CHAPTER XLI

THE NEWER 'MATTER'

> The twofold nature of light as a light-wave and as a light-quantum is thus extended to electrons and, further, to atoms: their wave-nature is asserting itself more and more, theoretically and experimentally, as concurrent with their corpuscular nature. (481)
>
> A. SOMMERFELD

> The concepts of wave amplitude, electric and magnetic field strengths, energy density, etc., were originally derived from primitive experiences of daily life, such as the observation of water waves or the vibrations of elastic bodies. (215)
>
> W. HEISENBERG

> The problem of quantum theory centers on the fact that the particle picture and the wave picture are merely two different aspects of one and the same physical reality. (215)
>
> W. HEISENBERG

> To me it seems extraordinarily difficult to tackle problems of the above kind, as long as we feel obliged on epistemological grounds to repress intuition in atomic dynamics, and to operate only with such abstract ideas as transition probabilities, energy levels, etc. (466)
>
> E. SCHRODINGER

> . . . for visualization, however, we must content ourselves with two incomplete analogies—the wave picture and the corpuscular picture. (215)
>
> W. HEISENBERG

> Not every physicist is an epistemologist, and not everyone must or can be one. Special investigation claims a whole man, so does the theory of knowledge. (326)
>
> E. MACH

The following chapter was written in 1928 and since then the newer quantum mechanics has been developed much further, proved enormously fruitful, and has been repeatedly supported by experiments. The literature on this subject is steadily accumulating, the most important classical memoirs by the originators of this new scientific trend have been collected into book form and are now easily accessible. There is also a large number of excellent technical, as well as non-technical presentations. On reading in December 1932 what I had written in 1928, I find that although from some aspects the presentation may be considered unsatisfactory and antiquated, yet the epistemological side of the older presentation remains valid. So it seems advisable to retain this chapter and add only a few further $\overline{A}$ suggestions.

It is known that practically all creative and constructive physicists, who have produced revolutionary and lasting works, were interested in epistemology. There are many physicists who know as much physics as an Einstein, for instance, yet Einstein remains quite unique and his work is to a large extent responsible for the present revolutionary developments of physics. The reason is simple. Einstein has corrected a long established epistemological fallacy, which can be expressed in my language as the rejection of the structural fallacy of elementalism in a limited yet very important field of physics. He also established and applied new fundamental epistemological principles, which is another way of saying that he established new standards of evaluation in physics, as for instance, that we should never postulate entities which cannot possibly be observed, that the 'laws of nature' should be formulated in terms of generally invariant relations expressed in tensor equations , .

The weakness of the system of Einstein, resulting in many futile criticisms, lies in the fact that he eliminated elementalism in one vital region of physics, but he did not formulate the *general epistemological principle* of *non-elementalism,* which should be applied everywhere, daily life included. This he could not have accomplished without a still deeper enquiry into the mechanism of time-binding, which produces all science, and which leads to the discovery of the fundamental fallacy in the use of the 'is' of identity. Only after the elimination of this remainder in us of the primitive man, does *structure* become the only possible link between the objective and verbal worlds, and becomes also the only possible content of 'knowledge'. 'Similarity of structure' then demands the complete and general elimination from science and life of any elementalism.

The strength of the newer quantum mechanics lies in the fact that the younger physicists have accepted the new epoch-making einsteinian standards of evaluation or epistemological principles; the weakness lies in the fact that the scientists do not realize that the fallacy of elementalism is entirely general and vitiates *all* scientific outlooks. No one can produce satisfactory theories, nor evaluate, nor interpret them properly as long as he continues to use the few-valued and *elementalistic* 'logics' and 'psychologies', which are at present always found at the bottom of any 'evaluation' or 'interpretation'.

The latest work of Dirac goes very far in the direction of building $\bar{A}$ physics by establishing his language of transformations, states, observables. , ascribing *structure* to protons, magnetic poles. , but even Dirac does not seem to realize the general fundamental $\bar{A}$ issues involved. Dirac says: 'The description which quantum mechanics allows us to give is *merely* a manner of speaking which is of value in helping us to deduce and to remember the results of experiments and which never leads to wrong conclusions. *One should not try to give too much meaning to it.*'[1] (Italics are mine.) The italicized words show that even Dirac does not realize fully the mechanism of identification, as otherwise he would not have used these words in this form. If we entirely abandon identification, then a theory or a book, being verbal, represents nothing else but special language; there is no 'merely' about that either, structure being the only possible link between the non-verbal and verbal worlds. Instead of warning the reader 'that one should not try to give too much meaning to it', we must simply *insist* that the *only* 'meaning' should be looked for in structure. , the 'too much meaning' always indicating inappropriate evaluation and ultimately semantic disturbances.

Current physical literature shows that the main problems of 'interpretation' depend on the *solution* of the *m.o* problems of 'observation', 'reality', 'fact'. , and border on the *scientific* solution of the problems of pathological 'delusions', 'illusions', and 'hallucinations', all of which involve the fundamental issues of the elimination of *identification.* But once the $\bar{A}$ issues are structurally formulated and applied in practice, they result in a $\bar{A}$, *non-el,* ∞-valued orientation which involves the recognition of the *multiordinality* of terms. , which also solves the problems of quantum 'interpretations', the details of which I cannot enter into here.

Originally the quantum writers had an inclination to ascribe 'physical meaning' to waves. The present tendency of specialists is to regard the waves as 'purely symbolic', forgetting that experimentally something else besides the symbols 'bends around the corners'. From a $\bar{A}$ point of view, when the problems of the multiordinality of such terms as 'observation', 'fact', 'reality'. , are understood we will have to ascribe 'physical reality' to the waves, ascribe finer *structure* to the 'electron', . We would also have to abandon the *A* 'particle'-orientation and treat the 'electron', 'proton'. , in a $\bar{A}$, ∞-valued way as minute *fields,* which under the present experimental conditions behave as 'particles'. This $\bar{A}$ field-orientation suggests a great many possible interpretations, impossible in the *A* 'particle'-orientation.

Mathematically, the geometry of 'space filling' curves would have to be elaborated further so that we would better understand the *structure of plenums* and this knowledge should be applied to physics.

We should also perform a direct series of experiments with a more elaborate Faraday box. A small wooden laboratory should be isolated from the rest of the world by every available *energetic screen* and physical experiments repeated under such new dynamic conditions. Technically the winding of, say, a foot thickness of insulated wires for different currents would not present any difficulties except for the door which should be also in the circuit. The eventual probable results of experiments in such a laboratory under different conditions could be calculated in advance, and it may be fairly well anticipated that at least significant discrepancies between the calculations and actual experiments would appear, throwing new light on the structure of the space-time plenum, the eventual connection between gravitation and electromagnetism , . Arguments alone will not help in this field and only experiments will point the pathway.

Because of the extensive literature dealing with the new quantum mechanics it seems unnecessary to dwell upon it any further, except by expressing the hope that some mathematicians and physicists will master the $\bar{A}$ ∞-valued orientation and will revise the theories now existing.

Section A. Introductory.

The new researches in the structure of the materials of the universe proceeded under unique conditions. On the one hand, since Planck in 1900 originated the quantum theory, which earlier elaboration we now call the classical quantum theory, the amount of experimental facts pointing in the direction of *some* quantum theory had become very convincing, yet, on the other hand, the lack of a structurally satisfactory theory to co-ordinate these new experimental facts was becoming distressing.

It appeared as though there were either a lack of 'geniuses' able to produce the required new theories, or the geniuses existed, but were unable to function properly. They seemed to suffer from a semantic blockage due to some identification which successfully prevented broader and unhampered vision.

Presently the co-ordinating theories *were* produced. In the production of the new quantum mechanics we see at work the unconscious semantic liberating influence of the $\overline{E}$ and $\overline{N}$ systems which had been developed and which we have already analysed. The classical theories had come to a structural impasse, but with the advent of younger scientists, who were educated in the theoretical semantic freedom of these new systems, the semantic blockage due to ascribing 'objectivity' to 'matter', 'space', and 'time' was removed, creative forces were released, and these young scientists proceeded to construct the structural formulations needed. They are now rightly hailed as geniuses.

It is astonishing how these young post-einsteinian scientists of various lands and different inclinations produced, independently and practically simultaneously, various new quantum theories, using different methods and different mathematical, as well as national, languages. When these different theories were studied and compared they were found to amount practically to one theory, but expressed by different mathematical languages. Now the use of different mathematical languages to express one group of experimental facts brings an additional benefit in that it gives diversified verbal structural information about the problems at hand. Since these developments are very recent and progress has been extremely rapid it is hard to keep track of the status of the problem.

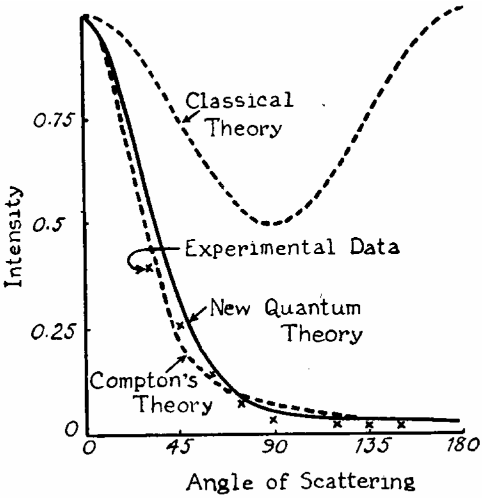
In this account of the newer quantum mechanics I will emphasize only the structural and semantic side, treating the different theories as behaviour of their respective authors, and as illustrating the issues above mentioned. From this point of view, we are not interested in discussing to what extent the given theories are 'true' or 'false', which means no more than similar or dissimilar to the world in structure. We are interested directly in those semantic aspects of human behaviour which have been neglected. When 'Smith' puts a black mark on white paper, it is to be called *human behaviour,* and behaviour *unique* for man. Our analysis does not involve the question of the validity of his doings. But since he did it, let us analyse his doings.

