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MAURICE ALLAIS

MARKETS, RISK AND MONEY

Essays in Honor of Maurice Allais

edited by

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TABLE OF CONTENTS

Introduction, by Bertrand R. Munier ix

Chapter 1: Fifty Years of Maurice Allais' Economic Writings: Seeds for Renewal in Contemporary Economic Thought, by Bertrand R. Munier 1

Chapter 2: Maurice Allais, a Belatedly Recognized Genius, by Thierry de Montbrial 51

Chapter 3: Self-Organizing Markets, by Jacques Lesourne 59

Chapter 4: A Theory of Spatial General Equilibrium in a Fuzzy Economy, by Claude Ponsard 91

Chapter 5: Maurice Allais, Unrecognized Pioneer of Overlapping Generations Models, by Edmond Malinvaud 111

Chapter 6: Some Possible Uses of Household Assets Accounts in National Accounting: The Case of American Households, by André Babeau 129

Chapter 7: Should We Get Rid of Economic Calculus?, by Marcel Boiteux 155

Chapter 8: Economics as Psychology: A Cognitive Assay of the French and American Schools of Risk Theory, by Lola L. Lopes 165

Chapter 9: On Maurice Allais' and Ole Hagen's *Expected Utility Hypotheses and the Allais Paradox: Contemporary Discussions of Decisions under Uncertainty with Allais' Rejoinder*. 'Rational' Decision Making versus 'Rational' Decision Modelling?, by Mark J. Machina 179

Chapter 10: Probability of Ruin: A Useful Alternative to the Expected Utility Hypothesis in Firm Decision Making, by James B. Ramsey 195

Chapter 11: Complexity and Strategic Decision under Uncertainty. How Can We Adapt the Theory?, by Bertrand R. Munier 209

Chapter 12: The Generalized Irreversibility Effect, by Xavier Freixas 233

Chapter 13: Comparison of Two Decision-Aid Models Applied to a Nuclear Power Plant Siting Example, by B. Roy and D. Bouyssou 249

Chapter 14: Risk in Serial Conflicts, Evolution, Intelligence and Chaos, by Werner Leinfellner 279

Chapter 15: Money, Credit and Monetary Dynamics in Maurice Allais' Work, by Jean-Jacques Durand 303

Biographical Data 325

Bibliography 1943–1991 329

BERTRAND R. MUNIER

INTRODUCTION

> Maurice Allais is a fountain of original and independent discoveries ... Had Allais' earliest writings been in English, a generation of economic theory would have taken a different course.
>
> Paul Samuelson*

When the Nobel Prize was awarded to Maurice Allais many people, even among the economists, wondered whether a single contribution like the 'Allais Paradox', as remarkable as it may be, could be sufficient to justify such a recognition. No better proof could have been given of the fact that Allais' writings were little known, with the exception of the quoted piece!

This book helps in the first place to bridge this gap. It provides keys to the thinking and the scientific environment of Maurice Allais, under the form of essays on subjects related to Allais' thought, as well as a general and precise survey of his writings, a complete bibliography and detailed biographical indications. The authors who have written this volume are best known in the international scientific community and belong to Allais' 'network' of scientific relations or friends; in France primarily, but also throughout the world.

Writing in 1986 in a related book in French,[1] I ventured to say that Maurice Allais had already produced "... a scientific work about which one can wonder, by comparison, why it has escaped the attention of the Nobel Committee". This was indeed – at the time – a rather strong statement, one which you do not make about just any author, however interesting and valuable his contributions ... But I had come to the conclusion that the economic writings of Maurice Allais had until then been underestimated, as well as too quickly discarded. They needed a thorough recognition, based on a serious analysis, in France as well as among the international community.

As Maurice Allais has meanwhile been awarded the 1988 Prize in Economic Science in memory of Alfred Nobel, there might be some

B. Munier (ed.), Markets, Risk and Money, ix–xii.

questioning about the usefulness of the present book, which appears several years later. I shall take up this view and show that it does not stand serious investigation.

Although the fame of the so-called Allais Paradox and of one or two other contributions has spread considerably since 1988, the deep theoretical appraisal of economics on which Allais' contributions rest remains little known to a large part of the professional economists themselves, to most analysts of social and economic activity, whether in business or in governmental agencies, as well as to graduate students in economics, psychology and political science. Even the specific works which have been singled out by the Nobel committee as the basis for its decision to award the 1988 Prize contain numerous contributions which remain, indeed, either attributed to other authors, or still unexplored, whether regarding macroeconomic equilibrium, inflation and the evaluation of the credit mechanism, for example, or regarding the role of the collective memorizing of economic activity and the implications this may have for the microeconomic design of market models and for economic policy. Yet today's world is full of problems and dangers about which Allais' writings have something to say.

But Allais' writings are not all easily accessible, particularly to the wide English-speaking community of economists. It was thus necessary to provide an account in English of Allais' economic thought, to show how it differs from the traditional as well as the most recent conventional models. This volume contains a set of contributions in English which should make it easy to gain a more precise insight into the inventive and specific appraisal by Maurice Allais of the economic and social spheres of action. Allais' writings are in this respect a deeply and widely extending mine of potential discovery, to the extent where Allais' intuitions and art of modeling are widely distanced from the conventional ones.

The reader will then determine what could be done to try solving the difficulties of the modern world in ways which would differ from the ones which have been explored hitherto. In this last respect, it is scarcely relevant that Allais' analyses have been convened with events lying 'far' in the past. In contrast to what decision theory and game theory imprint on our way of thinking, it is definitely the case that History crosses anew some nodes of its net, some road junctions already explored.

Other apparent paradoxes, much deeper than the so-called 'Allais' Paradox', will be found in the structure of the whole Allaisian contribution.

What will probably strike the reader during the process of discovering Allais' writings to which the book invites him, is the fact that such a multifaceted personality, with such an eclectic curiosity, has nevertheless produced a *synthetic* view of economics. The explanation probably lies in the fact that Maurice Allais has always striven, and most of the time managed with success, to discover *universal constants, but never at the expense of caricature.* If necessary, the space in which a given concept had been first defined has been adapted to this type of search; if necessary, a given analysis has been further developed at a higher level of control, in order to accommodate constant *functions* with more flexible values of the application *parameters*. The inspiration here is clearly closer to physics than is usual among economists. Examples can be found in the restatement of the quantity theory of money (and its parenthood with relativity theory) as well as in the reconstruction of the concept of attitude towards risk. There lies the tricky alliance, in Allaisian economics, of strong empirical validation with almost-universal models; there also the alliance of the latter with a constant suspicion of reductionist models of economic activity and of psychological processes.

Another of the apparently paradoxical aspects of Allais' thinking lies in the alliance of the quest for universal constants and yet of changing views as to what the very foundations of economics are. The years 1966–1967 constitute landmarks in this respect, although pretending that the models designed after that period (the markets economics) had not been present, to a lesser degree and at different levels of interpretation, in the previous writings of Allais would amount to an overstatement. Nevertheless, the changes in Allais' views have been sufficiently far-reaching to preclude the simple classification of Allais among the neoclassical economists.

Such views may appear amazing to those who have studied Allais' thinking through second-hand analyses. Such a second-hand literature is difficult to avoid in an English-speaking environment, since the greater part of Allais' writings have not been translated as of now.[2] This book is primarily intended to counterbalance such a Vulgate-type of interpretation of Allais' contribution to economic science.

Needless to say, the authors who have contributed to this volume have also conceived of these pages as ways to honor Maurice Allais as a professor as well as an economist. This latter quality would, however, have already been sufficient by itself to justify the following pages.

After all, Maurice Allais is one of the major economists of this century and his writings deserve, like the ones of all Nobel Prize winners and the ones of other great economists, to be thoroughly studied and evaluated in a most detailed way in all scientific communities, including the English-speaking one.

Ecole Normale Supérieure de Cachan
Fall, 1992.

NOTES

* 'A Chapter in the History of Ramsey's Optimal Feasible Taxation and Optimal Public Utilities Prices', in: *Economic Essays in Honour of Jorgen Gelting*, 1983, pp. 164–165.
[1] Boiteux, M., de Montbrial, Th., and Munier, B.R.: 1986, *Marchés, Capital et Incertitude*, Essais en l'Honneur de Maurice Allais, Paris, Economica.
[2] Such a translation has been undertaken by Kluwer Academic Publishers and should appear progressively in the immediate future. When this work has been completed, the present book will serve as a guide to the labyrinth of these collected writings.

BERTRAND R. MUNIER*

1. FIFTY YEARS OF MAURICE ALLAIS' ECONOMIC WRITINGS: SEEDS FOR RENEWAL IN CONTEMPORARY ECONOMIC THOUGHT

Until he discovered the situation in the United States of 1933, Maurice Allais has been fascinated by history, even in his youth, as well as by physics and mechanics a little later. But the trip was to dramatically change the focus of the young man's interest. He had recently graduated first in his class from the Ecole Polytechnique, one of the most competitive engineering schools in France, originally founded by Napoleon to educate an élite of engineers and scientists. To the young engineer, who had little training in economics, it seemed impossible that responsible men and women in charge of the American economy had let things deteriorate to the point he could observe when looking around on the American streets. The only explanation he could see was that these men and women did not know enough about economics. This conclusion made the field attractive to Maurice Allais: there was something to be searched for, some 'truths' indeed to be discovered! The fact that these potential discoveries could help improve society made the challenge still more attractive, because young Maurice had been the son of a family of very modest means.

When Maurice Allais was born, on May 31, 1911, his father and his mother ran a very modest retail dairy in Paris. After the death of his father in the Great War, his mother soon had to open a baby linen shop to earn a living for her son and herself. Under these difficult conditions, Maurice prepared for entrance to the Ecole Polytechnique. The year he graduated first in his class, in 1933, he received two prizes from the French Academy of Sciences – the Laplace Prize and the Rivot

* Professor of Economics and Management Science and Director of GRID at the Ecole Normale Supérieure de Cachan, Cachan, France. This survey borrows not only from Maurice Allais' writings, but also from Jean-Michel Grandmont's and Jacques Drèze's reports (1989) as well as from personal discussions with Maurice Allais and from other previous surveys from the same author (1986, 1989, 1991). The usual caveat holds.

B. Munier (ed.), Markets, Risk and Money, 1–50.

Prize – and he quite naturally, under such circumstances, considered starting a career as a researcher in physics. He entered the élite finishing school, the 'Ecole Nationale Supérieure des Mines de Paris', with this perspective in mind.

Fortunately for economic science, by the time he graduated from that famous school (1936), there was no position open for researchers in this field in France. The young man had no choice but to take a job as a mining engineer, which he found in Nantes, on the western coast of France. He thus thought he would turn to more practical matters, and Maurice is passionately convinced that only effective activities are worth looking at. Potentially effective activities, too, of course: and I confess that not everyone is convinced that he can always distinguish impractical propositions! I really do think that Maurice Allais was born to do research, whatever the field, and that it would have been a very inefficient allocation of resources if he had been compelled to do anything else!

Nevertheless, he had to have a job. The one which was offered him consisted mainly in keeping an eye on mines, quarries and parts of the public transportation system with respect to their regulatory environment; it also had immediate economic implications. On top of that, the social situation clearly worsened in France, more than it had done before in the years 1935–1936 – with a time lag with respect to England, the U.S.A. and central Europe – and the economic and social questioning, loosely speaking, became pressing. Maurice Allais thought again of his trip to the U.S.A. three years earlier and decided, this time for good, to start digging into economic science, although to him, the true scientific problems remained in physics. The future 1988 Nobel Laureate in Economic Science thus embarked on a career in economics almost without noticing he was doing so, with the conviction he was devoting much of his time to urgent and serious practical problems which were of concern to him, but which he first thought he was studying in the capacity of an amateur.

The final shift in his career took place during World War II. After a brief episode during which he was an artillery lieutenant in the Alps (1939–1940), Maurice Allais returned to his civil job in Nantes under German occupation of France and became very much concerned about the economic reconstruction of his country which was to come once the war would be over. Only then did he really become a professional economist and in 1944 he was made a Professor of Economics at the

Ecole Nationale Supérieure des Mines de Paris, from which he had graduated less than a decade earlier. In 1946, he was awarded a permanent position by the National Center for Scientific Research (Centre National de la Recherche Scientifique, CNRS), from which he retired only in 1981.

Meanwhile, the amateur work had taken him far ahead of many other economists of the time! He had bought some books in economics before the war, but his reading of the great authors started only in July 1940 on his return to Nantes. In one year he read the fundamental work of Léon Walras, of Vilfredo Pareto (whom he describes as being the greatest of all) and of Irving Fisher – the two last having for the reset of his life the deepest impact on his thinking – while he would later progressively depart, particularly in the late sixties, from Léon Walras' thought.

Going over from reading to writing did not take Maurice Allais a long time: even when the war broke out he was considering writing a book on probability and risk theory. Indeed, during his leisure time in the late 'thirties he had been interested by horse racing and had become firmly convinced that the fundamental question of human psychology under risky situations boils down to balancing the expectation of gains with the likelihood of ruin along some sort of locus of efficient bets. Unfortunately, the war broke out. The writing of the book was postponed (and never did take place, in fact). Thus, the scientific writing period of Maurice Allais effectively started on July 1, 1941, as he completed the 'Introduction' to a monumental future work in economics, called 'In Search of an Economic Discipline' (A la Recherche d'une Discipline Economique), the first part of which was to be devoted to pure economics. This survey thus covers exactly fifty years of writings in economic science.

The numerous contributions of Maurice Allais are currently becoming, if not thoroughly known, at least slightly more studied. The insufficient attention which had been paid to them in the past – which I had denounced, in 1978 when insisting with colleagues on his receiving the Gold Medal Award of the CNRS, and later on with explicit reference to a possible Nobel Committee decision in a survey published in French (Munier, 1986, pp. 33–34) – is a matter poorly recollected today. The Nobel Prize Award of 1988 helped! It remains to show, however – which is what I endeavor here – that Maurice Allais' writings have been much more innovative than is usually believed and can be regarded as containing the seeds for important advances in economic science which are yet to come.

Like many great forerunners, Maurice Allais sought to reassemble the 'broken frames' of economics, in a way that looked to him more meaningful with respect to the observed facts. His methodology therefore results from two main lines of efforts: (a) an effort towards never departing from observation, which will lead him to condemn theories, not observations, when there is a discrepancy between them; and (b) a remarkable effort towards synthesis. These traits will be found in more than one part of Maurice Allais' writings. If we depart from contributions to physics and to political science, which we shall not (and could not, anyway) analyze here, we can divide these scientific economic writings into four main areas of interest, each one endowed with theoretical aspects, but also with practical implications:

- the theory of markets and welfare;
- the theory of capital and economic growth;
- the theory of decision and risk;
- the theory of money and monetary dynamics.

I. MARKETS AND WELFARE

In this domain, two series of contributions have been made by Maurice Allais: his 1943 book, *In Search of a Science of Economics, Part One: Pure Economics* (*A la Recherche d'une Discipline Economique, Première Partie: Economie Pure*) is an amazingly rich and comprehensive contribution to equilibrium and economic efficiency theory. The book was originally intended to be the first part of the monumental work envisioned in 1941 (see above), the second part of which should have been devoted to the theory of interest and monetary theory, the third to international economics, the fourth to disequilibrium theory and the fifth and last to prescriptive considerations concerning the reconstruction of the European economy (Maurice Allais has been an outspoken pro-Europe advocate) after the war. No wonder that none of the publishers contacted ever accepted the gamble of launching such a project at the time! The 'first part' was thus merely typewritten and duplicated on June 15, 1943, and only later, in 1952, reproduced by the Imprimerie Nationale (the French Official Publications Printing Service) with a new (printed) foreword by the author, under the title *Treatise of Pure Economics* (*Traité d'Economie Pure*). It is an already enormous five-volume work (982 pages overall) which we designate below as Allais (1943–1952) (a new edition, the first real edition of the work as a book, has recently come out – see bibliography – but we did not have it at the time of writing). Much of the intended content of the second, third and fifth parts were published in 1947 as another book: *Economics and Interest* (*Economie et Intérêt*). Both of these books were expressly mentioned in the Nobel Academy Award. However, as the second deals with capital and growth, it will be examined only in the second part of this survey.

The other contributions to markets and welfare analysis are contained in small books, like *Pure Economics and Social Welfare* (*Economie Pure et Rendement Social*, 1945), in articles which did not add fundamental points to these two books, but also in a series of several seminal articles (from 1967 onwards) and in another major work, *General Theory of Surpluses* (*La Théorie Générale des Surplus*), which was published in 1981 as a special issue of the journal *Les Cahiers de l'ISMEA* and reprinted in 1989 as a book. The latter series of contributions together constitute a serious revision of Maurice Allais' view of basic economic activity. This change of mind progressively emerged as Maurice Allais worked on applications in the 'fifties and in the 'sixties, although 1967 can be regarded as a turning point in this respect.

I.1. *The 1943–1967 Period*

Allais (1943–1952) contains a number of original contributions, some of them still little widespread today. At the time, they represented decisive improvements and appeared to a few talented young economists as really remarkable. Thus, Gérard Debreu wrote, in 1948 (our translation):

> In the spring of 1946, the impressive basic work of Maurice Allais came into my hands by mere chance (. . .). I, however, started to read the foreword and, moved by the determination of the author, found at its end the courage to impose upon myself, within two or three months of often painful work, to assimilate the work. I came out of it passionately fond of economics and started to understand the significance of my first work. I do not hesitate to say, to put it bluntly, that my knowledge of economics today was basically acquired during these three months (in Allais (1943–1952), Foreword, pp. 12–13).

TABLE I

Selected writings of Maurice Allais on markets and general efficiency.

1943–1952:	*A la Recherche d'une Discipline Economique*, Première Partie: L'Economie Pure, Paris, Ateliers Industria, 1943. 852 pp. *Traité d'Economie Pure*, Paris, Imprimerie Nationale, 5 vols., 915 pp. (63 printed pages of introduction + preceding book), 1952. First entirely printed edition: Paris, Editions Clément Juglar, 1994.
1945:	*Economie Pure et Rendement Social*, Paris, Sirey, 72 pp.
1947a:	'Le Problème de la Coordination des Transports et la Théorie Economique', *Revue d'Economie Politique* **57**(2), 212–271.
1953a:	*La Gestion des Houillères Nationalisées et la Théorie Economique*, Paris, Imprimerie Nationale. 126 pp.
1965a:	'Options de la Politique Tarifaire dans les Transports', in collaboration with del Viscovo, Duquesne de la Vinelle, Oort and Seidenfus, in: *Travaux du Comité d'Experts pour l'Etude des Options de la Politique Tarifaire dans les Transports*, Brussels, EEC Studies, Transportation Series, No. 1. 206 pp.
1969a:	'Le Condizioni dell'Efficienza nell'Economia', in: *Programmazione e Progresso Economico*, Centro Studi e Ricerche su Problemi Economico Sociali, Milan, Franco Angeli, Editore, pp. 13–153, 169–181, 207–208 and 221–242.
1971:	'Les Théories de l'Equilibre Economique Général et de l'Efficacité Maximale, Impasses Récentes et Nouvelles Perspectives', *Revue d'Economie Politique* **81**(3), 331–409. English translation in: Schwödiauer, G. (Ed.), *Equilibrium and Disequilibrium in Economic Theory*, Dordrecht, Reidel, 1977, pp. 129–201.
1981–1989:	'La Théorie Générale des Surplus', *Economies et Sociétés* **15**(1–2–3) (Band I) and (4–5) (Band II), 1981, 716 pp. *La Théorie Générale des Surplus*, Grenoble, Presses Universitaires de Grenoble, 2nd ed. 1989. 716 pp.
1984a:	'The Concepts of Surplus and Loss and the Reformulation of the Theories of Stable Economic Equilibrium and Maximum Efficiency', in: Baranzini, M. and Scazzieri, R., *Foundations of Economics, Structures of Inquiry and Economic Theory*, London, Basil Blackwell, pp. 135–174.
1987a:	'Economic Surplus and the Equimarginal Principle', in: *The New Palgrave, A Dictionary of Economics*, London, Macmillan, Vol. 2, pp. 62–69.

TABLE I (continued)

1989:	'L'Economie des Infrastructures de Transport et les Fondements du Calcul Economique', *Revue d'Economie Politique* **99**(2), Special Issue, edited by Bertrand R. Munier and G. Terny, on: *Vingt Ans de Calcul Economique Public: Bilan et Perspectives*, pp. 159–203.
1994:	*Traité d'Economie Pure*, Paris, Editions Clément Juglar (final edition with references to the 1943 and the 1952 versions of the book, notes and comments from the author)

Nevertheless, the richness of this work by Allais has been overlooked for a long time. Let us hereafter take the points which we consider most important. Generally speaking, we find here two kinds of contributions: first, a much more rigorous treatment of some ideas which until then were in circulation, but defined in such an imprecise way that errors or apparent 'paradoxes' could emerge: Allais then, on his own, reconstructed the concepts and their theoretic roles with his own methods. Second, we also find concepts which are entirely new, due to Allais himself. Most of these appeared already in 1943, although some of them would be given their full importance only later, in 1967 (see I.2.).

I.1.1. We may take as an example of the first type the concepts of substitutability and complementarity in use within economic theory. Clearly, these concepts were not introduced by Allais. However, their usual definition was – and is – not very satisfactory. Cardinalists usually admit that two goods h and k are complementary or substitutes according to whether the sign of

$$\frac{\partial U}{\partial x_h \partial x_k}$$

is positive or negative, respectively, where U stands for the utility function and the xs for quantities consumed. This definition is intuitively satisfactory, but has the heavy inconvenience of being sensitive to some monotonically increasing transformations of the utility function. Take for example $U = x_1^{0.5} \cdot x_2^{0.5}$ and $W = a \cdot U^3 + b \cdot U$ with $\mathrm{d}W/\mathrm{d}U > 0$ (which entails $b > -3aU^2$). Further, take as an example $x_1 = 3$, $x_2 = 3$, and $b = -24$: it is then sufficient that $0 < a < 16/3\sqrt{3}$ in order to have h and k appear as substitutes if one considers the function

U, and as complementary goods if one considers the function W. This is why Hicks suggested – and almost all ordinalist economics accepted – using the sign of the Slutsky matrix coefficients

$$\left(\frac{\partial x_k}{\partial p_h}\right)_{U=\text{constant.}}$$

But these coefficients – apart from the fact that they are not directly observable – do not really characterize the preference pattern of the agent, for they also depend on an institutional characteristic: the budget constraint. (Yet almost the entire profession teaches that 'truth' without even a hint of the difficulties involved). Allais did avoid these difficulties as early as 1943, even though they were not known (except for the very first one) at that time. He defines coefficients:

$$(h,k) = \frac{U_{hk}}{U_h U_k},$$

which he calls 'normalized marginal desirability' (désirabilité marginale réduite) as well as the weighted sum $[h,k]$ of their differences to their median value μ, where the weights are chosen as the 'total desirability' of the goods $h = 1, 2, \ldots, l$ i.e. $x_h U_h$. One thus gets:

$$(1) \qquad [h,k] = \sum_{h=1}^{h=l} x_h U_h \cdot [(h,k) - \mu]\,.$$

Allais shows that a negative sign in (1) defines substitutability, whereas a positive sign defines complementary (§65, pp. 137–152 of the 1952 edition; see also §176 bis, pp. 434–443).

Intuitively, this can be regarded as meaning that one should consider two goods h and k as 'complementary' if and only if they show more complementarity, in the cardinalist sense recalled above, than the pair of goods which is median in this respect. This may seem difficult to teach, but has three advantages:

– it does not depend in any way upon the institutional environment considered, in contrast to the usual Slutsky–Hicks notion;
– it shows in a straightforward manner that complementarity and substitution are not absolute concepts (except for the unrealistic cases of perfect substitutability or perfect complementarity), but rather relative ones;

– it does not depend on any monotonically increasing transformation of U, as Maurice Allais rightly pointed out (*ibid.*, n. 6, p. 138).

Indeed we can consider again our above example, $W = f(U)$, where $f(U) = 3U^3 + b \cdot U$. We find:

$$\begin{aligned}(1,2) &= \frac{W_{12}}{W_1 W_2} = \frac{(3aU^2 + b)U_{12}}{(3aU + b)^2 U_1 U_2} + \frac{6aU}{(3aU^2 + b)^2} \\ &= \frac{\mathrm{d}^2 f/\mathrm{d}U^2}{\mathrm{d}f/\mathrm{d}U} + \frac{(1,2)}{\mathrm{d}f/\mathrm{d}U}\end{aligned}$$

i.e. we find that $(1,2)$ is defined up to a positive affine transformation. Given the above definition, the same holds for $[1,2]$. More generally, this means that such expressions as the (h,k) and the $[h,k]$ above will not change sign if we choose another expression for the utility function. This completes the proof.

The characterization of complementarity and substitutability thus arrived at should be attractive for the reasons already mentioned. It represents an important improvement upon the economic theory which is currently still in use. Nevertheless, few applications have been worked out to date (see Barten and Bettendorf, 1988, for an exception).

For normalized values of the preference relationships between the goods or services, Maurice Allais finally suggests the coefficient

$$(2) \qquad K_{hk} = \frac{[h,k]}{\sqrt{[h,h][k,k]}}$$

which, he shows, will vary between -1 (perfectly substitutable goods) and +1 (perfectly complementary goods).

I.1.2. We can take, as another example of Allaisian rigorous reconstruction, the two equivalence theorems between market equilibrium (under pure competition and perfect market conditions) and Pareto optimality. These theorems had already been exhibited in a static, timeless, environment, but had not been proved in a rigorous way, even in these static models. Allais recognized this fact and offered two alternative, lengthy but rigorous proofs for these theorems (Allais, 1943, pp. 617–627, §265–266 and pp. 628–635, §267–268), taking account of second-order conditions within a static as well as within an intertemporal model (see below for details and comments on the second proof). In fact, he combined both propositions into a single one: competitive prices and an adequately chosen distribution of income constitute a necessary and sufficient condition for the economy to be at one particular point of the Pareto efficient frontier. Not without a glance of regret for physics. Allais called this a 'theory of maximum social efficiency' ('théorie du rendement social maximum'). Later on, to avoid any misunderstanding, he used the term 'maximum economic efficiency'.

I.1.3. In Search of a Science of Economics is not, whatever it may seem, a purely theoretical book. It also deals with the problems encountered in the real world. Thus, Maurice Allais emphasizes nonconvex technologies and gives necessary and sufficient conditions to manage plants in public utilities sectors in a way that will produce the competitive prices required by welfare efficiency (Allais, 1947a, 1953a). Remember that, in these sectors, there is a long and brilliant tradition of engineer-economists in France, on the one hand; and that the post-War period has been one of particularly pressing needs in this area, which Maurice Allais, due to his first position as a mining engineer, could observe from the forefront. Some of Allais' recommendations have remained famous in economists' circles, for they not only triggered acrimonious comments, but also social protest! (Munier, 1991, pp. 187–188.) Yet they were quite correct.

This already tells us that Allais is certainly not advocating that free markets alone will solve everything, and that any government intervention could only be damaging. In fact, many reservations against such ideas appear in his different writings.

First, Allais admits – and even explicitly and strongly advocated in 1943– that natural monopolies should be governmental property. In any case, he would at least demand strict regulation of such firms (Allais, 1943–1952, pp. 656–657, §273).

Second, Allais emphasizes that one has to fight sources of unearned incomes (*revenus non gagnés*). These include all possible sorts of pure rents, including seniorage on money issuing. Allais advocated several policy measures to fight such unearned incomes.

For pure rent, Allais first took Walras' position: nationalize all land (*ibid.*, pp. 383–389, §163) and have the government rent it to farmers and other users (how practical this proposal might have turned to be, if put into practice, remains to be seen). Consistently, and for more general reasons of the same type, he refused, in 1947, to be a cosignatory of the declaration of the Mount Pelerin Society with Friedrich von Hayek. After 1948, however, Allais admitted that such a nationalization of land could impair political as well as economic freedom and changed his opinion. Thus, (Allais, 1977), he argues that a tax should be levied on all privately owned physical durable goods (as explained below) to reconcile private possession of land with the need to avoid rents being privately appropriated.

The irony is that, in predominantly socialist post-War France, Allais was considered a staunch free-market ('libéral') economist. On the other hand, no conservative journal would agree, at that time, to publish his proposal for a general tax on all physical assets to replace every income tax! Another irony was that his paper on the subject was finally published by the socialist party's newspaper *Le Populaire*! This at least proves Allais' considerable political independence.

On the other hand, Allais never changed opinion on seniorage: the issuing on money should be exclusively the central bank's responsibility and all the seniorage should thus go to the government. This is another issue on which Allais is completely opposed to Friedrich von Hayek (and in possibly substantial disagreement with Milton Friedman), for the consequence of such a view (Allais, 1987c) is that private banks should not be in a situation in which they could create money.

A third major reservation about market performances is that, even if capitalist economies were free of institutional imperfections like the ones mentioned above, they would not attain Pareto optimality. Beyond inadequate income distribution, which calls for some distributive function of public intervention, Allais advocates another type of intervention, to increase efficiency in the allocation of resources. He doubts that economic agents can perceive unbiased intertemporal rates of substitution (Allais, 1943–1952, pp. 661–662, §277). In this respect, Allais is as close to Richard Musgrave or to Herbert Simon as to any of the neoclassical economists, and he certainly would not subscribe to anything like a rational expectations hypothesis.

But these reservations about markets, which belong intimately to Allais' view of the economic world, do not contradict the fact that Allais fiercely opposed collectivism, as well as all types of government interventions which could turn out to be counterweights of decentralized market mechanisms, i.e. a very large fraction of such interventions in post-War France. One has to make clear, in addition, that until at least the early 1960s, such an attitude in France led to his being labeled as the bad pupil in the class. Being little gifted for retreat, or even for compromise, Allais expressed his views even more forcefully and squarely. This earned him a reputation of authoritarian behavior, which often bears on many strong personalities in a more or less unjustified way.

I.1.4. In order to deal with problems of the real world, the question of how the economy moves towards a stable equilibrium situation has been an important problem in Allais' research. It is little known that the first theorems on global stability in a *tâtonnement process* were proved by Allais in 1943 in the fairly general framework of his model. Negishi mentioned it, however, in his survey of the stability of economic equilibrium (Negishi, 1962). We have to recall that, although Allais himself admitted to have "given a proof of an abstract character and of a limited significance" (Allais, 1943–1952, p. 486) and only "sketched the rigorous proof which could be established by considering the statistical distributions of incomes and preference patterns" (*ibid.*, p. 488, n. 5), the proof he gave has represented a remarkable improvement upon both the Walrasian and the Hicksian treatments of the question. Hicks had simply argued that, since an equilibrium is a 'maximum situation', no one would care to deviate from it and that an equilibrium was therefore 'stable'. In contrast, Maurice Allais understood quite clearly that the stability question was much more 'complex' (he devoted a short section to draw upon the difficulties to be met for a general proof (*ibid.*, pp. 493–494)), and that what was at stake was a proof that, if the economy was outside equilibrium, market forces would drive it by themselves to equilibrium. As for Walras, he had basically argued that, if equilibrium were destroyed on one single market, the way excess demand evolves in relation to the price of that one market would drive the price back to equilibrium on that market and have influences on the other markets, which would cancel each other on the whole. Allais realized that this certainly was an acceptable approach, but that it did imply conditions which he himself termed 'restrictive' and which amounted to what is called in modern economics 'gross substitutability', i.e. a hypothesis that, if the price of a good k were to increase, the sum of the individual demands for other goods would also tend to increase. Besides, he also had to use the fact that all Walrasian supply and demand functions are homogeneous in prices and income – which he had shown before. Finally, he implicitly assumed that all functions were continuously differentiable.

Maurice Allais thought that one possibility for the gross substitutability condition to hold was to assume that the economy is 'in a neighborhood of equilibrium' – another new concept at the time – and that resources and preferences would not be too different from one individual to the other. He thus proved two lemmas using these hypotheses and then showed that what he called the 'characteristic function', i.e. the sum of the absolute values of discrepancies between supplies and demands, can only be decreasing because of market forces.

May this 'characteristic function' be assimilated to a Lyapunov function (Belloc and Moreaux, 1987; Grandmont, 1989)? Quite clearly, Allais never referred to Lyapunov in his book. Besides, Grandmont (Grandmont, 1989, p. 21) pointed out that the 'characteristic function' which Maurice Allais used is not differentiable everywhere (which a Lyapunov function should be), although Maurice Allais, in Grandmont's view, may be forgiven for having overlooked that technical point. For these reasons, and because time is not introduced into Allais' equations on stability as an explicit variable (but Allais mentioned successive 'periods' of adjustment in his tâtonnement process), also because Allais neither used the word substitutability in any of the sections quoted above, nor gave a justification of his hypotheses which would sufficiently differ from Walras' analysis, E. Roy Weintraub (Weintraub, 1991) dismisses the claim that Allais was the first in proving the stability of equilibrium within a tâtonnement process in the modern way (Arrow and Hurwicz, 1958). This might turn out to be a somewhat severe judgment. What I claim to have shown here is only that Allais' contribution was a radical improvement upon both Walras and Hicks and that he made the first – except for one technical flaw, indeed – the conceptual steps to the type of proofs which today make an explicit use of Lyapunov's second method. This seems more important than the identity of the mathematical tool to which he resorted.

I.1.5. In any case, our 'amateur' economist was not satisfied with a *tâtonnement* process, which is far too static for him. Maurice Allais considers such hypotheses to be artificial and thus of very limited interest (Allais, 1943–1952, p. 640, No. 270, nn. 3 and 9). He thus designed an out-of-equilibrium process at the same time as he gave his second proof of the equivalence theorem.

To give a pedagogical account of this model, let us consider an exchange economy outside equilibrium and assume temporarily that we may make use of a continuous Bergson social welfare function

$$W(U_1, U_2, \ldots, U_m)$$

(which Allais dispenses with, but at the expense of other concepts, which we want to introduce only later). Every one of the m agents of this economy has then a vector p^i of 'subjective' prices for the l goods or services on the markets, represented by his relative marginal rates of substitution in terms of the 'numéraire' good. Assume the 'numéraire' is the good l. Then, indexing the agents with i and the goods of services with h, one has

$$(3) \qquad p^i = \frac{\partial U_i / \partial x_{ih}}{\partial U_i / \partial x_{il}}, \qquad h = 1, 2, \ldots, \ldots, l.$$

We can define the mean of the subjective prices, for every good h, as:

$$\overline{p_h} = \frac{1}{m} \sum_{i=1}^{i=m} p_h^i.$$

At the time this economy is considered, let $W = W^\circ$. We can write the first-order total differential of W as:

$$(4) \qquad \mathrm{d}W = \sum_{i=1}^{i=m} \sum_{h=1}^{h=l} \frac{\partial W}{\partial U_i} \frac{\partial U_i}{\partial x_{ih}} \mathrm{d}x_{ih}.$$

Replacing $\partial U_i / \partial x_{ih}$ by its value in (3) and assuming (also temporarily) that, for every i, $(\partial W / \partial U_i)(\partial U_i / \partial x_{il}) = \lambda$ (so-called 'optimal distribution of income', with $\lambda > 0$ for obvious reasons), we can write

$$(5) \qquad \frac{\mathrm{d}W}{\lambda} = \sum_{\substack{h=1 \\ i=1}}^{\substack{h=l \\ i=m}} p_h^i \, \mathrm{d}x_{ih}.$$

As long as subjective price differences ($p_h^i \neq p_h^j$) prevail, there are motives for exchanges involving the commodities like h. But every individual i will exchange only if $\mathrm{d}U_i > 0$, i.e. if

$$\sum_{h=1}^{h=l} \frac{\partial U_i}{\partial x_{ih}} \mathrm{d}x_{ih} > 0$$

or, by (3)

$$(6) \qquad \sum_{h=1}^{h=l} \frac{\partial U_i}{\partial x_{il}} \, p_h^i \, \mathrm{d}x_{ih} > 0.$$

Under such conditions, the increase in social welfare (5) will be positive, being the sum of individual welfare index increases (6). But this cannot go on for ever! The process considered here is in fact constrained in two ways:

On one hand, there are limitations of the availability of goods in our exchange economy, for say every h, $\sum_{i=1}^{i=m} x_{ih} < \omega_h$ and thus the economy moves in the space of goods, within a set which is bounded from above (as it is obviously also bounded from below and continuous by assumption, it is a compact set).

On the other hand, prices must effectively be paid in exchanging commodities, not only 'subjective' prices, and thus some type of budget constraints exist. If i is a buyer of commodity h, he wants to pay less than or at most p_h^i for a unit of that commodity bought: if he is a seller of it, he wants to get at least p_h^i, possibly more, for any unit of that commodity sold. It makes sense to assume (third and last temporary assumption to make) that average prices $\overline{p_h}$ are the ones effectively paid. But then budget constraints are such that, for every i, the $\mathrm{d}x_{ih}$ exchanged have to meet $\sum_{h=1}^{h=l} \overline{p_h} \, \mathrm{d}x_{ih} = 0$. Summing over i, in vector notation,

$$(7) \qquad \bar{p} \cdot \sum_{i=1}^{i=m} \mathrm{d}x_i = 0$$

and thus, at any out-of-equilibrium period t, we can combine (5), (6) and (7) to write

$$(8) \qquad \frac{\mathrm{d}W}{\lambda} = \sum_{i=1}^{i=m} \left(p^i - \bar{p}\right) \, \mathrm{d}x_i > 0.$$

But from one period to the other, subjective as well as effectively paid prices will change. A reasonable pattern to assume is given by

$$(9) \qquad \mathrm{d}x_{it} = C\left(p_{t-1}^i - \overline{p_{t-1}}\right)$$

where C is a diagonal positive matrix. Finally, at any out-of-equilibrium period:

$$\text{(10)} \qquad \frac{\mathrm{d}W}{\lambda} = \sum_{i=1}^{i=m} \left(p_{t-1}^i - \overline{p_{t-1}}\right) C \left(p_{t-1}^i - \overline{p_{t-1}}\right) > 0.$$

This last expression of welfare increase at some out-of-equilibrium period being a positive defined quadratic form, it is obviously differentiable and monotonically increasing. Now, let us consider the maximum value of W on the set of attainable states, say W^*. That maximum value exists because the set is compact, as we have shown, and W is continuous. Thus, the function $W^* - W$ is continuous, positive, bounded from below and decreasing. It is a Lyapunov function: the dynamic system of that economy will therefore be led to stable points which are easy to characterize: the points where, for every i, $p_\tau^i = p_\tau$ at some period τ.

Now, recall that this has been obtained under three temporary assumptions, which make the little story just told in this paragraph a 'centralized version' of Maurice Allais' out-of-equilibrium dynamic model. Let us look at the way this latter model deals with these three assumptions.

1. The existence of a collective welfare function can be dispensed with: some type of price mechanism like (9) above, together with the other assumptions on preferences, is sufficient to ensure that all p_t^i will converge to each other at some stable point. This stable point will be on the border of the image in the space of utilities of the compact set of attainable states in the space of commodities. Many such stable points exist along that border (in fact, it is a continuous locus if the social welfare function is continuous, or, more generally, if all individual utility functions are continuous). Allais thus introduced this concept first into economic theory and called it the locus of 'maximum possibilities in the space of utilities'. This idea, completely new in 1943, is today familiar to any student who has taken economics 100 under the name of 'utility frontier'.

2. The assumption of a single effective price at each period of exchange is useless. In a *truly decentralized* economy, there need not be anything like a market price. In fact, let us replace (9) by the following:

$$\text{(11)} \qquad \mathrm{d}x_{it}^j = \frac{C}{m}\left(p_{t-1}^i - p_{t-1}^j\right)$$

where dx^j_{it} denotes the exchanges between agents i and j at period t. Note that, if we sum over j, for a given i, we get exactly (9).

3. As for the assumption of some 'optimal' distribution of income at every period t, it certainly construes the evolution of the economy towards one particular efficient point and does not have any special justification, except that it makes computations of dW straightforward. It is related to the specification of W, a concept which Allais dispenses with in the present model. Thus, Maurice Allais' model will not make any such assumption on income distribution. This does not imply, by any means, that income distribution is neutral in Allais' eyes: it may, on the contrary, call for some government intervention, as will be discussed below.

I.2. *The Period Following 1967*

To Allais, in 1943, the model which we just presented was only a way to prove the equivalence theorems between equilibrium and maximum efficiency. However, the more he worked, during the 'fifties and the 'sixties, in different fields of application, the more his dissatisfaction with post-Walrasian economics grew. While doing applied work, Allais had been particularly uneasy about total and average cost computations in practice, about the uniqueness of market price of every input and output, and about convexity hypotheses. Following his conviction that theories are to be adapted to facts, and not conversely, he changed his economic *Weltanschauung*.

In the Fall of 1967 he expressed his views on the shortcomings of Walrasian economics in a paper published in Italian (Allais, 1969a). In 1971, at the conference of French-speaking economists held in Lausanne in Walras' honor, he expressed his rejection of the Walrasian model even more forcefully (Allais, 1971). He argued that the assumptions of convex technologies and of a single price for every good on its market were both unrealistic.

Allais then recalled his 1943 model of agents searching, outside of equilibrium, for a maximum surplus by exchanging commodities at mutually agreed-upon prices. This model appeared to him then as more than a simple dynamic procedure for attaining an equilibrium state. Rather, if it is a basis for a new paradigm of economic activity, which Allais calls markets (with an 's') economics (*l'Economie de* *marchés*). Instead of having two theorems in welfare analysis, as in the standard approach, the new model will in fact display three: two equivalence theorems between equilibrium (or rather stable state) and state of maximum efficiency, and a convergence theorem. But let us give a more precise account of the model framework before examining the theorems.

A state of the economy will be represented as in standard general equilibrium theory by a set of m consumption vectors (x_i), with consumers indexed by $i = 1, \ldots, \ldots, m$, a set of n production vectors (y_j), with producers indexed by $j = 1, \ldots, \ldots, n$, each of these vectors having quantities of goods as components, indexed as $h = 1, \ldots, \ldots, l$ and obeying the usual convention (negative value for services offered by consumers and taken as inputs by producers, positive value for goods or services offered by producers and consumed by consumers. Let an attainable state be defined in the usual way, i.e.:

$$(12) \qquad \forall h \quad \sum_{i=1}^{i=m} x_{ih} = \sum_{j=1}^{j=n} y_{jh} + \sum_{i=1}^{i=m} \omega_{ih}$$

where the ω_{ih} represent the initial endowments of the consumers. The set of attainable states is denoted by $\{E_\omega\}$. At a given date, this is a compact set under reasonable assumptions.

Let ΔE_1 be a feasible change of a given state E_1 and let E_2 represent the new state: $E_2 = E_1 + \Delta E_1$. According to (12) we have:

$$(13) \qquad \forall h, \quad \sum_{i=1}^{i=m} x_{ih} = \sum_{j=1}^{j=n} y_{jh}$$

because of the sign convention just recalled. In addition, for every producer j:

$$(14) \qquad f_j \left(y_{1j1}^1 + \Delta y_{j1}^1, \ldots, \ldots, y_{jl}^1 + \Delta_{yl}^1 \right) = 0$$

and for every consumer i:

$$(15) \qquad U_i + \Delta U_i = U_i \left(x_{i1}^1 + \Delta x_{i1}, \ldots, \ldots, x_{il}^1 + \Delta x_{il} \right)$$

where exponents designate a given state of the economy.

Let us now consider a third state of the economy such that quantities $\Delta\sigma_{i1}$ are taken away from the $i = 1, 2, \ldots, \ldots, m$ consumers in such a way that every consumer i enjoys anew the same level of utility as in state E_1:

$$\forall i, \quad U_i\left(x_{i1}^1 + \Delta x_{i1} - \Delta\sigma_{i1}, \ldots, \ldots, x_{il}^1 + \Delta x_{il}\right)$$
$$(16) \qquad = U_i\left(x_{i1}^1, x_{i2}^1, \ldots, \ldots, x_{il}^1\right).$$

Allais describes the state E_3 as 'isohedonous' to the state E_1. When going over from E_1 to E_3, one 'releases' a quantity

$$(17) \quad \sum_{i=1}^{i=m} \Delta\sigma_{i1} = \Delta\sigma_1$$

of good 1, which is assumed to be a good used by all consumers (the 'numéraire' meets this last condition naturally). This quantity is called by Allais 'distributable surplus' in state E_1. By the transition from E_1 to E_3 this 'surplus' is in effect being distributed. Transitions considered may be finite or infinitesimal.

Not all attainable states can be reached, starting from a given state E_1, but only a subset of them, say $\{E_1\}$, which is compact. There therefore exists a maximal distributable surplus in good 1 when starting from a given state like E_1, say $\Delta_1^*\sigma_1$. We can thus define a function

$$(18) \quad \Delta^*\sigma_1(E)$$

over $\{E_\omega\}$. Allais calls this function the 'loss function' associated with the economy considered. The locus of attainable states of the economy where the loss function is equal to zero is the utility frontier.

Let us now come back to the model of a markets (with an 's') economy as already suggested. In this model, every agent tries to find one or several agents ready to accept, at prices agreeable to them, a bilateral or multilateral exchange (possibly involving production decisions). Such exchanges will release positive distributable surpluses which will be shared by all agents partaking in the exchange, in proportions depending upon the prices effectively paid by every agent.

A theorem similar to the one proved after Equations (3)–(10) can thus be established, which shows that there exist stable points, corresponding to the new notion of general economic equilibrium in favor of which Allais has argued since 1967.

One first-order necessary condition for such a stable state E_S is that there is no positive distributable surplus left, i.e. that the loss function in that state takes only nonpositive values: $\Delta^*\sigma_1(E_S) \leq 0$.

On the other hand, we see, from the standard definition of a Pareto optimal situation, that the first-order necessary characterization of such a state will be the same.

Similar observations could be made regarding second-order conditions ($\Delta^{2*}\sigma_1 \leq 0$).

It immediately follows that (first theorem of equivalence) any stable state of general (Allaisian) economic equilibrium is also a state of maximal economic efficiency.

Maurice Allais asserts that, conversely (second theorem of equivalence), any state of maximum efficiency (any Pareto optimal state) is a state of stable general (Allaisian) economic equilibrium. This statement, however, calls for some qualification (otherwise it would assert that, on Figure 1, all Pareto optimal states could be reached whatever the initial endowment and the other characteristics of the economy, which would clearly not be generally true). It seems, in particular, that the condition of free reallocation of initial endowments is required. But there can be no doubt that some rigorous theorem(s) can be proved, which makes for the two equivalence theorems of the markets economy.

As for the dynamic 'convergence' of the state of an economy to a stable state of general economic equilibrium, the theorem sketched in I.1.5 above applies *mutatis mutandis* and represents an important improvement upon standard, i.e. post-Walrasian, general equilibrium theory, which does not have anything really equivalent.

Interesting questions can be raised here, like, for instance:

Question 1. Does the set of Allaisian stable states of the markets economy include the set of Walrasian equilibria of that same economy?

The answer is here straightforwardly yes (see Figure 2), except if some odd constraint were introduced: the dynamic evolution of a markets economy *can* follow the hyperplane of the budget constraint of the corresponding Walrasian economy. On the other hand, if no specific reason is given for this to happen, *there are* a priori *infinitesimal chances that this particular equilibrium will obtain*, even discarding every possible uncertainty. This is not a minor observation.

Question 2. Does the set of Allaisian stable states of a markets economy contain the core (in the game-theoretic sense of the word) of the same economy?

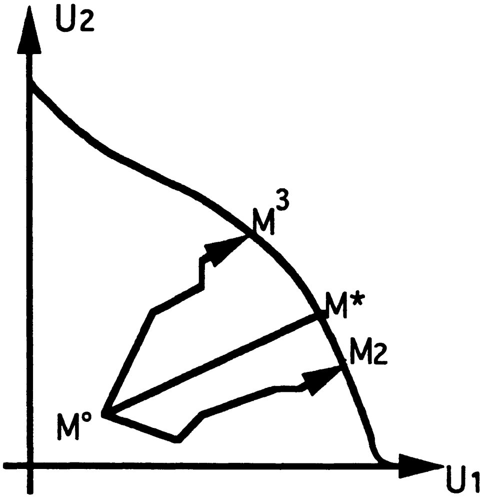

Fig. 1. Disequilibrium dynamics in a two-consumer markets economy. When the income distribution is constrained to be 'optimal' (Equations (3)–(10)), there is a well-defined correspondence between the initial endowment and some point(s) like M^* of the utility frontier. In the Allaisian general model of a markets economy, points like M^2 and M^3 can be reached through itineraries like the ones represented on this figure.

There is no rigorously fixed answer here, except in the two-agent case, where the set of Allaisian stable states coincides (except for the limiting points, if definition inequalities are used in the standard way) with the core. But as soon as the number of agents becomes greater than two, the set of Allaisian equilibria presumably (we do not know of any theorem proved yet) becomes larger than the core and includes it. Examples can be worked out (Malinvaud, 1969, pp. 132–136). See also Figure 2.

Question 3. What about the role of free competition? Free competition in markets economics is a necessary condition for general efficiency, but it has more impact on business through the general pressure which competitive situations put on human beings than through the effect of the price system. Indeed, input prices themselves are (consistently) not uniquely defined, except in a stable state. This has, in fact, been a constant view of competition for Maurice Allais.

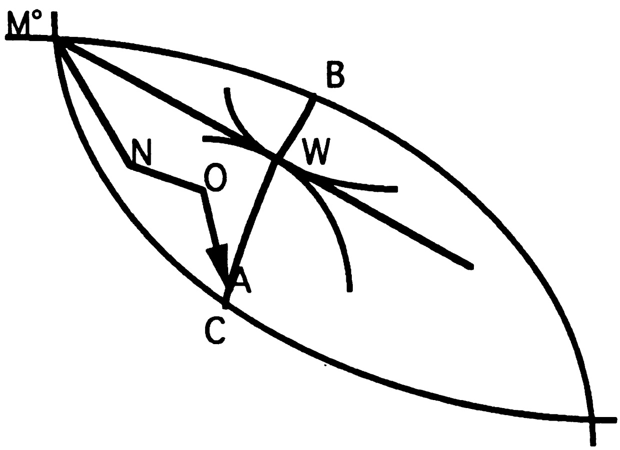

Fig. 2. A two-person exchange markets economy. Whereas Allaisian stable states like A (which is reached through the exchanges $\overrightarrow{M^\circ N}$, $\overrightarrow{NO}$ and $\overrightarrow{OA}$) are generally different from the Walrasian equilibria (assumed here to be the unique point W on Figure 2), they all belong to the core (the curve CB of the economy. Unfortunately, this result does not hold when more than two persons are considered in the markets economy.

Clearly, markets economics gives rise to interesting generalizations and insights. For example, (Allais, 1989, p. 421) losses in general welfare, when moving slightly away from the utility frontier because of taxes or tariffs, can be shown to be only "of the second order and in any case relatively small when compared to the gains resulting from the stronger struggling for efficiency which results in turn from liberalization of exchanges and thus from competitive pressure". Thus Allais rejects any kind of rigid doctrinal position here and admits some moderate and temporary tariffs, to achieve bearable conditions of shifting labor from one activity into another. Similarly, in markets economics, what is usually called a second-best optimum corresponds to an equilibrium with additional constraints to the ('natural') attainable state constraint, but the characterization of which need not be done in any different way than for any other equilibrium.

Question 4. Why is this view of economic activity more interesting than the Walrasian one? One could insist on the fact that most continuity hypotheses and all convexity hypotheses of the standard theory, which are disturbing and restrictive, can be dispensed with. But one more important answer here underlines that the Allaisian view of markets economics is no mechanical view of economic activity. Unlike in the Walrasian world, a single point of departure (a vector of initial endowments) can give rise to many points of arrival (the Allaisian stable states). Conversely, the same Allaisian stable state situation could be arrived at from many different possible vectors of initial endowments. In other words, there is no *a priori* 'one best path' from one state of the economy to another, even if we discard all types of uncertainty. This view of economics makes a system out of the economy which is closer to general systems in biology or in the human sciences, than to those of classical mechanics (Prigogine and Stengers, 1988; Lesourne, this book and 1991) which inspired Walras' work as is well known.

II. CAPITAL AND GROWTH

The reader should first recall what capital theory consisted of in the '30s. The available models were mostly intuitive and oversimplified. In particular, they dealt either with interdependencies between agents or with interdependencies between periods. Allais undertook to study interdependencies between *both* agents and periods.

Already in (Allais, 1943–1952) the determination of intertemporal equilibria and of the interest rate had been shown to depend heavily on the completeness or incompleteness of markets, even under the assumption of perfect expectations. But Allais' general views on interest, intertemporal allocation of resources and growth were expressed in another major book, also explicitly mentioned in the Nobel Foundation award, *Economics and Interest* (*Economie et Intérêt*) (Allais, 1947b), in particular in Chapters 7, 8, 9, 10 and in Annex II. Four main articles (Allais, 1960, 1962, 1965b, 1967) completed his contributions on capital and growth during the 1960s.

TABLE II

Selected writings of Maurice Allais on capital theory and growth.

1947b:	*Economie et Intérêt*, Paris, Imprimerie Nationale. 2 vols., 800 pp.
1960:	'Influence du Coefficient Capitalistique sur le Revenu réel par Tête', *Bulletin de l'Institut International de Statistique* **38**(2), 1–70 (Tokyo).
1962:	'The Influence of the Capital-Output Ratio on Real National Income', *Econometrica* **30**(4), 700–728. This paper was given as the Walras–Bowley Lecture at the 1961 meeting of the Econometric Society, and is also reproduced in: *Readings in Welfare Economics*, American Economic Association, Vol. XII, 1969. pp. 682–714.
1965b:	'The Role of Capital in Economic Development', in: *The Econometric Approach to Development Planning*, Amsterdam, North-Holland, and Chicago, Rand MacNally, 1965, pp. 697–1002.
1967:	'Some Analytical and Practical Aspects of the Theory of Capital', in: Malinvaud, E. and Bacharach, M., *Activity Analysis in the Theory of Growth and Planning*, London, MacMillan, and New York, St. Martin's Press.

II.1. *Discovery of the Overlapping Generations Problem*

Perhaps the most important contribution of Allais in capital theory (Allais, 1947b, Ch. 10 and Annex II) went unnoticed until E. Malinvaud (Boiteux, de Montbrial and Munier, 1986, Ch. 6) drew attention to it. It refers to the essential point that all consumers, at a given date, are not all at the same stage of their life cycle: young people make choices which will have consequences for their future life, while old people benefit from their former decisions. But the young consumers of today will eventually become old ones, interacting with new young consumers in an economic environment which will depend upon today's investments. Modeling this important fact leads to nothing other than what economists have come to call the 'overlapping generations model', usually attributed to Samuelson (Samuelson, 1958), whereas Allais had explicitly studied it eleven years earlier (Malinvaud, 1986, 1987).

The fundamental point Allais made in 1947 is that, contrary to the conclusion drawn from models relying on a single 'representative' consumer, the intertemporal preferences of consumers and the intertemporal production functions of firms are not sufficient to determine the interest rate, because the consumption/investment ratio does not affect one generation of economic agents only. At any given point of time, for example, old people may not decide on the distribution of the global income which should prevail between themselves and the younger generation. This basic indeterminacy can only be waived if there is some collective rule of fair division between generations. To be sure (our translation),

> there are cases where, from a common interest point of view, the spontaneous equilibrium to which free competition leads should be discarded, and this is simply because competition cannot take into account individual preferences but for a given generation at a given point of time, not for the other generations, indeed not even for the considered generation at a later point of time (Allais, 1947b, p. 592).

Government intervention in such cases can help to retrieve the distribution of spending

> which would have been obtained spontaneously ... if individuals would have had a clear and objective knowledge of their future preferences as well as of the advantages brought by the maximization of social productivity (Allais, 1947b, p. 593).

This is a recurrent point in Allais' thought: income distribution has to be carefully watched and controlled, which brings us back to the point discussed in I.1.5 and I.2, above.

In the 120 pages of his Annex II in *Economics and Interest* (1947) Allais investigates how the interest rate depends both on the consumers' preferences within each generation *and* on the discounted resources of income each generation can count upon during its life. Equation (19) of the chapter by Malinvaud in this book expresses the fundamental indeterminacy discovered by Allais, in the framework of a simplified model and in the case of a stationary economy.

II.2. *The 'Golden Rule' of Accumulation*

The investigation of stationary economies has been a deep concern of Allais in his major book in capital theory. It thus led Allais to another major contribution. In Chapter 7 (and also in Annex II) of (Allais, 1947b) he undertakes to show that, for a stationary economy ('régime permanent'), the optimal value of the interest rate should be zero. This condition of maximization of real national income – unheard of at the time – is nothing but the 'golden rule' of capital accumulation, in the special case of a stationary economy. The general case was established independently by E.S. Phelps (Phelps, 1961) and by Maurice Allais in the 1961 Walras–Bowley lecture at the Econometric Society meeting, which was published a year later (Allais, 1962). The economy is then not necessarily in a stationary state, but may be growing. The optimal interest rate then appears as equal to the rate of growth.

Let us recall here the proof of 1947 in the stationary case. It refers to three specific concepts:

- the concept of the characteristic period Θ_0, which reflects the decrease in marginal factor productivity as time elapses, independently of the place and date considered;
- the concept of 'global initial factor income' Y_{NW} ('revenu originaire national'), which is the sum of the values of the primary factors of production used in all production processes per unit of time, at a given period of observation. In a stationary economy, this income will be constant and equal to the sum of salaries and rents;
- the concept of 'characteristic function' φ, initially introduced by Stanley Jevons.

Then let V be the value at a given time t_0 of a given quantity produced. In equilibrium, that value will be equal to the sum of the costs incurred in terms of primary factors and of interest, which is

$$V = \sum_p (1+i)^p \cdot \Delta_{-p}$$

where Δ_{-p} stands for the costs in primary factors at time t_{0-p}. One can define

$$\text{(19)} \qquad \Theta = \frac{\sum_p (t_0 - t_{t-p})\Delta_{-p}}{\sum_p \Delta_{-p}}$$

where Θ is the average time to wait, once primary factors are being introduced into the production process, before the final product can be obtained. Θ will therefore be called 'average period of production'.

The original value $\Delta = \sum_p \Delta_{-p}$ of the capital necessary to a given production process, everything else being equal, grows with Θ. Thus, Θ may be regarded as characterizing the capital intensiveness of an economy. In addition, a function $\varphi(\theta)$ will denote, at the global level,

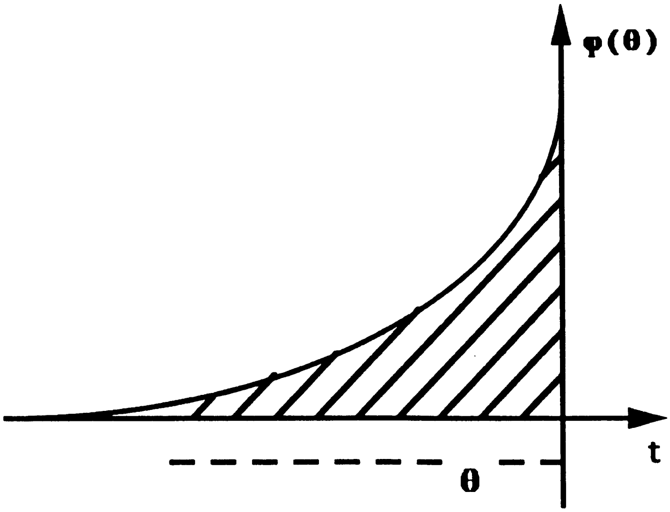

Fig. 3. The characteristic curve in Allais' analysis of capital. θ is the lag between the date where capital inputs are introduced and the date t where the resulting output is obtained.

the value of initial factor income spent per unit of time, θ units of time before time t at which the corresponding output is available for direct consumption. This definition implies that the function $\varphi(\cdot)$ characterizes the allocation over past years of the value of inputs needed to make a given consumption available at time t. In a stationary economy, global initial factor income equals precisely the value of the production available for consumption. One will then write:

$$(20) \qquad Y_{NW} = \int_0^{+\infty} \varphi(\theta)\, \mathrm{d}\theta$$

where θ is taken as a continuous variable. On Figure 3, the hatched area under the graph of $\varphi(\cdot)$ is equal to global initial factor income. In a stationary economy, the function $\varphi(\theta)$ thus also characterizes the allocation of global initial factor income between the different stages of production. It therefore is called the 'characteristic function'. Its graph is the 'characteristic curve'.

Allais very quickly noticed (Allais, 1947b, pp. 126, 188, etc.) that this function is nonambiguously defined only if the interest rate is given, for obvious reasons (already mentioned in II.1 above). On graph 3, if the rate of interest were to decrease, as global initial factor income can be considered as given, as a first approximation, the hatched area would remain constant and the slope of the curve would become closer to the horizontal at every point. This function has thus to be denoted by $\varphi(\theta, i)$, which we shall do, following Allais, hereafter.

One of the interesting features of this model of production is to allow for an analytic expression of the variation of national income Y_N as a functional of the characteristic function, for a given global initial factor income. Indeed, as the value at time t of the global initial factor income $\varphi(\theta)\, d\theta$ spent at time $(t - \theta)$ is $e^{i\theta}\varphi(\theta)\, d\theta$, national income in a stationary economy can be expressed as:

$$(21) \qquad Y_N = \int_0^{+\infty} e^{i\theta}\varphi(\theta)\, \mathrm{d}\theta$$

which can be approximated by

$$(22) \qquad Y_N = \int_0^{+\infty} \left(1 + i\theta + \frac{i^2\theta^2}{2} + \dots\right) \varphi(\theta)\, \mathrm{d}\theta.$$

As the continuous time version of (19) can be expressed as:

$$(23) \qquad \Theta_N = \frac{\int_0^{+\infty} \theta\varphi(\theta)\, \mathrm{d}\theta}{\int_0^{+\infty} \varphi(\theta)\, \mathrm{d}\theta}$$

we can combine (20), (22) and (23) to obtain the convenient approximation:

$$(24) \qquad Y_N \sim Y_{NW} + i\Theta_N Y_{NW}.$$

The global surplus will be nothing other than the total differential of consumption at constant prices (see e.g. Equation (5) above), which, in a stationary economy, will be the total differential of national income at constant prices, $\mathrm{d}Y_N$. We can obtain it from (21) above as:

$$(25) \qquad \mathrm{d}Y_N = \mathrm{d}i \int_0^{+\infty} e^{i\theta} \frac{\partial\varphi(\theta, i)}{\partial i}\, \mathrm{d}\theta.$$

From (24), we can also obtain an approximated formulation:

$$(26) \quad dY_N \sim Y_{NW} \frac{d\Theta_N}{di}\, i\, di.$$

From this formulation of the global economic surplus in a stationary economy, we can see that, this surplus having to be zero in a state of maximum efficiency (Pareto optimal state), the necessary and sufficient condition for such a state to obtain in the production sector of a stationary economy is that the interest rate i be zero.

In this Allaisian framework it can be shown that

$$(27) \quad \frac{1}{\Theta_N} = i + \frac{1}{\Theta_0}\,.$$

Thus, an infinite interest rate implies an average period of production equal to zero. An interest rate equal to $-(1/\Theta_0)$ would entail an infinite average period of production. As for $i = 0$, one can see immediately that it implies $\Theta = \Theta_0$. The latter then appears as the optimal value of the average period of production in a stationary economy.

Several other analytical and practical implications of the model were derived by Allais (Allais, 1962, 1965b, 1967). In particular, the assertion that economic welfare and progress are due rather to the level of education and knowhow and to the pressure of the competitive environment on economic agents, than to the quantity of capital per capita has been constantly supported by Maurice Allais as a consequence of his theory of capital and growth.

II.3. *The Cash Balance Model*

As the great synthesis on interest rate theory developed in Chapter 10 of *Economics and Interest* points out, an interest rate plays four different roles in turn:

1. it influences the value of accumulated capital at a given time;
2. it influences the savings–consumption ratio;
3. it influences the capitalistic intensiveness of production processes; and
4. it represents the price of holding liquid assets.

Maurice Allais' contributions to the first three aspects have already been mentioned. In the last one, studied in Chapter 8 of this decidedly inexhaustible book, Allais managed to develop the essentials of

Baumol's celebrated model of cash transaction demand in two footnotes (Allais, 1947b, nn. 11, pp. 238–239, and 12, pp. 240–241). This contribution also went unnoticed until the present author mentioned it (Munier, 1986, p. 33) and the remark spread. An account of this fact appeared in (Baumol and Tobin, 1989).

To finish on a slightly personal note, I would like to mention that this remarkable book (Allais, 1947b) went almost as unnoticed in France as it did abroad (language reasons obviously played a role in the latter case). I thus recall that, being a student in economics, I had not been able to find the book in the library of my own university. Taking advantage of a trip to Paris, in the late 'sixties, I went to the main economics library of the Sorbonne and found it. Unfortunately, I was not allowed to borrow it, not being a registered student at that institution. I could only photocopy it on a machine to which students had access within the library. But no copy card could then be purchased and one needed coins to operate the machine: one coin per page. But recall that the books has 800 pages! This is why I could read the entire book only later, in 1977, when Maurice Allais gracefully offered me a copy!

III. DECISION THEORY UNDER RISK

Soon after World War II Allais heard of von Neumann and Morgenstern's 1947 second edition of the *Theory of Games and Economic Behavior*. He confronted with it the observations on rationality in risk-taking he had made before the war. Recall that a friend of his had talked him into betting on horse races, following some rule of thumb as well as the advice of a specialized newspaper. Allais made some simulations on past data and tried some real bets. On this first basis, he observed that the advice given was not sufficient to constitute a solution to the real decision problem raised. This problem could be reduced to: either wager small amounts and receive average positive but modest gains; or stake large sums and hope for important rewards, but run a substantial risk of being ruined. To Allais, the true rationality problem was thus to determine some trade-off between expected gains and the probability of ruin.

As for himself, though, he ran simulations on a longer period of past data, took into account the taxes on this kind of income, and quickly stopped betting! But he recalled the observation on the rationality in betting.

TABLE III

Selected writings of Allais on decision theory under risk.

1953b:	'Généralisation des Théories de l'Equilibre Général et du Rendement Social au Cas du Risque', *Actes du Colloque International sur le Risque*, Paris, Editions du CNRS, Vol. 40 of the "Colloque Internationaux" series, pp. 81–109.
1953c:	'L'Extension des Théories de l'Equilibre Economique Général et du Rendement Social au Cas du Risque', *Econometrica* **21**, 269–290.
1953d:	'La Psychologie de l'Homme Rationnel devant le Risque, la Théorie et l'Expérience', *Journal de la Société de Statistique de Paris* (Janvier–Mars), 47–73.
1953e:	'Le Comportement de l'Homme Rationnel devant le Risque, Critique des Postulats et Axiomes de l'Ecole Américaine', *Econometrica* **21**(4), 503–546.
1953f:	'Fondements d'une Théorie Positive des Choix Comportant un Risque et Critique des Postulats et Axiomes de l'Ecole Américaine', *Actes du Colloque International sur le Risque*, Paris, Editions du CNRS, Vol. 40 of the 'Colloques Internationaux' series, pp. 257–332.
1957:	'Method of Appraising Economic Prospects of Mining Exploration over Large Territories, Algerian Sahara Case Study', *Management Science* **3**(4), 285–347.
1979:	'The So-Called Allais Paradox and Rational Decisions under Uncertainty', in: Allais, M. and Hagen, O. (Eds.), *Expected Utility Hypotheses and the Allais Paradox*, Dordrecht, Reidel, pp. 437–681.
1983:	'Fréquence, Probabilité et Hasard', *Journal de la Société de Statistique de Paris* **124**(2), 70–102, and (3), 144–221.
1984b:	'The Foundations of the Theory of Utility and Risk', in: Hagen, O. and Wenstøp, F. (Eds.), *Progress in Decision Theory*, Dordrecht, Reidel, pp. 3–131.
1986a:	'Determination of Cardinal Utility According to an Intrinsic Invariant Model', in: Daboni, L., Montesano, A., and Lines, M. (Eds.), *Recent Development in the Foundations of Utility and Risk Theory*, Dordrecht, Reidel, pp. 83–120.
1987b:	'Allais' Paradox', in: *The New Palgrave, A Dictionary of Economics*, London, Macmillan, Vol. 1, pp. 78–80.

TABLE III (continued)

1988:	'The General Theory of Random Choices in Relation to the Invariant Cardinal Utility Function and the Specific Probability Function: The (U, θ) Model, A General Overview', in: Munier, B. (Ed.), *Risk, Decision and Rationality*, Dordrecht, Reidel, pp. 231–289.
1991:	'Cardinal Utility: History, Empirical Findings and Applications, An Overview', *Theory and Decision* **31**(2–3), Special Issue on the FUR conferences.

III.1. *The Controversy of the 'Fifties*

Regarding von Neumann and Morgenstern's (vNM's) work, Maurice Allais immediately had the intuition that an important aspect of the psychology of risk was missing in the new theory: the trade-off between probability of ruin and expectation of earnings – a decisive point in his view – was not included in that theory. Maurice Allais very likely overlooked at first that the vNM utility was defined on a set of lotteries, not on assets or income like D. Bernoulli's utility function. But he has to be all the more forgiven for that, because vNM themselves, during a certain period, drew a similarity between their own concept and the one of Jevons, and the profession was at the time generally confused.

As soon as September 1951, Maurice Allais raised objections (at the European meeting of the Econometric Society in Louvain) to regarding expected utility as a rational rule from a prescriptive point of view or as a good descriptor of economic behavior. His general theme was *first* that there exists an 'absolute satisfaction' index (a cardinal utility in the sense of Jevons) which would associate psychological estimates to monetary gains and that, *second*, a rational decision rule under risk should not only take account of the expectation of these 'psychological values' but should find some way of also taking account of the probability of ruin which the dispersion of the gains – and thus of their associated psychological values – around their mean could entail.

The majority of the economic profession objected that the vNM utility somehow took account of that dispersion and of the risks it implied, although the idea that the curvature of the utility graph was linked to risk aversion was only loosely expressed then (Pratt's and Arrow's contributions were to come only fifteen years later). But the 'neo-Bernoullian' economists, as Allais came to call them (they were then almost the entire profession!), were divided as to the next point to make against Allais' assertions:

- some of them maintained that cardinal utility could only be identical to vNM's utility (this was notably vNM's own opinion);
- others maintained that cardinal utility was mere nonsense and simply did not exist.

The discussion between Allais and either fraction of the 'neo-Bernoullians' thus turned into a dialogue of the deaf. But one must admit that the question is not that simple. When we say in everyday economics that the expected utility of a lottery is the probability-weighted sum of the utilities of the possible gains, we are being rigorous if we are within a Bernoulli–Allais framework (a utility function defined on gains and losses, i.e. on the real line); we are not so if we are within a vNM framework (a utility function defined on lotteries). Indeed, in the latter case what we ought to say is: the utility of the lottery is equal (due to the set of axioms vNM use) to the probability-weighted sum of the utilities of the lotteries respectively assigning probability one to each of the possible outcomes ('elementary' lotteries). This lack of rigor in everyday language makes it easy to confuse both concepts, but they are indeed quite different. Whereas, in the Bernoulli–Allais case, the 'curvature' of the utility curve represents only the diminishing marginal valuation of income, in the vNM conception it claims to represent the attitude towards risk as well (Pratt, 1964). Yet the same term of 'cardinal utility' has been applied to either of these constructs. This term thus needs to be clarified and qualified whenever it is used, which was not realized in the 1950s and in the 1960s. Hence the impaired clarity of the discussion at the time.

In order to come out of that dead end, Maurice Allais tried to find counterexamples *which would not depend on the cardinality or the ordinality of the utility*: this was the origin of his 1952 experiment and thus of the famous 'Allais paradox'.

