Zeno of Elea

Arthur Fairbanks, ed. and trans.
The First Philosophers of Greece

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Page 112-119.

Fairbanks's Introduction

[Page 112] Zeno of Elea, son of Teleutagoras, was born early in the-fifth century B.C. He was the pupil of Parmenides, and his relations with him were so intimate that Plato calls him Parmenides's son (Soph. 241 D). Strabo (vi. 1, 1) applies to him as well as to his master the name Pythagorean, and gives him the credit of advancing the cause of law and order in Elea. Several writers say that he taught in Athens for a while. There are numerous accounts of his capture as party to a conspiracy; these accounts differ widely from each other, and the only point of agreement between them has reference to his determination in shielding his fellow conspirators. We find reference to one book which he wrote in prose (Plato, Parm. 127 c), each section of which showed the absurdity of some element in the popular belief.

Literature: Lohse, Halis 1794; Gerling, de Zenosin Paralogismis, Marburg 1825; Wellmann, Zenos Beweise, G.-Pr. Frkf. a. O. 1870; Raab, D. Zenonische Beweise, Schweinf. 1880; Schneider, Philol. xxxv. 1876; Tannery, Rev. Philos. Oct. 1885; Dunan, Les arguments de Zenon, Paris 1884; Brochard, Les arguments de Zenon, Paris 1888; Frontera, Etude sur les arguments de Zenon, Paris 1891

Simplicius's account of Zeno's arguments, including the translation of the Fragments

30 r 138, 30. For Eudemos says in his Physics, 'Then does not this exist, and is there any one ? This was the problem. He reports Zeno as saying that if any one explains to him the one, what it is, he can tell him what things are. But he is puzzled, it seems, because each of the senses declares that there are many things, both absolutely, and as the result of division, but no one establishes the mathematical point. He thinks that what is not increased by receiving additions, or decreased as parts are taken away, is not one of the things that are.' It was natural that Zeno, who, as if for the sake of exercise, argued both sides of a case (so that he is called double-tongued), should utter such statements raising difficulties about the one; but in his book which has many arguments in regard to each point, he shows that a man who affirms multiplicity naturally falls into contradictions. Among these arguments is one by which he shows that if there are many things, these are both small and great - great enough to be infinite in size, and small enough to be nothing in size. By this he shows that what has neither greatness nor thickness nor bulk could not even be. (Fr. 1)9 'For if, he says, anything were added to another being, it could not make it any greater; for since greatness does not exist, it is impossible to increase the greatness of a thing by adding to it. So that which is added would be nothing. If when something is taken away that which is left is no less, and if it becomes no greater by receiving additions, evidently that which has been added or taken away is nothing.' These things Zeno says, not denying the one, but holding that each thing has the greatness of **[Page 115]** many and infinite things, since there is always something before that which is apprehended, by reason of its infinite divisibility; and this he proves by first showing that nothing has any greatness because each thing of the many is identical with itself and is one.

Ibid. 30 v 140, 27. And why is it necessary to say that there is a multiplicity of things when it is set, forth in Zeno's own book? For again in showing that, if there is a multiplicity of things, the same things are both finite and infinite, Zeno writes as follows, to use his own words: (Fr. 2) 'If there is a multiplicity of things; it is necessary that these should be just as many as exist, and not more nor fewer. If there are just as many as there are, then the number would be finite. If there is a multiplicity at all, the number is infinite, for there are always others between any two, and yet others between each pair of these. So the number of things is infinite.' So by the process of division he shows that their number is infinite. And as to magnitude, he begins, with this same argument. For first showing that (Fr. 3) 'if being did not have magnitude, it would not exist at all,' he goes on, 'if anything exists, it is necessary that each thing should have some magnitude and thickness, and that one part of it should be separated from another. The same argument applies to the thing that precedes this. That also will have magnitude and will have something before it. The same may be said of each thing once for all, for there will be no such thing as last, nor will one thing differ from another. So if there is a multiplicity of things, it is necessary that these should be great and small--small enough not to have any magnitude, and great enough to be infinite.'

Ibid. 130 v 562,.3. Zeno's argument seems to deny that place exists, putting the question as follows: (Fr. 4) **[Page 116]** 'If there is such a thing as place, it will be in something, for all being is in something, and that which is in something is in some place. Then this place will be in a place, and so on indefinitely. Accordingly there is no such thing as place.'

Ibid. 131 r 563, 17. Eudemos' account of Zeno's opinion runs as follows: 'Zeno's problem seems to come to the same thing. For it is natural that all being should be somewhere, and if there is a place for things, where would this place be? In some other place, and that in another, and so on indefinitely.'

Ibid. 236 v. Zeno's argument that when anything is in a space equal to itself, it is either in motion or at rest, and that nothing is moved in the present moment, and that the moving body is always in a space equal to itself at each present moment, may, I think, be put in a syllogism as follows: The arrow which is moving forward is at every present moment in a space equal to itself, accordingly it is in a space equal to itself in all time; but that which is in a space equal to itself in the present moment is not in motion. Accordingly it is in a state of rest, since it is not moved in the present moment, and that which is not moving is at rest, since everything is either in motion or at rest. So the arrow which is moving forward is at rest while it is moving forward, in every moment of its motion.

237 r. The Achilles argument is so named because Achilles is named in it as the example, and the argument shows that if he pursued a tortoise it would be impossible for him to overtake it. 255 r, Aristotle accordingly solves the problem of Zeno the Eleatic, which he propounded to Protagoras the Sophist.11 Tell me, Protagoras, said he, does one grain of millet make a noise when it falls, or does the **[Page 117]** ten-thousandth part of a grain? On receiving the answer that it does not, he went on: Does a measure of millet grains make a noise when it falls, or not? He answered, it does make a noise. Well, said Zeno, does not the statement about the measure of millet apply to the one grain and the ten-thousandth part of a grain? He assented, and Zeno continued, Are not the statements as to the noise the same in regard to each? For as are the things that make a noise, so are the noises. Since this is the case, if the measure of millet makes a noise, the one grain and the ten-thousandth part of a grain make a noise.

Zeno's arguments as described by Aristotle

Phys. iv. 1; 209 a 23. Zeno's problem demands some consideration; if all being is in some place, evidently there must be a place of this place, and so on indefinitely. 3; 210 b 22. It is not difficult to solve Zeno's problem, that if place is anything, it will be in some place; there is no reason why the first place should not be in something else, not however as in that place, but just as health exists in warm beings as a state while warmth exists in matter as a property of it. So it is not necessary to assume an indefinite series of places.

vi. 2; 233 a 21. (Time and space are continuous . . . the divisions of time and space are the same.) Accordingly Zeno's argument is erroneous, that it is not possible to traverse infinite spaces, or to come in contact with infinite spaces successively in a finite time. Both space and time can be called infinite in two ways, either absolutely as a continuous whole, or by division into the smallest parts. With infinites in point of quantity, it is not possible for anything to come in contact in a finite time, but it is possible in the case of the infinites **[Page 118]** reached by division, for time itself is infinite from this standpoint. So the result is that it traverses the infinite in an infinite, not a finite time, and that infinites, not finites, come in contact with infinites.

vi. 9 ; 239 b 5. And Zeno's reasoning is fallacious. For if, he says, everything is at rest [or in motion] when it is in a space equal to itself, and the moving body is always in the present moment then the moving arrow is still. This is false for time is not composed of present moments that are indivisible, nor indeed is any other quantity. Zeno presents four arguments concerning motion which involve puzzles to be solved, and the first of these shows that motion does not exist because the moving body must go half the distance before it goes the whole distance; of this we have spoken before (Phys. viii. 8; 263 a 5). And the second is called the Achilles argument; it is this: The slow runner will never be overtaken by the swiftest, for it is necessary that the pursuer should first reach the point from which the pursued started, so that necessarily the slower is always somewhat in advance. This argument is the same as the preceding, the only difference being that the distance is not divided each time into halves. . . . His opinion is false that the one in advance is not overtaken; he is not indeed overtaken while he is in advance; but nevertheless he is overtaken, if you will grant that he passes through the limited space. These are the first two arguments, and the third is the one that has been alluded to, that the arrow in its flight is stationary. This depends on the assumption that time is composed of present moments ; there will be no syllogism if this is not granted. And the fourth argument is with reference to equal bodies moving in opposite directions past equal bodies in the stadium with equal speed, some from the end of the stadium, others from **[Page 119]** the middle; in which case he thinks half the time equal to twice the time. The fallacy lies in the fact that while he postulates that bodies of equal size move forward with equal speed for an equal time, he compares the one with something in motion, the other with something at rest.

Passages relating to Zeno in the Doxographists

Plut. *Strom.* 6 ; *Dox.* 581. Zeno the Eleatic brought out nothing peculiar to himself, but he started farther difficulties about these things. Epiph. *adv.* Baer. iii. 11; *Dox.* 590. Zeno the Eleatic, a dialectician equal to the other Zeno, says that the earth does not move, and that no space is void of content. He speaks as follows:-That which is moved is moved in the place in which it is, or in the place in which it is not; it is neither moved in the place in which it is, nor in the place in which it is not ; accordingly it is not moved at all.

Galen, *Hist. Phil.* 3; Dox. 601. Zeno the Eleatic is said to have introduced the dialectic philosophy. 7 ; *Dox*. 604. He was a skeptic.

Aet. i. 7; *Dox*. 303. Melissos and Zeno say that the one is universal, and that it exists alone, eternal, and unlimited. And this one is necessity [Heeren inserts here the name Empedokles], and the material of it is the four elements, and the forms are strife and love. He says that the elements are gods, and the mixture of them is the world. The uniform will be resolved into them he thinks that souls are divine, and that pure men who share these things in a pure way are divine. 28; 320. Zeno et al. denied generation and destruc- tion, because they thought that the all is unmoved.

Zeno of Elea

by

John Burnet

Life

According to Apollodorus, Zeno flourished in 01. LXXIX. (464-460 B.C.). This date is arrived at by making him forty years younger than Parmenides, which is in direct conflict with the testimony of Plato. We have seen already (§ 84) that the meeting of Parmenides and Zeno with the young Socrates cannot well have occurred before 449 B.C., and Plato tells us that Zeno was at that time "nearly forty years old." He must, then, have been born about 489 B.C., some twenty-five years after Parmenides. He was the son of Teleutagoras, and the statement of Apollodorus that he had been adopted by Parmenides is only a misunderstanding of an expression of Plato's *Sophist*. He was, Plato further tells us, tall and of a graceful appearance.

Like Parmenides, Zeno played a part in the politics of his native city. Strabo, no doubt on the authority of Timaeus, ascribes to him some share of the credit for the good government of Elea, and says that he was a Pythagorean. This statement can easily be explained. Parmenides, we have seen, was originally a Pythagorean, and the school of Elea was naturally regarded as a mere branch of the larger society. We hear also that Zeno conspired against a tyrant, whose name is differently given, and the story of his courage under torture is often repeated, though with varying details.

Writings

Diogenes speaks of Zeno's "books," and Souidas gives some titles which probably come from the Alexandrian librarians through Hesychius of Miletus. In the *Parmenides* Plato makes Zeno say that the work by which he is best known was written in his youth and published against his will. As he is supposed to be forty years old at the time of the dialogue, this must mean that the book was written before 460 B.C., and it is very possible that he wrote others after it. If he wrote a work against the " philosophers," as Souidas says, that must mean the Pythagoreans, who, as we have seen, made use of the term in a sense of their own. The *Disputations* (*Erides*) and the *Treatise on Nature* may, or may not, be the same as the book described in Plato's *Parmenides*.