In speaking about theories as complex and technical as the newer quantum mechanics, it is practically impossible to give a satisfactory account in a non-technical way. Such theories have not yet been worked out thoroughly enough, many points are still not clear, and no proper evaluation is possible at present. These difficulties are really immaterial to our present purpose, because these theories are *empirical facts on record,* and they throw significant light on human behaviour. It is not here proposed to make reflections on the world around us, but to analyse a certain linguistic, structural form of human behaviour.

From the point of view of structure and *s.r,* the classical quantum mechanics would be quite enough for *sanity* and adjustment; it is enough to realize that the still older theories of 'matter', 'space', and 'time' are *el, structurally fallacious* and represent only primitive identifications.

The advent of such a crop of geniuses and of several theories expressed quite differently, yet nearly equivalent, is an event of deep human semantic significance. It helps to understand the working of the human nervous system, and is in accord with the present general theory.

From the point of view of the physicist, these new theories are a marked structural improvement over the classical theory, a fact which can best be illustrated by a diagram of a special case. In Fig. 1, the crosses indicate the experimental data, the curves indicate the results as predicted by the classical theory, by the Compton theory, and by the new quantum mechanics. It should be noticed that the new quantum theory appears much more in accordance with the experimental data than the older theories. This fact is of great structural and semantic importance to us as well as to the physicist.



(After *The New Quantum Mechanics* by G. Birtwistle, p. 192)

FIG. 1

Section B. The nature of the problem.

At this point we may explain briefly the nature of the problem that was demanding solution.

We have become familiar with the use of co-ordinates. This procedure has been generalized and has given rise to 'generalized co-ordinates'. These are defined as *arbitrary* variables which represent not merely lengths but may also represent angles, surfaces, volumes. , though they must be capable of representing the $3n$ orthogonal co-ordinates. If, in a special case, we make the number of generalized co-ordinates equal to the number of degrees of freedom which the system has, these s generalized co-ordinates can be regarded as independent of each other. If we denote by q_i the generalized co-ordinate, q_i (i = 1 to s), then the orthogonal co-ordinates of any of the n particles can be represented as definite functions of the generalized co-ordinates, so that

$$x_h = f_h\,(q_1, q_2, q_3, \ldots q_s)\,. \qquad (1)$$

We know that kinetic energy is represented by $mv^2/2$ where m represents the mass and v the velocity, or the 'time' derivative of the 'space' travelled.

If we want to find the value for the energy we must differentiate each of the $3n$ equations (1) with respect to 'time', which gives the components of the velocity, square them, multiply them by the corresponding masses and add them together to find the double value for the energy.

For simplicity we will denote the 'time' derivatives by the chosen letter, but with a dot over it (newtonian method). Thus,

$$x_h=\partial x_h/\partial q_1\, q\cdot_1+\partial x_h/\partial q_2\, q\cdot_2+\ldots+\partial x_h/\partial q_s\, q_s \qquad (2)$$

Squaring (2), we have

$$\begin{aligned} x_h^2=&(\partial x_h/\partial q_1)^2\, q_1^2+(\partial x_h/\partial q_2)^2\, q_2^2+\ldots+(\partial x_h/\partial q_s)^2\, q_s^2 \\ &+2(\partial x_h/\partial q_1)(\partial x_h/\partial q_2)q_1q_2+2(\partial x_h/\partial q_1)(\partial x_h/\partial q_3)q_1q_3+\ldots \qquad (3) \\ &+2(\partial x_h/\partial q_2)(\partial x_h/\partial q_3)q_2q_3+\ldots \end{aligned}$$

The last expression (3) can be simplified:

$$x_h^2=\Sigma_{i=1}^{i=s}\,\Sigma_{k=1}^{k=s}\,(\partial x_h/\partial q_i)(\partial x_h/\partial q_k)\, q_i\, q_k\,. \quad (4)$$

It is easy to see that if in (4) we put $k=i$ we will have square members, and when $i\neq k$, every term will occur twice and so the above abbreviation (4), covers the formula (3).

If we write similar expressions for all of the $3n$ orthogonal co-ordinates, multiply by the corresponding masses and then add them together we obtain twice the value for the kinetic energy $2L$.

$$2L=\Sigma_{i=1}^{i=s}\,\Sigma_{k=1}^{k=s}\, c_{ik}\, q\cdot_i\, q\cdot_k\,, \text{ where}$$

$$c_{ik}=\Sigma_{h=1}^{h=n}\,[(\partial x_h/\partial q_i)(\partial x_h/\partial q_k)+(\partial y_h/\partial q_i)(\partial y_h/\partial q_k)+(\partial z_h/\partial q_i)(\partial z_h/\partial q_k)]\,.$$

The coefficients in the expansion of c_{ik} depend only on the values of the generalized co-ordinates and are independent of the value of the time-derivatives. The time-derivatives can be properly called *generalized velocities*, and we may denote them by q_i.[2]

In establishing formulae for the quantum theory we want to be as general as possible and not restrict ourselves to vibrational energy only. But we want to take into consideration *any* arbitrary point-mass, independently of whether we assume this point to be charged or not.

We define the momentum or impulse as the product of the mass and the velocity, or, $p=mv$. If, instead of denoting our co-ordinates by x, y, and z, we use the generalized co-ordinates q_i, we would have for the magnitude and direction of the velocities the time-derivatives of the co-ordinates; namely, q_i, where $q_1=x\cdot=dx/dt$, $q_2=y\cdot=dy/dt$, .

If p_1, p_2, p_3, represent the corresponding components of the momentum or impulse, then we would have $p_i=mq_i$. (5)

We should notice that the dynamical triplet of impulse co-ordinates occurs conjointly with the geometrical triplet of the co-ordinates of position. The second law of motion tells us that 'the change in momentum is proportional to the impressed force and takes place in the direction in which that force acts'. If we assume that the force K is derivable from the potential energy E_{pot}, (a function of q_i), then we have $p_i=K_i=-\partial E_{pot}/\partial q_i$. (6)

The kinetic energy (E_{kin}) is represented by

$$E_{kin}=m/2\ (q_1^2+q_2^2+q_3^2)=(p_1^2+p_2^2+p_3^2)/2m$$

where by (5), $q_i^2=p_i^2/m^2$. We call the total energy, which is represented as the sum of the kinetic and the potential energy, as expressed in terms of the generalized co-ordinates and momenta, the hamiltonian function H. Then we have:

$$H(q, p)=E_{kin}+E_{pot}, \ \partial H/\partial q_i=\partial E_{pot}/\partial q_i, \ \partial H/\partial p_i=\partial E_{kin}/\partial p_i=p_i/m. \quad (7)$$

From (5), (6), and *(7)*, we get the fundamental equations of motion,

$$\partial q_i/\partial t=\partial H/\partial p_i, \text{ and } \partial p_i/\partial t=-\partial H/\partial q_i.$$

The above hamiltonian, or canonical, form of the equations is remarkable because it preserves its form if any arbitrary co-ordinates are introduced; it is invariant under the transformation of co-ordinates. The equations hold not only for an individual point-mass but also for any arbitrary mechanical system. For arbitrary co-ordinates and systems the momentum or impulse p is defined by

$$\partial E_{kin}/\partial q_i,$$

so that the kinetic energy is expressed as a function of the q_i's and their derivatives the q_i's.

To help visualization we can construct and consider the p and q as rectangular co-ordinates in two dimensions in the phase plane of our system. In this plane the sequence of those graph-points that correspond to the successive states of motion of the system represent the phase paths or phase-orbits. The characteristic structural feature of the quantum theory is that it selects a discrete family of phase-orbits from the infinity of possible orbits.

We next consider a point-mass m that is bound elastically to its position of rest, and which can move to either side of the central position only in the direction $x=q$, or its reverse, when experiencing a restoring force. We call such point-mass a linear oscillator. If we wish to emphasize that our oscillator is capable only of definite vibrations, on account of its elastic attachment, we call it a 'harmonic oscillator'. If the vibration number, or the frequency of the oscillator, which is represented by the number of its free vibrations per unit of 'time', is denoted by ν, then the vibration is represented by $x=q=a \sin 2\pi\nu t$.

The impulse becomes $p=mv=mq=2\pi\nu ma\ \cos 2\pi\nu t$. The phase-orbit is represented by an ellipse in the p-q-plane and is given by the equation $q^2/a^2+p^2/b^2=1$, where the minor axis $b = 2\pi\nu ma$.

In our family of orbits the phase area between two orbits is equal to the quantum of action h. Sommerfeld regards h as an elementary region or element of the phase area, and considers it as the definition of the Planck quantum of action h. If W_n represents the energy of the oscillator when it describes the n-th orbit, then $W_n=nh\nu$. In these orbits the energy appears as a whole multiple of the elementary quantum of energy; $\varepsilon=h\nu$, and $W_n=n\varepsilon$.

