10.7) are simply $\varphi(\dots, Y, \dots)$.

So suppose $\mathfrak{F}$ is descriptive and prove our claim by induction on the construction of φ. The basis of induction is trivial. Let $\varphi = \psi \vee \chi$ and suppose that a point x does not belong to the right side of (10.7), i.e.,

$$x \notin \psi(\dots, \bigcap\{X \in P : Y \subseteq X\}, \dots) \cup \chi(\dots, \bigcap\{X \in P : Y \subseteq X\}, \dots).$$

By the induction hypothesis, we have $x \notin \bigcap\{\psi(\dots, X, \dots) : Y \subseteq X \in P\}$ and $x \notin \bigcap\{\chi(\dots, X, \dots) : Y \subseteq X \in P\}$. So there are sets $X', X'' \in P$ such that $Y \subseteq X' \cap X''$, $x \notin \psi(\dots, X', \dots)$ and $x \notin \chi(\dots, X'', \dots)$. By the monotonicity of ψ and χ, we then have $x \notin \psi(\dots, X' \cap X'', \dots)$ and $x \notin \chi(\dots, X' \cap X'', \dots)$, whence $x \notin \bigcap\{(\psi \vee \chi)(\dots, X, \dots) : Y \subseteq X \in P\}$. Thus, the set in the left-hand side of (10.7) is a subset of that in the right-hand side. To prove the converse inclusion, we observe first that

$$\bigcap\{(\psi \vee \chi)(\dots, X, \dots) : Y \subseteq X \in P\} \supseteq \bigcap\{\psi(\dots, X, \dots) : Y \subseteq X \in P\} \cup \bigcap\{\chi(\dots, X, \dots) : Y \subseteq X \in P\},$$

as follows from the set-theoretic inclusion

$$\bigcap_{i \in I}(X_i \cup Y_i) \supseteq \bigcap_{i \in I} X_i \cup \bigcap_{i \in I} Y_i,$$

and then we use the induction hypothesis.

The case $\varphi = \psi \wedge \chi$ is considered analogously. Let $\varphi = \Box\psi$. As was mentioned above, $\Box$ distributes over intersections. So we obtain

$$\bigcap\{\Box\psi(\dots, X, \dots) : Y \subseteq X \in P\} = \Box \bigcap\{\psi(\dots, X, \dots) : Y \subseteq X \in P\}$$

and then use the induction hypothesis.

The case $\varphi = \Diamond\psi$ is treated similarly, but this time we use Esakia's lemma and the fact that $\{\psi(\dots, X, \dots) : Y \subseteq X \in P\}$ either contains $\emptyset$, and so both sides of (10.7) become $\emptyset$, or is downward directed. (Indeed, if $X', X'' \in P$, $Y \subseteq X'$ and $Y \subseteq X''$ then $X' \cap X'' \in P$, $Y \subseteq X' \cap X''$ and, by monotonicity, $\psi(\dots, X' \cap X'', \dots) \subseteq \psi(\dots, X', \dots) \cap \psi(\dots, X'', \dots)$.) ❑

It follows from this lemma and considerations above that, given a modal formula $\varphi = \psi \to \chi$ with strongly positive ψ and positive χ, we can construct a first order formula $\phi(x)$ (in R and $=$) with one free individual variable x such that, for every descriptive or Kripke frame $\mathfrak{F}$ and every point a in $\mathfrak{F}$, $(\mathfrak{F}, a) \models \varphi$ iff $\phi(x)$ is satisfied in $\mathfrak{F}$ at a, or in symbols, $\mathfrak{F} \models \phi(x)[a]$. We will not, however, present this result as a theorem because by purely syntactic manipulations with modal and first order formulas we can get a stronger one.

Notice that using the monotonicity of positive formulas, the equivalence (10.4) can be generalized to the following one: for every $\mathfrak{F} = \langle W, R, P\rangle$, every positive $\chi_i(\dots, p, \dots)$, $i = 1, \dots, n$, and every $x_1, \dots, x_n \in W$,

$$\forall X \in P\ (Y \subseteq X \to \bigvee_{i \le n} x_i \in \chi_i(\dots, X, \dots)) \text{ iff}$$

$$\bigvee_{i \le n} x_i \in \bigcap\{\chi_i(\dots, X, \dots) :\ Y \subseteq X \in P\}. \qquad (10.8)$$

Say that a modal formula ψ is *untied* if it can be constructed from negative formulas and strongly positive ones using only $\wedge$ and $\Diamond$. (We remind the reader that *negative* formulas are built from the negations of variables with the help of $\bot$, $\top$, $\wedge$, $\vee$, $\Box$ and $\Diamond$. If $\nu(p_1, \dots, p_n)$ is negative then $\neg\nu(p_1, \dots, p_n)$ is equivalent in $\mathbf{K}$ to a positive formula, namely to $\nu^*(\neg p_1, \dots, \neg p_n)$; see Exercise 3.21.)

Lemma 10.29 *Let $\psi(p_1, \dots, p_n)$ be an untied formula and $\mathfrak{F} = \langle W, R, P\rangle$ a frame. Then for every $x \in W$ and all $X_1, \dots, X_n \in P$,*

$$x \in \psi(X_1, \dots, X_n) \textit{ iff } \exists y_1, \dots, y_l(\vartheta \wedge \bigwedge_{i \le n} T_i \subseteq X_i \wedge \bigwedge_{j \le m} z_j \in \nu_j(X_1, \dots, X_n))$$

where the formula in the right-hand side, effectively constructed from ψ, has only one free individual variable x, ϑ is a conjunction of formulas of the form uRv, T_i are suitable R-terms and $\nu_j(p_1, \dots, p_n)$ are negative formulas.

Proof An easy induction on the construction of ψ from negative and strongly positive formulas is left to the reader. ❑

We are now in a position to formulate and prove the main result of this section.

Theorem 10.30. (Sahlqvist's theorem) *Suppose that φ is a formula which is equivalent in $\mathbf{K}$ to a conjunction of formulas of the form $\Box^k(\psi \to \chi)$, where $k \ge 0$, χ is positive and ψ is constructed from propositional variables and their negations, $\bot$ and $\top$ with the help of $\wedge$, $\vee$, $\Box$ and $\Diamond$ in such a way that no ψ's subformula of the form $\psi_1 \vee \psi_2$ or $\Diamond\psi_1$, containing an occurrence of a variable without $\neg$, is in the scope of some $\Box$. Then one can effectively construct a first order formula $\phi(x)$ in R and $=$ having x as its only free variable and such that, for every descriptive or Kripke frame $\mathfrak{F}$ and every point a in $\mathfrak{F}$,*

$$(\mathfrak{F}, a) \models \varphi \textit{ iff } \mathfrak{F} \models \phi(x)[a].$$

Proof Since for $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_m$ we have $(\mathfrak{F}, a) \models \varphi$ iff $(\mathfrak{F}, a) \models \varphi_i$ for every $i \in \{1, \ldots, m\}$, we may begin with finding first order equivalents for φ_i and then take the conjunction of them.

To construct a first order equivalent for a formula $\Box^k(\psi \to \chi)$ defined in the formulation of our theorem, we observe first that one can equivalently reduce ψ to a disjunction $\psi_1 \vee \ldots \vee \psi_m$ of untied formulas, and hence $\Box^k(\psi \to \chi)$ is equivalent in $\mathbf{K}$ to $\Box^k(\psi_1 \to \chi) \wedge \ldots \wedge \Box^k(\psi_m \to \chi)$. So all we need is to find a first order equivalent for an arbitrary formula $\Box^k(\psi \to \chi)$ with untied ψ and positive χ. Let $p_1, \ldots p_n$ be all the variables in ψ and χ and $\mathfrak{F} = \langle W, R, P \rangle$ a descriptive or Kripke frame. Then, for any $x \in W$, we have:

$$
\begin{aligned}
(\mathfrak{F}, x) \models \Box^k(\psi \to \chi) &\text{ iff } \forall X_1, \ldots, X_n \in P\ x \in \Box^k(\psi \to \chi)(X_1, \ldots, X_n) \\
&\text{ iff } \forall X_1, \ldots, X_n \in P\ \forall y\ (xR^k y \to \\
&\qquad (y \in \psi(X_1 \ldots X_n) \to y \in \chi(X_1, \ldots, X_n))) \\
\text{(by Lemma 10.29)} &\text{ iff } \forall X_1, \ldots, X_n \in P\ \forall y\ (xR^k y \to (\exists y_1, \ldots, y_l\ (\vartheta \wedge \\
&\qquad \bigwedge_{i \le n} T_i \subseteq X_i \wedge \bigwedge_{j \le m} z_j \in \nu_j(X_1, \ldots, X_n)) \to \\
&\qquad y \in \chi(X_1, \ldots, X_n))) \\
&\text{ iff } \forall X_1, \ldots, X_n \in P\ \forall y, y_1, \ldots, y_l\ (\vartheta' \wedge \bigwedge_{i \le n} T_i \subseteq X_i \wedge \\
&\qquad \bigwedge_{j \le m} z_j \in \nu_j(X_1, \ldots, X_n) \to y \in \chi(X_1, \ldots, X_n))
\end{aligned}
$$

where $\vartheta' = xR^k y \wedge \vartheta$.

Let $\pi_j(p_1, \ldots, p_n) = \nu_j^*(\neg p_1, \ldots, \neg p_n)$ (recall that ν_j^* is the dual of ν_j and π_j is a positive formula). Then, by the laws of classical predicate logic, we can continue this chain of equivalences as follows:

$$
\begin{aligned}
&\text{iff } \forall y, y_1, \ldots, y_l\ (\vartheta' \to \forall X_1, \ldots, X_n \in P\ (\bigwedge_{i \le n} T_i \subseteq X_i \to \\
&\bigvee_{j \le m+1} z_j \in \pi_j(X_1, \ldots, X_n)))
\end{aligned}
$$

(where $\pi_{m+1}(p_1, \ldots, p_n) = \chi(p_1, \ldots, p_n)$ and $z_{m+1} = y$)

$$
\text{iff } \forall y, y_1, \ldots, y_l\ (\vartheta' \to \bigvee_{j \le m+1} z_j \in \pi_j(T_1, \ldots, T_n)),
$$

as follows from (10.8), the intersection lemma and (10.3). The rest is an immediate consequence of Lemma 10.25. ❑

The formulas φ described in the formulation of Theorem 10.30 are called *Sahlqvist formulas.* As a consequence of this theorem we obtain our first general completeness result for modal logics.

Theorem 10.31 *Suppose that L is a $\mathcal{D}$-persistent normal modal logic and Γ any set of Sahlqvist formulas. Then the logic $L \oplus \Gamma$ is also $\mathcal{D}$-persistent. Besides, if L is elementary then $L \oplus \Gamma$ is elementary as well.*

This result can easily be extended to quasi-normal logics. Let us call a logic $L \in \mathrm{Ext}\mathbf{K}$ *$\mathcal{D}$-persistent* if for every descriptive frame $\mathfrak{F}$ with actual world w, $\langle \mathfrak{F}, w\rangle \models L$ implies $\langle \kappa\mathfrak{F}, w\rangle \models L$. L is *elementary* if there is a set Φ of first order formulas (in R and $=$) with only one free variable x such that, for every Kripke frame $\mathfrak{F}$ with actual world w, $\langle \mathfrak{F}, w\rangle \models L$ iff $\mathfrak{F} \models \phi(x)[w]$ for all $\phi \in \Phi$. It should be clear that Theorem 10.31 will hold if we replace in it $\oplus$ by $+$ and regard L as a quasi-normal modal logic.

10.4 Logics of finite width

Our second completeness result holds for both normal modal and superintuitionistic logics. However, in the modal case it concerns only logics with transitive frames, i.e., extensions of $\mathbf{K4}$, and so all frames in this section are assumed to be transitive. We will prove it first for modal logics and then use the preservation theorem to transfer it to superintuitionistic ones.

This result can be formulated both syntactically and semantically. Its syntactical form states simply that, for every $n \geq 1$, all normal extensions of the logic $\mathbf{K4BW}_n$ are Kripke complete. In order to reformulate this semantically, we observe that Corollary 3.43 can be generalized to refined frames. Namely, we have

Proposition 10.32 *A rooted refined frame $\mathfrak{F} = \langle W, R, P\rangle$ validates $\boldsymbol{bw}_n$ iff $\mathfrak{F}$ is of width $\leq n$.*

Proof ($\Rightarrow$) Suppose otherwise. Then $\mathfrak{F}$ contains an antichain $x_0, \dots, x_n$. Since $\mathfrak{F}$ is differentiated, there exist disjoint sets $X_0, \dots, X_n \in P$ such that, for every $i, j \in \{0, \dots, n\}$, $x_i \in X_j$ iff $i = j$. Using the tightness of $\mathfrak{F}$, one can show that there are sets $Y_0, \dots, Y_n \in P$ such that $x_i \in Y_i$ and $Y_i \cap Y_j{\downarrow} = \emptyset$ for every $j \neq i$.

Now we put $Z_i = X_i \cap Y_i \in P$ and define a valuation $\mathfrak{V}$ on $\mathfrak{F}$ by taking, for every $i = 0, \dots, n$, $\mathfrak{V}(p_i) = Z_i$. Using the fact that $Z_0, \dots, Z_n$ are disjoint and do not see each other, the reader can readily show that $\boldsymbol{bw}_n$ is false under $\mathfrak{V}$ at the root of $\mathfrak{F}$, which is a contradiction.

($\Leftarrow$) follows from Corollary 3.43. ❑

Thus, a semantic counterpart of the completeness result formulated above may look like this: a modal logic is Kripke complete whenever it is characterized by a class of transitive general frames of width $\leq n$, for some $n \geq 1$. If a logic L satisfies this condition and is not characterized by any class of frames of width $< n$ then L is said to be *of width n*. $\mathbf{K4BW}_n$ is the smallest logic of width n.

We are going to prove this result in three moves. First we show that every finite width logic is characterized by a class of Noetherian frames of finite width. Frames of this sort have the *finite cover property* in the sense that every set of points in them has a finite cover. Then, removing some points from these frames, we establish that every finite width logic is $\mathcal{AFC}$-complete, where $\mathcal{AFC}$ is the class of all atomic transitive frames with the finite cover property. And finally we observe that every normal modal logic above **K4** is $\mathcal{AFC}$-persistent, which together with the preceding statement gives the Kripke completeness of all logics of finite width.

To justify the first move, we require the following generalization of König's lemma. Say that a sequence $x_0, x_1, \dots$ of points in $\mathfrak{F} = \langle W, R \rangle$ is *nondescending* if $x_i R x_j$ for no i and j such that $i > j$.

Lemma 10.33 *Suppose a frame $\mathfrak{F} = \langle W, R \rangle$ has no infinite antichains. Then every infinite nondescending sequence of distinct points in $\mathfrak{F}$ contains an infinite ascending subsequence.*

Proof Let $x_0, x_1, \dots$ be an arbitrary infinite nondescending sequence of distinct points in $\mathfrak{F}$. Observe first that there must exist some i such that the subsequence $X_i = \{x_j : j > i \text{ and } x_i R x_j\}$ is infinite. For otherwise, if there is no such i, we can inductively define an infinite antichain $x_{i_0}, x_{i_1}, \dots$ in $\mathfrak{F}$ by putting $i_0 = 0, \dots, i_{k+1} = 1 + \max\{\{i_k\} \cup \{i : x_i \in X_{i_k}\}\}$, etc.

Now we construct by induction an infinite ascending subsequence $x_{i_0}, x_{i_1}, \dots$ of $x_0, x_1, \dots$. Let x_{i_0} be the first point in the original sequence with infinite X_{i_0}, and if x_{i_n} has been already defined in such a way that X_{i_n} is infinite, then we let $x_{i_{n+1}}$ be the first point in the (infinite nondescending) sequence X_{i_n} with infinite $X_{i_{n+1}}$. ❑

Theorem 10.34 *Every finitely generated differentiated frame without infinite antichains is Noetherian.*

Proof Let $\mathfrak{F} = \langle W, R, P \rangle$ be a finitely generated differentiated frame without infinite antichains. Call a point $x_0 \in W$ *deep* if there is an infinite ascending chain $x_0, x_1, \dots$ of distinct points in $\mathfrak{F}$. So our goal is to prove that $\mathfrak{F}$ contains no deep points. Suppose otherwise.

For each $x \in W$, let U_x be the set of points accessible from x which are not deep. Call a point x *static* if $U_x = U_y$ for every deep $y \in x\uparrow$. It follows from Lemma 10.33 that every infinite ascending chain contains a static point. Indeed, otherwise there is a chain $x_0 R x_1 R \dots$ for which $U_{x_0} \supset U_{x_1} \supset \dots$, and so we can construct a sequence $y_0, y_1, \dots$ such that $y_i \in U_{x_i} - U_{x_{i+1}}$. It should be clear that the sequence is nondescending and so contains an infinite ascending subsequence, contrary to all y_i being not deep.

Let $\mathcal{G} = \{G_1, \dots, G_n\}$ be a set of P's generators. We write $x \sim_{\mathcal{G}} y$ if, for every $i = 1, \dots, n$, $x \in G_i$ iff $y \in G_i$, and denote by $[x]_{\mathcal{G}}$ the set $\{y \in W : x \sim_{\mathcal{G}} y\}$. For $x \in W$, let $V_x = \{[y]_{\mathcal{G}} : xRy \text{ and } y \text{ is deep}\}$. Say that a deep point x is *stationary* if $V_x = V_y$ for every deep $y \in x\uparrow$. Since $V_x \supseteq V_y$ whenever xRy and each V_x is finite ($|V_x| \leq 2^n$, to be more exact), every infinite ascending chain in $\mathfrak{F}$ contains a stationary point.

It follows that $\mathfrak{F}$ contains a point x which is both static and stationary, i.e., $U_x = U_y$ and $V_x = V_y$ for every deep $y \in x\uparrow$. Now, by induction on the construction of a set $X \in P$ from $G_1, \dots, G_n$ using, say, $\cap$, $-$ and $\downarrow$ it is not hard to show that $y \in X$ iff $z \in X$ for every deep $y, z \in x\uparrow$ such that $y \sim_{\mathcal{G}} z$. (The only nontrivial case is $X = Y{\downarrow}$. Suppose $y \in w{\downarrow}$ for some $w \in Y$. If w is not deep then $z \in w{\downarrow} \subseteq Y{\downarrow}$, since $U_y = U_z$. And if w is deep then, since $V_y = V_z$, there is a deep $v \in z{\uparrow}$ such that $w \sim_{\mathcal{G}} v$, from which, by the induction hypothesis, $v \in Y$ and so again $z \in Y{\downarrow}$.) But this leads to a contradiction. Indeed, x sees infinitely many deep points. Hence at least two of them, say y and z, are $\mathcal{G}$-equivalent and so $\forall X \in P(y \in X \leftrightarrow z \in X)$, contrary to $\mathfrak{F}$ being differentiated. ❑

As a consequence of Theorem 10.34 we obtain

Theorem 10.35 *Every differentiated finitely generated frame without infinite antichains has the finite cover property and contains no infinite clusters.*

Proof Suppose $\mathfrak{F}$ is a differentiated finitely generated frame and X a non-empty set of its points. By Theorem 10.34, $\mathfrak{F}$ contains no infinite ascending chains, and so every cluster in $\mathfrak{F}$ is finite and every point in X sees a final point in X or is final in X itself. Therefore, any subset of X containing one representative of each cluster generated by a final point in X is a cover for X. It is finite because it is an antichain. ❑

Each logic $L \in \mathrm{NExt}\mathbf{K4}$, as we know, is characterized by its finitely generated refined frames whose clusters are finite. If L is of finite width then these frames turn out to possess one more nice trait: they have the finite cover property. Our second move is to prove that atomic frames with the finite cover property and without infinite clusters are enough. To this end we will show first that certain points in general frames are practically useless and may be safely thrown out.

Let $\mathfrak{F} = \langle W, R, P \rangle$ be an arbitrary frame. A point $x \in W$ is said to be *eliminable* in $\mathfrak{F}$ if it has a proper successor in every set $X \in P$ containing x. If $\mathfrak{F}$ has the finite cover property then each eliminable point in $\mathfrak{F}$, if any, has a noneliminable successor in every set in P it belongs to. But actually, this fact holds for every descriptive frame $\mathfrak{F}$:

Theorem 10.36 *Suppose that $\mathfrak{F} = \langle W, R, P \rangle$ is a descriptive frame and $X \in P$. Then the set of final points in X is non-empty and forms a cover for X. In particular, every eliminable point in X has a noneliminable successor in X.*

Proof Suppose otherwise. This means that some x in X sees no final point in X. Let U be a maximal chain in X starting from x (i.e., for every chain $V \subseteq X$ beginning with x, $U \subseteq V$ implies $U = V$); its existence can be readily proved with the help of Zorn's lemma. Of course, U has no maximal point.

Now consider the family $\mathcal{X}$ of all sets $Y \in P$ such that Y contains all the points in U above some $y \in U$; more exactly, we let

$$\mathcal{X} = \{Y \in P : \ \exists y \in U \ y{\uparrow} \cap U \subseteq Y\}.$$

Clearly, $\mathcal{X}$ is not empty, since $X \in \mathcal{X}$, and has the finite intersection property. Hence, there is a $u \in \bigcap \mathcal{X}$. But then u is a maximal point in U. Indeed, $u \in X$ and so what we need is to establish that yRu for every $y \in U$. By the tightness of $\mathfrak{F}$, it suffices to show that $\forall Y \in P \ (y \in \Box Y \to u \in Y)$, which is quite clear, since $y \in \Box Y$ implies $Y \in \mathcal{X}$.

Thus we arrive at a contradiction which proves our theorem. ❑

Now, given a frame $\mathfrak{F} = \langle W, R, P\rangle$ in which each eliminable point x has a noneliminable successor in every set $X \in P$ containing x, we construct a new frame $\mathfrak{G} = \langle V, S, Q\rangle$ by taking

$$V = \{x \in W : x \text{ is noneliminable in } \mathfrak{F}\},$$

$$S = R \cap V^2, \quad Q = \{X \cap V : X \in P\}.$$

The fact that Q is closed under the Boolean operations and $\downarrow$ follows from the equalities (10.9)–(10.11) below which hold for every $X, Y \in P$:

$$(X \cap Y) \cap V = (X \cap V) \cap (Y \cap V), \tag{10.9}$$

$$(W - X) \cap V = V - (X \cap V), \tag{10.10}$$

$$X{\downarrow}R \cap V = (X \cap V){\downarrow}S. \tag{10.11}$$

The first two of them are trivial and (10.11) is proved like this. Suppose that $x \in X{\downarrow}R \cap V$, i.e., x is a noneliminable point in $\mathfrak{F}$ having a successor y in X. Let z be a noneliminable successor of y in X. Then $z \in X \cap V$, $x \in y{\downarrow} \subseteq z{\downarrow}$ and so $x \in (X \cap V){\downarrow}S$. The converse inclusion is obvious.

It follows from (10.9)–(10.11) that the map $X \mapsto X \cap V$, for $X \in P$, is a homomorphism of $\mathfrak{F}^+$ onto $\mathfrak{G}^+$. Moreover, if $X \neq Y$ then $X \cap V \neq Y \cap V$ for every $X, Y \in P$. (For if $x \in X - Y \in P$ then there is a noneliminable point in $X - Y$.) Thus, $\mathfrak{F}^+ \cong \mathfrak{G}^+$. It is easy to see also that $\mathfrak{G}$ is refined, though not necessarily compact. Clearly $\mathfrak{G}$ contains no eliminable points. Frames with this property are called *reduced*. As a consequence of Theorem 10.36 we then obtain

Proposition 10.37 *Every logic $L \in$ NExt**K4** is characterized by the class of its finitely generated reduced refined frames.*

Proposition 10.38 *Suppose $\mathfrak{F} = \langle W, R, P\rangle$ is a refined reduced frame with the finite cover property and without infinite clusters. Then $\mathfrak{F}$ is atomic.*

Proof Let x be an arbitrary point in $\mathfrak{F}$. Since $\mathfrak{F}$ is reduced, x is a final point in some $X \in P$. Using the fact that $\mathfrak{F}$ is differentiated and $C(x)$ is finite, one can construct a set $X_0 \in P$ which contains x and does not contain points from $C(x) - \{x\}$.

Let $y_1, \dots, y_m$ be all the final points in the set $X - C(x)$. By the same argument there is a set $Y_0 \in P$ such that $x \in Y_0$ and $y_1, \dots, y_m \notin Y_0$. Moreover, since x sees none of $y_1, \dots, y_m$, using the tightness of $\mathfrak{F}$ we can find a set $Y \in P$ containing $y_1, \dots, y_m$ and such that $x \notin Y{\downarrow}$.

Now consider the set $Z = (X \cap X_0 \cap Y_0) - Y{\downarrow}$, which clearly belongs to P and contains x. Suppose z is a point in Z different from x. Since z is final neither in X nor in $X - C(x)$, it must see at least one of y_i. But then $z \in Y{\downarrow}$, which is a contradiction. Therefore, $Z = \{x\}$. ❑

As a consequence of Theorem 10.35 and Propositions 10.32, 10.37 and 10.38 we obtain

Theorem 10.39 *Every finite width logic is characterized by a class of finitely generated refined atomic frames with the finite cover property.*

Remark Taking finitely generated universal frames, we see that in the preceding theorem a countable class of at most countable frames is enough.

Our final move is to show that every logic in NExt**K4** is persistent with respect to the class of atomic frames having the finite cover property. This result is a direct consequence of the following lemma and Theorem 9.43.

Theorem 10.40 *Suppose that* $\mathfrak{G} = \langle V, S, Q\rangle$ *is an atomic frame with the finite cover property validating a canonical formula* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. *Then* $\kappa\mathfrak{G}$ *validates* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ *as well.*

Proof Suppose otherwise. Then there exists a cofinal subreduction of $\kappa\mathfrak{G}$ to $\mathfrak{F} = \langle W, R\rangle$ satisfying (CDC) for $\mathfrak{D}$. For every point $x \in W$ we fix a finite cover V_x for $f^{-1}(x)$ in $\mathfrak{G}$. Since $\mathfrak{G}$ is atomic, $V_x \in Q$ for all $x \in W$.

Now we define a new partial map g from V onto W by putting

$$g(y) = \begin{cases} x & \text{if } y \in V_x \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In other words, g is obtained from f by restricting dom$f$ to the set $\bigcup_{x\in W} V_x$. It is easy to check that g is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Therefore, $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, which is a contradiction. ❑

Since every normal extension of **K4** is axiomatized by canonical formulas, we immediately derive

Theorem 10.41 *Every logic in* NExt**K4** *is persistent with respect to the class of atomic frames having the finite cover property.*

Putting together Theorems 10.39 and 10.41, we finally obtain the desirable completeness result.

Theorem 10.42. (Fine's theorem) *Every finite width logic is Kripke complete. More precisely, every modal logic of width* n *is characterized by a class of Noetherian Kripke frames of width* $\leq n$.

In fact, using the remark above, we can derive even a somewhat stronger theorem.

Theorem 10.43 *If a logic* $L \in$ NExt**K4** *is characterized by a class of frames without infinite antichains then it is also characterized by an at most countable class of at most countable Kripke frames.*

It is worth noting that unlike Theorem 10.31, Fine's theorem speaks only about Kripke completeness. Finite width logics are not necessarily canonical and characterized by elementary classes of frames, witness the logic **GL.3** (whether in finite or infinite language), for which the proofs of Theorems 6.5 and 6.7 go through. If the language is infinite then the proof of Theorem 6.6 shows that **GL.3** is not strongly complete either. It is of interest, however, that the following theorem holds.

Theorem 10.44 *Every finite width logic L in a finite language is strongly Kripke complete.*

Proof Suppose that the language of L has $m < \omega$ variables and t is an L-consistent tableau. Then t is realized at a point a in the canonical model $\mathfrak{M}_L(m) = \langle \mathfrak{F}_L(m), \mathfrak{V}_L(m) \rangle$. Let

$$V = \{a\} \cup \{x \in W_L(m) : aR_L(m)x \text{ and } x \text{ is noneliminable in } \mathfrak{F}_L(m)\},$$

$S = R_L(m) \cap V^2$, and $\mathfrak{U}(p) = \mathfrak{V}_L(m)(p) \cap V$. We claim that $\mathfrak{G} = \langle V, S \rangle$ is a (Kripke) frame for L and t is realized at a in $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$.

First, by induction on the construction of φ we show that $(\mathfrak{M}, x) \models \varphi$ iff $(\mathfrak{N}, x) \models \varphi$, for every $x \in V$. The basis of induction and the cases of $\varphi = \psi \odot \chi$ for $\odot \in \{\to, \wedge, \vee\}$ are trivial. So suppose $\varphi = \Box\psi$. If $(\mathfrak{M}, x) \not\models \Box\psi$ then there is a noneliminable point $y \in x\uparrow$ such that $(\mathfrak{M}, y) \not\models \psi$, whence $y \in V$ and, by the induction hypothesis, $(\mathfrak{N}, y) \not\models \psi$, from which $(\mathfrak{N}, x) \not\models \Box\psi$. The converse implication is evident. It follows in particular that t is realized at a in $\mathfrak{N}$.

So if a is noneliminable then we are done. Let a be eliminable. Then the cluster $C(a)$ is simple in $\mathfrak{G}$ (see Exercise 10.18). Suppose that $\mathfrak{G} \not\models \varphi$ for some $\varphi \in L$. By Theorem 9.43, there is $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ such that $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\mathfrak{H} \not\models \varphi$ whenever $\mathfrak{H} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, for every frame $\mathfrak{H}$. By Theorem 9.39, there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Since $\mathfrak{G}$ has the finite cover property, we may assume $f^{-1}(x)$ to be a finite antichain, for every x in $\mathfrak{F} = \langle W, R \rangle$. Let b be the root of $\mathfrak{F}$. Since $C(a)$ is a simple cluster, $C(b)$ is simple as well. For otherwise $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and so φ are refuted in the generated subframe $\mathfrak{G}'$ of $\mathfrak{G}$ consisting of only noneliminable points, which is a contradiction. So we may assume that $f^{-1}(b) = \{a\}$. Let $\mathfrak{a} \subseteq f^{-1}(W)$ be a finite antichain such that $f^{-1}(W - \{b\}) \subseteq \mathfrak{a}\uparrow$. Since all points in $\mathfrak{a}$ are noneliminable and $\mathfrak{a} \subseteq a\uparrow$, there must be a noneliminable point $c \in a\uparrow$ such that $\mathfrak{a} \subseteq c\uparrow$. But then we can extend f by putting $f(c) = b$ and get again a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. This means that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and so φ are refuted at c in $\mathfrak{G}'$, which is a contradiction. Thus, $\mathfrak{G} \models L$. ❑

In the intuitionistic case the definition of logic of width n remains the same as in the modal one. It is not hard to see that a superintuitionistic logic L is of width n iff $\boldsymbol{bw}_n \in L$ and $\boldsymbol{bw}_{n+1} \notin L$; so $\mathbf{BW}_n = \mathbf{Int} + \boldsymbol{bw}_n$ is the minimal si-logic of width n.

If L is a si-logic of width n then, as follows from Theorems 9.68 and 9.70, both τL and σL are also of width n. Moreover, by Theorem 9.56, if $M \in$ NExtS4 is of width n then its si-fragment ρM is of the same width. Thus we obtain the following intuitionistic variant of Fine's theorem:

Theorem 10.45 *Every superintuitionistic logic of width n is characterized by a class of Noetherian Kripke frames of width $\leq n$.*

Of course, the intuitionistic counterparts of Theorems 10.44 and 10.43 also hold.

10.5 The degree of Kripke incompleteness of logics in NExtK

So far, when dealing with Kripke completeness, we were interested only in whether a given logic is complete or not. Yet, there is another natural question concerning this property. If a logic L (in NExt**K** or Ext**Int**) is Kripke incomplete then at least two distinct logics have the same Kripke frames, namely L and the logic characterized by the class of Kripke frames for L. The problem is to determine how many distinct logics may share the same class of Kripke frames. In this section we obtain a complete solution to this problem for logics in NExt**K**. It is based on the lattice-theoretic notion of splitting.

Say that a logic L_1 in a complete lattice $\mathfrak{L}$ of logics (e.g. NExt**K**) *splits* $\mathfrak{L}$ if there is L_2 in $\mathfrak{L}$ such that, for every L in $\mathfrak{L}$, either $L \subseteq L_1$ or $L \supseteq L_2$ (but not both, i.e., $L_2 \not\subseteq L_1$). Clearly, the logic L_2, if it exists, is determined uniquely by L_1; we call it the *splitting* of $\mathfrak{L}$ by L_1 and denote it by $\mathfrak{L}/L_1$. Of course, L_1 is also uniquely determined by L_2; (L_1, L_2) is called a *splitting pair* in $\mathfrak{L}$.

In fact splittings were already introduced in Sections 4.3 and 9.4 under the name of prime logics. Indeed, we have the following:

Proposition 10.46 *A logic L_2 is a splitting of $\mathfrak{L}$ iff L_2 is prime in $\mathfrak{L}$.*

Proof For definiteness we assume $\mathfrak{L}$ to be a complete lattice of normal modal logics.

($\Rightarrow$) Suppose $L_2 = \bigoplus_{i\in I} L_i$ and $L_2 = \mathfrak{L}/L_1$. For each $i \in I$, we have either $L_i \subseteq L_1$ or $L_i \supseteq L_2$. If $L_i \supseteq L_2$ for some i, then we are done, because in this case $L_i = L_2$. Otherwise, $L_i \subseteq L_1$ for all i, whence $L_2 \subseteq L_1$, which is a contradiction.

($\Leftarrow$) Put $L_1 = \bigoplus\{L' \in \mathfrak{L} : L' \not\supseteq L_2\}$ and show that (L_1, L_2) is a splitting pair. Take any L in $\mathfrak{L}$. If $L \not\supseteq L_2$ then, by the definition, $L \subseteq L_1$. So suppose $L_1 \supseteq L \supseteq L_2$. Then $L_2 = L_2 \cap \bigoplus\{L' \in \mathfrak{L} : L' \not\supseteq L_2\}$. By Theorem 4.6, we have $L_2 = \bigoplus\{L_2 \cap L' : L' \not\supseteq L_2\}$, from which $L_2 = L_2 \cap L'$ for some $L' \not\supseteq L_2$, because L_2 is prime. But then $L_2 \subseteq L'$, which is again a contradiction. ❑

Example 10.47 (1) $\mathbf{D} = \mathbf{K} \oplus \Diamond\top = \mathrm{NExt}\mathbf{K}/\mathrm{Log}\bullet$. Indeed, if $\bullet$ is a frame for $L \in \mathrm{NExt}\mathbf{K}$ then $L \subseteq \mathrm{Log}\bullet$. Otherwise (by the generation and disjoint union theorems, see the proof of Makinson's theorem) $\Diamond\top \in L$ and so $\mathbf{D} \subseteq L$.

(2) By Proposition 10.46 and Theorem 9.46, a logic is a splitting of NExt**K4** or Ext**Int** iff it can be represented in the form $\mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot)$ or $\mathbf{Int} + \beta^\sharp(\mathfrak{F}, \bot)$, respectively.

If each logic in a family $\{L_i : i \in I\} \subseteq \mathfrak{L}$ splits the lattice $\mathfrak{L}$ then the logic $L = \bigoplus_{i\in I} \mathfrak{L}/L_i$ ($L = \sum_{i\in I} \mathfrak{L}/L_i$ in the intuitionistic case) is called a *union-splitting* of $\mathfrak{L}$ and denoted by $L = \mathfrak{L}/\{L_i : i \in I\}$. In this case for every L' in $\mathfrak{L}$ we clearly have $L' \supseteq L$ iff $L' \not\subseteq L_i$ for all $i \in I$.

Example 10.48 It is easy to check that $\mathbf{S4} = \mathbf{K4} \oplus \alpha(\bullet) = \mathbf{K4} \oplus \alpha^\sharp(\bullet, \bot) \oplus \alpha^\sharp(\overset{\circ}{\underset{\bullet}{\uparrow}}, \bot) = \mathrm{NExt}\mathbf{K4}/\{\mathrm{Log}\bullet, \mathrm{Log}\,\overset{\circ}{\underset{\bullet}{\uparrow}}\}$. By Example 10.47, the frame logics and only they are union-splittings of NExt**K4** and Ext**Int**.

The connection of splittings with finite rooted frames revealed by the examples above is not mere chance.

Theorem 10.49 *Suppose a logic $L_0 \in \mathrm{NExt}\mathbf{K}$ is finitely approximable and L splits $\mathrm{NExt}L_0$. Then there is a finite rooted frame $\mathfrak{F}$ such that $L = \mathrm{Log}\mathfrak{F}$.*

Proof Let $\mathcal{C}$ be the class of all finite rooted frames for L_0. Since L_0 is finitely approximable, we have $L_0 = \bigcap\{\mathrm{Log}\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\} \subseteq L$. And since $\mathrm{NExt}L_0/L \not\subseteq L$, there is $\mathfrak{F} \in \mathcal{C}$ such that $\mathrm{Log}\mathfrak{F} \subseteq L$. As will be shown in Section 12.1, all extensions of a tabular logic are also tabular. Therefore, L can be represented as $\bigcap_{i=1}^{n} \mathrm{Log}\mathfrak{F}_i$, for some finite rooted $\mathfrak{F}_i$, and so, by the same argument, there is i such that $L = \mathrm{Log}\mathfrak{F}_i$. ❑

To simplify our notation and terminology, we will write $L_0/\mathfrak{F}$ instead of $\mathrm{NExt}L_0/\mathrm{Log}\mathfrak{F}$ and say that $\mathfrak{F}$ splits $\mathrm{NExt}L_0$ and $L_0/\mathfrak{F}$ is the splitting of $\mathrm{NExt}L_0$ by $\mathfrak{F}$. The union-splitting $\mathrm{NExt}L_0/\{\mathrm{Log}\mathfrak{F} : \mathfrak{F} \in \mathcal{F}\}$ will be denoted by $L_0/\mathcal{F}$.

The semantic meaning of (union-) splittings is quite clear:

Proposition 10.50 *$L_0/\mathcal{F}$ is the smallest normal extension of L_0 without frames in $\mathcal{F}$.*

This observation and the next theorem show why splittings may be of great importance for solving our problem. Say that a Kripke complete (finitely approximable) logic L is *strictly Kripke complete* (respectively, *strictly finitely approximable*) in a lattice of logics $\mathfrak{L}$ if no other logic in $\mathfrak{L}$ has the same Kripke (finite) frames as L.

Theorem 10.51 *Every Kripke complete (finitely approximable) union-splitting $L = L_0/\mathcal{F}$ is strictly Kripke complete (or, respectively, strictly finitely approximable) in $\mathrm{NExt}L_0$.*

Proof Let L' be a logic in $\mathrm{NExt}L_0$ with the same Kripke (finite) frames as L. Then obviously $L' \subseteq L$. On the other hand, the frames in $\mathcal{F}$ do not validate L' and so, by Proposition 10.50, $L \subseteq L'$. ❑

The following property of splittings will be useful in Section 12.2.

Theorem 10.52 *Suppose that $L = L_0/\mathcal{F}$ for some class $\mathcal{F}$ of finite rooted frames. Then all immediate predecessors of L in $\mathrm{NExt}L_0$ are contained in the set $\{L \cap \mathrm{Log}\mathfrak{F} : \mathfrak{F} \in \mathcal{F}\}$. Moreover, if $\mathfrak{F} \in \mathcal{F}$ does not validate $\mathrm{Log}\mathfrak{G}$ for any $\mathfrak{G} \in \mathcal{F} - \{\mathfrak{F}\}$, then $L \cap \mathrm{Log}\mathfrak{F}$ is an immediate predecessor of L in $\mathrm{NExt}L_0$.*

Proof If L' is an immediate predecessor of L in $\mathrm{NExt}L_0$ then, by Proposition 10.50, $\mathfrak{F} \models L'$ for some $\mathfrak{F} \in \mathcal{F}$. Therefore, $L' \subseteq L \cap \mathrm{Log}\mathfrak{F} \subset L$ and so $L' = L \cap \mathrm{Log}\mathfrak{F}$.

Suppose now that $\mathfrak{F} \not\models \mathrm{Log}\mathfrak{G}$ for any $\mathfrak{G} \in \mathcal{F} - \{\mathfrak{F}\}$, and $L \cap \mathrm{Log}\mathfrak{F} \subseteq L' \subset L$. Then, since $L = L_0/\mathcal{F}$, we have $L' \subseteq \mathrm{Log}\mathfrak{F}'$ for some $\mathfrak{F}' \in \mathcal{F}$. Hence $\mathfrak{F}' = \mathfrak{F}$ and $L' = L \cap \mathrm{Log}\mathfrak{F}$. ❑

As we saw above, any finite rooted frame splits NExt**K4**. Now let us find out which frames may split NExt**K**. To this end we need some more frame-based formulas. Suppose $\mathfrak{F} = \langle W, R\rangle$ is a finite frame with root r. Let

$$\delta(\mathfrak{F}) = \bigwedge\{p_x \to \Diamond p_y : \ xRy\} \wedge \bigwedge\{p_x \to \neg\Diamond p_y : \ \neg xRy\} \wedge$$
$$\bigwedge\{p_x \to \neg p_y : \ x \neq y\} \wedge \bigvee\{p_x : \ x \in W\}$$

and, for every $m < \omega$, $\delta^m(\mathfrak{F}) = \bigwedge_{i=0}^{m} \Box^i \delta(\mathfrak{F})$. The meaning of the formulas $\delta^m(\mathfrak{F})$ is that a frame $\mathfrak{G}$ satisfies the set $\{\delta^m(\mathfrak{F}), p_r : m < \omega\}$ at a point x iff there is a generated subframe $\mathfrak{G}'$ of $\mathfrak{G}$ reducible to $\mathfrak{F}$. Indeed, the implication $(\Leftarrow)$ is clear and to prove $(\Rightarrow)$ it suffices to notice that the map f from $\mathfrak{G}'$ to $\mathfrak{F}$ defined by $f(v) = y$ iff $v \models p_y$ is a reduction.

Say that a frame $\mathfrak{F}$ is *cycle free* if $x \in x{\uparrow}^{\omega}$ for no x in $\mathfrak{F}$, i.e., the diagram of $\mathfrak{F}$ contains no cycles, including reflexive points. Clearly, a finite frame $\mathfrak{F}$ is cycle free iff $\mathfrak{F} \models \Box^n\bot$ for some $n < \omega$.

Theorem 10.53 *A finite rooted frame $\mathfrak{F}$ splits* NExt**K** *iff $\mathfrak{F}$ is cycle free.*

Proof $(\Rightarrow)$ Suppose that $\mathfrak{F}$ splits the lattice NExt**K**. By Corollary 3.29, we have $\mathbf{K} = \bigcap\{\mathrm{Log}\mathfrak{G} : \mathfrak{G}$ is a finite rooted cycle free frame$\}$. Then there is a finite rooted cycle free $\mathfrak{G}$ such that $\mathrm{Log}\mathfrak{G} \subseteq \mathrm{Log}\mathfrak{F}$ and so $\mathfrak{F} \models \Box^n\bot$ for some $n < \omega$.

$(\Leftarrow)$ Let $\mathfrak{F} \models \Box^n\bot$. We show that $(\mathrm{Log}\mathfrak{F}, \mathbf{K} \oplus \Box^n\bot \wedge \delta^{n-1}(\mathfrak{F}) \to \neg p_r)$ is a splitting pair. Denote it for brevity by (L_1, L_2). Take any logic $L \in$ NExt**K** and a frame $\mathfrak{G}$ characterizing it. Clearly $\mathfrak{F}$ contains no chains of length $> n$. Then we have $L_2 \not\subseteq L$ iff $\Box^n\bot \wedge \delta^{n-1}(\mathfrak{F}) \wedge p_r$ is satisfied in $\mathfrak{G}$ at some point x iff the subframe $\mathfrak{G}'$ of $\mathfrak{G}$ generated by x is reducible to $\mathfrak{F}$. Thus we have either $L_2 \subseteq L$ or $\mathfrak{F} \models L$ and so $L \subseteq L_1$. ❑

Theorem 10.54 *Every union-splitting of* NExt**K** *is finitely approximable.*

Proof We prove the finite approximability of $L = \mathbf{K}/\mathcal{F}$, $\mathcal{F}$ a class of finite rooted cycle free frames, using a variant of filtration.

Suppose $\varphi(p_1, \dots, p_n) \notin L$. We are going to filtrate the canonical model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ for L in the language with the variables $p_1, \dots, p_n$. To select a suitable "filter", let us first consider points in $\mathfrak{M}$ at which $\Box^m\bot$ is true and $\Box^{m-1}\bot$ is false for some $m < \omega$. We call them *points of type m* (having in mind that the maximal ascending chain starting from such a point is of length m). The key observation in the proof is

Lemma 10.55 *For every $m \geq 1$, there are finitely many points of type m in $\mathfrak{M}$.*

Proof The proof proceeds by induction on m. Clearly, $\mathfrak{M}$ contains $\leq 2^n$ points of type 1 (= dead ends); for otherwise $\mathfrak{M}$ would not be differentiated. And if

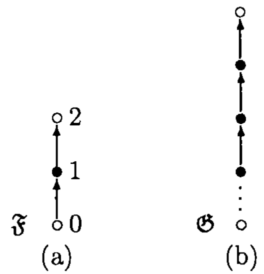


FIG. 10.2.

there are k points of type $\leq l$ then, by the same reason, we may have at most 2^{n+k} points of type $l+1$. ❑

Let $\Delta \subseteq \mathbf{Sub}\varphi$. We consider two cases. (a) All the points in $\mathfrak{M}$, at which of all formulas in $\mathbf{Sub}\varphi$ only those in Δ are true, are of type $\leq m$, for some $m < \omega$. In this case we put $m_\Delta = m$. (b) Case (a) does not hold, i.e., for every m there is a point x in $\mathfrak{M}$ such that, for every $\psi \in \mathbf{Sub}\varphi$, $x \models \psi$ iff $\psi \in \Delta$ and $x \models \Diamond^m \top$; then we put $m_\Delta = 0$. Finally, put

$$k = \max\{m_\Delta : \ \Delta \subseteq \mathbf{Sub}\varphi\} \text{ and } \Sigma = \mathbf{Sub}(\varphi \wedge \Box^k \bot).$$

We are ready now to filtrate $\mathfrak{M}$, a part of it to be more exact. Namely, we divide $\mathfrak{F} = \langle W, R \rangle$ into two parts: W_1 containing all the points in $\mathfrak{F}$ of type $\leq k$ and $W_2 = W - W_1$. By Lemma 10.55, W_1 is finite. For every $x, y \in W$, put $x \sim y$ if either $x, y \in W_1$ and $x = y$ or $x, y \in W_2$ and $x \sim_\Sigma y$. Having defined the equivalence classes $[x] = \{y \in W : \ x \sim y\}$ for $x \in W$, we can construct the corresponding finest filtration $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ of $\mathfrak{M}$ as was done in Section 5.3 (in fact we filtrate only points in W_2 and leave those in W_1 untouched) and prove that, for every $\psi \in \Sigma$, $(\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{N}, [x]) \models \psi$. Thus we have a finite model $\mathfrak{N}$ refuting φ.

It follows also that a point $[x]$ in $\mathfrak{N}$ is of type $m \leq k$ iff x has type m in $\mathfrak{M}$. Moreover, it turns out that $\mathfrak{N}$ contains no $[x]$ of type $l > k$. Indeed, otherwise $x \not\models \Box^k \bot$ and so Case (a) does not hold for $\Delta = \{\psi \in \mathbf{Sub}\varphi : \ x \models \psi\}$. This means that for every $m < \omega$ there is $y \in [x]$ such that $y \models \Diamond^m \top$ and so arbitrary long chains (of not necessarily distinct points) start from $[x]$, contrary to $[x]$ being of type l.

Thus $\mathfrak{G}$ contains two parts: the upper part consisting of points of type $\leq k$, which is clearly the generated subframe $\langle W_1, R \upharpoonright W_1 \rangle$ of $\mathfrak{F}$, and the lower one consisting of points without types, i.e., involved in some cycles. It follows that $\mathfrak{G} \models L$. For otherwise, according to the proof of Theorem 10.53, we have $\mathfrak{G} \not\models \Box^n \bot \wedge \delta^{n-1}(\mathfrak{F}') \to \neg p_r$ for some $\mathfrak{F}' \in \mathcal{F}$ (r being the root of $\mathfrak{F}'$) and $n = d(\mathfrak{F}')$, which means that the subframe $\mathfrak{G}'$ of $\mathfrak{G}$ generated by some x is reducible to $\mathfrak{F}'$. But then either $\mathfrak{G}'$ is a generated subframe of $\mathfrak{F}$, contrary to $\mathfrak{F} \models L$, or $\mathfrak{G}'$ contains a cycle, contrary to $\mathfrak{F}'$ being cycle free. ❑

It is to be noted that Theorem 10.54 does not hold for NExt**K4**.

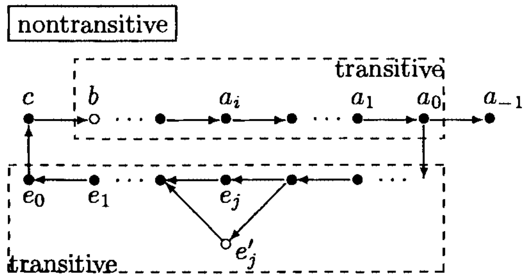


FIG. 10.3.

Example 10.56 Let us consider the logic $L = \mathbf{K4} \oplus \alpha^\sharp(\mathfrak{F}, \bot)$ and the formula $\alpha(\mathfrak{F}, \bot)$, where $\mathfrak{F}$ is the frame depicted in Fig. 10.2 (a). The frame $\mathfrak{G}$ shown in Fig. 10.2 (b) separates $\alpha(\mathfrak{F}, \bot)$ from L. Indeed, $\mathfrak{F}$ is a cofinal subframe of $\mathfrak{G}$ which, by Theorem 9.39, gives $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$. To show that $\mathfrak{G} \models \alpha^\sharp(\mathfrak{F}, \bot)$, suppose f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$. Then, by (R1), $f^{-1}(1)$ contains only one point, say x; by (R2), $f^{-1}(0)$ also contains only one point, namely, the root of $\mathfrak{G}$. So the whole infinite set of points between x and the root is outside of dom$f$, which means that f does not satisfy (CDC) for $\{\{1\}\}$.

On the other hand, suppose $\mathfrak{H}$ is a finite rooted frame refuting $\alpha(\mathfrak{F}, \bot)$ at its root. Then all final points in $\mathfrak{H}$ are reflexive. Besides, $\mathfrak{H}$ must contain a non-degenerate cluster C having an irreflexive immediate successor x. So by mapping C to 0, x to 1 and all the other points above C to 2 we obtain a reduction of the subframe of $\mathfrak{H}$ generated by C to $\mathfrak{F}$, from which $\mathfrak{H} \not\models L$.

It follows that L is not finitely approximable. Moreover, the very same argument shows that $\mathbf{K4.3} \oplus \alpha^\sharp(\mathfrak{F}, \bot)$ is not finitely approximable either.

We are in a position now to prove the main result of this section. Say that a logic $L \in \mathfrak{L}$ has *degree of Kripke incompleteness* $\varkappa$ in $\mathfrak{L}$ if exactly $\varkappa$ distinct logics in $\mathfrak{L}$ have the same Kripke frames as L. Strictly complete logics are those having degree of incompleteness 1. By Theorems 10.54 and 10.51, every union-splitting is strictly Kripke complete. All the other logics in NExt**K** turn out to have degree of incompleteness $2^{\aleph_0}$. Before proving this in general it is useful to consider two special cases, namely the logics Log$\bullet$ and Log$\circ$ (why are they not union-splittings?).

Example 10.57 We are going to construct a continual family of logics L_I, for $I \subseteq \omega - \{0\}$, the only rooted Kripke frame for which is $\bullet$. Define L_I to be the logic of the frame $\mathfrak{F}_I = \langle W_I, R_I, P_I \rangle$ with the underlying Kripke frame shown in Fig. 10.3, where the subframes in dashed boxes are transitive, a_0 sees all points e_i and e'_j, for $i < \omega$, $j \in I$, $e'_i \in W_I$ iff $i \in I$, and P_I consists of the sets of the form $X \cup Y$ such that X is a finite or cofinite subset of $\{a_{-1}, c, e_i, e'_j : i < \omega, j \in I\}$ and Y is either a finite subset of $\{a_i : i < \omega\}$ or is of the form $\{b\} \cup Y'$, where Y' is a cofinite subset of $\{a_i : i < \omega\}$ (check that P_I is closed under $-$, $\cap$ and $\downarrow$).

Observe that all points in $\mathfrak{F}_I$ save b are characterized by variable free formulas, for instance:

$$\alpha_{-1} = \Box\bot,\ \alpha_0 = \Diamond\Box\bot,$$

$$\alpha_{i+1} = \Diamond\alpha_i \wedge \neg\Diamond^2\alpha_i,\ \gamma = \Diamond^2\alpha_0 \wedge \neg\Diamond\alpha_0,$$

$$\epsilon_0 = \Diamond\gamma,\ \epsilon_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^2\epsilon_i,\ \epsilon'_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^+\epsilon_{i+1}$$

(α_i is true only at a_i, γ at c, ϵ_i at e_i, ϵ'_j at e'_j). It follows in particular that $\mathfrak{F}_I$ is 0-generated. Let $i \in I - J$. Then $\neg\epsilon'_i \in L_J - L_I$ and so there is a continuum of distinct L_I.

Since $\bullet$ is a generated subframe of $\mathfrak{F}_I$ for every I, we have $\bullet \models L_I$. We show now that if $\mathfrak{F}$ is a rooted Kripke frame for L_I then $\mathfrak{F}$ is $\bullet$. Suppose otherwise. Then root u of $\mathfrak{F}$ sees at least one point. Since

$$\alpha_{-1} \vee \alpha_0 \vee \Diamond\alpha_0 \vee \Diamond^2\alpha_0 \vee \Diamond^3\alpha_0 \in L_I,$$

we have $u \models \alpha_0 \vee \Diamond\alpha_0 \vee \Diamond^2\alpha_0 \vee \Diamond^3\alpha_0$ and so there is a point in $\mathfrak{F}$ at which α_0 is true. Using the fact that $\alpha_0 \to \Diamond^2\gamma \in L_I$, we can find a point x in $\mathfrak{F}$ such that $x \models \gamma$. Now observe that

$$\gamma \to \Box(\Box_0(\Box_0 p \to p) \to p) \in L_I,$$

where $\Box_0\varphi = \Box(\Diamond\alpha_0 \to \varphi)$. (Here we use the fact that each $X \in P_I$ contains some a_i, for $i > 0$, whenever $b \in X$.) So $x \models \Box(\Box_0(\Box_0 p \to p) \to p)$ *for any valuation* in $\mathfrak{F}$. By the definition of γ, there is $y \in x\uparrow$ such that $y \models \Diamond\alpha_0$ and also $y \models \Box_0(\Box_0 p \to p) \to p$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = y\uparrow$. Then clearly $y \models \Box_0(\Box_0 p \to p)$, from which $y \models p$ and so $y \in y\uparrow$. Now define another valuation $\mathfrak{V}'$ so that $\mathfrak{V}'(p) = y\uparrow - \{y\}$. Since y is reflexive, we again have $y \models \Box_0(\Box_0 p \to p)$, whence $y \models p$, which is a contradiction.

Thus $\bullet$ is the only rooted Kripke frame for L_I and Log$\bullet$ has degree of Kripke incompleteness $2^{\aleph_0}$ in **NExtK**.

Example 10.58 To prove that Log$\circ$ also has degree of Kripke incompleteness $2^{\aleph_0}$, we take the logics L'_I of the frames $\mathfrak{F}'_I = \langle W_I, R'_I, P_I\rangle$ in which $R'_I = R_I \cup \{\langle a_{-1}, a_{-1}\rangle\}$, i.e., the dead end in Fig. 10.3 is replaced by a reflexive point. This replacement makes it impossible to use variable free formulas. We overcome this obstacle with the help of the formulas

$$\alpha_0 = (\Diamond \bigwedge_{i=0}^{6} \Box^i q \wedge \neg \bigwedge_{i=0}^{6} \Box^i q) \vee (\Diamond \bigwedge_{i=0}^{6} \Box^i \neg q \wedge \neg \bigwedge_{i=0}^{6} \Box^i \neg q),$$

$$\alpha_{i+1} = \Diamond\alpha_i \wedge \neg\Diamond^2\alpha_i,\ \gamma = \Diamond^2\alpha_0 \wedge \neg\Diamond\alpha_0,$$

$$\epsilon_0 = \Diamond\gamma,\ \epsilon_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^2\epsilon_i,\ \epsilon'_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^+\epsilon_{i+1},$$

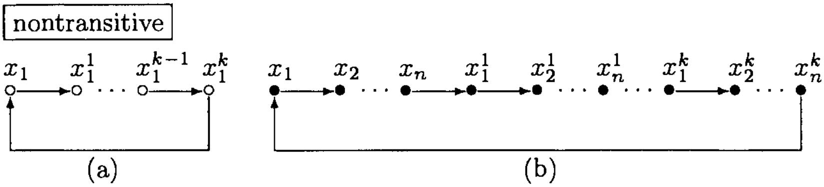


FIG. 10.4.

$$\delta = \neg \bigwedge_{i=0}^{4} \Box^i q \wedge \neg \bigwedge_{i=0}^{4} \Box^i \neg q$$

and by observing that $\bigwedge_{i=0}^{5} \Box^i p \to \Box^6 p \in L'_I$. The formulas above characterize points in $\mathfrak{F}'_I$ in the sense that if, under some valuation, δ is true somewhere in $\mathfrak{F}'_I$ then α_i, γ, ϵ_i and ϵ'_j are true only at the points a_i, c, e_i, e'_j, respectively; α_{-1} is characterized by $\neg\delta$. It follows that

$$\delta \to \gamma \vee \Diamond\gamma \vee \Diamond^2\gamma \vee \Diamond^3\gamma \in L'_I,\ \gamma \to \Box(\Box_0(\Box_0 p \to p) \to p) \in L'_I.$$

Since $\neg\epsilon'_j \in L'_J - L'_I$, for $j \in I - J$, there is a continuum of L'_I.

Let us show that $\circ$ is the only rooted Kripke frame for L'_I. Suppose otherwise, i.e., there is a rooted Kripke frame $\mathfrak{F}$ for L'_I different from $\circ$. Clearly $\mathfrak{F} \neq \bullet$, because $\Diamond\top \in L'_I$. Therefore, $\mathfrak{F}$ contains a root, say u, and some other point besides. Putting $\mathfrak{V}(q) = \{u\}$, we have $u \models \delta$ and so there is a point x in $\mathfrak{F}$ such that $x \models \Box(\Box_0(\Box_0 p \to p) \to p)$ under any valuation for p. The rest of the argument is the same as in Example 10.57.

Theorem 10.59. (Blok's theorem) *Suppose L is a normal modal logic. If $L = \mathbf{For}\mathcal{ML}$ or L is a union-splitting in* NExt$\mathbf{K}$ *then L is strictly Kripke complete. Otherwise L has degree of Kripke incompleteness $2^{\aleph_0}$ in* NExt$\mathbf{K}$.

Proof Suppose that L is not a union-splitting and L' is the greatest union-splitting (the sum of all union-splittings) contained in L. By Theorem 10.54, L' is finitely approximable and, since $L' \neq L$, there is a finite rooted frame $\mathfrak{F} = \langle W, R\rangle$ validating L' and refuting some $\varphi \in L$. Clearly, $\mathfrak{F}$ can be chosen to be minimal in the sense that its every proper generated subframe is a frame for L. It should be also clear that $\mathfrak{F}$ is not cycle free (for otherwise L' would not be the greatest union-splitting contained in L). Let $x_1 R x_2 R \dots R x_n R x_1$ be the shortest cycle in $\mathfrak{F}$ and $k = md(\varphi) + 1$.

We construct a new frame $\mathfrak{F}'$ by extending the cycle $x_1, \dots, x_n, x_1$ as is shown in Fig. 10.4 ((a) for $n = 1$ and (b) for $n > 1$). More precisely, we add to $\mathfrak{F}$ copies $x_i^1, \dots, x_i^k$ of x_i for each $i \in \{1, \dots, n\}$, organize them into the nontransitive cycle shown in Fig. 10.4 and draw an arrow from x_i^j to $y \in W - \{x_1, \dots, x_n\}$ iff $x_i R y$. Denote the resulting frame by $\mathfrak{F}' = \langle W', R'\rangle$ and let $x' = x_n^k$. By the construction, $\mathfrak{F}$ is a reduct of $\mathfrak{F}'$. It follows from Proposition 3.2 that for all models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ and $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}'\rangle$ such that

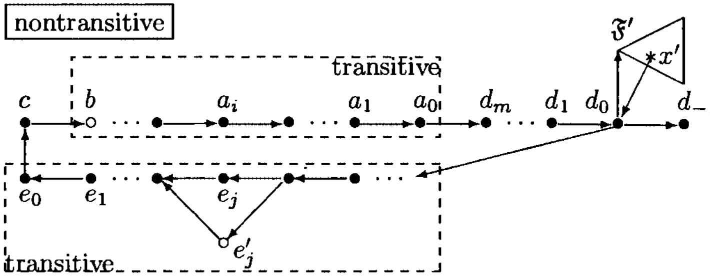


FIG. 10.5.

$$\mathfrak{V}'(p) = \mathfrak{V}(p) \cup \{x_i^j : x_i \in \mathfrak{V}(p),\ j \leq k\},\ p \in \mathbf{Var}\varphi,$$

and for every $x \in W$, $\psi \in \mathbf{Sub}\varphi$, $(\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{M}', x) \models \psi$. In particular, we can hook some other model on x', and points in W will not feel its presence by means of φ's subformulas.

The frame to be hooked on x' is similar to those in Examples 10.57 and 10.58. It depends on whether $\bullet \models L$ or $\circ \models L$. We consider only the former alternative leaving the latter to the reader as an exercise.

Fix some $m > |W'|$. For each $I \subseteq \omega - \{0\}$, let $\mathfrak{F}_I = \langle W_I, R_I, P_I \rangle$ be the frame whose diagram is shown in Fig. 10.5 (d_0 sees the root of $\mathfrak{F}'$, all points e_i and e'_j, for $i < \omega$, $j \in I$, and is seen from x'; the subframes in dashed boxes are transitive, $e'_i \in W_I$ iff $i \in I$) and P_I consists of sets of the form $X \cup Y$ such that X is a finite or cofinite subset of $W_I - \{b, a_i : i < \omega\}$ and Y is either a finite subset of $\{a_i : i < \omega\}$ or is of the form $\{b\} \cup Y'$, where Y' is a cofinite subset of $\{a_i : i < \omega\}$. It is not hard to see that the points a_i, c, e_i and e'_i in $\mathfrak{F}_I$ are characterized by the following variable free formulas:

$$\alpha_0 = \Diamond(\delta_m \wedge \Diamond(\delta_{m-1} \wedge \ldots \wedge \Diamond\delta_0)\ldots) \wedge \neg\Diamond^2(\delta_m \wedge \Diamond(\delta_{m-1} \wedge \ldots \wedge \Diamond\delta_0)\ldots),$$

$$\alpha_{i+1} = \Diamond\alpha_i \wedge \neg\Diamond^2\alpha_i,\ \gamma = \Diamond^2\alpha_0 \wedge \neg\Diamond\alpha_0,$$

$$\epsilon_0 = \Diamond\gamma,\ \epsilon_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^2\epsilon_i,\ \epsilon'_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^+\epsilon_{i+1},$$

where

$$\delta_0 = \Diamond\Box\bot,\ \delta_1 = \Diamond\delta_0 \wedge \neg\delta_0,\ \delta_2 = \Diamond\delta_1 \wedge \neg\delta_1 \wedge \neg\Diamond^+\delta_0,$$

$$\delta_{k+1} = \Diamond\delta_k \wedge \neg\delta_k \wedge \neg\Diamond^+\delta_{k-1} \wedge \ldots \wedge \neg\Diamond^+\delta_0.$$

(Here we use the fact that $m > |W'|$.) Define L_I as the logic of all frames for L and $\mathfrak{F}_I$. Since $\neg(\epsilon'_i \wedge \Diamond^{m+6}\neg\varphi) \in L_J - L_I$ for $i \in I - J$ (φ is refuted at the root of $\mathfrak{F}'$), the cardinality of the family $\{L_I : I \subseteq \omega - \{0\}\}$ is that of the continuum.

Let us show now that L_I shares the same Kripke frames with L. Clearly, $L_I \subseteq L$ and so we must prove that every Kripke frame for L_I validates L. Suppose otherwise. Then we have a rooted Kripke frame $\mathfrak{G}$ such that $\mathfrak{G} \models L_I$ but $\mathfrak{G} \not\models \psi$, for some $\psi \in L$. Since ψ is in L, it is valid in all frames for L, in particular, $\bullet \models \psi$. And since $\psi \notin L_I$, ψ is refuted in $\mathfrak{F}_I$. Moreover, by the construction of $\mathfrak{F}_I$, it is refuted at a point from which the root of $\mathfrak{F}'$ can be reached by a number of steps. Therefore, the following formulas are valid in $\mathfrak{F}_I$ and so belong to L_I and are valid in $\mathfrak{G}$:

$$\neg\psi \to \bigvee_{i=0}^{l} \Diamond^i \gamma, \tag{10.12}$$

$$\neg\psi \to \bigwedge_{i=0}^{l} \Box^i(\gamma \to \Box(\Box_0(\Box_0 p \to p) \to p)), \tag{10.13}$$

where the variable p does not occur in ψ and l is a sufficiently big number so that any point in $\mathfrak{F}_I$ is accessible by $\leq l$ steps from every point in the selected cycle and every point at which ψ may be false, and as before $\Box_0\chi = \Box(\Diamond\alpha_0 \to \chi)$.

According to (10.12), $\mathfrak{G}$ contains a point at which γ is true. By the construction of γ, this point has a successor at which, by (10.13), $\Box_0(\Box_0 p \to p) \to p$ and $\Diamond\alpha_0$ are true. Thus, we find ourselves in exactly the same contradictory situation as in Example 10.57, which proves that $\mathfrak{G} \models L$. ❑

This construction can be used to obtain one more important result.

Theorem 10.60 *Every union-splitting* $\mathbf{K}/\mathcal{F}$ *has* $\varkappa \leq \aleph_0$ *immediate predecessors in* NExt$\mathbf{K}$, *where* $\varkappa$ *is the number of frames in* $\mathcal{F}$ *which are not reducts of generated subframes of other frames in* $\mathcal{F}$. *Every consistent logic different from union-splittings has* $2^{\aleph_0}$ *immediate predecessors in* NExt$\mathbf{K}$. *(***For** $\mathcal{ML}$ *has 2 immediate predecessors in* NExt$\mathbf{K}$.*)*

Proof The former claim follows from Theorem 10.52. As to the latter, we demonstrate the idea of the proof assuming that $L \subseteq \mathrm{Log}\bullet$ and L is finitely axiomatizable over L_I constructed in the proof of the preceding theorem (which in fact is always the case). The general case is left to the reader.

By Zorn's lemma, NExtL_I contains an immediate predecessor L_I' of L. Besides, $L_I \oplus L_J = L$ whenever $I \neq J$. Indeed,

$$L_I \oplus L_J = (L \cap \mathrm{Log}\mathfrak{F}_I) \oplus (L \cap \mathrm{Log}\mathfrak{F}_J) = L \cap (\mathrm{Log}\mathfrak{F}_I \oplus \mathrm{Log}\mathfrak{F}_J)$$

and if $i \in I - J$ then, for every $\chi \in L$ and a sufficiently big l,

$$\neg \bigvee_{k=0}^{l} \Diamond^k \epsilon_i' \to \chi \in \mathrm{Log}\mathfrak{F}_I, \quad \neg\epsilon_i' \in \mathrm{Log}\mathfrak{F}_J,$$

from which $\chi \in \mathrm{Log}\mathfrak{F}_I \oplus \mathrm{Log}\mathfrak{F}_J$ and so $L \subseteq \mathrm{Log}\mathfrak{F}_I \oplus \mathrm{Log}\mathfrak{F}_J$. It follows that $L_I' \neq L_J'$ whenever $I \neq J$. ❑

It is worth noting that tabular logics, proper extensions of $\mathbf{D}$ and extensions of $\mathbf{K4}$ are not union-splittings in NExt$\mathbf{K}$.

10.6 Exercises and open problems

Exercise 10.1 Show that canonicity is preserved under sums of logics.

Exercise 10.2 Show that canonicity is preserved under ρ and τ.

Exercise 10.3 Show that Kripke completeness is not preserved under sums. (Hint: see Section 6.5.)

Exercise 10.4 Show that **S4.1** is not $\mathcal{R}$-persistent. (Hint: consider the general frame associated with the model $\langle\langle\omega, \leq\rangle, \mathfrak{V}\rangle$, where $\mathfrak{V}(p_i) = \{n : i \leq n\}$.)

Exercise 10.5 Describe the ultrafilter extensions of the frames $\langle\omega, \geq\rangle$, $\langle\omega, <\rangle$, $\langle\mathbb{Z}, <\rangle$, $\langle\mathbb{Q}, <\rangle$.

Exercise 10.6 Show that for a Kripke frame $\mathfrak{F}$, $\widehat{\mathfrak{F}}$ is a reduct of some ultrapower of $\mathfrak{F}$.

Exercise 10.7 Show that, for every $i \in I$, $\widehat{\mathfrak{F}}_i$ is a generated subframe of the ultrafilter extension of $\sum_{i\in I}\mathfrak{F}_i$.

Exercise 10.8 Prove that the logics $L = \mathbf{K}\oplus\{\Box p \to p,\ \Box(\Box p \to \Box q)\vee\Box(\Box q \to \Box p),\ \Diamond p\wedge\Box(p \to \Box p) \to p,\ \Box\Diamond p \to \Diamond\Box p\}$ and $\mathbf{Triv} = \mathbf{K}\oplus\Box p \leftrightarrow p$ are distinct, but their classes of Kripke frames are defined by the same first order condition $\forall x\forall y\ (xRy \leftrightarrow x = y)$, with respect to which **Triv** is complete. Therefore, L is elementary, though neither Kripke complete nor $\mathcal{D}$-persistent.

Exercise 10.9 Show that the interval between the logics of the preceding exercise contains infinitely many logics.

Exercise 10.10 Let $\varphi = \Diamond\Box(p \vee q) \to \Diamond(\Box p \vee \Box q)$ and

$$\phi = \forall x\forall y(xRy \to \exists z(xRz \wedge \forall u(zRu \to yRu) \wedge \forall u\forall v(zRu \wedge zRv \to u = v))).$$

Prove that

(i) $\mathfrak{F} \models \phi$ implies $\mathfrak{F} \models \varphi$;
(ii) φ and ϕ are equivalent on the class of at most countable Kripke frames;
(iii) φ and ϕ are equivalent on the class of descriptive frames;
(iv) φ is not first order definable.

Exercise 10.11 Give a complete proof of the Fine–van Benthem theorem. (Hint: Let $\mathfrak{F}$ be an arbitrary Kripke frame for L and u an ultrafilter over W. Define Φ' as the union of Φ (see the proof of Theorem 10.19) and all formulas of the form

$P_X(x)$, for $X \in u$ (x is a fixed individual variable),
$\forall y\ (xR^n y \to (P_{W-X}(y) \leftrightarrow \neg P_X(y)))$,
$\forall y\ (xR^n y \to (P_{X\cap Y}(y) \leftrightarrow P_X(y) \wedge P_Y(y)))$,
$\forall y\ (xR^n y \to (P_{X\downarrow}(y) \leftrightarrow \exists z\ (yRz \wedge P_X(z))))$.

Check that Φ' has a model, say, a frame $\mathfrak{F}^* \in \mathcal{C}$, define for $\mathfrak{F}^*$ the set Φ'' as in the proof of Theorem 10.19, take a frame $\mathfrak{F}'$ in which Φ'' is satisfied at a point a and then show that the subframe of $\mathfrak{F}'$ generated by a is reducible to the subframe of $\widehat{\mathfrak{F}}$ generated by u.)

Exercise 10.12 Prove the following variant of Sahlqvist's theorem. Let φ be a formula constructed from variables, their negations, $\top$ and $\bot$ using $\wedge$, $\vee$, $\Box$, and $\Diamond$ in such a way that either (1) no positive occurrence of a variable is in a subformula of the form $\psi\wedge\chi$ or $\Box\psi$ within the scope of some $\Diamond$, or (2) no negative occurrence of a variable is in a subformula of the form $\psi \wedge \chi$ or $\Box\psi$ within the scope of some $\Diamond$. Then one can effectively construct a first order equivalent for φ. If L is $\mathcal{D}$-persistent then $L \oplus \varphi$ is also $\mathcal{D}$-persistent, and if L is elementary then so is $L \oplus \varphi$.

Exercise 10.13 Construct a continuum of logics above **S4** axiomatizable by Sahlqvist formulas. (Hint: consider the formulas

$$\alpha_{-1} = \Box(\neg r_0 \wedge \neg s_0),$$

$$\alpha_0 = \Box(\neg t \wedge \neg r_1) \wedge r_0 \wedge \Diamond\alpha_{-1}, \ \beta_0 = \Box(\neg t \wedge \neg s_1 \wedge s_0),$$

$$\alpha_1 = \Box(p \wedge \neg q) \wedge t \wedge \Diamond\alpha_0 \wedge r_1, \ \beta_1 = \Box(\neg p \wedge q) \wedge t \wedge \Diamond\beta_0 \wedge s_1,$$

$$\alpha_{i+2} = \Diamond\alpha_{i+1} \wedge \Diamond\beta_i \wedge \Box\neg s_{i+1} \wedge r_{i+2},$$

$$\beta_{i+2} = \Diamond\beta_{i+1} \wedge \Diamond\alpha_i \wedge \Box\neg r_{i+1} \wedge s_{i+2},$$

$$\gamma_n = \Diamond\Box(p \wedge q) \wedge \alpha_n, \ \delta_n = \Diamond\Box(\neg p \wedge \neg q) \wedge \beta_n, \ \epsilon_n = \Diamond\gamma_n \wedge \Diamond\delta_n.)$$

Exercise 10.14 Show that the intersection of Sahlqvist logics is also a Sahlqvist logic.

Exercise 10.15 Show that the McKinsey formula $\boldsymbol{ma}$ is not first order definable on the class of finite frames.

Exercise 10.16 Prove that frame formulas are first order definable on the class of irreflexive transitive frames. Show, however, that this is not the case on the class of all transitive frames.

Exercise 10.17 Show that the reduced frame of $\mathfrak{F}_{\mathbf{Grz}}(n)$ contains no proper clusters and $\mathfrak{F}_{\mathbf{GL}}(n)$ contains no reflexive points.

Exercise 10.18 Let $\mathfrak{F}$ be a refined finitely generated frame of finite width. Show that for every point x in $\mathfrak{F}$, either all points in $C(x)$ are noneliminable or all points in $C(x)$ are eliminable and x is reflexive.

Exercise 10.19 Suppose L is the decidable union-splitting of NExtL_0 by a finite set of finite frames. Show that in this case we can effectively decide, given a formula φ, whether $L = L_0 \oplus \varphi$.

Exercise 10.20 Prove that if $L = L_0/\mathcal{F}$ is finitely axiomatizable then L has finitely many immediate predecessors in NExtL_0 and that otherwise there are precisely $\aleph_0$ immediate predecessors.

Exercise 10.21 Show that NExt$L$ has an axiomatic basis iff every logic in NExt$L$ is a union-splitting of NExt$L$.

Exercise 10.22 Suppose a logic $L_0 \in \text{NExt}\mathbf{K4}$ is finitely approximable. Prove that the following conditions are equivalent:

(i) all union-splittings of $\text{NExt}L_0$ are finitely approximable;

(ii) all logics in $\text{NExt}L_0$ are finitely approximable;

(iii) all logics in $\text{NExt}L_0$ are union-splittings of $\text{NExt}L_0$;

(iv) $\text{NExt}L_0$ has an axiomatic basis.

Exercise 10.23 Show that for each logic $L \in \text{NExt}\mathbf{K}$, a finite rooted frame $\mathfrak{F}$ for L splits $\text{NExt}L$ iff there is $m < \omega$ such that, for every (general) frame $\mathfrak{G}$ for L, $\delta^m(\mathfrak{F}) \wedge p_r$ is satisfied in $\mathfrak{G}$ only if $\delta^n(\mathfrak{F}) \wedge p_r$ is satisfied in $\mathfrak{G}$ for all $n < \omega$. In this case $L/\mathfrak{F} = L \oplus \delta^m(\mathfrak{F}) \to \neg p_r$.

Exercise 10.24 Prove that if $\boldsymbol{tra}_m \in L$, for some $m < \omega$, then all finite rooted frames for L split $\text{NExt}L$.

Exercise 10.25 Prove that every normal modal logic containing $\Box^n\bot$ is locally tabular.

Exercise 10.26 Show that $\mathbf{T}$ is not a splitting of $\text{NExt}\mathbf{K}$.

Exercise 10.27 Prove that $\circ$ is the only finite rooted frame that splits $\text{NExt}\mathbf{T}$.

Exercise 10.28 Show that the logics L_I constructed in Example 10.57 are immediate predecessors of $\text{Log}\bullet$ in $\text{NExt}\mathbf{K}$.

Exercise 10.29 Prove that every consistent normal extension of $\mathbf{T}$ has degree of incompleteness $2^{\aleph_0}$ in $\text{NExt}\mathbf{T}$.

Exercise 10.30 Construct a continuum of Post complete quasi-normal modal logics having no Kripke frames at all.

Problem 10.1 *Are canonical logics $\mathcal{D}$-persistent?*

Problem 10.2 *Are canonicity and $\mathcal{D}$-persistence preserved under intersections of logics?*

Problem 10.3 *Does the converse of the Fine–van Benthem theorem hold?*

Problem 10.4 *Are finitely axiomatizable Sahlqvist logics in* $\text{NExt}\mathbf{K4}$ *decidable?*

Problem 10.5 *What is the degree of Kripke incompleteness of logics in the lattices* $\text{NExt}\mathbf{K4}$, $\text{NExt}\mathbf{S4}$, $\text{Ext}\mathbf{Int}$*?*

10.7 Notes

In this chapter we considered only results concerning the completeness with respect to (infinite, in general) Kripke frames. The completeness with respect to finite frames is the subject of the next chapter.

Theorem 10.3 was proved by Bellissima (1988); later on we shall mention some other results from this paper. Theorem 10.5 belongs to Wolter (1993). The notion of complex logic was introduced by Goldblatt (1989).

That every Kripke complete and elementary logic is $\mathcal{D}$-persistent was first proved by Fine (1975b). Theorem 10.19 also appeared first in Fine (1975b), but the proof contained a little gap. The presentation in Section 10.2 follows van Benthem (1979b, 1980), where the notion of ultrafilter extension was introduced and the proof of Theorem 10.19 was completed (see Exercise 10.11). Theorem 10.20 is due to Chagrova (1990), Theorem 10.23 and Exercises 10.4, 10.10 to Fine (1975b). Exercises 10.6–10.8 and 10.15–10.16 were taken from van Benthem (1989, 1978).

Theorems 10.30 and 10.31 were proved by Sahlqvist (1975). The starting point of Sahlqvist's research was the conjecture of Lemmon and Scott (1977) that formulas of the form

$$\Diamond^{m_1}\Box^{n_1}p_1 \wedge \ldots \wedge \Diamond^{m_k}\Box^{n_k}p_k \to \varphi,$$

where φ is positive, axiomatize logics that are complete with respect to first order conditions which can be "read off" from the axioms. Independently a solution to this conjecture was obtained by Goldblatt (1976b). Other proofs of Sahlqvist's theorem were given by van Benthem (1983) (who formulated it as in Exercise 10.12), Sambin and Vaccaro (1989), Kracht (1993a) (who characterized also the elementary conditions corresponding to Sahlqvist formulas), and Jónsson (1994). Here we followed the proof by Sambin and Vaccaro; Lemma 10.27 is due to Esakia (1974). The result of Exercise 10.13 was obtained in Chagrov and Zakharyaschev (1995b) where a Sahlqvist calculus above **S4** which is not finitely approximable was also constructed. Above **T** a calculus of that sort was presented by Hughes and Cresswell (1984) (see Exercise 6.11). Exercise 10.14 is due to Kracht (1995). It is not hard to construct an undecidable polymodal Sahlqvist calculus; the transfer theorem of Kracht and Wolter (1997) provides us then with an undecidable Sahlqvist calculus in NExt**K**.

Venema (1991) extended Sahlqvist's theorem to logics with non-standard inference rules like Gabbay's (1981) irreflexivity rule. An intuitionistic analog of Sahlqvist's theorem has been proved by Ghilardi and Meloni (1997). We present here a somewhat simplified version of their result. Let $\overline{p}$, $\overline{q}$, $\overline{r}$, $\overline{s}$ denote tuples of propositional variables and $\overline{\psi}$, $\overline{\chi}$ tuples of formulas of the same length as $\overline{r}$ and $\overline{s}$, respectively. Suppose $\varphi(\overline{p},\overline{q},\overline{r},\overline{s})$ is an intuitionistic formula in which the variables $\overline{r}$ occur positively and the variables $\overline{s}$ occur negatively, and which does not contain any $\to$, except for negations and double negations of atoms, in the premise of a subformula of the form $\varphi' \to \varphi''$. Assume also that $\overline{\psi}(\overline{p},\overline{q})$ and $\overline{\chi}(\overline{p},\overline{q})$ are formulas such that $\overline{p}$ occur positively in $\overline{\psi}$ and negatively in $\overline{\chi}$, while $\overline{q}$ occur negatively in $\overline{\psi}$ and positively in $\overline{\chi}$. Then the logic

$$\mathbf{Int} + \varphi(\overline{p},\overline{q},\overline{\psi}(\overline{p},\overline{q}),\overline{\chi}(\overline{p},\overline{q}))$$

is canonical.

The material of Section 10.4 was taken mainly from Fine (1974c), where the method of dropping points from the canonical models was developed in order to prove Theorems 10.42 and 10.44. Si-logics of finite width were studied by Sobolev (1977a). Some decidability results concerning logics of finite width can be found in Chapter 16. As follows from Theorem 10.44, there are strongly complete modal logics that are not $\mathcal{D}$-persistent. But these logics are formulated in finite languages. Recently Wolter [1996b] has constructed a logic of that sort in the infinite language.

The question concerning the degree of Kripke incompleteness was raised by Fine (1974b) and solved for the lattices NExt**K**, NExt**D** and NExt**T** by Blok (1978, 1980b). Similar results concerning the degree of incompleteness with respect to neighborhood frames were obtained by Dziobiak [1978] for NExt**T** and NExt($\mathbf{D} \oplus \Box^n p \to \Box^{n+1} p$), and quite recently Chagrova has proved that the situation with the degree of neighborhood incompleteness in the whole class NExt**K** is exactly the same as in Blok's theorem. Theorem 10.60 is also due to Blok (1978).

The notion of splitting was introduced in lattice theory by Whitman (1943). McKenzie (1972) considered splitting varieties of lattices. In modal logic splittings were used by Blok (1978), Rautenberg (1977, 1979, 1980), Kracht (1990, 1993c) and Wolter (1993). The result of Exercise 10.19 was proved by Jankov (1968a) and Rautenberg (1979), that of Exercise 10.23 by Kracht (1990). Exercises 10.21 and 10.22 are due to Wolter (1993) and Exercise 10.24 to Rautenberg (1980). Rafter (1994) gave a partial characterization of canonical union-splittings. Later he showed that a continuum of union-splittings are canonical and as many are not.

11

FINITE APPROXIMABILITY

Let us now go one step down the hierarchy of frame classes and consider a stronger form of completeness, viz., completeness with respect to the class of finite frames or, in other terms, finite approximability. We have already met with one way of proving this property—the filtration method, requiring a special *ad hoc* technique in each particular case. Now we will show a few other methods which provide us in fact with general syntactical and semantic sufficient conditions of finite approximability.

11.1 Uniform logics

We begin with two results connecting the finite approximability of modal logics with the distribution of the operators $\Box$ and $\Diamond$ over their axioms. The first result to be obtained in this section concerns those normal extensions of deontic logic **D** whose additional axioms are uniform in the following sense.

We say φ is a *uniform formula of degree* 0 if $md(\varphi) = 0$, i.e., φ contains no modal operators at all. φ is a *uniform formula of degree* $n+1$ if there are a uniform formula $\psi(p_1, \dots, p_m)$ of degree 0 and uniform formulas $\chi_1, \dots, \chi_m$ of degree n such that $\varphi = \psi(\bigcirc_1\chi_1, \dots, \bigcirc_m\chi_m)$ where each $\bigcirc_i$ is either $\Box$ or $\Diamond$. In other words, a uniform formula of degree $n+1$ is a Boolean combination of formulas of the form $\Box\chi$ or $\Diamond\chi$ such that χ is a uniform formula of degree n. For example, both the McKinsey and Geach formulas are uniform formulas of degree 2, while the Löb and Grzegorczyk ones are not uniform.

The set of all uniform formulas of degree n is denoted by $\mathbf{U}_n$ and $\mathbf{U}$, the set of *uniform formulas*, is the union of all $\mathbf{U}_n$.

A remarkable property of uniform formulas of degree n is that their truth-values at a point x in a model are completely determined by the truth-values of their variables at the points accessible from x by n steps. More exactly, the following proposition holds (compare it with Proposition 3.2).

Proposition 11.1 *Suppose φ is a uniform formula of degree n and $\mathfrak{M}$, $\mathfrak{N}$ are models based upon the same frame and such that, for some point x, $(\mathfrak{M}, y) \models p$ iff $(\mathfrak{N}, y) \models p$ for any $y \in x{\uparrow}^n$ and any $p \in \mathbf{Var}\varphi$. Then $(\mathfrak{M}, x) \models \varphi$ iff $(\mathfrak{N}, x) \models \varphi$.*

Proof The proof proceeds by induction on n. The basis of induction is trivial, and the inductive step is justified by another induction on the construction of the uniform formula φ of degree $n = m+1$. The basis of the second induction is the case when $\varphi = \Box\psi$ or $\varphi = \Diamond\psi$ with $\psi \in \mathbf{U}_m$. Let $\varphi = \Box\psi$. Then we have:

$$(\mathfrak{M}, x) \models \varphi \text{ iff } \forall z \in x{\uparrow}\ (\mathfrak{M}, z) \models \psi$$

$$\text{iff } \forall z \in x\!\uparrow\ (\mathfrak{N}, z) \models \psi$$
$$\text{iff } (\mathfrak{N}, x) \models \varphi.$$

The only nontrivial transition here (from the first line to the second) is ensured by the induction hypothesis of the first induction and the fact that $z\!\uparrow^m \subseteq x\!\uparrow^{m+1}$ for every $z \in x\!\uparrow$. The case $\varphi = \Diamond\psi$ is considered analogously. The inductive step of the second induction presents no difficulty. ❑

A normal modal logic L is called *uniform* if it can be represented in the form $L = \mathbf{D} \oplus \Gamma$ where $\Gamma \subseteq \mathbf{U}$. In this section we prove that all uniform logics are finitely approximable.

To construct a finite frame separating a uniform logic L from a formula $\varphi \notin L$, we reduce $\neg\varphi$ to a form which is analogous to the full disjunctive normal form in $\mathbf{Cl}$ (see Exercise 1.2) and gives in fact a description of some finite models for $\neg\varphi$.

Let $\mathbf{Var} = \{p_1, \dots, p_r\}$ be a finite set of propositional variables. By induction on n we define a set $\mathbf{NF}_n$ of *normal forms* (in $\mathbf{Var}$) *of degree* n: $\mathbf{NF}_0$ is the set of all formulas of the form

$$\neg_1 p_1 \wedge \ldots \wedge \neg_r p_r,$$

where each $\neg_i$, for $i = 1, \dots, r$, is either blank or $\neg$, and $\mathbf{NF}_{n+1}$ is the set of all formulas of the form

$$\theta \wedge \neg_1 \Diamond\theta_1 \wedge \ldots \wedge \neg_s \Diamond\theta_s,$$

where $\theta \in \mathbf{NF}_0$, $\theta_1, \dots, \theta_s$ are all the distinct normal forms in $\mathbf{NF}_n$ and each $\neg_i$ is either blank or $\neg$. $\mathbf{NF}$, the set of *normal forms* in $\mathbf{Var}$, is the union of all $\mathbf{NF}_i$ for $i < \omega$.

Theorem 11.2 *Every modal formula φ with $\mathbf{Var}\varphi \subseteq \mathbf{Var}$ and $md(\varphi) \leq n$ is equivalent in $\mathbf{K}$ either to $\bot$ or to a disjunction of normal forms in $\mathbf{Var}$ of degree n.*

Proof We proceed by induction on n. The basis of induction is simply the theorem on the full disjunctive normal forms in $\mathbf{Cl}$ (see Exercise 1.2).

Now, suppose $md(\varphi) \leq k+1$. Replacing each $\Box$ in φ with $\neg\Diamond\neg$, we can reduce φ to an equivalent formula which is a Boolean combination of propositional variables and formulas $\Diamond\psi$ with $md(\psi) \leq k$. By the induction hypothesis and the $\mathbf{K}$-equivalences $\Diamond\bot \leftrightarrow \bot$ and $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$, each such $\Diamond\psi$ is equivalent either to $\bot$ or to a disjunction $\Diamond\theta_1 \vee \ldots \vee \Diamond\theta_m$ where $\theta_1, \dots, \theta_m$ are normal forms of degree k. Therefore, φ is equivalent to a formula of the form

$$\psi(p_1, \dots, p_r, q_1, \dots, q_s)\{\Diamond\theta_1/q_1, \dots, \Diamond\theta_s/q_s\},$$

where ψ contains no modal operators and $\theta_1, \dots, \theta_s$ are all the distinct formulas in $\mathbf{NF}_k$. Finally, reducing $\psi(p_1, \dots p_r, q_1, \dots, q_s)$ to the full disjunctive normal form and substituting $\Diamond\theta_1, \dots, \Diamond\theta_s$ for $q_1, \dots, q_s$ in it, respectively, we obtain an equivalent formula which is either $\bot$ or a disjunction of normal forms in $\mathbf{Var}$ of degree $k+1$. ❑

It is worth noting that for any distinct normal forms θ' and θ'' (in **Var**) of the same degree the implication $\theta' \to \neg\theta''$ is true in every model and so belongs to **K**. It follows that for every normal form θ in **Var** of degree n and every modal formula φ with $\mathbf{Var}\varphi \subseteq \mathbf{Var}$ and $md(\varphi) \leq n$ we have either $\theta \to \varphi \in \mathbf{K}$ or $\theta \to \neg\varphi \in \mathbf{K}$. Indeed, by Theorem 11.2, φ is equivalent in **K** either to $\bot$, in which case $\theta \to \neg\varphi \in \mathbf{K}$, or to a disjunction δ of normal forms of degree n. If θ is a disjunct of δ then $\theta \to \varphi \in \mathbf{K}$; otherwise $\theta \to \neg\varphi \in \mathbf{K}$.