III.2. *Allais' General Theory of Decision under Risk*

To test his intuitions, Allais designed a rather long questionnaire, which consisted of questions and tests.

- Series I and X of questions were related to the general characteristics of the psychology towards risk of the respondents.
- Series VI of questions intended to assemble data on cardinal utility: the results will be described hereafter.
- Series VII, VIII and IX of questions intended to assemble information on vNM utility indexes: results showed that these indexes were unstable for *every* individual questioned from one situation to another.
- Series II to V of questions were in fact tests to investigate whether, independently of the cardinality or ordinality of their preferences, the subjects followed one or the other form of the 'independence axiom' or, on the contrary, if their attitude towards security would lead them to violate that axiom in such or such a situation. Within these questions were tests IIA to IID of the questionnaire, which became famous under the somewhat misleading name of 'Allais paradox'.

It is interesting to list these series of questions, because, already in 1952, they characterize the general Allaisian model of decision under risk, as it appears from Allais' writings between 1952 and 1991. This general model should by no means be reduced to the Allais paradox. In fact, it rests on five basic axioms:

- *Ordering*: all possible choices can be ordered by the decision maker according to their respective 'psychological values'.
- *Dominance*: if prospect A offers gains which, for each of its component events, are greater than the gains offered by prospect B for the corresponding event, then prospect A is necessarily preferred to prospect B.
- *Combination*: prospects combine according to the basic axioms of total probability and conditional probability.
- *Homogeneity and invariance*: the psychological value index is homogeneous and invariant (it will be shown that it can be fitted to every individual up to a single parameter).
- *Cardinal isovariation*: if all cardinal utilities of the components of a prospect increase by a same quantity, the cardinal utility of this prospect increases by the same quantity.

Then, if one denotes by $u(C+g)$ the index of cardinal utility, where C is the personal valuation of the assets of the decision maker and g

some certain gain or loss on top of C, the certainty equivalent V of a given prospect can be given by:

$$(28) \quad u(C+V) = \bar{u} + r\left[\Psi(u-\bar{u})\right]$$

with

$$(29) \quad \bar{u} = \mu_1 = \int_{-\infty}^{+\infty} u\Psi(u)\,\mathrm{d}u$$

and

$$(30) \quad r = f(\mu_2, \ldots, \mu_k, \ldots)$$

where $\Psi(u)$ is the probability density function of gains in the prospect, μ_1 the moment of order 1 of that density, etc., and r a coefficient exhibiting the propensity to risk ($r > 0$) or the aversion to risk ($r < 0$) of the individual. In this formulation, r is a *functional of* $\Psi(\cdot)$. Note that it is *not simply a function of the individual's assets* as in vNM's theory (i.e. for most decision situations in practice an individual constant). On the other hand, Maurice Allais has shown that, when the probability of ruin is small, the value of r becomes negligible and expected utility becomes an acceptable approximation.

III.3. *The Allais Paradox*

The tests IIA and IID of the 1952 experiment evoked hereabove were more specifically designed to investigate whether a subjects' behavior obeyed the 'sure thing principle', i.e. the form given by Savage to the idea of the independence axiom of vNM. Indeed, Allais quite soon felt that the fundamental flaw of the neo-Bernoullian theory lies there: by requiring individual preferences to reflect both psychological evaluation of income and attitude towards risk, it finds itself compelled to admit (through the independence axiom) the idea that the different terms of the mathematical expectation of utility are *independent of each other*, whereas intuition tells us that a decisive part of our judgment, at least for a majority of us, focuses on the strength or weakness of the complementarity of these terms. Hence the inconsistencies of the neo-Bernoullian theory, at least in certain situations for a majority of individuals. Hence also the crucial importance of testing this 'independence' axiom.

We recall these tests (the Allais paradox) and comment briefly on them. Consider two pairs of prospects defined as follows:

a_1:	certainty of receiving	100 million Francs
a_2:	10% chance of receiving	500 million Francs
	89% chance of receiving	100 million Francs
	1% chance of receiving	0

and

a_3:	11% chance of receiving	100 million Francs
	89% chance of receiving	0
a_4:	10% chance of receiving	500 million Francs
	90% chance of receiving	0

Ask then the subjects to state what their one-shot decision would be in each of these pairs. (*Hint*: take a moment for introspection and ask what your own choice would be between a_1 and a_2, on one hand, and between a_3 and a_4 on the other).

Professors Allais and Darmois organized a conference in Paris in 1952 on mathematical economics and risk. The above questions were put to several participants by Maurice Allais, and in particular to authors or supporters of the 'neo-Bernoullian' expected utility theory (L.J. Savage, B. de Finetti, K.J. Arrow, M. Friedman and J. Marschak) as well as to other persons. Most of these individuals, including Savage himself, preferred a_1 to a_2 in the first pair and a_4 to a_3 in the second one, whereas, to be consistent with expected utility maximization, a person who prefers a_1 to a_2 should also prefer a_3 to a_4, and any expected utility maximizer who prefers a_2 to a_1 should prefer a_4 to a_3.

Later on, Allais submitted the questionnaire containing, among others, the above questions to a number of colleagues and students. About 65% of them made similar choices.

Where is the 'paradox'? Strictly speaking, nowhere. It is simply that these results conflict with the generally accepted view of the expected utility hypothesis. This can be straightforwardly shown by the simple calculation of expected utility for a_1, a_2, a_3 and a_4, where the probabilities of the outcomes are decomposed into combinations of 0.01, 0.10 and 0.89 probabilities. The reader can thus verify that preferring a_1 to a_2 on the ground of any type of expected utility entails: $0.01U(100) +$

$0.10U(100) > 0.01U(0) + 0.10U(500)$, and that preferring a_4 to a_3 entails: $0.01U(0) + 0.10U(500) > 0.01U(100) + 0.10U(100)$, an obvious contradiction to preferring a_1 to a_2.

Why, then, does such a large majority fall into the trap of the paradox? There is no 'trap', says Allais; but an example – of seemingly quite general value – of the complementarity between the consequences of different events, which shows that decision makers change their attitude towards risk in the vicinity of certainty. A very strong risk aversion is thus associated with the choice between a_1 (which is a *certain* option, differing from a_2 only through some event's consequences) and a_2. Such a strong feeling does not come into play when choosing between a_3 and a_4.

Allais had put the finger on what is now termed the 'certainty effect' in experimental decision science. But he suspected, without being able to prove it immediately, that the lesson was more general. One could formulate it in this way: *attitudes towards risk change not only from an individual to another, but also for a given individual between different patterns of risk.* This raises a challenge to vNM utility theory, for attitude towards risk is exhibited in that framework only by the curvature of the utility function, a characteristic parameter of the individual. In other words, in vNM's theory, attitudes towards risk can change from one individual to another, but *not* between different patterns of risk for the *same* individual.

This proposition has since then been corroborated by many experimental findings and empirical observations, obtained in different countries by different researchers of different backgrounds and cultures. In addition, these experiments reveal that attitude towards risk of a given individual changes *in a systematic way* (Machina, 1983), not just through random errors or misperceptions.

Meanwhile, Maurice Allais (1984b, 1986b, 1988, 1990a) made his views more explicit and testable in several important contributions. He insisted that one has to clearly distinguish between marginal valuation of income (or of wealth), on one hand, and attitude towards risk on the other, and to assign to each concept a separate analytical tool. Note that this is sound methodological practice, as the law of requisite variety (which economists know under the name of Tinbergen's principle) shows (Ashby, 1947).

III.4. *Cardinal Utility: Experimental Findings*

In 1984, at the FUR II conference in Venice, and in 1988 at the FUR IV conference in Budapest, Maurice Allais presented experimental investigations of the cardinal utility function. He showed (Allais, 1986) that the adjustment obtained (with a single parametric degree of freedom) on the 1952 data very well fitted five other sets of answers obtained in a 1975 complementary questionnaire. On the whole, the function is approximately log-linear for a wide interval of values, with asymptotic behavior for the very large gains (satiety effect) and a rigorously log-linear expression in the vicinity of 0 (what Allais interprets as a real balance effect). In Budapest (FUR IV) Maurice Allais used a different questionnaire (dating back to 1987) to investigate the loss side of the cardinal utility function. The latter is shown to decrease very fast with losses (Allais, 1991), the utility function also retaining its concavity for losses.

III.5. *A New Operational Formulation of Allaisian Theory*

In his 1986 (U, θ) model (Allais, 1988), Maurice Allais suggests that, in the same way as the (Bernoulli–Allais) utility function $U(\cdot)$ modifies the scale of the monetary outcomes (to reflect marginal valuation of income), a 'specific' probability function $\theta(\cdot)$ distorts the scale of the cumulative probability distribution (to reflect attitude towards risk in the specific risk-situation faced by the individual). A lottery P is then valued as a functional $A(P)$ which will be some expression of the 'distorted' probabilities, linearly weighted by the corresponding utilities. This linear weighting has been Maurice Allais' hypothesis to start with, as early as the 'fifties and, of course, when he went on working on this formulation of his general model. One can write:

$$\text{(31)} \quad \begin{aligned} A(P) = {} & u_1 + \theta(1 - p_1)(u_2 - u_1) \\ & + \theta(1 - p_1 - p_2)(u_3 - u_2) \\ & + \ldots + \theta(p_n)(u_n - u_{n-1}) \end{aligned}$$

which, when $\theta(\cdot)$ reduces to the identity function (i.e. in the very special case in which there is no probability distortion), returns to expected utility. When the probability of ruin is close to 0, $\theta(\cdot)$ is close to identity, in other words.

Similar expressions (but using a different utility function in each case) have been independently found by J. Quiggin in 1982 (Quiggin, 1982), by M. Yaari in 1984 (Yaari, 1987) and by others, notably Segal. J. Quiggin worked explicitly on the former suggestion by Allais of the linear weighting by the cardinal utilities (then still unexplored by Allais), whereas M. Yaari used a totally different axiomatic type approach of his own. All these models, however different they may be, have been subsumed under the heading of the 'anticipated utility' hypothesis (Munier, 1992). This could be the ultimate result of Maurice Allais' contribution to decision under risk. It can be regarded as one particular and elegant analytical expression of the original idea of Allais – recall the pre-War experience on horse-race betting – to balance the expected payoff in psychological value and the probability of ruin.

III.6. *Applications*

Direct applications of the same idea were recommended by Maurice Allais in practical studies. In particular, Allais suggested that, when the question to investigate is a public policy problem, the official authorities should make their decision in what would be termed today an interactive manner, i.e. on direct visualization of the values of the probability of ruin respectively associated with the values of expected consequences (gains, costs, etc.).

The best known example of this method is a study intended to rationalize mining exploration in the Algerian Sahara desert. The study he conducted in the mid-fifties (Allais, 1957) won him the Lanchester Prize in Operations Research, awarded by Johns Hopkins University.

To put his idea into practice, Maurice Allais had to determine the statistical distribution of the global value of mining sites which could be discovered in nonsedimentary Sahara, and then to compute the probability of profitability of the corresponding exploration strategy as well as its variation as a function of the area explored. Then, by taking into consideration the expected costs of the associated exploration, the probability of ruin could be exhibited.

Maurice Allais managed to show that such a statistical distribution of the global value of potential sites could be established, however unexpected this might appear, by considering nonsedimentary areas already explored in the world (it is an approximately log-normal distribution). He also showed that the necessary computations could be performed by looking at the same facts: the probability of discovery follows a Poisson distribution, the scalar parameters involved follows a Laplace–

Gauss law. But, in the end, the main idea behind the presentation of the statistical results is the search for a reasonable compromise between the expectation of gains and the probability of ruin.

The same view of risk psychology was also used, under a more restrictive specification, to extend the equivalence theorems between general equilibrium and maximum efficiency to a risky environment (Allais, 1953b, 1953c). Indeed, at the same 1952 conference already mentioned, K.J. Arrow and M. Allais presented alternative models of this question. Allais' model displays two goods, one certain (x) and one random (y), the latter being assumed to be normally distributed. Preference fields are such that every utility index is a function of expectation and standard deviation. Prices are determined in the standard general equilibrium way. The model shows that equivalence theorems between market equilibria and Pareto optima can then be generalized. The model is then extended to more general cases, where possibilities of buying insurance contracts or lottery tickets are accounted for. A distinction is drawn between globally ineliminable risks and globally nonexistent risks (like lotteries), which plays a major role in the discussion. The French version is measurably longer (Allais, 1953b), but slightly more explicit than the (Allais, 1953c) version.

In Arrow's celebrated model (Arrow, 1953), money has a specific usefulness only in the (admittedly likely) case where equities are less numerous than the states of the world, whereas such a restriction would not be needed in Allais' model. But Maurice Allais' main contributions to monetary theory are to be found elsewhere.

IV. MONEY AND MONETARY DYNAMICS

Although the point of departure of Allais' writings in monetary theory is again *Economics and Interest* (Allais, 1947b), most contributions of Maurice Allais in this domain were written and published in the 'fifties and extended in the 'sixties and 'seventies. They can definitely be subsumed under the quantitative theory tradition, but with two decisive qualifications.

1. On one hand, the demand side of the theory is extensively developed, in such a way that it becomes much less mechanical than in past contributions. One could venture to say that Maurice Allais, still fond of physics, wants to look into that discipline for a more

sophisticated inspiration for his theory than the one of classical mechanics: he finds it in relativity theory. Hence the concepts of 'psychological' time and 'psychological interest rate'.

2. On the other hand, lags in the realm of money management have been overlooked, in Maurice Allais' view, and will be rehabilitated in his monetary writings. Unlike physicists, economists have largely ignored reaction periods: here, Maurice Allais introduces some important considerations into economics.

TABLE IV

Selected writings of Allais on money and monetary dynamics.

1953g:	*Illustration de la Théorie Monétaire des Cycles Economiques par des Modèles Non Linéaires*, Paris, Centre d'Analyse Economique, Ecole Nationale Supérieure des Mines de Paris. Mimeo.
1954:	'Illustration de la Théorie Monétaire des Cycles Economiques par des Modèles Non Linéaires' (shortened version), *Econometrica* **22**(1), 116–120.
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1965c:	'Reformulation de la Théorie Quantitative de la Monnaie', *Bulletin SEDEIS*, Paris, special supplement to No. 928, 186 pp.
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1972:	'Forgetfulness and Interest', *Journal of Money, Credit and Banking* (1), 40–71.
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TABLE IV (continued)

1977:	*L'Impôt sur le Capital et la Réforme Monétaire*, Paris, Hermann. 370 pp.
1986b:	'The Empirical Approaches of the Hereditary and Relativistic Theory of the Demand for Money: Results, Interpretation, Criticisms and Rejoinders', *Economia della Scelte Pubbliche*, Fondazione Luigi Einaudi, **1**(2), 3–83.
1987c:	'The Credit Mechanism and Its Implications', in: Feiwel, G. (Ed.), *Essays in Honor of Kenneth Arrow*, London, Macmillan, pp. 491–561.
1989:	'Les Conditions Monétaires d'une Economie de Marchés', talk given at the Association Française de Science économiques, November 9, 1989. Mimeo.
1990a:	*Pour la Réforme de la Fiscalife*, Paris, Clément Juglar. 131 pp.
1990b:	*Pour l'Indexation*, Paris, Clément Juglar. 183 pp.

IV.1. *The Demand for Money and the Interest Rate*

Maurice Allais' theory of the demand for money has been given the name of HRL theory by its author. H stands for 'hereditary', R for 'relativistic', and L for 'logistic'. This lengthy appellation calls for immediate explanation. Let us first get rid of the logistic aspect, which has to do with the (logistic) form (determined by Allais on the basis of his hypotheses) of some function in the model, and which is therefore of relatively minor importance.

The two fundamental concepts which explain the hereditary and the relativity aspects of the theory are respectively the concept of 'rate of oblivion' (or of 'forgetfulness') $\chi(t)$ and of 'psychological time' t'.

On the one hand, individual behavior in HRL theory is conditioned by the memorizing of the past, not by expectation of the future. Collective memory is decisive in economics. Allais builds a theory in this respect, which lets oblivion work with respect to the past in the way the interest rate works with respect to the future (axiom P_1), in a perfectly symmetric perspective and, of course not without some connection with each other.

On the other hand, we usually count in physical time in economics, whereas this is completely arbitrary for human activity. Maurice Allais will then consider, not unlike in relativity theory, two different types of time: the standard physical one, denoted by t and the 'psychological' one denoted by t'. Only the latter will allow us to disclose the real 'constants' of economic activity. Thus, it is within the reference frame of psychological time t' that the rate of oblivion and the 'pure' discount rate (which Allais will thus call 'psychological rate of interest', i) will be constant and equal. On another hand, the rate of oblivion in standard physical time is variable: the more rapidly economic magnitudes evolve within that frame of reference, the more rapidly we forget (the higher our rate of oblivion). Thus, in a hyperinflation we forget very quickly what happened last month, or even yesterday, whereas in a slowly growing economy with stable prices, our monetary behavior is conditioned to a greater extent by what happened a month ago. In a stationary economy, the psychological and the standard rate of interest will both be equal and thus also equal to the rate of oblivion.

To give more formal definitions of all these variables and derive the relations between them, Maurice Allais then has recourse to the following set of further axioms:

P$_2$: Let $x(t)$ be the relative rate of change of global expenditure D at time t:

$$x(t) = \frac{1}{D}\frac{\mathrm{d}D}{\mathrm{d}t} \,.$$

Then define $Z(t)$ as a 'coefficient of psychological expansion'. At any period t, this coefficient can be expressed as:

$$(32) \qquad Z(t) = \int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \exp^{-}\left[\int_{\tau}^{t} \chi(u)\,\mathrm{d}u\right] \mathrm{d}\tau.$$

P$_3$: Relative desired cash balances $\Phi_D = M_D/D$ are an invariant function Ψ of $Z(t)$, which Allais denotes by $\Psi(Z) = \Phi_D/\Phi_0$. Furthermore, the limiting conditions are specified: $\Psi(-\infty) = 1 + b$, $\Psi(+\infty) = 0$, where b is a constant.

P$_4$: The connection between physical time t and psychological time t' is defined by

$$(33) \qquad \chi\,\mathrm{d}t = \chi'\,\mathrm{d}t'$$

where the unit of time t' is defined by the constancy of χ'. This would call for interesting justifications and discussions, which we leave to the reader.

P$_5$: The velocity of circulation of the desired cash balances is constant when measured with respect to psychological time:

$$(34) \qquad V'_D = \frac{D'}{M'_D} = \text{constant.}$$

From these two last postulates and from the preceding relations, it follows that $M'_D = M_D$, that $D'\mathrm{d}t' = D\mathrm{d}t$ and that the oblivion rate χ is a function of Z, and it can be expressed as:

$$(35) \qquad \chi = \frac{\chi'}{\Phi_0} V'_D \Psi(Z).$$

It thus depends on t only through $Z(t)$.

P$_6$ (logistic formulation): It is assumed that:

$$\frac{1}{\Psi}\frac{\mathrm{d}\Psi}{dt} = -\alpha\left[1 - \frac{\Psi}{1+b}\right]\frac{\mathrm{d}Z}{dt} \qquad \alpha = \text{constant.}$$

It follows, by integration and by the limit conditions given above for $\Psi(t)$, that:

$$(36) \qquad \Psi(Z) = \frac{1+b}{1+be^{\alpha z}}$$

Ψ is thus a logistic function of Z.

P$_7$: It is assumed that:

$$\lim_{Z\to+\infty} \frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{1}{\Psi}\frac{\mathrm{d}\Psi}{\mathrm{d}t},$$

which, according to **P$_6$**, entails that $\alpha = 1$.

P$_8$: Around $Z = 0$, the derivative $\mathrm{d}\Psi/\mathrm{d}t$ is a constant.

P$_9$: $\forall t$, $\chi(t) = i(t)$ is a (fundamental) postulate of intertemporal psychological symmetry.

P$_{10}$: Approximate equality of desired and real cash balances:

$$\left\| \frac{M - M_D}{M} \right\| < \varepsilon \tag{37}$$

with ε as small as required but always positive.

From these conditions, two basic equations can be derived, which allow to construct tests of the theory and forecasting models:

1. a 'hereditary' equation of desired cash balances:

$$M_D = \Phi_0 D \Psi(Z), \quad \text{with} \quad \Psi(Z) = \frac{2}{1 + e^z} \tag{38}$$

2. a 'relativity' equation:

$$i(t) = \frac{\chi_0}{\Psi(Z_t)} = \chi(t). \tag{39}$$

Allais shows also that the following differential equation can be used to run tests of the theory:

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} + \frac{\chi_0}{2}(1 + e^{Z^t}).Z_t = \frac{1}{D_t}\frac{\mathrm{d}D_t}{\mathrm{d}t}. \tag{40}$$

For further indications as to how to technically devise tests of the HRL theory, see the chapter by J.-J. Durand in this book.

This – admittedly somewhat subtle – model of money demand allows a prediction of both the interest rate (see the 'relativity' equation above) and the stock of money (see the 'hereditary' equation) in a given economy. The predictive power of the model appears quite strong in comparison with other existing models. Numerous tests have been performed by Allais over different periods and data series (Allais, 1965c, 1966, 1972, 1974). An interesting test on U.S. money stock data (M_2) (Bethenod, 1979) is worth looking at, as is a test on U.S. interest rate (prime commercial paper) data (Durand, 1977). As these tests have been performed, at least for those which have been run prior to the 1970s, with only one or (more generally) two (depending on the variables selected for the tests) degrees of freedom, such good results have stirred some suspicion. Some analysts have mentioned potential spuriousness in the correlations; others, circular reasoning (Darby, 1970; Scadding, 1972). Maurice Allais (1970, 1975, 1986b) has convincingly countered these arguments. For further discussion, see the chapter by J.-J. Durand in this book.

This theory of the demand for money amounts, as the reader may now grasp, to a specific restatement of the quantity theory of money. The subtlety of Maurice Allais has consisted in understanding that many of his predecessors had confused the reference frames of physical time and of psychological time. In particular, in the reference frame of physical time, the velocity of money circulation, far from being the mere *constant* that traditional quantity theory of money has hypothesized, is a stable *function*, incorporating memorized past events in the way we have seen.

Allais' theory of the money supply M_t^S is quite symmetric to the theory of the demand for money M_t^D. As both quantities depend from past variations of D, Allais shows that, by introducing these two fomulations into a single equation (equation 41 hereunder), it is possible to build a monetary model of business cycles.

IV.2. *Dynamic Theory of Business Cycles*

The second aspect of Allais' monetary contributions which we would like to mention here is, indeed, a dynamic theory of business cycles (Allais, 1953g, 1954, 1956a, 1956b, 1968) based on the consideration of the gap between money stock and money demand, through what Allais (1968, Vol. 1, pp. 75–83, Vol. 2, pp. 132–134) has called his 'fundamental equation of monetary dynamics'.

If we denote by V the velocity of circulation, E the indebtedness of the economic agents outside the banking sector (this is a factor which was important in the XIXth century and which is once again starting to be nonnegligible), T a reaction lag and M the money supply, this equation is:

$$\begin{aligned}\frac{1}{D}\frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{VT^2}\left(\frac{M-M_D}{M}\right)+\frac{1}{VT}\left(\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}t}\right)\\ &\quad +\frac{1}{T}\frac{E}{D}\left(\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}t}\right).\end{aligned} \tag{41}$$

Allais' monetarism holds that money supply management is all the more difficult to perform because it is a *lagged regulation* process. It would be difficult enough with one center of money creation. If, in addition, private banks do make profits in issuing money, as they do nowadays, there is every reason to fear that they will ignore the duty of regulation, and regulating the economy will become impossible. Note that another reason for private banks not being allowed to create money is that part of their profits is then rent of seniorage: we have seen that Allais does not want rents to accrue to private agents, in line with an old tradition in economic theory. He consequently explains (Allais, 1977, 1987c, 1989) the monetary conditions for an efficiently working markets economy (note the 's', again). The commercial banking sector should be organized so as to allow for the depositing of funds by the public, but keep money creation outside the reach of commercial banks. The creation of money should stay within the sole hands of the government, which would allow it to regulate the economy and at the same time receive all the seniorage. This latter fact would be quite a happy one, for income tax could be done away with. To make up for the difference, a general tax on all durable physical assets could be raised (Allais, 1990a). A general indexation of all future commitments should finally be introduced (Allais, 1990b). This would, among other advantages, do away with the tacit collusion of the inflation lobby.

CONCLUDING REMARKS

Maurice Allais' economic writings have certainly not been exhaustively analyzed in this chapter. It would have been necessary to mention a number of other contributions, in particular on the relationship between growth and inflation (Allais, 1969b), on all sort of economic policies (Third World questions, international relations liberalization, etc.) and on methodological questions (statistical methods, testing, philosophy of probability, etc.). Again, I have not even mentioned the noneconomic contributions, which were outside the scope of this survey. My excuse for these omissions is that a complete bibliography is given at the end of this volume and also that concentrating on the most important writings and on the contributions most likely to be recalled in the future is not necessarily a loss: one can be more explicit and hopefully more articulate and – who knows – perhaps more convincing.

I have tried, indeed, to convince the reader of this survey, not that Allais' contributions are perfect, but that they really are worth reading because they are inspired, full of ideas, even in the eyes of present economists. Young economists would be well advised to read them: what I have shown here of these writings is not exhaustive; the rest might reveal as many fruitful and little circulated ideas as what has been surveyed above.

Maurice Allais' economic writings in the fifty years which have just elapsed should thus enjoy a particular status in the history of economic thought. On one hand, they have led to important discoveries which have sometimes been attributed, at least temporarily, to others, and they are thus full of illustrations of Merton's theory of scientific progress. On the other hand, they have also started some unusual research programs and probably contain the seeds of useful and most original future contributions to economic science.

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THIERRY DE MONTBRIAL*

2. MAURICE ALLAIS, A BELATEDLY RECOGNIZED GENIUS

In a collective work such as this it is natural that some writers should take a more personal approach than others to the subject being honored. This is what I wish to do in the following pages. Who can remember without emotion the courses and seminars Maurice Allais gave at the Ecole des Mines? The walls strewn with blackboards covered in equations and graphs, the assistant ready to start the tape recorder as soon as the thoughts of the master began to flow, the quiet assurance of a man convinced that he held the key to many mysteries. A host of observations, half serious, half humorous – in fact often profound – embellished his lectures, for example:

> If you have not understood a text that you have just read, reread it; if, having reread it you have still not understood, read it yet again; if you still don't understand, question yourself on your intellectual capacity. If you consider objectively that this is not in question, you may conclude that it is the author of the text who is at fault.

For a man like myself, used to regarding printed matter with reverence, it was an important lesson. If a member of the audience arrived late for the lecture, Maurice Allais interrupted himself for several minutes and with great kindness tried to demonstrate, from the law of large numbers, that the only way to avoid being late was to always try to be early.

Everyone knows the role played by Maurice Allais in the training of quantitatively oriented economists in France. He has inspired and encouraged numerous careers and some of his former students are today well known in their fields. Yet, even if he had a privileged relationship with some of his students – and I think that this was true in my case – Maurice Allais has always been a man on his own. Like other powerfully creative people, scientists or artists, he has suffered from his isolation. Although honored with the gold medal of the French C.N.R.S. in 1978, his work had never received the international acclaim it deserved until Maurice Allais was awarded the 1988 Nobel Prize in Economic Science. It seems that this long lack of recognition can be explained by the specific

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character of his genius. First of all, his style is extremely personal and not easily grasped. He resorts to his own vocabulary. Even his way of using mathematics displays a certain originality! His numerical methods are also unique. It is because of this that some of the few economists who have studied in detail his imposing volume *Hereditary, Relativistic and Logistic Theory of Money* (the 'HRL' Theory) have not understood his econometric techniques and pointed out that the perfect correspondence between the facts and the results obtained indicates a circularity in the arguments. The work is significant, voluminous and diversified. The articles, theses and studies are all interconnected. Thus considerable effort is required to gain a complete understanding of Allais' work, a commitment that almost no economist of international repute has up to now, in my opinion, been prepared to make. This is why I am convinced that his work will long continue to be a goldmine, open to the exploration of future generations of researchers. It will perhaps suffer the same fate as the works of Walras, Pareto or Edgeworth, the importance of which has only been completely understood long after their publication. It is not simply a question of recognising that Allais was the author, or co-author, of a number of 'discoveries' which today comprise part of the accepted body of knowledge in Economic Science. I quote for example:

- the first general demonstration of the equivalence of maximum efficiency situations (Pareto optimum in current terminology) and market equilibrium, studied previously by Pareto, Barone and Lange;
- the theory of economic losses beyond the optimum;
- the 'golden rule' – wrongly accredited to the American Phelps – which states that the optimum capital – i.e. the maximum growth of consumption per capita – is achieved when the real interest rate is equal to the rate of economic growth (the population growth rate in conventional literature, the growth rate of 'original revenue' according to Maurice Allais);
- the formula expressing the demand for monetary funds as a function of the revenue, the interest rate, and the investment costs which the Anglo-Saxon literature attributes to Baumol, although the article written by the latter was published five years after Allais' work appeared;[1]
- the explanation of economic cycles by retarded regulation, nonlinear models allowing 'limit cycles', an idea generally attributed to Goodwin alone;

– the modern form of the quantitative theory of money, derived from a 'hereditary' formula, almost always exclusively associated with the Chicago School.

In certain cases Allais' claim to these discoveries is indisputable. In others, the same result has been achieved separately by two or even three researchers. This has often occurred in the history of science: a classic example is Mariotte's Law, which is called Boyle's Law in the English-speaking world. In certain cases, too, it is possible that Allais' formulations are better than the conventional statements. I am thinking, for example, of the use that he makes in his theory of capital of the idea of 'characteristic function', and of his own version of the golden rule: the capitalistic optimum. But all of these examples already belong to the history of economic thought. Too often researchers have paid insufficient attention to the work of Allais. It is certainly true of his theory of risk. He very strongly refutes the theory of the neo-Bernoullian school, i.e. propositions which enable comparison of risky choices to be reduced to the mathematical expectation of a von Neumann utility function. Almost all theoreticians today use the von Neumann–Morgenstern formula, which is mathematically very convenient, without sufficiently taking into account whether or not it adequately represents the behavior of economic data. The theory of capital is also a field where, in my opinion, Allais has paved the way for further research, which will have to be resumed, just as the founders of the modern theory of temporary stability rediscovered the third and the fourth parts of *Value and Capital* by John Hicks (1st edition, 1938) in the early 'seventies. Ever since his book *Economics and Interest* was published in 1947, Allais has striven in the wake of Irving Fisher, to bring about a synthesis of the real and monetary aspects of interest, and to introduce uncertainty in an explicit way.

His recent work, in conjunction with the HRL theory (particularly on the psychological equivalence of oblivion and interest) have extended the intuitions of 1947. It is certain that the theoreticians of today and tomorrow, with the courage to immerse themselves in these works, will discover a great deal. I have quoted Hicks, but one can obviously also think of Keynes, whose *General Theory* – if the truth be told, not easy to understand either, and full of obscurities – was completely reinterpreted in the late 'sixties by authors of whom Leijonhufvud is one of the most notable.

It also seems to me that the theory of economic markets (with an 's') developed by Maurice Allais from 1966 onwards, should merit a more profound examination by the profession. The author makes every effort to state a system of rules which allow an economy to develop progressively, by effective use of all its possible surpluses, towards a state of maximum efficiency. He considers that he has thus freed the optimum theory from mathematical hypotheses which are unrealistic, in his eyes, like that of general convexity. In this new perspective, prices play only a secondary role. I must again cite the monumental *General Theory of Surpluses* (*Théorie Générale des Surplus*) published in 1981, which presents a 'positive theory of surplus' as well as a critical analysis of the literature.

In all his works, Maurice Allais has attempted to confront theory with facts. Some epistemologists think that his conviction that ideology can be dissociated from economists is optimistic, and some statisticians think that he underestimates the difficulties of empirical economics. For a complete judgement on such matters, a critical analysis of his works is required such as, up to now, had not been attempted. I would simply like to recall the fundamental idea that in the social sciences, as well as in natural sciences, the validity of a theory depends on how well it fits the facts. What distinguishes Allais in this field from most contemporary economists is that he wants to apply this rule to all cases, even the most abstract ones, and he appears to believe in the existence of economic 'universal constants', such as those in the physical sciences.

In contrast to many of his fellow theoreticians. Allais has, in addition, always thought that the economist, equipped with all his current knowledge, should express his opinion – even commit himself – on the problems of his time. Thus, an important part of his publications is found to be devoted to applied economics (especially the economics of energy and transport pricing and to some other serious economic policy problems. For example, I quote three representative works:

1. *The Liberalization of International Economic Relations*, 1972
2. *French Inflation and Growth – Myths and Realities*, 1974
3. *Tax on Capital and Monetary Reform*, 1977.

These works, and others in the same vein, are obviously infinitely easier to understand than the books on research and it is a pity that they have not been sufficiently studied and applied. It is true that Maurice Allais has never been a media figure. Upon reading these books, it becomes apparent that his ideas are varied and not confined to the strict liberalism of which he is sometimes accused. His social philosophy is undeniable of the *libéral* type (in the French sense, i.e. that it favors both a free, decentralized society and a free-market economy) and, according to him, falls within the same lines of thought as Alexis de Tocqueville's, Leon Walras', Vilfredo Pareto's and John Maynard Keynes' views. But Allais' *libéralisme* is not to be interpreted as doing as *laissez-faire, laissez-passer*. The cornerstone of his thinking is that free trade implies appropriate social and institutional organization, with 'structural planning' (*Manifesto for a Free Society*, 1959). For example, he considers that completely free trade is only possible between sufficiently integrated countries. In particular, excessive exchanges rate variability and salary disparity between two countries are incompatible with free trade. Allais supports a unified Europe, but this will, in his eyes, require strong commitment and a single currency must be accepted.

> For myself, who lost a father in the First World War, how could the mere abandoning of sovereignty be weighed against the great and varied advantages of Federal Union?

he exclaimed during the course of a debate, reported in the *Annales des Mines* (No. 3, 1984).

As a supporter of structural planning, he is a demanding reformer. As an illustration, one should read *Tax on Capital and Monetary Reform* (1977), of which I would like to outline the essential arguments. The author declares himself to be convinced that "we live in times that, from many points of view, are similar to those which preceded or accompanied the decline of the Roman Empire". At the present, French society is not a completely *libéral* society but "is to a large extent founded on a singular mixture or corporatism and collectivism". That does not exclude the fact that "what still remains which is politically and economically *libéral* in our society belongs to a past age". According to the title of a celebrated book by Friedrich von Hayek, we are on *The Road to Serfdom*. But Allais refuses to consider this an inevitable evolution. His project aims at leading the French economy

> to a situation which, from the point of view of income distribution and social justice, will correspond entirely to the aspirations and conclusions of the great social reformers of the past, from the church Fathers to Proudhon and Marx,

while simultaneously becoming more efficient. The author claims as his own idea that allocation of income is not ethically acceptable in France. He asserts that the present tax system aggravates the situation because of its complexity, obscurity and injustices. It is the middle class that in the end bears the brunt of taxation.

Maurice Allais then takes up again the old theme of *libéral* economists regarding the distinction between unearned and earned income. To the first category belongs cash income from real estate, i.e. income arising out of ownership irrespective of the activities of these owners. An example of this is the increased land revenues resulting from improvements made by local authorities or the state. The 'false rights', resulting from fluctuations of the real value of debts and credits in a period of price instability, are also factors which cause unwarranted gains or losses of fortune without reason. These are 'unearned incomes', says Allais, which have always aroused strong opposition to an economy based on free markets. And in fact, "the most valid objection to a society organized on the basis of a free markets economy is founded on the income distribution". He then shows that society could be organized in such a way to preclude unearned income. But "since the time that the hypothesis of eliminating unearned income was posited, all nonsocialist thinkers have arrived at the same conclusion: a fair salary is one that balances supply and demand". At this point, Allais rebels against 'egalitarian demagogy'. Income inequality is perfectly legitimate, provided it reflects services rendered. Such a point of view is not incompatible with a policy of social transfers provided that it does not interfere with price mechanisms. Continuing his demonstration he proposes a 'threefold structure' of taxation. This depends on the following elements:

1. An annual tax of 2% on the value of nonconsumable assets, which would produce about 8% of the national income.
2. Resources deriving from state appropriation of all profits produced by the creation of money, evaluated at about 4.4% of the national income.
3. A general tax on consumable goods (including customs duties) which should raise 16.9% of the national income instead of 18.5% as at present.

The tax on company profits, progressive income tax, tax on inheritance, and the present tax on property and on capital gains ('plus values') would be eliminated.

The author shows the advantages of his propositions in detail, and refutes one by one all the objections which have been made, particularly to the tax on capital. But it should be understood that tax reform, founded on a capital tax, is only conceivable within the framework of a general reform ensuring effective elimination of unearned income and large unmerited profits, that presently occur in the French economy.

This income policy, as it should be called, should be integrated into the framework of a very rigorous monetary policy aiming at implementing the state appropriation of profits connected with the creation of money (as mentioned above) and to enable the government to effectively control the growth of money supply. Hence, the author takes up a position favorable to total coverage of cash deposits, leading particularly to a distinct separation of 'deposit banks' and 'lending banks'. On these questions Allais' theses are very similar to those of the Chicago school, while remaining distinguishable from them.

Having thoroughly criticized the system of partial indexing, the author finally pronounces himself in favor of general indexing for all future undertakings, which is only fair and removes the motivation of a large group of French people to maintain inflation, which has harmful effects, as Allais has powerfully demonstrated elsewhere.

Continuing the subject of monetary policy, Allais is a confirmed monetarist, i.e. he believes that the money supply should be controlled and that its expansion rate can be calculated to make economic growth compatible with price stability (in fact he advocates a 2% inflation rate). But nonetheless he agrees with those who consider that inflation is caused by inequality in the sharing out of the national income. The following passage deserves to be quoted in full:

> In the framework of a complex economy where decentralization of decision making is a necessary condition of efficiency, in the final analysis, the only possible regulation of prices and incomes rests only on the regulation of the money supply. But if one adopts the regulation of expansion rates of the money supply as the principle of regulation of prices and salary levels, the economy can only function if salaries and prices are fixed in such a way as to establish an effective balance between supply and demand. This implies that all economic agents accept the application of the fundamental rules for a decentralized markets economy.
>
> But, such an acceptance is only conceivable if the distribution of incomes, leading to the functioning of a markets economy, appears to be ethically acceptable. One therefore resorts to tax reform and tax on capital. In the absence of confiscation of pure rents from capital, income distribution of a markets economy is resisted. It is this resistance which engenders inflation.

The principal lesson of the book is perhaps the fact that there is a *libéral* project as revolutionary (revolutionary is how Raymond Aron qualifies – in his Preface – the propositions of Allais) as, and without doubt scientifically better founded than, the socialist project.

But in the face of so many demands, one might think that perhaps Allais is right to observe that "True *libéraux* and true socialists are the exception. Well-off socialists and self-serving *libéraux* abound". Besides, the author has no illusions, he warns us from the beginning, on the possibility that the policies he advocates would be adopted and applied. We come back then to the point of departure; if the French are revolutionary in speech only, will serfdom be inevitable at the end of the road?

I have decided to end this contribution in honor of Maurice Allais with a more explicit analysis of one of his books in which is best expressed the complex character of a thought which is sufficiently strong, coherent and well-argued to resist the passage of time.

Ecole Polytechnique,
France

NOTE

[1] The publication of Boiteux, M., de Montbrial, Th., and Munier, B., *Capital, Marchés et Incertitude*, Paris, Economica, 1986, triggered, directly and indirectly, new perceptions on such issues. On the point made here, see Baumol, W.J. and Tobin, J.: 1989, 'The Optimal Cash Balance Proposition: Maurice Allais' Priority', *Journal of Economic Literature* **27**, 1160–1168.

J. LESOURNE*

3. SELF-ORGANIZING MARKETS[1]

The market plays a central part in Maurice Allais' writings. Hence, the theoretical research concerning its operation, its creation, and its destruction seems to me a logical follow-up to the works of the author of *A la Recherche d'une Discipline Economique*. It explains why, as a tribute to a man whose teaching has deeply influenced me, I have chosen to devote this paper to the operation of a market under the impulse of economic agents' spontaneous behaviour.

The junior economist who has just learned that on a market the price level is determined by the intersection of demand and supply curves is convinced that he or she has acquired a basic and final knowledge, like the physicist who has just been presented to the second law of thermodynamics. In both cases, however, the macroscopic proposition hides a world of complexity. And there may be an analogy which is not superficial between Maxwell's demon, which selects molecules, and Walras' auctioneer, who collects demands and supplies before announcing the price.

The usual length of a paper, of course, prevents me from dealing with the subject in all its breadth. Nevertheless, to give the reader a good understanding of the type of issues found in this research field, I have thought it useful to split the text into two parts, the first being devoted to the assumptions which characterize the models and hence structure the field, and the second presenting one of the models with G. Laffond and I have developed in recent years.

THE SET OF ASSUMPTIONS

In all the models considered in this first part, the operation of the market results from similar mechanisms: buyers and sellers appear on the market, get in touch, obtain information, formulate expectations, elaborate plans, negotiate, sign contracts and thus continuously modify each

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B. Munier (ed.), Markets, Risk and Money, 59–89. Original © Economica (Paris) 1986. This English translation © 1995 Kluwer Academic Publishers.

other's information, expectations, and strategies, generating a dynamic process which may lead, for instance, either to stable states (prices and quantities being reproduced from one period to the next) or to more or less periodic oscillations, or to endless fluctuations. Such an analysis, which draws on stochastic processes, enables us also to consider the genesis of new markets out of the initial market (for instance as a consequence of the creation of new economic agents or of the modification of the commodity exchanged or of a geographical move of buyers or sellers). As it operates, the market may also be hit by external shocks or submitted to endogenous evolutions (arrival of new agents or change in buyers' or sellers' requirements) such that its survival may be at stake, though it does not imply the historical path to be determined. Resilient for some values of the parameters, the existence of a market may then reveal itself to be extremely fragile for other values of these parameters.

The common framework is the following: at the initial time, contacts start between buyers and sellers who, as time goes on, will exchange, subject to money payments, given quantities of one or several commodities. Time is divided into successive periods. The assumptions necessary to build such models concern the commodities exchanged, the agents and the operation of the market. They will be considered in that order.

The Commodities Exchanged

1. The first question is obviously whether or not the agents know perfectly the characteristics of the commodities exchanged. If this information is perfect, the agents' only goal will be to obtain the best contract conditions, but if this is not so, their situation will be much more delicate since they will have either to buy information on these characteristics or to estimate these characteristics from what they know or what is revealed by the evolution of the market.

Examples of markets with imperfect information on commodities are now well known: the amateurs who want to buy a good bottle of Bordeaux wine accept paying a higher price since they assume that, under the influence of informed consumers' demands, quality increases with price; a firm which wants to recruit a professional takes into account his degrees (which are supposed to reveal a minimum ability) and his last wages (which probably reveal the opinion of his employer on his ability); the plant manager who is going to hire a worker often decides to submit him to a test before any final decision[2].

2. Under the assumption that the commodities' characteristics are perfectly known by the agents, the second question is to check whether or not these characteristics will remain identical as the market operates. Traditional theory has always been interested in the first alternative, but the second one is equally plausible: the professional ability of an individual is changed by the jobs he occupies; the firms try to adapt their products to the preferences expressed by consumers; a used car depreciates as the number of kilometers driven increases. In contrast to a frequent practice, it is not sufficient to say that one passes from one market to another when the commodity changes, since the firm may recruit a more or less trained individual or the buyer may be interested in a new or a used car. In these cases, the successive markets are generated by the operation of the first market itself.

3. Let us assume now that the commodities exchanged have constant characteristics. The third question has long been considered by economists since it concerns the nature of the contracts between agents. Conventional categories are, for instance, spot contracts (one dollar for one pound of oranges), fixed term contracts (recruitment of a worker for six months), open-ended contracts (granting of a bank overdraft, recruitment of a worker for an indefinite period), forward deals (delivery of a ton of copper in three months' time), insurance contracts (payment of a compensation in case of an accident), etc., this list being, of course, not exhaustive. The literature on contracts has displayed considerable progress in the last two decades, but the problems raised by the operation of markets are complex enough for priority to be given to the simplest models, i.e. to the models with spot or fixed term contracts.

4. There remains, as far as commodities are concerned, a last question: the commodities supplied by the various sellers may be strictly identical or imperfectly substitutable, this last case being the rule: cars of differing makes, labour services offered by individuals having the same ability but coming from inside or outside the firm. In the paper, we shall retain as an assumption that the commodities offered by the sellers are identical.

The Agents

1. On simple markets, only two categories of agents operate, buyers and sellers of elementary economic theory, but reality displays a much broader spectrum of situations: agents may specialize in the gathering or diffusion of information (advertising agencies, market research corporations, staff selection firms), in purchases or sales for others (brokers), buyers (or sellers) may form coalitions to negotiate some or all contract conditions (of which trade unions constitute a magnificent example), a group of buyers and sellers may decide to apply special rules among them (for instance external candidates are not on an equal footing with members of the staff when a firm fills a job). A theory of market dynamics should be able to explain the genesis of these various agents and rules. However, in many models, buyers and suppliers are the only agents liable to be present on the market and they cannot form coalitions. Such an assumption will be made hereafter.

2. With the beginning of research on market dynamics, another question has emerged. To present it, let us suppose that the suppliers are constantly present on the market and are ready to sign contracts, period after period. Then two extreme hypotheses are conceivable with respect to the buyers:

- They are also constantly present on the market and they try during each period to obtain the commodities they want to consume. In other words, the set of buyers is given and the purchase are repeated from one period to the next.
- The buyers arrive on the market in successive waves, remain present until they have purchased the wanted commodity and then leave the place;[3] in other words, what is given is the set of the buyers arriving in each period (a generation of buyers) while the purchases of these buyers are not repeated from one period to the next.

Obviously, in the first situation, an equilibrium of the market will be reached – if there is any – when, in every period, every buyer purchases the same amount from the same sellers, the market state being reproduced identically from one period to the next while, in the second situation, the equilibrium implies the arrival, at every period, of identical generations of buyers, equal in- and outflows of buyers and sales, the distribution of which is stationary between the various buyers' generations.

3. Let us come back to the sellers: for a long time, competition theory has studied market dynamics when potential entry is either authorized or not allowed. Hence a new dichotomy arises between models describing the market.

4. If the number of sellers is assumed given, the next question concerns the sellers' cost functions.[4] But the issue can be considered briefly since the spectrum of possible assumptions is well known: at one extreme, the quantities available to each seller are given and the costs are neglected, or the seller is assumed to be a retailer whose cost is only a unit purchase cost. At the other extreme, a cost function is associated to each seller. But it is important to stress that the assumption made will have a deep influence on the market's dynamic behaviour.

5. As far as the agents are concerned, a last assumption has to be introduced: to simplify their models, authors[5] have often postulated that each buyer is only interested in one unit of the commodity (a typical example is the labour market since each labour supplier can have only one full-time job per period). When such an assumption is made, one may also admit that each seller also supplies only one unit of the commodity.

The model developed in the second part belongs to the following class: the sets of sellers and buyers are given; these agents are constantly on the market; the commodity exchanged is unique, has constant characteristics and is perfectly known to everybody; the contracts are for immediate delivery; each seller supplies per period only one unit of the commodity while each buyer is willing to purchase one unit also.

The Dynamics of the Market

To sketch a complete typology of existing or conceivable models would require too long a development here, but it is easy to present the most fundamental issues.

1. The first one concerns the existence of adjustment costs which may interfere with the market dynamics. Example of such costs can easily be found: intra-urban daily trips from home to work, removal costs, lay-off compensations, training expenditures, psychological disutilities induced by change, etc. Such costs may indeed freeze the operation of a market or trap a market in a state very different from the traditional equilibrium. They induce phenomena very similar to those of heat dissipation through friction in thermodynamics.

2. The information issue is much more delicate. The difficulties are well known: if two individuals A and B look for information on an event E, the behaviour of each of them is a piece of information for the other. Therefore they may have an interest in modifying their behaviour to cheat or help the other, a possibility that this other must take into account. It is reasonable, however, to start with models which deal with information in a much simpler way and neglect those subtleties though they are essential to game theory. In these models, the information transferred are correct and the agents do not interpret their respective behaviour to induce information from it.

3. In such a framework, the first task of a model builder is to define very precisely the information available to each agent, either initially or during the operation of the market. For instance, an extreme assumption may be: initially, the only piece of information available to a buyer (a seller) is the existence of sellers (buyers); at the end of period t, the only additional piece of information available to a buyer (a seller) is whether he has or has not bought (sold) a unit of the commodity during period t and in the case of a purchase (a sale) from (to) whom and at which price. Other assumptions are often made: in many models, it is assumed for instance that each supplier knows his demand curve or the distribution of the prices offered by all the sellers.

4. Once the freely available information has been defined, the operation of a market results from the interaction of three processes:

- an information search process;
- a process of negotiation between agents on the basis of the information possessed;
- a process of revision of the agents' expectations and strategies.

Depending on the model, the market descriptions may either imply a simultaneous intervention of these various processes or distinguish successive steps during which agents get information, negotiate or revise their strategies. Some models give more importance to information, others to negotiation, others to strategy revisions.

5. The search for information may be described through a great variety of assumptions. Often, models entitle only one category of agents to a search activity; buyers or sellers.[6] As for the search itself, it may take numerous shapes. Let us mention some of them when workers look for a job:

(i) During every period, the worker freely draws at random a sample of jobs. If any job may be discovered, the information will be said to be extensive, but it may not be so if some jobs are not liable to be discovered by an individual due to his past or present situation.

(ii) During every period, the worker may know the status of all the jobs if he accepts the payment of an information cost. The complete situation of the market is revealed to him as soon as he has paid the entrance fee.

(iii) During every period, the payment of an information cost only entitles the payer to draw a sample at random, as in the first case.

(iv) The individual may select *a priori* the size of the sample, but the information cost he has to pay initially is an increasing (and concave) function of the size.

(v) The status of each job may be revealed to the individual on the condition that he pays an information cost characteristic of this job. The individual has to select once and for all the jobs he wants to get information on (in which case he has to pay the sum of the corresponding information costs) or he may initiate a sequential search, deciding at each step whether it is worth paying the price to discover the status of a given additional job.

(vi) The payment of an information cost entitles the individual to draw a job at random and this cost increases with the number of jobs already explored since the individual has to discover a job, the access to which becomes more and more difficult.

6. Negotiation mechanisms may also be characterized by various sets of assumptions. For instance in some models of the labour market, the firms proposals are definite, while the workers become sequentially active on the market. An active worker compares the proposals he knows and applies to one job at most. If the holder of this job keeps it, the worker has to wait for the next period to become interested in other jobs. In other models, he could do it immediately and several workers would be simultaneously active. The existence of preferential relations between buyers and sellers constitutes also an important feature of a market evolution: the housewife revisits the retailer from whom she bought last time; the firm gives priority, when several offers are equal, to a former supplier.

7. The way in which agents adapt their requests plays also its part in a model definition. Of course, the various theories of individual behaviour can be drawn upon: at one end, the agent is passive and only reacts to past information without building expectations; at the other end, he is able to estimate probability distributions of consequences on an infinite horizon and to maximize a utility function; in between, he may evaluate on the basis of simple rules his decision consequences on a limited horizon, adopting a limited rationality behaviour. For instance, a seller's strategy may be limited to the choice either of a price, or of a minimum price or of a price probability distribution (the buyer drawing at random the price offered[7]).

Hence, the plentiful variety of assumptions necessary to the definition of a market dynamics model explains the numerous paths explored in the literature and the fact that till now only partial results are available.

After this presentation of the set of assumptions, the second part of the paper will illustrate the approach through the analysis of a particular model, characterized – among others – by the following hypotheses:

- Contracts lasting only one period and for delivery during this period.
- Identical commodities supplied by the various sellers.
- A given set of buyers, constantly present on the market and willing to acquire a unit of the commodity during every period.
- A given set of sellers, each of them having one unit of the commodity during every period. No adjustment costs.
- A search for information by buyers drawing freely at random a sample of sellers during every period.

- A price offered by each seller which is independent from the buyer during every period.

Such a model may be considered as representing a labour market, but it has been built more for theoretical reasons than for empirical purposes.

A MODEL OF THE LABOUR MARKET

The model will be analyzed in three steps, devoted respectively to its description and to its consequences when information is extensive or structured by the jobs.[8] To shorten the text, the proofs are not given.[9]

Model Description

The description of the model will successively concern the agents, the search for information, the negotiation process, and the adaptation of requirements. But, for convenience, we shall postpone three assumptions on the operation of the market, two of them dealing with the search for information and the last one with the negotiation process.

The agents. On the market, are constantly present a set $M = \{k\}_{1 \leq k \leq m}$ of workers and a set $N = \{i\}_{1 \leq i \leq n}$ of jobs. Time is a discrete variable.

For any $t \in \mathbb{N}$, period t is the employment period and hence the wages payment period.

To describe the market situation during period t, two mappings are introduced:

A mapping i_t from M into $N \cup \{0\}$. It associates to any $k \in M$ the job $i_t(k)$ occupied by individual k during period t ($i_t(k) = 0$ expressing that k is unemployed during period t). $N_t = i_t(M) \cap N$ is the set of the jobs occupied during period t.

A mapping s_t defined on N_t. It associates to any job $i \in N_t$ the wages $s_t(i)$ paid for this job during period t.

Under these conditions, the *apparent state of the market at period t* f_t is defined by:

(1) $\quad f_t = (s_t, i_t).$

The two mappings are supposed to verify assumptions 1 and 2:

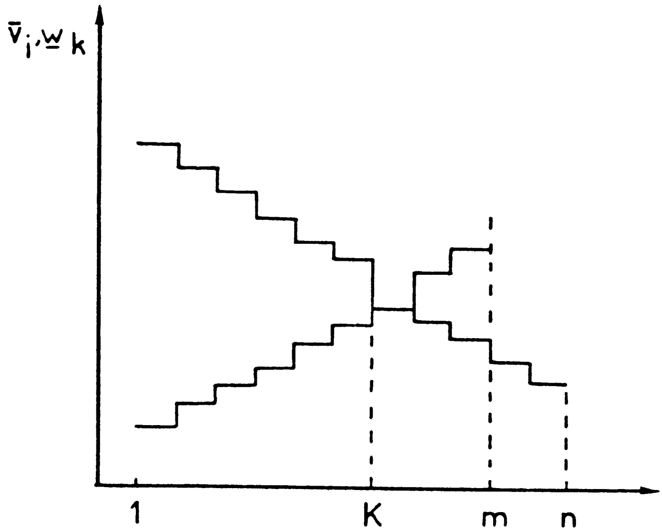

Fig. 1.

ASSUMPTION 1. $\forall k \in M, [i_t(k) \neq 0] \Rightarrow [\forall k \neq k', i_t(k') \neq i_t(k)]$

This assumption means that two different individuals cannot occupy the same job. Therefore, it is possible to introduce the set:

(2) $M_t = i_t^{-1}(N_t)$

of individuals occupied during period t and the mapping k_t from N into $M \cup \{0\}$ defined by:

(3) $k_t(i) = 0 \quad \text{if} \quad i \notin N_t \; \{k_t(i)\} = i_t^{-1}(\{i\}) \quad \text{if} \quad i \in N_t$

and which associates to any $i \in N$ the individual who occupies this job during period t ($k_t(i) = 0$ means that job i is unoccupied).

ASSUMPTION 2. $\forall i \in N_t, s_t(i) \in \mathbb{N}$

According to this assumption, wages have to be integers, which will later play a part in the adaptation of agents requirements.

We shall suppose also: (1) that there exists for any individual a minimum wage under which he refuses to work; (2) that there exists for any firm a maximum wage above which it prefers the job not to be occupied; (3) that the individuals are ordered by decreasing values of their maximum wage; (4) that there exists a rank both for jobs and for individuals for which the maximum and the minimum wages coincide (see Figure 1).

Hence Assumption 3:

ASSUMPTION 3.

(i) $\forall k \in M, \exists \underline{w}(k) \in \mathbb{N}$ *such that* $\forall t \in \mathbb{N}$, $k \in M_t$ *implies* $s_t(i_t(k)) \geq \underline{w}(k)$;

(ii) $\forall_i \in N, \exists \bar{v}(i) \in \mathbb{N}$ *such that* $\forall t \in N, i \in \mathbb{N}_t$ *implies* $s_t(i) \geq \bar{v}(i)$;

(iii) $k < k'$ *implies* $\underline{w}(k) < \underline{w}(k')$
$i < i'$ *implies* $\bar{v}(i) > \bar{v}(i')$;

(iv) $\exists K \leq \min\{m, n\}$ *such that* $\underline{w}(K) = \bar{v}(K) = p$.

If M^1 denotes the set the first K individuals and N^1 the set of the first K jobs, elementary economic theory teaches that the market is in equilibrium when the individuals in M^1 are employed and the jobs in N^1 are occupied at the unique wage level p.

The search for information. Let us suppose that the contracts linking a worker to a job are valid for one period only. Then, during period t, contracts are signed for period $(t + 1)$. These signings result from two-step operations. During a first subperiod, every individual $k \in M$ visits a subset $I_t(k)$ of jobs, which is embedded in the following assumption:

ASSUMPTION 4. *For any $k \in M$ and any $t \in \mathbb{N}$, there exists a set $B_t(k)$ of subsets of jobs. During t – first subperiod – any individual $k \in M$ selects an element $I_t(k) \in B_t(k)$, all the elements of $B_t(k)$ being liable to be selected.*

If $J_t(k)$ is the union of the sets $I_t(k)$ when $I_t(k)$ scans $B_t(k)$, $J_t(k)$ is the set of jobs that k may discover (and visit) during period t.

ASSUMPTION 5. *For any $t \in \mathbb{N}$, for any $k \in M^{t+1}$, $i_{t+1}(k) \in$*

$I_t(k) \cup \{i_t(k)\}$.

This assumption implies that an individual knows always the job he occupies.

The negotiation. The second subperiod of any period t is devoted to negotiation. In this model – contrary to what is done in others[10] – we shall not describe the negotiation in detail. We shall only assume that agents' strategies will be represented, during every period, by a reservation price which may of course differ from the minimum and maximum prices introduced in Assumption 3.

ASSUMPTION 6. (1) *For every* $t \in \mathbb{N}$ *and* $k \in M$, $\exists w_{t+1}(k) > \underline{w}(k)$ *such that* $k \in M^{t+1}$ *implies*

$$s_{t+1}\left[i_{t+1}(k)\right] \geq w_{t+1}(k).$$

(2) *For every* $t \in \mathbb{N}$ *and* $i \in N$, $\exists x_{t+1}(i) \leq \bar{v}(i)$ *such that* $i \in N^{t+1}$ *implies*

$$s_{t+1}(i) \leq x_{t+1}(i).$$

We shall assume that there exists a *privileged link* between any firm $i \in N_t$ and the worker k it employs in period t. The latter will prefer, under equality of wage offers, to remain in the same job. Hence the following assumption:

ASSUMPTION 7. *For any* $t \in \mathbb{N}$ *and* $k_0 \in M_t$, *let us write* $i_0 = i_t(k_0)$ *and* $i_1 = i_{t+1}(k_0)$; *four cases and four only are possible*:

(i) $k_0 \in M_{t+1}$ *and* $i_0 \in N_{t+1}$: $[s_{t+1}(i_0) \geq s_{t+1}(i_1)]$ *and* $[i_0 = i_1]$;
(ii) $k_0 \in M_{t+1}$ *and* $i_0 \notin N_{t+1}$: $[s_{t+1}(i_1) > x_{t+1}(i_0)]$;
(iii) $k_0 \notin M_{t+1}$ *and* $i_0 \notin N_{t+1}$: $[s_{t+1}(i_0) < w_{t+1}(k_0)]$;
(iv) $k_0 \notin M_{t+1}$ *and* $i_0 \notin N_{t+1}$: $[x_{t+1}(i_0) < w_{t+1}(k_0)]$.

At the end of period t, when the negotiation is finished, no individual remains interested by the conditions offered by a firm he knows. This situation may be expressed in various ways. We have chosen the following assumption:

ASSUMPTION 8. *For any* $t \in \mathbb{N}$ *and* $i \in I_t(k_0)$, *let us write* $i' = i_{t+1}(k)$ *and suppose* $i \neq i'$; *four cases and four only are possible*:

(i) $k \in M_{t+1}$ *and* $i \in N_{t+1}: s_{t+1}(i') \geq s_{t+1}(i)$;
(ii) $k \in M_{t+1}$ *and* $i \notin N_{t+1}: s_{t+1}(i') \geq x_{t+1}(i)$;
(iii) $k \notin M_{t+1}$ *and* $i \in N_{t+1}: w_{t+1}(k) \geq s_{t+1}(i)$;
(iv) $k \notin M_{t+1}$ *and* $i \notin N_{t+1}: w_{t+1}(k) > x_{t+1}(i)$.

It remains to introduce a last rule to characterize the negotiation process. The one which has been selected expresses that the wages paid to individuals only react to the pressure of supply or demand.

Let us express first that wages can only decrease under the pressure of labour supply.

To do so, we shall denote $\underline{r}_t(k)$ a lower bound of the wages individual k can obtain for period $(t+1)$ at the end of the negotiation.

- If $k \notin M_t$, the individual may remain unemployed and $\underline{r}_t(k) = w_{t+1}(k)$;
- If $k \in M_t$ and $w_{t+1}(k) \geq \min\{x_{t+1}(i_t(k)), s_t(i_t(k))\}$, the individual may become unemployed, even if there is no candidate for his job: $\underline{r}_t(k) = w_{t+1}(k)$;
- If $k \in M_t$ and $w_{t+1}(k) < \min\{x_{t+1}(i_t(k)), s_t(i_t(kk))\}$, the situation is more complex. Let us denote by A the set of individuals in such a situation:

$$A = \{k \in M_t / \min\{x_{t+1}(i_t(k)), s_t(i_t(k))\} > w_{t+1}(k)\} \tag{4}$$

and call a *negotiation chain* a sequence $c = (k_1, k_2, \dots k_p)$ such that:

$$\begin{cases} \{k_2, k_3, \dots, k_p\} \in A & \text{and} \quad k_\alpha \neq k_\beta \\ & \text{for } 1 \leq \alpha \leq \beta \leq p \\ i_t(k_{\alpha+1}) \in I_t(k_\alpha) & \text{for} \quad \alpha = 1, 2, \dots, (p-1). \end{cases} \tag{5}$$

In other words, individual k_α knows the job of his follower in the sequence. Therefore, for any $\alpha = 1, 2, \dots (p-1)$, k_α is a potential candidate for the job $i_{\alpha+1} = i_t(k_{\alpha+1})$. Then we shall introduce k_α's *minimum wage in the chain c*, $r_c(k_\alpha)$.

Denoting:

$$(6) \qquad r_t(k) = \min\{x_{t+1}(i_t(k)), s_t(i_t(k))\}$$

we can write:

(i) for k_1: $r_c(k_1) = r_t(k_1)$ since in chain c there is no candidate for the job i_1 of k_1 and k_1 may be sure to obtain $r_t(k_1)$;

(ii) for $k_{\alpha+1}$:

$$\begin{cases} r_c(k_{\alpha+1}) = r_t(k_{\alpha+1}) & \text{if } r_c(k_\alpha) \geq r_t(k_{\alpha+1}) \quad (7) \\ r_c(k_{\alpha+1}) = \max\{r_c(k_\alpha), w_{t+1}(k_{\alpha+1})\} & \text{if } r_c(k_\alpha) < r_t(k_{\alpha+1}) \quad (8) \end{cases}$$

we shall then define for any $k \in A$, $r_t(k)$ as the minimum of $r_c(k)$ over all chains c including k. Appendix 1 proposes an algorithm for the computation of $r_t(k)$ for any $k \in M$.

The definition of $r_t(k)$ obviously implies Assumption 9:

ASSUMPTION 9. *For any* $k \in M_{t+1}$, $s_{t+1}(i_{t+1}(k)) \geq r_t(k)$.

Let us also express that wages can only increase under the pressure of labour demand.

To do so, we shall denote by $v_t(i)$ an upper bound of the wages firm i may be compelled to offer for period $(t+1)$ at the end of the negotiation.

- If $i \notin N_t$, the job may remain unoccupied and:

$$\bar{v}_t(i) = x_{t+1}(i).$$

- If $i \in N_t$ and $x_{t+1} \leq \max\{w_{t+1}(k_t(i)), s_t(i)\}$, the job may remain unoccupied and the maximum wages which i may be led to offer on the market is independent of competition conditions:

$$\bar{v}_t(i) = x_{t+1}(i).$$

- If $i \in N_t$ and $x_{t+1}(i) > \max\{w_{t+1}(k_t(i)), s_t(i)\}$, the situation is more complex. Let us denote by B the set of jobs in such a situation:

$$(9) \qquad B = \{i \in N_t / \max\{w_{t+1}(k_t(i)), s_t(i)\} < x_{t+1}(i)\}$$

and call a negotiation chain $c = (i_1, \ldots, i_p)$ if:

$$(10)\quad \begin{cases} \{i_1, i_2, \ldots, i_{p-1}\} \subset B \text{ and } i_\alpha \neq i_\beta \\ \qquad\qquad\qquad\qquad\quad \text{if } \ 1 \leq \alpha \leq \beta \leq p \\ i_{\alpha+1} \in I_t(k_t(i_\alpha)) \qquad \text{for } \ \alpha = 1, \ldots, (p-1). \end{cases}$$

In other words, individual $k_\alpha = k_t(i_\alpha)$ who occupies job i_α knows job $i_{\alpha+1}$. Hence the latter job is a potential competitor of i_α in chain c. Therefore we shall introduce i_α *'s maximum negotiation wage in chain* c.

Denoting:

$$(11)\qquad v_t(i) = \max\{w_{t+1}(k_t(i)), s_t(i)\}$$

we can write:

$$v_c(i_p) = v_t(i_p)$$

since i_p is not in competition with any other job of the chain for the potential employment of $k_p = k_t(i_p)$.

$$(12)\quad \begin{cases} v_c(i_\alpha) = v_t(i_\alpha) & \text{if } v_t(i_\alpha) \geq v_c(i_{\alpha+1}) \\ v_c(i_\alpha) = \min\{v_c(i_{\alpha+1}), x_{t+1}(i_\alpha)\} & \text{if } v_t(i_{\alpha+1}) > v_t(i_\alpha). \end{cases}$$

We shall then define for any $i \in B$, $\bar{v}_t(i)$ as the maximum on all chains c of $v_c(i)$ when $i \in c$. Appendix 2 proposes an algorithm for the computation of $\bar{v}_t(i)$ for any $i \in N$.

The definition of $\bar{v}_t(i)$ obviously implies Assumption 10:

ASSUMPTION 10. *For any* $i \in N_{t+1}$, $s_{t+1}(i) \leq \bar{v}_t(i)$.

It is clear at this stage that the apparent market state is insufficient to describe the situation during period t. Hence, we shall introduce what we call the *real market state* e_t:

$$(13)\qquad e_t = (s_t, i_t, w_t, x_t).$$

But, before going further, it is essential to check that, for given e_t and functions x_{t+1}, w_{t+1}, and I_t, states e_{t+1} compatible with our assumptions do exist. This results from the following theorem.

THEOREM 1. *If* e_t *is known and verifies Assumptions 1–10, if* x_{t+1}, s_{t+1} *and* I_t *are known and verify Assumptions 4 and 6, then there exist* *real states* e_{t+1} *verifying assumptions 1–10.*

Requirements adaptation. Since there is no information cost in the model, the assumptions on the adaptation of agents' requirements may be straightforward. We shall suppose that individuals and firms adapt passively to the market. These adaptations will be by steps of one unit but will depend for the individuals on their employment status and for the firms on the occupation status of their job.

Hence the two following assumptions[11]:

ASSUMPTION 11. *If* $k \notin M_t$, *then* $w_{t+1}(k) = \max\{w_t(k) - 1, \underline{w}(k)\}$; *if* $k \in M_t$, *then* $w_{t+1}(k) = w_t(k)$.

ASSUMPTION 12. *If* $i \notin N_t$, *then* $x_{t+1}(i) = \min\{x_t(i) + 1, \bar{v}(i)\}$; *if* $i \in N_t$, *then* $x_{t+1}(i) = x_t(i)$.

Additional assumptions on market operation. We shall study in this paper two models which differ as for the possibilities for information search.

In the model with extensive information, any individual may discover any job during every period (Assumption 13a).

ASSUMPTION 13a. $\forall t \in \mathbb{N}, \forall k \in M, J_t(k) = N$.

In the model with information structured by jobs, an individual can only discover during a period a set of jobs which depends on the job he occupies (Assumption 13b):

ASSUMPTION 13b. $\forall t \in \mathbb{N}, \forall k \in M, J_t(k) = L(i_t(k))$.

To specify the search process, we shall denote:

$$(14) \qquad B_t = \prod_{k \in M} B_t(k)$$

and E the set of possible real market states. Assumption 14 expresses that individual choices are random but that any element in B_t may be selected by the individuals.

ASSUMPTION 14. *For* $\forall t \in \mathbb{N}$, $e_t \in E$, *there exists a probability distribution* P_{t,e_t} *on* B_t *and* $\varepsilon > 0$ *such that for* $\forall I_t \in B_t$, $I_t = \{I_t(k)\}_{k \in M}$:

$$P_{t,e_t}(I_t) > \varepsilon.$$

Once the jobs have been discovered (and consequently when I_t is known), period t negotiation moves the market from e_t to e_{t+1}. $T(e_t, I_t)$ *will designate the set of states which may be reached from* e_t *in one period.* By definition:

$$(15) \qquad e_{t+1} \in T(e_t, I_t).$$

Since the model does not formally describe the negotiation process, it is impossible to fully characterize e_{t+1} for e_t and I_t given. We shall suppose that the result of the negotiation is randomly determined.

ASSUMPTION 15. *For* $\forall t \in \mathbb{N}_t$, $e_t \in E$, $I_t \in B_t$, *there exists a probability distribution* Q_{t,e_t,I_t} *on* $T(e_t, I_t)$ *which determines the random selection of* e_{t+1} *on* $T(e_t, I_t)$.

Assumption 15 concludes the model description.

We are now going to discuss *three issues raised by the exploration of the model.*

The issue of stability. Do stable real states exist, i.e. states which, taking into account the process described, can only generate themselves for the following period? In such a case, the market, having reached such a state, will remain indefinitely in this state.

Let us introduce the following notation for convenience:

$$(16) \qquad \begin{cases} a_t(k) = \underline{w}_k \text{ if } k \notin M_t \;\; a_t(k) = s_t(i_t(k)) \text{ if } k \in M_t \\ b_t(i) = \bar{v}(i) \text{ if } i \notin N_t \;\; b_t(i) = s_t(i) \text{ if } i \in N_t. \end{cases}$$

The following theorem may then be easily proved.

THEOREM 2. *A real state is stable iff:*

$$x_{t+1}(i) = \bar{v}(i) \quad \forall i \notin N_t, w_{t+1}(k) = \underline{w}(k) \quad \forall k \notin M_t$$

$$a_t(k) \geq b_t(i) \quad \forall k \in M_t \quad \forall i \in J_t(k)$$

$$\underline{w}(k) > \bar{v}(i) \quad \forall k \notin M_t \quad \forall i \in J_t(k) \cap (N - N_t).$$

This theorem can also be expressed in the following way:

COROLLARY. *A real state e_t is stable iff:*

$$x_{t+1}(i) = \bar{v}(i) \quad \forall i \notin N_t, w_{t+1}(k) = \underline{w}(k) \quad \forall k \notin M_t$$

$$s_t[i_t(k)] \geq s_t(i') \quad \textit{for} \quad \forall k \in M_t, \forall i' \in J_t(k) \cap N_t$$

$$s_t[i_t(k)] \geq \bar{v}(i') \quad \textit{for} \quad \forall k \in M_t, \forall i' \in J_t(k) \cap (N - N_t)$$

$$\underline{w}(k) \geq s_t(i') \quad \textit{for} \quad \forall k \in M_t, \forall i \in J_t(k) \cap N_t$$

$$\underline{w}(k) > \bar{v}(i') \quad \textit{for} \quad \forall k \notin M_t, \forall i' \in J_t(k) \cap (N - N_t).$$

The issue of convergence. Whatever the initial state, does the market converge in probability towards a stable state in a finite time?