We call *stationary states* of the oscillator those states which the oscillator may pass through without cessation and without loss of energy, or, without radiation.

When an oscillator retains its stationary state, its energy is constant and its graph appears as an ellipse of the family in the phase plane. However, when the energy of the oscillator changes and jumps over to a smaller orbit, it emits energy. When it passes to a larger orbit it absorbs energy. The emission and absorption of energy occurs in multiples of the energy quantum, ε.

The graphs of the system in the phase plane are restricted to certain ‘quantised’ orbits. Between each orbit and its successor there is an elementary region, of area h. The n-th orbit, if closed, has as area nh. Or, expressed symbolically, $\int pdq=nh$. This integral is called the phase integral and is taken along the n-th orbit.

The quantum hypothesis can be structurally formulated so that the phase integral must be a whole multiple of the quantum of action h. This form of the classical quantum postulate is more general than the original formulation of Planck, although it includes the latter as a particular case.

In case of a rotating point-mass, a similar analysis gives us $E_{kin}=p\dot{q}/2$ and when $\nu=\dot{q}/2\pi$, $E_{kin}=nh/2\ \dot{q}/2\pi=nh\nu/2$ where ν represents the rotation frequency of the rotator, or the number of full revolutions per unit of ‘time’, and takes the place of the vibration number of the oscillator.

In the classical theory, the quantised states were distinguished from all other possibilities by the characteristic whole numbers, and so we had a network. In a quantum orbit the ‘electron’, if undisturbed, was supposed to move permanently without resistance and not to emit radiation. The phase-space, representing the manifold of the possible states, including non-stationary states, is crossed, mesh-like, by the graph curves of the stationary orbits. The size of the meshes is determined by Planck’s constant h.[3]

Section C. Matrices.

The older quantum mechanics forms an elaborate system, and we have a large accumulation of numerical data on record. Some of these data corroborated the older theories nicely, but some data were in contradiction to the classical theory. The problem was not to discard the numerical data, which, whatever they mean structurally, represent quite solidly established data, but to find new equations which would be satisfied by these facts. Now 'new equations' really mean *languages of new structure,* and therefore new formulations had to be discovered.

Dealing with tables which give special theoretical data, it was natural to start with a calculus which deals with such numerical special tables. Such a calculus had been developed long ago, and was called the matrix calculus. Later on, when matrices themselves were treated as complex quantities, and still later, as operators, we were enabled to pass to the more developed calculi which use ordinary differential equations. The new quantum theories give us a unique case, in which several mathematical methods have been used at once and of which the results are fairly in accord.

At this point it is advisable to give a few structural explanations of these mathematical notions, including the matrix calculus. If we have two equations of the first degree with two variables; namely, $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$, the solution of these equations takes the form: $x=(b_2c_1-b_1c_2)/(a_1b_2-a_2b_1)$; $y=(c_2a_1-c_1a_2)/(a_1b_2-a_2b_1)$
The common denominator of the two solutions can be written in a two-dimensional table

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \tag{1}$$

which is understood as the product of the upper left-hand number and the lower right-hand number, minus the product of the lower left-hand and upper right-hand numbers.

Similarly, the numerators of these solutions can also be represented in the form of two-dimensional tables; namely,

$$b_2c_1-b_1c_2 = \begin{vmatrix} c_1, & b_1 \\ c_2, & b_2 \end{vmatrix} \text{ (2) and } \qquad c_2a_1 - c_1a_2 = \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix} \tag{3}$$

to which the above-mentioned rule applies. Expressions like (1), (2), (3), are called determinants of the second order.

The numbers in the first, second. , horizontal lines are called the first, second. , *rows,* respectively; the vertical lines are called first, second. , *columns.*

The above definitions and method can be applied to any number of equations with an equal number of variables, and in each case our determinant would have n^2 numbers, n rows and n columns.

We may use another notation which employs one letter for the coefficients of our variables, with indexes or suffixes to indicate that their values are different. Let us consider n^2 elements in the table:

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & \ldots, & a_{1,n} \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & \ldots, & a_{2,n} \\ \ldots & & & & \\ a_{n,1}, & a_{n,2}, & a_{n,3}, & \ldots, & a_{n,n} \end{vmatrix} . \tag{4}$$

The expression (4) is called a determinant of the *n*-th order.

The notation by suffixes is very convenient and is very much used these days. The first suffix denotes the row, the second the column in which the element is situated. Usually the comma dividing the two numbers in the index is dropped and the coefficients are written simply a_{11}, instead of $a_{1,1}$. In general the element $a_{i,k}$, or a_{ik}, represents the element in row *i* and column *k*.

The elements Iying in the diagonal joining the upper left-hand to the lower right-hand number are called the principal diagonal. In our example we notice that the elements in the diagonal are such that $i = k$.

We have definite rules by which we can arrive at the solution of our equations, once the coefficients, which are the elements of the determinant, are given. In general, the determinants are treated as a functional form.

If *m* and *n* are positive integers, a manifold, or system of *mn* ordered quantities or elements arranged in *m* horizontal and *n* vertical rows, will be called a *rectangular matrix* and we may use the notation A = ($a(mn)$):

$$\begin{matrix} a(11) & a(12) & a(13) & \dots \\ a(31) & a(32) & a(33) & \dots \\ \dots & \dots & \dots & \dots \end{matrix} \text{, or } (a(nm)).$$

The numbers *m* and *n* are called the orders of the matrix. If $m=n$ the matrix is called a *square* matrix. Without loss of generality we can treat any rectangular matrix in which $m \neq n$ as a square matrix by supplementing the missing rows and columns with zeros.

A matrix of the type,

$$1 = \begin{matrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ .. & .. & .. & .. & \dots \end{matrix} \text{, or } (\delta(nm)),$$

where $\delta(nm) = 1$ for $n=m$, and $\delta(nm)=0$ for $n \neq m$, is called a unit matrix. The matrix

$$\begin{matrix} a(11) & 0 & 0 & 0 & \dots \\ 0 & a(22) & 0 & 0 & \dots \\ 0 & 0 & a(33) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix} \text{, or } (a(nm)\,\delta(nm)),$$

is called a diagonal matrix. In the new quantum mechanics a diagonal matrix is independent of *t* and represents a constant of the classical theory. The reverse is not necessarily true. The operation of differentiation can be expressed in terms of multiplication of matrices with the aid of the unit matrix.[4]

Equations in which matrices are equated are called matrix equations. If the equations involve only one unknown matrix, which does not occur more than once as a factor, such equations are called matrix equations of the first degree.

The m scalar equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = c_m$$

are together equivalent to one single matrix equation. There are several ways in which the notation can be simplified.

The difference between a determinant and a matrix is subtle, but important. By a determinant we understand, by definition, a certain homogeneous polynomial of the n-th degree, in the n^2 elements a_{ij}. Accordingly, a determinant gives a definite number when calculated.

But in many instances we are interested in the *table*, or the n^2 elements arranged in a certain order but *not* combined into a polynomial. Such an array, or table, is called a matrix. Thus, from this point of view, a matrix does not represent a definite quantity, but a system of quantities, and so a matrix is *not* a determinant.

We can illustrate this difference by an example. If we take a determinant of the second order

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$$

and change the rows into columns, or vice versa, thus:

$$\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}$$

the value of both determinants will be equal; namely,

$$a_1b_2 - a_2b_1,$$

by the definition rule already given; yet the *matrices* of the two determinants are *different.*

Although different, a determinant nevertheless defines a matrix, called the matrix of the determinant; conversely, a matrix defines a determinant, called the determinant of the matrix.

We have said that a matrix does not represent a quantity, while a determinant does. At this stage, and from this point of view, we may say so legitimately. However, we might eventually treat a matrix as a quantity also; but for this purpose we should have to enlarge the meaning of the term ‘quantity’.

In our present use of the term ‘quantity’ we mean the real and complex quantities of ordinary algebra.

It may be said that mathematicians have had a peculiar tendency, which has proven of great value in the development of mathematics, gradually to extend the meaning of terms in order to embrace new notions as they arise. For instance, we have enlarged the primitive meaning applied to positive integers to embrace negative numbers, which formerly would not have been considered as quantities. Similarly, if we here use the ordinary notion of algebraic quantity, then a matrix is not a quantity but a system of quantities. The problem is, how shall we enlarge this meaning to include the matrices ?

Mathematics recognizes that this generalization of *mathematical* notions is extremely useful and *legitimate*. This structural issue appears to be of very general application, as all of us exhibit a tendency towards it. It is a purely mathematical and useful tendency in mathematics, but it leads to disastrous results when applied to daily-life abstractions, as explained in Part VII. In this connection we should recall the difference between the mathematical contentless abstractions and the abstractions with physical content, with which we are generally concerned in science and life.

Let us now follow up the method by which a matrix can be considered as a quantity. If we have objects of two or more kinds which can be counted or measured, and if we consider an aggregate of such objects, say 5 horses, 3 cows and 2 sheep, we could denote such a complex quantity by the symbol (5,3,2). In this case, the first place in our symbol would be reserved for horses, the second for cows, and the third for sheep.