Of course, normal forms are too lengthy to be used in practice: each θ in $\mathbf{NF}_{n+1}$ contains $|\mathbf{NF}_n| + r$ conjuncts and $|\mathbf{NF}_n|$ is calculated recursively as

$$|\mathbf{NF}_0| = 2^r \text{ and } |\mathbf{NF}_n| = |\mathbf{NF}_0|\, 2^{|\mathbf{NF}_{n-1}|}.$$

However, they provide us with another theoretical tool for constructing models.

First we define a binary relation $<$ on **NF** by putting $\theta' < \theta''$ iff $\Diamond\theta'$ is a conjunct of θ'', for every $\theta', \theta'' \in \mathbf{NF}$. We write $\theta' <^n \theta''$ if there are $\theta_1, \dots, \theta_{n-1}$ such that $\theta' < \theta_1 < \dots < \theta_{n-1} < \theta''$; $\theta' <^0 \theta''$ means $\theta' = \theta''$. Now, with each normal form θ we associate a model $\mathfrak{M}_\theta = \langle \mathfrak{F}_\theta, \mathfrak{V}_\theta \rangle$ on a frame $\mathfrak{F}_\theta = \langle W_\theta, R_\theta \rangle$ which are defined as follows:

$$W_\theta = \{\theta' \in \mathbf{NF} :\ \theta' <^n \theta, \text{ for some } n \geq 0\},$$

$$\theta' R_\theta \theta'' \text{ iff } \theta' > \theta'',$$

$$\mathfrak{V}_\theta(p) = \{\theta' \in W_\theta :\ p \text{ is a conjunct of } \theta'\}.$$

Theorem 11.3 *For each normal form θ and each $\theta' \in W_\theta$, $(\mathfrak{M}_\theta, \theta') \models \theta'$.*

Proof An easy induction on the degree of θ' is left to the reader as an exercise (see also the proof of Theorem 11.6.) ❑

Note that Theorem 11.3 yields another proof of the finite approximability of **K**. Indeed, if $\varphi \notin \mathbf{K}$ then we reduce $\neg\varphi$ to a disjunction of normal forms. Since $\neg\varphi$ is not equivalent to $\bot$ (for otherwise φ would be equivalent to $\top$, contrary to $\varphi \notin \mathbf{K}$), this disjunction is not empty. Let θ be one of its disjuncts. Then, according to Theorem 11.3, we have $(\mathfrak{M}_\theta, \theta) \models \theta$ and so $(\mathfrak{M}_\theta, \theta) \not\models \varphi$. However, this proof does not go through for logics $L \supset \mathbf{K}$, because $\mathfrak{F}_\theta$ is not in general a frame for L. For example, no frame $\mathfrak{F}_\theta$ validates **D**, since it is finite and all its points are irreflexive.

For **D** the argument above will remain correct, if we somewhat modify the definitions of normal form and the model $\mathfrak{M}_\theta$. Observe first that the following proposition holds.

Proposition 11.4 *Suppose that, for some $n \geq 0$, $\mathbf{NF}_n = \{\theta_1, \dots, \theta_s\}$. Then $\Diamond\theta_1 \vee \dots \vee \Diamond\theta_s \in \mathbf{D}$.*

Proof The formula $\theta_1 \vee \dots \vee \theta_s$ is valid in **Cl** and so $\Diamond\theta_1 \vee \dots \vee \Diamond\theta_s \in \mathbf{D}$, since $\Diamond\theta_1 \vee \dots \vee \Diamond\theta_s \leftrightarrow \Diamond(\theta_1 \vee \dots \vee \theta_s) \in \mathbf{K}$ and $\Diamond\top \in \mathbf{D}$. ❑

It follows that every normal form $\theta \wedge \neg\Diamond\theta_1 \wedge \ldots \wedge \neg\Diamond\theta_s$ is equivalent to $\perp$ in $\mathbf{D}$, and so we can define $\mathbf{D}$-*suitable* normal forms like this. Every normal form of degree 0 is $\mathbf{D}$-suitable, and a normal form θ of degree $n > 0$ is $\mathbf{D}$-suitable if every $\theta' < \theta$ is $\mathbf{D}$-suitable and there is at least one $\theta' < \theta$. Alternatively, this means that in the inductive step of the original definition of normal form we require at least one $\neg_i$ to be blank. The next theorem is proved similarly to Theorem 11.2, using Proposition 11.4.

Theorem 11.5 *Every modal formula φ with* $\mathbf{Var}\varphi \subseteq \mathbf{Var}$ *and $md(\varphi) \leq n$ is equivalent in* $\mathbf{D}$ *either to $\perp$ or to a disjunction of* $\mathbf{D}$-*suitable normal forms in* $\mathbf{Var}$ *of degree n.*

As to the frame $\mathfrak{F}_\theta$, we can make it serial by adding to it a reflexive point accessible from the final points in W_θ. More exactly, given a normal form θ, define a model $\mathfrak{N}_\theta = \langle \mathfrak{G}_\theta, \mathfrak{U}_\theta \rangle$ on a frame $\mathfrak{G}_\theta = \langle V_\theta, S_\theta \rangle$ by taking

$$V_\theta = W_\theta \cup \{\top\},$$

$$\theta' S_\theta \theta'' \text{ iff either } \theta' R_\theta \theta'' \text{ or } md(\theta') = 0 \text{ and } \theta'' = \top,$$

$$\mathfrak{U}_\theta(p) = \mathfrak{V}_\theta(p).$$

It should be clear that if θ is $\mathbf{D}$-suitable, $\top$ is the reflexive last point in $\mathfrak{G}_\theta$, and so $\mathfrak{G}_\theta$ is serial.

Theorem 11.6 *For every normal form θ and every $\theta' \in V_\theta$, $(\mathfrak{N}_\theta, \theta') \models \theta'$.*

Proof By induction on the degree of θ'. The basis of induction is trivial.

Suppose $\theta' = \theta_0 \wedge \neg_1\Diamond\theta_1 \wedge \ldots \wedge \neg_s\Diamond\theta_s$ is of degree $n+1$. By the definition of $\mathfrak{U}_\theta$, $(\mathfrak{N}_\theta, \theta') \models \theta_0$. If $\neg_i$ is blank then $\theta_i < \theta'$, whence $\theta' S_\theta \theta_i$ and $(\mathfrak{N}_\theta, \theta') \models \Diamond\theta_i$, since, by the induction hypothesis, $(\mathfrak{N}_\theta, \theta_i) \models \theta_i$. And if $\neg_i$ is $\neg$ then $(\mathfrak{N}_\theta, \theta') \models \neg\Diamond\theta_i$, for otherwise $\theta' S_\theta \theta''$ and $(\mathfrak{N}_\theta, \theta'') \models \theta_i$ for some $\theta'' \in V_\theta$. By the definition of S_θ, θ'' is either a normal form of degree n or $\top$. The former case means $\theta'' = \theta_i$, since, by the induction hypothesis, $(\mathfrak{N}_\theta, \theta'') \models \theta''$ and since distinct normal forms cannot be simultaneously true at the same point; but this contradicts the definition of $<$. And in the latter case $md(\theta') = 0$, which is also impossible. ❑

Thus, the argument used above for proving the finite approximability of $\mathbf{K}$ remains valid for $\mathbf{D}$ too. Moreover, we will show now that it goes through for all uniform logics as well.

Suppose L is a uniform logic. Call a normal form θ L-*suitable* if $\mathfrak{G}_\theta$ is a frame for L. It should be clear that this definition agrees with the definition of $\mathbf{D}$-suitability.

Theorem 11.7 *Suppose L is a uniform logic. Then every modal formula φ with* $\mathbf{Var}\varphi \subseteq \mathbf{Var}$ *and $md(\varphi) \leq n$ is equivalent in L either to $\perp$ or to a disjunction of L-suitable normal forms in* $\mathbf{Var}$ *of degree n.*

Proof By Theorem 11.5, φ is equivalent in $\mathbf{D}$ to $\bot$ or a disjunction of $\mathbf{D}$-suitable normal forms of degree n. So it suffices to show that every $\mathbf{D}$-suitable normal form θ such that $\theta \to \bot \notin L$ is L-suitable. (If $\neg\theta \in L$ then θ is equivalent to $\bot$ in L.)

Suppose otherwise. Let θ be an L-consistent and $\mathbf{D}$-suitable normal form of the least possible degree under which it is not L-suitable. Then a uniform formula $\psi \in L$ of some degree m is refuted at the point θ in $\mathfrak{G}_\theta$, i.e., there is a model $\mathfrak{M} = \langle \mathfrak{G}_\theta, \mathfrak{V} \rangle$ such that $(\mathfrak{M}, \theta) \not\models \psi$.

For every $p \in \mathbf{Var}\psi$, let $\Gamma_p = \{\theta' \in \theta{\uparrow}^m\colon\ (\mathfrak{M}, \theta') \models p\}$ and let δ_p be the disjunction of all the formulas in Γ_p (if $\Gamma_p = \emptyset$ then $\delta_p = \bot$). Observe that for every $\theta' \in \theta{\uparrow}^m$ we have:

$$
\begin{aligned}
(\mathfrak{N}_\theta, \theta') \models \delta_p \ &\text{iff } \theta' \text{ is a disjunct of } \delta_p\\
&\text{iff } \theta' \in \Gamma_p\\
&\text{iff } (\mathfrak{M}, \theta') \models p.
\end{aligned}
$$

Therefore, by Proposition 11.1, the formula $\psi' = \psi\{\delta_p/p : \ p \in \mathbf{Var}\psi\}$ is false at θ in $\mathfrak{N}_\theta$. Now, if $md(\psi') > n$ then $m > n$ and so $\delta_p = \bot$ for every $p \in \mathbf{Var}\psi$, i.e., ψ' is variable free. But according to Exercise 3.19, ψ' is then equivalent in $\mathbf{D}$ to $\top$ or $\bot$, contrary to $\mathfrak{G}_\theta \not\models \psi'$ and the consistency of L. And if $md(\psi') \le n$ then, as we have observed, either $\theta \to \psi' \in \mathbf{K}$, which is impossible, since $(\mathfrak{N}_\theta, \theta) \not\models \theta \to \psi'$, or $\theta \to \neg\psi' \in \mathbf{K}$, from which $\psi' \to \neg\theta \in \mathbf{K}$ and so $\neg\theta \in L$, contrary to the L-consistency of θ. ❑

As a consequence of this theorem we obtain our final result.

Theorem 11.8 *Every uniform logic is finitely approximable.*

In particular, the McKinsey logic $\mathbf{K} \oplus \Box\Diamond p \to \Diamond\Box p = \mathbf{D} \oplus \Box\Diamond p \to \Diamond\Box p$ turns out to be finitely approximable.

11.2 Si-logics with essentially negative axioms and modal logics with $\Box\Diamond$-axioms

A formula is said to be *essentially negative* if every occurrence of a variable in it is in the scope of some $\neg$. For example, the Skvortsov formula in Exercise 2.16 is essentially negative. The following three facts:

- Glivenko's theorem,
- the local tabularity of $\mathbf{Cl}$, and
- a possibility of transforming a derivation of any formula φ in any logic in such a way that it should not contain variables having no occurrences in φ

enable us to reduce the derivability problem in a superintuitionistic logic with a finite set of essentially negative additional axioms to the derivability problem in $\mathbf{Int}$. Indeed, suppose ψ is an essentially negative formula, i.e., $\psi = \psi'(\neg\chi_1, \dots, \neg\chi_n)$ for some formulas $\psi'(q_1, \dots, q_n)$, $\chi_1, \dots, \chi_n$, and φ is an arbitrary formula. How can we decide whether or not $\varphi \in \mathbf{Int} + \psi$?

Let $p_1, \dots, p_m$ be all the variables in φ. If $\varphi \in \mathbf{Int} + \psi$ then there exists a substitutionless derivation of φ in $\mathbf{Int} + \psi$ in which substitution instances of the axiom ψ contain no variables different from $p_1, \dots, p_m$. Each of these substitution instances has the form $\psi'(\neg\chi_1', \dots, \neg\chi_n')$ where every χ_i', for $i = 1, \dots, n$, is some substitution instance of χ_i containing only (some of) $p_1, \dots, p_m$. By Glivenko's theorem (Corollary 2.49, to be more exact) and in view of the local tabularity of $\mathbf{Cl}$, there are $\leq 2^{2^m}$ pairwise non-equivalent in $\mathbf{Int}$ such substitution instances of $\neg\chi_i$, for each $i = 1, \dots, n$. Therefore, there exist only finitely many pairwise non-equivalent in $\mathbf{Int}$ substitution instances of ψ containing $p_1, \dots, p_m$, say $\psi_1, \dots, \psi_k$, and we can effectively construct them. Then, by the deduction theorem,

$$\varphi \in \mathbf{Int} + \psi \text{ iff } \psi_1, \dots, \psi_k \vdash_{Int} \varphi \text{ iff } \psi_1 \wedge \dots \wedge \psi_k \to \varphi \in \mathbf{Int}$$

and so we obtain a decision algorithm for $\mathbf{Int} + \psi$, because $\mathbf{Int}$ is decidable.

Let us observe now that in the argument above we used only two specific properties of $\mathbf{Int}$, namely its decidability and Glivenko's theorem, which holds for every consistent si-logic. Thus, actually we have proved

Theorem 11.9 *Suppose L is a decidable si-logic and ψ an essentially negative formula. Then the logic $L + \psi$ is also decidable.*

The proof of Theorem 11.9 can be easily supplemented to a proof of the following:

Theorem 11.10 *Suppose L is a finitely approximable si-logic and ψ an essentially negative formula. Then the logic $L + \psi$ is also finitely approximable.*

Proof We continue the argument above, taking L instead of $\mathbf{Int}$. Suppose $\varphi \notin L + \psi$. Then $\psi_1 \wedge \dots \wedge \psi_k \to \varphi \notin L$ and so there is a finite model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ with root x such that $\mathfrak{M} \models L$, $x \models \psi_1 \wedge \dots \wedge \psi_k$, and $x \not\models \varphi$. As was shown above, every formula in $L + \psi$ of the variables $p_1, \dots, p_m$ belongs to $L + \psi_1 \wedge \dots \wedge \psi_k$. Therefore, changing (if necessarily) the valuation $\mathfrak{V}$ in $\mathfrak{M}$ so that $\mathfrak{V}(q) = \mathfrak{V}(p_1)$ for every variable q different from $p_1, \dots, p_m$, we obtain that $x \models \psi$ and so $\mathfrak{M}$ is a finite model for $L + \psi$ refuting φ. ❑

It follows in particular that the si-logics, obtained by adding to $\mathbf{Int}$ Rose's non-realizable formula (see Section 2.9) or the Skvortsov formula or both, are decidable and finitely approximable.

Results similar to Theorem 11.9 and 11.10 hold for extensions of $\mathbf{K4}$ as well. However, in this case instead of essentially negative formulas we take so called *□◇-formulas* in which every occurrence of a variable is in the scope of a modality □◇. Instead of $\mathbf{Cl}$ we take $\mathbf{S5}$, which is locally tabular by Corollary 5.19. Finally, instead of Glivenko's theorem we use

Lemma 11.11 *For every modal formulas φ and ψ,*

$$\Diamond\varphi \leftrightarrow \Diamond\psi \in \mathbf{S5} \textit{ iff } \Box\Diamond\varphi \leftrightarrow \Box\Diamond\psi \in \mathbf{K4}.$$

Proof It suffices to show that $\Diamond\varphi \to \Diamond\psi \in \mathbf{S5}$ iff $\Box\Diamond\varphi \to \Box\Diamond\psi \in \mathbf{K4}$. ($\Leftarrow$) is a consequence of $\mathbf{K4} \subset \mathbf{S5}$ and $\Diamond p \leftrightarrow \Box\Diamond p \in \mathbf{S5}$.

($\Rightarrow$) Suppose $\Box\Diamond\varphi \to \Box\Diamond\psi \notin \mathbf{K4}$. Then there is a finite model $\mathfrak{M}$, based on a transitive frame, and a point x in it such that $x \models \Box\Diamond\varphi$ and $x \not\models \Box\Diamond\psi$. It follows from the former relation that every final cluster accessible from x, if any, is non-degenerate and contains a point where φ is true. The latter relation means that x sees a final cluster C at all points of which ψ is false. Now, taking the generated submodel of $\mathfrak{M}$ based on C, we clearly obtain a model for $\mathbf{S5}$ refuting $\Diamond\varphi \to \Diamond\psi$. ❑

Thus, we have everything that is required to prove the following two theorems.

Theorem 11.12 *Suppose L is a decidable normal (or quasi-normal) extension of* $\mathbf{K4}$ *and ψ a $\Box\Diamond$-formula. Then the logic $L \oplus \psi$ (respectively, $L + \psi$) is also decidable.*

Proof Similar to the proof of Theorem 11.9 with the help of Theorem 4.7 and Exercise 3.5. ❑

Theorem 11.13 *Suppose L is a finitely approximable normal (or quasi-normal) extension of* $\mathbf{K4}$ *and ψ a $\Box\Diamond$-formula. Then the logic $L \oplus \psi$ (respectively, $L + \psi$) is finitely approximable too.*

Proof Similar to the proof of Theorem 11.10 (in the normal case $\varphi \notin L \oplus \psi$ means that $\Box^+(\psi_1 \wedge \ldots \wedge \psi_k) \to \varphi \notin L$). ❑

It follows in particular that the quasi-normal logics $\mathbf{K4} + \Box\Diamond p \to \Diamond\Box p = \mathbf{K4} + \Box\Diamond p \to \neg\Box\Diamond\neg p$ and $\mathbf{S4.1}'$ are decidable and finitely approximable. It is to be noted that extending a finitely approximable logic with infinitely many $\Box\Diamond$-axioms does not in general preserve finite approximability (see Exercise 11.3).

11.3 Subframe and cofinal subframe logics

Another way towards general completeness results is to use the information about logics' frames which is contained in their canonical axioms. In Section 7.3 we saw that si-logics with disjunction free extra axioms are finitely approximable. According to Theorem 9.44, all these logics are axiomatizable by canonical formulas without closed domains—we called them subframe and cofinal subframe formulas. Now we consider modal logics in NExt$\mathbf{K4}$ with canonical axioms of that sort. With the help of the modal companion and preservation theorems the results obtained below can readily be transferred to the corresponding si-logics.

A logic $L \in$ NExt$\mathbf{K4}$ is called a *subframe logic* if it can be represented in the form

$$L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i) : \ i \in I\}.$$

The class of all subframe logics is denoted by $\mathcal{SF}$. A logic L of the form

$$L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \bot) : \ i \in I\}$$

is called a *cofinal subframe logic*, and the class of all such logics is denoted by $\mathcal{CSF}$.

Example 11.14 As is shown by Table 9.6, the majority of the standard modal logics are in $\mathcal{SF}$ or $\mathcal{CSF}$. Every extension of **S4.3** is axiomatizable by canonical formulas which are based on chains of non-degenerate clusters and so have no closed domains. Therefore, NExt**S4.3** = Ext**S4.3** is a (proper) subclass of $\mathcal{CSF}$.

Theorem 11.15 (i) *Suppose* $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \bot) : i \in I\}$. *Then for every canonical formula* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ *iff* $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i, \bot)$ *for some* $i \in I$, *i.e., iff* $\mathfrak{F}$ *is cofinally subreducible to* $\mathfrak{F}_i$ *for some* $i \in I$.

(ii) *Suppose that* $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i) : i \in I\}$. *Then for every* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ *iff* $\alpha(\mathfrak{F}, \mathfrak{D}) \in L$ *iff* $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i)$ *for some* $i \in I$, *i.e., iff* $\mathfrak{F}$ *is subreducible to* $\mathfrak{F}_i$ *for some* $i \in I$.

Proof (i) If $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ then $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i, \bot)$ for some $i \in I$, since clearly $\mathfrak{F} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$.

Now suppose that $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i, \bot)$ for some $i \in I$, i.e., there is a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{F}_i$. Suppose also that $\mathfrak{G}$ is a frame refuting $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Then there is a cofinal subreduction g of $\mathfrak{G}$ to $\mathfrak{F}$. By Theorem 9.21, the composition fg is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}_i$ and so, by the refutability criterion (Theorem 9.39), $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i, \bot)$. Thus, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is valid in every general frame for L, and hence $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$.

(ii) is proved analogously. ❑

As an immediate consequence of Theorem 11.15 and the completeness theorem for the canonical formulas (Theorem 9.43) we obtain

Corollary 11.16 *Every finitely axiomatizable subframe or cofinal subframe logic is decidable.*

Moreover, this result may be generalized to

Theorem 11.17 *Suppose* $L \in$ NExt**K4** *(or* $L \in$ Ext**Int***) is recursively axiomatizable by subframe or cofinal subframe formulas. Then* L *is decidable.*

Proof Let L be recursively axiomatizable by some cofinal subframe formulas. According to Theorem 11.15, $\alpha(\mathfrak{G}, \mathfrak{D}, \bot) \in L$ iff there is a cofinal subreduct $\mathfrak{F}$ of $\mathfrak{G}$ such that $\alpha(\mathfrak{F}, \bot)$ is an axiom of L. So our decision algorithm may be as follows. Given a formula $\alpha(\mathfrak{G}, \mathfrak{D}, \bot)$, we construct all rooted cofinal subreducts $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ of $\mathfrak{G}$ and then check whether at least one of the formulas $\alpha(\mathfrak{F}_1, \bot), \dots, \alpha(\mathfrak{F}_n, \bot)$ is an axiom of L. If the outcome of this check is positive then $\alpha(\mathfrak{G}, \mathfrak{D}, \bot) \in L$; otherwise $\alpha(\mathfrak{G}, \mathfrak{D}, \bot) \notin L$.

The case of a subframe L is considered in the same manner. ❑

However, there are undecidable recursively axiomatizable logics in $\mathcal{SF}$ and $\mathcal{CSF}$. Let $\mathfrak{F}_n = \langle W_n, R_n \rangle$, for $n = 3, 4, \dots$, be the sequence of frames shown in Fig. 11.1.

Lemma 11.18 *For no* $n \neq m$, $\mathfrak{F}_n$ *is subreducible to* $\mathfrak{F}_m$.

Proof Clearly $\mathfrak{F}_n$ is not subreducible to $\mathfrak{F}_m$ if $n < m$. So suppose that $n > m$ and f is a subreduction of $\mathfrak{F}_n$ to $\mathfrak{F}_m$. Since both a_1 and b_1 have three pairwise

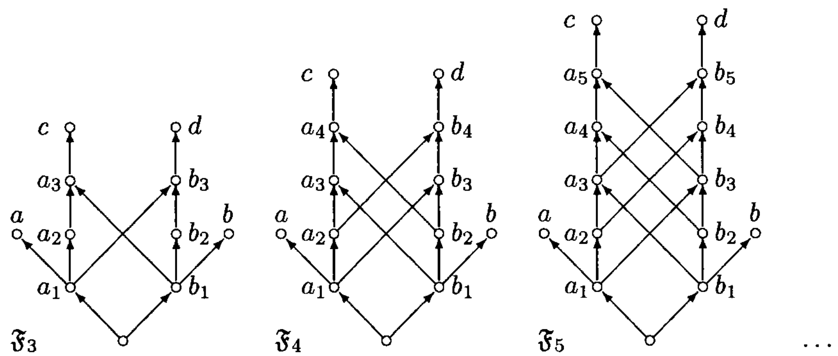


FIG. 11.1.

inaccessible successors in $\mathfrak{F}_m$, every point in $f^{-1}(a_1)$ and $f^{-1}(b_1)$ must see an antichain of three points as well. Therefore, without loss of generality we may assume that $f^{-1}(a_1) = \{a_1\}$ and $f^{-1}(b_1) = \{b_1\}$. It should be clear also that $f^{-1}(a) = \{a\}$ and $f^{-1}(b) = \{b\}$. Since $a_1 R_m a_2$ and not $b_1 R_m a_2$, we must have $f^{-1}(a_2) = \{a_2\}$; symmetrically, $f^{-1}(b_2) = \{b_2\}$. And by the same argument, for each i such that $1 \leq i \leq m$, $f^{-1}(a_i) = \{a_i\}$ and $f^{-1}(b_i) = \{b_i\}$. But then we come to a contradiction. For b_{m-1} does not see c in $\mathfrak{F}_m$, while in $\mathfrak{F}_n$ b_{m-1} sees all the points which are accessible from a_m except a_m itself, and so no point in $\mathfrak{F}_n$ can be mapped by f to c without violating (R1). ❑

As a consequence of Lemma 11.18 and Theorem 11.15 we obtain the following:

Theorem 11.19 (i) *The cardinality of both $\mathcal{SF}$ and $\mathcal{CSF}$ is that of the continuum.*

(ii) *There is a continuum of undecidable logics in $\mathcal{SF}$ and $\mathcal{CSF}$, with infinitely many of them being recursively axiomatizable (but not by canonical formulas).*

Proof (i) Let I be a set of natural numbers, $L_I = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i) : i \in I\}$ and $n \notin I$. Clearly, $\mathfrak{F}_n \not\models \alpha(\mathfrak{F}_n)$. On the other hand, by Lemma 11.18, $\mathfrak{F}_n \models \alpha(\mathfrak{F}_i)$ for every $i \in I$. Therefore, $\alpha(\mathfrak{F}_n) \notin L_I$ and so $L_I \neq L_J$ whenever $I \neq J$.

(ii) Take any recursively enumerable set I of natural numbers which is not recursive. The logic L_I is then undecidable, for otherwise, since $\alpha(\mathfrak{F}_n) \in L_I$ iff $n \in I$, the set I would recursive. By Craig's theorem (see Section 16.2), L_I is recursively axiomatizable. ❑

Since all the frames $\mathfrak{F}_n$ are partial orders, Theorem 11.19 holds for the classes of si-logics with implicative and disjunction free extra axioms. It means in particular that there is a continuum of si-logics axiomatizable by purely implicative formulas.

Another immediate consequence of Theorem 11.15 is the following:

Theorem 11.20 *All subframe and cofinal subframe logics are finitely approximable.*

Proof Suppose L is in $\mathcal{SF}$ or $\mathcal{CSF}$ and $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$. Then by Theorem 11.15, $\mathfrak{F}$ is a frame for L and, as we know, $\mathfrak{F} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. ❑

The terms "subframe logic" and "cofinal subframe logic" are justified by the following frame-theoretic characterization of these logics. Say that a class $\mathcal{C}$ of frames is *closed under (cofinal) subframes* if every (cofinal) subframe of $\mathfrak{F}$ is in $\mathcal{C}$ whenever $\mathfrak{F} \in \mathcal{C}$.

Theorem 11.21 (i) *A logic in* NExt**K4** *is a subframe logic iff it is characterized by a class of frames that is closed under subframes.*

(i) *A logic in* NExt**K4** *is a cofinal subframe logic iff it is characterized by a class of frames that is closed under cofinal subframes.*

Proof (ii) Suppose L is a cofinal subframe logic. We show that the class of all frames for L is closed under cofinal subframes. Let $\mathfrak{G}$ be a frame for L and $\mathfrak{H}$ a cofinal subframe of $\mathfrak{G}$. Then $\mathfrak{H} \models L$, since otherwise $\mathfrak{H} \not\models \alpha(\mathfrak{F}, \bot)$ for some $\alpha(\mathfrak{F}, \bot) \in L$ and so, by Theorem 9.21 and the refutability criterion, $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$ which is a contradiction.

Now suppose that L is characterized by some class of frames $\mathcal{C}$ that is closed under cofinal subframes. We show that $L = L'$ where

$$L' = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}, \bot) : \mathfrak{F} \not\models L\}.$$

Indeed, if $\mathfrak{F}$ is a finite rooted frame and $\mathfrak{F} \not\models L$ then $\alpha(\mathfrak{F}, \bot) \in L$, for otherwise $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$ for some $\mathfrak{G} \in \mathcal{C}$, and hence there is a cofinal subframe $\mathfrak{H}$ of $\mathfrak{G}$ which is reducible to $\mathfrak{F}$; but $\mathfrak{H} \in \mathcal{C}$ and so, by the reduction theorem, $\mathfrak{F}$ is a frame for L, which is a contradiction. Thus, $L' \subseteq L$.

To prove the converse inclusion, suppose $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$. Then $\mathfrak{F} \not\models L$, and hence $\alpha(\mathfrak{F}, \bot) \in L'$. Therefore, by Theorem 11.15, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L'$.

(i) is proved analogously. ❑

Corollary 11.22 *If a logic $L \in$* NExt**K4** *is characterized by a class of frames that is closed under cofinal subframes then L has the finite model property.*

Corollary 11.23 $\mathcal{SF} \subset \mathcal{CSF}$.

Proof The fact that $\mathcal{SF} \subseteq \mathcal{CSF}$ is an immediate consequence of Theorem 11.21. However, there is a continuum of cofinal subframe logics that are not subframe ones. Indeed, there is a continuum of logics axiomatizable by canonical formulas of the form $\alpha(\mathfrak{F}_i, \bot)$, where $\mathfrak{F}_i$ is the frame defined in Fig. 11.1. And none of them is a subframe logic, since the class of frames for such a logic is not closed under subframes. For if we add to $\mathfrak{F}_i$ a new point which is seen from all the points in $\mathfrak{F}_i$ and denote the result by $\mathfrak{G}_i$ then clearly $\mathfrak{G}_i \models \alpha(\mathfrak{F}_j, \bot)$ for any j, but $\mathfrak{F}_i$, being a subframe of $\mathfrak{G}_i$, refutes $\alpha(\mathfrak{F}_i, \bot)$. ❑

Corollary 11.24 *$\mathcal{CSF}$ is a complete sublattice of* NExt**K4**. *$\mathcal{SF}$ is a complete sublattice of $\mathcal{CSF}$.*

Proof Suppose $L_i \in \mathcal{CSF}$ for $i \in I$. Then for each $i \in I$, there is a set Δ_i of cofinal subframe formulas such that $L_i = \mathbf{K4} \oplus \Delta_i$. Therefore, we have $\bigoplus_{i\in I} L_i = \mathbf{K4} \oplus \bigcup_{i\in I} \Delta_i \in \mathcal{CSF}$.

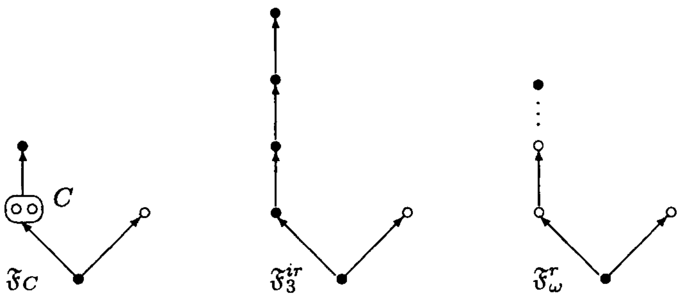


FIG. 11.2.

As to the intersection $L = \bigcap_{i\in I} L_i$, it is clear that L is characterized by the class $\bigcup_{i\in I}\{\mathfrak{F} : \mathfrak{F} \models L_i\}$ which is closed under cofinal subframes. Therefore, by Theorem 11.21, $L \in \mathcal{CSF}$.

The class $\mathcal{SF}$ is considered analogously. ❑

Translating Theorem 11.21 into si-logics we obtain a nice frame-theoretic criterion of axiomatizability by implicative and disjunction free formulas.

Theorem 11.25 (i) *A si-logic is axiomatizable by implicative formulas iff it is characterized by a class of frames closed under subframes.*

(ii) *A si-logic is axiomatizable by disjunction free formulas iff it is characterized by a class of frames closed under cofinal subframes.*

Now we give a frame-theoretic criterion of elementarity, $\mathcal{D}$-persistence and strong Kripke completeness of logics in $\mathcal{SF}$ and $\mathcal{CSF}$.

Let $\mathfrak{F}_C = \langle W_C, R_C\rangle$ be a frame containing a cluster C. For an ordinal ξ, $0 < \xi \le \omega$, we denote by $\mathfrak{F}_\xi^{ir} = \left\langle W_\xi, R_\xi^{ir}\right\rangle$ the frame that is obtained from $\mathfrak{F}_C$ by replacing C with an ascending chain of ξ irreflexive points. More exactly, we put

$$W_\xi = (W - C) \cup \{i : 0 \le i < \xi\}$$

and, for all $x, y \in W_\xi$,

$$\begin{array}{rll} xR_\xi^{ir}y \text{ iff } & xR_Cy & \text{or} \\ & \exists i, j < \xi\ (x = i\ \wedge\ y = j\ \wedge\ i < j) & \text{or} \\ & \exists i < \xi \exists z \in C\ (x = i\ \wedge\ zR_Cy) & \text{or} \\ & \exists i < \xi \exists z \in C\ (y = i\ \wedge\ xR_Cz). & \end{array}$$

$\mathfrak{F}_\xi^r = \left\langle W_\xi, R_\xi^r\right\rangle$ is the result of replacing C in $\mathfrak{F}_C$ with an ascending chain containing ξ reflexive points, i.e.,

$$R_\xi^r = R_\xi^{ir} \cup \{\langle i, i\rangle : 0 \le i < \xi\}.$$

Fig. 11.2 illustrates the given definition.

We say that a logic L has the *finite embedding property* if a Kripke frame $\mathfrak{F}$ validates L whenever each finite subframe of $\mathfrak{F}$ is a frame for L. L is said to be *universal* if there is a set Φ of universal first order sentences in R and $=$ (which are of the form $\forall x \dots \forall y\ \phi$, where ϕ contains no quantifiers) such that, for every Kripke frame $\mathfrak{F}$, $\mathfrak{F} \models L$ iff $\mathfrak{F} \models \Phi$.

Theorem 11.26 *The following conditions are equivalent for each subframe logic L:*

(1) *L is universal;*
(2) *L is elementary;*
(3) *L is $\mathcal{D}$-persistent;*
(4) *L is $\mathcal{R}$-persistent;*
(5) *L is canonical;*
(6) *L is strongly Kripke complete;*
(7) *for every finite rooted frame $\mathfrak{F}_C$ with a non-degenerate cluster C*

$$\forall \xi < \omega\ \mathfrak{F}_\xi^{ir} \models L \text{ implies } \mathfrak{F}_C \models L$$

and

$$\forall \xi < \omega\ \mathfrak{F}_\xi^{r} \models L \text{ implies } \mathfrak{F}_C \models L;$$

(8) *L has the finite embedding property.*

Proof The implication (1) $\Rightarrow$ (2) is trivial and (2) $\Rightarrow$ (3) follows from Theorems 10.19 and 11.20.

(3) $\Rightarrow$ (4). Let $\mathfrak{F}$ be a refined frame for L. According to the proof of Theorem 8.51, $\kappa\mathfrak{F}$ is (isomorphic to) a subframe of $\kappa(\mathfrak{F}^+)_+$. Since $\mathfrak{F} \models L$ and L is $\mathcal{D}$-persistent, we then have $(\mathfrak{F}^+)_+ \models L$ and $\kappa(\mathfrak{F}^+)_+ \models L$, from which, by the proof of Theorem 11.21, $\kappa\mathfrak{F} \models L$.

The implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (7). Suppose that $\mathfrak{F}_C = \langle W_C, R_C \rangle$ is a finite rooted frame with a non-degenerate cluster C and $\forall \xi < \omega\ \mathfrak{F}_\xi^{ir} \models L$. We must prove that $\mathfrak{F}_C \models L$.

Let $\{a_i : i \in I\}$ be all the points in W_ω. With each a_i we associate a variable p_i different from p_j for any $j \neq i$ and construct from them the canonical formulas $\alpha(\mathfrak{F}_\xi^{ir})$ for all ξ such that $0 < \xi < \omega$. Now take the tableau

$$(\emptyset, \{\alpha(\mathfrak{F}_\xi^{ir}) : 0 < \xi < \omega\})$$

and show that it is L-consistent. Suppose otherwise. Then we have some $\xi < \omega$ for which

$$\alpha(\mathfrak{F}_1^{ir}) \vee \alpha(\mathfrak{F}_2^{ir}) \vee \ldots \vee \alpha(\mathfrak{F}_\xi^{ir}) \in L.$$

But on the other hand, since $\mathfrak{F}_\zeta^{ir}$ is a subframe of $\mathfrak{F}_\xi^{ir}$, for $\zeta \leq \xi$, and by the proof of Theorem 9.39, there is a valuation $\mathfrak{V}$ in $\mathfrak{F}_\xi^{ir}$ such that all the formulas $\alpha(\mathfrak{F}_\zeta^{ir})$, for $\zeta \leq \xi$, are false at the root of $\mathfrak{F}_\xi^{ir}$ under $\mathfrak{V}$, which is a contradiction because $\mathfrak{F}_\xi^{ir} \models L$.

By (6), there is a model $\mathfrak{M}$ on a Kripke frame $\mathfrak{G} = \langle V, S\rangle$ such that all $\alpha(\mathfrak{F}_\xi^{ir})$, for $0 < \xi < \omega$, are simultaneously false at some point in $\mathfrak{M}$ and $\mathfrak{G} \models L$. Define a map f from V onto W_ω by taking

$$f(x) = \begin{cases} a_i & \text{if } x \not\models p_i \text{ and, for each } \xi < \omega, \text{ the} \\ & \text{premise of } \alpha(\mathfrak{F}_\xi^{ir}) \text{ is true at } x \\ \text{undefined otherwise.} \end{cases}$$

Using the proof of Theorem 9.39, it is not hard to check that f is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}_\omega^{ir}$. On the other hand, we can easily construct a reduction g of $\mathfrak{F}_\omega^{ir}$ to $\mathfrak{F}_C$. Indeed, if $C = \{b_0, \dots, b_n\}$ then we may take

$$g(x) = \begin{cases} x & \text{if } x \in W_C - C \\ b_i & \text{if } x = m \text{ and } i = \mathrm{mod}_{n+1}(m). \end{cases}$$

By Theorem 9.21, there is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}_C$ and so $\mathfrak{F}_C \models L$, for otherwise $\mathfrak{G} \not\models L$, which is a contradiction.

The case with $\mathfrak{F}_\xi^r$ is considered in exactly the same way.

(7) $\Rightarrow$ (8). Suppose otherwise, i.e., there is a Kripke frame $\mathfrak{G}$ such that every finite subframe of $\mathfrak{G}$ validates L but $\mathfrak{G} \not\models L$. Then there exists a subreduction f of $\mathfrak{G}$ to a finite rooted frame $\mathfrak{F} = \langle W, R\rangle$ such that $\mathfrak{F} \not\models L$. Starting with $\mathfrak{F}$ we construct by induction a finite rooted frame which is not a frame for L but is embeddable in $\mathfrak{G}$, contrary to our assumption. At the very beginning we mark by some signs all the clusters in $\mathfrak{F}$, which means that all of them are to be analyzed in the sequel.

Suppose now that we have already constructed a finite rooted frame $\mathfrak{H} = \langle V, S\rangle$ and a subreduction g of $\mathfrak{G}$ to $\mathfrak{H}$ such that $\mathfrak{H} \not\models L$ and $g^{-1}(x)$ is a singleton for each x belonging to an unmarked cluster in $\mathfrak{H}$. (At the first step $\mathfrak{H} = \mathfrak{F}$.)

Let $C = \{a_0, \dots, a_k\}$ be a marked cluster in $\mathfrak{H}$ all immediate predecessors $C_1, \dots, C_m$ of which are unmarked and let $b_1 \in C_1, \dots, b_m \in C_m$. By the induction hypothesis, $g^{-1}(b_i) = \{x_i\}$ for some $x_1, \dots, x_m$ in $\mathfrak{G}$. Choose a minimal number of disjoint sets $A_1, \dots, A_n$ of points in $\mathfrak{G}$ such that

- for each $i \in \{1, \dots, m\}$ there is $j \in \{1, \dots, n\}$ such that $A_j \subseteq x_i\uparrow$

and, for each $i \in \{1, \dots, n\}$, either

- $A_i = \{y_0, \dots, y_k\}$, $g(y_j) = a_j$ for $j = 0, \dots, k$, and A_i is a subset of a cluster in $\mathfrak{G}$

or

- A_i is an infinite ascending chain $y_0, y_1, \dots$ all the points of which are either simultaneously irreflexive or simultaneously reflexive and $g(y_j) \in C$ for $j \geq 0$.

The existence of such $A_1, \dots, A_n$ follows from the fact that g is a subreduction of $\mathfrak{G}$ to $\mathfrak{H}$. (See Fig. 11.3.) Our next action depends on the number of these $A_1, \dots, A_n$. Notice by the way that $1 \leq n \leq m$.

Case 1. $n = 1$.

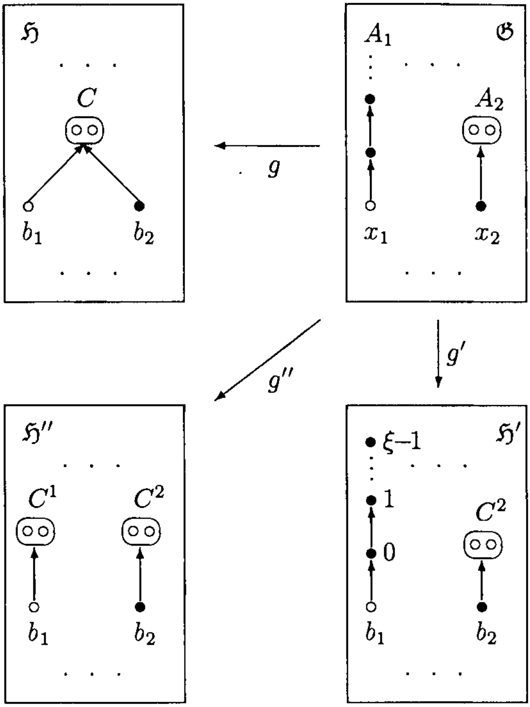


FIG. 11.3.

1.1. If $A_1 = \{y_0, \dots, y_k\}$, i.e., if A_1 is a part of a cluster in $\mathfrak{G}$, then we put $\mathfrak{H}' = \mathfrak{H}$, mark in $\mathfrak{H}'$ all the clusters that were marked in $\mathfrak{H}$ except C and define a partial map g' from $\mathfrak{G}$ onto $\mathfrak{H}'$ by taking

$$g'(x) = \begin{cases} g(x) & \text{if } x \in (\mathrm{dom}g - g^{-1}(C)) \cup A_1 \\ \text{undefined otherwise.} \end{cases}$$

It is clear that $\mathfrak{H}' \not\models L$, g' is a subreduction of $\mathfrak{G}$ to $\mathfrak{H}'$ and $g'^{-1}(x)$ is a singleton for each x belonging to an unmarked cluster in $\mathfrak{H}'$. Notice also that the number of marked clusters in $\mathfrak{H}'$ is less than that in $\mathfrak{H}$.

1.2. Suppose A_1 is an infinite ascending chain $y_0, y_1, \dots$ of irreflexive points. Then C is non-degenerate and, since $\mathfrak{H} = \mathfrak{H}_C \not\models L$, there is, by (7), some $\xi < \omega$ such that $\mathfrak{H}^{ir}_\xi \not\models L$. In this case we put $\mathfrak{H}' = \mathfrak{H}^{ir}_\xi$, mark in $\mathfrak{H}'$ all the clusters that were marked in $\mathfrak{H}$ (the new points $0, \dots, \xi - 1$ remain unmarked) and define a partial map g' from $\mathfrak{G}$ onto $\mathfrak{H}'$ by taking

$$g'(x) = \begin{cases} g(x) & \text{if } x \in \mathrm{dom}g - g^{-1}(C) \\ i & \text{if } x = y_i,\ 0 \le i < \xi \\ \text{undefined otherwise.} \end{cases}$$

Again g' is a subreduction, $\mathfrak{H}' \not\models L$, $g'^{-1}(x)$ is a singleton for each x belonging to an unmarked cluster in $\mathfrak{H}'$ and the number of marked clusters in $\mathfrak{H}'$ is less than that in $\mathfrak{H}$.

1.3. The case when A_1 is an ascending chain of reflexive points is considered in the same way but using the second part of (7), i.e., $\mathfrak{H}^r_\xi$ instead of $\mathfrak{H}^{ir}_\xi$.

Case 2. Suppose now $n > 1$. Then we first form a new frame $\mathfrak{H}'' = \langle V'', S'' \rangle$ by taking (see Fig. 11.3)

$$V'' = (V - C) \cup C^1 \cup \ldots \cup C^n,$$

where

$$C^i = \{a^i_0, \ldots, a^i_k\},\ i = 1, \ldots, n,$$

and, for all $x, y \in V''$,

$$\begin{array}{lll} xS''y \text{ iff } & x, y \in V - C \ \wedge\ xSy & \text{or} \\ & \exists i, j\ (x = a^i_j \ \wedge\ a_j S y) & \text{or} \\ & \exists i, j, l\ (y = a^i_j \ \wedge\ x \in b_l\overline{\downarrow} \ \wedge\ A_i \subseteq x_l\uparrow) & \text{or} \\ & \exists i, j, l\ (x = a^i_j \ \wedge\ y = a^i_l \ \wedge\ C \text{ is non-degenerate}). & \end{array}$$

Mark in $\mathfrak{H}''$ all the clusters that were marked in $\mathfrak{H}$ and $C^1, \ldots, C^n$ as well. After that we define a map g'' from $\mathfrak{G}$ onto $\mathfrak{H}''$ by taking

$$g''(x) = \begin{cases} g(x) & \text{if } x \in \operatorname{dom} g - g^{-1}(C) \\ a^i_j & \text{if } x = y_l \in A_i \text{ and } \operatorname{mod}_{k+1}(l) = j \\ \text{undefined otherwise.} & \end{cases}$$

It is not difficult to see that g'' is a subreduction of $\mathfrak{G}$ to $\mathfrak{H}''$. Moreover, $\mathfrak{H}'' \not\models L$, since $\mathfrak{H}''$ is reducible to $\mathfrak{H}$, and $g''^{-1}(x)$ contains only one point if $C(x)$ is an unmarked cluster in $\mathfrak{H}''$. But the number of marked clusters in $\mathfrak{H}''$ has become greater than that in $\mathfrak{H}$. However, we need not worry. For we can now analyze the new clusters $C^1, \ldots, C^n$, which clearly satisfy the condition of Case 1 and so we shall eventually construct a frame $\mathfrak{H}'$ having all the desirable properties and less marked clusters than $\mathfrak{H}$. Fig. 11.3 will help the reader to complete the details.

The implication (8) $\Rightarrow$ (1), completing the circle, is a consequence of the well known theorem of Tarski (1954) from classical model theory. Roughly, it is proved in the following way. Let $\mathcal{C}$ be the set of all finite rooted frames which do not validate L. With each $\mathfrak{F} \in \mathcal{C}$ we can associate a universal first order sentence $\phi_{\mathfrak{F}}$ such that a Kripke frame $\mathfrak{G}$ is a (classical) model for $\phi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not a subframe of $\mathfrak{G}$. It is easy to see now that a Kripke frame $\mathfrak{G}$ validates L iff $\mathfrak{G}$ is a (classical) model for the set $\{\phi_{\mathfrak{F}} :\ \mathfrak{F} \in \mathcal{C}\}$. ❑

Example 11.27 $\mathbf{Grz} = \mathbf{K4} \oplus \alpha(\bullet) \oplus \alpha(\circ\circ)$ is neither elementary nor $\mathcal{D}$-persistent nor strongly complete, since every finite linearly ordered reflexive frame validates **Grz**, while the two point cluster is not a frame for it. Neither $\mathbf{GL} = \mathbf{K4} \oplus \alpha(\circ)$ meets this properties. For each finite linearly ordered irreflexive frame validates **GL**, while any non-degenerate cluster does not.

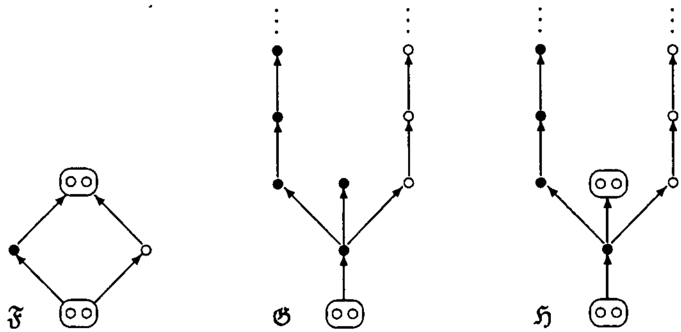


FIG. 11.4.

Theorem 11.26 can be generalized in two directions. First, we can extend it to the class $\mathcal{CSF}$. Say that a subreduction f of a frame $\mathfrak{G}$ to a finite frame $\mathfrak{F}$ is a *quasi-embedding* of $\mathfrak{F}$ in $\mathfrak{G}$ if $f^{-1}(x)$ is a singleton for every point x whose cluster $C(x)$ is not final in $\mathfrak{F}$. In such a case $\mathfrak{F}$ is called *quasi-embeddable* in $\mathfrak{G}$. For example, the frame $\mathfrak{F}$ in Fig. 11.4 is quasi-embeddable in $\mathfrak{G}$ and cofinally quasi-embeddable in $\mathfrak{H}$.

A logic L has the *finite cofinal quasi-embedding property* if a Kripke frame $\mathfrak{F}$ validates L whenever every finite frame which is cofinally quasi-embeddable in $\mathfrak{F}$ validates L.

Theorem 11.28 *The following conditions are equivalent for each cofinal subframe logic L:*

(1) *L is elementary;*

(2) *L is $\mathcal{D}$-persistent;*

(3) *L is canonical;*

(4) *L is strongly Kripke complete;*

(5) *for every finite rooted frame $\mathfrak{F}_C$ with a non-degenerate non-final cluster C, $\forall \xi < \omega\ \mathfrak{F}_\xi^{ir} \models L$ implies $\mathfrak{F}_C \models L$ and $\forall \xi < \omega\ \mathfrak{F}_\xi^{r} \models L$ implies $\mathfrak{F}_C \models L$;*

(6) *L has the finite cofinal quasi-embedding property.*

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are proved in the same way as the corresponding implications in Theorem 11.26.

(6) $\Rightarrow$ (1). Given a finite rooted frame $\mathfrak{F}$, one can construct a first order formula ϕ (in R and $=$) with the free variables $x_1, \dots, x_n$ such that a Kripke frame $\mathfrak{G}$ satisfies ϕ iff $\mathfrak{F}$ is cofinally quasi-embeddable in $\mathfrak{G}$ (for details see Exercise 11.12). Then $\mathfrak{G} \not\models L$ iff there is a finite rooted frame $\mathfrak{F} \not\models L$ which is cofinally quasi-embeddable in $\mathfrak{G}$ iff $\mathfrak{G} \models \exists x_1 \dots \exists x_n \phi$. ❑

Example 11.29 The logic $\mathbf{K4.1} = \mathbf{K4} \oplus \alpha(\bullet, \bot) \oplus \alpha(\circ\circ, \bot)$ is elementary, $\mathcal{D}$-persistent and strongly complete. Indeed, let $\mathfrak{F}_C$ be a finite frame with a non-final non-degenerate cluster C. Then $\mathfrak{F}_C \not\models \alpha(\bullet, \bot)$ iff $\mathfrak{F}_C$ has a dead end iff both $\mathfrak{F}_\xi^{ir} \not\models \alpha(\bullet, \bot)$ and $\mathfrak{F}_\xi^{r} \not\models \alpha(\bullet, \bot)$ hold for any finite ξ. Similarly, $\mathfrak{F}_C \not\models \alpha(\circ\circ, \bot)$

iff $\mathfrak{F}_C$ has a final proper cluster iff both $\mathfrak{F}_\xi^{ir} \not\models \alpha(\circ\circ, \bot)$ and $\mathfrak{F}_\xi^{r} \not\models \alpha(\circ\circ, \bot)$ hold for any finite ξ.

Remark Note that elementary logics in $\mathcal{CSF}$ are not necessarily universal, and $\mathcal{D}$-persistent logics in $\mathcal{CSF}$ are not necessarily $\mathcal{R}$-persistent, witness **S4.1** (see Exercise 10.4).

As an immediate consequence of Theorems 11.26, 11.28 and the preservation theorem we obtain

Theorem 11.30 *Every si-logic L with disjunction free extra axioms is elementary (universal, if L is axiomatizable by implicative formulas), $\mathcal{D}$-persistent and strongly complete.*

Another way of generalizing Theorem 11.26 is to extend it to the class of *subframe logics* in NExt**K**, which may be defined just as logics that are characterized by classes of (general) frames closed under subframes. (Such are, for instance, the logics **T**, **KB**, **K5**, $\mathbf{Alt}_n$ in Table 4.2.)

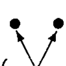
Theorem 11.31 *The following conditions are equivalent for each subframe logic $L \in$* NExt**K**:

(1) *L is universal and Kripke complete;*
(2) *L is elementary and Kripke complete;*
(3) *L is $\mathcal{D}$-persistent;*
(4) *L is $\mathcal{R}$-persistent;*
(5) *L is strongly Kripke complete;*
(6) *L has the finite embedding property and is Kripke complete.*

Proof We give only a sketch of the proof; details are left to the reader. All the implications except (5) $\Rightarrow$ (6) are established in the same way as in Theorem 11.26. Suppose L is strongly Kripke complete but does not have the finite embedding property. Then there is a rooted Kripke frame $\mathfrak{G} = \langle V, S\rangle$ such that $\mathfrak{G} \not\models L$ and all finite subframes of $\mathfrak{G}$ validate L. One can show that without loss of generality we may assume $\mathfrak{G}$ to be countable. Let a_i, $i < \omega$, be all the points in $\mathfrak{G}$ and a_0 the root. Consider the tableau $t = (\Gamma, \emptyset)$, where Γ consists of all formulas of the form p_0, $\Box^n(p_i \to \Diamond p_j)$ if $a_i S a_j$, $\Box^n(p_i \to \neg\Diamond p_j)$ if $\neg a_i S a_j$, $\Box^n(p_i \to \neg p_j)$ for $i \neq j$. Since every finite subframe of $\mathfrak{G}$ is a frame for L, t is L-consistent and so realizable in a Kripke frame $\mathfrak{H}$ for L. It is not hard to check that in this case $\mathfrak{H}$ is subreducible to $\mathfrak{G}$, which is a contradiction. ❑

It turns out, however, that subframe logics in NExt**K** are not in general finitely approximable and even Kripke complete.

Example 11.32 Let L be the logic of the frame $\mathfrak{F}$ constructed in Example 8.52. Since every rooted subframe $\mathfrak{G}$ of $\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{F}$, $\mathfrak{G} \models L$ and so L is a subframe logic. We show now that L has the same Kripke frames as the logic

$$\mathbf{GL.3} = \mathbf{K4} \oplus \alpha(\circ) \oplus \alpha($$  $$).$$

Suppose $\mathfrak{G}$ is a rooted Kripke frame for **GL.3** refuting a formula $\varphi \in L$. Then clearly $\mathfrak{G}$ contains a finite subframe $\mathfrak{H}$ refuting φ. Since $\mathfrak{H}$ is a finite chain of irreflexive points, it is isomorphic to a generated subframe of $\mathfrak{F}$. Therefore, $\mathfrak{F} \not\models \varphi$ contrary to our assumption. Thus $\mathfrak{G} \models L$.

Conversely, suppose $\mathfrak{G}$ is a Kripke frame for L. Then $\mathfrak{G}$ is irreflexive. For otherwise $\mathfrak{G}$ refutes the formula $\varphi = \Box^2(\Box p \to p) \wedge \Box(\Box p \to p) \to \Box p$ which is valid in $\mathfrak{F}$. Let us show now that $\mathfrak{G}$ is transitive. Suppose otherwise. Then $\mathfrak{G}$ refutes the formula $\Box p \to \Box(\Box p \vee (\Box q \to q))$ which is valid in $\mathfrak{F}$, because ω is a reflexive point. Finally, since $\mathfrak{G} \models \varphi$, $\mathfrak{G}$ is Noetherian and since $\mathfrak{F} \models \alpha($  $)$, we may conclude that $\mathfrak{G}$ is a frame for **GL.3**.

It follows that the subframe logic L is Kripke incomplete. Indeed, it shares the same class of Kripke frames with **GL.3** but is different from it, because $\Box p \to \Box\Box p \in \mathbf{GL.3} - L$.

11.4 Quasi-normal subframe and cofinal subframe logics

Let us now briefly consider quasi-normal logics containing **K4** which can be axiomatized by normal and quasi-normal canonical formulas without closed domains. Those quasi-normal logics that can be represented in the form

$$(\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i) : i \in I\}) + \{\alpha(\mathfrak{F}_j) : j \in J\} + \{\alpha^\bullet(\mathfrak{F}_k) : k \in K\} \tag{11.1}$$

are called, as in the normal case, (*quasi-normal*) *subframe logics* and those of the form

$$(\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \bot) : i \in I\}) + \{\alpha(\mathfrak{F}_j, \bot) : j \in J\} + \{\alpha^\bullet(\mathfrak{F}_k, \bot) : k \in K\} \tag{11.2}$$

are called (*quasi-normal*) *cofinal subframe logics*. The classes of quasi-normal subframe and cofinal subframe logics are denoted by $\mathcal{QSF}$ and $\mathcal{QCSF}$, respectively. The example of Solovay's logic $\mathbf{S} = \mathbf{K4} + \alpha(\circ) + \alpha(\bullet)$ shows that Theorem 11.20 cannot be extended to $\mathcal{QSF}$ and $\mathcal{QCSF}$. Yet we are going to prove that all finitely axiomatizable quasi-normal subframe and cofinal subframe logics are decidable.

We use the following notation. For a frame $\mathfrak{F} = \langle W, R\rangle$ with irreflexive root u and $0 < \xi < \omega$, $\mathfrak{F}^{ir}_\xi$ and $\mathfrak{F}^r_\xi$ denote the frames that are obtained from $\mathfrak{F}$ by replacing u with the descending chains $0, \dots, \xi - 1$ of irreflexive and reflexive points, respectively; $\mathfrak{F}^{ir}_{(\omega+1)^*} = \left\langle W_{(\omega+1)^*}, R^{ir}_{(\omega+1)^*}, P_{(\omega+1)^*}\right\rangle$ denotes the frame that is obtained from $\mathfrak{F}$ by replacing u with the infinite descending chain $0, 1, \dots$ of irreflexive points and then adding the irreflexive root ω, with $P_{(\omega+1)^*}$ containing all subsets of $W - \{u\}$, all finite subsets of natural numbers $\{0, 1, \dots\}$, all (finite) unions of these sets and all complements to them in the space $W_{(\omega+1)^*}$ (see Fig. 11.5). Note that if $\omega \in X \in P_{(\omega+1)^*}$ then X contains all natural numbers starting from some $n \geq 0$. Observe also that $\mathfrak{F}$ is a quasi-reduct of every frame of the form $\mathfrak{F}^{ir}_\xi$, $\mathfrak{F}^r_\xi$ or $\mathfrak{F}^{ir}_{(\omega+1)^*}$.

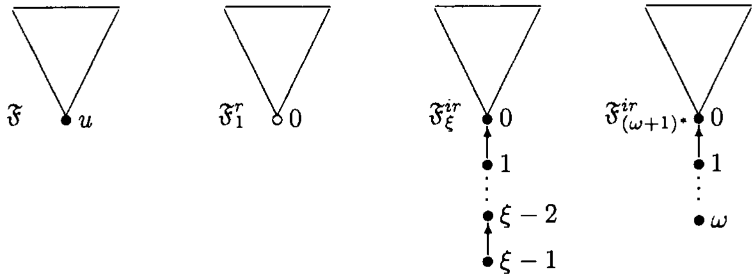


FIG. 11.5.

The following theorem characterizes the canonical formulas belonging to logics in $\mathcal{QSF}$ and $\mathcal{QCSF}$. Its proof, as that of Theorem 11.15, uses Theorem 9.21, which can be readily generalized to compositions of (cofinal) quasi-subreductions.

Theorem 11.33 *Suppose L is a subframe or cofinal subframe quasi-normal logic. Then*

(i) *for every finite frame $\mathfrak{F}$ with root u, $\alpha(\mathfrak{F},\mathfrak{D},\bot) \in L$ iff $\langle\mathfrak{F},u\rangle \not\models L$ and*

(ii) *for every finite $\mathfrak{F}$ with irreflexive root u, $\alpha^\bullet(\mathfrak{F},\mathfrak{D},\bot) \in L$ iff $\langle\mathfrak{F},u\rangle \not\models L$, $\langle\mathfrak{F}_1^r,0\rangle \not\models L$ and $\left\langle\mathfrak{F}^{ir}_{(\omega+1)^*},\omega\right\rangle \not\models L$.*

Proof (i) is proved similarly to Theorem 11.15. Details are left to the reader.

(ii) If $\alpha^\bullet(\mathfrak{F},\mathfrak{D},\bot) \in L$ then none of $\langle\mathfrak{F},u\rangle$, $\langle\mathfrak{F}_1^r,0\rangle$ and $\left\langle\mathfrak{F}^{ir}_{(\omega+1)^*},\omega\right\rangle$ validates L, since all of them are quasi-reducible to $\langle\mathfrak{F},u\rangle$ and so, by the refutability criterion, refute $\alpha^\bullet(\mathfrak{F},\mathfrak{D},\bot)$.

To prove the converse suppose that a frame $\mathfrak{G} = \langle V,S,Q\rangle$ with actual world w (which is the root of $\mathfrak{G}$) refutes $\alpha^\bullet(\mathfrak{F},\mathfrak{D},\bot)$ and show that $\langle\mathfrak{G},w\rangle \not\models L$. By the refutability criterion, there is a cofinal quasi-subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ such that $f(w) = u$. Consider the set $U = f^{-1}(u) \in Q$. Without loss of generality we may assume that $U = U\overline{\downarrow}$. There are three possible cases.

Case 1. The point w is irreflexive and $\{w\} \in Q$. Then the restriction of f to $\mathrm{dom}f - (U - \{w\})$ is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (AWC) and so, by the refutability criterion and Theorem 9.21, $\langle\mathfrak{G},w\rangle \not\models L$.

Case 2. There is a subset $X \subseteq U$ such that $w \in X \in Q$ and, for every $x \in X$, there exists $y \in X \cap x\uparrow$. Then the restriction of f to $\mathrm{dom}f - (U - X)$ is clearly a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}_1^r$ satisfying (AWC) and so again $\langle\mathfrak{G},w\rangle \not\models L$.

Case 3. If neither of the preceding cases holds then, for every $X \subseteq U$ such that $w \in X \in Q$, the set $D_X = X - X\downarrow$ of dead ends in X is a cover for X, i.e., $X \subseteq D_X\overline{\downarrow}$, and $w \in X - D_X \in Q$. Indeed, since Case 1 does not hold, $w \notin D_X$, for otherwise $\{w\} = D_X \in Q$. And if we assume that $X - D_X\overline{\downarrow} \neq \emptyset$ then $Y = (X - D_X\overline{\downarrow})\downarrow \subseteq U$, $w \in Y \in Q$ and $Y = Y\downarrow$, i.e., Case 2 holds, which is a contradiction. Put

$$X_0 = D_U,\ \ldots,\ X_{n+1} = D_{U-(X_0\cup\ldots\cup X_n)},\ \ldots\ ,X_\omega = U - \bigcup_{\xi<\omega} X_\xi.$$

Each of these sets, save possibly X_ω, is an antichain of irreflexive points and belongs to Q. Besides, $X_\zeta \subset X_n\downarrow = \bigcup_{n<\xi\le\omega} X_\xi$ for every $n < \zeta \le \omega$. Therefore, the map g defined by

$$g(x) = \begin{cases} f(x) \text{ if } x \in V - U \\ \xi \quad\ \text{ if } x \in X_\xi,\ 0 \le \xi \le \omega \end{cases}$$

is a cofinal quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ satisfying (AWC).

Suppose for definiteness that L is represented in the form (11.1). Since $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle$ does not validate L, it refutes at least one of its axioms, and again we have to consider three possible cases.

(a) $\mathfrak{F}^{ir}_{(\omega+1)^\bullet} \not\models \alpha(\mathfrak{F}_i)$ for some $i \in I$, i.e., there is a subreduction h of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_i$. Since $\{\omega\} \notin P_{(\omega+1)^\bullet}$, either $\omega \notin \mathrm{dom}h$ or the root $h(\omega)$ of $\mathfrak{F}_i$ is reflexive. Then the composition hg is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}_i$, from which $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i)$ and so $\langle \mathfrak{G}, w\rangle \not\models \Box\alpha(\mathfrak{F}_i)$, i.e. $\langle \mathfrak{G}, w\rangle \not\models L$.

(b) $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha(\mathfrak{F}_j)$ for some $j \in J$, i.e., there is a subreduction h of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_j$ satisfying (AWC). Then $h(\omega)$ is reflexive and so hg is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}_j$ satisfying (AWC). Therefore, $\langle \mathfrak{G}, w\rangle \not\models \alpha(\mathfrak{F}_j)$.

(c) $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha^\bullet(\mathfrak{F}_k)$ for some $k \in K$, i.e., there is a quasi-subreduction h of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_k$ satisfying (AWC). But then hg is a quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}_k$ satisfying (AWC), whence $\langle \mathfrak{G}, w\rangle \not\models \alpha^\bullet(\mathfrak{F}_k)$ and $\langle \mathfrak{G}, w\rangle \not\models L$.

Thus, every frame with actual world refuting $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ is not a frame for L, which means that $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot) \in L$. ❑

Corollary 11.34 *All subframe and cofinal subframe quasi-normal logics above* **S4** *are finitely approximable.*

Example 11.35 As an illustration let us use Theorem 11.33 to characterize those normal and quasi-normal canonical formulas that belong to Solovay's logic **S**.

Clearly, either $\alpha(\circ)$ or $\alpha(\bullet)$ is refuted at the root of every rooted Kripke frame. So all normal canonical formulas are in **S**. Every quasi-normal formula $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ associated with $\mathfrak{F}$ containing a reflexive point is also in **S**, since $\Box\alpha(\circ)$ is refuted at the roots of $\mathfrak{F}$, $\mathfrak{F}^r_1$ and $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$. But no quasi-normal formula $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ built on irreflexive $\mathfrak{F}$ belongs to **S**, because $\mathfrak{F}^{ir}_{(\omega+1)^\bullet} \models \alpha(\circ)$ (for $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ contains neither an infinite ascending chain nor a reflexive point) and $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \models \alpha(\bullet)$, since $\{\omega\} \notin P_{(\omega+1)^\bullet}$.

The obtained characterization together with the completeness theorem for the canonical formulas provide us with another decision algorithm for **S**. Notice also that incidentally we have proved the following completeness theorem for **S**.

Theorem 11.36 **S** *is characterized by the class*

$$\{\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle : \mathfrak{F} \textit{ is a finite rooted irreflexive frame}\}.$$

Theorem 11.33 reduces the decision problem for a logic L in $\mathcal{QSF}$ or $\mathcal{QCSF}$ to the problem of verifying, given a finite frame $\mathfrak{F}$ with root u, whether or not the frames $\langle \mathfrak{F}, u\rangle$, $\langle \mathfrak{F}_1^r, 0\rangle$ and $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega\right\rangle$ refute at least one axiom of L. The first two frames present no difficulty for a finitely axiomatizable L. And our aim now is to show that the condition $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega\right\rangle \not\models L$ can also be verified in finitely many steps.

Lemma 11.37 *Suppose L is a quasi-normal (cofinal) subframe logic represented in the form* (11.1) *(respectively,* (11.2)*) and $\mathfrak{F} = \langle W, R\rangle$ is a finite frame with irreflexive root u. Then $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega\right\rangle \not\models L$ iff one of the following conditions is satisfied:*

(i) *$\mathfrak{F}^{ir}_\xi$ is (cofinally) subreducible to $\mathfrak{F}_i$ for some $i \in I$ and some $\xi \leq |\mathfrak{F}_i|$;*

(ii) *for some $j \in J$, $\mathfrak{F}_j$ has a reflexive root and $\mathfrak{F}$ is (cofinally) subreducible to $\mathfrak{F}_j$, with (AWC) being satisfied;*

(iii) *$\mathfrak{F}^{ir}_\xi$ is (cofinally) quasi-subreducible to $\mathfrak{F}_k$ for some $k \in K$ and some $\xi \leq |\mathfrak{F}_k|$, with (AWC) being satisfied.*

Proof Let us suppose for definiteness that L is represented in the form (11.2); the form (11.1) is considered analogously.

($\Rightarrow$) If $\mathfrak{F}^{ir}_{(\omega+1)^\bullet} \not\models \alpha(\mathfrak{F}_i, \bot)$ for some $i \in I$, then there is a cofinal subreduction f of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_i$. The map

$$g(x) = \begin{cases} f(x) & \text{if } x \text{ belongs to a final cluster in } f^{-1}(f(x)) \\ \text{undefined otherwise} \end{cases}$$

is also a cofinal subreduction of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_i$, with $g(\xi) \neq g(\zeta)$ for any distinct $\xi, \zeta \leq \omega$. Let $\mathfrak{F}'$ be the result of removing from $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ all those points $\xi \leq \omega$ that are not in dom$g$. Clearly, $\mathfrak{F}'$ is isomorphic to $\mathfrak{F}^{ir}_\xi$ for some $\xi \leq |\mathfrak{F}_i|$ and g is a cofinal subreduction of $\mathfrak{F}'$ to $\mathfrak{F}_i$.

If $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega\right\rangle \not\models \alpha(\mathfrak{F}_j, \bot)$ for some $j \in J$, then there is a cofinal subreduction f of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_j$ satisfying (AWC). Since $\{\omega\} \notin P_{(\omega+1)^\bullet}$, the root $v = f(\omega)$ of $\mathfrak{F}_j$ is reflexive and so $f^{-1}(v)$ contains a reflexive point which belongs to $W - \{u\}$. But then the map

$$g(x) = \begin{cases} f(x) & \text{if } x \in W - \{u\} \\ v & \text{if } x = u \end{cases}$$

is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_j$ satisfying (AWC).

Finally, if $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega\right\rangle \not\models \alpha^\bullet(\mathfrak{F}_k, \bot)$ for some $k \in K$, then there is a cofinal quasi-subreduction f of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_k$ satisfying (AWC). Let v be the root of $\mathfrak{F}_k$. By the definition of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$, every $X \in P_{(\omega+1)^\bullet}$ containing ω also contains some $\xi < \omega$. Let ζ be the minimal number such that $f(\zeta) = v$. Then the map

$$g(x) = \begin{cases} v & \text{if } x = \zeta \\ f(x) & \text{if } x \text{ belongs to a final cluster in } f^{-1}(f(x)) \\ \text{undefined otherwise} & \end{cases}$$

is a cofinal quasi-subreduction of $\mathfrak{F}^{ir}_{\zeta+1}$ to $\mathfrak{F}_k$ satisfying (AWC). It remains, as we have already done before, to remove from $\mathfrak{F}^{ir}_{\zeta+1}$ all those points $\xi < \zeta$ that are not in dom$g$, thus obtaining a frame which is isomorphic to some $\mathfrak{F}^{ir}_{\xi}$, $\xi \leq |\mathfrak{F}_k|$, and cofinally quasi-subreducible by g to $\mathfrak{F}_k$ with $g(\xi - 1) = v$.

($\Leftarrow$) If the first condition holds then $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle$ refutes $\Box\alpha(\mathfrak{F}_i, \bot)$. The cofinal subreduction f of the second condition can be extended to the map

$$g(x) = \begin{cases} f(x) & \text{if } x \in W - \{u\} \\ v & \text{if } x = \xi \leq \omega \end{cases}$$

(v is the reflexive root of $\mathfrak{F}_j$) which is a cofinal subreduction of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_j$ with $g(\omega) = v$, and hence $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha(\mathfrak{F}_j, \bot)$. And the third condition gives in the same way a cofinal quasi-subreduction of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_k$ satisfying (AWC), from which $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha^\bullet(\mathfrak{F}_k, \bot)$. ❑

As a consequence of Theorem 11.33, Lemma 11.37 and the completeness theorem for the canonical formulas we obtain

Theorem 11.38 *All finitely axiomatizable subframe and cofinal subframe quasi-normal logics are decidable.*

It is not hard also to give a frame-theoretic characterization of the classes $\mathcal{QSF}$ and $\mathcal{QCSF}$ similar to Theorem 11.21. Let us say that a frame $\mathfrak{F}$ with actual world u is a *(cofinal) subframe* of a frame $\mathfrak{G}$ with actual world w if $\mathfrak{F}$ is a (cofinal) subframe of $\mathfrak{G}$ and $u = w$.

Theorem 11.39 *L is a (cofinal) subframe quasi-normal logic iff L is characterized by a class of frames with actual worlds that is closed under (cofinal) subframes.*

Proof Exercise. ❑

11.5 The method of inserting points

We conclude this chapter with two more sufficient conditions for finite approximability. Unlike the results of the two preceding sections, they concern logics whose canonical axioms may contain closed domains.

The first condition is based upon the observation that no subreduction can map a reflexive point to an irreflexive one and also upon the following:

Lemma 11.40 *Suppose $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{G}, \mathfrak{C}, \bot)$ are canonical formulas such that*

- *there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and*

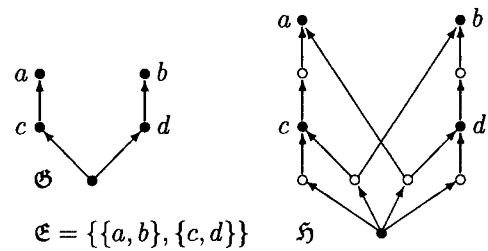


FIG. 11.6.

- *an antichain* $\mathfrak{e} \subseteq \mathrm{dom} f{\uparrow}$ *is in* $\mathfrak{C}$ *whenever* $f(\mathfrak{e}{\uparrow}) = \mathfrak{d}{\uparrow}$ *for some* $\mathfrak{d} \in \mathfrak{D}$.

Then $\alpha(\mathfrak{G},\mathfrak{C},\bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{F},\mathfrak{D},\bot)$.

Proof Let $\mathfrak{H}$ be a frame refuting $\alpha(\mathfrak{G},\mathfrak{C},\bot)$. Then there exists a cofinal subreduction g of $\mathfrak{H}$ to $\mathfrak{G}$ satisfying (CDC) for $\mathfrak{C}$. We show that the composition $h = fg$, which is a cofinal subreduction of $\mathfrak{H}$ to $\mathfrak{F}$, satisfies (CDC) for $\mathfrak{D}$.

Suppose $\mathfrak{d} \in \mathfrak{D}$, $x \in \mathrm{dom} h{\uparrow}$ and $h(x{\uparrow}) = \mathfrak{d}{\uparrow}$. Let $\mathfrak{e}$ be an antichain in $\mathfrak{G}$ such that $g(x{\uparrow}) = \mathfrak{e}{\uparrow}$. Then we have $\mathfrak{e} \subseteq \mathrm{dom} f{\uparrow}$, $f(\mathfrak{e}{\uparrow}) = \mathfrak{d}{\uparrow}$ and so $\mathfrak{e} \in \mathfrak{C}$. Therefore, by (CDC), $x \in \mathrm{dom} g$. But then $g(x) \in \mathrm{dom} f{\uparrow}$ and $f(g(x){\uparrow}) = \mathfrak{d}{\uparrow}$, since $g(x{\uparrow}) = g(x){\uparrow}$. So by (CDC), $g(x) \in \mathrm{dom} f$ and hence $x \in \mathrm{dom} h$. Thus, h satisfies (CDC) for $\mathfrak{D}$, which implies $\mathfrak{H} \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$. Since $\mathfrak{H}$ was an arbitrary refutation frame for $\alpha(\mathfrak{G},\mathfrak{C},\bot)$, it follows that $\alpha(\mathfrak{G},\mathfrak{C},\bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{F},\mathfrak{D},\bot)$. ❑

Remark In the proof above we did not use the cofinality condition. Consequently, Lemma 11.40 will remain true if we replace $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ and $\alpha(\mathfrak{G},\mathfrak{C},\bot)$ in it with $\alpha(\mathfrak{F},\mathfrak{D})$ and $\alpha(\mathfrak{G},\mathfrak{C})$, respectively, and regard f as a plain subreduction.

Theorem 11.41 *A logic*

$$L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i,\mathfrak{D}_i,\bot) : \ i \in I\} \oplus \{\alpha(\mathfrak{F}_j,\mathfrak{D}_j) : \ j \in J\}$$

is finitely approximable provided that either

(i) *for every* $i \in I \cup J$, *all points in* $\mathfrak{F}_i$ *are irreflexive*

or

(ii) *for every* $i \in I \cup J$, *all points in* $\mathfrak{F}_i$ *are reflexive.*

Proof (i) Suppose that all points in $\mathfrak{F}_i$, for every $i \in I \cup J$, are irreflexive and $\alpha(\mathfrak{G},\mathfrak{C},\bot)$ is an arbitrary canonical formula. We construct from $\mathfrak{G}$ a new finite frame $\mathfrak{H}$ by inserting into it new *reflexive* points. Namely, suppose $\mathfrak{e}$ is an antichain in $\mathfrak{G}$ such that $\mathfrak{e} \notin \mathfrak{C}$. Suppose also that $C_1, \dots, C_n$ are all the clusters in $\mathfrak{G}$ such that $\mathfrak{e} \subseteq C_i{\uparrow}$ and $\mathfrak{e} \cap C_i = \emptyset$, for $i = 1, \dots, n$, but no successor of C_i in $\mathfrak{G}$ possesses this property. Then we insert in $\mathfrak{G}$ new reflexive points $x_1, \dots, x_n$ so that each x_i could see only the points in $\mathfrak{e}$ and their successors and could be seen only from the points in C_i and their predecessors. The same we simultaneously do for all antichains $\mathfrak{e}$ in $\mathfrak{G}$ of that sort. The resulting frame is denoted by $\mathfrak{H}$ (see Fig. 11.6). Since no new point was inserted just below an antichain in $\mathfrak{C}$, the inversion of the natural embedding of $\mathfrak{G}$ in $\mathfrak{H}$ is a cofinal subreduction of $\mathfrak{H}$ to $\mathfrak{G}$ satisfying (CDC) for $\mathfrak{E}$. So $\mathfrak{H} \not\models \alpha(\mathfrak{G}, \mathfrak{E}, \bot)$.

Suppose now that $\alpha(\mathfrak{G}, \mathfrak{E}, \bot) \notin L$ and show that $\mathfrak{H}$ is a frame for L. If this is not the case then either $\mathfrak{H} \not\models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, for some $i \in I$, or $\mathfrak{H} \not\models \alpha(\mathfrak{F}_j, \mathfrak{D}_j)$, for some $j \in J$. We consider only the former case, since the latter one is treated similarly.

Thus, we have a cofinal subreduction f of $\mathfrak{H}$ to $\mathfrak{F}_i$ satisfying (CDC) for $\mathfrak{D}_i$. Since all the points in $\mathfrak{F}_i$ are irreflexive, no point that was added to $\mathfrak{G}$ belongs to $\mathrm{dom} f$. So f may be regarded as a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}_i$ satisfying (CDC) for $\mathfrak{D}_i$. We clearly may assume also that the subframe of $\mathfrak{G}$ generated by $\mathrm{dom} f$ is rooted (for otherwise we can take a suitable restriction of f).

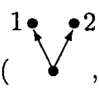
Let $\mathfrak{e}$ be an antichain in $\mathfrak{G}$ belonging to $\mathrm{dom} f{\uparrow}$ and such that $f(\mathfrak{e}{\uparrow}) = \mathfrak{d}{\uparrow}$ for some $\mathfrak{d} \in \mathfrak{D}_i$. If $\mathfrak{e} \notin \mathfrak{E}$ then there is a reflexive point x in $\mathfrak{H}$ such that $x \in \mathrm{dom} f{\uparrow}$ and x sees only $\mathfrak{e}{\uparrow}$ and, of course, itself. But then $f(x{\uparrow}) = f(\mathfrak{e}{\uparrow}) = \mathfrak{d}{\uparrow}$ and so, by (CDC), $x \in \mathrm{dom} f$, which is impossible. Therefore, $\mathfrak{e} \in \mathfrak{E}$ and so, by Lemma 11.40, $\alpha(\mathfrak{G}, \mathfrak{E}, \bot) \in L$, contrary to our assumption.

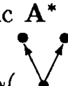
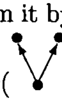
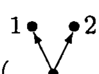
Thus, if $\alpha(\mathfrak{G}, \mathfrak{E}, \bot) \notin L$ then the finite frame $\mathfrak{H}$ validates all the axioms of L and refutes $\alpha(\mathfrak{G}, \mathfrak{E}, \bot)$, which means that L is finitely approximable.

(ii) Once again, given a canonical formula $\alpha(\mathfrak{G}, \mathfrak{E}, \bot)$, we construct in the same way the frame $\mathfrak{H}$, the only difference being that this time we insert into $\mathfrak{G}$ not reflexive but *irreflexive* points. And again we clearly have $\mathfrak{H} \not\models \alpha(\mathfrak{G}, \mathfrak{E}, \bot)$.

Suppose now that $\mathfrak{H} \not\models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$ for some $i \in I$, i.e., there is a cofinal subreduction f of $\mathfrak{H}$ to $\mathfrak{F}_i$ satisfying (CDC) for $\mathfrak{D}_i$. The difference between this case and (i) is that now new irreflexive points may belong to $\mathrm{dom} f$. But if x is such a point and $f(x) = y$ then there is $z \in x{\uparrow}$ such that $f(z) = y$, since y is reflexive. So there must be a reflexive point z in $\mathfrak{G}$ such that $z \in x{\uparrow}$ and $f(x) = f(z)$, for otherwise we could construct an infinite chain of irreflexive points in $\mathfrak{H}$, contrary to its finiteness. Therefore, the restriction of f to $\mathfrak{G}$ is a cofinal subreduction of $\mathfrak{G}$ (as well as of $\mathfrak{H}$) to $\mathfrak{F}_i$ satisfying (CDC) for $\mathfrak{D}_i$. The situation now is the same as in the previous case and so we are done. ❑

Example 11.42 According to Theorem 11.41 (i) the logic

$L = \mathbf{K4} \oplus \alpha($  $, \{\{1\}, \{1,2\}\})$

is finitely approximable. However, Artemov's logic $\mathbf{A}^* = L \oplus \mathbf{GL} = L \oplus \alpha(\circ)$ does not enjoy this property, because the formula $\alpha($  $)$ is separated from it by the frame shown in Fig. 11.7, but every finite irreflexive frame refuting $\alpha($  $)$ refutes $\alpha($  $, \{\{1\}, \{1,2\}\})$ as well. So the finite approximability is not in general preserved under sums of logics.

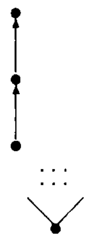


FIG. 11.7.

The scope of the method developed above is not bounded only by canonical axioms associated with homogeneous (i.e., irreflexive or reflexive) frames. Now we use the technique of inserting new points to prove that every normal extension of **K4** with modal reduction principles is finitely approximable.

We remind the reader that a modal reduction principle is a formula of the form $Mp \to Np$, where M and N are strings of $\Box$ and $\Diamond$. By Exercise 3.15, every modality Mp is equivalent in **K4** to a formula having one of the following six types:

$$\Box^n\Diamond\Box p,\ \Box^n\Diamond p,\ \Box^n p,\ \Diamond^n\Box\Diamond p,\ \Diamond^n\Box p,\ \Diamond^n p.$$

Using this fact, **K4**'s formulas $\Box p \to \Box^2 p$, $\Diamond^2 p \to \Diamond p$ and the equivalences of Exercise 3.15, we prove

Lemma 11.43 *For every set Γ of modal reduction principles there is a finite subset $\Delta \subseteq \Gamma$ such that* $\mathbf{K4} \oplus \Gamma = \mathbf{K4} \oplus \Delta$. *In other words, every normal extension of* **K4** *with modal reduction principles is finitely axiomatizable.*

Proof If Γ is infinite then it contains infinitely many modal reduction principles of the same type. Suppose, for instance, that the set Σ of all formulas in Γ of the type

$$\varphi(n,m) = \Diamond^n\Box p \to \Box^m\Diamond p,$$

for $m, n > 0$, is infinite. Define a partial order $\leq$ on Σ by taking

$$\varphi(n,m) \leq \varphi(k,l) \text{ iff } n \leq k \text{ and } m \leq l.$$

Clearly, the set Θ of minimal elements in Σ with respect to $\leq$ is finite. We show that $\mathbf{K4} \oplus \Sigma = \mathbf{K4} \oplus \Theta$. Suppose $\varphi(k,l) \in \Sigma$. Then there is $\varphi(n,m) \in \Theta$ such that $\varphi(n,m) \leq \varphi(k,l)$. Using $\Box p \to \Box^2 p$ and $\Diamond^2 p \to \Diamond p$, it is not hard to construct a derivation of $\Box^l\Diamond p$ in **K4** from the assumptions $\varphi(n,m)$ and $\Diamond^k\Box p$. Hence $\mathbf{K4} \oplus \Sigma \subseteq \mathbf{K4} \oplus \Theta$. The converse inclusion is trivial. In the same way we consider the other modal reduction principles whose premises begin with $\Diamond$ and conclusions with $\Box$.

Suppose now that we have an infinite set Σ of formulas of the type

$$\varphi(n,m) = \Box^n\Diamond p \to \Diamond^m\Box p.$$

Since $\varphi(n,m)$ is refuted in any frame with dead ends, $\mathbf{D4} \subseteq \mathbf{K4} \oplus \varphi(n,m)$ and so

$$\Box(p \to q) \to (\Box p \to \Diamond q) \in \mathbf{K4} \oplus \varphi(n,m). \tag{11.3}$$

Again, let Θ be the set of minimal formulas in Σ with respect to $\leq$ defined above. As before, to prove $\mathbf{K4} \oplus \Sigma \subseteq \mathbf{K4} \oplus \Theta$ we take $\varphi(k,l) \in \Sigma$ and choose $\varphi(n,m) \in \Theta$ such that $\varphi(n,m) \leq \varphi(k,l)$. Using (11.3) we derive from $\varphi(n,m)$ a formula $\varphi(k',l') \geq \varphi(k,l)$. Then, assuming $\Box^k \Diamond p$, we ascend to $\Box^{k'} \Diamond p$, get $\Diamond^{l'} \Box p$ and descend to $\Diamond^l \Box p$. The rest types of modal reduction principles of the form $\Box \boldsymbol{M} p \to \Diamond \boldsymbol{N} p$ are treated in exactly the same way.

If Γ contains an infinite subset Σ of formulas of the type

$$\varphi(n,m) = \Box^n \Diamond p \to \Box^m p, \text{ for } n > m,$$

then, as in the previous case, we have $\mathbf{K4} \oplus \Sigma = \mathbf{K4} \oplus \Theta$, where Θ is the (finite) set of minimal elements in Σ with respect to $\leq$. To prove this it suffices to show that, for every $k \geq n > m$,

$$\varphi(k,m) \in \mathbf{K4} \oplus \varphi(n,m). \tag{11.4}$$

Observe first that $\varphi(n + (n-m), n) \in \mathbf{K} \oplus \varphi(n,m)$, which together with Exercise 3.15 yields $\varphi(n + (n-m), m) \in \mathbf{K4} \oplus \varphi(n,m)$. Therefore, in view of $n > m$, we can find $l > k$ such that $\varphi(l,m) \in \mathbf{K4} \oplus \varphi(n,m)$ and then, using the axiom of $\mathbf{K4}$, easily derive (11.4).

Finally, if there is an infinite subset $\Sigma \subseteq \Gamma$ of formulas of the type

$$\varphi(n,m) = \Box^n \Diamond p \to \Box^m p, \text{ for } n \leq m,$$

then we define $\leq$ on Σ by taking $\varphi(n,m) \leq \varphi(k,l)$ iff $n \leq k$, $m \leq l$ and $m - n \leq l - k$ and proceed as before. The remaining cases are considered analogously. ❑

Now let us elucidate the constitution of refutation frames for those modal reduction principles that are not $\Box\Diamond$-formulas. In the following lemmas we denote by $\mathfrak{R}$ the frame $\begin{smallmatrix} \circ\, 1 \\ \uparrow \\ \circ\, 0 \end{smallmatrix}$, by $\mathfrak{C}_m$ the chain of $m+1$ irreflexive points and by $\mathfrak{D}^\sharp_m$ the set of all antichains in $\mathfrak{C}_m$. $\mathcal{C}_m$ denotes the (finite) class of all rooted 1-generated Kripke frames $\mathfrak{G}$ such that

- there is at most one reflexive point in $\mathfrak{G}$ and it is of depth 1;
- the longest chain of irreflexive points in $\mathfrak{G}$ is of length $m+1$.

For $m \geq n > 0$, $\mathcal{C}^n_m$ is the subclass of $\mathcal{C}_m$ whose frames $\mathfrak{G}$ satisfy one more condition:

- every chain of $n+1$ irreflexive points has a reflexive successor in $\mathfrak{G}$.

Given $\mathfrak{G} \in \mathcal{C}_m$, we denote by $\mathfrak{D}^\flat$ the set of all antichains $\mathfrak{d}$ in $\mathfrak{G}$ such that the subframe of $\mathfrak{G}$ generated by $\mathfrak{d}$ contains an irreflexive point of depth 1.