The issue of surplus evolution. This third issue needs some explanations. The surplus notion is well known when a market is in equilibrium. For instance, on Figure 2, the surplus is represented by the shaded area.

Extending this notion, we shall define *the surplus associated with any market state e_t* by the expression:

$$S_t = \sum_{i \in N_t} \bar{v}(i) - \sum_{k \in M_t} \underline{w}(k). \tag{17}$$

When the market follows a random walk out of the initial state, this surplus is a random variable.

But we shall also introduce the *instantaneous surplus* which refers to the reservation prices of period t:

$$\Sigma_t = \sum_{i \in N_t} x_t(i) - \sum_{k \in M_t} w_t(k). \tag{18}$$

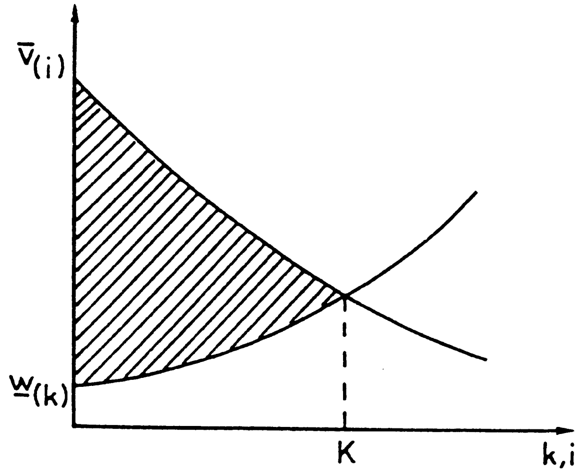

Fig. 2.

Σ_t is also a random variable. The evolution of these surpluses throughout the market dynamic process will help us to understand its operation.

These issues will be considered first in the model with extensive information and then in the model with information structured by jobs.

Model with Extensive Information

The properties of the model with extensive information are condensed in four theorems:

THEOREM 3. *Under the preceding assumptions and when information is extensive, a state e_t is stable iff:*

$$\begin{cases} N_t = N^1 & M_t = M^1 \\ s_t(i) = p & \text{for } \forall i \in N_t \\ w_t(k) = \underline{w}(k) & \forall k \in M_t, s_t(i) = \bar{v}(i) \quad \forall i \notin N_t. \end{cases}$$

In other words, the only stable states are the traditional equilibria in which the first K individuals are employed and the first K jobs occupied. In these states, all the wages paid are equal to the wages p of traditional equilibria. As for the reservation prices, they are equal to their minimum for the unemployed individuals and to their maximum for the unoccupied jobs.

THEOREM 4. *Under the preceding assumptions and when information is extensive the market state converges in probability in a finite time towards a stable state.*

This means that, whatever the initial state, the conjunction of the processes of information search, negotiation and request adaptation will lead the market to a stable state in a finite time. When such a state is reached, the dynamic process stops and the market state is reproduced from one period to the next.

THEOREM 5. *Under the preceding assumptions and when information is extensive, the surplus converges in probability towards a maximum equal to:*

$$\bar{S} = \sum_{i \in N^1} \bar{v}(i) - \sum_{k \in M^1} \underline{w}(k)$$

which implies that the surplus is maximum in a stable state.

This theorem enables an interpretation of the market evolution in terms of collective utility.[12] Initially, before the start of the transactions there exists in the market *a potential utility* which results from the difference between the wages which some firms are ready to pay and the wages which some individuals are willing to have. Progressively, the market operation extracts from this potential utility a surplus which is divided between the individuals employed $[s_t(i_t(k)) - w(k)]$ and the jobs occupied $[v(i) - s_t(i_t(k)]$. The process goes on until the market has totally transformed into a surplus, i.e. into agents' utility increases, the initial potential utility. This illustrates the deep analogy existing between the market model developed here and the thermodynamics models with reversible transformations.

THEOREM 6. *Under the preceding assumptions and when information is extensive, the instantaneous surplus does not decrease throughout the market dynamic process.*

It is worth noticing that this theorem relies on the assumptions made on requests adaptations. With other assumptions – sufficient, however, to ensure convergence – the instantaneous surplus could successively increase and decrease.

Model with Information Structured by Jobs

Up to now we have only verified that the model explains the operation of a traditional market when any individual is able to discover any job.

In such a case the initial state does not matter, since the stable states are all equivalent up to a permutation of individuals between jobs. This is no longer true when information is structured by jobs. The market still converges towards a stable state but the features of this state with respect to the individuals employed and the jobs occupied do depend on the initial state and on the dynamic walk of the market. To each stable state there is then a corresponding attractor (a trap) such that if the market state enters this trap, it can only reach the associated stable state.

We shall study in turn the properties of stable states, convergence, the evolution of the surplus and traps.

Properties of stable states. Let us call *i's spectrum* the set $L(i)$ of jobs which can be discovered by the individual occupying i and introduce the two following groups of notation:

$$(19)\quad L_x(i) = \{i' \in L(i)/\bar{v}(i) \geq x\}$$

$$(20)\quad L_x^n(i) = L_x[L_x^{n-1}(i)]$$

$$(21)\quad \hat{L}_x(i) = \bigcup_{n=1}^{\infty} L_x^n(i)$$

$\hat{L}_x(i)$ describes the information structure of the economy starting from a wages level x. If $i' \in \hat{L}_x(i)$, an individual k may pass from i to i' occupying only jobs for which firms are ready to offer at least x.

$$\begin{cases} L_t(i) = L(i) & (22) \\ L_t^n(i) = L[L_t^{n-1}(i) \cap N_t] & (23) \\ \hat{L}_t(i) = \bigcup_{n=1}^{\infty} L_t^n(i) & (24) \end{cases}$$

$\hat{L}_t(i)$ describes the information structure independently from any wages level x, but in a given market state e_t. If $i' \in \hat{L}_t(i)$, an individual may pass from i to i' through a chain of occupied jobs 'known to each other'.

THEOREM 7. *Under the preceding assumptions and if information is structured by jobs, a state e_t is stable iff:*

$$\begin{aligned} x_t(i) &= \bar{v}(i) \quad \forall i \notin N_t, & w_t(k) &= \underline{w}(k) \quad \forall k \notin M_t \\ s_t(i) &\geq b_t(i') \quad \forall i \in N_t, & &\forall i' \in \hat{L}_t(i) \\ \underline{w}(k) &\geq \bar{v}(i) \quad \forall k \notin M_t, & &\forall i \in L(0) \cap (N - N_t) \\ \underline{w}(k) &\geq s_t(i) \quad \forall k \in M_t, & &\forall i \in L(0) \cap N_t. \end{aligned}$$

But now the stable states are no longer limited to the traditional equilibria. To explore the properties of these states, we introduce two definitions:

- A stable state is said to be *concentrated* if, in this state, all the wages paid are equal.
- A stable state is said to be *efficient* if, in this state, the first K individuals are employed and the first K jobs occupied ($M_t = M^1$, $N_t = N^1$).

To study the *concentrated states* we shall denote by:

$$\begin{cases} N(x) = \{i \in N / \bar{v}(i) > x\} & (25) \\ M(x) = \{k \in M / \underline{w}(k) \leq x\}. & (26) \end{cases}$$

Then to any $x \leq p$ may be associated $g(x) > p$ such that:

$$\#M(x) = \#N[g(x)]. \qquad (27)$$

Therefore, if:

$$x_0 = \max\{\bar{v}(i), i \in L(0)\} \qquad (28)$$

it is possible to prove:

THEOREM 8. *If $x_0 \geq p$, a necessary and sufficient condition for all the stable states to be concentrated is that:*

1. $\forall i \notin N^1, \exists j \in N, \bar{v}(j) > v(i)$ *and* $j \in L(i)$

2. $\hat{L}_p(i) = N^1$ *for* $\forall i \in N^1$
3. *If* $K \notin L(0)$ *then* $L_{p+1}(i) = N^1 - \{K\}$ *for* $\forall i \in N^1 - \{K\}$.

If $x_0 > p$ *and if* $y_0 = g(x_0)$, *a necessary and sufficient condition for all the stable states to be concentrated is that:*

1. $\forall i \notin N^1, \exists j \in N, \bar{v}(j) > \bar{v}(i)$ *and* $j \in L(i)$
2. $\forall y \in \{p, p+1, \ldots, y_0\}, \hat{L}_y(i) = N(y)$ *for* $\forall i \in N(y)$

It can be shown that, when all the stable states are concentrated: (1) the wages paid in a stable state are greater than or equal to p; (2) in a stable state $N_t \subset N^1$; (3) if $x_0 \geq p$ and $K \in L(0)$ then any stable state is also efficient.

As for *the efficient states*, it is possible to prove the following theorem:

THEOREM 9. *A necessary and sufficient condition for all the stable states to be efficient is that:*

1. $\#M[\bar{v}(i)] \leq \#L_{\bar{v}(i)}(i) \cap N^1$ *for* $\forall i \notin N^1$
2. $\hat{L}_p(0) = N^1$
3. $K \in L(0)$

Generally, the set of stable states will include states which are not concentrated and/or not efficient (and hence different from the traditional equilibria).

Convergence. As for the model with extensive information, the operation of the market leads to a stable state.

THEOREM 10. *Under the preceding assumptions and when information is structured by jobs, the market state converges in probability to a stable state in a finite time.*

Surplus evolution. When information is structured by jobs, the surplus is no longer always equal to $\bar{S}$ in a stable state, but one can prove the following theorem:

THEOREM 11. *If e_t is a stable state of the model with information structured by jobs and $e_{t'}$ another state such that:*

$$i_{t'}(k) \in L[i_t(k)] \cup i_t(k) \textit{ for } \forall k \in M_t, \textit{ then } S_{t'} \leq S_t.$$

In other words, it is impossible to find a state $e_{t'}$ with a surplus greater than that in e_t while respecting the information constraints of e_t, which means that in e_t the surplus is locally maximum.

THEOREM 12. *Under the preceding assumptions and if information is structured by jobs, the instantaneous surplus does not decrease throughout the market dynamic process.*

Traps. Figure 3 illustrates their appearance. First, we shall say that two real stable states are *indistinguishable* if the individuals employed and the jobs occupied are identical. Such stable states will be said to correspond to the same condensed state of the market. Let us assume then that the market dynamic leads to two condensed stable states only, g_1 and g_2, g_1 being the traditional equilibrium. States like g_3 are such that a dynamic evolution starting from them may lead either to g_1 or to g_2. Others like g_4 or g_5 are such that the evolution leads to g_1 only or to g_2 only. In other words, g_4 is in the attraction pool – the trap – of g_1 while g_5 is in the trap of g_2. These two traps are necessarily non-intersecting subsets and each of them has a border such that if the market crosses it, it is unavoidably attracted either by g_1 or by g_2.

More precisely let us call $\Gamma(e)$ the set of possible followers of state e at the next period. A subset $A \subset E$ of states will be said to be *absorbing* if:

$$\Gamma(A) \subset A.$$

If S_1 is a subset of stable states, S_1 is absorbing. Let us then denote $H(S_1)$ the set of $A \subset E$ which are absorbing and such that $A \cap E_s = S_1$ where E_s is the set of stable states. It can be proved that $H(S_1) \neq \emptyset$ and is inductive (if C and D are elements of $M(S_1)$ so is $C \cup D$); it contains a maximal element $A(S_1)$. This element will be called the *trap of* S_1.

$A(S_1)$ is characterized by the relation:

$$(29) \qquad [e \in A(S_1)] \Leftrightarrow \left[\left\{ \bigcup_{n \in \mathbb{N}} \Gamma^n(e) \right\} \cap (E_s - S_1) = \emptyset \right].$$

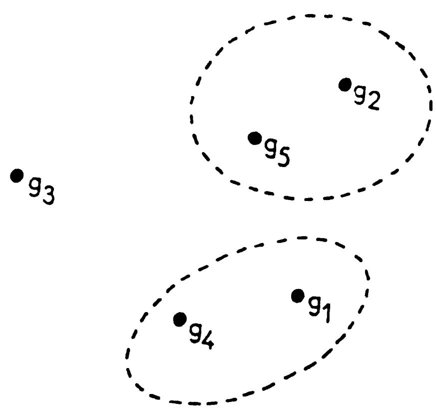

Fig. 3.

As for the *border of* S_1 it is the set $F(S_1)$ of states $e \in E$ such that:

$$e \notin A(S_1) \quad \text{and} \quad \Gamma(e) \cap A(S_1) \neq \emptyset.$$

An interesting problem would then be to characterize the sets $A(S_1)$, $F(S_1)$, $A(E_S - S_1)$, $F(E_S - S_1)$ when S_1 is the set of efficient states, since the solution of this problem would enable us to announce whether a market, starting from a state e_t, would surely converge to an efficient state (in spite of the imperfection of information), would be definitely unable to correctly allocate individuals and jobs, or could still give birth to the two situations. Unfortunately, it seems difficult to obtain a general answer to this question. However, an interesting result is the following.

Let us denote:

$$\begin{cases} A = \left\{ i \in N / L_{\bar{v}(i)}(i) = \{i\} \right\} \\ B = \{ i \in N / \hat{L}_{\bar{v}(i)}(i) \subset A \}. \end{cases}$$

Theorem 13 can then be proved.

THEOREM 13. *If* e_t *is such that:*

1. $A_1 = \{i \in A / L_{x_t}(i) \subset \{i\}\} \neq \emptyset$
2. $B_1 = \{i \in B / \hat{L}_{\bar{v}(i)} \subset A_1\} \cap N_t \neq \emptyset$,
then e_t *is in the trap of efficient states.*

With information structured by jobs, a condensed stable state cannot be reached from any initial market state and several condensed stable states can be reached from the same initial market state. Therefore, *when information is not extensive, the initial state, the samples drawn at random by the individuals during their search, and the hazards of negotiations, all have an influence on the final state which will prevail.*

The preceding models illustrate two *causes of economic losses* by comparison with a perfect market which would in one period allocate individuals and jobs as in the traditional equilibrium:

1. A stable state is only reached after a random time t. Hence the necessity to discount the future utilities associated to this stable state. The corresponding loss is only partially compensated by the utilities generated by the fact that during the transition, some individuals are employed and some jobs occupied.
2. The stable state reached may not be efficient. Hence a (random) loss due to the imperfection of information.

In reality, two other causes generate also losses which the model does not take into account.

1. Individuals looking for jobs and firms giving information on these jobs have to bear costs. These costs must be deducted from collective utility, but their very existence modifies market dynamics.
2. Individuals passing from one job to another and firms changing their staff have to bear adjustment costs. These costs must be deducted from collective utility, but in this case, too, their very existence modifies market dynamics.

CONCLUSION

At a more fundamental level, the research briefly presented in this paper can be interpreted in terms of systems theory since we may consider as a system the market composed of the sets of buyers and sellers and of the commodity exchanged. But this system may be viewed at different levels of complexity.

At the most elementary level, the market is described by economists *as a system transforming inputs into outputs*. The inputs are obviously the quantities of the commodity possessed initially by the individuals. The outputs are the quantities possessed by other agents after the exchange has taken place. The environment is represented by the agents' incomes and by the prices of the other commodities. Such a representation is only valid when market is in equilibrium.

Quite different is the vision of the market when – second level – it is perceived as *a cybernetic system*: interest is centered on the supply decrease (or the demand increase) generated by a price decline. The major characteristic of the market becomes its stability or, in other words, its ability to dampen, at the level of the quantities exchanged, the fluctuations of the parameters of supply and demand curves. In this vision, the market has become a cybernetic machine.

At the third level, *self-organization* becomes essential. As in chemical reactions between molecules the market evolution results from random search processes and from the influences of contacts between agents on their subsequent behaviour. Therefore, the outcome is no longer the same stable state and the final organization of the market depends on its history. This history may even (in models which have not been presented here) lead to its destruction or give rise to other markets. But, in contrast to chemical molecules, individuals anticipate, build strategies and adapt their behaviour.

It would be possible to consider a fourth level, the market agents being considered themselves as self-organizing systems. The market would then be conceived as a *society of systems*[13] and a new set of problems would appear such as those which concern communication languages, behaviour norms, coalition building.

But for such an approach to find its roots in economics, it is essential to develop models built on the concepts of economic theory. This paper has tried to illustrate the type of research which has then to be explored.

APPENDIX 1

The following algorithm allows the computation of $r_t(k)$ for $\forall k \in M$. Let us denote:

$$\begin{cases} r_t(k) = \max\{w_{t+1}(k), \min[x_{t+1}(i_t(k), s_t(i_t(k))]\} \\ \qquad\qquad \text{if} \quad k \in M_t \\ r_t(k) = w_{t+1}(k) \\ \qquad\qquad \text{if} \quad k \notin M_t \end{cases}$$

$$\begin{cases} N(1) = \min\{r_t(k), k \in M\} \\ X(1) = (M - M_t) \cup \{k \in M_t / r_t(k) = N(1)\} \\ a_t(k) = r_t \quad \text{for all} \quad k \in X(1) \end{cases}$$

Let us assume that $X(\alpha)$, $N(\alpha)$ and $a_t(k)$ are known for all $k \in X(\alpha)$ and denote:

$$Y(\alpha) = \{k \in X(\alpha)/a_t(k) = N(\alpha)\}$$

$$Z(\alpha) = \{k \in M_t/i_t(k) \in \bigsqcup_{k \in Y(\alpha)} I_t(k)\}.$$

Two cases are possible:

1. $Z(\alpha) - X(\alpha) \neq \emptyset$.

In that case, we take:

$$\begin{cases} N(\alpha + 1) = N(\alpha) \\ X(\alpha + 1) = X(\alpha) \cup Z(\alpha) \\ a_t(k) = \max\{w_{t+1}(k), N(\alpha)\} \text{ for } \forall k \in Z(\alpha) - X(\alpha) \end{cases}$$

2. $Z(\alpha) - X(\alpha) \neq \emptyset$.

In that case, we take:

$$\begin{cases} N(\alpha + 1) = N(\alpha) + 1 \\ X(\alpha + 1) = X(\alpha) \sqcup \{k \in M_t - X(\alpha)/r_t(k) \\ \qquad\qquad = N(\alpha + 1)\} \\ a_t(k) = N(\alpha + 1) \quad \text{for} \quad \forall K \in X(\alpha + 1) - X(\alpha). \end{cases}$$

Then: $N(\alpha + 1) \geq N(\alpha)$ and $X(\alpha + 1) \supset X(\alpha)$ and $N(\alpha + 1) \neq N(\alpha)$ or $X(\alpha + 1) \neq X(\alpha)$. As a consequence, for all $\alpha > m \max\{r_t(k), k \in M\}$, $X(\alpha) = M$ and we have defined $a_t(k)$ for all $k \in M$.

But the following theorem can then easily be proved.

THEOREM. *For* $\forall k \in M$, $\alpha_t(k) = \underline{r}_t(k)$.

APPENDIX 2

The following algorithm enables us to compute $\bar{v}_t(i)$ for $\forall t \in N$. Let us denote:

$$\begin{cases} v_t(i) = x_{t+1}(i) & \text{if } i \notin N_t \\ v_t(i) = \min\{x_{t+1}(i), \max[s_t(i), w_{t+1}(k_t(i))]\} & \text{if } i \in N_t \end{cases}$$

$$\begin{cases} N(1) = \max\{v_t(i), i \in N\} \\ X(1) = \{i \in N_t / v_t(i) = N(1)\} \cup (N - N_t) \\ a_t(i) = v_t(i) \quad \text{for} \quad \forall i \in X(1). \end{cases}$$

Let us assume that $X(\alpha)$, $N(\alpha)$ and $a_t(i)$ are known for all $i \in X(\alpha)$ and denote:

$$\begin{cases} Y(\alpha) = \{i \in X(\alpha) / a_t(i) = N(\alpha)\} \\ Z(\alpha) = \{i \in N_t / I_t(k_t(i)) \cap Y(\alpha) \neq \emptyset\} \end{cases}$$

Two cases are possible:

1. $Z(\alpha) - X(\alpha) \neq \emptyset$.

In this case, we take:

$$N(\alpha + 1) = N(\alpha)$$

$$\begin{cases} X(\alpha + 1) = X(\alpha) \cup Z(\alpha) \\ a_t(i) = \min\{x_{t+1}(i), N(\alpha)\} \\ \qquad \text{for all} \quad i \in X(\alpha + 1) - X(\alpha) \end{cases}$$

2. $Z(\alpha) - X(\alpha) = \emptyset$.

In this case, we take:

$$\begin{aligned} N(\alpha + 1) &= N(\alpha) - 1 \\ X(\alpha + 1) &= X(\alpha) \cup \{i \in N_t - X(\alpha) / v_t(i) \\ &= N(\alpha + 1)\} \\ a_t(i) &= N(\alpha + 1) \quad \text{for all} \quad i \in X(\alpha + 1) - X(\alpha). \end{aligned}$$

Then $N(\alpha + 1) \leq N(\alpha)$ and $X(\alpha + 1) \supset X(\alpha)$ and $N(\alpha + 1) \neq N(\alpha)$ or $X(\alpha + 1) neq X(\alpha)$. As a consequence for $a \geq n \cdot \max\{v_t(i), i \in N\}$, $X(\alpha) = N$ and we have defined $a_t(i)$ for all $i \in N$.

But the following theorem can then easily be proved:

THEOREM. *For all* $i \in N$, $\bar{v}_t(i) = a_t(i)$.

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NOTES

[1] The present paper is based on research conducted at CNAM in cooperation with H. Caron-Salmona, G. Laffond, and E. Renault.
[2] For models where buyers know imperfectly the commodities exchanged see, among others, Akerlof (1970), Hey and McKenna (1981), Smallwood and Conlisk (1979). This approach leads in particular to the introduction of 'signalling' and 'moral hazard' (Riley, 1979; Rothschild and Stiglitz, 1976; Spence, 1974).
[3] See Diamond's model (1971).
[4] See McMinn (1980), Reinganum (1979). In these papers, the differences between sellers' unit production costs may generate an equilibrium with price dispersion.
[5] For instance McMinn (1980), Reinganum (1979), Salop and Stiglitz (1977).
[6] For instance, only buyers search in Diamond (1971), Fisher (1970), McMinn (1980), Reinganum (1979), Salop and Stiglitz (1977); only sellers in Butters (1977).
[7] For instance, price distributions offered by the sellers are introduced in Butters (1977).
[8] The precise definition of the terms is given in due course.
[9] They may be obtained from Laboratoire d'Económétrie, 2 Rue Conté, Paris 75003, France.
[10] See for instance J. Lesourne and G. Laffond: 'Market Dynamics and Search Processes', paper presented at the Econometric Society European meeting, Athens, August, 1979.
[11] See, for instance, J. Lesourne and G. Laffond (1975).
[12] See Lesourne and Laffond (1979).
[13] See J. Lesourne, *Les Systèmes du Destin*, Ch. 4, Dalloz, Paris, 1976.

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CLAUDE PONSARD†*

4. A THEORY OF SPATIAL GENERAL EQUILIBRIUM IN A FUZZY ECONOMY

> It would not be rational to admit a system of axioms and not accept its implications (principle of consistency), but it is still an open question whether rationality should be defined on the basis of *criteria* relating only to random choice, or following *criteria* which are independent of all consideration of random choice. Maurice Allais in *Expected Utility Hypotheses and the Allais Paradox*, 1979, p. 10

1. INTRODUCTION

The masterly work of Allais in economics is so plentiful that his contribution to the theory of spatial general equilibrium has rather been forgotten. To tell the truth, it is a forerunner's analysis (1943) and a question of minor importance in the Allais corpus as a whole. However, this pioneering contribution is worth quoting in relation to the history of spatial economic theory (Ponsard, 1983, p 118) as the author initiated the integration of the space concept into the framework of welfare theory.

The purpose of the present paper is now to answer the following question: is the integration of fuzzy behavior in the theory of spatial general equilibrium efficient enough to make a significant contribution?

Indeed, economic agents pursue more or less precise, and sometimes incompatible, objectives. The constraints which limit their means are elastic, being either subjectively more or less constraining or objectively fuzzy. Such a spatial behavior must be analyzed as an optimization programming of fuzzy objective functions with elastic constraints. The

* Before I could finish editing this book, Claude Ponsard died after a short and terrible sickness. He was buried in Dijon on March 30, 1990. One of his last pleasures on this earth was to see the Nobel Prize awarded to Maurice Allais. May this chapter be also dedicated to the memory of Claude Ponsard.

B. Munier (ed.), Markets, Risk and Money, 91–110.

theories of consumers' and producers' spatial equilibria are particular specifications of a fuzzy economic calculation.

As to the conditions for the compatibility of such imprecise types of behavior, the first step in the analysis is to show that the fuzziness of individual behavior does not bar the way to a state of general equilibrium in a spatial economy.

The first part of this paper is devoted to the description of the analysis framework and a short survey of the spatial partial equilibrium theories. The matter of the second part is the statement of a theorem on the conditions for the existence of a spatial general equilibrium consistent with imprecise individual behavior.

Remark. In order to avoid any ambiguity in the notation of mathematical symbols, ordinary (non-fuzzy) concepts are underlined, while fuzzy concepts are not. For example, $A \subset \underline{E}$ is read: A is a fuzzy subset of the ordinary reference set $\underline{E}$, this rule being applied to sets only, but not to their elements, without any possible confusion.

2. ANALYSIS FRAMEWORK AND SPATIAL PARTIAL EQUILIBRIUM

2.1. *The Characteristics of the Economic Space*

Consider an economic space including n goods i $(i = 1, \ldots, n)$, m consumers j $(j = 1, \ldots, m)$ and r producers k $(k = 1, \ldots, r)$. The set of agents is finite, but large enough so that the economy has a competitive structure.

The locations of economic agents are given with some freely chosen spatial distribution. We denote, without any ambiguity, the consumers' residences with the help of the same indexes j, one and only one individual residence corresponding to each consumer. Thus, different values of these indexes are allocated to distinct residences which are located in a given place or in several places. In the same manner, producers' plants are denoted with the help of indexes k.

Although the following model is inspired by Debreu's theory (1959), the definition of an economic good which is chosen is not that of merchandise with the meaning stated by that author. It is well known that such merchandise is described as a product or a service which is characterized by various properties, especially the date at which it will be available and the place where it will be accessible. Then, the value of merchandise in the future is equated with the value of the same merchandise at present through a discount rate. In the same manner, the value of distant merchandise is supposed to be equated with the value of the same merchandise which is available in a given place through an exchange rate.

But, if it is possible to state the present value of merchandise available in the future with the help of a discount rate, it is not accurate to state the local value of distant merchandise with the help of an exchange rate. Indeed, the actualization calculation is based upon the axiom of the depreciation of the future, which lays the foundation of the discount rate. For lack of a similar axiom in spatial analysis, Debreu's exchange rate has no theoretical foundation. Henceforth, this definition of merchandise does not allow us to explicitly analyze the impact of space on economic equilibrium. So the usual definition of an economic good must be retained. We call 'located goods' the non-spatial goods endowed with the coordinates of their supply or demand places in the economic space.

The economic space is then characterized by a price system which allows us to state the nominal value of physical products and services at a given instant. The delivered price of an ith good unit which is demanded by the jth consumer (located at the place j) from the kth producer (located at the place k) is denoted by p_{ijk}. It is equal to the sum of the factory price and the transportation price from k to j:

$$p_{ijk} = p_{ik} + p_{i'kj}$$

where p_{ik} is the factory price of the good i at place k and $p_{i'kj}$ the transportation price of the good i from k to j, with i' designing the transportation service of the good i.

A price vector, denoted by p, $p = [p_{ijk}]$, describes a spatial price system. The set of prices is denoted by $\underline{P}$. Thus: $p \in \underline{P}$. We make the assumption that one price at least is not null. Subsequently, this condition will be necessary to satisfy a generalized Walras law.

2.2. *The Description of the Fuzzy Economy*

2.2.1. Imprecise preferences, fuzzy utility, sets of located possible and efficient consumptions (Ponsard, 1981a, 1981b, 1985). Let $\underline{X}_j$ be the set of the located possible consumptions of the jth consumer. The elements x_j of $\underline{X}_j$ are vectors of $\mathbb{R}^{nmr}$ whose compoundings x_{ijk} state the quantities of the good i which are demanded or supplied by the jth consumer to the kth producer. They are non-negative or non-positive according as they are demanded or supplied, respectively.

To simplify, we put $nmr = l$. The elements x_j of $\underline{X}_j$ are vectors of $\mathbb{R}^l$. The jth consumer has an imprecise preference–indifference system described by a fuzzy total preorder. Under some conditions which insure the existence of a topological totally preordered space, a continuous function of fuzzy utility, denoted by μ_{U_j}, is stated. It is such that:

$$\mu_{U_j}\colon \underline{X}_j \longmapsto [0,1],$$

$$\forall x_j \in \underline{X}_j, x_j \longmapsto \mu_{U_j}(x_j) \in [0,1].$$

Thus we define a fuzzy subset of $\underline{X}_j$, denoted by U_j, such that:

$$U_j = \left\{x_j, \mu_{U_j}; \forall x_j \in \underline{X}_j\colon \mu_{U_j}(x_j) \in [0,1]\right\}$$

which describes a fuzzy objective in the jth consumer's economic calculation.

The jth consumer disposes of a given budget, represented by w_j with $w_j \in \mathbb{R}^n$. For a set $\underline{X}_j$, a spatial price system p and a wealth w_j, the budget set is defined by:

$$\underline{B}_j = \{x_j; \forall x_j \in \underline{X}_j\colon p \cdot x_j \leq w_j\}$$

where $p \cdot x_j \leq w_j$ designates the budget constraint.

But, except for some particular cases, it is elastic, so that the located efficient consumptions set (for a given constraint $\underline{B}_j$ is no longer reduced to the classical frontier of the consumption technical optima. It is a fuzzy subset of $\underline{X}_j$, since each element x_j belongs 'more or less' to $\underline{X}_j$. Thus we construct a mapping, denoted by μ_{C_j}, such that

$$\mu_{C_j}\colon \underline{X}_j \longmapsto [0,1]$$

$$\forall x_j \in \underline{X}_j, x_j \longmapsto \mu_{c_j}(x_j) \in [0,1]$$

with

$$\begin{aligned} \mu_{C_j}(x_j) &= 1 \quad \text{if} \quad p \cdot x_j = w_j \\ &= 0 \quad \text{if} \quad p \cdot x_j > w_j \\ \mu_{C_j}(x_j) &\in \,]0,1[\quad \text{if} \quad p \cdot x_j < w_j. \end{aligned}$$

The mapping μ_{C_j} depends on w_j. It is monotonic and decreasing for increasing values of w_j: $(w'_j > w_j) \Rightarrow [(w'_j - p \cdot x_j) > (w_j - p \cdot x_j)]$

$$\Rightarrow \mu'_{C_j}(x_j) \leq \mu_{C_j}(x_j).$$

Thus we define a fuzzy subset of $\underline{X}_j$, denoted by C_j, such that:

$$C_j = \left\{x_j, \mu_{C_j}; \forall x_j \in \underline{X}_j \colon \mu_{C_j}(x_j) \in [0,1]\right\}$$

which describes a fuzzy constraint in the jth consumer's economic calculation.

2.2.2. Producers' fuzzy objectives, sets of possible and efficient productions (Ponsard, 1982a). Let $\underline{Y}_k$ be the set of the possible productions of the kth producer. The elements y_k of $\underline{Y}_k$ are vectors of $\mathbb{R}^l$ whose compoundings y_{ijk} state the quantities of the good i which are supplied or demanded by the kth producer to the jth consumer. They are non-negative or non-positive according as they are supplied or demanded, respectively.

Transportations are production activities, just like all the others. For a production y_k in $\underline{Y}_k$ and a spatial price system p, the producer's profit, denoted by $\underline{G}_k$, is by definition:

$$\underline{G}_k = p \cdot y_k.$$

The objective of the kth producer is the maximization of the fuzzy utility associated with the profit.

We construct a mapping, denoted by μ_{G_k}, which is the profit fuzzy utility mapping. It is such that:

$$\mu_{G_k} \colon \underline{Y}_k \longmapsto [0,1]$$

$$\forall y_k \in \underline{Y}_k, y_k \longmapsto \mu_{G_k}(y_k) \in [0,1]$$

with

$$\begin{aligned} \mu_{G_k}(y_k) &= 0 \quad \text{in the inaction assumption} \\ \mu_{G_k}(y_k) &\in \,]0,1[\quad \text{in all the other cases.} \end{aligned}$$

We find again, as a particular case, the classical assumption if the profit utility is maximal ($\mu_{G_k}(y_k) = 1$) for the maximum amount of profit.

Thus we define a fuzzy subset of $\underline{G}_k$, denoted by G_k, which describes the fuzzy objective of the kth producer. It is such that:

$$G_k = \{y_k, \mu_{G_k}; \forall y_k \in \underline{Y}_k \colon \mu_{G_k}(y_k) \in [0,1]\}\,.$$

Now, instead of partitioning the set $\underline{Y}_k$ into two classes, for a given technological constraint, that of efficient productions and its complemental class in $\underline{Y}_k$, we admit that all the elements of $\underline{Y}_k$ are 'more or less" efficient productions. Thus we construct a membership function of the elements of $\underline{Y}_k$, denoted by μ_{H_k}, such that:

$$\mu_{H_k} \colon \underline{Y}_k \longmapsto [0,1]$$

$$\forall y_k \in \underline{Y}_k, y_k \longmapsto \mu_{H_k}(y_k) \in [0,1]$$

with

$\mu_{H_k}(y_k) = 0$	in the assumption of the production of wastes
$\mu_{H_k}(y_k) = 1$	in the classical case of maximal efficiency
$\mu_{H_k}(y_k) \in]0,1[$	in all the other cases.

We define a fuzzy subset of $\underline{Y}_k$, denoted by H_k, such that:

$$H_k = \{y_k, \mu_{H_k}; \forall y_k \in \underline{Y}_k \colon \mu_{H_k}(y_k) \in [0,1]\}$$

which describes a fuzzy constraint in the kth producer's economic calculation.

2.2.3. Allocation of wealth and incomes. The initial allocation of the stocks of goods i is given. Let w_j be the vector of the jth consumer resources and $w = \sum_{j=1}^{m} w_j$ the initial wealth in the economy.

The distribution of profit between the consumers is defined by the parts of the net revenues of firms which are received by the consumers. These parts are assumed to be non-negative and their sum equal to one for the r firms in the economy.

With these hypotheses, each consumer has three different origins for his income: income proceeding from its profit parts, from the sale of its initial dotations, and from the sale of the services which are included in its consumption set.

We hypothesise that each agent can supply a quantity of each good in order to avoid discontinuity solutions.

2.3. *Definition of the Spatial General Equilibrium in a Fuzzy Economy*

2.3.1. Fuzzy economic calculation. Let $\underline{E}, \underline{E} = \{x\}$, be the set of *a priori* possible alternatives. A fuzzy decision, denoted by D, in $\underline{E}$ is by definition the intersection of the fuzzy subset $F, F \subset \underline{E}$, describing the aimed objective, and of the fuzzy subset C, $C \subset \underline{E}$, describing the constraint:

$$D = F \cap C.$$

We have a membership function, denoted by μ_D, such that:

$$\mu_D \colon \underline{E} \longmapsto [0, 1]$$

$$\forall x \in \underline{E}, x \longmapsto \mu_D(x) = \mu_F(x) \wedge \mu_C(x)$$

with the following conditions:

$$\begin{aligned} \mu_D(x) &\simeq 1 \quad \text{iff } x \text{ is good for } F \text{ and } C \\ \mu_D(x) &\simeq 0 \quad \text{iff } x \text{ is bad for } F \text{ and } C. \end{aligned}$$

An optimal decision is such that:

$$\sup_{x \in \underline{E}} \mu_D(x) = \sup_{x \in \underline{E}} [\mu_F(x) \wedge \mu_C(x)].$$

This formulation calls for a very important remark in the framework of spatial partial equilibrium theories: objective and constraint have the same role in the decision making process. Indeed, their relationships are symmetric since two fuzzy subsets of the same reference set are concerned and the operation $\cap$ is commutative.

Formally, a fuzzy economic calculation has a structure which is similar to the structure of a fuzzy mathematical programming. Tanaka, Okuda and Asai (1974) prove that the solution for the problem of finding the best possible decision is to settle an element x of $\underline{E}$ such that:

$$\sum_{x \in \underline{E}} \mu_D(x) = \sum_{x \in \underline{A}} \mu_F(x), \quad \text{with} \quad \underline{A} \subset \underline{E}$$

and

$$\underline{A} = \{x; x \in \underline{E} \colon \mu_C(x) \geq \mu_F(x)\}$$

i.e. $\underline{A}$ is a subset of $\underline{E}$ such that the value of the constraint membership function is at least equal to the value of the objective membership function.

A good understanding of this result requires us to state the three following comments:

2.3.1.1. The conditions for the function $\sup_{x \in \underline{A}} \mu_F(x)$ to be continuous are little restrictive. Among them, we find on the one hand the condition that the (non-fuzzy) sets of possible consumption and production be compact, and on the other hand the condition that the fuzzy subsets which describe objectives satisfy the condition of fuzzy strict convexity. Mathematically, it would be indifferent to place the fuzzy strict convexity condition on the constraint rather than on the objective, since they play the same part in the decision making. But for economic reasons, it is more appropriate to place it on the objective. Indeed, in the consumer and producer spatial equilibrium theory, it guarantees the continuity property of the fuzzy utility functions. Moreover, in the producer equilibrium theory, the assumption of increasing returns does not raise a problem since the fuzzy strict convexity condition is not placed on the technological constraint.

2.3.1.2. Generally the solution is not unique. In particular, the analysis of the unicity conditions points out that the fuzzy strong convexity condition must take the place of the fuzzy strict convexity condition.

2.3.1.3. In the particular case where the objective is precise and the constraint alone is fuzzy, then the fuzzy economic calculation can be solved by a different and much simpler method (Ponsard, 1982b).

2.3.2. Consumers' spatial partial equilibrium. Consumers maximize their fuzzy utilities with their elastic constraints of budget and taking into account the equilibrium prices, which are viewed as parameters. Here, the multicriteria viewpoint on consumers' behavior is laid aside (Ponsard, 1986).

The jth consumer chooses a consumption x_j in a non-empty fuzzy subset X_j of $\underline{X}_j$ such that:

$$X_j = U_j \cap C_j$$

where X_j describes the demand fuzzy subset of the jth consumer.

At equilibrium, the consumption x_j is such that:

$$\sup_{x_j \in \underline{X}_j} \mu_{X_j}(x_j) = \sup_{x_j \in \underline{A}} \mu_{U_j}(x_j)$$

with

$$\underline{A} = \left\{ x_j; x_j \in \underline{X}_j : \mu_{C_j}(x_j) \geq \mu_{U_j}(x_j) \right\}.$$

As the continuity of the function $\sup_{x_j \in \underline{A}} \mu_{U_j}(x_j)$ implies that U_j be strictly convex, a result is that, $\forall x_j \in \underline{A}$, $\sup_{x_j \in \underline{X}_j} \mu_{X_j}(x_j)$ is strictly quasiconcave and X_j is strictly convex. Therefore $\underline{X}_j$ must also be strictly convex. Additionally, $\underline{X}_j$ has to be compact.

In the goods supply space, the places where the several commodities are demanded or supplied and the respective quantities purchased or sold are obtained since $x_j = [x_{ijk}]$ where k is the index of supply places of the goods indexed by i which are elements of the vector x_j.

Now let $x = \sum_{j=1}^{m} x_j$ be a total consumption and $\underline{X} = \sum_{j=1}^{m} \underline{X}_j$ be the total consumption set, with which a fuzzy subset X, $X \subset \underline{X}$, is associated. It is defined as follows.

The total consumption is such that:

$$\frac{\mu_X(x)}{x} = \frac{\text{Me}\left[\sup_{x_j \in \underline{X}_j} \mu_{X_j}(x_j)\right]}{\sum_{j=1}^{m} x_j}$$

where Me[·] is the median of the distribution of the values of the membership functions of the individual demands to X_j at the equilibrium. The above expression is not a quotient. It means that the sum of individual consumptions (under the line) is endowed with a membership function, denoted by μ_X, (above the line) such that:

$$\mu_X : \underline{X} \longmapsto [0,1]$$

$$x \longmapsto \mu_X(x) = \text{Me}\left[\sup_{x_j \in \underline{X}_j} \mu_{X_j}(x_j)\right].$$

The median is an operator for aggregating the x_j individual demand membership functions to X_j. It has the superiority of being non-additive, which is coherent with the fuzzy nature of the variables and the subjectivity of items. In economic terms, it describes the behavior of the median consumer as a representative agent of the consumers set.

Thus we have:

$$X = \left\{ x, \mu_X; \forall x \in \underline{X} \colon \mu_X(x) = \mathrm{Me} \left[\sup_{x_j \in \underline{X}_j} \mu_{X_j}(x_j) \right] \right\}.$$

2.3.3. Producers' Spatial Partial Equilibrium. Producers maximize their fuzzy utilities of profit with their elastic technological constraints and taking into account the equilibrium prices which are viewed as parameters.

The kth producer chooses a production y_k in a non-empty fuzzy subset y_k of $\underline{Y}_k$ such that:

$$Y_k = G_k \cap H_k$$

where Y_k describes the supply fuzzy subset of the kth producer. At equilibrium, the production y_k is such that:

$$\sup_{y_k \in \underline{Y}_k} \mu_{Y_k}(y_k) = \sup_{y_k \in \underline{A}} \mu_G(y_k)$$

with

$$\underline{A} = \{y_k; y_k \in \underline{Y}_k \colon \mu_{H_k}(y_k) \geq \mu_{G_k}(y_k)\}.$$

As the continuity of the function $\sup_{y_k \in \underline{A}} \mu_G(y_k)$ implies that G_k be strictly convex, a result is that $\forall y_k \in \underline{A}$, $\sup_{y_k \in \underline{Y}_k} \mu_{Y_k}(y_k)$ is strictly quasiconcave and Y_k is convex. Therefore, $\underline{Y}_k$ must also be strictly convex. Additionally, $\underline{Y}_k$ has to be compact.

In the goods demand space, the places where the several products are supplied or demanded and the respective quantities sold or purchased are obtained since $y_k = [y_{ijk}]$ where j is the index of demand places of the goods indexed by i which are elements of the vector y_k.

Now let $y = \sum_{k=1}^r y_k$ be a total production and $\underline{Y} = \sum_{k=1}^r \underline{Y}_k$ be the total production set, to which a fuzzy subset Y, $Y \subset \underline{Y}$, is associated. It is defined as follows.

The total production is such that:

$$\frac{\mu_Y(y)}{y} = \frac{\mathrm{Me}\left[\sup_{y_k \in \underline{Y}_k} \mu_{Y_k}(y_k)\right]}{\sum_{k=1}^r y_k}$$

where Me[·] is the median of the distribution of the values of the membership functions of the individual supplies that Y_k at the equilibrium. The above expression means that the sum of individual productions (under the line) is endowed with a membership function, denoted by μ_Y, (above the line), such that:

$$\mu_Y : \underline{Y} \longmapsto [0,1]$$

$$y \longmapsto \mu_Y(y) = \text{Me}\left[\sup_{y_k \in \underline{Y}_k} \mu_{Y_k}(y_k)\right].$$

Thus we have:

$$Y = \left\{ y, \mu_Y; \forall y \in \underline{Y}: \mu_Y(y) = \text{Me}\left[\sup_{y_k \in \underline{Y}_k} \mu_{Y_k}(y_k)\right]\right\}$$

with the above-mentioned interpretation applied to the median producer.

2.3.4. Excess Demand Point-to-Set Mapping and Markets Equilibrium. With the previous definitions, the excess demand in a fuzzy economy, denoted by e, is equal to:

$$e = x - y - w.$$

As $x \in \underline{X}$ and $y \in \underline{Y}$, an excess demand $x - y - w$ is an element of a set denoted by $\underline{E}$, with $\underline{E} = \underline{X} - \underline{Y} - \{w\}$.

Furthermore, since $x \in X$, $X \subset \underline{X}$, and $y \in Y$, $Y \subset \underline{Y}$, then e is also an element of a fuzzy subset, denoted by E, $E \subset \underline{E}$. The membership function μ_E is defined from $\underline{E}$ to $[0,1]$ and is dependent on $(x - y)$. It is such that $\mu_E(e) = 0$ if $(x - y) \leq 0$ (excess demand is non-positive), $\mu_E(e) = 1$ if $y = 0$ (excess demand is maximum) and $\mu_E(e) \in]0,1[$ in all the other cases, the function beng monotonic continuous, and increasing for increasing values of $(x - y)$.

We then define an excess demand point-to-set mapping e such that:

$$e : \underline{P} \longmapsto \underline{E}$$

$$p \longmapsto e(p) = x(p) - y(p) - w$$

where $x(p)$ is the value of the demand point-to-set mapping for a price p, $y(p)$ the value of the supply point-to-set mapping for a price p and w the initial resources.

The point-to-set mapping e is continuous by virtue of the continuity property of the objective functions of consumers and producers. In order to simplify the notation, the symbol e designates the excess demand and the excess demand point-to-set mapping. No ambiguity is possible in the context.

At equilibrium we must verify that $e \leq 0$. We must search for the conditions which have to be fulfilled by $e(p)$ in order for p to be such that $e(p) \leq 0$ exists.

3. CONDITIONS OF SPATIAL GENERAL EQUILIBRIUM IN A FUZZY ECONOMY

Mathematical tools are stated by Butnariu (1982) and Kaleva (1985) who express the existence conditions of fixed points for fuzzy mappings. The following statement is slightly different from theirs.

3.1. *Fuzzy Point-to-Set Mappings and Fixed Points*

3.1.1. Let X be a non-empty set and $\underline{L}$ a lattice with at least two elements, the set of all mappings from $\underline{X}$ in $\underline{L}$ is denoted by $\underline{L}^{\underline{X}}$. A fuzzy subset A of $\underline{X}$ is an element of $\underline{L}^{\underline{X}}$ such that:

$$A = \{x, \mu_A; \forall x \in \underline{X}\colon \mu_A(x) \in \underline{L}\}\,.$$

The set of all fuzzy subsets of $\underline{X}$ is denoted by $\mathcal{P}(\underline{X})$.

3.1.2. A fuzzy point-to-set mapping over a set $\underline{X}$ is a mapping, denoted by Γ, from $\underline{X}$ to $\mathcal{P}(\underline{X})$ which associates an element of $\mathcal{P}(\underline{X})$ to any x of $\underline{X}$:

$$\Gamma\colon \underline{X} \longmapsto \mathcal{P}(\underline{X})$$

$$\forall x \in \underline{X}, x \longmapsto \Gamma(x) = A.$$

We have:

$$\Gamma(x) = \{y, \mu_x; y \in \underline{X}, y \in \Gamma(x)\colon \mu_x(y) \in \underline{L}\}\,.$$

Thus x is a parameter. Giving a particular value to y, we can write:

$$\Gamma_x(y) = \{y, \mu_\Gamma; y \in \underline{X}\colon \mu_\Gamma(x, y) \in \underline{L}\}$$

where μ_Γ is the membership function of the couple (x, y) to the fuzzy subset $\Gamma_x(y)$.

3.1.3. An element of x^* of $\underline{X}$ is called fixed point of the fuzzy point-to-set mapping Γ iff its membership function to A is at least equal to the membership function of any element y of $\underline{X}$, i.e. iff $\mu_{x^*}(x^*) \geq \mu_{x^*}(y)$, $\forall y \in \underline{X}$. In other words, a fixed point of a fuzzy point-to-set mapping Γ over $\underline{X}$ is an element x^* such that $\mu_\Gamma(x^*, x^*) \geq \mu_\Gamma(x^*, x)$, $\forall x \in \underline{X}$.

This definition is a generalization of the usual one. Indeed, if $\underline{L} = \{0, 1\}$, we again find the definition of an ordinary point-to-set mapping: $y \in \Gamma(x) \Leftrightarrow \mu_x(y) = 1$. Hence, if $\mu_{x^*}(x^*) \geq \mu_{x^*}(y)$, then $\mu_{x^*}(x^*) = 1$ and $x^* \in \Gamma(x^*)$.

3.1.4. Let $\underline{C}$ be a subset of $\underline{X}$. The fuzzy point-to-set mapping Γ is closed iff the membership funciton $\mu_\Gamma(x, y)$ is upper semicontinuous over $\underline{C}$.

3.1.5. The fuzzy point-to-set mapping Γ is convex iff its membership function is quasiconcave.

3.1.6. The fuzzy subset $\Gamma_x(y)$ is normal iff $\sup_{y \in \underline{X}} \mu_\Gamma(x, y) = 1$, $\forall x \in \underline{X}$.

3.1.7. If $\underline{X}$ is a real topological vector space, locally convex and Hausdorff separated, $\underline{C}$ a non-empty, convex and compact subset of $\underline{X}$, and Γ a fuzzy point-to-set mapping over $\underline{C}$, a usual point-to-set mapping, denoted by $\hat{\Gamma}$, can be associated with Γ by:

$$\hat{\Gamma}_x(y) = \left\{ y, y \in \underline{C}\, \Gamma_x(y) = \bigcup_{z \in \underline{C}} \Gamma_x(z) \right\}.$$

3.1.8. A fuzzy point-to-set mapping Γ over $\underline{C}$, $\underline{C} \subset \underline{X}$, is said to be a fuzzy function iff, $\forall x \in \underline{C}$, $\hat{\Gamma}_x \neq \emptyset$. It is said to be a 'very fuzzy' function iff $\hat{\Gamma}_x$ is a mapping over $\underline{C}$.

3.1.9. A fuzzy point-to-set mapping over $\underline{C}$, denoted by Γ', can be associated with a fuzzy function Γ, by:

$$\Gamma'_x(y) = \begin{cases} \Gamma(x,y) & \text{if } \quad y \in \hat{\Gamma}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\hat{\Gamma}' = \hat{\Gamma}$.

Hence it results immediately that Γ and Γ' have the same fixed points and Γ' is a 'very fuzzy' function.

3.1.10. A fuzzy point-to-set mapping Γ over $\underline{C}$, $\underline{C} \subset \underline{X}$, is said to be F-continuous iff, $\forall x \in \underline{C}$, and for any open set $\underline{D}$ in $\underline{X}$, we have:

$$\Gamma_x(y) \subseteq \underline{D} \cap \underline{C} \Rightarrow \exists \underline{V} \in \underline{\surd}(y)/\Gamma_x(\underline{V}) \subseteq \underline{D} \cap \underline{C}$$

where $\underline{\surd}(y)$ is the set of all neighborhoods of y in $\underline{X}$.

3.1.11. BUTNARIU–KALEVA THEOREM. *Let $\underline{X}$ be a real topological vector space, locally convex and Hausdorff separated, and $\underline{C}$ be a non-empty, convex and compact subset of $\underline{X}$. If Γ is a closed and convex fuzzy point-to-set mapping over $\underline{C}$, and if Γ_x is normal, $\forall x \in \underline{C}$, then Γ has a fixed point in $\underline{C}$.*

Proof. The fuzzy point-to-set mapping Γ has a fixed point x^* iff the point-to-set mapping $\hat{\Gamma}$ has a fixed point, i.e. iff $x^* \in \hat{\Gamma}(x^*)$. Indeed x^* is a fixed point of Γ iff $\mu_{x^*}(x^*) \geq \mu_{x^*}(y)$, which implies that $x^* \in \Gamma(x^*)$. Therefore it suffices to verify that $\hat{\Gamma}$ fulfils the assumptions of Kakutani's theorem.

First, $\hat{\Gamma}_x \neq \emptyset$. Indeed $\hat{\Gamma}_x$ is an upper semicontinuous point-to-set mapping. It has a maximum y and effectively attains it in the compact subset $\underline{C}$ of $\mathbb{R}^n$. Thus $\exists y \in \underline{C}/y \in \hat{\Gamma}_x$ and $\hat{\Gamma}_x \neq \emptyset$.

Then $\hat{\Gamma}_x$ is convex. Indeed, let (x, y) be elements of $\underline{C}$ and $r \in \underline{L}$, with $\underline{L} = [0, 1]$. If $y \in \hat{\Gamma}_x$ and $z \in \hat{\Gamma}_x$, then $\Gamma_x(ry + (1-r)z \geq \wedge(\Gamma_x(y), \Gamma_x(z))$ by virtue of the quasiconcavity of the membership function of the point-to-set mapping Γ. Moreover, $\Gamma_x(ry+(1-r)z) = \sup_{w \in \underline{C}} \Gamma_x(w)$ by virtue of the definition of $\hat{\Gamma}$. Therefore, $(ry + (1-r)z) \in \hat{\Gamma}_x(z)$ and the property is proved. Besides, $\forall x \in \underline{C}$, $\hat{\Gamma}_x$ is compact. Indeed, let (x_i), $i \in \underline{I}$, a series in $\hat{\Gamma}_x$ whose limit point is x^*. Then $\Gamma_x(x_i) \geq \Gamma_x(y)$, $\forall y \in \underline{X}$, and $\forall i \in \underline{I}$. From that,

$\Gamma_x(x^*) = \lim_{i \in \underline{I}} \sup \Gamma_x(x_i) \geq \Gamma_x(y), \forall y \in \underline{C}$, by virtue of the upper semicontinuity property of Γ_x. Consequently, $\hat{\Gamma}_x$ is a closed subset of the compact set $\underline{C}$. So $\hat{\Gamma}_x$ is compact.

Finally, the set graph $\cup_{x \in \underline{C}}(\{x\} \times \hat{\Gamma}(x))$ is closed. Indeed, let (x_i) and (y_i), $i \in \underline{I}$, be two series in $\underline{C}$ such that (x_i, y_i) be elements of the set graph, $\forall i \in \underline{I}$. Since $\underline{C}$ is compact, $\lim(x_i) \in \underline{C}$. Moreover, $\Gamma_{x_i}(y_i) \geq \Gamma_{x_i}(y), \forall y \in \underline{C}, \forall i \in \underline{I}$.

If $x_i \longmapsto x^*$ and $y_i \longmapsto y^*$ in $\underline{X}$, therefore in $\underline{C}$ since $\underline{C}$ is compact, then:

$$\hat{\Gamma}_{x^*}(y^*) = \lim_{i \in \underline{I}} \sup \Gamma_{x_i} \geq \lim_{i \in \underline{I}} \sup \Gamma_{x_i}(y), \quad \forall y \in \underline{C}.$$

Since Γ is upper semicontinuous, we have:

$$\Gamma_{x^*}(x^*) \geq \Gamma_{x^*}(y), \quad \forall y \in \underline{C},$$

i.e. (x^*, y^*) is an element of the graph and, by definition, the graph is closed. ■

3.2. *Walras' Generalized Law*

3.2.1. Consider an excess demand continuous point-to-set mapping e which associates the subset $e(p)$ of $\underline{E}$ with any price vector p of $\underline{P}$, such that: $e(p) = x(p) - y(p) - w$.

Walras' Law (1874–1877) can be generalized to the case of a competitive spatial economy with fuzzy behavior of agents. It claims that, for all the goods i, the sum of the excess demand values is null or negative, i.e.:

$$\sum_{i=1}^{n} p_i e_i(p) \leq 0, \quad \forall p \in \underline{P}.$$

This condition means that excess demands have to be non-positive and the goods whose excess demand is negative must have a null price. Consequently, according to the definition of a spatial general equilibrium in a fuzzy economy (Section 2.3.4.), a competitive equilibrium will be reached if there exists a price vector $p^* \in \underline{P}$ and an excess demand vector $e^* \in e(p^*)$ such that $e^* \leq 0$ and $\mu_E(e^*) = 0$.

3.2.2. If Walras' Generalized Law is not verified, a groping process modifies the price levels. In all the markets of goods i such that $e_i(p) \neq 0$, prices p_{ijk} are rectified by a factor which depends on the excess demand, denoted by ε. Prices remain constant in other cases.

Thus, at any step of the groping process, prices p'_{ijk} are substituted for prices p_{ijk} so as:

$$p'_{ijk} = \begin{cases} p_{ijk} & \text{if } ei(p) = 0 \\ p_{ijk} + \varepsilon & \text{if } e_i(p) > 0 \\ \max(0, p_{ijk} - \varepsilon) & \text{if } e_i(p) < 0. \end{cases}$$

We have a continuous function, denoted by g, such that:

$$g\colon \underline{E} \longmapsto \underline{P}$$

$$e \longmapsto g(e) = \{p'/p' \cdot e < p \cdot e\}.$$

In order to verify subsequently the conditions of the fixed point theorems, prices are normalized, i.e. they are such that $\sum_{i,j,k} p_{ijk} = 1$.

Let $\overline{\underline{P}} = \{\overline{p}; p \in \underline{P} : \sum_{i,j,k} p_{ijk} = 1\}$ be the set of normalized prices. The set $\overline{\underline{P}}$ is a simplex. Thus a continuous function, denoted $\overline{g}$, is associated with the function g. It is such that:

$$\overline{g}\colon \underline{E} \longmapsto \overline{\underline{P}}$$

$$e \longmapsto \overline{g}(e) = \{\overline{p}'/\overline{p}' \cdot e < \overline{p} \cdot e\}$$

This transformation causes no loss of generality. The prices of goods in the balanced or excess supply markets are relatively brought down as for the prices of goods in the excess demand markets.

In the same manner, we denote be $\overline{e}$ the continuous point-to-set mapping, associated with the excess demand point-to-set mapping e, which has $\overline{\underline{P}}$ as domain and $\underline{E}$ as co-domain.

The following theorem states the conditions for which the groping process tends to a price equilibrium in all the markets.

3.2.3. THEOREM. *Walras' Generalized Law is verified if the function denoted by h, from the price simplex $\overline{\underline{P}}$ in $\overline{\underline{P}}$ is continuous.*

Proof. Consider the function h such that:

$$h\colon \overline{\underline{P}} \longmapsto \overline{\underline{P}}$$

$$\overline{p} \longmapsto h(\overline{p}) = (\overline{g} \circ \overline{e})(\overline{p}).$$

The function h is continuous since $\overline{g}$ and $\overline{e}$ are continuous mappings.

Then the assumptions of the Brouwer theorem are fulfilled since the simplex $\underline{\overline{P}}$ is a non-empty, convex and compact set. The function h is a continuous transformation of the simplex $\underline{\overline{P}}$ in $\underline{\overline{P}}$. Therefore, there exists a fixed point $\overline{p}^* \in \underline{\overline{P}}$ which is constant in the transformation $\overline{p} \mapsto e(\overline{p}) \mapsto \overline{p}'$. ■

3.3. *Existence of a Spatial General Equilibrium in a Fuzzy Economy*

Now we are ready to express a theorem which states the existence conditions of a spatial competitive equilibrium in a fuzzy economy. It is a generalization of Debreu's theorem (1959) to the case of a spatial economy which is characterized by fuzzy behavior of agents.

An excess demand fuzzy point-to-set mapping on $(\underline{\overline{P}} \times \underline{E})$, denoted by φ, is defined from $(\underline{\overline{P}} \times \underline{E})$ to $\mathcal{P}(\underline{\overline{P}} \times \underline{E})$ such that $\varphi(\overline{p}, e) \times \{h(\overline{p}, \overline{e}(\overline{p})\}$. Then the following theorem holds.

THEOREM. *Let $(\underline{\overline{P}} \times \mathbb{R}^l)$ be a real topological vector space, locally convex and Hausdorff separated, and $(\underline{\overline{P}} \times \underline{E})$ be a subset of $(\underline{\overline{P}} \times \mathbb{R}^l)$. If the excess demand fuzzy point-to-set mapping φ on $(\underline{\overline{P}} \times \underline{E})$ is closed and has images $\varphi(\overline{p}, e)$ which are non-empty, normalized, convex, and verifies Walras' Generalized Law, then a competitive equilibrium exists in the spatial fuzzy economy, i.e. there exists a price vector $p^* \in \underline{P}$ and an excess demand vector $e^* \in e(p^*)$ such that $e^* \leq 0$ and $\mu_E(e^*) = 0$.*

Proof. Let $\underline{E} \subset \mathbb{R}^l \cdot \underline{E}$ is non-empty, convex and compact since, by definition, $\underline{E} = \underline{X} - \underline{Y} - \{w\}$, and the possible consumption and production sets, $\underline{X}$ and $\underline{Y}$ respectively, are non-empty, strictly convex and compact by virtue of the continuity conditions of the objective functions in the consumers and producers fuzzy economic calculations (see 2.3.1.1., 2.3.2. and 2.3.3.).

Now let φ be the excess demand fuzzy point-to-set mapping on $(\underline{\overline{P}} \times \underline{E})$. The fuzzy point-to-set mapping φ is closed. Indeed, its graph:

$$G_\varphi = \left\{(x, y) \in (\underline{\overline{P}} \times \underline{E})^2 / y \in \varphi(x) = G_h \times G_{\overline{e}}\right\}$$

is closed because it is the product of two closed sets.

Moreover, the fuzzy point-to-set mapping φ is convex on $(\overline{\underline{P}} \times \underline{E})$ since it is the Cartesian product of two convex point-to-set mappings.

Lastly, the normality property of $\varphi(\overline{p}, e)$ is verified: $\sup \mu_\varphi[(\overline{p}, e), (h(\overline{p}), \overline{e}(\overline{p}))] = 1$, where μ_φ is the membership function of the couple in brackets to the fuzzy subset $\varphi(\overline{p}, e)$, since μ_φ being a continuous numerical function on a compact set it is bounded and it actually attains its bounds.

Therefore, the assumptions of the Butnariu–Kaleva theorem are verified and the fuzzy point-to-set mapping φ has a fixed point in $(\overline{\underline{P}} \times \underline{E})$.

By virtue of Walras' Generalized Law, the fixed point corresponds to a competitive equilibrium state. Indeed, there exists a vector $(\overline{p}^*, e^*)$ of $(\overline{\underline{P}} \times \underline{E})$ such that:

$$(\overline{p}^*, e^*) \in \varphi(\overline{p}^*, e^*) = \left(h(\overline{p}^*), \overline{e}(\overline{p}^*)\right).$$

Then we have:

$$[\overline{p}^* \in h(\overline{p}^*) \text{ and } e^* \in e(\overline{p}^*)]$$
$$\Rightarrow \forall p \in \underline{P}\colon p \cdot e^* \leq p^* \cdot e^* \leq 0.$$

Whence: $e^* \leq 0$ and $\mu_E(e^*) = 0$. ■

4. CONCLUSION

The theorem on the conditions for the existence of a spatial general equilibrium in a fuzzy economy holds whatever the economic space configuration may be. As long as the competitive structure of markets is kept, it is applicable as well to cases of a space *à la* Thünen (concentrated demand, scattered supply), of a space *à la* Hotelling (scattered demand, concentrated supply),[1] or of a space *à la* Lösch (scattered demand and supply). Finally, it covers, as a particular case, the model of a classical a-spatial economy (demand and supply concentrated in a single point). But this result holds only when space is characterized in a poor manner. It has been described with the help of a set of points which are separated by given distances and all the prices, including the transportation prices, are settled in competitive markets. A richer characterization of space would initiate notable difficulties. We find again the idea that space is not economically neutral and that the integration of the space concept in general equilibrium theory raises specific problems (Mougeot, 1978).

However, if the use of fuzzy concepts were to lead only to more general results than the classical ones, it would be rather uninteresting. In fact, fuzzy models are fit for solving problems which have no solution, or less accurate solutions, in the framework of classical economics (Ponsard, 1988). For example, early discussions on the existence and the effects of a perfect or a weakened rationality of the individuals have a solution: a fuzzy decision can be optimal with regard to any level of rationality. In the consumer's equilibrium theory, a continuous function of fuzzy utility can be associated with a preference fuzzy binary relation under little restrictive conditions. In the same way, in the producer's equilibrium theory, the assumption of increasing returns does not raise a specific problem since the convexity condition is not imposed on the technological constraint. Finally, based on these new foundations, the general equilibrium theory is stated without the help of assumptions such as the existence of a continuum of agents or a perfect competition which are meaningless in spatial analysis. Moreover, the classical theory has some solutions for problems raised by incomplete or weakly rational preferences, but it never states the conditions for the compatibility of the behavior attendant on them. The use of fuzzy concepts enables us to analyze the behavior of agents endowed with such non-standard preferences in the framework of a theory of general equilibrium.

Of course, much complementary research must be looked at. In particular, the stability and optimality properties of the spatial general equilibrium in a fuzzy economy will have to be studied. In the same way, the assumption of competition in markets will have to be replaced by that of an oligopolistic economy (Ponsard, 1987). The theory will have to be formalized in terms of fuzzy games and cores. Finally, in the present analysis, the locations of the economic agents are given. The difficult problem of the existence of a spatial general equilibrium in a fuzzy economy with endogenous locations is not yet solved.

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NOTE

[1] We know today that Hotelling's conclusions on the 'concentration of demand' were erroneous ones. When (for the last time November 24, 1989) I met Claude Ponsard

at the colloque annuel de l'Institut de Mathématiques Economiques (I.M.E.) in Dijon, we decided that one could keep the expression of a space '*à la* Hotelling' in its widely accepted meaning among spatial economists and quantitative geographers (Editor's note).

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EDMOND MALINVAUD

5. MAURICE ALLAIS, UNRECOGNIZED PIONEER OF OVERLAPPING GENERATIONS MODELS*

Economy and Interest,[1] published in 1947, was the result of an intense effort of reflexion pursued by Maurice Allais on economic theory after the publication in 1943 of the work first entiteld 'In search of an economic discipline' which later became the *Treatise on Pure Economics*.[2]

This was also the text on which the brilliant students who clustered around him during subsequent years were privileged to be trained. This in many respects elaborate and difficult book makes important contributions to the theory of capital. For many years it has hardly been studied in reference to this subject, probably because its results were expected to be found in other more didactic and systematic books published more recently. Thus nobody seems to have noticed until now that, *Economy and Interest* contains a modelization which has shown itself to be, in the course of the past decade, most appropriate in the treatment of numerous questions concerning either the theory of capital or the theory of money.

If we view time as an indefinite procession of periods, consumption in this model concerns individuals belonging to successive generations, the lifetimes of which overlap. Custom has today attributed the introduction of the model to an article published by Paul Samuelson in 1958.[3] However, not only does the model already appear explicitly in an appendix to the work which was published eleven years earlier, but it seems to have played an important part in the development of ideas which Maurice Allais reveals in various chapters.[4]

My intention here is to rehabilitate Allais as the pioneer of the theory, thereby, hopefully, forcing the academics to pay proper attention to *Economy and Interest*. To this end I shall take the short model studied in an appendix of the work and show how it led Maurice Allais to acknowledge very clearly an important feature of Overlapping Generation Models, i.e. the data of technical feasibilities as well as consumer preferences do not suffice for the determination of the rate of interest and the allocation of resources. But before starting, let us examine what made the Pure Theory of Capital so important.

B. Munier (ed.), Markets, Risk and Money, 111–128.

1. THE PURE THEORY OF CAPITAL

Most of the main issues traditionally studied by political economy concerned capital; they so required that the intertemporal aspect of phenomena be explicitly taken into account, together with the interdependences that are part and parcel of these phenomena. The intertemporal aspect is the source of very serious complications that one has some difficulty in overcoming. Success can only be achieved with a great effort of abstraction. Such is the object of the Theory of Capital.

Speaking of Pure Theory one wants to convey the idea of a strong refinement of reality. More precisely, the fundamental hypothesis of the abstractions assumes the existence of a complete system of prices and rates of interest which are taken as given by the various agents and are such as to clear markets. In short, on may speak of perfect competition.

The fact that this hypothesis is generally used in the Theory of Capital does not necessarily mean that theoreticians believe in the ideal working of markets, but it reveals a selective choice in their research. It is the most convenient hypothesis for simultaneously treating a few of the most essential aspects of the problem, such as interdependence, time span, and the structure of prices and interest rates.

But, even whilst adhering to this convenient hypothesis hard problems remain. They have to be clarified in the simplified context of the Pure Theory before taking other complications into account.

In this respect the attitude of Joan Robinson, who died recently, was particularly revealing. Everyone knows that she felt little ideological sympathy for economic liberalism and for the theory according to which the spontaneous functioning of markets would assure the efficiency of production and distribution. Everyone acknowledges her contributions to the Theory of Imperfect Competition. However, in her reflexions on the Theory of Capital and in the discussions on the subject, in which she engaged during the 'sixties and 'seventies, she always agreed to reason from the convenient hypothesis of Perfect Competition.

Temporal spacing is initially introduced into the model in the form of a finite number of periods, the two-dimensional multiplicity of goods and multiplicity of periods then being treated in a similar way. For the relevance of such a representation, the economic activity then needs to be considered as operating between two given dates, more or less distant from each other: the initial and terminal dates. The activity, therefore, starts from certain data characterizing the initial date, principally of the capital existing at this date, which seems quite natural. But it should also lead to certain results at the terminal date, principally to leave a capital of which the volume and the composition will have been specified in advance. To impose this last condition seems unsatisfactory, since only knowledge of the ensuing activity would determine what terminal capital would be suitable.

That is why it was soon realised that the Theory of Capital should also consider alternative formulations in which time would be treated as unlimited and in which, consequently, the requirement for terminal conditions would disappear. The most convenient alternative consists in supposing that the environment remains the same through time and therefore in concentrating on stationary equilibria which occur without change from one period to the next, *ad infinitum*. A slightly more general variation would be to study "proportional growth" at a regular rate.

There is another reason for not treating the multiplicity of periods in the same way as that of goods, and that is to know not only that the periods are ordered but also that economic activity is composed of various interdependent operations, each one of which affects a period or a number of successive periods rather than all periods. Current production operations are rapidly fulfilled; with certain exceptions equipment deteriorates and becomes worn out, with the result that it can only be used for a certain length of time; individuals themselves have only a limited lifetime in which to match their productive forces and their consumption (this is their 'life cycle').

The model appropriate to this aspect of things has rapidly been discovered for the productive sector. Most often one distinguishes current inputs from productive capital, thereby assuming production functions connecting the production obtained during a period to the capital invested at the beginning of the period and to the inputs consumed during the period. The capital is, moreover, subject to a specific law of amortization. It is sometimes prefered to link the production of a period to the quantities of primary factors which have been affected to it, either during the same period or during several previous periods. So at one particular time, the capital is made up of all the primary inputs which are invested in the productive process and are thus, as it were, on the way of maturation. In fact, *Economy and Interest* elects to use sometimes one and sometimes the other of these models.

For a number of questions concerning the pure theory of capital, it is sufficient to consider the productive sector and to study the conditions under which it transforms, most efficiently, primary factors that it uses into goods available for consumption. Such is the case of the determination of certain characteristics of prices and rates of discount, or still of what Maurice Allais called in 1947 'theory of social productivity' and which was later spoken of by other economists as the 'golden rule'. One can thus understand that even the purest models of the theory of capital have, for a long time, not had to be explicit about consumer representation.[5]

But, in this respect, a particular representation has actually surfaced in the last decade. Individuals are considered explicitly as belonging to distinct generations. We all have only a limited lifespan and make our choices for a 'life cycle' in order to benefit, in the best possible way, from the economic conditions with which we will be confronted during our existence. At a given moment several successive generations coexist, and take part in economic operations which may affect two or more of them simultaneously.