In mathematics, we do not specify horses, or cows, or sheep, but consider sets of quantities, and distinguish them by the position which they have in our symbolism. We may denote such a complex quantity by a single letter, $A=(a,b,c,)$. (For instance, we denote a fraction by a single letter, although a fraction is specified by *two* numbers.)

In such an instance, we should call a complex quantity equal to another when, and only when, the components are respectively equal. And a complex quantity is said to vanish only in case all the components vanish.

Ordinary mathematical operations can be applied to such complex quantities. For instance, we may define a sum or difference of two complex quantities

$$A'=(a_1,b_1,c_1) \text{ and } A''=(a_2,b_2,c_2) \text{ as}$$
$$A'+/-A''=(a_1+/-a_2,b_1+/-b_2,c_1+/-c_2)\ .\ ,$$

a definition which is entirely satisfactory theoretically, and also practically, as can be verified from our example.

From this point of view we may consider a matrix as a complex quantity with mn or m^2 components. A matrix would then represent a complex quantity, as a special case under the general method sketched above.

We could then define our further operations. A matrix would be said to be zero when all elements are equal to zero. Two matrices would be said to be equal when they have equal numbers of rows and columns and every element of one is equal to the corresponding element of the other.

By setting up some such rules we could develop a calculus of matrices, and matrices would be considered as complex numbers. In general, the algebraic rules would be found to be applicable to matrices, which would further justify us in treating matrices as complex numbers.

One of the notable exceptions in our operations would be found in the application of the classical operation of multiplication and its dependencies. In ordinary algebra and arithmetic, multiplication is what is called 'commutative' which means that $2 \times 3 = 3 \times 2 = 6$, or $a \times b = b \times a$.

In defining the multiplication of matrices we have no *a priori* grounds for determining why one definition or restriction should be preferable to another.

Only practice can show which definition is more workable or more fruitful in results. In the matrix calculus the definition of Cayley is generally accepted, as it has led to the most workable results. It was based on considerations of the composition of *linear* transformations.

The definition is approximately as follows: The product **ab** of two square matrices of the n-th order gives a square matrix of the n-th order in which the element which lies in the i-th row and j-th column is obtained by multiplying each element of the i-th row of **a** by the corresponding element of the j-th column of **b** and adding the results.

If we denote by a_{ij} and b_{ij} the elements in the i-th row and j-th column of **a** and **b** respectively, then by definition the element (i,j) of our product **ab** would give,

$$a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}, \qquad (5)$$

and the (i,j) element in the matrix **ba** would be

$$a_{1j}b_{i1}+a_{2j}b_{i2}+\ldots+a_{nj}b_{in}. \qquad (6)$$

In general, the quantities (5) and (6) are not equal and therefore we see that the multiplication of matrices is, in general, *not commutative.* The *order* in which we perform our multiplication is of importance and **ab** *is not* generally equal to **ba**, (**ab** ≠ **ba**).[5]

It should be noticed that the vector calculus has made us familiar with new operations which differ from arithmetical operations. For instance, the sum of two vectors differs in general from the arithmetical sum and is defined by the law of the parallelogram (see Chapter XXXIII). This definition is more general, and the arithmetical definition expresses only the particular case in which the vectors have one direction. Similarly, the non-commutative law of multiplication corresponds more closely to vector multiplication than to arithmetical multiplication.

We will not go further into the details of the matrix calculus, which is a well-developed mathematical discipline with a large literature, but will emphasize some methodological points of importance.

One of the main applications of the theory of matrices is found in the subject of *linear transformations.*

In mathematics, instead of using the given variables, we very often introduce new variables which are functions of the old. Such transformations, or change of variables, are particularly simple and important when the functions in question are homogeneous and *linear.*

If $x_1, x_2, \ldots, x_n$ represent the original variables, and $x'_1, \ldots, x'_n$ the new variables, we have, by definition, the formulae of transformations:

$$\begin{gathered} x'_1 = a_{11}x_1 + \cdots + a_{1n}x_n \\ \cdots\cdots\cdots \\ x'_n = a_{n1}x_1 + \ldots + a_{nn}x_n. \end{gathered}$$

The square matrix made up of the coefficients is called the matrix of the transformation and the determinant is called the determinant of the transformation and is completely determined by the matrix. We have already seen the importance of linear equations and linear transformations in physics and therefore in the investigation of the world around us. The theory of matrices is connected with such transformations, hence the importance of the theory of matrices for physics.

For our purpose another characteristic of the matrix calculus is of interest and that is the fact that in physics we usually have a large number of empirical numerical data which enter as coefficients in equations and which can always be put in the form of a two-dimensional array of numbers, or a table, which we have just called a matrix.

It appears that every physical quantity, however complicated, can be represented by such a table giving the values of the parameters which determine its character. From the definition of the term ‘variable’ as *any* value out of a possible range of values, we might treat our variables in two distinct ways, one from the point of view of *function* or operations, the other from the *extensional* point of view, when the function or operations are unknown, although the particular values of the variable are given. The matrix calculus takes this last point of view.

In physical research work we deal for the most part with arrays of numbers or unique and specific, mostly asymmetrical, relations which the experiments give us. Our usual problem is to find the structure, the function, and the operations which are satisfied by the given experimental relations.

We see that the dual approach to our solutions is due entirely to the definition which we have accepted for the variable. We have two issues: Either to find the values of the variable which satisfy the given function and operations or, having particular values of the variable (experimental), to find the function and operations.

Obviously every physical quantity can be represented by a matrix, which may be a. sequence, and every mathematical theorem can be reduced to a property of matrices. Once the proper mathematical theories are worked out it will be always possible to pass from one form of representation to the other.[6]

In the older mechanics the functions were rather obvious and so the use of the matrix calculus was not so imperative. In the newer mechanics, the opposite is the case. We have a large amount of numerical experimental relations, but the functions and operations connecting these variables are unknown and the problem is to find them. From this point of view the matrix calculus represents an *extensional* calculus, a calculus of *observation.* In using the descriptive term ‘observation’ we must add that some objections have been advanced to such a use of the term. The answer is that the term ‘observation’, like most of our most important terms, must be considered a *multiordinal term.* Once this is understood the objections to the use of the term do not hold.

There is no limitation as to what the elements of a matrix may represent; they may be functions, functions of functions, .

Section D. The operator calculus.

The use of the *operator calculus is* interesting, structurally and psycho-logically, in that attention is concentrated, not on the numerical quantities, but on the semantic *operations of combining* them. The calculi used in the newer quantum mechanics are peculiar, because, while they retain the numerical data, and as far as possible, the classical equations, they alter the operations by which these quantities are combined, or the interpretation of the equations.

As an illustration of such a procedure we can take two different formulae for the addition of velocities; one from the classical mechanics, where

$$V_{13} = V_{12} + V_{23}\ , \qquad (1)$$

and the other, the formula for velocity as given by the Einstein theory; namely,

$$V_{13}=(V_{12}+V_{23})/(1+V_{12}\times V_{23}/c^2) \qquad (2)$$

In these formulae V_{12} represents the velocity of body 1 relative to body 2 , . In formula (1) the sign '+' symbolizes the ordinary arithmetical operation of addition. As we already know, this formula has proven too simple to represent accurately the experimental data, and Einstein has replaced it by the more elaborate formula (2).

The above statement is the usual way of speaking about the modification in formulation which has taken place in physics since Einstein. But we could equally well say that the formula has not been altered except that the '+' has no longer the old meaning and does not now represent the arithmetical operation of addition. Both points of view lead ultimately to one value, V_{13}, and the *computations* are similar in both cases.

We should notice especially the great freedom with which we can treat mathematical entities. Our voluntary selection of the point of view becomes important. A similar freedom of selection of interpretation appears to a still larger extent in all *verbal* problems, a fact of considerable structural and semantic importance in any theory of sanity, as we have already seen.

Further illustration of this freedom can be seen in the way in which the ordinary notion of multiplication is re-interpreted in the operator calculus. Let us denote by q and f, two numerical quantities, and qf as their product. But we could view this problem differently. We could say that *qf results* from a semantic *operation q* performed on f, or a semantic *operator* ($q\times$) acting upon f which transformed f into qf. We could denote the *operation* of multiplying by q or the operator ($q\times$) by a single symbol Q. Quite obviously the operator Q *is* not the number q; in other words, the semantic operation of multiplying, say by two, is *not* the number 2.

The operation of multiplying integer 1 by integer 2 gives the result 2. Similarly, the operation of multiplying 1 by q, or, in our new language, the application of the operator Q to the integer 1, gives q. In symbols, $Q1=q$. If we take any arbitrary function f, the result of the operation of Q upon f is written $Qf = qf$. If we were to follow the operation Q by the differential operation d/dx the result would be* :

$$d/dx(Qf)=(dq/dx)\, f + q\, df/dx=[dq/dx \times]f+Qdf/dx$$

But as f is arbitrary, it may be omitted from the equations and the result written in the operator form as

$$d/dx(Q)-Qd/dx=[dq/dx \times]=dQ/dX$$

The symbol dQ/dX, defined by this equation, should be read as 'the operation of multiplying by dq/dx'. Similarly, $d(QP)/dX = PdQ/dX+QdP/dX$.

In translating the ordinary equations into the language of operators nothing new is introduced. This translation involves only a change of *'mental' focus.* Instead of concentrating our attention on the numerical values we concentrate on the operations of combining them. Since the great problems of the quantum mechanics consist in finding new methods of computation, or of combining numerical values, such a change of attitude may prove to be structurally useful.