Lemma 11.44 (i) *If $n > m \geq 0$ then*

$$\mathbf{K4} \oplus \Box^n\Diamond\Box p \to \Box^m p = \mathbf{K4} \oplus \alpha(\mathfrak{R}, \bot) \oplus \{\alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot) : \ \mathfrak{G} \in \mathcal{C}_m\}.$$

(ii) *If $m \geq n > 0$ then*

$$\mathbf{K4} \oplus \Box^n\Diamond\Box p \to \Box^m p = \mathbf{K4} \oplus \alpha(\mathfrak{R}, \bot) \oplus \{\alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot) : \ \mathfrak{G} \in \mathcal{C}^n_m\}.$$

(iii) *If $n > m \geq 0$ then*

$$\mathbf{K4} \oplus \Box^n\Diamond p \to \Box^m p = \mathbf{K4} \oplus \alpha(\circ\circ, \bot) \oplus \alpha(\mathfrak{R}, \bot) \oplus \{\alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot) : \ \mathfrak{G} \in \mathcal{C}_m\}.$$

(iv) *If $m \geq n > 0$ then*

$$\mathbf{K4} \oplus \Box^n\Diamond p \to \Box^m p = \mathbf{K4} \oplus \alpha(\circ\circ, \bot) \oplus \alpha(\mathfrak{R}, \bot) \oplus \{\alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot) : \ \mathfrak{G} \in \mathcal{C}^n_m\}.$$

(v) *If $n > m \geq 0$ then*

$$\mathbf{K4} \oplus \Box^n p \to \Box^m p = \mathbf{K4} \oplus \alpha(\mathfrak{C}_m, \mathfrak{D}^\sharp_m).$$

Proof (i) Suppose $\Box^n\Diamond\Box p \to \Box^m p$ is refuted under a valuation $\mathfrak{V}$ at the root of a refined frame $\mathfrak{F} = \langle W, R, P\rangle$, generated by the set $\mathfrak{V}(p)$, and show that $\mathfrak{F}$ also refutes one of the axioms in the right-hand part of the equality to be established. Consider two cases.

Case 1. There is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{R}$. Then $\mathfrak{F} \not\models \alpha(\mathfrak{R}, \bot)$.

Case 2. Assume now that $\mathfrak{F}$ is not subreducible cofinally to $\mathfrak{R}$. Then $\mathfrak{F}$ contains at most one reflexive point of finite depth and it is of depth 1. Indeed, it follows from our assumption that every reflexive point x of finite depth > 1 has an irreflexive successor y of depth 1. But then, since $x \models \Box\Diamond\Box p$, we must have also $y \models \Diamond\Box p$, which is impossible. So all reflexive points of finite depth, if any, lie at depth 1, and since p is true at all of them and $\mathfrak{F}^{\leq 1}$ is a generated subframe of $\mathfrak{F}^{\leq 1}_{\mathbf{K4}}(1)$, there exists at most one point of that type.

In this situation, to refute $\Box^n\Diamond\Box p \to \Box^m p$ the frame $\mathfrak{F}$ must contain at least one chain of $m+1$ irreflexive points. Take a minimal generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ containing such a chain. Then clearly we have $\mathfrak{G} \in \mathcal{C}_m$ and $\mathfrak{F} \not\models \alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot)$.

Thus we have proved that

$$\mathbf{K4} \oplus \Box^n\Diamond\Box p \to \Box^m p \subseteq \mathbf{K4} \oplus \alpha(\mathfrak{R}, \bot) \oplus \{\alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot) : \ \mathfrak{G} \in \mathcal{C}_m\}.$$

To establish the converse inclusion, suppose first that a frame $\mathfrak{F}$ refutes $\alpha(\mathfrak{R}, \bot)$. This means that there is a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{R}$. Without loss of generality we may assume that f is a reduction of a generated subframe of $\mathfrak{F}$ to $\mathfrak{R}$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(p) = f^{-1}(1)$. Then it is easy to check that $x \not\models \Box^n\Diamond\Box p \to \Box^m p$, for every $x \in f^{-1}(0)$.

Suppose now that $\mathfrak{F} \not\models \alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot)$, for some $\mathfrak{G} \in \mathcal{C}_m$. Then without loss of generality we may assume that there is a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$

satisfying (CDC) for $\mathfrak{D}^\flat$ and such that the root y of $\mathfrak{F}$ is in $\mathrm{dom} f$. Let $a_0, \dots, a_m$ be a longest chain of irreflexive points in $\mathfrak{G}$. Clearly, $f(y) = a_0$. Define a valuation in $\mathfrak{F}$ so that $x \not\models p$ iff $x \in f^{-1}(a_m)$ and prove that then we shall have $y \not\models \Box^n\Diamond\Box p \to \Box^m p$. Notice first that $y \not\models \Box^m p$ and so it suffices to show that $y \models \Box^n\Diamond\Box p$. Suppose otherwise. Then there is an ascending chain $y, y_1, \dots, y_n$ such that $y_n \not\models \Diamond\Box p$. Since $n > m$ and by (CDC), this is possible only if $f(y_n\uparrow)$ contains the reflexive point in $\mathfrak{G}$ (for otherwise $y_1, \dots, y_n$ are irreflexive points in $\mathrm{dom} f$ and so $f(y_1), \dots, f(y_n)$ is a chain of irreflexive points in $\mathfrak{G}$). But then $y_n \models \Diamond\Box p$, which is a contradiction.

The remaining items are proved analogously; we leave them to the reader as an exercise. ❑

For points x and y in a frame $\mathfrak{F} = \langle W, R \rangle$ such that xRy, let

$$l(x,y) = \sup\{k+1 : \exists x_1, \dots, x_k \in W \ xRx_1 \dots Rx_kRy\}.$$

If there are arbitrarily long chains (of not necessarily distinct points) connecting x and y, in particular, if x or y or a point between them is reflexive, then $l(x,y) = \infty$.

It is not hard to see that the following lemma holds.

Lemma 11.45 *For every Kripke frame $\mathfrak{F}$, $\mathfrak{F} \not\models \alpha(\mathfrak{C}_m, \mathfrak{D}^\sharp_m)$ iff there are points x and y in $\mathfrak{F}$ such that $m \le l(x,y) < \infty$.*

The crucial step in establishing the finite approximability of logics whose axioms are modal reduction principles is

Lemma 11.46 *Every logic $L \in$ NExt**K4** axiomatizable by modal reduction principles of the types $\Box^n\Diamond\Box p \to \Box^m p$, $\Box^n\Diamond p \to \Box^m p$, $\Box^n p \to \Box^m p$ is finitely approximable.*

Proof We use virtually the same technique of inserting reflexive points as in the proof of Theorem 11.41.

By Lemma 11.44, L can be axiomatized by canonical axioms of the form $\alpha(\mathfrak{R}, \bot)$, $\alpha(\circ\circ, \bot)$, $\alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot)$ and $\alpha(\mathfrak{C}_m, \mathfrak{D}^\sharp_m)$ (where $\mathfrak{G} \in \mathcal{C}_m$, for some m). Fix such an axiomatization. By Theorem 11.41 and Lemma 11.44, L is finitely approximable if all its axioms are of the form $\Box^n p \to \Box^m p$. So let us assume that $L \supseteq \mathbf{K4} \oplus \alpha(\mathfrak{R}, \bot)$. Take an arbitrary canonical formula $\alpha(\mathfrak{H}, \mathfrak{E}, \bot)$.

For every antichain $\mathfrak{e}$ in $\mathfrak{H}$ such that $\mathfrak{e} \notin \mathfrak{E}$ and $\mathfrak{e}\underline{\uparrow}$ contains an irreflexive point of depth 1, we insert new reflexive points between $\mathfrak{e}$ and its immediate predecessors in the same way as was done in the proof of Theorem 11.41. We are going to show now that either $\alpha(\mathfrak{H}, \mathfrak{E}, \bot) \in L$ or the constructed finite frame—call it $\mathfrak{H}'$—separates $\alpha(\mathfrak{H}, \mathfrak{E}, \bot)$ from L. Clearly $\mathfrak{H}' \not\models \alpha(\mathfrak{H}, \mathfrak{E}, \bot)$. So if $\mathfrak{H}' \models L$ then we are done. Suppose $\mathfrak{H}'$ is not a frame for L. Then three cases are to be considered.

Case 1. $\mathfrak{H}' \not\models \alpha(\mathfrak{R}, \bot)$, i.e., there is a cofinal subreduction f of $\mathfrak{H}'$ to $\mathfrak{R}$. Then $\mathfrak{H}$ is also cofinally subreducible to $\mathfrak{R}$, because every new reflexive point has an irreflexive successor of depth 1 and so cannot belong to $\mathrm{dom} f$. By Theorem 11.15, it follows that $\alpha(\mathfrak{H}, \mathfrak{C}, \bot) \in L$. For the same reason, if $\alpha(\circ\circ, \bot)$ is an axiom of L and $\mathfrak{H}' \not\models \alpha(\circ\circ, \bot)$ then $\alpha(\mathfrak{H}, \mathfrak{C}, \bot) \in L$.

Case 2. Suppose that $\alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot)$ is an axiom of L, for some $\mathfrak{G} \in \mathcal{C}_m$, and $\mathfrak{H}' \not\models \alpha(\mathfrak{G}, \mathfrak{D}^\flat, \bot)$. This means that there is a cofinal subreduction f of $\mathfrak{H}'$ to $\mathfrak{G}$ satisfying (CDC) for $\mathfrak{D}^\flat$ and such that the subframe of $\mathfrak{H}'$ generated by dom$f$ is rooted. Since the only reflexive point in $\mathfrak{G}$, if any, is of depth 1, no new reflexive point is in dom$f$ and so the map f may be considered as a cofinal subreduction of $\mathfrak{H}$ to $\mathfrak{G}$ satisfying (CDC) for $\mathfrak{D}^\flat$. Let $\mathfrak{e}$ be an antichain in $\mathfrak{H}$ such that $\mathfrak{e} \subseteq \mathrm{dom}f\!\uparrow$ and $f(\mathfrak{e}\underline{\uparrow}) = \mathfrak{d}\underline{\uparrow}$, for some $\mathfrak{d} \in \mathfrak{D}^\flat$. Since for every closed domain $\mathfrak{d} \in \mathfrak{D}^\flat$, $\mathfrak{d}\underline{\uparrow}$ contains an irreflexive point of depth 1 in $\mathfrak{G}$, $\mathfrak{e}\underline{\uparrow}$ must also contain a final irreflexive point. So if $\mathfrak{e} \notin \mathfrak{C}$ then there is a reflexive point in $\mathfrak{H}'$ just below $\mathfrak{e}$, contrary to f satisfying (CDC) for $\mathfrak{D}^\flat$. Hence $\mathfrak{e} \in \mathfrak{C}$ and, by Lemma 11.40, $\alpha(\mathfrak{H}, \mathfrak{C}, \bot) \in L$.

Case 3. If $\mathfrak{H}' \not\models \alpha(\mathfrak{C}_m, \mathfrak{D}^\sharp_m)$ then, by Lemma 11.45, there are points a and b in $\mathfrak{H}$ such that $l(a, b) = m$ in both $\mathfrak{H}$ and $\mathfrak{H}'$. By the construction of $\mathfrak{H}'$, this means, in particular, that every antichain $\mathfrak{e} \subseteq a\!\uparrow$, having a point in $b\underline{\downarrow}$, is in $\mathfrak{C}$ whenever $\mathfrak{e}\underline{\uparrow}$ contains an irreflexive point of depth 1. Using our assumption that $\alpha(\mathfrak{N}, \bot) \in L$, we show that in this case $\alpha(\mathfrak{H}, \mathfrak{C}, \bot) \in L$ as well.

Suppose otherwise. Since

$$\mathbf{K4} \oplus \Box^n\Diamond\Box p \to \Box^m p = \mathbf{K4} \oplus \Diamond^m p \to \Diamond^n\Box\Diamond p,$$

L is a Sahlqvist logic. So it is $\mathcal{D}$-persistent and there is a finitely generated refined frame such that its underlying Kripke frame $\mathfrak{F} = \langle W, R\rangle$ validates L and refutes $\alpha(\mathfrak{H}, \mathfrak{C}, \bot)$. Let h be a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$. Our aim now is either to subreduce cofinally $\mathfrak{F}$ to $\mathfrak{N}$ or to find points x, y in $\mathfrak{F}$ with $m \le l(x, y) < \infty$, which will mean that either $\mathfrak{F} \not\models \alpha(\mathfrak{N}, \bot)$ or $\mathfrak{F} \not\models \alpha(\mathfrak{C}_m, \mathfrak{D}^\sharp_m)$.

Let us consider first the maximal generated subframe $\mathfrak{F}'$ of $\mathfrak{F}$ whose final points are reflexive. If there is a reflexive point of depth > 1 or an infinite ascending chain of irreflexive points in $\mathfrak{F}'$ then clearly $\mathfrak{F}$ is cofinally subreducible to $\mathfrak{N}$. So suppose this is not the case. If there is a point in $\mathfrak{F}'$ of depth $> m + 1$ then, by Lemma 11.45, we are done.

Thus $\mathfrak{F}'$ is of depth $\le m + 1$. We show that, for every $x \in h^{-1}(a)$, there is $y \in h^{-1}(b) \cap x\!\uparrow$ such that $m \le l(x, y) < \infty$. Take any $x \in h^{-1}(a)$. By the definition of subreduction, we clearly must have some $y \in h^{-1}(b) \cap x\!\uparrow$ with $m \le l(x, y)$. Suppose $l(x, y) = \infty$. Then there is a chain $xRx_1 \dots Rx_nRy$ such that all x_i are not in $\mathfrak{F}'$ and n exceeds the number of points in $\mathfrak{H}$. Let $\mathfrak{e}_i$ be an antichain in $\mathfrak{H}$ such that $h(x_i\!\uparrow) = \mathfrak{e}_i\underline{\uparrow}$. Since x_i sees an irreflexive point of depth 1, $\mathfrak{e}_i$ also sees or contains such a point and so $\mathfrak{e}_i \in \mathfrak{C}$. Therefore all x_i are in dom$h$, which is possible only if $h(x_i)$, for some i, is reflexive, i.e., we have a reflexive point between a and b. But then $l(a, b) = \infty$, which is a contradiction. ❑

In fact, the modal reduction principles that do not belong to the scope of Lemma 11.46 either axiomatize logics of finite depth or are deductively equal to

$\Box\Diamond$-formulas. This follows from the next two lemmas.

Lemma 11.47 *For every $n > 0$, $\mathbf{K4} \oplus \Diamond^n\Box\Diamond p \to \Box^m p$, $\mathbf{K4} \oplus \Diamond^n\Box p \to \Box^m p$ and $\mathbf{K4} \oplus \Diamond^n p \to \Box^m p$ are logics of finite depth.*

Proof It is enough to show that the axioms of these logics are refuted in an arbitrary finite rooted frame $\mathfrak{F}$ of depth $\max\{m, n\} + 2$. Define a valuation in such an $\mathfrak{F}$ so that $x \models p$ iff x is of depth 1. It should be clear that under this valuation $\Box^m p$ is false at the root z of $\mathfrak{F}$. By the definition, there is a point y of depth 1 which is accessible from z by n steps. And since $y \models \Box\Diamond p \wedge \Box p \wedge p$, it follows that $z \models \Diamond^n\Box\Diamond p \wedge \Diamond^n\Box p \wedge \Diamond^n p$. ❑

Lemma 11.48 (i) *For every $n, m > 0$,*

$$\mathbf{K4} \oplus \Box^n\Diamond\Box p \to \Diamond^m p = \mathbf{K4} \oplus \Box^n\Diamond p \to \Diamond^m p = \mathbf{K4} \oplus \Box^n p \to \Diamond^m p =$$
$$\mathbf{K4} \oplus \alpha(\bullet, \bot) = \mathbf{K4} \oplus \Diamond\top = \mathbf{D4}.$$

(ii) *For every $n, m > 0$,*

$$\mathbf{K4} \oplus \Box^n p \to \Box^m\Diamond\Box p = \mathbf{K4} \oplus \Box^n p \to \Box^m\Diamond p = \mathbf{K4} \oplus \Box^m\Diamond\top.$$

Proof (i) follows from the obvious fact that the modal reduction principles under consideration are refuted by frames with dead ends and validated by finite serial frames.

(ii) We prove only the latter equality. Clearly, it is sufficient to show that

$$\Box^n p \to \Box^m\Diamond p \in \mathbf{K4} \oplus \Box^m\Diamond\top.$$

Since the logic $\mathbf{K4} \oplus \Box^m\Diamond\top$ is finitely approximable, we take a finite frame $\mathfrak{F}$ for it and prove that $\mathfrak{F} \models \Box^n p \to \Box^m\Diamond p$. Suppose otherwise, i.e., under some valuation $x \models \Box^n p$ and $x \not\models \Box^m\Diamond p$, for some x in $\mathfrak{F}$. Then there is a point y of depth 1 accessible from x by m steps and such that $y \not\models \Diamond p$. Since $y \models \Box^n p$, y is irreflexive. But then we must have $x \not\models \Box^m\Diamond\top$, which is a contradiction. ❑

Now we have everything we need to prove

Theorem 11.49 *Every logic $L \in$ NExt**K4** axiomatizable by modal reduction principles is finitely approximable and decidable.*

Proof Observe first that

$$\mathbf{K4} \oplus \Diamond^m p \to \Diamond^n\Box\Diamond p = \mathbf{K4} \oplus \Box^n\Diamond\Box p \to \Box^m p,$$

$$\mathbf{K4} \oplus \Box^n\Diamond\Box p \to \Diamond^m\Box p = \mathbf{K4} \oplus \Box^m\Diamond p \to \Diamond^n\Box\Diamond p,$$

etc. So L is (finitely, by virtue of Lemma 11.43) axiomatizable by modal reduction principles mentioned in Lemmas 11.46, 11.47 and $\Box\Diamond$-formulas ($\Diamond\top$, as well as any other variable free formula, is also a $\Box\Diamond$-formula). The claim of our theorem follows then from Lemmas 11.46, 11.47 and Theorems 11.13, 8.85. ❑

11.6 The method of removing points

Unlike Theorem 11.41 and Lemma 11.46, the sufficient condition of the finite approximability to be obtained in this section is proved by the more conventional technique of removing points from, say, universal models. Such a technique was used in the selective filtration method and Fine's method of maximal points (Section 10.4). Another example of that sort is the method of step-wise refinement with removing Σ-remaindered points, exploited in the proof of Theorem 9.34, which actually establishes the finite approximability of cofinal subframe logics. Here we are going to tune this method by adopting a subtler strategy of removing points to cover a wider class of canonical axioms with a rather complex structure of closed domains.

Suppose we have a logic

$$L = \mathbf{K4} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$$

and a canonical formula $\alpha = \alpha(\mathfrak{H}, \mathfrak{C}, \bot)$ which is not in L. Then there exists a rooted frame $\mathfrak{F} = \langle W, R, P\rangle$ for L such that $\mathfrak{F} \not\models \alpha$, i.e., there is a cofinal subreduction h of $\mathfrak{F}$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$. Construct the countermodel $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ for α as it was done in the proof of Theorem 9.39. Without loss of generality we may assume that

- $\mathrm{dom} h\underline{\uparrow} = \mathrm{dom} h\overline{\downarrow} = W$;
- if a is a reflexive point in $\mathfrak{H}$ then a point $x \in W$ is in $h^{-1}(C(a))$ whenever $h(x\underline{\uparrow}) = a\uparrow$;
- $\mathfrak{F}$ is generated by the sets $\mathfrak{V}(p_i)$, p_i a variable in α.

Let $\Sigma = \mathbf{Sub}\alpha$. It is easy to check that all points $x, y \notin \mathrm{dom} h$ such that $h(x\uparrow) = h(y\uparrow)$ are Σ-equivalent in $\mathfrak{M}$. Now we construct a sequence

$$\mathfrak{M}_0 = \mathfrak{M}, \dots, \mathfrak{M}_i = \langle \mathfrak{F}_i, \mathfrak{V}_i\rangle, \mathfrak{M}_{i+1} = \langle \mathfrak{F}_{i+1}, \mathfrak{V}_{i+1}\rangle, \dots$$

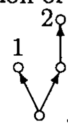
of models in almost the same way as in the proof of Theorem 9.34. The only difference concerns removing points. Suppose we have already constructed $\mathfrak{M}_i$ and its reduct $[\mathfrak{M}_i]$ (we use the same notations as in the proof of Theorem 9.34). Now we throw away points of two sorts.

First, for every proper cluster C of depth $i + 1$ such that some $x \in C$ is Σ-remaindered in $[\mathfrak{F}_i]^{\leq i}$, we remove from C all the points except x. It should be clear from the construction of $\mathfrak{M}$ that every removed point is also Σ-remaindered in $[\mathfrak{F}_i]^{\leq i}$ and that the set of all such points is in $[P_i]$. Let $[\mathfrak{M}_i'] = \langle [\mathfrak{F}_i'], [\mathfrak{V}_i']\rangle$ be the resulting submodel of $[\mathfrak{M}_i]$.

Second, we call a point x in $[W_i']^{>i+1}$ *redundant* in $[\mathfrak{M}_i']$ if it is Σ-remaindered in $[\mathfrak{F}_i']^{\leq i+1}$ and, for every $j \in I$ and every cofinal subreduction g of $[\mathfrak{F}_i']^{\leq i+1}$ to the subframe of $\mathfrak{G}_j$ generated by some $\mathfrak{d} \in \mathfrak{D}_j$ such that $\mathfrak{d} \subseteq g(x\uparrow)$ and g satisfies (CDC) for $\mathfrak{D}_j$, there is a point $y \in x\uparrow$ in $[\mathfrak{F}_i']^{\leq i+1}$ such that $g(y\uparrow) = \mathfrak{d}\underline{\uparrow}^{\bullet}$. Let X be the maximal set of redundant points in $[\mathfrak{M}_i']$ which is upward closed in $[W_i']^{>i+1}$. Since $[\mathfrak{F}_i']^{\leq i+1}$ is finite and every point in it is an atom, it is not hard to see that $X \in [P'_i]$ (this is left to the reader). We define $\mathfrak{M}_{i+1} = \langle \mathfrak{F}_{i+1}, \mathfrak{V}_{i+1} \rangle$ as the submodel of $[\mathfrak{M}'_i]$ induced by the set of points in $[\mathfrak{F}'_i]$ different from those in X.

It should be clear that $\mathfrak{M}_i$ (and hence $\mathfrak{M}_0$) is Σ-subreducible to $\mathfrak{M}_{i+1}$, and so $\mathfrak{M}_{i+1} \not\models \alpha$. Besides, as follows from the definition of redundant points, if $\mathfrak{F}_{i+1} \not\models \alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$, for some $j \in I$, then $\mathfrak{F}_i \not\models \alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$. Hence $\mathfrak{F}_{i+1} \models L$. So the finite approximability of L will be established if we manage to prove that our modified process of refining and removing eventually terminates (i.e., $W_i^{>i} = \emptyset$ for some $i > 0$).

It is not hard to see that for some $\mathfrak{F}$, L and α the process never stops, even though L is finitely approximable. On the other hand, there are many axioms $\alpha(\mathfrak{G}, \mathfrak{D}, \bot)$ such that too deep points in $\mathfrak{F}_i$ cannot be mapped to points in closed domains in $\mathfrak{D}$ by cofinal subreductions to $\mathfrak{G}$, which induces eventual halting of the process. Here is a simple example illustrating this phenomenon.

Example 11.50 Let L be the smallest modal companion of the Scott logic **SL**, i.e., $L = \mathbf{S4} \oplus \alpha(\mathfrak{G}, \{\{1,2\}\}, \bot)$, where $\mathfrak{G}$ is the frame . Suppose also that $\alpha = \alpha(\mathfrak{H}, \mathfrak{C}, \bot) \notin L$, $\mathfrak{F}$ separates α from L and that our "algorithm", when being applied to $\mathfrak{F}$, α and L, works infinitely long. Then the frame $\mathfrak{F}_\omega = \langle W_\omega, R_\omega \rangle$, where

$$W_\omega = \bigcup_{0<i<\omega} W_i^{\leq i},\ R_\omega = \bigcup_{0<i<\omega} R_i^{\leq i},$$

is of infinite depth. By König's lemma, there is an infinite descending chain

$$\dots x_i R_\omega x_{i-1} \dots R_\omega x_2 R_\omega x_1$$

in $\mathfrak{F}_\omega$ such that x_i is of depth i, for every $i < \omega$. Since there are only finitely many pairwise non-Σ-equivalent points in $\mathfrak{M}$, there must be some $n > 0$ such that, for every $k \geq n$, each point in $C(x_k)$ is Σ-remaindered in $\mathfrak{F}_k^{<k}$. And since $\mathfrak{F}_1^{\leq 1}$ is finite, there is $m \geq n$ starting from which all x_i see the same points of depth 1.

Let us consider now the frame $\mathfrak{F}_m$ and ask ourselves why points in the m-cyclic set X, folded at step $m+1$ into $C(x_{m+1})$, were not removed at step m. X is upward closed in $W_m^{>m}$ and every point in it is Σ-remaindered in $\mathfrak{F}_m^{\leq m}$. So the only reason for keeping some $x \in X$ in the frame is that $\mathfrak{F}_m^{\leq m}$ is cofinally subreducible to $\mathfrak{G}^{\leq 1}$, x sees inverse images of both points in $\mathfrak{G}^{\leq 1}$ but none of its successors in $\mathfrak{F}_m^{\leq m}$ does. By the cofinality condition, these inverse images can be taken from $\mathfrak{F}_1^{\leq 1}$. But then they are also seen from x_m, which is a contradiction.

Thus sooner or later our algorithm will construct a finite frame separating L from α, which proves that both L and **SL** are finitely approximable.

Theorem 11.52 to be proved below is based essentially upon the same idea as Example 11.50, though it uses a more sophisticated construction. To formulate it we require some new notions.

A point x in a frame $\mathfrak{G}$ is called a *focus* of an antichain $\mathfrak{a}$ in $\mathfrak{G}$ if $x \notin \mathfrak{a}$ and $x\uparrow = \{x\} \cup \mathfrak{a}\uparrow$.

Suppose $\mathfrak{G}$ is a finite frame and $\mathfrak{D}$ a set of closed domains in $\mathfrak{G}$. Define by induction on n the notions of an *n-stable point* in $\mathfrak{G}$ (relative to $\mathfrak{D}$) and an *n-stable antichain* in $\mathfrak{D}$. A point x is *1-stable* in $\mathfrak{G}$ iff either x is of depth 1 in $\mathfrak{G}$ or the cluster $C(x)$ is proper. A point x is *$n+1$-stable* in $\mathfrak{G}$ (relative to $\mathfrak{D}$) iff it is not m-stable, for any $m \leq n$, and either there is an n-stable point in $\mathfrak{G}$ (relative to $\mathfrak{D}$) which is not seen from x or x is a focus of an antichain in $\mathfrak{D}$ containing an $n-1$-stable point and no n-stable point. And we say an antichain $\mathfrak{d}$ in $\mathfrak{D}$ is n-stable iff it contains an n-stable point in the subframe $\mathfrak{G}'$ of $\mathfrak{G}$ generated by $\mathfrak{d}$ (relative to $\mathfrak{D}$) and no m-stable point in $\mathfrak{G}'$ (relative to $\mathfrak{D}$), for $m > n$. A point or an antichain is *stable* if it is n-stable for some n.

It should be clear from the definition that if a point in an antichain is stable then the remaining points in the antichain are also stable.

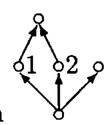
Example 11.51 (1) Suppose $\mathfrak{G}$ is a finite rooted generated subframe of one of the frames shown in Fig. 11.8 (a)–(c). Then, regardless of $\mathfrak{D}$, each point in $\mathfrak{G}$ different from its root is n-stable where n is the number located near the point. Every antichain $\mathfrak{d}$ in $\mathfrak{G}$, containing at least two points, is also n-stable, with n being the maximal degree of stability of points in $\mathfrak{d}$.

(2) If $\mathfrak{G}$ is a rooted generated subframe of the frame depicted in Fig. 11.8 (d) and $\mathfrak{D}$ is the set of all two-point antichains in $\mathfrak{G}$ then every point in $\mathfrak{G}$ is n-stable (relative to $\mathfrak{D}$), where n stays near the point. However, for $\mathfrak{D} = \emptyset$ no point in $\mathfrak{G}$, save those of depth 1, is stable.

(3) If $\mathfrak{G}$ is a finite tree of clusters then every antichain in $\mathfrak{G}$, different from a non-final singleton, is either 1- or 2-stable in $\mathfrak{G}$ regardless of $\mathfrak{D}$. More generally, if no point in $\mathfrak{d}$ sees all the points of depth 1 in $\mathfrak{d}\uparrow$, in particular, if $\mathfrak{d}$ has no upper bound in $\mathfrak{G}$, then $\mathfrak{d}$ is also either 1- or 2-stable. Every antichain containing a point x with proper $C(x)$ is 1- or 2-stable as well, whatever $\mathfrak{G}$ and $\mathfrak{D}$ are.

(4) Every antichain is stable in every irreflexive frame $\mathfrak{G}$ relative to the set $\mathfrak{D}^\sharp$ of all antichains in $\mathfrak{G}$. However, this is not so if $\mathfrak{G}$ contains reflexive points, because reflexive singletons are open domains and do not belong to $\mathfrak{D}^\sharp$. For instance, the antichains $\{a\}$ in

and $\{1, 2\}$ in



are not stable.

Now we are in a position to make a crucial step in the justification of our method.

Theorem 11.52 *Suppose $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ and there is $d > 0$ such that, for any $i \in I$, every closed domain $\mathfrak{d} \in \mathfrak{D}_i$ is n-stable in $\mathfrak{G}_i$ (relative to $\mathfrak{D}_i$), for some $n \leq d$. Then L is finitely approximable.*

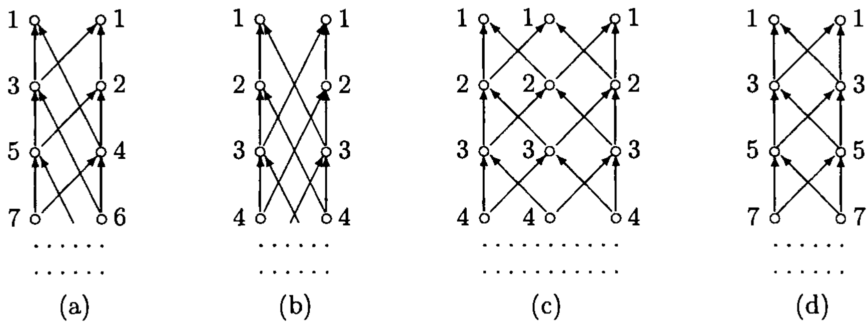


FIG. 11.8.

Proof It is enough to show that the algorithm defined above comes to a stop for every $\alpha = \alpha(\mathfrak{H}, \mathfrak{C}, \bot) \notin L$ and $\mathfrak{F}$ separating α from L. Suppose otherwise, i.e., given some $\alpha \notin L$ and $\mathfrak{F}$, the algorithm works infinitely long. Then the frame $\mathfrak{F}_\omega$, defined as in Example 11.50, is of infinite depth.

For each point x in $\mathfrak{F}_\omega$, we denote by $N(x)$ the number of pairwise non-Σ-equivalent points in $x{\uparrow}$. Since $N(x)$ cannot exceed $2^{|\Sigma|}$, there exist $k \leq 2^{|\Sigma|}$ and $n_1 \geq 1$ such that, for every $n \geq n_1$, $\mathfrak{F}_\omega$ contains at least one point x of depth n with $N(x) = k$ and no point y of depth n with $N(y) < k$. Indeed, let l_n be the minimal number $N(x)$ among all x in $\mathfrak{F}_\omega$ of depth n. The sequence $l_1, l_2, \dots$ is clearly non-decreasing and so there must be n_1 such that all l_i starting from l_{n_1} are the same. Then we can take $k = l_{n_1}$.

Put

$$X_1 = \{x \in W_\omega^{>n_1} : \ N(x) = k\}.$$

It follows from the given definition that every point in X_1 is Σ-remaindered in $W_\omega^{\leq n_1}$ and that if $x \in y{\downarrow}$, for some $x \in X_1$ and y of depth $> n_1$, then $y \in X_1$.

Now we define by induction an infinite descending sequence of non-empty sets $X_1 \supset X_2 \supset \dots$ and an infinite ascending sequence of integers $n_1 < n_2 < \dots$. Let X_r and n_r be already defined and, for each x in $\mathfrak{F}_\omega$, let

$$N_r(x) = \left|\{y \in x{\uparrow}\colon\ y \in W_\omega^{\leq n_r}\}\right|.$$

Since $\mathfrak{F}_\omega^{\leq n_r}$ is finite, there exist k and $n_{r+1} > n_r$ such that, for each $n > n_{r+1}$, $\mathfrak{F}_\omega$ contains at least one point $x \in X_r$ of depth n with $N_r(x) = k$ and no point $y \in X_r$ of depth n with $N_r(y) < k$. Then we put

$$X_{r+1} = \{x \in W_\omega^{>n_{r+1}} : \ x \in X_r \text{ and } N_r(x) = k\}.$$

By transitivity, we obviously have that if $x \in y{\downarrow}$, for some $x \in X_{r+1}$ and $y \in W_\omega^{>n_{r+1}}$, then $y \in X_{r+1}$, with x and y seeing exactly the same points of depth $\leq n_r$.

Our construction is completed now, and we are ready to derive a contradiction. Take $s = n_{d+1} + 1$, where d is the constant supplied by the assumption of our theorem, and consider an arbitrary point $x \in X_{d+1}$ of depth $s+1$. The question leading to a contradiction is why the set Y, folded at step $s+1$ into $C(x)$, was not removed at step s. Y is upward closed in $W_s^{>s}$ and its every point is Σ-remaindered in $\mathfrak{F}_s^{\leq n_1}$. So the only reason why a point $y \in Y$ was not removed at step s is that there exists a cofinal subreduction g of $\mathfrak{F}_s^{\leq s}$ to the subframe of some $\mathfrak{G}_i$ generated by some $\mathfrak{d} \in \mathfrak{D}_i$, $i \in I$, such that $\mathfrak{d} \subseteq g(y\uparrow)$ and (CDC) for $\mathfrak{D}_i$ is satisfied, but there is no $z \in y\uparrow$ of depth $\leq s$ with $\mathfrak{d} \subseteq g(z\uparrow)$.

Let f be the restriction of g to $y\uparrow$. It should be clear that f is also a cofinal subreduction of $\mathfrak{F}_s^{\leq s}$ to $\mathfrak{d}\uparrow$ satisfying (CDC) for $\mathfrak{D}_i$.

By induction on r we show now that, for each r-stable point a (relative to $\mathfrak{D}_i$) in $\mathfrak{d}\uparrow$ and each $u \in f^{-1}(a)$, there exists a point $v \in u\uparrow$ of depth $\leq n_r$ such that $f(v) = a$. In other words, this means that $f^{-1}(a)$ has a cover in $\mathfrak{F}_s^{\leq n_r}$.

The point a is 1-stable iff it is of depth 1 or the cluster $C(a)$ is proper. In the former case $f^{-1}(a)$ has a cover in $\mathfrak{F}_s^{\leq 1}$ because f is cofinal. As for the latter case, observe first that since a point $u \in X_m$, for $m \geq 1$, is Σ-remaindered in $\mathfrak{F}_s^{\leq n_1}$, the cluster $C(u)$ cannot be proper. So, as follows from the definition of reduction, in the case when $C(a)$ is proper $f^{-1}(a)$ also has a cover in $\mathfrak{F}_s^{\leq n_1}$.

Suppose our claim holds for points whose degree of stability in $\mathfrak{d}\uparrow$ is $\leq r$, a is an $r+1$-stable point in $\mathfrak{d}\uparrow$ (relative to $\mathfrak{D}_i$) and $f(u) = a$ but u is of depth $> n_{r+1}$. Since $u \in y\uparrow$, we must have $u \in X_{r+1}$. So u sees the same points of depth $\leq n_r$ as y, in particular, inverse f-images of all r-stable points in $\mathfrak{d}\uparrow$ (relative to $\mathfrak{D}_i$). But then a sees all r-stable points in $\mathfrak{d}\uparrow$ and so the only possibility for a to be $r+1$-stable is to be a focus of an antichain $\mathfrak{e} \in \mathfrak{D}_i$ whose points are at most $r-1$-stable. By the induction hypothesis, u sees inverse f-images of all points in $\mathfrak{e}$ located in $\mathfrak{F}_s^{\leq n_{r-1}}$, and they are seen also from any successor v of u of depth n_{r+1}, which certainly exists. But then, by (CDC), $f(v) = a$.

Thus all the points in $\mathfrak{d}$ have inverse f-images in $\mathfrak{F}_s^{\leq n_d}$. Take any point $z \in y\uparrow$ of depth s. Since z sees the same points of depth $\leq n_d$ as y, we must then have $\mathfrak{d} \subseteq f(z\uparrow)$, which is a contradiction. ❑

Using the modal companion and preservation theorems we can transfer this result to si-logics:

Theorem 11.53 *If, for some $d > 0$, a logic $L \in$ ExtInt can be axiomatized by a (finite or infinite) set of intuitionistic canonical formulas $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ in which every closed domain $\mathfrak{d} \in \mathfrak{D}$ is n-stable in $\mathfrak{F}$ (relative to $\mathfrak{D}$) for some $n \leq d$, then L is finitely approximable.*

As an immediate consequence we obtain

Corollary 11.54 *If a logic $L \in$ NExt**K4** (or $L \in$ ExtInt) is finitely axiomatizable by canonical formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ (or, respectively, $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$) in which every $\mathfrak{d} \in \mathfrak{D}$ is stable in $\mathfrak{F}$ (relative to $\mathfrak{D}$), then L is finitely approximable and decidable.*

Example 11.51 shows a number of applications of these results. For instance, we get the following

Theorem 11.55 *Every normal extension of a cofinal subframe logic with*

- *canonical formulas, every closed domain in which contains a point generating a proper cluster and/or*
- *canonical formulas based upon reflexive trees of clusters and/or*
- *a finite number of frame formulas based upon irreflexive frames*

is finitely approximable.

Now we use Corollary 11.54 to prove that every normal extension of **S4** with a formula in one variable is finitely approximable. To this end we require two lemmas. Until the end of this section we will assume all frames to be quasi-orders.

A pair $(\mathfrak{a}, \mathfrak{b})$ of antichains in a frame $\mathfrak{F}$ is called a *cut* of $\mathfrak{F}$ if $\mathfrak{b}$ consists of focuses for $\mathfrak{a}$ and, for every point x in $\mathfrak{F}$, either $x \in \mathfrak{a}\uparrow$ or $x \in \mathfrak{b}\downarrow$. For example, every pair $(\mathfrak{a}, \mathfrak{b})$, where $\mathfrak{a}$ and $\mathfrak{b}$ contain the points labeled by n and $n+2$, respectively, is a cut of the frame in Fig. 11.8 (d).

Lemma 11.56 *Suppose $\mathfrak{F}$ is a finite frame generated by an antichain $\mathfrak{d}$ and $\mathfrak{D}$ a set of antichains in $\mathfrak{F}$ containing $\mathfrak{d}$. If $\mathfrak{d}$ is not stable in $\mathfrak{F}$ relative to $\mathfrak{D}$ then there is a cut $(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{F}$ such that $\mathfrak{a} \notin \mathfrak{D}$ and all clusters $C(x)$, for $x \in \mathfrak{b}\downarrow$, are simple.*

Proof Let b be a point in $\mathfrak{F}$ such that it is not stable itself but has only stable immediate successors relative to $\mathfrak{D}$. It must exist, since points in $\mathfrak{d}$ are not stable, while the final points in $\mathfrak{F}$ are stable. Take an antichain $\mathfrak{a}$ for which b is a focus. Then, for any x in $\mathfrak{F}$, either $x \in \mathfrak{a}\uparrow$ or $x \in \mathfrak{a}\downarrow$. Indeed, suppose otherwise. Since $x \notin \mathfrak{a}\downarrow$, x must be stable in $\mathfrak{F}$, because the points in $\mathfrak{a}$ are stable. And since $x \notin \mathfrak{a}\uparrow$, we have also $x \notin b\uparrow$, and so b must be stable, which is a contradiction.

Let $\mathfrak{b}$ be a maximal antichain of focuses for $\mathfrak{a}$ containing b and let x be a point in $\mathfrak{F}$ such that $x \notin \mathfrak{a}\uparrow$. Then, as was shown above, $x \in \mathfrak{a}\downarrow - \mathfrak{a}$. To prove that $x \in \mathfrak{b}\downarrow$, suppose otherwise. Since no focus of $\mathfrak{a}$ is accessible from x, there is a point $y \in x\uparrow - \mathfrak{a}\uparrow$ which does not see all the points in $\mathfrak{a}$ and so is stable. But this leads to a contradiction, since b does not see y and so must be also stable.

Since b is a focus for $\mathfrak{a}$ and not stable, $\mathfrak{a} \notin \mathfrak{D}$. And if the cluster $C(x)$, for some $x \in \mathfrak{b}\downarrow$, is proper then b must be 1- or 2-stable, contrary to its choice.

❑

Lemma 11.57 *For every formula $\varphi(p)$, one can effectively construct canonical formulas $\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, $i = 1, \dots, n$, such that*

$$\mathbf{S4} \oplus \varphi = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : \ i = 1, \dots, n\} \tag{11.5}$$

and every antichain in $\mathfrak{D}_i$ is stable in $\mathfrak{F}_i$ relative to $\mathfrak{D}_i$.

Proof According to Theorem 9.34, the logic $\mathbf{S4} \oplus \varphi$ can be effectively represented in the form (11.5), with all $\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$ being associated with refined 1-generated finite models $\mathfrak{M}_i$ based on quasi-ordered frames $\mathfrak{F}_i$. We show that an arbitrary antichain $\mathfrak{d} \in \mathfrak{D}_i$ is stable in $\mathfrak{F}_i$ relative to $\mathfrak{D}_i$.

Suppose otherwise. Then, by Lemma 11.56, there is a cut $(\mathfrak{a}, \mathfrak{b})$ of the subframe $\mathfrak{G}$ of $\mathfrak{F}_i$ generated by a $\mathfrak{d} \in \mathfrak{D}_i$ such that $\mathfrak{a} \notin \mathfrak{D}_i$ and the clusters in $\mathfrak{b}\downarrow$ are simple. Consider two cases.

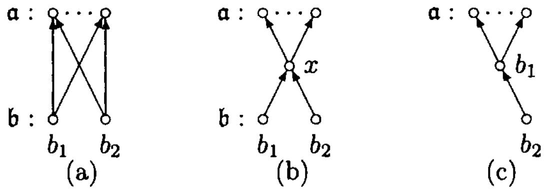


FIG. 11.9.

Case 1: $\mathfrak{b}$ contains only one point, say, b. Then, since $\mathfrak{F}_i$ is 1-generated and refined, $\mathfrak{b}$ may have only one immediate predecessor, which in turn has at most one immediate predecessor, etc. In other words, $\mathfrak{b}{\downarrow}$ is a chain in $\mathfrak{G}$ and so $\mathfrak{d}$ is a reflexive singleton, which is a contradiction.

Case 2: $\mathfrak{b}$ contains at least two points. In fact, in this case, since $\mathfrak{F}_i$ is 1-generated and refined and there are no proper clusters in $\mathfrak{b}\overline{\downarrow}$, $\mathfrak{b}$ consists of exactly two points, say, b_1 and b_2. Since $\mathfrak{a} \notin \mathfrak{D}_i$, the antichain $\mathfrak{a}$ is an open domain in $\mathfrak{M}_i$, which means that we can insert a new point x between $\mathfrak{a}$ and $\mathfrak{b}$ (see Fig. 11.9 (a), (b)) and extend to it the valuation in $\mathfrak{M}_i$ in such a way that the truth-values of all φ's subformulas at all points in $\mathfrak{F}_i$ will remain the same as before. Without loss of generality we may assume that $x \models p$, $b_1 \models p$ and $b_2 \not\models p$. It follows that in the extended model x and b_1 are $\mathbf{Sub}\varphi$-equivalent and so we can draw an arrow from b_2 to b_1 in the model $\mathfrak{M}_i$ (see Fig. 11.8 (c)) without changing the truth-values of φ's subformulas, i.e., for every point y in the resulting model $\mathfrak{M}_i'$ and every $\psi \in \mathbf{Sub}\varphi$, we shall have

$$(\mathfrak{M}_i, y) \models \psi \text{ iff } (\mathfrak{M}_i', y) \models \psi.$$

Let $\mathfrak{N}$ be the submodel of $\mathfrak{M}_i'$ generated by $\mathfrak{d}$ and $\mathfrak{N}'$ the refinement of $\mathfrak{N}$. Since $\mathfrak{N}$ is finite and in view of Theorem 8.69, it is reducible to $\mathfrak{N}'$ by a reduction f. Clearly, the pair $(f(\mathfrak{a}), \{f(b_1)\})$ is a cut of $\mathfrak{N}'$ and so we find ourselves in the framework of Case 1, i.e., $f(b_1){\downarrow}$ is a chain in $\mathfrak{N}'$. By the reduction theorem and the definition of open domains, it follows immediately that $\mathfrak{d}$ is an open domain in $\mathfrak{M}_i$, which is a contradiction. ❑

As a consequence of Corollary 11.54 and Lemma 11.57 we obtain

Theorem 11.58 *Every normal extension of* **S4** *axiomatizable by a finite number of formulas in one variable is finitely approximable and is decidable.*

Corollary 11.59 *Every si-logic axiomatizable by formulas in one variable is finitely approximable and decidable.*

Proof Follows from Theorems 7.67, 11.58 and the preservation theorem. ❑

Exercises 11.27 and 11.28 show that Theorem 11.58 is the best possible result as far as the number of variables in logics' axioms is concerned.

11.7 Exercises and open problems

Exercise 11.1 Show that for every formula φ with $md(\varphi) \leq n$ there is a unique disjunction of normal forms of degree n which is equivalent to φ ($\bot$ is assumed to be the empty disjunction).

Exercise 11.2 Show that $\Diamond\varphi \in \mathbf{S5}$ iff $\Box\Diamond\top \to \Box\Diamond\varphi \in \mathbf{K4}$.

Exercise 11.3 Show that in general Theorem 11.13 does not hold if we add to L infinitely many $\Box\Diamond$-axioms. (Hint: consider the formulas

$$\alpha_0 = \Box\Diamond p \wedge \Box\Diamond\neg r,\ \ \beta_0 = \Box\Diamond q \wedge \Box\Diamond\neg r,$$

$$\alpha_1 = \Box\Diamond\neg q \wedge \neg\Box\Diamond\neg r \wedge \neg\Box\Diamond\neg p,\ \ \beta_1 = \Box\Diamond\neg p \wedge \neg\Box\Diamond\neg r \wedge \neg\Box\Diamond\neg q,$$

$$\alpha_{n+2} = \Diamond\alpha_{n+1} \wedge \Diamond\beta_n \wedge \neg\Diamond\beta_{n+1},\ \ \beta_{n+2} = \Diamond\beta_{n+1} \wedge \Diamond\alpha_n \wedge \neg\Diamond\alpha_{n+1},$$

$$\gamma_n = \Diamond\alpha_{n+1} \wedge \Diamond\beta_{n+1} \wedge \neg\Diamond\alpha_{n+2} \wedge \neg\Diamond\beta_{n+2},$$

$$\delta = \Box(\alpha_1 \to \Diamond\alpha_0 \wedge \neg\Diamond\beta_0) \wedge \Box(\beta_1 \to \Diamond\beta_0 \wedge \neg\Diamond\alpha_0),$$

$$\iota_n = \neg\Diamond\gamma_{n-1} \wedge \Diamond\gamma_n \wedge \Diamond\gamma_{n+1},\ \ \varphi_n = \delta \to \Box(\iota_n \to \Diamond\iota_{n+1}),$$

(see Fig. 6.5) and show that the logic $\mathbf{Grz} \oplus \{\varphi_n : n < \omega\}$ is not compact.)

Exercise 11.4 (i) A logic $L = L_0 + \varphi$ (or $L = L_0 \oplus \varphi$) has the *simple substitution property* if for every $\psi(p_1, \dots, p_n)$, $\psi \in L$ iff $\varphi_1 \wedge \dots \wedge \varphi_m \to \psi \in L$, where $\varphi_1, \dots, \varphi_m$ are all possible substitution instances of φ obtained by replacing its variables by some of $p_1, \dots, p_n$. Show that if L_0 is finitely approximable then L is finitely approximable as well.

(ii) Prove that if φ is a conservative in $\mathrm{NExt}L$ formula (see Section 14.1) and L is finitely approximable then $L \oplus \varphi$ is finitely approximable too.

Exercise 11.5 Show that a logic $L \in \mathrm{NExt}(\mathbf{S4} \oplus \boldsymbol{bd_n})$ has the simple substitution property iff $\bigvee_{i<j\leq m} \Box(p_i \leftrightarrow p_j) \in L$ for some m.

Exercise 11.6 Show that if a logic $L \in \mathrm{NExt}\mathbf{S4}$ is finitely approximable then $L \oplus \mathbf{Grz}$ is finitely approximable too.

Exercise 11.7 Prove that (i) L is a cofinal subframe logic iff, for every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \bot) \in L$ whenever $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$, and that (ii) L is a subframe logic iff, for every $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}) \in L$ whenever $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$.

Exercise 11.8 Show that (i) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \bot) : i \in I\}$ iff $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is in $\mathbf{K4} \oplus \alpha(\mathfrak{F}_i, \bot)$ for some $i \in I$, and that (ii) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i) : i \in I\}$ iff $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{F}_i)$ for some $i \in I$.

Exercise 11.9 Using Corollary 11.22 prove that if φ is a Boolean combination of modalities then $\mathbf{S4} \oplus \varphi$ is finitely approximable. Does this result hold if we replace $\mathbf{S4}$ by $\mathbf{K4}$?

Exercise 11.10 Prove that all logics in $\mathrm{Ext}\mathbf{S4.3}$ are finitely axiomatizable.

Exercise 11.11 Show that **K4Z** and **Dum** are not elementary, while **K4H** is.

Exercise 11.12 Let $\mathfrak{F} = \langle W, R\rangle$ be a finite rooted frame and $a_0, \dots, a_n$ all the points in $\mathfrak{F}$. With each a_i in a non-final cluster or in a final one having no predecessors associate the individual variable x_i, and if the final cluster $C(a_i)$ has the immediate predecessors $C(a_j), \dots, C(a_k)$ then associate with a_i the variables $x_i^j, \dots, x_i^k$. The variables thus associated with points in $\mathfrak{F}$ will be denoted by x_i^s, where s is either blank or $0 \le s \le n$. Let

$$\vartheta_k(x) = \exists y_1 \dots \exists y_k(\bigwedge_{i\neq j} y_i \neq y_j \wedge R(x, y_1) \wedge R(y_1, y_2) \wedge \dots \wedge R(y_{k-1}, y_k))$$

(which means "x sees a chain of k distinct points") and

$$\vartheta(x) = \neg\exists y R(x, y) \vee \exists z(R(x, z) \wedge \neg\exists y R(z, y))$$

(which means "x is a final irreflexive point itself or sees such a point").

Define $\phi_{\mathfrak{F}}$ to be the conjunction of the following formulas under all admissible values of their parameters:

(0) $R(x_i, x_j^s)$: a_iRa_j, s is either blank or $s = i$ and the cluster $C(a_i)$ is not final in $\mathfrak{F}$;

(1) $\neg R(x_i^s, x_j^t)$: not a_iRa_j;

(2) $x_i^s \neq x_j^t$: $i \neq j$, $0 \le i < j \le n$;

(3) $\vartheta_k(x_i^s)$: $C(a_i)$ is a final non-degenerate cluster in $\mathfrak{F}$ containing k points;

(4) $\neg\exists x \bigwedge_{a_i\in X} R(x_i^s, x)$: X is an antichain in $\mathfrak{F}$ such that $\widehat{X} = \emptyset$, where $\widehat{X} = \{y : X \subseteq y{\downarrow}\}$;

(5) $\forall x(\bigwedge_{a_i\in X} R(x_i^s, x) \to \vartheta_k(x))$: X is an antichain in $\mathfrak{F}$ such that all final clusters in $\widehat{X}$ are non-degenerate and the smallest of them contains $k \ge 1$ points;

(6) $\forall x(\bigwedge_{a_i\in X} R(x_i^s, x) \to \vartheta(x))$: X is an antichain in $\mathfrak{F}$ such that each final cluster in $\widehat{X}$ is degenerate;

(7) $\forall x(\bigwedge_{a_i\in X} R(x_i^s, x) \to \vartheta(x) \vee \vartheta_k(x))$: X is an antichain in $\mathfrak{F}$ such that $\widehat{X}$ contains both degenerate and non-degenerate clusters and k is the number of points in the smallest non-degenerate one.

Prove that a Kripke frame $\mathfrak{G}$ satisfies the formula $\phi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is cofinally quasi-embeddable in $\mathfrak{G}$.

Exercise 11.13 Show that every cofinal subframe logic is elementary on the class of finite frames.

Exercise 11.14 Show that all subframe and cofinal subframe logics whose canonical axioms are built on irreflexive frames are elementary, and the cardinality of this class is that of the continuum.

Exercise 11.15 Show that there is a continuum of cofinal subframe logics of depth 3.

Exercise 11.16 Prove that every (cofinal) subframe logic can be axiomatized by an independent set of (cofinal) subframe formulas, and such an axiomatization is unique.

Exercise 11.17 Prove that a logic in $\mathcal{CSF} \cap \mathrm{NExt}\mathbf{Grz}$ is elementary iff it is of finite depth, and that the classes $\{L \in \mathcal{SF} : \mathbf{K4} \subset L \subseteq \mathbf{GL}\}$ and $\{L \in \mathcal{SF} : \mathbf{S4} \subset L \subseteq \mathbf{Grz}\}$ contain only non-elementary logics.

Exercise 11.18 Given an intuitionistic disjunction free formula, construct a first order equivalent for it with the prefix of the form $\forall\exists$.

Exercise 11.19 Prove that a (cofinal) subframe logic L is elementary iff, for every Kripke frame $\mathfrak{F}$, $\mathfrak{F} \models L$ implies $\widehat{\mathfrak{F}} \models L$.

Exercise 11.20 Show that $\mathbf{T} = \mathrm{SNExt}\mathbf{K}/\mathrm{Log}\bullet$, where $\mathrm{SNExt}\mathbf{K}$ is the lattice of subframe logics above $\mathbf{K}$.

Exercise 11.21 Show that every subframe logic in $\mathrm{NExt}\mathbf{Alt}_n$ is finitely approximable.

Exercise 11.22 Let $L = \mathbf{Alt}_3 \oplus p \to \Box\Diamond p \oplus \Box q \wedge \neg q \to \neg(\Diamond p \wedge \Diamond\neg p)$ and

$$\chi_1 = q_1 \wedge \neg q_2 \wedge \neg q_3,\ \chi_2 = \neg q_1 \wedge q_2 \wedge \neg q_3,\ \chi_3 = \neg q_1 \wedge \neg q_2 \wedge q_3,$$

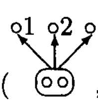
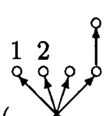
$$\varphi = \Box q \wedge \neg q \wedge \chi_1,\ \psi = (\chi_1 \to \Diamond\chi_2) \wedge (\chi_2 \to \Diamond\chi_3) \wedge (\chi_3 \to \Diamond\chi_1).$$

Show that $\psi \not\models_L^* \neg\varphi$, but for every finite model $\mathfrak{M}$ based on a frame for L, $\mathfrak{M} \models \psi$ implies $\mathfrak{M} \models \neg\varphi$. (Hint: consider the frame

$$\langle \omega, \{\langle m, m\rangle : m > 0\} \cup \{\langle m, n\rangle : |m - n| = 1\}\rangle\,.)$$

Exercise 11.23 Show that the following logics are not finitely approximable:

$\mathbf{K4} \oplus \alpha^\sharp($  $)$, $\mathbf{Gl} \oplus \alpha^\sharp($  $)$,

$\mathbf{S4} \oplus \alpha($  $, \{\{1,2\}\})$, $\mathbf{Int} + \boldsymbol{bw}_4 + \beta($  $, \{\{1,2\}\})$.

Show that the smallest modal companion of the latter logic can be axiomatized by a Sahlqvist formula.

Exercise 11.24 Prove that the modal logic of the frame shown in Fig. 8.1 (b) is not finitely approximable and finitely axiomatizable, but all its proper normal extensions are.

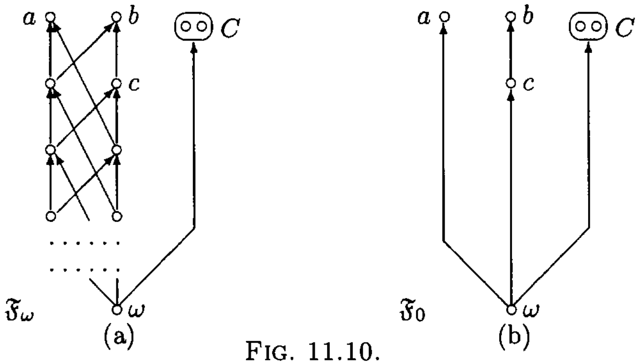


FIG. 11.10.

Exercise 11.25 Let $\mathfrak{F}_\omega$ be the frame shown in Fig. 11.10 (a). Denote by $\{\mathfrak{F}_i : i < \omega\}$ the class of its finite subframes such that each $\mathfrak{F}_i$ contains ω, the cluster C and a finite generated subframe of the "Nishimura ladder" including the points a, b, c. For example, the smallest subframe of $\mathfrak{F}_\omega$ of that sort—call it $\mathfrak{F}_0$—is depicted in Fig. 11.10 (b). Fix a point in C, say d, and denote by $\mathfrak{D}_0$ the set of all non-trivial antichains in $\mathfrak{F}_0$ containing d. And, for every $i < \omega$, let $\mathfrak{D}_i^\sharp$ be the set of all non-trivial antichains in $\mathfrak{F}_i$. Show that

(i) for each $i < \omega$, there is a formula φ_i in one variable such that

$$\mathbf{S4} \oplus \alpha(\mathfrak{F}_i, \mathfrak{D}_i^\sharp, \bot) = \mathbf{S4} \oplus \varphi_i;$$

(ii) for every distinct $i, j < \omega$, $\mathfrak{F}_i \models \alpha(\mathfrak{F}_j, \mathfrak{D}_j^\sharp, \bot)$.

Exercise 11.26 Prove that

(i) there is an infinite ascending chain of logics in NExt**S4**, each of which is axiomatizable by a formula in one variable;

(ii) there is a logic in NExt**S4** which is not finitely axiomatizable but has an infinite set of one variable axioms;

(iii) the cardinality of the class of logics in NExt**S4** with one variable axioms is that of the continuum.

Exercise 11.27 Show that there is a logic in NExt**S4** which is axiomatizable by formulas in one variable and not finitely approximable.

Exercise 11.28 Construct a modal formula φ in two variables such that $\mathbf{S4} \oplus \varphi$ is not finitely approximable.

Problem 11.1 *Can one replace finite approximability in Theorems 11.10 and 11.13 with Kripke (or some other kind of) completeness?*

Problem 11.2 *Are the logics of the form* $\mathbf{K} \oplus \Box^n p \to \Box^m p$ *and* $\mathbf{K} \oplus \boldsymbol{tra}_n$ *finitely approximable?*

Problem 11.3 *Are the logics in* NExt**K** *axiomatizable by modal reduction principles finitely approximable, decidable?*

11.8 Notes

The method of constructing models from the normal forms was developed by Fine (1975a); Section 11.1 presents the main results of this paper. Cresswell (1983) modified Fine's method to prove the finite approximability of the McKinsey logic **KM** using the canonical models.

The essentially negative axioms were considered by McKay (1971), who used Glivenko's theorem to show that the addition of such an axiom preserves the decidability of si-logics. Rybakov (1978b, 1992) proved the modal analog of Glivenko's theorem (Lemma 11.11) and applied it to $\Box\Diamond$-axioms. That infinitely many $\Box\Diamond$-axioms do not preserve finite approximability (Exercise 11.3) was shown by Rybakov (1978b). Logics with the simple substitution property (Exercise 11.4) were introduced by Sasaki (1989). The characterization of finite depth logics in NExt**S4** with this property (Exercise 11.5) was obtained in Sasaki *et al.* (1994). The result of Exercise 11.4 (ii) was proved by Maksimova (1987).

The subframe logics in NExt**K4** were introduced and studied by Fine (1985). The cofinal subframe logics in NExt**K4**, Ext**K4** and Ext**Int** were considered in Zakharyaschev (1996a). Wolter (1993) investigated subframe logics in NExt**K** (Theorem 11.31, Example 11.32 and Exercises 11.20, 11.21 were taken from his dissertation). Recently he has constructed a finitely axiomatizable subframe logic which is not decidable using the bimodal logic of this sort found by Spaan (1993). Exercises 11.7–11.17 are due to Zakharyaschev (1996a, 1997). The finite approximability of logics in NExt**S4.3** was first established by Bull (1966) with the help of the algebraic technique; Fine (1971) gave a semantic proof and showed that all these logics are finitely axiomatizable and so decidable. That intuitionistic disjunction free formulas are $\forall\exists$-definable (Exercise 11.18) was proved by Chagrova (1986) and Rodenburg (1986); Shimura (1993) gave a direct proof that the si-logics with this kind of axioms are canonical. Exercise 11.19 is due to van Benthem (1989) and Exercise 11.22 to Wolter (1994). Theorem 11.36 was actually proved by Visser (1984) (in terms of so called tail models); see also Chagrov (1985b). Minimal tense extensions of cofinal subframe logics were investigated by Wolter (1995, 1996a).

The methods of proving finite approximability presented in Sections 11.5 and 11.6 were developed by Zakharyaschev (1993, 1997). However, some of the results in these sections were obtained earlier using different techniques. That finite approximability is not in general preserved under sums of si-logics was observed by Blok (1976). Modal reduction principles were studied by van Benthem (1976b) who showed that all of them are first order definable on transitive frames and described those that are first order definable on the class of all frames. Problem 11.2, as far as we know, was raised by Segerberg. That extensions of **GL** by a finite number of frame formulas are finitely approximable was proved independently by Kracht (1993c). Moreover, he showed that the addition of such formulas preserves the finite approximability in NExt**GL**. Exercise 11.24 is due to Kracht (1993b). The finite approximability of si-logics with extra axioms in one variable was first established by Sobolev (1977b), who gave in fact a rather general syntactical sufficient condition of the finite approximability of si-logics and also constructed a si-logic with a two-variable axiom which is not finitely approximable. An extension of **Grz** with infinitely many one-variable axioms which is not finitely approximable and even not compact was constructed by Shehtman (1980). Earlier Shehtman (1977) presented incomplete calculi in NExt**Grz** and Ext**Int** with axioms in two variables. An example of a finitely axiomatizable Sahlqvist logic above **S4** that is not finitely approximable (see Exercise 11.23) was given in Chagrov and Zakharyaschev (1995b).

12

TABULARITY

Now we consider tabular and locally tabular modal and superintuitionistic logics. The main question we try to answer here is how to determine whether a given logic is tabular or locally tabular.

12.1 Finite axiomatizability of tabular logics

First we establish that every tabular modal (no matter normal or not) and si-logic is finitely axiomatizable. This will be done with the help of a syntactic criterion of tabularity which uses the following formulas:

$$\alpha_n = \neg(\varphi_1 \wedge \Diamond(\varphi_2 \wedge \Diamond(\varphi_3 \wedge \ldots \wedge \Diamond\varphi_n)\ldots)),$$

$$\beta_n = \bigwedge_{m=0}^{n-1} \neg\Diamond^m(\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n),$$

where for $1 \le i \le n$, $\varphi_i = p_1 \wedge \ldots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \ldots \wedge p_n$. The reader can check that a frame $\mathfrak{F} = \langle W, R\rangle$ refutes α_n at a point x_1 iff a chain of length n starts from x_1, i.e., $x_1 R x_2 R \ldots R x_n$ for some distinct $x_1, \ldots, x_n$. $\mathfrak{F}$ refutes β_n at x_1 iff there is a chain $x_1 R x_2 R \ldots R x_m$ of length $m < n$ such that x_m is of branching n, i.e., $x_m R y_1, \ldots, x_m R y_n$ for some distinct $y_1, \ldots, y_n$. The conjunction $\alpha_n \wedge \beta_n$ will be denoted by $\boldsymbol{tab}_n$.

Theorem 12.1 (i) *A logic $L \in$ Ext$\mathbf{K}$ is tabular iff, for some $n < \omega$, $\boldsymbol{tab}_n \in L$.*

(ii) *There is a recursive function $f(n)$ such that every rooted frame validating $\boldsymbol{tab}_n$ contains $\le f(n)$ points.*

Proof (i) Suppose L is tabular, i.e., $L = \mathrm{Log}\mathfrak{F}$ for some finite frame $\mathfrak{F}$ of cardinality $n-1$. Then clearly $\mathfrak{F} \models \boldsymbol{tab}_n$, from which $\boldsymbol{tab}_n \in L$.

Suppose now that $\boldsymbol{tab}_n \in L$. This means that only chains of length $\le n-1$ can start from every point (every distinguished point, if L is not normal) in the canonical model for L, and each point in those chains is of branching $\le n-1$. Indeed, let $x_1 R \ldots R x_n$ be a chain starting from a (distinguished) point x_1. Since $x_i \ne x_j$, for $1 \le i < j \le n$, by the definition of the canonical model, there are formulas ψ_{ij} such that $x_i \models \psi_{ij}$ and $x_j \not\models \psi_{ij}$. Then taking $\chi_i = \neg \bigwedge_{j=1}^n \psi_{ij}$, we have, for $1 \le i \le n$,

$$x_i \models \chi_1 \wedge \ldots \wedge \chi_{i-1} \wedge \neg\chi_i \wedge \chi_{i+1} \wedge \ldots \wedge \chi_n,$$

and so

$$x_1 \models \varphi_1' \wedge \Diamond(\varphi_2' \wedge \Diamond(\varphi_3' \wedge \ldots \wedge \Diamond\varphi_n') \ldots),$$

where $\varphi_i' = \varphi_i\{\chi_1/p_1, \ldots, \chi_n/p_n\}$. Therefore, $x_1 \models \alpha_n\{\chi_1/p_1, \ldots, \chi_n/p_n\}$, which is a contradiction. The claim concerning branching is proved analogously.

It follows immediately that the number of points in every subframe generated by a (distinguished) point in the canonical model for L does not exceed the number of points in the $n-1$-ary tree of depth $n-1$, that is

$$f(n) = 1 + (n-1) + (n-1)^2 + \ldots + (n-1)^{n-2}.$$

So L is complete with respect to a class of finite models of cardinality $\leq f(n)$, which means (see the proof of Theorem 8.47) that L is characterized by a finite class of finite frames, i.e., tabular.

(ii) It suffices to take the function $f(n)$ defined above. ❑

Corollary 12.2 (i) $L \in$ NExt**K** *is tabular iff* $\boldsymbol{alt}_n \wedge \boldsymbol{tra}_n \in L$, *for some* $n < \omega$.

(ii) $L \in$ Ext**Int** *is tabular iff* L *is of finite width and depth, i.e.,* $\boldsymbol{bw}_n \wedge \boldsymbol{bd}_n \in L$ *for some* $n < \omega$.

Proof Exercise. ❑

Corollary 12.3 *Every tabular modal or si-logic* L *has finitely many extensions and all of them are tabular.*

Proof That all these extensions are tabular follows from Theorem 12.1 (i). And by (ii), there exist only finitely many distinct rooted frames for L and so only finitely many extensions of L. ❑

We can prove now the main results of this section.

Theorem 12.4 (i) *Every tabular logic* $L \in$ Ext**K** *is finitely axiomatizable.*

(ii) *Every tabular si-logic is finitely axiomatizable.*

Proof (i) According to Theorem 12.1 (i), L is an extension of $\mathbf{K} + \boldsymbol{tab}_n$, for some $n < \omega$. By Corollary 12.3, we have a chain

$$\mathbf{K} + \boldsymbol{tab}_n = L_1 \subset L_2 \subset \ldots \subset L_{k-1} \subset L_k = L$$

of logics such that

$$\{L' \in \text{Ext}\mathbf{K} : \; L_i \subset L' \subset L_{i+1}\} = \emptyset,$$

for every $i = 1, \ldots, k-1$. It remains to notice that if L' is finitely axiomatizable, $L' \subset L''$ and there is no logic located properly between L' and L'' then L'' is also finitely axiomatizable (e.g. $L'' = L' + \varphi$, for any $\varphi \in L'' - L'$).

(ii) is proved analogously. ❑

12.2 Immediate predecessors of tabular logics

Let L be a tabular logic and L' an arbitrary logic. The question we consider in this section is how to determine whether $L = L'$. To be more specific, we always

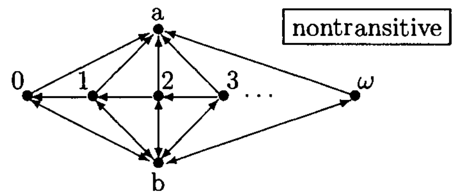


FIG. 12.1.

will assume that all our logics are finitely axiomatizable (normal or quasi-normal) extensions of some basic logic L_0 (in particular, $L' = L_0 \oplus \varphi$ or $L' = L_0 + \varphi$).

Since L is tabular, it is not difficult to check the inclusion $L' \subseteq L$ (at least, in principle). The converse inclusion is more problematic. To verify it after establishing $L' \subseteq L$ we may use the following simple observation.

Suppose L has only finitely many immediate predecessors (in NExtL_0 or ExtL_0), say $L_1, \dots, L_n$, and all of them are tabular. Then $L = L'$ iff $L' \subseteq L$ and $L' \not\subseteq L_1, \dots, L' \not\subseteq L_n$, which reduces our question to the decidability problem for tabular logics.

For example, by Makinson's theorem, the logic $\mathbf{K} \oplus \bot$ has exactly two immediate predecessors in NExt**K**, namely $\mathbf{K} \oplus \Box\bot = \mathrm{Log}\bullet$ and $\mathbf{K} \oplus p \leftrightarrow \Box p = \mathrm{Log}\circ$, which are tabular. Therefore, we obtain an algorithm for deciding whether a modal formula axiomatizes the inconsistent logic. However, in the class NExt**K** this is the only known positive result of that sort. By Theorem 10.60, every consistent tabular logic has a continuum of immediate predecessors in NExt**K**. In particular, we have the following:

Theorem 12.5 *The logic* $\mathbf{K} \oplus \Box\bot$ *has infinitely many tabular immediate predecessors in* NExt**K**.

Proof Let L be the logic of the frame $\mathfrak{F} = \langle W, R, P \rangle$, where

$$W = \{a, b\} \cup \{n : \ n \le \omega\},$$

$$R = \{\langle m, a \rangle, \langle m, b \rangle, \langle b, m \rangle, \langle m, n \rangle : \ n < m \le \omega\}$$

(see Fig. 12.1 in which the subframe containing the natural numbers and ω is transitive) and P is the family of finite subsets of W without ω and cofinite subsets of W containing ω. Notice that $\mathfrak{F}$ is descriptive and each of its points except ω is definable by a variable free formula. Namely, for $n < \omega$, we have

$$\{a\} = \{x : \ x \models \Box\bot\}, \quad \{b\} = \{x : \ x \models \neg(\Box\bot \vee \Diamond\Box\bot)\},$$

$$\{0\} = \{x : \ x \models \Diamond\{a\} \wedge \neg\Diamond\Diamond\{a\}\},$$

$$\{n+1\} = \{x : \ x \models \Diamond\{n\} \wedge \neg\Diamond(\neg\{b\} \wedge \Diamond\{n\})\}.$$

It follows that $\mathfrak{F}$ is the 0-generated universal frame for L.

Now take any proper normal extension L' of L and consider its 0-generated universal frame $\mathfrak{G}$. Clearly, $\mathfrak{G}$ is (isomorphic to) a generated subframe of $\mathfrak{F}$. But $\mathfrak{F}$ has no generated subframe different from itself and $\bullet$. Therefore, either $L = L'$ or $L' = \mathrm{Log}\bullet = \mathbf{K} \oplus \Box\bot$. Since the former alternative is impossible, L is an immediate predecessor of $\mathbf{K} \oplus \Box\bot$.

To construct an infinite sequence of tabular immediate predecessors of $\mathbf{K}\oplus\Box\bot$ it suffices to take the logics of the frames $\langle\{a, b, 0, 1, \dots, n\}, R\restriction\{a, b, 0, 1, \dots, n\}\rangle$. ❑

Similar results can be proved for tabular logics in the class Ext**K4** (see Exercises 12.4 and 12.5). We show, however, that in NExt**K4** our criterion works perfectly well. To this end we require the following:

Theorem 12.6 (i) *Each finitely axiomatizable logic $L \in$ NExt**K4** of finite depth is a finite union-splitting, i.e., can be represented in the form*

$$L = \mathbf{K4} \oplus \{\alpha^\sharp(\mathfrak{F}_i, \bot) : \ i \in I\}$$

with finite I.

(ii) *Each finitely axiomatizable logic $L \in$ Ext**S4** of finite depth can be represented in the form $L = \mathbf{S4} + \{\alpha^\sharp(\mathfrak{F}_i, \bot) : \ i \in I\}$ with finite I.*

Proof (i) Let $L = \mathbf{K4}\oplus\varphi$ be a logic of depth n and m the number of variables in φ. We show that L coincides with the logic

$$L' = \mathbf{K4} \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \ |\mathfrak{G}| \le \sum_{i=1}^{n+1} 2^m c_m(i), \ \mathfrak{G} \not\models \varphi\}$$

(the function $c_m(i)$ was defined in Theorem 8.82). Indeed, the inclusion $L \supseteq L'$ is obvious. Suppose now that $\varphi \notin L'$. Then there is a rooted refined m-generated frame $\mathfrak{F}$ for L' refuting φ. Clearly, $\mathfrak{F}$ is of depth $\le n$, since otherwise $\alpha^\sharp(\mathfrak{G}, \bot)$ is an axiom of L' for every rooted generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ of depth $n + 1$ (φ is refuted in such frames because L is of depth n), and so $\mathfrak{F} \not\models L'$, which is a contradiction. But then $\alpha^\sharp(\mathfrak{F}, \bot)$ is an axiom of L', contrary to our assumption.

(ii) A similar proof is left to the reader as an exercise. ❑

We are in a position now to prove

Theorem 12.7 (i) *Every tabular logic $L \in$ NExt**K4** has finitely many immediate predecessors and they are also tabular.*

(ii) *Every tabular logic $L \in$ Ext**S4** has finitely many immediate predecessors and they are also tabular.*

Proof (i) Suppose L is the logic of a finite transitive frame $\mathfrak{F}$. By Theorem 12.6, L is a finite union-splitting. Take any independent axiomatization of L by frame formulas, say

$$L = \mathbf{K4} \oplus \{\alpha^\sharp(\mathfrak{F}_i, \bot) : \ i = 1, \dots, n\}.$$

By Theorem 10.52 the logics $L_i = \mathrm{Log}(\mathfrak{F} + \mathfrak{F}_i)$, for $i = 1, \dots, n$, are all the distinct immediate predecessors of L. By the definition, they are tabular.

(ii) is proved in the same way. ❑

Moreover, we have the following lattice-theoretic criterion of tabularity in NExt**K4** and Ext**S4**:

Theorem 12.8 (i) *A logic in* NExt**K4** *is tabular iff it has finitely many normal extensions.*

(ii) *A logic in* Ext**S4** *is tabular iff it has finitely many extensions.*

Proof Exercise. ❑

With the help of the Blok–Esakia theorem and the fact that the map ρ preserves tabularity we immediately obtain

Theorem 12.9 (i) *Every tabular si-logic has finitely many immediate predecessors and they are also tabular.*

(ii) *A si-logic is tabular iff it has finitely many extensions.*

12.3 Pretabular logics

The tabularity criteria, obtained so far, are not effective and moreover, as will be shown in Section 17.3, no effective tabularity criterion exists in general. However, for sufficiently strong logics, e.g. those in NExt**S4** and Ext**Int**, the tabularity problem turns out to be decidable. The effective criterion to be proved below uses the following notion.

We say that a logic $L \in$ (N)ExtL_0 is *pretabular* in the lattice (N)ExtL_0, if L is not tabular but every proper extension of L in (N)ExtL_0 is tabular. In other words, a pretabular logic in (N)ExtL_0 is a maximal non-tabular logic in (N)ExtL_0.

Theorem 12.10 *In the lattices* Ext**K**, NExt**K** *and* Ext**Int**, *every non-tabular logic is contained in a pretabular one.*

Proof By Theorem 12.1 and Corollary 12.2, a logic is non-tabular iff it does not contain the formula $\boldsymbol{tab_n}$ ($\boldsymbol{bw_n} \wedge \boldsymbol{bd_n}$ in the intuitionistic case), for any $n < \omega$. It follows that the union of an ascending chain of non-tabular logics is a non-tabular logic as well. The standard use of Zorn's lemma completes the proof. ❑

Thus, pretabular logics provide typical, in a sense, examples of non-tabular logics in a given lattice.

If there is a good description of all pretabular logics in a lattice, we have at our disposal an effective (modulo the description) tabularity criterion for the lattice. Indeed, take for definiteness the lattice NExt**K4**. How can we determine, given a formula φ, whether $\mathbf{K4} \oplus \varphi$ is tabular? We may launch two parallel processes: one of them generates all derivations in $\mathbf{K4} \oplus \varphi$ and stops after finding a derivation of $\boldsymbol{tab_n}$, for some $n < \omega$; another process checks if φ belongs to a pretabular logic in NExt**K4** and stops if this is the case. The termination of the first process means that $\mathbf{K4} \oplus \varphi$ is tabular, while that of the second one shows that it is not tabular.

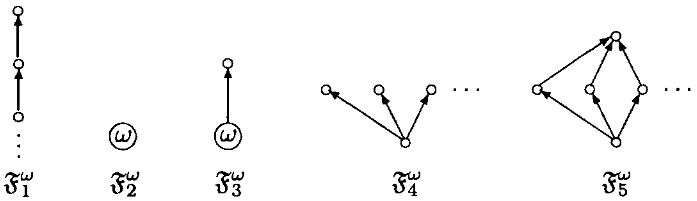


FIG. 12.2.

Unfortunately, it is impossible to describe in an effective way all pretabular logics in (N)Ext**K** and even (N)Ext**K4**: in Section 13.2 we shall construct a continuum of them. However, for smaller lattices like NExt**GL** or NExt**S4** such descriptions can be found. We shall use the following:

Theorem 12.11 *Every non-tabular logic $L \in$* NExt**K4** *has a non-tabular finitely approximable normal extension.*

Proof Since L is non-tabular and characterized by the class of its rooted finitely generated refined frames, we have either a sequence $\mathfrak{F}_i$, $i = 1, 2, \dots$, of rooted finite frames for L of depth i, or a sequence $\mathfrak{F}_i$ of rooted finite frames for L of width $\geq i$. In both cases the logic $\mathrm{Log}\{\mathfrak{F}_i : i < \omega\} \supseteq L$ is non-tabular and finitely approximable. ❑

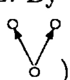
As an immediate consequence we obtain

Corollary 12.12 *Every pretabular logic in* NExt**K4** *is finitely approximable.*

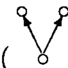
Let us begin with pretabular normal extensions of **S4**.

Theorem 12.13 *There are exactly five pretabular logics in* NExt**S4**, *viz., the logics of the frames depicted in Fig. 12.2 (where ω is an ω-point cluster).*

Proof First we show that the logics of the frames in Fig. 12.2 are really pretabular. For $0 < n < \omega$, we denote by $\mathfrak{F}_1^n$ a chain of n simple clusters, by $\mathfrak{F}_2^n$ a cluster with n points; $\mathfrak{F}_3^n$, $\mathfrak{F}_4^n$ and $\mathfrak{F}_5^n$ are defined analogously by restricting the infinite cluster and antichains in the frames $\mathfrak{F}_3^\omega$, $\mathfrak{F}_4^\omega$ and $\mathfrak{F}_5^\omega$ in Fig. 12.2 to n-point cluster and antichains, respectively.

Let $L = \mathrm{Log}\mathfrak{F}_1^\omega$. Clearly, L is not tabular. Denote by L' a normal pretabular extension of L. By Corollary 12.12, L' is finitely approximable. And since $\alpha($$) \in L$ and $\alpha($$) \in L$, finite rooted frames for L' are of the form $\mathfrak{F}_1^n$, for $n < \omega$. It follows immediately that the same canonical formulas belong to L and L', from which $L = L'$.

Suppose now that $L = \mathrm{Log}\mathfrak{F}_3^\omega$. Since $\alpha(\mathfrak{Cl}_n) \notin L$, for any cluster $\mathfrak{Cl}_n$ with $n < \omega$ points, L is not tabular. What are finite rooted frames for L? Every such frame is either a single point or a chain of two clusters, the last one being simple.

This follows from the fact that the formulas $\alpha($  $)$, $\alpha($  $)$ and $\alpha($$, \bot)$ are valid in $\mathfrak{F}_3^\omega$ and so belong to L. And since L is finitely approximable (as a logic of depth 2) and $\mathfrak{F}_3^{n+1}$ is reducible to $\mathfrak{F}_3^n$, any proper extension of L is tabular.

The logics of the frames $\mathfrak{F}_2^\omega$, $\mathfrak{F}_4^\omega$ and $\mathfrak{F}_5^\omega$ are considered in the same manner.

Let us show now that NExtS4 contains no other pretabular logics than those mentioned above. Suppose L is a normal pretabular extension of **S4**. If L is of infinite depth then among its frames there are finite chains (of simple clusters) of any length $n < \omega$, from which $L \subseteq \mathrm{Log}\mathfrak{F}_1^\omega$. Since L is pretabular, it follows that $L = \mathrm{Log}\mathfrak{F}_1^\omega$.

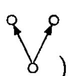
Suppose that L is a logic of finite depth. If for any $n < \omega$ there is a frame for L containing a final cluster with $\geq n$ points then, by the generation and reduction theorems, every finite cluster validates L and so, in view of its pretabularity, $L = \mathrm{Log}\mathfrak{F}_2^\omega$. If for any $n < \omega$, frames for L contain non-final clusters with $\geq n$ points, then we can reduce their subframes generated by such clusters to frames of the form $\mathfrak{F}_3^n$, $n < \omega$. Hence, $L = \mathrm{Log}\mathfrak{F}_3^\omega$.

It remains to consider the case when all clusters in finite rooted frames for L (of finite depth) contain $< n$ points, for some $n < \omega$. Then in these frames there must be points of branching $\geq n$, for every $n < \omega$. Say that a point x in a finite frame $\mathfrak{F}$ is of *outer* (*inner*) *branching* n if n is the number of pairwise inaccessible immediate successors of x belonging to final (respectively, non-final) clusters in $\mathfrak{F}$. Two cases are possible now.

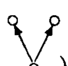
Suppose first that finite rooted frames for L contain points of outer branching $\geq n$, for every $n < \omega$. Clearly, we can reduce their subframes generated by such points to frames of the form $\mathfrak{F}_4^n$, which means that $L = \mathrm{Log}\mathfrak{F}_4^\omega$. And if finite rooted frames for L have points of arbitrarily great inner branching then these points generate subframes that are reducible to $\mathfrak{F}_5^n$, $n < \omega$, and so $L = \mathrm{Log}\mathfrak{F}_5^\omega$. ❑

It is not difficult to axiomatize the logics of the frames in Fig. 12.2.

Corollary 12.14 *The following logics and only they are pretabular in the lattice* NExtS4*:*

$\mathrm{Log}\mathfrak{F}_1^\omega = \mathbf{S4} \oplus \alpha($  $) \oplus \alpha($$)$,

$\mathrm{Log}\mathfrak{F}_2^\omega = \mathbf{S4} \oplus \alpha($  $)$,

$\mathrm{Log}\mathfrak{F}_3^\omega = \mathbf{S4} \oplus \alpha($  $) \oplus \alpha($  $) \oplus \alpha($$, \bot)$,

$\mathrm{Log}\mathfrak{F}_4^\omega = \mathbf{S4} \oplus \alpha($  $) \oplus \alpha($$)$,

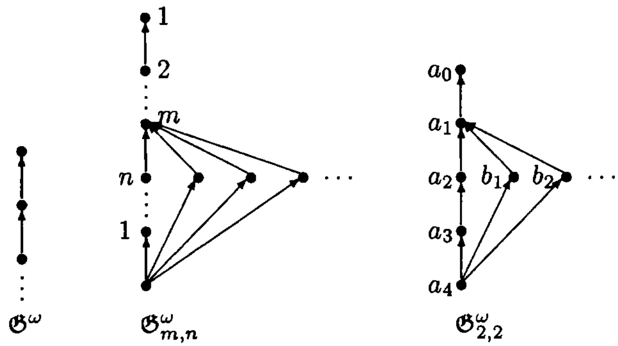


FIG. 12.3.

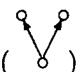
$\mathrm{Log}\mathfrak{F}_5^\omega = \mathbf{S4} \oplus \alpha($  $, \bot) \oplus \alpha(\mathfrak{Ch}_4) \oplus \alpha($ $)$,

where $\mathfrak{Ch}_4$ *is a chain of four points.*

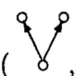
Proof Exercise. ❑

Using the Blok–Esakia theorem and the fact that the maps ρ and σ preserve tabularity (and so pretabularity), we obtain a description of pretabular si-logics.

Theorem 12.15 *There are three pretabular logics in* ExtInt, *namely*

$\mathrm{Log}\mathfrak{F}_1^\omega = \mathbf{Int} + \beta($  $)$,

$\mathrm{Log}\mathfrak{F}_4^\omega = \mathbf{Int} + \beta($  $)$,

$\mathrm{Log}\mathfrak{F}_5^\omega = \mathbf{Int} + \beta($  $, \bot) + \beta(\mathfrak{Ch}_4)$.

Let us consider now pretabular logics in the lattice NExt**GL**.

Theorem 12.16 *The set of pretabular logics in* NExt**GL** *is denumerable. It consists of the logics* $\mathrm{Log}\mathfrak{G}^\omega$ *and* $\mathrm{Log}\mathfrak{G}^\omega_{m,n}$, *for* $m \geq 0$, $n \geq 1$, *where* $\mathfrak{G}^\omega$ *and* $\mathfrak{G}^\omega_{m,n}$ *are the frames depicted in Fig. 12.3. If* $\langle m, n\rangle \neq \langle k, l\rangle$ *then* $\mathrm{Log}\mathfrak{G}^\omega_{m,n} \neq \mathrm{Log}\mathfrak{G}^\omega_{k,l}$.

Proof That all these logics are not tabular and $\mathrm{Log}\mathfrak{G}^\omega$ is pretabular can be proved in the same way as in the proof of Theorem 12.13. We show only that $\mathrm{Log}\mathfrak{G}^\omega_{2,2}$ is pretabular; other logics $\mathrm{Log}\mathfrak{G}^\omega_{m,n}$ are considered analogously. Denote by $\mathfrak{G}^n_{2,2}$ the frame obtained from $\mathfrak{G}^\omega_{2,2}$ (shown in Fig. 12.3) by deleting the points b_i for $i > n$. $\mathfrak{G}^n_{k,l}$ is defined in the same way. For $n < \omega$, let

$$\gamma_n = \Box^{n+1}\bot \wedge \Diamond^n\top.$$

Then for all points a_i and b_k in $\mathfrak{G}^{\omega}_{2,2}$, we have:

$$a_i \models \gamma_i, \quad a_i \not\models \gamma_j \text{ if } 0 \le i \le 4, \; j \ne i;$$
$$b_k \models \gamma_2, \quad b_k \not\models \gamma_l \text{ if } k \ge 1, \; l \ne 2.$$

For a variable free formula φ, put

$$v(\varphi) = \Box^+(\varphi \to p) \vee \Box^+(\varphi \to \neg p).$$

The meaning of this formula is that $v(\varphi)$ is valid in a rooted transitive Kripke frame $\mathfrak{F}$ iff $\mathfrak{F}$ contains at most one point where φ is true. It should be clear that the following formulas are valid in $\mathfrak{G}^{\omega}_{2,2}$ and so belong to $\mathrm{Log}\mathfrak{G}^{\omega}_{2,2}$:

$$\alpha(\circ), \; \Box^5\bot, \; \Box^+ \bigvee_{i=0}^{4} \gamma_i, \; v(\gamma_i) \; (0 \le i \le 4, \; i \ne 2), \; \Box^+(\gamma_3 \to v(\gamma_2)).$$

If we call a point at which γ_i is true a *point of type* i, then this means that every rooted frame for $\mathrm{Log}\mathfrak{G}^{\omega}_{2,2}$ is irreflexive and of depth ≤ 5; each of its points is of one of the types 0,1,2,3,4, where a point of type $i \ne 2$, if any, is unique, and a point of type 3, if any, sees only one point of type 2.

It follows that the class $\mathcal{C}$ of finite rooted frames for $\mathrm{Log}\mathfrak{G}^{\omega}_{2,2}$ consists of irreflexive chains of length ≤ 5 and the frames $\mathfrak{G}^{n}_{2,2}$, for $n = 1, 2, \ldots$. Since $\mathfrak{G}^{n+1}_{2,2}$ is reducible to $\mathfrak{G}^{n}_{2,2}$ and the chains of length ≤ 4 are generated subframes of $\mathfrak{G}^{n}_{2,2}$, every non-tabular finitely approximable normal extension of $\mathrm{Log}\mathfrak{G}^{\omega}_{2,2}$ must have $\mathcal{C}$ as the class of its finite rooted frames. And since $\mathrm{Log}\mathfrak{G}^{\omega}_{2,2}$ is finitely approximable itself (as a logic of finite depth), it is pretabular.

Now take any pretabular logic $L \in \mathrm{NExt}\mathbf{GL}$. If L is of infinite depth then clearly $L = \mathrm{Log}\mathfrak{G}^{\omega}$.

Suppose L is of finite depth. What are finite frames characterizing it? Observe first that L is characterized by a class of finite rooted frames of the same depth. Indeed, suppose $\{\mathfrak{F}_i : i < \omega\}$ is a sequence of pairwise non-isomorphic finite rooted frames such that $L = \mathrm{Log}\{\mathfrak{F}_i : \; i < \omega\}$ and let $d = \max\{d(\mathfrak{F}_i) : \; i < \omega\}$ (so that $\Box^{d-1}\bot \notin L$). If the sequence contains only finitely many frames of depth d, then the rest of the frames in it determine a non-tabular extension L' of L. And since $\Box^{d-1}\bot \in L'$, we arrive at a contradiction with the pretabularity of L. Therefore, the sequence contains infinitely many frames of depth d. Let L'' be the logic determined by these frames. Clearly, L'' is not tabular (otherwise the frames of depth $< d$ determine a non-tabular proper extension of L), from which $L = L''$.