The simplest model, which is most often sufficient to deal with the principal problems, supposes that each consumer lives during two periods exactly; the first during which he works and consumes, the second during which he is inactive and lives on his savings. If x^j and x^v represent the vectors of his consumption during the two periods, that of his youth and that of his old age, the consumer is assumed to make choices represented by a utility function $S(x^j, x^v)$. For the same reasons of simplicity, one supposes that, most often, each generation comprises the same number of consumers, and that all have the same utility function. But, obviously, a general theory can be conceived which does not make such particular hypotheses.[6] In the same way, it is possible to take account of the transmission of wealth between successive generations by means of inheritance.

2. APPENDIX II OF ECONOMY AND INTEREST

With regard to these recent developments, we turn our attention to Appendix II of *Economy and Interest.* It takes up almost 120 pages of a work totalling 800. With two other much shorter appendices, it is introduced by three pages which reveal very clearly the importance that Maurice Allais attributed to it (pp. 631–633). It can be summed up as follows:

The preceding general study has been able to show what were the principal aspects of the phenomenon of interest ... However the various proofs have only been ... partial ... Such an approach would not on its own be considered as sufficient ..., because the separate examination of the different elements of a mechanism would not be sufficient to show how they function in conjunction with each other ... In fact, the study of the mechanism, as a whole, can only be carried out with simplified models ... Such is the aim of the appendices ... Their interest cannot be over emphasized ... Naturally such studies of simplified models could not constitute a method of proof ... but they enable certain theories to be eliminated with absolute confidence.[7]

The benefits gained by studying particular cases in detail is well known to mathematicians. Such a study is doubly valuable in the development of a theory; on the one hand it helps to enumerate all the facts that the theory must take account of, whereas a general reflexion tends to forget some of them, and on the other hand it frequently reveals the restrictive character of certain hypotheses which *a priori* appear harmless.

In economic theory the examination of particular short models with strict specifications has often played an even more important role. The results obtained from these models have been considered as applying qualitatively to a much wider field. For lack of anything better, their general validity has often been taken as a working hypothesis. Certainly, it should only be an intermediate stage in the development of the theory; we could not base a final proof on what only consists of a limited indication. However, the study of these short models is often revealed, *a posteriori*, as having led to a much better understanding of phenomena than the one literary reasoning founded on intuition had led to beforehand.

In the present case, the appendices, and especially Appendix II, are often invoked by Maurice Allais in the course of his main text, to justify certain essential affirmations of the theory which he is demonstrating. Retrospectively, it would seem that this method of working has caused no fundamental errors.

The principal object of Appendix II lies in the explicit determination of a stationary competitive equilibrium, and in the study of factors affecting the equilibrium value of the interest rate. Additionally, the appendix enables certain propositions on the optimality of equilibrium and the role of money to be verified.

There is no question here of giving a complete account of the appendix but simply of showing how it defines perfectly clearly the Overlapping Generations Model and how it already made clear certain properties that were only perceived much later by other economists studying the model. To do this it is proper to return to the specification in the appendix without changing it nor trying to generalize. (Some simplifications will even result from a convenient choice of units, thanks to which some unnecessary parameters will be eliminated.)

3. THE PRODUCTIVE SYSTEM

Two goods can be produced: the quantity of work L_1 enables the quantity:

$$(1) \qquad K = aL_1 \qquad L_1 \geq 0$$

of the production good to be obtained, a being a constant.[8] This good can be used during the subsequent period to take part in the production of the quantity Q of the consumer good. It is then totally destroyed.[9]

More precisely the production function is:

$$(2) \qquad Q = L_2^{1/2}(K + U)^{1/2} \quad L_2 \geq 0 \quad K \geq 0$$

L_2 being the quantity of work employed in the sector of consumer goods, from which:

$$(3) \qquad L = L_1 + L_2$$

for the total quantity of work. This is an exogenous datum, in the same way as the positive quantity U which, for Maurice Allais, represents land service.

In this model of the productive system the endogenous variables are L_1, L_2, K and Q. If it happens that the stationary equilibrium implies $K = 0$ (we will see that this is the case, for example, if the real rate of interest is not negative and if $U > aL$), then the quantity of consumer good available $Q = \sqrt{LU}$ can be considered as predetermined, which was the hypothesis of Paul Samuelson. But the 'simplified model' of Maurice Allais is less restrictive.

If we take the consumer good as numeraire, the price system of a stationary equilibrium is represented by: q = price of production good, s = wage rate, r = interest factor, ($r - 1$ is the rate of interest) and possibly u = price of land service (to introduce u explicitly is convenient for the determination of the rent uU). The assumption is made that the producer of the consumer good must acquire the production good in the period which precedes that of his employment. Thus the cost of the use of the production good is rq.

The competitive equilibrium implies, for the production of the first good:

(4) $s = aq \quad \text{if } K > 0 \quad (s \geq aq \text{ if not}).$

Likewise for the production of the second good, supposed positive

(5) $$s = \frac{Q}{2L_2}$$

(6) $$rq = u = \frac{Q}{2(K+u)}.$$

It is obvious that the six preceding equations allow one degree of freedom for the determination of the seven endogenous variables L_1, L_2, K, Q, q, s, r. This is quite natural since the productive system is the only thing to have been considered for the moment. It is interesting to see how the equilibrium varies according to the rate of interest.

A calculation which is not necessary to include here shows that two zones of variation in r must be distinguished:

– if $r \geq aL/U$ then $L_1 = K = 0$, $L_2 = L$, $Q = \sqrt{LU}$

$$s = Q/2L, \quad rq = u = Q/2U$$

– if $0 \leq r \leq aL/U$, then the following equations apply

(7) $$K = \frac{aL - rU}{1+r} \quad L_2 = \frac{r(U + aL)}{a(1+r)}$$

(8) $$Q = \frac{U + aL}{1+r} \cdot \sqrt{\frac{r}{a}}$$

(9) $$q = \frac{1}{2\sqrt{ar}} \quad s = \frac{1}{2}\sqrt{\frac{a}{r}}$$

In particular the available quantity of consumer goods can be assumed to vary from one stationary equilibrium to another according to the rate of interest. Figure 1 represents this function for the case where $aL > U$.

Before going further it is interesting to note the effect of the 'golden rule': the quantity of consumer goods produced in a static equilibrium is a maximum when the interest rate is zero ($r = 1$). This is a

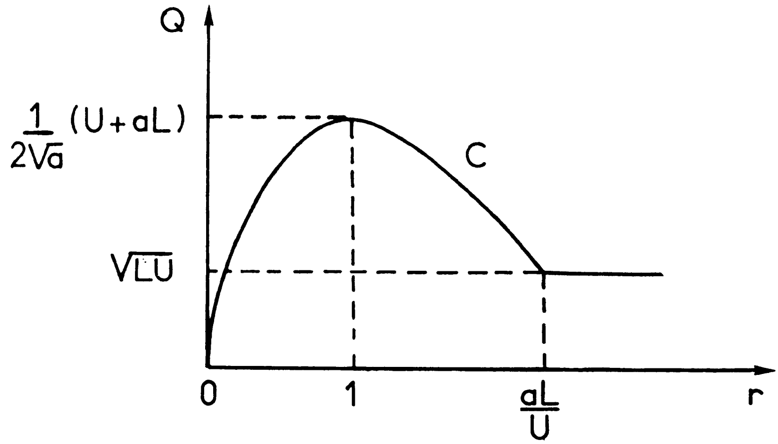

Fig. 1.

property to which Maurice Allais devoted much time in *Economy and Interest*, hence before publication of the first discussions of the golden rule, but restricting himself to stationary equilibria, a particular case of proportional growths to which this rule applies. He speaks then of 'social productivity' and clearly demonstrates the relationship between this idea and that of Paretian optimality.

4. STATIONARY EQUILIBRIA

The consideration of the productive system alone is not sufficient to determine the equilibrium. A more complete model is necessary in order to understand how the real interest rate is determined in the market whilst keeping to the point of view of the Pure Theory of Capital, that is to say, particularly in neglecting monetary phenomena.

Economists' intuition leads them to conclude that the interest rate results from confrontation between productive feasibilities and consumers' choice. In reference to Figure 1, for example, we are tempted to say that the interest rate would be all the higher as consumers were to show a stronger preference for the present. Being given the consumers' utility function should thus determine the appropriate point on the curve in Figure 1. But if examined in depth we understand that this intuition depends on the study of a finite number of time periods, often of only two periods. The Overlapping Generations Models show, correctly, that the phenomenon is more complex.

In order to understand the distinction properly, let us first suppose that there would be, effectively, only two periods, that the available work L should be divided, during the first period, between the production $K = aL_1$, of production goods and that of $Q_1 = \sqrt{L_2 U}$ of consumer goods. During the second period, obviously, production goods are no longer produced and the available quantity of consumer goods is $Q_2 = \sqrt{L(K + U)}$.

It can easily be calculated that the analogue of Equation (8) summing up the working of the productive system is then written:

$$(10) \qquad Q_1^2 = \frac{r^2 U^2 (U + aL)}{a(aL + r^2 U)} \quad Q_2^2 = \frac{aL^2 (U + aL)}{aL + r^2 U} \, .$$

Let us further suppose that the utility function of the representative consumer is:

$$(11) \qquad S(Q_1, Q_2) = \log Q_1 + \alpha \log Q_2 .$$

Maximizing this utility with the condition of a given total present resource R, that is to say with the condition:

$$(12) \qquad Q_1 + \frac{1}{r} Q_2 = R$$

the consumer chooses to divide his resource such that:

$$(13) \qquad Q_2 = \alpha r Q_1 .$$

The consumption during the second period is, relative to that of the first period, that much higher as the interest rate is higher and the preference for the present is weaker (α is higher).

Comparison of Equations (10), (12), and (13) determine what the interest factor r and the present resource R should be. In particular it is easy to obtain:

$$(14) \qquad r = \frac{a^2 L^2}{\alpha U^2} .$$

So the interest rate actually appears as an increasing function of the preference for the present. It obviously also depends on productive feasibilities which are expressed here by the ratio aL/U.[10]

We return now to the 'simplified model' of Maurice Allais and assume successive generations have a constant population n (in the following formulae we will let $n = 1$ for simplicity). Each generation lives for two periods, provides a quantity of work L during the first period while it consumes x_j, and stays inactive during the second period while it consumes x_v.

Obviously

$$(15) \qquad Q = x_j + x_v.$$

To choose their consumption plans, all generations conform to the same utility function:

$$(16) \qquad S(x_j, x_v) = \log x_j + \alpha \log x_v.$$

In a stationary equilibrium they are both subject to the same budgetary restrictions:

$$(17) \qquad x_j + \frac{1}{r} x_v = R$$

where R is their total resource, discounted at the time of their youth.

Based on these data, from a simple calculation we obtain:

$$(18) \qquad x_v = r\alpha x_j$$

$$(19) \qquad Q = \frac{1 + \alpha r}{1 + \alpha} \cdot R.$$

It appears then that taking account of the preferences of consumers is no longer sufficient to remove the indeterminacy resulting from only considering the productive sector. The supplementary Equation (19) in Q and r brings into effect the new variable R. In other words, in order to determine the equilibrium, the specification must be more precise and a rule must be given defining how the present resource R is obtained.[11]

Basically, this is not a new situation. It is well known that competitive equilibrium depends not only on technical feasibilities and psychological preferences but also on the distribution of rights between individuals. The indeterminacy being discussed at present results fundamentally from what has not been specified: the distribution of rights between generations. However, the sensitivity of the interest rate to this specification is certainly contrary to the intuition of most economists.

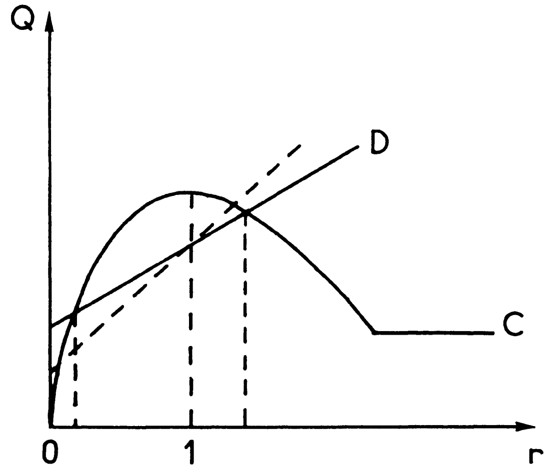

Fig. 2.

Discussion of the effects of the rule used to define present income is an important part of *Economy and Interest*; so it is necessary to pause here in order to review various specifications chosen from among the many which spring to mind. Rather than giving formulae applying to stationary equilibria, attention will be concentrated on graphical determinations keeping the same case as that to which Figure 1 applies, $aL > U$ (the converse case does not lead to qualitatively different conclusions).

A. The Real Resource is an Exogenous Datum

R being exogenous, Equation (19) then defines an increasing linear relationship between R and Q. The equilibrium corresponds to an intersection between the straight line D representing this relationship and the curve C resulting from only considering the productive system (see Figure 2).

If $R < (U + aL)/2\sqrt{a}$, there are two equilibria, one of which carries a positive interest rate and the other a negative rate. If the preference for the present decreases (i.e. if α increases) the interest rate decreases in each of the two equilibria (replacing the continuous straight line D by the dotted straight line).

There is, however, another possibility, where, without being too high,

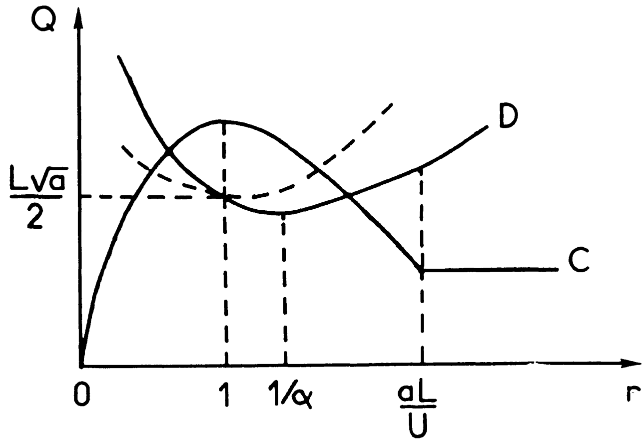

Fig. 3.

R will exceed $(U + aL)/2\sqrt{a}$. The interest rate will then be negative in each of the two equilibria. Further, the equilibrium leading to the real interest rate nearest to zero will have this paradoxical characteristic, that the interest rate will be a decreasing function of the preference for the present (increasing function of α).

B. Real Resource Is Equal to Work Revenue Only

To Equation (19) must be added

$$(20) \qquad R = sL$$

taking into account the value of s found by studying the productive system, we see that (19) must then be replaced by

$$(21) \qquad \begin{cases} Q = \frac{1+\alpha r}{1+\alpha} \cdot \frac{L}{2}\sqrt{\frac{a}{r}} & \text{if } r \leq \frac{aL}{U} \\ Q = \frac{1+\alpha r}{1+\alpha} \cdot \frac{\sqrt{LU}}{2} & \text{if } r > \frac{aL}{U} \,. \end{cases}$$

The curve defined by (21) must pass through the point $r = 1$, $Q = L\sqrt{a}/2$, it reaches its minimum for $r = 1/\alpha$. Figure 3 shows that there must then exist two stationary equilibria, one carrying a positive interest rate, the other a negative one. In both equilibria, the interest rate decreases when the preference for the present diminishes (replacing curve D in a continuous line by the dotted line).

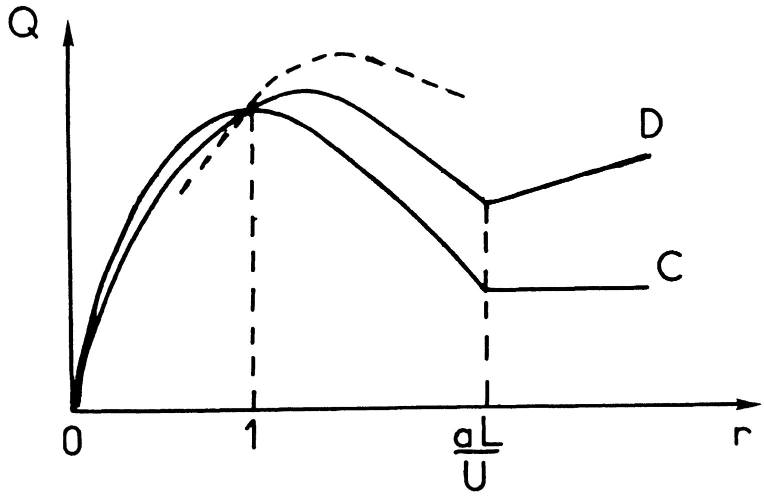

Fig. 4.

C. The Present Resource Is Equal to Aggregate Income

If all the revenues created by production are distributed to young consumers, then one can write $R = Q$ and Equation (19) shows that the only possible solution is $r = 1$. The interest rate should be nil whatever the preference for the present. In order to make a comparison with the other cases discussed here, a similar diagram must be established to those presented elsewhere. It is sufficient then to consider the formation of revenues given by the equation:

$$(22) \quad R = sL + uU + (r - 1)qK.$$

The equations define what the study of productive system implies for s, q, U and K leading to the replacement of this equality by:

$$(23) \quad \begin{cases} Q = \frac{1+\alpha r}{1+\alpha} \cdot \frac{(aL+U)}{1+r}\sqrt{\frac{r}{a}} & \text{if } r \leq \frac{aL}{U} \\ Q = \frac{1+\alpha r}{1+\alpha} \cdot \sqrt{LU} & \text{if } r > \frac{aL}{U} \,. \end{cases}$$

It is easy to verify that the curve D defined by (23) has no other point of intersection with C than corresponding to $r = 1$, because the point at the origin obviously has no meaning (Figure 4). Thus, the stationary equilibrium carries a nil interest rate whatever the preference for the present.

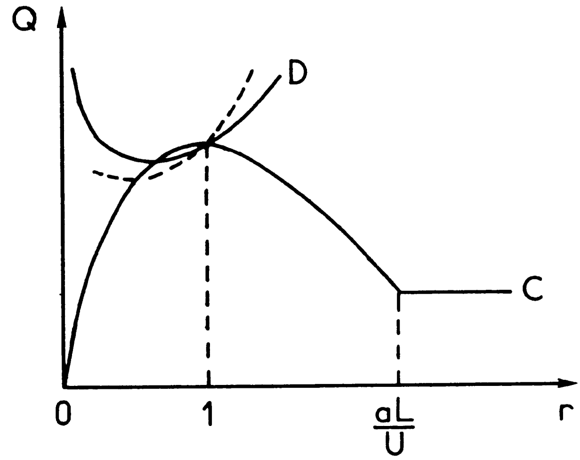

Fig. 5.

D. Rents Are Distributed to the Young

To Equation (19) must be added

(24) $R = sL + uU.$

Taking account of the relationship defining s and u from the equilibrium of the productive system, Equality (19) implies:

$$(25) \quad \begin{cases} Q = \frac{1+\alpha r}{1+\alpha} \cdot \frac{rU+aL}{2\sqrt{ar}} & \text{if } r \leq \frac{aL}{U} \\ Q = \frac{1+\alpha r}{1+\alpha} \cdot \sqrt{LU} & \text{if } r > \frac{aL}{U}. \end{cases}$$

So there are in general two equilibria. The first corresponds to $r = 1$ and therefore carries a nil interest rate. The second carries a negative interest rate if

$$(26) \quad \alpha > \frac{aL - U}{aL + 3U}$$

and a positive interest rate in the contrary case. The interest rate at this second equilibrium increases when the preference for the present increases. (Figure 5 shows the case where the inequality (26) applies.)

E. Rents Are Distributed to the Old

So the current resource is expressed by:

$$(27) \quad R = sL + \frac{u}{r} U$$

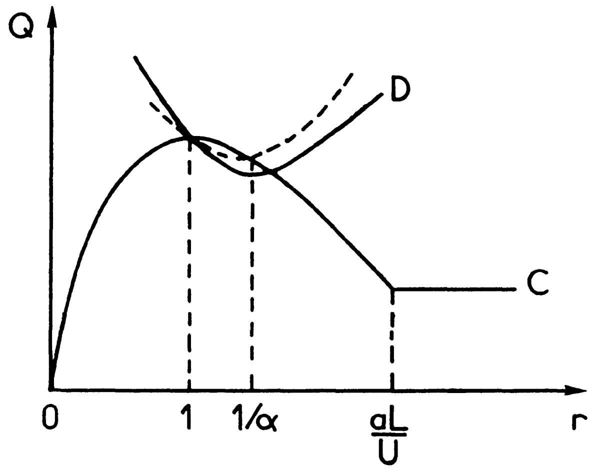

Fig. 6.

from which follows definition of the curve D:

$$(28)\quad 28\begin{cases} Q = \frac{1+\alpha r}{1+\alpha}\cdot\frac{U+aL}{2\sqrt{ar}} & \text{if } r \le \frac{aL}{U} \\ Q = \frac{1+\alpha r}{1+\alpha}\cdot\frac{(1+r)\sqrt{LU}}{2r} & \text{if } r > \frac{aL}{U}\,. \end{cases}$$

There are in general two equilibria of which the first carries a nil interest rate and the second a positive rate so long as

$$(29)\quad \alpha < 1.$$

that is to say that there is preference for the present. If this preference decreases, the interest rate at the second equilibrium also decreases (Figure 6).

The five preceding cases are sufficient to show the diversity of the possible situations. In examining them, it can be understood how Maurice Allais has been able to obtain a wide range of numerical values for the interest rate, according to the cases he considered.

5. MAURICE ALLAIS AND PAUL SAMUELSON

The above presentation of stationary equilibria was obviously not sufficient to account for the long Appendix II. Particularly interesting with regard to modern developments of the Theory of Overlapping Generation Models are the three following contributions to this Appendix.

In the first place, Maurice Allais establishes a connection between the formation of revenues and, not only the intervention of public transfers, but also the structure of property. It particularly distinguishes the case of 'private ownership of land' from the case of 'community ownership'. The specification that he used to establish this connection does not appear obvious to me. But this is an important aspect of any complete theory.

Secondly, he introduces in a second part of the Appendix the need for cash of individuals, jointly with various hypotheses about the evolution of the money supply. The way he treats this aspect must be compared with that which is advocated by certain recent works on money theory.

Finally, a third part of the Appendix shows how the model can be adapted in order to take account of the propensity to possess and to bequeath. The approach is totally similar to that which one finds in certain modern publications.

So it appears today that Maurice Allais and later Paul Samuelson have each introduced the Overlapping Generation Model in a perfectly explicit way. Considered from the point of view of present day theories, their contributions, however, appear complementary.

Whereas Maurice Allais principally focused on the interplay between the production sector and the consumer sector, Paul Samuelson was interested in the conditions of trade between individuals belonging to different generations, thus not having concomitant life cycles. Whereas Maurice Allais was aware of the multiplicity of possible specifications for the formation of revenue, Paul Samuelson only considered the case where the young were holders of all the material resources that the community generates during each period (case C) of the preceding section). He then brought out his famous *Demographic Theory of Interest Rates*. Independently of individual preferences, the interest rate at equilibrium is equal to the rate of increase of the population; it is nil if the population is stationary. But Paul Samuelson did not realise that the case where interest rate was independent of consumer preferences was rather special. Maurice Allais, however, did not really recognise the existence of this case.

The variety of the possible specifications for the formation of revenue has an important role in *Economy and Interest* since it is the foundation of an interventionist attitude. Maurice Allais explains several times that the state has the power and the duty to see that accumulation of capital is accomplished under favorable conditions.

In this respect, the following passage from Chapter 5 (p. 145) is typical:

> The theory shows that for a given psychological and technical structure, the economic equilibrium is determined ... However, when the state intervenes, for example, by an action on the capital market such as borrowing or lending, a supplementary degree of freedom is introduced and there is an infinity of permanent regimes corresponding to a given psychological and technical structure: each one of these regimes corresponding to a different capitalistic equipment and a different interest rate for a given intervention of the state. (These statements are confirmed by studying simplified models in Appendix II).

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NOTES

[ast] It is only after he wrote the original of this chapter in French that Professor Malinvaud published a note in English on these findings: 'The Overlapping Generations Model in 1947', *Journal of Economic Literature* **XXV**(1), 103–105. The English reader will find the full development of the argument for the first time in English in this chapter.

[1] Imprimerie Nationale, Paris, 1947.

[2] *In Search of an Economic Discipline* has been published by Maurice Allais himself. It was reprinted under the title *Treatise on Pure Economics* by the Imprimerie Nationale.

[3] Typical in this respect is the phrase which begins the series of articles by Y. Balasko and K. Shell devoted to the theoretical questions regarding this model: 'The overlapping generations model' was introduced by Samuelson in 1958, (*Journal of Economic Theory*, December, 1980). The article referred to is: P.A. Samuelson, 'An Exact Consumption–Loan Model of Interest, with or without the Social Contrivance of Money', *Journal of Political Economy* (1958), 467–482.

[4] It is only in researching what would be my contribution to this volume of essays that I understood how Maurice Allais was a pioneer in this respect. However, I studied *Economy and Interest* closely in 1948 and then had the advantage of being taught by Maurice Allais. Ten years later I had meditated quite a lot on the propositions put forward by Paul Samuelson in his famous article. But I had not made the connection between the two until now.

[5] For models in the literature, relative to the theory of capital before 1950, see Chapter 9 of *Economy and Interest* and Section 21 of my article 'Capital Accumulation and Efficient Allocation of Resources', *Econometrica* (1953), 233–268.

[6] See, for example, the articles of Balasko and Shell already quoted.

[7] The reader may also like to know the reference on p. 633: 'We are happy to thank

... for these observations and suggestions M. Debreu ... who has accepted to verify ... the calculations of these appendices.

[8] Here and subsequently I am not using the notation of M. Allais but which seems to me to be in current use.

[9] Since each generation only participates in economic operations for two periods, these are long. In this context, we are supposing that the production good is used during a single period.

[10] For simplicity it is assumed here that equilibrium implies $K > O$. The reader may easily proceed to a fuller discussion.

[11] The volume of output Q is obviously equal in each period to the aggregate real income of this period. But aggregate income must be clearly distinguished from the discounted real resource R available to the young generation. Income is actually equal to $R + (r - 1)(R - x_j)$, the second term being income from saving.

ANDRE BABEAU

6. SOME POSSIBLE USES OF HOUSEHOLD ASSETS ACCOUNTS IN NATIONAL ACCOUNTING: THE CASE OF AMERICAN HOUSEHOLDS

In the developed countries, after the emergence of national accounting in terms of flux during the years after the Second World War, the publication of household assets over fairly long periods has been taking place. In France, following the publication of accounts for 1971, 1972, and 1976, INSEE [3, 4] intends to bring out a 1970–1979 series in the relatively near future. In the United States, N. and R. Ruggles [6] have established at the request of the Bureau of Economic Analysis at the Department of Commerce, a 1947–1980 series of household asset accounts, which have recently been published. Other countries are moving in the same direction.

There is no doubt that these household accounts will be used in many ways, for example our macroeconomic models which, up to now, have exclusively made use of flow variables, will certainly, in the future also include numerous stock variables. We do not intend her to enumerate all the uses of stock accounts; at the most we will offer three uses of the series of accounts, however modest; utilizations concerned with the analysis of:

1. factors of development in the relationship between household assets and income;
2. the part played by different factors in the global increase in wealth;
3. the development of the part played by different types of household assets (real, financial, etc.) in the total household assets.

Each section of this article will be devoted to one of these themes. Each section will start by introducing an 'empirical frame of analysis', then some examples of the application of the proposed documents will be suggested. As the series published in the United States is the most comprehensive that we know of, it will be used for these examples.

We would like to inform the reader straight away that he will find only very few theoretical references in the following pages. Not that we

B. Munier (ed.), Markets, Risk and Money, 129–154.

underestimate the importance of theory in the future use of household assets accountancy, but, as these have emerged only recently, it seems more interesting to present some simple uses which are without doubt compared to those which are to come, like the scales of the beginner to the process of the virtuoso pianist.

1. THE DEVELOPMENT OF THE RATIO BETWEEN HOUSEHOLD ASSETS AND INCOME

The development of the macroeconomic ratio of household assets to the domestic revenue has not, up to now, been the subject of much study. Thus, Modigliani [5] in his development of the theory of the life cycle, studies, in particular, the ratio which exists between the saving ratio and the household assets/income ratio. Babeau [2] extends the analysis to the behavior of indebtedness and adopts a purely descriptive procedure. In a similar situation, but from another viewpoint, we wonder whether a reformulation of the household assets/income ratio, no longer involving the elements composing domestic financial resources (savings, debts, etc.), but the use of these resources (investments, financial accounts, loan repayments, etc.), would not reveal an interesting aspect, particularly by the connection which could then be established with the composition of household assets.

If this point of view is admitted, the first problem which arises is that of choice of nomenclature for the flow of employment of financial resources and for domestic capital assets. This choice, like many others of the same type, is limited by two conditions. Firstly, that of simplicity; in the type of analysis which we are using everything which is not fairly simple, as Paul Valery said, will be unusable. Secondly, that of realism – it would be useless to suggest a nomenclature which did not take account of the main specific features of various kinds of assets.

In simplifying without significantly changing, four specific features seem to be retained at the macroeconomic level of the households sectors, which we want to observe:

(a) the possibility for assets to record variation in nominal prices;
(b) the existence of a 'capital consumption' corresponding to the 'erosion' of physical capital in the sense of national accounting;
(c) the possibility of increasing the assets 'in volume' by new investments or placements;

(d) finally, the possibility of contracting a loan to acquire desired assets, the planned loans therefore being especially medium- and long-term.

These four criteria lead us to distinguish the four types of assets, given in the table below, where the value 1 means that the criterion is satisfied and 0 that it is not.

Classification of domestic capital assets following four criteria.

	a	b	c	d
Renewable physical assets	1	1	1	1
Land, agricultural and non-agricultural	1	0	0	1
Assets with fixed nominal denomination	0	0	1	0
Fluctuating financial assets	1	0	1	0

This classification obviously does not claim to be comprehensive, neither from the point of view of criteria taken into account (one could still distinguish between income-producing and non-income-producing assets), nor from the point of view of considered assets, (for example, gold, jewellery, works of art are not explicitly taken into account).

It can be seen that the definition of a category or the classification of an asset in either category liable to modification. For example, durable goods other than dwellings could, according to the country considered, be included or not in the renewable real assets; obligations, on the other hand, could be classified among the fixed assets if their value of reimbursement is considered, or as fluctuating assets if their value on the secondary market is taken into account. Finally, despite the heterogeneity introduced into what follows, we will not distinguish land and agricultural land from other physical assets. In fact, in the capital assets of industrial countries, land and agricultural land represent only a relatively modest part of the total. Another solution could be adopted if the capital assets of less developed countries were to be studied. Finally, we are left with three types of assets.

(1) *Physical assets* comprise residential assets, household goods derived from individual effort and possibly other non-consumable goods.

We will denote:

P_{bi} = gross household asset in this category;
P_{ni} = capital asset net of outstanding loans;
g_i = gross household investment rate measured with reference to their gross disposable income;
P_i = annual price variation of this type of asset;
d_i = rate of capital consumption measured with reference to P.

(2) *Assets with fixed nominal denomination* comprise essentially cash, cash at call, term deposits but possibly other nominally denominated assets such as bonds, etc.

We will denote:

P_l = household assets in this category;
g_l = rate of placement in these assets measured as a ratio of gross disposable income;
p_l = price variation of this type of asset which hypothetically will be zero.

(3) *Fluctuating financial assets* mainly comprise shares but also possibly other securities.

We will denote:

P_v = household assets in this category;
g_v = rate of investment with reference to gross disposable income;
p_v = nominal price variation of these assets.

Finally, we will denote:

P_n = the net total household assets.

Given:

$$P_n = P_{ni} + P_l + P_v.$$

Let P_b equal gross total households assets. Given:

$$P_b = P_{bi} + P_l + P_v.$$

And we have:

$$\alpha_i = \frac{P_{bi}}{P_b}, \quad \alpha_l = \frac{P_l}{P_b}, \quad \alpha_v = \frac{P_v}{P_b}$$

and naturally:

$$\alpha_i + \alpha_l + \alpha_v = 1$$

p = annual price variation of total household assets, i.e.:

$$p = \alpha_i p_i + \alpha_v p_v + \alpha_l p_l = \alpha_i p_i$$

Y_b is available gross household income in the usual sense of national accounting;
Y_n is disposable net income after capital consumption;
p_y is price variation of gross domestic product;
r is nominal rate of increase of Y_b.

Finally, we will denote the household assets/income ratios by:

$$R_b = \frac{P_b}{Y_b}, \quad R_n = \frac{P_n}{Y_n}.$$

1.1. *Presentation of the Frame of Analysis*

Given that data on national accounting is naturally secretive, but taking into account calculations which can be carried out, starting from the equations below, practically nothing is lost by considering time as continuous and much is gained from the point of view of simplicity and of presentation of the calculations.

In using the given notation the instantaneous variation of gross household asset $\dot{P}_b$ is written:

$$(1) \qquad \dot{P}_b = (g_i + g_l + g_v)Y_b + (p_i - d_i)P_{bi} + p_v P_v$$

denoting:

$$\hat{R}_b = \frac{\dot{R}_b}{R_b} = \hat{P}_b - \hat{Y}_b$$

where

$$\hat{P}_b = \frac{\dot{P}_b}{P_b} \quad \text{and} \quad \hat{Y}_b = \frac{\dot{Y}_b}{Y_b}.$$

Then

$$\hat{R}_b = \frac{g_i + g_l + g_v}{R_b} + \alpha_i(p_i - d_i) + \alpha_v p_v - r$$

from which we get:

$$\text{(2)} \qquad \dot{R}_b = g_i + g_l + g_v + [\alpha_i(p_i - d_i) + \alpha_v p_v - r]\, R_b.$$

Equations (1) and (2) can be treated as first-order differential equations of constant coefficients.

Let us find the solution to Equation (2). One can write:

$$\dot{R}_b = [\alpha_i(p_i - d_i) + \alpha_v p_v - r] \times \left(R_b + \frac{g_i + g_l + g_v}{\alpha_i(p_i - d_i) + \alpha_v p_v - r}\right).$$

Putting:

$$A = r - \alpha_i(p_i - d_i) - \alpha_v p_v$$

we finally have:

$$\text{(3)} \qquad R_b = \frac{g_i + g_l + g_v}{A} + ke^{-At}$$

where k is a constant of integration the value of which can be found by making $t = 0$.

From which

$$k = R_{bo} - \frac{g_i + g_l + g_v}{A}$$

where R_{bo} is the value of R_b at time 0.

The development of R_b with time depends, as is shown by Equation (3), on the sign of A. If A is negative, R_b tends to infinity. If A is positive, the permanent rule is obtained:

$$\text{(4)} \qquad \bar{R}_b = \frac{g_i + g_l + g_v}{A}$$

of which the value is positive since in macroeconomic terms it is reasonable to put $g_i + g_l + g_v > 0$.[2] The economic significance of $A > 0$ is easy to show. We can put $p_i - d_i = p_i'$ where p_i' is the price of assets observed on markets which already takes account of the erosion of physical assets. This is the case for secondary markets on which assets are traded. So the expression $(\alpha_i p_i' + \alpha_v p_v)$ corresponds to a variation in the price of household assets, integrating the phenomenon of capital consumption, and $(r - \alpha_i p_i' - \alpha_v p_v)$ expresses the variation of real income when this variation in the price of household assets is taken as a reference. The condition $A > 0$ therefore indicates the necessity for the increase of such a real income. If the real revenue, as we understand it, were to decrease this would mean that the nominal income would increase less quickly than the household assets. In these conditions, even for $g_i = g_l = g_v = 0$ it is not difficult to understand why $\bar{R}_b$ tends to infinity.

Under a permanent regime (for $A > 0$), the ratio $\bar{R}_b$, therefore reaches the given value, from Equation (4) according to $g_i, g_l, g_v, p_i, p_v, d_i, \alpha_i$ and α_v. At this point, household asset and income progress to the rate r, and we have not only $Y_b = Y_{bo}e^{rt}$ where Y_{bo} is the value of the income at the point in time 0 but also $P_b = P_{bo}e^{-rt}$ where P_{bo} is the value of P_b at time 0.

The value $\bar{R}_b$ assisted with a permanent regime is, however, only reached after a great number of years. It can, in fact, be shown that the relative difference $(R_b - \bar{R}_b)/R_b$ decreases with e^{-At} and that it is therefore reduced by half at the end of a period of $\log 2/A$ years. For all valid values of A this period is generally between 10 and 20 years. The reduction of the relative difference by 3/4 occurs in a period of 20–40 years, etc. Hence the developments involved are of very long period. Nevertheless, for comparisons with average values of R_b, obtained over periods longer than a decade, $\bar{R}_b$ can make an interesting reference.

Equation (4) also enables us to confirm that the various factors really perform in the way one could expect. Besides, all things being equal, the ratio of the permanent rule is all the higher if g_i, g_l, p_i, and p_v are high and r and d_i are low.

For a stationary economy, where the population, productivity and prices are constant we obtain, since:

$$r = p_i = p_r = 0$$

$$\bar{R}_b = \frac{g_i + g_l + g_v}{\alpha_i d_i} \,. \tag{5}$$

In fact, this equation is not very meaningful because if $\dot{P}_b = 0$ then we must have $d_i P_{bi} = (g_i + g_l + g_v) Y_b$ and for $g_l + g_v > 0$ this implies $d_i P_{bi} > g_i Y_b$.

The physical part of household assets then tends to zero. On the other hand R_b tends towards infinity since the condition $A > 0$ is not satisfied.

If we wish to emphasize the role of rate of increase of real income, in the sense indicated above, of:

$$r' = A = r - \alpha_i(p_i - d_i) - \alpha_i p_v$$

we simply write:

$$(6) \qquad \bar{R}_b = \frac{g_i + g_l + g_v}{r'}$$

where the permanent regime ratio is the quotient: force of investment (in the widest sense), to the rate of increase of real income r'.

However, the real variation of income is more often calculated from the price variation the gross domestic product than from the price variation of household assets. We would thus have:

$$r'' \# r - p_y.$$

It then follows that for $\bar{R}_b$:

$$(7) \qquad \bar{R}_b = \frac{g_i + g_l + g_v}{r'' - [\alpha_i(p_i - p_y - d_i) + \alpha_v(p_v - p_y) - \alpha_i p_y]}\ .$$

The term in brackets in the denominator corresponds to the relative price variation of household assets with reference to the price of the GDP. The permanent rule ratio is thus even stronger than the force of investment and the development of the relative price of household assets more positive and even weaker than the variation of real income.

Instead of considering the permanent rule R_b we can examine its short-term fluctuations when the revenue oscillates about the tendency described by rate r. Besides, all things being equal, a simple consideration of Equation (3) shows that in this case R_b fluctuates in a 'contracyclic' manner in relation to the income. A higher rate of growth of income than r lowers, *coeteris paribus* the value of R_b and a weaker rate raises it.

However, we might wish to look still further, at fluctuations of a *real* rate of growth (for example about a tendency $r'' = r - p_y$). On the other hand we can observe that the *coeteris paribus* is here not very realistic and that it is very possible that the fluctuations about r'' should be associated with the fluctuations of g and with the relative prices of household assets. In these conditions it could be, for example, (an example not completely chosen at random) that the decline of the real rate of growth is, from the point of view of development of R_b, more than compensated by the accompanying reduction of g and by the decline of the relative price of assets with respect to the price of the domestic commodity. In these conditions, we would obtain 'procyclic' variations of R_b. But this calls implicitly for a theoretical formalization whose specifications, in that case, would have to be agreed.

1.2. *Empirical Results Concerning the Development of the Ratio between Household Assets and Income in the United States, and Its Causes*

We will use the results, shown below, of work carried out in the United States on the accounts of household assets by Nancy and Richard Ruggles [6].[3] The various parameters and calculated dimensions shown below have the meanings already assigned to them in the preceding pages. However, let us specify the composition of capital:

- *Physical Assets*
  dwellings;
  land – agricultural and non-agricultural;
  durables, inventories of clothes, food products.
- *Assets with Fixed Nominal Denomination*
  cash, sight and term deposits;
  treasury bonds;
  other fixed nominal income assets.
- *Business Ownership Assets (Equities Held)*
  shares – net corporate assets of agricultural and non-agricultural individually owned business, parts of trusts.

It can be seen that bonds are classified here as fixed nominal assets, i.e. that their par value of reimbursement is taken into account and not their price on the secondary market. Individual business accounting, on the other hand, only refers to net assets. The total assets obtained from the sum of the three types of credit represents an asset free of gross liability for individual assets but not in the case of corporate assets.

Tables I, II and III give values assumed by various parameters or dimensions such as have been previously defined. We will study, in turn, characteristics of the three sub-periods occurring within the period 1947–1980, the short-term development factors of the ratio between household assets and income, and finally the theoretical values assumed by this ratio for each one of the three sub-periods and possible long-term

TABLE I

United States: 1947–1957.

Years	r	d_i	g_i	g_l	g_v	p_i	p_v	p	p_y	R_b
1947	nd	7.3	18.0	2.9	2.1	9.6	5.9	5.4	nd	4.38
1948	11.8	7.0	18.4	1.4	3.4	3.1	1.1	1.4	7.3	4.12
1949	1.3	6.7	17.6	1.6	1.5	-1.8	1.4	0.1	-0.2	4.20
1950	11.5	7.1	20.5	2.6	2.5	7.2	11.8	7.7	3.0	4.23
1951	2.3	7.0	19.6	4.0	2.6	4.5	8.1	5.2	7.6	4.52
1952	14.7	6.9	16.8	5.3	1.9	1.4	2.5	1.6	1.7	4.15
1953	6.2	7.0	16.8	4.7	1.6	0.5	-1.4	-0.5	1.7	4.03
1954	3.7	7.2	16.3	3.5	1.9	0.9	14.2	6.6	1.8	4.26
1955	6.9	7.3	18.4	4.6	1.1	2.2	10.1	5.4	2.0	4.34
1956	7.0	7.4	17.0	5.5	1.4	3.5	5.2	3.6	3.5	4.34
1957	5.6	7.3	15.9	4.8	0.9	1.1	-3.8	-1.5	3.2	4.16
Average	6.5	7.1	17.8	3.7	1.9	2.9	5.0	3.3	2.7	4.25

tendencies.

1.2.1. Characteristics of the three sub-periods. The division into three sub-periods is partly due to the development of the relationship R_b (see Graph 1), but we perceive on examination that each sub-period also has its own aspect concerning the values assumed by various parameters.

In the course of the first sub-period (1947–1957) the average value of R_b is relatively low. This weakness is partly due to modest investments and placements $(g_i + g_l + g_v)$ and partly to price developments only moderately favorable to household assets.

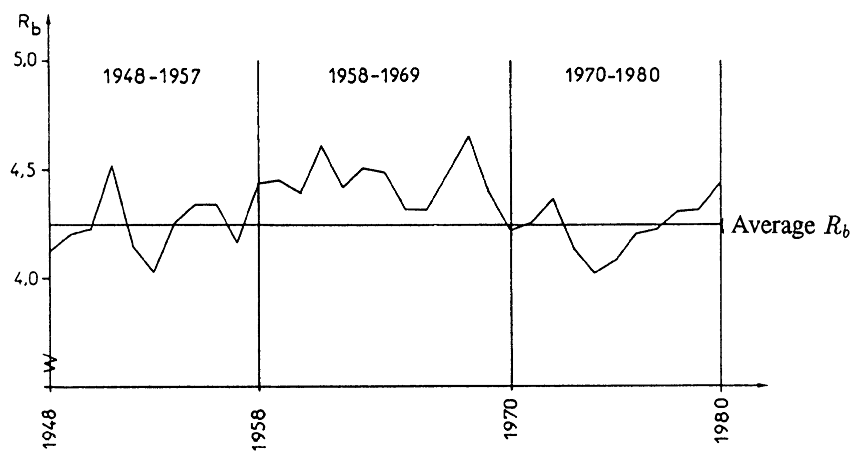
In the course of the second sub-period (1958–1969) R_b rose somewhat not because of an increase in investments and placements, but rather because of a more favorable price development of corporate ownership assets, assets which then represent more than 45% of the total assets.

TABLE II

United States: 1958–1969.

Years	r	d_i	g_i	g_l	g_v	p_i	p_v	p	p_y	R_b
1958	4.2	7.4	17.2	4.5	3.2	1.8	17.3	8.3	1.8	4.44
1959	5.8	7.4	16.5	5.6	0.2	1.7	5.3	3.0	2.2	4.45
1960	3.8	7.2	15.6	4.9	0.3	0.1	0.7	0.3	1.1	4.40
1961	3.8	7.2	14.4	4.6	1.8	0.5	13.1	6.2	0.7	4.61
1962	5.0	7.1	15.3	6.2	-0.7	0.5	-4.2	-1.8	1.3	4.42
1963	5.1	7.1	15.9	6.1	0.7	-0.5	9.8	4.3	1.5	4.51
1964	7.6	7.2	16.2	7.5	1.3	1.7	6.0	3.4	0.7	4.49
1965	7.1	7.0	16.4	7.3	1.3	0.6	8.4	4.1	2.5	4.32
1966	7.7	6.9	16.0	6.2	0.2	3.3	-2.5	-0.2	2.8	4.32
1967	6.9	6.9	15.3	8.4	2.1	2.6	13.4	6.9	3.2	4.49
1968	8.0	7.1	16.6	8.7	1.4	6.0	12.2	7.6	4.3	4.65
1969	7.9	7.1	16.3	6.0	-0.9	4.2	-4.8	-0.9	4.9	4.39
Average	6.1	7.1	16.0	6.3	0.9	1.9	6.1	3.4	2.2	4.17

Graph 1. Development of R_b, United States: 1948–1980.



The third period (1970–1980) saw the appearance of a new weakness in the average value of R_b due, not to the decline of global investments and placements (which, on the contrary is increased) but rather to a

TABLE III

United States: 1970–1980

Years	r	d_i	g_i	g_l	g_v	p_i	p_v	p	p_y	R_b
1970	8.5	7.1	14.7	6.7	-0.1	3.5	0.9	1.6	5.3	4.22
1971	8.3	7.2	16.5	8.2	-0.8	2.8	10.6	5.5	4.7	4.26
1972	8.0	7.3	18.0	10.7	1.5	5.5	10.7	6.4	4.0	4.37
1973	12.5	7.1	17.7	10.9	-1.2	8.7	-2.4	2.0	5.1	4.14
1974	8.9	7.0	15.5	9.3	-0.2	10.0	-4.5	2.0	8.1	4.02
1975	9.8	7.0	14.9	9.8	0.8	5.1	14.9	7.2	9.2	4.08
1976	8.4	7.1	16.7	10.7	0.7	7.7	12.1	7.6	4.8	4.21
1977	9.4	7.1	18.2	10.7	0.2	8.6	4.4	4.9	5.6	4.23
1978	11.8	7.0	18.5	11.4	0.3	12.5	10.9	8.8	7.3	4.31
1979	12.5	6.7	17.4	11.3	-0.5	7.1	15.7	8.2	8.4	4.32
1980	11.1	6.8	14.9	9.9	-0.2	10.5	18.8	10.8	8.7	4.44
Average	9.9	7.0	16.6	10.0	-0.4	7.5	8.4	5.9	6.5	4.24
Average 1947–1980	7.5	7.1	16.8	6.7	0.8	4.0	6.5	4.2	3.8	4.32

particularly rapid increase of nominal income. If the part played by corporate ownership assets in the total assets clearly diminishes in the course of this sub-period, by reason of a disinvestment by households, the development of prices of these assets seems to play an important part in the fluctuations of R_b in the course of this sub-period. Thus the decline of p_v in 1973 and 1974 appears to account for a great deal in the obvious weakness of R_b, seen over these two years. Conversely, the raised rates of p_v from 1975 onwards, largely explain, without doubt, the spectacular ascent of R_b between 1974 and 1980. But we will now study more systematically the short term factors of development of R_b.

1.2.2. The short-term factors of development of the household assets/income ratio. It is a question here of explaining the evolution from one year to the next of observed values of R_b. Table IV gives the coefficients

TABLE IV

Coefficients of variation of parameters and magnitudes.

	r	d_i	g_i	g_l	g_v	p_i	p_v	p	R_b
1947–1980	0.57	0.03	0.08	0.43	1.53	0.89	1.05	0.82	0.03
1947–1957	0.65	0.03	0.08	0.37	0.36	1.06	1.08	1.06	0.03
1958–1969	0.86	0.02	0.04	0.21	1.24	0.96	1.19	1.13	0.02
1970–1980	0.18	0.03	0.08	0.14	1.84	0.38	0.87	0.45	0.03

of variation of parameters and dimensions of Tables I, II and III for the total period and for each of the three sub-periods considered.

We can see, firstly, that the coefficient of variation of R_b is very weak and that the relationship shows no clear tendency in the course of the 34 years of the period studied (see Graph 1).

Of all the parameters appearing in Equation (4), capital consumption d_i is by far the most stable, with a coefficient of variation a little less, at 3% of the order of that of R. Neither is any tendency discernible, and the stability from one sub-period to another is as great as for R_b.

The rate of investment in real assets g_i is also quite stable with, however, a coefficient of variation which is more than double that of d_i and of R_b. No long term tendency is apparent, but we must, however, take note of the high values obtained during the course of the first sub-period, showing a special effort of investment made by households in the period immediately after the War.

The coefficient of variation of g_l, rate of investment in corporate ownership assets, is 43% over the period; it is therefore relatively high in relation to the first coefficients of variation examined. A part of this coefficient of variation is due to the increasing tendency of g_l, of which the average value is 3.7% for the first sub-period, 6.3% for the second and 10% for the third.

The coefficient of variation of g_v, rate of placement in financial assets (stocks and shares) is, even more clearly, higher than the previous one. There is certainly a part of this coefficient which also derives from the tendency (decreasing with time in this case), since the average values are 1.9%, 0.9%, and -0.4% (coefficient of linear correlation = -0.73). So, in the course of the third sub-period, it is established that, conforming

TABLE V

Short-term factors influencing the household asset/income ratio over the period 1947–1980*.

	1	2	3	4	5	6	7
Order of the variables at the beginning	r	p_i	p_v	g_l	g_i	d_i	g_v
Sign of coefficient of regression	–	+	+	(+)	(+)	(–)	(–)
Coefficient of multiple determination corrected**	0.098	0.210	0.269	0.247	0.222	0.185	0.160

* Concerning the regression of $(R_b - R_b - 1)/(R_b - 1)$ over the seven variables considered: we use a stepwise model of maximization of the coefficient of determination. This is adjusted to take account, each time of the number of observations and the number of regressors. The sign of the coefficient of regression is given in brackets when it is not significant to 5%.

** Concerning R_c^2 such that: $R_c^2 = (R^2 - k/(n-1))((n-1)(n-k-1))$ where k is the number of variables in the regression and n the number of observations.

to the statistics published by the New York Stock Exchange, American households have disinvested in corporate assets, and particularly in assets of listed companies. For all that, once the tendency is removed, the coefficients of variation of each sub-period remain the highest of all those we have to examine.

The coefficients of variation of price parameters are also rather high. For p_i the coefficient is partly explained by the differences in average between sub-periods. Without actually acknowledging a trend (it is without doubt more of a strong fluctuation) we see, however, a clear acceleration of price rise of real assets in the course of the last sub-period. The coefficient of variation of p_v is higher than that of p_i without doubt there seems to be a certain tendency to rise, the figures being 5.0%, 6.1% and 8.4%; but, having removed this tendency, there still remain high coefficients of variation for each sub-period.

Finally, the coefficient of variation of r, even if it is not among the highest, is considerable for the period and for two sub-periods out of three.

After this examination, one can presume that the factors that most influence R_b in the short term are the parameters with the highest coefficients of variation. To confirm this, we have regressed the relative annual variation of R_b on r, d_i, g_i, g_l, g_v, p_i and p_v, by means of a

TABLE VI

Household asset/income ratios, observed and theoretical.

	1947–1980	1947–1957	1958–1969	1970–1980
R_b	4.32	4.25	4.47	4.24
$\bar{R}_b$	4.19	4.18	4.64	4.51
Difference (in %)	-3.0	-1.6	+3.8	+6.4

computer program of the type 'stepwise' (see Table V). The only really significant influences are, on the one hand, those of price variation (p_v has a more marked influence than p_i) which contribute to the rise in value of the relationship R_b, and, on the other hand, those of variation of income which contribute, on the contrary, to its decline. As for the other factors, they exert no significant influence, but it will be noted that the sign obtained for d_i, g_i, and g_l is the right one.

1.2.3. Value of the theoretical household assets/income ratio and long-term factors of development of the ratio's observed value. Table VI shows a comparison between actual and theoretical values of $\bar{R}_b$, insofar as they have been able to be calculated from Equation (4), and taking *average* values for the various parameters, for the whole period or for the three sub-periods. The percentage variation is calculated with reference to the actual value of R_b.

This table shows a good correspondence between theoretical and actual values since the variations are not more than 6% for the sub-periods and for the whole period the variation is only 3%.

This good correspondence of theoretical data with observed data is without doubt due, to a large extent, to the stability of R_b during the course of the considered period. It has, in fact, been shown elsewhere that, in the case where R_b was fluctuating particularly, the variation between theoretical and observed values were clearly larger [2].

The stability of the household assets/income ratio during the period is not due, however, to the constancy of the parameters appearing in Equation (4). But on examination it appears that compensations occur in the course of time, for example, in the numerator of the equation the

TABLE VII

Coefficient of linear correlation between factors influencing the level of R_b.*

	d_i	g_l	g_v	p_i
r	−0,303	0,487		0,639
g_i				0,297
g_l			−0,691	0,589
g_v				− 0,301

* Only coefficients of correlation, up to the limit of 10% of confidence, have been considered.

decrease in g_v is compensated (and even beyond) by the increase in g_l. In the course of the third period the increase in force of investment, in the widest sense, which would have been able to contribute to a large increase in R_b is compensated by the acceleration of growth.

These compensations have been more clearly determined by calculating the coefficient of linear correlation between the factors occurring in Equation (4) which defines R_b (cf. Table VII). In this table we notice the particularly marked substitution between g_l and g_v. On the other hand it appears that the price development of company ownership assets is not correlated with any other variable although the development of prices of real assets seems to be marginally connected to the nominal development of revenues (without doubt through the medium of the general rhythm of inflation).

Naturally, these 'compensations' could not have been fortuitous and it requires a much more general theoretical model to account for it. Such a model would bring water to the mill of those who believe in the stability, in the very long term, of the value of the household assets/income ratio. For us this stability does not seem necessary and we have shown elsewhere [2] that, in a country like France, the ratio was liable to undergo variations of long duration. It seems to us, therefore, more important, at the point we have reached in development of knowledge in this subject, to try to analyze the role of different factors in the variation of the ratio, rather than to build up a theoretical model claiming to demonstrate a stability which, empirically, is not at all proven and which was temporarily held for a fact only because our instruments of measure are so imperfect.

2. THE INFLUENCE OF VARIOUS FACTORS IN THE VARIATION OF HOUSEHOLD ASSETS

We shall first show the elementary formulae of decomposition enabling us to isolate the part played by each factor in the growth of household assets; then we will show the principal results obtained during the period, with the aid of these formulae.

2.1. *Decomposition Formulae*

In A. Babeau [2] we have been shown the respective parts of savings, of liability, and of price movements in the growth of household assets. Comparably, looking at the employment of domestic financial resources, the influence of each of the uses in the variation of household assets can be simply explained. If we take e to be the rate of nominal variation of gross household assets, we have:

$$w = \frac{\dot{P}_{\mathrm{b}}}{P_{\mathrm{b}}} = \frac{(g_{\mathrm{i}} + g_{\mathrm{l}} + g_{\mathrm{v}})Y_{\mathrm{b}} + \alpha_{\mathrm{i}}(p_{\mathrm{i}} - d_{\mathrm{i}})P_{\mathrm{b}} + \alpha_{\mathrm{v}} p_{\mathrm{v}} P_{\mathrm{b}}}{P_{\mathrm{b}}}$$

or

$$w = \left(\frac{g_{\mathrm{i}}}{R_{\mathrm{b}}} - \alpha_{\mathrm{i}} d_{\mathrm{i}}\right) + \frac{g_{\mathrm{l}}}{R_{\mathrm{b}}} + \frac{g_{\mathrm{v}}}{R_{\mathrm{b}}} + (\alpha_{\mathrm{i}} p_{\mathrm{i}} + \alpha_{\mathrm{v}} p_{\mathrm{v}}), \tag{8}$$

an equation in which we can easily isolate, in the growth of household assets, the respective impacts of the net investment $(g_{\mathrm{i}}/R_{\mathrm{b}} - \alpha_{\mathrm{i}} d_{\mathrm{i}})$, of the placements in liquidities $g_{\mathrm{i}}/R_{\mathrm{b}}$, of the placements in securities $(g_{\mathrm{v}}/R_{\mathrm{b}})$ and of the price movements.

There again, it could be interesting after an analysis of rate of growth of nominal household assets to consider an analysis of its real rate of growth defined by:

$$w' = \frac{1+w}{1+p_{\mathrm{y}}} - 1 \# w - p_{\mathrm{y}}$$

which becomes

$$\begin{aligned} w' &= \left(\frac{g_{\mathrm{i}}}{R_{\mathrm{b}}} - \alpha_{\mathrm{i}} d_{\mathrm{i}}\right) + \frac{g_{\mathrm{l}}}{R_{\mathrm{b}}} \\ &\quad + \left[\alpha_{\mathrm{i}}(p_{\mathrm{i}} - p_{\mathrm{y}}) + \alpha_{\mathrm{v}}(p_{\mathrm{v}} - p_{\mathrm{y}}) - \alpha_{\mathrm{i}} p_{\mathrm{y}}\right], \end{aligned} \tag{9}$$

an equation which shows clearly the influence of the real growth of household assets in the relative price movements of each of the three components of household assets.

In the same way, each of these components, P_{bi}, P_{l}, and P_{v}, could have its growth analyzed, specifying the influenced of investments or placements and that of nominal or relative price movements.

In Equations (8) and (9) there is no explicit reference to the permanent state but these equations remain valid for the situation where $w = r$ for a long period and where consequently R remains stable, with a value given by Equation (4).

2.2. *Respective Importance of the Various Factors in the Growth of Household Assets*

Table VIII provides the percentages of price variables and investment variables which make up the growth of gross household assets and which were calculated from Equations (8) and (9). For the whole of the period it will be seen that the most important influence is that of investment or placement variables particularly when we consider their relative prices (almost 90% of the total growth of household assets). The main components of this sector are the placements in fixed nominal value claims (liquidities, bonds) which exert the greatest weight. The net investments are in second position and as for the part played by placements in corporate bonds, it seems to be particularly modest.

The role of relative prices is not more than a total of 10% of growth of household assets. But it is an algebraic sum of influences and the order of these influences is reversed here, with respect to that which we have just seen for volumes. It is the prices of the business ownership assets which

TABLE VIII

Respective impacts of rates of investment or placement and of price movements in the growth of gross household assets (in %).

	Volume				Price				Total
	g_i*	g_l	g_v	Sub total	p_i	p_v	p_l	Sub total	
1947–1980									
• nominal growth	18.9	21.6	2.7	43.2	18.9	37.9	0.0	56.8	100.0
• relative growth	38.9	44.4	5.6	88.9	ε	33.3	-22.2	11.1	100.0
1947–1957									
• nominal growth	28.7	14.0	7.0	49.6	14.9	35.5	0.0	50.4	100.0
• relative growth	49.6	23.3	12.1	85.0	1.9	28.4	-15.3	15.0	100.0
1958–1969									
• nominal growth	19.9	22.3	3.2	45.4	9.8	44.8	0.0	54.6	100.0
• relative growth	30.5	34.1	4.8	69.4	-2.4	44.1	-11.1	30.6	100.0
1970–1980									
• nominal growth	12.8	24.8	-0.9	36.7	30.4	32.9	0.0	63.3	100.0
• relative growth	40.6	78.6	-3.0	116.2	12.8	23.6	-52.6	-16.2	100.0

* This concerns the investment *net* of capital consumption: $((g_i/R_b) - \alpha_i d_i)$.

have developed most favorably, while the contribution from development of relative prices of real assets comes second, although almost nil. Finally, as expected, the contribution of development of relative prices of placements in assets with fixed nominal value is strongly negative.

Hence the proposed analysis shows, from the point of view of development of the composition of household assets, which we will study below, a certain compensation between price variables and volume variables. The placements in corporate ownership assets are low but the favorable development of prices contributes to the maintenance of their part in the total household assets. Conversely, placements in liquidities and loans are large, but the negative development of their relative prices contributes to the deceleration of the growth of their role in the total household assets. We could naturally argue from this observation in favor of 'the effect of real cash balance' and the 'desired structure' of household assets.

But such a generalization would be, without doubt, premature. With regard to surplus, a more thorough examination of each of the sub-periods considered shows a certain specific feature in each of them. For example, the sub-period 1958–1969 is distinguished by an influence especially favorable to the development of prices of corporate ownership assets, and besides, it is known that there was a sharp rise in the Dow Jones index, (at least over a part of this period). But in contrast, the contribution from the development of relative prices of physical assets was clearly negative. In the course of the sub-period 1970–1980 the acceleration of inflation resulted in a negative total contribution of relative price movements of assets. The favorable development of prices of physical assets and of company ownership assets was not sufficient to compensate the decline of the relative price of liquidities and loans, of which the volume increased heavily, by contrast, because of particularly high values of g_l.

We now possess all the elements for studying the development of the role played by each type of asset in the total household assets.

3. DEVELOPMENT OF THE STRUCTURE OF GROSS CAPITAL

The simple consideration of the development of the role played by each of the three types of assets, considered in the total household assets, has already taught us a great deal, but the analysis of developments can be made more interesting, if we inquire beforehand into the conditions of existence of a permanent state of the structure of the wealth of the households.

3.1. *Conditions for a Permanent State of Structure of Assets*

Equation (4) above shows that for the ratio R_b to tend to a value of permanent state, it is necessary that, not only the parameters g_i, g_l, g_v, d_i, p_i, and p_v should be constant, but that each of the assets composing the total household asset should also be constant. If, in effect, the parts α_i, α_v and α_l vary with time, there will be no permanent rule possible for R_b.[3] We must therefore study more closely under what conditions the α's could remain constant and what, in this case, is the expression giving their value in the permanent rule.

In order that the parts α remain constant, it is necessary and sufficient that each type of capital asset grows in the rhythm of the total household assets. So we have:

$$w = \frac{g_i Y_b}{P_{bi}} + (p_i - d_i) = \frac{g_v Y_b}{P_v} + p_v = \frac{g_l Y_b}{P_l}$$

$$(10) \qquad = \frac{(g_i + g_v + g_l)}{P_b} + \alpha_i(p_i - d_i) + \alpha_v p_v.$$

Naturally, this situation can arise apart from the case of the permanent rule, for the household assets/income ratio. If, however we take this case, we have $w = r$ and it is then easy to calculate the values of the parts of the permanent rule. Then we obtain:

$$(11) \qquad \bar{\alpha}_i = \frac{g_i}{(r - p_i + d_i)\bar{R}_b} \quad \bar{\alpha}_v = \frac{g_v}{(r - p_v)\bar{R}_b} \quad \bar{\alpha}_l = \frac{g_l}{r\bar{R}_b}.$$

The parts of the permanent rule are thus, as we would expect, proportional to the rates g_i, g_v, and g_l and inversely proportional to the value of the household assets/income ratio and to the difference between the rates of nominal variation of revenue and that of the price of the considered asset.[5]

It will be recalled here what was said earlier; that there is no permanent rule for a static situation.

$r = p_i = p_v = 0$. The condition $\dot{p} = 0$ which is expressed by $d_i P_{bi} = (g_i + g_v + g_l)Y_b$ cannot be satisfied in the long term. It implies, in fact, for $g_l + g_v > 0$ that $d_i P_{bi} > g_i Y_b$ and the real household assets therefore decrease with time. α_i tends to zero and R_b tends to infinity where $A = 0$.

Finally, it will be observed that in order for the parts expressed in the Equations (11) to make sense, it is necessary for, either values of g_i, g_v, and g_l to be positive and values of $(r - p_i + d_i)$, $(r - p_v)$ and r also to be positive, or else for values of g_i and g_l to be negative and values of $(r - p_i + d_i)$, $(r - p_v)$ and r also to be negative. We will confirm in the following Tables IX–XI that these equations provided very biased estimations of the values observed for each of the parts, as much for the whole period as for each sub-period.

3.2. *Observed Development of the Structure of Household Assets*

In the course of the period, the part of each type of household asset had developed perceptibly. The part α_i of physical assets changed from barely more than 28% in 1947 to more than 40% in 1980; the part of

TABLE IX
Development of the structure of household assets 1947–1957 in percent.

Years	Part of physical assets α_i	Part of nominally denominated assets α_j	Part of corporate shares α_v	Total
1947	28.5	24.1	47.4	100.0
1948	30.5	23.2	46.3	100.0
1949	31.2	22.9	45.9	100.0
1950	32.7	21.0	46.3	100.0
1951	33.5	20.1	46.4	100.0
1952	34.1	20.4	45.5	100.0
1953	35.1	21.0	43.9	100.0
1954	33.8	19.9	46.3	100.0
1955	33.7	19.3	47.0	100.0
1956	34.2	19.3	46.5	100.0
1957	35.5	20.3	44.2	100.0
Average	33.0	21.1	45.9	100.0
Coefficient of variation	0.06	0.07	0.02	
Theoretical average	39.1	13.4	29.8	

specific assets of nominal value increased also, changing from 20% in 1957 to almost 25% in 1979. As for x_v, the part of company assets is obviously the great loser of the period, changing from more than 47% in 1947 to less than 34% in 1978. If these developments seem enormous, they are not however, obtained under the same conditions and the nature of the factors involved varies at the same time from one type of asset to another, and probably, for the same type of asset, from one sub-period to the other.

For α_i, the growth in the course of the first sub-period, from 28–35%, was due to a phenomenon of increase in *volume* of physical assets. It was the high value of household investments in this type of asset which explains the substantial growth of P_{bi} (especially the spread of principal home ownership). We notice that, on the contrary, the price of P_{bi}

TABLE X
Development of the structure of household assets 1958–1969 in percent.

Years	α_i	α_j	α_v	Total
1958	33.4	19.2	47.4	100.0
1959	33.5	19.4	47.1	100.0
1960	33.8	20.0	46.2	100.0
1961	32.1	19.4	48.5	100.0
1962	33.3	20.7	46.0	100.0
1963	32.1	20.7	47.2	100.0
1964	32.0	21.0	47.0	100.0
1965	31.4	21.1	47.5	100.0
1966	33.1	21.9	45.0	100.0
1967	31.9	21.6	46.5	100.0
1968	31.8	21.2	47.0	100.0
1969	34.0	22.2	43.8	100.0
Average	32.7	20.7	46.6	100.0
Coefficient of variation	0.03	0.05	0.03	
Theoretical average	31.7	23.1	nd	

increased less quickly than the price of the total household assets. In the course of the second sub-period, α_i levelled out due to the fact that g_i subsided somewhat (16% instead of 17.8%). The rise of α_i resumed in the course of the third sub-period, less because of g_i (which only rose slightly) than because of the value reached by p_i (7.5% against less than 6% for the price of the total household assets).

Concerning the development of α_l, it was distinguished by a decline, more or less continuous up to 1958, then the upward trend began in the course of the second sub-period because of an increase in g_l (expansion of savings deposits? purchase of fixed income securities?) and from the fact that the price rise of the total household assets remained moderate. Finally, the tendency continued in the course of the third sub-period, because of the new increase in g_l which attained an average of 10%. However, the increase in α_l was then decelerated by a rapid rise in the price of total household assets (p reached almost 6%).

TABLE XI

Development of the structure of household assets 1970–1980 in percent.

Years	α_i	α_j	α_v	Total
1970	34.9	22.8	42.3	100.0
1971	34.5	22.8	42.7	100.0
1972	34.7	23.0	42.3	100.0
1973	37.3	24.3	38.4	100.0
1974	40.1	25.2	34.7	100.0
1975	39.0	25.1	35.9	100.0
1976	39.0	24.9	36.1	100.0
1977	40.4	25.2	34.4	100.0
1978	41.7	24.8	33.5	100.0
1979	41.2	24.6	34.2	100.0
1980	40.7	23.8	35.1	100.0
Average	38.5	24.3	37.2	100.0
Coefficient of variation	0.07	0.04	0.09	
Theoretical average	41.6	23.8	nd	
Observed average 1947–1980	34.7	22.0	43.3	100.0
Coefficient of variation	0.09	0.09	0.11	
Theoretical average 1947–1980	36.7	20.7	20.9	

The part α_v did not decrease during the first two sub-periods. It was maintained at a high level because of positive values of g_v and favorable price development (p_v is of the order of 5 or 6%). But the decline appeared as early as 1970 because of rather regularly negative values for g_v, and despite the price developments which, on average, remained favorable. This movement implies, at one and the same time, a diminution of household investments in their own individual companies and a certain falling off with regard to shares in listed and non-listed companies.

We thus see that, on the whole, if the factors of price help, here also, to explain the developments in the short term of the roles of various types of assets in the household assets, then in the long term it would seem to be a good thing that the phenomena of volume should resume the advantage, and that they should be the principal determinants of these developments of very long period, that are observed in the structure of household assets. But in order to support this affirmation, it would be proper to make further observations by extending this study to countries other than the United States.

4. CONCLUDING REMARKS

We hope to have shown in the preceding pages the usefulness of the proposed analytic framework in taking advantage of the information which is becoming available in household asset accountancy. We emphasize the very different status of parameters involved in Equation (4), expressing the permanent regime value of the ratio household assets/income: d_i, rate of capital consumption of physical household assets is a technical parameter; g_i, g_v, and g_l, rates of investment or placement are parameters of behavior; finally r, p_i, and p_v are characteristics of the economic environment. The transposition of the proposed equations from household accounts to the accounts of other economic sectors, naturally, requires certain adaptations both of the adopted formulation of the problem and of certain concepts used. But this is perfectly feasible.

Among the empirical results obtained for American households we will remember:

1. The role of variation of price, and especially of prices of company assets in short, and very short term variations in the household assets/income ratio.
2. The stability in the United States, over the period studied, of the household assets/income ratio which is due to a certain number of 'compensations' between values of parameters involved in Equation (4). This equation provides, moreover, a good estimation of actual average values for the whole period and for each sub-period.
3. The important role of volume variables (rate of investment or placement) in the growth of household assets; the increase in relative prices of assets provides only a reduced contribution.
4. The clear deformation in the structure of household assets, to the profit of physical assets, liquid or financial assets, and fixed assets

and therefore to the detriment of corporate assets of which the role is clearly decreasing, despite a favorable development in their relative price.

These first results already show the role of variables which were practically ignored until recently in macroeconomic analysis, for instance, in considering rate of household investment in real assets, rate of investment in corporate assets and in liquidities, development of relative prices of real assets and of financial assets. These different variables could be promoted to important roles if the tendency of close association with macroeconomic models, flow variables and stock variables is confirmed.

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NOTES

[1] As opposed to the relationship capital/product (see Allais [1]).
[2] In rejecting this hypothesis, we obviously have a more comprehensive discussion of all the possible cases of development of R_b.
[3] For a brief presentation in French of these works see A. Babeau [2].
[4] This is only true as long as one considers p_i and p_v as exogenous. If p is treated as exogenous and supposed constant, there is no further need to consider the development of α.
[5] We will prove, on the other hand, in making $R_b = (g_i+g_l+g_v)/(r-\alpha_i(p_i-d_i)-\alpha_v p_v)$, that Equations (11) really lead to $\bar{\alpha}_i + \bar{\alpha}_v + \bar{\alpha}_l = 1$.