It should be noticed here that once matrices are considered, and treated, as quantities, or unique and specific relations, by similar reasoning they can be treated as operators. This problem is of fundamental structural and semantic importance because in the quantum theory we deal with matrices which have infinite numbers of terms and since this complexity presents great technical difficulties it is of enormous advantage to be able to pass to some more developed methods of calculation.

At the preliminary stage in the operator calculus we have assumed that multiplication was commutative, that $QP=PQ$; but in the further and more general development of the theory, this does not hold.

In general we must assume in the operator calculus that multiplication is not commutative, that $QP \neq PQ$. For instance, if $Q=(q\times)$ and $P=d/dq$, the two operations are certainly not commutative. Naturally, the validity of $2\times2=4$ is not doubted, but generalized non-commutative multiplication has a definite asymmetrical and so structural geometrical interpretation, to be found in the vector calculus. When we associate with each numerical quantity its own operation of multiplication we thus obtain a more general calculus. Operators may be regarded as compound, or built up from the elementary arithmetical operations of addition and multiplication. They represent, so to say, functions of these operations.[7]

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* The operator D_x, also written d/dx, is called a differential operator. If applied to a product (uv) the results are given by the formula $D_z(uv)=uD_zv+vD_zu$.

Section E. The new quantum mechanics.

The main problem of the quantum theory is to determine these functions of the operations, so that the solution of certain equations (hamiltonian) may represent the experimental facts. The original equations of the new mechanics of Heisenberg, Born, and Jordan were frankly founded on an empirical basis. As Dirac puts it, in seeking for the new equations, the classical equations were to be retained as far as possible and only the operations by which these quantities are combined were to be altered.

To gain this freedom to alter multiplication the data were first interpreted as matrices. Then it was found, by Born and Wiener, that the matrices could be interpreted as a special kind of operator, which furnished means to calculate the matrices. Carl Eckart independently developed a simple operator calculus for the solution of the quantum problems. In this present work I follow closely the paper of Eckart.[7]

The origin of the new quantum mechanics was an epoch-making paper by Werner Heisenberg, in July, 1925. The older quantum theory had postulated the existence of stationary states of the atom, which were calculated with the aid of the older mechanics. In the new mechanics the equations have similar form as in the classical theory, but the variables no longer obey the commutative law of multiplication. In general pq is not equal to qp, ($pq=qp$ and $pq-qp=0$ in the classical theory) but $pq-qp=(h/2\pi i)1$, where h represents the Planck constant, q the generalized co-ordinate, p the momentum, 1 stands for the unit matrix, and π and i have the usual meaning. The fact that multiplication is not commutative in our calculus allows us to give a definite value to the above difference and by introducing the Planck constant h, we are enabled to introduce the quantum conditions in our calculations.

The quantum conditions of the older theory led to an algebraic equation. By using the classical equation with a non-commutative multiplication law for the variables it is possible to perform calculations in the new and wider scheme of dynamics. The difference between pq and qp is expressed in terms of the Planck constant h. When h is made to approach zero, pq approaches qp, and so we pass to the classical mechanics. Thus we see that the classical mechanics appear only as a particular case of this more general theory.

In introducing his theory, Heisenberg pointed out that the older mechanics uses quantities which are *never observable,* and *can never be observed,* such as, for instance, orbital frequencies and amplitudes, or the position and 'time' of revolution of an 'electron'. , which, as such, have no physical meaning. He proposes to use *observable* data, such as the frequencies and intensities of the radiations , . Now these frequencies are always differences between two terms given by integers. If T_n and T_m are two such terms, the observable frequency is theoretically represented by $\nu_{nm}=T_n-T_m$. Such numbers as ν_{nm} characterize the atom as far as it is observable. It was natural that such a collection ('sum' in this case has no longer any physical meaning) of terms could best be represented by a matrix. In the classical theory a dynamical quantity was represented structurally by a trigonometrical Fourier series; in the new, it is represented by a two-dimensional table of values; that is, by a matrix giving the frequencies and the intensities of radiations.

An important and interesting structural issue now appears. It is that the Heisenberg theory gives a new formulation for the hamiltonian equations of motion, whereby their form is preserved, yet they are made applicable both to periodic and to *non-periodic motion.* It becomes possible to fuse the classical mechanics with the quantum mechanics. The distinction between ‘quantised’ and ‘unquantised’ motion loses all meaning, and a fundamental equation, $pq—qp=(h/2\pi i)\ 1$, is formulated which is valid for *all motion.*

The Heisenberg theory is also characterized by its thoroughly behaviouristic, actional, functional, and operational character. The number of unjustifiable assumptions is the lowest in existence and most of the identifications are eliminated. According to Heisenberg, electrons and atoms do not have the ‘same’ kind of ‘reality’ as ordinary *objects* of lower order abstractions. This conclusion, which underlies his whole work, is of particular importance structurally. As we know, differences in character separate different orders of abstraction and since the quantum phenomena belong to a higher order of abstraction they must differ from objects which belong to a lower order of abstraction. In this theory the feelings of ‘space’ and ‘time’ are no longer applicable to the ‘inside’ of the atom—as might be expected.

The distinction between ‘inner’ and ‘outer’ electrons in an atom becomes meaningless, since it is impossible to recognize a particular entity among a series of similar entities. In accordance with the new ‘space-time’ outlook we gain a physical basis for the absolute individuality of some eventual unit.

Because of its structure, the Heisenberg theory is a very fundamental one and there is little doubt that the Heisenberg *methods* will be elaborated further and will be kept as a permanent *checking method* in physics. A theory which is thoroughly behaviouristic, with a minimum of assumptions, will probably remain both a most important instrument of research and an inspiration to physicists and mathematicians.

The Heisenberg theory, again, because of its structure and method, does not lend itself easily to visualization. This is not against the theory. The pictorial representations of lower order abstractions are not to be relied upon. Besides, visualization depends on the *lower centres* and therefore must be represented by a *macroscopic* representation of a continuous (rather than a discrete) character, such as waves, .

If we were to try to describe the Heisenberg theory pictorially, which is obviously difficult to do, we would have to give a *negative* description. We should have to say that what we observe must be considered only as radiations from the location which the atom was supposed to occupy.[8]

There remains but to mention some more characteristics of the Heisenberg theory which seem to have very far-reaching structural and semantic bearings

This theory appears frankly statistical and introduces fundamental probability assumptions. The moment we realize that the human organism is essentially an *abstracting* affair and that the abstracting is performed on different levels, or in different orders, it becomes obvious that statistical methods and probability notions become fundamental.

In the earlier days we used to assume that statistical laws were laws with exceptions. Such an outlook was conditioned by our dealings with macroscopic events. Now, we analyse such macroscopic events in terms of microscopic and sub-microscopic events, the statistical laws become accurate laws, not for individuals but for *groups* of individuals. Because we abstract in different orders, we deal *only* with statistical data, mass effects of different 'packets' of nervous excitement, as is best illustrated by different thresholds in different nervous tissues.

The processes in the higher centres, *more remote from the external world,* deal with a special material, no more with statistical data of 'packets' and averages, but with what we used to call 'inferences', 'inductions'. , which give only the *probability* of happenings. But as we have already seen, probability has become a well-developed structural mathematical discipline, which has not yet made much effect upon our primitive-made macroscopic metaphysics and language. It should be noticed that the highest activities of the nervous centres are based on statistical data furnished by the lower centres. So we see that, to the best of our knowledge about ourselves and the world around us, a modern structural and semantic outlook, in science or in life, must be based on statistical and probability methods.

In space-time every point has a date, and therefore in the language of space-time all points are different and do not repeat themselves. Such structural outlook is, of course, again conditioned, and leads toward the statistical and probability methods. The main psycho-logical importance of the new methods is to diminish affective tension, which is always wasteful and harmful. Inferences may involve belief. When *belief* is *too strong,* although this is never justified according to the best modern knowledge, we very easily fall into identification, delusions, illusions, and the like. It should be emphasized that the last-mentioned pathological states are always compound. They involve at least two components. One of these consists of some ignorance somewhere; the other of strong affective belief in the 'truth' of our mistaken notions. The stronger the affective tension is, the more dangerous the semantic disturbance becomes.

The Heisenberg theory has succeeded in formulating (verbal) structural methods which are best suited to represent the experimental facts which underlie physics, as well as being structurally in accord with the working of the human nervous system. That is why I venture to assert that this theory will never be abandoned as a checking and research instrument.

In an hypothetical experiment in the quantum field, we may assume what may be called a gamma-ray microscope. If we were to illuminate an 'electron' by gamma-rays, the rays would disturb the experiment, and in our fundamental equation, $pq\text{-}qp=(h/2\pi i)$ 1, by which the 'position' was to be determined, the 'momentum' would thereby be disturbed. This change of 'momentum' would be greater the shorter the wave-length of the rays used; and the shorter the wave-length of the ray, the more accurate the determination of 'position' would be. Hence, the more exactly a co-ordinate q could be found the less exactly could its momentum p be found, and vice versa.

So we have to introduce corrections for errors, and have to introduce 'mean values' and 'probability functions', which we can develop and compute. Lately, Bohr has further developed the probability aspects of the newer quantum mechanics but I have not seen this work. Heisenberg introduces 'probability packets' which correspond to the 'wave packets' of Schrödinger.