Now we use the classification of points in frames by means of the formulas γ_i introduced above.

Lemma 12.17 *Suppose L is a pretabular logic in* NExt**GL** *characterized by a class $\{\mathfrak{F}_k : \; k < \omega\}$ of finite rooted frames of depth d. Then*

(i) *for every $i \le d-1$ except possibly only one $j < d-1$, each frame $\mathfrak{F}_k$, $k < \omega$, contains exactly one point of type i and*

(ii) *all the points of type j, except one of them, are accessible only from the root of $\mathfrak{F}_k$.*

Proof Since $d(\mathfrak{F}_k) = d$, for all $k < \omega$, every point in $\mathfrak{F}_k$ is of one of the types $0, 1, \dots, d-1$ and for each $i \leq d-1$, there is a point in $\mathfrak{F}_k$ of type i. And since these frames are pairwise non-isomorphic, at least for one $j \leq d-1$ and every $n < \omega$, there is $\mathfrak{F}_k$ containing $\geq n$ points of type j.

Observe that for every $i < d-1$, there is $n < \omega$ such that every point of type i in every $\mathfrak{F}_k$ sees at most n points of type j. For otherwise we could take the infinite subsequence of the (non-isomorphic) rooted subframes of $\mathfrak{F}_k$, generated by points of type i, and then the logic determined by this subsequence would be a non-tabular proper extension of L.

Notice also that for every $i < d-1$ different from j, each $\mathfrak{F}_k$ contains only one point of type i, and if $\mathfrak{F}_k$ contains a point of type j that is seen not only from the root (which means $j < d-2$), then this point is unique. Indeed, if this is not the case then $v(\gamma_i) \notin L$, for some $i \neq j$. On the other hand, using the observation above, we can construct an infinite sequence of non-isomorphic reducts of $\mathfrak{F}_k$ containing arbitrarily many points of type j and satisfying the desirable properties. This sequence determines then a non-tabular proper extension L' of L, since $v(\gamma_i) \in L'$, which is a contradiction. ❑

Now, returning to the proof of our theorem, we see that all finite rooted frames for L have the form $\mathfrak{G}^l_{m,n}$ for some fixed $m \geq 0$ and $n \geq 1$. Therefore, $L = \mathrm{Log}\mathfrak{G}^\omega_{m,n}$. The last claim of the theorem is obvious. ❑

Using the semantic description of pretabular logics in NExt**GL**, it is not hard to find finite sets of (canonical) formulas axiomatizing them.

Theorem 12.18 *All pretabular logics in* NExt**GL** *are finitely axiomatizable and so decidable.*

Proof Exercise. ❑

The technique developed in the proofs of Theorems 12.13 and 12.16 can be used for finding pretabular logics in NExt**D4**. We invite the reader to prove the following:

Theorem 12.19 *There exist ten pretabular logics in* NExt**D4**, *viz., the logics of the frames depicted in Fig. 12.2 and 12.4. All these logics are finitely axiomatizable and so decidable.*

Other applications of this technique for describing pretabular logics in the classes (N)Ext$\mathbf{K4BD}_n$, Ext**GL** can be found among the exercises in Section 12.5.

12.4 Some remarks on local tabularity

The notion of local tabularity turns out to be much more complex than the close notion of tabularity and, besides, it is not so well studied. The title of this section corresponds to our moderate knowledge in this area.

Let us consider first modal logics. Observe at once that we have

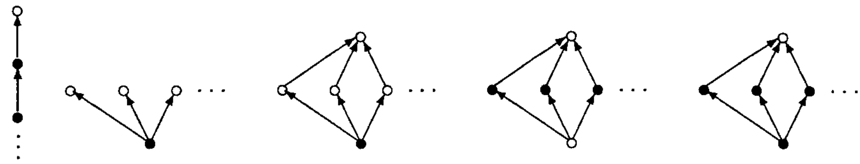


FIG. 12.4.

Proposition 12.20 *A logic L =* ExtK *is locally tabular iff* $\ker L \in$ NExtK *is locally tabular.*

Proof Follows from Theorem 7.4. ❑

So we confine ourselves to considering here only normal modal logics. Using the results of Section 8.6, we can easily obtain the following criterion of local tabularity in the lattice NExt**K4**.

Theorem 12.21 *A logic $L \in$* NExt**K4** *is locally tabular iff L is of finite depth.*

Proof ($\Rightarrow$) Suppose L is a logic of infinite depth, i.e., it has finite frames of any depth $< \omega$. Consider the sequence of formulas α_n defined by

$$\alpha_1 = p, \quad \alpha_{n+1} = p \vee \Box(p \to \Box\alpha_n)$$

and show that these formulas are pairwise non-equivalent in L. Take any distinct n and m, say $n > m$, and any finite frame $\mathfrak{F} = \langle W, R\rangle$ of depth $2n - 1$. Let $x_{2n-1}R \dots Rx_1$ be a chain of points in $\mathfrak{F}$ from distinct clusters. Define a valuation in $\mathfrak{F}$ so that $x \not\models p$ iff $x = x_{2k-1}$ for some $k \leq n$. Then clearly we have that for every $i, k \leq n$, $x_{2k-1} \not\models \alpha_i$ iff $k \geq i$ and so $x_{2m-1} \not\models \alpha_m$, $x_{2m-1} \models \alpha_n$. Therefore, $\alpha_m \leftrightarrow \alpha_n \notin L$.

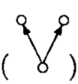
($\Leftarrow$) According to the results of Section 8.6, finitely generated descriptive frames for logics of finite depth are finite. Therefore, $\mathfrak{A}_L(n)$ is finite for every $n < \omega$, which means that L is locally tabular. ❑

Since the formulas α_n in the proof above contain only one variable, we have

Corollary 12.22 *A logic $L \in$* NExt**K4** *is locally tabular iff the algebra $\mathfrak{A}_L(1)$ is finite.*

Every logic $L \in$ NExt**S4**, which is not locally tabular, is clearly validated by the infinite descending chain of reflexive points. And since this chain characterizes **Grz.3**, we arrive at

Theorem 12.23 *A logic $L \in$* NExt**S4** *is not locally tabular iff $L \subseteq$* **Grz.3**.

The logic **Grz.3** = **S4** $\oplus\, \alpha($$) \oplus \alpha($  $)$ is decidable, and so we can always effectively determine, given a formula φ, whether **S4** $\oplus\, \varphi$ is locally tabular. Since **Grz.3** is not locally tabular itself but all its proper extensions possess this property, we may call it a *pre-locally tabular logic.* Of course, pre-local tabularity as well as pretabularity depends on the choice of a lattice of logics.

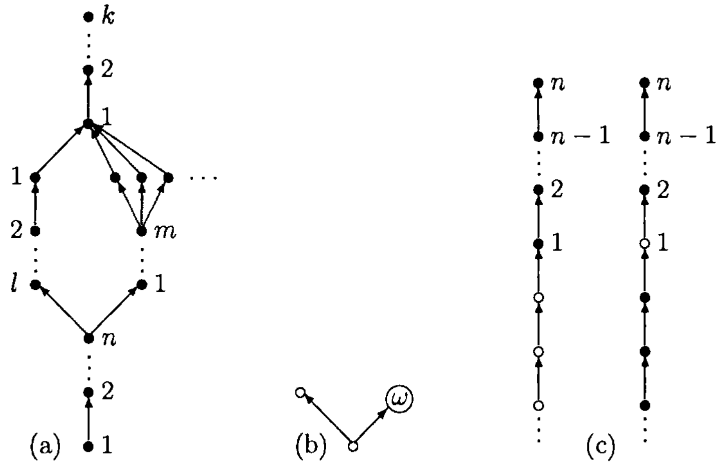


FIG. 12.5.

Thus, we have the following facts: in the class NExt**S4** there is only one pre-locally tabular logic and every normal extension of **S4** that is not locally tabular is contained in the pre-locally tabular logic. The latter fact can probably be extended to the class NExt**K4** (this is our conjecture). As to the former one, we have

Theorem 12.24 *There is a continuum of pre-locally tabular logics in* NExt**K4**.

Proof All the logics constructed in the proof of Theorem 13.15 are pre-locally tabular. ❑

The situation with locally tabular and pre-locally tabular logics in Ext**Int** turns out to be quite different from that in NExt**S4**. First, there is no connection between the local tabularity and finite depth, though all finite depth si-logics are locally tabular, of course. For instance, the logic ρ**Grz.3** = **LC** is locally tabular (see Section 8.7). And second, there is a continuum of pre-locally tabular logics in Ext**Int** (see Exercise 12.14).

12.5 Exercises and open problems

Exercise 12.1 Show that, for every $n < \omega$, $\mathbf{K} \oplus \boldsymbol{tab}_n = \mathbf{K} + \boldsymbol{tab}_n$.

Exercise 12.2 Show that tabular logics form filters in the lattices Ext**Int**, NExt**K**, and Ext**K**.

Exercise 12.3 Prove analogues of Theorem 12.1, Corollary 12.3 and Theorem 12.4 for m-modal logics, $m < \omega$. (Hint: use the formulas α_n and β_n defined as follows: α_n is the conjunction of all formulas of the form

$$\neg(\varphi_1 \wedge \Diamond_{i_1}(\varphi_2 \wedge \Diamond_{i_2}(\varphi_3 \wedge \ldots \wedge \Diamond_{i_n}\varphi_{n+1})\ldots)),$$

for $i_j \in \{1, \ldots, m\}$, $1 \leq j \leq n$, and β_n is the conjunction of all formulas of the form

$$\neg\Diamond_{i_k} \ldots \Diamond_{i_k}(\Diamond_{i_{k+1}}\varphi_1 \wedge \ldots \wedge \Diamond_{i_{k+n+1}}\varphi_{n+1}),$$

for $k \leq m$, $i_j \in \{1, \ldots, m\}$, $1 \leq j \leq k + n + 1$, where

$$\varphi_i = p_1 \wedge \ldots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \ldots \wedge p_{n+1}.$$

Exercise 12.4 Prove that every tabular consistent normal modal logic has infinitely many tabular normal immediate predecessors.

Exercise 12.5 Prove that every tabular logic in Ext**K4** (Ext**K**) has infinitely many tabular immediate predecessors in Ext**K4** (Ext**K**).

Exercise 12.6 Prove that **GL.3** $+ \Box p \to p$ is the only pretabular logic of infinite depth in Ext**GL**.

Exercise 12.7 Show that the set of pretabular logics of finite depth in Ext**GL** is denumerable and consists of the logics of the frames shown in Fig. 12.5 (a) with distinguished roots, where $l, m, n \geq 0$ are fixed for each logic.

Exercise 12.8 Prove that every pretabular logic in Ext**GL** is finitely axiomatizable.

Exercise 12.9 Show that the sets of pretabular logics in NExt**GL** and Ext**GL** are disjoint.

Exercise 12.10 Show that the set of pretabular logics in (N)Ext**K4BD**$_n$ is finite for every $n < \omega$ and that all of them are finitely axiomatizable.

Exercise 12.11 Prove that all extensions of every pretabular logic L in NExt**S4** are normal and so L is also pretabular in Ext**S4**.

Exercise 12.12 Show that besides the logics mentioned in the preceding exercise, there is only one pretabular logic of finite depth in Ext**S4**, namely the logic of the frame in Fig. 12.5 (b) with distinguished root.

Exercise 12.13 Show that there are countably many pre-locally tabular logics in NExt**K4.3**, namely the logics of each of the frames in Fig. 12.5 (c).

Exercise 12.14 Show that there is a continuum of pre-locally tabular logics in Ext**Int**.

Exercise 12.15 Show that **KC** is the intersection of all pre-locally tabular si-logics.

Exercise 12.16 Prove the analog of Theorem 12.23 for NExt**GL**.

Problem 12.1 *Is it true that every non-locally tabular logic in* NExt**K** *(*Ext**Int***) is contained in a pre-locally tabular one?*

Problem 12.2 *Is the problem "***K4** $\oplus \varphi$ *is of finite depth" decidable?*

Problem 12.3 *Is the problem "***K4** $\oplus \varphi$ *is of finite width" decidable?*

12.6 Notes

The problem of determining whether a given logic is tabular has attracted logician's attention since Gödel (1932) proved that **Int** is not tabular and Dugundji (1940), using the same idea, demonstrated the non-tabularity of all Lewis' logics. (The term "tabularity", as far as we know, was introduced by Kuznetsov in view of the fact that tabular logics can be defined by "truth-tables" similar to that for **Cl**.) Later, analogous facts were discovered by Drabbé (1967) with respect to the filters of distinguished elements in matrices characterizing logics: Lewis' systems **S1**–**S3** cannot be determined by matrices with a fixed finite number of distinguished elements.

The finite axiomatizability of tabular superintuitionistic and normal (poly) modal logics follows from a rather general algebraic result of Baker (1977). Note, however, that for si-logics this was proved (but not published) in the mid 1960s by de Jongh. That all tabular quasi-normal modal logics are finitely axiomatizable was first established by Blok and Köhler (1983). The idea of the proof of Theorem 12.1 can be easily extended to polymodal, in particular tense logics; see Chagrov (1996).

That every tabular logic in Ext**Int** has a finite number of immediate predecessors, with all of them being also tabular, was discovered by Kuznetsov (1971). The same fact for NExt**K4** is proved analogously; see Blok (1980c). Blok (1978) proved that every consistent tabular logic in NExt**K** has a continuum of immediate predecessors.

The idea of using pretabular logics for constructing effective criteria of tabularity of si-logics was proposed by Kuznetsov. Maksimova (1972, 1975b) found all pretabular logics in Ext**Int** and NExt**S4**; for the latter class the same result was obtained by Esakia and Meskhi (1977). Pretabular logics in NExt**K4** were investigated by Blok (1980c); some discrepancies in this paper were corrected in Chagrov (1989, 1996), where pretabular logics in Ext**S4** and Ext**GL** were also considered. We used ideas of the latter paper for presenting the material of Section 12.3. A rather difficult problem is to describe pretabular logics in the class of normal extensions of the Brouwerian system $\mathbf{T} \oplus p \to \Box\Diamond p$; it is known only that there are infinitely many of them; see Meskhi (1983).

That all (not necessarily normal) modal logics of finite depth are locally tabular was proved by Segerberg (1971). Maksimova (1975a) showed the converse. Corollary 12.22, asserting that to disprove local tabularity formulas in one variable are enough, was noticed by Maksimova (1989c).

The problem of local tabularity for si-logics turns out to be much more complicated: unlike NExt**S4**, where there is only one pre-locally tabular logic, Ext**Int** contains a continuum of them, as was proved by Mardaev (1984). Mardaev (1987) strengthened this result. He showed that, for every tabular si-logic $L \supseteq \mathbf{KC}$, there is a continuum of pre-locally tabular logics $\{L_i : i \in I\}$ and a continuum of finitely pre-approximable logics $\{M_i : i \in I\}$ such that $\mathbf{KC} \subseteq L_i \subseteq M_i \subseteq L$, for $i \in I$. That **KC** is involved here is explained by the fact, discovered by Kuznetsov, that every pre-locally tabular si-logic is an extension of **KC**; see

Tsytkin (1987). These results show that the notions of pre-local tabularity and finite pre-approximability in **ExtInt** cannot be used to obtain effective criteria of local tabularity and finite approximability. As we shall see in Section 17.3, the property of finite approximability turns out to be undecidable, and nothing is known about algorithms recognizing local tabularity.

One more interesting property—an antipode of tabularity—is antitabularity: we call a consistent logic *antitabular* if all models (frames, algebras, matrices) for it are infinite. Although there are no such logics in **ExtInt** and **NExtK**, in general there exists a lot of them. For instance, all logics used in the proof of Theorem 13.15 are antitabular. It is easy to construct a continual family of normal antitabular tense logics; Chagrov (1982) showed that there are antitabular modal companions of **Int** containing **S3**.

13

POST COMPLETENESS

This chapter considers some properties of modal and superintuitionistic logics connected with Post completeness.

13.1 m-reducibility

A logic L is said to be *m-reducible* if, for every formula $\varphi(q_1, \dots, q_n) \notin L$, there exist formulas $\psi_1(p_1, \dots, p_m), \dots, \psi_n(p_1, \dots, p_m)$ such that

$$\varphi(\psi_1(p_1, \dots, p_m), \dots, \psi_n(p_1, \dots, p_m)) \notin L.$$

A logic is called *reducible* if it is m-reducible for some $m < \omega$.

Theorem 13.1 *The following conditions are equivalent for every logic L in* **NExtK** *(***ExtInt** *or* **ExtK***):*

(i) *L is m-reducible;*

(ii) *L is characterized by the algebra $\mathfrak{A}_L(m)$ (by one of its m-generated Tarski–Lindenbaum matrices, if $L \in$* **ExtK***);*

(iii) *every proper (normal, if $L \in$* **NExtK***) extension of L contains a formula in m variables that is not in L.*

Proof The equivalence of (i) and (ii) is clear.

We prove the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) only for $L \in$ **ExtK**. Let $L = \mathrm{Log}\,\langle \mathfrak{A}_{L'}(m), \nabla_L \rangle$, where L' is a normal logic contained in L, and let L'' be a proper extension of L. Suppose that every formula in m variables in L' belongs to L. Then the matrix $\langle \mathfrak{A}_{L'}(m), \nabla_{L''} \rangle$ is isomorphic to $\langle \mathfrak{A}_{L'}(m), \nabla_L \rangle$, whence

$$L \subset L'' \subseteq \mathrm{Log}\,\langle \mathfrak{A}_{L'}(m), \nabla_{L''} \rangle = \mathrm{Log}\,\langle \mathfrak{A}_{L'}(m), \nabla_L \rangle = L,$$

which is a contradiction.

Suppose now that (iii) holds but $L \neq \mathrm{Log}\,\langle \mathfrak{A}_{L'}(m), \nabla_L \rangle$, for any normal logic L' contained in L. Then $\mathrm{Log}\,\langle \mathfrak{A}_{L'}(m), \nabla_L \rangle$ is a proper extension of L. On the other hand, by the definition, it contains no formula in m variables that is not in L, contrary to (iii). ❑

Remark This theorem has an unexpected consequence: if L is a normal logic every proper normal extension of which contains a formula in m variables that is not in L, then all (not only normal!) proper extensions contain formulas in m variables that are not in L.

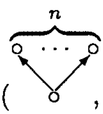
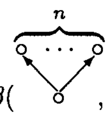
Theorem 13.2 *No logic in the intervals*

$$[\mathbf{K4}, \mathbf{S5}],\ [\mathbf{K4}, \mathbf{Grz} \oplus \boldsymbol{bd}_2],\ [\mathbf{K}, \mathbf{GL} \oplus \boldsymbol{bd}_2],\ [\mathbf{Int}, \mathbf{Int} + \boldsymbol{bd}_2]$$

is reducible.

Proof Let us consider first the interval $[\mathbf{K4}, \mathbf{S5}]$. To show that logics in it are not m-reducible for any $m < \omega$, it is sufficient to find formulas $\varphi_m \notin \mathbf{S5}$ all substitution instances in m variables of which are in $\mathbf{K4}$. Denote by $\mathfrak{Cl}_n$ the n-point cluster and put $\varphi_m = \alpha(\mathfrak{Cl}_{2^m+1}, \bot)$. Clearly $\varphi_m \notin \mathbf{S5}$. However, φ_m is valid in the universal frame of rank m for $\mathbf{K4}$, because all final clusters in it contain $\leq 2^m$ points.

For the other intervals we use in the same manner the formulas

$\alpha($  $, \bot)$, $\neg(\bigwedge_{i=1}^{2^m+1} \Diamond(\Box\bot \wedge \neg p_i \wedge \bigwedge_{j \neq i} p_j))$ and $\beta($  $, \bot)$,

respectively, where in the first formula $n = 2^{2^m} + 2^m$ and in the last one $n = 2^m + 1$. ❑

This trick with final clusters in the universal frames does not go through for $\mathbf{KC} = \mathbf{Int} + \neg p \vee \neg\neg p$.

Theorem 13.3 $\mathbf{KC}$ *is* 2*-reducible.*

Proof We require some auxiliary facts.

Lemma 13.4 *Every finitely generated pseudo-Boolean algebra* $\mathfrak{A}$ *is generated by a finite chain of elements in* $\mathfrak{A}$.

Proof The proof is conducted by induction on the number of $\mathfrak{A}$'s generators. The basis of induction is trivial.

Suppose the claim of our lemma holds for $m-1$-generated algebras and consider a pseudo-Boolean algebra $\mathfrak{A}$ with generators $a_1, \dots, a_m$. By the induction hypothesis, the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, generated by $a_1, \dots, a_{m-1}$, is generated also by a chain $b_1 < b_2 < \dots < b_n \neq \top$. Put $b_{n+1} = \top$ and show that $\mathfrak{A}$ is generated by the chain

$$b_1 \wedge a_m \leq b_1 \leq b_1 \vee (b_2 \wedge a_m) \leq b_2 \leq \dots \leq b_n \leq b_n \vee (b_{n+1} \wedge a_m).$$

Since this chain contains all $b_1, \dots, b_n$, it generates $\mathfrak{B}$. So it suffices to prove that it generates the element a_m as well.

Observe that, for $1 \leq i \leq n$, we have

$$(b_i \vee (b_{i+1} \wedge a_m)) \wedge (b_i \to b_i \wedge a_m) = b_{i+1} \wedge a_m.$$

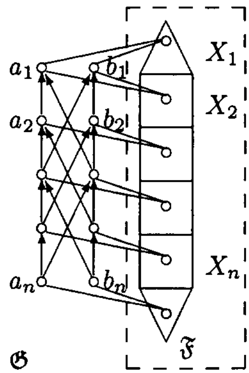


FIG. 13.1.

Taking $i = 1$, we obtain that the chain generates the element

$$(b_1 \vee (b_2 \wedge a_m)) \wedge (b_1 \to b_1 \wedge a_m) = b_2 \wedge a_m.$$

Using it in the equality above for $i = 2$, we then get

$$(b_2 \vee (b_3 \wedge a_m)) \wedge (b_2 \to b_2 \wedge a_m) = b_3 \wedge a_m,$$

etc. Thus in n steps we shall generate $b_{n+1} \wedge a_m = a_m$. ❑

Corollary 13.5 *Suppose a finitely generated pseudo-Boolean algebra $\mathfrak{A}$ refutes a formula $\varphi(p_1, \dots, p_m)$. Then there exist formulas*

$$\chi_1(q_1, \dots, q_n), \dots, \chi_m(q_1, \dots, q_n)$$

and a valuation $\mathfrak{V}$ in $\mathfrak{A}$ such that $\varphi(\chi_1, \dots, \chi_m)$ is refuted under $\mathfrak{V}$ in $\mathfrak{A}$ and $\mathfrak{V}(q_1), \dots, \mathfrak{V}(q_n)$ is a chain of elements generating $\mathfrak{A}$.

Now we are in a position to prove Theorem 13.3. By Theorem 5.33, **KC** is characterized by the class of finite rooted frames with last elements. Let $\mathfrak{F} = \langle W, R \rangle$ be such a frame refuting a formula $\varphi(p_1, \dots, p_m)$ under a valuation $\mathfrak{V}$. By Corollary 13.5, we have formulas $\chi_i(q_1, \dots, q_n)$, $i = 1, \dots, m$, such that $\varphi(\chi_1, \dots, \chi_m)$ is refuted in the model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and the sets $X_i = \mathfrak{V}(q_i)$, for $i = 1, \dots, n$, form a chain with respect to $\subseteq$. Without loss of generality we may assume that $X_1 \subset X_2 \subset \ldots \subset X_n \neq W$. Construct from $\mathfrak{F}$ and X_i, $1 \leq i \leq n$, a new frame $\mathfrak{G}$ as is shown in Fig. 13.1. Here a_1 and b_1 see all the points in X_1, and every point in $X_i - X_{i-1}$ sees a_{i-1} and b_{i-1} but not a_i and b_i, for $i \geq 2$. The points a_n and b_n are seen only from the points in $W - X_n$. Put $U = \{a_i, b_i : 1 \leq i \leq n\}$.

Take the formulas in the two variables p and q "describing" the points in U as in Section 6.5:

$$\alpha_0 = q,\ \beta_0 = p,\ \alpha_1 = p \to q,\ \beta_1 = q \to p,$$

$$\alpha_{n+1} = \beta_n \to \alpha_n \vee \beta_{n-1},\ \beta_{n+1} = \alpha_n \to \beta_n \vee \alpha_{n-1}\quad (n \geq 1)$$

and define a valuation $\mathfrak{U}$ in $\mathfrak{G}$ by putting

$$\mathfrak{U}(p) = \{a_1\} \cup X_1 = a_1\!\uparrow,\ \ \mathfrak{U}(q) = \{b_1\} \cup X_1 = b_1\!\uparrow .$$

Then by induction on $i \geq 1$ we can show that in the model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$

$$\{x :\ x \not\models \alpha_i\} = a_i\!\downarrow,\ \{x :\ x \not\models \beta_i\} = b_i\!\downarrow .$$

Finally, we define the formulas in the variables p and q which will be substituted instead of $q_1, \dots, q_n$:

$$\begin{aligned}\gamma_i &= \alpha_{i+1} \wedge \beta_{i+1} \to \alpha_i \vee \beta_i,\ \text{for } 1 \leq i \leq n-1,\\ \gamma_n &= \alpha_n \vee \beta_n.\end{aligned}$$

Denote by δ^* the result of replacing the variables q_i in a formula δ with γ_i.

Lemma 13.6 *For every formula δ in the variables $q_1, \dots, q_n$,*
(i) *there is $x \in W$ such that $(\mathfrak{N}, x) \models \delta^*$ iff $(\mathfrak{N}, y) \models \delta^*$ for all $y \in U$;*
(ii) *for every $x \in W$, $(\mathfrak{M}, x) \models \delta$ iff $(\mathfrak{N}, x) \models \delta^*$.*

Proof We prove (i) and (ii) by simultaneous induction on the construction of δ. The basis of induction and the cases $\delta = \delta_1 \wedge \delta_2$ and $\delta = \delta_1 \vee \delta_2$ are obvious. Let $\delta = \delta_1 \to \delta_2$.

First we establish (i). Suppose $x \models \delta^*$ for some $x \in W$, but there is $y \in U$ for which $y \not\models \delta^*$, i.e., there is a point $z \in y\!\uparrow$ such that $z \models \delta_1^*$ and $z \not\models \delta_2^*$. Then either $z \in U$ or $z \in X_1$.

If $z \in U$ then δ_1^* is true at the last point in $\mathfrak{G}$ (which belongs to W). Since $x \models \delta^*$, the formula δ_2^* is also true at the last point in $\mathfrak{G}$, which is a contradiction, because by the induction hypothesis, we should then have $z \models \delta_2^*$. If $z \in X_1$ then we may assume z to be the last point in $\mathfrak{G}$. (For as is easy to verify by induction, every formula in p and q has the same truth-values at all points in X_1 under $\mathfrak{U}$.) But this is impossible, since $x \models \delta^*$ implies $z \models \delta^*$.

The converse implication is trivial because the last point in $\mathfrak{G}$ belongs to W.

Let us now prove (ii). If $(\mathfrak{M}, x) \not\models \delta$ then $(\mathfrak{N}, x) \not\models \delta^*$ follows immediately from the induction hypothesis. Suppose $(\mathfrak{N}, x) \not\models \delta^*$ for some $x \in W$. Then there is $y \in x\!\uparrow$ such that $(\mathfrak{N}, y) \models \delta_1^*$ and $(\mathfrak{N}, y) \not\models \delta_2^*$. If $y \in W$ then $(\mathfrak{M}, x) \not\models \delta$ is a direct consequence of the induction hypothesis. Let $y \in U$. Then by the induction hypothesis for (i), $(\mathfrak{N}, z) \models \delta_1^*$ and $(\mathfrak{N}, z) \not\models \delta_2^*$, where z is the last point in $\mathfrak{G}$. Therefore, by the induction hypothesis for (ii), $(\mathfrak{M}, x) \not\models \delta$. ❑

Thus, by Lemma 13.6, we have

$$\mathfrak{G} \not\models \varphi(\chi_1(\gamma_1^*, \dots, \gamma_n^*), \dots, \chi_m(\gamma_1^*, \dots, \gamma_n^*)),$$

from which $\varphi(\chi_1(\gamma_1^*,\dots,\gamma_n^*),\dots,\chi_m(\gamma_1^*,\dots,\gamma_n^*)) \notin \mathbf{KC}$, since $\mathfrak{G} \models \mathbf{KC}$. It remains to observe that this formula contains no variable different from p and q. ❑

Theorem 13.7 **KC** *is not* 1*-reducible.*

Proof Suppose otherwise. Then by Theorem 13.1, **KC** is the logic of $\mathfrak{F}_{\mathbf{KC}}(1)$ (see Fig. 8.14 displaying the universal frame for **KC** of rank 2). Since $\mathfrak{F}_{\mathbf{KC}}(1)$ is of depth 2, we must have $\boldsymbol{bd}_2 \in \mathbf{KC}$, which is impossible. ❑

Theorem 13.8 $\mathbf{KC} + \boldsymbol{bd}_2$ *is not reducible.*

Proof Assuming otherwise, we would have that $\mathbf{KC} + \boldsymbol{bd}_2$ is tabular, which certainly is not the case, because for every $n \geq 1$, $\beta(\mathfrak{F}_5^n) \notin \mathbf{KC} + \boldsymbol{bd}_2$, where $\mathfrak{F}_5^n$ is shown in Fig. 12.2. ❑

Let us now briefly consider the reducibility of modal companions of si-logics. According to Theorem 13.2, for every consistent si-logic L its smallest modal companion $\boldsymbol{\tau}L$ is not reducible, i.e., $\boldsymbol{\tau}$ does not preserve the reducibility. However, $\boldsymbol{\sigma}$ does.

Theorem 13.9 *If L is an m-reducible si-logic then $\boldsymbol{\sigma}L$ is also m-reducible.*

Proof Let M be a proper normal extension of $\boldsymbol{\sigma}L$. Then $\boldsymbol{\rho}M \supset L$ and so, by Theorem 13.1, there is $\varphi(p_1,\dots,p_n) \in \boldsymbol{\rho}M - L$. It follows that $T(\varphi) \in M - \boldsymbol{\sigma}L$. By Theorem 13.1, this means that $\boldsymbol{\sigma}L$ is m-reducible. ❑

Corollary 13.10 **Grz.2** *is* 2*-reducible.*

13.2 0-reducibility, Post completeness and general Post completeness

0-reducible logics are "almost the same" as Post complete ones. Recall that a logic L is called *Post complete* in a lattice of logics (containing L) if L is consistent and does not have proper consistent extensions in the lattice. Of course, Post completeness of L depends essentially on the chosen lattice of logics (it corresponds to the coatomicity of L in the lattice). The following generalization of the notion of Post completeness is not connected with the choice of a lattice; it is an intrinsic property of logics.

Say that a logic L is *generally Post complete* if it is consistent and does not have proper consistent extensions closed under the inference rules that are admissible in L. It should be clear that every Post complete logic, say in the lattices Ext**K**, NExt**K**, Ext**Int**, is generally Post complete.

The following two theorems give various characterizations of generally Post complete and simply Post complete logics.

Theorem 13.11 *For every consistent modal or si-logic L, the following conditions are equivalent:*

(i) *L is* 0*-reducible;*

(ii) *L is characterized by one of its* 0*-generated Tarski–Lindenbaum matrices;*

(iii) *L is characterized by some* 0*-generated matrix;*

(iv) *an inference rule is admissible in L iff all variable free substitution instances of it are admissible in L;*

(v) *L is generally Post complete.*

Proof Notice first that the equivalence of (i) and (ii) was proved in Theorem 13.1. The condition (i) is also a special case of (iv), because any formula $\varphi \in L$ may be regarded as the rule $\bot \to \bot/\varphi$ admissible in L. The equivalence (ii) $\Leftrightarrow$ (iii) follows from the almost obvious fact that every 0-generated matrix characterizing L is isomorphic to the 0-generated Tarski–Lindenbaum matrix for L. The implication (ii) $\Rightarrow$ (iv) is also clear (see the proof of Theorem 7.7).

So it remains to establish that (iii) $\Leftrightarrow$ (v). Suppose L is generally Post complete. Take the 0-generated submatrix $\langle \mathfrak{A}, \nabla \rangle$ of the Tarski–Lindenbaum matrix for L. Since quasi-identities are clearly preserved under the formation of submatrices and the quasi-identities corresponding to the admissible rules in L are true in the Tarski–Lindenbaum matrix for L (see Theorem 7.7), we then have $L = \mathrm{Log}\,\langle \mathfrak{A}, \nabla \rangle$.

Conversely, let $L = \mathrm{Log}\,\langle \mathfrak{A}, \nabla \rangle$ for some non-degenerate 0-generated matrix $\langle \mathfrak{A}, \nabla \rangle$. As was observed above, we may assume that $\langle \mathfrak{A}, \nabla \rangle$ is the 0-generated Tarski–Lindenbaum matrix for L. Suppose L' is a consistent extension of L inheriting all the admissible rules in L. Then we have $L \subseteq L' \subseteq \mathrm{Log}\,\langle \mathfrak{A}, \nabla' \rangle$, where $\nabla' = \{\|\varphi\|_L : \varphi \in L'\}$. Clearly, $\nabla \subseteq \nabla'$. Suppose $\nabla' \neq \nabla$, i.e., there is $\|\varphi\|_L \in \nabla' - \nabla$. Then the rule $\varphi/\bot$ is admissible in L and so in L' as well. It follows that $\varphi \notin L'$, which is a contradiction. Thus, $L \subseteq L' \subseteq \mathrm{Log}\,\langle \mathfrak{A}, \nabla' \rangle = \mathrm{Log}\,\langle \mathfrak{A}, \nabla \rangle = L$ and so $L' = L$. ❑

Theorem 13.12 *For every modal or si-logic L, the following conditions are equivalent:*

(i) *L is Post complete in* Ext**K** *(or* Ext**Int***);*

(ii) *L is consistent and the variety of matrices for L is generated by any of its non-degenerate matrices;*

(iii) *L is characterized by a 0-generated matrix $\langle \mathfrak{A}, \nabla \rangle$ in which ∇ is an ultrafilter.*

Proof (i) $\Rightarrow$ (ii). Suppose otherwise, i.e., $\langle \mathfrak{A}, \nabla \rangle$ is a non-degenerate matrix for L but $\mathrm{Var}\,\langle \mathfrak{A}, \nabla \rangle \neq \mathrm{Var}L$. Since $\bot \notin \nabla$, $\mathrm{Log}\,\langle \mathfrak{A}, \nabla \rangle$ is then a proper consistent extension of L, contrary to L being Post complete.

(ii) $\Rightarrow$ (iii). Let $\langle \mathfrak{A}, \nabla \rangle$ be a non-degenerate 0-generated matrix in $\mathrm{Var}L$, say the 0-generated submatrix of some non-degenerate matrix for L, which must exist because L is consistent. We show that ∇ is an ultrafilter in $\mathfrak{A}$. Suppose otherwise. This means that for some variable free formula φ, we have $\varphi \notin L$ and $\neg\varphi \notin L$. Then, by the deduction theorem, $L + \varphi$ is a proper consistent extension of L any Tarski–Lindenbaum matrix of which is non-degenerate and does not generate $\mathrm{Var}L$, contrary to (ii).

(iii) $\Rightarrow$ (i). Suppose L is characterized by a 0-generated matrix $\langle \mathfrak{A}, \nabla \rangle$ with an ultrafilter ∇ and $\varphi(p_1, \dots, p_n) \notin L$, i.e., $\langle \mathfrak{A}, \nabla \rangle \not\models \varphi(p_1, \dots, p_n)$. Then there are variable free formulas $\psi_1, \dots, \psi_n$ such that $\varphi(\psi_1, \dots, \psi_n)$ is refuted by $\langle \mathfrak{A}, \nabla \rangle$, i.e., $\varphi(\psi_1, \dots, \psi_n) \notin \nabla$. Since ∇ is an ultrafilter, we have $\neg\varphi(\psi_1, \dots, \psi_n) \in \nabla$ and so $\neg\varphi(\psi_1, \dots, \psi_n) \in L$. It follows that $L + \varphi(\psi_1, \dots, \psi_n)$ is inconsistent. Thus, L has no proper consistent extension. It remains to notice that since $\bot \notin \nabla$, L is consistent and so Post complete. ❑

This theorem shows, in particular, the place of Post complete logics among generally Post complete ones. Another indication to the place is given by

Theorem 13.13 *For every generally Post complete modal logic L, L is Post complete in* Ext**K** *iff L is structurally complete.*

Proof Exercise. (Hint: the implication ($\Rightarrow$) is established with the help of the proof of Theorem 1.25; to show ($\Leftarrow$), use Theorem 13.11 (ii) and Theorem 13.12 (iii) in order to find a variable free inference rule which is admissible but not derivable in L.) ❑

The results about Post completeness above concerned only Ext**K**. The reason is that there are very few Post complete logics in Ext**Int** and NExt**K**. As we already know, **Cl** is the only Post complete (and the only generally Post complete—check!) extension of **Int**. As to NExt**K**, as a consequence of Makinson's theorem we have

Theorem 13.14 *There are only two Post complete logics in* NExt**K**, *viz.*, Log$\circ$ *and* Log$\bullet$.

Let us consider now the family of (generally) Post complete logics in the lattice of extensions of an arbitrary quasi-normal logic L. By Theorem 13.11, the logic of the matrix $\langle \mathfrak{A}_{L'}(0), \nabla_L(0)\rangle$, where $L' = \ker L$, is the smallest generally Post complete extension of L. The generally Post complete extensions of L are the logics of the matrices of the form $\langle \mathfrak{A}_{L'}(0), \nabla\rangle$, where ∇ is a proper filter containing $\nabla_L(0)$, while the Post complete extensions of L are the logics of the matrices $\langle \mathfrak{A}_{L'}(0), \nabla\rangle$ in which ∇ is an ultrafilter containing $\nabla_L(0)$. Using this observation, we can prove

Theorem 13.15 (i) *There is a continuum of generally Post complete logics in* NExt**K4**.

(ii) *There is a continuum of Post complete logics in* Ext**K4**.

Proof (i) For $N \subseteq \omega$, denote by $\mathfrak{F}(N)$ the transitive Kripke frame of the form shown in Fig. 13.2 in which the only reflexive points are $2m+1$, $4m+2$, $4n+4$, for $m < \omega$, $n \in N$. (The frame in Fig. 13.2 corresponds to N such that $0, 2 \notin N$ and $1 \in N$.) The reader can readily check that $\mathfrak{F}(N)$ is a generated subframe of $\kappa\mathfrak{F}_{\mathbf{K4}}(0)$. Denote by $\mathfrak{A}(N)$ the 0-generated subalgebra of $\mathfrak{F}(N)^+$. Since each point in $\mathfrak{F}_{\mathbf{K4}}(0)$ is definable by a variable free formula, Log$\mathfrak{A}(N_1) =$ Log$\mathfrak{A}(N_2)$ only if $N_1 = N_2$. Thus, the cardinality of the class of generally Post complete logics in NExt**K4** is that of the continuum.

(ii) Let $\nabla(N)$ be a non-principal ultrafilter in $\mathfrak{A}(N)$. It is easy to see that such an ultrafilter is unique: it is the set of all cofinite subsets in $\mathfrak{F}(N)$ (since $\mathfrak{A}(N)$ consists of finite and cofinite subsets in $\mathfrak{F}(N)$, this set is an ultrafilter; on

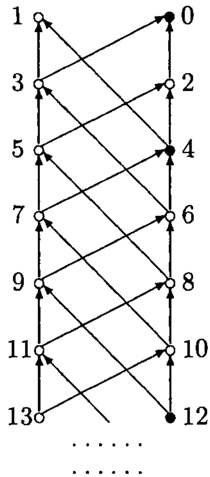


FIG. 13.2.

the other hand, any non-principal ultrafilter must contain all cofinite subsets in $\mathfrak{F}(N)$). Take distinct $N_1, N_2 \subseteq \omega$ and $i = 4n+4$, for $n \in N_2 - N_1$ (or $n \in N_1 - N_2$). Suppose the reflexive point i in $\mathfrak{F}(N_2)$ is defined by a variable free formula φ_i. Then $\neg\Diamond\varphi_i \in \mathrm{Log}\,\langle\mathfrak{A}(N_1), \nabla(N_1)\rangle$ and $\Diamond\varphi_i \in \mathrm{Log}\,\langle\mathfrak{A}(N_2), \nabla(N_2)\rangle$, i.e., the logics $\mathrm{Log}\,\langle\mathfrak{A}(N_1), \nabla(N_1)\rangle$ and $\mathrm{Log}\,\langle\mathfrak{A}(N_2), \nabla(N_2)\rangle$ are distinct if $N_1 \neq N_2$. It follows that there is a continuum of Post complete quasi-normal extensions of **K4**. ❑

Which logics have exactly one Post complete extension? The importance of such logics is emphasized by

Theorem 13.16 *Every consistent logic $L \in$ Ext**K** is the intersection of some logics having only one Post complete extension in* Ext**K**.

Proof Observe first that the following simple result holds.

Lemma 13.17 *A modal logic L has exactly one Post complete extension iff, for every variable free formula φ, either $\varphi \in L$ or $\neg\varphi \in L$.*

Proof ($\Rightarrow$) Suppose $\varphi \notin L$ and $\neg\varphi \notin L$, for some variable free φ. Then the logics $L + \neg\varphi$ and $L + \varphi$ have distinct Post complete extensions.

($\Leftarrow$) Follows from Theorem 13.12 (iii). ❑

Call a matrix $\langle\mathfrak{A}, \nabla\rangle$ *maximal* if ∇ is an ultrafilter in $\mathfrak{A}$. As a consequence of Lemma 13.17 we obtain that a logic characterized by a maximal matrix has only one Post complete extension.

The crucial step in the proof of our theorem is

Lemma 13.18 *Every non-degenerate non-maximal matrix $\langle\mathfrak{A}, \nabla\rangle$ is a submatrix of the direct product of maximal extensions of $\langle\mathfrak{A}, \nabla\rangle$.*

Proof For each $a \notin \nabla$, denote by $\nabla(a)$ some ultrafilter in $\mathfrak{A}$ such that $\nabla \subseteq \nabla(a)$ and $a \notin \nabla(a)$, and form the direct product $\prod_{a \notin \nabla} \langle \mathfrak{A}, \nabla(a) \rangle$. The reader can readily check that the matrix $\langle \mathfrak{A}, \nabla \rangle$ is embedded in this product by the map $b \mapsto f_b$, where f_b maps $\{a : a \notin \nabla\}$ to b. ❑

To complete the proof of Theorem 13.16, suppose L is a consistent logic and $\langle \mathfrak{A}, \nabla \rangle$ its characteristic matrix in which ∇ is not an ultrafilter. By Lemma 13.18, $\langle \mathfrak{A}, \nabla \rangle$ is a submatrix of the direct product $\prod_{i \in I} \langle \mathfrak{A}, \nabla_i \rangle$ of its maximal extensions. Therefore,

$$L = \mathrm{Log}\,\langle \mathfrak{A}, \nabla \rangle \supseteq \mathrm{Log} \prod_{i \in I} \langle \mathfrak{A}, \nabla_i \rangle = \bigcap_{i \in I} \mathrm{Log}\,\langle \mathfrak{A}, \nabla_i \rangle\,.$$

On the other hand, we clearly have $\mathrm{Log}\,\langle \mathfrak{A}, \nabla \rangle \subseteq \mathrm{Log}\,\langle \mathfrak{A}, \nabla_i \rangle$ for every $i \in I$, and so

$$L \subseteq \bigcap_{i \in I} \mathrm{Log}\,\langle \mathfrak{A}, \nabla_i \rangle\,.$$

It follows that $L = \bigcap_{i \in I} \mathrm{Log}\,\langle \mathfrak{A}, \nabla_i \rangle$ and, as was observed above, every $\mathrm{Log}\langle \mathfrak{A}, \nabla_i \rangle$ has only one Post complete extension. ❑

Remark According to Lemma 13.18, every non-degenerate variety of matrices is generated by its maximal matrices.

Theorem 13.16 shows that the study of any modal logic reduces, in a sense, to the study of logics having in Ext**K** a single Post complete extension. So it is worth considering classes of logics having exactly one Post complete extension in Ext**K**, which is common for all of them. The following theorem shows that such a class always contains a smallest logic.

Theorem 13.19 *Suppose L' is a Post complete extension of L in* Ext**K**. *Then the logic $L + \{\varphi \in L' : \varphi$ is variable free$\}$ is the smallest logic among those extensions of L that have L' as their only Post complete extension.*

Proof Follows from the fact that two modal logics have the same Post complete extensions in Ext**K** iff they contain the same variable free formulas. ❑

Above **K4**, Theorem 13.19 can be strengthened in the following way.

Theorem 13.20 *Suppose L' is a tabular Post complete extension in* Ext**K** *of a logic $L \supseteq$* **K4**. *Then there is a variable free formula φ such that $L + \varphi$ is the smallest logic among the extensions of L having L' as their only Post complete extension.*

Proof Suppose L' is characterized by a finite matrix $\langle \mathfrak{A}, \nabla \rangle$. By Theorem 13.12, we may assume $\langle \mathfrak{A}, \nabla \rangle$ to be 0-generated. Let $\varphi_1, \dots, \varphi_n$ be some variable free formulas such that (a) $\varphi_1 = \bot$, (b) φ_i, for $i \geq 2$, are constructed from previous formulas in this sequence using one of the connectives $\wedge$, $\vee$, $\rightarrow$, $\Box$, (c) φ_i, for $1 \leq i \leq n$, have different values a_i in $\mathfrak{A}$ and (d) $|\mathfrak{A}| = n$. In other words, these formulas describe a process of generating $\mathfrak{A}$ from $\bot$. Now we define φ as the conjunction of the following formulas, for $1 \leq i, j, k \leq n$:

$$\Box^+(\varphi_i \leftrightarrow \varphi_j \odot \varphi_k) \text{ if } a_i = a_j \odot a_k,\ \odot \in \{\wedge, \vee \rightarrow\},$$
$$\Box^+(\varphi_i \leftrightarrow \Box\varphi_j) \text{ if } a_i = \Box a_j,$$
$$\varphi_i \text{ if } a_i \in \nabla.$$

By the definition, $[\varphi) = \nabla$ and, in particular, $\langle \mathfrak{A}, \nabla \rangle \models \varphi$.

Given a variable free formula ψ with value a_i in $\mathfrak{A}$, put $\psi^* = \varphi_i$.

Lemma 13.21 *For every variable free ψ, $\Box^+(\psi^* \leftrightarrow \psi) \in L + \varphi$.*

Proof The proof proceeds by induction on the construction of ψ. The basis of induction is trivial. Suppose $\psi = \psi_1 \odot \psi_2$, for $\odot \in \{\wedge, \vee, \rightarrow\}$, $\Box^+(\psi_1^* \leftrightarrow \psi_1) \in L + \varphi$, $\Box^+(\psi_2^* \leftrightarrow \psi_2) \in L + \varphi$, $\psi^* = \varphi_i$, $\psi_1^* = \varphi_j$, $\psi_2^* = \varphi_k$ and $a_i = a_j \odot a_k$. In particular, we have $\Box^+(\psi^* \leftrightarrow \psi_1^* \odot \psi_2^*) \in L+\varphi$. So to prove $\Box^+(\psi^* \leftrightarrow \psi) \in L+\varphi$, it is sufficient to show that

$$\Box^+(\psi_1^* \odot \psi_2^* \leftrightarrow \psi_1 \odot \psi_2) \in L + \varphi,$$

which is established using the induction hypothesis and the formulas

$$\Box^+(p_1 \leftrightarrow p_2) \wedge \Box^+(p_3 \leftrightarrow p_4) \rightarrow \Box^+(p_1 \odot p_3 \leftrightarrow p_2 \odot p_4)$$

belonging to **K**.

Suppose now that $\psi = \Box\psi_1$, $\Box^+(\psi_1^* \leftrightarrow \psi_i) \in L + \varphi$, $\psi^* = \varphi_i$, $\psi_1^* = \varphi_j$ and $a_i = \Box a_j$. In particular, $\Box^+(\psi^* \leftrightarrow \Box\psi_1^*) \in L + \varphi$ and so to prove $\Box^+((\Box\psi_1)^* \leftrightarrow \Box\psi_1) \in L + \varphi$, it suffices to show that

$$\Box^+(\Box\psi_1^* \leftrightarrow \Box\psi_1) \in L + \varphi.$$

The latter is established using the induction hypothesis and the formula

$$\Box^+(p_1 \leftrightarrow p_2) \rightarrow \Box^+(\Box p_1 \leftrightarrow \Box p_2)$$

which is in **K4**. ❑

It follows that for every variable free formula ψ, we have $\langle \mathfrak{A}, \nabla \rangle \models \psi$ iff $\psi \in L + \varphi$. Indeed, if $\langle \mathfrak{A}, \nabla \rangle \models \psi$ then $\psi^* = \varphi_i$ for some $a_i \in \nabla$. By the definition of φ, we then have $\psi^* \in L + \varphi$, from which $\psi \in L + \varphi$. Conversely, suppose $\psi \in L+\varphi$. Then $\psi^* \in L+\varphi$ and, by the deduction theorem, $\varphi \rightarrow \psi^* \in L$. Therefore, $\langle \mathfrak{A}, \nabla \rangle \models \varphi \rightarrow \psi^*$ and, since $[\varphi) = \nabla$, we obtain $\langle \mathfrak{A}, \nabla \rangle \models \psi^*$, and hence $\langle \mathfrak{A}, \nabla \rangle \models \psi$.

Thus, $L + \varphi$ contains the same variable free formulas as L', which means that $L + \varphi$ has the unique Post complete extension L'. ❑

Call a logic *antitabular* if it is consistent but does not have finite models. It should be clear that a consistent logic is antitabular iff all its Post complete extensions are not tabular. Using Theorem 13.20 we obtain

Theorem 13.22 *If a logic $L \supseteq \mathbf{K4}$ has infinitely many Post complete extensions then it also has an antitabular extension.*

Proof Observe first that every Post complete logic is either tabular or antitabular. Let L_i, for $i \in I \subseteq \omega$, be all the distinct tabular Post complete extensions of L. If I is finite then we are done. Suppose I is infinite. By Theorem 13.20, there are variable free formulas φ_i such that $L + \varphi_i$ is the smallest extension of L having L_i as its only Post complete extension. Note that $\neg\varphi_j \in L + \varphi_i$ for $i \neq j$.

Now define $L' = L + \{\neg\varphi_i : \ i \in I\}$. If L' is consistent then, as any other logic, it has a Post complete extension which, by the definition of φ_i, must be different from all L_i. Therefore, L' is antitabular.

Suppose that L' is inconsistent, i.e., there is a derivation of $\bot$ in L'. Then we have $\neg\varphi_1, \ldots, \neg\varphi_n \vdash_L \bot$ for some n, whence, by the deduction theorem, $\varphi_1 \vee \ldots \vee \varphi_n \in L$ and so $\varphi_1 \vee \ldots \vee \varphi_n \in L_{n+1}$. On the other hand, we have $\neg\varphi_1 \in L_{n+1}, \ldots, \neg\varphi_n \in L_{n+1}$, and hence $\neg(\varphi_1 \vee \ldots \vee \varphi_n) \in L_{n+1}$, contrary to L_{n+1} being consistent. ❑

Unlike 2-reducibility (see Theorem 13.8), 0-reducibility turns out to be inherited by finitely approximable extensions of a given logic above **K4**.

Theorem 13.23 *Every finitely approximable extension of a generally Post complete logic in* Ext**K4** *is also generally Post complete.*

Proof We consider only normal logics because for quasi-normal ones the proof is analogous. The observations at the beginning of this section show that every generally Post complete logic $L \in$ NExt**K4** is characterized by a 0-generated algebra and extends the logic $\mathrm{Log}\mathfrak{F}_{\mathbf{K4}}(0)$. Since here we are interested in finitely approximable logics, let us consider finite frames for $\mathrm{Log}\mathfrak{F}_{\mathbf{K4}}(0)$.

Let a_i, $i < \omega$, be some enumeration of points in $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(0)$ and α_i a variable free formula defining a_i in $\mathfrak{F}_{\mathbf{K4}}(0)$ (i.e., $x \models \alpha_i$ iff $x = a_i$). Put

$$v(\alpha_i) = \Box^+(\alpha_i \to p) \vee \Box^+(\alpha_i \to \neg p).$$

The meaning of $v(\alpha_i)$ is that it is valid precisely in those transitive rooted frames that contain at most one point where α_i is true. Then clearly $\mathfrak{F}_{\mathbf{K4}}(0) \models v(\alpha_i)$ for every $i < \omega$. This observation provides us with the following:

Lemma 13.24 (i) *No finite rooted frame for* $\mathrm{Log}\mathfrak{F}_{\mathbf{K4}}(0)$ *has non-trivial reducts.*

(ii) *The class of finite rooted frames for* $\mathrm{Log}\mathfrak{F}_{\mathbf{K4}}(0)$ *coincides with the class of rooted generated subframes of* $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(0)$.

(iii) *Every normal finitely approximable extension of* $\mathrm{Log}\,\mathfrak{F}_{\mathbf{K4}}(0)$ *is characterized by a class of rooted generated subframes of* $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(0)$ *closed under the formation of rooted generated subframes, with this correspondence being 1–1.*

Thus, if $L \supseteq \mathrm{Log}\mathfrak{F}_{\mathbf{K4}}(0)$ is finitely approximable then it is characterized by a class of finite frames in which every point is definable by a variable free formula. It follows that L is 0-reducible and so, by Theorem 13.11, generally Post complete. ❑

Exercise 13.12 shows that the requirement of finite approximability in Theorem 13.23 is essential.

Lemma 13.24 has one more interesting application. Together with the construction of Theorem 13.15 it provides us with a continuum of pretabular logics in NExt**K4**.

Theorem 13.25 *There is a continuum of pretabular logics in* NExt**K4**.

Proof It suffices to show that the logics $L = \mathrm{Log}\mathfrak{A}(N)$, defined in the proof of Theorem 13.15 (i), are pretabular in NExt**K4**. It should be clear that they are not tabular. Suppose L' is a pretabular extension of L in NExt**K4**. By Corollary 12.12, L' is finitely approximable and, since $\mathfrak{A}(N)$ is 0-generated, all its finite rooted frames are, by Lemma 13.24, generated subframes of $\mathfrak{F}(N)$. Since L' is not tabular, it has finite frames of any depth. By the construction of $\mathfrak{F}(N)$, its every generated subframe of depth n contains all $\mathfrak{F}(N)$'s generated subframes of depth $\leq n-2$. Therefore, the classes of finite rooted frames for L and L' coincide and so, since L is finitely approximable by its definition, $L = L'$. ❑

13.3 Exercises and open problems

Exercise 13.1 Show that, for every logic L in the intervals mentioned in Theorem 13.2 and every $m < \omega$, the logic $\mathrm{Log}\mathfrak{A}_L(m)$ is not n-reducible for any $n < m$.

Exercise 13.2 Prove or disprove that σ**KC** is 1-reducible.

Exercise 13.3 Show that **Grz.3** is 1-reducible.

Exercise 13.4 Prove that if there are variable free formulas φ_i, $i < \omega$, such that $\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_n} \to \varphi_j \notin L$ for $j \notin \{i_1, \ldots, i_n\}$, then L has a continuum of generally Post complete extensions.

Exercise 13.5 Prove that if there are variable free formulas φ_i, $i < \omega$, such that $\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_n} \to \varphi_{j_1} \vee \ldots \vee \varphi_{j_m} \notin L$ for $\{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_m\} = \emptyset$, then L has a continuum of Post complete extensions.

Exercise 13.6 Prove that the intersection of generally Post complete logics is also generally Post complete. Is this true for sums of logics?

Exercise 13.7 Show that every logic L in the interval [**K4.3**, **GL.3**] has countably many Post complete extensions in Ext$L$ and a continuum of generally Post complete extensions.

Exercise 13.8 What is the number of (generally) Post complete (normal) extensions of $\mathbf{K} \oplus \Box^n\bot$ and $\mathbf{K4BD}_n$?

Exercise 13.9 Prove that a modal logic has $n < \omega$ Post complete extensions in Ext**K** iff it has $2^n - 1$ generally Post complete extensions.

Exercise 13.10 Give an example of a logic which is characterized by a maximal matrix but is not Post complete.

Exercise 13.11 Prove that all logics in Ext**GL.3** are generally Post complete.

Exercise 13.12 Construct a generally Post complete logic having a normal extension which is not generally Post complete.

Exercise 13.13 Prove that **GL** has the same Post complete extensions as **GL.3**, namely, the logics of the roots of finite irreflexive transitive chains and also $L_\omega = \mathbf{GL.3} + \boldsymbol{re}$.

Problem 13.1 *Prove or disprove that* $\mathbf{Int} + \bigvee_{i=1}^{n} \neg(p_i \wedge \bigwedge_{j \neq i} \neg p_j)$ *is* m*-reducible for* $m = [\log_2 n] + 1$.

Problem 13.2 *Does the map* $\sigma L \mapsto L$ *preserve* m*-reducibility?*

Problem 13.3 *Are there logics with countably many generally Post complete extensions?*

Problem 13.4 *Do Theorems 13.20 and 13.22 hold for logics above* **K***?*

Problem 13.5 *Does the equation* $\mathrm{Log}\mathfrak{F}_{\mathbf{K4}}(0) = \mathbf{K4} \oplus \{v(\alpha_i) : \ i < \omega\}$ *hold?*

13.4 Notes

The notion of reducibility appeared first in McKinsey and Tarski (1948), where it was proved that **S4**, **S5** and **Int** are not reducible. Later similar facts were established for a few other logics. Theorem 13.2 is due to Chagrov (1993). That **KC** is 2-reducible was noted by Mardaev (1987) and the key lemma in the proof of this result (Lemma 13.4) was proved by Blok (1977).

Although the lattices NExt**K** and Ext**Int** are similar as far as the number of Post complete logics in them is concerned, the algorithmic problem of determining, given a formula φ, whether $\mathbf{K} \oplus \varphi$ is Post complete is undecidable (Chagrov 1996), while the problem of Post completeness in Ext**Int** turns out to be decidable.

In view of Makinson's theorem, when dealing with Post complete modal logics we primarily consider logics without the postulated rule RN. The first results concerning Post completeness of modal logics were obtained by McKinsey (1944), who proved that **S4** has only one Post complete extension and above **S2** there are infinitely many of them. The main problem of many subsequent papers concerning Post completeness was to determine the set of Post complete extensions of certain logics and estimate its cardinality. Some results of that sort can be found among the exercises in Section 13.3; see also Segerberg (1972, 1976) and Blok and Köhler (1983).

A considerable step in understanding the nature of Post complete logics was made by Makinson and Segerberg (1974), who established a connection between the number of Post complete extensions of a given logic and the number of ultrafilters in the modal algebras determining it. A similar observation was made in Sambin and Valentini (1980). The most complete exposition of the current researches of Post completeness, in particular computing the number of Post complete extensions of normal modal logics can be found in Bellissima (1990).

Note that in the polymodal case the method of studying Post completeness is basically the same, however the description of Post complete logics is of course more complicated.

The notion of generally Post complete logic was introduced and investigated in Chagrov (1985b); the theorems characterizing Post complete and generally Post complete logics in Section 13.2 were taken from this paper. The remaining results in this section were proved in Chagrov (1989, 1994b), where it is shown, in particular, that the requirement of finite approximability is essential in Theorem 13.23.

14

INTERPOLATION

Recall that a logic L is said to have the *Craig interpolation property* if, for every implication $\alpha \to \beta$ in L, there exists a formula γ, called an *interpolant* for $\alpha \to \beta$ in L, such that $\alpha \to \gamma \in L$, $\gamma \to \beta \in L$ and $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$. In this chapter we present the most important semantic methods of proving and disproving the interpolation property of modal and superintuitionistic logics.

14.1 Interpolation theorems for certain modal systems

First we extend the construction used in the proof of Craig's interpolation theorem for $\mathbf{Cl}$ in order to prove the interpolation property of a few standard modal logics. As in that proof, our plan is, given that $\alpha \to \gamma$ and $\gamma \to \beta$ are not in L for any γ with $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$, to "saturate" the inseparable tableau $t_0 = (\{\alpha\}, \{\beta\})$ to complete inseparable tableaux which describe a model for L realizing t_0. The difference is that for $\mathbf{Cl}$ it was sufficient to construct a single complete inseparable extension of t_0, while in the modal case to define the Kripke model we need, a set of such tableaux with an accessibility relation between them may be required. We should warn the reader that although we use the same terminology as in the proof of Theorem 1.28, some notions will be defined in a slightly different way.

Theorem 14.1 **S4** *has the interpolation property.*

Proof Suppose $\alpha \to \gamma \notin \mathbf{S4}$ and $\gamma \to \beta \notin \mathbf{S4}$ for any formula γ whose variables occur in both α and β, and show that in this case $\alpha \to \beta \notin \mathbf{S4}$.

We shall be considering tableaux of the form $t = (\Gamma, \Delta)$ in which all formulas in Γ contain only variables occurring in α and formulas in Δ contain only variables from β. Say that t is *inseparable* (relative to α and β) if there is no formula γ such that $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$ and $\bigwedge_{i=1}^{n} \varphi_i \to \gamma \in \mathbf{S4}$, $\gamma \to \bigvee_{i=1}^{m} \psi_i \in \mathbf{S4}$ for some $\varphi_1, \dots, \varphi_n \in \Gamma$, $\psi_1, \dots, \psi_m \in \Delta$. The tableau t is called *complete* (relative to α and β) if for every φ and ψ with $\mathbf{Var}\varphi \subseteq \mathbf{Var}\alpha$ and $\mathbf{Var}\psi \subseteq \mathbf{Var}\beta$, one of the formulas φ and $\neg\varphi$ is in Γ and one of ψ and $\neg\psi$ is in Δ.

Lemma 14.2 *Every inseparable tableau* $t_0 = (\Gamma_0, \Delta_0)$ *can be extended to a complete inseparable tableau.*

Proof Let $\varphi_1, \varphi_2, \dots$ and $\psi_1, \psi_2, \dots$ be enumerations of all formulas whose variables occur in α and β, respectively. Define tableaux $t'_n = (\Gamma'_n, \Delta'_n)$ and $t_{n+1} = (\Gamma_{n+1}, \Delta_{n+1})$ inductively by taking, for $n = 0, 1, \dots$,

$$t'_n = \begin{cases} (\Gamma_n \cup \{\varphi_n\}, \Delta_n) & \text{if this pair is inseparable} \\ (\Gamma_n \cup \{\neg\varphi_n\}, \Delta_n) & \text{otherwise,} \end{cases}$$

$$t_{n+1} = \begin{cases} (\Gamma'_n, \Delta'_n \cup \{\psi_n\}) & \text{if this pair is inseparable} \\ (\Gamma'_n, \Delta'_n \cup \{\neg\psi_n\}) & \text{otherwise.} \end{cases}$$

Finally, we put $t^* = (\Gamma^*, \Delta^*)$, where $\Gamma^* = \bigcup_{n<\omega} \Gamma_n$, $\Delta^* = \bigcup_{n<\omega} \Delta_n$.

We show now that the tableau t^* is complete and inseparable relative to α and β. That t^* is complete follows directly from the definition. Suppose t^* is separable. Then for some formulas $\varphi_1, \ldots, \varphi_n \in \Gamma^*$, $\psi_1, \ldots, \psi_m \in \Delta^*$ and some formula γ containing only those variables that occur in both α and β, we have $\bigwedge_{i=1}^n \varphi_i \to \gamma \in \mathbf{S4}$ and $\gamma \to \bigvee_{i=1}^m \psi_i \in \mathbf{S4}$. Since $n, m < \omega$, there exists $k < \omega$ such that $\varphi_1, \ldots, \varphi_n \in \Gamma_k$ and $\psi_1, \ldots, \psi_m \in \Delta_k$, which means that t_k is separable.

So it remains to show that if $t = (\Gamma, \Delta)$ is inseparable, $\mathbf{Var}\varphi \subseteq \mathbf{Var}\alpha$ and $\mathbf{Var}\psi \subseteq \mathbf{Var}\beta$ then

- one of the tableaux $(\Gamma \cup \{\varphi\}, \Delta)$ or $(\Gamma \cup \{\neg\varphi\}, \Delta)$ is inseparable and
- one of the tableaux $(\Gamma, \Delta \cup \{\psi\})$ or $(\Gamma, \Delta \cup \{\neg\psi\})$ is inseparable.

We prove only the former claim, leaving the latter to the reader. Suppose, on the contrary, that both tableaux are separable, i.e., there are formulas γ_1, γ_2 in variables occurring in both α and β such that, for some $\varphi_1, \ldots, \varphi_n \in \Gamma$, $\psi_1, \ldots, \psi_m \in \Delta$, we have

$$\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi \to \gamma_1 \in \mathbf{S4}, \ \gamma_1 \to \psi_1 \vee \ldots \vee \psi_m \in \mathbf{S4},$$

$$\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \neg\varphi \to \gamma_2 \in \mathbf{S4}, \ \gamma_2 \to \psi_1 \vee \ldots \vee \psi_m \in \mathbf{S4}.$$

Then we obtain

$$(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi) \vee (\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \neg\varphi) \to \gamma_1 \vee \gamma_2 \in \mathbf{S4},$$

$$\gamma_1 \vee \gamma_2 \to \psi_1 \vee \ldots \vee \psi_m \in \mathbf{S4},$$

which in view of

$$(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi) \vee (\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \neg\varphi) \leftrightarrow \varphi_1 \wedge \ldots \wedge \varphi_n \in \mathbf{S4}$$

gives us

$$\varphi_1 \wedge \ldots \wedge \varphi_n \to \gamma_1 \vee \gamma_2 \in \mathbf{S4}, \ \gamma_1 \vee \gamma_2 \to \psi_1 \vee \ldots \vee \psi_m \in \mathbf{S4}.$$

Since $\mathbf{Var}\gamma_1 \vee \gamma_2 \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$, this contradicts t being inseparable. ❑

Now we define a frame $\mathfrak{F} = \langle W, R \rangle$ by taking W to be the set of all complete and inseparable extensions of the inseparable tableau $(\{\alpha\}, \{\beta\})$ and, for tableaux $t_1 = (\Gamma_1, \Delta_1)$, $t_2 = (\Gamma_2, \Delta_2)$ in W, $t_1 R t_2$ iff $\Box\varphi \in \Gamma_1$ implies $\varphi \in \Gamma_2$. Using the axioms $\Box p \to p$ and $\Box p \to \Box\Box p$ of $\mathbf{S4}$, one can readily check that R is a quasi-order on W, i.e., $\mathfrak{F}$ is a frame for $\mathbf{S4}$.

Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking for every variable $p \in \mathbf{Var}(\alpha \to \beta)$, $\mathfrak{V}(p) = \{(\Gamma, \Delta) \in W : \text{either } p \in \Gamma \text{ or } p \in \mathbf{Var}\beta \text{ and } p \notin \Delta\}$. Put $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$.

Lemma 14.3 *For every* $t = (\Gamma, \Delta)$ *in* $\mathfrak{F}$ *and all formulas* φ *and* ψ *with* $\mathbf{Var}\varphi \subseteq \mathbf{Var}\alpha$, $\mathbf{Var}\psi \subseteq \mathbf{Var}\beta$,

$$(\mathfrak{M}, t) \models \varphi \textit{ iff } \varphi \in \Gamma, \ (\mathfrak{M}, t) \not\models \psi \textit{ iff } \psi \in \Delta.$$

Proof By induction on the construction of φ and ψ. The basis of induction follows from the definition of $\mathfrak{V}$ and the completeness and inseparability of t. The cases of the Boolean connectives present no difficulty. So suppose $\varphi = \Box\varphi_1$.

If $t \models \Box\varphi_1$ then, for every $t' = (\Gamma', \Delta') \in t\uparrow$, we have $t' \models \varphi_1$ and so, by the induction hypothesis, $\varphi_1 \in \Gamma'$. Suppose $\Box\varphi_1 \notin \Gamma$. Then, since t is complete, $\neg\Box\varphi_1 \in \Gamma$. Consider the tableau $t_0 = (\Gamma_0, \Delta_0)$, where

$$\Gamma_0 = \{\neg\varphi_1\} \cup \{\chi : \ \Box\chi \in \Gamma\}, \ \Delta_0 = \{\neg\chi : \ \neg\Box\chi \in \Delta\}.$$

We show that t_0 is inseparable. Suppose otherwise. Then there is a formula γ with $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$ such that, for some formulas $\Box\chi_1, \dots, \Box\chi_n \in \Gamma$, $\neg\Box\chi_{n+1}, \dots, \neg\Box\chi_m \in \Delta$,

$$\neg\varphi_1 \wedge \chi_1 \wedge \dots \wedge \chi_n \to \gamma \in \mathbf{S4}, \ \gamma \to \neg\chi_{n+1} \vee \dots \vee \neg\chi_m \in \mathbf{S4}.$$

Using now the formulas

$$\Box(p \wedge q_1 \wedge \dots \wedge q_n \to r) \to (\Diamond p \wedge \Box q_1 \wedge \dots \wedge \Box q_n \to \Diamond r),$$

$$\Box(r \to p_1 \vee \dots \vee p_k) \to (\Diamond r \to \Diamond p_1 \vee \dots \vee \Diamond p_k),$$

belonging to every modal logic and the fact that **S4** is closed under necessitation, we obtain

$$\neg\Box\varphi_1 \wedge \Box\chi_1 \wedge \dots \wedge \Box\chi_n \to \Diamond\gamma \in \mathbf{S4},$$

$$\Diamond\gamma \to \neg\Box\chi_{n+1} \vee \dots \vee \neg\Box\chi_m \in \mathbf{S4},$$

contrary to t being inseparable.

Let $t' = (\Gamma', \Delta')$ be a complete inseparable extension of t_0. By the definition of t_0, we have tRt' and so $\varphi_1 \in \Gamma'$, contrary to $\neg\varphi_1 \in \Gamma_0 \subseteq \Gamma'$ and t' being inseparable.

Suppose now that $\Box\varphi_1 \in \Gamma$. Then for every $t' = (\Gamma', \Delta')$ such that tRt', we have $\varphi_1 \in \Gamma$ and so, by the induction hypothesis, $t' \models \varphi_1$. Consequently, $t \models \Box\varphi_1$.

The formula ψ is treated in the dual way. ❑

To complete the proof of our theorem, it remains to observe that, in view of Lemma 14.3, $\mathfrak{M} \not\models \alpha \to \beta$ and so $\alpha \to \beta \notin \mathbf{S4}$. ❑

Notice that specific properties of **S4** were used in the proof above only to establish that $\mathfrak{F}$ is a frame for **S4**. The rest of our considerations is suitable for any other normal modal logic (the normality was exploited in the proof of Lemma 14.3). Therefore, if we exclude using the axioms $\Box p \to p$ and/or $\Box p \to \Box\Box p$ then by the same argument we shall obtain

Theorem 14.4 *The logics* **K**, **K4**, **T** *have the interpolation property.*

Observe also that the construction of the models in the proof of Theorems 14.1 and 14.4 resembles the construction of the canonical models. For instance, we could use them to establish the Kripke completeness of the logics under consideration. Indeed, if $\varphi \notin L$ then $\top \to \varphi \notin L$ and so $\top \to \varphi$ does not have an interpolant in L; the constructed model will be then a model for $L \in \{\mathbf{S4}, \mathbf{K}, \mathbf{K4}, \mathbf{T}\}$ refuting φ. Moreover, using a somewhat subtler argument we could construct finite models and prove thereby the finite approximability of those logics. Such a construction will be described in Section 14.5, where we establish the interpolation property of **GL**.

For a logic $L \in \mathrm{Ext}\mathbf{K}$ ($L \in \mathrm{Ext}\mathbf{Int}$), we say that a formula $\alpha(p)$ is *conservative* in $\mathrm{Ext}L$ if

$$\alpha(\bot) \wedge \alpha(p) \wedge \alpha(q) \to \alpha(p \to q) \wedge \alpha(\Box p) \in L$$

(in the intuitionistic case the conjunct $\alpha(\Box p)$ should be replaced with the formula $\alpha(p \wedge q) \wedge \alpha(p \vee q)$). If $L \in \mathrm{NExt}\mathbf{K4}$, we call $\alpha(p)$ *conservative* in $\mathrm{NExt}L$ if

$$\Box^+(\alpha(\bot) \wedge \alpha(p) \wedge \alpha(q)) \to \alpha(p \to q) \wedge \alpha(\Box p) \in L.$$

Theorem 14.5 (i) *If L has the interpolation property and formulas α_i, for $i \in I$, are conservative in* $\mathrm{Ext}L$, *then $L+\{\alpha_i : i \in I\}$ also has the interpolation property.*

(ii) *If $L \in$* $\mathrm{NExt}\mathbf{K4}$ *has the interpolation property and formulas α_i, for $i \in I$, are conservative in* $\mathrm{NExt}L$, *then $L \oplus \{\alpha_i : i \in I\}$ also has the interpolation property.*

Proof We prove only (ii); the proof of (i) can be obtained by omitting all $\Box^+$ and replacing all $\oplus$ with $+$. Suppose $\varphi \to \psi \in L \oplus \{\alpha_i : i \in I\}$. Then there is a finite $J \subseteq I$, say $J = \{1, \dots, l\}$, such that $\varphi \to \psi \in L \oplus \{\alpha_i : i \in J\}$ and so, as easily follows from the definition of conservative formulas (see Exercise 14.1) and the deduction theorem for **K4**,

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(\bot) \wedge \alpha_j(p_1) \wedge \ldots \wedge \alpha_j(p_n)) \to (\varphi \to \psi) \in L,$$

where $p_1, \dots, p_m, p_{m+1}, \dots, p_k$ and $p_{m+1}, \dots, p_k, p_{k+1}, \dots, p_n$ are all the variables in φ and ψ, respectively. It follows that

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(\bot) \wedge \alpha_j(p_1) \wedge \ldots \wedge \alpha_j(p_k)) \wedge \varphi \to$$
$$(\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(p_{m+1}) \wedge \ldots \wedge \alpha_j(p_n)) \to \psi) \in L.$$

Since L has the interpolation property, there is $\chi(p_{m+1}, \dots, p_k)$ such that

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(\bot) \wedge \alpha_j(p_1) \wedge \ldots \wedge \alpha_j(p_k)) \wedge \varphi \to \chi \in L$$

and

$$\chi \to (\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(p_{m+1}) \wedge \ldots \wedge \alpha_j(p_n)) \to \psi) \in L,$$

which is equivalent to

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(p_{m+1}) \wedge \ldots \wedge \alpha_j(p_n)) \to (\chi \to \psi) \in L.$$

Then we obtain $\varphi \to \chi \in L \oplus \{\alpha_i : i \in I\}$ and $\chi \to \psi \in L \oplus \{\alpha_i : i \in I\}$, i.e., χ is an interpolant for $\varphi \to \psi$ in $L \oplus \{\alpha_i : i \in I\}$. ❑

Corollary 14.6 *There is a continuum of logics in* NExt**K4** *having the interpolation property.*

Proof According to Theorem 13.15, there are a continuum of logics in NExt**K4** axiomatizable by variable free formulas which clearly are conservative. ❑

Lemma 14.7 *The formulas $\Box\Diamond p \to \Diamond\Box p$, $\Box\Diamond p \leftrightarrow \Diamond\Box p$ and $\Box p \leftrightarrow \Diamond p$ are conservative in* NExt**S4**.

Proof Exercise. ❑

As a consequence of Theorem 14.5 and Lemma 14.7 we obtain another

Corollary 14.8 *The logics* **S4.1**, **S4** $\oplus\, \Box\Diamond p \leftrightarrow \Diamond\Box p$ *and* **Triv** *have the interpolation property.*

The following result shows that the interpolation property is preserved while passing from a modal logic in NExt**S4** to its superintuitionistic fragment.

Theorem 14.9 *If $L \in$* NExt**S4** *has the interpolation property then ρL has this property as well.*

Proof Suppose that $\alpha \to \beta \in \rho L$. Then $\mathsf{T}(\alpha) \to \mathsf{T}(\beta) \in L$ and so there is an interpolant γ' for $\mathsf{T}(\alpha) \to \mathsf{T}(\beta)$ in L, which means that $\mathsf{T}(\alpha) \to \gamma' \in L$ and $\gamma' \to \mathsf{T}(\beta) \in L$. Since $\mathsf{T}(\varphi) \leftrightarrow \Box\mathsf{T}(\varphi) \in$ **S4** (see Exercise 3.25) and $\mathsf{T}(\varphi(p_1, \ldots, p_n)) \leftrightarrow \mathsf{T}(\varphi(\Box p_1, \ldots, \Box p_n)) \in$ **S4** for every intuitionistic formula $\varphi(p_1, \ldots, p_n)$, we have $\mathsf{T}(\alpha) \to \Box\gamma'' \in L$ and $\Box\gamma'' \to \mathsf{T}(\beta) \in L$, where γ'' is obtained from γ' by prefixing $\Box$ to each of its variables. By induction on the construction of a modal formula $\psi(p_1, \ldots, p_n)$ one can readily show also that there exists an intuitionistic formula $\varphi(p_1, \ldots, p_n)$ such that

$$\Box\psi(\Box p_1, \ldots, \Box p_n) \leftrightarrow \mathsf{T}(\varphi(p_1, \ldots, p_n)) \in \mathbf{S4}.$$

Now take an intuitionistic formula γ such that $\Box\gamma'' \leftrightarrow \mathsf{T}(\gamma) \in$ **S4** and $\mathbf{Var}\gamma = \mathbf{Var}\gamma''$. Then we obtain $\mathsf{T}(\alpha) \to \mathsf{T}(\gamma) \in L$ and $\mathsf{T}(\gamma) \to \mathsf{T}(\beta) \in L$, from which

$\Box(\mathsf{T}(\alpha) \to \mathsf{T}(\gamma)) \in L$ and $\Box(\mathsf{T}(\gamma) \to \mathsf{T}(\beta)) \in L$, and finally, $\alpha \to \gamma \in \rho L$ and $\gamma \to \beta \in \rho L$. ❑

Since $\rho\mathbf{S4} = \mathbf{Int}$, $\rho(\mathbf{S4}\oplus\Box\Diamond p \leftrightarrow \Diamond\Box p) = \mathbf{KC}$, $\rho\mathbf{Triv} = \mathbf{Cl}$, as a consequence of Theorems 14.1, 14.9 and Corollary 14.8 we obtain

Corollary 14.10 *The logics* **Int**, **KC** *and* **Cl** *have the interpolation property.*

14.2 Semantic criteria of the interpolation property

Say that a class $\mathcal{C}$ of algebras is *amalgamable* if for every algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ in $\mathcal{C}$ such that $\mathfrak{A}_0$ is embedded in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by isomorphisms f_1 and f_2, respectively, there exist $\mathfrak{A} \in \mathcal{C}$ and isomorphisms g_1 and g_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ into $\mathfrak{A}$ with $g_1(f_1(x)) = g_2(f_2(x))$, for any x in $\mathfrak{A}_0$.

Theorem 14.11 *A si-logic L has the interpolation property iff the variety* Var$L$ *is amalgamable.*

Proof ($\Rightarrow$) Suppose L has the interpolation property and f_1, f_2 are isomorphisms of $\mathfrak{A}_0$ into $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ pseudo-Boolean algebras for L (with universes A_0, A_1, A_2). Without loss of generality we will assume $\mathfrak{A}_0$ to be a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, i.e., that f_1 and f_2 are the identity maps: $f_1(x) = f_2(x) = x$ for all $x \in A_0$. With each element $a \in A_i$, $i = 0, 1, 2$, we associate a variable p_a^i in such a way that, for $a \in A_0$, $p_a^0 = p_a^1 = p_a^2$. Denote by $\mathcal{L}_i$ the (intuitionistic) language with the variables p_a^i, for $a \in A_i$, $i = 0, 1, 2$, and let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. We will not distinguish between terms and formulas in the languages we have just introduced and denote them by the same symbols. Also we will assume that $\mathcal{L}$ is the language of our logic L.

Let us fix the valuation $\mathfrak{V}_i$ of $\mathcal{L}_i$ in $\mathfrak{A}_i$, defined by $\mathfrak{V}_i(p_a^i) = a$, and put, for $i = 1, 2$,

$$\Sigma_i = \{\varphi \in \mathbf{For}\mathcal{L}_i : \ \mathfrak{V}_i(\varphi) = \top\}.$$

It is clear that $L \cap \mathbf{For}\mathcal{L}_i \subseteq \Sigma_i$ and that Σ_i is closed under *modus ponens*. Let Σ be the closure of $\Sigma_1 \cup \Sigma_2 \cup L$ under *modus ponens*. We show that, for every $\varphi \in \mathbf{For}\mathcal{L}_i$, $\psi \in \mathbf{For}\mathcal{L}_j$ such that $\{i, j\} = \{1, 2\}$,

$$\varphi \to \psi \in \Sigma \text{ iff } \exists\chi \in \mathbf{For}\mathcal{L}_0\ (\varphi \to \chi \in \Sigma_i \wedge \chi \to \psi \in \Sigma_j). \tag{14.1}$$

The "if" part of (14.1) is obvious, since Σ is closed under MP and so under the rule $\varphi \to \chi, \chi \to \psi / \varphi \to \psi$.

Suppose now that $\varphi \to \psi \in \Sigma$. This means that there is a substitutionless derivation of $\varphi \to \psi$ in L from some finite sets of assumptions $\Gamma_i \subseteq \Sigma_i$ and $\Gamma_j \subseteq \Sigma_j$. By the deduction theorem, we then have

$$\bigwedge \Gamma_i \wedge \bigwedge \Gamma_j \to (\varphi \to \psi) \in L$$

and so

$$\bigwedge \Gamma_i \wedge \varphi \to (\bigwedge \Gamma_j \to \psi) \in L.$$

Since L has the interpolation property, there is a formula $\chi \in \mathbf{For}\mathcal{L}_0$ such that

$$\bigwedge \Gamma_i \wedge \varphi \to \chi \in L, \ \bigwedge \Gamma_j \to (\chi \to \psi) \in L,$$

from which, by MP, $\varphi \to \chi \in \Sigma_i$ and $\chi \to \psi \in \Sigma_j$. This establishes the "only if" part of (14.1).

Notice, by the way, that putting $\varphi = \top$ in (14.1), we obtain that $\Sigma \cap \mathbf{For}\mathcal{L}_j = \Sigma_j$, for $j = 1, 2$.

Now we construct an algebra $\mathfrak{A}$ by taking the set $\{\|\varphi\| : \varphi \in \Sigma\}$ as its universe A, where $\|\varphi\| = \{\psi : \varphi \leftrightarrow \psi \in \Sigma\}$ and putting $\bot = \|\bot\|$, $\|\varphi\| \odot \|\psi\| = \|\varphi \odot \psi\|$, for $\odot \in \{\wedge, \vee, \to\}$. This definition is correct because $\mathbf{Int} \subseteq L \subseteq \Sigma$. It should be also clear that $\mathfrak{A} \in \mathrm{Var}L$.

Define maps g_i from $\mathfrak{A}_i$ into $\mathfrak{A}$, for $i = 1, 2$, by taking $g_i(a) = \|p^i_a\|$. By the definition, g_i is an injection. Let us show that g_i is a homomorphism. First, we have $g_i(\bot) = \|p^i_\bot\| = \|\bot\| = \bot$, because $\mathfrak{V}_i(p^i_\bot) = \bot$. Second, suppose $c = a \odot b$ in $\mathfrak{A}_i$, for $\odot \in \{\wedge, \vee, \to\}$. Then $\mathfrak{V}_i(p^i_a \odot p^i_b) = \mathfrak{V}_i(p^i_c)$ and so

$$g_i(a \odot b) = \|p^i_{a\odot b}\| = \|p^i_a \odot p^i_b\| = \|p^i_a\| \odot \|p^i_b\| = g_i(a) \odot g_i(b).$$

Thus, g_i is an embedding of $\mathfrak{A}_i$ in $\mathfrak{A}$. And for $a \in A_0$, we have

$$g_1(f_1(a)) = g_1(a) = \|p^0_a\| = g_2(a) = g_2(f_2(a)).$$

($\Leftarrow$) Assuming $\mathrm{Var}L$ to be amalgamable, we show that L has the interpolation property. To this end we require the following:

Lemma 14.12 *Suppose $\mathfrak{A}_0$ is a subalgebra of pseudo-Boolean algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, $a \in A_1$, $b \in A_2$ and there is no $c \in A_0$ such that $a \leq_1 c \leq_2 b$ (where $\leq_i$ is the partial order and A_i the universe in $\mathfrak{A}_i$). Then there are prime filters ∇_1 in $\mathfrak{A}_1$ and ∇_2 in $\mathfrak{A}_2$ such that $a \in \nabla_1$, $b \notin \nabla_2$ and $\nabla_1 \cap A_0 = \nabla_2 \cap A_0$.*

Proof We remind the reader that a set of elements in $\mathfrak{A}_i$ is a filter (ideal) iff it can be represented in the form $[X)_i$ (respectively, $(X]_i$) for some $X \subseteq A_i$. Take the sets

$$X = \{x \in A_0 : \ a \leq_1 x\}, \ Y = \{y \in A_0 : \ y \leq_2 b\}.$$

By the condition of the lemma, $X \cap Y = \emptyset$. We are going to extend Y to some ideal Δ_2 in $\mathfrak{A}_2$ in such a way that $b \in \Delta_2$ and $X \cap \Delta_2 = \emptyset$. To this end consider the family

$$\mathcal{F}_2 = \{\Delta \subseteq A_2 : \ \Delta = (\Delta]_2, \ \{b\} \cup Y \subseteq \Delta, \ X \cap \Delta = \emptyset\}.$$

$\mathcal{F}_2$ is not empty, because $(b]_2 \in \mathcal{F}_2$. The union of any chain (with respect to $\subseteq$) of $\mathcal{F}_2$'s elements is again in $\mathcal{F}_2$. So, by Zorn's lemma, $\mathcal{F}_2$ contains a maximal element, which we denote by Δ_2.

The ideal Δ_2 turns out to be prime, i.e., $x \wedge y \in \Delta_2$ implies $x \in \Delta_2$ or $y \in \Delta_2$. Indeed, suppose $x \wedge y \in \Delta_2$ but $x \notin \Delta_2$ and $y \notin \Delta_2$. Since Δ_2 is maximal in $\mathcal{F}_2$, we then have

$$X \cap (\{x\} \cup \Delta_2]_2 \neq \emptyset \neq X \cap (\{y\} \cup \Delta_2]_2,$$

i.e., Δ_2 contains elements u and v such that $a \leq_1 x \vee u$, $a \leq_1 y \vee v$, with $x \vee u$ and $y \vee v$ being in $\mathfrak{A}_0$. It follows that

$$a \leq_1 (x \vee u) \wedge (y \vee v) = (x \wedge y) \vee (x \wedge v) \vee (u \wedge y) \vee (u \wedge v) \in \Delta_2.$$

And since $(x \vee u) \wedge (y \vee v)$ is in $\mathfrak{A}_0$, it belongs to X, contrary to $X \cap \Delta_2 = \emptyset$.

By Proposition 7.27, $\nabla_2 = A_2 - \Delta_2$ is a prime filter in $\mathfrak{A}_2$. Put $\nabla_0 = \nabla_2 \cap A_0$ and $\Delta_0 = \Delta_2 \cap A_0$. By the definition, we have $X \subseteq \nabla_0$, $Y \subseteq \Delta_0$ and $\nabla_0 \cap \Delta_0 = \emptyset$. Now we extend the set $\{a\} \cup \nabla_0$ to obtain the filter ∇_1 we need. Consider the family

$$\mathcal{F}_1 = \{\nabla \subseteq A_1 : \ \nabla = [\nabla)_1, \ \{a\} \cup \nabla_0 \subseteq \nabla, \ \nabla \cap \Delta_0 = \emptyset\}.$$

To prove that it is not empty, it suffices to show that $[\{a\} \cup \nabla_0)_1 \in \mathcal{F}_1$, which in turn follows from $[\{a\} \cup \nabla_0)_1 \cap \Delta_0 = \emptyset$. So suppose that $x \in [\{a\} \cup \nabla_0)_1 \cap \Delta_0$. Then for some $z \in \nabla_0$, we have $a \wedge z \leq_1 x$, i.e., $a \leq_1 z \to x$ and $x \in \Delta_0$. By the definition of X, $z \to x \in X \subseteq \nabla_0$, which together with $z \in \nabla_0$ yields $x \in \nabla_0$. Therefore, $x \in \nabla_0 \cap \Delta_0$, which is a contradiction.

Thus, $\mathcal{F}_1$ is not empty. The union of any chain of $\mathcal{F}_1$'s elements also belongs to $\mathcal{F}_1$. So by Zorn's lemma, $\mathcal{F}_1$ contains a maximal (with respect to $\subseteq$) element. Denote it by ∇_1 and show that the filter ∇_1 is prime. Suppose $x \vee y \in \nabla_1$ but $x \notin \nabla_1$ and $y \notin \nabla_1$. Then

$$[\{x\} \cup \nabla_1)_1 \cap \Delta_0 \neq \emptyset \neq [\{y\} \cup \nabla_1)_1 \cap \Delta_0,$$

i.e., ∇_1 contains u_x and u_y such that, for some $v_x, v_y \in \Delta_0$, we have

$$x \wedge u_x \leq_1 v_x, \ \ y \wedge u_y \leq_1 v_y.$$

It follows that

$$(x \wedge u_x) \vee (y \wedge u_y) \leq_1 v_x \vee v_y \in \Delta_0.$$

The left part of this inequality can be transformed in the following way:

$$(x \wedge u_x) \vee (y \wedge u_y) = (x \vee y) \wedge (x \vee u_y) \wedge (u_x \vee y) \wedge (u_x \vee u_y).$$

Here every conjunct belongs to ∇_1 and so the whole conjunction is in ∇_1, from which $v_x \vee v_y \in \nabla_1$. Thus, we have obtained that $v_x \vee v_y \in \nabla_1 \cap \Delta_0$, contrary to $\nabla_1 \cap \Delta_0 = \emptyset$.