It is not likely that Zeno wrote dialogues, though certain references in Aristotle have been supposed to imply this. In the *Physics* we hear of an argument of Zeno's, that any part of a heap of millet makes a sound, and Simplicius illustrates this by quoting a passage from a dialogue between Zeno and Protagoras. If our chronology is right, it is quite possible that they may have met; but it is most unlikely that Zeno should have made himself a personage in a dialogue of his own. That was a later fashion. In another place Aristotle refers to a passage where "the answerer and Zeno the questioner" occurred, a reference which is most easily to be understood in the same way. Alcidamas seems to have written a dialogue in which Gorgias figured, and the exposition of Zeno's arguments in dialogue form must always have been a tempting exercise.

Plato gives us a clear idea of what Zeno's youthful work was like. It contained more than one "discourse," and these discourses were subdivided into sections, each dealing with some one presupposition of his adversaries. We owe the preservation of Zeno's arguments on the one and many to Simplicius. Those relating to motion have been preserved by Aristotle; but he has restated them in his own language.

Dialectic

Aristotle in his *Sophist* called Zeno the inventor of dialectic, and that, no doubt, is substantially true, though the beginnings at least of this method of arguing were contemporary with the foundation of the Eleatic school. Plato gives us a spirited account of the style and purpose of Zeno's book, which he puts into his own mouth:

In reality, this writing is a sort of reinforcement for the argument of Parmenides against those who try to turn it into ridicule on the ground that, if reality is one, the argument becomes involved in many absurdities and contradictions. This writing argues against those who uphold a Many, and gives them back as good and better than they gave; its aim is to show that their assumption of multiplicity will be involved in still more absurdities than the assumption of unity, if it is sufficiently worked out.

The method of Zeno was, in fact, to take one of his adversaries' fundamental postulates and deduce from it two contradictory conclusions. This is what Aristotle meant by calling him the inventor of dialectic, which is just the art of arguing, not from true premisses, but from premisses admitted by the other side. The theory of Parmenides had led to conclusions which contradicted the evidence of the senses, and Zeno's object was not to bring fresh proofs of the theory itself, but simply to show that his opponents' view led to contradictions of a precisely similar nature.

Zeno and Pythagoreanism

That Zeno's dialectic was mainly directed against the Pythagoreans is certainly suggested by Plato's statement, that it was addressed to the adversaries of Parmenides, who held that things were "a many." Zeller holds, indeed, that it was merely the popular form of the belief that things are many that Zeno set himself to confute; but it is surely not true that ordinary people believe things to be "a many" in the sense required. Plato tells us that the premisses of Zeno's arguments were the beliefs of the adversaries of Parmenides, and the postulate from which all his contradictions are derived is the view that space, and therefore body, is made up of a number of discrete units, which is just the Pythagorean doctrine, We know from Plato that Zeno's book was the work of his youth. It follows that he must have written it in Italy, and the Pythagoreans are the only people who can have criticized the views of Parmenides there and at that date.

It will be noted how much clearer the historical position of Zeno becomes if we follow Plato in assigning him to a later date than is usual. We have first Parmenides, then the pluralists, and then the criticism of Zeno. This, at any rate, seems to have been the view Aristotle took of the historical development.

What Is the Unit?

The polemic of Zeno is clearly directed in the first instance against a certain view of the unit. Eudemus, in his *Physics*, quoted from him the saying that "if any one could tell him what the unit was, he would be able to say what things are." The commentary of Alexander on this, preserved by Simplicius, is quite satisfactory. "As Eudemus relates," he says, "Zeno the disciple of Parmenides tried to show that it was impossible that things could be a many, seeing that there was no unit in things, whereas 'many' means a number of units." Here we have a clear reference to the Pythagorean view that everything may be reduced to a sum of units, which is what Zeno denied.

The Fragments

The fragments of Zeno himself also show that this was his line of argument. I give them according to the arrangement of Diels.

(1) If what is had no magnitude, it would not even be.... But, if it is, each one must have a certain magnitude and a certain thickness, and must be at a certain distance from another, and the same may be said of what is in front of it; for it, too, will have magnitude, and something will be in front of it. It is all the same to say this once and to say it always; for no such part of it will be the last, nor will one thing not be as compared with another. So, if things are a many, they must be both small and great, so small as not to have any magnitude at all, and so great as to be infinite. R. P. 134.

(2) For if it were added to any other thing it would not make it any larger; for nothing can gain in magnitude by the addition of what has no magnitude, and thus it follows at once that what was added was nothing. But if, when this is taken away from another thing, that thing is no less; and again, if, when it is added to another thing, that does not increase, it is plain that what was added was nothing, and what was taken away was nothing. R. P. 132.

(3) If things are a many, they must be just as many as they are, and neither more nor less. Now, if they are as many as they are, they will be finite in number.

If things are a many, they will be infinite in number; for there will always be other things between them, and others again between these. And so things are infinite in number. R. P. 133.

The Unit

If we hold that the unit has no magnitude -- and this is required by what Aristotle calls the argument from dichotomy, -- then everything must be infinitely small. Nothing made up of units without magnitude can itself have any magnitude. On the other hand, if we insist that the units of which things are built up are something and not nothing, we must hold that everything is infinitely great. The line is infinitely divisible; and, according to this view, it will be made up of an infinite number of units, each of which has some magnitude.

That this argument refers to points is proved by an instructive passage from Aristotle's *Metaphysics*. We read there --

If the unit is indivisible, it will, according to the proposition of Zeno, be nothing. That which neither makes anything larger by its addition to it, nor smaller by its subtraction from it, is not, he says, a real thing at all; for clearly what is real must be a magnitude. And, if it is a magnitude, it is corporeal; for that is corporeal which is in every dimension. The other things, i.e. the plane and the line, if added in one way will make things larger, added in another they will produce no effect; but the point and the unit cannot make things larger in any way.

From all this it seems impossible to draw any other conclusion than that the "one" against which Zeno argued was the "one" of which a number constitute a "many," and that is just the Pythagorean unit.

Space

Aristotle refers to an argument which seems to be directed against the Pythagorean doctrine of space, and Simplicius quotes it in this form:

If there is space, it will be in something; for all that is is in something, and what is in something is in space. So space will be in space, and this goes on *ad infinitum*, therefore there is no space. R. P. 135.

What Zeno is really arguing against here is the attempt to distinguish space from the body that occupies it. If we insist that body must be *in* space, then we must go on to ask what space itself is in. This is a "reinforcement" of the Parmenidean denial of the void. Possibly the argument that everything must be "in" something, or must have something beyond it, had been used against the Parmenidean theory of a finite sphere with nothing outside it.

Motion

Zeno's arguments on the subject of motion have been preserved by Aristotle himself. The system of Parmenides made all motion impossible, and his successors had been driven to abandon the monistic hypothesis in order to avoid this very consequence. Zeno does not bring any fresh proofs of the impossibility of motion; all he does is to show that a pluralist theory, such as the Pythagorean, is just as unable to explain it as was that of Parmenides. Looked at in this way, Zeno's arguments are no mere quibbles, but mark a great advance in the conception of quantity. They are as follows:

(1) You cannot cross a race-course. You cannot traverse an infinite number of points in a finite time. You must traverse the half of any given distance before you traverse the whole, and the half of that again before you can traverse it. This goes on *ad infinitum*, so that there are an infinite number of points in any given space, and you cannot touch an infinite number one by one in a finite time.

(2) Achilles will never overtake the tortoise. He must first reach the place from which the tortoise started. By that time the tortoise will have got some way ahead. Achilles must then make up that, and again the tortoise will be ahead. He is always coming nearer, but he never makes up to it.

The "hypothesis" of the second argument is the same as that of the first, namely, that the line is a series of points; but the reasoning is complicated by the introduction of another moving object. The difference, accordingly, is not a half every time, but diminishes in a constant ratio. Again, the first argument shows that, on this hypothesis, no moving object can ever traverse any distance at all, however fast it may move; the second emphasizes the fact that, however slowly it moves, it will traverse an infinite distance.

(3) The arrow in flight is at rest. For, if everything is at rest when it occupies a space equal to itself, and what is in flight at any given moment always occupies a space equal to itself, it cannot move.

Here a further complication is introduced. The moving object itself has length, and its successive positions are not points but lines. The first two arguments are intended to destroy the hypothesis that a line consists of an infinite number of indivisibles; this argument and the next deal with the hypothesis that it consists of a finite number of indivisibles.

(4) Half the time may be equal to double the time. Let us suppose three rows of bodies, one of which (A) is at rest while the other two (B, C) are moving with equal velocity in opposite directions (Fig. 1). By the time they are all in the same part of the course, B will have passed twice as many of the bodies in C as in A (Fig.2).

Therefore the time which it takes to pass C is twice as long as the time it takes to pass A. But the time which B and C take to reach the position of A is the same. Therefore double the time is equal to the half.

According to Aristotle, the paralogism here depends on the assumption that an equal magnitude moving with equal velocity must move for an equal time, whether the magnitude with which it is equal is at rest or in motion. That is certainly so, but we are not to suppose that this assumption is Zeno's own. The fourth argument is, in fact, related to the third just as the second is to the first. The Achilles adds a second moving point to the single moving point of the first argument; this argument adds a second moving line to the single moving line of the arrow in flight. The lines, however, are represented as a series of units, which is just how the Pythagoreans represented them; and it is quite true that, if lines are a sum of discrete units, and time is similarly a series of discrete moments, there is no other measure of motion possible than the number of units which each unit passes.

This argument, like the others, is intended to bring out the absurd conclusions which follow from the assumption that all quantity is discrete, and what Zeno has really done is to establish the conception of continuous quantity by a *reductio ad absurdum* of the other hypothesis. If we remember that Parmenides had asserted the one to be continuous (fr. 8), we shall see how accurate is the account of Zeno's method which Plato puts into the mouth of Socrates.

Paradoxes of Multiplicity and Motion

Kant's, Hume's, and Hegel's Solutions to Zeno's Paradoxes.

The Contemporary Solution to Zeno's Paradoxes.

Zeno was an Eleatic philosopher, a native of Elea (Velia) in Italy, son of Teleutagoras, and the favorite disciple of Parmenides. He was born about 488 BCE., and at the age of forty accompanied Parmenides to Athens. He appears to have resided some time at Athens, and is said to have unfolded his doctrines to people like Pericles and Callias for the price of 100 minae. Zeno is said to have taken part in the legislation of Parmenides, to the maintenance of which the citizens of Elea had pledged themselves every year by oath. His love of freedom is shown by the courage with which he exposed his life in order to deliver his native country from a tyrant. Whether he died in the attempt or survived the fall of the tyrant is a point on which the authorities vary. They also state the name of the tyranny differently. Zeno devoted all his energies to explain and develop the philosophical system of Parmenides. We learn from Plato that Zeno was twenty-five years younger than Parmenides, and he wrote his defense of Parmenides as a young man. Because only a few fragments of Zeno's writings have been found, most of what we know of Zeno comes from what Aristotle said about him in *Physics*, Book 6, chapter 9.