It is difficult to speak briefly, and yet in a satisfactory way, about these new developments, and particularly difficult to give credit properly to different authors. All their works are interwoven and at present they all really work together in spite of the fact that historically some of these theories have been developed independently.

What we call today, for the sake of brevity, the Heisenberg theory, because of its originator, has been further developed by Heisenberg, Born, Jordan, and others. Later, when the wave-mechanics appeared, all the new theories were finally fused into a very elaborate and impressive structure.

Historically, P. A. M. Dirac worked at the theory from a different mathematical point of view, utilizing what is called the 'Poisson bracket' method. In this treatment the difficulties of the matrix calculus were avoided. He introduced dynamical variables which he called the q numbers. These do not obey the commutative law of multiplication although the c numbers (classical) do. He also considered the difference of the non-commutative products $xy\text{-}yx$, where x and y are functions, respectively, of the co-ordinates $q_1 \ldots q_s$, and of the momenta $p_1 \ldots p_s$ of a multiple periodic system with s degrees of freedom.

Dirac has generalized the matrix theory and the Schrödinger equations. His work seems to be most important, in physics and mathematics, but it is not possible for us to consider it here in any detail.[9]

Let us recall once more that there are fundamental differences between the different orders of abstraction, and that we all have to abstract in different orders. From this point of view it is natural that *every* theory, even if expressed at present in a form which cannot be *visualized*, like the Heisenberg theory or the original Dirac theory, has sooner or later to be expressed in a structural form which can be visualized. These problems have really nothing, or at most very little, to do with the world around us. They are concerned with the neurological structure which produces *all* theories.

Theories of a structure such as that of the Heisenberg theory are extremely important, as already explained, but in them we lose the help of 'intuition'. Now 'intuition' (lower centres) has two quite different effects—sometimes it leads us astray, but on other occasions it helps greatly.

An ‘intuitive’ theory has a creative aspect, but always ought later to be revised and scrutinized by non-intuitive means. In fact, because of our nervous structure, we should always strive to produce *both aspects* of theories—strive *consciously*—for thus we facilitate progress. Historically, we can never completely avoid producing both types of theories, as they are inherent in our nervous structure and in the different orders of abstractions we produce.

It is precisely in the newer quantum mechanics that a typical example of this simple neurological fact is found. The non-intuitive handling of data was introduced by Heisenberg: the translation of the matrix calculus into operational and ‘Poisson brackets’ methods; and, finally, the new ‘wave mechanics’ of de Broglie, Schrödinger, and others, gives us a perfect translation into intuitive methods.

It should be noticed that according to the old notions such two methods, the intuitive and the non-intuitive, were not supposed to be a *neurological necessity*. We still assumed that they were separated ‘absolutely’, and even today in many quarters we argue as if they were absolutely separable. If we accept the principle of non-elementalism, we realize that this distinction is verbal only and that the invention of verbal means has little or nothing to do with the world around us, but that it depends on human structural ingenuity.

Investigation of the ordered cyclic nerve currents shows unmistakably that such sharp differentiation is unjustifiable; and we must conclude, in accordance with historical experience, that translation from one method to the other must be a necessity, and so will be accomplished some day in every field. It is true that at present the Einstein theory has not been translated with entire success into terms of lower order abstractions. This is a task which is facilitated by this present work. The newer quantum mechanics gives us an unparalleled example of such translation, and hence our main interest should be concentrated on this structural aspect.

On neurological grounds it seems certain that visualization involves in some way the lower nerve centres, which again, by evolutionary necessity, involve macroscopic forms of representation. Our macroscopic experience led us to *geometrical intuitions*. These were framed three-dimensionally in *‘absolute emptiness’*, and were impossible in higher dimensions. The old structure represented a static empty ‘space’ in which nothing could happen and which was thus unfit to represent this world around us where something is going on everywhere.

The new structure represents *‘fulness’* or a plenum. We can visualize it as a network of intervals or world-lines and then, by the notions of the differential calculus, as already explained, we pass easily to the visualization of the many-dimensional space-time world of Minkowski-Einstein. Now, in such a world, the curves are represented by functions, and, vice versa, functions represent curves. Thus it is obvious that analytical ‘non-intuitive’ methods have ‘intuitive’ structural geometrical counterparts. From this point of view the method of operators represents a passing step from the non-intuitive to intuitive methods. As soon as we have functions, we can represent a functional calculus as an operational calculus. This involves a more behaviouristic semantic attitude and so leads to the possibility of translation of either method into the other. Ultimately these are psycho-logical transformations and translations.

We should not be surprised to find that in the development of the newer quantum mechanics the operational calculus plays just such a role.

Now our older macroscopic experience, which affected our lower nerve centres, gave rise to the elementary geometrical structural notions of ‘lines’, ‘surfaces’, ‘volumes’, . For the building of physics we had to introduce ‘time’, ‘motion’, . In older days we did not realize that these give us forms of representation and that it is optional with us which forms we will accept as fundamental or use as a starting point.

The old descriptive apparatus posited structurally an absolute and immutable ‘space’ (emptiness), ‘time’, ‘matter’ out of which we built up a verbal definition for ‘motion’. The semantic attitude of all of us, scientists included, depended upon identification. We ascribed lower-centre significance to higher-centre abstractions. We did not discriminate enough between the macroscopic and small scale events. So we had ‘geometrical optics’ in which we ‘perceived’ a ray of light (in a dusty room, let us say) as a ‘straight line’. Further investigation disclosed that the ‘rays’ on one level of abstraction were *waves* on another, but they were not perceived as waves by the lower centres

But through our lower centres we had acquaintance with some waves, such as in water; so representation for waves was developed. A wave-theory still remained intuitively workable, even when we dealt with waves which we could not see. Now the equations of waves are well known. It is then possible to translate a non-intuitive matrix mechanics, when we treat matrices as operators, into a functional calculus which has an intuitive geometrical wave representation.

This is precisely what has happened and now, perhaps for the first time in human history, we have all the aspects of a theory being worked out simultaneously, with mutual co-operation of all workers and the use of methods which are mutually complementary from the neurological side. There seems little risk in predicting that because of these neurological factors the newer quantum mechanics will give extremely rapid and far-reaching results. When scientists become aware of the structural semantic and neurological issues involved, perhaps such achievements will be multiplied *consciously*, instead of being a kind of coincidence.

Personally, I am convinced that these new achievements are not simply coincident. It seems that the abolishment of the old, *el*, static ‘absolute space’ and ‘absolute time’ *has relieved the younger scientists from a semantic blockage*. This release was due to the bold stroke of genius of Einstein in refusing to use the vicious aristotelian ‘is’ of identity. As soon as we realize that words are not the objective levels, we gain an *unconscious semantic freedom* in handling words, as words. At once this freedom is bound to produce many different forms of representation for events, according to the personal make-up of the individual workers. And of course these forms can be translated into one another.

Section F. The wave mechanics.

We have not sufficient space at our disposal to discuss more fully the new wave mechanics. I found that short of a small volume, no explanations, readily intelligible, could be made.

In mathematics and physics, which represent the most developed sciences, we consciously and unconsciously strive for more and more general formulations. The work of Einstein, showing that the classical mechanics was only a particular case of a more general mechanics, has given a healthy stimulation to such a fruitful line of work.

As the quantum phenomena could not be accounted for by the old mechanics, it was natural that physicists should try schemes of new mechanics which included classical mechanics as a particular case. Thus, Sommerfeld, through his methods of the application of the Einstein theory to the quantum mechanics and his generalizations of phase-space, and his treatment of the relation of wave-optics to ray-optics and of the relation of mechanics to ray-optics, came close to the discovery of the wave mechanics.[10]

The new wave mechanics originated in 1924 in Paris, with the thesis of Louis de Broglie, published in 1925, and republished in a book form in 1926.

The controversy between the corpuscular light theory of Newton (emission theory) and the wave theory of Huygens is well known. The emission theory had its support in the ‘rectilinear’ propagation of light, which followed from the inertia of light particles. Also, it explained the reflection and the refraction. , of light, but failed in other respects. It is true that the wave theory also had its weak structural spots. In it the ‘rectilinear’ propagation of light remained a complete mystery and it completely failed to account for the dispersion of light, until this was explained on the electron theory.

Both theories assumed the periodicity of light phenomena, but the acceptance of one theory was generally held to mean the rejection of the other. It did not occur to many that both theories might be correct but only partial structural aspects of a more general theory.

With the advent of the quantum theory of Planck (1900) new methods were found. In 1905 Einstein propounded his theory of ‘light quanta’ successfully. He assumed that radiation occurs in discrete quanta of energy $h\nu$, where ν represents the frequency. From this point of view the quantum had the characteristic discreteness of a corpuscle, and yet the frequency characteristic of a wave. We see that the new theory involved a kind of a blend of the two older theories.

De Broglie generalized still further the above notions. His theory is in a way the result of the theory of Einstein. As we already know, Einstein shows the connection of mass and energy, so that the conservation of mass becomes also the conservation of energy, and vice versa. Starting with these premises de Broglie concluded that if any element, in the most general sense, be it an electron or proton or light quantum or what not, has energy W, there must be in the system a periodic phenomenon of frequency ν, defined by $W=h\nu$. From this point of view all forms of energy, radiation included, must have an atomic structure and the atoms of energy must be grouped around certain points, forming what we call ‘electrons’, ‘light quanta’, .