Observe now that, by the definition, $a \in \nabla_1$ and $b \notin \nabla_2$. So it remains to check that $\nabla_1 \cap A_0 = \nabla_2 \cap A_0$. Suppose $x \in \nabla_1 \cap A_0$. Then $x \notin \Delta_0$, whence $x \notin \Delta_2$ and so $x \in \nabla_2 \cap A_0$. Conversely, if $x \in \nabla_2 \cap A_0$ then $x \in \nabla_0$ and so $x \in \nabla_1$, because $\nabla_0 \subseteq \nabla_1$, from which $x \in \nabla_1 \cap A_0$. ❑

We are in a position now to prove the part ($\Leftarrow$) in Theorem 14.11. Suppose $\varphi(p_1,\dots,p_m,q_1,\dots,q_n)$ and $\psi(q_1,\dots,q_n,r_1,\dots,r_l)$ are formulas for which there is no formula $\chi(q_1,\dots,q_n)$ such that $\varphi \to \chi \in L$ and $\chi \to \psi \in L$ or, which is the same, $\mathfrak{A} \models \varphi \to \chi$ and $\mathfrak{A} \models \chi \to \psi$ for any $\mathfrak{A} \in \mathrm{Var}L$. We show that in this case there exists an algebra $\mathfrak{A} \in \mathrm{Var}L$ refuting $\varphi \to \psi$.

Let $\mathfrak{A}'_0$, $\mathfrak{A}'_1$ and $\mathfrak{A}'_2$ be the free algebras in $\mathrm{Var}L$ generated by the sets $\{c_1,\dots,c_n\}$, $\{a_1,\dots,a_m,c_1,\dots,c_n\}$ and $\{c_1,\dots,c_n,b_1,\dots,b_l\}$, respectively. According to this definition, $\mathfrak{A}'_0$ is a subalgebra of both $\mathfrak{A}'_1$ and $\mathfrak{A}'_2$. By Lemma 14.12, there are prime filters ∇_1 in $\mathfrak{A}'_1$ and ∇_2 in $\mathfrak{A}'_2$ such that $\varphi(a_1,\dots,a_m,c_1,\dots,c_n) \in \nabla_1$ and $\psi(c_1,\dots,c_n,b_1,\dots,b_l) \notin \nabla_2$. Put $\mathfrak{A}_1 = \mathfrak{A}'_1/\nabla_1$, $\mathfrak{A}_2 = \mathfrak{A}'_2/\nabla_2$. Then

$$\|\varphi(a_1,\dots,a_m,c_1,\dots,c_n)\|_{\nabla_1} = \top,\ \ \|\psi(c_1,\dots,c_n,b_1,\dots,b_l)\|_{\nabla_2} \neq \top.$$

Construct an algebra $\mathfrak{A}_0$ by taking $A_0 = \{\|a\|_{\nabla_1} : a \in A'_0\}$. By the definition, $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$, i.e., is embedded in $\mathfrak{A}_1$ by the map $f_1(x) = x$. We show that $\mathfrak{A}_0$ is embedded in $\mathfrak{A}_2$ by the map $f_2(\|x\|_{\nabla_1}) = \|x\|_{\nabla_2}$.

For every $\odot \in \{\wedge, \vee, \to\}$ and every $\|a\|_{\nabla_1}, \|b\|_{\nabla_1} \in A_0$ we have

$$\begin{aligned} f_2(\|a\|_{\nabla_1} \odot \|b\|_{\nabla_1}) = f_2(\|a \odot b\|_{\nabla_1}) = \|a \odot b\|_{\nabla_2} = \\ \|a\|_{\nabla_2} \odot \|b\|_{\nabla_2} = f_2(\|a\|_{\nabla_1}) \odot f_2(\|b\|_{\nabla_1}). \end{aligned}$$

Besides, $f_2(\|\bot\|_{\nabla_1}) = \|\bot\|_{\nabla_2} = \bot \in A_2$. Thus, f_2 is a homomorphism. Let us show that it is injective:

$$\begin{aligned} \|a\|_{\nabla_1} = \|b\|_{\nabla_1} \text{ iff } & a \leftrightarrow b \in \nabla_1 \\ \text{iff } & a \leftrightarrow b \in \nabla_2 \text{ (since } \nabla_1 \cap A'_0 = \nabla_2 \cap A'_0) \\ \text{iff } & \|a\|_{\nabla_2} = \|b\|_{\nabla_2}, \text{ i.e., } f_2(\|a\|_{\nabla_1}) = f_2(\|b\|_{\nabla_1}). \end{aligned}$$

Since $\mathrm{Var}L$ is amalgamable, there are an algebra $\mathfrak{A}$ for L and isomorphisms g_1 and g_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ into $\mathfrak{A}$, respectively, such that $g_1(f_1(x)) = g_2(f_2(x))$, for every $x \in A_0$. Define a valuation $\mathfrak{V}$ in $\mathfrak{A}$ by taking

$$\begin{aligned} \mathfrak{V}(p_i) &= g_1(\|a_i\|_{\nabla_1}), && \text{for } i = 1,\dots,m, \\ \mathfrak{V}(q_j) &= g_1(\|c_j\|_{\nabla_1}) = g_2(\|c_j\|_{\nabla_2}), && \text{for } j = 1,\dots,n, \\ \mathfrak{V}(r_k) &= g_2(\|b_k\|_{\nabla_2}), && \text{for } k = 1,\dots,l. \end{aligned}$$

Then

$$\mathfrak{V}(\varphi(p_1,\dots,p_m,q_1,\dots,q_n)) = \top,\ \ \mathfrak{V}(\psi(q_1,\dots,q_n,r_1,\dots,r_l)) \neq \top,$$

from which $\mathfrak{A} \not\models \varphi \to \psi$ and so $\varphi \to \psi \notin L$. ❑

It is worth noting that the property of amalgamability can be strengthened without violating Theorem 14.11. Say that a class $\mathcal{C}$ of algebras is *superamalgamable* if the condition of amalgamability is satisfied in $\mathcal{C}$ for every $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, f_1, f_2, and if $x \in A_i$, $y \in A_j$, $\{i,j\} = \{1,2\}$, then

$$g_i(x) \le g_j(y) \text{ implies } \exists z \in A_0 \ (x \le_i f_i(z) \text{ and } f_j(z) \le_j y).$$

Let us supplement the proof of ($\Rightarrow$) in Theorem 14.11 to establish that Var$L$ is superamalgamable. Suppose $a \in A_i$, $b \in A_j$, $\{i,j\} = \{1,2\}$, and $g_i(a) \le g_j(b)$. Then $g_i(a) \to g_j(b) = \top$ and so $\|p^i_a \to p^j_b\| = \top$, i.e., $p^i_a \to p^j_b \in \Sigma$. By (14.1), we have $\chi \in \mathbf{For}\mathcal{L}_0$ with $\mathfrak{V}(\chi) = c$ such that $a \le_i c = f_i(c)$ and $c = f_j(c) \le_j b$. Thus, we obtain

Theorem 14.13 *A si-logic L has the interpolation property iff* Var$L$ *is superamalgamable.*

Observe also that in the proof of ($\Leftarrow$) in Theorem 14.11 we actually used the amalgamability of the class of L's algebras satisfying the condition

$$x \vee y = \top \text{ implies } x = \top \text{ or } y = \top.$$

Such algebras are called *well-connected.* (The condition was ensured by the fact that the filters ∇_1 and ∇_2 were prime.) Thus, we have another variant of the criterion for the interpolation property.

Theorem 14.14 *A si-logic L has the interpolation property iff its class of well-connected pseudo-Boolean algebras is amalgamable.*

Let us now turn to modal logics. The situation here is a bit more complicated. First of all, we have

Theorem 14.15 *A normal modal logic L has the interpolation property iff* Var$L$ *is superamalgamable.*

Proof Similar to the proofs of Theorems 14.11 and 14.13. ❑

However, the amalgamability corresponds to a different variant of the interpolation property. Say that a normal modal logic L has the *interpolation property for derivability* if, for every formulas φ and ψ such that $\varphi \vdash^*_L \psi$, there is a formula χ containing only common variables in φ and ψ and such that $\varphi \vdash^*_L \chi$ and $\chi \vdash^*_L \psi$.

Theorem 14.16 *A normal modal logic L has the interpolation property for derivability iff* Var$L$ *is amalgamable.*

Proof Similar to the proof of Theorem 14.11. ❑

In Section 14.4 we shall see examples of logics which have the interpolation property for derivability but do not have the Craig interpolation property.

14.3 Interpolation in logics above LC and S4.3

In this section we give a complete description of "linear" modal and si-logics with the interpolation property and in the next one extend it to the whole classes ExtInt and NExtS4. First we consider si-logics with linear frames.

Theorem 14.17 *The logic* $\mathbf{LC} = \mathbf{Int} + (p \to q) \vee (q \to p)$ *has the interpolation property.*

Proof According to Theorem 14.14, it suffices to show that the class of well-connected algebras in VarLC is amalgamable. This class coincides with the class of all linearly ordered pseudo-Boolean algebras. Indeed, if $(a \to b) \vee (b \to a) = \top$ in a well-connected algebra then $a \to b = \top$ or $b \to a = \top$, i.e., either $a \leq b$ or $b \leq a$.

Let $\mathfrak{A}_0$ be a subalgebra of linear algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$. If one of these algebras is degenerate then the rest are also degenerate and the condition of amalgamability is trivially satisfied. So suppose these algebras are non-degenerate.

We construct $\mathfrak{A}$ in the following way. As its universe A we take $A_1 \cup A_2$. Since the operations in a pseudo-Boolean algebra are completely determined by the partial order $\leq$ in it, it suffices to define $\leq$ in $\mathfrak{A}$ so that $\langle A, \leq \rangle$ be a linear order with greatest and least elements and $\langle A_1, \leq_1 \rangle$, $\langle A_2, \leq_2 \rangle$ could be embedded in $\langle A, \leq \rangle$ preserving $\top$ and $\bot$. For $x, y \in A$, put

$$\begin{aligned} x \leq' y \text{ iff } & (x, y \in A_1 \wedge x \leq_1 y) \vee (x, y \in A_2 \wedge x \leq_2 y) \vee \\ & (x \in A_1 \wedge y \in A_2 \wedge \exists z \in A_1 \cap A_2\ (x \leq_1 z \wedge z \leq_2 y)) \vee \\ & (x \in A_2 \wedge y \in A_1 \wedge \exists z \in A_1 \cap A_2\ (x \leq_2 z \wedge z \leq_1 y)). \end{aligned}$$

It is easily checked that $\langle A, \leq' \rangle$ is a partial order with the greatest and least elements $\top$ and $\bot$ and, for every $x, y \in A_i$, $i = 1, 2$, we have $x \leq' y$ iff $x \leq_i y$. Now we supplement $\leq'$ to a linear order $\leq$. This can be done, for instance, like this: take any well-ordering of $A \times A$ and, starting with $\leq'$, add to it by transfinite induction the next pair $\langle x, y \rangle$ from $A \times A$ if x and y are still incomparable after the preceding step and then form the transitive closure of the new relation. The resulting linear order will clearly satisfy the properties we need. ❑

Denote by $\mathbf{LC}_n$ the logic of the n-point linear frame (= the logic of the $n+1$-element linear pseudo-Boolean algebra).

Theorem 14.18 $\mathbf{LC}_2$ *has the interpolation property.*

Proof There are only two well-connected non-degenerate $\mathbf{LC}_2$-algebras, namely 2- and 3-element chains. A simple direct check of all possible cases (see, for example, Fig. 14.1 in which dash arrows show embeddings) establishes that this class of algebras is amalgamable. ❑

Thus, we have four "linear" si-logics (including the inconsistent one) having the interpolation property. And it turns out that that is all.

Theorem 14.19 *Any logic* $L \in \mathrm{Ext}\mathbf{LC}$ *different from* $\mathbf{LC}$, $\mathbf{LC}_2$, $\mathbf{Cl}$ *and* $\mathbf{For}\mathcal{L}$ *does not have the interpolation property.*

Proof Suppose L has the interpolation property and differs from $\mathbf{LC}_2$, $\mathbf{Cl}$ and $\mathbf{For}\mathcal{L}$. We show then that $L = \mathbf{LC}$.

It follows from our assumption that there is a well-connected algebra for L containing at least 4 elements and so there is a 4-element algebra (e.g. a suitable subalgebra of the former one). We are going to prove by induction on n that

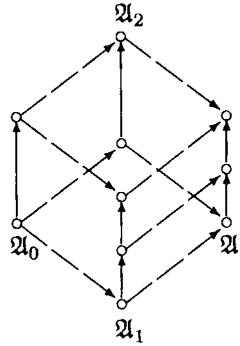


FIG. 14.1.

Var$L$ contains n-element well-connected algebras for every $n < \omega$. This is true for $n = 1, 2, 3, 4$.

Suppose Var$L$ contains an n-element linear algebra (and so m-element ones as well, for $1 \le m < n$) and show that an $n+1$-element linear algebra belongs to Var$L$ too. Let $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ be the pseudo-Boolean algebras defined by the following linear orderings of their elements:

- $\mathfrak{A}_0$ is $\bot < a < \top$,
- $\mathfrak{A}_1$ is $\bot < a < b < \top$,
- $\mathfrak{A}_2$ is $\bot < c_1 < \ldots < c_{n-3} < a < \top$.

By the definition, $\mathfrak{A}_0$ is a subalgebra of both $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Since L has the interpolation property, there must be a well-connected algebra $\mathfrak{A}$ for L containing $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as its subalgebras. This means that $\mathfrak{A}$ contains an $n+1$-element subalgebra determined by the order

$$\bot < c_1 < \ldots < c_{n-3} < a < b < \top.$$

Thus, the class of all finite linear algebras, characterizing **LC**, is contained in Var$L$. It follows that $L \subseteq \mathbf{LC}$ and so $L = \mathbf{LC}$. ❑

Let us consider now extensions of **S4.3**. By Theorem 14.9 and the results of this section, of all logics in NExt**S4.3** only modal companions of **LC**, **LC**$_2$, **Cl** and **For**$\mathcal{L}$ may have the interpolation property.

Theorem 14.20 *No logic in* $\rho^{-1}\mathbf{LC}$ *has the interpolation property.*

Proof We show that there is a formula $\alpha \to \beta$ which belongs to all logics in $\rho^{-1}\mathbf{LC} = [\mathbf{S4.3}, \mathbf{Grz.3}]$ but does not have an interpolant in any of them. Let

$$\alpha(p,q,r) = \Box((p \to \Box r) \wedge (\Box(q \to \Box r) \to \Box r) \wedge (\Box r \to p \vee q)),$$

$$\beta(p,q,r') = \Box((q \to \Box r') \wedge (\Box(p \to \Box r') \to \Box r') \wedge (\Box r' \to p \vee q)) \to p \vee q.$$

It is not difficult to verify that $\alpha \to \beta$ is valid in every finite frame for **S4.3** and so belongs to **S4.3**. It remains to show that there is no formula $\gamma(p,q)$ such that $\alpha \to \gamma \in$ **Grz.3** and $\gamma \to \beta \in$ **Grz.3**. Let $\mathfrak{F} = \langle W, R\rangle$ be the frame depicted in Fig. 8.3 (a) and $\mathfrak{F}' = \langle W', R'\rangle$ its subframe obtained by removing the point ω. Clearly, both $\mathfrak{F}$ and $\mathfrak{F}'$ are frames for **Grz.3**. Put $\mathfrak{A}_0 = \mathfrak{F}'^+$, $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{F}^+$ and define embeddings f_i of $\mathfrak{A}_0$ in $\mathfrak{A}_i$, for $i = 1, 2$, as follows.

Let ∇_1, ∇_2 be non-principal ultrafilters in $\mathfrak{A}_0$ such that $a = \{2n : n < \omega\} \in \nabla_1 - \nabla_2$. To show that such ultrafilters exist, consider the filter ∇ of cofinite sets in $\mathfrak{A}_0$. The filters $[\nabla \cup \{a\})$ and $[\nabla \cup -\{a\})$ are then non-degenerate. (For otherwise, if say $\emptyset \in [\nabla \cup \{a\})$, we would have $b \cap a = \emptyset$ for some cofinite set b in $\mathfrak{A}_0$, which is impossible.) And we can take as ∇_1 and ∇_2 any ultrafilters containing $[\nabla \cup \{a\})$ and $[\nabla \cup -\{a\})$, respectively, which clearly are non-principal and satisfy the property we need.

Define f_i by taking, for any x in $\mathfrak{A}_0$,

$$f_i(x) = \begin{cases} x \cup \{\omega\} & \text{if } x \in \nabla_i \\ x & \text{otherwise} \end{cases}$$

and show that it is an embedding of $\mathfrak{A}_0$ in $\mathfrak{A}_i$. Clearly, f_i is an injection. So it suffices to prove that it preserves $\cap$, $-$ and $\Box$.

Consider $f_i(x \cap y)$, for $x, y \subseteq W'$. If $x \cap y \in \nabla_i$ then $x \in \nabla_i$ and $y \in \nabla_i$, i.e., $f_i(x) = x \cup \{\omega\}$, $f_i(y) = y \cup \{\omega\}$ and so

$$f_i(x \cap y) = (x \cap y) \cup \{\omega\} = (x \cup \{\omega\}) \cap (y \cup \{\omega\}) = f_i(x) \cap f_i(y).$$

And if $x \cap y \notin \nabla_i$ then $x \notin \nabla_i$ or $y \notin \nabla_i$, i.e., either $\omega \notin f_i(x) = x$ or $\omega \notin f_i(y) = y$, and so $f_i(x \cap y) = x \cap y = f_i(x) \cap f_i(y)$.

Now take $f_i(W' - x)$, $x \subseteq W'$. If $W' - x \in \nabla_i$ then $x \notin \nabla_i$, i.e., $\omega \notin f_i(x) = x$. Then $f_i(W' - x) = (W' - x) \cup \{\omega\} = W - x = W - f_i(x)$. And if $W' - x \notin \nabla_i$ then $x \in \nabla_i$, i.e., $f_i(x) = x \cup \{\omega\}$, and so

$$f_i(W' - x) = W' - x = W - (x \cup \{\omega\}) = W - f_i(x).$$

Finally, consider $f_i(\Box_0 x)$, for $x \subseteq W'$ (here the subscript near $\Box$ indicates in which algebra this $\Box$ operates). There are three types of elements of the form $\Box_0 x$ in $\mathfrak{A}_0$: $\{\omega + 1, m : m < \omega\}$, $\{m : m < \omega\}$, and $\{0, 1, \dots, n\}$ $(n < \omega)$. The former two sets x are cofinite and besides $x = \Box_0 x$. In this case we have

$$f_i(\Box_0 x) = \Box_0 x \cup \{\omega\} = \Box_1(x \cup \{\omega\}) = \Box_1 f_i(x).$$

If $\Box_0 x = \{0, \dots, n\}$ then $n + 1 \notin x$. Therefore, $\Box_1 x = \Box_1(x \cup \{\omega\}) = \{0, \dots, n\}$ and so $f_i(\Box_0 x) = \Box_1 f_i(x)$ no matter whether x is in ∇_i.

Define valuations $\mathfrak{V}_i$ in the algebras $\mathfrak{A}_i$, for $i = 0, 1, 2$, by taking

$$\mathfrak{V}_0(p) = \{2n + 1 : n < \omega\},\ \mathfrak{V}_0(q) = \{2n : n < \omega\} (= a),$$

$$\mathfrak{V}_1(p) = \{2n+1 : n < \omega\},\ \mathfrak{V}_2(p) = \{2n+1 : n < \omega\} \cup \{\omega\},$$
$$\mathfrak{V}_1(q) = \{2n : n < \omega\} \cup \{\omega\},\ \mathfrak{V}_2(q) = \{2n : n < \omega\},$$
$$\mathfrak{V}_1(r) = \{n : n < \omega\},\ \mathfrak{V}_2(s) = \{n : n < \omega\}.$$

Notice that

$$f_1(\mathfrak{V}_0(p)) = \mathfrak{V}_1(p),\ f_1(\mathfrak{V}_0(q)) = \mathfrak{V}_1(q),$$
$$f_2(\mathfrak{V}_0(p)) = \mathfrak{V}_2(p),\ f_2(\mathfrak{V}_0(q)) = \mathfrak{V}_2(q)$$

and so, for any formula $\varphi(p,q)$, we have

$$f_1(\mathfrak{V}_0(\varphi(p,q))) = \mathfrak{V}_1(\varphi(p,q)),\ f_2(\mathfrak{V}_0(\varphi(p,q))) = \mathfrak{V}_2(\varphi(p,q)).$$

Suppose now that $\gamma(p,q)$ is an interpolant for $\alpha \to \beta$ in **Grz.3**, i.e., $\alpha \to \gamma \in$ **Grz.3** and $\gamma \to \beta \in$ **Grz.3**. Then both formulas must be valid in the algebras under consideration, in particular,

$$\begin{aligned}
&\mathfrak{V}_1(\alpha) = \\
&\Box_1((\mathfrak{V}_1(p) \supset_1 \Box_1\mathfrak{V}_1(r)) \cap (\Box_1(\mathfrak{V}_1(q) \supset_1 \Box_1\mathfrak{V}_1(r)) \supset \Box_1\mathfrak{V}_1(r))) \cap \\
&(\Box_1\mathfrak{V}_1(r) \supset_1 \mathfrak{V}_1(p) \cup \mathfrak{V}_1(q)) = \\
&\Box_1(W \cap (\Box_1(W - \{\omega\}) \supset_1 \Box_1\mathfrak{V}_1(r)) \cap W) = \\
&\Box_1((W - \{\omega, \omega+1\}) \supset (W - \{\omega, \omega+1\})) = W \subseteq \mathfrak{V}_1(\gamma),
\end{aligned}$$

i.e., $\mathfrak{V}_1(\gamma) = W$, and

$$\begin{aligned}
&\mathfrak{V}_2(\gamma) \subseteq \mathfrak{V}_2(\beta) = \\
&\mathfrak{V}_2(\Box((q \to \Box s) \land (\Box(p \to \Box s) \to \Box s) \land (\Box s \to p \lor q))) \supset_2 \mathfrak{V}_2(p \lor q) = \\
&W \supset_2 \mathfrak{V}_2(p \lor q) = \mathfrak{V}_2(p \lor q) = W - \{\omega+1\} \neq W,
\end{aligned}$$

i.e., $\mathfrak{V}_2(\gamma) \neq W$. Since f_i is an embedding, we have $\mathfrak{V}_0(\gamma) = f_1^{-1}(\mathfrak{V}_1(\gamma)) = f_1^{-1}(W) = W'$ and so $\mathfrak{V}_2(\gamma) = f_2(\mathfrak{V}_0(\gamma)) = f_2(W') = W$, contrary to $\mathfrak{V}_2(\gamma) \neq W$. ❑

Remark Note that the formula α used in the proof above was "boxed". This means that in fact we have established a stronger result: no logic in ρ^{-1}**LC** has the interpolation property for derivability.

Let us consider now the set $\rho^{-1}\mathbf{Cl} = \{\mathrm{Log}\mathfrak{Cl}_n : 1 \leq n \leq \omega\}$ and $\rho^{-1}\mathbf{LC}_2 = \{\mathrm{Log}\mathfrak{Cl}_m^n : 1 \leq n, m \leq \omega\}$, where $\mathfrak{Cl}_n$ is the n-point cluster and $\mathfrak{Cl}_m^n$ is the chain of two clusters, the first with m and the last with n points. We remind the reader that $\mathrm{Log}\mathfrak{Cl}_\omega = \mathbf{S5}$, $\mathrm{Log}\mathfrak{Cl}_1 = \mathbf{Triv}$.

Theorem 14.21 *In $\rho^{-1}\mathbf{Cl}$ only* **S5**, $\mathrm{Log}\mathfrak{Cl}_2$ *and* **Triv** *have the interpolation property. In $\rho^{-1}\mathbf{LC}_2$ only* $\mathrm{Log}\mathfrak{Cl}_n^1$ *and* $\mathrm{Log}\mathfrak{Cl}_1^n$, *for $n = 1, 2, \omega$, have the interpolation property.*

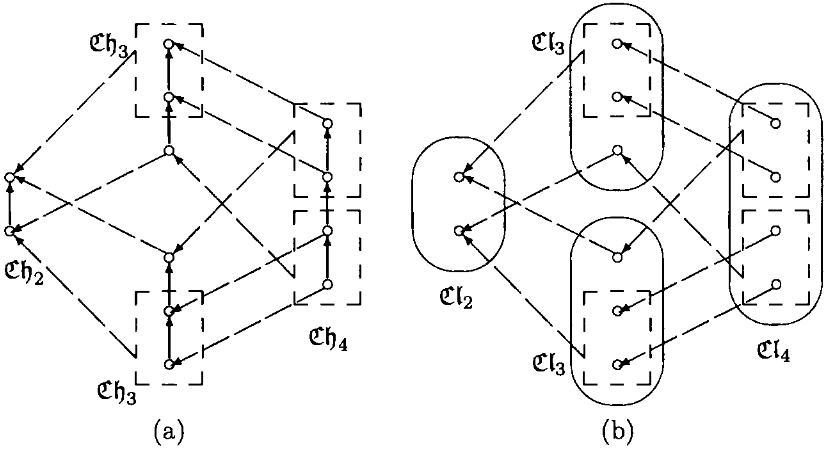


FIG. 14.2.

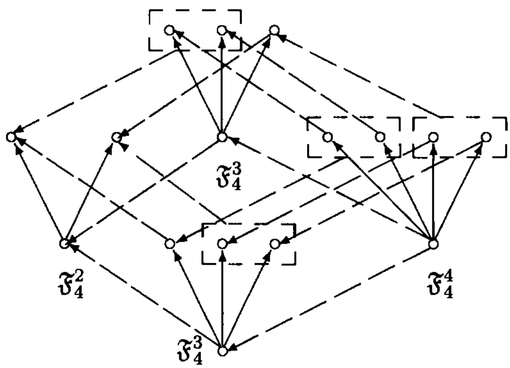


FIG. 14.3.

Proof The proof involves neither new ideas nor methods as compared with the proofs of Theorems 14.17–14.19. For example, in the same way as in the proof of Theorem 14.19 we constructed n-point linear frames with the help of the 3-point one and the amalgamation property (for $n = 4$ the construction is shown in Fig. 14.2 (a), where $\mathfrak{Ch}_n$ is the linear frame with n points and dash arrows indicate reductions), starting with the 3-point cluster and using the (super)amalgamability we can "grow" arbitrary finite clusters (see Fig. 14.2 (b)). ❑

14.4 Interpolation in ExtInt and NExtS4

Now we describe all si-logics and normal extensions of **S4** with the interpolation property. Since the proofs are of the same sort as those of Theorems 14.17–14.19 (though technically somewhat more involved), we confine ourselves here only to

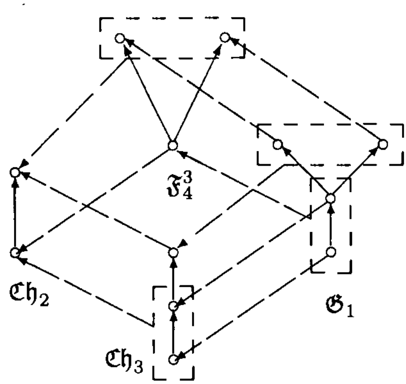
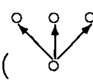
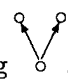


FIG. 14.4.

a rough sketch, hoping that the interested reader will be able to complete the details.

Theorem 14.22 *A si-logic has the interpolation property iff it is one of*

$$\mathbf{Int},\ \mathbf{KC},\ \mathbf{Int} + \boldsymbol{bd}_2,\ \mathbf{Int} + \boldsymbol{bd}_2 + \beta($$  $$) = \mathrm{Log}$$  ,

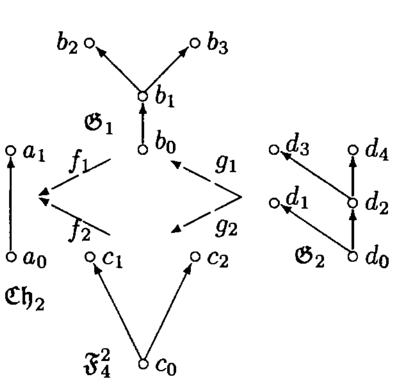
$$\mathbf{LC},\ \mathbf{LC}_2,\ \mathbf{Cl},\ \mathbf{For}\mathcal{L}.$$

Proof The fact that all these logics have the interpolation property was partly already proved (see Corollary 14.10 and Theorems 14.17, 14.18). The interpolation property of $\mathbf{Int} + \boldsymbol{bd}_2 = \mathrm{Log}\mathfrak{F}_4^\omega$ and $\mathrm{Log}\mathfrak{F}_4^2$ (see Fig. 12.2) is established in the same way as Theorems 14.17 and 14.18. (Here and below we use the notations introduced in Section 12.3.)

Suppose now that L is a si-logic with the interpolation property different from the eight logics listed above. By Theorem 14.19, $L \notin \mathrm{Ext}\mathbf{LC}$ and so at least one of the frames $\mathfrak{F}_4^2$ or $\mathfrak{F}_5^2$ in Fig. 12.2 validates L.

Suppose $\mathfrak{F}_4^2 \models L$ and consider two cases: $\mathfrak{Ch}_3 \not\models L$ and $\mathfrak{Ch}_3 \models L$. In the former case L is of depth 2 and, since $L \notin \{\mathbf{Cl}, \mathrm{Log}\mathfrak{F}_4^2, \mathbf{For}\mathcal{L}\}$, $\mathfrak{F}_4^3$ must be a frame for L. By the standard amalgamation argument we can prove then that $\mathfrak{F}_4^n \models L$ for every $n < \omega$. For $n = 4$ the construction is shown in Fig. 14.3 (of course, we do not obtain $\mathfrak{F}_4^4$ immediately; first we get a (general) frame $\mathfrak{F} \models L$ reducible to both copies of $\mathfrak{F}_4^3$ in the proper way and then show that $\mathfrak{F}$ is reducible to $\mathfrak{F}_4^4$). But then $L = \mathrm{Log}\mathfrak{F}_4^\omega = L + \boldsymbol{bd}_2$, which is a contradiction.

Thus $\mathfrak{Ch}_3 \models L$ holds. Notice that starting with $\mathfrak{F}_4^2$ and $\mathfrak{Ch}_3$ and using the amalgamation property we can construct $\mathfrak{F}_4^3$ and so $\mathfrak{F}_4^n$, for every $n < \omega$. Indeed, first we obtain the frame $\mathfrak{G}_1$ as is shown in Fig. 14.4, then $\mathfrak{G}_2$ as in Fig. 14.5 and finally $\mathfrak{G}_3$ as in Fig. 14.6, which is clearly reducible to $\mathfrak{F}_4^3$. And using $\mathfrak{Ch}_3$ and $\mathfrak{F}_4^n$, $n < \omega$, we can construct any finite tree (which can easily be shown by induction



$$
\begin{aligned}
f_1(b_0) &= a_0\\
f_1(\{b_1, b_2, b_3\}) &= a_1\\
f_2(c_0) &= a_0\\
f_2(\{c_1, c_2\}) &= a_1\\
g_1(d_0) &= b_0\\
g_1(\{d_1, d_3\}) &= b_2\\
g_1(d_2) &= b_1\\
g_1(d_4) &= b_3\\
g_2(d_0) &= c_0\\
g_2(d_1) &= c_1\\
g_2(\{d_2, d_3, d_4\}) &= c_2
\end{aligned}
$$

FIG. 14.5.

on the number of points in trees). But the class of finite trees characterizes **Int**, i.e., $L = \mathbf{Int}$, which is again a contradiction.

It follows that $\mathfrak{F}_5^2 \models L$ and so $\mathfrak{Ch}_3 \models L$, because $\mathfrak{Ch}_3$ is a reduct of $\mathfrak{F}_5^2$. Therefore, the class of frames for L contains all finite chains $\mathfrak{Ch}_n$, $n < \omega$. From $\mathfrak{Ch}_3$ and $\mathfrak{F}_5^2$ by the amalgamation property we can construct $\mathfrak{F}_5^3$ (see Fig. 14.7). Now observe that $\mathfrak{Ch}_4$ and $\mathfrak{F}_5^3$ are obtained from $\mathfrak{Ch}_3$ and $\mathfrak{F}_4^3$ by adding to them last points. And if we add last points to the frames in Fig. 14.3–14.6 and connect them by reduction arrows then we can construct arbitrary finite trees with an added top point. This means that $L \subseteq \mathbf{KC}$. But since $\mathfrak{F}_4^2 \not\models L$, we must have $L = \mathbf{KC}$, contrary to our choice of L.

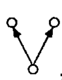
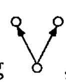
We have considered all possible cases and everywhere arrived at a contradiction. Therefore, such a logic L does not exist. ❑

Let us turn now to NExtS4. For a si-logic L and $n, m \leq \omega$, we denote by $M(L, m, n)$ the modal logic above **S4** characterized by the class of frames $\mathfrak{F}$ such that $\rho\mathfrak{F}$ is a finite frame for L, final clusters in $\mathfrak{F}$ contain at most m points and the remaining (non-final) clusters at most n points. Although the following two theorems, presented here without proofs, do not give an exhaustive description of logics in NExtS4 with the interpolation property (for derivability), they provide us with finite lists of logics containing all of them.

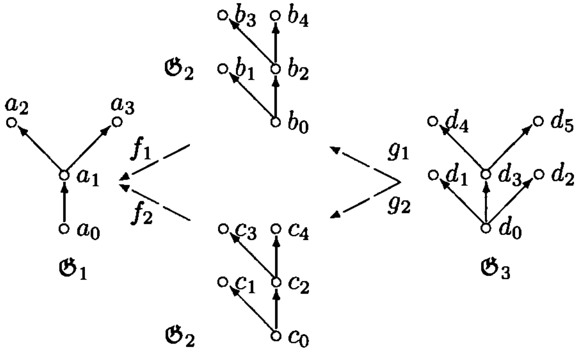
Theorem 14.23 (i) *The following logics have the interpolation property:*

$$M(\mathbf{Int}, n, \omega),\ M(\mathbf{KC}, n, \omega),$$

$$M(\mathbf{Int} + \boldsymbol{bd}_2, n, 1),\ M(\mathbf{Int} + \boldsymbol{bd}_2, 1, n),$$

$M(\text{Log}$  $, n, 1)$, $M(\text{Log}$  $, 1, n)$,

$M(\mathbf{LC}_2, n, 1)$, $M(\mathbf{LC}_2, 1, n)$, *for* $n = 1, 2, \omega$,



$g_1(d_0) = b_0$
$g_1(d_1) = b_1$
$g_1(\{d_2, d_5\}) = b_4$
$g_1(d_3) = b_2$
$g_1(d_4) = b_3$
$g_2(d_0) = c_0$
$g_2(d_1) = c_1$
$g_2(\{d_2, d_5\}) = c_3$
$g_2(d_3) = c_2$
$g_2(d_4) = c_4$

$f_1(b_0) = a_0,\ f_1(\{b_1, b_3\}) = a_2,\ f_1(b_2) = a_1,\ f_1(b_4) = a_3$
$f_2(c_0) = a_0,\ f_2(\{c_1, c_4\}) = a_3,\ f_2(c_2) = a_1,\ f_2(c_3) = a_2$

FIG. 14.6.

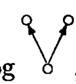
$$\mathbf{S5},\ \mathbf{Log}\mathfrak{Cl}_2,\ \mathbf{Triv},\ \mathbf{Grz},\ \mathbf{Grz.2},\ \mathbf{For}\mathcal{ML}.$$

(ii) *Each normal logic above* **S4** *having the interpolation property and different from the logics mentioned in* (i) *is contained in the following list:*

$$M(\mathbf{Int}, 1, 2),\ M(\mathbf{Int}, 2, 1),\ M(\mathbf{Int}, 2, 2),\ M(\mathbf{Int}, \omega, 1),\ M(\mathbf{Int}, \omega, 2),$$

$$M(\mathbf{KC}, 1, 2),\ M(\mathbf{KC}, 2, 1),\ M(\mathbf{KC}, 2, 2),\ M(\mathbf{KC}, \omega, 1),\ M(\mathbf{KC}, \omega, 2).$$

Theorem 14.24 (i) *The following logics have the interpolation property for derivability, but do not have the (plain) interpolation property:*

$M(\mathbf{Int} + \boldsymbol{bd}_2, m, n)$, $M($Log  $, m, n)$, $M(\mathbf{LC}_2, m, n)$,

where $m, n \in \{2, \omega\}$.

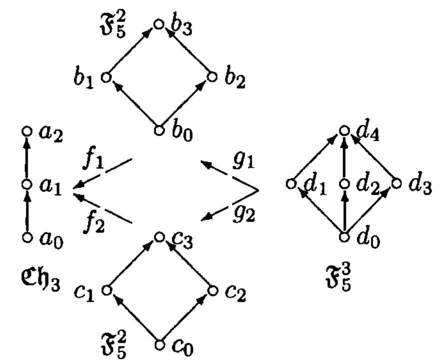
(ii) *Each normal extension of* **S4** *having the interpolation property for derivability and different from the logics mentioned in* (i) *is contained in the list of Theorem 14.23* (ii).

14.5 Interpolation in extensions of GL

Theorem 14.25 **GL** *has the interpolation property.*

Proof Suppose $\alpha \to \beta$ has no interpolant in **GL**. Our goal is to construct a finite irreflexive transitive frame refuting $\alpha \to \beta$.

Let $t = (\Gamma, \Delta)$ be a finite tableau all formulas in which are constructed from variables and their negations using the connectives $\wedge$, $\vee$, $\Box$, $\Diamond$. Without loss



$$\begin{array}{ll} & g_1(d_0) = b_0 \\ & g_1(d_1) = b_1 \\ f_1(b_0) = a_0 & g_1(\{d_3, d_4\}) = b_3 \\ f_1(\{b_1, b_3\}) = a_2 & g_1(d_2) = b_2 \\ f_1(b_2) = a_1 & g_2(d_0) = c_0 \\ f_2(c_0) = a_0 & g_2(d_2) = c_1 \\ f_2(\{c_2, c_3\}) = a_2 & g_2(\{d_1, d_4\}) = c_3 \\ f_2(c_1) = a_1 & g_2(d_3) = c_2 \end{array}$$

FIG. 14.7.

of generality we will assume α and β to be formulas of that sort. Say that t is *separable* (relative to α and β) if there is a formula γ with $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$ such that $\bigwedge \Gamma \to \gamma \in \mathbf{GL}$ and $\gamma \to \bigvee \Delta \in \mathbf{GL}$.

It should be clear that if $t = (\Gamma, \Delta)$ is a finite inseparable tableau then taking the closure of it under the saturation rules (SR1)–(SR4) (see Section 1.2) we can obtain a finite inseparable tableau satisfying (S1)–(S4). It will be denoted by $[t] = (\ulcorner\Gamma\urcorner, \llcorner\Delta\lrcorner)$.

Now we construct by induction a finite rooted model for $\mathbf{GL}$ refuting $\alpha \to \beta$. As its root we take the tableau $(\ulcorner\alpha\urcorner, \llcorner\beta\lrcorner)$. If we have already put in our model a tableau $t = (\Gamma, \Delta)$ and it has not been considered yet, then for every $\Diamond\varphi \in \Gamma$ and every $\Box\psi \in \Delta$, we add to the model the tableaux $t_1 = (\Gamma_1, \Delta_1)$ and $t_2 = (\Gamma_2, \Delta_2)$ in which

$$\Gamma_1 = \ulcorner\{\chi, \Box\chi, \Box(\neg\varphi)', \varphi : \ \Box\chi \in \Gamma\}\urcorner, \ \Delta_1 = \llcorner\{\chi, \Diamond\chi : \ \Diamond\chi \in \Delta\}\lrcorner,$$

$$\Gamma_2 = \ulcorner\{\chi, \Box\chi : \ \Box\chi \in \Gamma\}\urcorner, \ \Delta_2 = \llcorner\{\chi, \Diamond\chi, \Diamond(\neg\psi)', \psi : \ \Diamond\chi \in \Delta\}\lrcorner,$$

where $(\neg\varphi)'$ and $(\neg\psi)'$ are formulas equivalent to $\neg\varphi$ and $\neg\psi$, respectively, and containing $\neg$ only prefixed to variables (and no $\to$, of course).

Lemma 14.26 *If t is inseparable then t_1 and t_2 are also inseparable.*

Proof We consider only t_1, because t_2 is treated in the dual way. Suppose t_1 is separable, i.e., there is a formula γ containing only common variables in α and β and such that $\bigwedge \Gamma_1 \to \gamma \in \mathbf{GL}$ and $\gamma \to \bigvee \Delta_1 \in \mathbf{GL}$. Then with the help of the formulas $\Box(p \wedge q \to r) \to (\Box p \wedge \Diamond q \to \Diamond r)$ and $\Box(p \to q) \to (\Diamond p \to \Diamond q)$ belonging to any modal logic, we obtain

$$\Box \bigwedge \{\chi, \Box\chi : \ \Box\chi \in \Gamma\} \wedge \Diamond(\Box\neg\varphi \wedge \varphi) \to \Diamond\gamma \in \mathbf{GL},$$

$$\Diamond\gamma \to \Diamond \bigvee \{\chi, \Diamond\chi : \ \Diamond\chi \in \Delta\} \in \mathbf{GL}.$$

And since $\Diamond(\Box\neg\varphi \wedge \varphi)$, $\Box(\chi \wedge \Box\chi)$ and $\Diamond(\chi \vee \Diamond\chi)$ are equivalent in **GL** to the formulas $\Diamond\varphi$, $\Box\chi$ and $\Diamond\chi$, respectively, we have

$$\bigwedge\{\Box\chi : \Box\chi \in \Gamma\} \wedge \Diamond\varphi \to \Diamond\gamma \in \mathbf{GL},\ \ \Diamond\gamma \to \bigvee\{\Diamond\chi : \Diamond\chi \in \Delta\} \in \mathbf{GL},$$

whence $\bigwedge\Gamma \to \Diamond\gamma \in \mathbf{GL}$ and $\Diamond\gamma \to \bigvee\Delta \in \mathbf{GL}$, contrary to t being inseparable. ❑

Put $tR't_1$ and $tR't_2$. The process of adding new tableaux must eventually terminate, since each step reduces the number of formulas of the form $\Diamond\varphi$ and $\Box\psi$ in the left and right parts of tableaux, respectively: having appeared once such a formula vanishes at the next step and in view of $\Box(\neg\varphi)'$, $\Diamond(\neg\psi)'$ and Lemma 14.26 cannot appear again. Let W be the set of all tableaux constructed in this way and R the transitive closure of R'. Clearly, the resulting frame $\mathfrak{F} = \langle W, R\rangle$ is transitive and irreflexive and so $\mathfrak{F} \models \mathbf{GL}$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking, for each variable p,

$$\mathfrak{V}(p) = \{(\Gamma, \Delta) \in W :\ p \in \Gamma\}.$$

To show that $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$ refutes $\alpha \to \beta$, by induction on the construction of φ one can readily prove that, for every $t = (\Gamma, \Delta) \in W$, if $\varphi \in \Gamma$ then $(\mathfrak{M}, t) \models \varphi$ and if $\varphi \in \Delta$ then $(\mathfrak{M}, t) \not\models \varphi$. ❑

Unlike NExt**S4**, there are much more logics with the interpolation property in NExt**GL**. More precisely, we have the following strengthening of Corollary 14.6:

Theorem 14.27 NExt**GL** *contains a continuum of logics with the interpolation property.*

Proof By Theorems 14.25 and 14.5, it suffices to present a continuum of logics in NExt**GL** axiomatizable by conservative formulas. For $i < \omega$, we put

$$\alpha_i = \Box^+(\Diamond^{i+1}\top \wedge \Box^{i+2}\bot \to \Box^{i+1}p \vee \Box^{i+1}\neg p).$$

Lemma 14.28 *Each formula* α_i *is conservative in* NExt**GL**.

Proof We need to show that

$$\Box^+\alpha_i(\bot) \wedge \Box^+\alpha_i(p) \wedge \Box^+\alpha_i(q) \to \alpha_i(p \to q) \in \mathbf{GL}, \tag{14.2}$$

$$\Box^+\alpha_i(\bot) \wedge \Box^+\alpha_i(p) \to \alpha_i(\Box p) \in \mathbf{GL}. \tag{14.3}$$

Suppose (14.2) does not hold, which means that this formula is false at a point x in some model for **GL**, i.e.,

$$x \models \Box^+\alpha_i(\bot) \wedge \Box^+\alpha_i(p) \wedge \Box^+\alpha_i(q), \tag{14.4}$$

$$x \not\models \alpha_i(p \to q). \tag{14.5}$$

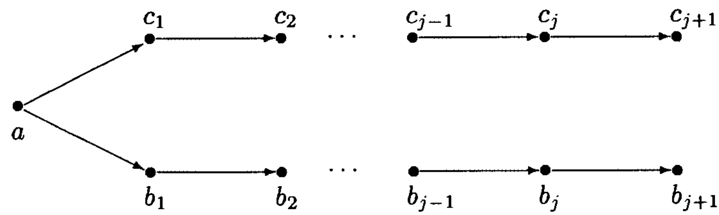


FIG. 14.8.

It follows from (14.5) that there is $y \in x\underline{\uparrow}$ such that

$$y \models \Diamond^{i+1}\top \wedge \Box^{i+2}\bot, \tag{14.6}$$

$$y \not\models \Box^{i+1}(p \to q) \vee \Box^{i+1}\neg(p \to q) \tag{14.7}$$

and so, for some $y_1, y_2 \in y\uparrow^{i+1}$, we have

$$y_1 \models p,\ y_1 \not\models q \tag{14.8}$$

and $y_2 \not\models \neg(p \to q)$, i.e., $y_2 \not\models p$ or $y_2 \models q$. If $y_2 \not\models p$ then, by (14.6) and (14.8), we must have $x \not\models \alpha_i(p)$, contrary to (14.4). And if $y_2 \models q$ then, using (14.6) and (14.8), we obtain $x \not\models \alpha_i(q)$, which is again a contradiction.

To prove (14.3) it is sufficient to notice that $\alpha_i(\Box p) \in \mathbf{GL}$. Indeed, we have $\Box^{i+2}\bot \to \Box^{i+1}\Box p \in \mathbf{GL}$ and so $\Diamond^{i+1}\top \wedge \Box^{i+2}\bot \to \Box^{i+1}\Box p \vee \Box^{i+1}\neg\Box p \in \mathbf{GL}$. ❑

For every $N \subseteq \omega$, put

$$\mathbf{GL}(N) = \mathbf{GL} \oplus \{\alpha_i :\ i \in N\}.$$

Since the model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F}$ is the frame shown in Fig. 14.8, $j \notin N$ and $\mathfrak{V}(p) = \{b_{j+1}\}$, separates α_j (refuted at a) from $\mathbf{GL}(N)$, $\mathbf{GL}(N_1) \neq \mathbf{GL}(N_2)$ whenever $N_1 \neq N_2$. Thus, we have a continuum of normal extensions of $\mathbf{GL}$ which, by Lemma 14.28, Theorems 14.5 and 14.25, have the interpolation property. ❑

On the other hand we have

Theorem 14.29 NExt**GL** *contains a continuum of logics without the interpolation property.*

Proof Let α_i, for $i < \omega$, be the formulas introduced in the proof of Theorem 14.27. For $N \subseteq \omega - \{0, 1, 2, 3, 4\}$, we put

$$\mathbf{GL}(N) = \mathbf{GL} \oplus \{\alpha_i :\ i \in N\} \oplus \beta \vee \gamma,$$

where

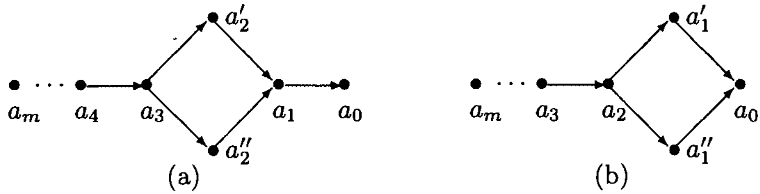


FIG. 14.9.

$$\beta = \Box^+(\Box^3\bot \to \Box(\Box^2\bot \land \Diamond\top \to p) \lor \Box(\Box^2\bot \land \Diamond\top \to \neg p)),$$

$$\gamma = \Box^+(\Box^4\bot \to \Box(\Box^3\bot \land \Diamond^2\top \to q) \lor \Box(\Box^3\bot \land \Diamond^2\top \to \neg q)).$$

Observe that the frames of the form shown in Fig. 14.8 validate both β and γ and so, for every $j \notin N$, $j > 4$, we have $\alpha_j \notin \mathbf{GL}(N)$. Therefore, $\mathbf{GL}(N_1) \neq \mathbf{GL}(N_2)$ whenever $N_1 \neq N_2$. It remains to prove that $\mathbf{GL}(N)$ does not have the interpolation property.

We show that the formula $\neg\beta \to \gamma$, which clearly is in $\mathbf{GL}(N)$ (because it is equivalent to $\beta \lor \gamma$) has no interpolant in $\mathbf{GL}(N)$. Suppose otherwise. Then there is a variable free formula δ such that

$$\neg\delta \to \beta \in \mathbf{GL}(N),\ \delta \to \gamma \in \mathbf{GL}(N).$$

According to the classification of the variable free formulas in **GL** given in Theorem 8.87, δ has one of the forms

$$\delta = \bot \lor \varphi_{i_1} \lor \ldots \lor \varphi_{i_n} \text{ or } \delta = \neg(\bot \lor \varphi_{i_1} \lor \ldots \lor \varphi_{i_n}),$$

where $\varphi_i = \Box^{i+1}\bot \land \Diamond^i\top$.

Suppose $\delta = \bot \lor \varphi_{i_1} \lor \ldots \lor \varphi_{i_n}$. Then the model $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle$, where $\mathfrak{F}_1$ is the frame in Fig 14.9 (a) with $m = \max\{i_1, \ldots, i_n\} + 32$ and $\mathfrak{V}(q) = \{a_2'\}$, separates the formula $\delta \to \gamma$ (refuted at a_m) from $\mathbf{GL}(N)$, which is a contradiction. And if $\delta = \neg(\bot \lor \varphi_{i_1} \lor \ldots \lor \varphi_{i_n})$ then the model $\mathfrak{M}_2 = \langle \mathfrak{F}_2, \mathfrak{V}_2 \rangle$, where $\mathfrak{F}_2$ is shown in Fig 14.9 (b) with $m = \max\{i_1, \ldots, i_n\} + 33$ and $\mathfrak{V}(p) = \{a_1'\}$, separates $\neg\delta \to \beta$ from $\mathbf{GL}(N)$, which is again a contradiction. ❑

Now let us consider extensions of **S**.

Theorem 14.30 **S** *has the interpolation property.*

Proof Although the axiom $\Box p \to p$ of **S** is not conservative in Ext**GL** (check this!), the proof is similar to that of Theorem 14.5 (i).

Suppose $\varphi \to \psi \in \mathbf{S}$. Then by Theorem 5.61, we have

$$\bigwedge_{\Box\chi \in \mathbf{Sub}(\varphi \to \psi)} (\Box\chi \to \chi) \to (\varphi \to \psi) \in \mathbf{GL}$$

and so

$$\bigwedge_{\Box\chi\in\mathbf{Sub}\varphi}(\Box\chi\to\chi)\wedge\varphi\to(\bigwedge_{\Box\chi\in\mathbf{Sub}\psi}(\Box\chi\to\chi)\to\psi)\in\mathbf{GL}.$$

By Theorem 14.25, this formula has an interpolant α in **GL**, i.e.,

$$\bigwedge_{\Box\chi\in\mathbf{Sub}\varphi}(\Box\chi\to\chi)\wedge\varphi\to\alpha\in\mathbf{GL},\quad \bigwedge_{\Box\chi\in\mathbf{Sub}\psi}(\Box\chi\to\chi)\to(\alpha\to\psi)\in\mathbf{GL},$$

from which $\varphi\to\alpha\in\mathbf{S}$ and $\alpha\to\psi\in\mathbf{S}$. ❑

Theorem 14.31 Ext**S** *contains a continuum of logics with the interpolation property.*

Proof Exercise. (Hint: use the formulas α_i which were defined in the proof of Theorem 14.27). ❑

Theorem 14.32 *Suppose L is a modal logic with the interpolation property and having only one Post complete extension. Then L is Halldén complete.*

Proof Suppose that formulas φ and ψ have no common variables and $\varphi\vee\psi\in L$. Then $\neg\varphi\to\psi\in L$ and so there is a variable free formula χ such that $\neg\chi\to\varphi\in L$ and $\chi\to\psi\in L$. Since L has only one Post complete extension, we must have either $\chi\in L$ or $\neg\chi\in L$. Therefore, $\varphi\in L$ or $\psi\in L$. ❑

As a consequence of Theorems 14.31 and 14.32 we obtain

Corollary 14.33 *There is a continuum of Halldén complete logics in* Ext**S**. *In particular,* **S** *itself is Halldén complete.*

Theorem 14.34 Ext**S** *contains a continuum of logics which are not Halldén complete and so a continuum of logics without the interpolation property.*

Proof Exercise. (Hint: use the proof of Theorem 14.29 and Theorem 14.32). ❑

14.6 Exercises and open problems

Exercise 14.1 Suppose $L\in$ Ext**K** or $L\in$ Ext**Int** and $\alpha(p)$ is a conservative formula in Ext$L$. Show that for every formula $\varphi(p_1,\ldots,p_n)\in L+\alpha$,

$$\alpha(\bot)\wedge\alpha(p_1)\wedge\ldots\wedge\alpha(p_n)\to\varphi\in L.$$

(Hint: consider a substitutionless derivation of φ in $L+\alpha$ containing only the variables occurring in φ.)

Exercise 14.2 Say that a formula $\alpha(p)$ is *conservative* in NExt$L\subseteq$ NExt**K** if, for some n,

$$\bigwedge_{i\le n}\Box^i(\alpha(\bot)\wedge\alpha(p)\wedge\alpha_q)\to\alpha(p\to q)\in L,\quad \bigwedge_{i\le n}\Box^i(\alpha(\bot)\wedge\alpha(p))\to\alpha(\Box p)\in L.$$

Prove Theorem 14.5 (ii) for $L\in$ NExt**K**.

Exercise 14.3 Say that a logic $L \in \text{NExt}\mathbf{S4}$ has the *weak interpolation property* if every formula $\alpha \to \beta \in L$ has an interpolant in L whenever each occurrence of a variable in it is prefixed by $\Box$. Prove that L has the weak interpolation property iff ρL has the (plain) interpolation property.

Exercise 14.4 Show that the class of finite algebras for **Grz.3** is superamalgamable.

Exercise 14.5 Give canonical axiomatizations of the logics mentioned in Theorems 14.23 and 14.24.

Exercise 14.6 Say that a logic L has the *Lyndon interpolation property* if for every $\alpha \to \beta \in L$, there exists γ such that $\alpha \to \gamma \in L$, $\gamma \to \alpha \in L$ and the variables occurring in γ positively (negatively) have also positive (negative) occurrences in both α and β. Show that **K**, **K4**, **T** and **S4** have the Lyndon interpolation property.

Exercise 14.7 Prove that a pseudo-Boolean algebra is subdirectly irreducible iff it is well-connected.

Problem 14.1 *Do the si-logics* **LC**, $\mathbf{BD}_2$, $\mathbf{BD}_2 + (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$ *have the Lyndon interpolation property?*

Problem 14.2 *Which of the logics in* NExt**S4** *with the Craig interpolation property do have the Lyndon interpolation property?*

Problem 14.3 *Which logics in Theorems 14.23 and 14.24 (ii) do have the interpolation property and the interpolation property for derivability, respectively?*

Problem 14.4 *Construct a continuum of Halldén complete extensions of* **S** *without the interpolation property.*

Problem 14.5 *Describe the logics with the interpolation property in the classes* NExt**D**, NExt**D4**, Ext**D4**, Ext**S4**.

14.7 Notes

The interpolation theorems for **K**, **K4**, **T**, **S4** are due to Gabbay (1972a). Gabbay (1971b) gave semantic proofs of the interpolation property of **Int** and some of its extensions. The proofs presented in Section 14.1 are slight modifications of the proofs given by Maksimova (1982b) to show that the predicate variants of these logics have the (stronger) Lyndon interpolation property; see Exercise 14.6. This property was established also for some si-logics. Problem 14.1 lists the si-logics for which the situation is still unclear. Maksimova (1982b) gave also examples of logics in NExt**S4** which have the Craig interpolation property but do not have the Lyndon interpolation property. Here is one of them.

Example 14.35 Let L be the logic of the cluster $\mathfrak{Cl}_2$ with points a and b. By Theorem 14.21, it has the Craig interpolation property. Consider the formula

$$\Diamond p \wedge \neg p \wedge \Box(\neg p \vee q) \to \neg q \vee \Box q$$

which is clearly in L. Suppose γ is a Lyndon interpolant for this formula in L. Then γ contains only one variable q, and it occurs only positively. Define a valuation in $\mathfrak{Cl}_2$ so that all variables are true at a and false at b. It is easy to check that this valuation refutes one of the formulas

$$\Diamond p \wedge \neg p \wedge \Box(\neg p \vee q) \rightarrow \gamma, \quad \gamma \rightarrow \neg q \vee \Box q.$$

The rest of the material in Section 14.1 was also taken from Maksimova (1982b). However, the term "conservative" appeared first in Maksimova (1987). The result of Exercise 14.3 was announced in Maksimova (1980) and that of Exercise 14.4 was proved by Maksimova (1982b).

The semantic criteria of the interpolation property of Section 14.2 were taken from Maksimova (1977, 1979). Maksimova used those criteria to describe all si-logics with the interpolation property and to estimate the number of such logics in NExt**S4**. Theorem 14.16 was proved by Maksimova (1979) only for normal extensions of **S4**; later it was considerably generalized by Czelakowski (1982).

Theorem 14.20 was proved by Maksimova (1982a). This proof was generalized in Maksimova (1989c) to show that no logic in NExt**K4** of finite width and infinite depth, for instance **GL.3**, has the interpolation property.

That variable free formulas can be used to construct modal logics with the interpolation property seems to be noticed first by Rautenberg (1983). Maksimova (1987) generalized considerably this observation by introducing the conservative formulas. She also noticed that the addition of a finite set of conservative formulas preserves finite approximability, and that finiteness here is essential. The "positive" part of Section 14.5 is due to Smoryński (1978) (Theorem 14.25) and to Maksimova (1989a) (Theorem 14.27), and the "negative" one was obtained using some observations of Chagrov (1990b).

To conclude, we note two open directions of studies concerning the interpolation property. First, the big (continual) families of logics with this property were constructed with the essential help of variable free formulas. In this connection it would be of interest to investigate the interpolation property in the classes NExt**D** and NExt**D4**. Another direction is to describe quasi-normal extensions of **S4** or **D4** with the interpolation property.

Pitts (1992) used the cut-elimination technique to prove the so called *uniform Craig interpolation theorem* for **Int** which means that, for every formula $\alpha(p_1, \ldots, p_k, q_1, \ldots, q_l)$ there is a unique (up to the equivalence in **Int**) formula $\beta(q_1, \ldots, q_l)$ such that $\alpha \rightarrow \beta \in \mathbf{Int}$ and if $\alpha \rightarrow \gamma(q_1, \ldots, q_l) \in \mathbf{Int}$ and $\gamma \rightarrow \delta(q_1, \ldots, q_l, r_1, \ldots, r_m) \in \mathbf{Int}$, then $\beta \rightarrow \delta \in \mathbf{Int}$. Using semantical methods Shavrukov (1993) proved the uniform Craig interpolation theorem for **GL**. Beklemishev (1989) gave a complete description of provability logics with interpolation.

In Maksimova (1992a, 1992b) the reader can find more results concerning interpolation and some other related properties. It is proved in particular that a normal modal logic has interpolation iff it has the Beth definability property.

15

THE DISJUNCTION PROPERTY AND HALLDÉN COMPLETENESS

Recall that a modal logic L has the (*modal*) *disjunction property* if, for every $n \geq 1$ and all formulas $\varphi_1, \ldots, \varphi_n$,

$$\Box\varphi_1 \vee \ldots \vee \Box\varphi_n \in L \text{ implies } \varphi_i \in L, \text{ for some } i \in \{1, \ldots, n\}.$$

A si-logic L has the *disjunction property* if, for all φ and ψ

$$\varphi \vee \psi \in L \text{ implies } \varphi \in L \text{ or } \psi \in L. \tag{15.1}$$

And a (modal or superintuitionistic) logic L is said to be *Halldén complete* if (15.1) holds for all φ and ψ containing no common variables.

15.1 Semantic equivalents of the disjunction property

First we prove a semantic criterion of the modal disjunction property for logics in NExt**K**.

Theorem 15.1 *Suppose a logic $L \in$ NExt**K** is characterized by a class $\mathcal{C}$ of descriptive rooted frames closed under the formation of rooted generated subframes. Then L has the disjunction property iff, for every $n \geq 1$ and every $\mathfrak{F}_1, \ldots, \mathfrak{F}_n \in \mathcal{C}$ with roots $x_1, \ldots, x_n$, there is a rooted frame $\mathfrak{F}$ for L with root x such that $\mathfrak{F}_1 + \ldots + \mathfrak{F}_n$ is (isomorphic to) a generated subframe of $\mathfrak{F}$ with $\{x_1, \ldots, x_n\} \subseteq x\uparrow$.*

Proof ($\Rightarrow$) Let $\mathfrak{F}_L = \langle W_L, R_L, P_L \rangle$ be a universal frame for L, big enough to contain $\mathfrak{F}_1 + \ldots + \mathfrak{F}_n$ as its generated subframe. Assuming that $\mathfrak{F}_L$ is associated with a suitable canonical model for L, we show that there is a point t in $\mathfrak{F}_L$ such that $t\uparrow = W_L$.

Consider the tableau

$$t_0 = (\emptyset, \{\Box\varphi : \ \exists(\Gamma, \Delta) \in W_L \ \varphi \in \Delta\}).$$

Clearly, it is L-consistent (for otherwise $\Box\varphi_1 \vee \ldots \vee \Box\varphi_n \in L$ for some formulas $\varphi_1, \ldots, \varphi_n \notin L$, contrary to L having the disjunction property). Let t be a maximal L-consistent extension of t_0. By the definition of R_L, we then have tR_Lt', for every $t' \in W_L$.

($\Leftarrow$) Suppose otherwise. Then there are formulas $\varphi_1, \ldots, \varphi_n \notin L$ such that $\Box\varphi_1 \vee \ldots \vee \Box\varphi_n \in L$. Take frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n \in \mathcal{C}$ refuting $\varphi_1, \ldots, \varphi_n$ at their roots, respectively, and let $\mathfrak{F}$ be a rooted frame for L containing $\mathfrak{F}_1 + \ldots + \mathfrak{F}_n$ as a generated subframe and such that its root x sees the roots of $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$. Then all the formulas $\Box\varphi_1, \ldots, \Box\varphi_n$ are refuted at x and so $\Box\varphi_1 \vee \ldots \vee \Box\varphi_n \notin L$, which is a contradiction. ❑

It should be clear that if we need to use only the sufficient condition of Theorem 15.1 then the requirement that frames in $\mathcal{C}$ are descriptive is redundant.

Remark Since $\Box\varphi_1 \vee \Box\varphi_2 \vee \ldots \vee \Box\varphi_n \to \Box\varphi_1 \vee \Box(\Box\varphi_2 \vee \ldots \vee \Box\varphi_n) \in \mathbf{K4}$, a logic $L \in \mathrm{Ext}\mathbf{K4}$ has the disjunction property iff, for all φ and ψ, $\Box\varphi \vee \Box\psi \in L$ implies $\varphi \in L$ or $\psi \in L$. So, for such L we may assume that in Theorem 15.1 $n \leq 2$. And clearly a logic $L \in \mathrm{NExt}\mathbf{S4}$ has the disjunction property iff, for all φ and ψ, $\Box\varphi \vee \Box\psi \in L$ implies $\Box\varphi \in L$ or $\Box\psi \in L$.

As a direct consequence of the proof above we obtain

Corollary 15.2 *A consistent logic $L \in$ NExt**K** has the disjunction property iff the canonical frame $\mathfrak{F}_L = \langle W_L, R_L \rangle$ contains a point x such that $x{\uparrow} = W_L$.*

Corollary 15.3 *If a logic $L \in$ NExt**K** has the disjunction property then the rule $\Box p/p$ is admissible in L.*

Theorem 15.1 is a good tool for proving and disproving the disjunction property of logics with transparent semantics.

Example 15.4 (i) Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ be serial Kripke frames with roots $x_1, \ldots, x_n$. Then the frame obtained from $\mathfrak{F}_1 + \ldots + \mathfrak{F}_n$ by adding to it a point x seeing all $x_1, \ldots, x_n$ is also serial. Therefore, **D** has the modal disjunction property.

(ii) Since no rooted symmetrical frame can contain a proper generated subframe, no consistent logic in NExt**KB** has the disjunction property.

The reader can find more examples in the next section and among the exercises in Section 15.5.

Similarly to Theorem 15.1 and Corollary 15.2 one can prove the following semantic equivalents of the disjunction property for si-logics.

Theorem 15.5 (i) *Suppose a si-logic L is characterized by a class $\mathcal{C}$ of descriptive rooted frames. Then L has the disjunction property iff, for every $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$, $\mathfrak{F}_1 + \mathfrak{F}_2$ is a generated subframe of a rooted frame for L.*

(ii) *A si-logic has the disjunction property iff its canonical frame is rooted.*

Example 15.6 The disjoint union of two Medvedev frames $\mathfrak{B}_n$ and $\mathfrak{B}_m$ is clearly a generated subframe of $\mathfrak{B}_{n+m}$. So Medvedev's logic **ML** has the disjunction property.

A more interesting and complex example is provided by

Theorem 15.7 *The Kreisel–Putnam logic* **KP** *has the disjunction property.*

Proof We remind the reader that **KP** is characterized by the class of finite rooted frames $\mathfrak{F} = \langle W, R\rangle$ satisfying the condition

$$\forall x, y, z\ (xRy \wedge xRz \wedge \neg yRz \wedge \neg zRy \rightarrow \exists u\ (xRu \wedge uRy \wedge uRz \wedge \forall v\ (uRv \rightarrow \exists w\ (vRw \wedge (yRw \vee zRw))))). \qquad (15.2)$$

If $\mathfrak{F}$ is such a frame then, as is easy to see, for each non-empty $X \subseteq W^{\leq 1}$, the generated subframe of $\mathfrak{F}$ based on the set $W - (W^{\leq 1} - X)\downarrow$ is rooted; we denote its root by $r(X)$.

Let $\mathfrak{F}_1 = \langle W_1, R_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2\rangle$ be finite rooted frames satisfying (15.2). We construct from them a frame $\mathfrak{F} = \langle W, R\rangle$ by taking

$$W = W_1 \cup W_2 \cup U,$$

where $U = \{X_1 \cup X_2 :\ X_1 \subseteq W_1^{\leq 1},\ X_2 \subseteq W_2^{\leq 1},\ X_1, X_2 \neq \emptyset\}$, and, for every $x, y \in W$,

$$xRy \text{ iff } (x, y \in W_i \wedge xR_iy) \vee (x, y \in U \wedge x \supseteq y) \vee (x = X_1 \cup X_2 \in U \wedge y \in W_i \wedge r(X_i)R_iy).$$

It follows from the given definition that $\mathfrak{F}_1 + \mathfrak{F}_2$ is a generated subframe of $\mathfrak{F}$, $W_1 \cup W_2$ is a cover for $\mathfrak{F}$ and $W_1^{\leq 1} \cup W_2^{\leq 1}$ is its root. So our theorem will be proved if we show that (15.2) holds.

Suppose $x, y, z \in W$ satisfy the premise of (15.2). Since (15.2) holds for $\mathfrak{F}_1$, $\mathfrak{F}_2$ and since $\mathfrak{F}_i \subsetneq \mathfrak{F}$, we can assume that $x = X_1 \cup X_2 \in U$. Let $Y_1 \cup Y_2$ and $Z_1 \cup Z_2$ be the sets of final points in $y\uparrow$ and $z\uparrow$, respectively, with $Y_i, Z_i \subseteq W_i$, $i = 1, 2$. By the definition of R, we have $Y_i, Z_i \subseteq X_i$. Consider the point $u = (Y_1 \cup Z_1) \cup (Y_2 \cup Z_2)$. Clearly xRu, uRy and uRz. Suppose now that $v \in u\uparrow$. Let w be any final point in $v\uparrow$. Then $v \in (Y_1 \cup Z_1) \cup (Y_2 \cup Z_2)$ and so either yRw or zRw. ❑

To transfer the disjunction property from modal logics to their si-fragments and back we prove the following:

Theorem 15.8 *The maps ρ, τ and σ preserve the disjunction property.*

Proof That ρ preserves the disjunction property follows from the obvious fact that for every modal companion M of a si-logic L, $\varphi \vee \psi \in L$ iff $\mathsf{T}(\varphi \vee \psi) \in M$ iff $\mathsf{T}(\varphi) \vee \mathsf{T}(\psi) \in M$ (recall that $\mathsf{T}(\varphi)$ and $\mathsf{T}(\psi)$ may be regarded as boxed).

Suppose now that a si-logic L has the disjunction property and is characterized by a class $\mathcal{C}$ of rooted descriptive frames. By Theorem 9.68, σL is characterized by the class $\sigma\mathcal{C}$. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be arbitrary frames in $\mathcal{C}$ and $\mathfrak{F}$ a frame for L containing $\mathfrak{F}_1 + \mathfrak{F}_2$ as a generated subframe. Then, by Lemma 9.67, $\sigma\mathfrak{F}$ is a frame for σL in which $\sigma\mathfrak{F}_1 + \sigma\mathfrak{F}_2$ is a generated subframe as easily follows from the definition of $\sigma\mathfrak{F}$. Hence σL has the disjunction property.

To prove that the disjunction property is preserved under τ, we define an operator τ_ω as follows. Given an intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$, it returns the frame $\tau_\omega\mathfrak{F} = \langle \omega W, \omega R, \omega P\rangle$ in which $\langle \omega W, \omega R\rangle$ is the direct product of

$\langle W, R\rangle$ and the ω-point cluster $\langle \omega, \omega^2\rangle$ and ωP is the Boolean closure of the set $\{I \times X : I \subseteq \omega, X \in \boldsymbol{\sigma} P\}$ (see Section 8.3).

If L is characterized by a class $\mathcal{C}$ of descriptive rooted frames then, by Corollary 9.71, $\boldsymbol{\tau} L$ is characterized by $\boldsymbol{\tau}_\omega \mathcal{C}$. Let $\boldsymbol{\tau}_\omega \mathfrak{F}_i = \langle \omega W_i, \omega R_i, \omega P_i\rangle$, for $i = 1, 2$, be any frames in $\boldsymbol{\tau}_\omega \mathcal{C}$ and $\mathfrak{F} = \langle W, R, P\rangle$ a rooted intuitionistic frame containing $\mathfrak{F}_1 + \mathfrak{F}_2$ as a generated subframe. Clearly the underlying Kripke frame of $\boldsymbol{\tau}_\omega \mathfrak{F}_1 + \boldsymbol{\tau}_\omega \mathfrak{F}_2$ is a generated subframe of the underlying Kripke frame of $\boldsymbol{\tau}_\omega \mathfrak{F}$. So it remains to show that for $i = 1, 2$, $\omega P_i = \{X \cap \omega W_i : X \in \omega P\}$. But this follows from the definition of ωP and the equalities $\omega W_i - X = (\omega W - X') \cap \omega W_i$, $X \cap Y = (X' \cap Y') \cap \omega W_i$ which hold for every $X, Y \subseteq \omega W_i$ and $X', Y' \subseteq \omega W$ such that $X = X' \cap \omega W_i$, $Y = Y' \cap \omega W_i$. ❑

15.2 The disjunction property and the canonical formulas

In this section we use the apparatus of the canonical formulas to prove several sufficient and necessary conditions of the disjunction property for logics in NExt**S4** and Ext**Int**. First we obtain a complete description of cofinal subframe logics in NExt**S4** with the disjunction property. We assume that every logic $L \in \mathcal{CSF} \cap$ NExt**S4** is represented by its independent canonical axiomatization

$$L = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \bot) : i \in I\}. \qquad (15.3)$$

All frames in this section are assumed to be quasi-orders.

Say that a finite rooted frame $\mathfrak{F}$ with ≥ 2 points is *simple* if its root cluster and at least one of the final clusters are simple.

Suppose $\mathfrak{F} = \langle W, R\rangle$ is a simple frame, $a_0, a_1, \dots, a_m, a_{m+1}, \dots, a_n$ are all its points, with a_0 being the root, $C(a_1), \dots, C(a_m)$ all the distinct immediate cluster-successors of a_0 and a_n a final point with simple $C(a_n)$. For every $k = 1, \dots, n$, we define a formula ψ_k by taking

$$\psi_k = \bigwedge_{a_i R a_j, i \neq 0} \varphi_{ij} \wedge \bigwedge_{i=1}^{n} \varphi_i \wedge \varphi'_\bot \to p_k$$

where φ_{ij}, φ_i were defined in Section 9.4 and $\varphi'_\bot = \Box(\bigwedge_{i=1}^{n} \Box p_i \to \bot)$. Now we associate with $\mathfrak{F}$ the formula $\gamma(\mathfrak{F}) = \Box p_0 \vee \Box \psi_1$ if $m = 1$, and the formula $\gamma(\mathfrak{F}) = \Box \psi_1 \vee \ldots \vee \Box \psi_m$ if $m > 1$.

Lemma 15.9 *For every simple frame $\mathfrak{F}$, $\gamma(\mathfrak{F}) \in \mathbf{S4} \oplus \alpha(\mathfrak{F}, \bot)$.*

Proof By Theorem 11.20, it suffices to show that $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ implies $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$, for any finite frame $\mathfrak{G}$. So suppose $\gamma(\mathfrak{F})$ is refuted in a finite frame $\mathfrak{G}$ under some valuation. Define a partial map f from $\mathfrak{G}$ onto $\mathfrak{F}$ by taking, for any x in $\mathfrak{G}$,

$$f(x) = \begin{cases} a_0 & \text{if } x \not\models \gamma(\mathfrak{F}) \\ a_i & \text{if } x \not\models \psi_i,\ 1 \leq i \leq n \\ \text{undefined otherwise} & \end{cases}$$

and show that it is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$.

Suppose $f(x) = a_i$ and a_iRa_j. If $i \neq 0$ then in exactly the same way as in the proof of Theorem 9.39 we can find $y \in x\uparrow$ such that $f(y) = a_j$. And if $i = 0$, $j \neq 0$ then there is $k \in \{1, \dots, m\}$ such that a_kRa_j. Since $x \not\models \Box\psi_k$, we have a point $z \in x\uparrow$ such that $f(z) = a_k$ and then, as was shown above, there is $y \in z\uparrow$ with $f(y) = a_j$. It follows in particular that f is a surjection.

Now let $f(x) = a_i$, $f(y) = a_j$ and $y \in x\uparrow$. If $i = 0$ then clearly a_iRa_j. So suppose $i \neq 0$. If $j \neq 0$ then in the same way as in the proof of Theorem 9.39 we show that a_iRa_j. But in fact this is the only possible case. Indeed, if $j = 0$ then, for $m = 1$ we have $x \models \Box p_0$ (because $x \models \varphi_i$ and $a_0 \notin a_k\uparrow$), contrary to $y \not\models \Box p_0$, and if $m > 1$ then there is $k \in \{1, \dots, m\}$ such that $a_k \notin a_i\uparrow$ but, since a_0Ra_k, we must have a point $z \in y\uparrow$ with $f(z) = a_k$, which leads to a contradiction between $x \models \Box p_k$ and $z \not\models p_k$.

Thus f is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$. However it is not necessarily cofinal. So we extend f by putting $f(x) = a_n$, for every x of depth 1 in $\mathfrak{G}$ such that $f(x\downarrow) = \{a_0\}$. Clearly the improved map is still a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$, and using the formula $\varphi'_\bot$ it is easy to show that it is cofinal.

It follows that $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$ and so $\gamma(\mathfrak{F}) \in \mathbf{S4} \oplus \alpha(\mathfrak{F}, \bot)$. ❑

Lemma 15.10 *Suppose $i \in \{1, \dots, m\}$ and $\mathfrak{G}$ is the subframe of $\mathfrak{F}$ generated by a_i. Then $\alpha(\mathfrak{G}, \bot) \in \mathbf{S4} \oplus \psi_i$.*

Proof Exercise. ❑

We are in a position now to prove a criterion of the disjunction property for the cofinal subframe logics in NExtS4.

Theorem 15.11 *A consistent cofinal subframe logic $L \in$ NExt**S4** has the disjunction property iff no frame $\mathfrak{F}_i$ in its independent axiomatization (15.3) is simple, for $i \in I$.*

Proof ($\Rightarrow$) Suppose on the contrary that $\mathfrak{F}_i$ is simple, for some $i \in I$. Since the axiomatization (15.3) is independent, every proper generated subframe of $\mathfrak{F}_i$ validates L (for otherwise there would be an axiom $\alpha(\mathfrak{F}_j, \bot)$ of L, for $j \neq i$, with $\mathfrak{F}_i$ being subreducible cofinally to $\mathfrak{F}_j$, which is a contradiction). By Lemma 15.9, $\gamma(\mathfrak{F}_i) \in L$ and so, by virtue of L having the disjunction property, either $p_0 \in L$ or $\psi_j \in L$. However, both alternatives are impossible: the former means that L is inconsistent, while the latter, by Lemma 15.10, implies $\alpha(\mathfrak{G}, \bot) \in L$ where $\mathfrak{G}$ is the subframe of $\mathfrak{F}_i$ generated by an immediate successor of $\mathfrak{F}_i$'s root.

($\Leftarrow$) Given two finite rooted frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$ for L, we construct the frame $\mathfrak{F}$ as shown in Fig. 15.1. Clearly, $\mathfrak{G}_1 + \mathfrak{G}_2 \subsetneq \mathfrak{F}$. So to apply Theorem 15.1, it suffices to show that $\mathfrak{F} \models L$. Suppose otherwise, i.e., there exists a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some $i \in I$. Let x_i be the root of $\mathfrak{F}_i$. Since $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are not subreducible cofinally to $\mathfrak{F}_i$ and since L is consistent, $f^{-1}(x_i) = \{x\}$. By the cofinality condition, it follows in particular that $y \in \mathrm{dom} f$. But then $\mathfrak{F}_i$ is simple, which is a contradiction. ❑

Using the preservation theorem and Theorem 9.44, we immediately obtain

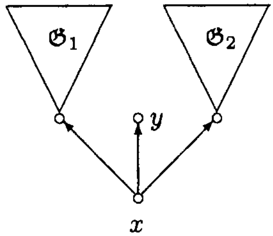


FIG. 15.1.

Corollary 15.12 *No consistent proper extension of* **Int** *with disjunction free axioms has the disjunction property.*

It is worth noting that the proof of Theorem 15.11 provides us with a somewhat stronger result. In fact the proof of ($\Rightarrow$) yields

Proposition 15.13 *If $L \in$* NExt**S4**, *$\mathfrak{F}$ is a simple frame, $\alpha(\mathfrak{F}, \bot) \in L$ and $\alpha(\mathfrak{G}, \bot) \notin L$ for any proper $\mathfrak{G} \subsetneq \mathfrak{F}$ then L does not have the disjunction property.*

Transferring this observation to the intuitionistic case, we obtain

Theorem 15.14 *If a consistent si-logic L has the disjunction property then the disjunction free fragments of L and* **Int** *are the same.*

Now we prove two simple sufficient conditions of the disjunction property for si-logics whose canonical axioms may contain closed domains. These conditions are far from being optimal and can be extended in various directions. First we use the simplest possible construction.

Theorem 15.15 *Suppose a si-logic L can be axiomatized by canonical formulas $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ or $\beta(\mathfrak{F}, \mathfrak{D})$ such that the set X of immediate successors of $\mathfrak{F}$'s root contains ≥ 3 points and $\mathfrak{d} \in \mathfrak{D}$, for every antichain $\mathfrak{d}$ containing a subset of X with $\geq |X|/2$ points. Then L has the disjunction property.*

Proof Let $\mathfrak{F}_1 = \langle W_1, R_1, P_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, P_2 \rangle$ be rooted frames for L. Construct a frame $\mathfrak{F}_0 = \langle W_0, R_0, P_0 \rangle$ by adding to $\mathfrak{F}_1 + \mathfrak{F}_2$ a root a_0 and defining P_0 as the pseudo-Boolean closure of $\{Y_1 \cup Y_2 : Y_1 \in P_1, Y_2 \in P_2\}$. By induction on the construction of a set $Y \in P_0$ one can readily show that $Y \cap W_i \in P_i$, for $i = 1, 2$, and so $\mathfrak{F}_1 + \mathfrak{F}_2$ is a generated subframe of $\mathfrak{F}_0$.

To show that $\mathfrak{F}_0 \models L$, suppose otherwise. Then $\mathfrak{F}_0$ refutes an axiom $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ (or $\beta(\mathfrak{F}, \mathfrak{D})$) of L, i.e., there is a cofinal (or plain) subreduction f of $\mathfrak{F}_0$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Let a be the root of $\mathfrak{F}$. Since $\mathfrak{F}_i \models L$, for $i = 1, 2$, $f^{-1}(a) = \{a_0\}$.

Now take that i for which W_i contains inverse f-images of all points in some antichain $\mathfrak{a} \subseteq X$ with $|\mathfrak{a}| \geq |X|/2$ and let $\mathfrak{d}$ be the antichain in $\mathfrak{F}$ such that $f(W_i) = \mathfrak{d}\uparrow$. By the condition of our theorem, $\mathfrak{d} \in \mathfrak{D}$ and so, by (CDC), the root a_i of $\mathfrak{F}_i$ must be in dom$f$. But then $f(a_i) = a$, which is a contradiction. ❑

Corollary 15.16 *Every si-logic axiomatizable by formulas $\beta^\sharp(\mathfrak{F},\bot)$ (or formulas $\beta^\sharp(\mathfrak{F})$) such that the root of $\mathfrak{F}$ sees ≥ 3 immediate successors has the disjunction property.*

The second sufficient condition uses a more complicated construction.

Theorem 15.17 *Suppose a si-logic L is axiomatized by formulas $\beta(\mathfrak{F},\mathfrak{D},\bot)$ with $\mathfrak{F}$ of depth ≥ 3 and $\mathfrak{D}$ containing an antichain $\mathfrak{d} \subseteq \mathfrak{F}^{\leq 1}$ having no focus in $\mathfrak{F}$. Then L has the disjunction property.*

Proof Let $\mathfrak{F}_1 = \langle W_1, R_1, P_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, P_2\rangle$ be rooted finitely generated refined frames for L. With each antichain $\mathfrak{a}$ in $(\mathfrak{F}_1+\mathfrak{F}_2)^{\leq 1}$ such that $|\mathfrak{a}| \geq 2$ we associate a new point $x_\mathfrak{a}$; the set of all such points is denoted by V. Construct a frame $\mathfrak{F}_0 = \langle W_0, R_0, P_0\rangle$ by taking

$$W_0 = \{a_0\} \cup W_1 \cup W_2 \cup V,$$

$$\begin{aligned} xR_0y \text{ iff } x = a_0 \vee \exists i \in \{1,2\}\ (x,y \in W_i \wedge xR_iy)\vee \\ \exists x_\mathfrak{a} \in V\ (x = x_\mathfrak{a} \wedge (y = x_\mathfrak{a} \vee y \in x_\mathfrak{a})) \end{aligned}$$

and defining P_0 as the pseudo-Boolean closure of $\{Y_1 \cup Y_2 :\ Y_1 \in P_1,\ Y_2 \in P_2\}$. $\mathfrak{F}_1+\mathfrak{F}_2$ is then a generated subframe of $\mathfrak{F}_0$. Moreover, since the original frames are finitely generated and refined, $(\mathfrak{F}_1+\mathfrak{F}_2)^{\leq 1}$ is a cover for $\mathfrak{F}_0$.

Assume now that $\mathfrak{F}_0$ refutes an axiom $\beta(\mathfrak{F},\mathfrak{D},\bot)$ of L. Let f be a cofinal subreduction of $\mathfrak{F}_0$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Since $\mathfrak{F}$ is of depth ≥ 3 and $\mathfrak{F}_i \models L$ for $i = 1,2$, the root a_0 is in dom$f$. Take an antichain $\mathfrak{d} \in \mathfrak{D}$ having no focus in $\mathfrak{F}$ and consisting of only points of depth 1. Let $\mathfrak{a}$ be an antichain in $\mathfrak{F}_0^{\leq 1}$ such that $f(\mathfrak{a}) = \mathfrak{d}$. Since $x_\mathfrak{a}$ is a focus for $\mathfrak{a}$, we must have, by (CDC), that $x_\mathfrak{a} \in$ dom$f$. But then $f(x_\mathfrak{a})$ is a focus for $\mathfrak{d}$, which is a contradiction.

Thus $\mathfrak{F}_0 \models L$ and so L has the disjunction property. ❑

15.3 Maximal si-logics with the disjunction property

The disjunction property of a si-logic means that formulas in the logic represent only constructive principles of reasoning. Since **Cl** is not constructive in this sense, it is of interest to find maximal (consistent) si-logics with the disjunction property. That they exist follows from Zorn's lemma (see Exercise 15.8). Here is a concrete example of such a logic.

Theorem 15.18 *The Medvedev logic* **ML** *is a maximal si-logic with the disjunction property.*

Proof Suppose on the contrary that there exists a proper consistent extension L of **ML** having the disjunction property. Then we have a formula $\varphi \in L - \mathbf{ML}$. We show first that there is an essentially negative substitution instance φ^* of φ such that $\varphi^* \notin \mathbf{ML}$.

Since $\varphi(p_1,\dots,p_n) \notin \mathbf{ML}$, there is a Medvedev frame $\mathfrak{B}_m$ refuting φ under some valuation $\mathfrak{V}$. With every point x in $\mathfrak{B}_m$ we associate a new variable q_x and extend $\mathfrak{V}$ to these variables by taking $\mathfrak{V}(q_x)$ to be the set of final points in $\mathfrak{B}_m$ that are not accessible from x. By the construction of $\mathfrak{B}_m$, we have $y \models \neg q_x$ iff $y \in x\uparrow$, from which

$$\mathfrak{V}(\bigvee_{x\in\mathfrak{V}(p_i)} \neg q_x) = \mathfrak{V}(p_i).$$

Let $\varphi^* = \varphi(\bigvee_{x\in\mathfrak{V}(p_1)} \neg q_x, \dots, \bigvee_{x\in\mathfrak{V}(p_n)} \neg q_x)$. It follows that $\mathfrak{V}(\varphi^*) = \mathfrak{V}(\varphi)$ and so $\varphi^* \notin \mathbf{ML}$.

Thus, we may assume that φ is an essentially negative formula. Recall now that $\mathbf{KP} \subseteq \mathbf{ML}$ (see Exercise 5.32) and so $\mathbf{ML}$ contains the formulas

$$\boldsymbol{nd}_k = (\neg p \to \neg q_1 \vee \dots \vee \neg q_k) \to (\neg p \to \neg q_1) \vee \dots \vee (\neg p \to \neg q_k),$$

which, as is easy to see, belong to $\mathbf{KP}$. Let us consider the logic

$$\mathbf{ND} = \mathbf{Int} + \{\boldsymbol{nd}_k : \ k \geq 1\}.$$

It should be clear that $\mathbf{ND} \subseteq \mathbf{KP} \subseteq \mathbf{ML}$ (in fact both inclusions here are proper). Using the fact that the outermost $\to$ in $\boldsymbol{nd}_k$ can be replaced with $\leftrightarrow$ and that $(\neg p \to \neg q) \leftrightarrow \neg(\neg p \wedge q) \in \mathbf{Int}$, one can readily show that every essentially negative formula is equivalent in $\mathbf{ND}$ to the conjunction of formulas of the form $\neg\chi_1 \vee \dots \vee \neg\chi_l$.

So $L - \mathbf{ML}$ contains a formula of the form $\neg\chi_1 \vee \dots \vee \neg\chi_l$. Since L has the disjunction property, $\neg\chi_i \in L$ for some i. But then, by Glivenko's theorem, $\neg\chi_i \in \mathbf{ML}$, which is a contradiction. ❑

It turns out, however, that $\mathbf{ML}$ is not the unique maximal logic with the disjunction property in Ext$\mathbf{Int}$. Moreover, the following result holds.

Theorem 15.19 *There is a continuum of maximal si-logics with the disjunction property.*

Proof It is sufficient to show that there is a continuum of si-logics such that (i) each of them has a consistent extension with the disjunction property and (ii) no pair of them has a common consistent extension with the disjunction property.

For each $n > 8$, let

$$\varphi^n = \bigvee_{i=1}^{n} (q_1 \vee (q_1 \to q_2 \vee (q_2 \to q_3 \vee (q_3 \to \neg\chi_i^n)))),\ \psi^n = \bigwedge_{k=1}^{7} \psi_k^n \to \psi_8^n,$$

where

$$\chi_i^n = p_i \wedge \bigwedge_{i\neq j} \neg p_j,\ \psi_1^n = \neg \bigwedge_{i=1}^{n} \neg\chi_i^n,$$

$$\psi_2^n = \bigwedge_{1\leq i<j\leq n} (\bigwedge_{i\neq k\neq j} \neg\chi_k^n \to \neg\chi_i^n \vee \neg\chi_j^n),$$

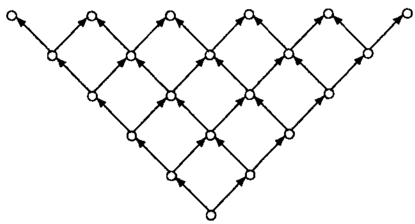


FIG. 15.2.

$$\psi_3^n = (\bigwedge_{k=5}^{n} \neg\chi_k^n \to \bigvee_{k=1}^{4} \neg\chi_k^n) \to$$
$$(\neg\chi_1^n \vee \bigwedge_{i=2}^{4} \neg\chi_i^n) \wedge (\neg\chi_3^n \vee (\neg\chi_2^n \wedge \neg\chi_4^n)) \wedge$$
$$\bigwedge_{1\le i<j<k\le 4} (\bigwedge_{t\notin\{i,j,k\}} \neg\chi_t^n \to \neg\chi_i^n \vee \neg\chi_j^n \vee \neg\chi_k^n),$$

$$\psi_4^n = (\bigwedge_{k\notin\{2,4,5\}} \neg\chi_k^n \to \neg\chi_2^n \vee \neg\chi_4^n \vee \neg\chi_5^n) \to$$
$$(\neg\chi_1^n \vee \neg\chi_3^n \to \neg\chi_2^n \vee \neg\chi_4^n),$$

$$\psi_5^n = \bigvee_{k\notin\{1,5\}} \neg\chi_k^n \to \neg\chi_1^n \vee \neg\chi_5^n,$$

$$\psi_6^n = \bigwedge_{i=4}^{n-3}((\bigwedge_{k\notin\{i-1,i+1,i+2\}} \neg\chi_k^n \to \neg\chi_{i-1}^n \vee \neg\chi_{i+1}^n \vee \neg\chi_{i+2}^n) \to$$
$$\neg\chi_{i-1}^n \vee \neg\chi_{i+1}^n) \wedge (\neg\chi_n^n \to \neg\chi_{n-3}^n \vee \neg\chi_{n-1}^n),$$

$$\psi_7^n = (\bigwedge_{k=1}^{n-4} \neg\chi_k^n \to \bigvee_{k=n-3}^{n} \neg\chi_k^n) \to \neg\chi_{n-2}^n \vee \neg\chi_n^n,$$

$$\psi_8^n = (\bigwedge_{k=5}^{n} \neg\chi_k^n \to \bigvee_{k=1}^{4} \neg\chi_k^n) \vee \neg\chi_5^n.$$

Observe that, as follows from the construction of φ^n, no consistent si-logic containing φ^n has the disjunction property.