Zeno's contribution to Eleatic philosophy is entirely negative. He did not add anything positive to the teachings of Parmenides, but devoted himself to refuting the views of the opponents of Parmenides. Parmenides had taught that the world of sense is an illusion because it consists of motion (or change) and plurality (or multiplicity or the many). True Being is absolutely one; there is in it no plurality. True Being is absolutely static and unchangeable. Common sense says there is both motion and plurality. This is the Pythagorean notion of reality against which Zeno directed his arguments. Zeno showed that the common sense notion of reality leads to consequences at least as paradoxical as his master's.

Paradoxes of Multiplicity and Motion

Zeno's arguments can be classified into two groups. The first group contains paradoxes against multiplicity, and are directed to showing that the 'unlimited' or the continuous, cannot be composed of units however small and however many. There are two principal arguments:

1. If we assume that a line segment is composed of a multiplicity of points, then we can always bisect a line segment, and every bisection leaves us with a line segment that can itself be bisected. Continuing with the bisection process, we never come to a point, a stopping place, so a line cannot be composed of points.
2. The many, the line, must be both limited and unlimited in number of points. It must be limited because it is just as many (points) as it is, no more, and less. It is therefore, a definite number, and a definite number is a finite or limited number. However, the many must also be unlimited in number, for it is infinitely divisible. Therefore, it's contradictory to suppose a line is composed of a multiplicity of points.

The second group of Zeno's arguments concern motion. They introduce the element of time, and are directed to showing that time is no more a sum of moments than a line is a sum of points. There are four of these arguments:

1. If a thing moves from one point in space to another, it must first traverse half the distance. Before it can do that, it must traverse a half of the half, and so on ad infinitum. It must, therefore, pass through an infinite number of points, and that is impossible in a finite time.
2. In a race in which the tortoise has a head start, the swifter-running Achilles can never overtake the tortoise. Before he comes up to the

point at which the tortoise started, the tortoise will have got a little way, and so on ad infinitum.

3. The flying arrow is at rest. At any given moment it is in a space equal to its own length, and therefore is at rest at that moment. So, it's at rest at all moments. The sum of an infinite number of these positions of rest is not a motion.
4. Suppose there are three arrows. Arrow B is at rest. Suppose A moves to the right past B, and C moves to the left past B, at the same rate. Then A will move past C at twice the rate. This doubling would be contradictory if we were to assume that time and space are atomistic. To see the contradiction, consider this position as the chains of atoms pass each other:
   A1 A2 A3 ==>
   B1 B2 B3
   C1 C2 C3 <==
   Atom A1 is now lined up with C1, but an instant ago A3 was lined up with C1, and A1 was still two positions from C1. In that one unit of time, A2 must have passed C1 and lined up with C2. How did A2 have time for two different events (namely, passing C1 and lining up with C2) if it had only one unit of time available? It takes time to have an event, doesn't it?

Both groups of Zeno's arguments, those against multiplicity and those against motion, are variations of one argument that applies equally to space or time. For simplicity, we will consider it only in its spatial sense. Any quantity of space, say the space enclosed within a circle, must either be composed of ultimate indivisible units, or it must be divisible *ad infinitum*. If it is composed of indivisible units, these must have magnitude, and we are faced with the contradiction of a magnitude which cannot be divided. If it is divisible *ad infinitum*, we are faced with the contradiction of supposing that an infinite number of parts can be added up to make a merely finite sum.

Kant's, Hume's, and Hegel's Solutions to Zeno's Paradoxes

According to Kant, these contradictions are immanent in our conceptions of space and time, so space and time are not real. Space and time do not belong to things as they are in themselves, but rather to our way of looking at things. They are forms of our perception. It is our minds which impose space and time upon objects, and not objects which impose space and time upon our minds. Further, Kant drew from these contradictions the conclusion that to comprehend the infinite is beyond the capacity of human reason. He attempted to show that, wherever we try to think the infinite, whether the infinitely large or the infinitely small, we fall into irreconcilable contradictions.

As might be expected, many thinkers have looked for a way out of the paradoxes. Hume denied the infinite divisibility of space and time, and declared that they are composed of indivisible units having magnitude. But the difficulty that it is impossible to conceive of units having magnitude which are yet indivisible is not satisfactorily explained by Hume.

Hegel believed that any solution which is to be satisfactory must somehow make room for both sides of the contradiction. It will not do to deny one side or the other, to say that one is false and the other true. A true solution is only possible by rising above the level of the two antagonistic principles and taking them both up to the level of a higher conception, in which both opposites are reconciled. Hegel regarded Zeno's paradoxes as examples of the essential contradictory character of reason. All thought, all reason, for Hegel, contains immanent contradictions which it first posits and then reconciles in a higher unity, and this particular contradiction of infinite divisibility is reconciled in the higher notion of *quantity*. The notion of quantity contains two factors, namely the *one* and the *many*. Quantity means precisely a many in one, or a one in many. If, for example, we consider a quantity of anything, say a heap of wheat, this is, in the first place, one; it is one whole. Secondly, it is many, for it is composed of many parts. As one it is continuous; as many it is discrete. Now the true notion of quantity is not one, apart form many, nor many apart from one. It is the synthesis of both. It is a many *in* one. The antinomy we are considering arises from considering one side of the truth in a false abstraction from the other. To conceive unity as not being in itself multiplicity, or multiplicity as not being unity, is a false abstraction. The thought of the one involves the thought of the many, and the thought of the many involves the thought of the one. You cannot have a many without a one, any more than you can have one end of a stick without the other.

Now, if we consider anything which is quantitatively measured, such as a straight line, we may consider it, in the first place, as one. In that case it is a continuous divisible unit. Next we may regard it as many, in which case it falls into parts. Now each of these parts may again be regarded as one, and as such is an indivisible unit; and again each part may be regarded as many, in which case it falls into further parts; and this alternating process may go on for ever. This is the view of the matter which gives rise to Zeno's contradictions. But it is a false view. It involves the false abstraction of first regarding the many as something that has reality apart from the one, and then regarding the one as something that has reality apart from the many. If you persist in saying that the line is simply one and not many, then there arises the theory of indivisible units. If you persist in saying it is simply many and not one, then it is divisible *ad infinitum*. But the truth is that it is neither simply many nor simply one; it is a many *in* one, that is, it is a *quantity*. Both sides of the contradiction are, therefore, in one sense true, for each is a factor of the truth. But both sides are also false, if and in so far as, each sets itself up as the whole truth.

The Contemporary Solution to Zeno's Paradoxes

Kant's, Hume's and Hegel's solutions to the paradoxes have been very stimulating to subsequent thinkers, but ultimately have not been accepted.

There is now general agreement among mathematicians, physicists and philosophers of science on what revisions are necessary in order to escape the contradictions discovered by Zeno's fruitful paradoxes. The concepts of space, time, and motion have to be radically changed, and so do the mathematical concepts of line, number, measure, and sum of a series. Zeno's integers have to be replaced by the contemporary notion of real numbers. The new one-dimensional continuum, the standard model of the real numbers under their natural (less-than) order, is a radically different line than what Zeno was imagining. The new line is now the basis for the scientist's notion of distance in space and duration through time. The line is no longer a sum of points, as Zeno supposed, but a set-theoretic union of a non-denumerably infinite number of unit sets of points. Only in this way can we make sense of higher dimensional objects such as the one-dimensional line and the two-dimensional plane being composed of zero-dimensional points, for, as Zeno knew, a simple sum of even an infinity of zeros would never total more than zero. The points in a line are so densely packed that no point is next to any other point. Between any two there is a third, all the way 'down.' The infinity of points in the line is much larger than any infinity Zeno could have imagined. The non-denumerable infinity of real numbers (and thus of points in space and of events in time) is much larger than the merely denumerable infinity of integers. Also, the sum of an infinite series of numbers can now have a finite sum, unlike in Zeno's day. With all these changes, mathematicians and scientists can say that all of Zeno's arguments are based on what are now false assumptions and that no Zeno-like paradoxes can be created within modern math and science. Achilles catches his tortoise, the flying arrow moves, and it's possible to go to an infinite number of places in a finite time, without contradiction.

No single person can be credited with having shown how to solve Zeno's paradoxes. There have been essential contributions starting from the calculus of Newton and Leibniz and ending at the beginning of the twentieth century with the mathematical advances of Cauchy, Weierstrass, Dedekind, Cantor, Einstein, and Lebesque.

Philosophically, the single greatest contribution was to replace a reliance on what humans can imagine with a reliance on creating logically consistent mathematical concepts that can promote quantitative science.

Zeno's Paradox of the Race Course

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1. The Paradox

Zeno argues that it is impossible for a runner to traverse a race course. His reason is that

"motion is impossible, because an object in motion must reach the half-way point before it gets to the end" (Aristotle, *Physics* 239b11-13).

Why is this a problem? Because the same argument can be made about **half** of the race course: it can be divided in half in the same way that the entire race course can be divided in half. And so can the half of the half of the half, and so on, *ad infinitum*.

So a crucial assumption that Zeno makes is that of **infinite divisibility**: the distance from the starting point (*S*) to the goal (*G*) can be divided into an infinite number of parts.

2. Progressive vs. Regressive versions

How did Zeno mean to divide the race course? That is, **which half** of the race course Zeno mean to be dividing in half? Was he saying (a) that before you reach *G*, you must reach the point halfway from the halfway point to *G*? This is the **progressive** version of the argument: the subdivisions are made on the right-hand side, the goal side, of the race-course.

Or was he saying (b) that before you reach the halfway point, you must reach the point halfway from *S* to the halfway point? This is the **regressive** version of the argument: the subdivisions are made on the left-hand side, the starting point side, of the race-course.

If he meant (a), the progressive version, then he was arguing that the runner could not **finish** the race. If he meant (b), the regressive version, then he was arguing that the runner could not even **start** the race. Either conclusion is repugnant to reason and common sense, and it seems impossible to ascertain which version Zeno had in mind.

But it turns out that it really doesn't matter which version Zeno had in mind. For although this may not be obvious, the conclusions of the two versions of the argument are **equivalent**. Let us see why.

Since Zeno was generalizing about **all** motion, his conclusion was either (a) that **no** motion could be completed or (b) that **no** motion could be begun. But in order to begin a motion, one has to complete a smaller motion that is a part of it. For consider any motion, *m*, and suppose that *m* has been begun. It follows that some smaller initial portion of *m* has been completed; for if no such part of *m* has been completed, *m* could not have yet begun. Hence, if no motion can be completed, then none can be begun.

It is even more obvious that if no motion can be begun, then none can be completed. So the conclusion of (a) ("no motion can be completed") entails, and is entailed by, the conclusion of (b) ("no motion can be begun"). That is, the two conclusions are logically **equivalent**. Hence we needn't worry about how Zeno wanted to place the halfway points.

3. Terminology

R	the runner
S	the starting point (= Z_0)
G	the end point
Z_1	the point halfway between S and G
Z_2	the point halfway between Z_1 and G
Z_n	the point halfway between Z_{n-1} and G

Z-run	a run that takes the runner from one *Z*-point to the next *Z*-point

4. Zeno's Argument formulated

1. In order to get from *S* to *G*, *R* must make infinitely many *Z*-runs.
2. It is impossible for *R* to make infinitely many *Z*-runs.
3. Therefore, it is impossible for *R* to reach *G*.

5. Evaluating the argument

a. Is it **valid**? ***Yes***: the conclusion follows from the premises.

b. Is it **sound**? I.e., is it a valid argument with **true premises**? This is what is at issue.

c. One might try to object to the first premise, (1), on the grounds that one can get from *S* to *G* by making one run, or two (from *S* to Z_1 and from Z_1 to *G*). But this is not an adequate response. For according to the definitions above, the runner, if he passes from *S* to *G*, will have passed through all the *Z*-points. But to do that is to make all the *Z*-runs.