Applying the Lorentz-Einstein transformation, he finds a rather startling fact—that the frequency associated with any assumed mass m_0; namely, $\nu_0=m_0c^2/h$, represents no more and no less than a *periodic* phenomenon, analogous to a stationary wave, which spreads around the point of which the mass is a singularity.[11]

In other words, a ‘mass particle’ at rest is the centre of a pulsation throughout the spread, or otherwise it is a singularity of the pulsation. The quantity which pulsates is called ψ and $\psi\psi^-$; is interpreted as the *electric density,* where ψ^- is the conjugate of the complex quantity ψ.

In the Minkowski representation, the above astonishing result becomes quite simple, and we can see clearly how simultaneous pulsations become travelling waves. In Fig. 2, we give a two-dimensional diagram of space-time. OX_0 is the ‘space’ co-ordinate, OT_0 the ‘time’ (ict_0) co-ordinate. ψ_1, ψ_2, ψ_3, represent the traces of the surfaces of constant phase which are perpendicular to OT_0. The Lorentz-Einstein transformation is equivalent to the transformation from rectangular system X_0OT_0 to the rectangular system XOT, forming an angle θ. In the new system the lines ψ_1, ψ_2, ψ_3 . . . , are no longer parallel to the X axis and so represent a moving *wave front.* In this new system the ψ-lines represent the moving wave front for different points, *P*, *P'*. , on one phase line and have *different* values of *t.* The smaller the velocity (*v*) of the particle, the smaller the angle θ and the greater the distance *PQ* travelled by the phase in a given ‘time’, *P’Q* or Δt, which means the greater the phase velocity *u*.

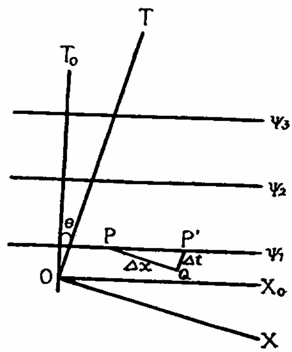

FIG. 2
(From *Wave Mechanics*, by H. F. Biggs)

The frequency of these pulsations or waves is the ‘total energy of the particle’ divided by Planck’s constant, *h*. In symbols, $h\nu=mc^2$+potential energy, where

$$m=\frac{m_0}{\sqrt{1-\beta^2}},\ \beta=\frac{v}{c}.$$

The problem before us is to connect structurally, the waves with two observations, one of the radiations, the other of what we call the ‘material particles’.

Our interest at present lies only in the connection with the latter. According to this theory, the ‘region occupied by the particle’ is only a region where a set of ψ-waves, which vary continuously in direction and in frequency in a small range, reinforce each other to give a *wave-group* travelling with what we usually call ‘the velocity of the particle’.

As the waves have different frequencies, they travel with different velocities and so we have to face the problem of the dispersing medium, where, according to classical theories, the region of reinforcement has a velocity different from that of the phase. The ordinary expression for ‘group velocity’ gives, on the wave interpretation, the magnitude of the ‘velocity of the particle’.[12]

That which, in the classical theories, we called the ‘motion of the particle’ is represented by the motion of the region of reinforcement of ψ-waves. The direction of motion is represented by the direction of a ray-, or wave-normal. The particular ray selected as the position of the ‘path of the particle’ is represented by the ray cut in coinciding phase by a set of ψ-waves with slightly varying directions. The ‘position of the particle’ is given by the small region occupied by the *group* of waves of slightly different frequencies and velocities.

In this connection we should notice an important point which is necessitated by the methods of generalization and of translation from macroscopic to sub-microscopic events and vice versa. Schrödinger, by formulating the differential equations for the ψ-waves, has brought out clearly the important structural point that wave mechanics bears a similar relationship to classical mechanics of particles as wave-optics bears to ray-optics.

Here again we have the macroscopic phenomena capable of being treated by mathematical methods different from those used for the small-scale phenomena.

In classical mechanics the state of a system whose co-ordinates were $q_1 \ldots q_s$, and whose momenta were $p_1 \ldots p_s$, was represented by a point in a $2s$-dimensional q-spread and the changes in the system were represented by the passage of the point along some curve, a ‘ray’, so to say.

Schrödinger regards the classical mechanics as only an approximation, while rigorous treatment must be made by the aid of wave mechanics.

The large-scale, or macroscopic, mechanical processes correspond to a wave signal in the q-spread and can be regarded as a point in comparison with the geometrical structure of the path. In small-scale phenomena, such as the atomic processes, a rigorous wave formulation must be used.

This analysis can be carried further and Hamilton knew well and used the analogy between mechanics and geometrical optics. The Hamilton variation principle, $\delta\int L dt=0$, is Fermat’s principle for a wave motion; the Hamilton-Jacobi equation expresses Huygens’ principle for wave motion; and the new wave mechanics expresses the Kirchhoff analysis of physical optics. As Huygens, principle could deal with the problems of physical optics up to a certain point, so the Hamilton-Jacobi equations could deal with atomic problems up to a certain point. At the exact wave analysis of Kirchhoff was needed to clear up the finer points of physical optics, so the new wave mechanics is required for the exact solution of the atomic problems.[13]

A detailed analysis shows that classical mechanics was associated with geometrical optics (ray-optics). Obviously, a more exact system of mechanics would be one associated with wave-optics, which would give the classical results in all cases where the wave-length was negligible in comparison with the dimensions of the path. Schrödinger suggests that a correct extension of the analogy would be to regard the wave system as *sine-waves.* In this connection it should be recollected that Fourier has shown that any given form of waves can be represented by the superposition of *sine-waves,* and that therefore a *sine-wave* (see Chapter XXXII) may be considered as a general formulation.

The ray methods in physics worked only to some extent, that is to say, in the cases where the radii of curvature of such rays and the dimensions of the spreads were large in comparison with the wave-length. When this is not the case we have to consider waves and not rays. Naturally, dealing with atomic dimensions, which-are very small, instead of using the paths of the particles or the ψ-rays, we have to use the ψ-waves. It appears that this difference was the main, rather puzzling distinction between the classical mechanics and the quantum mechanics, between the macroscopic theories and the sub-microscopic ones.

The above realization, and its formulation into a mathematical theory, seems to be an important and extremely fruitful generalization, which probably will be retained as a method.

One of the puzzling features of the quantum theory was the structural appearance of the whole-number laws of the ‘orbits’. That some such whole-number relation is justified seems to be well established, yet it contradicted the older ‘continuous’ mechanics. A new theory, to be at all satisfactory, should be able to fit these whole-number empirical data. The first test of the new wave mechanics, and also its first success, was precisely in this field.

If a ray of the ψ-wave was supposed to run around in a circle for a stationary state, the circumference must be a *whole multiple* of the wave-length, or

$$2\pi r = n\lambda = n\,\frac{h}{mv}\text{, and } mvr = n\,\frac{h}{2\pi}\text{,}$$

where n is an integer. We see that the quantum condition of the Bohr theory, that the angular momentum must be a whole multiple of $h/2\pi$, is only the result of the requirement that the wave-function ψ shall be single-valued, which is another way of saying that the circumference ($2\pi r$) must contain a whole number of wave-lengths. It may be compared perhaps with waves travelling around a circular loop of string. If they travelled both ways we would have stationary waves.

At this point a very important structural feature of the new wave mechanics makes its appearance. In the above interpretation the ‘velocity of the electron’ has lost its physical meaning, it becomes simply the *wave length* of the ψ-waves. In the wave mechanics, as well as in the matrix mechanics, there is no meaning to the older ‘position of an electron on its orbit’. So the wave mechanics again embodies the advantages of the matrix mechanics by not postulating entities which can never be observed. The whole numbers, as Schrödinger remarks, ‘appear as naturally as do “integers” in the theory of vibrating strings’. In the theory of string vibrations these whole numbers are determined by certain boundary conditions which have to be satisfied by the solution of a differential equation. In the new wave mechanics there is also a differential equation representing the Schrödinger wave equation.[14]

Section G. Structural aspects of the new theories.

We should notice the important distinction between the two structural types of these theories. The extensional matrix theory can hardly be visualized, with all the consequent advantages and disadvantages. The wave mechanics can be visualized. From what we already know of the structure and working of the nervous system, we see that the wave mechanics will have a *creative* element and the matrix mechanics will remain an important *checking* method.

At present, all these new theories seemingly have blended or perhaps it would be better to say that they have been translated from one language to another and all the workers in this field work from all angles.

It should be mentioned also, that Einstein, Bose, Jordan, and others, work from the point of view of *statistics,* and that these methods, too, are being retranslated and connected with the rest of the new theories.

The new wave mechanics evades the difficulties of the matrix calculus and brings the new mechanics within the scope of the highly developed analysis of the theory of differential equations. It also enlists the creative aspects of 'intuition', 'visualization', .

Concluding our consideration of the subject, three remarkable aspects of the wave mechanics must be referred to. We are already acquainted with the term 'action'. It appears that the main point of the passing from the old mechanics to the new was the stroke of genius of de Broglie, when he divided action by the fundamental constant h with some definite numerical factor which then gives us the *phase.* In the expression for ψ, the energy appears as the 'time' component of a space-time vector whose 'space' components are those of the momentum. When this vector is divided by h, its components become the *frequency,* or the number of waves which each axis cuts per centimetre.