For each set N of natural numbers > 8, let

$$L(N) = \mathbf{Int} + \{\psi^n \to \varphi^n : \ n \in N\} + \{\psi^n : \ n \notin N, \ n > 8\}.$$

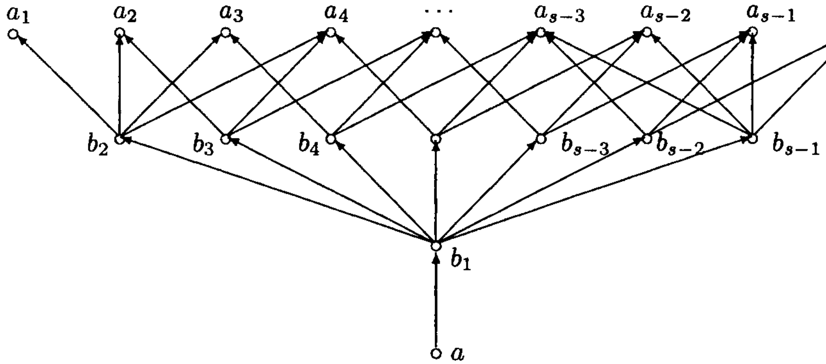


FIG. 15.3.

Lemma 15.20 *If $N_1 \neq N_2$, then $L(N_1)$ and $L(N_2)$ have no common consistent extension with the disjunction property.*

Proof Without loss of generality we may assume that there is $n \in N_1 - N_2$. Then $\psi^n \to \varphi^n \in L(N_1)$ and $\psi^n \in L(N_2)$. It follows that $\varphi^n \in L(N_1) + L(N_2)$ and so no consistent extension of $L(N_1) + L(N_2)$ has the disjunction property. ❑

Lemma 15.21 *$L(N)$ has a consistent extension with the disjunction property.*

Proof Let $\mathfrak{F}_m$ be the frame of the form shown in Fig. 15.2 with $m \geq 1$ final points. For every $s \in N$ and every $s+1$-tuple $\langle a, a_1, \dots, a_s \rangle$ of points in $\mathfrak{F}_m$ such that $a_1, \dots, a_s$ are distinct and final in $a\uparrow$, we add to $\mathfrak{F}_m$ new points $b_1, \dots, b_{s-1}$ and extend the accessibility relation to them by drawing the arrows shown in Fig. 15.3. The resulting frame is denoted by $\mathfrak{F}_m(N)$.

Now we put

$$L = \mathrm{Log}\{\mathfrak{F}_m(N) : \ m < \omega\}.$$

Since $\mathfrak{F}_m(N) + \mathfrak{F}_k(N)$ is clearly a generated subframe of $\mathfrak{F}_{m+k}(N)$, by Theorem 15.5 L has the disjunction property. So it remains to show that $L(N) \subseteq L$, i.e., that all axioms of $L(N)$ are valid in all frames of the form $\mathfrak{F}_m(N)$.

Suppose that ψ^n is refuted in $\mathfrak{F}_m(N)$ under some valuation. Then there is a point x such that $x \models \psi_i^n$, for $i = 1, \dots, 7$, and $x \not\models \psi_8^n$. We are going to show that in this case $n \in N$ and so ψ^n cannot be an axiom of $L(N)$.

Notice first that x does not belong to $\mathfrak{F}_m$. For otherwise, since $x \not\models \psi_8^n$, we would have five distinct final points $a_1, \dots, a_5 \in x\uparrow$ such that $a_j \models \chi_j^n$, for $j = 1, \dots, 5$. Since $x \models \psi_1^n$, each final successor of x validates χ_i^n for some $i \in \{1, \dots, n\}$. Therefore, there are two adjacent final points c and d in $x\uparrow$ at which distinct χ_i^n and χ_j^n are true. But then

$$e \not\models \bigwedge_{i \neq k \neq j} \neg\chi_k^n \to \neg\chi_i^n \vee \neg\chi_j^n,$$

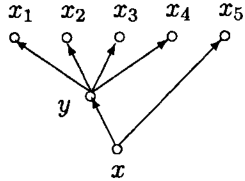


FIG. 15.4.

where e is the immediate predecessor of c and d in $\mathfrak{F}_m$. Since $e \in x\uparrow$ and $x \models \psi_2^n$, we arrive at a contradiction.

Now let us take a closer look at the condition $x \not\models \psi_8^n$. It means that there are points $x_1, \dots, x_5, y$ in $\mathfrak{F}_m(N)$ which together with x form the diagram shown in Fig. 15.4. Comparing it with Fig. 15.3 and recalling that x does not belong to $\mathfrak{F}_m$, we conclude that x can be identified only with b_1 in Fig. 15.3. Using this observation we show that $n = s$, from which $n \in N$, as required.

Among $b_2, \dots, b_{s-1}$ only b_2 and b_{s-1} have four successors and can refute the first disjunct in ψ_8^n. Let us first assume that

$$b_{s-1} \models \bigwedge_{k=5}^{n} \neg\chi_k^n, \quad b_{s-1} \not\models \bigvee_{k=1}^{4} \neg\chi_k^n.$$

Then each of the formulas $\chi_1^n, \dots, \chi_4^n$ is true at exactly one of $a_{s-3}, \dots, a_s$ and so

$$b_{s-2} \not\models \bigwedge_{1 \le i < j < k \le 4} \Big(\bigwedge_{t \notin \{i,j,k\}} \neg\chi_t^n \to \neg\chi_i^n \vee \neg\chi_j^n \vee \neg\chi_k^n \Big).$$

Since $b_1 \models \psi_3^n$, we obtain then

$$b_{s-2} \not\models \bigwedge_{k=5}^{n} \neg\chi_k^n \to \bigvee_{k=1}^{4} \neg\chi_k^n,$$

which is impossible, because b_{s-2} has only three successors.

Thus we are forced to conclude that

$$b_2 \models \bigwedge_{k=5}^{n} \neg\chi_k^n, \quad b_2 \not\models \bigvee_{k=1}^{4} \neg\chi_k^n.$$

As before, it follows that exactly one of $\chi_1^n, \dots, \chi_4^n$ is true at each a_i, $1 \le i \le 4$. Now consider b_3. Since it has only three successors, the condition $b_1 \models \psi_3^n$ leaves only one possibility: $b_3 \not\models \neg\chi_2^n \vee \neg\chi_4^n$ and $b_3 \models \neg\chi_1^n \vee \neg\chi_3^n$. But then the conclusion of ψ_4^n is not true at b_3 and so b_3 must refute the premise. It follows that $a_5 \models \chi_5^n$.

Observe now that either χ_1^n or χ_3^n is true at a_3. Since $b_1 \models \psi_5^n$ and $n > 8$, we may have only $a_3 \models \chi_3^n$. Then $b_4 \not\models \neg\chi_3^n \vee \neg\chi_5^n$. By virtue of $b_1 \models \psi_6^n$, we also have $b_4 \not\models \chi_6^n$, which is possible only if $a_6 \models \chi_6^n$. In the same way, using the condition $b_1 \models \psi_6^n$, we can show that $a_i \models \chi_i^n$, for $i = 7, \dots, n-1$. And the last conjunct of ψ_6^n ensures that b_{n-2} sees a final point x at which χ_n^n is true. Since no distinct χ_i^n and χ_j^n can be simultaneously true at a point, $s \geq n$. It follows also that $b_{n-2} \neq b_{s-1}$ and so $x = a_n$.

Since b_{n-1} sees both a_n and a_{n-2}, we have $b_{n-1} \not\models \neg\chi_{n-2}^n \vee \neg\chi_n^n$. And since $b_1 \models \psi_7^n$, we then have also $b_{n-1} \not\models \bigvee_{k=n-3}^n \neg\chi_k^n$, which means that b_{n-1} sees at least four distinct final points. So $b_{n-1} = b_{s-1}$ and consequently $n = s$.

It remains to show that $\mathfrak{F}_m(N) \models \psi^s \to \varphi^s$ for every $m < \omega$ and every $s \in N$. Suppose that φ^s is refuted at some x in $\mathfrak{F}_m(N)$ under some valuation. This means that s chains of length ≥ 4 start from x and at their final points $a_1, \dots, a_s$ the formulas $\chi_1^s, \dots, \chi_s^s$ are true, respectively. It follows also that x is a point in $\mathfrak{F}_m$ which sees the configuration shown in Fig. 15.3. It is not hard to check now that in such a situation $b_1 \not\models \psi^s$, from which $x \not\models \psi^s$. ❑

It follows from Lemmas 15.20 and 15.21 that there is a continuum of logics satisfying (i) and (ii) and so a continuum of maximal si-logics with the disjunction property. ❑

15.4 Halldén completeness

In this section we show various methods for establishing Halldén completeness of logics in Ext**K** and Ext**Int**. Let us begin with a lattice-theoretic criterion of this property.

Theorem 15.22 *A superintuitionistic or quasi-normal modal logic L is Halldén incomplete iff there are logics $L_1, L_2 \in \mathrm{Ext}L$ such that $L_1 \not\subseteq L_2$, $L_2 \not\subseteq L_1$ and $L = L_1 \cap L_2$.*

Proof ($\Rightarrow$) If L is Halldén incomplete then there are formulas $\varphi_1, \varphi_2 \notin L$ with $\mathbf{Var}\varphi_1 \cap \mathbf{Var}\varphi_2 = \emptyset$ and $\varphi_1 \vee \varphi_2 \in L$. Consider the logics $L_i = L + \varphi_i$, for $i = 1, 2$. Clearly, L_1 and L_2 are incomparable with respect to $\subseteq$ and $L \subseteq L_1 \cap L_2$. To prove the converse inclusion, take any formula $\psi \in L_1 \cap L_2$. Then there are substitution instances φ_1^i and φ_2^j of φ_1 and φ_2, respectively, for $i = 1, \dots, m$, $j = 1, \dots, n$, such that

$$\bigwedge_{i \leq m} \varphi_1^i \to \psi \in L, \quad \bigwedge_{j \leq n} \varphi_2^j \to \psi \in L.$$

It follows that

$$(\bigwedge_{i \leq m} \varphi_1^i) \vee (\bigwedge_{j \leq n} \varphi_2^j) \to \psi \in L$$

and so

$$\bigwedge_{i \leq m, j \leq n} (\varphi_1^i \vee \varphi_2^j) \to \psi \in L.$$

Since φ_1 and φ_2 have no common variables, $\varphi_1^i \vee \varphi_2^j \in L$. Hence $\psi \in L$.

($\Leftarrow$) If $L_1 \not\subseteq L_2$, $L_2 \not\subseteq L_1$ and $L = L_1 \cap L_2$ then there are formulas $\varphi_1 \in L_1 - L_2$ and $\varphi_2 \in L_2 - L_1$ without common variables. Then we clearly have $\varphi_1, \varphi_2 \notin L$ and $\varphi_1 \vee \varphi_2 \in L$. ❑

Example 15.23 Since the lattices ExtS5, Ext**LC**, Ext**BD**$_2$ are linearly ordered by inclusion, all logics in them are Halldén complete.

It is to be noted, however, that Theorem 15.22 does not hold if we consider only normal modal logics and take NExt$L$ instead of Ext$L$ (see Exercise 15.16).

Now we obtain a semantic criterion.

Theorem 15.24 *Suppose a logic $L \in$ Ext**K** is characterized by a class C of descriptive rooted frames with distinguished roots. Then L is Halldén complete iff, for every frames $\langle \mathfrak{F}_1, d_1 \rangle$ and $\langle \mathfrak{F}_2, d_2 \rangle$ in C, there is a frame $\langle \mathfrak{F}, d \rangle$ for L reducible to both $\langle \mathfrak{F}_1, d_1 \rangle$ and $\langle \mathfrak{F}_2, d_2 \rangle$.*

Proof ($\Rightarrow$) Suppose the frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are $\varkappa'$- and $\varkappa''$-generated, respectively. Then they are (isomorphic to) generated subframes of the universal $\varkappa'$- and $\varkappa''$-generated frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$ for $\ker L$. Without loss of generality we may assume that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are associated with the canonical models for $\ker L$ in disjoint languages $\mathcal{ML}_1$ and $\mathcal{ML}_2$, respectively. The frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are reducts of the universal $(\varkappa' + \varkappa'')$-generated frame $\mathfrak{G}$ for $\ker L$, associated with the canonical model for $\ker L$ in the language $\mathcal{ML} = \mathcal{ML}_1 \cup \mathcal{ML}_2$. Let g_i, for $i = 1, 2$, be the natural reduction of $\mathfrak{G}$ to $\mathfrak{G}_i$, i.e., for every $t = (\Gamma, \Delta)$ in $\mathfrak{G}$, $g_i(t) = (\Gamma \cap \mathbf{For}\mathcal{ML}_i, \Delta \cap \mathbf{For}\mathcal{ML}_i)$.

Consider the points $d_1 = (\Gamma_1, \Delta_1)$ and $d_2 = (\Gamma_2, \Delta_2)$ in $\mathfrak{G}_1$ and $\mathfrak{G}_2$, respectively. Put $d' = (\Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2)$ and show that this tableau is L-consistent. Suppose otherwise. Then there are formulas $\varphi_1 \in \Gamma_1$, $\varphi_2 \in \Gamma_2$, $\psi_1 \in \Delta_1$, $\psi_2 \in \Delta_2$ such that $\varphi_1 \wedge \varphi_2 \to \psi_1 \vee \psi_2 \in L$. But this is (classically) equivalent to $(\varphi_1 \to \psi_1) \vee (\varphi_2 \to \psi_2) \in L$. Since $\varphi_1 \to \psi_1$ and $\varphi_2 \to \psi_2$ have no variables in common, we must then have $\varphi_1 \to \psi_1 \in L$ or $\varphi_2 \to \psi_2 \in L$, contrary to $\langle \mathfrak{F}_1, d_1 \rangle$ and $\langle \mathfrak{F}_2, d_2 \rangle$ validating L.

Let d be a maximal L-consistent extension of d' in the language $\mathcal{ML}$. Then clearly $g_i(d) = d_i$ for $i = 1, 2$. So the restriction f_i of g_i to the subframe $\mathfrak{F}$ of $\mathfrak{G}$ generated by d is a reduction of $\mathfrak{F}$ to $\mathfrak{F}_i$ with $f_i(d) = d_i$. It remains to observe that $\langle \mathfrak{F}, d \rangle$ validates L.

($\Leftarrow$) Suppose that $\varphi_1 \notin L$ and $\varphi_2 \notin L$, for some formulas φ_1 and φ_2 with no variables in common. Let $\langle \mathfrak{F}_i, d_i \rangle$, for $i = 1, 2$, be a frame in C refuting φ_i under a valuation $\mathfrak{V}_i$ of φ_i's variables. Take a frame $\langle \mathfrak{F}, d \rangle$ for L reducible to both $\langle \mathfrak{F}_1, d_1 \rangle$ and $\langle \mathfrak{F}_2, d_2 \rangle$ by reductions f_1 and f_2. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking, for $p \in \mathbf{Var}\varphi_i$, $\mathfrak{V}(p) = f_i^{-1}(\mathfrak{V}_i(p))$. Then f_i is a reduction of the model $\langle \mathfrak{F}, \mathfrak{V} \rangle$ (restricted to φ_i's variables) to $\langle \mathfrak{F}_i, \mathfrak{V}_i \rangle$ and so, by the reduction theorem, we have $d \not\models \varphi_1 \vee \varphi_2$. Therefore, $\varphi_1 \vee \varphi_2 \notin L$. ❑

Notice that the proof of ($\Leftarrow$) does not use the fact that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are descriptive and rooted. So if we need only the sufficient condition of Theorem 15.24, the requirement that frames in C are descriptive and rooted is redundant.

Theorem 15.25 *A consistent logic $L \in \mathrm{Ext}\mathbf{K}$ is Halldén complete iff it is characterized by a frame with a single distinguished point.*

Proof ($\Rightarrow$) Consider the tableau $t' = (\emptyset, \Delta)$ where Δ is a set of formulas such that (a) $L \cap \Delta = \emptyset$, (b) every formula that is not in L can be obtained from a formula in Δ by renaming its variables and (c) distinct formulas in Δ have distinct variables. Since L is Halldén complete, t' is L-consistent and has a maximal L-consistent extension t. Then the frame $\langle \mathfrak{F}_{\mathrm{ker}\, L}(\omega), t\rangle$ characterizes L.

($\Leftarrow$) follows from Theorem 15.24 ❑

For normal modal logics the proof of Theorem 15.24 yields the following:

Theorem 15.26 *Suppose a logic $L \in \mathrm{NExt}\mathbf{K}$ is characterized by a class $\mathcal{C}$ of descriptive rooted frames closed under the formation of rooted generated subframes. Then L is Halldén complete iff, for all frames $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$ and with roots x_1, x_2, respectively, there is a frame $\mathfrak{F}$ for L reducible to $\mathfrak{F}_1$ and $\mathfrak{F}_2$ by reductions f_1 and f_2, respectively, such that $f_1(x) = x_1$ and $f_2(x) = x_2$ for some x in $\mathfrak{F}$.*

Example 15.27 $\mathbf{S4.3}$ is characterized by the frame $\langle \mathbb{Q}, \leq\rangle$, $\mathbb{Q}$ the set of rationals. Since for every $x, y \in \mathbb{Q}$, there is an isomorphism f of $\langle \mathbb{Q}, \leq\rangle$ onto itself with $f(x) = y$, $\mathbf{S4.3}$ is Halldén complete.

For si-logics Theorems 15.26 and 15.25 transform into

Theorem 15.28 (i) *Suppose a si-logic L is characterized by a class $\mathcal{C}$ of rooted descriptive frames. Then L is Halldén complete iff, for every frames $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$, there is a rooted frame $\mathfrak{F}$ for L containing generated subframes reducible to $\mathfrak{F}_1$ and $\mathfrak{F}_2$.*

(ii) *A si-logic L is Halldén complete iff it is characterized by a rooted frame.*

Proof Exercise. ❑

Halldén completeness is obviously preserved while passing from a modal logic in $\mathrm{NExt}\mathbf{S4}$ to its si-fragment. However, this is not so in the case of the converse transition even for the maps $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$.

Theorem 15.29 *There is a Halldén complete si-logic having no Halldén complete modal companions.*

Proof Consider the si-logic of the frame $\mathfrak{F}$ shown in Fig. 15.5. By Theorem 15.28, it is Halldén complete (but, as any other tabular logic, does not have the disjunction property). Let $M \in \boldsymbol{\rho}^{-1}L$. Construct the formulas $\alpha(\mathfrak{F}_1, \bot)$ and $\alpha^\sharp(\mathfrak{F}_2, \bot)$, for $\mathfrak{F}_1$ and $\mathfrak{F}_2$ depicted in Fig. 15.5 so that they would not have common variables. Since $\mathfrak{F} \models \boldsymbol{\sigma}L \supseteq M$, $\mathfrak{F} \not\models \alpha(\mathfrak{F}_1, \bot)$ and $\mathfrak{F} \not\models \alpha^\sharp(\mathfrak{F}_2, \bot)$, neither of those formulas is in M.

On the other hand, by Corollary 9.71, the smallest modal companion $\boldsymbol{\tau}L \subseteq M$ of L is characterized by the frame $\langle \omega, \omega^2\rangle \times \mathfrak{F}$. Since it clearly validates $\alpha(\mathfrak{F}_1, \bot) \vee \alpha^\sharp(\mathfrak{F}_2, \bot)$, this disjunction is in M and so M is not Halldén complete.

❑

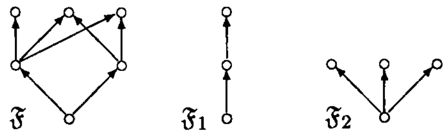


FIG. 15.5.

We conclude this section with two sufficient conditions of Halldén completeness for logics in **NExtGrz** formulated in terms of the canonical formulas. Recall that every logic $L \in$ **NExtGrz** can be represented in the form $L = \mathbf{Grz} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}$ with partially ordered $\mathfrak{F}_i$.

Theorem 15.30 *If a Kripke complete logic $L \in$ **NExtGrz** can be axiomatized by canonical formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ such that the root of $\mathfrak{F}$ has only one immediate successor then L is Halldén complete.*

Proof Suppose $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$ are partially ordered Kripke frames for L with roots a_1 and a_2, respectively. Construct a frame $\mathfrak{F}_0 = \langle W_0, R_0 \rangle$ by gluing a_1 and a_2 into a single point a, i.e., by taking

$$W_0 = \{a\} \cup (W_1 - \{a_1\}) \cup (W_2 - \{a_2\}),$$

$$xR_0y \text{ iff } x = a \vee \exists i \in \{1, 2\}\ (x, y \in W_i \wedge xR_iy).$$

It should be clear that $\mathfrak{F}_0$ is reducible to both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ (here essential is that these frames are Noetherian partial orders). So to apply Theorem 15.26 we must show that $\mathfrak{F}_0$ validates L.

Assume that $\mathfrak{F}_0$ refutes an axiom $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ of L. Then there is a cofinal subreduction of $\mathfrak{F}_0$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are frames for L, $f(a)$ is the root of $\mathfrak{F}$. Suppose $\mathfrak{F}_i$ contains an inverse f-image of the immediate successor of $f(a)$. Then the restriction of f to $\mathfrak{F}_i$ is clearly a cofinal subreduction of $\mathfrak{F}_i$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$, whence we have $\mathfrak{F}_i \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, which is a contradiction. ❑

Theorem 15.31 *Suppose a normal extension L of **Grz** can be axiomatized by canonical formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ or $\alpha(\mathfrak{F}, \mathfrak{D})$ such that the set X of immediate successors of $\mathfrak{F}$'s root contains ≥ 3 points and $\mathfrak{d} \in \mathfrak{D}$ for every antichain $\mathfrak{d}$ containing a subset of X with $\geq |X|/2$ points. Then L is Halldén complete.*

Proof Similar to the proof of Theorem 15.15. ❑

15.5 Exercises and open problems

Exercise 15.1 Reformulate Theorem 15.1 for quasi-normal modal logics and use it to show that **S** and **S4.1**$'$ have the disjunction property.

Exercise 15.2 Find a formula violating the disjunction property in all consistent logics in **NExtKB**.

Exercise 15.3 Which of the standard modal and si-logics have the disjunction property? Which of them are Halldén complete?

Exercise 15.4 Show that the logics of finite depth or of finite width do not have the disjunction property and that there is a continuum of such logics.

Exercise 15.5 Prove that the logics $\mathbf{ND}_k$ and $\mathbf{ND}$ have the disjunction property.

Exercise 15.6 Show that the class of si-logics with the disjunction property is not closed under intersections and sums. Show that the class of Halldén complete si-logics is not closed under intersections and sums.

Exercise 15.7 Prove that the interval $[\mathbf{Int}, L]$ contains a continuum of logics with the disjunction property and as many without it, for every $L \supset \mathbf{Int}$.

Exercise 15.8 Prove that every consistent logic with the disjunction property is contained in a maximal consistent logic with the disjunction property.

Exercise 15.9 Prove that every logic with the disjunction property is the intersection of an infinite descending chain of logics and has no immediate successors.

Exercise 15.10 Show that the implication free fragment of every si-logic with the disjunction property coincides with that of $\mathbf{Int}$.

Exercise 15.11 Construct a Kripke incomplete and an undecidable si-calculi with the disjunction property. (Hint: use the following observations. Suppose that

$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i) : i = 1, \dots, n\}$$

is a Kripke incomplete or undecidable si-logic. Then the logic

$$\widehat{L} = \mathbf{Int} + \{\beta(\widehat{\mathfrak{F}}_i, \widehat{\mathfrak{D}}_i, \bot) : i = 1, \dots, n\},$$

where $\widehat{W_i} = W_i \cup \{0, 1, 2, 3\}$, $\widehat{R}_i$ is the reflexive and transitive closure of the relation

$$R_i \cup \{\langle x, 0\rangle : x \in W_i\} \cup \{\langle 0, j\rangle : 1 \le j \le 3\}$$

and $\widehat{\mathfrak{D}}_i = \mathfrak{D}_i \cup \{\{1, 2\}\}$, has the disjunction property and retains the "negative" property of L.)

Exercise 15.12 Show that $\mathbf{Int}$ is the only consistent si-logic having the following generalized disjunction property: for any $n \ge 2$ and any formulas φ_i, ψ_i, $1 \ge i \ge n$, if $\bigwedge_{i=1}^{n}(\varphi_i \to \psi_i) \to \bigvee_{i=1}^{n} \varphi_i$ then $\bigwedge_{i=1}^{n}(\varphi_i \to \psi_i) \to \varphi_i$ for some i.

Exercise 15.13 Show that each consistent si-logic with the disjunction property has infinitely many modal companions without the disjunction property.

Exercise 15.14 Show that a normal modal logic L is Halldén complete iff, for all modal algebras $\mathfrak{A}$ and $\mathfrak{B}$ for L, there are an algebra $\mathfrak{C}$ for L and embeddings f and g of $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathfrak{C}$, respectively, such that $f(a) \le g(b)$ for no a in $\mathfrak{A}$ different from $\bot$ and b in $\mathfrak{B}$ different from $\top$.

Exercise 15.15 Show that every Post complete logic in Ext**Int** and (N)Ext**K** is Halldén complete.

Exercise 15.16 Show that the modal logic of the frame $\mathfrak{F}$ in Fig. 15.5 is not represented as an intersection of two incomparable normal logics.

Exercise 15.17 Let DP and HC denote the classes of logics that have the disjunction property and are Halldén complete, respectively. Show that there is a continuum of logics in each of the following classes: Ext**Int** $\cap$ DP, Ext**Int** $\cap$ HC $\cap$ $-$DP, Ext**Int** $-$ HC, NExt**Grz** $\cap$ HC $\cap$ DP, NExt**Grz** $\cap$ $-$HC $\cap$ DP, NExt**Grz** $\cap$ HC $\cap$ $-$DP, NExt**Grz** $\cap$ $-$HC $\cap$ $-$DP.

Exercise 15.18 Show that all normal consistent extensions of **GL** except Log$\bullet$ are Halldén incomplete.

Exercise 15.19 Show that **S** is Halldén complete.

Exercise 15.20 Say that a si-logic L has the property DP* if, for all formulas φ and ψ, $\varphi \vee \psi \in L$ implies $\neg\neg\varphi \in L$ or $\neg\neg\psi \in L$. Show that L has DP* iff $\neg p \vee \neg\neg p \notin L$.

Exercise 15.21 Say that a logic L is *Maksimova complete* if, for every formulas $\varphi_1 \to \psi_1$ and $\varphi_2 \to \psi_2$ with no variables in common, $\varphi_1 \wedge \varphi_2 \to \psi_1 \vee \psi_2 \in L$ implies $\varphi_1 \to \psi_1 \in L$ or $\varphi_2 \to \psi_2 \in L$. (It should be clear that a modal logic is Halldén complete iff it is Maksimova complete.) Show that a si-logic is Maksimova complete iff for any two rooted descriptive frames for L there exists a rooted frame for L reducible to both of them.

Exercise 15.22 Show that if formulas φ and ψ have no variables in common then, for every si-logic L, $\varphi \to \psi \in L$ implies $\neg\varphi \in L$ or $\psi \in L$.

Problem 15.1 *Does there exist a decidable maximal si-logic with the disjunction property?*

Problem 15.2 *Does there exist a finitely axiomatizable maximal si-logic with the disjunction property?*

Problem 15.3 *Is it true that a si-logic has an extension with the disjunction property iff its disjunction free fragment coincides with that of* **Int**?

Problem 15.4 *Is it true that τL is Halldén complete iff σL is Halldén complete?*

Problem 15.5 *Suppose L is Kripke complete and $\mathcal{C}$ the class of rooted frames for L. Is it true that in Theorems 15.1 and 15.5 we can always take $\mathfrak{F}$ to be a Kripke frame?*

Problem 15.6 *Are si-logics with extra axioms in one variable Halldén complete?*

15.6 Notes

The study of the disjunction property of si-logics was started by Łukasiewicz (1952) who conjectured that this property is a characteristic one for **Int** in the sense that no proper consistent extension of **Int** is constructive. The conjecture was refuted by Kreisel and Putnam (1957) who proved that both **KP** and **SL** have the disjunction property (the proof of Theorem 15.7 is due to Gabbay (1970a)). Medvedev (1963) and Varpakhovskij (1965) showed that **ML** and the realizability logic are constructive too. Gabbay and de Jongh (1974) constructed an infinite family of si-logics with the disjunction property, namely the logics $\mathbf{T}_n$ of finite n-ary trees. Ono (1972) showed that all $\mathbf{B}_n$ posses this property as well. Anderson (1972) described the constructive si-logics with extra axioms in one variable: he proved that among the consistent logics of that sort only those of the form $\mathbf{Int} + \boldsymbol{nf}_{2n+2}$, for $n \geq 5$, $n \neq 6$, and possibly $\mathbf{Int} + \boldsymbol{nf}_{14}$, have the disjunction property. Wroński (1974) completed the picture by showing that $\mathbf{Int} + \boldsymbol{nf}_{14}$ is constructive. (Another proof of this result was found by Sasaki (1992).) Finally, Wroński (1973) showed that there is a continuum si-logics with the disjunction property.

Theorem 15.1 was in essence proved in Hughes and Cresswell (1984); an algebraic variant of Theorem 15.5 is due to Maksimova (1986). That $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$ preserve the disjunction property was noted by Gudovschikiv and Rybakov (1982) and Zakharyaschev (1991). The material of Section 15.2 was taken from Zakharyaschev (1987) and Chagrov and Zakharyaschev (1993). Theorem 15.14 was independently proved by Minari (1986); a purely semantic proof can be found in Zakharyaschev (1994). Problem 15.3 was formulated by Minari (1986).

That **ML** is a maximal si-logic with the disjunction property was proved by Levin (1969); the proof of Theorem 15.18 is due to Maksimova (1986). Kirk (1982) noted that there is no greatest consistent si-logic with the disjunction property. Maksimova (1984) showed that there are infinitely many maximal constructive si-logics, and Chagrov (1992a) proved that in fact there is a continuum of them; see also Ferrari and Miglioli (1993, 1995a, 1995b). Galanter (1990) claims that each si-logic characterized by the class of frames of the form

$$\langle \{W : W \subseteq \{1, \dots, n\},\ W \neq \emptyset,\ |W| \notin N\}, \supseteq \rangle ,$$

where $n = 1, 2, \dots$ and N is some fixed infinite set of natural numbers, is maximal in the class of consistent si-logics with the disjunction property.

Theorem 15.22 was proved by Lemmon (1966c), Theorems 15.25 and 15.28 (ii) by Wroński (1976). The sufficient condition of Theorem 15.26 (formulated in terms of Kripke frames) was used by van Benthem and Humberstone (1983). Theorems 15.29–15.31 are taken from Chagrov and Zakharyaschev (1993). Exercise 15.14 is due to Maksimova (1995) who proved also algebraic characterizations for some other properties closely related to Halldén completeness. More results and references can be found in Chagrov and Zakharyaschev (1991).

Part V

Algorithmic problems

In this part we consider logics and their properties from the algorithmic point of view, i.e., we are interested in the existence of algorithms which are able to decide mass problems concerning them. Almost all algorithmic problems we have dealt with so far were solved positively by means of presenting concrete decision procedures. However, the "real algorithmic science" appears only when we need to prove that there is *no* algorithm deciding a particular problem and to estimate the efficiency of existing algorithms.

16

THE DECIDABILITY OF LOGICS

The first and perhaps most important algorithmic question arising immediately after creating a logic is the question of its decidability: is there an algorithm which is capable of deciding, given an arbitrary formula, whether it belongs to the logic or not?

16.1 Algorithmic preliminaries

So far when we considered algorithmic problems—mainly the decidability problem for various logics—we could do without a precise definition of the concept of algorithm, simply presenting some informal decision procedures. In any case the reader will most likely agree that those procedures can be realized as computer programs. But now we will be also interested in obtaining "negative" algorithmic results which assert that there are no algorithms deciding such-and-such problems. Clearly in this case we must formulate exactly what objects we are going to prove as not existing.

Of course, our intuitive idea of algorithm is too vague (and perhaps has too many traits of a rather psychological nature) to be transformed directly to a formal definition. However, many decades of using various formal versions of the notion of algorithm show that most people have more or less the same algorithmic intuition, because all of them turned out to be in a sense equivalent. So intuitive algorithmic constructions may be regarded as precisely those which can be realized in terms of one of such formalizations. This statement, known as *Church's thesis*, is clearly unprovable (though it can be disproved in principle).

By accepting Church's thesis we gain in two respects:

- to show that an algorithm exists, it suffices to present its convincing and intuitively clear description without being involved in details of any specific formalization;
- to show that an algorithm does not exist, it suffices to prove that no algorithm in a specific formal system can perform the desirable actions, i.e., to prove the absence of a mathematical object.

In this book we will use only one algorithmic formalism which is called *Minsky machines*. It has been chosen for purely technical reasons as the most convenient (from our standpoint) for being simulated by modal and intuitionistic formulas. The reader not familiar with algorithm theory and not willing to take on trust the facts formulated below without proofs should consult first a good textbook, say Cutland (1980) or Mal'cev (1970).

Algorithmically computable arithmetical partial functions are called *partial recursive functions*. The word "partial" here means that the domain of a function may be smaller than the whole set of natural numbers. Completely defined partial recursive functions are called *total recursive functions* or simply *recursive functions*. We will regard the terms "algorithm" and "partial recursive function" as synonymous. The fact that we consider only arithmetical functions is not essential. For there are various ways of reducing algorithmic operations on constructive objects (e.g. formulas or derivations) to those on natural numbers—we mean effective enumerations. However we prefer to deal with syntactical and semantic objects directly. In the former case we will assume that our languages are based on the set $\mathbf{Var} = \{p_0, p_1, \dots\}$ of variables and in the latter that frames, relations, valuations, etc. are defined by algorithms (we shall make this more precise if required). Thus we allow using such terms as "a partial recursive function from the set of pairs $\langle$formula, frame$\rangle$ into $\{0, 1\}$" and similar.

A set X is called *recursive* (or *decidable*) if there is an algorithm which, given an object x from the class under consideration, recognizes whether $x \in X$ or not. X is said to be *recursively enumerable* if one of the following equivalent conditions is satisfied:

- X is the domain of a partial recursive function;
- X is either the range of a total recursive function or empty.

The latter condition justifies the term "enumerable" in the sense that a recursive function, say f, enumerates the members of non-empty X, possibly with repetitions: $X = \{f(0), f(1), f(2), \dots\}$.

We have already used the fact that there are recursively enumerable sets (of natural numbers) which are not recursive; concrete examples will be shown later on. These two kinds of sets are connected as follows: X is recursively enumerable iff it can be represented in the form $X = \{x : \exists y\, \langle x, y\rangle \in Y\}$, for some recursive set Y of pairs. We also have the following simple proposition which may be used for proving the decidability of logics.

Proposition 16.1 *Suppose Y is a recursive set and $X \subseteq Y$. Then X is recursive iff both X and $Y - X$ are recursively enumerable.*

Proof ($\Rightarrow$) Change a decision algorithm for X in such a way that instead of the answer "no" (for the input elements from $Y - X$) it would give no answer at all entering, for instance, an endless loop. The domain of the resulting algorithm will then coincide with X, which means that X is recursively enumerable. By inserting in the original algorithm an endless loop instead of the answer "yes" we clearly obtain an algorithm whose domain is $Y - X$.

($\Leftarrow$) Here is a decision procedure for X. First check whether a given object x is in Y. If it is, run two algorithms enumerating X and $Y - X$, respectively, and wait until x appears. ❑

Now we define the algorithmic formalism that will be used in what follows for establishing various undecidability results concerning modal and si-logics.

A *Minsky machine* is a finite set (program) of instructions for transforming triples $\langle s, m, n\rangle$ of natural numbers, called *configurations*. The intended meaning of the components in the current configuration $\langle s, m, n\rangle$ is as follows: s is the number (label) of the current machine state or, which is the same, the number of the instruction to be executed at the next step, and m, n represent the current state of information. Each instruction has one of the four possible forms:

$$s \to \langle t, 1, 0\rangle\ ,\ \ s \to \langle t, 0, 1\rangle\ ,$$

$$s \to \langle t, -1, 0\rangle\ (\langle t', 0, 0\rangle),\ \ s \to \langle t, 0, -1\rangle\ (\langle t', 0, 0\rangle).$$

The last of them, for instance, means: transform $\langle s, m, n\rangle$ into $\langle t, m, n-1\rangle$ if $n > 0$ and into $\langle t', m, n\rangle$ if $n = 0$. The meaning of the others is defined analogously.

If $\boldsymbol{P}$ is a Minsky machine then the notation $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ means that starting with $\langle s, m, n\rangle$ and applying the instructions in $\boldsymbol{P}$, in finitely many steps (possibly, in 0 steps) we can reach the configuration $\langle t, k, l\rangle$. In particular, we always have $\boldsymbol{P} : \langle s, m, n\rangle \to \langle s, m, n\rangle$. If the relation $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ does not hold, we write $\boldsymbol{P} : \langle s, m, n\rangle \not\to \langle t, k, l\rangle$.

Of all possible states of a machine two are distinguished: s_1 is regarded as the only *initial state*, at which the machine starts working, and s_0 as the only *final state*, at which the machine halts. Of course, the program contains no instruction with the number s_0. If no instruction can be applied to a current non-final configuration then we will think of our machine as working forever (or being out of order and returning no result). All Minsky machines are assumed to be deterministic, i.e., they may not contain distinct instructions with the same numbers.

Now, which arithmetical partial functions are computable by Minsky machines? The answer is the following statement which, in view of our definition of partial recursive functions and the known fact that Minsky machines are equivalent to any other universal algorithmic formalism, can be called the *Church–Minsky thesis*:

- an arithmetical partial function $f(x)$ is a partial recursive function iff there is a Minsky machine $\boldsymbol{P}$ such that, for every natural x, if $f(x)$ is defined then $\boldsymbol{P} : \langle s_1, 2^x, 0\rangle \to \left\langle s_0, 2^{f(x)}, 0\right\rangle$ and if $f(x)$ is undefined then the machine, having started at $\langle s_1, 2^x, 0\rangle$, never comes to the final state.

Using this statement, by the standard argument we can prove the undecidability of various problems concerning Minsky machines. First we have the undecidability of the *configuration problem*:

Theorem 16.2 *There is no algorithm which, given a program $\boldsymbol{P}$ and configurations $\langle s, m, n\rangle$ and $\langle t, k, l\rangle$, can decide whether $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ holds.*

This theorem may be used for establishing a lot of our further undecidability results, but not all of them. It will be much more convenient to use a variant of the configuration problem with fixed suitable $\boldsymbol{P}$ and $\langle s, m, n\rangle$, called the *second configuration problem*:

Theorem 16.3 *There exist a program $\boldsymbol{P}$ and a configuration $\langle s, m, n\rangle$ such that there is no algorithm which is capable of deciding, given a configuration $\langle t, k, l\rangle$, whether $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$.*

Proof Let X be a recursively enumerable non-recursive set and $g(x)$ a recursive function enumerating X, with $g(0) = a$. Define a partial recursive function $f(x)$ as follows. Given x, we compute $g(0), g(1), \dots$ until we get $x = g(m_1)$ for some number m_1 and then continue computing $g(m_1 + 1), g(m_1 + 2), \dots$ until $g(m_2) \notin \{g(0), \dots, g(m_1)\}$ for some m_2. When (and if) this process stops, we put $f(x) = g(m_2)$ (otherwise $f(x)$ is undefined). Clearly $f(x)$ is a partial recursive function and $X = \{a, f(a), f(f(a)), \dots\}$.

Let $\boldsymbol{P}'$ be a Minsky program computing $f(x)$. Define another program $\boldsymbol{P}$ by renaming s_0 in $\boldsymbol{P}'$ into s' (not occurring in $\boldsymbol{P}'$) and adding two new instructions:

$$s' \to \langle s'', 0, 1\rangle\,,\ s'' \to \langle s_1, 0, -1\rangle\, (\langle s_1, 0, 0\rangle),$$

where s'' is a new state. Notice that $\boldsymbol{P}$ does not have a final state and, having started at the configuration $\langle s_1, 2^a, 0\rangle$, it works forever. But more important is that

$$X = \{a\} \cup \{x :\ \boldsymbol{P} : \langle s_1, 2^a, 0\rangle \to \langle s', 2^x, 0\rangle\}.$$

Thus, if the second configuration problem for $\boldsymbol{P}$ and $\langle s_1, 2^a, 0\rangle$ were decidable, the set X would be recursive. ❑

We shall also require two variants of the *halting problem.*

Theorem 16.4 *There is a Minsky machine $\boldsymbol{P}$ such that no algorithm can recognize, given an arbitrary configuration $\langle s, m, n\rangle$, whether $\boldsymbol{P}$ comes to the final state having started at $\langle s, m, n\rangle$.*

Theorem 16.5 *There is a configuration $\langle s, m, n\rangle$ such that no algorithm can recognize, given a Minsky machine $\boldsymbol{P}$, whether $\boldsymbol{P}$ comes to the final state having started at $\langle s, m, n\rangle$.*

To prove Theorem 16.4 it suffices to take a recursive function enumerating a non-recursive set and use the Church–Minsky thesis. As to Theorem 16.5, one can exploit the following statement.

Call a property of Minsky machines *non-trivial* if there are machines both with this property and without it. A property is called *invariant* if equivalent machines have (or do not have) the property simultaneously. Here by equivalent machines we mean those which, having started at the same initial configuration, come to the same final configuration or never stop. Thus an invariant property depends not on the intrinsic organization of programs, but on what they compute.

Theorem 16.6. (The Rice–Uspensky theorem) *For every non-trivial invariant property of Minsky machines, there is no algorithm which, given an arbitrary program, can decide whether is satisfies the property or not.*

16.2 Proving decidability

Observe first that "most logics" are undecidable. For there are "only" countably many algorithms (they may be considered as words in a fixed finite alphabet) but uncountably many logics. Moreover, for the same reason "most logics" are not even recursively enumerable. Fortunately the "most interesting logics" form a countable family and so this cardinality argument does not go through for them.

In this section we analyze from the recursion-theoretic point of view the method of proving decidability we have used so many times before.

To begin with, we enumerate formulas. Every formula in **For**$\mathcal{ML}$ may be regarded as a word (a string of symbols) in the alphabet

$$p,\ \wedge,\ \vee,\ \to,\ \bot,\ \Box,\ |,\ (,\ )$$

where $|$ is a symbol for generating subscripts: p_0 is represented as p, p_1 as $p|$, p_2 as $p||$, etc. Of course, using two or more special signs instead of $|$ we could write formulas shorter. But in principle this does not matter: for any finite alphabet we can effectively determine whether a given string of symbols is a formula. Writing down all possible strings—first of length 1, then of length 2, etc.—and discarding those that are not formulas, we can effectively enumerate all formulas in **For**$\mathcal{ML}$ or **For**$\mathcal{L}$. Thus we obtain

Lemma 16.7 **For**$\mathcal{ML}$ *and* **For**$\mathcal{L}$ *are recursively enumerable (without repetitions). Moreover, these sets are recursive.*

Now we consider enumerations of formulas in logics.

Lemma 16.8 *Every logic L with a recursively enumerable set of axioms is also recursively enumerable.*

Proof Notice first that every derivation in L may be regarded as a word in the alphabet $\mathcal{A}$ of L's language with the extra symbol "," used for separating formulas in derivations. So we have a recursive enumeration of L's axioms, say, $\varphi_0, \varphi_1, \varphi_2, \ldots$ and a recursive enumeration $w_0, w_1, w_2, \ldots$ of all words in $\mathcal{A}$. Now, for every $n \geq 0$ we select from $w_0, \ldots, w_n$ all those derivations in L which use only axioms in the list $\varphi_0, \ldots, \varphi_n$. (To check whether a formula ψ is a substitution instance of an axiom φ, it suffices to write down all the substitution instances of φ of length not greater than that of ψ and compare them with ψ.) Since every derivation uses only finitely many axioms, sooner or later it will be found. Thus we recursively enumerate all the derivations in L and thereby L itself. ❑

Strange as it may seem at first sight, there is no difference between recursively enumerable axiomatizations and recursive ones.

Lemma 16.9 *Every recursively enumerable logic L is recursively axiomatizable (i.e., has a recursive set of axioms).*

Proof Let $\varphi_1, \varphi_2, \ldots$ be a recursive enumeration of L. For every $n \geq 1$, put

$$\psi_n = \underbrace{\varphi_n \wedge \ldots \wedge \varphi_n}_{n}.$$

Since the rules $p/p \wedge p$ and $p \wedge p/p$ are admissible (and derivable) in all modal and si-logics, $\{\psi_n : 1 \leq n < \omega\}$ is a set of axioms for L. This axiomatization is recursive because to verify whether a formula ψ is an axiom it suffices to represent ψ as a conjunction $\chi_1 \wedge \ldots \wedge \chi_k$ in all possible ways (there are finitely many of them), generate φ_k and compare ψ with ψ_k. ❑

Putting together Lemmas 16.8 and 16.9 we obtain

Theorem 16.10. (Craig's theorem) *For every logic L the following conditions are equivalent:*

(i) *L has a recursively enumerable set of axioms;*

(ii) *L has a recursive set of axioms;*

(iii) *L is recursively enumerable.*

Remark It should be clear that Theorem 16.10 remains true if we take axiom schemes rather than axioms. Also we can consider in Theorem 16.10 axiomatizations of L over some fixed *recursively enumerable* logic $L_0 \subseteq L$; without the requirement of recursive enumerability only (i) and (ii) are equivalent.

To apply Proposition 16.1 for establishing the decidability of a logic L we must be able to enumerate recursively not only L itself but also its complementation, i.e., the set of formulas which do not belong to L. In the majority of the decidability proofs above we managed to do without this, using effective characterizations of (finite) frames for L, upper bounds for the size of minimal frames separating L from formulas out of L and the following:

Theorem 16.11 *Suppose a logic L is characterized by a recursive class C of finite frames and there is a recursive function $f(x)$ such that every $\varphi \notin L$ is refuted in a frame $\mathfrak{F} \in C$ with $|\mathfrak{F}| \leq f(l(\varphi))$. Then L is decidable.*

Proof Given φ, we construct all finite frames with $\leq f(l(\varphi))$ points, discard those that are not in C and check whether φ is refuted in at least one of the remaining frames. ❑

However, actually we do not need upper bounds to establish decidability. Usually even finite approximability is enough. For we clearly have

Lemma 16.12 (i) *The class of all finite algebras (matrices, frames) is recursively enumerable.*

(ii) *If L is characterized by a recursively enumerable class of finite algebras (matrices, frames) then the set of formulas which do not belong to L is recursively enumerable.*

Using this observation we obtain

Theorem 16.13. (Harrop's theorem) *Every finitely axiomatizable and finitely approximable logic L is decidable.*

Proof By Theorem 16.10, L is recursively enumerable. From the recursively enumerable sequence of all finite frames we can remove all those that are not frames for L simply by checking whether they validate L's axioms. Thus L is characterized by a recursively enumerable class of finite frames and so L's complementation is recursively enumerable too. ❑

Theorem 11.19 (ii) shows that the requirement of finite axiomatizability in Harrop's theorem cannot be replaced with that of recursive axiomatizability: there are undecidable recursively axiomatizable subframe logics. The reason for this phenomenon is that the class of finite frames characterizing a given recursively axiomatizable subframe logic is not necessarily even recursively enumerable. On the other hand, the single requirement that a logic is characterized by a recursive class of finite frames does not mean that the logic is decidable either.

Theorem 16.14 *There is a logic which is characterized by a recursive set of finite frames and which is not recursively enumerable.*

Proof We require the following:

Lemma 16.15 *There is a recursive set X of natural numbers such that the set $\{|x - y| : x, y \in X, x \neq y\}$ is not recursive.*

Proof Let $f(x)$ be a recursive function whose range is not recursive. Notice that the set

$$Y = \{10^{10^{f(n)+1}} : n < \omega\}$$

is recursively enumerable but not recursive. Put

$$X = \{10^{4n+1}, 10^{4n+1} + 10^{10^{f(n)+1}} : n < \omega\}.$$

It is not hard to see that X is decidable. On the other, hand we have

$$Y = \{|x - y| : x, y \in X, x \neq y\} \cap \{10^n : n < \omega\}.$$

Since the intersection of recursive sets is also recursive, it follows that the set $\{|x - y| : x, y \in X, x \neq y\}$ cannot be recursive. ❑

Now take the set X constructed in the proof of Lemma 16.15 and, for every $n < \omega$, define a frame $\mathfrak{F}_n$ as is shown in Fig. 16.1, where $\{i, \dots, j\} = \{x \in X : x < n\}$. Clearly the set $\mathcal{C} = \{\mathfrak{F}_n : n < \omega\}$ is recursive. We show that the logic $L = \mathrm{Log}\mathcal{C}$ is not recursively enumerable. By Lemma 16.12 (ii) and Proposition 16.1, it suffices to prove that L is not decidable.

For every $n < \omega$, we put

$$\varphi_n = \neg(\Diamond(\Diamond^2\psi \wedge \neg\Diamond(\Diamond\psi \wedge \neg\Diamond^2\psi)) \wedge \Diamond(\Diamond^2\chi_n \wedge \neg\Diamond(\Diamond\chi_n \wedge \neg\Diamond^2\chi_n))),$$

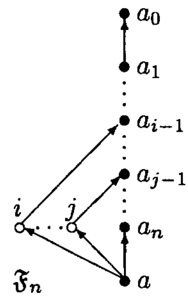


FIG. 16.1.

where $\psi = p \wedge \Box\neg p$ and $\chi_n = \Diamond^n\psi \wedge \neg\Diamond^{n+1}\psi$. The reader can readily check that

$$\varphi_n \in L \text{ iff } n \in \{|x-y| : \ x, y \in X, \ x \neq y\}.$$

By Lemma 16.15, it follows that L is undecidable. ❑

However, we do not know whether the class of all finite frames for the logic constructed in the proof above is recursive. In this connection the following result is worth noting.

Theorem 16.16 *Suppose L is a recursively axiomatizable finitely approximable logic in* NExt**K4** *or* Ext**Int**. *Then L is decidable iff the class of all finite frames for L is decidable.*

Proof ($\Leftarrow$) follows from Theorem 16.10 and Lemma 16.12 and ($\Rightarrow$) is a consequence of the fact that a finite rooted frame $\mathfrak{F}$ validates L iff $\alpha^\sharp(\mathfrak{F}, \bot) \notin L$ ($\beta^\sharp(\mathfrak{F}, \bot) \notin L$). ❑

Lemma 16.12 (ii) can be extended to logics characterized by classes of frames or algebras effectively determined by algorithms. Say that a (pseudo-Boolean or modal) algebra is *recursive* if its universe is a recursive set and the operations are realized by some algorithms (in particular, there are algorithms computing $\top$ and $\bot$). Thus a recursive algebra may be thought of as a suitable collection of algorithms. A class of recursive algebras is called *recursively enumerable* if there is an algorithm enumerating the collections of algorithms corresponding to those algebras. A matrix $\langle\mathfrak{A}, \nabla\rangle$ is *recursive* if both $\mathfrak{A}$ and ∇ are recursive. *Recursive frames* can be defined in the same manner.

Lemma 16.17 *If a logic L is characterized by a recursively enumerable class C of recursive algebras (matrices, frames) then the set of formulas that are not in L is also recursively enumerable.*

Proof Let $\varphi_0, \varphi_1, \ldots$ be an effective enumeration of formulas, $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ an effective enumeration of algebras in C and, for every $i < \omega$, let $a^i_0, a^i_1, \ldots$ be an effective enumeration of elements in $\mathfrak{A}_i$. An algorithm enumerating all formulas that are not in L may be as follows. For every $n < \omega$ and every $i, j \leq n$ we compute the value of φ_i in $\mathfrak{A}_j$ under all possible assignments of the elements $a_0^j, \dots, a_n^j$ to φ_i's variables. And if this value is different from $\top$, the algorithm returns φ_i as its next value. ❑

Using this lemma we obtain the most general criterion of decidability.

Theorem 16.18 *A logic is decidable iff it is recursively axiomatizable and characterized by a recursively enumerable class of recursive algebras (matrices).*

Proof ($\Leftarrow$) follows from Theorem 16.10 and Lemma 16.17. And to prove ($\Rightarrow$) it suffices to observe that the Tarski–Lindenbaum algebra for a decidable logic is also decidable (we can fix an effective enumeration of all formulas and construct the universe of $\mathfrak{A}_L$ from the formulas φ such that φ has the smallest number in $\|\varphi\|_L$). ❑

16.3 Logics containing K4.3

Now we use the observations of the preceding section to show how a good completeness result can be used for establishing decidability even in the absence of finite approximability.

We consider the class NExt**K4.3** of normal modal logics of width 1. According to Theorem 6.2, not all of them are finitely approximable, and so the standard way of proving decidability by using Harrop's theorem does not go through for logics in the class. Let us recall, however, that the essential point in the proof of Harrop's theorem was not that a logic is complete with respect to the class of *finite* frames but that (i) this class is recursively enumerable and (ii) we can always check effectively whether a frame in the class validates a given formula. By Fine's theorem of Section 10.4, all logics in NExt**K4.3** are complete for the class of Kripke frames of width 1 without infinite ascending chains, i.e., Noetherian chains of clusters. This class does not meet the conditions (i) and (ii). What we are going to prove below is that it contains a subclass which satisfies both (i) and (ii) and is still big enough to ensure completeness.

We will use the apparatus of canonical formulas. Each logic $L \in$ NExt**K4.3** can be represented in the form

$$L = \mathbf{K4.3} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}, \tag{16.1}$$

where all $\mathfrak{F}_i$ are chains of clusters. For a finitely axiomatizable L representation (16.1) with finite I can be constructed effectively, given a finite set of L's axioms.

The following simple example explains in terms of canonical formulas why logics of width 1 are not necessarily finitely approximable.

Example 16.19 Let us consider the logic $L = \mathbf{K4.3} \oplus \alpha(\mathfrak{F}, \{\{1\}\}, \bot)$ and the formula $\alpha(\mathfrak{F}, \bot)$ where $\mathfrak{F}$ is the frame depicted in Fig. 16.2 (a). The frame $\mathfrak{G}$ shown in Fig. 16.2 (b) separates $\alpha(\mathfrak{F}, \bot)$ from L. Indeed, $\mathfrak{F}$ is a cofinal subframe of $\mathfrak{G}$ which, by the refutability criterion (Theorem 9.39), gives $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$. To establish that $\mathfrak{G} \models \alpha(\mathfrak{F}, \{\{1\}\}, \bot)$, suppose f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$.

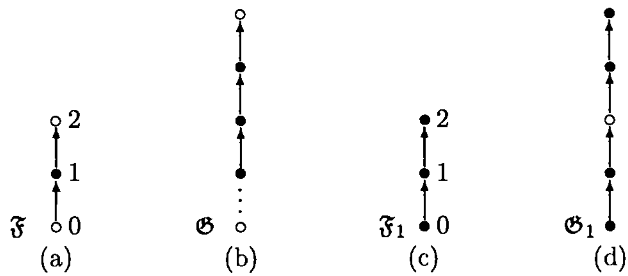


FIG. 16.2.

Then $f^{-1}(1)$ contains only one point, say x; $f^{-1}(0)$ also contains only one point, namely the root of $\mathfrak{G}$. So the whole infinite set of points between x and the root is outside of $\mathrm{dom} f$, which means that f does not satisfy (CDC) for $\{\{1\}\}$.

On the other hand, if $\mathfrak{H}$ is a finite refutation frame (of width 1) for $\alpha(\mathfrak{F},\bot)$ then $\mathfrak{H}$ contains a non-degenerate cluster C having an irreflexive immediate successor x, and by mapping C to 0, x to 1 and all the points above x to 2 we obtain a cofinal subreduction of $\mathfrak{H}$ to $\mathfrak{F}$ satisfying (CDC) for $\{\{1\}\}$, from which $\mathfrak{H} \not\models L$.

Returning to our completeness problem, let us observe that the refutability criterion for canonical formulas may be somewhat simplified if we deal only with Noetherian chains of clusters. Say that a subreduction f of one frame to another is *injective* if $f(x) \neq f(y)$ for every distinct $x, y \in \mathrm{dom} f$.

Theorem 16.20 *For any Noetherian chain of clusters $\mathfrak{G}$ and any canonical formula $\alpha(\mathfrak{F},\mathfrak{D},\bot)$, $\mathfrak{G} \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$ iff there is an injective cofinal subreduction g of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.*

Proof ($\Rightarrow$) Suppose $\mathfrak{G} \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$. Then there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. We reduce f to a map g so that $g^{-1}(x)$ will be a singleton for every point x in $\mathfrak{F}$. Observe first that $f^{-1}(x)$ must be a singleton if x is irreflexive (here we use the fact that $\mathfrak{G}$ is a chain of clusters.) Suppose now that x is a reflexive point in $\mathfrak{F}$. Since $\mathfrak{G}$ contains no infinite ascending chains, $f^{-1}(x)$ has a finite cover and so there is a reflexive point $u_x \in f^{-1}(x)$ such that $f^{-1}(x) \subseteq u_x\!\downarrow$. Fix such a u_x for each reflexive x in $\mathfrak{F}$ and define g by taking, for any y in $\mathfrak{G}$,

$$g(y) = \begin{cases} f(y) & \text{if either } f(y) \text{ is irreflexive or} \\ & f(y) \text{ is reflexive and } y = u_{f(y)} \\ \text{undefined otherwise.} & \end{cases}$$

It should be clear from the definition that g is an injective cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$.

Suppose $y \in \mathrm{dom} g\!\uparrow$ and $g(y\!\uparrow) = x\underline{\uparrow}$ for some $\{x\} \in \mathfrak{D}$. Then x is irreflexive and we must have $y \in \mathrm{dom} f\!\uparrow$, $f(y\!\uparrow) = x\underline{\uparrow}$, from which $y \in \mathrm{dom} f$, since f satisfies (CDC) for $\mathfrak{D}$. It follows that $f(y)$ is irreflexive (for otherwise $x \in f(y)\!\uparrow$

and there is $z \in y\uparrow$ such that $f(z) = f(y)$ which together with $f(y\uparrow) = \underline{x\uparrow}$ implies $f(y) \in x\uparrow$, contrary to x being irreflexive) and so $y \in \mathrm{dom}g$. Thus $\overline{g}$ satisfies (CDC) for $\mathfrak{D}$.

($\Leftarrow$) follows from the refutability criterion. ❑

Theorem 16.20 may be interpreted in the following way. Every Noetherian chain of clusters refuting $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ can be obtained from $\mathfrak{F}$ by inserting some Noetherian chains of clusters just below clusters $C(x)$ in $\mathfrak{F}$ such that $\{x\} \notin \mathfrak{D}$ and by enlarging some non-degenerate clusters in $\mathfrak{F}$.

We show now that if a formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is not in $L \in$ NExt**K4.3** then it can be separated from L by a frame constructed from $\mathfrak{F}$ by inserting in open domains between its adjacent clusters either finite descending chains of irreflexive points possibly ending with a reflexive one or infinite descending chains of irreflexive points and without using the operation of enlarging $\mathfrak{F}$'s non-degenerate clusters.

Let $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ be a canonical formula built upon a chain of clusters $\mathfrak{F}$ and $C(x_0), \dots, C(x_n)$ all distinct clusters in $\mathfrak{F} = \langle W, R\rangle$ ordered in such a way that $C(x_0) \subset \overline{C(x_1)\downarrow} \subset \dots \subset \overline{C(x_n)\downarrow}$. By a *type* for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ we will mean any n-tuple $t = \langle \xi_1, \dots, \xi_n\rangle$ such that, for $i \in \{1, \dots, n\}$, either $\xi_i = m$ or $\xi_i = m+$, for some $m < \omega$, or $\xi_i = \omega$, with $\xi_i = 0$ if $\{x_i\} \in \mathfrak{D}$.

Given a type $t = \langle \xi_1, \dots, \xi_n\rangle$ for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, we define a *t-extension* of $\mathfrak{F}$ as the frame $\mathfrak{G}$ that is obtained from $\mathfrak{F}$ by inserting between each pair $C(x_{i-1})$, $C(x_i)$ of $\mathfrak{F}$'s adjacent clusters either a descending chain of m irreflexive points, if $\xi_i = m < \omega$, or a descending chain of $m + 1$ points of which only the last (lowest) one is reflexive, if $\xi_i = m+$, or an infinite descending chain of irreflexive points, if $\xi_i = \omega$. More formally the t-extension $\mathfrak{G} = \langle V, S\rangle$ of $\mathfrak{F}$ may be defined as follows. For $1 \le i \le n$, we first put

$$V_i = \begin{cases} \{a^i_j : \ 0 < j \le m\} & \text{if } \xi_i = m < \omega \\ \{a^i_j, b^i : \ 0 < j \le m\} & \text{if } \xi_i = m+ \\ \{a^i_j : \ 0 < j < \omega\} & \text{if } \xi_i = \omega, \end{cases}$$

$$S_i = \{\langle b^i, b^i\rangle, \langle b^i, a^i_j\rangle, \langle a^i_j, a^i_k\rangle : \ b^i, a^i_j, a^i_k \in V_i, \ j > k\}.$$

And then

$$V = W \cup \bigcup_{i=1}^{n} V_i$$

and S is the transitive closure of the relation

$$R \cup \bigcup_{i=1}^{n} S_i \cup \bigcup_{i=1}^{n} \{\langle x_{i-1}, x\rangle, \langle x, x_i\rangle : \ x \in V_i\}.$$

Example 16.21 The frame $\mathfrak{G}$ in Fig. 16.2 (b) is the t-extension of $\mathfrak{F}$ in Fig. 16.2 (a), for every $t = \langle \omega, n\rangle$, $0 \le n < \omega$, which clearly is a type for $\alpha(\mathfrak{F}, \bot)$. $\mathfrak{G}_1$ in Fig. 16.2 (d) is the $\langle 0, 1+\rangle$-extension of $\mathfrak{F}_1$ in Fig. 16.2 (c), with $\langle 0, 1+\rangle$ being a type for $\alpha(\mathfrak{F}_1, \{\{1\}\}, \bot)$.

It should be clear that, for every type t for $\alpha(\mathfrak{F},\mathfrak{D},\bot)$, the t-extension of $\mathfrak{F}$ refutes $\alpha(\mathfrak{F},\mathfrak{D},\bot)$.

The following trivial observation will be used several times below.

Lemma 16.22 *Suppose $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ is a canonical formula and f a cofinal subreduction of a frame $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Suppose also that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$ containing* $\mathrm{dom} f$. *Then f is also a cofinal subreduction of $\mathfrak{H}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.*

Proof Exercise. ❑

Theorem 16.23 *Suppose $L \in$* **NExtK4.3** *and $\alpha(\mathfrak{F},\mathfrak{D},\bot) \notin L$. Then $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ is separated from L by the t-extension of $\mathfrak{F}$, for some type t for $\alpha(\mathfrak{F},\mathfrak{D},\bot)$.*

Proof Since $\alpha(\mathfrak{F},\mathfrak{D},\bot) \notin L$, we have a Noetherian chain of clusters $\mathfrak{G} = \langle V, S\rangle$ separating $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ from L. By Theorem 16.20, there is an injective cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. By the generation theorem, without loss of generality we may assume that f maps the root of $\mathfrak{G}$ to the root of $\mathfrak{F}$.

Let $\mathfrak{G}_0 = \langle V_0, S_0\rangle$ be the (cofinal) subframe of $\mathfrak{G}$ obtained by removing from $\mathfrak{G}$ all those points that are not in $\mathrm{dom} f$ but belong to clusters containing some points in $\mathrm{dom} f$, or formally

$$V_0 = V - \bigcup_{x\in\mathrm{dom} f} (C(x) - \mathrm{dom} f)$$

and S_0 is the restriction of S to V_0. By Lemma 16.22, the very same map f is an injective cofinal subreduction of $\mathfrak{G}_0$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$, and so $\mathfrak{G}_0 \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$. It should be also clear that $\mathfrak{G}_0$ is a reduct of $\mathfrak{G}$, and hence $\mathfrak{G}_0 \models L$.

Let $C(x_0),\dots,C(x_n)$ be all the distinct clusters in $\mathfrak{G}_0$ such that

$$\mathrm{dom} f = \bigcup_{i=0}^{n} C(x_i)$$

and $C(x_0) \subset C(x_1)\bar{\downarrow} \subset \dots \subset C(x_n)\bar{\downarrow}$. By induction on i we define now a sequence of frames $\mathfrak{G}_0 \supseteq \dots \supseteq \mathfrak{G}_n$ such that, for each $i \le n$,

(a) f is an injective cofinal subreduction of $\mathfrak{G}_i$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$;

(b) between $C(x_{i-1})$ and $C(x_i)$ the frame $\mathfrak{G}_i$ contains either a finite descending chain of irreflexive points possibly ending with a reflexive one or an infinite descending chain of irreflexive points;

(c) $\mathfrak{G}_i \models L$.

Suppose $\mathfrak{G}_{i-1} = \langle V_{i-1}, S_{i-1}\rangle$ has been already constructed and $i \le n$. Take the chain $\mathfrak{C}_i = \langle W_i, R_i\rangle$ of clusters located between $C(x_{i-1})$ and $C(x_i)$, i.e., $W_i = C(x_i)\downarrow - (C(x_i) \cup C(x_{i-1})\bar{\downarrow})$ and R_i is the restriction of S_{i-1} to W_i. Three cases are possible.

Case 1. $\mathfrak{C}_i$ is a finite chain of irreflexive points. Then nothing should be done with $\mathfrak{G}_{i-1}$: we just put $\mathfrak{G}_i = \mathfrak{G}_{i-1}$.

Case 2. $\mathfrak{C}_i$ contains a non-degenerate cluster $C(x)$ having only finitely many distinct successors $y_1, \dots, y_m$ in $\mathfrak{C}_i$, all of them being irreflexive. Then we put $\mathfrak{G}_i = \langle V_i, S_i \rangle$, where $V_i = (V_{i-1} - W_i) \cup \{y_1, \dots, y_m, x\}$ and S_i is the restriction of S_{i-1} to V_i. The conditions (a) and (b) are clearly satisfied by $\mathfrak{G}_i$. To show (c) it suffices to observe that $\mathfrak{G}_i$ is a reduct of $\mathfrak{G}_{i-1}$ (just map all the points in $\mathfrak{G}_i$ to themselves and those removed from $\mathfrak{G}_{i-1}$ to x) and use the reduction theorem.

Case 3. Suppose Cases 1 and 2 do not hold. Then, since $\mathfrak{C}_i$ is a Noetherian chain of clusters, it must contain an infinite descending chain Y of irreflexive points such that every point in $W_i - Y$ sees all the points in Y.

Put $\mathfrak{G}_i = \langle V_i, S_i \rangle$ where $V_i = (V_{i-1} - W_i) \cup Y$ and S_i is the restriction of S_{i-1} to V_i. Again $\mathfrak{G}_i$ clearly satisfies (a) and (b). To prove (c) suppose $\mathfrak{G}_i \not\models \alpha(\mathfrak{H}, \mathfrak{C}, \bot)$ for some $\alpha(\mathfrak{H}, \mathfrak{C}, \bot) \in L$. Then there is an injective cofinal subreduction g of $\mathfrak{G}_i$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$. Consider g as a cofinal subreduction of $\mathfrak{G}_{i-1}$ to $\mathfrak{H}$ and show that it also satisfies (CDC) for $\mathfrak{C}$. Indeed, the closed domain condition could be violated only by a point in $W_i - Y$. So suppose $z \in W_i - Y$ and $g(z{\uparrow}) = w{\uparrow}$, for some $\{w\} \in \mathfrak{C}$. Since $g^{-1}(w)$ is a singleton and $Y \subseteq z{\uparrow}$, there is a point $y \in Y$ such that $g(y{\uparrow}) = w{\uparrow}$ and $y \notin \mathrm{dom} g$, contrary to g satisfying (CDC) for $\mathfrak{C}$ as a subreduction of $\mathfrak{G}_i$ to $\mathfrak{H}$. It follows from the definition that $\mathfrak{G}_n$ is the t-extension of $\mathfrak{F}$, for some type t for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Hence $\mathfrak{G}_n \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Besides, we have $\mathfrak{G}_n \models L$. ❑

Thus, a frame separating a formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$ from $L \in$ NExt**K4.3** can be found in the class of t-extensions of $\mathfrak{F}$ (t being a type for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$), which is clearly recursively enumerable. We show now that there is an algorithm which, given a formula $\alpha(\mathfrak{H}, \mathfrak{C}, \bot)$ and a type t for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, decides whether $\alpha(\mathfrak{H}, \mathfrak{C}, \bot)$ is valid in the t-extension of $\mathfrak{F}$.

Let k be the number of irreflexive points in $\mathfrak{H}$, $t = \langle \xi_1, \dots, \xi_n \rangle$ a type for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\mathfrak{G}$ the t-extension of $\mathfrak{F}$ formed according to the formal definition above. Construct a cofinal subframe $\mathfrak{G}_k$ of $\mathfrak{G}$ by "cutting off" the infinite descending chains inserted in $\mathfrak{F}$ (if any) just below their $k+1$th points and let X be the set of all these $k+1$th points. In other words, $\mathfrak{G}_k$ is the s-extension of $\mathfrak{F}$, the type $s = \langle \zeta_1, \dots, \zeta_n \rangle$ defined by

$$\zeta_i = \begin{cases} k+1 \text{ if } \xi_i = \omega \\ \xi_i \quad \text{otherwise,} \end{cases}$$

and $X = \{a^i_{k+1} : \xi_i = \omega\}$. It follows from the definition that $\mathfrak{G}_k$ is finite.

Theorem 16.24 $\mathfrak{G} \not\models \alpha(\mathfrak{H}, \mathfrak{C}, \bot)$ *iff there is an injective cofinal subreduction f of $\mathfrak{G}_k$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$ and such that $X \cap \mathrm{dom} f = \emptyset$.*

Proof ($\Rightarrow$) Let $\mathfrak{G} \not\models \alpha(\mathfrak{H}, \mathfrak{C}, \bot)$. By Theorem 16.20, there is an injective cofinal subreduction f of $\mathfrak{G} = \langle V, S \rangle$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$. By Lemma 16.22, without loss of generality we may assume that if $\xi_i = \omega$ then $V_i \cap \mathrm{dom} f =$

$\{a_1^i, \dots, a_m^i\}$, for some $m \le k$. Now, by "cutting off" all the infinite descending chains V_i, for $\xi_i = \omega$, just below a_{k+1}^i, we obtain $\mathfrak{G}_k$, with f being an injective cofinal subreduction of $\mathfrak{G}_k$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$ and $X \cap \mathrm{dom} f = \emptyset$.

($\Leftarrow$) If f is an injective cofinal subreduction of $\mathfrak{G}_k$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$ and $X \cap \mathrm{dom} f = \emptyset$ then clearly f is also an injective cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{H}$. We show that it satisfies (CDC) for $\mathfrak{C}$ as well. Only the points in the "cut off" tails of infinite descending chains $V_i = \{a_1^i, a_2^i, \dots\}$ should be verified. So suppose $m > k+1$ and $f(a_m^i\uparrow) = x\underline{\uparrow}$, for some $\{x\} \in \mathfrak{C}$. Since $\{a_{k+1}^i, \dots, a_m^i\} \cap \mathrm{dom} f = \emptyset$, we must then have $f(a_{k+1}^i\uparrow) = x\underline{\uparrow}$ and, by (CDC), $a_{k+1}^i \in \mathrm{dom} f$, contrary to $X \cap \mathrm{dom} f = \emptyset$. ❑

Thus, given a type t for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and a canonical formula $\alpha(\mathfrak{H}, \mathfrak{C}, \bot)$, only finitely many steps is required to verify whether the t-extension of $\mathfrak{F}$ refutes $\alpha(\mathfrak{H}, \mathfrak{C}, \bot)$, and so we can prove the decidability of finitely axiomatizable logics in NExt**K4.3** using Harrop's argument.

Theorem 16.25 *All finitely axiomatizable normal extensions of* **K4.3** *are decidable.*

Proof Exercise. ❑

16.4 Undecidable calculi and formulas above K4

We are going to construct an undecidable finitely axiomatizable logic in NExt**K4** by simulating the behaviour of Minsky machines in frames, describing points in those frames by modal formulas and using the undecidability of the configuration problems.

With each Minsky program $\boldsymbol{P}$ and configuration $\langle s, m, n\rangle$ we associate the transitive frame $\mathfrak{F}$ depicted in Fig. 16.3. The meaning of points in $\mathfrak{F}$ is as follows. Its subframe consisting of the points b_0 (the only reflexive point in $\mathfrak{F}$), $b_1, \dots, b_4$ and r will be used for characterizing all points in $\mathfrak{F}$ by variable free formulas. The points of the form $e(t, k, l)$ represent configurations $\langle t, k, l\rangle$ such that $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$; $e(t, k, l)$ sees the points a_t^0, a_k^1, a_l^2 representing the components of $\langle t, k, l\rangle$. Note that $\mathfrak{F}$ contains the points a_j^i for any $i = 0, 1, 2$ and $j \ge -1$, but only those $e(t, k, l)$ for which $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$.

Here are variable free formulas characterizing points in $\mathfrak{F}$ in the sense that each of these formulas, denoted by Greek letters with subscripts and/or superscripts, is true in $\mathfrak{F}$ only at the point denoted by the corresponding Roman letter with the same subscripts and/or superscripts:

$$\beta_0 = \Diamond\top \wedge \Box\Diamond\top,\ \beta_1 = \Box\bot,\ \beta_2 = \Diamond\top \wedge \Box^2\bot,$$

$$\beta_3 = \Diamond\beta_0 \wedge \Diamond\beta_1 \wedge \neg\Diamond^2\beta_1,\ \beta_4 = \Diamond\beta_2 \wedge \Diamond\beta_3 \wedge \neg\Diamond^2\beta_2 \wedge \neg\Diamond^2\beta_3,$$

$$\rho = \Diamond\beta_4,\ \gamma_1 = \Diamond\beta_2 \wedge \neg\Diamond^2\beta_2 \wedge \neg\Diamond\beta_3,\ \gamma_2 = \Diamond\gamma_1 \wedge \neg\Diamond^2\gamma_1 \wedge \neg\Diamond\beta_3,$$

$$\gamma_3 = \Diamond\gamma_2 \wedge \neg\Diamond^2\gamma_2 \wedge \neg\Diamond\beta_3,\ \delta_1 = \Diamond\beta_3 \wedge \neg\Diamond^2\beta_3 \wedge \neg\Diamond\beta_2,$$

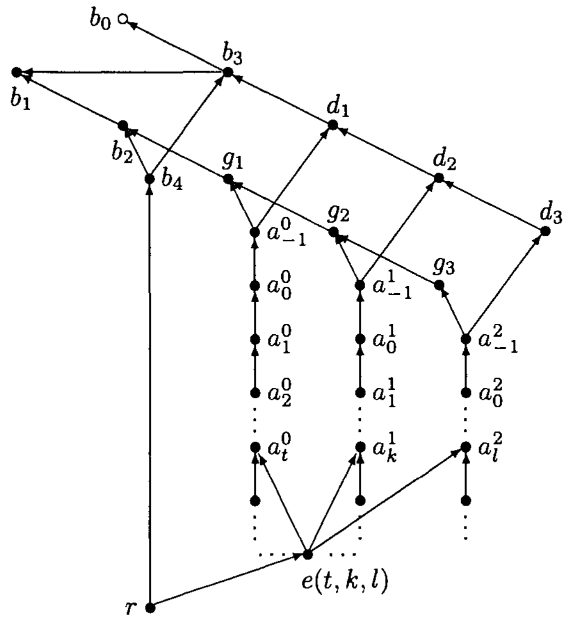


FIG. 16.3.

$$\delta_2 = \Diamond\delta_1 \wedge \neg\Diamond^2\delta_1 \wedge \neg\Diamond\beta_2,\ \ \delta_3 = \Diamond\delta_2 \wedge \neg\Diamond^2\delta_2 \wedge \neg\Diamond\beta_2,$$

$$\alpha^i_{-1} = \Diamond\gamma_{i+1} \wedge \Diamond\delta_{i+1} \wedge \neg\Diamond^2\gamma_{i+1} \wedge \neg\Diamond^2\delta_{i+1} \quad (i \in \{0,1,2\}),$$

$$\alpha^i_j = \Diamond\alpha^i_{-1} \wedge \bigwedge_{i\neq k} \neg\Diamond\alpha^k_{-1} \wedge \Diamond^j\alpha^i_{-1} \wedge \neg\Diamond^{j+1}\alpha^i_{-1} \quad (i \in \{0,1,2\},\ j \geq 0).$$

The formulas characterizing the points $e(t,k,l)$ are denoted by $\epsilon(t,\alpha^1_k,\alpha^2_l)$ and defined as

$$\epsilon(t,\alpha^1_k,\alpha^2_l) = \bigwedge_{i=0}^{t} \Diamond\alpha^0_i \wedge \neg\Diamond\alpha^0_{t+1} \wedge \Diamond\alpha^1_k \wedge \neg\Diamond^2\alpha^1_k \wedge \Diamond\alpha^2_l \wedge \neg\Diamond^2\alpha^2_l.$$

Lemma 16.26 *For every triple $\langle t,k,l\rangle$ of natural numbers,*

(i) $\{x :\ x \models \epsilon(t,\alpha^1_k,\alpha^2_l)\} = \begin{cases} \{e(t,k,l)\} & \textit{if } \boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle \\ \emptyset & \textit{otherwise.} \end{cases}$

(ii) $\mathfrak{F} \models \rho \wedge \Diamond\epsilon(s,\alpha^1_m,\alpha^2_n) \to \Diamond\epsilon(t,\alpha^1_k,\alpha^2_l)$ *iff* $\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$.

Proof Straightforward. ❑

By Theorem 16.3, it follows immediately that, for appropriate program $\boldsymbol{P}$ and configuration $\langle s,m,n\rangle$, the logic of $\mathfrak{F}$ is undecidable. This fact, however, is not so interesting because such examples are easily obtained by the cardinality argument. What we really need is an undecidable *calculus*, and the connection of $\mathfrak{F}$ with $\boldsymbol{P}$ and $\langle s, m, n\rangle$ will help us to construct one.

As in Sections 6.1 and 6.3, we should be able to describe by means of formulas the movement down the chains $a_0^1, a_1^1, \dots$ and $a_0^2, a_1^2, \dots$. To this end we require the following formulas representing an arbitrary fixed position in these chains:

$$\pi_1 = \Diamond\alpha_{-1}^1 \wedge \neg\Diamond\alpha_{-1}^0 \wedge \neg\Diamond\alpha_{-1}^2 \wedge p_1 \wedge \neg\Diamond p_1,\ \pi_2 = \pi_1(\Diamond p_1/p_1),$$

$$\tau_1 = \Diamond\alpha_{-1}^2 \wedge \neg\Diamond\alpha_{-1}^0 \wedge \neg\Diamond\alpha_{-1}^1 \wedge p_2 \wedge \neg\Diamond p_2,\ \tau_2 = \tau_1(\Diamond p_2/p_2).$$

Lemma 16.27 *For every valuation in $\mathfrak{F}$ and every point x,*

(i) *if $x \models \pi_1$ then, for some $i \geq 0$,*

$$\{y :\ y \models \pi_1\} = \{a_i^1\},\ \{y :\ y \models \pi_2\} = \{a_{i+1}^1\};$$

(ii) *if $x \models \pi_2$ then, for some $i \geq 1$,*

$$\{y :\ y \models \pi_2\} = \{a_i^1\},\ \{y :\ y \models \pi_1\} = \{a_{i-1}^1\};$$

(iii) *if $x \models \tau_1$ then, for some $i \geq 0$,*

$$\{y :\ y \models \tau_1\} = \{a_i^2\},\ \{y :\ y \models \tau_2\} = \{a_{i+1}^2\};$$

(iv) *if $x \models \tau_2$ then, for some $i \geq 1$,*

$$\{y :\ y \models \tau_2\} = \{a_i^2\},\ \{y :\ y \models \tau_1\} = \{a_{i-1}^2\}.$$

Proof Follows directly from the definition. ❑

Now, using π_1, π_2, τ_1, τ_2 we define formulas representing an arbitrary fixed configuration: for $t \geq 0$ and $i, j \in \{1, 2\}$,

$$\epsilon(t, \pi_i, \tau_j) = \bigwedge_{k=0}^{t} \Diamond\alpha_k^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\pi_i \wedge \neg\Diamond^2\pi_i \wedge \Diamond\tau_j \wedge \neg\Diamond^2\tau_j,$$

$$\epsilon(t, \pi_1, \alpha_0^2) = \bigwedge_{k=0}^{t} \Diamond\alpha_k^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\pi_1 \wedge \neg\Diamond^2\pi_1 \wedge \Diamond\alpha_0^2 \wedge \neg\Diamond^2\alpha_0^2,$$

$$\epsilon(t, \alpha_0^1, \tau_1) = \bigwedge_{k=0}^{t} \Diamond\alpha_k^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\alpha_0^1 \wedge \neg\Diamond^2\alpha_0^1 \wedge \Diamond\tau_1 \wedge \neg\Diamond^2\tau_1.$$

The first formula represents an arbitrary configuration provided that for $i = 2$ (or $j = 2$) its second (respectively, third) component is not 0. The other two formulas represent configurations whose second and third components are equal to 0, respectively.

The meaning of these formulas in $\mathfrak{F}$ should be clear from the construction and Lemma 16.27; their syntactic meaning is clarified by

Lemma 16.28 *For all formulas φ and ψ, let $\varphi \equiv \psi$ mean that $\varphi \leftrightarrow \psi \in \mathbf{K}$ and $\varphi^* = \varphi\{\Diamond^k \alpha_0^1/p_1, \Diamond^l \alpha_0^2/p_2\}$. Then*

(i) $\pi_1^* \equiv \alpha_k^1$, $\pi_2^* \equiv \alpha_{k+1}^1$;

(ii) $\tau_1^* \equiv \alpha_l^2$, $\tau_2^* \equiv \alpha_{l+1}^2$;

(iii) $(\epsilon(t, \pi_i, \tau_j))^* \equiv \epsilon(t, \alpha_{k+(i-1)}^1, \alpha_{l+(j-1)}^2)$, *for* $i, j \in \{1, 2\}$;

(iv) $(\epsilon(t, \pi_1, \alpha_0^2))^* \equiv \epsilon(t, \alpha_k^1, \alpha_0^2)$;

(v) $(\epsilon(t, \alpha_0^1, \tau_1))^* \equiv \epsilon(t, \alpha_0^1, \alpha_l^2)$.

Proof Exercise. ❑

We are in a position now to write down formulas AxI simulating instructions I of Minsky machines:

- If I has the form $t \to \langle t', 1, 0\rangle$ then we put

$$AxI = \rho \wedge \Diamond\epsilon(t, \pi_1, \tau_1) \to \Diamond\epsilon(t', \pi_2, \tau_1);$$

- If I is $t \to \langle t', 0, 1\rangle$ then

$$AxI = \rho \wedge \Diamond\epsilon(t, \pi_1, \tau_1) \to \Diamond\epsilon(t', \pi_1, \tau_2);$$

- If I is $t \to \langle t', -1, 0\rangle$ $(\langle t'', 0, 0\rangle)$ then

$$\begin{aligned} AxI = {} & (\rho \wedge \Diamond\epsilon(t, \pi_2, \tau_1) \to \Diamond\epsilon(t', \pi_1, \tau_1)) \wedge \\ & (\rho \wedge \Diamond\epsilon(t, \alpha_0^1, \tau_1) \to \Diamond\epsilon(t'', \alpha_0^1, \tau_1)); \end{aligned}$$

- If I is $t \to \langle t', 0, -1\rangle$ $(\langle t'', 0, 0\rangle)$ then

$$\begin{aligned} AxI = {} & (\rho \wedge \Diamond\epsilon(t, \pi_1, \tau_2) \to \Diamond\epsilon(t', \pi_1, \tau_1)) \wedge \\ & (\rho \wedge \Diamond\epsilon(t, \pi_1, \alpha_0^2) \to \Diamond\epsilon(t'', \pi_1, \alpha_0^2)). \end{aligned}$$

The formula simulating $\boldsymbol{P}$ as a whole is

$$AxP = \bigwedge_{I \in \boldsymbol{P}} AxI.$$

Lemma 16.29 *Suppose $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$. Then*

$$\rho \wedge \Diamond\epsilon(s, \alpha_m^1, \alpha_n^2) \to \Diamond\epsilon(t, \alpha_k^1, \alpha_l^2) \in \mathbf{K4} \oplus AxP.$$

Proof The proof proceeds by induction on the number of instructions used to compute $\langle t, k, l\rangle$ starting from $\langle s, m, n\rangle$. The basis of induction is trivial and the step of induction follows from Lemma 16.28, according to which

$$\rho \wedge \Diamond\epsilon(s', \alpha_{m'}^1, \alpha_{n'}^2) \to \Diamond\epsilon(t', \alpha_{k'}^1, \alpha_{l'}^2) \in \mathbf{K4} \oplus AxP$$

whenever $\langle t', k', l'\rangle$ is obtained from $\langle s', m', n'\rangle$ by applying a single instruction in $\boldsymbol{P}$. ❑

To obtain the converse of Lemma 16.29, it suffices to observe that the following lemma holds:

Lemma 16.30 $\mathfrak{F} \models AxP$.

Proof Straightforward using Lemma 16.26. ❑

As a consequence of Lemmas 16.26, 16.29 and 16.30 we derive

Lemma 16.31 *For every* $\boldsymbol{P}$, $\langle s,m,n\rangle$ *and* $\langle t,k,l\rangle$,

$$\rho \wedge \Diamond\epsilon(s,\alpha_m^1,\alpha_n^2) \to \Diamond\epsilon(t,\alpha_k^1,\alpha_l^2) \in \mathbf{K4} \oplus AxP \textit{ iff } \boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle\,.$$

Recall now that we have effectively constructed our formulas for given arbitrary $\boldsymbol{P}$ and $\langle s,m,n\rangle$. So Theorem 16.2 provides us with

Theorem 16.32 (i) *There is no algorithm which, given modal formulas* φ *and* ψ, *could decide whether* $\psi \in \mathbf{K4} \oplus \varphi$.

(ii) *There is no algorithm which, given* φ *and* ψ, *could decide whether* ψ *is valid in all transitive Kripke frames validating* φ.

This result can be considerably strengthened by fixing appropriate φ or ψ. If we take $\boldsymbol{P}$ and $\langle s,m,n\rangle$ for which the second configuration problem is undecidable then, by Lemma 16.31, we obtain the following:

Theorem 16.33 *There is a program* $\boldsymbol{P}$ *such that the calculus* $\mathbf{K4} \oplus AxP$ *is undecidable; besides, there is no algorithm which, given a formula* ψ, *could decide whether* ψ *is valid in all Kripke frames for* $\mathbf{K4} \oplus AxP$.

Say that a formula ψ is *undecidable* in (N)Ext$L$ if no algorithm can recognize, for an arbitrary φ, whether $\psi \in L + \varphi$ (respectively, $\psi \in L \oplus \varphi$). To find an undecidable formula in NExt**K4** we require two more lemmas.

Lemma 16.34 *For every triple* $\langle t,k,l\rangle$ *such that* $\boldsymbol{P} : \langle s,m,n\rangle \not\to \langle t,k,l\rangle$,

$$\mathfrak{F} \models (\rho \wedge \Diamond\epsilon(s,\alpha_m^1,\alpha_n^2) \to \Diamond\epsilon(t,\alpha_k^1,\alpha_l^2)) \to \neg\rho.$$

Proof Follows immediately from Lemma 16.26. ❑

Lemma 16.35 $\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$ *iff*

$$\neg\rho \in \mathbf{K4} \oplus AxP \oplus (\rho \wedge \Diamond\epsilon(s,\alpha_m^1,\alpha_n^2) \to \Diamond\epsilon(t,\alpha_k^1,\alpha_l^2)) \to \neg\rho.$$

Proof ($\Rightarrow$) follows from Lemma 16.31 by modus ponens and ($\Leftarrow$) from Lemmas 16.30, 16.34 and the fact that $r \not\models \neg\rho$. ❑

As a direct consequence of Lemma 16.35 and Theorem 16.3 we obtain

Theorem 16.36 *The formula* $\neg\rho$ *is undecidable in* NExt**K4**.

Remark According to Theorem 16.36, even variable free formulas may be undecidable in NExt**K4**. However, there is no undecidable calculus in NExt**K4**

with variable free axioms. The undecidable calculus in Theorem 16.33 was constructed by adding to **K4** an axiom in two variables p_1 and p_2. In fact even one variable is enough: we can identify p_1 and p_2 and change the substitution in Lemma 16.28 to $(\Diamond^+\alpha_0^1 \to \Diamond^k\alpha_0^1) \wedge (\Diamond^+\alpha_0^2 \to \Diamond^l\alpha_0^2)/p_1$.