Alternatively, one might object to (1) on the grounds that passing through all the *Z*-points (even though there are infinitely many of them) does not constitute making an infinite number of *Z*-runs. The reason might be that after you keep halving and halving the distance, you eventually get to distances that are so small that they are no larger than points. But points have no dimension, so no "run" is needed to "cross" one. But this is a mistake. For **every** *Z*-run, no matter how tiny, covers a finite distance (>0). No *Z*-run is as small as a point.

So we have established that the first premise is **true**. (Note: this does not establish that *R* can actually get from *S* to *G*. It only establishes that **if** he does, he will make all the *Z*-runs.)

d. The crucial premise is (2). Why can't *R* make infinitely many *Z*-runs? Our difficulty here is that Zeno gives no explicit argument in support of (2).

1. Supporting the second premise

There are three possible reasons that might be given in support of (2).

a. To make all the *Z*-runs *R* would have to run **infinitely far.**
b. To make all the *Z*-runs *R* would have to run **forever** (i.e., for an infinite length of time).
c. To make all the *Z*-runs *R* would have to do something it is **logically impossible** to do. (I.e., the claim that *R* makes all the *Z*-runs leads to a logical contradiction.)

Which of these reasons did Zeno have in mind? Aristotle assumed that (b) was what Zeno intended (and he based his refutation on that assumption). More recent critics have suggested that Zeno's argument can be made much more interesting if we use (c) to support his second premise. We will consider both (b) and (c) later. But since there is some reason to think that Zeno believed (a), we will begin there.

To see why one might think that Zeno had (a) in mind, we will examine a related argument that he actually gave: his argument against plurality. We will then return to the race course.

Zeno: Argument against Plurality

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1. Introduction

The argument is contained in **4**=B1 and **3**=B2 (from Simplicius' commentary on Aristotle's *Physics*). But there is a problem with the text, and some of the argument is garbled or lost. Fortunately, we can reconstruct it. Zeno attempts to show that the assumption that **there are many things** leads to a contradiction: viz., that each thing is *both* infinitely small and infinitely large.

There are two limbs to the argument. The pluralist's assumption, "There are many things," leads to these two conclusions:

A. Each thing is "so small as not to have size."
B. Each thing is "so large as to be unlimited."

Simplicius's text does not preserve (A) completely. It starts with (A), and then is garbled and switches over to (B). But we can reconstruct the argument for (B).

2. The Argument

Simplicius (in **4**=B1) preserves one key principle ("if it exists, each thing must have some size and thickness"). It is a premise that Zeno thinks his materialist/pluralist opponents must accept. **3**=B2 contains an argument in support of this principle ("Suppose that *x* has no size. Then when *x* is added to a thing it does not increase the size of that thing, and when *x* is subtracted from a thing, that thing does not decrease in size. Clearly, *x* is nothing, i.e., does not exist."). So the argument begins with this premise:

1. What exists has size (magnitude).

Zeno also seems to be making the following two assumtions:

2. What has size can be divided into (proper) **parts** that exist.
3. The *part of* relation is **transitive**, **irreflexive**, and **asymmetical**.

> **Proper** parts: *x* is a proper part of *y* iff *x* is a part of *y* and *y* is not a part of *x*
>
> **Transitive**: if *x* is a part of *y* and *y* is a part of *z*, then *x* is a part of *z*.
>
> **Irreflexive**: *x* is not a part of *x*.
>
> **Asymmetical**: if *x* is a part of *y*, then *y* is not a part of *x*.

4. The rest of his argument is preserved in **4**=B1. Roughly paraphrased, it runs:
5. Pick any existing physical object, *x*.
6. *x* has size. [from 1 and 4]
7. *x* has parts. [from 2 and 5]
8. Let *x'* be one of those parts; then *x'* "must be apart from the rest" of *x*. That is, one part of *x* must **protrude**, or "be in front" of the rest of *x*, as Zeno goes on to say.

Now Zeno says that the same argument applies to *x'*!

9. So some part of *x'* (call it *x''*) protrudes from the rest of *x'*, and so on, *ad infinitum*.

> Since Zeno is assuming, reasonably enough, that the *part of* relation is transitive (i.e., that the parts of the parts of *x* are also parts of *x*) it follows that *x* is composed of an infinite number of parts (since *x'*, *x''*, *x'''*, etc., *ad infinitum*, are all parts of *x*).

10. So *x* has infinitely many parts. [from 8 and 3]

Zeno immediately infers that such an object (with infinitely many parts) must be infinitely large.

11. So *x* is infinitely large. [from 9 and ?]

1. Evaluation of the argument

Everything is fine up to step (9). But (9) does not entail (10). Zeno seems to be implicitly assuming (what I'll call) the **Infinite Sum Principle**: viz., that **the sum of an infinite number of terms is infinitely large**. (9), together with the Infinite Sum Principle, entails (10). And from (10) it follows (by Universal Generalization) that **every** magnitude is infinitely large, which is the conclusion of the second limb.

The Infinite Sum Principle appears to be correct. But is it? What makes it *seem* correct is the observation that you can make something as large (a finite size) as you want out of parts as small as you want, and it takes only a finite number of them to do this! To see that this is so, consider the following: pick any magnitude, *y*, as large as you like; and pick any small magnitude, *z*, as small as you like (but $z > 0$). It is obvious that you can obtain a magnitude at least as large as *y* by adding *z* to itself a *finite* number of times. That is:

"*y* "*z* $*x* (*x* · *z* ≥ *y*)

For every *y* and for every *z*, there is at least one *x* such that *x* times *z* is greater than or equal to *y*.

No matter how small *z* is, if you have enough things of at least that magnitude (but still only finitely many) you get a *total* magnitude at least as large as *y*. So, the reasoning goes, if you had an infinite number of *z*'s, you'd get an infinitely large sum.

This may seem convincing, but it doesn't support the Infinite Sum Principle. For this argument has been assuming that of our infinitely many parts, there is a **smallest**. (More precisely, there is one than which none is smaller.) What our argument actually supports is only an **amended** version of the Infinite Sum Principle.

The sum of an infinite number of terms, **one of which is the smallest**, is infinite.

This amended principle is **true**. But it won't help Zeno. For in his series there is no smallest term! That is, x' is smaller than x, and x'' is smaller than x', etc. We have an infinite series of continually decreasing terms. And the sum of such a series may be finite.

E.g.: 1/2 + 1/4 + 1/8 + . . .+ . . .= 1.

[If this last point seems puzzling, you need to learn a little about infinite sequences, limits of infinite sequences, infinite series, and sums of infinite series. Please take a moment to study the mathematical background to Zeno's paradoxes.]

2. Review

Zeno's argument is based on two principles:

- Infinite Divisibility Principle
- Infinite Sum Principle

He gives a compelling argument for the first, but does not even mention the second. From these he infers his conclusion that **every magnitude is infinitely large**.

This argument is **valid**, but **unsound**. For the Infinite Sum Principle is false.

We can fix the Infinite Sum Principle by restricting it to infinite sets with smallest elements. The amended principle is true, and so the resulting argument's premises are both true. But this amended argument is **invalid**. For the amended principle requires that there be **smallest** parts, and the Infinite Divisibility Principle does not guarantee that there are such parts - it allows the parts to get smaller and smaller, *ad infinitum*.

We can make Zeno's argument valid, but then one of its premises is false. Or we can make both of its premises true, but then it is invalid. Either way, Zeno's argument is unsound.

The Race Course: Part 2

1. Our look at the plurality argument suggests that Zeno may have thought that to run all the *Z*-runs would be to run a distance that is **infinitely long**. If this is what he thought, he was mistaken.

   The reason the sum of all the *Z*-intervals is not an infinitely large distance is that there is no smallest *Z*-interval. And Zeno does not establish that there is some smallest *Z*-run. (If there were a smallest *Z*-run, he wouldn't have been able to show that R had to make infinitely many *Z*-runs.)

2. What about Aristotle's understanding of Zeno? Here is what he says [RAGP **8**]:

   "*Zeno's argument makes a false assumption when it asserts that it is impossible to traverse an infinite number of positions or to make an infinite number of contacts one by one* ***in a finite time***" (*Physics* 233a21-24).

3. Aristotle points out that there are two ways in which a quantity can be said to be infinite: in **extension** or in **divisibility**. The race course is infinite in divisibility. But, Aristotle goes on, "the time is also infinite in this respect."

   Hence, there is a sense in which *R* has an infinite number of distances to cross. But in that sense he also has an infinite amount of time to do it in. (A finite distance is infinitely divisible, then why isn't a finite time also infinitely divisible?)

4. So Zeno cannot establish (2) for either of the first two reasons we considered: to make all the *Z*-runs, *R* does not have to run infinitely far. Nor does *R* have to keep running forever.

Logical Impossibility: Infinity Machines & Super-Tasks

1. On this reading, Zeno's argument attempts to show that it is **logically impossible** for *R* to reach *G*. That is, Zeno's puzzle is not

that the runner has to run **too far**, or that the runner has to run for **too long a time**, but that the claim that the runner has completed all the *Z*-runs **leads to a contradiction**.

2. Following James Thomson ["Tasks and Super-Tasks," on reserve], let us define a **super-task** as an infinite sequence of tasks. Can one perform a super-task? Bertrand Russell thought that one could, as Thomson explains ["Tasks and Super-Tasks," p. 93]:

   "Russell suggested that a man's skill in performing operations of some kind might increase so fast that he was able to perform each of an infinite sequence of operations after the first in half the time he had required for its predecessor. Then the time required for all of the infinite sequence of tasks would be only twice that required for the first. On the strength of this Russell said that the performance of all of an infinite sequence of tasks was only medically impossible."

   But Thomson argues that to assume that a super-task has been performed in accordance with Russell's "recipe" leads to a logical contradiction.

   a. Thomson's Lamp example ["Tasks and Super-Tasks," pp. 94-95]:

      "There are certain reading lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button the lamp goes off. So if the lamp was originally off, and you pressed the button an odd number of times, the lamp is on, and if you pressed the button an even number of times the lamp is off. Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next half minute, and so on, according to Russell's recipe. After I have completed the whole infinite sequence of jabs, i.e. at the end of the two minutes, is the lamp on or off? It seems impossible to answer this question. It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction."

b. Applying this to the race course ["Tasks and Super-Tasks," pp. 97-98. By 'Z' here Thomson means the set of Z-points.]:

"... suppose someone could have occupied every Z-point without having occupied any point external to Z. Where would he be? Not at any Z-point, for then there would be an unoccupied Z-point to the right. Not, for the same reason, between Z-points. And, ex hypothesi, not at any point external to Z. But these possibilities are exhaustive. The absurdity of having occupied all the Z-points without having occupied any point external to Z is exactly like the absurdity of having pressed the lamp-switch an infinite number of times...."

3. This gives us an argument that can be set out like this:
   a. Suppose *R* makes all the *Z*-runs.
   b. Then *R* cannot be to the left of *G*. [Reason: if *R* is to the left of *G*, there are still *Z*-points between *R* and *G*, and so not all of the *Z*-runs have been made.]
   c. So *R* has reached *G*.
   d. But, since no *Z*-run reaches *G*, *R* has not reached *G*.