These are the methods by which we can use differential equations, whereby the older discontinuities disappear and the particle is represented as a group of reinforcing waves.[15]

From this point of view we also come to the conclusion that the 'conservation of energy', which was very valuable in the old days, is perhaps only a gross macroscopic generalization and will give place to a newer and more fundamental notion of the conservation of *frequency* or 'times'.[16]

It has been already mentioned that the newer mechanics must be represented in accordance with *statistical* data, *probabilities,* with due attention paid to the theory of errors, . While these requirements have very little to do with the world around us, they are unconditionally required by our nervous structure, which is, after all, the general author of all our 'knowledge' and 'theories'. Let us be candid about it; there is no such thing as 'knowledge' outside of a nervous system, and therefore the neurological requirements, as already mentioned, become paramount. The newer theories brilliantly satisfy this requirement.

As an example, we may perhaps mention an aspect of the wave mechanics theory which is not at present settled, but which remains just as interesting.

Schrödinger shows that in highly excited states a suitably chosen group of waves represents a *'wave packet'*, which behaves like a point-mass of the ordinary mechanics. It oscillates with frequency ν_0 in a rectilinear path. The number and breadth of the waves which form the packet vary with the 'time', but the width of the packet remains constant. The remarkable part about the shape of the curve is that it represents the *Gauss error curve.*

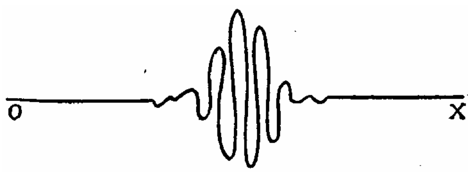

FIG. 3
(From *The New Quantum Mechanics,* p. 115, by George Birtwistle)

Heisenberg has shown that this result is only accidentally true, but for our purpose of illustration this is quite enough.

The newer quantum mechanics has shown once more the necessity for a re-analysis of our fundamental notions. From a space-time point of view, which seems to be a permanent acquisition of science, since it is a language and method of structure closer related to the external world than the older languages and methods, it seems beyond dispute that even macroscopic phenomena are the results of *repeated observation.* Now, such a point of view, although it is extremely plausible, and close to neurological and physical data, necessitates a complete reconstruction of our describing apparatus, which is not adapted to such an outlook.

The problem of 'observation' enters. Bohr suggests, and rightly, that this vocabulary is strictly connected with the older 'causal' vocabulary. One of the main points of the present work is to draw attention to the *multiordinal mechanism and terms* and to show that the analysis of these problems cannot be even attempted without first analysing the structure of our languages, of our 'knowledge', and the neurological and semantic aspects which such analysis involves.

When this analysis is carried through we see that the problems of 'continuity' and 'discontinuity' lose their absolute character. They become *verbal* problems, to be solved through the ingenuity of some one who will suggest the solutions.

The newer quantum mechanics give us ample material for work on these problems, but they also illustrate a much more general and important problem, which is the subject matter of the present book; namely, that all 'knowledge' is *structural,* strictly dependent on the nervous structure and functioning, and the language we use. 'Method' is that aspect of the search for structure which deals with the most expedient means for finding structure. Since words *are not* the things we speak about, the study of linguistic structure becomes a most important research method. The more languages (theories) we have for analysis and structural comparison, the more glimpses do we get at the structure of the world. The newer quantum mechanics give us enormous material of a linguistic, structural, and semantic character.

It is natural that this wisdom can come only from the study of the structure of the highest developed languages in existence, which are mathematical languages. If we want progress in any line of human endeavour, this progress is always dependent on the languages we use, since what we call 'progress' is a co-operative affair and therefore dependent on means of communications and languages.

From the point of view of structure, we deal with a world of *absolute individuals* and therefore our languages must be such as to reflect such individuality. We already know that this involves an extensional attitude and methods, which historically have produced mathematics as the only language which as yet reflects in structure the world around us.

With the newer quantum mechanics, the old 'discontinuity' resolves itself into an essential *individuality,* as noticed by Bohr, perfectly foreign to the older theories.

History proves that we were slow in arriving at that point. Our tragedies began when the 'intensional' biologist Aristotle took the lead over the 'extensional' mathematical philosopher Plato, and formulated all the primitive identifications, subject-predicativism. , into an imposing system, which for more than two thousand years we were not allowed to revise under penalty of persecution. Mathematics was not particularly encouraged, but at least, not persecuted, so that it was developed into the present day great linguistic system. The theory of function involves semantic factors of non-identification.

The invention of the differential and the integral calculi, represents the two great structural and psycho-logical aspects of analysis and differentiation, versus synthesis and integration.

The application of these methods led us to differential geometries, to methods of treating 'fulness', and to 'contact' methods. 'Fulness' necessitated geometries of higher dimensions, impossible in 'absolute emptiness', and so the fusion of geometry with physics became possible. The four-dimensional world of Minkowski and the theory of Einstein finally achieved this fusion. The next step was the invention of the new quantum mechanics, where all these important, nay, all-important, structural, semantic, and linguistic achievements find their culmination. The old primitive metaphysics become too 'materialistic' for an enlightened age.

Without legislating about the 'truth' or 'falsehood' of the newer mechanics, as a matter of *human behaviour* these theories are the best indications and examples of the structure of human 'knowledge', which I have attempted to formulate in this work as a general theory.

The *A*-system was strictly interconnected with primitive-made structural assumptions or metaphysics, reflected in the structure of the older languages and in the *el* notions about language, 'psychology', 'logic', and the pre-scientific anthropomorphic astronomy, physics, and other disciplines. Of late, science has developed in spite of all handicaps and persecutions, and has begun to depart structurally and semantically from the path of aristotelianism and the dark ages. Every science has had to build its own language and this fact completely condemns the *A* language, which, it is shocking to notice, we continue to preserve in our daily life.

Should we wonder that we have shown hardly any progress at all in our purely human affairs and notions ? We should wonder, rather, that we have been able to survive until now—though with needless difficulties and suffering. More wars, more revolutions, more insanity, more morons, more struggle and competition, and more unhappiness are what we are entitled to *expect and predict* as the outcome of this structurally and semantically impossible situation.

As the organism works as-a-whole, such things as 'pure intellect' or 'pure emotions' represent structurally *el* fictions and scientists should realize that their professed detached scientific attitude is profoundly and fundamentally unjustified. All science has 'emotional' components, which play most important roles in life. If we live in a modern world, but keep the 'emotional attitudes' of primitive bygone days, then naturally we are bound to be semantically unbalanced, and cannot be adjusted to a fundamentally primitive 'civilization' in the midst of great technical achievements.[*] When scientists understand that, then the layman will have a different attitude toward science. He will understand that science is not a privilege of the few, something without effect upon all and every one. He will realize that while he lives in a modern world, *made so* by science, *structural ignorance* of the fundamentals as discovered by science leaves him with primitive structural assumptions or metaphysics, which by necessity build for him a delusional world leading to semantic unbalance and ultimately to 'mental' and nervous ills.

From the *non-elementalistic* point of view, the only escape is to realize that ignorance in an adult is, and must be, pathological, because 'knowledge' is to be considered as a normal characteristic of human *nervous tissue.*

A special structural and methodological brief and simplified account of scientific achievements, such as I have attempted in this work, must be a part of a theory of *sanity.* Sanity means *adjustment* and without the minimum of the best structural knowledge of each date concerning this world, such adjustment is impossible.

It is not necessary that the reader should fully understand all technical details of a theory, to be *aware* (instinctive, affective. ,) of the *existence* of the structural and semantic problems and to realize that some of the most competent and skilled professionals are working at these theories. Such awareness has great pacifying semantic influence; it eliminates the older affective tensions which were due to identifications, absolutism, dogmatism, flights into mysticism, and other similar pathological disturbances.

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[*] See my *Manhood of Humanity* (E. P. Dutton, New York).

A *non-aristotelian system* must deal with all these structural and semantic issues. It is hoped that once a $\bar{A}$-system is presented to the public, scientists and laymen will become more interested in the structural and semantic issues emphasized here, and that new and wider researches will be undertaken.

History shows that such hopes are not illusory. The greatest men of science have always had wide human aims and interests. From the *non-elementalistic* point of view, they probably became productive geniuses because of this broad human urge. From the point of view of psychiatry, it is well known that 'mental' ills involve usually anti-social affective attitudes. When we see men with distinctly anti-social tendencies, no matter how they rationalize them, they are invariably ill in some way. A fully healthy individual is never anti-social.

That science should include structural and semantic factors of sanity may be a startling notion, but only at first ! In the present analysis this turns out to be, rather unexpectedly, a necessity. But on second consideration we should rather expect it. Science and mathematics show the working of the 'human mind' at its best. Accordingly, we can learn from science and mathematics how this 'human mind' should work, *to be at its best.* Then we should make an analysis of science and mathematics from some wider structural and semantic point of view,—the task which has been undertaken in the present work.

At this early stage it is, of course, of comparatively little importance to what extent this analysis turns out to be satisfactory. The main point is that it has been *originated.* If the present author fails, others, perhaps even because of his failure, may be stimulated to do it better. The great and vital thing is that it should be done, by someone.