It is worth noting that in Ext**K4** even the formula $\bot$ turns out to be undecidable (though it is clearly decidable in NExt**K4**). Indeed, declare the root r in the frame $\mathfrak{F}$ to be its only actual world. Then we have

Lemma 16.37 *For every triple* $\langle t,k,l\rangle$ *such that* $\boldsymbol{P} : \langle s,m,n\rangle \not\to \langle t,k,l\rangle$,

$$\langle \mathfrak{F}, r\rangle \models (\rho \wedge \Diamond\epsilon(s,\alpha_m^1,\alpha_n^2) \to \Diamond\epsilon(t,\alpha_k^1,\alpha_l^2)) \to \bot.$$

Using this result we obtain

Lemma 16.38 $\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$ *iff*

$$\bot \in \mathbf{K4} + AxP + (\rho \wedge \Diamond\epsilon(s,\alpha_m^1,\alpha_n^2) \to \Diamond\epsilon(t,\alpha_k^1,\alpha_l^2)) \to \bot.$$

Using once again the undecidability of the second configuration problem we finally arrive at

Theorem 16.39 *The formula* $\bot$ *is undecidable in* Ext**K4**, *i.e., no algorithm can recognize, given a formula* φ, *whether the logic* $\mathbf{K4} + \varphi$ *is consistent.*

The only property of $\bot$ we used while proving Theorem 16.39 was that it is refuted at r. This means that any formula refutable at r is undecidable in Ext**K4**. In particular, undecidable are all formulas φ such that $\mathbf{K4}+\varphi \supseteq \mathbf{S4}$. By replacing r with a reflexive actual world (which is not essential for our construction), we see that all formulas φ axiomatizing over **K4** extensions of **GL** are also undecidable in Ext**K4**.

16.5 Undecidable calculus and formula in ExtInt

This section should be considered as a system of instructions for transferring the results of the preceding one to superintuitionistic logics. We confine ourselves only to describing the construction, formulating lemmas and theorems and pointing out their modal prototypes. The reader who has understood the construction of the undecidable modal calculus should encounter no fundamental difficulties in the intuitionistic case.

Fix an arbitrary Minsky machine $\boldsymbol{P}$ and an arbitrary configuration $\langle s,m,n\rangle$ and let $\mathfrak{F}$ be the intuitionistic Kripke frame depicted in Fig. 16.4, with the point $e(t,k,l)$ occurring in it iff $\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$. To characterize the points in $\mathfrak{F}$ we require the following formulas:

$$\alpha_{-3}^0 = p_6 \to \bot, \quad \beta_{-3}^0 = q_6 \to \bot,$$

$$\alpha_{-2}^0 = p_5 \to p_6 \vee \alpha_{-3}^0, \quad \beta_{-2}^0 = q_5 \to q_6 \vee \beta_{-3}^0,$$

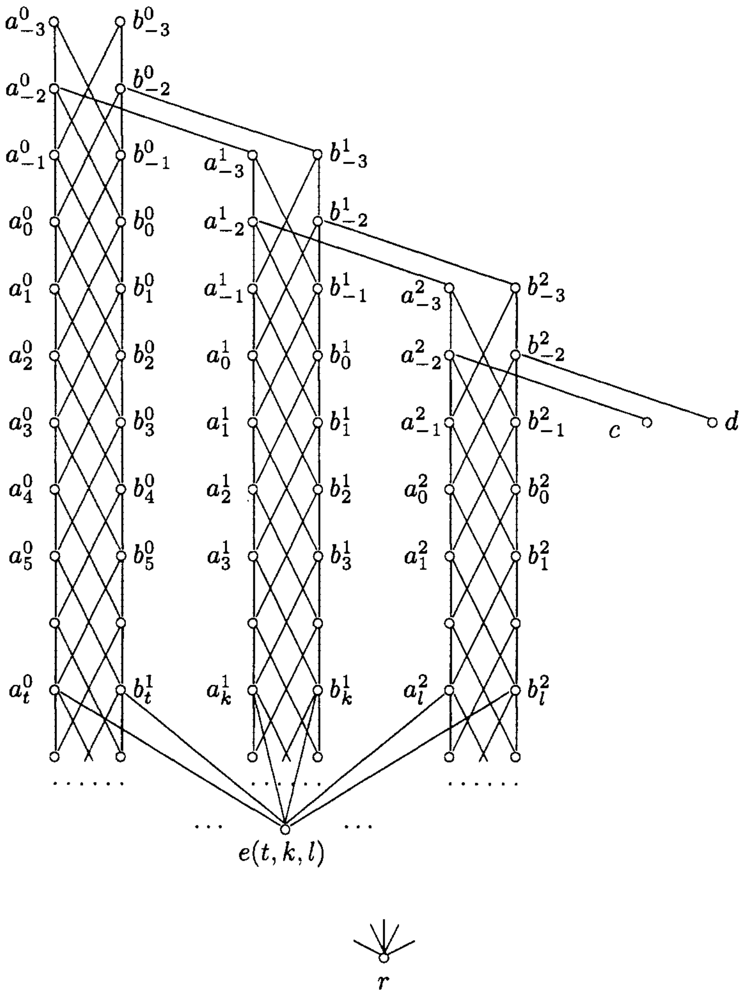


FIG. 16.4.

$$\alpha^1_{-3} = p_4 \to p_5 \vee \alpha^0_{-2}, \quad \beta^1_{-3} = q_4 \to q_5 \vee \beta^0_{-2},$$

$$\alpha^1_{-2} = p_3 \to p_4 \vee \alpha^1_{-3}, \quad \beta^1_{-2} = q_3 \to q_4 \vee \beta^1_{-3},$$

$$\alpha^2_{-3} = p_2 \to p_3 \vee \alpha^1_{-2}, \quad \beta^2_{-3} = q_2 \to q_3 \vee \beta^1_{-2},$$

$$\alpha^2_{-2} = p_1 \to p_2 \vee \alpha^2_{-3}, \quad \beta^2_{-2} = q_1 \to q_2 \vee \beta^2_{-3},$$

$$\gamma = \neg q_6 \to p_1 \vee \alpha^2_{-2}, \quad \delta = \neg p_6 \to q_1 \vee \beta^2_{-2},$$

$$\rho = \gamma \vee \delta.$$

It is not hard to see that a Kripke frame refutes ρ iff it contains a subframe of the form shown in Fig. 16.5 such that the points c and d have no common successors

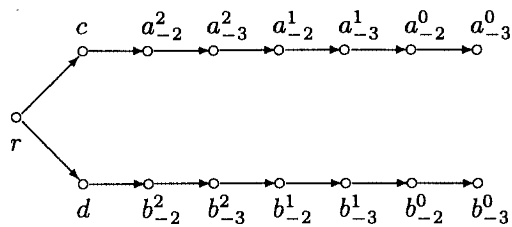


FIG. 16.5.

in it. Clearly, $\mathfrak{F}$ contains only one (up to the evident symmetry) subframe of that sort: its points are denoted by the same symbols as the corresponding points of the frame in Fig. 16.5. So if ρ is refuted in $\mathfrak{F}$ under some valuation, the points in $\mathfrak{F}$ can be characterized as follows:

$$\{x:\ x \not\models \alpha^i_j\} = \{a^i_j\},\ \{x:\ x \not\models \beta^i_j\} = \{b^i_j\}\ (i \in \{0,1,2\},\ j \geq -3),$$

$$\{x:\ x \not\models \gamma\} = \{g\},\ \{x:\ x \not\models \delta\} = \{d\},$$

$$\{x:\ x \not\models \epsilon(t, \alpha^1_k, \alpha^2_l)\} = \begin{cases} \{e(t,k,l)\} & \text{if } \boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle \\ \emptyset & \text{if } \boldsymbol{P} : \langle s,m,n\rangle \not\to \langle t,k,l\rangle, \end{cases}$$

where $x \not\models \varphi \to \psi$ means $x \models \varphi$ and $x \not\models \psi$, and for $j \geq -2$, $t,k,l \geq 0$,

$$\alpha^0_{j+1} = \beta^0_j \to \alpha^0_j \vee \beta^0_{j-1},\ \beta^0_{j+1} = \alpha^0_j \to \beta^0_j \vee \alpha^0_{j-1},$$

$$\alpha^1_{j+1} = \alpha^2_{-3} \wedge \beta^2_{-3} \wedge \beta^1_j \to \alpha^1_j \vee \beta^1_{j-1} \vee \alpha^1_{-3} \vee \beta^1_{-3},$$

$$\beta^1_{j+1} = \alpha^2_{-3} \wedge \beta^2_{-3} \wedge \alpha^1_j \to \beta^1_j \vee \alpha^1_{j-1} \vee \alpha^1_{-3} \vee \beta^1_{-3},$$

$$\alpha^2_{j+1} = \gamma \wedge \delta \wedge \beta^2_j \to \alpha^2_j \vee \beta^2_{j-1} \vee \alpha^2_{-3} \vee \beta^2_{-3},$$

$$\beta^2_{j+1} = \gamma \wedge \delta \wedge \alpha^2_j \to \beta^2_j \vee \alpha^2_{j-1} \vee \alpha^2_{-3} \vee \beta^2_{-3},$$

$$\epsilon(t, \alpha^1_k, \alpha^2_l) = \alpha^0_{t+1} \wedge \beta^0_{t+1} \wedge \alpha^1_{k+1} \wedge \beta^1_{k+1} \wedge \alpha^2_{l+1} \wedge \beta^2_{l+1} \to \\ \alpha^0_t \vee \beta^0_t \vee \alpha^1_k \vee \beta^1_k \vee \alpha^2_l \vee \beta^2_l.$$

(In fact, the first two conjuncts and the last two disjuncts in α^i_j and β^i_j, for $i = 1,2$, are redundant; they are added only to simplify the proof a bit.)

The intuitionistic counterparts of the formulas π_i and τ_i from the preceding section are:

$$\pi_{-2} = r,\ \pi'_{-2} = s,\ \pi_{-1} = p,\ \pi'_{-1} = q,$$

$$\pi_{i+1} = \alpha^2_{-3} \wedge \beta^2_{-3} \wedge \pi'_i \to \pi_i \vee \pi'_{i-1} \vee \alpha^1_{-3} \vee \beta^1_{-3},$$

$$\pi'_{i+1} = \alpha^2_{-3} \wedge \beta^2_{-3} \wedge \pi_i \to \pi'_i \vee \pi_{i-1} \vee \alpha^1_{-3} \vee \beta^1_{-3},$$

$$\tau_{-2} = r',\ \tau'_{-2} = s',\ \tau_{-1} = p',\ \tau'_{-1} = q',$$
$$\tau_{i+1} = \gamma \wedge \delta \wedge \tau'_i \to \tau_i \vee \tau'_{i-1} \vee \alpha^2_{-3} \vee \beta^2_{-3},$$
$$\tau'_{i+1} = \gamma \wedge \delta \wedge \tau_i \to \tau'_i \vee \tau_{i-1} \vee \alpha^2_{-3} \vee \beta^2_{-3}\ (i \geq -1).$$

Using them we define, for $i, j \in \{1, 2\}$ and $t \geq 0$,

$$\epsilon(t, \pi_i, \tau_j) = \alpha^0_{t+1} \wedge \beta^0_{t+1} \wedge \pi_{i+1} \wedge \pi'_{i+1} \wedge \tau_{j+1} \wedge \tau'_{j+1} \to \alpha^0_t \vee \beta^0_t \vee \pi_i \vee \pi'_i \vee \tau_j \vee \tau'_j,$$

$$\epsilon(t, \pi_1, \alpha^2_0) = \alpha^0_{t+1} \wedge \beta^0_{t+1} \wedge \pi_2 \wedge \pi'_2 \wedge \alpha^2_1 \wedge \beta^2_1 \to \alpha^0_t \vee \beta^0_t \vee \pi_1 \vee \pi'_1 \vee \alpha^2_0 \vee \beta^2_0,$$

$$\epsilon(t, \alpha^1_0, \tau_1) = \alpha^0_{t+1} \wedge \beta^0_{t+1} \wedge \alpha^1_1 \wedge \beta^1_1 \wedge \tau_2 \wedge \tau'_2 \to \alpha^0_t \vee \beta^0_t \vee \alpha^1_0 \vee \beta^1_0 \vee \tau_1 \vee \tau'_1.$$

Finally, we define the formulas simulating the instructions I of a Minsky machine $\boldsymbol{P}$:

- if I is of the form $t \to \langle t', 1, 0\rangle$ then

$$AxI = \epsilon(t', \pi_2, \tau_1) \to \epsilon(t, \pi_1, \tau_1) \vee \rho;$$

- if I is $t \to \langle t', 0, 1\rangle$ then

$$AxI = \epsilon(t', \pi_1, \tau_2) \to \epsilon(t, \pi_1, \tau_1) \vee \rho;$$

- if I is $t \to \langle t', -1, 0\rangle$ $(\langle t'', 0, 0\rangle)$ then

$$AxI = (\epsilon(t', \pi_1, \tau_1) \to \epsilon(t, \pi_2, \tau_1) \vee \rho) \wedge (\epsilon(t'', \alpha^1_0, \tau_1) \to \epsilon(t, \alpha^1_0, \tau_1) \vee \rho);$$

- if I is $t \to \langle t', 0, -1\rangle$ $(\langle t'', 0, 0\rangle)$ then

$$AxI = (\epsilon(t', \pi_1, \tau_1) \to \epsilon(t, \pi_1, \tau_2) \vee \rho) \wedge (\epsilon(t'', \pi_1, \alpha^2_0) \to \epsilon(t, \pi_1, \alpha^2_0) \vee \rho),$$

and the formula simulating the behavior of $\boldsymbol{P}$ itself:

$$AxP = \bigwedge_{I \in \boldsymbol{P}} AxI,$$

Denote by φ^* the result of substituting the formulas α^1_{i-3}, β^1_{i-3}, α^1_{i-2}, β^1_{i-2}, α^2_{j-3}, β^2_{j-3}, α^2_{j-2}, β^2_{j-2} instead of the variables r, s, p, q, r', s', p', q' in φ, respectively.

Lemma 16.40 *The following equivalences are in* **Int***:*
(i) $(\epsilon(t, \pi_k, \tau_l))^* \leftrightarrow \epsilon(t, \alpha^1_{i+k-1}, \alpha^2_{j+l-1})$;
(ii) $(\epsilon(t, \pi_1, \alpha^2_0))^* \leftrightarrow \epsilon(t, \alpha^1_i, \alpha^2_0)$;
(iii) $(\epsilon(t, \alpha^1_0, \tau_1))^* \leftrightarrow \epsilon(t, \alpha^1_0, \alpha^2_j)$.

Lemma 16.41 $\mathfrak{F} \models AxP$.

As a consequence of these two lemmas we have

Corollary 16.42 $\epsilon(t, \alpha^1_k, \alpha^2_l) \to \epsilon(s, \alpha^1_m, \alpha^2_n) \vee \rho \in$ **Int** $+ AxP$ *if and only if* $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$.

Now if we take a machine $\boldsymbol{P}$ for which the configuration problem is undecidable, Corollary 16.42 will mean that the calculus **Int** $+ AxP$ is also undecidable. Thus we obtain

Theorem 16.43 *There is a program* $\boldsymbol{P}$ *such that the calculus* **Int** $+ AxP$ *is undecidable; besides, there is no algorithm which, given a formula* ψ, *could decide whether* ψ *is valid in all Kripke frames for* **Int** $+ AxP$.

Observe also that the following statement holds.

Lemma 16.44 *For every triple* $\langle t, k, l\rangle$ *such that* $\boldsymbol{P} : \langle s, m, n\rangle \not\to \langle t, k, l\rangle$,

$$\mathfrak{F} \models (\epsilon(t, \alpha^1_k, \alpha^2_l) \to \epsilon(s, \alpha^1_m, \alpha^2_n) \vee \rho) \to \rho.$$

From this and the preceding lemmas in the same way as Lemma 16.35 we derive

Lemma 16.45 $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ *iff*

$$\rho \in \mathbf{Int} + AxP + (\epsilon(t, \alpha^1_k, \alpha^2_l) \to \epsilon(s, \alpha^1_m, \alpha^2_n) \vee \rho) \to \rho.$$

Thereby, we prove

Theorem 16.46 *The formula* ρ *is undecidable in* **ExtInt**.

16.6 The undecidability of the semantical consequence problem on finite frames

When constructing undecidable calculi in Sections 16.4 and 16.5, we were forced to use infinite frames simply because every finitely approximable calculus is decidable. So for the present nothing can be said about the decidability of the semantical consequence on finite frames, i.e., about the decidability of the relation $\varphi \models_{fin} \psi$, which means that ψ is valid in all finite frames validating φ. In this section we modify the construction of Section 16.4 to prove the undecidability of $\models_{fin}$. For purely technical reasons it will be more convenient for us to deal with only finite transitive irreflexive frames, i.e., to consider the relation $\boldsymbol{la} \wedge \varphi \models_{fin} \psi$.

Let $\mathfrak{F}$ be the transitive irreflexive frame shown in Fig. 16.6. Its intended meaning will be defined a bit later and meanwhile we introduce some formulas

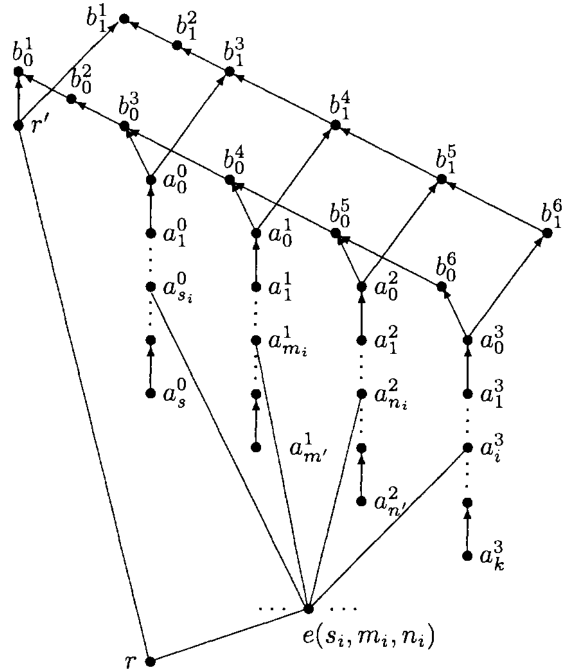


FIG. 16.6.

characterizing points in $\mathfrak{F}$. (We do not actually need $\mathfrak{F}$ in this part of the proof, but it is helpful to have it at hand.) Put

$$\rho' = \Box^2\bot \to \Box p \vee \Box\neg p, \;\; \rho = \Box\rho',$$

Clearly $x \not\models \rho$ and $y \not\models \rho'$ under some valuation in $\mathfrak{F}$ iff $x = r$, $y = r'$ and either $b_0^1 \models p$, $b_1^1 \not\models p$ or $b_0^1 \not\models p$, $b_1^1 \models p$. For the reason of symmetry we will consider only the former case. Now we put

$$\beta_0^1 = \Box\bot \wedge p, \;\; \beta_1^1 = \Box\bot \wedge \neg p,$$

$$\beta_i^j = \Box^j\bot \wedge \Diamond^{j-1}\beta_i^1 \wedge \neg\Diamond\beta_{1-i}^1 \;\; (i \in \{0,1\}, \; 1 < j \leq 6),$$
$$\alpha_0^i = \Diamond\beta_0^{i+3} \wedge \Diamond\beta_1^{i+3} \wedge \Box^{i+4}\bot \;\; (0 \leq i \leq 3),$$
$$\alpha_j^i = \Diamond^j\alpha_0^i \wedge \neg\Diamond^{j+1}\alpha_0^i \wedge \bigwedge_{k \neq i} \neg\Diamond\alpha_0^k \;\; (0 \leq i \leq 3, \; j > 0),$$

$$\epsilon(t, \alpha_k^1, \alpha_l^2) = \bigwedge_{i=0}^{t} \Diamond\alpha_i^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\alpha_k^1 \wedge \neg\Diamond^2\alpha_k^1 \wedge \Diamond\alpha_l^2 \wedge \neg\Diamond^2\alpha_l^2$$

where $t, k, l \geq 0$ and again the Greek letters, denoting formulas, correspond to the Roman letters for points in $\mathfrak{F}$. The formulas describing an arbitrary position in the chains $a_0^i, a_1^i, \ldots$, for $i = 1, 2, 3$, and an arbitrary configuration are defined in the same way as in Section 16.4:

$$\pi_1 = \Diamond^+\alpha_0^1 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^2 \wedge \neg\Diamond\alpha_0^3 \wedge p_1 \wedge \neg\Diamond p_1,$$

$$\pi_2 = \Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^2 \wedge \neg\Diamond\alpha_0^3 \wedge \Diamond p_1 \wedge \neg\Diamond^2 p_1,$$

$$\tau_1 = \Diamond^+\alpha_0^2 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^3 \wedge p_2 \wedge \neg\Diamond p_2,$$

$$\tau_2 = \Diamond\alpha_0^2 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^3 \wedge \Diamond p_2 \wedge \neg\Diamond^2 p_2,$$

$$\sigma_1 = \Diamond^+\alpha_0^3 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^2 \wedge p_3 \wedge \neg\Diamond p_3,$$

$$\sigma_2 = \Diamond\alpha_0^3 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^2 \wedge \Diamond p_3 \wedge \neg\Diamond^2 p_3,$$

$$\epsilon(t, \pi_i, \tau_j) = \bigwedge_{k=0}^{t} \Diamond\alpha_k^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\pi_i \wedge \neg\Diamond^2\pi_i \wedge \Diamond\tau_j \wedge \neg\Diamond^2\tau_j,$$

$$\epsilon(t, \pi_1, \alpha_0^2) = \bigwedge_{k=0}^{t} \Diamond\alpha_k^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\pi_1 \wedge \neg\Diamond^2\pi_1 \wedge \Diamond\alpha_0^2 \wedge \neg\Diamond^2\alpha_0^2,$$

$$\epsilon(t, \alpha_0^1, \tau_1) = \bigwedge_{k=0}^{t} \Diamond\alpha_k^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\alpha_0^1 \wedge \neg\Diamond^2\alpha_0^1 \wedge \Diamond\tau_1 \wedge \neg\Diamond^2\tau_1,$$

where $t \geq 0$, $i, j \in \{1, 2\}$. The following lemma is proved in the same way as Lemma 16.28.

Lemma 16.47 *For all formulas φ and ψ, let $\varphi \equiv \psi$ mean that $\Box^+(\varphi \leftrightarrow \psi) \in$ **GL** and $\varphi^* = \varphi\{\Diamond^k\alpha_0^1/p_1, \Diamond^l\alpha_0^2/p_2, \Diamond^m\alpha_0^3/p_3\}$. Then*

(i) $\pi_1^* \equiv \alpha_k^1$, $\pi_2^* \equiv \alpha_{k+1}^1$;

(ii) $\tau_1^* \equiv \alpha_l^2$, $\tau_2^* \equiv \alpha_{l+1}^2$;

(iii) $\sigma_1^* \equiv \alpha_m^3$, $\sigma_2^* \equiv \alpha_{m+1}^3$;

(iv) $(\epsilon(t, \pi_i, \tau_j))^* \equiv \epsilon(t, \alpha_{k+(i-1)}^1, \alpha_{l+(j-1)}^2)$, *for* $i, j \in \{1, 2\}$;

(v) $(\epsilon(t, \pi_1, \alpha_0^2))^* \equiv \epsilon(t, \alpha_k^1, \alpha_0^2)$;

(vi) $(\epsilon(t, \alpha_0^1, \tau_1))^* \equiv \epsilon(t, \alpha_0^1, \alpha_l^2)$.

The formulas AxI simulating instructions we are going to use now have an essential difference from those in Section 16.4: they not only reflect the transformation of configurations but also calculate the number of steps in computations. They are as follows:

- If I is $t \to \langle t', 1, 0\rangle$ then we put

$$AxI = \neg\rho \wedge \Diamond(\epsilon(t, \pi_1, \tau_1) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1) \wedge \Diamond\sigma_2 \to \Diamond(\epsilon(t', \pi_2, \tau_1) \wedge \Diamond\sigma_2 \wedge \neg\Diamond^2\sigma_2);$$

- If I is $t \to \langle t', 0, 1\rangle$ then

$$AxI = \neg\rho \wedge \Diamond(\epsilon(t, \pi_1, \tau_1) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1) \wedge \Diamond\sigma_2 \to \Diamond(\epsilon(t', \pi_1, \tau_2) \wedge \Diamond\sigma_2 \wedge \neg\Diamond^2\sigma_2);$$

- If I is $t \to \langle t', -1, 0\rangle$ ($\langle t'', 0, 0\rangle$) then

$$AxI = (\neg\rho \wedge \Diamond(\epsilon(t, \pi_2, \tau_1) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1) \wedge \Diamond\sigma_2 \to \Diamond(\epsilon(t', \pi_1, \tau_1) \wedge \Diamond\sigma_2 \wedge \neg\Diamond^2\sigma_2)) \wedge (\neg\rho \wedge \Diamond(\epsilon(t, \alpha_0^1, \tau_1) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1) \wedge \Diamond\sigma_2 \to \Diamond(\epsilon(t'', \alpha_0^1, \tau_1) \wedge \Diamond\sigma_2 \wedge \neg\Diamond^2\sigma_2));$$

- If I is $t \to \langle t', 0, -1\rangle$ ($\langle t'', 0, 0\rangle$) then

$$AxI = (\neg\rho \wedge \Diamond(\epsilon(t, \pi_1, \tau_2) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1) \wedge \Diamond\sigma_2 \to \Diamond(\epsilon(t', \pi_1, \tau_2) \wedge \Diamond\sigma_2 \wedge \neg\Diamond^2\sigma_2)) \wedge (\neg\rho \wedge \Diamond(\epsilon(t, \pi_1, \alpha_0^2) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1) \wedge \Diamond\sigma_2 \to \Diamond(\epsilon(t'', \pi_1, \alpha_0^2) \wedge \Diamond\sigma_2 \wedge \neg\Diamond^2\sigma_2)).$$

And again for a Minsky program $\boldsymbol{P}$ we define $AxP = \bigwedge_{i\in P} AxI$. But this time we are after another fish. What we really need is the following two formulas:

$$\varphi(\boldsymbol{P}) = \boldsymbol{la} \wedge AxP \wedge \lambda \wedge \upsilon,$$

and

$$\psi(s_1, m, n) = \neg\rho \wedge \Diamond(\epsilon(s_1, \alpha_m^1, \alpha_n^2) \wedge \Diamond\alpha_1^3 \wedge \neg\Diamond^2\alpha_1^3) \to \neg\Diamond(\epsilon(s_0, \pi_1, \tau_1) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1),$$

where

$$\lambda = \neg(\neg\rho \wedge \Diamond(\Diamond\alpha_0^3 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^2 \wedge \Box^+ r \wedge \neg q) \wedge \Diamond(\Diamond\alpha_0^3 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^2 \wedge \Box^+ q \wedge \neg r)),$$

$$\upsilon = \neg(\neg\rho \wedge \Diamond(q \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1 \wedge \Diamond\alpha_0^1 \wedge \Diamond\alpha_0^2 \wedge \Diamond\alpha_0^3) \wedge \Diamond(\neg q \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1 \wedge \Diamond\alpha_0^1 \wedge \Diamond\alpha_0^2 \wedge \Diamond\alpha_0^3))$$

and s_1 and s_0 are the initial and final states, respectively. The meaning of λ is that if a frame validates $\boldsymbol{la} \wedge \lambda$ and at a point x the formula ρ is false under some valuation then the set of points in $x\uparrow$ at which $\Diamond\alpha_0^3 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^2$ is true is strictly linearly ordered by the accessibility relation of the frame. Our main technical result is

Lemma 16.48 *$\varphi(\boldsymbol{P}) \models_{fin} \psi(s_1, m, n)$ iff the program $\boldsymbol{P}$, having started with the configuration $\langle s_1, m, n\rangle$, never comes to the final state s_0.*

Proof Let us begin with ($\Leftarrow$); it is this part of the lemma that uses specific features of finite frames. Suppose the machine $\boldsymbol{P}$ starts at $\langle s_1, m, n\rangle$ and works forever. To show that $\varphi(\boldsymbol{P}) \models_{fin} \psi(s_1, m, n)$, assume otherwise. Then there is a finite frame $\mathfrak{G} = \langle V, S\rangle$ such that

$$\mathfrak{G} \models \varphi(\boldsymbol{P}) \tag{16.2}$$

but, for some $a \in V$ under some valuation in $\mathfrak{G}$, $a \not\models \psi(s_1, m, n)$, i.e.,

$$a \not\models \rho, \tag{16.3}$$

$$a \models \Diamond(\epsilon(s_1, \alpha_m^1, \alpha_n^2) \wedge \Diamond\alpha_1^3 \wedge \neg\Diamond^2\alpha_1^3), \tag{16.4}$$

$$a \models \Diamond(\epsilon(s_0, \pi_1, \tau_1) \wedge \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1). \tag{16.5}$$

(16.4) means that there is $b \in a\uparrow$ such that

$$b \models \epsilon(s_1, \alpha_m^1, \alpha_n^2), \tag{16.6}$$

$$b \models \Diamond\alpha_1^3, \tag{16.7}$$

$$b \not\models \Diamond^2\alpha_1^3, \tag{16.8}$$

while (16.5) implies that for some $c \in a\uparrow$,

$$c \models \epsilon(s_0, \pi_1, \tau_1), \tag{16.9}$$

$$c \models \Diamond\sigma_1, \tag{16.10}$$

$$c \not\models \Diamond^2\sigma_1. \tag{16.11}$$

It follows from (16.7) that there is a point, call it a_1^3, such that bSa_1^3 and $a_1^3 \models \alpha_1^3$, i.e.,

$$a_1^3 \models \Diamond\alpha_0^3 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^2, \tag{16.12}$$

$$a_1^3 \not\models \Diamond^2\alpha_0^3. \tag{16.13}$$

Similarly, by (16.10) we have a point $x \in c\uparrow$ such that $x \models \sigma_1$, i.e.,

$$x \models \Diamond\alpha_0^3 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^2, \tag{16.14}$$

$$x \models p_3 \wedge \neg\Diamond p_3. \tag{16.15}$$

By (16.2), (16.3) and the property of λ mentioned above, the set of points accessible from a at which the conditions of the form (16.12), (16.14) are satisfied form a strict chain $a_k^3 S a_{k-1}^3 \dots S a_2^3 S a_1^3$, whose last point is, by (16.13), a_1^3. The points in the chain may be characterized by the formulas α_i^3 in the sense that $a_i^3 \models \alpha_i^3$ and $a_i^3 \not\models \Diamond\alpha_k^3$, for $1 \le i \le k$.

By (16.14), $x = a_l^3$ for some $l \in \{1, \dots, k\}$, and so, by (16.14), (16.15) and (16.11), $a_l^3 \models \sigma_1$, $c \models \Diamond\alpha_l^3$, $c \not\models \Diamond^2\alpha_l^3$. Thus, we have managed to identify x with some a_l^3 and now, using the conjuncts of $\varphi(\boldsymbol{P})$ and the finiteness of $\mathfrak{G}$, we will go down step by step from a_1^3 to $x = a_l^3$.

Suppose $\boldsymbol{P}$ starts working at $\langle s_1, m, n\rangle$ and produces the infinite computation $\langle s_1, m, n\rangle = \langle s_1, m_1, n_1\rangle, \langle s_2, m_2, n_2\rangle, \langle s_3, m_3, n_3\rangle, \dots$ in which $s_i = s_0$ for no $i > 0$. In this computation only the first l steps are of importance for us. Notice that for every $i \in \{1, \dots, k-1\}$, (16.2) yields

$$a \models \Diamond(\epsilon(s_i, \alpha_{m_i}^1, \alpha_{n_i}^2) \wedge \Diamond\alpha_i^3 \wedge \neg\Diamond^2\alpha_i^3) \rightarrow$$

$$\Diamond(\epsilon(s_{i+1}, \alpha_{m_{i+1}}^1, \alpha_{n_{i+1}}^2) \wedge \Diamond\alpha_{i+1}^3 \wedge \neg\Diamond^2\alpha_{i+1}^3).$$

Using (16.4) and MP, we obtain then, for $1 \le i \le k-1$,

$$a \models \Diamond(\epsilon(s_{i+1}, \alpha_{m_{i+1}}^1, \alpha_{n_{i+1}}^2) \wedge \Diamond\alpha_{i+1}^3 \wedge \neg\Diamond^2\alpha_{i+1}^3)$$

and in particular

$$a \models \Diamond(\epsilon(s_l, \alpha_{m_l}^1, \alpha_{n_l}^2) \wedge \Diamond\alpha_l^3 \wedge \neg\Diamond^2\alpha_l^3).$$

The latter condition means that there is a point $d \in a\!\uparrow$ such that

$$d \models \epsilon(s_l, \alpha_{m_l}^1, \alpha_{n_l}^2), \tag{16.16}$$

$$d \models \Diamond\alpha_l^3, \tag{16.17}$$

$$d \not\models \Diamond^2\alpha_l^3. \tag{16.18}$$

It follows from (16.17) and (16.18) that dSa_l^3 and $\neg dSa_{l+1}^3$. Since among $a_1^3, \dots, a_k^3$ there is only one point where σ_1 is true, with the help of (16.14) and (16.15) we obtain that $d \models \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1$.

Now let us put the conditions we need together. From (16.9), (16.10) and (16.11) we derive

$$c \models \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1 \wedge \Diamond\alpha_0^1 \wedge \Diamond\alpha_0^2 \wedge \Diamond\alpha_0^3 \tag{16.19}$$

and from (16.16) and (16.17)

$$d \models \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1 \wedge \Diamond\alpha_0^1 \wedge \Diamond\alpha_0^2 \wedge \Diamond\alpha_0^3. \tag{16.20}$$

Since by (16.9) and (16.16),

$$c \models \bigwedge_{i=0}^{s_0} \Diamond\alpha_i^0 \wedge \neg\Diamond\alpha_{s_0+1}^0, \quad d \models \bigwedge_{i=0}^{s_l} \Diamond\alpha_i^0 \wedge \neg\Diamond\alpha_{s_l+1}^0$$

and $s_l \neq s_0$, the points c and d must be distinct. Therefore, we may define a valuation in $\mathfrak{G}$ so that $c \models q$, $d \not\models q$. Together with (16.19), (16.20) and (16.3) this gives $a \not\models v$, contrary to (16.2). Thus indeed we have $\varphi(\boldsymbol{P}) \models_{fin} \psi(s_1, m, n)$.

Now we prove the contraposition of ($\Leftarrow$). Suppose that $\boldsymbol{P}$ starts at $\langle s_1, m, n \rangle$ and reaches the final state s_0 via the computation

$$\langle s_1, m, n \rangle = \langle s_1, m_1, n_1 \rangle, \ldots, \langle s_k, m_k, n_k \rangle = \langle s_0, m_k, n_k \rangle .$$

We are going to separate $\varphi(\boldsymbol{P})$ from $\psi(s_1, m, n)$ by the finite frame $\mathfrak{F}$ shown in Fig.16.6, where $s = \max\{s_1, \ldots, s_k\}$, $m' = \max\{m_1, \ldots, m_k\}$ and $n' = \max\{n_1, \ldots, n_k\}$. First, it is readily checked that $\mathfrak{F} \models \varphi(\boldsymbol{P})$. On the other hand, we can define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ so that $\mathfrak{V}(p) = \{b_0^1\}$, $\mathfrak{V}(p_1) = \{a_{m_k}^1\}$, $\mathfrak{V}(p_2) = \{a_{n_k}^2\}$, $\mathfrak{V}(p_3) = \{a_k^3\}$ and then $r \not\models \psi(s_1, m, n)$, i.e., $\mathfrak{F} \not\models \psi(s_1, m, n)$. $\square$

Now recall that we associated $\varphi(\boldsymbol{P})$ and $\psi(s_1, m, n)$ with a program $\boldsymbol{P}$ and a configuration $\langle s_1, m, n \rangle$ in an effective way. So Theorems 16.4 and 16.5 provide us with the following results.

Theorem 16.49 (i) *There is a formula φ such that the problem of recognizing, for an arbitrary formula ψ, whether $\varphi \models_{fin} \psi$ is algorithmically undecidable.*

(ii) *There is a formula ψ such that the problem of recognizing, for an arbitrary formula φ, whether $\varphi \models_{fin} \psi$ is algorithmically undecidable.*

Thus the semantical consequence problem on finite frames is undecidable. Moreover, since the set $\{\langle \varphi, \psi \rangle : \varphi \not\models_{fin} \psi\}$ is clearly recursively enumerable, we also have

Corollary 16.50 (i) *The set $\{\langle \varphi, \psi \rangle : \varphi \models_{fin} \psi\}$ is not recursively enumerable.*

(ii) *There is a formula φ such that the set $\{\psi : \varphi \models_{fin} \psi\}$ is not recursively enumerable.*

(iii) *There is a formula ψ such that the set $\{\varphi : \varphi \models_{fin} \psi\}$ is not recursively enumerable.*

16.7 Admissible and derivable rules

Admissible and derivable rules are used for simplifying the construction of derivations. Derivable rules may replace some fragments of fixed length in derivations, thereby shortening them linearly. Admissible rules, which are not derivable, in principle may reduce derivations even more drastically. In this section we consider the algorithmic problem of recognizing whether a given inference rule is admissible or derivable in certain modal and si-logics.

To begin with, let us observe that even in tabular logics the admissibility problem is not trivial.

Example 16.51 Let $L = \text{Log}$. We show that L is not structurally complete, namely that the inference rule $\Diamond p \wedge \Diamond \neg p / \bot$ is admissible but not derivable in the tabular logic L.

Since L is consistent, this rule is admissible iff $\Diamond\varphi \wedge \Diamond\neg\varphi \notin L$ for any formula φ. Suppose on the contrary that $\Diamond\varphi\wedge\Diamond\neg\varphi \in L$ for some φ. Since $L \subset \mathbf{Triv}$, both formulas $\Diamond\varphi$ and $\Diamond\neg\varphi$ are in $\mathbf{Triv}$ and so $\varphi, \neg\varphi \in \mathbf{Triv}$ because $\Diamond p \to p \in \mathbf{Triv}$, contrary to $\mathbf{Triv}$ being consistent. Thus our rule is admissible.

By the deduction theorem for $\mathbf{S4} \subset L$, the rule $\Diamond p \wedge \Diamond\neg p/\bot$ is derivable iff $\Diamond(\Box\neg p \vee \Box p) \in L$, which is not the case because $\Diamond(\Box\neg p \vee \Box p)$ is clearly refuted in the two point cluster.

Thus the decidability of a logic and the deduction theorem cannot help us in general to recognize admissible rules. Yet the admissibility problem for tabular logics turns out to be decidable.

Theorem 16.52 *For every tabular logic L, there exists an algorithm deciding whether a given inference rule is admissible in L.*

Proof We consider only $L \in \mathrm{NExt}\mathbf{K}$; other logics are treated analogously. That a rule $\varphi(p_1,\dots,p_n)/\psi(p_1,\dots,p_n)$ is not admissible in the logic L determined by a finite algebra $\mathfrak{A}$ means that there are formulas $\chi_1(q_1,\dots,q_m),\dots,\chi_n(q_1,\dots,q_m)$ such that

$$\varphi' = \varphi(\chi_1,\dots,\chi_n) \in L, \quad \psi' = \psi(\chi_1,\dots,\chi_n) \notin L, \tag{16.21}$$

i.e., $\mathfrak{A} \models \varphi'$ and $\mathfrak{A} \not\models \psi'$. Without loss of generality we may assume $m \leq |\mathfrak{A}|$ (for otherwise we could identify some of the variables $q_1,\dots,q_m$). Since $\mathfrak{A}$ is finite, there are only finitely many pairwise non-equivalent in L formulas in $\leq |\mathfrak{A}|$ variables, and we can effectively construct them. Therefore, trying all possible n-tuples $\chi_1,\dots,\chi_n$ of these formulas, we either satisfy (16.21), and then φ/ψ is not admissible, or do not satisfy it, which means that the rule is admitted by L. ❑

Notice that the criterion of admissibility for tabular logics used in the proof above can be clearly extended to arbitrary logics in the following way.

Theorem 16.53 *A rule φ/ψ is admissible in a logic L in* NExt**K** *or* Ext**Int** *iff the quasi-identity $\varphi = \top \to \psi = \top$ is true in $\mathfrak{A}_L(n)$ for every $n < \omega$ iff for any $n < \omega$ and any valuation $\mathfrak{V}$ in $\mathfrak{A}_L(n)$, $\mathfrak{V}(\varphi) = \top$ implies $\mathfrak{V}(\psi) = \top$.*

Proof Exercise. ❑

In general this criterion is not effective. However, we can try to "effectivize" it using the effective description of the upper part of the universal frames $\mathfrak{F}_L(n)$ obtained in Section 8.7, at least for some well-behaved logics.

First we show that dealing with normal modal logics, it is sufficient to consider inference rules of a rather special form. Let $\varphi(q_1,\dots,q_{2n+2})$ be a formula containing no $\Box$ and $\Diamond$ and represented in the full disjunctive normal form (see Exercise 1.2). Say that an inference rule is *reduced* if it has the form

$$\varphi(p_0,p_1,\dots,p_n,\Diamond p_0,\Diamond p_1,\dots,\Diamond p_n)/p_0.$$

Theorem 16.54 *For every rule φ/ψ one can effectively construct a reduced rule φ'/ψ' such that φ/ψ is admissible in a logic $L \in$ NExt**K** iff φ'/ψ' is admissible in L.*

Proof Observe first that if φ and ψ do not contain p then φ/ψ is admissible in L iff $\varphi \wedge (\psi \leftrightarrow p)/p$ is admissible in L. So we can consider only rules of the form φ/p_0. Besides, without loss of generality we may assume that φ does not contain $\Box$ (recall that $\Diamond\varphi = \neg\Box\neg\varphi$).

With every non-atomic subformula χ of φ we associate the new variable p_χ. For convenience we also put $p_\chi = p_i$ if $\chi = p_i$ and $p_\chi = \bot$ if $\chi = \bot$. We are going to show now that the rule

$$\begin{aligned} p_\varphi \wedge \bigwedge\{p_\chi \leftrightarrow p_{\chi_1} \odot p_{\chi_2} : \chi = \chi_1 \odot \chi_2 \in \mathbf{Sub}\varphi,\ \odot \in \{\wedge, \vee, \rightarrow\}\} \wedge \\ \bigwedge\{p_\chi \leftrightarrow \Diamond p_{\chi_1} :\ \chi = \Diamond\chi_1 \in \mathbf{Sub}\varphi\}/p_0 \end{aligned} \tag{16.22}$$

is admissible in L iff φ/p_0 is admissible in L. For the sake of brevity denote the antecedent of (16.22) by φ''.

($\Rightarrow$) Since every substitution instance of φ''/p_0 is admissible in L, the rule $\varphi \wedge \bigwedge_{\chi\in\,\mathbf{Sub}\varphi}(\chi \leftrightarrow \chi)/p_0$ (obtained from it by replacing each p_χ with χ) and so φ/p_0 are also admissible in L.

($\Leftarrow$) Suppose φ/p_0 is admissible in L and a substitution instance $\varphi''\boldsymbol{s}$ of φ'' is in L. Let $\boldsymbol{s} = \{\alpha_\chi/p_\chi : \chi \in \mathbf{Sub}\varphi\}$. By induction on the construction of χ one can readily show that $\alpha_\chi \leftrightarrow \chi\boldsymbol{s} \in L$. Indeed, the basis of induction is trivial and the step of induction follows immediately from the equivalent replacement theorem. Therefore, α_φ is equivalent in L to $\varphi\boldsymbol{s}$. Since $\varphi''\boldsymbol{s} \in L$, we must have in particular $p_\varphi\boldsymbol{s} = \alpha_\varphi \in L$, from which $\varphi\boldsymbol{s} \in L$ and so $p_0\boldsymbol{s} \in L$. Thus φ''/p_0 is admissible in L.

The rule φ''/p_0 is not reduced, but it is easy to make it so simply by representing φ'' in its full disjunctive normal form φ', treating subformulas $\Diamond p_i$ as variables. ❑

From now on we will deal with only reduced rules different from $\bot/p_0$ (which is clearly admissible in any logic). Let $\bigvee_j \varphi_j/p_0$ be a reduced rule, in which each disjunct φ_j is a conjunction of the form

$$\neg_0 p_0 \wedge \ldots \wedge \neg_m p_m \wedge \neg^0\Diamond p_0 \wedge \ldots \wedge \neg^m\Diamond p_m, \tag{16.23}$$

where each $\neg_i$ and $\neg^j$ is either blank or $\neg$. It will be convenient for us to identify such conjunctions with the sets of their conjuncts. Now, given the non-empty set W of conjunctions of the form (16.23) occurring in the premise of the rule under consideration, we define a frame $\mathfrak{F} = \langle W, R\rangle$ and a model $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$ on it by taking

$$\begin{aligned} \varphi_i R\varphi_j \text{ iff } \forall k \in \{0,\ldots,m\}(\neg\Diamond p_k \in \varphi_i \rightarrow \neg\Diamond p_k \in \varphi_j \wedge \neg p_k \in \varphi_j) \wedge \\ \exists k \in \{0,\ldots,m\}(\neg\Diamond p_k \in \varphi_j \wedge \Diamond p_k \in \varphi_i), \end{aligned}$$

$$\mathfrak{V}(p_k) = \{\varphi_i \in W : p_k \in \varphi_i\}.$$

It should be clear that $\mathfrak{F}$ is finite, transitive and irreflexive.

We are in a position now to formulate a criterion for admissibility of reduced rules in **GL**.

Theorem 16.55 *A reduced rule $\bigvee_j \varphi_j/p_0$ is not admissible in* **GL** *iff there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ defined as above on a set W of conjunctions of the form (16.23) and such that*

(i) *$\neg p_0 \in \varphi_i$, for some $\varphi_i \in W$;*

(ii) *$\varphi_i \models \varphi_i$, for every $\varphi_i \in W$;*

(iii) *for every antichain $\mathfrak{a}$ in $\mathfrak{F}$ there is $\varphi_j \in W$ such that, for each $k \in \{0, \dots, m\}$, $\varphi_j \models \Diamond p_k$ iff $\varphi_i \models \Diamond^+ p_k$ for some $\varphi_i \in \mathfrak{a}$.*

Proof ($\Rightarrow$) Suppose $\psi_0, \dots, \psi_m$ are formulas in variables $q_1, \dots, q_n$ such that $\bigvee_j \varphi_j^* \in$ **GL** and $p_0^* \notin$ **GL**, where χ^* denotes the formula $\chi\{\psi_0/p_0, \dots, \psi_m/p_m\}$. This is equivalent to $\mathfrak{M}_{\mathbf{GL}}(n) \models \bigvee_j \varphi_j^*$ and $\mathfrak{M}_{\mathbf{GL}}(n) \not\models p_0^*$ (recall that $\mathfrak{M}_{\mathbf{GL}}(n)$ is the n-universal model for **GL** introduced in Section 8.7). Define W to be the set of those disjuncts φ_j in $\bigvee_j \varphi_j$ whose substitution instances φ_j^* are satisfied in $\mathfrak{M}_{\mathbf{GL}}(n)$. Clearly $W \neq \emptyset$. Let us check conditions (i)–(iii).

(i) Take a point x in $\mathfrak{M}_{\mathbf{GL}}(n)$ at which p_0^* is false. Since $\bigvee_j \varphi_j^*$ is true in $\mathfrak{M}_{\mathbf{GL}}(n)$, we must have $x \models \varphi_i^*$ for some i. One of the formulas p_0^* or $\neg p_0^*$ is a conjunct of φ_i^*. Clearly it is not p_0^*. Therefore, $\neg p_0 \in \varphi_i$.

(ii) It suffices to show that, for every $\varphi_i \in W$ and $k \in \{0, \dots, m\}$, $\varphi_i \models \Diamond p_k$ iff $\Diamond p_k \in \varphi_i$ (that $\varphi_i \models p_k$ iff $p_k \in \varphi_i$ follows from the definition of $\mathfrak{V}$). Suppose $\varphi_i \models \Diamond p_k$. Then there is $\varphi_j \in W$ such that $\varphi_i R \varphi_j$ and $\varphi_j \models p_k$. By the definition of $\mathfrak{V}$, this means that $p_k \in \varphi_j$ and so, by the definition of R, $\Diamond p_k \in \varphi_i$. Conversely, suppose that $\Diamond p_k \in \varphi_i$. Then $x \models \varphi_i^*$ and, in particular, $x \models \Diamond p_k^*$ for some x in $\mathfrak{M}_{\mathbf{GL}}(n)$. Let y be a final point in the set $\{z \in x{\uparrow} : z \models p_k^*\}$. Since $\mathfrak{M}_{\mathbf{GL}}(n)$ is irreflexive, we have $y \models p_k^*$, $y \not\models \Diamond p_k^*$ and $y \models \varphi_j^*$ for some $\varphi_j \in W$. It follows that $\varphi_i R \varphi_j$ and $\varphi_j \models p_k$, from which $\varphi_i \models \Diamond p_k$.

(iii) Let $\mathfrak{a}$ be an antichain in $\mathfrak{F}$. For every $\varphi_i \in \mathfrak{a}$, let x_i be a final point in the set $\{y \in W_{\mathbf{GL}}(n) : y \models \varphi_i^*\}$. It should be clear that the points $\{x_i : \varphi_i \in \mathfrak{a}\}$ form an antichain $\mathfrak{b}$ in $\mathfrak{F}_{\mathbf{GL}}(n)$ and so, by the construction of $\mathfrak{F}_{\mathbf{GL}}(n)$, there is a point y in $\mathfrak{F}_{\mathbf{GL}}(n)$ such that $y{\uparrow} = \mathfrak{b}\underline{\uparrow}$. Then the formula $\varphi_j \in W$ we are looking for is any one satisfying the condition $y \models \varphi_j^*$, as can be easily checked by a straightforward inspection.

($\Leftarrow$) Let $\mathfrak{M}$ be a model meeting (i)–(iii). To prove that $\bigvee_j \varphi_j/p_0$ is not admissible in **GL** we require once again the n-universal model $\mathfrak{M}_{\mathbf{GL}}(n)$, but this time we take n to be the length (the number of symbols) of the rule. By induction on the depth of points in $\mathfrak{M}$ one can readily show that $\mathfrak{M}$ is a generated submodel of $\mathfrak{M}_{\mathbf{GL}}(n)$ (recall that we defined $\mathfrak{M}$ as a model of the language with the variables $p_0, \dots, p_m$).

Our aim is to find formulas $\psi_0, \dots, \psi_m$ such that $\mathfrak{M}_{\mathbf{GL}}(n) \models \bigvee_j \varphi_j^*$ and $\mathfrak{M}_{\mathbf{GL}}(n) \not\models p_0^*$ (here again $\chi^* = \chi\{\psi_0/p_0, \dots, \psi_m/p_m\}$). Loosely, we are going to extend the properties of $\mathfrak{M}$ to the whole model $\mathfrak{M}_{\mathbf{GL}}(n)$. We take the sets

$\{\varphi_i\}$ in $\mathfrak{F}_{\mathbf{GL}}(n)$ and augment them inductively in such a way that we could embrace all points in $\mathfrak{F}_{\mathbf{GL}}(n)$ and retain the property (ii).

Fix any point φ_{i_0} of depth 1 in the model $\mathfrak{M}$ and denote by $\varphi_{i_0}^0$ the set $(\mathfrak{F}_{\mathbf{GL}}^{\leq 1}(n) - \mathfrak{F}^{\leq 1}) \cup \{\varphi_{i_0}\}$. For the remaining φ_i in W, we put $\varphi_i^0 = \{\varphi_i\}$. This is the basis of our inductive construction.

Suppose we have already constructed sets φ_i^l for $l \geq 0$. Take any set X of $l+1$ points in W and associate with it a single formula φ_j satisfying (iii) for the antichain consisting of all the first (minimal) points in X with respect to R. Define an auxiliary set $[X]$ as follows. Using the abbreviation

$$X^\dagger = -\bigcup_{\varphi_i \in W} \varphi_i^l \cap \bigcap_{\varphi_i \in X} \varphi_i^l{\downarrow} \cap -\bigcup_{\varphi_i \notin X} \varphi_i^l{\downarrow},$$

we put $[X] = X^\dagger$ if the condition

$$\forall k \in \{0, \dots, m\}(\neg\Diamond p_k \in \varphi_j \to \neg p_k \in \varphi_j) \tag{16.24}$$

holds and $[X] = X^\dagger - X^\dagger{\downarrow}$ otherwise. Now we define φ_j^{l+1} by adding to φ_j^l all sets $[X]$ with which φ_j was associated (if φ_j was not associated with any set then $\varphi_j^{l+1} = \varphi_j^l$). Clearly this process terminates as far as all possible subsets of points in W are exhausted, i.e., in $N = |W|$ steps.

It follows directly from the construction that we have

Lemma 16.56 (i) *$\varphi_j^l \subseteq \varphi_j^{l+1}$ for any $\varphi_j \in W$ and $l < N$;*

(ii) *$\varphi_j^l \in P_{\mathbf{GL}}(n)$ for any $\varphi_j \in W$ and $l \leq N$;*

(iii) *$\varphi_i^l \cap \varphi_j^l = \emptyset$ for any distinct $\varphi_i, \varphi_j \in W$ and $l \leq N$.*

Lemma 16.57 *For every x in $\mathfrak{M}_{\mathbf{GL}}(n)$ and every $l < N$, if $x \notin \bigcap_{\varphi_j \in W} \varphi_j^l$ then $x \in \bigcap_{\varphi_i \in X} \varphi_i^l{\downarrow}$ for some set X of $l+1$ points in W.*

Proof The proof proceeds by induction on l. Suppose $l = 0$ and $x \notin \bigcap_{\varphi_j \in W} \varphi_j^0$. By the definition of $\varphi_{i_0}^0$ and φ_i^0 for $i \neq i_0$, all points in $\mathfrak{F}_{\mathbf{GL}}^{\leq 1}(n)$ are in $\bigcup_{\varphi_j \in W} \varphi_j^0$. So x is of depth > 1. But then x sees a point of depth 1, say $y \in \varphi_i^0$ for some $\varphi_i \in W$, from which $x \in \varphi_i^0{\downarrow}$.

Let us assume now that the claim of our lemma holds for $l < N - 1$ and $x \notin \bigcap_{\varphi_j \in W} \varphi_j^{l+1}$. By Lemma 16.56 (i), it follows that $x \notin \bigcap_{\varphi_j \in W} \varphi_j^l$ and so, by the induction hypothesis, $x \in \bigcap_{\varphi_i \in X} \varphi_i^l{\downarrow}$ for some set X of $l+1$ points in W. Let φ_j be the point associated with X and consider the set $[X]$.

Suppose (16.24) holds for φ_j. This means that if $x \notin \varphi_k^l{\downarrow}$ for all $\varphi_k \notin X$ then $x \in [X]$ and so $x \in \varphi_j^{l+1}$, which is a contradiction. Therefore, $x \in \varphi_k^l{\downarrow}$ for some $\varphi_k \notin X$. Put $Y = X \cup \{\varphi_k\}$. Then we have $x \in \bigcap_{\varphi_i \in Y} \varphi_i^l{\downarrow}$ and, by Lemma 16.56 (i), $x \in \bigcap_{\varphi_i \in Y} \varphi_i^{l+1}{\downarrow}$.

Suppose now that (16.24) does not hold. If $x \notin \varphi_k^l{\downarrow}$ for all $\varphi_k \notin X$ then either $x \in [X]$, which as we know leads to a contradiction, or $x \in X^\dagger{\downarrow}$, i.e., there is $y \in x{\uparrow}$ such that $y \notin \bigcup_{\varphi_j \in W} \varphi_j^l$, $y \in \varphi_i^l{\downarrow}$ for all $\varphi_i \in X$ and $y \notin \varphi_i^l{\downarrow}$ for

$\varphi_i \notin X$. Of all possible y with these properties we take a final one with respect to $R_{\mathbf{GL}}(n)$. By the definition of $[X]$, we have $y \in [X]$ and so $y \in \varphi_j^{l+1}$. Then $x \in \varphi_j^{l+1}\downarrow$ and, since φ_j is associated with X and does not satisfy (16.24), there is p_k such that $p_k \in \varphi_j$ but $p_k \notin \varphi_i$ for all $\varphi_i \in X$, i.e., $\varphi_j \notin X$. Therefore, we can put $Y = X \cup \{\varphi_j\}$, which provides $x \in \bigcap_{\varphi_i \in Y} \varphi_i^l\downarrow \subseteq \bigcap_{\varphi_i \in Y} \varphi_i^{l+1}\downarrow$. ❑

Lemma 16.58 $W_{\mathbf{GL}}(n) = \bigcup_{\varphi_j \in W} \varphi_j^N$.

Proof Take any $x \in W_{\mathbf{GL}}(n)$. If $x \in \bigcup_{\varphi_j \in W} \varphi_j^{N-1}$ then, by Lemma 16.56 (i), $x \in \bigcup_{\varphi_j \in W} \varphi_j^N$ and we are done. So suppose that $x \notin \bigcup_{\varphi_j \in W} \varphi_j^{N-1}$. Then by Lemma 16.57, $x \in \bigcup_{\varphi_j \in W} \varphi_j^{N-1}\downarrow$. Let us consider $[W]$ and suppose that for φ_j associated with W condition (16.24) is satisfied. Then clearly $x \in [W]$ and so $x \in \varphi_j^N$. Suppose now that (16.24) does not hold. As in the proof of Lemma 16.57, we then have p_k, for some $k \leq m$, such that $\neg\Diamond p_k \in \varphi_j$, $p_k \in \varphi_j$ and, by (iii), $\neg p_k \in \varphi_i$ for all $\varphi_i \in W$, which is a contradiction. ❑

According to Lemma 16.56 (ii) and (iii), the sets φ_i^N can be represented as $\mathfrak{V}_{\mathbf{GL}}(n)(\alpha_i)$ for some formulas α_i in n variables such that distinct α_i and α_j cannot be true at any point in $\mathfrak{M}_{\mathbf{GL}}(n)$ simultaneously. For $i \leq m$ we put

$$\psi_i = \bigvee\{\alpha_j : \ \varphi_j \in W, \ p_i \in \varphi_j\}.$$

Lemma 16.59 *For every point x in φ_j^N, $x \models \varphi_j^*$.*

Proof The routine induction on the minimal l such that $x \in \varphi_j^l$ is left to the reader as an exercise. ❑

We are ready now to complete the proof of Theorem 16.55. It follows from Lemma 16.59 that, for every x in φ_j^N, $x \models \bigvee_j \varphi_j^*$ and so, by Lemma 16.58, $\mathfrak{M}_{\mathbf{GL}}(n) \models \bigvee_j \varphi_j^*$, i.e., $\bigvee_j \varphi_j^* \in \mathbf{GL}$. And by (i), there is $\varphi_j \in W$ such that $\neg p_0 \in \varphi_j$. Therefore, $x \models \neg p_0^*$ for $x \in \varphi_j^N$ and so $p_0^* \notin \mathbf{GL}$, which means that the rule $\bigvee_j \varphi_j / p_0$ is not admissible in $\mathbf{GL}$. ❑

A remarkable feature of the criterion we have just proved is that it can be effectively checked. Thus we have

Theorem 16.60 *There is an algorithm which, given an inference rule, can decide whether it is admissible in* **GL** *or not.*

The effective admissibility criteria similar to Theorem 16.55 can be proved for many other logics. We confine ourselves here only to formulating such a criterion for **Grz**.

Suppose a reduced rule $\mathbf{r} = \bigvee_j \varphi_j / p_0$ is given. First we delete from its antecedent all disjuncts containing $\neg\Diamond p_k$ and p_k for some k. The resulting rule $\mathbf{r}'$ will be admissible in **Grz** iff $\mathbf{r}$ is admissible in **Grz** because, for any φ, ψ, χ,

$$\varphi \vee (\psi \wedge \neg\Diamond\chi \wedge \chi) \leftrightarrow \varphi \in \mathbf{K} \oplus \Box p \to p.$$

Now, if $\mathbf{r}' = \bot/p_0$ then everything is clear. Otherwise we take a non-empty set W of disjuncts in $\mathbf{r}'$ and construct the frame $\mathfrak{F} = \langle W, R\rangle$ and the model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ in almost the same way as above: the only difference is that now we take

$$\begin{aligned}\varphi_i R \varphi_j \text{ iff } & (\forall k \in \{0,\dots,m\}(\neg\Diamond p_k \in \varphi_i \to \neg\Diamond p_k \in \varphi_j) \wedge \\ & \exists k \in \{0,\dots,m\}(\neg\Diamond p_k \in \varphi_j \wedge \Diamond p_k \in \varphi_i)) \vee \\ & \varphi_i = \varphi_j.\end{aligned}$$

Theorem 16.61 *A reduced rule $\mathbf{r} = \bigvee_j \varphi_j/p_0$, which is different from $\bot/p_0$ and has no disjuncts containing both $\neg\Diamond p_k$ and p_k for some k, is not admissible in* **Grz** *iff there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ defined as above on a set W of disjuncts in* $\mathbf{r}$ *and such that*

(i) $\neg p_0 \in \varphi_i$, *for some* $\varphi_i \in W$;

(ii) $\varphi_i \models \varphi_i$, *for every* $\varphi_i \in W$;

(iii) *for every antichain* $\mathfrak{a}$ *in* $\mathfrak{F}$ *there is* $\varphi_j \in W$ *such that, for each variable* p_k *in* $\mathbf{r}$, $\varphi_j \models \Diamond p_k$ *iff* $\varphi_i \models \Diamond^+ p_k$ *for some* $\varphi_i \in \mathfrak{a}$.

Proof The proof is conducted by the same scheme as the proof of Theorem 16.55. ❑

As a consequence we obtain

Theorem 16.62 *The admissibility problem in* **Grz** *is decidable.*

We show now that the admissibility problem in **Int** can be reduced to the same problem in **Grz** and so is also decidable. To this end we require the following generalization of Theorem 3.83 in which we assume for simplicity that the Gödel translation T prefixes $\Box$ to every subformula of a given intuitionistic formula (see Exercise 3.25).

Theorem 16.63 *A rule φ/ψ is admissible in* **Int** *iff the rule $\mathsf{T}(\varphi)/\mathsf{T}(\psi)$ is admissible in* **Grz**.

Proof ($\Leftarrow$) Let $p_1,\dots,p_n$ be all variables in φ and ψ. Suppose that the rule $\mathsf{T}(\varphi)/\mathsf{T}(\psi)$ is admissible in **Grz** and $\varphi(\chi_1,\dots,\chi_n) \in$ **Int**. Then by Theorem 3.83, $\mathsf{T}(\varphi(\chi_1,\dots,\chi_n)) \in$ **Grz**. Since $\Box\Box p \leftrightarrow \Box p \in$ **Grz**, we have

$$\mathsf{T}(\varphi(\chi_1,\dots,\chi_n)) \leftrightarrow \mathsf{T}(\varphi)\{\mathsf{T}(\chi_1)/p_1,\dots,\mathsf{T}(\chi_n)/p_n\} \in \mathbf{Grz}.$$

It follows that $\mathsf{T}(\psi(\chi_1,\dots,\chi_n)) \in$ **Grz** and so, again by Theorem 3.83, we obtain $\psi(\chi_1,\dots,\chi_n) \in$ **Int**.

($\Rightarrow$) This part of the proof requires two auxiliary lemmas.

Lemma 16.64 *For every modal formula $\varphi(p_1,\dots,p_n)$ there is an intuitionistic formula $\psi(p_1,\dots,p_n)$ such that*

$$\Box\varphi(\Box p_1,\dots,\Box p_n) \leftrightarrow \mathsf{T}(\psi) \in \mathbf{Grz}.$$

Proof The proof proceeds by induction on the construction of φ. The basis of induction is clear because $\Box\Box p \leftrightarrow \Box p \in \mathbf{Grz}$. The same concerns the step of induction for $\varphi = \Box\varphi'$. So suppose $\varphi = \varphi'(\Box\varphi_1, \dots, \Box\varphi_m, p_1, \dots, p_n)$ and φ' contains no occurrence of $\Box$. Then putting $\psi_{m+i} = p_i$ and using the fact that $\Box\Box p \leftrightarrow \Box p \in \mathbf{Grz}$, the equivalent replacement theorem and the induction hypothesis, we obtain that

$$\Box\varphi(\Box p_1, \dots, \Box p_n) \leftrightarrow \Box\varphi'(\mathsf{T}(\psi_1), \dots, \mathsf{T}(\psi_{m+n})) \in \mathbf{Grz},$$

for some intuitionistic formulas $\psi_1, \dots, \psi_{m+n}$ in the variables $p_1, \dots, p_n$. Now we transform φ' into its conjunctive normal form and get either $\top = \neg\bot$ or $\bigwedge_i(\bigvee_j \neg p_j \vee \bigvee_k p_k)$. In the former case we have

$$\Box\varphi(\Box p_1, \dots, \Box p_n) \leftrightarrow \mathsf{T}(\neg\bot) \in \mathbf{Grz}$$

and in the latter

$$\Box\varphi(\Box p_1, \dots, \Box p_n) \leftrightarrow \bigwedge_i \Box(\bigwedge_j \mathsf{T}(\psi_j) \to \bigvee_k \mathsf{T}(\psi_k)) \in \mathbf{Grz}$$

and so $\Box\varphi(\Box p_1, \dots, \Box p_n) \leftrightarrow \mathsf{T}(\bigwedge_i(\bigwedge_j \psi_j \to \bigvee_k \psi_k)) \in \mathbf{Grz}$. ❑

Lemma 16.65 *Suppose that* $\varphi(p_1, \dots, p_n) \notin \mathbf{Grz}$. *Then there exist formulas* $\chi_1(q_1, \dots, q_m), \dots, \chi_n(q_1, \dots, q_m)$ *such that* $\varphi(\chi'_1, \dots, \chi'_n) \notin \mathbf{Grz}$, *where* $\chi'_i = \chi_i(\Box q_1, \dots, \Box q_m)$.

Proof Follows from the fact that **Grz** is finitely approximable and that every set of points in a finite partially ordered frame can be represented as a Boolean combination of upward closed sets. ❑

We are in a position now to complete the proof of Theorem 16.63. Suppose a rule φ/ψ is admissible in **Int**, $p_1, \dots, p_n$ are all the variables in it and assume also that, for some $\chi_1, \dots, \chi_n$,

$$\mathsf{T}(\psi)(\chi_1(q_1, \dots, q_m), \dots, \chi_n(q_1, \dots, q_m)) \notin \mathbf{Grz}.$$

By Lemma 16.65, there are substitution instances χ'_i of χ_i such that

$$\mathsf{T}(\psi)(\chi'_1(\Box r_1, \dots, \Box r_k), \dots, \chi'_n(\Box r_1, \dots, \Box r_k)) \notin \mathbf{Grz}.$$

In accordance with Lemma 16.64, for each χ'_i we choose an intuitionistic formula χ''_i such that

$$\Box\chi'_i(\Box r_1, \dots, \Box r_k) \leftrightarrow \mathsf{T}(\chi''_i(r_1, \dots, r_k)) \in \mathbf{Grz}.$$

Since every variable in $\mathsf{T}(\psi)$ is "boxed" and $\Box\Box p \leftrightarrow \Box p \in \mathbf{Grz}$, we then have

$$\mathsf{T}(\psi)\{\mathsf{T}(\chi''_1)/p_1, \dots, \mathsf{T}(\chi''_n)/p_n\} \notin \mathbf{Grz}$$

and so $\mathsf{T}(\psi(\chi_1'', \dots, \chi_n'')) \notin \mathbf{Grz}$, from which $\psi(\chi_1'', \dots, \chi_n'') \notin \mathbf{Int}$ and hence $\varphi(\chi_1'', \dots, \chi_n'') \notin \mathbf{Int}$. Now we apply the same chain of arguments but in the reverse order to $\varphi(\chi_1'', \dots, \chi_n'')$ and finally get

$$\mathsf{T}(\varphi)(\chi_1(q_1, \dots, q_m), \dots, \chi_n(q_1, \dots, q_m)) \notin \mathbf{Grz},$$

which establishes the admissibility of $\mathsf{T}(\varphi)/\mathsf{T}(\psi)$ in $\mathbf{Grz}$. ❑

As a consequence of Theorems 16.62 and 16.63 we obtain

Theorem 16.66 *The admissibility problem for inference rules in* **Int** *is decidable.*

For the logics considered above and many others as well the derivability problem for inference rules is solved trivially using the deduction theorem. We remind the reader that according to Theorem 3.51 and Exercise 3.5, if necessitation RN is not a postulated inference rule in a logic L then

$$\Gamma, \varphi \vdash_L \psi \text{ iff } \Gamma \vdash_L \varphi \to \psi$$

and if $L \in \mathrm{NExt}\mathbf{K4}$ and RN is a postulated rule in L then

$$\Gamma, \varphi \vdash_L^* \psi \text{ iff } \Gamma \vdash_L \Box^+\varphi \to \psi.$$

Thus, in these cases the derivability problem for inference rules reduces to the decidability problem, i.e., we have

Theorem 16.67 (i) *If the rule RN is not postulated in a logic L then the derivability problem for inference rules in L is decidable iff L is decidable.*

(ii) *The derivability problem for inference rules in a logic $L \in$* NExt**K4** *is decidable iff L is decidable.*

The same result holds of course for modal logics containing $\boldsymbol{tra}_n$, for some $n < \omega$. In general, in view of the existential quantifier in the deduction theorem for logics in NExt**K**, the situation is more complicated. However, for some systems the deduction theorem can be "effectivized", as was done in Theorem 3.57. Another method of establishing the decidability of the derivability problem in a logic L is to show that L is globally finitely approximable. Using one of these ways one can prove the following:

Theorem 16.68 *The derivability problem for inference rules is decidable in the logics* **K**, **D**, **T**, $\mathbf{T} \oplus p \to \Box\Diamond p$.

Proof Exercise. ❑

Algorithms recognizing admissible or derivable inference rules in a logic L can be used as decision algorithms for L as well: $\varphi \in L$ iff the rule $\top/\varphi$ is admissible in L iff $\top/\varphi$ is derivable in L. However, the converse does not hold. The aim of the rest of this section is to present corresponding examples

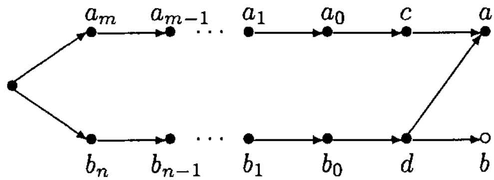


FIG. 16.7.

Let X be a recursive set of pairs of natural numbers such that the projection $X' = \{n : \exists m \langle m, n\rangle \in X\}$ is not recursive. Denote by $\mathfrak{F}(m,n)$ the transitive frame shown in Fig. 16.7 and consider the normal modal logic

$$L_1 = \mathrm{Log}\{\mathfrak{F}(m,n) : \left\langle \frac{m-1}{2}, \frac{n-1}{2} \right\rangle \notin X\}.$$

Theorem 16.69 (i) *The derivability problem for inference rules in L_1 is decidable.*

(ii) *The admissibility problem for inference rules in L_1 is undecidable.*

Proof (i) Since $L_1 \supseteq \mathbf{K4}$, it suffices, by Theorem 16.67, to show that L_1 is decidable. And this is a consequence of the following:

Lemma 16.70 *For every formula φ, $\varphi \notin L_1$ iff $\mathfrak{F}(m,n) \not\models \varphi$ for some m and n such that* $\max\{m,n\} \le l(\varphi) + 1$ *and* $\mathfrak{F}(m,n) \models L_1$.

Proof In Section 18.3 we will be proving similar results in full details. So we leave this one to the reader as an exercise. A little hint is that all frames of the form $\mathfrak{F}(i, 2n)$ and $\mathfrak{F}(2m, i)$ validate L_1. ❑

(ii) We require the following variable free formulas:

$$\alpha = \Box\bot,\ \beta = \Diamond\top \wedge \Box\Diamond\top,$$

$$\gamma = \Diamond\alpha \wedge \neg\Diamond^2\alpha \wedge \neg\Diamond\beta,\ \delta = \Diamond\alpha \wedge \Diamond\beta \wedge \neg\Diamond^2\alpha$$

which characterize, respectively, the points a, b, c, d in $\mathfrak{F}(m,n)$. Now put

$$\mathbf{r}_n = \varphi_n(p)/\psi(p)$$

where

$$\varphi_n(p) = \neg(\Diamond(\neg\Diamond\delta \wedge \Diamond\gamma \wedge p) \wedge \neg\Diamond(\neg\Diamond\delta \wedge \Diamond(\Diamond\gamma \wedge p)) \wedge \Diamond(\neg\Diamond\gamma \wedge \Diamond^{2n+1}\delta) \wedge \neg\Diamond(\neg\Diamond\gamma \wedge \Diamond^{2n+2}\delta)),$$

$$\psi(p) = \neg(\neg\Diamond\delta \wedge \Diamond\gamma \wedge p).$$

Since the rule $\mathbf{r}_n$ is defined effectively by n, (ii) will follow from the fact that X' is not recursive and

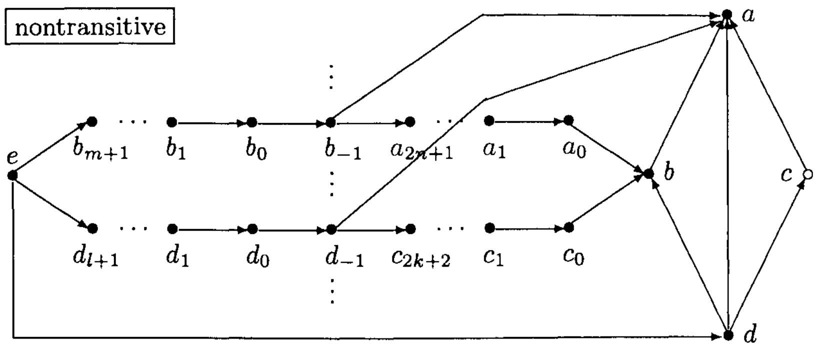


FIG. 16.8.

Lemma 16.71 *The rule* $\mathbf{r}_n$ *is admissible in* L_1 *iff* $n \notin X'$.

Proof Suppose $n \in X'$. This means that there is m such that $\langle m, n \rangle \in X$ and so $\mathfrak{F}(m, n)$ is not a frame for L_1. It is not hard to see then that $\varphi_n(\Diamond^{2m+1}\gamma) \in L_1$. However, $\psi(\Diamond^{2m+1}\gamma) \notin L_1$ because this formula is refuted in all frames $\mathfrak{F}(2k, i)$ for sufficiently big k (e.g. $k = 2m + 4$). Thus $\mathbf{r}_n$ is not admissible in L_1.

Now let $n \notin X'$ and show that $\mathbf{r}_n$ is admissible in L_1. Suppose $\psi(\chi) \notin L_1$ for some formula χ. Then for some m and k such that $\langle \frac{m-1}{2}, \frac{k-1}{2} \rangle \notin X$ and some valuation, $\neg\Diamond\delta \wedge \Diamond\gamma \wedge \chi$ is true at a point x in $\mathfrak{F}(m, k)$. It follows that $x = a_i$ for some i. Let i be the minimal number for which $a_i \models \neg\Diamond\delta \wedge \Diamond\gamma \wedge \chi$.

Since $n \notin X'$, $\mathfrak{F}(i, 2n + 1)$ is a frame for L_1. Define a valuation in this frame so that the same variables be true at the points of the set $a_i\!\uparrow$ in $\mathfrak{F}(i, 2n + 1)$ and $\mathfrak{F}(m, k)$. Then we shall have $\mathfrak{F}(i, 2n + 1) \not\models \varphi_n(\chi)$ and so $\varphi_n(\chi) \notin L_1$. ❑

This completes the proof of Theorem 16.69. ❑

Let us consider now the frame $\mathfrak{G} = \langle V, S \rangle$ depicted in Fig. 16.8. $\mathfrak{G}$ is not transitive; the arrows show all the accessibilities in it. Chains of points a_i, b_i, c_i and d_i satisfy the following conditions. If $\langle m, n \rangle \in X$ then $\mathfrak{G}$ contains a chain of the form

$$b_{m+1} S b_m S \dots S b_{-1} S a_{2n+1} S \dots S a_1 S a_0$$

and besides, for every pair $\langle k, l \rangle$, $\mathfrak{G}$ contains a chain of the form

$$d_{l+1} S d_l S \dots S d_{-1} S c_{2k+2} S \dots S c_1 S c_0.$$

Put $L_2 = \mathrm{Log}\mathfrak{G}$.

Theorem 16.72 (i) L_2 *is decidable.*

(ii) *The derivability problem for inference rules in* L_2 *is undecidable.*

Proof (i) (Sketch) It is not hard to observe that although $\mathfrak{G}$ is infinite (and L_2 is not finitely approximable), in order to refute a formula $\varphi \notin L_2$ it is sufficient to consider only the part of $\mathfrak{G}$ with the chains of a_i, b_i, c_i and d_i of length $\leq l(\varphi) + 1$

and, since X is recursive, we can effectively check whether $\mathfrak{G}$ contains chains of that sort.

(ii) We introduce formulas characterizing points in $\mathfrak{G}$ (follow the diagram of $\mathfrak{G}$):

$$\alpha = \Box\bot, \ \beta = \Diamond\top \wedge \Box^2\bot,$$

$$\gamma = \Diamond\alpha \wedge \Diamond^2\alpha \wedge \neg\Diamond\beta, \ \delta = \Diamond\alpha \wedge \Diamond\beta \wedge \Diamond\gamma, \ \epsilon = \Diamond\delta,$$

$$\lambda_0 = \Diamond\beta \wedge \neg\delta, \ \lambda_{i+1} = \Diamond\lambda_i \wedge \neg\Diamond\alpha, \ \lambda'_{i+1} = \Diamond\lambda_i \wedge \Diamond\alpha \ (i \geq 0).$$

Now put

$$\mathbf{r}_n = \Box\neg\lambda'_{2n+2}/\Box\neg\delta$$

and show that the rule $\mathbf{r}_n$ is derivable in L_2 iff $n \in X'$.

Suppose $\mathbf{r}_n$ is derivable in L_2. By the deduction theorem, we then have some $m < \omega$ such that

$$\bigwedge_{i=1}^{m} \Box^i\neg\lambda'_{2n+2} \to \Box\neg\delta \in L_2$$

or, which is equivalent,

$$\mathfrak{G} \models \Diamond\delta \to \bigvee_{i=1}^{m} \Diamond^i\lambda'_{2n+2}.$$

Since e is the only point in $\mathfrak{G}$ at which $\Diamond\delta$ is true, $e \models \bigvee_{i=1}^{m} \Diamond^i\lambda'_{2n+2}$ and so we have a chain $eSb_{l+1}Sb_lS \dots Sb_{-1}Sa_{2n+1}S \dots Sa_1Sa_0$ for some $l \leq m$. By the construction of $\mathfrak{G}$, this means that $\langle l, n\rangle \in X$, from which $n \in X'$.

It is easy to see that all the steps in this argument are reversible. Consequently, $\mathbf{r}_n$ is admissible in L_2 whenever $n \in X'$. ❑

16.8 Exercises and open problems

Exercise 16.1 Show that all logics in NExt**K4.3** containing $\boldsymbol{den}_n$ for some $n < \omega$ are finitely approximable.

Exercise 16.2 Show that there are undecidable recursively axiomatizable logics in NExt**K4.3**.

Exercise 16.3 For $k < \omega$, say that a type $t = \langle \xi_1, \dots, \xi_n\rangle$ for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is a *k-type* if, for every ξ_i such that $\xi_i = m < \omega$ or $\xi_i = m+$, we have $m \leq k$. Suppose L is a finitely axiomatizable normal extension of **K4.3** and k the maximal number of irreflexive points in the frames underlying the formulas in some finite canonical axiomatization of L. Prove that, for any canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff for every $k+1$-type t for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, the t-extension of $\mathfrak{F}$ is not a frame for L.

Exercise 16.4 Let $\varphi \equiv_{fin} \psi$ mean that φ and ψ are valid in the same finite frames. Prove the analogues of Theorem 16.49 and Corollary 16.50 for the relation $\equiv_{fin}$.

Exercise 16.5 Show that there is a purely implicative undecidable formula in Ext**Int**.

Exercise 16.6 Prove an analog of Theorem 16.14 for si-logics.

Exercise 16.7 Prove that $(p \to q) \vee (q \to p)$ and all formulas in one variable are decidable in Ext**Int**, and all variable free formulas are decidable in NExt**GL** and Ext**GL**.

Exercise 16.8 Prove that the normal modal logic of the two point irreflexive chain is not structurally complete.

Exercise 16.9 Prove that every structurally complete normal modal logic either contains $\mathbf{D} = \mathbf{K} \oplus \Diamond\top$ or coincides with $\mathbf{K} \oplus \Box\bot$.

Exercise 16.10 Give an example of a structurally incomplete tabular si-logic.

Exercise 16.11 Show that the rule $\Box p/p$ is admissible but not derivable in **GL**.

Exercise 16.12 Prove the decidability of the derivability problem for inference rules in every tabular logic.

Problem 16.1 *Are finitely axiomatizable modal and si-logics of finite width decidable?*

Problem 16.2 *Can Theorem 16.16 be extended to logics in* NExt**K***?*

Problem 16.3 *Are the realizability logic and* **ML** *decidable?*

Problem 16.4 *Is the admissibility problem in* **K** *decidable?*

16.9 Notes

The material of Section 16.2 is rather standard and mostly well known (not only to modal logicians). Say, Craig's (1953) theorem holds in a very wide class of formal systems; counterexamples for it have been found only among equational logics having no relation to "real" logics.

Till the end of the 1960s the decidability of various non-tabular logics was established mainly with the help of Theorem 16.11, i.e., by proving the finite approximability with an effective upper bound for the size of the minimal refutation frames (algebras, matrices). Harrop's theorem is more general. However, its deficiency is that now we cannot *a priory* estimate the effectiveness of the algorithm it provides. The examples of finitely approximable recursively axiomatizable logics that are not decidable, presented earlier in the book, and Theorem 16.14, proved in Chagrov (1994a), answer the natural questions concerning possible generalizations. The last theorem of Section 16.2 is the strongest (and so practically useless) generalization of Harrop's theorem. Some results on the connection between the decidability of finitely approximable logics and recursive bounds of the size of refutation models can be found in Ulrich (1982, 1983, 1984).

At the end of the 1960s a method of embeddings into various rich and yet decidable theories was developed to prove the decidability of modal and si-logics that are not finitely approximable; consult Gabbay (1971a, 1975, 1976). The most popular tools were Rabin's (1969) and Büchi's (1962) theorems. Gabbay (1975) used Rabin's theorem to establish the decidability of $\mathbf{K} \oplus \Diamond^n\Box p \to \Box p$, $\mathbf{K} \oplus \Box p \to \Box^n p$, $\mathbf{K} \oplus \Box^n\Diamond p \to \Diamond p$ and some other logics. One of the strongest results obtained by this method is Sobolev's (1977a) theorem, according to which all si-calculi of width 2 and all si-calculi of finite width containing the formula

$$(((p \to q) \to p) \to p) \vee (((q \to p) \to q) \to q)$$

are decidable. Note, however, that such decidability results can be proved also without using rich theories. The proof that all calculi in NExt**K4.3** are decidable, taken from Zakharyaschev and Alekseev (1995), shows an alternative way to establish decidability by proving first a good completeness result. Wolter (1996c) extended Theorem 16.25 to tense linear calculi (which in general are not even Kripke complete).

The question on the approximability of logics by recursive algebras was raised by Kuznetsov in the 1960s. The fact that recursive pseudo-Boolean algebras are not enough to characterize all si-logics was discovered by Chagrov and Tsytkin (1987), and Chagrov (1994a) strengthened this result to si-logics of widths 3 and to other types of recursive semantics; for example, he showed that there is a si-logic of width 3 (by Fine's theorem, it is Kripke complete) which is characterized neither by recursive algebras, nor by recursive Kripke frames. These results were obtained by using the cardinality argument (see Notes to Chapter 4). They give no solution to the analogous problems concerning calculi. The following problems raised by Kuznetsov are still open:

- Is it true that every (si-) calculus is characterized by recursive algebras?
- Is it true that every (si-) calculus characterized by recursive algebras is decidable?

It would be of interest also to clarify the relation between recursive algebras and frames. In general, however, this field of studies remains still *terra incognita.*

The first undecidable modal and si-calculi were constructed by Thomason (1975c), Isard (1977) and Shehtman (1978b). Shehtman (1982) gave examples of undecidable bimodal and tense calculi whose axioms are reductions of modalities. He notes also that the decidability problem for normal modal calculi axiomatizable by modal reduction principles is open.

Since undecidable calculi can be used as a base for obtaining "negative" solutions to various algorithmic problems, it is of interest to find the simplest possible calculi of that sort. For example, Chagrov (1994c) constructed undecidable calculi in Ext**Int**, NExt**S4** and NExt**GL** with axioms in four, three and one variable, respectively. For comparison we remind the reader that all calculi in NExt**S4** with one-variable axioms are finitely approximable and so decidable (see Section 11.6). On the other hand, Sobolev (1977b) constructed a si-calculus with two-variable axioms that is not finitely approximable, and Shehtman (1977) even an incomplete one. It is unknown whether there exist undecidable si-calculi with axioms in two or three variables; the same concerns calculi in NExt**S4** with two-variable axioms.

Having used Minsky machines to construct undecidable calculi, we followed the idea of Isard (1977), developed further by Chagrova (1989a, 1989b) who gave an example of an undecidable elementary si-calculus.

The notion of undecidable formula was introduced in Chagrov (1994c), where numerous examples of such formulas in various classes of logics were given. Here is the simplest known undecidable formula in Ext**Int**:

$$\neg(p \wedge q) \vee \neg(\neg p \wedge q) \vee \neg(p \wedge \neg q) \vee \neg(\neg p \wedge \neg q).$$

The algorithmic problem of semantical consequence on finite frames was solved negatively by Chagrov (1990a) practically for all natural classes of frames, including intuitionistic ones. The presentation of Section 16.6 follows Chagrov and Chagrova (1995).

Note by the way that the decidability problem for such interesting logics as the realizability logic and **ML** is still open in spite of numerous attempts to solve it.

Thomason (1975a) showed that there is a modal formula φ such that the set of formulas which are valid in all frames for φ is Π^1_1-complete. Note, however, that this result as well as similar results of Thomason (1975b) essentially use nontransitive frames. It would be of interest to transfer them to the transitive and intuitionistic cases.

One of the reasons to study admissible and derivable in a given logic inference rules are various applications. We have already mentioned the possibility of using such rules to shorten derivations. Another application is connected with the problem of finite axiomatizability, because it essentially depends on the set of postulated inference rules. Without going into details, note, for instance, that although Medvedev's logic is not finitely axiomatizable, as was shown by Maksimova *et al.* (1979), there is still a hope to find a finite axiomatization for it by adding some sort of (non-structural) rules; see (Medvedev, 1979). An active study of non-structural rules was initiated by Gabbay (1981b); see (Venema, 1993).

The decidability of the admissibility problem for inference rules in **GL**, **Grz** and **Int** was proved by Rybakov (1984b, 1985a, 1985b, 1986a, 1986b, 1987a, 1987b, 1989, 1990a, 1990b, 1990c, 1993). For other logics similar results were obtained in Rybakov (1981, 1984c, 1984a). In particular in the latter paper it was shown that the admissibility problem is decidable in all extensions of **S4.3**, which is a generalization of Fine's (1971) result according to which all these logics are decidable. The same ideas have been recently extended by Rybakov (1994) to **K4** and some of its extensions.

The key role in all these papers is played by the universal models, which, as we saw in Chapter 8, have a clear structure in the case of finitely approximable logics in NExt**K4** and Ext**Int**. We know no other logic for which the admissibility problem has been solved positively. Even for **K**, whose universal models can be described in some way, this problem has not been solved yet. Another open problem here is to construct a decidable calculus the admissibility problem for which is undecidable.

A decidable logic whose admissibility problem is undecidable was constructed by Chagrov (1992b). We know nothing about examples of that sort in **ExtInt**. Spaan (1993) proved that the logic

$$\mathbf{Alt}_2 \oplus \bigwedge_{1 \le i \le 4} \Diamond\Diamond p_i \to \bigvee_{1 \le i < j \le 4} \Diamond\Diamond(p_i \wedge p_j)$$

is decidable (actually, it is a subframe logic; see Exercise 11.21) but the derivability problem for inference rules in it is undecidable. Kracht and Wolter (1997) showed that the derivability problem for inference rules is undecidable in the class of decidable logics.

17

THE DECIDABILITY OF LOGICS' PROPERTIES

In this chapter we return to the main question of Part IV—how to determine whether a given logic satisfies a given property—and consider it from the algorithmic point of view.

17.1 A trivial solution

The ideal solution to the algorithmic problem of recognizing a property $\mathcal{P}$ of logics in a given family should present an algorithm which, given an effective definition of a logic L in the family, could determine whether L satisfies $\mathcal{P}$ or not. In this case it is appropriate to call the property $\mathcal{P}$ decidable in that family. As we saw in Section 16.2, there are distinct (though equivalent and effectively related with each other) algorithmic ways of defining logics: an algorithm enumerating a logic's formulas, an algorithm enumerating its axioms, an algorithm recognizing them. So without loss of generality we can speak only about recursively axiomatizable logics, but freely use any of these ways.

Unfortunately, this "ideal solution" is inaccessible, as is shown by the following:

Theorem 17.1. (Kuznetsov's theorem) *No non-trivial property of recursively axiomatizable logics is decidable in any lattice of logics considered above.*

Proof We will use the undecidability of the halting problem for Minsky machines. Also we need an effective procedure enumerating pairs of natural numbers. Recall that $l(n)$ and $r(n)$ denote the left and the right components of the pair with number n, respectively (i.e., n is the number of the pair $\langle l(n), r(n)\rangle$).

Let $\mathcal{P}$ be a non-trivial property of logics in some lattice. Since $\mathcal{P}$ is not trivial, the lattice contains a logic different from the inconsistent one. Suppose for definiteness that the inconsistent logic satisfies $\mathcal{P}$ and a logic L with a recursive enumeration $\varphi_0, \varphi_1, \dots$ of its formulas does not have $\mathcal{P}$.