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MARCEL BOITEUX

7. SHOULD WE GET RID OF ECONOMIC CALCULUS?

Among all the merits one can attribute to Maurice Allais is the one of having developed economic calculus in France, by initiating successive generations of engineers and statisticians into theories of value and decision.

A quantitative analysis leaned against a certain degree of rationality is now widely used in big businesses as well as in relations between firms on the one side and government institutions on the other.

Sales at marginal cost, actualization and maximization of discounted profits, all constitute various experiences of it. A new discipline was progressively set up, which did not fail as time went by to give rise to violent criticism, if not to the creation of a counter-discipline.

Is it really reasonable to believe that one can quantify such phenomena as complicated and influenced by human nature as those of economics? So many mistakes were made in the name of reason, which could have been avoided with a little more intuition! Some people rebel against those technocrats who claim to speak the truth when developing mathematical theories, to which the average man cannot even have access: *vis à vis* the problems of our daily life, intuition and good sense claim their rights.

The aim of this work is not to exchange scientific views on economic calculus, on its problems and recent developments. Pragmatically, we intend to see here if economic calculus, at least in its most common applications, is of great use.

* * *

In that respect, starting by opposing intuition to reason in a field where people rather appeal to good sense, may not be the worst way to deal with it.

Well, it does sound as if we have embarked on a wrong-headed debate, since a rational analysis of the situation could not in any case work without a little intuition.

B. Munier (ed.), Markets, Risk and Money, 155–163.

1. Through structuring logical sequences within a mass of data, calculus enables intuition to play on hypotheses. Such a method is worthwhile at the only condition that hypotheses appear more familiar than conclusions, as far as these hypotheses are common to other problems for which people already have a certain experience. The aim is not to oppose logic to intuition but to use the first in order to move the intuition field so that it does not look like a revealed secret anymore but like the outcome of an acquired and mutual experience.

2. Those logical sequences never quite reflect realities but rather idealize and adapt them to mathematics. The changeover from the reality to the model is based on intuition, too; i.e. on the intuition that consists in not deviating too much from reality.

Besides, the choice of the model obviously favors the kind of idealization that is at best suited to calculus, so that you sometimes have to make a very difficult choice between a model that perfectly – or almost perfectly – links the conclusion you are looking for to well-known hypotheses, but stands very far from reality, and a more accurate model one can only deal with by odds and ends.

3. Insofar as the model itself idealizes and so distorts reality, agreement on hypotheses do not necessarily override agreement on conclusions: calculus helps deciding, it does not dictate decisions. Technocrats have good reasons to rebel against those who prefer miscalculation to right figures. That does sound inconsistent as far as hypotheses have been agreed on whereas the conclusions are, as for them, rejected. On the other hand, the technocrat has no other choice but bow to one's arguments when one shows him new evidence, except if he succeeds in buying them on his own account.

To tell the truth, experience shows that reason has been of a better profit to people than instinct. We shall therefore agree to it.

ECONOMIC CALCULUS IN THE PRIVATE SECTOR

The preceding paragraphs apply to any reasoned analysis of the decision process, and therefore to economic calculus, which, as far as it is concerned, has no other purpose but moving the intuition field by going back, through a model as accurate as possible, from the decision to be taken to the hypotheses that justify this decision.

Therefore, a firm which aims at making as much profit as possible within a given period – in a time when there will be financial adjustments between the firm and the rest of the economy on the basis of a well-defined interest rate – can work out its policy – sales, production capacity, investments, etc. – by maximizing its discounted profit on the basis of hypotheses that open the door to discussion.

If the given data of the problem are risky (i.e. known only in probability), the firm will maximize the mathematical expectation of its discounted profit subject to maintaining to a given minimum its probability to disappear before the law of large numbers enables it to recover from the bad luck of our everyday life. And if the firm works towards an end that, even in the long run, is not only money, as far as it finds a compromise between money and the rest, it will maximize the discounted present value of this mixture of money and other aggregates.

However, if some of the given data of the problem are not risky but rather uncertain (that is to say that you cannot integrate them in any probability scheme) – should it be a question of prices, of interest rates, of constraints concerning technology or labor regulations, clients reactions to the sales policy of the firm as well as to the sales policy of its present and future competitors – and if this 'other thing' which goes beyond the value of the discounted profit, is hard to modelize, then things are not so clear.

Things are not so clear, but, except under very special circumstances, the solution which leads to a minimum profit on the basis of likely hypotheses has all the same little chance of succeeding.

It would thus be advisable to do some research on the matter, should it be on the only purpose of avoiding such a situation to happen: among all likely evolutions, one may perhaps favor one or the other, all being dealt with as if they were certain, and then try to see which decisions are likely to work in more than one case, whereas others, which were maybe splendid in some cases, appear to be fully inadequate in others. In this way, one discovers the elements of a strategy, the definition of which depends on data processing at least as much as on calculation itself.

In this research, intuition, if not shrewdness, will no doubt play a determinant part. Is it thus not advisable to check what intuition suggests and try to look for hypotheses for which reason justifies what instinct proposes? Backward calculus – in which the decision is not the result of a theory but where, on the contrary, a decision is taken while checking that the hypotheses which could justify it are likely to work – is at least as useful as direct calculus.

Besides, when the policy of the firm has been well enough defined for the near future as well as for some scenarios of a more distant future, there is no waste of time in tuning it more finely. Let us explain: when you have to build a factory, it is worth trying to save money subject to given goals. In that very case, the good old calculation technique is worthy once again.

There are people who will say that the elaboration process of a strategy is far more important than optimizing each possible path of the strategy. For sure. If economic calculus can be of no use but only be this second optimization phase – and this in a very shaky way, to the extent that global optimization must take into account the uncertainty bearing on the paths of future evolution – if calculus can save nothing but a few pennies in a decision tree where an error of judgment is worth several million dollars, then doubtless something is going wrong!

Some people think that economic calculus will provide them with the right decision. Economic calculus can help in reaching a decision, but does not dictate it.

On another hand, admitting that the elaboration of a strategy for the firm is the result of an unstructured mixture of brilliant ideas, among which the privilege of the boss is to have transcendent decisive thoughts, may be going too far and honoring instinct too much.

Again, suppose even that it is true: the chairman will nevertheless have to explain his decision to his Board of Directors, to his bank or to his Secretary of State. On that score, economic calculus has the merit of being a communication language. In fact, when you do not go further than instinct or revelation, debate comes down to nothing more than a clash of convictions: everyone sticks to his opinion and, usually, the most obstinate will have the best of it. On the contrary, when the decision is supported by economic calculus – in the widest sense of the phrase – the debate enables us to check the likelihood of the hypotheses, if needed the processes of reasoning, and everyone's position moves according to the various arguments. Should it only be to start a discussion which would enable both parties to find their feet, show the hypotheses and basic options on which the dispute lies and then have a look at the real problems, economic calculus is essential to the decision process: that is the only way to agree on how disagreements arise at a level where, when you have got the problem rid of its logic and consistency, everyone is entitled to have his own opinion. Should the opinion of the boss prevail at that very moment – an opinion based on his personality's options, for which he was selected to get the position he is holding – it would be his privilege and he will assume personal responsibility for it.

ECONOMIC CALCULUS IN THE PUBLIC SECTOR

In the public and parapublic sectors, economic calculus tends to take another dimension in the way that it intends to support decisions of public interest. All that has been said before is still worthwhile, with the difference that you do not even look for the optimization of a discounted profit any longer (possibly in correlation with other aggregates), but for the optimization of the collective interest.

In addition to the criticisms made of economic calculus in the private sector, one must consider those brought about by the requirement of modeling this collective interest.

Serving as a background to economic calculus, one may then find what people call in a somewhat abusive manner 'the economic theory', which is to be found again in such developments as Pareto optimality, or value theory, and which Professor Allais illustrated and taught under the designation of *théorie des choix.*

This is no time to develop a theory. We shall then stick to the elementary account, according to which, in a perfectly competitive economy insuring unicity of every market price and eliminating the firms showing a deficit, every single decision whose purpose is not to maximize the (discounted) profit of a firm appears as more detrimental to some than advantageous to others. As a result, the market price of every single good or service must be the same as if one had perfect competition and, according to this price system, every decision which does not maximize the profit of a given firm is a bad one.

In the absence of pure competition, then, prices made by public companies should be equal to marginal production costs[1] – i.e., what is called 'marginal cost pricing'. Besides, between two ways of equally satisfying a customer, the cheapest should always be selected. Finally, it follows that to sustain given production figures of a firm over several years, one had better choose the lowest discounted cost, as far as the actualization rate (or, more accurately, the intertemporal set of such rates) characterizes the preference for today in this economy and is the one on which both the marginal rate of profitability of capital and the interest rate of financial investments adjust.

* * *

Among the many criticisms brought about by the confrontation with daily life of a perfectly pure theory, the discussion of which could be the substance of several volumes, let us briefly mention those few which affect more particularly the validity of economic calculus.

1. The theory is based on the rational behavior of the consumer, who is supposed to know perfectly what he has to know and behave in a consistent way. That is not obviously the case. That is why the consumer should be better informed and, if it is not possible, he should have his spontaneous behavior modified by an alteration of prices.
   Enlightening him on market data, yes indeed! Make him happy, in spite of all he can say? This is much more difficult to advocate. Besides, no one ever knows where the superiority complex of a higher civil servant stops as far as 'he' knows what the consumer really wants.

2. The theory starts from the principle that everything that gives birth to an individual feeling of satisfaction or frustration is kept in the accounts and can be of market value. Such is not, for instance, the case with environment goods, such as clean water, pure air and unspoiled areas.
   All that is true, but not destructive.
   Rules can set some limits to contain, in a somewhat bearable way, the pollution on the environment. Better, we could be charged for it, either directly or by paying the victims compensation in cash.

3. Getting to a situation in which one cannot improve the situation of the first anymore without affecting the second does not prove that social inequality between people is good. A change of prices, even inducing wrong choices, could be preferable if it leads to a better repartition of the satisfactions.
   Certainly! If taxes cannot cope with it, suppose even they stay neutral, we all must come round in the end. All the same, one has

to consider how wrong choices could become, in order to find what should be called the less bad way to get what one wants. What is said to be the last bad is not always the easiest.

4. Some sectors working 'with increasing returns'[2] such as railways, electricity or commercial networks, show a deficit when carrying out sales at a marginal cost. If the gap is not filled by government intervention firms must increase their prices.
   This is true, although the rents induced by inflation have considerably reduced the problem.

5. Actual competition is far from perfect. Firms invest a great deal of energy trying to create situational rents through advertising and oligopolies through industrial mergers. Nationalism often leads the authorities to shut their eyes to these blows struck against fair competition in order to enable their own leading firms to benefit before entering the international game. If, as a result, prices no longer reflect the social utility within each country, the public sector should then have a look at these deviations in order to be able to take a compensating action and restore as much as possible the optimality of choices.
   Granted. But, if such an action modifies the prices, it does not bias them once and for all. Besides, if the public sector should modify its economic calculus, by how much should it? Where? When and how?

6. The long-lasting recession, the deficit in the balance of payments and unemployment do not constitute the best of equilibrium conditions, so that what the theory teaches, which is not always welcome, even when firms are working at full capacity and full employment, is being questioned.

Considering the theory, the first objections were already hard to resist but this last one really breaches the dam.

In order to answer it, one must add other constraints to the theory of value and examine how prices, incomes and behavior could be modified. That is the aim of the theory of 'second best' optimality. However, interpreting the results is often difficult and applying this theory requires a knowledge – be it only in an approximate way – of the values of various elasticities.

Then what? Improve the theory as well as the practical availability of data, to be sure. But, meanwhile, what action can we advise?

Macroeconomic models flourish today, which introduce in their formulation basic imbalances of our economies and suggest corrections. However, the numerical models of macroeconomics (with their global figures) and the theoretical models of microeconomics (with their locally valid prescriptions) are separated by a vast intellectual desert, still almost unexplored.

Besides, macroeconomic equations reflect short-term, at best medium-term behavior – i.e. the only ones that can be checked – whereas decisions resulting from a microeconomic calculus are often structuring decisions to be evaluated in the long run.

Moreover, if you admit that long run imbalances take over from each other and compensate each other, so that a balanced growth with medium rate gives a rough estimate of the distant future, does it not sound better to refer to this model rather than admit that today's imbalances will last forever? This may no doubt constitute a good reason to keep as basic rules the ones taught by economic calculus. However, that is not enough to prevent us from making local corrections.

Which ones? 'Corrections decided upon by political power' say those who like formulas. But the question is: what should experts advise the so-called political power to do?

There is no real answer to this question. However, to be well aware that we do not really know what to do counts for a good deal and makes us behave more humbly. That being said, even if you cannot determine the 'fictitious' prices and so substitute them for the actual prices by optimizing a social utility function in a second best model, you can try some simulations in macroeconomic models, provided their deficiencies have been carefully examined. You can then try to disaggregate the 'fictitious' prices you have obtained, in order to take into account the great variety of goods and services which the macroeconomic model reduce to a single variable. You will then analyze the perverse effects that the use of these 'fictitious' prices in the public sector only will induce in the economy and correct them through a taxation scheme in the private sector and through an *ad hoc* change in the costs evaluation procedures of the public sector.

No doubt this is quite flimsy. One should therefore by most careful in making these corrections.

The reader should, of course, say that this nevertheless amounts to supporting economic calculus. But this calculus – I stand ready to admit that – will have to be abandoned as soon as someone comes to know of a better procedure. Meanwhile, we must live.

Electricité de France,
Honorary Chairman of the Board

NOTES

[1] More accurately, a state monopoly should fix its sales level in such a way that the price at which each of its goods is sold equals the marginal production cost prevailing at this level of sales.

[2] A much more subtle concept than people think when attributing it in a wrong sense to various industrial activities.

LOLA L. LOPES

8. ECONOMICS AS PSYCHOLOGY: A COGNITIVE ASSAY OF THE FRENCH AND AMERICAN SCHOOLS OF RISK THEORY

Economics is not psychology, nor is it meant to be. But there is no escaping the psychological issues that run through economic risk theory. Likewise, psychological risk theory is bound by its economic origins. These common threads are nowhere more in need of unraveling than in the debate between the French and the American schools of risk theory.

It is best to begin by drawing a line between the two schools. Three candidate demarcations come to mind: (1) ability to account for what has come to be called 'the Allais paradox';[1] (2) meaningfulness of a cardinal measure of utility or psychological value; and (3) importance of the distribution of psychological values in risky choice. Although each of these factors has been prominent in Allais' (1979a, b) criticism of the American school, I believe that only one of them is psychologically critical.

If one takes the Allais paradox to be central to the French view, then theories that predict the paradoxical pattern of choices (e.g., Kahneman and Tversky, 1979; Machina, 1982) might be construed as lying within the French camp. In my view, such a classification would not be warranted. As I will argue below, Allais offered his problems to illustrate the operation of psychological mechanisms that are disallowed by both classical and modern expected utility theory. Theories that explain the problematical choice pattern by other means may be interesting in their own right, but they are not in the spirit of the French school.

In both the 1952 memoir and the 1979 commentary and rejoinder to critics, Allais emphasized the psychological reality of a cardinal index of utility that exists independently of risky choice. This view harks back to Bernoulli's (1738/1967) original use of the utility construct and to the well-established psychological notion that physical reality is connected to experience by a psychophysical function that relates objective and subjective magnitudes. In economics, however, the status of cardinal (or psychophysical) utility has been less secure. Savage (1954, pp. 94–96) called this notion 'mystical' and 'nonsensical', and Arrow (1951, p. 425)

B. Munier (ed.), Markets, Risk and Money, 165–178.

called it 'meaningless'. It might be reasonable, therefore, to classify theories in which utility is given a psychophysical interpretation (e.g., Kahneman and Tversky, 1979) as being in the French school.

I would, however, reject this classification also on the grounds that, despite the importance of cardinal utility in Allais' thinking, he specified in 1952 (1979a, pp. 52–53, 104) and reiterated in 1979 (1979b, p. 460) that risk necessarily involves the dispersion of psychological values. Thus, even when it may be assumed that neither monetary value nor probability has been distorted, distributional factors can bring about the experiences of pleasure or aversion that we associate with risk.

The remaining possibility is the one that I use to classify theories as French or American. Theories are in the French school if they consider risky choice to involve one or more aspects of the distribution of possible outcomes. These include probabilities of achieving goals or avoiding loss or ruin as well as higher order moments or other measures of dispersion. Note that I have used the term 'possible outcomes' rather than psychological values. I have done this to underscore Allais' critical emphasis on distributional factors. Such factors are just as important for an intendedly objective person who makes decisions systematically on the basis of monetary values as they are for a more ordinary person who makes decisions intuitively on the basis of psychological values.

My comparison of the two schools of risk theory will focus on three primary issues: (1) the theoretical role of risk in risky choice; (2) the behavioral implications of expectation maximization as a prototype for the process of risky choice; and (3) the normative evaluation of human decision-making abilities. This is far from an exhaustive list of possible comparisons, and I have omitted some issues that Allais considers to be central to the economic debate (e.g., his views on probability, 1979b). The issues that I have selected, however, are the ones that, in my opinion, have the strongest implications for the way that psychologists think about risk.

1. WHAT IS THE ROLE OF RISK IN RISKY CHOICE?

Despite the fact that the American school of risk theory is mainly associated with the modern expected utility theory of von Neumann and Morgenstern (1947), Savage (1954), and others, the mathematical roots of the theory go back to Daniel Bernoulli's (1738/1967) original use of the utility construct in 1738. At that time, mathematically sophisticated people believed that decision making under risk both could and should be viewed as maximization of expected value. There were, however, certain paradoxical situations that suggested that something was amiss with the expected value criterion. One of these was the now-famous St. Petersburg game in which a coin is tossed until it lands tails, at which point the player is paid 2^n monetary units, where n is the toss on which tails occurs. The expected value of the game is infinite, but most people consider it to have only minor value.

In his analysis of the paradox, Bernoulli attacked the idea that individuals encountering identical risks respond identically. He argued that a pauper, having found a lottery ticket with an expected value of 10 000 ducats, would be well advised to sell that ticket for 9 000 ducats, and a rich man, in turn, would be well advised to buy it. This led him to propose that the psychological value of money is a marginally decreasing function of objective value, in which case, the paradox is solved since the mathematical expectation of the psychological values, what we call today the expected utility, is not only finite, but small.

Bernoulli's solution of the St. Petersburg paradox was the progenitor of modern day sensory psychophysics and, as such, deserves a lasting place in the history of psychology. But equally important is that Bernoulli's theory provided a structural prototype for psychological theories of risk and a strategy for assimilating behavioral data into the structure. Faced with the failure of the expected value criterion to account for human preferences, Bernoulli retained the assumption that people choose among risks by maximizing a mathematical expectation and modified only the presumed inputs to the process.

Modern descendants of Bernoulli's theory (Edwards, 1962; Kahneman and Tversky, 1979; Karmarkar, 1978) have extended the distinction between objective and subjective values from utilities to probabilities. In fact, Kahneman and Tversky (1984, p. 344) speak of a 'psychophysics of chances' through which the psychological impact of probability comes to differ from objective probability by processes that are basically perceptual. What remains the same, however, is the structural rule for integrating the probability and value information into an overall index.

In one of the Sherlock Holmes stories, the great detective identifies a villain by noticing in the failure of a dog to bark the absence of something that should have been there. Psychophysical approaches to risky choice are also identifiable by the absence of what, on the face of it, ought to be present. The missing bark in this case is a psychological construct corresponding to risk. Obviously, whether one considers this to be a virtue or a flaw depends on one's view of risk taking. At a minimum, however, psychophysical theories are faced with three awkward facts. First, the operational language that they use to describe behavior does not mesh with the mechanism presumed to generate behavior. For example, the term *risk aversion* has nothing to do theoretically either with risk or with aversion. Second, psychophysical theories are isolated conceptually from other psychological theories having to do with risk. These include axiomatic approaches to risk measurement (Luce, 1980; Pollatsek and Tversky, 1970) as well as more descriptively oriented studies of perceived risk (Slovic, Fischhoff and Lichtenstein, 1980). Third, psychophysical theories fail to give an account of the psychological experience of risk. For all that these theories are concerned, experienced risk might be an epiphenomenon having nothing to do with how we choose among risks.

It is, perhaps, unfair to burden present-day economic theory with all this psychological baggage since, in the modern view, utility functions are not considered so much to cause preferences as to provide a means for summarizing preferences that accord with certain axioms (Luce and Raiffa, 1957). Nevertheless, sizeable gaps exist between theories in which risk attitude is expressed *indirectly* in terms of a hypothetical utility function and alternative theories in which an independently defined construct corresponding to risk functions *directly*. For example, portfolio analysis (Markowitz, 1959) is commonly used to help both individual and institutional investors meet their goals with respect to balancing investment risks and returns. One of the few bridges connecting portfolio theory with modern utility theory is the so-called *quadratic utility function* in which preferences among portfolios are assumed to depend only on expected returns and variances. Although this bridge appears at first glance to be solid, it has numerous weaknesses including that risk is not particularly well captured by variance and that, even at best, the function is only usable within narrow constraints (Borch, 1968). Likewise, utility functions containing catastrophic segments are a poor substitute for the constrained optimization models that are used extensively in agricultural economics to capture 'safety-first' rules (see, e.g., Anderson, 1979).

The French school differs precisely on this point. Although Allais has emphasized that choices involving risk are undoubtedly affected by psychological distortions of both monetary value and probability, he has stated equally emphatically that it is the dispersion of psychological values that is specific to the psychology of risk. To quote the 1952 memoir, "It may be a greater error to neglect the dispersion of psychological values than it is to treat them on the same footing as monetary values" (1979a, p. 55). In other words, according to Allais, one cannot have a psychological theory of risky choice without its having in it something that corresponds to a psychological concept of risk.

2. WHAT DO PEOPLE *DO* WHEN THEY CHOOSE AMONG RISKS?

2.1. *The Classical Challenge*

The American school of risk theory rests historically and conceptually on the premise that risky choice is isomorphic at some level with maximizing the mathematical expectation of a random variable. The first serious challenge to this idea was the St. Petersburg paradox (discussed above), for which Bernoulli's solution was only one of many proposed. Two others from Buffon also preserved the expectation principle. The first was that probabilities below a critical threshold (he suggested 0.0001) are, for practical purposes, equivalent to zero (Daston, 1980). This solves the paradox since, in computing the expectation, the infinitely many terms involving payoffs that occur with probabilities below this threshold each become zero. The second was that sellers would be unable to pay infinitely high prizes (Samuelson, 1977). This also solves the paradox since, again speaking computationally, the infinitely many terms beyond the maximum payoff grow vanishingly small as n increases.

These solutions (along with Bernoulli's) are alike not only in retaining the expectation principle, but also in the locus of their psychological effect. In each, the expectation of the game is made small by operations that reduce the magnitudes of terms with large payoffs and small probabilities. Thus, the solutions suggest that the paradox arises from psychological mechanisms that distort or otherwise modify these particular values.

Solutions that retain the expectation principle can be contrasted with solutions that replace the principle by one based on likely outcomes to the player. As it happens, Buffon also considered something along these lines. What he did was to simulate results of the game by having a child toss a coin 2000 times. On the basis of the observed sequences, Buffon estimated the worth of the game to the player and found it to be quite small (Samuelson, 1977). Some 230 years later, not knowing about Buffon's experiment (and also not knowing that Samuelson had declared Buffon's exercise to be 'nonsensical'), I ran a similar experiment involving hundreds of millions of trials on a computer (Lopes, 1981). My results confirmed that the game is no bargain for the player. Allais (1979b), meanwhile, had also produced an analysis involving likely outcomes to the player, though his approach was through the mathematical theory of ruin. He showed that even if a player can purchase the game for a very small price (say, \$33), and even if the player has a very large fortune (say, \$1 million), the probability is very large (in this case, 0.9999) that the player will be ruined if settlement must be made after every game.

The latter solutions differ fundamentally from those that retain the expectation principle in that the locus of their psychological effect involves small payoffs that occur with large probability. They differ also in that no operations are implied that distort or modify given outcomes and probabilities. On the contrary, all three boil down to recognition of the fact that the St. Petersburg game is objectively very likely to pay only a small amount.

These two classes of solutions for the paradox define two classes of underlying psychological mechanisms. Distinguishing between them empirically should be straightforward. One method would compare the effects of manipulations that alter high payoff/low probability outcomes versus low payoff/high probability outcomes. Another would have subjects describe how they come up with a value for the game. Although I have not performed either experiment formally, classroom observation over many years indicates that people price the game by estimating what they are likely to win and then offering that amount or something similar.

2.2. *The Modern Challenge*

Modern utility theory was also stimulated by a challenge to the expectation principle, this time directed at von Neumann and Morgenstern's (1947) use of an expected utility criterion in game theory. Their axioms for expected utility theory were intended to specify sufficient conditions for applying such a criterion to single choices. In the ensuing years, however, the expected utility criterion has come to be seen as necessary for rationality, and application of the criterion to both long and short runs has come to be a central tenet of the American school.

Allais, in contrast, considers mathematical expectation to be irrelevant for the isolated case even if provision is made for psychological values. As he says,

> . . . the neo-Bernoullian formulation is of no greater, and no lesser, interest than any formulation purporting to represent a set of numbers by a single figure. The median or the geometric mean of psychological values could equally well have been taken. That would have been just as interesting (1979a, p. 104).

Although Allais accepts the asymptotic validity of the expected utility criterion for cases in which the probability of ruin is fairly low, he states unequivocally that

> *There is a yawning gulf between the conduct that is rational in relation to a random choice related to a nonrepeatable event, and the conduct that is rational when the event will recur very often and in similar circumstances* (1979b, p. 490).

A question of some interest is whether ordinary people also see a difference between short-run and long-run risk taking. At least two studies (Lichtenstein, Slovic, and Zink, 1969; Montgomery and Adelbratt, 1982) have investigated whether people would maximize expected value if they knew more about it. In the studies, subjects were shown sets of gambles differing in expected value and were asked for their preferences for a single play. Then the expected value principle was explained and the gambles were presented for choice again along with information about each gamble's expected value. The results of both experiments revealed that subjects did not consider expected value to be relevant for single plays. Instead, they preferred gambles that gave them a reasonably high probability of winning an acceptably large amount of money. When, however, subjects were instructed that they would be allowed to play the gambles a number of times, preferences shifted toward those predicted by the expected value principle (Montgomery and Adelbratt, 1982).

The related issue of whether violations of expected utility theory occur for both single and repeated trials has been studied by Keren and Wagenaar (1987). They showed that for gambles similar to those in the second of Allais' paradoxes (Allais, 1979a, p. 92), the paradoxical choice pattern holds if the gambles are to be played just once but is reduced substantially when the gambles are presented for repeated play.

It may seem surprising that few studies have investigated either people's responses to long-run situations or their understanding of the expectation principle. But one cannot overestimate the influence that von Neumann and Morgenstern (1947) had on the psychological study of risk. The earliest research in this field was motivated less by interest in people's responses to risk than by aesthetic appreciation of the new theory. Perhaps because the intent of that theory was to justify expectation maximization for single plays, psychologists ignored alternative explanations for phenomena such as risk aversion and also ignored ample evidence that the cognitive basis for risky choice is qualitatively different from expectation maximization, particularly for single plays. Considering that before 1947 most economists rejected the principle of expectation maximization for single plays, it is more than a little odd that psychologists should have scarcely considered that naive subjects might do the same.

In analyzing expected utility theory psychologically, one needs to separate the roles played by utility and expectation. Expected utility theory requires that a person's choices under risk derive from a cognitive operation that is isomorphic, *at least in terms of output*, to computing a mathematical expectation. But subjects do not consider mathematical expectation to be relevant to unique choices. They prefer, instead, to maximize their chances of attaining particular goals (for a related analysis see Lopes, 1987). Such operations do not correspond to computation of mathematical expectation either procedurally or in terms of output. For repeated choices, on the other hand, subjects both understand and approve of the expectation principle. Ironically, however, they are not particularly risk averse for repeated trials and may, in fact, choose to maximize expected value for very long runs. Thus, expected utility theory is of limited psychological validity for either unique or repeated trials, failing for unique trials because the expectation principle is violated and being unnecessary for repeated trials because there is little risk aversion to be explained.

3. WHAT *SHOULD* PEOPLE DO WHEN THEY CHOOSE AMONG RISKS?

Allais (1979b, p. 518; 1984, p. 114) has complained – with considerable justice, I believe – that too much attention has been paid to his famous paradox and too little to his contribution to the economic theory of risk. Certainly that is true in psychology where little is known of Allais' views beyond what is conveyed by the paradox itself. Much of the problem, of course, is that Allais' original paper was not available in English until recently. But equally important is that the Allais paradox is always presented to psychologists in the context of Savage's (1954) historic response. A particularly interesting case in point is an experiment by Slovic and Tversky (1974) that investigated whether people's responses to the Allais paradox and the related Ellsberg paradox (Ellsberg, 1961) can be shifted in the direction predicted by expected utility theory. What Slovic and Tversky did was to elicit subjects' spontaneous preferences and then give them written counterarguments presenting either the viewpoint of Dr. A. (for Allais) or Dr. S. (for Savage). The basic result was that most subjects stayed with the disallowed pattern when new preferences were elicited. In fact, of the shifts of preference that occurred, more were toward Allais' viewpoint than were away from it.

These results should not surprise those of us who share the preferences of naive subjects. We have also read the counterarguments and we do not change our minds. But what is surprising is that even in the face of subjects' rejection of Savage's argument, Allais is still not accorded a clear victory. On the contrary. Although Slovic and Tversky conclude with a hypothetical dialog in which Dr. A. and Dr. S. argue their divergent views, it is not Dr. A. who gets the last word. The final speech, spoken by Dr. S., runs as follows (Slovic and Tversky, 1974, pp. 373–374):

> It is not my belief in the axiom that is at issue, but rather the arguments on which it is based. Your objections are well taken. Yet I have observed that, in general, the deeper the understanding of the axiom, the greater the readiness to accept it. Were it not for the cogency of the argument, I doubt that this would be the case.

The belief that violations of expected utility theory reflect errors and misunderstandings on the part of subjects also arises in what are called information processing theories of risky choice. Researchers in this tradition (e.g., Payne, 1973; Slovic and Lichtenstein, 1968) stress the role that is played in the risky choice process by limitations in people's ability to process complex information. On this view, a subject's preferences among gambles reflect simple comparison operations applied lexicographically to payoffs and probabilities according to whatever the subject considers to be important in the given context. Often this involves choosing on the basis of probabilities if the probability differences are reasonably large, and choosing on the basis of outcome magnitudes if the probability differences are small (Payne and Braunstein, 1971; Lichtenstein and Slovic, 1971).

The information processing viewpoint is clearly correct in its main outline. Not only do people's choice processes differ computationally from what expected utility theory suggests, but the choices themselves differ qualitatively from what the theory requires. Thus, for example, lexicographic processing has been shown to lead to intransitivities of gamble preferences (Tversky, 1969) and also to the preference reversal phenomenon (Slovic and Lichtenstein, 1983) in which subjects' bids for gambles are ordinally inconsistent with choices among the same gambles. Yet the more general validity of the information processing approach is flawed, in my opinion, by the tacit assumption that, were it not for these processing limitations, subjects' choices would be consistent with expected utility theory. This is an exceedingly strong assumption since it leaves no option but to conclude that human beings are, except for their psychophysical responses to money and probability and their limited capacity to process information, expected value maximizers.

The focus that most psychologists bring to risky choice has been conditioned almost entirely by the theoretical and aesthetic dogmas of the American school of risk theory. The French school shifts that focus from psychophysics and processing limitations to individual planning and risk management. It also removes the normative onus of always seeing human behavior in the light of what it is not. By taking the view that rationality consists of choosing appropriate means to pursue reasonable ends (Allais, 1979a, p. i; Sen, 1986), risky choice is revealed as a form of problem solving, inviting analysis of the goals that people have in risky choice and the means-end relations that they see between goals and available actions.

Risky choice is sometimes also the stuff of high drama, as much affected by emotion and motivation as by cognition and perception. Oddly, however, psychologists have been more hesitant to tackle these factors than economists. Some notable cases in point are Allais' (1979a) catalog of the secondary factors that influence choice, Hagen's (1969) linking of the emotions of hope and fear to distributional skewness, Pope's (1983) analysis of the role of uncertainty in the pre-outcome period, and the large body of work in agricultural economics on subsistence farming and the safety-first principle (Anderson, 1979). But the crowning irony is that normatively oriented economists such as Machina (1981, 1982) have been more willing to revise their views on human irrationality than descriptively oriented psychologists such as Kahneman and Tversky (1979). Thus, Machina's counsel to people whose preferences in the Allais paradox run counter to the independence axiom is "... don't let anyone who doesn't happen to share these preferences convince you that you're 'irrational'" (1981, p. 173). Kahneman and Tversky, on the other hand, having built a psychological theory that accounts for the Allais paradox, remark "These departures from expected utility theory must lead to normatively unacceptable consequences, such as inconsistencies, intransitivities, and violations of dominance" (1979, p. 277).

In 1957, Luce and Raiffa published an account of von Neumann and Morgenstern's theory that was aimed at social scientists. In the preface they remarked that "probably the most important prerequisite [for readers] is that ill-defined quality: mathematical sophistication" (Luce and Raiffa, 1957, p. viii). A few years earlier, in the closing pages of his now-famous memoir, Allais summarized his feelings as follows: "Of course it is quite disappointing to have to exert so much effort to prove the illusory character of a formulation whose oversimplification is evident to anyone with a little psychological intuition" (Allais, 1979a, p. 106). To my mind, these are, indeed, the poles that have defined the two schools of risk theory – mathematical sophistication on the American side and psychological intuition on the French side. Allais' psychological analysis of risky choice is, in my opinion, both substantively and methodologically sophisticated, particularly as regards the empirical difficulties of distinguishing effects due to distributional variables from effects due to the psychological distortion of probability and monetary value. Unfortunately, Allais' refusal to oversimplify what is, in fact, an extremely complicated problem has not endeared him to his critics (see, e.g., Amihud, 1979, p. 187).

Although economics and psychology have different methods and different missions, economic risk theory and psychological risk theory are linked by common interest in how people make decisions under risk. Psychological risk theory currently bears the strong imprint of the American economic school – structurally, empirically, and philosophically. Had the views of the French school been equally accessible to psychologists at the outset, we might today have a very different psychology of risk. Certainly, I hope that tomorrow's psychological theory will incorporate the insights of the French school. I hope also that tomorrow's economic theory will be invigorated by increased attention to the psychological issues in risky choice.

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NOTES

[1] In the more famous of Allais' problems (1979a, pp. 88–92), a subject is offered a choice between gambles (a) and (b), and another choice between gambles (a') and (b'):

(a) \$1 million for sure	(b) 0.10 to win \$5 million 0.89 to win \$1 million 0.01 to win nothing
(a') 0.11 to win \$5 million 0.89 to win nothing	(b') 0.10 to win \$5 million 0.90 to win nothing

The paradox involves choice of (a) and (b'). Although this pattern is commonly observed, it violates expected utility theory.

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MARK J. MACHINA

9. ON MAURICE ALLAIS' AND OLE HAGEN'S *EXPECTED UTILITY HYPOTHESES AND THE ALLAIS PARADOX: CONTEMPORARY DISCUSSIONS OF DECISIONS UNDER UNCERTAINTY WITH ALLAIS' REJOINDER.* 'RATIONAL' DECISION MAKING VERSUS 'RATIONAL' DECISION MODELLING?*

I. AN EXTRAORDINARY CONTROVERSY

It is hard to think of a more notorious, long-standing, and often outright confused controversy in modern decision theory than the continuing debate on the meaning of 'rationality' in choice under uncertainty. Centered around the 'expected utility' theory of risk taking first proposed over two centuries ago by mathematician Daniel Bernoulli, this debate has seen the dramatic reversal of Samuelson's and others' opinion of the theory from 'logically arbitrary' to 'logically compelling', repeated charges and countercharges of 'nonscientific theories', 'anti-scientific attitudes', and circular definitions of 'rationality', the appearance of a major article in the debate with an editorial warning to the reader that it was being published 'on the author's responsibility', and the reaction of Savage who, upon being shown that the preferences he expressed in a survey violated his own 'rationality' postulate, concluded upon reflection that it was his preferences, and not the postulate, which were in error!

Although this debate has at one time or another engaged some of the most respected mathematicians/statisticians, psychologists, and economists of our time (de Finetti, Edwards, Friedman, Marschak, Morgenstern, Samuelson, Savage, Tversky, Wold, etc.), the individual most

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responsible for its origin and continuation over 35 years is the French economist Maurice Allais. Although the revival of expected utility theory in the 'forties due to its axiomatization by von Neumann and Morgenstern generated much controversy, much if not most of this reflected a confusion over the meaning of the conclusion of their (logically valid) argument – it was Allais who led the opposition to the *premises* of their theory of 'rational' behavior under uncertainty. Similarly, it was Allais' famous 'Paradox' which until recently continued to provide the major refuting evidence to the theory, and which is still discussed in textbooks on the subject, either as a classic example of scientific refutation or as an accidental and correctable example of 'irrational behavior', depending on the particular author's outlook.

On the other hand, Allais, primarily through sins of omission, has done much to hinder the debate, especially in regard to the propagation of his own views. While it was he who initiated and organized the famous 1952 Paris conference on risk taking where seminal contributions from Arrow, Savage, and Samuelson were first presented, the proceedings of this conference were published only in French (and even then given limited distribution), and while many of the other participants' contributions (including the above three) eventually found their way into English, Allais' equally important criticism of the 'neo-Bernoullian' or 'American' school represented by the above authors was published in the United States in summary version only, again in French. Yet while the theoretical critique he presented in that summary received at least some distribution, the empirical results and analysis of his extensive survey on risk preferences, which were to provide the empirical support for these arguments and were due to be published 'shortly' in 1952, did not appear until this collection of papers, published in 1979.

This volume thus constitutes a most welcome addition to the literature on risk taking and 'rational' behavior under uncertainty. Its 681 pages contain: (i) the first English translation of the full 1952 Allais 'memoir' expounding his views and criticisms of the American school and his own alternative theory; (ii) current views (pro and con) on the debate by some of the original participants (de Finetti, Marschak, Morgenstern) as well as more recent entrants; and (iii) an in-depth statement by Allais of his current views, including a partial analysis of the results of his 1952 survey. While both mammoth and scholarly (Allais' contributions alone have 486 footnotes and cross-references), it provides a fascinating and well-balanced combination of mathematical analysis, philosophical discussion, new empirical evidence, personal reminiscences, and heated debated. I shall try to give some of its flavor by presenting a critical, but I hope fair, treatment of some of the key issues of the debate and the contributions of this volume.

II. THE EXPECTED UTILITY HYPOTHESIS AND THE ALLAIS CRITIQUE

The controversy is best understood by beginning with the case of individual choice under certainty, where there is almost universal agreement on the meaning of 'rationality'. If A is a set of conceivable alternatives, say alternative bundles of n commodities so that $A \subseteq R^n$, then a (complete, reflexive) binary relation $\succeq$ on $A \times A$ (read 'is at least as preferred as') is considered 'rational' if and only if it is transitive. In such a case (and granted an additional technical assumption of a topological nature) it is always possible to mathematically summarize or 'represent' $\succeq$ by a real-valued 'preference function' $V(x_1, \ldots, x_n)$ on A, in the sense that $(x_1, \ldots, x_n) \succeq (x'_1, \ldots, x'_n)$ if and only if $V(x_1, \ldots, x_n) \geq V(x'_1, \ldots, x'_n)$, so that a rational individual may be modelled *as if* trying to maximize the value of $V(\cdot)$ over the currently attainable (e.g., affordable) subset of A. Besides (essentially) guaranteeing its existence, however, 'rationality' places no further restrictions on the representation $V(\cdot)$: any further restrictions are either testable hypotheses on preferences which a rational individual may or may not satisfy, or else assumptions, such as differentiability, made for analytic convenience. It is not 'irrational', for example, to hate asparagus.

Although weak, the assumption of rationality is clearly not without implications. An increase in disposable income, for example, can never make the individual worse off, but it is important to note that this is *not* because $V(\cdot)$ must be increasing in all or any of its arguments, merely that the maximum value of any function can never decrease as the attainable (i.e., affordable) set increase. It is also crucial for our purposes to note another aspect of preference functions. In the previous century, economists originally assigned a psychological reality to the function $V(\cdot)$, calling the units it was measured in 'units of satisfaction' or 'utils'. Since 'satisfaction', like location along a line, has no natural origin or unit of measure, $V(\cdot)$ was regarded as a 'cardinal' function: by proper choice of origin and units we might equally legitimately represent the individual's satisfaction by any positive affine transformation

$a + bV(\cdot)$ of $V(\cdot)$ ($b > 0$); however, $V(\cdot)$ could not be subjected to *nonlinear* transformations without changing some 'real' aspects of the individual's preferences. Economists have since come to realize that, as a representation of preferences, $V(\cdot)$ is actually 'ordinal': any monotonically increasing transformation $f(V(\cdot))$ will represent the same preference ranking $\succeq$ and hence the same choice behavior, and indeed, given the individual's awareness of $\succeq$, there is no need to assume that he or she consciously thinks in terms of any actual preference function at all.

It is possible, and might seem natural, to extend this approach to the case of choice under uncertainty, where the objects of choice are probability distributions or 'lotteries' over outcomes. Consider the set P of all alternative lotteries over the set $\{\$i\}_{i=0}^{m}$, so that a typical element in P may be represented by $(p_0, \ldots, p_m)$, where p_i is the probability assigned by the lottery to the outcome of winning $\$i$ (so P is the unit simplex in R^{m+1}). By analogy with the certainty case, 'rationality' would appear to require only that the preference ranking $\succeq^*$ over P be transitive, and hence (given the above-mentioned topological assumption) representable by a preference function $V^*(p_0, \ldots, p_m)$. Actually, since we are now working with money directly rather than with commodity bundles, we would also want to impose the 'more money is better' implication of the previous paragraphs and require that any shift of probability mass from an outcome $\$i$ to a higher outcome $\$j$ be preferred. This property, termed 'monotonicity', is equivalent to $\partial V^*(p_0, \ldots, p_m)/\partial p_i \leq \partial V^*(p_0, \ldots, p_m)/\partial p_j$ whenever $i < j$, and is ordinal in that it is preserved under monotonically increasing transformations of $V^*(\cdot)$. Beyond this, however, rationality would again seem to warrant neither further restrictions on $V^*(\cdot)$ nor that the individual aware of his or her $\succeq^*$ ranking think in terms of any particular $V^*(\cdot)$ at all.

What then is the 'expected utility hypothesis'? It is that $\succeq^*$ be represented by a preference function which is linear, and hence of the form $V^*(p_0, \ldots, p_m) \equiv \sum u_i p_i$ for some fixed set of coefficients $\{u_i\}_{i=0}^{m}$. The phrase 'expected utility' comes from the fact that if we call the coefficient u_i of p_i the 'utility' of receiving the outcome $\$i$, then $V^*(\cdot)$ consists of the mathematical expectation of 'utility' implied by the lottery $(p_0, \ldots, p_m)$. A useful account of the expected utility and related models is given in this volume by Günter Menges.

Now, while linearity is typically a useful first approximation to any function, economists and statisticians were at first hard put to see why a rational individual must *necessarily* have a linear $V^*(\cdot)$ (see the elegant statement of the 'pre-1950 Samuelson' in this regard). Yet today most professionals indeed *do* view linearity as a *sine qua non* of rational behavior towards risk, so much so that Pratt could with full conviction write "I am all in favor of any argument which will convince anyone not already convinced that maximizing expected utility is the only behavior worth rational consideration."

Although it is possible, with strong enough additional assumptions, to interpret expected utility maximization as a 'rule of long-run success' (a derivation is given in this volume by A. Camacho), the primary reason for this direct about-face on what decision modellers viewed as 'rational' behavior was the discovery by Ramsey, von Neumann and Morgenstern, Marschak, Rubin, Samuelson, Savage, and others that 'linearity in the probabilities' was equivalent to what has been termed the 'strong independence axiom'. One of several equivalent statements of this axioms reads 'a lottery $(p_0, \ldots, p_m)$ will be preferred to $(p'_0, \ldots, p'_m)$ if and only if $\lambda \cdot (p_0, \ldots, p_m) + (1-\lambda) \cdot (p''_0, \ldots, p''_m)$ is preferred to $\lambda \cdot (p'_0, \ldots, p'_m) + (1-\lambda) \cdot (p''_0, \ldots, p''_m)$, for all $\lambda \in (0, 1)$ and $(p_0, \ldots, p_m)$, $(p'_0, \ldots, p'_m)$, and $(p''_0, \ldots, p''_m)$ in P'. The argument for the 'rationality' of this prescription is straightforward: the choice among the latter pair of prospects is equivalent in terms of final probabilities to being presented with a coin which has a $(1-\lambda)$ chance of landing tails (in which case you will 'win' the lottery $(p''_0, \ldots, p''_m)$) and being asked *before the flip* whether you would prefer to win the lottery $(p_0, \ldots, p_m)$ or $(p'_0, \ldots, p'_m)$ in the event of a head. Now, either the coin will land tails, in which case your choice will not have mattered, or else it will land heads, in which case you are in effect back to a decision between $(p_0, \ldots, p_m)$ and $(p'_m, \ldots, p'_m)$ and you should clearly make the same choice as you did before. Many have found this principle to be compelling. Indeed, Friedman and Savage felt that "the Greeks must surely have had a name for it" and Marschak comes close in this volume to suggesting that it be taught in curricula along with the principals of arithmetic and logic.

Besides the discovery and interpretation of the independence axiom, one other factor had a bearing on the eventual acceptance of the theory. Recall that the preference relation $\succeq^*$ of an expected utility maximizer can be represented by the set of linear coefficients $\{u_i\}$. In this case, it is straightforward to show that any positive affine transformation

$\{a+bu_i\}$ $(b > 0)$ will represent the same preference relation, but that no nonlinear transformation of the u_i's will. Thus the economists of the late 'forties and early 'fifties, who had none too recently cast off the notion of cardinality in the certainty case, were once again asked to believe in it as part of a new and self-proclaimed 'rational' theory of choice under uncertainty! This quite understandably caused some confusion and resistance until it become generally understood that the objects of choice were not outcomes but rather lotteries over outcomes and (thus) that even linear preference functions over P could be subjected to nonlinear (but monotonic) transformations of the form $f(\sum u_i p_i)$ without changing preferences, and that in any event the independence axiom was defined directly on the ranking $\succeq^*$ so that there was still no need to posit the actual psychological reality of any $V^*(\cdot)$ function, much less the set of coefficients $\{u_i\}$. The theory in its final form thus consisted of the beliefs that: (i) satisfying the independence axiom is a necessary condition for rationality and the preferences of such a rational individual could be represented by the expectation of a cardinal 'utility index' $\{u_i\}$; but that (ii) there was no reason to assume that any cardinal index actually exists in the mind of the individual. Imagine then the profession's reaction upon being told by Allais that *both* these views were wrong!

Allais' view are complex and multifaceted, and the otherwise well-read will be amazed to see how much of the subsequent debate, as well as the theory of behavior toward risk in general, is anticipated in his 1952 memoir. His main points, however, are:

(i) the actual psychological reality of a cardinal index $\{s_i\}$ (distinguished from $\{u_i\}$) giving the 'psychological values' of the outcomes $\{\$i\}$ (and more generally, of nonmonetary outcomes as well), and which "can be defined operationally by considering either psychologically equivalent variations ... or minimum perceptible thresholds (Weber–Fechner)";

(ii) in choosing among actual lotteries, individuals take into account not only the expectation of psychological value implied by each lottery, but the variance and possibly higher moments of s_i as well, so that while the u_i's of an expected utility maximizer may be inferred directly from choice over lotteries, the s_i's of an 'Allais-type' individual in general cannot be; and

(iii) it is *perfectly rational* for individuals to take into account more than just the mean of the s_i's, and indeed, except for monotonicity (which he calls 'absolute preference') and transitivity, 'rationality' as such imposes no restrictions whatsoever on preferences over lotteries.

Many of the differences, and most of the misunderstanding, between Allais and his critics stem from this fundamental difference over whether preference rankings or psychological values are the underlying 'real' generators of choice, and some of the deeper philosophical and linguistic aspects of this difference of approach are discussed in this volume by Werner Leinfellner and Edward Booth. One particularly long-lived and well-known argument has concerned whether an aversion to risk (which both sides agree is consistent with rationality) may result in a 'rational' individual choosing to violate linearity in the probabilities. To the Americans, an individual is 'risk averse' if he or she always prefers receiving the expected monetary value of any lottery to the lottery itself. This condition on $\succeq^*$ is completely consistent with the independence axiom and (given the latter) is equivalent to the condition that the utility indices derived from $\succeq^*$ form an increasing concave sequence (i.e., $0 \leq (u_{i+1} - u_i) \leq (u_i - u_{i-1})$ for all i). Thus to the Americans risk aversion is completely compatible with, and requires no deviation from, linearity – the individual's aversion to risk is completely captured by the shape of the $\{u_i\}$ index. To Allais the $\{s_i\}$ index exists independently of and logically prior to risk preferences, and since it is the s_i's rather than the actual monetary values $\$i$ which measure the true psychological benefit of the outcomes, a risk averter would naturally choose to take into account the dispersion as well as the mean of the s_i's in ranking lotteries. Thus the $\{s_i\}$ index reflects nothing about attitudes toward risk, and it is perfectly rational for an individual to want to maximize something other than the linear form $\sum s_i p_i$. Furthermore, since Allais argues that no (Allais-type) individual can satisfy the independence axiom *except* by maximizing $\sum s_i p_i$, it follows that rational individuals may, for reasons of risk aversion, choose to depart from expected utility maximization.

However, since introspection reveals that I would choose among alternative lotteries by conjuring up neither a well-defined ranking *nor* a cardinal index of psychological value, I turn from the above argument to those aspects of the debate which are of practical importance to: (i) the decision modeller, who as descriptive scientist is concerned only with the differing observable implications of the two models and the available evidence; and (ii) the decision maker, who would like to be thought of as 'rational', but who wants to know if it is okay, in the words of the pre-1950 Samuelson, to "satisfy his [or her] preferences and let the axioms satisfy themselves."

III. IMPLICATIONS FOR DECISION MODELLERS

Allais has sought to operationalize his notion of psychological value by means of either minimum perceptible thresholds or 'psychologically equivalent variations' in wealth. On the former approach, it would seem that the minimum perceptible difference between two sums of money would depend on whether they were presented as two piles of bills and coins or two account balances. In the nonmonetary case (say slices of pie) it would seem that the same fuzziness of perception that resulted in imperceptible differences would also render these 'minimum thresholds' themselves too fuzzy and unstable to be of use in deriving this index. In a contribution to this volume, Peter Fishburn has shown that much of expected utility theory may be derived in a model that allows for fuzziness of perception (e.g., intransitive indifference) and it may similarly be possible for Allais to do the same with this theory.

Allais' other method of deriving the cardinal $\{s_i\}$ index, which he actually uses in his 1952 survey, consists of direct questions of the form 'for what value of i is your intensity of preference for \$$i$ over \$100 the same as your intensity of preference for \$100 over \$50?' Personally, I would respond to this question by asking what it meant. Would I rather obtain \$100 after having hoped for \$$i$ or obtain \$50 after having hoped for \$100? Surely the former – \$100 is better than \$50 regardless of i. Would I prefer \$100 to an even chance of \$$i$ or \$50? Since an Allais-type individual would consider the variance of psychological value and not just the mean in this situation, this also will not work.

Yet Allais' subjects did provide answers to such questions, and while these questions do not seem to correspond to any actual or hypothetical choice behavior, they nevertheless do elicit verbal (or written) behavior. Nor is this theory of psychological value irrefutable: if I really had such an $\{s_i\}$ index this would place restrictions on my answers to such questions. Although Allais does not seem to have tested these restrictions directly, he concluded that subjects responded 'consistently' to these questions, revealing $\{s_i\}$ indices which were approximately log-linear over large ranges of outcome values.

However, Allais' theory of psychological value is only a theory of risk taking to the extent that the $\{s_i\}$ index is specifically linked to a preference ranking $\succeq^*$ or a preference function $V^*(\cdot)$, and the exact nature of this link has caused some confusion. On the one hand, de Finetti's contribution to this volume repeats the earlier argument that nothing besides the expectation of the index ought matter, since risk preferences are already captured in the shape of the index. This is true for the u_i's of an expected utility maximizer, which are derived from $\succeq^*$, but not for Allais' s_i's, which are derived from verbal behavior in a riskless context. On the other hand, Allais' assertion that any individual satisfying the independence axiom must exhibit $\{u_i\} = \{a + bs_i\}$ $(b > 0)$ also deserves careful scrutiny: in response to de Finetti's 1952 counterexample $V^*(\cdot) \equiv \sum f(s_i)p_i$, Allais has reproved the result via the addition of an additional 'axiom of isovariation'. The amazing strength of this result, linking non-risk-related survey behavior to preferences over lotteries, leads one to wonder whether 'isovariation' might not be a lot stronger than it first appears.

Yet while the link between the survey behavior and risk preferences remains to be fully explored, I feel that the more fruitful approach to the psychology of risk would be to concentrate on the nature of $\succeq^*$ or $V^*(\cdot)$ directly. Over the years, Allais has been charged with providing little in this regard, and hence of fostering an 'unscientific' theory (this was Friedman's view in 1950 and is repeated by Yakov Amihud in this volume). It is true that Allais offers many 'psychological factors of choices involving risk' without always suggesting how strongly, and in which direction, he expects them to operate. Yet he does offer at least some well-defined, refutable hypotheses (among them absolute preference), and his primary empirical assertion – namely that rational agents will not always choose according to the independence axiom even after it has been explained to them – is clearly scientifically legitimate (not to mention important, if verified).

Since this volume still only offers 'selected findings' of Allais' extensive 1952 survey, his primary (though not sole) empirical contribution to the debate remains his well-known counterexample to the independence axiom, the so-called 'Allais Paradox'. Before proceeding, the reader may wish to note his or her preferences over the lotteries:

$$a_1: \left\{ \text{100\% chance of \$1 000 000} \right. \quad \text{versus} \quad a_2: \left\{ \begin{array}{l} \text{10\% chance of \$5 000 000} \\ \text{89\% chance of \$1 000 000} \\ \text{1\% chance of \$0} \end{array} \right.$$

and

$$a_3: \begin{cases} 10\% \text{ chance of } \$5\,000\,000 \\ 90\% \text{ chance of } \$0 \end{cases} \quad \text{versus} \quad a_4: \begin{cases} 11\% \text{ chance of } \$1\,000\,000 \\ 89\% \text{ chance of } \$0 \end{cases}$$

Allais and several researchers since him have found that the modal (if not majority) choice of subjects has been for a_1 over a_2 and a_3 over a_4, which can be shown to violate the independence axiom (i.e., there is no set $\{u_i\}$ of utilities which can generate these choices). This was one of the examples with which Allais 'tricked' Savage (who initially chose a_1 and a_3) and similar examples were offered in the early 'fifties by Allais and Georges Morlat.

The main objections to the Allais Paradox as 'evidence' have been: (i) that individuals would always, like Savage, change their preferences upon being shown how they violate the axiom; and (ii) that the example in question is an isolated case, and examples involving less extreme payoffs and probabilities would result in fewer violations, if any, of expected utility. On the first point, although experimenters (especially ones who believe in the rationality of the axiom themselves) typically are able to talk subjects out of violations. Slovic and Tversky as well as MacCrimmon have found that when subjects were presented with written arguments for *and against* conforming with the axiom, there is a roughly equal propensity for preferences to change in either direction.

On the second point, recent experiments have shown that, not only are such Allais-type violations replicable with less extreme payoffs and probabilities, but that the nature and direction of such departures from linearity are both systematic and predictable. This volume contains two major studies of this type. Coeditor Ole Hagen extends and formalizes many of Allais' ideas, offering well-defined refutable hypotheses on risk preferences as well as theoretical implications and empirical tests of them. In a characteristically careful and illuminating piece, Kenneth MacCrimmon (with coauthor Stig Larsson) provides an exhaustive compendium of the various alternative axiomatizations of expected utility as well as alternative decision rules, and presents evidence, both new and old, on the different types of violations of expected utility and the dependence of these violation propensities on the parameter values (probabilities and payoffs) involved. The outcome of these and other studies suggests that the two most systematic types of violations are: (i) that, relative to linearity, individuals are more sensitive to (i.e., proportionately overweight) the probability of the most extreme outcome when this probability is small than when it is large (called the 'common ratio effect'); and (ii) that the nature of the 'common consequence' on the tail side of the earlier coin example *does* indluence individuals' choices over which lottery they would prefer in the event of a head, with a more preferred common consequence leading to a more risk averse choice (the 'common consequence effect'). The conclusion is clear: preferences systematically depart from linearity, and if the proportion of the above types of violations steadily drops as less extreme probabilities and payoffs are used, this simply reflects the fact that linear functions provide better approximations to nonlinear ones 'in the small' than they do 'in the large'.

Elsewhere I have extended this last idea to show how much of what has been termed 'expected utility' analysis in fact does not require the independence axiom (i.e., linearity) at all. Recall the earlier result that risk aversion is equivalent to the sequence $\{u_i\}$ of utilities ('linear coefficients') being concave. Taking an arbitrary *non*linear $V^*(\cdot)$ and defining the cardinal sequence $\{\partial V^*(p_0, \ldots, p_m)/\partial p_i\}$ (i.e., the 'local linear coefficients') as the 'local utilities' at the point $(p_0, \ldots, p_m)$ in P, it may be shown that $V^*(\cdot)$ is made worse off by all mean-preserving increases in risk if and only if the *local* utilities form concave sequences at all points in P. Similar generalizations of 'expected utility' results may be obtained, as well as a simple condition on the functions $\{\partial V^*(\cdot)/\partial p_i\}$ which generates both the 'common ratio' and 'common consequence' effects.

The implications for decision modellers? To Allais, whose 1979 essay remains very close to his 1952 views, I would say "Your disciplines and intellectual descendants have gone beyond criticizing the 'rationality' of expected utility and are now formalizing your thoughts into well-defined and testable alternative models. Join them!" To the neo-Bernoullians: "The evidence against the independence axiom is mounting. You have always admitted, when pressed, that expected utility was a prescriptive and not a descriptive theory. Take that admission seriously and join in this search for better predictive models."

IV. IMPLICATIONS FOR DECISION MAKERS

Having already presented the main argument in favor of the rationality of the independence axiom, I offer here what I feel to be the strongest

TABLE I

Action	Outcome	
	0.05 Probability	0.95 Probability
b_1	α	γ
b_2	β	γ
b_3	α	δ
b_4	β	δ

counterargument, drawing on the arguments of Allais as well as comments by Tversky, Drèze, and Samuelson (the latter two, however, do not necessarily disagree with its rationality).

Consider the decision problem in Table I, where the (mutually exclusive) outcomes α, β, γ, and δ denote *completely specified consequences*, that is, exhaustive descriptions of every observable aspect of the world which would be attained under the given outcome. In particular, we assume that each of the descriptions α, β, γ, and δ pertain solely to what *will* be true if they occur, and not to aspects of any of the other infinite number of conceivable states of the world (that is, to what might *otherwise* have happened). In this case the prescription of the independence axiom is clear: since the possible outcome γ is common to actions b_1 and b_2, and δ is common to actions b_3 and b_4, I should choose over $\{b_1, b_2\}$ and $\{b_3, b_4\}$ solely on the basis of α and β, so that I ought to prefer b_1 to b_2 if and only if I prefer b_3 to b_4.

Now assume these four alternative consequences are identical in all respects except for the following: in each, your best friend has been in the hospital for an operation, and in

- α: comes out with a permanent limp, and receives flowers and a sympathy card from you;
- β: comes out with a permanent limp, and receives champagne and a box of cigars from you;
- γ: comes out in perfect health (no limp); and
- δ: dies during the operation.

If I were in a position where I had to choose *ex ante* between b_1

and b_2, I would choose b_1 since in this case the 0.05 probability event would, relative to what I had reasonable cause to hope for, be a most unfortunate and unhappy outcome. On the other hand, if my choices were between b_3 and b_4 I would certainly specify champagne and cigars in the unlikely and near miraculous event that my friend did not die. Furthermore, I do not feel that these choices would be 'irrational'.

I have constructed this particular example to highlight what I feel to be the key objection to the independence axiom: namely, that my attitudes toward ('utility of') a particular outcome need not be independent of what might otherwise be expected to happen and how likely these other possibilities are. Indeed, even if I were allowed to make my choice after learning the outcome of the operation, my decision between α and β might 'rationally' depend on whether I had expected γ or δ to have almost certainly happened instead, and if such 'complementary' across mutually exclusive outcomes is legitimate *ex post*, it could hardly be irrational *ex ante*.

One objection to this example is that it does not really violate the axiom because α and β are not really the 'same' outcomes in the $\{b_1, b_2\}$ decision as in the $\{b_3, b_4\}$ decision: the complete description of a consequence must include not only its physically observable aspects but also my 'state of mind' if it were to occur, and clearly my state of mind in both α and β would depend on whether I had been expecting γ or expecting δ. Arguing this, however, is to defend the axiom by rendering it observationally irrefutable: it allows me to defend *any* pair of choices in situations like the above table, the coin example, or the Allais Paradox.

A more useful objection might be that 'rationality' would at least require that the axiom be satisfied in the 'ethically neutral' case where the outcomes are purely monetary payoffs. However, this seems to fly in the face of the economist's typical view that money is only valued for the nonmonetary outcomes it affords us. In any event, even if the outcomes were purely monetary, say α a lottery ticket and β, γ and δ sure payments with γ very large and δ very small, my preferences for bearing further risk (i.e., α versus β) may well depend, *ex ante* or *ex post*, on whether the alternative outcome would be (or would have been) γ, in which case not getting γ would be a disappointment and I might be inclined not to gamble further, or whether it would be (would have been) δ, in which case I would consider myself lucky if I do not get it, and possibly feel willing to bear the uncertainty of an additional bet (α). It is important to note that it is not my estimation of the respective probabilities of the gamble α which are affected here, merely my willingness to bear them.

The argument over the rationality of the independence axiom may well go on forever. The implications for decision makers? "Make sure you understand the argument for the independence axiom, and the usefulness of structuring decisions to highlight common consequences as in Table I. But if you truly feel that your enjoyment of outcomes such as α or β will depend on what might have otherwise happened (or more to the point, what might still otherwise happen), then don't let anyone who doesn't happen to share these preferences convince you that you're 'irrational'".

V. AN IMPORTANT VOLUME

Besides those mentioned above, the contributors to this collection include Oskar Morgenstern on the need to continually be looking beyond our scientific theories (of risk taking or anything else) toward richer and more complete descriptions of reality, Karl Borch on the usefulness of the 'stochastic dominance' and related criteria in ordering uncertain prospects, Richard Cyert and Morris DeGroot on adaptive behavior when the individual cannot completely determine the 'utility' of an outcome without experiencing it, and Samuel Gorovitz on the neglect of very low probability events and the St. Petersburg Paradox. Each of these papers make original, if more specialized, contributions to the field of decision making under uncertainty.

Although it is a shame that it does not contain the current views of some of the other original participants in the debate (e.g., Kenneth Arrow, Milton Friedman, Paul Samuelson) or some of the psychologists who have made important contributions to the field (Ward Edwards, Daniel Kahneman, Amos Tversky), this volume nonetheless represents a watershed point in the economic theory of decision making under uncertainty. Indeed, it is probably almost single-handedly responsible for the current revival in the theoretical and empirical study of nonexpected utility models, and as such, continues (after too long a lapse) the intellectual impact of Maurice Allais.

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JAMES B. RAMSEY

10. PROBABILITY OF RUIN: A USEFUL ALTERNATIVE TO THE EXPECTED UTILITY HYPOTHESIS IN FIRM DECISION MAKING

In any science there are frequently periods of time during which alternative hypotheses hold the allegiance of competing groups of scholars. When the competing theories outperform each other with respect to different classes of phenomena, so neither is clearly and universally superior to the other, then both theories will be adhered to for long periods of time even though they may contradict each other in part. In such circumstances, neither theory is a valid interpretation of the data, but pragmatism dictates using each wherever part of the theory seems to provide useful predictions. Such has been the situation in physics for some years with respect to the debate between the quantum and wave theories.

Such also is the case in economics with respect to the analysis of individual behavior in the face of both risk and uncertainty. The dominant theory at the moment, as measured by the number of scholars proffering allegiance, is the expected utility hypothesis. In terms of individual decision-making, the chief alternative of long standing is Maurice Allais' proposal, and at present a minority opinion is taken by the probability of ruin proponents. Bolstering the Allais position is by now a long list of theoretical and empirical work criticizing the expected utility hypothesis. Some recent examples in this voluminous literature are Grether and Plott (1979), Kahneman and Tversky (1979), and Machina (1982).

One may conclude from this literature that within a strictly limited class of alternative prospects the expected utility hypothesis seems to perform in a predictively useful manner, but that on widening the class of alternative prospects in empirically important directions it fails, as was first cogently indicated by Maurice Allais (1953) and more recently in Allais and Hagen (1979).

All of this criticism of the expected utility hypothesis is at the first level, as it were, having to do with individual decision making within a

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static environment. If one wishes to extend the analysis to firm decision making, then even by the premises of the expected utility hypothesis itself the theory is inapplicable unless one can reduce the firm to the equivalent of the decision making of a single individual maximizing his own utility subject only to the usual constraints. Palpably, firms in general cannot be usefully analyzed by taking such an approach, so some alternative is needed. The chief difficulty, of course, in extending the hypothesis from individuals to firms is the inability to define, except under trivial circumstances, a single peaked preference function for group decision making so prevalent in that environment. Some predictively useful alternative is needed. To date, the practice has been to use the expected utility hypothesis to analyze firm decision making, even though all scholars agree the theory is in the strict logical sense irrelevant. The major objective of my contribution is to begin to explore the usefulness of a probability of ruin approach to firm decision making under uncertainty. Even the most casual observation of firm behavior indicates that simple straightforward maximization of expected present values is an inadequate theoretical tool, so that some different analytical framework is needed.

The essential idea underlying the probability of ruin approach is quite simple, even though the details may on occasion become fairly complicated. The firm is assumed to maximize its expected net present value, where the expectation operator allows for the potential truncation of the income stream due to ruin; i.e., at some lower bound on accumulated income the firm ceases to exist. No essential difference is introduced in the analysis if the firm is allowed to borrow to a limited extent on its expected earnings, so that even if its net worth as conventionally calculated by accountants is negative, expected net worth would be positive, and the firm would be able to continue to function. However, it is clear that firms do go bankrupt and that even for those which do not the threat of bankruptcy must be allowed for by the firms in their decision making.

Counterpoising this discussion is the idea that such behavior can occur only through a 'failure in the capital markets', in that the firm is unable to borrow fully against its net expected return exclusive of the probability of ruin. It is a fact that firms cannot and do not borrow fully against their own estimates of their net present values. But the apparently objective statement of market failure is really normative in that the 'ideal' capital market should ignore risk and uncertainty and *a fortiori* should ignore differences between subjective evaluations of risk. The neo-classical normative concept also ignores the importance of specialization in risk taking so that a moral hazard situation is created in that only the firm itself can distinguish between the effects of its own efforts and knowledge and those due to chance. Notwithstanding these caveats the market for the drilling of wildcat oil wells, for example, provides a rich source of instances of the use of contingency contracts and the efforts of the market to handle a range of problems usually cited as the sources of market failure. The analysis in this contribution is strictly positive and in the particular example used to illustrate the theory to be developed – the market for oil reserves – I will accept the world as it is and not wish it were otherwise.

From a theoretical viewpoint, the chief elements of analytical gain over the use of the expected utility hypothesis provided by a probability of ruin approach are that:

- no patently unrealistic assumptions about behavior or firm organization need be made;
- the theory leads to the concept of specialization in types of risk and recognizes that risk is not a simple single-dimensional concept;
- the theory emphasizes the extent to which firms can adjust to the optimal mix of types of risk; and
- the theory leads to the idea of an equilibrium (steady state) distribution of firms by size.

In the subsequent discussion the elementary notions are developed within the context of oil exploration. However, the reader should note that oil exploration is merely an example, albeit an important one. Other examples include the search for new markets, research into new processes and products, and even the search for inputs. The classical example and the only one well researched to date is insurance. The balance of this paper is in three sections – the development of individual firm decision making, the description of market equilibrium, and, finally, some discussion of the theoretical implications and some hints about dynamics. For further algebraic details the reader is referred to Ramsey (1979). In any event, the analysis presented here should be regarded as merely a beginning, and others are encouraged to join the search for a more fruitful approach to firm decision making under uncertainty.

1. THE ANALYSIS OF FIRM DECISION MAKING FACING RUIN

The prototypical exploration firm is a firm which specializes in a particular type of risk taking – that involved in the economic decisions with respect to the discovery of oil reserves. The mechanics involved in selecting the actual geographical coordinates of a well and the process of drilling itself are relatively unimportant in this analysis. What is of concern is the selection of types of exploratory areas in which to search, the choice of exploratory mode, the selection of an exploratory portfolio, and the intensity of exploratory activity.

The exploratory firm is regarded as a unified decision-making body engaged solely in oil exploration, where the financial objective is to maximize its net worth suitably defined. Variations in the funding of 'independent' exploration divisions of larger firms and variations in the initial size of firms as measured by expected net worth are equivalent to the 'entry' and 'exit' of firms of various sizes. This concept will be of importance in the next section.

The demand for oil reservoirs is considered to be competitive with no collusive elements. The supply of lease areas is assumed to be in a steady state as determined, for example, by a national government with a fixed rate of lease offering. The analysis essentially abstracts from the problems involved in the supply of new lease areas.

The multi-dimensional nature of 'risk' is captured in this particular version of the probability of ruin approach by concentrating on just two characteristics – the probability of a discovery given a unit level of exploratory activity and the mean value of the distribution of reservoirs given one is discovered. All variables are measured in terms of a 'standard unit of exploration'.

Thus, with E units of exploration the distribution of the number of discoveries n is $r(n \mid E\lambda)$, where λ is the mean value of n given the distribution $r(\cdot)$ and a unit level of exploration. The density function of the value (size, essentially) of a discovery is $p(y)$ with mean η.

Define Y_t, the discounted cumulative present value of oil discovered from period t_0 to period t, by:

$$(1) \qquad Y_t = \sum_{t=t_0+1}^{t} v^{i-t_0}\tilde{y}_i, \quad v = (1+r)^{-1}$$

$$\tilde{y}_i = \sum_{s=1}^{n_i} y_{si},$$

where r is the appropriate discount rate; n_i is the number of discoveries in period i; y_{si} is the value of each discovery in period i.

The distribution of $\tilde{y}_i$ is a weighted sum of convolution sums of $p(y)$, where the weights are given by the distribution $r(n \mid \lambda)$. The random variable Y_t is a stationary stochastic process.

In any period t, ruin is said to occur at period t if:

$$(2) \qquad W + Y_t - EV_t < 0,$$

where $V_t = \sum_{i=t_0+1}^{t} v^{i-t_0}$ and W is the initial wealth of the firm.

Equation (2) merely states that when the discounted expenditures on exploration, EV_ts exceed total discounted net worth in any period t, ruin occurs.