   Since (a) leads to a contradiction [(c) contradicts (d)], the argument continues, it is logically impossible for (a) to be true. Therefore,

   e. It is impossible for *R* to make all the *Z*-runs.
4. Does the argument work? There are two parts:

   i. Does (a) "*R* makes all the *Z*-runs" entail (c) "*R* reaches *G*"?
   ii. Does (a) "*R* makes all the *Z*-runs" entail (d) "*R* does not reach *G*"?

   It turns out that (as Paul Benacerraf has shown, see "Tasks, Super-tasks, and the Modern Eleatics," on reserve) neither of these entailments holds.

1. We'll start with (ii). The reason for supposing that *R* does not reach *G* is that **no *Z*-run reaches** *G*. So we must be assuming (as Thomson actually says) that *R* makes all the *Z*-runs **and no others**. So now we ask: how can *R* reach *G* if the only runs he makes are *Z*-runs, and no *Z*-run reaches *G*? How can one run to *G* without making a run that reaches *G*?

   Now it appears that what leads to a contradiction is the assumption that *R* **makes all the *Z*-runs and no others**. This allows for two possible replies to Zeno.

a. The weak reply: Zeno is entitled to assume that *R* makes all the *Z*-runs. But he is **not** entitled to assume that *R* makes all the *Z*-runs **and no others**. So he doesn't get his contradiction.

b. A stronger reply (Benacerraf): we cannot derive a contradiction even from the assumption that *R* makes all the *Z*-runs **and no others**.

1. Benacerraf's key claim: **From a description of the *Z*-series, nothing follows about any point outside the *Z*-series.**

   We can apply this point to both the **lamp** and the **race course**:

a. **The lamp**: Nothing about the state of the lamp after two minutes follows from a description of the lamp's behavior during the two-minute interval when the super-task was being performed. It does **not** follow that the lamp is on; it does **not** follow that the lamp is off. It could be either.

b. **The race course**: Nothing about whether and when the runner reaches *G* follows from the assumption that he has made all the *Z*-runs and no others.

2. This is because *G* is the **limit** point of the infinite sequence of *Z*-points. It is not itself a *Z*-point. If we assume that the runner makes all the *Z*-runs and no other runs, we have the following options about *G*. It can be either:

a. The **last** point *R* reaches, or

b. The **first** point *R* does not reach.

It must be one of these, but it does not have to be both. Benacerraf explains why ("Tasks, Super-Tasks and the Modern Eleatics," p. 117-118):

"... any point may be seen as dividing its line either into (a) the sets of points to the right of and including it, and the set of points to the left of it; or into (b) the set of points to the right of it and the set of points to the left of and including it: That is, we may assimilate each point to its right-hand segment (a) or to its left-hand segment (b). Which we choose is entirely arbitrary ..."

Consequently, **both** of the following situations are possible:

c. *R* makes all the *Z*-runs and no others, and reaches *G*.

d. *R* makes all the *Z*-runs and no others, and does not reach *G*.

All that "*R* makes all the *Z*-runs and no others" entails is that *R* reaches **every** point to the **left** of *G*, and **no** point to the **right** of *G*. It entails nothing about whether ***G* itself** is one of the points reached or one of the points not reached.

The difference between Thomson and Benacerraf can be put as follows. Let '*Z*' abbreviate '*R* makes all the *Z*-runs' and '*G*' abbreviate '*R* reaches *G*'. Then Thomson's claim is that *Z* entails *G* and *Z* also entails ¬*G*; so *Z* entails a contradiction, and is therefore logically impossible. Whereas Benacerraf replies that *Z* does **not** entail *G* and *Z* does **not** entail ¬*G*; hence Thomson has not shown that *Z* entails a contradiction.

3. Consider Benacerraf's vanishing genie: suppose the runner is a genie who **vanishes** as soon as he makes all the *Z*-runs. There is a temptation to say that there must be a **last** point he reaches before he vanishes. And that would have to be *G*. So how is it possible for him to make all the *Z*-runs without reaching *G*?

Benacerraf gives us a beautiful illustration of this possibility by adding one new wrinkle — a shrinking genie: ["Tasks, Super-Tasks and the Modern Eleatics," p. 119]:

"Ours is a reluctant genie. He shrinks from the thought of reaching 1. In fact, being a rational genie, he shows his repugnance against reaching 1 by shrinking so that the ratio of his height at any point to his height at the beginning of the race is always equal to the ratio of the unrun portion of the course to the whole course, He is full grown at 0, half-shrunk at ½, only 1/8 of him is left at 7/8, etc. His instructions are to continue in this way and to disappear at 1. Clearly, now, he occupied every point to the left of 1 (I can tell you exactly when and how tall he was at that point), but he did not occupy 1 (if he followed instructions, there was nothing left of him at 1). Of course, if we must say that he vanished at a point, it must be at 1 that we must say that he vanished, but in this case, there is no temptation whatever to say that he occupied 1. He couldn't have. There wasn't enough left of him."

4. The mistake in Thomson's argument (which tries to show that a contradiction can be derived from the assumption that the runner makes all the *Z*-runs and no others) is to assume that one and the same point, *G*, has to be **both** the last one that *R* reaches and the first one that he doesn't reach.

   But this assumption is mistaken. *G* divides the space *R* traverses from the space that he does not traverse. But *G* itself cannot be said to belong to **both** spaces (even though it is arbitrary which of the two we associate it with). Indeed, if there is such a thing as the last point *R* (or anyone) reaches, then there **cannot** be a first point that he does not reach.

   The reason is that (as Zeno is assuming) space is a **continuum**; points in space do not have next-door neighbors. There is no **next** point after *G*. Therefore, if *G* is **last** point *R* reaches, then there is no **first** point *R* does not reach. Consequently, *G* cannot be that point. So Thomson's argument fails.

Movement through a continuum, through infinitely divisible space, is indeed a puzzling phenomenon. But it does not lead to Zeno's paradox.

Zeno's Paradox of the Arrow

A reconstruction of the argument

(following Aristotle, *Physics* 239b5-7 = RAGP 10):

1. When the arrow is in a place just its own size, it's at rest.
2. At every moment of its flight, the arrow is in a place just its own size.
3. Therefore, at every moment of its flight, the arrow is at rest.

Aristotle's solution

- The argument falsely assumes that time is composed of "nows" (i.e., indivisible instants).
- There is no such thing as motion (or rest) "in the now" (i.e., at an instant).

Weakness in Aristotle's solution: it seems to deny the possibility of motion or rest "at an instant." But instantaneous velocity is a useful and important concept in physics:

The velocity of *x* at instant *t* can be defined as the limit of the sequence of *x*'s average velocities for increasingly small intervals of time containing *t*.

In this case, we can reply that if Zeno's argument exclusively concerns (durationless) instants of time, the first premise is false: "*x* is in a place just the size of *x* at instant *i*" entails neither that *x* is resting at *i* nor that *x* is moving at *i*.

Perhaps instants and intervals are being confused

"When?" can mean either "at what instant?" (as in "When did the concert begin?") or "during what interval?" (as in "When did you read *War and Peace*?").

1a. At every **instant** at which the arrow is in a place just its own size, it's at rest. (*false*)
2a. At every **instant** during its flight, the arrow is in a place just its own size. (*true*)

1b. During every **interval** throughout which the arrow stays in a place just its own size, it's at rest. (*true*)
2b. During every **interval** of time within its flight, the arrow occupies a place just its own size. (*false*)

Both versions of Zeno's premises above yield an unsound argument: in each there is a false premise: the first premise is false in the "instant" version (1a); the second is false in the "interval" version (2b). And the two true premises, (1b) and (2a), yield no conclusion.

A final reconstruction

In this version there is no confusion between instants and intervals. Rather, there is a fallacy that logic students will recognize as the "quantifier switch" fallacy. The universal quantifier, "at every instant," ranges over instants of time; the existential quantifier, "there is a place," ranges over locations at which the arrow might be found. **The order in which these quantifiers occur makes a difference!** (To find out more about the order of quantifiers, click here.) Observe what happens when their order gets illegitimately switched:

1c. If there is a place just the size of the arrow at which it is located at every instant between t_0 and t_1, the arrow is at rest throughout the interval between t_0 and t_1.
2c. At every instant between t_0 and t_1, there is a place just the size of the arrow at which it is located.

We will use the following abbreviations:

L(*p*, *i*)	The arrow is located at place *p* at instant *i*
R	The arrow is at rest throughout the interval between t_0 and t_1

The argument then looks like this:

1c. If there is a *p* such that for every *i*, *L*(*p*, *i*), then *R*.
$*p* "*i* *L*(*p*, *i*) ® *R*
2c. For every *i*, there is a *p* such that: *L*(*p*, *i*).
"*i* $*p* *L*(*p*, *i*)

But (2c) is not equivalent to, and does not entail, the antecedent of (1c):

There is a *p* such that for every *i*, *L*(*p*, *i*)
$***p "i L(p, i)***

The reason they are not equivalent is that the order of the quantifiers is different. (2c) says that the arrow always has some location or other ("at every instant *i* it is located at some place *p*") - and that is trivially true as long as the arrow exists! But the antecedent of (1c) says there is some location such that the arrow is always located **there** ("there is some place *p* at which it is located at every instant *i*") - and that will only be true provided the arrow does not move!

So one cannot infer from (1c) and (2c) that the arrow is at rest.

The Arrow and Atomism

Although the argument does not succeed in showing that motion is impossible, it does raise a special difficulty for proponents of an **atomic conception** of space. For an application of the Arrow Paradox to atomism, click here.

Zeno's Paradoxes: A Timely Solution

Peter Lynds

Zeno of Elea's motion and infinity paradoxes, excluding the Stadium, are stated (1), commented on (2), and their historical proposed solutions then discussed (3). Their correct solution, based on recent conclusions in physics associated with time and classical and quantum mechanics, and in particular, of there being a necessary trade off of all precisely determined physical values at a time (including relative position), for their continuity through time, is then explained (4). This article follows on from another, more physics orientated and widely encompassing paper entitled —Time and Classical and Quantum Mechanics: Indeterminacy vs. Discontinuity" (Lynds, 2003), with its intention being to detail the correct solution to Zeno's paradoxes more fully by presently focusing on them alone. If any difficulties are encountered in understanding any aspects of the physics underpinning the following contents, it is suggested that readers refer to the original paper for a more in depth coverage.

1. The Problems

2. General Comment

3. Their Historical Proposed Solutions

4. Zeno's Paradoxes: A Timely Solution

    (a) Time and Mechanics: Indeterminacy vs. Discontinuity
    (b) Einstein's Train
    (c) The solution 2500 years later

5. Closing Comment

1. The Problems

Achilles and the Tortoise

Suppose the swift Greek warrior Achilles is to run a race with a tortoise. Because the tortoise is the slower of the two, he is allowed to begin at a point some distance ahead. Once the race has started however, Achilles can never overtake his opponent. For to do so, he must first reach the point from where the tortoise began. But by the time Achilles reaches that point, the tortoise will have advanced further yet. It is obvious, Zeno maintains, that the series is never ending: there will always be some distance, however small, between the two contestants. More specifically, it is impossible for Achilles to preform an infinite number of acts in a finite time.

Distance behind the Tortoise: 5, 2.5, 1.25, 0.625, 0.3125, 0.015625, ...

Time: 1, 0.5, 0.25, 0.125, 0.625, 0.03125, ...