Now, given an arbitrary program $\boldsymbol{P}$, we define an effective procedure for enumerating axioms $\psi_0, \psi_1, \dots$ of some logic L':

$$\psi_n = \begin{cases} \varphi_n \text{ if } \boldsymbol{P} \text{ does not halt after} \\ \quad l(n) \text{ steps on the input } r(n) \\ \bot \text{ otherwise.} \end{cases}$$

Then we have the following implications:

$$\begin{gathered}\boldsymbol{P} \text{ does not halt on any input}\\ \Downarrow \\ \psi_n = \varphi_n \text{ for every } n\\ \Downarrow \\ L' = L \\ \Downarrow \\ L' \text{ does not have } \mathcal{P}\end{gathered}$$

and

$$\begin{gathered}\boldsymbol{P} \text{ halts on some input}\\ \Downarrow \\ \bot \text{ is an axiom of } L'\\ \Downarrow \\ L' \text{ is inconsistent} \\ \Downarrow \\ L' \text{ satisfies } \mathcal{P}.\end{gathered}$$

Thus, if we could effectively recognize $\mathcal{P}$ then we would also be able to decide the undecidable halting problem. ❑

Theorem 17.1 prompts us to change the definition of decidable property. We call a property $\mathcal{P}$ *decidable* in the lattice of logics (N)Ext$L$ if there exists an algorithm which, given a finite set Γ of axioms, can determine whether $L + \Gamma$ (respectively, $L \oplus \Gamma$) satisfies $\mathcal{P}$ or not.

In the next section we shall see that the decidability problem for properties in this sense, i.e., for properties of calculi, is not so trivial and frustrating.

17.2 Decidable properties of calculi

In this section we have collected those properties the decidability of which follows easily from the results obtained in Part IV.

We begin with the consistency problem in NExt$\mathbf{K}$. According to Makinson's theorem, the logic $\mathbf{K} \oplus \bot$ has exactly two immediate predecessors in NExt$\mathbf{K}$, viz., Log$\circ$ and Log$\bullet$. Hence, $\mathbf{K} \oplus \varphi \neq \mathbf{K} \oplus \bot$ iff $\varphi \in$ Log$\circ$ or $\varphi \in$ Log$\bullet$. So to decide whether a logic $\mathbf{K} \oplus \varphi$ is consistent it suffices just to check the conditions $\circ \models \varphi$ and $\bullet \models \varphi$, which can be done in finitely many steps. If at least one of them is satisfied then $\mathbf{K} \oplus \varphi$ is consistent, otherwise it is inconsistent. Thus we obtain

Theorem 17.2 *The property of consistency is decidable in* NExt$\mathbf{K}$.

Let us generalize this observation. In fact, the algorithm described above decides the problem of coincidence with a fixed logic, namely, the inconsistent one. And almost the same scheme works for recognizing the coincidence with any decidable logic L having in the lattice under consideration finitely many decidable immediate predecessors, say, $L_1, \dots L_n$. Indeed, a logic L' coincides with L iff $L' \subseteq L$, $L' \not\subseteq L_1, \dots, L' \not\subseteq L_n$, which can be effectively checked if L' is finitely axiomatizable. Using this scheme together with Theorems 12.7, 12.9 we obtain

Theorem 17.3 *The property of coincidence with a fixed tabular logic in* NExt**K4** *(*Ext**Int**, Ext**S4***) is decidable.*

This scheme is not applicable immediately to tabular logics in Ext**GL**. Yet, it can be "pressed out" to produce

Theorem 17.4 *The property of coincidence with a fixed tabular logic in* Ext**GL** *is decidable.*

Proof Exercise (see the proof of Theorem 17.6). ❑

Sometimes a similar argument can be used for proving the decidability of tabularity. Let us consider this property in Ext**Int**. As we saw in Section 12.3, every non-tabular si-logic is contained in one of the three pretabular logics in Ext**Int**, call them L_1, L_2 and L_3. So a calculus $\mathbf{Int} + \varphi$ is tabular iff $\varphi \notin L_1$, $\varphi \notin L_2$ and $\varphi \notin L_3$, which can be checked effectively, because L_1, L_2 and L_3 are decidable. Analogous tabularity criteria hold for NExt**S4** and Ext**S4**. Thus we have

Theorem 17.5 *The property of tabularity is decidable in* NExt**S4**, Ext**S4** *and* Ext**Int**.

To extend this result to NExt**GL** and Ext**GL** a more sophisticated argument is required.

Theorem 17.6 *Tabularity is decidable in* NExt**GL** *and* Ext**GL**.

Proof We consider only Ext**GL** and leave the easier case of NExt**GL** to the reader.

Observe first that if $\Box^n\bot \in \mathbf{GL} + \varphi$ for no $n < \omega$ then $L = \mathbf{GL} + \varphi$ is not tabular. Indeed, otherwise, by Corollary 12.3, the logic $L + \{\Diamond^n\top : \ n < \omega\}$ would be tabular, which is a contradiction, because it is consistent and does not have finite frames at all.

Suppose now that we have succeeded in establishing that $\Box^n\bot \in L$ for some $n < \omega$, i.e., $L \in \mathrm{Ext}(\mathbf{GL} + \Box^n\bot)$. According to Exercises 12.6–12.8, there are finitely many pretabular logics in $\mathrm{Ext}(\mathbf{GL} + \Box^n\bot)$ and all of them are decidable. So we may use the scheme above to check whether L is tabular. Thus, our problem reduces to the problem of verifying in an effective way whether $\Box^n\bot \in L$ for some $n < \omega$.

In Exercise 13.13 we described the Post complete extensions of **GL**. Let us denote them here by L_i: for $i < \omega$, L_i is the logic of i-point irreflexive chain with the distinguished root and $L_\omega = \mathbf{GL.3} + \boldsymbol{re}$ is the logic of the matrix of finite and cofinite sets in the frame $\langle \omega, > \rangle$ whose ultrafilter of distinguished elements consists of cofinite sets. It is not hard to see that $\Box^n\bot \in \mathbf{GL} + \varphi$ for some n iff $\varphi \notin L_\omega$. Indeed, if $\varphi \in L_\omega$ then $L_\omega \in \mathrm{Ext}L$, and so $\Box^n\bot \notin L$ for any n, because otherwise $\Box^n\bot \in L_\omega$, which is impossible. And if $\varphi \notin L_\omega$ then either L is inconsistent or it is consistent and its Post complete extensions are of the form L_i for $i < \omega$. Now observe that if L had infinitely many Post complete extensions then, by Theorem 13.22, L_ω would also be an extension of L. So L

has only finitely many extensions of the form L_i, say, $L_{i_1}, \dots, L_{i_m}$. But then $\Box^n\bot \in L$, where $n > \max\{i_1, \dots, i_m\}$, for otherwise $L + \Diamond^n\top$ is consistent, contrary to $\Diamond^n\top \notin L_{i_1}, \dots, \Diamond^n\top \notin L_{i_m}$. To complete the proof, it remains to recall that, by Theorem 11.38, L_ω is decidable. ❑

The part of the proof above concerning the problem whether $\Box^n\bot \in \mathbf{GL} + \varphi$ for some $n < \omega$ may be treated as determining whether $\mathbf{GL} + \varphi$ is locally tabular. The local tabularity of $\mathbf{GL} + \varphi$ is also equivalent to the existence of $n < \omega$ such that $\Box^n\bot \in \mathbf{GL} \oplus \varphi$, which in turn is equivalent to $\varphi \notin \mathbf{GL.3}$ (see Section 12.4). And the latter can be checked effectively because $\mathbf{GL.3}$ is decidable. In the same way one can recognize the local tabularity of calculi in NExt**S4** and Ext**S4**. Thus we obtain

Theorem 17.7 *Local tabularity is decidable in the classes* NExt**GL**, Ext**GL**, NExt**S4**, Ext**S4**.

The final decidability result in this section is left to the reader as an exercise:

Theorem 17.8 *The interpolation property is decidable in the classes* Ext**Int** *and* NExt**S4**.

Proof Use results of Section 14.4. ❑

17.3 Undecidable properties of modal calculi

In fact, in Section 16.4 we already met with undecidable properties. Such was the property of coincidence with the undecidable calculus or with the (finitely approximable and so decidable) calculus axiomatizable by the undecidable formula ρ. And the most important undecidable property found there was the consistency in Ext**K4**. The latter result can be extended to

Theorem 17.9 *Let L be a tabular extension of* **K4**. *Then the problem of coincidence with L is undecidable in* Ext**K4**.

Proof Consider the logic

$$L' = \mathbf{K4} + AxP + (\neg\nu \wedge \Diamond\epsilon(s, \alpha^1_m, \alpha^2_n) \to \Diamond\epsilon(t, \alpha^1_k, \alpha^2_l)) \to \nu,$$

where ν axiomatizes L over **K4** (by Theorem 12.4, L is finitely axiomatizable) and contains no occurrences of p_1, p_2, and the remaining formulas result from those in Section 16.4 by replacing every occurrence of ρ with $\neg\nu$.

Lemma 17.10 (i) $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ *implies* $L' = L$.

(ii) *If* $\boldsymbol{P} : \langle s, m, n\rangle \not\to \langle t, k, l\rangle$ *then L' is not tabular and so* $L' \neq L$.

Proof (i) In the same way as in the proof of Lemma 16.29 one can show that if $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ then

$$\neg\nu \wedge \Diamond\epsilon(s, \alpha^1_m, \alpha^2_n) \to \Diamond\epsilon(t, \alpha^1_k, \alpha^2_l) \in L'.$$

It follows by MP that $\mathbf{K4} + \nu \subseteq L'$. The converse inclusion is clear because all additional axioms of L' are either of the form $\neg\nu \wedge \varphi \to \psi$ or of the form $\varphi \to \nu$.

(ii) It suffices to observe that if $P : \langle s, m, n\rangle \not\to \langle t, k, l\rangle$ then all L's axioms are valid at the point r in the frame shown in Fig. 16.3, where all the formulas β_n, defined in Section 12.1, are refutable, and so, by Theorem 12.1, L' is not tabular. ❑

Theorem 17.9 follows immediately. ❑

Another consequence of Lemma 17.10 is

Theorem 17.11 *The property of tabularity is undecidable in* Ext**K4**.

Now we consider the tabularity problem in NExt**K**. Let $\mathfrak{F}$ be the nontransitive frame which is obtained from the (transitive) frame in Fig. 16.3 by making r reflexive and putting $b_1 R r$.

Theorem 17.12 *For every formula ν refutable at any point in $\mathfrak{F}$ save b_0 and such that $\Diamond\top \notin \mathbf{K} \oplus \nu$, the problem of coincidence with $\mathbf{K} \oplus \nu$ is undecidable in* NExt**K**.

Proof Without loss of generality we may assume that p_1 and p_2 do not occur in ν. In all formulas from Section 16.4 we replace every occurrence of β_0 with $\beta_0 \wedge \Diamond^2\Box\bot$, ρ with $\neg\nu$ and every formula of the form $\Diamond\epsilon(t,\pi,\tau)$ with $\Diamond^3\epsilon(t,\pi,\tau)$. For instance, the axiom $Ax(t \to \langle t', 1, 0\rangle)$ will look now like

$$\neg\nu \wedge \Diamond^3\epsilon(t,\pi_1,\tau_1) \to \Diamond^3\epsilon(t',\pi_2,\tau_1).$$

Using the former notations for the new formulas, we put

$$L = \mathbf{K} \oplus AxP \oplus (\neg\nu \wedge \Diamond^3\epsilon(s,\alpha^1_m,\alpha^2_n) \to \Diamond^3\epsilon(t,\alpha^1_k,\alpha^2_l)) \to \nu.$$

Lemma 17.13 (i) *$P : \langle s, m, n\rangle \to \langle t, k, l\rangle$ implies $L = \mathbf{K} \oplus \nu$.*

(ii) *If $P : \langle s, m, n\rangle \not\to \langle t, k, l\rangle$ then $\mathfrak{F} \models L$. In particular, $\nu \notin L$ and so $L \neq \mathbf{K} \oplus \nu$.*

Proof (i) is proved analogously to (i) in Lemma 17.10. To prove (ii) one can observe that after all the changes Lemma 16.30 still holds (here we use the fact that the set of points in $\mathfrak{F}$, where ν is refutable, coincides with the set of those points that see "in three steps" (via r) every point of the form $e(t', k', l')$). Details are left to the reader as an exercise. ❑

Theorem 17.12 is a direct consequence of Lemma 17.13. ❑

This rather general theorem has a number of interesting consequences.

Corollary 17.14 *Let L' be a tabular normal modal logic such that $\Diamond\top \notin L'$. Then the problem of coincidence with L' is undecidable in* NExt**K**.

Proof Observe that from every point in $\mathfrak{F}$ save b_0 arbitrary long chains are accessible. Therefore, by Theorem 12.1, the axiom of L', call it ν, satisfies the condition of Theorem 17.12. ❑

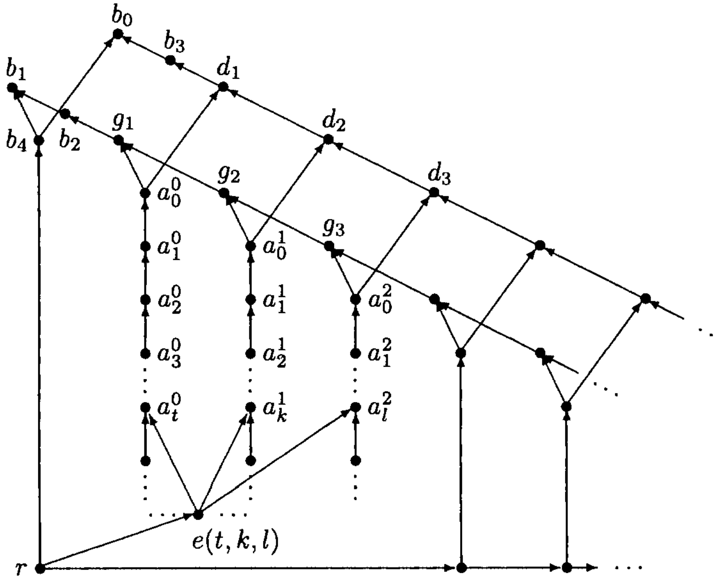


FIG. 17.1.

Corollary 17.15 *The tabularity problem is undecidable in* NExt**K**.

Proof It suffices to take ν as in the previous proof, L as in the proof of Theorem 17.12 and use Lemma 17.13. ❑

Corollary 17.16 *Let L be a finitely axiomatizable consistent normal extension of* **GL** *(e.g.,* **GL** *itself,* **GL.3**, Log•*). Then the problem of coincidence with L is undecidable in* NExt**K**.

Proof Exercise. ❑

Let us consider now other standard properties of modal logics. In the rest of this section we will be dealing only with the lattice NExt**GL**. From now on the notations will have a different meaning; a resemblance with the previous ones merely emphasizes an analogy.

To understand the axioms of the logic to be introduced below it is useful to bear in mind the (transitive) frame $\mathfrak{F} = \langle W, R, P\rangle$ whose underlying Kripke frame is shown in Fig. 17.1 and P is the family of finite and cofinite subsets of W. As before, $\mathfrak{F}$ contains only those points $e(t,k,l)$ for which $\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$. $\boldsymbol{P}$ and $\langle s,m,n\rangle$ are chosen in accordance with Theorem 16.3, so that the second configuration problem is undecidable for them.

It is easily checked that $\mathfrak{F} \models \mathbf{GL}$. Notice also that r is the only point in $\mathfrak{F}$ where the formula

$$\nu = \Box(\Box^2\bot \to \Box p \vee \Box\neg p)$$

is refuted: this happens iff $b_0 \models p,\ b_1 \not\models p$ or $b_0 \not\models p,\ b_1 \models p$.

Suppose now that ν is really refuted in $\mathfrak{F}$ under some valuation. Since b_0 and b_1 are symmetrical in $\mathfrak{F}$, without loss of generality we may assume that the valuation satisfies the former of the alternatives above. It is easy to see then that the points a_j^i in $\mathfrak{F}$ are characterized by the formulas:

$$\begin{aligned}\alpha_0^i = {} & \Diamond(\Diamond^{i+2}(\Box\bot \wedge p) \wedge \neg\Diamond^{i+3}(\Box\bot \wedge p) \wedge \neg\Diamond(\Box\bot \wedge \neg p)) \wedge \\ & \Diamond(\Diamond^{i+2}(\Box\bot \wedge \neg p) \wedge \neg\Diamond^{i+3}(\Box\bot \wedge p) \wedge \neg\Diamond(\Box\bot \wedge p)) \wedge \\ & \neg\Diamond^2(\Diamond^{i+2}(\Box\bot \wedge p) \wedge \neg\Diamond^{i+3}(\Box\bot \wedge p) \wedge \neg\Diamond(\Box\bot \wedge \neg p)) \wedge \\ & \neg\Diamond^2(\Diamond^{i+2}(\Box\bot \wedge \neg p) \wedge \neg\Diamond^{i+3}(\Box\bot \wedge \neg p) \wedge \neg\Diamond(\Box\bot \wedge p)),\end{aligned}$$

$$\alpha_j^i = \Diamond^j\alpha_0^i \wedge \neg\Diamond^{j+1}\alpha_0^i \wedge \bigwedge_{i\neq k} \alpha_0^k \quad (i \in \{0,1,2\},\ j \geq 1).$$

Using them, in exactly the same way as in Section 16.4 we define $\epsilon(t, \alpha_k^1, \alpha_l^2)$, π_1, π_2, τ_1, τ_2, $\epsilon(t, \pi_i, \tau_j)$, $\epsilon(t, \pi_1, \alpha_0^2)$, $\epsilon(t, \alpha_0^1, \tau_1)$ and then AxP with $\neg\nu$ instead of ρ and prove the literal analogues of Lemmas 16.26–16.31.

We require also the formulas

$$\nu' = \Box(\Box^3\bot \to \Box(\Box^2\bot \wedge \Diamond\top \to q) \vee \Box(\Box^2\bot \wedge \Diamond\top \to \neg q)),$$

$$\lambda_1 = q_1 \wedge \Box\neg q_1 \wedge \Diamond^5(\Box\bot \wedge p) \wedge \neg\Diamond(\Box\bot \wedge \neg p),$$

$$\mu_1 = q_2 \wedge \Box\neg q_2 \wedge \Diamond^5(\Box\bot \wedge \neg p) \wedge \neg\Diamond(\Box\bot \wedge p),$$

$$\lambda_2 = \lambda_1\{\Diamond q_1/q_1\},\ \mu_2 = \mu_1\{\Diamond q_2/q_2\},$$

$$\kappa_i = \Diamond\lambda_i \wedge \Diamond\mu_i \wedge \neg\Diamond^2\lambda_i \wedge \neg\Diamond^2\mu_i \ (i \in \{1,2\}).$$

Now, given a configuration $\langle t, k, l\rangle$, we define a logic L by taking

$$\begin{aligned}L = {} & \mathbf{GL} \oplus AxP \oplus (\neg\nu \wedge \Diamond\epsilon(s, \alpha_m^1, \alpha_n^2) \to \Diamond\epsilon(t, \alpha_k^1, \alpha_l^2)) \to \nu \oplus \\ & \nu \vee \nu' \oplus \nu \vee (\Diamond\kappa_1 \to \Diamond(\Diamond\kappa_2 \wedge \neg\Diamond^+\kappa_1)).\end{aligned}$$

Lemma 17.17 *Suppose* $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$. *Then:*
(i) $L = \mathbf{GL} \oplus \nu$;
(ii) *L is axiomatizable in* NExt**GL** *by a* **GL**-*conservative formula;*
(iii) *L is finitely approximable;*
(iv) *L is decidable;*
(v) *L is Kripke complete;*
(vi) *L has the interpolation property;*
(vii) *L has the disjunction property.*

Proof (i) proved in the same way as (i) in Lemma 17.10, (ii) follows from (i) by the argument in the proof of Lemma 14.28. By Exercise 11.4, (iii) is a consequence of (ii); (iv) and (v) follow from (iii) and (vi) from (ii) by Theorems 14.5 and 14.25. Finally, one can easily obtain (vii) by describing the class of finite frames for L and using (iii); we leave this to the reader as an exercise. ❑

Lemma 17.18 *Suppose* $\boldsymbol{P} : \langle s, m, n\rangle \not\to \langle t, k, l\rangle$. *Then:*
(i) $L \subset \mathbf{GL} \oplus \nu$;
(ii) *L is not axiomatizable in* NExt**GL** *by a* **GL**-*conservative formula;*
(iii) *L is not finitely approximable;*
(iv) *L is undecidable;*
(v) *L is Kripke incomplete;*
(vi) *L does not have the interpolation property;*
(vii) *L does not have the disjunction property.*

Proof It is sufficient to establish only (iv), (v), (vi) and (vii). To prove (iv), observe that $\mathfrak{F} \models L$, which together with the analog of Lemma 16.29 yields the following analog of Lemma 16.31: for every configuration $\langle t', k', l'\rangle$,

$$\boldsymbol{P} : \langle s, m, n\rangle \to \langle t', k', l'\rangle \text{ iff } \neg\nu \wedge \Diamond\epsilon(s, \alpha^1_m, \alpha^2_n) \to \Diamond\epsilon(t', \alpha^1_{k'}, \alpha^2_{l'}) \in L.$$

It remains to recall that the second configuration problem is undecidable for $\boldsymbol{P}$ and $\langle s, m, n\rangle$.

To justify (v) we use the last axiom of L, i.e., $\nu \vee (\Diamond\kappa_1 \to \Diamond(\Diamond\kappa_2 \wedge \neg\Diamond^+\kappa_1))$. If we try to refute the formula $\nu \vee \Diamond\kappa_1$ in a Kripke frame for L, then this axiom will require an infinite ascending chain of distinct points (as in Section 6.3), which will contradict $\boldsymbol{la} \in L$. On the other hand, with the help of $\mathfrak{F}$ one can show that $\nu \vee \Diamond\kappa_1 \notin L$.

(vi) is proved analogously to Theorem 14.27 by considering the axiom $\nu \vee \nu'$ (which is equivalent to $\neg\nu \to \nu'$). In the same manner one proves that $\nu \notin L$ (using $\mathfrak{F}$) and $\nu' \notin L$ (using the frame in Fig. 14.9), which in view of $\nu \vee \nu' \in L$ gives (vii). ❑

As an immediate consequence of Lemmas 17.17 and 17.18 we obtain

Theorem 17.19 *The following properties are undecidable in* NExt**GL**: *decidability, finite approximability, Kripke completeness, the interpolation property, the disjunction property, the axiomatizability by* **GL**-*conservative formulas, the property of coincidence with* $\mathbf{GL} \oplus \Box(\Box^2\bot \to \Box p \vee \Box\neg p)$.

Incidentally we have also got

Theorem 17.20 *The formula* $\Box(\Box^2\bot \to \Box p \vee \Box\neg p)$ *is undecidable in* NExt**GL**.

17.4 Undecidable properties of si-calculi

Here we confine ourselves only to demonstrating that the same scheme of proving the undecidability of calculi's properties is applicable to superintuitionistic logics as well. All the notations used below correspond to those introduced in Section 16.5.

Theorem 17.21 *The following properties are undecidable in* Ext**Int**: *decidability, finite approximability, axiomatizability by disjunction free formulas.*

Proof Consider the logic

$$L = \mathbf{Int} + AxP + (\epsilon(t, \alpha_k^1, \alpha_l^2) \to \epsilon(s, \alpha_m^1, \alpha_n^2) \vee \rho) \to \rho$$

introduced at the end of Section 16.5. If $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ then, as in Lemma 17.10 (i), we have $L = \mathbf{Int} + \rho$, where ρ contains only positive occurrences of $\vee$. Therefore, by Exercise 4.11, L is axiomatizable by a disjunction free formula and so, by McKay's theorem, it is finitely approximable and decidable. And if $\boldsymbol{P} : \langle s, m, n\rangle \not\to \langle t, k, l\rangle$ then using the frame in Fig. 16.4 one can show that L is undecidable (see the analogous proof of (iv) in Lemma 17.18) and so it is not finitely approximable and not axiomatizable by disjunction free formulas. ❑

Corollary 17.22 *The properties of decidability and finite approximability are undecidable in* NExt**Grz**.

Proof Follows from Theorem 17.21 and the preservation theorem. ❑

17.5 Exercises and open problems

Exercise 17.1 Extend the proof of Theorem 17.1 to the following classes of logics: (a) consistent si-logics; (b) consistent (normal) extensions of **S4**; (c) consistent normal extensions of **GL**.

Exercise 17.2 Show that, for every modal formula φ,

$$\varphi \in L_\omega \text{ iff } \bigwedge_{\Box\psi \in \mathbf{Sub}\varphi} (\Box\psi \to \psi) \to \varphi \in \mathbf{GL.3}.$$

Exercise 17.3 Prove that (a) every non-trivial property of recursively axiomatizable logics in the family $\{\mathrm{Log}\circ, \mathrm{Log}\bullet\}$ is decidable; (b) every non-trivial property of recursively axiomatizable logics in $\{\mathrm{Log}\circ, \mathrm{Log}\bullet, \mathrm{Log}(\circ + \bullet)\}$ is undecidable.

Exercise 17.4 Prove that the following properties are decidable:

(i) Halldén completeness in NExt**GL**.

(ii) The "weak disjunction property", i.e., if $\varphi \vee \psi \in L$ then $\neg\neg\varphi \in L$ or $\neg\neg\psi \in L$, in Ext**Int**.

(iii) The property "to be a modal companion of **Int**" in the classes NExt**S4** and Ext**S4**.

(iv) The axiomatizability by **Int**-conservative formulas in Ext**Int** and by **S4**-conservative formulas in NExt**S4**. (Hint: see Theorem 17.8.)

(v) The pretabularity in Ext**Int**, NExt**S4**, NExt**GL**, Ext**GL**.

(vi) The antitabularity in Ext**GL**.

Exercise 17.5 Prove that Halldén completeness is undecidable in the classes Ext**Int**, NExt**S4**, Ext**S4**, Ext**GL**.

Exercise 17.6 Prove that the property "to be a modal companion of **Int**" is undecidable in NExt**K4**.

Exercise 17.7 Prove the decidability of coincidence with **D** in NExt**K**. Generalize this result to all finite union-splittings of NExt**K**.

Exercise 17.8 Prove the decidability of the following formulas:
(i) the formulas in one variable in **ExtInt**;
(ii) the variable free formulas in **NExtGL**.

Exercise 17.9 Prove that the property "to be a modal companion of $\mathbf{Int} + \varphi$" is decidable in **NExtS4** iff $\mathbf{Int} + \varphi$ is decidable and φ is decidable in **ExtInt**.

Exercise 17.10 Prove that the property of axiomatizability by variable free formulas is undecidable in **NExtK4**.

Exercise 17.11 Prove that in **ExtS** the interpolation property and Halldén completeness are undecidable.

Exercise 17.12 Prove that the problem of first order definability of modal formulas is undecidable. (Hint: use for instance the formula

$$AxP \oplus ((\rho \wedge \Diamond\epsilon(s, \alpha_m^1, \alpha_n^2) \to \Diamond\epsilon(t, \alpha_k^1, \alpha_l^2)) \to \neg\rho) \wedge (\neg\rho \vee (\Box(\Box p \to p) \to \Box p))$$

where the formulas AxP, ρ and ϵ are taken from Section 16.3; in the case when $\boldsymbol{P} : \langle s, m, n \rangle \to \langle t, k, l \rangle$ this formula is equivalent to the variable free $\neg\rho$, which is certainly first order definable; and if $\boldsymbol{P} : \langle s, m, n \rangle \not\to \langle t, k, l \rangle$ then this formula is valid in the frame shown in Fig. 16.2, from the root of an ultrapower of which an infinite ascending chain is accessible and so $\rho \vee (\Box(\Box p \to p) \to \Box p)$ is refutable in it.)

Exercise 17.13 Prove that Kripke completeness and the axiomatizability by purely implicative formulas are undecidable in **ExtInt**.

Exercise 17.14 (i) Prove that the set of inconsistent calculi in **ExtK4** is not recursively enumerable.

(ii) Prove that the set of non-tabular calculi in **NExtK** is not recursively enumerable.

Problem 17.1 *Does Theorem 17.1 hold for the classes of consistent logics in* **ExtK**, **ExtK4**, **ExtGL**?

Problem 17.2 *Is there an algorithm which, given an effective procedure enumerating the complement of a logic L in* **ExtInt** *(***NExtK**, **ExtK**, *etc.), decides whether or not L satisfies a given non-trivial property?*

Problem 17.3 *Is the property "to be a (un)decidable formula" decidable in* **ExtInt**, **NExtK**, *etc.?*

Problem 17.4 *Is local tabularity decidable in* **ExtInt**?

Problem 17.5 *Is structural completeness decidable in* **ExtInt** *and* **NExtK4**?

Problem 17.6 *Are the sets of modal (in various standard classes) and si-calculi with (or without) the properties like Kripke completeness, finite approximability, etc. recursively enumerable?*

17.6 Notes

The jump from considering individual modal and si-logics to big classes of them gave rise to new settings of problems, in particular, algorithmic ones. When dealing with a separate logic, we are searching for answers to some standard list of questions: whether the logic is decidable, Kripke complete, finitely approximable, tabular, etc. For classes of logics these questions become mass algorithmic problems.

The pioneer paper in which such kinds of algorithmic problems were brought in sight and solved "negatively" was Linial and Post (1949), where it was shown in essence that the property "to be an axiomatization of **Cl**" is undecidable. Kuznetsov (1963) significantly extended this result: for every si-calculus C, the problem of recognizing, given an arbitrary list of formulas, whether it axiomatizes C with the rules MP and Subst (and without axioms of **Int**) is algorithmically undecidable. By a proper choice of C we can get various undecidable properties of propositional calculi. In particular, undecidable are the properties of consistency, completeness with respect to the classical truth-table (i.e., the property "to be an axiomatization of **Cl**") and some other properties which can be formulated in the form of the deductive equivalence to a certain fixed si-calculus. Note that Kuznetsov's theorem of Section 17.1 was not published by the author. We are grateful to L. Maksimova for informing us about it.

Until the late 1970s the efforts in the algorithmic direction of studies in modal logic were oriented mainly to obtaining positive results. To prove that a property is decidable one has, as a rule, to investigate deeply enough the property itself. One such property was tabularity. Kuznetsov's idea to use pretabular logics (first they were called "quasi-tabular") helped Maksimova (1972) to demonstrate that the property of tabularity is decidable in the class of si-logics, which became an impetus to consider the tabularity problem in other classes of logics. Another remarkable (with respect to its algorithmic behavior) property—the interpolation in Ext**Int** and NExt**S4**—was examined by Maksimova (1977, 1979, 1980). Note that for NExt**S4** it was shown only that a decision algorithm exists; its concrete form depends on the set of logics with the interpolation property in NExt**S4**, which is not completely characterized yet.

Only a few properties, besides those mentioned in Section 17.2 and exercises, are known to be decidable. One of them is "to have the same negation free fragment as **Int**": according to Jankov (1968a), it is equivalent to the property "to be included in **KC**", which is decidable in view of the decidability of **KC**. We do not know, however, whether the property "to have the same implicative fragment as **Int**" is decidable.

It is worth noting that there is an interesting correlation between the decidability of a formula φ, say in Ext**Int**, and the decidability of the calculus axiomatizable by φ: the property of coincidence with $L = \mathbf{Int} + \varphi$ is decidable iff both L and φ are decidable.

The first paper specially devoted to obtaining results on the undecidability of properties of calculi in NExt**K** was Thomason (1982), which, besides establishing that Kripke completeness is undecidable, shows in fact that the finite approximability in NExt**K** and the consistency of normal bimodal calculi are undecidable as well. The next step was made by Chagrova (1989d), who used the results of Chagrova (1989a, 1989c) to prove some theorems on the undecidability of properties of intuitionistic formulas related to their first order definability. She showed, for instance, the undecidability of the first order definability on countable frames and of the property "to be first order definable on countable frames but not on all frames"; see also Chagrov and Chagrova (1995). The main result asserting that the first order definability of intuitionistic formulas (or si-calculi) is undecidable was proved in a somewhat different way in Chagrova (1991).

Further progress in this direction was connected with the discovery of a general scheme for proving undecidability results of that sort. First it was applied in Chagrov (1990b, 1990c) and explicitly formulated in Chagrov and Zakharyaschev (1993). We followed this scheme in Sections 17.3 and 17.4. The results concerning the undecidability of tabularity are taken from Chagrov (1996), where it is proved, in particular, that the problem of coincidence with a fixed consistent tabular logic in NExt**K** is undecidable. Moreover, by combining the technique of Chagrov (1996) and the proof of Blok's theorem one can show that if L is a finitely axiomatizable consistent normal modal logic different from a union-splitting then the problem "$\mathbf{K} \oplus \varphi = L$?" is undecidable. For more applications of that scheme in various situations see Chagrov (1994c).

Another approach was taken by Kracht and Wolter (1997). Here the undecidability of properties is first shown for bimodal logics using word problems. Then, drawing on the simulation technique developed by Thomason (1974b, 1975c), the results are transferred to logics in NExt**K** similar to Thomason (1982).

Problem 17.4 was posed by Maksimova. As for Problem 17.5, Tsytkin (1987) and Rybakov (1995) proved that the property of hereditary structural completeness is decidable in Ext**Int** and NExt**K4**. Cresswell (1985) showed that no recursively enumerable family of algorithms consists only of decision algorithms for all decidable normal modal logics.

18

COMPLEXITY PROBLEMS

Suppose we have managed to construct a decision algorithm for a given modal or si-logic. Then we are facing the following questions. What are the complexity parameters (say, the required time and memory) of this algorithm? Does there exist a simpler algorithm? Since it is *a priori* impossible to estimate the efficiency of the decision procedure provided by Harrop's theorem, and the use of Büchi's and Rabin's theorems reduces the decision problem for a propositional logic to that for a second order theory (which is known to be very complicated), we consider here only those decision algorithms that are based on estimating the size of minimal frames separating formulas from logics.

18.1 Complexity function. Kuznetsov's construction

In Section 4.3 we introduced the notions of exponential, polynomial and linear approximability. More generally, for a finitely approximable logic L we consider the function

$$f_L(n) = \max_{\substack{l(\varphi)\leq n \\ \varphi\notin L}} \min_{\substack{\mathfrak{F}\models L \\ \mathfrak{F}\not\models\varphi}} |\mathfrak{F}|,$$

where $l(\varphi)$ is the length of φ, i.e., the number of subformulas in φ. $f_L(n)$ is called the *complexity function* of L. The exponential, polynomial and linear approximability of L mean then that there are positive constants c_1, c_2, c_3, respectively, such that the following conditions hold:

$$f_L(n) \leq 2^{c_1\cdot n}, \ f_L(n) \leq n^{c_2}, \ f_L(n) \leq c_3 \cdot n.$$

A necessary condition for an algorithm to be regarded as "acceptable" or "sufficiently efficient" is, as is well known, its polynomial parameters. Without going into details (consult Garey and Johnson, 1979) we accept this claim as a working thesis.

The simplest consistent modal and si-logics are the tabular ones, and among them **Cl**, **Triv** and **Verum** as the logics characterized by single-point frames. Although nobody knows if there exists a polynomial time decision algorithm for at least one of them, other tabular logics seem to be even more complex. To make our discussion more concrete, we introduce a notion of polynomial reducibility of one logic to another.

Say that a logic L_1 is *polynomially reducible* to a logic L_2 if there is a function g from the language of L_1 into that of L_2 which is computable by a polynomial time algorithm (of the length of the input formula) and such that

$$\varphi \in L_1 \text{ iff } g(\varphi) \in L_2.$$

Notice that **Cl** is polynomially reducible to any consistent modal or si-logic. In the former case one can use the identity function $g(\varphi) = \varphi$ and in the latter, by Glivenko's theorem, $g(\varphi) = \neg\neg\varphi$.

We say also that L_1 and L_2 are *polynomially equivalent* (with respect to their decision algorithms) if they are polynomially reducible to each other. It is easy to see that every two tabular logics are polynomially equivalent (consult Exercises 18.1 and 18.2). Polynomially equivalent logics may be regarded as similar as far as the complexity of their decision algorithms is concerned.

Example 18.1 As we showed in Section 4.3, **LC** is linearly approximable. This provides us with a decision algorithm for **LC** that works exponential time of the length of the input formula.

Thus, both **Cl** and **LC** are decidable by algorithms requiring exponential time. But are they polynomially equivalent? In Section 18.5 we shall show how questions of that sort can be answered in an easy, though indirect way. Here we give a direct construction for obtaining such results.

Let us recall that **Cl**, in spite of all the criticism against it, works perfectly well as far as finite objects are concerned. For instance, if we need to check whether a formula is true in a finite model, we can base our arguments on the laws of **Cl**. Let us formalize this observation as was proposed by Kuznetsov (1979). As an example we will consider **LC** and indicate the points where specific properties of this logic are essential.

Let φ be a formula of length n. According to Example 4.15, $\varphi \notin$ **LC** means that $\mathfrak{F} \not\models \varphi$ for some linearly ordered frame $\mathfrak{F}$ containing at most $n+1$ points. We describe this by means of classical formulas, understanding their variables in the following way. Suppose we have a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and x, y, z, for $1 \le x, y, z \le n+1$, are names (numbers) of points in $\mathfrak{F}$. With every pair $\langle x, y \rangle$ of points in $\mathfrak{F}$ we associate a variable p_{xy} whose meaning is "x sees y". And with every subformula ψ of φ and every point x we associate a variable q_x^ψ which means "ψ is true at x". Denote by α the conjunction

$$q_1^\varphi \wedge q_2^\varphi \wedge \ldots \wedge q_{n+1}^\varphi.$$

It means that φ is true in $\mathfrak{M}$. And let β be the conjunction of the following formulas under all possible values of their subscripts:

$$p_{xx},$$
$$p_{xy} \wedge p_{yz} \to p_{xz},$$
$$p_{xy} \wedge q_x^\psi \to q_y^\psi,$$
$$q_x^\perp \leftrightarrow \perp,$$
$$q_x^{\psi \wedge \chi} \leftrightarrow q_x^\psi \wedge q_x^\chi,$$

$$q_x^{\psi\vee\chi} \leftrightarrow q_x^\psi \vee q_x^\chi,$$
$$q_x^{\psi\to\chi} \leftrightarrow \bigwedge_{y=1}^{n+1}(p_{xy} \wedge q_y^\psi \to q_y^\chi).$$

(The first two formulas say that R is reflexive and transitive and the rest simulate the truth-relation in $\mathfrak{M}$.) Finally, we define a formula saying that our frame is linear:

$$\gamma = \bigwedge_{x\neq y} (p_{xy} \vee p_{yx}).$$

The formula $f(\varphi) = \beta \wedge \gamma \to \alpha$ is of length $\leq 1997 \cdot l^3(\varphi)$ (perhaps the reader can reduce the constant) and can be clearly constructed by an algorithm working at most linear time of the length of φ. It is readily seen that the following lemma holds:

Lemma 18.2 $\varphi \in$ **LC** *iff* $f(\varphi) \in$ **Cl**.

As a consequence we obtain

Theorem 18.3 **LC** *and* **Cl** *are polynomially equivalent.*

Later on in the same way we shall establish more general results. But now let us have a closer look at the construction we used. Notice that the properties of **LC** were essential only at the following two points:

- we estimated the size of the frame refuting φ as not exceeding $l(\varphi) + 1$;
- in the conjunct γ.

So, if we want to apply such a construction, say to **Int** instead of **LC**, then we need a polynomial upper bound for the complexity function of **Int** and do not need the conjunct γ at all (for other logics we may need another formula γ). Thus we obtain the following conditional "theorem": *if* **Int** *is polynomially approximable then it is polynomially equivalent to* **Cl**.

18.2 Logics that are not polynomially approximable

In fact, Kuznetsov's construction was originally created for **Int**, but it turned out that just for **Int** it cannot be used. We are going to show now that this logic, as well as many others, is not polynomially approximable.

Consider the sequence of formulas

$$\beta_n = \bigwedge_{i=1}^{n-1} ((\neg p_{i+1} \to q_{i+1}) \vee (p_{i+1} \to q_{i+1}) \to q_i) \to (\neg p_1 \to q_1) \vee (p_1 \to q_1).$$

It should be clear that $l(\beta_n) = O(n)$[12]. We show that every refutation frame $\mathfrak{F} = \langle W, R\rangle$ for β_n contains at least 2^n points. Suppose that under some valuation in $\mathfrak{F}$ the formula β_n is refuted at a point x. Then we have, for $1 \leq i \leq n-1$,

[12] We write $f(n) = O(g(n))$ if there is a constant $c > 0$ such that $f(n) \leq c \cdot g(n)$, for all $n \geq 0$.

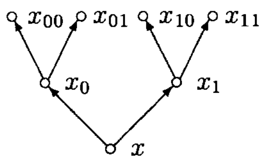


FIG. 18.1.

$$x \not\models (\neg p_1 \to q_1) \vee (p_1 \to q_1), \tag{18.1}$$

$$x \models (\neg p_{i+1} \to q_{i+1}) \vee (p_{i+1} \to q_{i+1}) \to q_i. \tag{18.2}$$

It follows from (18.1) that there are points x_0 and x_1 for which xRx_0, xRx_1, $x_0 \not\models q_1$, $x_1 \not\models q_1$, $x_0 \models \neg p_1$, and $x_1 \models p_1$. The later two conditions say that x_0 and x_1 do not have common successors in $\mathfrak{F}$. Thus (18.1) gives us the binary tree of depth 2; see Fig. 18.1.

Now let us use condition (18.2) for $i = 1$. Since $x_0 \not\models q_1$ and $x_1 \not\models q_1$, we have $x_0 \not\models (\neg p_2 \to q_2) \vee (p_2 \to q_2)$, $x_1 \not\models (\neg p_2 \to q_2) \vee (p_2 \to q_2)$ and so there are points x_{00}, x_{01}, x_{10}, x_{11} such that x_iRx_{ij}, $x_{ij} \not\models q_2$, $x_{i0} \models \neg p_2$, $x_{i1} \models p_2$ for all $i, j \in \{0, 1\}$. It follows in particular that no pair of these points has common successors in $\mathfrak{F}$. Thus, $\mathfrak{F}$ contains the full binary tree of depth 3 depicted in Fig. 18.1. Continuing in the same way, in n steps we shall extract from $\mathfrak{F}$ the full binary tree of depth $n + 1$ having 2^n final points.

It remains to observe that β_n is indeed refuted in such a tree, and so $\beta_n \notin$ **Int** (we leave this to the reader as an exercise). Thus, we obtain

Lemma 18.4 *There is a constant $n > 0$ such that $f_{\mathbf{Int}}(n) \geq 2^{c \cdot n}$.*

As is well known, an exponential function with base > 1 grows more rapidly than any polynomial. Therefore, **Int** cannot be polynomially approximable. Given arithmetic functions $f(n)$ and $g(n)$, we write $f(n) \asymp g(n)$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. In view of the exponential upper bound for $f_{\mathbf{Int}}(n)$ obtained in Theorem 2.32, we then have

Theorem 18.5 $\log_2 f_{\mathbf{Int}}(n) \asymp n$.

Notice that the formulas used in the proof of Lemma 18.4 contain only positive occurrences of $\vee$ which, by Exercise 4.11 and Corollary 15.12, do not belong to any consistent si-logic with the disjunction property. Thus we obtain a rather unexpected

Theorem 18.6 *No consistent si-logic with the disjunction property is polynomially approximable.*

In some cases we can obtain even stronger results. Notice that the proof of Lemma 18.4 establishes in fact that the minimal frame refuting β_n contains exponentially many *final* points. Let us have a look now at the interval [**KP**, **ML**]. As follows from Exercise 2.10, finite rooted frames for logics in it have the following property: for every partition of the set of final points in such a frame into two non-empty sets, there is a point in the frame which sees all points in one set and no point in the other. Together with the fact that β_n does not belong to **ML** this gives

Theorem 18.7 *There is a constant $c > 0$ such that for every logic L in the interval* [**KP**, **ML**],

$$f_L(n) \geq 2^{2^{c \cdot n}}.$$

Since one can extract from the proof of Theorem 5.44 a twice exponential upper bound for $f_{\mathbf{KP}}(n)$, we have

Corollary 18.8 $\log_2 \log_2 f_{\mathbf{KP}}(n) \asymp n$.

Now let us turn to modal logics. The translations we used to embed **Int** into **S4**, **GL**, **K4** transform β_n into modal formulas with similar semantic properties and the length $\leq c \cdot l(\beta_n)$, for some $c > 0$. Thus, we have the exponential lower bound for the complexity functions of all finitely approximable modal logics into which the si-logics considered above are embeddable. Taking into account the exponential upper bounds for these functions provided by the filtration method, we obtain, in particular, the following

Theorem 18.9 *If* $L \in \{\mathbf{S4}, \mathbf{Grz}, \mathbf{S4.1}, \mathbf{S4.2}, \mathbf{GL}, \mathbf{K4}\}$ *then* $\log_2 f_L(n) \asymp n$.

In this theorem we used the fact that frames for all logics under consideration are transitive. For **K** the same construction does not work. However, if in the modal translation of β_n we replace subformulas of the form $\Box\varphi$ by $\bigwedge_{i=0}^{2n} \Box^i\varphi$ then the resulting sequence of formulas will again possess the required property with the only exception: now the length of formulas in the sequence is not a linear but a square function of n. Using the fact the functions of the form $2^{\sqrt{c \cdot n}}$ still grow faster than polynomials, we obtain then

Theorem 18.10 **K** *is not polynomially approximable.*

18.3 Polynomially approximable logics

As follows from the results of the preceding section, a necessary condition for a logic to be polynomially approximable is that its frames do not contain arbitrarily big full finite binary trees. The simplest kinds of logics satisfying this condition are logics of finite depth and finite width. In this and the next sections we study the complexity of them. We begin with modal logics and then use the fact that their si-fragments cannot be more complex.

First we establish a fact the easy proof of which contains the main features of the other proofs in this section.

Lemma 18.11 *Suppose $\mathfrak{M}$ is a transitive model, x a point in $\mathfrak{M}$ and φ a formula. Then there is a submodel $\mathfrak{M}'$ of $\mathfrak{M}$ containing x and satisfying the following properties:*

- *the skeletons of $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic (more precisely, every cluster in $\mathfrak{M}$ has a representative in $\mathfrak{M}'$);*

- *each cluster in $\mathfrak{M}'$ contains at most $l(\varphi) + 1$ points;*
- *for every $\psi \in$ **Sub**φ and every y in $\mathfrak{M}'$, $(\mathfrak{M}, y) \models \psi$ iff $(\mathfrak{M}', y) \models \psi$.*

Proof Let $\Box\psi_1, \dots, \Box\psi_n$ be all "boxed" subformulas of φ (we remind the reader that $\Diamond$ was defined as an abbreviation). Clearly $n \leq l(\varphi)$. As the worlds of $\mathfrak{M}'$ we take x and, for every $i \in \{1, \dots, n\}$ and every cluster C in $\mathfrak{M}$, a point in C refuting ψ_i, if any. Besides if a cluster in $\mathfrak{M}$ contains no points of that sort, we put in $\mathfrak{M}'$ its arbitrary representative. The reader can readily verify by induction on the construction of φ that the resulting submodel satisfies the required properties. ❑

Thus, we can always assume that the size of clusters in the frames under consideration does not exceed the length of the refuting formula (+1, if necessary). It follows immediately that **S5** is linearly approximable. But in fact a more general result holds.

Theorem 18.12 (i) *Every logic $L \in \{\mathbf{K4BD}_n, \mathbf{S4} \oplus \boldsymbol{bd}_n : n < \omega\}$ is polynomially approximable, with the power of the corresponding polynomial $\leq n$.*

(ii) *Every logic $L \in \{\mathbf{Grz} \oplus \boldsymbol{bd}_n, \mathbf{GL} \oplus \boldsymbol{bd}_n, \mathbf{BD}_n : n < \omega\}$ is polynomially approximable, with the power of the corresponding polynomial $\leq n - 1$.*

Proof The proofs are similar for all the types of modal logics mentioned in the formulation of the theorem, and the result for $\mathbf{BD}_n$ follows from that for $\mathbf{Grz} \oplus \boldsymbol{bd}_n$. So we consider only the case $L = \mathbf{K4BD}_n$.

Suppose $\varphi \notin \mathbf{K4BD}_n$ and let $\Box\psi_i$, for $1 \leq i \leq m$, be all "boxed" subformulas of φ. Then there is a model $\mathfrak{M}$ of depth $\leq n$ refuting φ at its root x. Now, starting with $C(x)$, we mark by some labels some clusters in $\mathfrak{M}$. Namely, if C is a marked cluster then, for every $i \in \{1, \dots, m\}$, we mark exactly one cluster which is accessible from C and contains a point refuting ψ_i, if it exists. It should be clear that the total number of marked clusters does not exceed $1+m+m^2+\ldots+m^{n-1}$. Let $\mathfrak{M}'$ be the submodel formed by the marked clusters. One can readily prove by induction that $\mathfrak{M}'$ refutes φ at x and contains at most $n \cdot (l(\varphi) + 1)^n$ points. ❑

Let us consider now extensions of some standard logics with the formulas $\boldsymbol{bw}_n$ bounding width.

Theorem 18.13 *All logics $\mathbf{K4BW}_n$, $\mathbf{S4} \oplus \boldsymbol{bw}_n$, $\mathbf{Grz} \oplus \boldsymbol{bw}_n$, $\mathbf{GL} \oplus \boldsymbol{bw}_n$, $\mathbf{BW}_n$, for $n < \omega$, are linearly approximable.*

Proof Again we consider only the modal case. Let L be one of the modal logics mentioned in the formulation of our theorem and $\varphi \notin L$. Then φ is refuted at the root x of a model $\mathfrak{M}$ based upon a finite frame (for L) of width $\leq n$. Let $\Box\psi_i$, for $1 \leq i \leq m$, be all "boxed" subformulas of φ. For each i, we fix a maximal antichain of points in $\mathfrak{M}$ that refute ψ_i and do not see points from other clusters at which ψ_i is false. Now we form the submodel $\mathfrak{M}'$ of $\mathfrak{M}$ by putting into it x and all points from the selected antichains. It is not hard to check that the underlying frame of $\mathfrak{M}'$ validates L, $\mathfrak{M}'$ refutes φ and the number of points in $\mathfrak{M}'$ is not greater than $n \cdot l(\varphi) + 1$. ❑

A logic, all (normal) extensions of which are polynomially (exponentially, quadratically, linearly) approximable, is called *hereditarily polynomially* (respectively, *exponentially, quadratically, linearly*) *approximable*. For example, hereditarily exponentially approximable are **K4BD**$_3$ and **BD**$_3$, which follows from the description of the universal frames of finite rank for these logics given in Section 8.7. (In the next section we shall see that they are not hereditarily polynomially approximable.) Examples of hereditarily polynomially approximable logics are provided by the following:

Theorem 18.14 (i) **S4.3** *is hereditarily polynomially approximable.*
(ii) $\mathbf{S4} \oplus \boldsymbol{bd}_3 \oplus \mathsf{T}(\boldsymbol{wem})$ *is hereditarily quadratically approximable.*
(iii) $\mathbf{S4} \oplus \boldsymbol{bd}_2$ *is hereditarily quadratically approximable.*

Proof We prove only (i) leaving (ii) and (iii) to the reader. As was shown in Section 11.3, every extension of **S4.3** is characterized by a class of finite chains of (non-degenerate) clusters. Let $L \in \mathrm{Ext}\mathbf{S4.3}$ and $\varphi \notin L$. Then there is a model $\mathfrak{M}$ based upon a finite chain of clusters for L such that φ is false at its root. Using the same strategy as in the proof of Theorem 18.13 we select points in $\mathfrak{M}$ refuting "boxed" subformulas of φ and form the submodel based upon the set of selected points augmented by a point from the final cluster in $\mathfrak{M}$. The reader can readily check that the constructed submodel separates φ from L. ❑

18.4 Extremely complex logics of finite width and depth

As follows from the description of the universal frames for logics of finite depth (see Section 8.7), an upper bound for the complexity function of a logic of depth $k > 2$ is

$$2^{2^{\cdot^{\cdot^{2^{c \cdot n}}}}} \Big\} k-2$$

for some constant $c > 1$. Is it possible to reduce it? The answer is provided by the following theorem and its corollary:

Theorem 18.15 *For every $k > 2$, there is a si-logic L of depth k such that*

$$f_L(n) \geq 2^{2^{\cdot^{\cdot^{2^{n}}}}} \Big\} k-2 .$$

Proof For a set X, we denote by $\mathcal{P}_2X$ the collection of subsets of X containing at least two elements and by $\mathcal{P}X$ the standard power-set of X. Put $\mathcal{P}_2^m X = \mathcal{P}_2(\mathcal{P}_2^{m-1}X)$, with $\mathcal{P}_2^1 X = \mathcal{P}_2 X$; $\mathcal{P}^m X$ is defined analogously.

Let us consider the intuitionistic Kripke frames $\mathfrak{F}_n = \langle W_n, R_n\rangle$ in which, for $n > 2$,

$$W_n = \{a, a_1, \dots, a_n\} \cup \{b_x^s : x \in \mathcal{P}_2^s\{1, \dots, n\},\ 1 \leq s \leq k-2\},$$

and R_n is the reflexive and transitive closure of the relation R' defined by

$$cR'd \text{ iff } \exists x, y, i, s\ (c = a \vee (c = b_x^1 \wedge d = a_i \wedge i \in x) \vee$$

$$(c = b_y^{s+1} \wedge d = b_x^s \wedge y \in x \wedge 1 \leq s \leq k-3)).$$

The logic characterized by the class of these frames is denoted by L. To show that L is as required we shall use the formulas

$$\alpha(m) = \beta_1(m) \to \beta_2(m),$$

where

$$\beta_1(m) = \bigvee_{x \in \mathcal{P}^{k-2}\{1,\dots,m\}} \gamma_x^{k-2},$$

$$\begin{aligned}\gamma_x^1 = & \bigwedge_{i \notin x} \neg(p_i \wedge \neg p_{i-1} \wedge \dots \wedge \neg p_1 \wedge q_1) \wedge \\ & (p'_{k-2} \to p'_{k-1} \vee \neg p'_{k-1}) \wedge (q'_{k-2} \to q'_{k-1} \vee \neg q'_{k-1}) \to \\ & \bigvee_{i \in x} \neg(p_i \wedge \neg p_{i-1} \wedge \dots \wedge \neg p_1 \wedge q_1) \vee \neg p'_{k-1} \vee \neg q'_{k-1},\end{aligned}$$

for $x \in \mathcal{P}\{1,\dots,m\}$, and for other subscripts,

$$\begin{aligned}\gamma_x^{i+1} = & \bigwedge_{y \notin x} \gamma_y^i \wedge (p'_{k-1-i} \to p'_{k-i} \vee (p'_{k-i} \to \dots \to p'_{k-1} \vee \neg p'_{k-1}) \dots) \wedge \\ & (q'_{k-1-i} \to q'_{k-i} \vee (q'_{k-i} \to \dots \to q'_{k-1} \vee \neg q'_{k-1}) \dots) \to \\ & \bigvee_{y \in x} \gamma_y^i \vee (p'_{k-i} \to p'_{k-i+1} \vee (p'_{k-i+1} \to \dots \to p'_{k-1} \vee \neg p'_{k-1}) \dots) \vee \\ & (q'_{k-i} \to q'_{k-i+1} \vee (q'_{k-i+1} \to \dots \to q'_{k-1} \vee \neg q'_{k-1}) \dots),\end{aligned}$$

$$\begin{aligned}\beta_2(m) = & q_1 \vee p'_1 \vee q'_1 \vee (q_1 \wedge \neg p'_1 \wedge \dots \wedge \neg p'_{k-1} \wedge \neg q'_1 \wedge \dots \wedge \neg q'_{k-1} \to \\ & q_2 \vee (q_2 \to \dots \to q_{k-2} \vee (q_{k-2} \wedge (p_1 \to p_2) \wedge \dots \wedge (p_{m-1} \to p_m) \to \\ & \neg p_1 \vee \neg(p_2 \wedge \neg p_1) \vee \dots \vee \neg(p_m \wedge \neg p_{m-1})) \dots) \vee \\ & (p'_1 \wedge \neg q_1 \wedge \neg q'_1 \wedge \dots \wedge \neg q'_{k-1} \to \\ & p'_2 \vee (p'_2 \to \dots \to p'_{k-1} \vee \neg p'_{k-1})) \dots) \vee \\ & (q'_1 \wedge \neg q_1 \wedge \neg p'_1 \wedge \dots \wedge \neg p'_{k-1} \to \\ & q'_2 \vee (q'_2 \to \dots \to q'_{k-1} \vee \neg q'_{k-1})) \dots).\end{aligned}$$

By a straightforward, though somewhat tedious inspection, one can prove

Lemma 18.16 *For every $m > 2$, $\alpha(m) \in L$.*

Besides, we have

Lemma 18.17 *For every $m > 2$, $\beta_2(m) \notin L$.*

Proof We must show that, for some n, there is a valuation in $\mathfrak{F}_n$ under which $\beta_2(m)$ is false at the root of $\mathfrak{F}_n$. Put $n = m + 6$ and let $c_1 = b_{x_1}^{k-2}$, $d_1 = b_{x_2}^{k-2}$, $e_1 = b_{x_2}^{k-2}$, where

$$x_1 \in \mathcal{P}_2^{k-2}\{1, \dots, m\},$$

$$x_2 \in \mathcal{P}_2^{k-2}\{m+1, m+2, m+3\},$$

$$x_3 \in \mathcal{P}_2^{k-2}\{m+4, m+5, m+6\}.$$

The point c_1 is chosen so that it could see the point $b^1_{\{1,\dots,m\}}$. Then there are $c_2, \dots, c_{k-2} = b^1_{\{1,\dots,m\}}$ such that $c_1 R_n c_2 R_n \dots R_n c_{k-2} R_n a_i$, for $1 \leq i \leq m$. Besides, there are $d_2, \dots, d_{k-1}$ and $e_2, \dots, e_{k-1}$, for which $d_1 R_n d_2 R_n \dots R_n d_{k-1}$, $e_1 R_n e_2 R_n \dots R_n e_{k-1}$ and all points mentioned above are pairwise distinct. Notice also that the choice of x_1, x_2, x_3 ensures that the sets of successors of c_1, d_1, e_1 are disjoint.

Define a valuation in $\mathfrak{F}_n$ in such a way that $c_1 \models q_1, \dots, c_{k-2} \models q_{k-2}, a_i \models p_j$, $d_1 \models p'_1, \dots, d_{k-1} \models p'_{k-1}$, $e_1 \models q'_1, \dots, e_{k-1} \models q'_{k-1}$, where $1 \leq i \leq j \leq m$. It is not hard to check that under this valuation $\beta_2(m)$ is refuted at the root of $\mathfrak{F}_n$. ❑

Lemma 18.18 *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame for L refuting $\beta_2(m)$. Then*

$$|W| \geq \left. 2^{2^{\cdot^{\cdot^{2^m}}}} \right\} k-2.$$

Proof By Lemma 18.16, $\mathfrak{F} \not\models \beta_1(m)$ and so there are points in $\mathfrak{F}$ at which the disjuncts of $\beta_1(m)$ are not true. It is easy to see that distinct disjuncts are refuted at distinct points. ❑

To complete the proof of Theorem 18.15, it remains to observe that the length of $\beta_2(m)$ is $O(m)$. ❑

Corollary 18.19 (i) *For every $k > 2$, there is a logic $L \in$ NExt**Grz** of depth k such that*

$$f_L(n) \geq \left. 2^{2^{\cdot^{\cdot^{2^n}}}} \right\} k-2.$$

(ii) *For every $k > 2$, there is a logic $L \in$ NExt**GL** of depth k such that*

$$f_L(n) \geq \left. 2^{2^{\cdot^{\cdot^{2^n}}}} \right\} k-2.$$

For finitely approximable logics of finite width we have no *a priory* upper bounds for their complexity functions. And this is no accident. We are going to show now that there are finitely approximable logics of finite width whose complexity functions grow more rapidly than an arbitrarily given increasing arithmetic function.

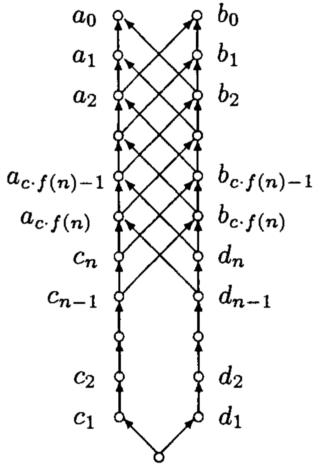


FIG. 18.2.

Theorem 18.20 *For every arithmetic function $f(n)$, there is a finitely approximable si-logic L of width 2 such that $f_L(n) \geq f(n)$.*

Proof Without loss of generality we may assume $f(n)$ to be a monotone non-decreasing function. Fix a sufficiently big constant c, say 1996, and define L to be the si-logic characterized by the class of finite frames $\mathfrak{F}_n$ shown in Fig. 18.2. Clearly, L is of width 2. Consider the formulas

$$\gamma(m) = \delta_1(m) \to \delta_2(m),$$

where

$$\delta_2(m) = \alpha_1 \vee \beta_1 \vee \delta_1'(m),$$

$$\delta_2'(m) = (p_1' \wedge \bigwedge_{i=1}^{n} q_i' \to p_2' \vee (p_2' \to \ldots \to p_n' \vee \neg p_n')\ldots) \vee$$
$$(q_1' \wedge \bigwedge_{i=1}^{n} p_i' \to q_2' \vee (q_2' \to \ldots \to q_n' \vee \neg q_n')\ldots),$$

$$\alpha_1 = r \to p \vee \neg p,\ \beta_1 = \neg r \to p \vee \neg p,\ \alpha_2 = \beta_1 \to \alpha_1 \vee \neg\neg p,$$

$$\beta_2 = \alpha_1 \to \beta_1 \vee \neg p,\ \alpha_{i+1} = \beta_i \to \alpha_i \vee \beta_{i-1},\ \beta_{i+1} = \alpha_i \to \beta_i \vee \alpha_{i-1},$$

$$\delta_1(m) = \bigvee_{i=1}^{c\cdot f(m)} (\alpha_i \vee \beta_i).$$

The following lemma is similar to Lemma 18.16, but its proof is not so cumbersome.

Lemma 18.21 *For every $m > 2$, $\gamma(m) \in L$.*

Proof Let us try to refute $\delta_1(m) \to \delta_2(m)$, for $m > 2$, in a frame $\mathfrak{F}_n$. If $\delta_2(m)$ is not true at x then x must see two incomparable chains of $\geq m$ points (they are required to refute $\delta_2'(m)$). By the definition of $\mathfrak{F}_n$, they are subchains of $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ (from which $m \leq n$). Besides, to refute $\alpha_1 \vee \beta_1$ we must have two two-point chains accessible from x and having no common successors; these can be only a_1, a_0 and b_1, b_0. Now we prove by induction that a_0, $a_1, \ldots$, $a_{c\cdot f(m)}$ and b_0, $b_1, \ldots$, $b_{c\cdot f(m)}$ are the final points refuting α_0, $\alpha_1, \ldots$, $\alpha_{c\cdot f(m)}$ and β_0, $\beta_1, \ldots$, $\beta_{c\cdot f(m)}$, respectively. Using the fact that $f(m)$ is monotone, we see that $\delta_1(m)$ is not true at x either. Thus $\mathfrak{F}_n \models \delta_1(m) \to \delta_2(m)$. ❑

Lemma 18.22 *Suppose $\mathfrak{F} = \langle W, R\rangle$ is a frame for L refuting $\beta_2(m)$. Then*

$$|W| \geq 2c \cdot f(n).$$

Proof As was shown in Lemma 18.21, $\delta_1(m)$ is refuted in $\mathfrak{F}$ and so $\mathfrak{F}$ contains at least $2c \cdot f(m)$ points. ❑

To complete the proof of our theorem, it remains to observe that the length of $\delta_2(m)$ is $O(m)$ and $\delta_2(m) \notin L$ for any m.. ❑

Corollary 18.23 (i) *For every arithmetic function $f(n)$, there is a finitely approximable logic $L \in$* NExt**Grz** *of width 2 such that $f_L(n) \geq f(n)$.*

(i) *For every arithmetic function $f(n)$, there is a finitely approximable logic $L \in$* NExt**GL** *of width 2 such that $f_L(n) \geq f(n)$.*

18.5 Algorithmic problems and complexity classes

Now let us turn to the relationship between the complexity of algorithmic problems for modal and si-logics and some standard complexity classes. First we consider the class $\boldsymbol{NP}$ of problems that can be solved by polynomial time algorithms on nondeterministic machines. Note that here we deal with only algorithmic problems of recognizing sets (or properties), i.e., those problems that can be formulated as the question "$x \in X$?", for some suitable set X. Such are, for instance, the problems "$\varphi \in L$?" and "$\varphi \notin L$?" for a fixed logic L.

We remind the reader that unlike *deterministic* machines (each next step of which is uniquely determined by the program and the current state of memory or configuration), a *nondeterministic machine* has in general an opportunity to choose its next step. For example, a nondeterministic Minsky machine may have two or more instructions with the same left part, and which of them will be executed is decided "by guess". It is easy to show that, given such a machine, one can construct an equivalent machine whose work consists of two stages: first the machine writes by guess some auxiliary word and then it deterministically calculates the required result using the word obtained at the first stage, i.e., it checks whether the guessed word is suitable for our purpose. The work of the machine is regarded as successful if *there is* a word having guessed which the machine produces then a positive (in some sense) result.

Let us consider for example the problem of finding a model for **Cl** satisfying a given formula, which is known as the *satisfiability problem* for propositional formulas. To solve it a standard deterministic algorithm (we do not distinguish here between algorithms and the machines realizing them) constructs (in one form or another) the truth-table for the formula and looks for T in the column under the main connective. Clearly, this is an exponential time algorithm (of the number of the formula's variables). Now we describe a nondeterministic algorithm solving the same problem. At the first stage it guesses a suitable valuation of the variables and at the second checks whether the formula is true under this valuation. Only quadratic time is required for this operation. Thus, the nondeterministic algorithm turns out to be much faster.

Although the concept of nondeterministic algorithm is just an abstraction, it is quite useful to understand how complex the problems requiring the exhaustive search are.

Denote by $\boldsymbol{P}$ the class of problems that can be solved by polynomial time deterministic algorithms. As we saw above, the satisfiability problem for Boolean (classical) formulas is in $\boldsymbol{NP}$. Does it belong to $\boldsymbol{P}$? This question is of great importance for complexity theory. As is shown by Cook's theorem below, it is equivalent to the question "$\boldsymbol{P} = \boldsymbol{NP}$?", known as the problem of eliminating the exhaustive search and regarded often as one of the main problems in this theory.

In spite of its boundlessness, the class $\boldsymbol{NP}$ contains problems that are in a sense most representative from the "polynomial point of view". Call a problem "$x \in X$?" $\boldsymbol{NP}$*-complete* if

- it belongs to $\boldsymbol{NP}$ and
- every problem "$y \in Y$?" in $\boldsymbol{NP}$ is polynomially reducible to the problem "$x \in X$?", i.e., there is a polynomial time function (algorithm) $f(y)$ such that $y \in Y$ iff $f(y) \in X$.

It is clear that if a problem "$x \in X$?" is in $\boldsymbol{NP}$ and some $\boldsymbol{NP}$-complete problem is polynomially reducible to it (in which case "$x \in X$?" is called $\boldsymbol{NP}$*-hard*) then "$x \in X$?" is also $\boldsymbol{NP}$-complete. Besides, to prove that a problem belongs to $\boldsymbol{NP}$, it is sufficient to reduce it polynomially to some problem in $\boldsymbol{NP}$.

For a detailed discussion of the given definitions the reader can consult the introduction to complexity theory (Garey and Johnson, 1979) where it is proved in particular that the satisfiability problem for Boolean formulas is $\boldsymbol{NP}$-complete. This result is known as Cook's theorem. It follows immediately that the *nonderivability problem* for **Cl**, i.e., "$\varphi \notin \mathbf{Cl}$?" is $\boldsymbol{NP}$-complete too.

$\boldsymbol{NP}$-completeness of the satisfiability problem partly justifies our changing from the original problem of searching for a polynomial time decision algorithm for a given logic to the problem of establishing its polynomial equivalence to **Cl**. Indeed, if $\boldsymbol{P} = \boldsymbol{NP}$ then **Cl** as well as any logic polynomially equivalent to **Cl** is decided by a polynomial time algorithm and vice versa. Thus, to justify the dubious equality $\boldsymbol{P} = \boldsymbol{NP}$ it would be sufficient to present a polynomial time decision algorithm for at least one logic that is polynomially equivalent to **Cl**.

However, in any case we know that **Cl** and all logics polynomially equivalent to it are decided by polynomial time nondeterministic algorithms.

Using the fact, observed in Section 18.1, we see that to establish $\boldsymbol{NP}$-completeness of the nonderivability problem for a consistent logic L it suffices to prove that the problem "$\varphi \notin L$?" belongs to $\boldsymbol{NP}$. Notice that Theorems 18.12 and 18.13 yield the following:

Lemma 18.24 *Suppose L is one of the logics* $\mathbf{K4BD}_n$, $\mathbf{S4} \oplus \boldsymbol{bd}_n$, $\mathbf{Grz} \oplus \boldsymbol{bd}_n$, $\mathbf{GL} \oplus \boldsymbol{bd}_n$, $\mathbf{BD}_n$, $\mathbf{K4BW}_n$, $\mathbf{S4} \oplus \boldsymbol{bw}_n$, $\mathbf{Grz} \oplus \boldsymbol{bw}_n$, $\mathbf{GL} \oplus \boldsymbol{bw}_n$, $\mathbf{BW}_n$. *Then the problem of nonderivability in L is in* $\boldsymbol{NP}$.

Proof Exercise. (Hint: use suitable modifications of Kuznetsov's construction.) ❑

As a consequence we obtain

Theorem 18.25 *Let L be one of the logics* $\mathbf{K4BD}_n$, $\mathbf{S4} \oplus \boldsymbol{bd}_n$, $\mathbf{Grz} \oplus \boldsymbol{bd}_n$, $\mathbf{GL} \oplus \boldsymbol{bd}_n$, $\mathbf{BD}_n$, $\mathbf{K4BW}_n$, $\mathbf{S4} \oplus \boldsymbol{bw}_n$, $\mathbf{Grz} \oplus \boldsymbol{bw}_n$, $\mathbf{GL} \oplus \boldsymbol{bw}_n$, $\mathbf{BW}_n$. *Then the problem of nonderivability in L is* $\boldsymbol{NP}$*-complete.*

Another complexity class we consider here is the class of problems that can be solved by polynomial space (whether deterministic or nondeterministic, see Garey and Johnson, 1979) algorithms. It is denoted by $\boldsymbol{PSPACE}$. Call a problem "$x \in X$?" $\boldsymbol{PSPACE}$*-complete* if

- it belongs to $\boldsymbol{PSPACE}$ and
- any problem "$y \in Y$?" in $\boldsymbol{PSPACE}$ is polynomially reducible to the problem "$x \in X$?", i.e., there is a polynomial time function $f(y)$ such that $y \in Y$ iff $f(y) \in X$.

It is known that $\boldsymbol{NP} \subseteq \boldsymbol{PSPACE}$. In particular, both problems "$\varphi \in$ **Cl**?" and "$\varphi \notin$ **Cl**?" are in $\boldsymbol{PSPACE}$ (check!). This is not so clear for the logics considered in Section 18.2. However, the (non)derivability problem for many of them not only belongs to $\boldsymbol{PSPACE}$ but is also $\boldsymbol{PSPACE}$-complete. The first step to show this is

Lemma 18.26 *Let* $L \in \{\mathbf{GL}, \mathbf{Grz}, \mathbf{Int}\}$. *Then the problem "$\varphi \in L$?" is in* $\boldsymbol{PSPACE}$.

Proof Suppose $L = \mathbf{GL}$. The proof of Theorem 14.25 provides us with a decision algorithm for **GL**. Indeed, $\varphi \in \mathbf{GL}$ iff the formula $\neg\varphi \to \bot$ has an interpolant in **GL**, and to check if this is the case, it is sufficient to construct a finite tree model according to the rules supplied by that proof. In general, this procedure requires exponential space because it constructs a tree of depth $O(l(\varphi))$ and branching $O(l(\varphi))$. However, we need not construct the whole tree at once: it is enough to demonstrate that each of its branches can be constructed. This can be done as follows.

Let us consider the transition from a tableau $(\Gamma, \llcorner\bot\lrcorner)$ to its immediate successors $(\Gamma_i, \llcorner\bot\lrcorner)$, for $1 \leq i \leq m$, where

$$\Gamma_i = \ulcorner\{\chi_i, \Box\chi_i, \Box(\neg\varphi)', \varphi : \ \Box\chi \in \Gamma\}\urcorner$$

and $\Diamond\chi_1, \ldots, \Diamond\chi_m$ are all formulas in Γ of the form $\Diamond\chi$. We need not accomplish this transition simultaneously to all these tableaux. First we can pass to $(\Gamma_1, \llcorner\bot\lrcorner)$ and try to realize it alone. Having succeeded, we then "clean" the memory and pass to $(\Gamma_2, \llcorner\bot\lrcorner)$, etc. Clearly, we again obtain a decision procedure for **GL** requiring $O(l^3(\varphi))$ memory: $O(l^2(\varphi))$ for writing and processing each tableau, the total number of which does not exceed $l(\varphi)$.

The logics **Grz** and **Int** may be treated in a similar way, or one can use the embeddings of them into **GL** defined in Section 3.9. ❑

Thus, to complete the proof that the derivability problem in the logics under consideration is ***PSPACE***-complete, it remains to show that some ***PSPACE***-complete problem is reducible to it, i.e., that it is ***PSPACE****-hard*. To this end we use the ***PSPACE***-complete *truth problem for QBF* (quantified Boolean formulas): given a Boolean formula $\varphi(p_1, \ldots, p_n)$ and a prefix $Q_1p_1 \ldots Q_np_n$, where each Q_i is either $\forall$ or $\exists$, to check whether the formula $Q_1p_1 \ldots Q_np_n\varphi(p_1, \ldots, p_n)$ is true. (The formulas $\forall p\psi(p)$ and $\exists p\psi(p)$ are regarded to be true iff $\psi(\mathrm{T}) \wedge \psi(\mathrm{F})$ and $\psi(\mathrm{T}) \vee \psi(\mathrm{F})$ are true, respectively.) It should be clear that it is sufficient to consider formulas that are in conjunctive normal form (note however that the transformation to this or other standard form can substantially change the length of the formula).

Let $\varphi = Q_1p_1 \ldots Q_np_n\psi(p_1, \ldots, p_n)$ be a Boolean formula with quantifiers and

$$\psi = \bigwedge_{t=1}^{m}(\bigvee_{s=1}^{i_t} p_{st} \vee \bigvee_{s=i_t+1}^{j_t} \neg p_{st})$$

(we assume that p_{st} is always in $\{p_1, \ldots, p_n\}$). Now we construct an implicative formula φ^*. To simulate quantifiers, we require the formulas

$$\mathbb{A}(q_1, q_2, q_3, q_4, q) = (q_1 \wedge q_2 \to q_3) \wedge (q_1 \wedge q_2 \to q_4) \wedge (q_1 \to q) \wedge (q_2 \to q) \to q,$$

$$\mathbb{E}(q_1, q_2, q_3, q_4, q) = (q_1 \wedge q_2 \to q_3) \wedge (q_1 \wedge q_2 \to q_4) \wedge (q_1 \wedge q_2 \to q) \to q.$$

The variables p_i will be simulated by the formulas of the form $\delta_i = q_i \to r_i$, $\overline{\delta}_i = r_i \to q_i$. Their intended meaning is as follows: if δ_i is refuted then $p_i = \mathrm{T}$ and if $\overline{\delta}_i$ is refuted then $p_i = \mathrm{F}$. To refute $\mathbb{A}(\delta_i, \overline{\delta}_i, \delta_j, \overline{\delta}_j, q)$, we need a point, say x, at which q is false. Then x sees two points, say x_1 *and* x_2, such that $x_1 \not\models \delta_i$ *and* $\alpha_2 \not\models \overline{\delta}_i$ (i.e., we check all the truth-values of p_i), which is ensured by the third and fourth conjuncts of the premise. The first and second conjuncts ensure that the sets of points above x refuting δ_i and $\overline{\delta}_i$ are upward closed. The case of refuting $\mathbb{E}(\delta_i, \overline{\delta}_i, \delta_j, \overline{\delta}_j, q)$ is described analogously but replacing *and* by *or* (i.e., only one truth-value of p_i is chosen).

With the prefix $Q_1p_1\dots Q_np_n$ we associate the formula $Q_n(\delta_1,\overline{\delta}_1,\dots,\delta_n,\overline{\delta}_n)$ in the following way:

if $Q_1 = \forall$ then $Q_1(\delta_1,\overline{\delta}_1) = \mathbb{A}(\delta_1,\overline{\delta}_1,\delta_2,\overline{\delta}_2,q)$;

if $Q_1 = \exists$ then $Q_1(\delta_1,\overline{\delta}_1) = \mathbb{E}(\delta_1,\overline{\delta}_1,\delta_2,\overline{\delta}_2,q)$;

if $Q_n = \forall$ then

$$\begin{aligned}Q_n(\delta_1,\overline{\delta}_1,\dots,\delta_n,\overline{\delta}_n) = {} & (\mathbb{A}(\delta_n,\overline{\delta}_n,\delta_{n+1},\overline{\delta}_{n+1},\delta_{n-1}) \to \delta_{n-1}) \wedge \\ & (\mathbb{A}(\delta_n,\overline{\delta}_n,\delta_{n+1},\overline{\delta}_{n+1},\overline{\delta}_{n-1}) \to \overline{\delta}_{n-1}) \to \\ & Q_{n-1}(\delta_1,\overline{\delta}_1,\dots,\delta_{n-1},\overline{\delta}_{n-1});\end{aligned}$$

if $Q_n = \exists$ then

$$\begin{aligned}Q_n(\delta_1,\overline{\delta}_1,\dots,\delta_n,\overline{\delta}_n) = {} & (\mathbb{E}(\delta_n,\overline{\delta}_n,\delta_{n+1},\overline{\delta}_{n+1},\delta_{n-1}) \to \delta_{n-1}) \wedge \\ & (\mathbb{E}(\delta_n,\overline{\delta}_n,\delta_{n+1},\overline{\delta}_{n+1},\overline{\delta}_{n-1}) \to \overline{\delta}_{n-1}) \to \\ & Q_{n-1}(\delta_1,\overline{\delta}_1,\dots,\delta_{n-1},\overline{\delta}_{n-1});\end{aligned}$$

Now consider $\psi(p_1,\dots,p_n)$ and construct a formula φ_k by induction on k:

$$\varphi_1 = (\bigwedge_{s=1}^{i_1} \delta_{s1} \wedge \bigwedge_{s=i_1+1}^{j_1} \overline{\delta}_{s1} \to \delta_n \wedge \overline{\delta}_n) \to Q_n(\delta_1,\overline{\delta}_1,\dots,\delta_n,\overline{\delta}_n),$$

$$\varphi_k = (\bigwedge_{s=1}^{i_k} \delta_{sk} \wedge \bigwedge_{s=i_k+1}^{j_k} \overline{\delta}_{sk} \to \delta_n \wedge \overline{\delta}_n) \to \varphi_{k-1}.$$

Finally, we put $\varphi^* = \varphi_m$. The properties of the constructed formulas we need are described by the following

Lemma 18.27 *$l(\varphi^*) = O(l(\varphi))$ and φ is true iff $\varphi^* \notin$ **Int**.*

Proof Exercise. ❑

Using this lemma we easily obtain

Lemma 18.28 *For every logic $L \in \{$**GL**, **Grz**, **Int**$\}$, the problem "$\varphi \in L$?" is **PSPACE**-hard.*

As a consequence of Lemmas 18.26 and 18.28 we finally have

Theorem 18.29 *For every logic $L \in \{$**GL**, **Grz**, **Int**$\}$, the problem "$\varphi \in L$?" is **PSPACE**-complete.*

Note that along with Lemma 18.28 we obtain the following analog of Theorem 18.6.

Theorem 18.30 *The problem of derivability in any consistent si-logic with the disjunction property is **PSPACE**-hard.*

18.6 Exercises and open problems

Exercise 18.1 Prove that the polynomial equivalence of logics is an equivalence relation, i.e., it satisfies the conditions of reflexivity, transitivity and symmetry.

Exercise 18.2 Prove that every consistent tabular logic is polynomially equivalent to **Cl**.

Exercise 18.3 Prove that $\log_2 f_{\mathbf{KC}}(n) \asymp n$ and that no logic in the interval $[\mathbf{Int}, \mathbf{KC}]$ is polynomially approximable.

Exercise 18.4 Show that one cannot reduce the power of the polynomials in Theorem 18.12. (Hint: consider the intuitionistic formulas

$$\varepsilon(m) = \bigwedge_{s=1}^{n-1} (\varepsilon_m^s \to \boldsymbol{bd}_s) \to \varepsilon_m^n,$$

where, for $1 \leq m \leq n$,

$$\varepsilon_m^s = \bigwedge_{i=1}^{m-1} (p_i^{s-1} \to p_{i+1}^{s-1}) \to (p_1^{s-1} \to \boldsymbol{bd}_{s-1}) \vee$$
$$(p_2^{s-1} \wedge \neg p_1^{s-1} \to \boldsymbol{bd}_{s-1}) \vee \ldots \vee (p_m^{s-1} \wedge \neg p_{m-1}^{s-1} \to \boldsymbol{bd}_{s-1}),$$

and prove that to refute $\varepsilon(m)$ a frame must contain the full m-ary tree of depth n. In the modal case add to the premise of the suitable translation of $\varepsilon(m)$ the conjunct

$$\Box(\bigwedge_{i=2}^{m} \Diamond(r_i \wedge \neg r_{i-1}) \wedge \Diamond r_1 \wedge \bigwedge_{i=2}^{m-1} \Box(r_{i-1} \to r_i)).)$$

Exercise 18.5 Let α_n be the formula introduced in Section 12.1. Show that the logic $\mathbf{K} \oplus \alpha_n$ is polynomially approximable and estimate the power of the corresponding polynomial.

Exercise 18.6 Show that all pretabular logics in Ext**Int** and NExt**K4** considered in Chapter 12 are linearly approximable.

Exercise 18.7 Show that all modal companions of tabular and pretabular si-logics are polynomially approximable and estimate the power of the corresponding polynomials.

Exercise 18.8 Prove that every normal finitely approximable extension of the logic $\mathbf{K4.3} \oplus \Box\Diamond\top$ is linearly approximable.

Exercise 18.9 Prove that for every arithmetic function $f(n)$ there is a finitely approximable normal extension L of **K4.3** such that $f_L(n) \geq f(n)$. (Hint: use the logic of the frames shown in Fig. 18.3.)

Exercise 18.10 Show that there is a constant $c > 0$ such that, for every cofinal subframe logic L, $f_L(n) \leq 2^{c \cdot n}$.

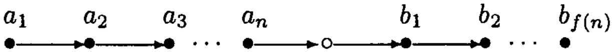


FIG. 18.3.

Exercise 18.11 (i) Prove that there is a logic $L \in$ NExt**S4** of depth $k \geq 2$ such that

$$f_L(n) \geq \left. 2^{2^{\cdot^{\cdot^{2^n}}}} \right\} k-2$$

and the si-fragment of L is polynomially approximable.

(ii) Prove that for every arithmetic function $f(n)$ there is a finitely approximable logic $L \in$ NExt**S4** of width 2 such that $f_L(n) \geq f(n)$ and ρL is linearly approximable.

Exercise 18.12 The *diameter* of a finite transitive frame $\mathfrak{F}$ is $\max\{n, m, k\}$, where n is the size of the maximal cluster in $\mathfrak{F}$, m the length of the longest chain of points from distinct clusters in $\mathfrak{F}$ and k the maximal number of immediate successors of points in $\mathfrak{F}$. Prove that for every logic L in the list **K4**, **S4**, **Grz**, **S4.1**, **S4.2**, **Int**, **KC**, if $\varphi \notin L$ then φ is separated from L by a frame whose diameter does not exceed $l(\varphi)$.

Exercise 18.13 Prove that the derivability problem for the logics **K**, **K4**, **S4** and **KC** is $\boldsymbol{PSPACE}$-complete.

Problem 18.1 *Prove or disprove the "preservation theorem": for every si-logic L, L is polynomially approximable iff τL is polynomially approximable iff σL is polynomially approximable.*

Problem 18.2 *Are* **Int** *and* **KP** *polynomially equivalent? Does* **KP** *belong to* $\boldsymbol{PSPACE}$*?*

Problem 18.3 *Is there a recursive upper bound for $f_{\mathbf{ML}}$? Is there a recursive upper bound for the size of the minimal refutation Medvedev frame?*

Problem 18.4 *One can easily show that* **Int** *in the language with one variable is linearly approximable. Is* **Int** *in the language with two variables linearly (or polynomially) approximable? Is this logic polynomially decidable? What about* **S4**, **Grz** *and other standard modal logics in the language with one variable?*

Problem 18.5 *Do there exist finitely axiomatizable and finitely approximable logics which are more complex than* **KP***?*

Problem 18.6 *Prove that if L is a consistent si-logic different from* **Cl** *and axiomatizable by formulas in one variable then $\log f_L(n) \asymp n$. What is the complexity of logics of the form* $\mathbf{S4} \oplus \varphi(p)$*?*

Problem 18.7 *How does the addition of an essentially negative axiom or $\Box\Diamond$-axiom to a logic affect its complexity function?*

Problem 18.8 *Do there exist logics (or calculi) with the C-complete (non) derivability problem, where C is an arbitrary member of the hierarchy of Meyer–Stockmeyer (for the definition consult Garey and Johnson, 1979)?*

Problem 18.9 *Is it true that every polynomially approximable calculus is polynomially equivalent to* **Cl**? *Or, which is equivalent, is it true that the nonderivability problem for such a calculus is NP-complete?*

18.7 Notes

The study of complexity problems is a relatively new direction in modal logic. Although upper bounds for the size of refutation algebras and frames were found for a number of standard logics, usually this was just an intermediate aim in the proofs of their decidability. Complexity problems for si-logics were first explicitly mentioned by Kuznetsov (1975). Approximately at the same time the studies of logical foundations of computer science stimulated some interest in complexity aspects of modal logics.

One of the questions raised by Kuznetsov (1975) was the problem of polynomial approximability of **Int** and its pretabular extensions (Kuznetsov himself observed that **LC** is linearly approximable). Kuznetsov (1979) showed that if this problem is solved positively for **Int** then **Int** and **Cl** are polynomially equivalent (see Section 18.1).

The result of Statman (1979), who proved that the derivability problem in **Int** is $PSPACE$-complete and so a positive solution to Kuznetsov's question would imply that $NP = PSPACE$, gave to the complexity direction in modal logic another impetus. Ladner (1977) showed that the derivability problem in the logics **K**, **T** and **S4** is $PSPACE$-complete. He proved also NP-completeness of the satisfiability problem in **S5** and that **S5** is linearly approximable.

It is to be noted that Statman (1979) and Ladner (1977), defining the length of formulas, took into account not only the number of propositional variables and connectives in them (i.e., the number of subformulas) but also the length of indices: compare for instance the formulas $p \to q \vee r$ and $p_{1997} \to p_{19971997} \vee p_{199719971997}$. Sometimes calculating the length of indices is redundant. We mean logics formulated in languages with finitely many variables. Unfortunately, very little is known about the complexity of such logics. We will mention here only one question. The Rieger–Nishimura lattice provides us with a linear time decision algorithm for **Int** in the language with one variable. However, nothing is known about **Int** in the languages with two or more variables. On the one hand, to prove the lower bound for $f_{\mathbf{Int}}(n)$ or that the derivability problem in **Int** is $PSPACE$-hard we used formulas involving infinitely many variables, which suggests that the fragments of **Int** with finitely many variables are possibly much simpler. On the other hand, even **Int** in two variables is rather rich: every negation free formula nonderivable in **Int** (observe that proving the lower bounds we could use only negation free formulas; we did not use $\neg$ to construct various "negative" examples either) has a substitution instance in two variables that is not in **Int**. And four variables are enough to construct an undecidable si-calculus.

So we think it would be of interest to study the two-variable fragment of **Int** with respect to both its complexity function and its relation to the standard complexity classes $\boldsymbol{P}$, $\boldsymbol{NP}$, $\boldsymbol{PSPACE}$, etc. The same concerns one-variable fragments of modal logics, in particular, **S4**, **Grz**. When constructing "very complex" logics we were forced to use infinitely many variables. For logics of finite depth that was stipulated by their local tabularity. But we do not have this restriction in the case of finite width logics and so one can conjecture that these logics in finite languages are polynomially (linearly) approximable. It would be of interest also to estimate the complexity of **KP** and **ML** with finitely many variables.

Note also that instead of polynomial reducibility Statman (1979) and Ladner (1977) used the (stronger) *log-space reducibility*. Whether these two types of reducibility are different in this context is an open problem. In any case, we do not know any examples of $\boldsymbol{PSPACE}$-complete problems with respect to polynomial but not log-space reducibility.

Kuznetsov (1975) claimed that if a si-calculus is polynomially approximable then it is polynomially equivalent to **Cl**. In February 1984 Kuznetsov (he died few months later) confessed to one of the authors that he could not reconstruct the proof and had doubts whether his original proof in 1974 was correct. That is why we formulate this claim as an open problem. For all polynomially approximable calculi known to us (including tense logics of Ono and Nakamura, 1980) Kuznetsov's claim holds. Namely, one can modify in a suitable way the construction in Section 18.1 by replacing in it the question about validity with that of satisfiability. This incidentally suggests that if $\boldsymbol{NP} = \boldsymbol{coNP}$ then Kuznetsov's problem is solved positively. Is the converse true, i.e., is this problem as hopeless as "$\boldsymbol{NP} = \boldsymbol{coNP}$?"?

Let us return to the questions raised by Kuznetsov (1975). That **Int** is not polynomially approximable was proved in Zakharyaschev and Popov (1979). Chagrov (1983) somewhat strengthened this result and proved also that minimal logics of finite width and depth as well as all modal companions of tabular and pretabular si-logics are polynomially approximable. The material of Sections 18.4 and 18.5 was taken from Chagrov (1985a). Note that the extremely complex logics constructed in Section 18.4 are not finitely axiomatizable. Moreover, **KP** is the most complex calculus we know.

A discussion of complexity problems in modal logics used in artificial intelligence can be found in Halpern and Moses (1992). Exercise 18.12) is due to Darjania (1979). Complexity aspects of polymodal logics are considered by Spaan (1993).

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