Let $Q(t, W)$ represent the probability of ruin in period t, but not before, with an initial wealth W, and let $F(y)$ denote the distribution of Y_1 with $t_0 = 0$ for convenience; then, $Q(t, W)$ can be defined recursively by:

$$\begin{aligned} Q(1, W) &= F(w_1), \quad w_t = EV_t - W, \\ (3) \qquad Q(t, W) &= \int_0^\infty Q(t-1, u) \mathrm{d}F(w_t + u). \end{aligned}$$

A moment's reflection will persuade the reader that given these definitions the probability of no ruin ever is $1 - \sum_1^\infty Q(t, W)$. The present value of an exploration expenditure level at rate E per period, with initial wealth W, facing a discovery environment characterized by λ for mean probability of a discovery and η for the expected value of a discovery, is given by:

$$(4) \qquad PV = \frac{E}{r}(\lambda\eta - 1)\left[1 - \sum_{t=1}^{\infty} Q(t, W) v^{t-1}\right],$$

where $v = (1+r)^{-1}$, and r is the appropriate discount rate. Equation (4) is easily derived from the statement of the problem summarized in Equations (1)–(3).[1] Equation (4) is interesting in its formulation in that the probability of ruin effect is exhibited entirely through the term

$(1 - \psi(w))$, where $\psi(w) = \sum_{t=1}^{\infty} Q(t, w)v^{t-1}$. Thus, $(E/r)(\lambda\eta - 1)$ is the discounted net present value of the exploration stream E if ruin is ignored. The ruin modified PV is less by the scale factor $(1 - \psi(w))$.

As a first approximation the firm is assumed to maximize PV with respect to E, the level of exploratory activity, given values for λ and η; that maximizing solution is given by:

$$(5) \qquad E^* = [1 - \psi]/(\partial\psi/\partial E);$$

in short, the firm maximizes PV by setting E equal to the reciprocal of the relative rate of change in probability of ruin, which is equivalent to equating the relative changes in the risk-free net present value and in the probability of ruin to changes in the value of E.

Equation (5) can be analyzed with respect to the effects on the optimum solution due to changes in W, F, λ, and η. The effect of W is best left to the discussion of the joint determination of W, λ and η. One immediate and interesting consequence of the optimal solution for E shown in Equation (5) is that an increase in interest rates leads to an increase in the rate of exploration. While surprising at first, the idea is intuitively plausible when one recognizes that a higher r both allows a longer run stream of exploration to be financed by any given initial wealth W and that the probability of ruin effect is reduced due to the heavier discounting of future expenditures.

An increase in both λ and η leads to an increase both in PV and in E^*, the optimal level of exploration; in short, all firms would like to obtain larger values of both λ and η. However, this observation leads to a most fruitful theoretical notion.

Let us assume, quite reasonably, in fact, that nature is niggardly in that there exists a continuous frontier of possibilities in the tradeoff between increases in λ and corresponding decreases in η. In short, there exists a continuous function $g(\cdot)$ such that $\eta = g(\lambda)$ and $g'(\lambda) < 0$. By substituting into PV the function $g(\cdot)$, one can convert the previous optimization problem into the maximization of PV with respect to only two control variables, E and λ (and though $g(\lambda)$, η). The partial differential of PV with respect to λ is:

$$(6) \qquad PV_\lambda = r^{-1}\left[MR(\lambda)(1 - \psi) - (\lambda g(\lambda) - 1)\partial\psi/\partial\lambda\right] = 0$$

which implies as a first-order condition that:

$$(6') \qquad \frac{MR(\lambda)}{\lambda g(\lambda) - 1} = (\partial\psi/\partial\lambda)/(1 - \psi),$$

where

$$MR(\lambda) = \frac{\partial(\lambda g(\lambda) - 1)}{\partial\lambda} = g(\lambda) + \lambda g'(\lambda),$$

and

$$\partial\hat{\psi}/\partial\lambda = \partial\psi/\partial\lambda + (\partial\psi/\partial\eta)g'(\cdot).$$

Equation (6') says that the optimal values of λ and E are obtained where the relative increase in risk-free present value is just offset by the relative increase in the probability of ruin. Notice that the left-hand side of (6') does not depend on W, whereas the right-hand side does. Let $m(\lambda) = MR(\lambda)/(\lambda g(\lambda) - 1)$ and $s(\lambda \mid w) = \partial\psi/\partial\lambda/(1 - \psi)$.

It can be shown that $s(\lambda \mid w)$ is a continuous downward sloping function whose value approaches zero for all values of λ as W goes to infinity. As W approaches infinity, the profit maximizing solution is to choose λ such that $m(\lambda)$ is zero and maximize undiscounted present value; in short, the probability of ruin is no longer an effective constraint on behavior.

The relationship between W and λ (for optimal E) is a more complex case and is really an empirical issue. Two basic situations can be identified. First, as W increases the optimizing points of intersection between $m(\lambda)$ and $s(\lambda \mid w)$ slide down the $m(\lambda)$ curve to give increasing values of λ for increasing values of W. The alternative situation gives the opposite result: as W increases the optimal value of λ decreases. This latter and perhaps more realistic situation prevails when $m(\lambda)$ becomes negative for relatively small values of λ and $s(\lambda \mid w)$ evaluated at small values for W becomes positive only for relatively large values of λ. These situations are illustrated in Figure 1. If $m_0(\lambda)$ prevails, an increase in W (W_0 to W_1) leads to an increase in λ. If $m_1(\lambda)$ prevails, an increase in W leads to a decrease in λ. In the former case $\eta(\lambda)$ decreases and in the latter increases. Note that the latter condition is consistent with the casual observation that 'big' exploration firms, i.e. large W, specialize in high risk (low λ), but high expected gain (high η) exploration relative to small (low W) firms.

If the firm face a series of exploratory opportunities which can be bounded by a piecewise continuous frontier which slopes down from high η/low λ to low η/high λ, then by engaging in joint ventures with other firms the firm can utilize any weighted combination of exploratory opportunities on the boundary. That is, for any given value of W the

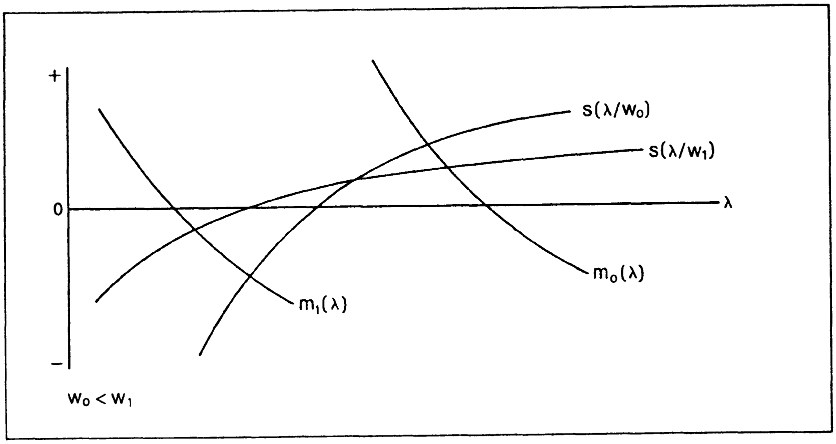

Fig. 1. Variation in first order conditions to shifts in value of W.

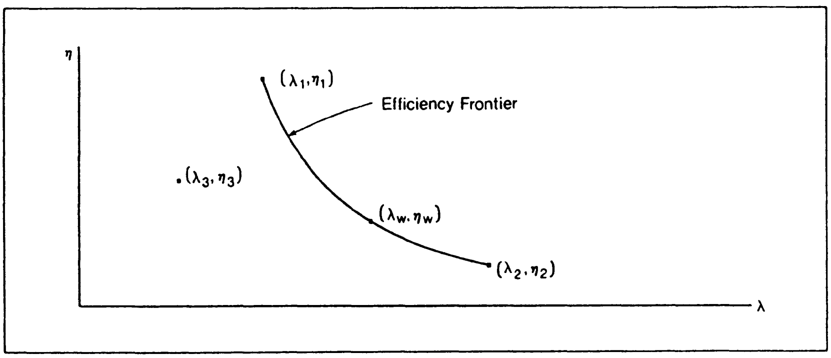

Fig. 2. Illustrating the exploratory efficiency frontier. Three types of exploratory area (λ_i, η_i) $i = 1, 2, 3$. (λ_3, η_3) unexplored; (λ_w, η_w) on frontier; $\lambda_w = w\lambda_1 + (1-w)\lambda_2$, for some w, $0 \leq w \leq 1$; $\eta_w = (w\lambda_1\eta_1 + (1-w)\lambda_2\eta_2)/\lambda_w$.

firm specializes in an optimal portfolio of exploratory opportunities at the optimal level of E^* for the portfolio as a whole. If, in addition, it is true that increases in W lead to decreases in the optimal choice for λ and hence to increases in the corresponding value for η, then firms will be distributed along the frontier by size, large firms specializing in low λs, high ηs and small firms specializing in high λs, low ηs. See Figure 2.

2. MARKET EQUILIBRIUM AND ELEMENTS OF INTER-FIRM DYNAMICS

The analysis of the previous section derived the result that, given the size of each firm in the market, each firm would specialize by selecting an optimal triplet E^*, λ^*, η^*. Let us define the individual firm's gain in wealth from engaging in oil exploration:

$$(7) \qquad H(E, \lambda, \eta \mid W) = r^{-1}E(\lambda\eta - 1)(1 - \psi) - W.$$

If, at the optimum values, $H(E^*, \lambda^*, \eta^* \mid W) \geq 0$ the firm will stay in the industry, but if $H(E^*, \lambda^*, \eta^* \mid W) < 0$, the firm will exit. Clearly, entry and exit can occur simultaneously; for example, firms with very large Ws might exit and firms with small Ws enter if, for example, there are too many large firms trying to compete for a small number of low λ (high η) areas and too few firms competing for the high λ (low η) areas. Equilibrium occurs in a period t when:

$$(8) \qquad \sup_{i \in I_t} \{H_i(E_i^*, \lambda_i^*, \eta_i^* \mid W_i)\} \leq 0$$

and

$$\inf_{i \in I_t} \{H_i(E_i^*, \lambda_i^*, \eta_i^* \mid W_i)\} \geq 0$$

where I_t is the number of firms in the industry in period t. In short, the pair of inequalities in (8) is nothing other than the usual 'zero profit constraint' for equilibrium to prevail. Implicitly, it has been assumed that no firm outside the industry will seek to enter if the most profitable firm in the industry is making only a normal return.

In equilibrium at any period t, the common internal market rate of return r^* is given by:

$$(9) \qquad r^* = \frac{E_i^*(\lambda_i^*\eta_i^* - 1)(1 - \psi_i)}{W_i}, \quad i = 1, 2, \ldots, I_t.$$

And if the condition on probability of ruin which ensures that λ^* decreases with W, then firms will be distributed over the frontier from top left to bottom right as W_i decreases from highest to lowest. See Figure 2.

Equilibrium poses certain constraints on the distribution of firms since by definition and for each firm $H(E_i^*, \lambda_i^*, \eta_i^* \mid W_i) = 0$, $i = 1, 2, \ldots, I_t$. For each value of W there is one and only one zero profit position on the frontier. Limitations in the number of areas of each type, that is, limitations in the number of areas of type (λ_i, η_i), $i = 1, 2, \ldots, \theta$, supposing there to be θ types constituting the efficiency boundary limits the number of firms with the appropriate wealth level which can include such areas in their portfolios, it being assumed that firms cannot compose portfolios with an infinite number of alternatives each included at infinitely small proportions. Thus, what is generated is an equilibrium distribution of firms, although not enough is known about the process to guess the shape of the distribution of firms over the efficiency frontier.

If the exploration market is not in equilibrium, then both

$$\sup_{i \in I_t} \{H_i(E_i^*, \lambda_i^*, \eta_i^* \mid W_i)\} > 0$$

and

$$\inf_{i \in I_t} \{H_i(E_i^*, \lambda_i^*, \eta_i^* \mid W_i)\} < 0$$

may occur, hence, one can observe both entry and exit.

In terms of the usual analysis of market entry/exit, the previous statement might appear to be clear, but in this model the statement on entry is ambiguous due to the difference in firm size for entry and exit and the consequent differences in the effects on equilibrium. Entry as measured simply by number of firms is no longer relevant. We might therefore measure net entry in terms of the net effect on exploration or on net increase in the expected value of exploration efforts. One may observe entry in terms of any one of the three definitions and simultaneously exit in terms of any of the others.

Only entry in terms of effects on total exploration rates or in terms of expected value of output have re-equilibrating effects to minor perturbations about the market equilibrium. Thus, entry in terms of the total sum of exploratory activity will tend to raise the cost of exploration in the short run so that λ_i will fall as a unit of exploration buys less exploratory activity. Secondly, if in the short run the supply of areas of a given type is not infinitely elastic, then η values will fall as firms engage in nonreplacement sampling from a fixed set of geographic regions.

With both or either of λ, η falling over the frontier, $H(E^*, \lambda^*, \eta^* \mid W)$ values will fall and entry will cease.

Entry in terms of the expected value of output in the short run leads initially to a decrease in the price of oil which implies a decrease in the value of η and a consequent decline in the optimal values of $H(E^*, \lambda^*, \eta^* \mid W)$.

Of more interest, perhaps, are the effects of exogenously induced shifts in the composition of exploratory regions in the distribution of firms by size. For example, if government action is taken to penalize the profits of large firms relative to small firms, for example by a differential excise tax, then the optimum portfolios of firms will be pushed down along the efficiency frontier towards the high λ/low η combinations. Dynamically, this is a disastrous policy in that technologically the high λ/low η areas are in part generated by the low λ/high η areas. The latter tend to be new exploratory regions and the former are composed in the main of regions which have already received a considerable amount of exploration. Consequently, λ values are high and η values low, due to the simultaneous accumulation of information and depletion of the largest reservoirs as a given geographical area is explored.

As a consequence, if the exploration of new, low λ/high η, regions is discouraged the frontier can no longer remain in a steady state since the decline in the development of new regions will over time reduce the supply of partially explored regions. As the process continues, the frontier will eventually collapse to the origin.

The essential element to note with respect to the maintenance of a steady state as postulated in the discussion of equilibrium is that the equilibrium is defined with respect to the availability of exploratory types as indexed by the joint parameters (λ, η). Such a steady state can be maintained only by a steady flow of a series of specific geographical regions into and through the system. Thus, a given geological basin will first appear as a low λ, high η region which will slide down the efficiency frontier (or even cease to be explored by moving inside the frontier) as exploration progresses. After maturity is reached with high λs/low ηs, then the region decays as the subsequent lowering of *both* λ and η pushes the region inside the frontier and it ceases to be of exploratory interest.

3. SOME SPECULATIVE COMMENTS

The example of the oil industry was used most intensively in the above discussion because it provides such a clear example of a number of the chief concerns of this paper. In order to see the possibilities for extending this analysis to other areas of economic interest, it will be useful to summarize the key issues involved in the discussion in the previous section.

The first and principal concept underlying the analysis is the idea of specialization in risk taking. In essence, there is no such thing as 'risk' without the modification, 'relative to some activity'. Basically, it is a problem caused by difficulties in the transmission of specialized knowledge. Someone capable in one form of risk taking, say oil exploration, is not necessarily capable in any other form of risk taking, say investment banking, life insurance, or retailing women's clothing.

The second major premise is that within a risk class different firms specialize in the way in which they handle that risk. In the example discussed above the specialization is mainly in the capital intensity of exploratory techniques as applied to different 'types' of exploratory regions and the ability of a firm to specialize in capital intensive/low probability of success ventures was tied to its net worth. However, the general notion is broader than this example. The notion is that within a general risk class there is a variety of opportunities for specialization and that a small number of firms, perhaps only one, specialize their techniques for handling risk to the particular requirements of the situation. For example, in investment banking, a bank might specialize in dealing with small clothing manufacturers and devise special procedures for evaluating and dealing with risk in that environment. (This example is especially illustrative in that recently a very successful Long Islands bank tried to enter such a market in the borough of Manhattan and promptly went bankrupt.)

It is the relevance of these first two premises which ensure that the probability of ruin concept is of practical importance and not just a theoretical potential. The inability of the firm to transmit its specialized knowledge to potential investors and lenders implies that it cannot expect to borrow fully against its own evaluation of the expected net present value of its efforts. The more specialized the knowledge the more difficult it will be for the firm to sell or borrow against its potential earnings and, therefore, the more important becomes the concept of ruin to firm decision making.

The first major result arising from these ideas is the notion of a nontrivial distribution of firms even within the context of a stationary steady state. Indeed, the above analysis would indicate that the existence of a single unique optimal size of firm within an industry is a special, or degenerate, case of the general result.

Finally, another interesting concept arising out of this analysis is that within an industry, more precisely within a risk class, the specialized activities of individual firms may create the conditions needed for the successful operation of other firms. This is a type of market synergy not usually considered in the analysis of markets. In the exploration example, the exploratory efforts of 'large firms' created the exploratory areas needed by smaller firms for their profitable operation. Note that this interaction is not collusive; indeed, the interactions need not be explicitly recognized by the participating firms. Secondly, while it is tautological to say that some firms are providing external benefits to others, this view does not capture the full import of this concept. The benefits can go both ways, so that each affects the others. More forcefully, it may often be the case that the viability of the market itself may depend on the presence of these synergistic elements.

Other areas of application of these ideas are even more theoretically interesting than oil exploration. Consider first as a natural extension the market for research. The major difference between research and exploration is that the simple characterization in terms of λ and η may be insufficient to capture the principal elements of differential risk. Another difficulty is in the creation of a suitable index of output. However, with these problems solved, the main elements of the theory would seem to be most appropriate. Specialization in risk is vital, specialization within risk classes is evident (compare, for example, the development of computers), and the synergistic aspects are clear. The basic research of some firms provides the preconditions for the success of other firms' innovations. Some firms specialize in fundamental research, others in quick payoff minor innovations.

The financial markets provide another obvious example of the potential applicability of these concepts.

At this times these ideas are in their infancy and require much further effort and thought to be developed successfully. Nevertheless, the potential payoff would seem to be considerable especially in light of the fact that such analysis using probability of ruin concepts bypasses the host of theoretical problems associated with the use of the expected utility hypothesis in firm decision making.

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NOTE

[1] The reader is reminded at this point that all details have been suppressed in this presentation and that the full analysis is found in Ramsey (1980).

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BERTRAND R. MUNIER

11. COMPLEXITY AND STRATEGIC DECISION UNDER UNCERTAINTY. HOW CAN WE ADAPT THE THEORY?

> The most dramatic and spectacular secret of success is novelty, and novelty is that which an infallible algorithm must, by definition, exclude
> G.L.S. Shackle, *Epistemics and Economics*, p. 426.

I. INTRODUCTION

Experience shows that those who try to apply the most accepted mathematical theory of decision under uncertainty in a prescriptive way, will encounter situations where this technique will be firmly rejected by those responsible for taking the decision, whether it concerns an individual, a company director for example, or a group, such as a Board of Directors or a technical study group. The opinion expressed by those involved in such cases, is that such things as utility functions in a risky universe do not exist, whatever name you might give them. The same opinion believes that we cannot completely rely on well-defined probabilities, once again, whatever the name given to them may be and whatever the technique used to define their distribution in the situation envisaged. It means, finally, that to retain the expected utility criterion constitutes neither a convincing procedure for choosing between several projects or multifaceted (and therefore multidimensional) strategies, nor a way of escaping from the inadequacies of the utilities and probabilities mentioned above.

A. *A Frequently Encountered Reaction to Be Avoided*

It is not enough to declare that the people under discussion are usually not sufficiently informed about the theory. Resistance to the theory, *if encountered frequently*, should be responded to by modifications to the theory, and not by accusations against the users.

B. Munier (ed.), Markets, Risk and Money, 209–232. *Original* © *Economica (Paris)* 1986. *This English translation* © 1995 *Kluwer Academic Publishers.*

It is not enough to declare that certain logical contradictions can be discerned in users' responses during investigations into their behavior. In fact, in this case, most often it is these very responses to the investigations which are called into question. The most recent implications of their high-powered logic were not perceived by the investigation which, on the other hand, much better understands the meaning of the criterion of expected utility. As for direct investigations, known as behavior observation, apart from all the sociological problems of investigating large numbers of people, pose the very sensitive question of interpretation of the observed behavior.[1] The most rewarding of the contributions from the famous article by Paul Samuelson [20] on 'The Revealed Preference' has perhaps been, in the light of later works such as that of Houttaker [10] and Corlett and Newman [4], to show that the same series of observations on individual questions can serve to validate two different sets of axioms (revealed preference with strong axiom, or complete predetermination of preferences) on the determinants of individual behavior. It is totally acceptable that one or the other of these sets of axioms could be used under the headings of a *working hypothesis suitable for descriptive ends, in situations* where investigations have revealed behavior compatible with the consequences of these sets of axioms.

It is not acceptable, however, under the pretext of this sole compatibility with empirical observations, to consider that the model defined by one of these sets of axioms is sufficiently validated to ask the person responsible for a decision to follow the specific prescriptions which result from this one model, if he wants 'to make a rational decision'. It is in order to give a clear definition of the frontier between the world of the descriptive and the world of the prescriptive that the expression 'decision aid' has appeared. There is here an exciting program of research to be carried out on economic epistemology.

B. *Three Possible Lines of Research*

So three approaches to the problem are offered for study.

1. Throw out the baby with the bath water and renounce the use of the mathematical theory of decision. So it is found convenient to refer to 'clinical' description models containing multiple sociological observations. Munier [17] has given a general glimpse of this research carried out in France recently.

2. Try to modify the axiomatic basis of the model in such a way as to facilitate its application.
3. Suggest alternative algorithms to that of expected utility, without trying to specify the axiomatic foundation but with the same prospect of a more suitable method of application than above.[2]

Suggestions (2) and (3) seem to us to be promising. On the one hand, they try to discover the relevant part contained in the traditional theory of decision – there is hardly any social science model which does not contain its own relevance – and in this sense there 'remains' something of the traditional theory in these categories of models. On the other hand, they endeavor to take account of observations arising from the *first* line of research to improve the mathematical decision models. The two types of work are proceeding side by side in a research groping its way from a set of axioms to algorithms or to procedures, and back again.

In this article we do not pretend to present a review of all the work undertaken in this field over the past fifteen to twenty years. We will limit ourselves to an original suggestion drawn from the second line of research indicated above. But firstly we will recall the current direction of axiomatic revisions so far as they concern uncertainty of the future.

C. *The Connections between Possible Axiomatic Revisions*

It could be thought that questioning the axioms of L.J. Savage corresponds, each time, to the consideration of any given particular situation and because of that, there would not be a general connection between them. A close examination, however brief, shows that this is not so.

Thus the pioneering work of M. Allais [1], and the recent reports that he has produced, some of which have been published [2], have dealt with Postulate No. 2 of Savage, which has been named the Postulate of the Sure Thing.[3] But no one has noticed that Allais' procedure puts the sixth proposition, which deals with continuity of preferences for possible acts in a risky universe, in a rather dubious light. In fact, the introduction of the second-order moment of a distribution, was, from the beginning of his work, only seen by Allais as a first approximation, having the virtue of being operational but expressing a choice of a multidimensional nature, for which consideration of the set of values of uncertain gain and of the set of related probabilities [1, pp. 22ff] or further, the (infinite) set of moments of distribution [2, p. 570] would be indispensable. But, *the more the number of dimensions increases, the more the proposition of continuity becomes restrictive.* Man's calculating capacity, not being infinite, but rather very limited, it would be astonishing if a simplifying mental procedure (such as lexicographic ordering) did not occur. Starting from a moment of order n, this would be sufficient to explode the sixth proposition of Savage, in all its known formulations. However, this sixth postulate is *crucial*[4] in Savage's work, since it alone enables the establishment of the very concept of subjective probability as is accepted today (de Finetti, Savage, etc.) i.e. the concept of 'revealed' preference in an uncertain universe (Savage) and the criterion of expected utility.

On the other hand, we can well challenge Postulate 6 without at the same time logically having to reject the Sure Thing Postulate (Postulate 2).

From these brief reflexions on the axiomatic structure of the most popular neo-Bernoullian Theory, we can draw two conclusions:

(a) Maurice Allais was particularly consistent in examining Proposition 2 and the Savagian concept of numerical subjective probability (this judgement of consistency must be understood to be totally independent of our opinion of Allais' views on probabilities).
(b) We can perfectly well understand the attitudes of those responsible for any given decision refusing to use subjective probabilities, utilities in an uncertain universe and criteria of expected utility to make their choice, seeing that there is some reason to think that only the postulate of continuity is in question, without necessarily also calling Postulate 2 into question.

Further on in this essay we shall isolate the sixth postulate, in order to suggest an interpretation of the situations concerned, and to propose a simple prescriptive model, the properties of which we could explore.

Hence our procedure is not along the same lines as that of Allais, as a result of the final remark made at the end of Section (B) but it is rather to be considered as running on parallel lines. Indeed, our model passes the test of the Allais paradox with success.

II. ESIC, A MODEL ADAPTED TO STRATEGIC DECISION TAKING

If we have already touched on the concepts of complexity, uncertainty and risk, interdependence, algorithm and procedure, it is because the experience which are referred to in the framework of this essay concern situations that we will try to describe with the aid of these concepts (A), before justifying (and in order to justify) the nature of the set of axioms on which we will build (B), and specifying the criteria that we will retain on this basis (C). Thus we will have proposed a complete process of Strategic Evaluation under Complex Uncertainty (*Evaluation Stratégique en Incertitude Complex*, ESIC).

A. *Descriptions of Situations and Behavior*

The celebrated distinction drawn by Frank Knight between risk and uncertainty is familiar to everyone. This distinguished random situations where, with regard to repeated and frequent experiences of the same phenomena, we have long series of frequencies (objective probabilities) called risk situations on the one hand, and on the other hand, situations of uncertainty where such information does not exist and for which, consequently, there are no rational criteria of insurance, e.g. decision, even in the case where vague qualitative information about the future is available.

In the light of the works and reflexions of subsequent authors such as Keynes [11, 12] or Savage [21], without even mentioning numerous authors of the '70s and '80s, the concept of risk, thus defined, is found to be virtually discarded in the economic and social sciences. For various reasons, most economists admit that identically repeated phenomena over long periods, to which Knight refers, only rarely exist in economic life.[5] Above all, almost all authors admit that, even if there are established frequencies of a truly, identically repeated experience of a phenomenon, and recognized as such by everyone, in a world where information increasingly circulates in 'real time', no logic can impose the use of past frequencies as probabilities. Allais himself accepted the idea from the beginning of the 'fifties, that one could not always keep numerical values of frequencies without modifying them ([1, p. 23] for example). In a much more radical way Savage, de Finetti and all the followers of the neo-Bernoullian school deny any meaning other than the singular one to frequencies, and announce in *The Foundations of Statistics* [21] a remarkable foundation for probability, which can henceforth do no more than result in a personal judgement.

So, most often, there is in the social and economic universe uncertainty and not risk in the sense of Knight. But the concept of uncertainty has not had the same chance as that of probability. Much less work has been devoted to it, with the result that ideas on the subject are far from being unanimous. It has been mentioned earlier that for Savage and the neo-Bernoullians, *all* information about the future, even qualitative, is in some way 'revealed' by the behavior of the decision maker in situations of uncertainty and is *always* transformable into *numerical* probability.

On the other hand, Keynes, Shackle and others judge this position to be excessively reductionist. Keynes wrote, for example, in the *General Theory* . . . :

> When there has to be a long delay for them to produce their full effect, our decisions to do something positive must be considered, for the most part, as a demonstration of our natural enthusiasm . . . and not as a deliberate means to numerical benefits multiplied by numerical probabilities [11, Ch. XII, Section VII].

He adds, a little further on:

> Let us not hasten to conclude that everything depends on irrational psychological fluctuations . . . What we would simply like to remind you is that human decisions involving the future in a personal, political or economic way, cannot be inspired by a strict mathematical forecast since the basis of such a forecast is nonexistent [*ibid*, p. 178].

The following year he specified the concept of uncertainty to which he refers as follows:

> By 'uncertain' knowledge . . . I do not wish, simply, to distinguish what is known as certain from what is simply probable . . . The sense in which I use the term is that in which the prospect of a European war is uncertain, in the same way that the price of copper and the interest rate in twenty years from now, or the obsolescence of a new invention or the position of owners of private wealth in the social system of 1970 are all uncertain. In these matters there is no scientific base on which to form no matter what meaningful probability. We simply do not know [12, pp. 216–217].

So Keynes refers to: (1) decisions of which the full effects require a long delay to be realized; and (2) to an uncertainty putting into practice knowledge too far beyond our reach or our horizons for us to be able to formulate a precise forecast. This is the type of situation in which we meet, in fact, the greatest resistance to numerical subjective probabilities and to the criteria of expected utility, as we mentioned at the beginning of this essay.

To surmount the difficulties raised by Keynes is possible, as we will see, if we model uncertainty as a product of *chance* $\times$ *complexity*.

Chance can be defined, according to Cournot, as the meeting with series of independent causes.

When what Le Moigne [15] and other distinguished system analysts call the 'operating system' can be modelled without too much waste of information by the economist, the uncertainty which remains is amenable to the neo-Bernoullian treatment for its probability content.

On the other hand, when the 'operating system' of the decision maker is too complex, in the sense that:

(a) the significant interdependent connections between the constitutional elements of the system are too numerous (and therefore too intermingled) for the observer to model them clearly;
(b) the behavior of each constitutional element of the system is liable to be counter-intuitive,

then uncertainty is no longer amenable to neo-Bernoullian postulates, *in particular* to Postulate 6 (continuity postulate). We notice that the element (a) of our definition of complexity makes reference to what is currently called a *complicated* system; the element (b) makes reference to 'unforeseeability' and to the excessive *variety* of certain elements in the system (cf. Shannon [23]).

This *complex uncertainty* is that which Shackle qualifies, without conceptualizing it specifically, as 'high uncertainty' in various passages of his work. It is exactly this that we have called elsewhere 'non-Savagian uncertainty' [5, 6].

As for the decisions aspired to by Keynes, they are clearly in the field of decisions which are usually called strategic in organization theory. For decisions of this type, the canonical model of Herbert Simon [24], specifying intelligence, design and selection, can always be followed. The essential thing for strategic decisions, hence not always programmable in the sense of Simon, is to make good use of the model and to consider these phases, not as a linear progression, but as 'gears within gears', let us say 'phases in one of many iterations'. Simon expresses it concisely: "At whatever stage, problems engender further problems which, in their turn, demand aspects of intelligence, conception, and selection etc." [24, p. 39]. This does not mean – on the contrary, in fact – that one discards all *algorithms out of the procedure* proper to the problem. The factor that is at once persuasive and efficient in the models of Simon is this capacity for the scientific alliance of man with machine. The quotation of Shackle [22, p. 426], with which we began this essay, is totally applicable to the decisions and situations which we are alluding to here. But, without detracting from the flexibility of human intelligence which a particular procedure enables us to use, what algorithm (set of axioms + criterion) can be devised for the selection phase in such a way as to aid human decision, respecting the characteristics of the environment in which the decision is to be taken (Figure 1)?

It is in conceiving it, in this sense, that we are able, for the class of problems that we are considering, despite difficulties, to keep an algorithm related to the neo-Bernoullian theory of decision, although much more general than that.

B. *An Adapted Set of Axioms*

We must reply to the question posed by the selection phase (Figure 1) for a class of problems that we are going to describe, in the light of our preceding remarks, as follows:

(α) Concerning a strategic decision under complex uncertainty, we are unable to illustrate all the ideas postulated by Savage nor use Postulate 6 (continuity postulate) and therefore neither can we use numerical probabilities.

(β) We are unable, in consequence, to resort to a Neumann utility function. *More generally*, for reasons stated in (α) and because 'a long delay is needed' (Keynes, see above) for the full consequences of our decisions to be felt, it is illusory to rely on any objective function whatsoever.

(γ) Under (complex) uncertainty, a decision maker "feels surrounded by possible actions leading to failure, and to others possibly leading to much greater loss than gain" [...] "but the current of affairs indispensable to the survival of men, in that case, makes it unreasonable to reaffirm decisions taken earlier" (Shackle [22, p. 78]).

Under these conditions we can suggest a set of axiom founded on concepts of states of the world and scenarios; of qualitative likelihood; of graph of likelihood image and of ordinal, multicriterial evaluation. We will show below that this set of axioms enables us to design a criterion *adapted to the class of decision problems prescribed by* α, β, γ *above*, and which, for this class of problems is a criterion of 'reasoned prudence'. This characteristic can only be desirable, in fact, for the types of decisions and environments envisaged here. We will refer, in

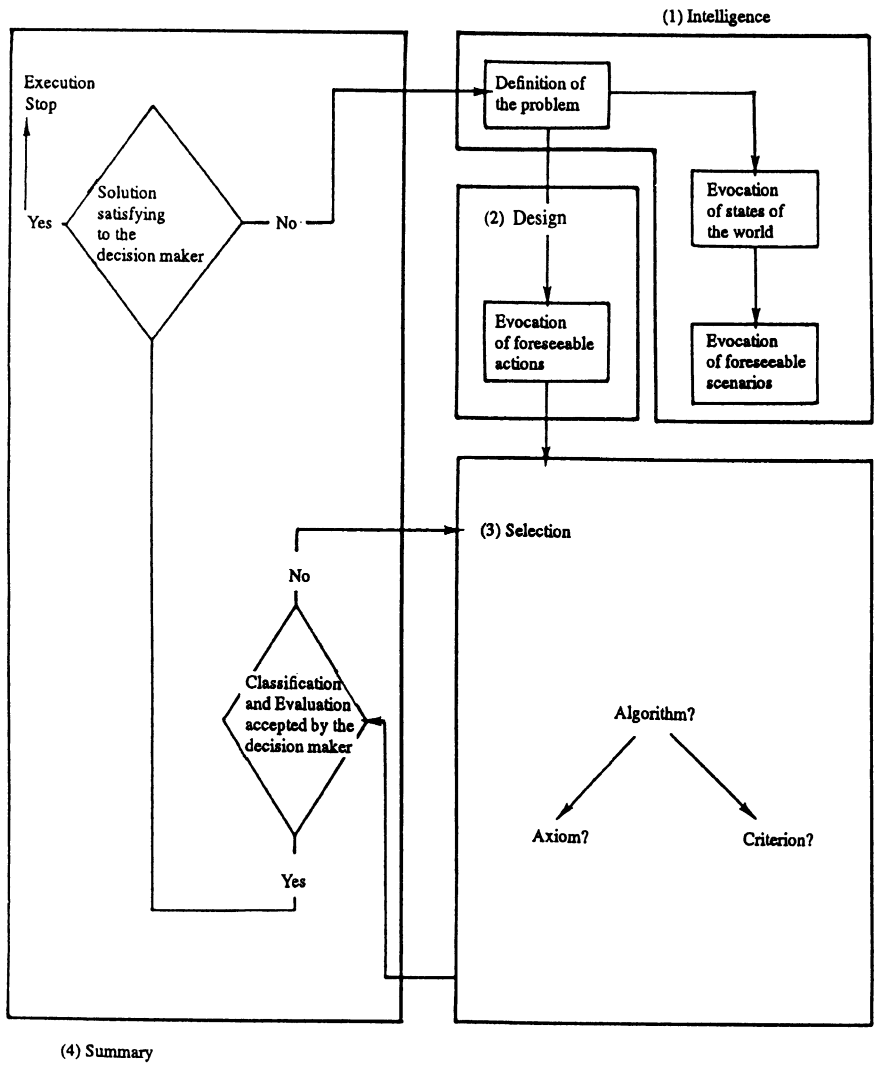


Fig. 1. ESIC: Process of decision in which is inserted the problem of designing an adapted algorithm.

the following paragraphs, to the idea of the algorithm at an instant of time t during the process of decision.

1. States of the World, Horizon, Scenarios

The *world* is defined here as the set of beings and objects in which the person responsible for the decision will be interested, by the impact which they will be liable to have on the consequences of his decision.

A state of the world is a description of admissible characteristics and attributes to these beings and objects during that period of time (horizon) in which the person responsible for the decision is interested.[6]

We denote the set of states of the world, such as exists at instant i of the process of decision, by Ω_t.

The states of the world are denoted by ω_t.

An *horizon* H_t is an element of a finite subset of N, consisting of indices $h = t, t+1, \ldots, H_t$. So the horizon H_t varies in N with the iteration in time t. We define:

$$f_t \colon A_t \times \prod_{h=t}^{H_t} \Omega_t \to C_t$$

where the set A_t represents the set of decisions a 'evoked' (in the sense that March and Simon use this term, that is to say, discovered in the design phase and extracted from the memory and/or the imagination of the decider) as far as the iteration t. The set C_t is that of consequences c attached at time t to all $(H_t + 1 - t) -$ tuple $(a, \omega_t, \ldots, \omega_{H_t})$.

$\forall a \in A_t$, we denote f_t^a the function $f_t(a, \cdot)$ defined by $\prod_{h=t}^{H_t} \Omega_t$ on C_t.

$\forall a$, f_t^a is naturally assumed to be surjective.

$C_t^a = f_t^a(\prod_{h=t}^{H_t} \Omega_t)$ is the set of possible consequences (or conditional results) from the action a which can be perceived at iteration t.

A *scenario* relative to a decision a is an equivalence class s_t^a of $\prod_{h=t}^{H_t} \Omega_t$ defined by $f_t^a(\omega_t, \ldots, \omega_{H_t}) = f_t^a(\omega'_t, \ldots, \omega'_{H_t}) = \ldots = f_t^a(s_t^a)$ (by construction, it is, in fact, possible to stretch f on its quotient set).

The quotient set of $\prod_{h=t}^{H_t} \Omega_t$ by the equivalence relation thus defined is a partition of $\prod_{h=t}^{H_t} \Omega_t$. The elements s_t^a of this partition are therefore *the admissible scenarios for a given action* a such that they are perceived at the instant t. The set of such scenarios is denoted by $\mathcal{S}_t^a$.

$\mathcal{S}_t = \cup_{a \in A_t} \mathcal{S}_t^a$ is therefore the set of all possible scenarios which can be perceived at iteration t.

2. Qualitative Likelihood

We will define a *qualitative likelihood* as a binary relation, $\leq_1$, defined over $\mathcal{S}_t$ read as 'less likely or as likely than' and enabling scenarios s_t to be compared two-by-two. The set $\mathcal{S}_t$ being able to vary after each iteration t, the relationship $\leq_1$ is therefore not mathematically defined over the same set from one instant to the other. For simplicity's sake we have eliminated the index t from the symbol for the relationship, since we are here putting ourselves in the frame of a given iteration t.

We have the following axioms:

P_1: $\leq_1$ is reflexive

P_2: $\leq_1$ is transitive

$P_3 \forall a \in A_t$: the restriction of $\leq_1$ over $\mathcal{S}_t^a$ is complete

P_4: $\leq_1$ is monotonic with regard to set inclusion, i.e.:

$$\forall S', S'' \in \mathcal{S}_t \colon S' \subset S'' \Rightarrow S' \leq_1 S''.$$

Three essential differences distinguish the concept of qualitative likelihood, thus defined, from that of 'qualitative probability' defined by Savage [21, p. 32].

(a) $\forall_t \mathcal{S}_t$ is not an algebra on $\prod_{h=t}^{H_t} \Omega_t$ – far from it. The qualitative probability of Savage is defined on the complete algebra of subsets of Ω (events).

(b) Savage considers no horizon, his model being independent of time and therefore static. Besides, the sets Ω and A are presumed completely known, as well as the functions f.

(c) The relationship $\leq_1$ which we are going to consider is, in general, not complete on $\mathcal{S}_t$, contrary to the very strong hypothesis of Savage. We must add that Savage conceived other hypotheses in place of our Postulate $P4$ above [21, p. 31].

Last, but not least, we certainly do not require for our concept of likelihood the transposition to the continuous case, although Savage did it for the concept of qualitative probability (Postulate $P6$) in such a way as to be able to prove the existence of a distribution of *numerical* probabilities. We will restrict ourselves to the *qualitative* world.

All these differences make the algorithm presented here much less restrictive and much more acceptable by practitioners than the neo-Bernoullian model of Savage.

3. Graph of Likelihood Image

For a given $\mathcal{S}_t^a$, let us denote G_{tl}^a the graph induced by $\leq_1$ on $\mathcal{S}_t^a$. We define the image of G_{tl}^a over C_t^a by:

$$L_{tl}^a = \{((c', c'') \mid c' = f_t^a(S_a'), c'' \\ = f_t^a(S_a''), (S_a', S_a'') \in G_{tl}^a\} .$$

L_{tl}^a is clearly the graph of a preorder on C_t^a. It is on the part of the 'decision maker' a preordinal classification of likelihoods of foreseeable conditional results for action a at iteration t.

It seems quite natural to consider that, if we choose action a, possible consequences of actions *other* than a should at the same time be considered as equally unlikely and as less likely than whatever the consequence of a may be. Therefore we complete the preorder on C_t^a by an inferior equivalence class composed of all the elements of C_t/C_t^a. We therefore obtain a preorder on C, of which the graph will be denoted G_t^a and which we will call by definition the graph of likelihood image at iteration t.

4. Multiple Criteria of Evaluation

Each foreseeable action at iteration t can be evaluated with the aid of a set I_t of 'points of view'. This set is also liable to be modified gradually in the course of successive iterations as the problem is 'informed' by the decision maker. We will state:

$P5$: There is a family I_t of m_t points of view i, each one of which infers a preorder $\leq_t^i$ on C_t at iteration t.

C. *A Criterion of 'Reasoned Prudence' in Selection*

With the aid of the elements which have just been described we are in the process of designing a simple, easily programmable criterion of selection, using all available information without it being necessary to introduce supplementary information in a way which would, of necessity, be artificial. An example will show that we have a moderately prudent criterion in the situations described in Section II.B above. Furthermore, it is possible – for example in a case where there is a risk of ruin – to increase the 'prudent' quality of the criterion. Here we re-encounter the ideas of Allais.

1. Design of the Criterion

Let us consider at time t a point of view i engendering a preorder $\leq_t^i$ over C_t of which we will denote by G_t^i the graph in $C_t \times C_t$. Intuitively, we could expect nothing better, from the point of view i, than an action a, of which the graph of likelihood image G_t^a would coincide with G_t^i. This would mean, in effect, that the most favorable consequence c^*, from this point of view i, would be at the same time the most likely. Conversely, the most unfavorable consequence c_* from this point of view would be, at the same time, the least likely. It is a question of a 'qualitative' or 'ordinal' interpretation of Pascal's famous wager.

It is therefore suggested that we define on A, the complete preorder ${}_A\leq_t^i$ in the following way:

$$a \;{}_A\leq_t^i\; a' \Leftrightarrow |G_t^i \Delta G_t^{a'}| \leq |G_t^i \Delta G_t^a|$$

where Δ stands for the symmetrical difference of preorders.

This clearly defines a selection criterion and, afterwards, allows the application of an adapted multicriterial procedure, say g_t, to give a 'resulting preorder' ${}_A\leq_t$ of m_t preorders ${}_A\leq_t^i$ each one interpreting a partial point of view.

We will write:

$$g_t \colon \{{}_A\leq_t^i\}_{I_t} \rightarrow {}_A\leq_t \,.$$

Some *very important* points must be made here.

(a) The algorithm that we have just described, and whose particularly interesting properties will be discussed below, must *in no way* be considered as a decision machine which one can simply follow, but, on the contrary, as a practical method of amplifying cognition and thought, offering several degrees of freedom in the course of a given process of strategic decision.

These degrees of freedom are particularly important during phases (4) or 'summings up' of successive iterations.

Thus the person responsible for the decision will examine the classification (resulting preorder) obtained. Will it surprise and shock him? Then he will choose the *short iteration loop* and will begin a new phase of selection, modifying, at least, the multicriterial parameters of evaluation (see Figure 2).

If, on the contrary, the classification obtained is accepted by the decision maker, it only remains for him to ask whether or not he is satisfied with the action classed as first choice.

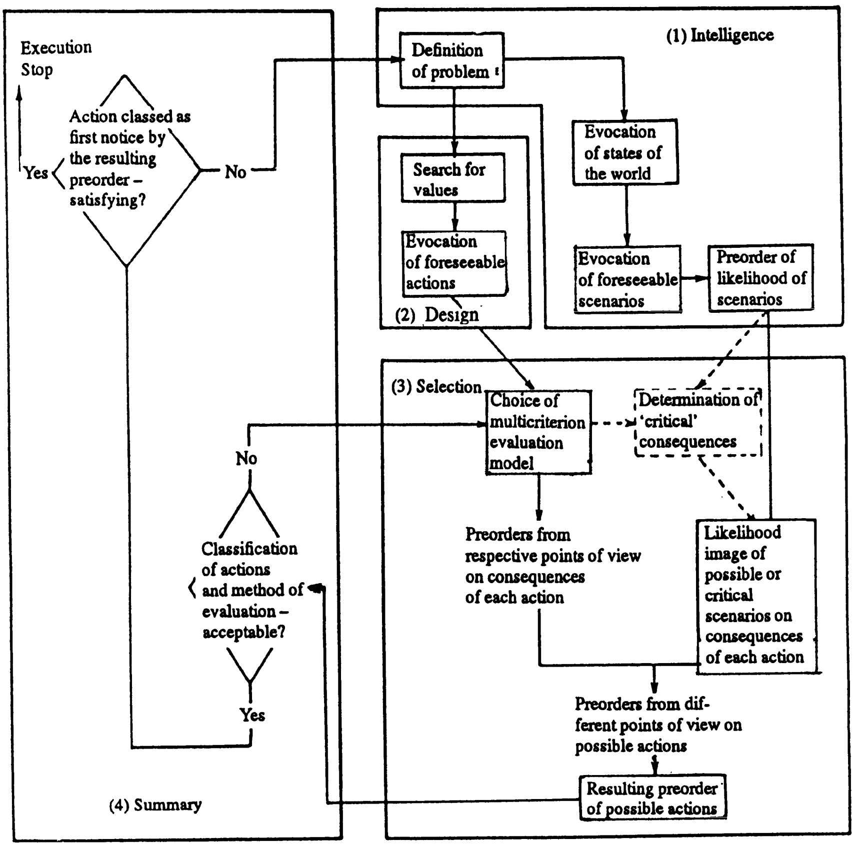


Fig. 2. ESIC: Complete process of strategic evaluation under complex uncertainty.

As long as he is not, he may run through a new stage following the *long iteration loop* (see Figure 2).

(b) The real virtues of the multicriterial procedures are demonstrated here and confer their status on these evaluation models, which is that of methods of learning on preferences in *weak rationality*. This is why methods such as Electre IV or, failing that, Electre III or II, seem perfectly appropriate (cf. Roy [19]).

(c) It will be seen that the treatment of uncertainty, proposed here for multicriterial choices, approaches the propositions of Fandel and Wilhelm [7]. But this one is an ordinal version of these propositions, which is more easily applicable because subject to much less restrictive hypotheses – or to demands for information.

	e_1	e_2
a_1	c_{11}	c_{12}
a_2	c_{21}	c_{22}

Fig. 3.

In our view, it is a simple method, but one which breaks a barrier for multicriterial methods based on outranking relations: up to now, no treatment under uncertain future has existed for these methods.

2. A Simple Example

Let us consider a very simple problem of choice of an action at some iteration[7] of a decision making process.

- Two actions are envisaged (a_1: status quo, a_2: risked innovation) depending on two scenarios e_1 and e_2 (Figure 3).
- A single point of view is judged sufficient (a particular case of the preference relation in standard utility theory) and allows the classification of four possible conditional consequences, c_{ij} $(i, j = 1, 2)$.
- The complete preference preorder on the c_{ij}s has a graph $G_{\leq}$ which can be represented as:

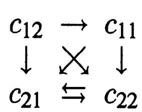
$$c_{21} \longrightarrow c_{12} \longrightarrow c_{11} \longrightarrow c_{22}.$$

- Likewise in applying the propositions P_1–P_4 the information on likelihoods can thus be represented:

$$e_1 \longrightarrow e_2$$

We are going to apply the proposed algorithm and show that it describes *reasoned prudence for the class of problems studied here and defined* in Section II.B above.

G^{a_1} (likelihood image of a_1) has a graph which can be shown diagrammatically as follows:



and G^{a_2} (likelihood image of a_2):

$$\begin{array}{ccc} c_{22} & \rightarrow & c_{21} \\ \downarrow & \times & \downarrow \\ c_{11} & \leftrightarrows & c_{12} \end{array}$$

Thus:

$$|G^{a_1} \Delta G_{\leq}| = 5 \quad \text{and} \quad |G^{a_2} \Delta G_{\leq}| = 7.$$

So the proposed criterion suggests that a_1 is selected since:

$$|G^{a_1} \Delta G_{\leq}| < |G^{a_2} \Delta G_{\leq}| \Leftrightarrow a_2 \;_A < a_1.$$

Let us suppose that the 'true' information, unknown to the decision maker, can be numerically represented in the following way for utilities:

$$c_{11} = -10, \quad c_{12} = +10, \quad c_{21} = +100, \quad c_{22} = -50.$$

$P(e_1)$, probability of scenario e_1, is now presumed to vary between 0 and 1.

It can be seen immediately from Figure 4 that the proposed criterion is a *criterion of intermediate prudence between the Maxmin criterion and the criterion of expected utility.*[8]

The Maxmin criterion supposes that neither probabilities nor likelihoods are known and practices a 'blind prudence'. Conversely, the criterion of expected utility supposes that likelihoods and probabilities are known perfectly. Our criterion, which we will call the criterion of parallel preorders, lies between these two criteria and is seen as a criterion of 'reasoned prudence'.

'Critical' Scenarios

We can easily show situations in which the decision maker will be induced to choose an action a_1, in preference to an action a_2 although:

- the most favorable scenario is dominated, from the point of view of likelihood, by another scenario if the action a_1 in question is chosen;
- the most favorable scenario is also the most likely if the action a_2 is chosen.

For example, scenarios have consequences that can be ordered as:

$$c(s_4) \leq c(s_3) \leq c(s_2) \leq c(s_1)$$

Structure of Likelihood	$P(e_1)$	Action selected by the criterion		
		of Maxmin	of parallel preorders	of expected utilities
	0			
		a_1	a_1	a_1
	6/17			
$e_1 \leq_l e_2$		a_1	a_1	a_2
	0.50			
$e_2 \leq_l e_1$		a_1	a_2	a_2
	1			

Fig. 4.

and their likelihoods are ordered as:

$$s_4 \leq_1 s_3 \leq_1 s_1 \leq_1 s_2 \qquad \text{for } a_1$$
$$s_3 \leq_1 s_4 \leq_1 s_2 \leq_1 s_1 \qquad \text{for } a_2.$$

We obtain (the index of time t remains implicit):

$$|G \leq \Delta G^{a_1}| = 2 \quad \text{and} \quad |G \leq \Delta G^{a_2}| = 3.$$

So the criterion of parallel preorders leads us to select, at the given iteration t, the action a_1.

In the course of the summary phase the decision maker may find the order $a_2 \; {}_A{\lesssim}_t \; a_1$ unacceptable because it makes the obtaining of $c(s_1)$ extremely important and is strongly affected by the prospect of $c(s_4)$ happening (risk of ruin, for example).

In the course of the summary phase he will establish this and take the *short iteration loop*. Instead of questioning the multicriterial procedure, however, it is the algorithm of choice under uncertainty that he will modify. Since his preferences are concentrated on certain 'critical' scenarios, he could limit the use of the algorithm in the following iteration, to the subset $\hat{C}_{t+1} \subset C_{t+1}$ composed only of 'critical scenarios' of the decision problem, say s_1 and s_4, where s_4 represents risk of 'ruin'.

This latter logic, which remains to be further explored rigorously, should enable us to make new connections with the problems raised by the famous 'Allais' paradox'.

Having started by questioning Postulate 6 (continuity postulate) of Savage, we are here discussing the criticism of Postulate 2 (independence postulate or sure thing postulate).

III. SOME IMPORTANT PROPERTIES OF THE CRITERION

It is important to prove that the properties of the criterion of parallel preorders fully support the usage recommended in this essay and in particular the *iterative* usage and the *preordinal* classification that we wish to obtain on A at each iteration.

In this respect the following properties are interesting:

Property A. Under certain conditions ${}_A\lesssim_t$ is a complete preorder on A_t.

Property B. The criterion of parallel preorders provides independent results to possible solutions a, not taken into consideration up to now (independence of irrelevant alternatives).

This property B, in fact, at successive iterations and with the progressive expansion of set A as t increases, means that we are not constrained to reconsidering the total classification obtained over A_t if, at the end of instant t we have judged this classification to be acceptable. We will see that it is liable to more restrictive conditions than the preceding property.

Properties A and B are established in theorem form in the Appendix.

IV. CONCLUSION

In matters of strategic decision under uncertainty the complex character of uncertainty does not permit a direct application of the standard algorithm of mathematical neo-Bernoullian theory. In the same way as Herbert Simon showed, so we should resort to a *process* of decision of which he has described four essential phases, repeated in the ESIC model proposed above.

We have shown that the selection phase of this process can and must, whenever possible, depend on a procedure of algorithmic classification, on the condition that this is regarded as a simple aid to the thought process, the parametrization of which should allow certain degrees of freedom.

We have also shown that the mathematical theory of decision helps the conception of a *new algorithm*, which we will call 'algorithm of parallel orders' adapted to the class of problems of strategic decision '*status quo* or risky innovation' and on which the process of strategic decision can be made to depend.

This algorithm represents a generalization of Savage's algorithm in two ways:

- the evaluation can perfectly well be *multicriterial* (weak rationality, need for learning process);
- the concept of probability is replaced by the more general concept of *likelihood* (complexity, qualitative treatment).

The design of this algorithm of selection which transforms 'reasoned prudence' in matters of strategic decision under uncertainty, also enables us to show that the *multicriterial methods based on an outranking relation are amenable to an integrated treatment of uncertainty.*

Finally, ESIC should be much more easily accepted by users from both private and public sectors, for all the reasons shown. In the type of *status quo* situation versus risky innovation described above, it should even be appreciated for some answers it gives to questions which have hitherto been unresolved.

APPENDIX

THEOREM A. *For a given point of view i, the criterion of parallel orders induces a complete preorder on the set A_t.*

Proof. The cardinal of the symmetric difference between two preorders is a particular distance between these two preorders. We can put:

$$|G_t^a \Delta G_{\leq}^i| = d(G_t^a, G_{\leq}^i).$$

From thence:

$$a \;_A\!\lesssim_t^i a' \Leftrightarrow d(G_t^{a'}, G_t^i) \geqq d(G_t^a, G_t^i)$$

and $_A\!\lesssim_t^i$ is clearly the inverse image of the natural order $\geqq$ on $\mathbb{R}$ induced by the application d.

COROLLARY. *From the time that the multicriterial procedure g_t is defined in such a way that it produces a preorder from m_t complete preorders, we know that $_A\!\lesssim_t$ will be a predetermination on A_t.*

As an example, the conditions that g_t produce a complete preorder $_A\!\lesssim_t$ based on m_t complete preorders $_A\!\lesssim_t^i$ when the chosen procedure is that of Electre II are as follows: in denoting C_k the 'concordance' set of value k and D_l the 'discordance' set of value l in $A_t \times A_t$ with $I_{kl} = C_k \cap \bar{D}_l$, the outranking relation R should be defined by:

$$aRa' \Leftrightarrow:$$

- $a = a'$
- or $(a, a') \in I_{kl}$
- or there is a series $a_1, a_2, \ldots, a_n$, having each one of its elements in A_t with
  $(a, a_1) \in I_{kl}$
  $(a_1, a_2) \in I_{kl}$
  $\vdots$
  $(a_{n-1}, a_n) \in I_{kl}$
  $(a_n, a') \in I_{kl}$

The proof of this theorem is obtained simply by verifying that with such a definition, aRb and bRc imply aRc, in each of the nine possible cases.

In this case, the outranking relation R is necessarily a preorder on A_t.

Important Note. The preorder $_A\!\lesssim_t$ can be an incomplete preorder on A_t and retain cases of incomparability.

It is possible (by an application of rank, for example) to complete this preorder. But then a totally artificially piece of information is implicitly introduced into the classification. Also we will prefer to take repeated short and/or long iteration loops until we acquire a better understanding of the problem and that a piece of information, other than gratuitous enables us to make ${}_A{\lesssim}_t$ complete on A_t.

In matters of theory of decision we are faced here, in a concrete and specific way with the difference between a procedural point of view and a purely algorithmic point of view.

THEOREM B. *The relationship $a\ {}_A{\lesssim}_t\ a'$ between two actions $a \in A$ is independent of the relationships between the actions a_1, a_2 etc. ($a_1 \neq a, a'$ and $a_2 \neq a, a'$) in A_t.*

Proof. Let us consider the following partition of $C_t \times C_t$:

$$\begin{aligned} C_t \times C_t &= \left\{(G_t^a \cap G_t^{a'}) \cup (G_t^a \Delta G_t^{a'}) \cup \overline{(G_t^a \cup G_t^{a'})}\right\} \\ &= \{D_1 \cup D_2 \cup D_3\} \end{aligned}$$

where $\overline{(*)}$ stands for the complement of $(*)$ in $C_t \times C_t$.

One can write:

$$\begin{aligned} G_t^i \Delta G_t^a &= \left\{(G_t^i \cap D_1) \Delta (G_t^a \cap D_1)\right\} \\ &\quad \cup \left\{(G_t^i \cap D_2) \Delta (G_t^a \cap D_2)\right\} \\ &\quad \cup \left\{(G_t^i \cap D_3) \Delta (G_t^a \cap D_3)\right\}. \end{aligned}$$

and a fully similar equality enables us to express $G_t^i \Delta G_t^{a'}$. But,

$$G_t^a \cap D_3 = G_t^{a'} \cap D_3 = \emptyset$$

and

$$G_t^a \cap D_1 = G_t^{a'} \cap D_1 = D_1.$$

The expressions $(G_t^i \Delta G_t^a)$ and $(G_t^i \Delta G_t^{a'})$ and their respective cardinals therefore do not differ on $D_1 \cup D_3$ but only on $D_2 = (G_t^a \Delta G_t^{a'})$, which clearly depends on nothing other than the structure of likelihood in admissible scenarios for a and a' independently of all actions a_1, a_2, etc., other than a and a'.

COROLLARY. *When there is a unique criterion (utility relation) Theorem B is always true. When there is a family I_t of criteria i, property B is true only if*

(a) $\forall t$, $I_t = I$ *(family of invariable criteria)*;
(b) $\forall t$, *the procedure* g_t: $\{\precsim_t^i\} \mapsto \{\precsim_t\}$ *proves independence with respect to irrelevant alternatives.*

These last two conditions are clearly restrictive.

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NOTES

[1] No language of observation is neutral: cf. Kuhn, *The Structure of Scientific Revolution.*
[2] The multicriterial models of the French school (Roy, Jacques-Lagreze) constitute, without doubt, a good example of this third attitude concerning algorithms for decision aid under certainty. In an uncertain universe the works of Fourgeaud–Lenclud–Sentis (1970–1971) or those of Michael Rizzi (1981) belong to the same current of thought. The latter approach has nevertheless serious drawbacks for the treatment of 'subjective probabilities' which violate the second proposition of Savage and which are therefore not probabilities in the usual sense, from which a certain confusion and risk of error arise during the use of probability calculations.
[3] We refer here to the order of these propositions in the collected work of Savage (1954). For a 'pedagogical' exposé of these axioms cf. our article 'Quelque Critiques de la Rationalité Economique dans l'Incertain', *Revue Economique* **35**(1), January 1984, 65–86.
[4] Certain authors have strongly emphasized that the work of Savage "had not so much consisted in establishing subjective probabilities from choices as in establishing numerical subjective probabilities from comparative subjective probabilities" (Anscombe and Aumann, 1963, p. 204). It is quite clearly stated that the proposition of continuity is the core of the work.
[5] The set of states of the world is of very high cardinal value even allowing that the set is finite. The set of admissible states is generally smaller. The difficulty in selecting the admissible states of the world has been strongly emphasized by Shackle [22, pp. 20–21] in his critique of subjective probabilities. We cannot here avoid this very specific criticism. This is why we should emphasize the absolute necessity of the time concept to reinforce intelligence (in the Anglo-Saxon sense) in the process of decision (cf. diagram above).
[6] Concerning the same iteration, the index t of this instant avoided in this paragraph, without risk of confusion.
[7] Figure 4 is based on the hypothesis $P(e_1) \leqq P(e_2) \Leftrightarrow e_1 \precsim e_2$. We can only accept this working hypothesis.

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12. THE GENERALIZED IRREVERSIBILITY EFFECT

The recent developments in environmental economics have often been based on the study of decisions concerning irreplaceable assets. These are defined as goods which are impossible to reproduce and which have no near substitute satisfying the same need. The Grand Canyon, the forest of Fontainebleau or Nôtre Dame de Paris are examples of irreplaceable assets. While the possibility exists of using these irreplaceable assets in a way that destroys them, as input in a production process, for example, specific problems of economic calculations are raised. In fact, the destruction of the asset creates a utility uniquely in the present, whereas keeping it determines:

1. a level of utility generally inferior during the present period; and
2. a superior future utility evaluated by taking into account the possibility of destruction of the irreplaceable asset in the future.

Consequently, apart from the classical problems of all dynamic models (choice of rate of discounting) a supplementary difficulty arises. The destruction of the irreplaceable asset reduces the total of all possible actions in the future. In fact this constitutes an irreversible decision in the sense that it is impossible to retrieve the initial asset. On the other hand, when there is conservation it will always be possible to go back afterwards on a decision taken at the present time and to assign to the asset a use that destroys it.

The study of the optimal level of an irreplaceable asset has been carried out by Arrow and Fisher [1] and by Henry [4, 5] and has resulted in evidence of an irreversibility effect. This concerns decisions involving uncertainty, a hypothesis natural here since the chosen model affecting the irreplaceable model requires a dynamic context, and therefore imperfect information concerning the future. It is said that there is an irreversibility effect when the fact of taking account of better information in the future has the effect of increasing the optimal level of irreplaceable assets, i.e. to favor its conservation. The fundamental result established by Henry has shown that under not very restrictive hypotheses the irreversibility effect will occur.

B. Munier (ed.), Markets, Risk and Money, 233–248. *Original* © *Economica (Paris)* 1986. *This English translation* © 1995 *Kluwer Academic Publishers.*

Consequently, this result calls into question the usual diagrams of economic calculations applied to irreversible decision. In fact the maximization of expected social utility leads to decisions which are not only erroneous, but even more biased in the same direction. The calculation with expectations leads to too great a destruction of the irreplaceable asset. A rigorous economic calculation should be more specific and take account of information which would be accessible in the future.

Given the consequences of the irreversibility effect, it seems natural to study its validity in a more general context. This is the aim of our article.

First, we will assume that the present utility increases with the quantity of irreplaceable asset destroyed. On this point our model is more general, since the preceding demonstrations of the irreversibility effect [5, 3] do not take account of the difference between the values of two successive assets. Thus, for example, they were not able to consider a mining resource as an irreplaceable asset.

Second, we will generalize the irreversibility effect for nonrenewable assets. Consequently, this effect should be taken into consideration in the study of natural resources, of which the reproduction is not controlled (forests, populations of certain kinds of fish, etc.).

Finally, it seems to us that our work allows a better understanding of the hypotheses necessary to the existence of the irreversibility effect.

The following section (I) describes the model used and gives an initial idea of the necessary and sufficient conditions for the irreversibility effect which is the basis of our analysis. Section II established a general view of the irreversibility effect. Finally, in Section III we have shown that the hypotheses of quasi-concavity, necessary to obtain our result, cannot be impaired.

I. THE MODEL

We will consider a model with two periods, the present and the future. We will denote the level of the nonrenewable asset at time t, by D_t, $t = 1, 2$; and $\bar{D}$ will be the level of this available asset in period 1, the result of decisions taken in the past.

The nonrenewable character of the asset is recognized by the fact that the available quantity at each period is the unique result of the quantity of asset which has not been destroyed in the course of the preceding period. Thus, from the quantity $\bar{D}$, the quantity $\bar{D} - D_1$ will be consumed whereas the level D_1 will be retained and will determine the level of the available asset in the subsequent period, which we will denote by $\Phi(D_1, \theta)$ where θ is a random variable.

This general formulation enables us to take account of several types of phenomena. Firstly, it enables us to take into account irreplaceable assets such as natural or historic sites, in which case the level of the asset at the beginning of the second period is that remaining from the first period:

$$\Phi(D_1, \theta) = D_1.$$

Afterwards, this formulation enables us to deal with the case of a stable Malthusian population increasing at a rate of $\alpha - 1$, in which case:

$$\Phi(D_1, \theta) = \alpha D_1, \quad \alpha > 0.$$

Finally, the parameter θ enables us to integrate chance into the increase of the nonreproducible asset. Its distribution will be presumed to be independent of D_1.

Further, we will suppose that for a given value of θ, the function $\Phi(D_1, \theta)$ is strictly increasing in D_1, so that any increase in D_1 will lead to an increase in the level of the available asset in period 2.

The introduction of this random factor enables us to take account of imperfect information on the level of the asset D_1. Thus, for a mining resource, D_1 can be the value of the known reserves and this value can become $D_1 + \theta$ for the next period. The expectation $E(D_1 + \theta)$ constitutes an evaluation of existing reserves of the resource.

The destruction of the irreplaceable asset enables us to obtain a quantity of product:

$$Y_1(\bar{D} - D_1) \text{ for period 1}$$

and

$$Y_2\left(\Phi(D_1, \theta) - D_2, \omega\right) \text{ for period 2}$$

where ω is a random variable reflecting the state of the economy. The two functions Y_t are increasing and quasi-concave.

The additive intertemporal function is the sum of a certain function $U^1(Y_1(\bar{D} - D_1), D_1)$ measuring the utility of the present period and of a random function $U^2(Y_2(\Phi(D_1, \theta) - D_2, \omega)D_2, \omega)$ representing the future utility when the state of the economy ω again influences the determination of the value of the utility in the second period. We note that $\tilde{U}^2$ possibly contains a discount factor. In each of the two periods the utility function in increasing in its first two variables and is strictly quasi-concave. Further, we will suppose that $\tilde{U}^2$ is continuously differentiable in its first two variables with nonzero derivatives, and that the marginal rate of substitution between the consumed asset and the remaining asset at the point $(Y_2(0,\omega), D_2)$ decreases with D_2:

$$\frac{\frac{\partial \tilde{U}}{\partial D_2}(Y_2(0,\omega), D_2)}{\frac{\partial \tilde{U}}{\partial Y_2}(Y_2(0,\omega), D_2)}$$

is a decreasing function of D_2.

This hypothesis means that when the value of the asset D_2 increases, the public decision maker is ready to sacrifice marginally more of this asset in order to obtain a first unit of the asset to be consumed.

Later on, we will simplify the notation by elimination of explicit reference to the functions Y_i and writing:

$$U^1(Z_1, D_1) \quad \text{and} \quad U^2(Z_2, D_2, \omega)$$

where Z_i is the part of the asset destroyed by consumption in period i. U^1 and U^2 are quasi-concave functions since functions Y_1 and Y_2 are themselves quasi-concave. The total number of possible states which the variable ω can take will be denoted Ω, and $(\Omega, \mathcal{F}, \mu)$ will be a probability space. Furthermore, for each value of the variables D_2 and Z_2, $U^2(Z_2, D_2, \omega)$ will be presumed μ-measurable.

Suppose that the decision maker receives a piece of information before choosing the value of the nonrenewable asset for the second period. Two information structures will be represented by $\mathcal{F}$-measurable functions, ρ and ν.

$$\rho\colon\ (\Omega, \mathcal{F}, \mu) \to (E\rho, \mathcal{B}_\rho)$$
$$\nu\colon\ (\Omega, \mathcal{F}, \mu) \to (E\nu, \mathcal{B}_\nu).$$

ν will afterwards represent 'better information', an idea that we will define later on (Section II).

An element x (and similarly y) corresponding to a value $x = \rho(\omega)$ (respectively $y = \nu(\omega)$ will be observed, then D_2 will be chosen in such a way as to maximize the conditional expectation of U^2.

$$\max \left\{ E^{\rho=x} U^2(Z_2, D_2, \omega) \mid Z_2 + D_2 = Y(D_1, \theta), D_2 \geq 0, Z_2 \leq 0 \right\}. \tag{1.1}$$

We denote $V^2(\rho, x, D_1, \theta)$ the optimum value of the above program. $V^2(\rho, x, D_1, \theta)$ is an indirect utility function. Elsewhere we will put:

$$V^2(\rho, D_1) = E_{x,\Theta}\left[V^2(\rho, x, D_1, \theta)\right]$$

which enables us to write the problem of choice for the first period as:

$$\max_{0 \leq D_1 \leq \bar{D}} U^1(D_1) + V^2(\rho, D_1) \tag{1.2}$$

for the information structure ρ, and:

$$\max_{0 \leq D_1 \leq \bar{D}} U^1(D_1) + V^2(v, D_1) \tag{1.3}$$

for the information structure ν.

Let us suppose that the solutions to these two problems are unique, denoting them by $\hat{D}_1$ and $\hat{D}_1'$, respectively.

The existence of the irreversibility effect implies that $\hat{D}_1' \geq \hat{D}_1$ when ν corresponds to a better information than ρ.

Firstly, let us suppose that the functions U^1 and V^2 are differentiable in D_1. We will then obtain first conditions:

$$U^1_{D_1}(\hat{D}_1) + V^2_{D_1}(\rho, \hat{D}_1) = 0 \tag{1.4}$$

$$U^1_{D_1}(\hat{D}_1) + V^2_{D_1}(\nu, \hat{D}_1) = 0. \tag{1.5}$$

The strict quasi-concavity of the objective function (1.3) implies:

$$U^1_{D_1}(D_1) + V^2_{D_1}(\nu, D_1) > 0 \quad \text{for} \quad D_1 < \hat{D}_1.$$

Consequently, if we want to show that $\hat{D}_1' \geq \hat{D}_1$, it is equivalent to establishing that:

$$U^1_{D_1}(\hat{D}_1) + V^2_{D_1}(\nu, \hat{D}_1) \geq 0 \tag{1.6}$$

or, using condition (1.4):

$$(1.7) \quad V^2_{D_1}(\nu, \hat{D}_1) - V^2_{D_1}(\rho, \hat{D}_1) \geq 0.$$

Thus, the irreversibility effect happens if, and only if, the function $V^2(\nu_1, D_1') - V^2(\rho, D_1)$ is locally increasing at the point $\hat{D}_1$. The condition (1.7) allows an interesting economic interpretation. Indeed, $V^2(\nu, D_1') - V^2(\rho, D_1)$ is the marginal value of the information at point D_1; the established condition (1.7) means then that the marginal value of information must increase with D_1. So this implies that the information and the irreplaceable asset are complementary goods, since an increase in one increases the marginal utility of the other. Thus, *the irreversibility effect is verified each time that the information and the irreplaceable asset are complementary goods.*

We will show here that this complementarity property is satisfied as soon as the utility expectation functions, conditioned by the difference information received (x or y) are quasi-concave functions.

In the calculations below we will not assume unique solutions the problems (1.2) and (1.3). Consequently, the irreversibility effect will be brought into play for the minimum and maximum values in the set of solutions to these problems. Let us denote respectively $\hat{D}_1$, $\hat{\hat{D}}_1$, the maxima of the sets of solutions to (1.2) and (1.3) and $\underline{\hat{D}}_1$, $\underline{\hat{\hat{D}}}_1$, the minima of the same sets.

The irreversibility effect then implies that we have simultaneously:

$$\underline{\hat{D}}_1 \leq \underline{\hat{\hat{D}}}_1, \quad \text{and} \quad \hat{D}_1 \leq \hat{\hat{D}}_1.$$

II. INCREASING INFORMATION AND IRREVERSIBILITY

Let us begin by specifically defining the concept of better information. In order to do this we will use a definition which generalizes those of Marschak and Radner [8]. We will denote by $\rho(\mu)$ and $\mu(\mu)$ the probability measures which are the images of μ by ρ and ν, respectively.