The Dichotomy

It is not possible to complete any journey, because in order to do so, you must firstly travel half the distance to your goal, and then half the remaining distance, and again of what remains, and so on. However close you get to the place you want to go, there is always some distance left. Furthermore, it is not even possible to get started. After all, before the second half of the distance can be travelled, one must cover the first half. But before that distance can be travelled, the first quarter must be completed, and before that can be done, one must traverse the first eight, and so on, and so on to infinitum.

Distance: 1, 1.5, 1.75, 1.875, 1.9375, 1.96885, 1.984425, ...

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Time: 1, 1.5, 1.75, 1.875, 1.9375, 1.96885, 1.984425, ... or Distance: 2, 1, 0.5, 0.25, 0.125, 0.625, 0.03125, 0.015425, ... Time: 2, 1, 0.5, 0.25, 0.125, 0.625, 0.03125, 0.015425, ...

William James' version of the Dichotomy Time can never pass, as to do so it is necessary for some time interval to go by, say 60 seconds. But before the 60 seconds, half of that, or 30 seconds, must firstly pass. But before that, a half of that time

must firstly pass, and so on, and so to infinitum. 60 s, but first 30 s, but first 15 s, but first 7.5 s, but first 3.75 s, but first 1.875 s, but first .... or Time: 30, 45, 52.5, 56.25, 58.125, 59.0625, 59.531225, 59.765612, ...

G. J. Whitrow's version of the Dichotomy A bouncing ball that reaches three quarters of its former height on each bounce, will bounce an infinite number of times, in the same way that distances and times decrease in the Dichotomy. The only difference is that Whitrow uses a factor of three-quarters where Zeno used one half. It also doesn't however matter what fraction is used. The only thing that would change if the balls initial velocity and the distance from the floor of the first bounce remained the same, would be the time in which an

infinite numbers of bounces took place. Height of bounce: 1, 0.75, 0.5625, 0.421875, 0.3164062, 0.2373046, ... Time: 1, 0.75, 0.5625, 0.421875, 0.3164062, 0.2373046, ... or Height of bounce: 1, 0.25, 0.0625, 0.015625, 0.0039062, 0.0009765, ... Time: 1, 0.25, 0.0625, 0.015625, 0.0039062, 0.0009765, ...

The Arrow

All motion is impossible, since at any given instant in time an apparently moving body (the arrow) occupies just one block of space. Since it can occupy no more than one block of space at a time, it must be stationary at that instant. The arrow cannot therefore ever be in motion as at each and every instant it is frozen still.

2. General Comment

It is doubtful that with his paradoxes, Zeno was attempting to argue that motion was impossible, as is sometimes claimed. Zeno would of known full well that in the cases of Achilles and the Tortoise and the Dichotomy (dichotomy in this relation meaning arithmetical or geometrical division), that the respective body apparently in motion would inevitably reach and pass the said impossible boundary in every day settings. Pointing this out does not refute Zeno's argument, as Diogenes the Cynic is apocryphally reputed to have thought he'd done by getting up and walking away. Rather, Zeno is saying through the use of dialectic and by showing that an idea results in contradiction, that an infinite series of acts cannot be completed in finite period of time. If we choose not to believe this we must demonstrate where the fallacy lies and how it is possible. As such, instead of being arguments against the possibility of motion, the paradoxes are critiques of our underlying assumptions regarding the idea of continuous motion in an infinitely divisible space and time. It is the same with the Arrow paradox. We of course know that motion and physical continuity are possible and an obvious feature of nature, so there has to be something wrong with the initial assumptions regarding the paradoxes. But what?

Although Zeno's paradoxes may at first seem like whimsical little puzzles and as though they could be quite easily disposed of without much thought and effort, they show themselves to be immeasurably subtle and profound, as Bertrand Russell once characterised them, when examined in detail, and over the centuries mathematicians, philosophers and physicists have continually argued about them at great length. These people can be divided into two camps: those that think there is no real problem, and those who believe that Zeno's paradoxes have not yet been solved (Morris, 1997).

3. Their Historical Proposed Solutions

Of Zeno's paradoxes, the Arrow is typically treated as a different problem to the others. In fact, all of the paradoxes are usually thought to be quite different problems, involving different proposed solutions, if only slightly, as is often the case with the Dichotomy and Achilles and the Tortoise, with the differentiation being that the first is thought to be expressed in terms of absolute motion, where as the second shows that the same argument applies to relative motion. Although it is not important to the argument, or its possible solution, this is actually incorrect, as any motion necessarily requires relative motion and that a body's position is changing in relation to something else. Therefore, like the paradox of Achilles and the Tortoise, the Dichotomy also involves relative motion, as its position is purported to change over time: in this case, presumably relative to a hypothetical fixed point on earth.

It is usually claimed that the Arrow paradox is resolved by either of two different lines of thought. Firstly, by way of a vague connection to special relativity, where it is argued:

> —The theory of special relativity answers Zeno's concern over the lack of an instantaneous difference between a moving and a non-moving arrow by positing a fundamental re-structuring of the basic way in which space and time fit together, such that there really *is* an instantaneous difference between a moving and a non-moving object, in so far as it makes sense to speak of "an instant" of a physical system with mutually moving elements. Objects in relative motion have different planes of simultaneity, with all the familiar relativistic consequences, so not only does a moving object look different to the world, but the world looks different to a moving object." [2]

However, such arguments are often asserted by those who don't seem to entirely understand relativity and/or its mathematical formalisation, and the reasoning underpinning them is usually of a non-descriptive nature. Indeed, it is difficult to see how special relativity is relevant to the problem at all.

The more popular and common proposed solution is that the arrow, although not in motion at any one instant, when it's trajectory is traced out, it can be seen to be move because it occupies different locations at different times. In other words, although not in motion at any one instant, the arrow is in motion at *all* instants in time (an infinite number of them), so is never at rest. This conclusion stems from calculus and continuous functions (as emphasised by Weierstrass and the —at-at theory of motion"), by pointing out that although the *value* of a function f(t) is constant for a given t, the *function* f(t) may be non-constant at t. Recently, some potential problems with the at-at theory have been noted and revolve around the question of whether it is compatible with instantaneous velocity. [3] Another proposed solution to the Arrow paradox is to deny instantaneous velocities altogether.[4 *]

[2] See, *Zeno and the Paradox of Motion*, by Kevin Brown. www.mathpages.com/home/iphysics.html [3] See, Frank Arntzenius, —Are there really instantaneous Velocities?". *The Monist* , vol 83, no 2, (2000). [4] Albert, D. *Time and Chance*. Chp. 1. Harvard University Press, (2000). For responses to Albert, see David Malament's, —On the Time Reversal Invariance of Classical Electromagnetic Theory" (forthcoming *in Stud. Hist. Phil. Mod. Phys*).

The paradoxes of Achilles and the Tortoise and the Dichotomy are often thought to be solved through calculus and the summation of an infinite series of progressively small time intervals and distances, so that the time taken for Achilles to reach his goal (overtake the Tortoise), or to traverse the said distance in the Dichotomy, is in fact, finite. The faulty logic in Zeno's argument is often seen to be the assumption that the sum of an infinite number of numbers is always infinite, when in fact, an infinite sum, for instance, $1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots$, can be mathematically shown to be equal to a finite number, or in this case, equal to 2.

This type of series is known as a geometric series. A geometric series is a series that begins with one term and then each successive term is found by multiplying the previous term by some fixed amount, say x. For the above series, x is equal to 1/2. Infinite geometric series are known to converge (sum to a finite number) when the multiplicative factor x is less than one. Both the distance to be traversed and the time taken to do so can be expressed as an infinite geometric series with x less than one. So, the body in apparent motion traverses an infinite number of "distance intervals" before reaching the said goal, but because the "distance intervals" are decreasing geometrically, the total distance that it traverses before reaching that point is not infinite. Similarly, it takes an infinite number of time intervals for the body to reach its said goal, but the sum of these time intervals is a finite amount of time.

So, for the above example, with an initial distance of say 10 m, we have,

$t = 1 + 1/2 + 1/2^2 + 1/2^3 + \ldots + 1/2^n$ Difference $= 10/2^n\ m$

Now we want to take the limit as n goes to infinity to find out when the distance between the body in apparent motion and its said goal is zero. If we define

$$S_n = 1 + 1/2 + 1/2^2 + 1/2^3 + \ldots + 1/2^n$$

then, divide by 2 and subtract the two expressions:

$$2n+1$$

$$S_n - 1/2\, S_n = 1 - 1/$$

or equivalently, solve for S_n:

$S\,n = 2\,(\,1 - 1\,/\,2^{\,n+1})$

So that now S n is a simple sequence, for which we know how to take limits. From the last expression it is clear that:

$\lim S\,n = 2$

as *n* approaches infinity.

Therefore, Zeno's infinitely many subdivisions of any distance to be traversed can be mathematically reassembled to give the desired finite answer.

A much simpler calculation not involving infinitely many numbers gives the same result:

For the Dichotomy:

• A body traverses 10 metres per second, so covers 20 meters in 2 seconds

*

Although correct to question the validly of instantaneous velocity, as we shall see shortly, the real answer to its possible plausibility comes from a different and much more direct source. Furthermore, rather than just being a question of instantaneous velocity, the same applies to the rest of physics and all instantaneous physical values and magnitudes.

For Achilles and the Tortoise:

- Achilles runs 10 metres per second, so covers 20 metres in 2 seconds
- The tortoise runs 5 metres per second, and has an advantage of 10 metres. Therefore, he also reaches the 20 metre mark after 2 seconds

4. Zeno's Paradoxes: A Timely Solution

The way in which calculus is often used to solve Achilles and the Tortoise and the Dichotomy through the summation of an infinite series by employing the mathematical techniques developed by Cauchy, Weierstrass, Dedekind and Cantor, certainly provides the correct answer in a strictly mathematical sense by giving up the desired numbers at the end of calculation. It is obviously dependent however on an object in motion having a precisely defined position at each given instant in time. As we will see shortly, this isn't representative of how nature works. Moreover, the summation of an infinite series here works as a helpful mathematical tool that produces the correct numerical answer by getting rid of the infinities, but it doesn't actually solve the paradoxes and show how the body's motion *is actually possible*. The same fault applies to the Arrow paradoxes proposed solution via Weierstrass' ―at-at theory of motion", as a continuous function is a *static* and completed indivisible mathematical entity, so by invoking this model we are essentially agreeing that physical motion does not truly exist, and is just some sort of strange subjective illusion. Furthermore, the above proposed solution also problematically posits the existence of an infinite succession of instants underlying a body's motion. In his book, *Zeno's paradoxes*, Wesley C. Salmon discusses the proposed functional solution:

> ―A function is a pairing of elements of two (not necessarily distinct) classes, the do-definition, if motion is a functional relation between time and position, then motion consists solely of the pairing of times with positions. Motion consists not of traversing an infinitesimal distance in an infinitesimal time (before Cauchy's definition of the derivative as certain limit, the derivative was widely regarded as a ratio of infinitesimal quantities. The use of the derivative to represent velocity thus implied that physical motion over a finite distance is compounded out of infinitesimal movements over infinitesimal distances during infinitesimal time spans); it consists of the occupation

> of a unique position at each given instant of time. This conception has been appropriately dubbed "the at-at theory of motion." The question, how does an object get from one point to another, does not arise. Thus Russel was led to remark, —Weierstrass, by strictly banishing all infinitesimals, has at last shown that we live in an unchanging world, and that the arrow, at every moment of its flight, is truly at rest. The only point where Zeno probably erred was in inferring (if he did infer) that, because there is no change, therefore the world must be in the same state at one time as at another. This consequence by no means follows..." [5]

What doesn't seem to be realised is that in all of the paradoxes (and proposed solutions to them), it is taken for granted that a body in relative motion has a determined and defined relative position at any given instant, and indeed, that there is an instant in time underlying a body's motion, whether it be an actual physical feature of time itself, and/or a meaningful and precise physical indicator at which the position of a body in motion would be determined, and as such, not constantly changing.

(a). Time and Mechanics: Indeterminacy vs. Discontinuity

Time enters mechanics as a measure of interval, relative to the clock completing the measurement. Conversely, although it is generally not realized, in *all* cases a time value indicates an interval of time, rather than a precise static instant in time at which the relative position of a body in relative motion or a specific physical magnitude would theoretically be precisely determined. For example, if two separate events are measured to take place at either 1 hour or 10.00 seconds, these two values indicate the

[5] For a collection of papers on this matter, and others relating to the paradoxes, see, *Zeno's Paradoxes.* W. C. Salmon (ed). Bobbs-Merrill, New York, (1970).

events occurred during the time intervals of 1 and 1.99999...hours and 10.00 and 10.009999...seconds, respectively. If a time measurement is made smaller and more accurate, the value comes closer to an accurate measure of an interval in time and the corresponding parameter and boundary of a specific physical magnitudes potential measurement during that interval, whether it be relative position, momentum, energy or other. Regardless of how small and accurate the value is made however, it cannot indicate a precise static instant in time at which a value would theoretically be precisely determined, because there is not a precise static instant in time underlying a dynamical physical process. If there were, the relative position of a body in relative motion or a specific physical magnitude, although precisely determined at such a precise static instant, it would also by way of logical necessity be frozen static at that precise static instant. Furthermore, events and all physical magnitudes would remain frozen static, as such a precise static instant in time would remain frozen static at the same precise static instant: motion would not be possible. (Incidentally, the same outcome would also result if such a precise static instant were hypothetically followed by a continuous sequence of further precise static instants in time, as by its very nature, a precise static instant in time does not have duration over interval in time, so neither could a further succession of them. This scenario is not plausible however in the first instance, as the notion of a *continuous progression* of precise static instants in time is obviously not possible for the same reason). Rather than facilitating motion and physical continuity, this would perpetuate a constant precise static instant in time, and as is the very nature of this ethereal notion i.e. a physical process frozen static at an "instant' as though stuck on pause or freeze frame on a motion screen, physical continuity is not possible if such a discontinuous chronological feature is an intrinsic property of a dynamical physical process, and as such, a meaningful (and actual physical) indicator of a time at which the relative position of a body in relative motion or a certain physical magnitude is precisely determined as has historically been assumed. That is, it is the human observer who subjectively projects and assigns a precise instant in time upon a physical process, for example, in order to gain a meaningful subjective picture or "mental snapshot' of the relative position of a body in relative motion. [6]

It might also be contended in a more philosophical sense that a general definition of static would entitle a certain physical magnitude as being *unchanging* for an extended interval of time. But if this is so, how then could time itself be said to be frozen static at a precise instant if to do so also demands it must be unchanging for an extended interval of time? As a general and sensible definition this is no doubt correct, as we live in a world where indeed there is interval in time, and so for a certain physical magnitude to be static and unchanging it would naturally also have to remain so for an extended duration, however short. There is something of a paradox here however. If there were a precise static instant underlying a dynamical physical process, everything, including clocks and watches would also be frozen static and discontinuous, and as such, interval in time would not be possible either. There could be no interval in time for a certain physical magnitude to remain unchanging. Thus this general definition of static breaks down when the notion of static is applied to time itself. We are so then forced to search for a revised definition of static for this special temporal case. This is done by qualifying the use of stasis in this particular circumstance by noting static and unchanging, with static and unchanging as not being over interval, as there could be no interval and nothing could change in the first instance. At the same time however, it should also be enough just to be able to recognize and acknowledge the fault and paradox in the definition when applied to time.

It might also be argued by analogy with the claim by some people that the so-called 'block universe model', i.e. a 4-dimensional model of physical reality incorporating time as well as space, is static or unchanging. This claim however involves the common mistake of failing to recognize that unless there is *another* time dimension, it simply doesn't make sense to say that the block universe is static, for there is no 'external' time interval over which it remains the same. If we then apply the same line of reasoning to the hypothetical case being discussed presently, we could say: It doesn't make sense to say that everything would be static at an instant, (with physical continuity and interval in time not being possible), as there would be no time interval for such an assertion to be relative to, referenced from, or over which such an instant would remain the same etc. This objection is valid. However, as it applies

[6] In a 1942 paper, Zeno of Elea's Attacks on Plurality, *Amer. J. Philology* **63,** 1-25; 193-206, H. Frankel hinted towards this same conclusion: —The human mind, when trying to give itself an accurate account of motion, finds itself confronted with two aspects of the phenomenon. Both are inevitable but at the same time they are mutually exclusive. Either we look at the continuous flow of motion; then it will be impossible for us to think of the object in any particular position. Or we think of the object as occupying any of the positions through which its course is leading it; and while fixing our thought on that particular position we cannot help fixing the object itself and putting it at rest for one short instant."

to the hypothetical case under investigation, it should also be clear that it is not any more applicable or relevant than being a semantical problem of the words one employs to best try to put across a point and as being a contradiction in terms, rather than pertaining to any contradiction in the actual (in this case, hypothetical) physics involved. One could certainly also assert that there were no interval in time, and so if one wishes, there were a precise static instant underlying a physical process, without it being dependent on there actually being interval: as is the case with the hypothetical absence of mass and energy, and the resulting absence of 3 spatial dimensions. [7]

(b). Einstein's Train

The absence of a precise static instant in time underlying a dynamical physical process means that a body (micro and macroscopic) in relative motion does not have a precisely determined relative position at any time.

The reason why can be demonstrated by employing Albert Einstein‘s famous 1905 train and the other theoretical device it is associated with, the thought experiment. An observer is watching a train traveling by containing a young Albert Einstein. At any given time as measured by a clock held by the observer, Einstein‘s train is in motion. If the observer measures the train to pass a precisely designated point on the track at 10.00 seconds, this value indicates the train passes this point during the measured time interval of 10.00 and 10.00999...seconds. As Einstein‘s train is in motion at all measured times, regardless of how great or small its velocity and how small the measured time interval

(i.e. 10.0000000-10.0000000999...seconds), Einstein‘s train does not have a precisely determined relative position to the track at any time, because it is not stationary at any time while in motion, for to have a precisely determined relative position at any time, the train would also need to be stationary relative to the track at that time. Conversely, the train does not have a precisely determined relative position at an ethereal precise static instant in time, because there is not a precise static instant in time underlying the train‘s motion. If there were, Einstein‘s trains motion would not be possible.

As the time interval measurement is made smaller and more accurate, the corresponding position the train can be said to "occupy‘ during that interval can also be made smaller and more accurate. Momentarily forgetting L_P, T_P and time keeping restrictions, these measurements could hypothetically

be made almost infinitesimally small, but the train does not have a precisely determined position at any time as it is in motion at all times, regardless of how small the time interval. For example, at 100km/hr,

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during the interval of 10s Einstein‘s train traverses the distance of 2.7cm. Thus, it is exactly due to the train not having a precisely determined relative position to the track at any time, whether during a time interval, however small, or at a precise static instant in time, that enables Einstein's train to be in motion. Moreover, this is not associated with the preciseness of the measurement, a question of re-normalizing infinitesimals or the result of quantum uncertainty, as the trains precise relative position is not to be gained by applying infinitely small measurements, nor is it smeared away by quantum considerations. It simply does not have one. There is a very significant and important difference.

If a photograph is taken (or any other method is employed) to provide a precise measurement of the trains relative position to the track, in this case it does appear to have a precisely determined relative position to the track in the picture, and although it may also be an extremely accurate measure of the time interval during which the train *passes* this position or a designated point on the track, the imposed time measurement itself is in a sense arbitrary (i.e. 0.000000001 second, 1 second, 1 hour etc), as it is impossible to provide a time at which the train is precisely in such a position, as it is not precisely in that or any other precise position at any time. If it were, Einstein's train would not, and could not be in motion.

On a microscopic scale, due to inherent molecular, atomic and subatomic motion and resulting kinetic energy, the particles that constitute the photograph, the train, the tracks, the light radiation that propagates from the train to the camera, as well as any measuring apparatus e.g. electron microscope, clock, yardstick etc, also do not have precisely determined relative position's at any time. Naturally, bodies at rest in a given inertial reference frame, which are not constituted by further smaller particles in relative motion, have a precisely defined relative position at all measured times. However, as this hypothetical special case is relevant to only indivisible and the most fundamental of particles, whose existence as independent "massive' objects is presently discredited by quantum physics and the intrinsic "smearing' effects of wave-particle duality and quantum entanglement, if consistent with these

[7] Please see Lynds (2003) for considerations regarding the resulting negation of the notion of a flowing and physically progressive time, and the reason for natures exclusion if it.

considerations, this special subatomic case would not appear to be applicable. Furthermore, and crucially, because once granted indeterminacy in precise relative position of a body in relative motion, also subsequently means indeterminacy in *all* precise physical magnitudes, including gravity, this also applies to the very structure of space-time, the dynamic framework in which all inertial spatial and temporal judgments of relative position are based.[8] As such, the previously mentioned possible special case isn't actually one, and the very same applies.

The only situation in which a physical magnitude would be precisely determined was if there *were* a precise static instant in time underlying a dynamical physical process and as a consequence a physical system were frozen static at that instant. In such a system an indivisible mathematical time value, e.g. 2s, would correctly represent a precise static instant in time, rather than an interval in time (as it is generally assumed to in the context of calculus, and traceable back to the likes of Galileo, and more specifically, Newton, thus guaranteeing absolute preciseness in theoretical calculations before the fact

i.e. $\Delta d/\Delta t=v$). Fortunately this is not the case, as this static frame would include the entire universe. Moreover, the universe's initial existence and progression through time would not be possible. Thankfully, it seems nature has wisely traded certainty for continuity.