DEFINITION. ν is finer than ρ if there is a function δ, $\nu(\mu)$-measurable, defined on E and such that $\delta(\nu) = \rho$, and, if for all x, $x \in \mathcal{B}_\rho$ the measure of conditional probability $\nu(\mu(* \mid \rho = x))$ defined on $E_\nu, \mathcal{B}_\nu)$ exists.

ν corresponds to a better structure of information than ρ, if it is finer than ρ. In fact, for each signal y having *a priori* a nonzero probability of being received, the knowledge of y, corresponding to the information structure ν, enables us to identify the signal $\delta(y)$ which would have been received with the information structure ρ.

The definition of better information proposed by Blackwell is based on the conditional distributions.

DEFINITION. Let $\{B_j\}, j = 1, \dots, N$ be a family of parts of Ω which produces a σ-algebra $\mathcal{F}$ and let $\tilde{x}_\rho$ and $\tilde{x}_\nu$ be two correlated random variables, then $\tilde{x}_\nu$ is sufficient for $\tilde{x}_\rho$ if there is a stochastic matrix $Q = (q_{ij}), i = 1, \dots, n, j = 1, \dots, m$, with

$$q_{i,j} \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} q_{i,j} = 1$$

such that

$$P_r(\tilde{x}_\rho = x_i \mid \omega \in B_j) = \sum_{k=1}^{m} q_{ik} P_r(\tilde{x}_\nu = x_k \mid \omega \in B_j)$$

where

$$P_r(\tilde{x} = x_i \mid \omega \in B_j)$$

is the probability that $\tilde{x} = x_i$ knowing that ω belongs to B_j. If $\tilde{x}_\nu$ is sufficient for $\tilde{x}_\rho$ then $\tilde{x}_\nu$ constitutes a better information structure than $\tilde{x}_\rho$.

When $\mathcal{F}$ is a finite set, it is possible to show that the two definitions above are equivalent (see, for example, Green and Stokey [2]). The following proposition is deduced from Blackwell's theorem in the framework of our model.

PROPOSITION 1. *Let ν be finer than ρ. Then*

$$V^2(\nu, D_1) \geq V^2(\rho, D_1) \quad \textit{forall } D_1.$$

The proof is identical to that of Marschak and Radner [8, pp. 54–55].

Let us define the function

$$\Delta(D_1, \nu, \rho) = V^2(\nu, D_1) - V^2(\rho, D_1).$$

The preceding proposition assures us that the function $\Delta(*, \nu, \rho)$ is positive. For a fixed value of D_1, $\Delta(D_1, \nu, \rho)$ is the difference between the value of the information ν and the value of the information ρ. Δ is therefore interpreted as the value of additional information when we go from ρ to ν. The condition for the asset D_1 and the value of additional information to be complementary therefore implies here that $\Delta(D_1, \nu, \rho)$ should be an increasing function of D_1. The following lemma shows, under a condition of quasi-concavity of conditional expectation functions, that Δ is an increasing function of D_1. Denoted by $J(D_2, D_1, x, \theta)$ the conditional expectation is defined as follows:

$$J(D_2, D_1, x, \theta) = E^{\rho=x} U^2 \left(\Phi(D_1, \theta) - D_2, \omega\right).$$

PROPOSITION 2. *If $J(D_2, D_1, y, \theta)$ is quasi-concave in D_2 for all D_1, y and θ, then $\Delta(D_1, \nu, \rho)$ is an increasing function of D_1 for all information structures ν finer than ρ.*

Proof. See Appendix.

The above proposition enables us to establish a generalization of the irreversibility effect. As $\hat{D}_1$ is the maximum solution to the problem (1.2) and $\hat{D}_1'$ is the maximum solution to the problem (1.3) we get:

$$U^1(\hat{D}_1) + V^2(\rho, \hat{D}_1) \leq U^1(\hat{D}_1) + V^2(\rho, \hat{D}_1) \tag{2.1}$$

and

$$U^1(\hat{D}_1) + V^2(\nu, \hat{D}_1) \leq U^1(\hat{D}_1) + V^2(\nu, \hat{D}_1). \tag{2.2}$$

Adding these two formulae, we obtain after simplification:

$$V^2(\nu, \hat{D}_1) - V^2(\rho, \hat{D}_1) \leq V^2(\nu, \hat{D}_1) - V^2(\rho, \hat{D}_1)$$

or

$$\Delta(\hat{D}_1, \nu, \rho) \leq \Delta(\hat{D}_1, \nu, \rho) \tag{2.3}$$

from which we deduce the following theorem.

THEOREM. *If the function $J(D_2, D_1, y, \theta)$ is quasi-concave in D_2 for all D_1, y and θ then the irreversibility effect is verified:*

$$\underline{\hat{D}}_1 \le \underline{\hat{\hat{D}}}_1 \quad \text{and} \quad \hat{D}_1 \le \hat{\hat{D}}_1.$$

Proof. Using the equations above, the proof of $\hat{D}_1 \le \hat{\hat{D}}_1$ follows immediately. In fact, suppose that $\hat{D}_1 > \hat{\hat{D}}_1$: as Δ is increasing according to Proposition 2, (2.3) is satisfied by equality, and consequently (2.1) and (2.2) are also satisfied by equality, but in this case $\hat{D}_1$ is the solution to the problem (1.3) on the same grounds as $\hat{\hat{D}}_1$. And this contradicts the fact that $\hat{D}_1$ is the maximum solution. The proof is the same for the minimum solutions $\underline{\hat{D}}_1$ and $\underline{\hat{\hat{D}}}_1$.

The above theorem constitutes a generalization of the irreversibility effect in the case of nonrenewable assets. The hypotheses are not very restrictive, but it is natural to wonder if it is possible to diminish them. The section above shows, with the aid of counterexamples that the irreversibility effect cannot be satisfied by weaker hypotheses.

III. THE LIMITS OF THE IRREVERSIBILITY EFFECT

The theorem we have established in the preceding section constitutes a sufficient condition for the validity of the irreversibility effect. It is clear that these conditions are not necessary since, if they were not satisfied for unlikely natural states, the irreversibility effect would be satisfied just the same. However, we will show here that it is not possible to obtain a more general result so that the hypotheses are the weakest possible given the structure of our model.

We will first construct an example to show that in the absence of quasi-concavity, the irreversibility effect is invalidated. Using Equations (2.1) to (2.3) of Section II shows that for the irreversibility effect to be invalidated it is sufficient to obtain a function $\Delta(D_1, \nu, \rho)$ locally decreasing in D_1 at the optimal point $\hat{D}_1$.

Considering the following example with two sates of nature ω_1 and ω_2 of respective probabilities $\nu(\{\omega_1\}) = p_1$ and $\mu(\{\omega_1\}) = p_2$.

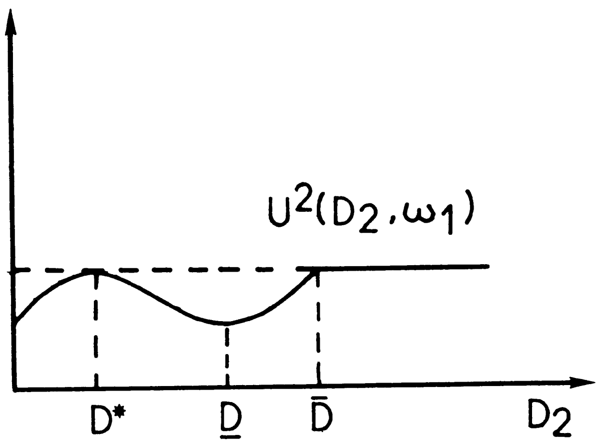

Fig. 1.

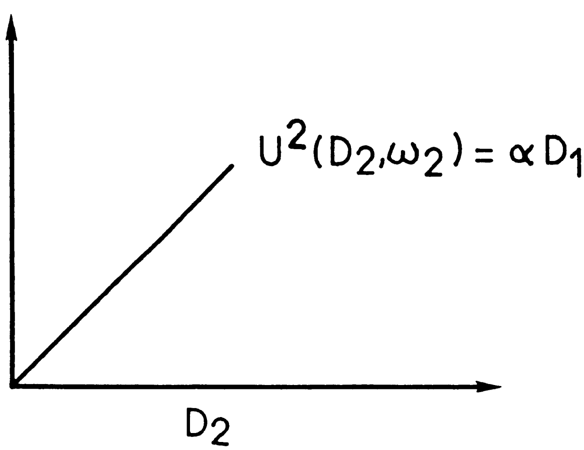

Fig. 2.

We will not specify the analytic form of the functions $U^2(\cdot, \omega_i$, but only their graphical representation.

$U^2(\cdot, \omega_i)$ does not satisfy the hypothesis of quasi-concavity. We will consider two information structures for the second period. ν corresponds to the perfect information structure with:

$$\nu(\omega_1) \neq \nu(\omega_2)$$

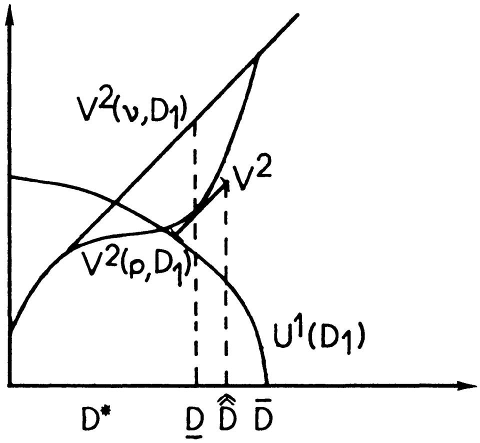

Fig. 3.

and ρ corresponds to the absence of information:

$$\rho(\omega_1) = \rho(\omega_2).$$

The optimal correspondences $\hat{D}_2(D_1, \rho)$ and $\hat{D}_2(D_1, \nu)$ are easily obtained. For low p_1 or high α we have:

$$\hat{D}_2(D_1, \rho) = \{D_1\} \tag{3.1}$$

Elsewhere the optimal solutions, when the information is complete, are given by

$$\begin{aligned} \hat{D}_2\left(D_1, \nu(\omega_1)\right) &= \{D_1\}, 0 < D_1 < D^* \\ \hat{D}_2\left(D_1, \nu(\omega_1)\right) &= \{D^*\}, D^* < D_1 < \bar{D} \\ \hat{D}_2\left(D_1, \nu(\omega_1)\right) &= \{D^*\}u \mid \bar{D}, D_1 \mid \bar{D} < D_1 \\ \hat{D}_2\left(D_1, \nu(\omega_2)\right) &= \{D_1\}. \end{aligned} \tag{3.2}$$

From expressions (3.1) and (3.2) we can calculate the values of $V^2(\nu, D_1)$ and $V^2(\rho, D_1)$ as they appear in Figure 3.

At the point $\underline{D}$ the slope of $V^2(\nu, D_1)$ is identical to that of $V^2(\rho, D_1)$ and for D_1 between $\underline{D}$ and $\bar{D}$, the slope of $V^2(\rho, D_1)$ is greater. This can be expressed as:

$$V^2_{D_1}(\nu, D_1) - V^2_{D_1}(\rho, D_1) > 0.$$

If $\hat{\hat{D}}_1$ belongs to the interval $]\underline{D}, \bar{D}[$, which is the case for the function U^1 in Figure 3, we have:

$$U^1_{D_1}(\hat{\hat{D}}_1) + V^2_{D_1}(\rho, \hat{\hat{D}}_1) > 0$$

and therefore, using first-order conditions of the type (1.5):

$$V^2_{D_1}(\rho, \hat{\hat{D}}_1) - V^2_{D_1}(\nu, \hat{\hat{D}}_1) > 0,$$

or $\hat{D}_1 > \hat{\hat{D}}_1$, which contradicts the irreversibility effect.

Consequently, the hypothesis of quasi-concavity is crucial to obtaining the irreversibility effect. It is this hypothesis, in fact, which creates the situation where the value of the irreplaceable asset is complementary to the information obtained.

CONCLUSION

Our analysis has enabled us to generalize the irreversibility effect, shown up to now for irreplaceable assets, to the case of nonrenewable assets. Consequently, in this case also, the economic calculations should take into account explicitly future information and eliminate the use of certainty equivalents which constitute a possible source of error in the allocation of resources [4].

The hypotheses which enable us to obtain a general irreversibility effect are essentially those used to demonstrate it in the case of irreplaceable assets. Nevertheless, we have had to add a condition of growth of the marginal rate of substitution with the nonrenewable asset when it is integrally retained, which constitutes a supplementary hypothesis used in the proof of Lemma 1. But from the economic point of view, this local condition is not a very important restriction.

APPENDIX

Proof of Proposition 1. Before proceeding to the proof of Proposition 1, it is necessary to establish Lemma 1 shown below. Let

$$J(D_2, D_1, x, \Theta) = E^{\rho=x} U^2 \left(\Phi(D_1, \theta) - D_2, D_2, \omega\right)$$

and let $\hat{D}_2(D_1, x, \theta)$ be the set of values of D_2, $0 \leq D_2 \leq \Phi(D_1, \theta)$, which maximize $J(D_2, D_1, x, \theta)$.

LEMMA 1. *If $J(D_2, D_1, x, \theta)$ is a quasi-concave function for all D_1, x and θ then one and only one of the conditions below is satisfied:*

(I) $\forall \varepsilon > 0, \forall D_2 \in \hat{D}_2(D_1 + \varepsilon, x, \theta), D_2 < \Theta(D_1 + \varepsilon, \theta)$
(II) $\exists \alpha > 0, \forall \varepsilon \in [0, \alpha], \Theta(D_1 + \varepsilon, \theta) \in \hat{D}_2(D_1 + \varepsilon, x, \theta)$

Proof. Let us say first of all that if (I) is satisfied, then (II) cannot be. Also, the proof will consist in showing that if (i) is invalidated (II) must be satisfied. If (I) is not satisfied that means that there is $\alpha > 0$ such that $\Phi(D_1 + \alpha, \theta) \in \hat{D}_2(D_1 + \varepsilon, x, \theta)$, which is to say that $D_2 = \Phi(D_1 + \alpha, \theta)$ is the solution to

$$\begin{cases} \max E^{\rho=x} U^2(Z_2, D_2, \omega) \\ Z_2 + D_2 \leq \Phi(D_1 + \alpha, \theta). \end{cases}$$

This problem (formally equivalent to that of a consumer whose income is $\Phi(D_1 + \alpha, \theta)$ when the price system is $(1, 1)$), only allows a solution $D_2 = \Phi(D_1 + \alpha, \theta)$ if the marginal rate of substitution between the good D_2 and the good Z_2 at the point $(0, \Phi(D_1 + \alpha, \theta))$ is greater than or equal to 1. Now, according to the hypothesis on the utility function U^2, for $\varepsilon > 0$, $0 < \varepsilon < \alpha$, the marginal rate of substitution at the point $(0, \Phi(D_1 + \varepsilon, \theta))$ is also greater than or equal to 1. But, as the objective function is quasi-concave, this implies that $\Phi(D_1 + \varepsilon, \theta)$ is also a solution, and (II) is satisfied.

Using the above Lemma, we can establish Proposition 1 which we will now state.

PROPOSITION 1. *If $J(D_2, D_1, x, \theta)$ is quasi-concave in D_2 for all D_1, x and θ then $\Delta(D_1, \nu, \rho)$ is an increasing function of D_1 for all information structure ν finer than ρ.*

Proof. As ν is finer than ρ, we can use the conditional density of the signal y knowing w, and write

$$\Delta\ (D_1, \nu, \rho) = \int_{\theta} \int_{E_\rho} \int_{E_\nu} E^{\nu=y} \left\{ U^2 \left(\hat{D}^2(D_1, y), \omega \right) \right.$$

$$- U^2\left(\hat{D}^2(D_1, x), \omega\right)\Big\} \nu(\mu)(\mathrm{d}y \mid \rho = x)\rho(\mu)(\mathrm{d}x)\,\mathrm{d}\theta$$

where

$$E^{\nu=y}U^2\left(\hat{D}_2(D_1, y)\omega\right)$$

is the optimum value of U^2: knowing that signal y has been received:

$$E^{\nu=y}U^2\left(\hat{D}^2(D_1, y, \theta)\omega\right)$$

$$= E^{\nu=y}\left(U^2\left(\Phi(D_1, \theta) - \hat{D}_2, \hat{D}_2, \omega\right)\right)$$

for $\hat{D}_2 \in \hat{D}_2(D_1, y, \theta)$.

Using this decomposition of Δ, it appears that a sufficient condition for Δ to be increasing in D_1 is that for all x the functions:

$$\begin{aligned}\Delta(D_1, x) &= E^{\rho=x}\Big\{U^2\left(\hat{D}_2(D_1, y, \theta), \omega\right) \\ &\quad - U^2\left(\hat{D}^2(D_1, x, \theta), \omega\right)\Big\}\end{aligned}$$

should be increasing in D_1, $\forall x$, and $\forall\theta$ (in order to simplify the notation, the parameter θ does not appear in the function Δ). We will now show that this condition is indeed satisfied, as

$$\forall\varepsilon > 0, \Delta(D_1 + \varepsilon, x) - \Delta(D_1 - x) \geq 0.$$

For each D_1 and each signal x or $y_1 D_2$ can satisfy either condition (I) or condition (II) of the above Lemma. We will show that in all cases the function $\Delta(D_1, x)$ is increasing.

(a) Suppose firstly, that at point D_1 and for a signal x corresponding to the information structure ρ, condition (I) is satisfied: the constraint $D_2 \leq \Phi(D_1, \theta)$ is not saturated at the optimum. This implies that it is not saturated for the level of asset $D_1 + \varepsilon$, with $\varepsilon > 0$. There is therefore equality between the sets of solutions associated with level D_1 and those associated with level $D_1 + \varepsilon$:

$$\hat{D}_2(D_1, x, \theta) = \hat{D}_2(D_1 + \varepsilon, x, \theta).$$

It follows that:

$$E^{\rho=x}U^2\left(\hat{D}_2(D_1 + \varepsilon, x, \theta), \omega\right)$$

$$= E^{\rho=x} U^2 \left(\hat{D}_2(D_1, x, \theta), \omega \right)$$

and that

$$\begin{aligned}\Delta(D_1 \; + \; \varepsilon, x) - \Delta(D_1, x) \\ &= E^{\rho=x} \left\{ U^2 \left(\hat{D}_2(D_1 + \varepsilon, \nu(\omega), \theta), \omega \right) \right. \\ &\quad \left. - \; U^2 \left(\hat{D}_2(D_1, \nu(\omega), \theta), \omega \right) \right\}\end{aligned}$$

and this expression is always positive or zero since the set of all possible choices for D_2 is greater for $D_1 + \varepsilon$ than for D_1.

(b) Suppose now that condition (II) applies for D_1 and for signal x. Let y be the signal received when the information structure is ν. For D_1 and y, again either condition (I) or condition (II) of the Lemma is satisfied.

(b.1) If condition (I) is satisfied for D_1 and y, the constraint affecting D_2 is not saturated for all $\varepsilon > 0$, since we have:

$$\hat{D}_2 < \Phi(D_1 + \varepsilon, \theta) \quad \forall \hat{\hat{D}}_2 \in \hat{D}_2(D_1 + \varepsilon, y, \theta).$$

Consequently,

$$\hat{D}_2(D_1, y, \theta) = \hat{D}_2(D_1 + \varepsilon, y, \theta),$$

which implies that the terms in y of the difference $\Delta(D_1 + \varepsilon, y) - \Delta(D_1, y)$ cancel each other.

Furthermore,

$$\hat{\hat{D}}_2 \leq \Phi(D_1, \theta) < \Phi(D_1 + \varepsilon, \theta),$$

the quasi-concavity of U^2 implies

$$\begin{aligned}E^{\nu=y} U^2 \left(\hat{D}_2(D_1, y, \theta), \omega \right) &\geq E^{\nu=y} U^2 \left(\Phi(D_1, \theta), \omega \right) \\ &\geq E^{\nu=y} \left(\Phi(D_1 + \varepsilon, \theta), \omega \right).\end{aligned}$$

Consequently the difference

$$\begin{aligned}\Delta \; (D_1 + \varepsilon, y) - \Delta(D_1, y) \\ &= E^{\nu=y} \left\{ U^2 \left(\Phi(D_1, \theta), \omega \right) - U^2 \left(\Phi(D_1 + \varepsilon, \theta), \omega \right) \right\}\end{aligned}$$

is positive.

(b.2) If condition (II) is satisfied for D_1 and y, then:

$$\exists\alpha > 0, \forall\varepsilon \in [0,\alpha], \Phi(D_1+\varepsilon,\theta) \in \hat{D}_2(D_1+\varepsilon, y, \theta).$$

This means that the choice of level of asset $D_2 = \Phi(D_1+\varepsilon,\theta)$ which is optimal after reception of the signal x is also optimal after reception of the signal y, and therefore:

$$\Delta(D_1+\varepsilon, y) - \Delta(D_1, y) = 0.$$

Consequently in all cases

$$\Delta(D_1+\varepsilon, y) - \Delta(D_1, y) \geq 0$$

and on integrating in y for a given x, we have

$$\Delta(D_1+\varepsilon, x) - \Delta(D_1, x) \geq 0$$

which completes the proof.

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13. COMPARISON OF TWO DECISION-AID MODELS APPLIED TO A NUCLEAR POWER PLANT SITING EXAMPLE

ABSTRACT. The aim of this paper is to examine on non-theoretical grounds the extent to which outranking and MAUT decision-aid approaches differ.

For this purpose, we chose a study using utility theory conducted by Keeney and Nair, dealing with a nuclear plant siting problem. We had to determine what the study would have been if it had been conducted with the use of the ELECTRE III model.

In this 'experiment', we are not interested in the practical problem for its own sake but in:
- the way to build criteria;
- the representation of a decision-maker's preferences;
- the use of the model and the nature of the derived prescription.

Confronting the two models, we study:
- the differences that they induce when facing a real problem and building a set of data,
- their respective part of arbitrariness, weakness, realism, robustness,
- the convergence or divergence of their results,

and insist upon the differences between 'descriptive' and 'constructive' approaches.

1. INTRODUCTION

1.1. *The Two Competing Models*

Let us consider a situation where a decision is necessary and where several criteria are involved. The analyst who has to help an actor in such a decision process by using as rigorous a method as possible, generally has the choice between several approaches, which involve several ways of viewing the real world and can lead to significantly different models. The objective of the present study is to compare two of these models that are frequently used, and thus to shed light on two different currents of thought that have been developing on either side of the Atlantic.

The first of the two models derives from multiattribute utility theory. This theory is based on a set of axioms referring to a highly coherent and complete preference system, considered as an objective reality, not influenced by the analyst. His task is therefore supposed to consist

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merely of delimiting such a preference system and making it explicit. To this end he has to consider that probability distributions can always be used to analyze the uncertainty affecting the evaluations of the various consequences of each solution, alternative, programme or possibility, what we will call actions, relevant to the decision problem. The analyst may then assess using this probabilistic description, partial utility functions u_i (the subscript i referring to an attribute or to a 'point of view'), and aggregate them into a global utility function u. It is then a logical consequence of the set of axioms (cf. von Neumann and Morgenstern, 1947; Fishburn, 1970 and, for a critical discussion, Allais, 1954) that the expected value of the global utility is a criterion representing the preference system. More precisely, for any two actions a, a':

$$E\left(u(a')\right) > E\left(u(a)\right) \;\Leftrightarrow\; a'Pa,$$
$$E\left(u(a')\right) = E\left(u(a)\right) \;\Leftrightarrow\; a'Ia$$

(where P and I represent respectively strict preference and indifference relations). Accordingly, we will call this expected utility criterion a 'true-criterion'.

The second model does not claim to deal with an objective reality to be 'described', but with the relationship with reality that the actors of the decision process have or wish to have. This model is thus a construction designed to illuminate possible decisions by means of pragmatic ideas and intentional actions. It is therefore difficult to connect this model with a set of axioms. In addition to probability distributions, it uses dispersion thresholds and discrimination thresholds as a way of defining what is uncertain but also what is imprecise and ill-defined in the evaluation of the consequences of the actions. This model no longer refers to a complete and coherent preference system. It considers instead that, given any two actions a and a', and given their evaluations in terms of different criteria, each of the following two statements

'a' is to be considered as at least as good as a' $(a'Sa)$,
'a is to be considered as at least as good as a'' (aSa')

can be either accepted, or refused, or, in ambiguous cases, appraised on a scale of credibility. Moreover, the acceptance or refusal of one of the two statements does not imply any information as to the acceptance or refusal of the other; if both statements are refused, the two actions are said to be incomparable.

The definition of such a relation S – which is called an outranking relation (see Roy, 1971) – involves not only the thresholds mentioned above, but also diverse variables ('indices of importance' and veto thresholds), whose function is to reflect the respective part to be played by each criterion. The formulas defining S are constructed in such a way as to respect certain qualitative principles, and, in particular, they rule out the possibility that a major disadvantage on one criterion could be compensated for by a large number of minor advantages on other criteria. They do not imply that S should necessarily be transitive or complete. The only justification for such formulas is the application of common sense to these principles.

In contrast with expected utility, S does not in general provide a clear ranking of the actions in the form of a complete preorder. In this approach, the systematic search for such a preorder cannot be justified, and, accordingly, the model only leads to the establishment of a *partial* preorder. A detailed 'robustness' analysis then allows one to determine which of the comparisons of actions are convincingly justified by the model in spite of the element of arbitrariness in the allocating of values to certain of the parameters (thresholds, indices of importance, etc.).

Further details of these models and their theoretical background can be found in Keeney and Raiffa (1976) and Roy (1977, 1978).

1.2. *The Methodology of the Comparison*

In order to compare the two models and, more generally, the two corresponding approaches, we examined a particular example, the siting of a nuclear power plant on the North-West Coast of the United States. The Washington Public Power Supply System (WPPSS) requested Woodward–Clyde Consultants to carry out a study on this subject a few years ago. In many ways, this study seems to be a very good example of the application of the first of the above-mentioned approaches. It has been described in a number of papers, most notably by Keeney and Nair (1976) and Keeney and Robillard (1977).

After an initial stage of the study involving a large number of alternatives and attributes, the set of potential sites was reduced to 9. In order to judge and compare them, 6 points of view were chosen, leading to 6 partial utility functions (and consequently 6 criteria if one is arguing in terms of expected values). Our aim was to carry out the work that could have been done using the outranking model – henceforth model

S – instead of the utility one – model U. The description below covers the different stages of the construction of model U, and for each one shows the corresponding stages in model S. The data of the situation will be given at the same time as the description, which will consist of three parts:

- the modelling of the partial preferences on each of the 6 points of view, in other words the construction of the criteria;
- the aggregation model defining the global preferences;
- the recommendations themselves.

Given that we could not obtain information either from experts or from the WPPSS management, we were often obliged to make deductions exclusively on the basis of the information available. As our aim was not to carry out another study but to compare the two models, this disadvantage had little influence on our work.

1.3. *The Objectives of the Comparison*

We had three objectives in comparing the two different models applied to the same decision situation:

(a) to emphasize the different ways in which the two models explored reality and drew on what are officially (and mistakenly) called 'data' (data are more often 'built' than 'given');
(b) to understand better the extent to which the two models are arbitrary, vulnerable, realistic or robust (all elements necessary for assessing their respective degrees of reliability);
(c) to appreciate better how and when the two models produce similar or different recommendations.

It would certainly have been interesting to attempt to place the comparison on another level: that of their contribution to the decision process, in other words, their acceptability to the different actors and their impact on the course of the process. However, this would have required an experimental study of a different nature from the present one.

The final section of this paper will be devoted to an assessment of the study in terms of these three objectives.

2. THE CRITERIA

2.1. *Introduction*

The designers of model U used 6 relevant points of view for comparing the sites, which we will accept for the purpose of the present study, assuming that the WPPSS was willing to impose them. The 6 points of view are:

1. the health and security of the population in the surrounding region;
2. the loss of salmonids in streams absorbing the heat from the power-station;
3. the biological effects on the surrounding region (excluding the salmonid lost);
4. the socio-economic impact of the installation;
5. the aesthetic impact of the power lines;
6. the investment costs and the operating costs of the power-station.

(Further details may be found in Keeney and Nair, 1976.)

The description of the consequences of an action s (the installation of a power-station on site s) connected with any one of the 6 points of view is clearly not simple. Here again, we based model S on the description carried out by Keeney and Nair in the perspective of model U. We will give details of this description in the next paragraph. But first we must emphasize what such a description consists of, and how one deduces from it a representation of the preferences in model U *vis-à-vis* each point of view. We must also indicate how model S differs in these respects. We will thus see that, in each approach, a distinctive sub-model of preference is constructed. This sub-model constitutes what is usually called a criterion; it will be denoted g_i for the point of view i.

In model U, it is an *a priori* condition that the consequences of an action s be describable in terms of 6 random variables $X_i(s)$ ($i = 1, \dots, 6$). Each variable is regarded as an attribute linked to the action in question. The carrying out of this action must be accompanied by a realization of $X_i(s)$ by means of a random draw according to its probability distribution. The particular value $x_i(s)$ thus realized must encapsulate on its own all the information to be taken into account concerning the point of view considered. The first step must therefore consists in determining this information in concrete fashion, in order to be able to define the attribute and then make the probability distribution explicit. But since the different distribution may be probabilistically dependent, the general case must be studied in terms of the joint distribution of the 6 random variables.

This explains why the preference system that the set of axioms refers to is based on the comparison of such multidimensional probability distributions. In the particular case we are considering, but also in general when dealing with real decision-aid problems, it is accepted in practice that:

- the random variables $X_i(s)$ are probabilistically independent;
- the preference system benefits from two simplifying hypotheses: preferential independence and utility independence (cf. Keeney and Raiffa (1976) and Keeney (1974)).

These two hypotheses[1] together with the classical axioms of utility theory renders the following procedure legitimate:

- the analyst questions the person who seems to possess the preference system to be represented, in order to assess a partial utility function $u_i(x)$ related to the point of view i;
- he makes explicit the marginal probability distribution of the attribute $X_i(s)$;
- he calculates the expected value of this partial utility for each of the actions: $g_i(s) = E[u_i(X_i(s))]$;
- in the preference system to be represented, the bigger $g_i(s)$ is, the better s is, other things being equal.

In this case, it is meaningful to compare two actions s and s' by referring only to point of view i. The comparison is carried out in terms of the numbers $g_i(s)$ and $g_i(s')$. The function g_i is then a true-criterion in the sense ascribed to this term in Section 1.1. (For further details, see Roy, 1985, Ch. 9.)

This possibility of comparing any two actions – other things being equal – is a prerequisite for model S. The points of view i must indeed be designed in such a way that these *ceteris paribus* comparisons constitute an appropriate departure point for the relationships that the analyst must establish between the actors (possibly the decision-makers) and their vision of reality. Since the preference system of these actors is no longer regarded as pre-existing in this reality, the existence and the definition of the criteria g_i can no longer be a direct consequence of its observable properties. These criteria should, in particular, be defined with relation to the nature of the information available on each point of view and by taking into account as much as possible the elements of imprecision, uncertainty and indetermination which affect this information. Obviously, there is nothing to prevent a given criterion from taking the form of an expected utility criterion. However, in many cases, probability distributions may appear insufficient for taking into account the whole significance of these elements. In addition, the framework of true-criterion may seem too narrow to describe the conclusions of such comparisons. Model S therefore leads one to substitute pseudo-criteria for the true-criteria of model U.

The pseudo-criterion induces on the set of actions a structure generalizing the semi-order one (Luce, 1956) by introducing two discrimination thresholds: q_i (the indifference threshold) and p_i (the preference threshold). For the point of view of criterion g_i, we have:

- s' indifferent to s iff $|g_i(s') - g_i(s)| \leq q_i$;
- s' strictly preferred to s iff $g_i(s') > g_i(s) + p_i$;
- s' weakly preferred to s iff $q_i < g_i(s') - g_i(s) \leq p_i$.

In the general case, the thresholds q_i and p_i may be dependent on $g_i(s)$ (or on $g_i(s')$). Further details may be found in Roy and Vincke (1984) and Jacquet-Lagrèze and Roy (1981).

In model U, the criteria g_i are defined as soon as one has assessed the utility functions u_i and chosen a probabilistic description for each of the attributes X_i. The procedure culminating in the determination of $g_i(s)$ and the two associated discrimination thresholds characterizing each of the pseudo-criteria of model S is completely different (cf. Roy, 1985, Ch. 8 and 9). It is based on an analysis of the consequences belonging to the point of view i and on our ability to model them, either as a single number constituting what we will call a 'single point evaluation' (which may or may not be allocated an imprecision threshold), or as several numbers constituting a 'non single point evaluation', each of these numbers possessing (potentially) an index of likelihood having the meaning, for example, of a probability. Since the only information available to us was the probabilistic description of model U, such a thorough analysis was not possible here. Consequently, we based the definition of the criteria involved in model S on common sense, although we tried to stay as close as possible to what we believe this part of study could have been in a real context, with experts and decision-makers. The type of reasoning used in the next sections is therefore more important than the precise numerical values elicited.

2.2. *Case of Two Criteria (Nos. 1 and 5) Based on Quantitative Single Point Evaluation*

Among the 6 attributes used to describe the consequences of the actions in model U, there were two, X_1 and X_5, which were not regarded as random numbers, but as numbers that were known with certainty. In other words, a site s is characterized in terms of these two points of view by two figures, $x_1(s)$, $x_5(s)$; and this is why we speak in this case of single-point quantitative evaluations. The evaluation on point of view No. 5 being in many ways simpler, we will choose this one to start with.

The figure $x_5(s)$ represents the length of the high-tension wires (needed to connect the power station to the grid) which will harm the environment if the power station is constructed. For the 9 potential sites, it varies from 0 to 12 miles.[2] Although the measure of this attribute was not regarded as a random variable, it proved necessary to define a utility function $u_5(x_5)$ in order to take this attribute into account in the global preference model. The assessment of this function was carried out using the classical 50–50 lottery technique (cf. Raiffa, 1968, and Keeney and Nair, 1976). The results obtained implied a linear expression:

$$u_5(x_5) = 1 - \frac{x_5}{50} .$$

It follows that the true-criterion g_5 of model U is simply

$$g_5(s) = 1 - \frac{x_5(s)}{50} .$$

Within model S, a criterion associated with this point of view could have been defined by letting $g_5(s) = x_5(s)$. Nevertheless, this number does not seem to be precise enough, for one to be able to say that, if two sites s and s' are characterized, respectively, by

$$x_5(s) = 10, \quad x_5(s') = 9,$$

then site s' can necessarily be regarded (other things being equal) as significantly better than site s. The difference of one mile may indeed not seem convincing, given the uncertainty in the situating of the powerlines and, especially, the arbitrariness inherent in the choice of the sections of line to be taken into consideration. We did not have access to the information necessary for evaluating the influence of these factors, and we consequently assumed that $x_5(s)$ was not known within an interval whose size grew with the distance involved but remaining no less than 1 mile for short distances. It seemed reasonable to choose a very low rate of growth: 3% (a rate of 10% would not have changed the results). This amounts to saying that $g_5(s) = x_5(s)$ is ill-determined over an interval of the form:

$$[g_5(s) - \eta_5(g_5(s))\,; g_5'(s) + \eta_5(g_5(s))]$$

with $\eta_5(s) = 1 + (3/100)\,g_5(s)$.

The function η_5 characterizes what is called a dispersion threshold (cf. Roy, 1985, Ch. 8). General formulas (cf. Roy and Bouyssou, 1983, Appendix 4) can be used to deduce the two discrimination thresholds which complete the definition of the pseudo-criterion g_5:

indifference threshold:

$$q_5(g_5(s)) = 1 + \frac{3}{100}\,g_5(s).$$

preference threshold:

$$p_5(g_5(s)) = 2.0618 + 0.0618\,g_5(s).$$

The certain number $x_1(s)$ is an official index: the 'site population factor'. This index provides a measure of the total population whose health and security might be affected by the construction of a power station on the site, and is expressed as a function of the distance of the population from the power station. The index varies in this case between 0.011 and 0.057. Still considering the 50–50 lottery technique, a linear form was again employed for the utility function. Given extreme values for x_1 of 0 and 0.2, we have:

$$u_1(x_1) = 1 - 5x_1,$$

and hence the true criterion of model U:

$$g_1(s) = 1 - 5x_1(s).$$

For model S, once again it would have been natural to set $g_1(s) = x_1(s)$. Even more than $x_5(s)$, $x_1(s)$ seems to be imprecise and arbitrary. This number is the outcome of an 'aggregation operation' whose aim is to represent a distribution characterizing a set of people located at various distances from the power station by means of a single number.

The problem is that this distribution may change with time. The type of this 'aggregation operation' is not the only one that can be imagined; and indeed the very way in which it is applied can result in variations. Accordingly, it seemed to be reasonable to adopt a dispersion threshold equal to $(10/100)x_1$. The indifference and preference thresholds characterizing the pseudo-criterion $g_1(s)$ have, under these conditions, the following values

$$q_1\left(g_1(s)\right) = 0.1\, g_1(s), \quad p_1\left(g_1(s)\right) = \frac{2}{9}\, g_1(s).$$

2.3. *The Case of Two Criteria (Nos. 3 and 4) Based on Non Single Point Qualitative Evaluations*

To define the attributes X_3 and X_4, Keeney and Nair introduced two qualitative scales having respectively 8 and 7 adjacent intervals. The nature of the biological or socio-economic impact, covered by each interval, was determined by means of relatively concrete and precise descriptions of the future situation. For each of the two attributes and for each site s, approximately 10 experts were asked to use such descriptions to characterize the outcome which, in their view, seemed most probable in the hypothesis of the power station being constructed on that site. The proportion of votes received by each interval was used to define the (subjective) probability distributions of $X_3(s)$ and $X_4(s)$.

Two utility functions $u_3(x_3)$ and $u_4(x_4)$ were then assessed (using a particular technique adapted to the qualitative nature of these scales, cf. Keeney and Nair (1976), $g_3(x_3)$ and $g_4(x_4)$ corresponding respectively to the expected utility of $X_3(s)$ and $X_4(s)$.

Once again, it is important to point out that we would have used a similar method to evaluate the biological and socio-economic impacts on the potential sites. The evaluation obtained by Keeney and Nair (a distribution of the experts' opinions, involving in general more than one interval of the scale in question) is called a 'non single point one'. In order to define $g_3(s)$ and $g_4(s)$, only one of the intervals considered by the experts must be chosen. We selected the interval nearest the centre, that is the one which divides the experts most equally into those who are at least as optimistic and those who are at least as pessimistic as this value. Given the nature of the scales in question, constant discrimination thresholds were adopted. After examining the distributions of the experts' opinions, we used

$$q_3 = 1, \qquad p_3 = 2$$
$$q_4 = 0, \qquad p_4 = 1.$$

2.4. *Case of a First Criterion (No. 2) Based on Non Single Point Quantitative Evaluations*

X_2 is more complex than the other attributes studied up till now. The total quantity Q of salmonids which might be destroyed following the construction of a power station was not relevant on its own to the appraisal of the 'loss of salmonids'. Given the sensitivity of certain ecological equilibria, the destruction of 10 000 salmonids in a river containing 20 000 cannot be regarded as equivalent as the loss of the 10 000 in a river containing 300 000. It was therefore necessary to analyze the consequences in terms of two factors:

- the total number Y of salmonids living in the river;
- the percentage Z of salmonids destroyed.

An exhaustive study (cf. Keeney and Robillard, 1977) led the authors to distinguish between large rivers ($Y > 300\ 000$) and small ones ($Y < 100\ 000$) there were no medium-sized rivers in this particular study. For the large rivers, the attribute studied, X_2, could be taken into account simply by using the absolute number $Q = Y \cdot Z$ by means of a utility function defined by:

$$u_2(X_2) = 0.568 + 0.432\, u_Q(Q)$$

with

$$U_Q(Q) = 0.7843 \left(e^{(0.00274\ (300-Q))} - 1\right)$$

(Q being expressed in thousands).

For the small rivers, on the other hand, it proved necessary to take Y and Z into account separately, by means of two partial utility functions $u_Y(Y)$ and $u_Z(Z)$ (cf. Roy and Bouyssou, 1983, Appendix 3), the utility of X_2 being deduced from them by:

$$u_2(X_2) = u_Y(Y) + u_Z(Z) - u_y(Y) \cdot u_Z(Z).$$

To calculate the expected value $g_2(s)$, the authors of model U assumed that:

– for each site s, Y took on a value $y(s)$, known with certainty;
– Z was a normal random variable with a standard deviation equal to half its expected value.

In order to implement model S, we would probably not have undertaken so complex a study to define criterion g_2. Doubts about the results of this work may be all the more justified given that:

– the probability distributions of variables Y and Z were not defined with as much care as the utility function, and
– the expected utility $g_2(s)$ (which orders the 9 sites in exactly the same way as the numbers $E(Q(s))$) does not seem to reflect very faithfully the qualitative principles adopted at the beginning of the utility analysis.

We would instead have tried to analyze why, given two rivers containing exactly y and y' salmonids, it was more damaging to destroy q of them in the first – assumed here to contain the least fish – than a slightly larger number q' in the second. Then we would have explored qualitative considerations to try to connect q' with q, y and y' in such a way that the damage done in the two rivers was of the same magnitude. One could, for instance, have examined whether a simple formula such as $q' = q \cdot (y'/y)^\alpha$ was capable – with α appropriately chosen between 0 and 1 – of representing the experts' opinions on such cases of equivalent amounts of damage. On the sole basis of the analysis done for model U, we considered it possible to define criterion g_2 from the above formula, by adopting two different versions of this criterion corresponding relatively to:

$$\alpha = \frac{1}{2}: \qquad g_2'(s) = \frac{q}{\sqrt{y}} = z\sqrt{y},$$
$$\alpha = 0: \qquad g_2''(s) = q = z \cdot y.$$

(The values of the criteria g_2 are calculated, in model S, by setting $z = \bar{z}(s)$.)

The above reasoning was effected without taking into account the difficulties of evaluating y and predicting z for each river. The large value adopted for the standard deviation of Z and the necessity of coping with the imprecision affecting y led us to adopt a broad dispersion threshold which we fixed as 0.5 $g_2'(s)$ and 0.5 $g_s''(s)$. We thus have

$$q_2' = 0.5\, g_2'(s), \qquad p_2' = 2\, g_2'(s),$$

$$q_2'' = -.5\, g_2''(s), \qquad p_2'' = 2\, g_2''(s).$$

2.5. *Case of a Second Criterion (No. 6) Based on Non Single Point Quantitative Evaluations*

The authors of model U considered that the investment and operating costs of a power station located on a site could be appraised relatively to the costs of the cheapest size s_2. The attribute $X_6(s)$ therefore reflects a differential cost. It was supposed that the insufficient knowledge affecting this cost could be modelled by treating $X_6(s)$ as a normal random variable with a standard deviation equal to a quarter of its expected value.[3] This expected value was estimated by the values $\bar{x}_6(s)$ varying from 0. to 17.7 (in millions of dollars per year, cf. Roy and Bouyssou, 1983, Appendix 3). Let us point out that it is sure that $X_6(s_2) = 0$.

The criterion $g_6(s)$ of the model U is the expected utility of this random differential cost. Again invoking the lottery technique, the utility function $u_6(x_6)$ was defined as

$$u_6(x_6) = 1 + 2.3(1 - e^{0.009\, x_6}).$$

Once again, we would probably have constructed model S in a different way. Since it is not the same actors who are responsible for the investment and running costs, we would perhaps have introduced a criterion for each of them. But because we cannot analyze these costs in detail in the present study, we will merely set

$$g_6(s) = \bar{x}_6(s).$$

Lacking a more objective foundation, we can use the following reasoning to determine dispersion threshold. Firstly, the values of $\bar{x}_6(s)$ which were suggested contain the assumption that the investment and running costs that are not included in the differential cost will actually lead to the same expenses on site s_2 as on any other site s. This is obviously a source of sufficient error to cast into doubt the whole idea that a site s' is more economical than a site s when $\bar{x}_6(s) - \bar{x}_6(s')$ is small. We decided, on the basis of this single hypothesis, that the 'real' differential cost had to be regarded as ill-determined on an asymmetric interval: $[\bar{x}_6(s) - 1; \bar{x}_6(s) + 2]$.

Secondly, the calculation of $\bar{x}_6(s)$ follows on from the evaluation of multiple factors which all involve specific expenses for site s. But the study carried out on each site remains brief until the construction is actually decided. In other words, these costs are not necessarily the only ones: they are relatively imprecise and possibly too optimistic. The margin of error resulting is asymmetric and its size is proportional to $\bar{x}_6(s)$ itself. The factors involved here seem to have no connection with the ones taken into account previously. We shall therefore assume that the effects can be added together. We have the following dispersion threshold:

$$\left[\bar{x}_6(s) - 1 - 0.1\,\bar{x}_6(s), \bar{x}_6(s) + 2 + 0.5\,\bar{x}_6(s)\right].$$

Thus

$$\begin{aligned} q_6\left(g_6(s)\right) &= 1.1 + 0.11\,g_6(s), \\ p_6\left(g_6(s)\right) &= 3.33 + 0.67\,g_6(s). \end{aligned}$$

3. AGGREGATION OF THE CRITERIA AND GLOBAL PREFERENCE

3.1. *Introduction*

Having in this way defined the true-criteria of model U and the pseudo-criteria of model S, we will now present the part of the model dealing with their aggregation. In the present section, we will briefly describe the parameters involved in the aggregation phase of each model. The following two sections will be devoted to the evaluation of these parameters.

Assuming that the WPPSS's preference system is a pre-existing entity, that it conforms to the axioms of utility theory, that the hypotheses of independence mentioned in Section 1.2 are acceptable, and that the responses to the questions posed in order to assess the partial utility functions were governed by this preference system implies (using a general theorem – cf. Keeney and Raiffa (1976) that this preference system is representable by means of a true-criterion $g(s)$ defined in terms of the criteria $g_i(s)$ by one of the following two expressions:

$$(1) \qquad g(s) = \sum_{i=1}^{i=6} k_i \cdot g_i(s) \quad \text{with} \quad \sum_{i=1}^{i=6} k_i = 1,$$

(2) $$g(s) = \frac{1}{k}\left[\prod_{i=1}^{i=6}(1 + k \cdot k \cdot g_i(s)) - 1\right]$$

with

(3) $$k \neq 0, \quad k \geq -1, \quad k = \prod_{i=2}^{i=6}(1 + k \cdot k_i) - 1.$$

This last expression of $g(s)$ was the one chosen by Keeney and Nair (we will see the reasons why in Section 3.2). In order to complete the characterization of model U, it is consequently sufficient to assess the coefficient k_i (whose values increase with the relative importance attached to criterion i, once the utility functions have been defined) and to deduce the value of k from them be solving Equation (3), which normally has only non-zero root greater than – 1 (cf. Keeney and Nair, 1976).

In model S – which corresponds to ELECTRE III (cf. Roy, 1978) – the aim is no longer to use the pseudo-criteria $g_i(s)$ to determine a true-criterion, or even a pseudo-criterion. The more modest aim is to compare each site s to site s'[4] on the basis of their values on each g_i, taking into account the thresholds q_i and p_i, and hence to adopt a position on the acceptance, the refusal or, more generally, the credibility of the proposition:

'site s is at least as good as site s''.

As we pointed out in Section 1.2, this credibility depends on pragmatic rules of simple common sense, rules which are mainly based on notions called concordance and discordance. These notions allow one:

- to characterize a group of criteria judged concordant with the proposition studied, and to assess the relative importance of this group of criteria within the set of the 6 criteria:
- to characterize amongst the criteria not compatible with the proposition being studied, those which are sufficiently in opposition to reduce the credibility resulting from the taking into consideration of the concordance itself, and to calculate the possible reduction that would result from this.

In order to be able to carry out such calculations, we must express in explicitly numerical fashion:

- the relative importance k_i accorded by the decision-maker to criterion i in calculating the concordance; let us merely indicate here that these numbers have virtually no influence except for the order that they induce (because of their addition) on the groups of criteria involved in the calculations of concordance;
- the minimum level of the discordance giving to criteria i the power of withdrawing all credibility from the proposition being studied, in the case when this criterion is the only one of the 6 which is not in concordance with the proposition: this minimum level is called the veto threshold of criterion i; it is not necessarily a constant, and therefore we will denote it $v_i[g_i(s)]$.

It is important to emphasize that model S is different from model U in that the indices of importance (and also the veto thresholds) are not values stemming from the observation of a pre-existing variable but values designed to convey deliberate positions adopted by the decision-maker, positions which are mainly of a qualitative nature. It follows that the techniques to be applied in order to evaluate the parameters we have just discussed for both models reflect two different attitudes towards reality (cf. Section 5.1) even more than the criteria do.

In each model, there is a considerable amount of arbitrariness affecting the value chosen. The recommendations must consequently take into account the robustness of the results towards these factors. They nevertheless depend strongly on the underlying model.

3.2. *Modulation of the Importance of the Criteria*

Within model U, the most classical method to assess the scaling constants k_i consists in comparing lotteries (see Raiffa, 1969 for a review of other available methods).

Let us denote $\tilde{x}_i$ and $\underset{\sim}{x}_i$ the respective values used to scale the partial utility function u_i between 0 and 1. We have $u_i(\underset{\sim}{x}_i) = 0$ and $u_i(\tilde{x}_i) = 1$. Let us consider the following two multidimensional lotteries. The first one, L_1, is a degenerate lottery resulting for sure in an 'imaginary site'[5] which receives the worst evaluations on all the criteria except j, where its evaluation is $\tilde{x}_j$. The second lottery, L_2, gives rise to another imaginary site whose evaluation is either the best possible on all the criteria with probability p, or the worst possible on all the criteria with probability $(1 - p)$.

The expected utility of L_2 is p; and the utility of L_1 of k_j in the multiplicative representation (2) – and indeed also in the additive one (1). If the decision-maker is able to determine that particular probability p which guarantees indifference between the two lotteries, we can state $k_j = p$.

By iterating this procedure, one can therefore – in principle – assess the 6 coefficients k_i, and hence k, the solution to Equation (3).

The lotteries to be compared here are multidimensional, unlike the ones used to assess the partial utility functions. Even with the help of sophisticated interview techniques to assess the probability p, it is difficult to escape the conclusion that this sort of comparison of imaginary sites is extremely complex, and that the decision-maker may be unable to reply to such questions in a reliable fashion. In order to try to avoid this obstacle, the designers of model U used a more indirect assessment technique comprising:

- an ordering of the coefficients k_i;
- an estimation of tradeoffs between attributes;
- an estimation of the coefficients k_i.

This procedure, which is described in detail in Roy and Bouyssou (1983, Appendix 6) and Keeney and Nair (1976), is still based on lottery comparisons of type L_1 and L_2. It is therefore vital not to attribute an illusory precision to the values of the k_i estimated in this way.

The designers of model U used in the end:

$$\begin{aligned} k_1 &= 0.358, \quad k_2 = 0.218, \quad k_3 = 0.013, \\ k_4 &= 0.104, \quad k_5 = 0.059, \quad k_6 = 0.400. \end{aligned}$$

One can observe that $\sum_{i=1}^{6} k_i = 1.152 \neq 1$, which justifies the choice of the multiplicative structure (cf. Keeney, 1974).

Solving Equation (3) then gives $k = -0.3316$.[6]

In model S, the only influence of the indices of importance is the ranking they impose on the different criteria or groups of criteria. If we had carried out the study, we would probably have tried to assess such a ranking interactively with the decision-makers of the WPPSS. We would then have tried to find various sets of indices of importance compatible with these merely ordinal considerations.

Without access to the decision-makers, we had to try to 'translate' the information conveyed by the utility function concerning the relative importance of the criteria into indices of importance, to attempt to produce a comparable system of values and hence to ensure that the comparison of the results of the two methods was still meaningful. The technique used is detailed in Roy and Bouyssou (1983, Appendix 7). Let us simply point out that the k_i in model U do not have an immediate interpretation in terms of the relative importance of the criteria (cf. Keeney and Raiffa, 1976 and Zeleny, 1981). The magnitude of the scale and the shape of the partial utility function both affect the k_i values. This relative importance seemed to us to be reflected more accurately by the range of variation of the difference ratios:

$$(4) \qquad R_{ij} = \frac{\partial g / \partial g_i}{\partial g / \partial g_j}, \quad i, j = 1, \ldots, 6,$$

where g is given by Formula (2) and the g_i are as defined in Part 2.

One can qualitatively interpret the value of R_{ij} as the gain needed on criterion j to compensate a loss on criterion i. For example, if R_{ij} is always greater than 1 for all possible values of g_j and g_i, it seemed reasonable to us to consider that criterion i was intrinsically more important than criteria j within model S. We examined the variation ranges of the ratios R_{ij} which led us to employ eight sets of indices of importance (cf. Roy and Bouyssou, 1983, Appendix 7) covering collectively the same value system as the one convoyed by model U. In fact, we considered that the k_i were so imprecise in model U and that this translation was so inherently arbitrary that it became unrealistic to try to maintain a single set of indices.

3.3. *The Veto Thresholds*

As veto thresholds convey deliberate and 'intentional' positions, they cannot be 'assessed'. This explains why we would probably have produced the same kind of work as the one reported here had the study been a real one. Once the decision-maker is satisfied with the qualitative principles underlying the partially compensatory character of model S, one can then ascribe numerical values to the different thresholds in empirical fashion, taking into account the relative importance of the criteria, the distribution of the site evaluations over the criteria, and the size of the various preference thresholds. Given an inevitable arbitrariness in the choice of these numerical values, one generally then carries out a systematic robustness analysis on these coefficients.

Model U being compensatory, it was not possible to deduce from the available information qualitative considerations that would have helped to determine the veto thresholds. Therefore, it is principally our particular perception of the problem which is reflected in this choice. However, the robustness analysis showed that the values chosen had little influence on the results within a fairly wide range of variation. It seemed reasonable in all cases to take the thresholds $v_j(g_j(s))$ as multiples of the preferences thresholds $p_j(g_j(s))$ (not that there is necessarily any fixed link between these two figures). We imagined that the less important the criterion the larger the value of the coefficient α_j such that $v_j(g_j(s)) = \alpha_j p_j(g_j(s))$. In particular, the veto thresholds for criteria 3 (biological impact), 5 (aesthetic impact) and 4 (socio-economic impact) were chosen so as to have no influence. On the first level of analysis, we used the following values:

$$
\begin{aligned}
v_1\left(g_1(s)\right) &= 6\, p_1\left(g_1(s)\right), \\
v_2\left(g_2(s)\right) &= 2.5\, p_2\left(g_2(s)\right), \\
v_3\left(g_3(s)\right) &= 4\, p_3\left(g_3(s)\right), \\
v_4\left(g_4(s)\right) &= 4\, p_4\left(g_4(s)\right), \\
v_5\left(g_5(s)\right) &= 20\, p_5\left(g_5(s)\right), \\
v_6\left(g_6(s)\right) &= 1.7\, p_6\left(g_6(s)\right).
\end{aligned}
$$

4. CONTENTS AND PRESENTATION OF THE RECOMMENDATIONS

4.1. *Introduction*

We have, in model U:

$$g(s) = \left[\prod_{i=1}^{6}(1 + kk_i g_i(s)) - 1\right](1/k).$$

The values of k and of the k_i were given in Section 3.2 and the form of the $g_i(s)$ in Part 2. One can therefore obtain the number $g(s)$ and using the principles of the true-criterion, rank the sites on the following basis:

$$s' \text{ preferred to } s \;\Leftrightarrow\; g(s') > g(s)$$
$$s' \text{ indifferent to } s \;\Leftrightarrow\; g(s') = g(s),$$

and hence deduce the recommendations.

In model S, the situation is different. As mentioned above, this model seeks to establish a fuzzy outranking relation between the actions, that is to evaluate the proposition 's' is at least as good as s' on a credibility scale. A distillation procedure is then used to rank the actions on the basis of this fuzzy relation (see Roy, 1978). Two total preorders thus emerge, which behave in opposite ways when confronted with those actions which are hard to compare with another group of actions (one of the preorders tends to put them before this group, and the other after).

The intersection of these two preorders leads to a partial preorder emphasizing the actions which have an ill-defined situation in the ranking. This incomparability must be accepted, since model S explicitly acknowledges the imprecise, and even arbitrary, nature of some of the data used. The quality and reliability of the recommendations depend therefore to a considerable extent on a systematic robustness analysis.

4.2. *The Results*

One can summarize the results of model U as in Table I.[7]

The ranking obtained is therefore a complete ordering.

The authors of model U carried out a sensitivity analysis on this ordering. Nevertheless, the fact that they disposed of an axiomatic basis and that they had obtained the various data (shapes of utility functions, values of the k_i) by questioning persons supposed to represent the decision-maker,[8] led them to effect an analysis only of 'marginal'[9] modifications of the data. This resulted in a virtually complete stability of the ordering *vis-à-vis* these modifications (cf. Keeney and Nair, 1976).

The robustness analysis is a crucial part of model S. We present in Roy and Bouyssou (1983, Appendices 9 and 10) the overall robustness analysis (which involves more than 100 different sets of parameters) and the results obtained. Knowing the arbitrariness of the evaluation of some of the parameters, we considered that an entire subset of the space of the parameters was in fact plausible, a subset which we checked systematically in order to make our conclusions as reliable as possible.

TABLE I

Rank	Site	$g(s)$
1	S_3	0.926
2	S_2	0.920
3	S_1	0.885
4	S_4	0.883
5	S_8	0.872
6	S_9	0.871
7	S_7	0.862
8	S_5	0.813
9	S_6	0.804

We will merely observe here that, of all the possible sources of variations, the form of criterion 2 selected (g_2' or g_2'') has the greatest influence. In Roy and Bouyssou (1983, Appendix 10), we showed that, with the exception of the form of criterion 2, the stability of the results is good when confronted with variations that cannot be considered marginal. The robustness analysis bore principally on the indices of importance (8 sets), the discrimination thresholds (criteria 2 and 6) and the veto threshold (criteria 2, 3 and 6) (cf. Roy and Bouyssou, 1983, Appendix 9).

The totality of these results may be presented, in very brief and qualitative form, as two graphs, corresponding respectively to the g_2' form and the g_2'' form of criterion 2 (the influence of the other parameters being less important). Figure 1 shows representative outranking graphs.

4.3. *The Recommendations*

It should be emphasized that the reason the WPPSS requested this study was to select which of the 9 sites were most likely to be chosen by the administration for the construction of the power station. The WPPSS was interested in two sorts of information:

- the sites which could be totally eliminated at this state of the decision process, from any further considerations;

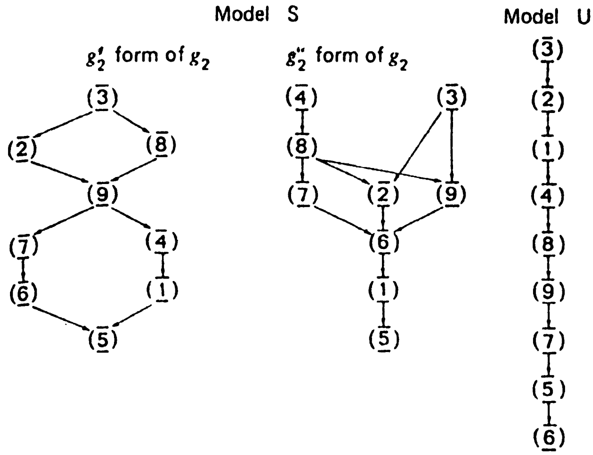

Fig. 1. Representation outranking graphs. The transitivity arcs have been omitted; two sites not connected by an arc (not considering the transitivity ones) are incomparable. The graph given for model U is a diagrammatical representation of Table I.

– the sites among those remaining that would be the most likely to be considered the best in future, more detailed studies.

The study of the ranking provided by model U shows that S_5 and S_6 can safely be eliminated from further stages of the study, and that S_3 and S_2 are in the leading positions with S_1 and S_4 just behind (cf. Table I and Figure 1).

The analysis of the results of model S (cf. Figure 1 and Roy and Bouyssou, 1983, Appendix 10) shows that there is a remarkable stability at the bottom of the ranking, with S_5, S_6 and S_1. Site S_3 is in the leading place, whatever form of criterion 2 is chosen. S_2, S_8 and S_4 are just behind, whereas S_7 and S_9 are to be found in a zone of instability in the middle.

Like the authors of model U, we would have recommended S_3, if the WPPSS had required that only one site be chosen. On the other hand, there is a major divergence between the two models concerning the position of S_1 and, to a certain extent, S_8 (we will come back to this point in Section 5.3).

Underlining the fact that the case has not been studied here for its sake, we will now try to give partial answers to the three questions mentioned in Section 1.3.

5. CONCLUSIONS

5.1. *The Origin and the Treatment of the Data*

In model U, the procedures used to assess the different parameters involved in the definition of the global utility function (partial utility functions $u_i(s)$, coefficients k_i) follow logically from the set of axioms underlying the analysis. These axioms imply that lottery comparisons can always be used to carry out this estimation.

This position is unassailable on the formal level, but the number of questions raised – and their complexity – imply that the decision-maker (or his representative – cf. Section 4.2) is obliged to collaborate closely with the analyst. The legitimacy of these techniques is inseparable from the hypothesis that a complete system of preference pre-exists in a form which is implicit but which is nevertheless in line with the axioms in the decision-maker's mind.[10] It must also be assumed that the replies given by this decision-maker or his representatives are in fact governed by such an implicit system, and that this system is not likely to be fundamentally altered during the dialogue with the analyst. The urgency of the decision problem to be solved and the analyst's experience then create the necessary conditions for the disclosure of these attitudes which are represented in terms of a utility function. When certain opinion brought up are in contradiction with the axioms defining the coherence, it is assumed that the normative character of the axioms (completeness, transitivity, independence) is sufficiently obvious for the decision-maker to *adapt* his views to them (cf. Morgenstern, 1979). In such a perspective – unlike that prevailing in most of the other social sciences – the axioms of the formal model are also behavioral axioms – and, when necessary, normative axioms. This attitude underlines most of the studies based on model U. It explains why analysts place such great confidence in the data they gather and why they virtually never fundamentally question them when the sensitivity analysis is carried out.

The same is true when evaluating the consequences of the actions. The probability distributions provided by the experts are thus rarely questioned, even when they are clearly imprecise and/or arbitrary (cf. criteria 2 and 6 of the power station study). Once again, 'marginal' sensitivity analyses are carried out that imply generally a high level of stability in the ranking obtained.

Model S has no axiomatic basis, and consequently it is often difficult to interpret certain parameters used in it (veto thresholds, indices of importance). Only considerations based on common sense allow the decision-maker and the analyst to give them a numerical value. This explains why the results produced by the model S are significant only when the analyst has carried out a major robustness analysis, systematically exploring the numerical values of the parameter compatible with the qualitative 'data' he started with. This procedure should not be considered as merely a palliative for the lack of axiomatic foundations and the lack of sophisticated techniques, for assessing the parameters, but constitutes instead one of the original features of the approach, which consists of trying to design a preference system and not of trying to represent an existing system in the most accurate way possible.

The difference observed between those two prescriptive approaches in the way they obtain the data are in fact connected with a much deeper division: the one between a model drawing validity from a 'descriptive' aim of representing a pre-existing relation and a model whose validity is based on a 'constructive' aim of designing an acceptable preference relation in collaboration with the decision-maker.[11] Sophisticated assessment procedures only draw meaning with relation to a given reality, which must be adhered to as closely as possible.

In order to be in a position to apply utility theory, it must also be assumed that all the imprecise, uncertain or arbitrary elements in the evaluation of actions on the various consequences can be taken into account by means of probability distributions. Such a hypothesis is necessary for the expected value of this distribution on a utility scale to be regarded as a true-criterion.

In those cases where the principal aim is to help the decision-maker cope with a risk, a probability distribution can afford a satisfactory modelling of the evaluation of an action. When analyzing the losses of salmonids in a river (criterion 2), one might try above all to study the risk of these species totally disappearing from it. If a well-established probability distribution is available for describing the phenomenon, expected utility may appear an adequate criterion.

In contrast, even if, *a priori*, it is possible to use probabilistic tools to model the cost of a power station (the definition of which is not free from ambiguity – cf. Section 2.5) by closely modelling each of those of its elements (rate of inflation, cost of construction material and fission material, etc.) that might influence the cost of the project, this information is probably not very useful to the decision-maker. What is important is not to know a probability distribution on cost with a possibly misleading precision, but to be able to say whether one action can be considered as significantly cheaper (or more expensive) than another. In this situation, arguing in terms of dispersion thresholds would seem necessary, as in all cases where one is dealing more with conceptual looseness and imprecision than with a really random phenomenon. Model S does not assume any *a priori* restrictions on the nature of the imprecision and uncertainty affecting the evaluation of actions, and seeks to translate these phenomena as a pseudo-criterion.

However, these approaches are not exclusive, and indeed one can imagine using model S with a criterion based on an expected utility surrounded by thresholds. Model S substitutes pseudo-criteria for true-criteria, and this is as much the result of a refusal to restrict 'nondeterminism' to randomness as the result of the role played by the idea of criterion in designing the preference relation. This model is intended to 'construct' rather than 'describe', and therefore starts from a criterion that allows one to compare two actions – other things being equal – unlike model U, where the fact of referring to a pre-existing reality (theoretically) obliges one to test hypotheses of independence associated with the preference structure before being able to talk of a criterion (cf. Section 2.1).

Because of this, the pseudo-criteria base the comparison of actions in model S, whereas the true-criteria of model U represent it.

The use of a pseudo-criterion follows on from the caution, and even the skepticism, with which the analyst using model S regards his methodology. He cannot use existing preferences as fixed points, and can only deduce that there is a convincing preference when the often approximative tools he is using leave him in no doubt – hence the use of a 'buffer-zone' embodied in the discrimination thresholds. As for model U, it supposes that a preference relation pre-exists, and that the information gathered using the function $u_i(x_i)$ is sufficiently reliable to allow it to be 'extrapolated' to more complex lotteries in exact fashion (for example, in the case of the cost, the function $u_6(x_6)$ is assessed from even-chance lotteries whereas the calculations are carried out using normal laws).

5.2. *Robustness and Fragility of the Approaches*

The distinction between a 'constructive' attitude and a 'descriptive' one illustrates the relative advantages and disadvantages of models U and S. If the decision-maker is clearly identified and possesses a sufficiently precise and stable preference structure, one can certainly adopt a purely descriptive attitude. Nevertheless, we consider that in most real decision-aid problems, an attitude of a constructive nature is inevitable.

Every decision forms part of the social structure of the organization, which is often complex and conflictual, meaning that often the only single decision-maker one can talk about is a fictional entity (see Roy, 1985, Ch. 2). It is then difficult to assume that a collective group of decision-makers possesses a pre-existing and coherent preference.

In fact, the designers of model U did not assess some parameters included in the global utility function by questioning the decision-maker(s) of WPPSS (cf. Section 4.2), but by using judgements provided by the study team itself. This practice, which does not seem unusual in studies based on model U (given the difficulty and the number of the questions asked) can cause reasonable doubt as to the reliability of the assessment procedures of the utility function: it implies that sensitivity analyses of the same scope as for model S have to be carried out.

Once one has accepted the advantages – and even the necessity – of a constructive approach, one can understand better the implications of an axiomatic basis for decision-aid models. For many people, the attraction of an axiomatic basis is the legitimacy it apparently confers to their work. But this legitimacy is valid only for the 'theory', and not for the 'model' which is an 'interpretation' and a putting into practice of the 'theory'. Model U is based on a formal theory for representing an existing preference system. It is hard to imagine what a design theory of a preference system could be – a theory that would underly model S. If the axiomatic basis legitimizes the theory, it does not follow that it does the same for the model. The legitimacy of the model must be sought in the effectiveness with which it enables the actors to arrive at convictions (possibly upsetting preconceptions) and to communicate with other people. A decision-aid model must not be merely a formal theory, but must form the basis for an interaction with reality and for an action on reality.

Finally, let us point out that model U can conceivably be used in a constructive perspective. This is in fact what is really done in most studies. However, model U should be considered in this case independently of its axiomatic basis: one should study the reliability of the assessment procedures of the partial utility functions and of the constants k_i as tools designed to construct and/or enrich the decision-maker's preference relation between the actions.

Many of the misunderstandings in comparing models S and U seem to stem from the fact that model U is designed in terms of a constructive attitude but only draws a particular legitimacy from its axiomatic basis if it derived from a descriptive attitude.

We do not believe that normative conclusions can be drawn from this study concerning models S and U as potential tools for decision-aid. Each model has advantages in certain domains – the usefulness of both has already been pointed out in numerous studies.

It should also be recognized that the choice of a model very often depends on 'cultural' factors and 'decision-making customs' which cannot be analyzed in a formal way.

More generally, our study shows that the problem of the validation and the legitimacy of decision-aid models requires a major re-thinking. The concept of 'predictive power' cannot apparently act as the basis for validity tests in this domain – unlike the situation in many other disciplines.

5.3. *Agreement among Recommendations*

In Section 4.3, we observed that, if there was a certain agreement in the recommendations on site S_3, there were also differences: the positioning of site S_1, in particular, was controversial. Model U ranked S_1 as among the best sites studied, while model S recommended that it be dropped from later stages of the study. In the same way, site S_8 is considered as a 'good' site in model S, but appears in the middle of the ranking in model U.

These disagreements in the two models reflect the contrasts in the qualitative principles underlying them, especially concerning the reliability of the differences between the evaluations on the different criteria and the more or less compensatory nature of their aggregation. Site S_1 (cf. Roy and Bouyssou, 1983, Appendices 3 and 5) is evaluated very highly on most of the criteria (g_3, g_4, g_5, g_6), but receives the worst possible evaluation on both health and security (g_1) and salmonid loss (g_2). Model S, being partially compensatory, ranks such a profile near the bottom whereas model U (perfectly compensatory) places the site among the best, because of its very good scores on many criteria.

Inversely, site S_8 may be interpreted as an average 'compromise' site (cf. Roy and Bouyssou, 1983, Appendices 3 and 5), and is well-placed in model S; but in model U, it appears lower down, behind other sites where good performances on certain criteria compensate very bad ones on others.

In addition, conclusions of too great a generality should not be drawn from the good agreement of the recommendations on site S_3. An intuitive examination of the evaluations of this action shows that it seems to be a good site in terms of the information available. It is therefore 'normal' for S_3 to be in the first place in both methods. A good part of the agreement obtained in thus peculiar to the problem studied (in another problem, a site of type S_1 could have appeared at the top in model U).

Given such a fundamental opposition in the qualitative principles underlying the two models, it is not all surprising that they culminate in dissimilar recommendations.

In our view, these inevitable disagreements do not imply that decision-aid is useless but simply that a single problem may have several valid responses. Given that two different decision-aid models cannot be implemented in the same decision process, the decision-maker must be conscious of the qualitative choices implied by the different models – often conveying the analysts' own ethical choices – before coming to personal conclusions on the choice to be made. In this domain, the many different approaches reflect in our view the complexity of the researcher's task much more than a scientific weakness.

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NOTES

[1] The theory includes tests designed to check their realism, but putting them into practice involves difficulties that make the results unconvincing.

[2] All the numerical data used in models U and S can be found in Roy and Bouyssou (1983).

[3] The costs are supposed to correspond to a standard type of construction which is considered fixed. No tradeoffs with criterion 1 are explicitly considered.
[4] This pairwise comparison remains feasible even when hundreds of alternatives are taken into account. See for example Roy *et al.* (1983).
[5] This imaginary site is also 'idealized' since its consequences are supposed to be perfectly determined by the probability distribution.
[6] The results in this paper are the ones we obtained by calculating from the data published in the articles quoted. They are slightly different from those given by Keeney and Nair (1976).
[7] See note 6.
[8] In fact, most frequently the research team themselves.
[9] Marginal, by opposition to a cross-linked variation of all the parameters in the model. Here, each parameter varies separately, within a variation range which is not necessarily small.
[10] In actual studies, the decision-maker is supposed to be able to express a set of fundamental attitudes compatible with the axioms. Comparing complex actions is then equivalent to an extrapolation of those attitudes, whose validity is guaranteed by the set of axioms (cf. Bouyssou, 1984).
[11] It is important to stress that the terms 'descriptive' and 'constructive' do not apply to the models themselves but to their justification and the attitude in which they are elaborated.