(c). The Solution over 2500 Years Later

To return to Zeno's paradoxes, the solution to *all* of the mentioned paradoxes then,[9] is that there isn't an instant in time underlying the body's motion (if there were, it couldn't be in motion), and as its position is constantly changing no matter how small the time interval, and as such, is at no time determined, it simply doesn't have a determined position. In the case of the Arrow paradox, there isn't an instant in time underlying the arrows motion at which it's volume would occupy just —one block of space", and as its position is *constantly* changing in respect to time as a result, the arrow is never static and motionless. The paradoxes of Achilles and the Tortoise and the Dichotomy are also resolved through this realisation: when the apparently moving body's associated position and time values are fractionally dissected in the paradoxes, an infinite regression can then be mathematically induced, and resultantly, the idea of motion and physical continuity shown to yield contradiction, as such values are not representative of times at which a body is in that specific precise position, but rather, at which it is *passing* through them. The body's relative position is constantly changing in respect to time, so it is never *in that* position at any time. Indeed, and again, it is the very fact that there isn't a static instant in time underlying the motion of a body, and that is doesn't have a determined position at any time while in motion, that allows it to be in motion in the first instance. Moreover, the associated temporal (*t*) and spatial (*d*) values (and thus, velocity and the derivation of the rest of physics) are just an *imposed static* (and in a sense, arbitrary) *backdrop*, of which in the case of motion, a body *passes* by *or through* while in motion, but has no inherent and intrinsic relation. For example, a time value of either 1 s or 0.001 s (which indicate the time intervals of 1 and 1.99999....s, and 0.001 and 0.00199999.... s, respectively), is never indicative of a time at which a body's position might be determined while in motion, but rather, if measured accurately, is a representation of the interval in time during which the body *passes* through a certain distance interval, say either 1 m or 0.001 m (which indicate the distance intervals of 1 and 1.99999....m, and 0.001 and 0.0019999....m, respectively). Therefore, the more simple proposed solution mentioned earlier to Achilles and the Tortoise and the Dichotomy by applying velocity to the particular body in motion, also fails as it presupposes that a specific body has precisely determined

[8] For further detail, please see Lynds (2003). [9] Zeno conceived another paradox, often referred to as the Stadium or Moving Rows. Unlike the paradoxes of Achilles and the Tortoise, the Dichotomy, the Arrow, and their variations however, the stadium is a completely different type of problem. It is usually stated as follows: Consider three rows of bodies, each composed of an equal number of bodies of equal size. One is stationary, while the other two pass each other as they travel with equal velocity in opposite directions. Thus, half a time is equal to the whole time. Although its exact details, and so also its interpretation, remain controversial, the paradox is generally thought to be a question of relative velocity, and to be addressed through reasoning underpinning Einstein's 1905 theory of special relativity. I would suggest however that, if the argument is to be accepted as it has been set forward above, it doesn't actually pose a paradox (and that Special Relativity has no direct relevance to it either), but rather that Zeno has failed to recognise that the time taken for the each moving row to pass the other would be *half* the time required to pass a row of the same length if it were stationary, rather than being (in any sense) equal, which in some ways, is the intuitive view. That is, Zeno couldn't decide if the time required was equal or a half, as both intuitively seemed to make equal sense.

position at a given time, thus guaranteeing absolute preciseness in *theoretical* calculations before the fact i.e. $\Delta d/\Delta t=v$. That is, a body in motion simply doesn't have a determined position at any time, as at no time is its position not changing, so it also doesn't have a determined velocity at any time.

Lastly, and to complete the mentioned paradoxes, William James' variation on the Dichotomy is resolved through the same reasoning and the realisation of the absence of a instant in time at which such an indivisible mathematical time value would theoretically be determined and static at that instant, and not constantly changing. That is, interval *as represented by a clock or a watch* (as distinct from an absent actual physical progression or flow of time) is *constantly* increasing, whether or not the time value as indicated by the particular time keeping instrument remains the same for a certain extended period i.e. at no time is a time value anything other than an interval in time and it is never a precise static instant in time as it assumed to be in the paradoxes.

5. Closing Comment

To close, the correct solution to Zeno's motion and infinity paradoxes, excluding the Stadium, have been set forward, just less than 2500 years after Zeno originally conceived them. In doing so we have gained insights into the nature of time and physical continuity, classical and quantum mechanics, physical indeterminacy, and turned an assumption which has historically been taken to be a given in physics, determined physical magnitude, including relative position, on its head. From this one might infer that we've been a bit slow on the uptake, considering it has taken us so long to reach these conclusions. I don't think this is the case however. Rather that, in respect to an instant in time, it is hardly surprising considering the extreme difficulty of seeing through something that one actually *sees* and *thinks* with. Moreover, that with his deceivingly profound and perplexing paradoxes, the Greek philosopher Zeno of Elea was a true visionary, and in a sense, over 2500 years ahead of his time.

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A Critique of Recent Claims of a Solution to Zeno's Paradoxes

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Abstract

In a recently published paper it is concluded that there is a necessary trade off of all precisely determined physical values at a time for their continuity in time. This conclusion was based on the premise that there is not a precise instant in time underlying a continuous dynamical physical process. Based on the conclusion stated above, it was further asserted that three of Zeno's paradoxes were solved. In the short critique following it is demonstrated that the conclusions in the paper were due to a *non sequitur* fallacy made in the reasoning employed. Causality issues found in the conclusion made are also explored. Both the conclusion and alleged solutions to Zeno's paradoxes are then termed invalid.

Introduction

Back at the time of Sir Isaac Newton, many believed that gravity obeyed some form of an inverse square law and some hand waving arguments were made in a way of proof. Newton's remarkable accomplishment was not only in stating a set of consistent laws for the motion of particles but also in discovering the mathematics that enabled him to derive the law of gravitation. Even so, Newton's accomplishment was challenged to the point that he had to tackle the inverse problem of gravitation, which involved proving that his law had as one of its solution the orbits of the planets as observed and documented by Kepler and other astronomers. Newton was successful in proving the inverse problem by inventing the calculus of variations and showing that a conic section was the solution of his equations for gravity. That was an extraordinary achievement of a genius mind. (It must be made clear here for later reference that one must not assume that because a law was derived from the analysis of some orbits will necessarily have as its solution those orbits. This is simply because an hypothesis, or even error, made during the derivation can result in it generating a wrong solution.)

All that was taking place back in the time of Newton and Leibniz, when critical thinking, philosophy and science were at an elevated and delicate balance, and *verba volant, scripta manent* meant something.

A Non Sequitur: Indeterminacy vs. Discontinuity

In a recently published paper[i], an attempt is made to prove that continuity implies indeterminacy, insists that there is no precise instant in time for a physical magnitude in continuous motion to be determined exactly. According to Lynds then

Continuity implies Indeterminacy (1)

Even if one accepts the argument made by Lynds using the concept of precise instants in time to assert the truth of conditional 1, this is something that has been known for a long time now. The concept has been debated since the time of ancient philosophers, later by Newton, Leibniz and Berkeley and in modern times placed into a rigid mathematical framework by great minds such as Cantor and Robinson with the Continuum Hypothesis and Non-Standard Analysis, respectively[ii]. But let us assume that the argument made by Lynds is acceptable and (1) is true.

Lynds then asserts that because of indeterminacy there is a trade-off of all precise physical values at any instant of time for their continuity in time. This can be described in short as:

Indeterminacy implies Continuity (2)

This new conditional, (2), is the inverse problem to be demonstrated, as in Newton's case discussed above. That is, given that continuity is deduced from the phenomena, one must prove that such continuity is the effect of indeterminacy. The conditional (2) cannot be simply deduced from conditional (1). It is known that if *A implies B* is true, then the conditional *B implies A* is not necessarily true, except in the case that A is a necessary and sufficient condition for B, that is A and B have an *equivalence* relationship. Deducing (2) from (1) alone is a *non sequitur* unless one can show that the cause of continuity is indeterminacy, in addition to showing that the cause of indeterminacy is continuity. Thus, an equivalence relation must be demonstrated. However, Lynds seems to assume that because he has demonstrated that if continuity is present then indeterminacy results, it follows that if it is assumed indeterminacy is present, then that implies continuity must also be present. However, the second part, the inverse problem, is the hard part to prove and it is not even touched in the paper.

The intricacies of a method used to demonstrate indeterminacy may be the reason why it may not imply continuity. The proof of the existence of indeterminacy was based on the assumption that there is not a precise instant in time for physical values to be determined. But what about if there really is one and we just cannot prove it? In that case, the implication (1) is wrong and nothing can be said about (2). Furthermore, even if there is a precise instant in time, indeterminacy can still be present for other reasons not explored in the arguments. This is why a proof of the inverse problem is necessary for a complete demonstration of the argument.

Stretched Causality

Lynds, using arguments employing the concept of a precise instant in time, has only demonstrated the truth of conditional (1). He does not demonstrate the truth of conditional (2) because he does not establish how indeterminacy facilitates continuity but resorts in some type of circular argumentation involving Zeno's paradoxes and a trade off. Furthermore, the proposition:

$$\text{Indeterminacy} \equiv \text{Continuity} \qquad (3)$$

is an equivalence relation arising form (1) and (2), and besides being a bold statement it establishes a form of a physical law with stretched causality. Specifically, if motion is continuous, it causes indeterminacy, and concurrently, the effect of indeterminacy is continuity via some undefined process, termed a trade off of physical values by Lynds. The concurrent nature of cause and effect suggests that one of the two notions must have a mathematical use only, if one must avoid a causality violation. One should always be very critical when a physical law involves stretched causality. In the case of Newton's second law, the famous equation $\mathbf{F} = m\mathbf{a}$, the skepticism was overlooked in the face of the predictive power and consistency of his system of laws. In the case of (3), one can hardly think of any predictive capacity useful in a better understanding of motion.

Zeno rests in peace

As relating to Zeno's paradoxes, Lynds seems to have a different comprehension of the arrow paradox than most. The arrow paradox is about the assertion of Zeno that the phenomenology of motion implies an illusion. Specifically, Zeno claimed that an arrow in motion cannot be distinguished from an arrow at rest or at another place in its path and therefore, if there is nothing deduced directly from the phenomena about the motion of an arrow, motion is an illusion. Zeno was not particularly concerned whether precise instants in time can be defined or whether something was traded-off for the arrow to be in motion. His paradox described a concern about the concept of motion at a higher level than that encountered in the dichotomy paradox. In the dichotomy paradox Zeno claimed that motion is impossible and not only that a goal will never be reached as Lynds and others misinterpret. Under some interpretations of the dichotomy paradox, motion cannot even commence. In the arrow paradox, in addition to the impossibility of motion, Zeno claims that motion is an illusion. Of course, the arrow paradox was later used to attack discrete atomism and to claim that if there is a void between adjacent atoms, motion cannot take place simply because there is no medium for it. However, Zeno seems to have considered that a trivial conclusion, as discrete atomism was an easy target and he concentrated on attacking pluralism.

A solution to Zeno's paradoxes is not possible just by proving (1), since for the most part those paradoxes deal with continuity issues, which Lynds assumes to exist in the first place. By introducing indeterminacy, a claim for a solution to Zeno's paradoxes is a clear *non sequitur*.

Zeno's paradoxes are not about the existence of precise instants in time and precise physical values, or some type of a trade off, but about the impossibility and illusion of motion in general. Nevertheless, despite the errors in the paper by Lynds, the positive side effect is a revived interest in an ancient paradox that is still unanswered. Current physics cannot answer Zeno's paradoxes and a bolder step is required than the one attempted by Lynds in order to obtain a solution. Zeno's paradoxes are not logical puzzles and a solution to them has not been offered by modern physics. The paradoxes challenge the naive Pythagorean perceptions of space-time and a complete solution would require a revision of physics and cosmology, not just analysis based on simple mechanics concepts, which result in the paradoxes being valid in the first place.

Conclusion

Lynds commits a *non sequitur* but this is not to say that (2) is necessarily false. It just says that Lynds has failed to prove it in a scientifically accepted way, either deductively or inductively. If (3) is true in a macroscopic world, the foundations of Newtonian mechanics are shaken and so are those of General Relativity.

The alternative is then an extension of Quantum Mechanics principles in the macroscopic world and a total revision of classical mechanics. As such, although wrong in his analysis and argumentation, Lynds may have provided the stimulation for investigating the viability of such a revision, something lacking at the moment empirical support but being a very popular speculation in science fiction and metaphysics. However, there is not anything new in such thinking or approach. It does not provide any definite breakthrough, in terms of a quantitative law, to serve as a basis for any extension or challenge to classical mechanics applications in a macroscopic world.

Therefore, nothing new was said in the paper by Lynds, whilst what was concluded was the result of a fallacious argumentation.

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