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WERNER LEINFELLNER

14. RISK IN SERIAL CONFLICTS, EVOLUTION, INTELLIGENCE AND CHAOS

1. RISK, SERIAL DECISIONS AND EVOLUTION

This paper deals with some consequences and generalizations of Allais' work and especially of his risk theory. Allais' dynamic view of the social sciences, of risk as a psychological, internal, and regulating factor of decision making and of the micro- and macroeconomy, his emphasis of the role of the history, our past experience, in decision making, and of the nonlinear character of the social sciences are just a few of his pioneering ideas. Allais concepts of risk in serial decision making and conflict solutions and in evolutionary processes had not only a decisive impact on the foundations of economics, but also on the explanation of how we may think.

This paper sets forth Allais' idea that intelligent thinking is far more than deductive, Boolean or rational thinking. Thinking is intelligent conflict-solving. 'I decide, therefore I am' or 'decerno ergo sum', is now more important than 'I think, therefore I am' or 'cogito ergo sum'. Rather, we think, and thereby survive, by solving the never-ending, serial social and environmental conflicts, provided we do not neglect past and future solutions and always take uncertainty and risk into account, otherwise humankind would not have survived. This postulates a new way of looking at intelligence which is modular, nonlinear. In philosophy and basic research on intelligence and in evolutionary processes we can no longer neglect Allais' views.

Von Neumann–Morgenstern's (NM) utility and game theory is the first theoretical reconstruction of how we make decisions. This theory offered decision makers, economists and social scientists many advantages, such as convenient methods to measure riskless utility under uncertainty. It freed utility from monetary values and used the maximization of expected utility, the classical probability calculus and its

B. Munier (ed.), Markets, Risk and Money, 279–302.

independence axiom as its methodological backbone. In brief, the NM utility theory became the cornerstone of game and decision theory and developed into a general theory which elegantly solved individual, social, political and economic conflicts of our society.

But the NM utility theory and the traditional game and decision theory remained a 'static' theory, dealing only with single riskless conflicts. It turned out that this static theory was unable to solve serial conflicts. Serial conflicts are most important for understanding learning, adaption in biological, psychological and social conflict solutions.

Two revolutionary steps changed the classical NM utility game and decision theory. The first was the 'dynamization' of the static game and decision theory, anticipated in Allais' papers of 1954 and 1955 about dynamic economic models [2], the second was Allais' radical imposition of risk on utility theory [1]. Long after Allais dynamic economic models of 1954–1956, dynamization of game and decision theory has been achieved by the theory of dynamic differential games. By this step decision theory became a dynamic theory, and could successfully solve serial, time dependent conflicts. Dynamic game theory gave us a new understanding of serial conflicts, or evolutionary processes, self-organization of living systems, learning processes and finally of animal and human intelligence. Today any kind of evolution can be regarded as a general learning process under uncertainty and risk. Evolution became identical with serial conflict solutions under uncertainty and 'risk-guided' memory accumulation for better solutions of future problems and conflicts. From this point of view evolution is, firstly any gradual or sudden change in the series of solutions of a long lasting problem or conflict, provided the past solutions are stored, inherited, or tradited for future use. Secondly, solutions of conflicts are regulated by Allais type risk-factors [1], which are superimposed on the maximization of one or more magnitudes, such as expected utility, stability, security. It was Allais who, with his completely new methods, created a supertheory of risk, which successfully supported the dynamic game and decision theory in explaining all possible forms of serial conflicts and evolutionary processes. For example, cultural and scientific developments can be explained by evolutionary processes. Culture can be seen as a never ending process of alterations of culturally or scientifically stored and transmitted relatively stable artifacts, characteristic behavior, institutions, concepts and mentifacts, within or across generations [2].

Biological, competitive and cooperative evolutionary processes begin phylogenetically with organic biological stages and end with societal, cultural or scientific evolution. If they are guided by unconscious decisions (conflict solutions), they show up as a change in the development of relatively stable cultural traits. This can be expressed easily by increase or decrease in the frequency of traits, qualities, properties, attitudes, rules, behavior, institutions, concepts, artifacts and mentifacts within an across generations.

One of the most important consequence of this dynamic aspects of evolution is that optimal solutions of conflicts between alternative behavior, cultural traits, arti- and mentifacts, and concepts can only be understood by their history. This is exactly what Allais called "the hereditary conditioning of men by past events", or "The presence is a function of the past" [3]. According to Mach [4], to solve a problem, we have to take into account all the previous, successful or wrong solutions of the problem. Therefore, if we find an optimal solution we can only regard it as a preliminary and relatively (or evolutionarily) stable solution. Any change of marginal conditions, environmental-random factors, and internal risk factors, etc., may force us to give up its optimality and continue to search for a better and new solution.

The second revolutionary change wrought by Allais in 1952 was to impose risk factors upon traditional utility, game and decision theory. Suddenly risk became the main guiding and dynamic factor of the motor of evolutionary processes and replaced the maximization of survival of species. As already mentioned, the dynamic nature of any evolutionary process is the ability of the decision maker to store and use past experience for solving future conflicts. This process is guided by internal risk factors. It is not the mysterious maximization of the survival of the species which helps the individual to solve the *ur*-conflict of survival. Despite the fact that maximization of survival has always been the paradigm of Darwinistic evolutionary theory, it has remained a mere theoretical magnitude, a real 'intervening' variable in genetic theories up to now.

In opposition to Darwin's competitive maximization of the species' survival this broader view of evolution rests on the following assumptions. (1) Evolution is a sequence of cooperative and competitive conflict solutions. (2) Evolution is pushed forward by internal, biological or psychic impulses (risk factors) of the individual or their genes, which consciously or unconsciously (genetically) steer and control the individual's decisions. (3) Biological evolution is evolution of intelligence.

Therefore, normal risk is an internal, inborn tendency of living beings which regulates all their life preserving decisions and conflict solutions automatically. Of course, human beings may manipulate their own risk behavior, because of their free will. But, normal individuals do not need to follow a goal, since their risk factors will automatically respond to random events and changes in their environment. Risk factors of the Allais type influence the individual's solutions whenever it has to solve a conflict under uncertainty, provided the individual is not consciously against it. The influence of risk is restricted only to serial processes, which are either intraindividual or an interindividual series of conflict solutions. Intraindividual conflict solutions begin early in infancy and continue throughout life. We have them at our disposal whenever we need them. Interindividual conflict solutions stretch over generations because they are genetically and culturally transmitted from generation to generation. They are both long-lasting learning processes which are steered by individual risk factors. Once again it is the risk which regulates the individual decisions, not the maximization of a mystical magnitude, the survival of the species.

Darwinian competitive survival of the species is therefore only one side of the story. It remains a theoretical concept, since it can only be confirmed indirectly in models of population genetics by the increase of the surviving generations. What really pushes the individual ahead is maximization of stability and security, which is regulated by the individual's inner risk attitude in uncertain conflicts. Therefore evaluation and conflict solution under uncertainty is always risk regulated. It is intelligent to end never the search for optimal individual solutions during serial social, economic, ethical and cultural conflicts. In the long run this will secure and stabilize the individual's or the group's survival.

Solutions of biological serial conflicts begin either competitive in the animal kingdom as fights for territorial rights or fights for the possession of females between rivals. Or are cooperative solutions, such as formation of families, hordes, etc. Human cultures create (in a self-organizing sense) for example, optimal solutions of incest avoidance (taboo), social conflicts, economic and political conflicts, and finally of conflicting scientific problems. Conflict solutions may, after some trials and possible setbacks result in relatively stable solutions. But, they may run through chaotic, intermediary phases, or stop and disappear or die out just like species. Series of conflict solutions may be disturbed by external and internal randomly occurring events, or may be interrupted and taken up again and continued. They are always guided by the individual's personal risk aversion, risk neutrality or risk loving factors (attitudes) and not by the theoretical concept of maximization of the species survival. Quite generally within the animals kingdom, animal risk aversion will contribute in a direct way to the survival of the species. Even under abnormal conditions their behavior is modified and guided by risk factors, according to the Allais risk theory which Bataglio *et al.* (1985) have shown convincingly [5].

One should never forget that serial conflict solutions are intelligent solutions which are stretched over a long period of time which can last generations, according to Lumsden and Wilson's Coevolution theory of cultural and genetic evolution.

If risk is a genetically fixed and regulating, internal factor, the whole present bulk of utility and decision theories and models can be subdivided. There exists a pre-Allais period (the classical von Neumann–Morgenstern and Marschak models) which includes uncertainty, but no real risk and the new risk theories, which began with the pioneering works of Maurice Allais in the 'fifties.

2. INTELLIGENCE, AN EVOLUTIONARY PROCESS VERSUS RISKLESS RATIONALITY

Intelligence is a serial, stochastic and cognitive process based on memory preservation, storage and availability. It solves serial conflicts, either cooperatively or competitively with the help of rule-based default hierarchies of preferences, regulated by internal risk factors of the individuals. In contrast to the 'static', axiomatic, and logical idol of deductive, riskless rationality it is historical and dynamic.

It is interesting that Aristotle did not know the word 'logic'. Instead he used '*analytikos*' [6], which comes from '*analysis*' = 'solution of a problem' which differs from *logismos* = deductive calculation.

Allais' psychologization of decision making showed that maximization of expected utility is regulated by the individual's risk attitude: risk loving, risk neutral and risk averting factors determine and modify the solutions of conflicts. In a similar manner Simon's [7] bounded rationality introduced internal factors such as optimization of satisfaction.

If rational thinking is a static way of thinking of terms of axiomatic, linear and analytic models, intelligent thinking is stochastic, nonlinear

TABLE I

Rational thinking	Intelligent thinking
static, analytic, deals with single conflicts	dynamic and evolutionary, deals with series of conflicts
logical and deductive	stochastic, rule-based default system, inductive
atomistic, ahistoric	holistic, historic, memorable
local models	local and global models
competitive	cooperative and competitive
isolated from its environment	interacting with its environment
analytic and riskless	cognitive and argumentative [8]
normalized, idealized models	renormalized, taking into account the changing risk
general and timeless	local and time-dependent
no uncertainty, riskless	including uncertainty and risk, descriptive and prescriptive default rules
pseudonormative: highest norm	descriptive and prescriptive, default-rule based
criterion: truth	criterion: risk guided optimization of stability, security
linear structure	nonlinear structure

and dynamic. Some of the differences between rational and intelligent thinking are given in Table I.

According to Allais, Simon, Selten, Holland and Leinfellner, intelligent decision making [9] is far more complex than the deductive-rational conflict solution. Thus, intelligence is a very complex cognitive function, the highest form of a human's way to solve serial conflicts. This is because it takes account of the environment's randomness (uncertainty) and our daily individual risk. Only intelligent thinking can solve ethical conflicts, which are always conflicts between egotistic and altruistic interests, by minimizing from pain and losses, and minimaxing social pleasures and gains [10]. Finally, as already mentioned, risk throws us back to the *ur*-conflict of human and social life (to the maximization of our survival and stability *vis à vis* uncertainty and risk).

Beginning in the 'fifties, almost unnoticeably, this new concept of intelligence replaced the older deductive and ahistorical form of rationality. Allais' school of utility, the Simon–Newells and Feigenbaum concept of artificial intelligence [11] no longer used the old, simple concept of a logical, riskless deductive and ahistorical rationality. Instead, intelligence was used as a serial conflict, or problem solving in the long run. Cognitive psychology and philosophy contributed to this development, and intelligence became identical with serial and complex conflict or problem solving of the same or similar type. Allais' risk theory, statistical decision theory, dynamic game theory and the theory of evolution have shown convincingly that the process of how to cope optimally with uncertainty and how to solve a never ending series of risky conflicts can be theoretically reconstructed. In intelligent conflict solutions it plays no role, whether these problems are conflicts of our daily and social lives, of our economy, or pure mathematical or even theoretical scientific problems. Sooner or later, the models of serial game and decision theory became a conflict solving theory, an evolutionary game theory *vis à vis* constantly changing uncertainty in our environment and changing risk.

Since the concept of intelligence or intelligent conflict solution used the imposition of risk factors on maximizing expected utility, survival and stability, it was able to explain how speechless, nonhuman intelligence of primates, animals and even cells [12] functioned. The intelligent behavior of animals turned out not only to be the evolutionary forerunners of human intelligence, but proved that the evolution of the species is identical with the evolution of intelligence.

Until today the way we think and solve conflicts could not be satisfactory explained by rational, ahistorical, deductive or riskless reasoning. We have to assume that human intelligence works and solves conflicts the way the different, competitive and cooperative models of utility theory, dynamic decision theory and dynamic game theory under risk and uncertainty, etc., describe it. As classical mechanics became a small part of quantum theory, deductive reasoning, which includes Morgenstern's and Marschak's theories became a minor part of a future unified theory of intelligence.

To sum up: intelligence is serial conflict solution. It is the capability to cope with uncertainty, risk and to solve optimally the sheer endless series of societal, environmental, etc., conflicts. Finally – at least until today – intelligence is the capability to solve the *ur*-conflict of life, which is the survival of life on a habitable earth [13].

3. WHAT IS INTELLIGENCE?

This question has been answered in more detail elsewhere by the present author [14]. Quite generally, intelligence is a conflict solving ability. This functional capacity of hierarchically structured living systems, works under uncertainty and risk and is an effective cooperation of four cognitive functions f_i, which form the recursive functional F:

f_1: An input organ (device) which allots at the same time preferences on all incoming sense data and patterns, and represents and stores them in memory systems.

f_2: A twofold memory function, an episodic and a semantic memory. (The latter is linguistic–semantic in human memory [15]).

f_3: A probabilistic, rule-based default function with inferencing capabilities and superimposed modifying risk factors.

f_4: A realizing effector.

If all these functions are 'artificial' man-made, technical devices, then we get the concept of an artificial intelligence. If the f_is are natural functions, we get the working scheme of a 'biological or human intelligence'. Therefore, computer intelligence or artificial intelligence is so far only a special case of biological intelligence; it is a partial material reconstruction of human intelligence.

1. This definition only characterizes what we call individual intelligence and not the collective cultural intelligence, for example, the cooperation of many brains or computers or scientific team works, think-tanks, etc.

2. According to today's more advanced view of intelligence it deals exclusively with series of conflict solutions of the same or similar type, where the outcome of one conflict within the series influences the solution of the next conflict. According to Mach [16], all scientific concepts are evolutionary mentifacts. For example, what Newton called 'space' underwent an evolutionary development from a Euclidean concept to a relativistic and to Hawking's concept of space [17]. Intelligence is a dynamic feedback process with loops and an iteration of the same or similar conflict until a relatively optimal solution with respect to certain environmental conditions is achieved. If empirical evidences or environmental conditions change the iteration begins again.

There are two basic types of optimal or relatively stable solutions: the cooperative and competitive solution. Both permit approximate and cyclic or periodic solutions. Such dynamic, serial solutions are controlled by imposed factors, for example, the risk factor in utility theory. It is the changing risk attitude (risk averting, risk loving or risk neutrality) which adapts itself to the randomly changing environment and which influences the next following solution. More specifically, the result (a single solution) of the conflict C_1 at the time T_1 is fed back and influences, controls, improves or changes the next solution of the same type or similar type of conflict C_2 at the time t_2, where $t_2 > t_1$, and so forth. Feed back processes work like amplifiers and multipliers, feedback control systems and servomechanisms.

Each conflict begins with unstable, open alternatives. The consecutive solutions form evolutionary paths, trajectories,(saddle points, sinks, attractors, bifurcations).

Intelligence is, as we have already mentioned, a holistic complex 'functional' system $F[(f_1, (f_2, (f_3, (f_4))))]$ which uses an iterated (recursive) application of all four functions, given already corroborated rules of how to solve optimally serial conflicts. For example, even a simple solution of the simple problem, the multiplication of two numbers, stays unsolved, as long as we do not remember the rules of how to perform the solution, or the algorithm of how to solve it.

One of the long-term strategies of computer research will be the task of how to build full-fledged risk factors into the software and default-based rules into the programs of our computers. For the time being we can forget this research and leave it to Hofstadter and Dennet in their book *The Mind's I* to install 'feelings for risk' into computers. The short-term research strategies which characterize the momentary cognitive status of artificial intelligence are: (1) parallel processing of pattern recognition for f_1; (2) memory increase for f_2; (3) simulation of inductive reasoning, learning and evaluating for f_3; (4) to simulate and imitate the functions 1–4 of the human intelligence. This reverses the decades old trend in AI, that we have to learn from a computer how the brain works. We certainly have to learn from the brain how our future computers will work.

4. PARADOXES, THE INDICATORS OF CHAOS

Today there are two views of paradoxes: the traditional, logical and the new, dynamic one. The latter regards paradoxes as indicators of chaotic states. A paradox in the traditionally static, logical and axiomatic systems is simply a statement that goes against a generally accepted opinion. For example, Allais' paradox is a statement which contradicts the established von Neumann–Morgenstern model of utility. A logical paradox consists of two contrary, or even contradictory assumptions, assumptions to which we are lead by apparently sound arguments. In our case the assumptions are the von Neumann–Morgenstern axioms (ANM), which are sound, if they are used in their idealized, riskless field of application. Here they do not seem to create any difficulties. It is only in the particular combination with Allais' axioms of risk (AR) in which the paradox occurs that leads to the well-known troublesome conclusion: (ANM) & AR) $\rightarrow$ Allais Paradox. This paradox consists in the apparent equivalence and simultaneous use of the two different assumptions, ANM and AR, which are not at all equivalent. The former ANM is the negation of the later AR, with respect to maximization of expected utility and the independence axiom. Other very well known paradoxes are the common ratio effect of Kahneman and Tversky, the Bergen paradox of Hagen, Grether and Plott's (1979) preference reversal effect. They show convincingly that the domain of application (D) of the von Neumann–Morgenstern's axioms is indeed too narrow, since D permits only uncertainty and risk neutral choices. Moreover, according to Allais the NM model covers only external uncertainty, but not the internal risk, which is what Allais has shown convincingly in his questionnaire in 1952. Morgenstern, with whom the author worked during the years 1963–1967 confirmed this view. When I asked him where he had taken his axioms from, he told me that he had obtained the NM axioms from a careful, statistical observation of past, risk neutral, invariant preference patterns for the purpose to use these axioms to predict future preference behavior.

The dynamic view regards paradoxes as chaotic phases in time-dependent dynamic systems (series of conflict solutions) with feedback. Transitions from almost riskless to risky behavior may cause perturbations and produce an intermediary, irregular behavior, as the answers to Allais' publication of the results of his questionnaire (1952) have shown. In other words, paradoxes in iterated conflicts may turn out to be nothing more than chaotic, intermediate phases of unpredictable, aperiodic behavior. For example, Hofstadter (1985) regards Gödels famous impossibility theorems as the result of a 'dynamization' of the static, logical system of the *Principia Mathematica* of Russell and Whitehead. Gödel's recursive method is, according to Hofstadter, an iteration with feedback, which turns logical paradoxes into chaotic phases. As is well known, Gödel used a self-referential (feedback) method for his proofs, a fact he mentioned explicitly in his famous article of 1931 [18]. In brief, 'dynamized' logical systems with feedback show chaotic phases which logicians have called paradoxes. If that is true, then the question arises whether Arrow's impossibility theorem, the prisoner's dilemma, and Allais' and Hagen's paradoxes also belong to the same class of dynamic systems, which slip from time to time into chaotic phases.

In addition, when we change from 'static' paradoxes to chaotic phases, at the same time we change from linear to nonlinear systems, as Machina pointed out [19]. Von Neumann expressed it once: "Even the character of equations may change simultaneously in all relevant aspects and both order and degree change."

Deductive, ahistorical and static rationality, which Hofstadter called the riskless 'Boolean dream' and Allais, in his Nobel lecture, called 'tautological reasoning", is nevertheless the core of all classical sciences. But classical sciences are a minority, nowadays. The majority of sciences, as well as intelligent thinking, are modular, nonlinear and cannot be confused with linear, axiomatic Boolean deduction or with one single, all-embracing logical method. Any predecision for one single particular set of axioms is a pseudo-normative, prescriptive, coercion [20]. If you include true risk into the static risk neutral NM model, then paradoxes will inevitably appear. If you iterate a risky conflict, then chaotic transitory phases may appear.

5. CHAOS THEORY AND RISK FACTORS

What is chaos? Traditional sciences regarded paradoxes and chaos as exceptions, not worth exploring. There are several very plausible reasons which explain why systems get temporarily out of order, such as intended or unintended randomly-occurring disturbances from the surroundings, which utility theorists call uncertainty. But they are unsatisfactory explanations of suddenly occurring chaotic states without any external disturbances and causes.

When in the 1970s Allais [21], Thom and scientists in the United States and Europe began to be interested in irregularities, disorder and chaos, time-dependent differential equations were used to represent the way systems changed continuously over time. Very early on, the 'error accumulation' explanation of disorder had been ruled out. In order to know the state of the system at any time in the future one had to know its initial state. If there exists a small error or uncertainty in our knowledge of the initial state, this error may be amplified exponentially in the temporal evolution of the system. Then Laplacian determinism is no longer possible. But that is not what we understand by chaos.

Often, external or internal random events (mutations) have been regarded as the causes and disturbing factors of disorder, as in the Darwinian type of evolution, which may push a system into irregular states. Most of these explanations work with external selecting factors, but cannot explain chaos, since it may occur without any external factors or disturbances. It seems that chaos is a new form of erratic behavior which is caused solely by internal factors. The question arises – which will be discussed in the remainder of this article – whether risky serial conflict solutions show chaotic phases, caused by Allais' psychological risk-factors.

Today chaos has become a label for a rapidly growing science and, it seems, where chaos begins, classical, linear science stops and nonlinear disciplines begin. Chaos is by no means total randomness and disorder, but rather some kind of strange, hitherto unknown brand of higher order. Chaos sets in when complex dynamic processes, which develop in time, suddenly show irregularities, erratic intermediate phases, without any external disturbances. When prosperity brings windfall profits, the rhythmic heartbeat becomes arhythmic and ends in fibrillation, or when a regularly dripping water tap begins dripping irregularly or uninsurable losses pile up – which it does, as everybody can check – we speak about chaotic phases.

Chaos occurs, as Feigenbaum and Lorenz have shown, only in complex processes, in time-evolving, hierarchical systems with feedback. But until today chaotic processes have been found mainly in the natural sciences; the social sciences seemed to be chaos free. This was a mistake, since today there are many more examples of feedback-guided processes of dynamically evolving systems in the social and economic sciences than in the natural sciences.

According to Gleick, chaotic processes resemble more a rising column of cigarette smoke breaking into wild whirls, or a sudden outbreak of panic at a convention. Fire in a hall may cause a chaotic panic: everybody will rush to the exit as Nash's theorem explains [22]. Since each individual has his/her own leeway how he/she will get out of the hall as soon as possible, nobody can predict how he/she will manage it or where he/she will flee to. In this case the behavior of the whole looks rather chaotic and unpredictable, nevertheless, for each individual, there is obviously a secret, risk-averting attractor, namely to get out and save him/herself as soon as possible without being harmed. It is the empirical, individual risk which changes and, during this chaotic period, influences the individual's behavior in an unpredictable way. A further example is when the growth of a population's wealth is influenced by last year's increase or decrease, or when learning processes are reinforced by past experience. Since serial risk taking, learning and intelligent problem solving are typical examples of complex, iterated feedback processes, we should begin to search for hidden chaos factors in the models of Allais, Hagen and Machina.

Today it seems that chaos is everywhere; chaotic processes become the rule and classical processes as, for example, decision making under certainty and without risk or Newtonian mechanics are the exception. Strangely enough, in our economic system business cycles, or crashes on the stock market, resemble far more chaotic weather changes or epileptic seizures than exceptions to the rule.

If racial discrimination, chaotic psychic moods and business cycles which hit us from time to time, like a bolt from the blue, can no longer be regarded as catastrophic exceptions, but rather as belonging to dynamic systems with feedback, then why not assume that chaotic, intermediary phases belong to serial conflict solutions under uncertainty and risk? It is obvious that these phases are caused by internal trouble makers: the Allais-chaos factors.

Since 1970 researchers such as Feigenbaum, Lorenz, Mandelbrot, Allais, Thom and Genz began to use computers to investigate chaotic systems like perturbation in fluids, the weather and fractals. They found that internal factors were responsible for irregular, chaotic patterns. These factors resemble the risk factors of Allais. They soon detected that the outbreak of chaotic phases depended solely on these internal chaos factors. At a certain critical value, a threshold value, turbulences or chaotic phases appeared out of the blue.

The disclosure that internal factors, which are hidden, lying dormant within orderly systems, may suddenly cause internal chaotic states motivated chaos theorists to find and define these factors. This discovery will certainly change our view of evolution and science, since these factors characterize a sort of new 'internal evolution', which cannot be found in the selection-adaptation scheme of the traditional view of 'external evolution'. Since risk theory of serial conflicts is a typical nonlinear mathematical systems we may assume that Allais, Hagen's and Machina's higher moments of the utility distribution harbor risk factors which may be responsible for chaotic intermediary phases in serial conflict solutions.

Fortunately, today's theory of chaos, complicated as it may be, shows drastically that behind the chaos there exists a new kind of indeterministic, complex order. Incredibly enough, on this new view chaos is democracy in nature, since it permits a certain chaotic leeway for the individuals of a system under natural laws, just as our laws permit the individuals a certain degree of freedom in democracies. In our example, the sudden outbreak of fire may cause a chaotic panic: everybody will risk in different ways to the nearest exit, using the free space available to him or her to get out of the hall as quickly as possible. During this flight the risk and the risk factor will change continuously. We know from Allais and Hagen's risk models that at the moment when the internal risk factors change, the skewness of the individual's utility distribution will change accordingly. The changes influence how the individuals will solve the pending conflict to get out of the room as soon as possible. During this period determinism will break down. Nobody can predict how he/she will manage it or where he/she will flee to. Therefore the behavior of the whole looks rather chaotic and unpredictable. Nevertheless there is obviously, for each individual, a secret attractor, his own risk attitude which is superimposed on the maximization of saving him or herself.

Attractors, as in this example, represent behavioral risk patterns of a very complex order, and to this day they have not yet been fully analyzed, with exception of Allais' risk theory and the discussion of his work in the *Theory and Decision Journal* and the *Theory and Decision Library*. Therefore it is no wonder that a 'social' chaos theory tries to discover these strange and normally hidden attractors which start and steer the chaotic processes. Chaos theory assumes that the complex chaotic 'order' is superimposed by attractors on the basic classical order which, in the natural sciences, is governed by natural laws and, in standard utility theory, by adopted rules and the maximization of utility. If we can put risk factors into a continuum which, for example, begins with avoiding risk ($r = 0$), risk neutrality ($r = 0.5$) and loving risk ($r = 1$), transitions from order to chaos may be dependent on irregular changes of the value of risk.

In models these kinds of factors (attractors) form mathematical fixed points, iterations of equilibria, trajectories, time-dependent evolutionary trajectories and chaotic phases. Where chaotic states begin, linear-deterministic mathematical methods and models are no longer able to give an adequate description of the societal reality. Indeterministic, nonlinear models begin to represent the reality. Indeterminism means that predictions and prognoses predict only intervals, leeways, group behavior, but not the individual's exact behavior. For example, we all know that we will have breakfast tomorrow, but we cannot predict with certainty that I myself will enjoy breakfast tomorrow.

Time series of dynamically changing systems can be represented by trajectories in representational phase spaces. The trajectories can become stable either by converging to a steady state, to a point in the expected value space, or can repeat themselves periodically or display what is new-, short- or long-lived chaotic behavior. In any representational mathematical model differential equations connect the different states in a multidimensional representational space in which a point represents the actual state of the system plus all the information about the system.

6. CHAOS, RISK THEORY AND INTELLIGENCE

Any complex dynamic, temporal process, for example, a series of conflict solutions whose final solution or output can be fed back into its input, possesses feedback loops and may become chaotic if the final solution is not risk-optimal. If such a loop exists, even if it is a simple one, we may expect both chaotic intermediary phases or long-lasting chaos. The difference between these states lies only in a single internal factor, the chaos factor, which we will call r. A very minute change in the value of this parameter could change the orderly behavior of a loopy system into a chaotic one, without any influence from outside.

Since serial risk taking over a period of time is actually such an iterated loopy process and risk a chaos factor, then chaotic phases or business cycles we have to expect in micro- and macroeconomic systems In social and political systems, according to Arrow's impossibility theorem, undemocratic rules and serial dictators may lead to chaotic and revolutionary states in the societies involved. In all these chaotic processes risk may change any minute and the changed risk factors influence the individual's next solution of the conflict. Since risk is a typical internal, individual biologically or psychologically guiding factor, it influences internally the evolution. This is quite contrary to the Darwinian view of an evolution regulated by external selection. The concept of external evolution is based on the idea that mutations arise at random and external selection favors the most advantageous. In brief, organisms do not have the ability to select advantageous mutations. But recent findings by Cairns (Harvard University) and Hall (University of Connecticut) suggest that, given a conflict, even living organisms, such as bacteria, have the ability to regulate and direct mutations internally by risk.

To sum up: on the one hand, the theory of serial conflict solutions under uncertainty and risk seems to be with the help of memory-storage and learning theory an adequate, new theoretical model of how intelligence works. Its greatest advantage is that it has guaranteed our survival on the planet Earth so far. It is genetically rooted in the way the animal and human cognitive intelligence copes with its global, societal and environmental, risky conflicts.

On the other hand, intelligence is not immune to chaotic phases. But chaotic phases have the advantage to create new, never expected, creative or innovative alternatives. They show up out of the blue, just like in Kekulé's famous dream which taught him how to scientifically interpret the cyclic structure of benzene, a dream which found aromatic, organic chemistry [23].

7. CHARACTERISTICS OF MATHEMATICAL MODELS OF CHAOS

Chaos theory as well as utility theory under uncertainty and iterated risk share functions of functions, or functionals to represent theoretically the sequential, time-dependent processes with feedback. This situation, as we will show in the following sections, is clearly given in Allais', Machina's and in Hagen's and many other models.

Nonserial risk theories used curves which illustrate the deviation of risk-loving and risk-averting from risk-neutral behavior, for example the Arrow–Pratt risk measure [24].

In serial-risk theories the nonlinear terms tend to be the features that scholars want to leave out to get a simplified understanding. For example, without risk a simple linear equation expresses the amount of goods you need to increase the utility. With risk in serial decisions the relationship gets complicated and smeared because the amount of utility change depends on the changes in risk. Nonlinearity means that changing the risk is the way of playing the game. One cannot assign a constant, never changing magnitude to risk, because the momentary risk depends on the value you may loose or win in a previous or future decision. It is a clear case of mutual interdependence of values, probabilities and risk, when utility depends on changes in risk. These intertwined relations and their sudden changes make nonlinearity hard to calculate, but also creates rich kinds of complex, evaluative behavior that never occur in linear systems. Analyzing the behavior of nonlinear equations is like walking through a maze, whose walls rearrange themselves with each step you take, as Gleick expressed it.

The theoretical or mathematical representation of serial conflicts and intelligent problem solving simply hinges on the iteration of a function. We get a function of a function, or a functional which represents theoretically the sequential process of conflict solutions under risk.

Iteration has therefore a twofold meaning: empirically it means the influence by past experience, for instance, in a typical learning process under uncertainty and risk. Theoretically it means that whenever we iterate a specific mathematical equation, as Feigenbaum did, we search for the internal factor, in our case the chaos factor which, at a certain value, initiates chaotic behavior [25].

To find mathematically such a factor in models of risk taking we have to iterate special equations of the following form:

$$f(x), f(f)x)), \ldots.$$

The whole problem of learning is to feed f's output always back into $f(f)x))$ and to iterate this procedure again and again. Then watch carefully to see if some kind of pattern emerges.

The nth iteration of a monotonic function, for example x^2, is harmless $(((x^2)^2)^2), \ldots, ^2$, which results in x^{2n}. If we use a nonmonotonic function i.e. a function whose curve is folded, it moves upward until it reaches its peak height and then bends back downward. The parabola with its equation, $y = 4rx(1 - x)$, where r does not exceed 1, is the most elementary one to hide a chaos factor. If we allow input values of x between 0 and 1 the output is between 0 and r. We may now feed back the output value into the next function as input, i.e. iterate it when we learn a new solution. If one iterates a folded function of this type the value of y obtained will go up or down between 0 and r. It turns out, astonishingly, that the differences with respect to degree of regularity simply depends on r. For example, for r below 0.89248617967..., the curves are regular, but for r at or beyond this value a chaotic sequence of values will be observed by the values of x, $f(x)f(f(x)))$, ..., independent of the initial values of x [26]. This theoretical 'chaos' model clearly shows why periodic orbits change into messy, aperiodic and chaotic ones. It also theoretically explains not only the transition to turbulence in a fluid but also paradoxical behavior in risky dynamically changing situations.

If we could use this type of equation, as is done later, serial utility theory under risk and uncertainty could offer a theoretical explanation of (i) how intelligent, serial problem or conflict solving by learning is influenced and controlled by internal psychological risk factors of Allais' type, and (ii) could explain and maybe forecast the onset of dangerous chaotic phases. (iii) At the same time this model could serve as a theoretical explanation and representation of how conscious and unconscious serial, intelligent conflict solutions are automatically regulated by inbuilt changing risk factors which adapt themselves to the changing situation or lead to chaotic phases.

For example, the decision maker consciously or unconsciously may get into a chaotic and messy phase of three states: the risk-neutral state in which he does not influence the maximization of expected utility at all; the state in which he exerts a negative influence (risk averting); and the state in which he has a positive influence on the final outcome (risk seeking). Thus the functional relational between this intended risk adaption and the previous decision, or simply the difference between the actual and the desired next outcome becomes nonlinear.

The strange attractor and risk factor can be represented mathematically in the phase space of serial solutions of conflicts. This is shown by the second and third moment of distributions in Allais and Hagen's system, and is hidden away in the moments of the Taylor expansion of $V(\cdot)$ [27].

8. APPENDIX: RISK ATTITUDES AND CHAOS, POSSIBLE CHAOS FACTORS IN THE MODELS OF ALLAIS, HAGEN AND MACHINA

The first time higher moments as risk factors were used was in Allais' Model of 1986–1987:

$$u(C+V) = EX + r|\phi(u-\bar{u})|$$

$$EX = r_1 = \int u\phi(u)\,\mathrm{d}u$$

$$r^* - f(r_1, r_2, \ldots, r_n)$$

$$(1) \qquad u = u(C+g),$$

C is the initial capital, u the cardinal utility, V the monetary value of the random variable, ϕ the probability density which defines the function $\phi(u)$. EX is the first moment of u and the r_i are the higher moments, the possible chaos factors, according to the Taylor expansion. For $r^* = 0$ Allais' model is equivalent to the NM model. Transitions from r_1 to r_2 will possibly create chaotic intermediary states. The r_i are the hidden attractors of serial-risk behavior, which we have sought.

If we want to measure the utility connected with a random variable (an uncertain prospect EX), we will usually start, in a similar way as in relativity theory, with the Euclidean spatio-temporal indices, with the local utility function. According to Allais, the higher moments characterize the distortion by psychological risk factors of the normal distribution EX, just as in relativity theory the intensity of the curvature vector of the space indicates the distortion of our normal Euclidean space [28]. The question remains open how to find a continuum of the higher moments.

In Allais, Hagen's, Machina's and Muneras' model each probability distribution has its higher moments. The NM theory only uses the expected average value EX; Allais and Hagen's models use the higher moments, for example, the second moment or the standard deviation (variance s). If two probability distributions X and Y are given then everybody will prefer X to Y, if:

$$(2) \qquad EX \succeq EY \rightarrow s_x^2 \leq s_y^2.$$

Additionally, Hagen uses the second and third moments to characterize the influence of psychological variables on the normal NM distribution:

$$(3) \qquad U = g(EX, s^2, M_i).$$

These central moments are invariant to linear transformations. The inclusion of the third moment does not achieve a global order, but permits only the inclusion of more psychological variables.

Hagen postulates that the utility U of a random variable is a function of risk imposed on the expected values and is a cardinal utility function, similar to Allais' formulation:

$$(4) \qquad U = EX + f(\sigma, z) + \varepsilon$$

f expresses the utility under uncertainty with the help of σ, the standard deviation of u and z, where:

$$(5) \qquad z = r_3 : \sigma^2$$

is the relation of the third moment and the variance, ε, is an error factor. It is clear that Hagen gives up linearity and the independence axiom.

Left-skewness implies that the decision maker is risk averse and right-skewness risk loving. Chaotic phases alternate between the two extremes in an irregular fashion.

We want to regard the risk (r) as a continuously changing factor, between 0 and 1, which begins with risk seeking attitudes ($r^* = 0$ in the extreme case), risk neutrality ($r^* = 0.5$), and ends with total risk aversion ($r^* = 1$). To illustrate the feedback character of serial conflict solutions we will take the already discussed equation from evolutionary game theory [29] and adapt it to a simple model for serial conflict solutions. y_i is the amount of utility (gains or losses) during a series of conflict solutions. Then, any next following conflict within the iterated series of conflict solutions is given by:

$$(6) \qquad y_i^{i+1} = ry_i - s(y_i)^2$$

y is simply a variable, r is a general risk factor and s is a measure of uncertainty. The iteration starts from y_1 over y_2 to y_3 and so forth, and is the representation of the iterated conflict solution. The behavior after a finite sequence of conflict solutions under uncertainty and changing risk is most interesting.

The risk factor (parameter) $r > 1$ indicates the change (increase, decrease) of risk and hope from one conflict to the other for one single individual. It would simply lead to an exponential increase of risk aversion of the decision maker, if the term $-s(y^i)^2y$ did not stop it. s simply represents uncertainty.

We can expect that there will be an equilibrium y^* (a fixed point) at a certain value of $r > 1$:

$$(7) \qquad y^* = ry^* - s(y^*)^2$$

which is exactly the von Neumann–Morgenstern normal distribution or the NM expected utility. We assume for all these cases that the uncertainty s has to be small relative to r.

Equation 6 can be easily transformed into the logistic form by substitution of $y = (r/s)x$ [30]:

$$(8) \qquad x_{i+1} = rx_i(1 - x_i), \quad 0 < x_i < 1.$$

This equation depends only on the internal risk factor r. To keep x between 0 and 1, r must be less than 4 or $r = 4$. There is an equilibrium or fixpoint at:

$$(9) \qquad x^* = (r - 1)/r.$$

Now, if r is small or exactly if $1 < r < 3$, the risk factor approaches x^* and we get a stable fixpoint or risk attractor. But if we continue to iterate the equation, there is a clear feedback in the sense of a direct causal influence of any x_i on x_j, since any temporally preceding risk r_i at the temporal point t_i will influence the consecutively following risk r_j at t_j, with $t_j > t_i$.

If r starts to become greater than 3, x^*, the risk averting attitude, becomes unstable and chaotic phases will appear sooner or later. If $x_1 = x^*$, then $x_i = x^*$, but now even small deviations from $x + 1 = x^*$ cause x_i to begin to deviate from x^*, in contrast to the behavior before. The equilibrium breaks down and an interval of deviation $[a, b]$ follows when bifurcation begins. At the transition from $r = 3$ to $r > 3$, the interval is given by:

$$(10) \qquad x_{a,b} = (r + 1)/2r \pm [(r - 3)(r + 1)]^{1/2}/(2r).$$

Bifurcation means that each conflict solution is characterized by the same value of risk (hope or risk aversion), and the same solution occurs again and again within the interval $[a, b]$. If we use the normalized $r^*(r_0 = 0, r^4 = 1)$, a sudden jump occurs at $r^* = 0.75$, since Equation (8) becomes nonlinear because it contains a quadratic term.

If we again increase r^*, the normalized risk factor, then more and more bifurcations will occur and a chaotic phase appears at $r^* = 0.892$, and in the risk behavior of our individual. The logistic Equation (8) is internally determined, albeit in a statistical sense, and it depends completely on the changing risk attitude.

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JEAN-JACQUES DURAND

15. MONEY, CREDIT AND MONETARY DYNAMICS IN MAURICE ALLAIS' WORK

The Nobel Prize in Economics has given recognition to Allais' work regarding the allocation of scarce resources. It does not refer explicitly to the financial aspect of this work. The latter cannot, however, be dissociated from it. Allais is in fact convinced that humanities and social sciences, as well as physics, need a unitarian theory regarding human behavior. So, the varied areas that he deals with point towards to same global conception. This concern for synthesis meets, under an original structure, with the study of real phenomena and monetary phenomena linked to the research for conditions of a maximum efficiency of the economy.

An unsteady monetary economy cannot lead either to the fulfilment of such an efficiency, nor to the equitable distribution of income. This evidence is to be the outcome of a general theory of monetary dynamics explaining the behavior of economic agents and the origin of economic fluctuations, through a coherent unit including all the monetary phenomena. This structure allows Allais to state precisely what the monetary conditions of an efficient working market economy can be; an economy that may be ethically acceptable.

The main lines of this synthesis are going to be introduced with an effort to bring forward the original materials. To this end two main trusts of ideas will be treated with particular emphasis:

- Allais has always been looking for the existence of invariables in human behavior. To this aim he has had to define original concepts based on the importance of the hereditary conditioning of economic operators;
- from these new bases, he has been able to bring out which responsibility money and credit could have in generating and amplifying the throwing out of balance.

B. Munier (ed.), Markets, Risk and Money, 303–323.

1. THE EMERGENCE OF NEW CONCEPTS

The Nobel Prize has recognized a pioneering work: Allais has proposed new concepts such as the rate of forgetfulness, the psychological time or even the time for economic agents to react. From these concepts evolve new leading ideas among which the hereditary process of forgetfulness and the conditioning through the past, the analogy between forgetfulness and interest, and lastly the hereditary spreading of monetary phenomena. All of which allow a better understanding of the behavior of operators in monetary matters. The latter obey structural regularities as striking as the ones that can be verified in physics. To convince us, Allais has looked at the same time for invariable and unitarian behavior functions facing the problems of the demand for money and the determination of interest rates.

1.1. *Invariable Functions*

Allais takes the line followed by Fisher with his formulation of distributed lags. But the significance given is much more valuable: we are conditioned by the past, more precisely by the memory we have of it. One of his most original contributions consists in the analogy between the forgetfulness process and the one of discounting: in the same way that the influence of the future on the present is progressively reduced by discounting, in the same way the influence of the past on the present is regressively reduced by the forgetfulness process. In other words,

> At time t, the past is taken into consideration in the same way as the future ... At any moment, an instantaneous rate of forgetfulness $\chi(t)$ can be defined for the collectivity considered which plays the same part in the process of memory as that played by the psychological rate of interest $i(t)$ in evaluating the present impact of the future ... [1]

This hereditary process varies in intensity according to circumstances. We have here a contribution which is too often not recognized compared to the Anglo-Saxon formulas. For instance, when faced with hyperinflation, we are more preoccupied by the present and less by the past as compared with a 'normal' situation and *a fortiori* to a depressive situation. The concept of psychological time explains this behavior: according to circumstances, we perceive differently the lasting character of phenomena. The rate of forgetfulness which represents the rapidity with which we forget the past, becomes variable to take this reality into account.[2]

For Allais this behavior explains the way we perceive the present situation through the memory we keep of the variations in global expenditure. It can be summarized explicitly as follows:

$$(1)\qquad x = \frac{1}{D}\frac{\mathrm{d}D}{\mathrm{d}t}$$

where D represents global expenditure.

The coefficient of psychological expansion, $Z(t)$ is determined by the cumulative influence of the past rates of variations of D, weighted by a memory coefficient which is itself variable $k(t,\tau)$. $Z(t)$ represents the psychological global effect of past rates of expansion $x(\tau)$ of global expenditure $D(\tau)$.

$$(2)\qquad Z(t) = \int_{-\infty}^{t} x(\tau)k(t,\tau)\,\mathrm{d}\tau$$

with

$$k(t,\tau) = \exp^{-\int_\tau^t \chi(u)\,\mathrm{d}u}$$

which represents this memory coefficient between the moments τ and t,[3] that is to say the effect at t of an equal growth rate of 1, $(t-\tau)$ units of time previously, $\chi(t)$ representing the rate of forgetfulness.

Equation (1) may also be written in a differential form:

$$(3)\qquad \frac{\mathrm{d}Z}{\mathrm{d}t} + \chi z = x.$$

The originality of this analysis is also due to the fact that Allais gives an operational content allowing us to quantify in a unique index the total psychological appreciation of the economic situation through an invariable function of time and space. The estimation of this index, which represents the coefficient of psychological expansion $Z(t)$, assumes that the form of the function determining the rate of forgetfulness itself depends on the economic situation through the invariable function that depends on the introduction of psychological time.

The relation

$$\chi'\,\mathrm{d}t' = \chi\,\mathrm{d}t$$

allows one to transfer from one time reference to another. Psychological time is defined by the condition that the rate of forgetfulness per unit of psychological time is a constant. In the same way we have:

$$D' \, \mathrm{d}t' = D \, \mathrm{d}t.$$

Allais also assumes that the function of relative desired money balances

$$\Phi_D = \frac{M_D}{D}$$

is constant in the psychological time referential

$$\Phi'_D = \frac{M'_D}{D'} = \text{ constant.}$$

As the desired money balances are a stock, we have:

$$M_D = M'_D$$

with Ψ the function of relative desired money balances:

$$\Psi = \frac{\Phi_D}{\Phi_0}$$

Ψ_0 being a constant equal to the value of Φ_D for $Z = 0$.
We get:

$$\frac{M'_D}{D'} = \frac{M_D}{D}\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{M_D}{D}\frac{\chi}{\chi'} = \Phi_0 \Psi \frac{\chi'}{\chi}$$

for $Z = 0$ we have $\Psi = 1$, $\chi' = \chi_0$, $M'_D/D' = \Phi_0$, that is to say:

$$\chi = \frac{\chi_0}{\Psi(Z)} \tag{4}$$

an independent relation of the form that will be kept for the function of desired money balances.

The rate with which we forget the past conforms to an invariable function which remains to be specified from our behavior on the subject of holding money balances. The latter will be specified with the help of a very simple main hypothesis: the relative desired money balances, that is to say the part of their assets relative to global expenditure that people wish to hold in the form of money balances, depends upon their psychological economic appreciation.

$$(5) \qquad \Phi_D = \frac{M_D}{D} = \Phi_0 \Psi(Z)$$

and is more particularly supposed to depend on a decreasing function of Z

$$(6) \qquad \Psi(Z) = \frac{1+b}{1+be^{\alpha Z}}\,.$$

This function possesses inferior and superior limits:

$$\Psi(Z) \to 0 \quad \text{when} \quad Z \to \infty.$$

When the growth of D becomes very high (hyperinflation) the purchasing power of money balances decreases and the intermediaries wish to hold lower and lower relative money balances. Indeed, such balances can be cancelled if price increases are unlimited.

$$\Psi(Z) \to 1 + b = \max \quad \text{when} \quad Z \to -\infty.$$

When an economic situation becomes increasingly depressed, the operators want higher and higher relative money balances so as to provide against the depression. But these money balances have their maximum limits determined by the total assets of the collectivity.

We can then write:

$$(7) \qquad \Psi(Z) = \frac{2}{1+e^{Z}}\,.$$

The relative desired money balances function is itself invariable.[5] Therefore, all would be happening as if the economic agents would react in the same way, confronted with similar phenomena, whatever the institutional framework. It is worth mentioning the importance of such a result on the whole field of social sciences.

1.2. *Unitarian Functions*

From the behavior of these invariable functions Allais proposes a very original formulation explaining at the same time the behavior of money holding and the way interest rates are determined. The formulation can be summarized in the equations presented in Table I.

TABLE I

Hereditary and relativistic formulation of the demand for money.

(5)	$M_D/D = \Phi_D = \Phi_0\Psi(Z)$
(6)	$\Psi(Z) = (1+b)/(1+be^{\alpha Z})$
(2)	$Z(t) = \int_{-\infty}^{t} x(\tau)\exp^{-\int_{\tau}^{t}\chi(u)\,du}\,d\tau$
(1)	$x = \frac{1}{D}\frac{dD}{dt}$
(4)	$\chi(t)/\chi_0 = 1/Psi(Z) \quad \chi(t) = i(t)$
(3)	$(dZ/dt) + z(t)Z(t) = x(t)$
(8)	$\alpha = 1; \quad b = 1; \quad \chi_0 = 0.004$ per month
(9)	$\lvert M - M_D\rvert/M_D < \varepsilon \quad 0 < \varepsilon \ll 1$
(10)	$M^* = M_D = \Phi_0 D\Psi(Z)$
(13)	$i(t) = i_0/\Psi(Z)$

Definitions and Notations

$D(t)$ = global expenditure

$M(t)$ = money balances actually held (the money supply)

$M_D(t)$ = desired money balances

$\Phi_D(t)$ = relative desired money balances

$x(t) = d\log D/dt$ = rate of increase of global expenditure

$i(t)$ = psychological rate of interest at instant t

$\chi(t)$ = rate of forgetfulness at instant t

$Z(t)$ = coefficient of psychological economic expansion

Φ_0, i_0, χ_0 = values of Φ_D, i, χ for $Z = 0$

$\Psi(Z) = \Phi_D/\Phi_0$ = function of desired money balances.

In spite of the fact that it is often compared to Milton Friedman's model, because of its hereditary aspect, Allais' model is very different, as the following remarks make clear.

The past exerts its influence with a variable intensity. Most assuredly, the existence of a variable rate of forgetfulness of course complicates the model which can at first appear to be very complex. But this hypothesis is much more realistic and partly explains the standard of results obtained compared to the ones of the Anglo-Saxon world.[6]

With the equilibrium, when $M = M_D$, the relation between money supply and global expenditure appears under the form

$$(8) \qquad M = \Phi_0 D \Psi(Z).$$

The function $\Psi(Z)$ being specified and stable, this relation sheds a new light on the interpretation of the equation of exchange and to innumerable discussions regarding the quantity theory of money. If it is true that the latter, as Fisher has pointed out, does not imply a constant V, the fact remains that it is an incomplete theory. V is here equal to

$$(9) \qquad V = \frac{D}{M} = \frac{1}{\Phi_0}\,\Psi(Z) \quad \text{if} \quad M + M_D$$

and would therefore be a functional of the psychological economic appreciation, i.e. a result allowing to reconcile the quantity and anti-quantity positions.

The analogy between forgetfulness and interest, i.e. the assumed symmetry between past and future allows us to write:

$$(10) \qquad \frac{\chi}{\chi_0} = \frac{i}{i_0}$$

where i is set for the psychological rate of interest, i.e. the rate with which the collectivity discounts the future and i_0 the value of this rate for $Z = 0$.[7] Considering the relations (4) and (12) we have:

$$(11) \qquad i = \frac{i_0}{\Psi(Z)}\,.$$

The unitarian and synthetic character of Allais' formulation appears clearly here: a similar equation system explains the factors determining money balances behavior and the psychological rate of interest in the same throw.

From relations (8) and (13) we derive:

$$(12) \qquad M_D = \Phi_0 D \frac{i_0}{i}\,.$$

The desired money balances are proportional to the global expenditure and inversely proportional to the psychological rate of interest. But we cannot conclude from (14) any relation of causality between the market rate of interest and the desired money balances. The latter depend only upon the level of the global expenditure and of its past variations. At most, it exists through Z a relation of interdependence between M_D and i that can explain results obtained in Keynesian formulations of the demand for money.[8]

From relations (8) and (13) we deduce

$$(13) \quad i = \frac{i_0}{2}\,(1 + e^Z).$$

The psychological rate of interest increases with the appraisal of the conjunctural economic situation and therefore with the global expenditure and vice versa. But if $Z \to -\infty$ we get:

$$i = \frac{i_0}{2} = 2.4\%$$

a minimal value which, considering the relations between M_D/D and i, may make one think of an analogy to the liquidity trap.

But it is here only a relation of interdependence which, on top of that, does not let the market rate of interest interfere. The psychological rate of interest is different from the market rate and by its nature escapes direct observation. It does not result from the market forces nor from the confrontation of supply and demand, but represents the fundamental trade-off made by economic agents between present and future. It simply establishes the common element for all market rates of interest.

As early as 1947, in *Economie et Intérêt*, Allais had already laid the foundations of a general theory of interest, based on the concept of the pure interest rate:[9] to each rate of interest observed on the market corresponds a pure rate of interest that can be defined as the price of capital services i.e. as the return on a risk-free bond with no liquidity advantage and with no management expenditures. For an investment yielding an observed nominal return of j, we have:

$$i_j = j + 1_j + \mu_j - \rho_j$$

where i_j, 1_j, μ_j and ρ_j respectively denote the corresponding pure rate of interest, the liquidity premium (i.e. the extra advantage of being able to negotiate this asset before its maturity), the premium for nominal capital appreciation and the corresponding risk premium.[10] The economic mechanism tends to equalize the various pure rates of interest, on the one hand, and the pure rates of interest with the psychological rate of interest on the other hand, so that the differences noticed on the market between nominal rates must be attributed to different premiums.

This analysis has not yet been completely exploited in the literature: it could in fact allow decisive improvements concerning the theory of the term structure of interest rates and the theory of monetary dynamics, all the more because the psychological rate of interest can be estimated, in the framework of Allais' model, whereas Wicksell's or Keynes' natural rate of interest, even though it lies at the origin of numerous analyses, has never been in a position to receive numerical applications.[11]

1.3. *An Operational Formulation*

Faithful to his method, Allais has confronted his model with empirical data. Different approaches have been used, but in the limited framework of this study we shall mainly insist on the general method which consists in comparing the money supply (money balances) set against the estimated desire money balances.[12] Not being able to confront Equation (2) directly with empirical data, Allais uses the associated differential Equation (3). After having replaced the rate of forgetfulness by its value in terms of Z from Equation (3), the successive values of $Z(t)$ are calculated by an approximate integration of the differential Equation (3) starting with a given initial value of Z_1 of Z. We get the derived estimate

$$M^* = \Phi_0 D \Psi(Z)$$

of the monetary supply using the observed value $D(t)$ of global expenditure. The determination of the parameters Φ_0 and Z_1 is done by the application of a principle of least squares in such a way that the quantity

$$e^2 = \frac{1}{N} \sum_{n=1}^{N} (\log M_n - \log M_n^*)^2$$

is minimized.

One must observe that once these constants have been determined, the value of M^* depends solely on the past variations of global expenditure and is entirely independent of M.

The estimation of the psychological rate of interest by relation (15) is then instantaneous.[13]

Unlike many authors, Allais has confronted his model with data taken from many countries and for very different periods, going from deflation to hyperinflation. The quality of the adjustments as well as the stability of the results is absolutely astonishing, if the diversity of the situations analyzed is taken into account, and if one remarks that one single model allows an account to be given of such data, using only two arbitrary parameters. Moreover, the estimations of the evolution of the psychological rate of interest obtained at any given time are always very close to the general evolution of the rate of interest of the periods analyzed, as it should be. The unitarian aspect of this model is in this way perfectly confirmed by empirical data, and one must notice that no single formulation, to this day, has allowed such adjustments to be obtained at the same time for such different economic situations and for simultaneous estimations of desired money balances and of the interest rate.

It is not surprising that such results have stirred up critics. Except for the objections which have no scientific interest, problems of spurious correlation of circularity have been evoked.

Is the concordance between monetary supply and the coefficient Z not partially spurious? Such is the question asked by Cagan[14] in the case where Z, consisting of a weighted average of past variations of global expenditure, tends towards the latter and that these two series can undergo the common influence of economic cycles. It is for Allais a remark which is valid for all models of the hereditary type and that cannot in itself undermine the results obtained, as long as the factor common to its variables is not clear. And in the absence of an explanation of the existence of an underlying structure, the formulation to be maintained is the one that leads to the best results.[15]

The existence of a circularity has also been mentioned. Scadding considers that Allais' formulation is the equivalent to calculating the velocity of money from its past values.[16] Nevertheless this demonstration compares the observed velocity of money to the desired velocity and neglects in its serial developments the terms of the second order so that Allais has easily been able to demonstrate that from this point of view there was no circularity in his model. This brings forward an implicit structure between money supply and global expenditure which nothing would allow us to postulate at first.[17] Allais has furthermore, on this occasion, proposed a new empirical approach that illustrates his answer to Scadding: the coefficient Z is directly estimated from the observed velocity of money and then allows the calculation of an estimate of the rate of increase of global expenditure with the help of the differential Equation (3), which only depends on a single arbitrary parameter.[18]

The quality of the results obtained, however, is not the only characteristic of Allais' formulation which is written in a global system giving a synthetic vision of the monetary dynamics. This is the second characteristic feature of Allais' monetary theory.

2. MONETARY DYNAMICS AND ECONOMIC INSTABILITY

Allais' monetary theory forms a whole: it does not limit itself to a simple model of demand for money and of interest rate. This model is written in a general theory of monetary dynamics which explains why money is at the center of disequilibrium and what could be the monetary conditions of an efficiently working markets economy.

2.1. *General Structure of Monetary Dynamics*

From 1953 Allais has elaborated a general model of monetary dynamics explaining the generation of business cycles from the interdependence between global expenditure and money supply. We are dealing with a nonlinear model with lagged adjustment implying the existence of cycles, the period and amplitude of which are determined by the structure of the model.[19] In its first version, the fluctuations of global expenditure were supposed to be proportional to the difference between money balances and desired money balances, the factor of proportionality being equal to the opposite of a parameter called reaction time which represents the average delay between the collecting of the income and the expenditure decisions. Here appears the first formulation in the literature which brings explicitly into play the role played by the difference $M - M_D$ in economic cycles. The latter would thus result essentially from monetary and not from real factors.

This formulation has then been generalized by introducing in particular a reaction time which is not any more constant but variable through an invariant function of Z. The model provides a unitarian theory, able to explain situations as different as stability close to equilibrium, conjunctural fluctuations or hyperinflation.[20] Finally, on top of the difference $M - M_D$, Allais has introduced the variations of the money supply as an element determining fluctuations of global expenditure.[21]

Finally, and taking into account the fact that the operators are seeking to adjust the money balances they hold with the money balances they desire, the fundamental equation of monetary dynamics is written:

$$TD(t+T/2) = TR(t-T/2) + M(t) - M_D(t) + M(t+T/2) - M(t-T/2)$$

where T, D, R, M and M_D respectively represent the average period separating the expenditure decisions, the global expenditure, the global income, the money supply and the desired money balances. Global expenditure of operators in each period is equal to the amount of the income of the previous period increased by the excess of money balances on the desired money balances and of the total value of payments financed by the creation of money.[22] Since any expenditure of an operator turns out to be a return for another operator, we have:

$$D(t) = R(t)$$

$$T(t+T/2)D(t+T/2) - T(t-T/2)D(t-T/2) = M(t) - M_D(t) + M(t+T/2) - M(t-T/2)$$

that is to say as a first approximation:[23]

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{T^2}(M - M_D) + \frac{1}{T}\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right).$$

The fluctuations of global expenditure are at the same time proportional to the difference between money balances and desired money balances and to the variation of the money supply, the coefficient of proportionality being itself variable with time.[24]

The conjunctural instability is by its very nature monetary. For a given value of desired money balances, the more variable the money supply, the more the fluctuations will be increased, inducing a variation of the coefficient Z and therefore of the desired money balances, of the reaction time and of global expenditure, and so on.[25]

Money appears well in the heart of disequilibria: its irresponsible creation by the credit mechanism lies at their very origin and this fact explains the care with which Allais analyzes the credit mechanism and its implications.[26]

In the tradition of Fisher, Allais studies the historical evolution of credit without which, for him, none of the understandings of monetary mechanisms can exist. Like Fisher, Allais points out how the mechanisms of the creation of money have been improved, being established today upon bookkeeping entries, and how the creation of money or its destruction (which depends to a large extent on the decisions of private operators) fluctuates a lot with the economic fluctuations, leading to a general instability of the economy. But this analysis goes further than Fisher's. A complete study of the credit mechanism must take time deposits into account. From this point of view, one of Allais' contributions is to have pointed out that by its very nature the banking activity in the present system relies on the difference between the maturities of assets and liabilities: the key property of a bank balance sheet is that the maturities of assets exceed those of liabilities. This system is potentially unstable.

As for time deposits, the sole consideration of maturity patterns do not necessarily lead to the creation of money *ex nihilo*. A fifteen-year loan financed by five successive deposits, each of three years durations, will not create any money if the five deposits constitute a saving. The bank in this case acts as a financial intermediary: it makes use of a saving which already existed to best serve the borrower. This operation boils down to a simple translation of purchasing power.

If, on the other hand, these deposits accounts play the role of a cash balance or are considered as such by their holder, there is a duplication of purchasing power and a creation of money because two operators, the lender of the deposit account and the borrower, consider the same amount of money to be available at the same time.

The most common case is most likely to be an intermediate one, with part of the deposit account acting as a money balance: the creation of money is then only partial.

For Allais, the present credit mechanism, resting upon the fractional reserve banking system, reinforces the confusion between money balances and saving accounts which should constitute two distinct concepts, the macroeconomic effects of which are fundamentally different. The financing of a credit by a saving balance leaves the global purchasing power unchanged, whereas such financing by money balances leads to an increase of the global purchasing power and to a creation of money. The money supply is in this way made up of the whole of the assets with which the economic agents consider that they can make payments without any delay or restriction. In the present framework of the institutional credit mechanism, the amount of money supply depends upon the subjective considerations concerning the effective liquidity of a claim. For each asset there must be a cash substitutability ratio held between 0 (the asset considered is a saving) and 1 (the asset considered is a cash balance). In this way we have a first approximation:

$$M = M1 + \int_t^H \sigma(t,\theta)p(t,\theta)\, \mathrm{d}\theta$$

where $p(t,\theta)\, \mathrm{d}\theta$ represents the time deposit accounts the maturity of which it falls between θ and $\theta\ \mathrm{d}\theta$, and $\sigma(t,\theta)$ the corresponding cash substitutability ratio.[27]

The money supply is a concept of psychological order that no official definition today allows to represent correctly because of their profoundly arbitrary character. This original analysis is quite prolific. Consideration of the substitutability ratio of the assets not included in $M1$ opens the way to recent research with the aim of finding indicators which more correctly represent the money supply, especially taking weighted monetary aggregates into account. It completely renews all the discussion regarding the nature and definition of money and shows why, for two centuries, i.e. since the banking and currency controversy, all the relevant debates on the concept of liquidity have never ended. Founded on the disassociation between cash balances and saving, it points out that the arguments of a restrictive concept of money or of an enlarged concept fail to recognize the profound nature of the credit system. It illustrates the disarray of the monetary authorities *vis-à-vis* the recent development of financial innovations which increase the ambiguity between money and saving. Finally, it allows us to better understand all the difficulties of the central banks in controlling monetary aggregates, the creation of which escapes from them for the largest part.

Thus, for Allais, the credit mechanism constitutes the main factor for the amplification of economic imbalances. Business cycles stem from wide variations in the evolution of the money supply. However, the credit responsibility stretches beyond that: the conditions of an efficiently working markets economy are being endangered.

2.2. *The Conditions of an Efficiently Working Markets Economy*

Under its present organization, the credit system leads to many distortions. In the first place, the differences in the maturity structure of assets and liabilities lead to the financing of long-term credits with short-term deposits. This transformation of liquidity leads to a very dangerous situations which is the more unstable as the difference of the maturity structure increases. It also leads the economy away from a situation of maximum economic efficiency: this method of financing levels out the price of short-term and long-term resources and artificially distorts the structure of interest rates by lowering the long-term rates as compared to the short-term rates. Saving is discouraged and moves towards short-term investments, while unproductive long-term investments are encouraged. The collapse of saving and the waste of capital result from this situation.

Secondly, and in a wider way, Allais is convinced that all the economic difficulties of the western world are the result of the misunderstanding of a fundamental fact: no decentralized system of a markets economy can work properly if an uncontrolled creation of money is temporarily allowed to escape from economic adjustments. The massive banking indebtedness, financed not be a real saving but by a creation of money, makes the world economy rest upon gigantic pyramids of debts making subsequent adjustments even more difficult and increasing the potential instability of the system as the adjustments are projected into the future due to a new indebtedness. It is this banking indebtedness which is responsible for the imbalances on stock exchanges and financial markets.[28]

Finally and thirdly, and form an even wider point of view, Allais is convinced that no markets economy can work properly, that is to say efficiently and in an ethically acceptable way, unless the real value of money remains stable as time passes. This is not the case today: the *ex nihilo* and nonstop creation of money leads to the emergence of a purchasing power with no counterpart whatever, except promises to pay in the future.[29] From this we have an increase in demand and a widespread inflation, which can lead to deflation when adjustments take place. This results also in a considerable alteration in income distribution due to the appropriation of the benefits of the creation of money. The latter is inequitably distributed and every acceptable adjustment which could take place in the sharing of surpluses is ruled out.

To recapitulate, the current organization of the credit system leads to basic instability, economic inefficiency and inequity. It must be reformed.

The reform proposed by Allais is written in the tradition of the Chicago School and especially of the 100% money plan presented by Fisher after the Great Depression. However, Allais' analysis goes further: it deals with a credit reform system which must also be accompanied by fiscal reform and widespread indexation.[30]

The credit reform must be able to suppress all the main defects of the present system of credit. Its principle is to give the government the exclusive privilege of creating money and therefore a complete control of the creation of money by making the latter impossible for the banks. This principle implies that all deposits likely to be used for payments must have a total coverage by basic money. To this end, two types of banks have to be differentiated. Deposit banks submitted to a 100% money system would only be able to receive current deposits covered entirely by basic money, and would not be allowed any loans. The purpose of the banks would be to make sure that the payments on behalf of their customers are made. The services rendered would be invoiced. This follows the principles of economic efficiency which implies that every service given must be paid for.

Lending banks would continue to trade in promises to pay, but would not be allowed to receive any current deposit and would be compelled to finance their loans by borrowing on at least the same terms. This is for Allais an essential step, to avoid creation of money in the sole interest of banks, and moreover to insure optimum efficiency. All the necessary investments could be financed by this system in which saving would be remunerated at its service value according to market conditions. Such a system can only work if the two kinds of banks are distinct. This implies that all the time deposits collected by lending banks cannot in any case take the place of cash balances. And in turn this implies a profound change in banking and depositors' habits, and in the type of assets offered by lending banks. The clear dissociation between cash balances and saving that would follow would be the condition of effectiveness of such a reform.[31]

The centralization of the creation of money in the hands of the government would at the same time secure a fiscal reform, the aim of which would be to tax unearned incomes, i.e. incomes which do not correspond to any given service. In the opposite, the taxation scheme would exempt the earned incomes. The benefit of the creation of money, transferred to the collectivity and associated with a tax on capital would allow, according to Allais, the complete abolition of the whole of the income tax.

Finally, and to insure a correct working of the economy, all future commitments should be indexed in real value through the use of a stable account unit in all contracts.

The conditions for an efficiently working markets economy could then be realized. The income distribution would be ethically acceptable and the instability of the present credit system would disappear. The control of the creation of money would in fact give the central banks the possibility to insure regular and compatible monetary expansion together with stability of prices.[32] Thus business cycles, which stem from the uncontrolled evolution of the money supply itself, deriving in turn for a large part from the credit mechanism, would be eliminated or considerably decreased.

Allais' monetary theory forms a logical and synthetic structure which proposes an overall vision of monetary phenomena. This is what makes it so rich and original.

It cannot be separated from the rest of his work. By allowing him to specify the monetary conditions of an efficiently working markets economy, it is written in the line of thought which has won him the Nobel Prize: research into the conditions of maximum efficiency in the economy.

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NOTES

[1] See Allais, 1970, p. 18, and 1972a, pp. 47–48.

[2] The rate of forgetfulness is influenced by business fluctuations: the more rapidly global expenditure is increased, the higher the rate of forgetfulness. In his first works on the subject (between 1950 and 1956) Allais assumed that the rate of forgetfulness was constant. See in particular Allais 1954, §4, pp. 11–13. It was the first time this concept was proposed in the literature.

[3] The memory coefficient acts in a way similar to the one played by the discounting coefficient. $Z(t)$ is the coefficient of psychological expansion. It represents the psychological global effect of past rates of expansion $x(\tau)$ of global expenditure $D(\tau)$. On the analogy between memory and discounting see Allais, 1986, pp. 6–8.

[4] Constant equals the value χ_0 of the rate of forgetfulness for $Z = 0$.

[5] The constants b and α are determined as equal to 1 with the help of two other postulates. See especially Allais, 1965, pp. 82–89, and 1966, pp. 1135–1137.

[6] Allais' model is based on the consideration of memory. Z is a memorized quantity

and not an anticipated one, in opposition to the Anglo-Saxons. (See for example Cagan, 1969, p. 429.) Allais is very critical when it comes to the meaning of qualified quantities of anticipation in the literature. See especially Allais, 1972a, pp. 46–47, note 22.

Moreover, when the Anglo-Saxon authors consider the real cash balance in the tradition of Walras, Allais considers the relative cash balance according to Pigou's tradition, reckoning that the usefulness of money balances depends more upon the total amount of payments to be made than upon the single value of assets. Allais, 1969, pp. 443–444.

[7] When considering the rates of interest observed during periods neither inflationary neither deflationary, Allais supposes that $\chi_0 = i_0 = 4.8\%$ per annum. See note 5.

[8] The psychological rate of interest being otherwise different from the market rates.

[9] Allais, 1947, pp. 253–263.

[10] The premium for nominal capital appreciation like the liquidity premium is an added advantage and must therefore be added to the observed interest rate. The risk premium, on the contrary, presents a disadvantage that must be subtracted from the observed rate. See Allais, 1972b, p. 5, and 1974, p. 287.

[11] Any attempt to keep the pure rate of interest different from the psychological rate of interest is bound to give rise to evolutions either in the increase or in the lowering of the dynamic process. The analogy with Wicksell is strong and the analysis even more powerful because of its operational character.

[12] For a presentation and a comparison of the different methods of confronting the hereditary and relativistic theory of the demand for money with empirical data, see especially Allais, 1986. These methods can be deduced from the different relations of the model under the assumption that the relative difference $(M - M_D/M_D)$ remains relatively small.

[13] It is possible to estimate the psychological rate of interest directly, without going through the estimation of M_D in introducing new hypotheses regarding the relations between the psychological rate of interest and the market rates. See Allais, 1974, Béthenod and Durand, 1979, Durand, 1980.

[14] Cagan, 1964, pp. 3–7, and 1969, pp. 427–432.

[15] For a synthetic presentation of Allais' answers to the critics see especially Allais, 1986.

[16] Scadding, 1972, p. 151.

[17] Allais, 1972c, and 1975, pp. 455–458.

[18] The last version of this method is available in Allais, 1985, pp. 905–948.

[19] Allais, 1953, 1954, 1955. This model is, with the one of Goodwin (who considers exclusively real quantities) the first econometric model of lagged economic regulation. Already in *Economie et Intérêt*, Allais had laid the foundations of a theory of economic cycles based on the conception of a lagged regulation. See Allais, 1947, Ch. VIII, pp. 318–334, and pp. 359–369.

[20] Allais, 1955. The 1955 formulation relates in this way the instability conditions to the origin and order of magnitude of the period: the weaker the rate of forgetfulness, the stronger is the tendency to stability and the higher is the limit of the cycle period. There would be three limits: one stable (stability close to equilibrium) and two unstable (conjunctural fluctuations and hyperinflation).

[21] Allais, 1967, Vol. I, pp. 75–86.

[22] By neglecting the extra banking indebtness of the economic operators.

[23] Using a Taylor's expansion formula.
[24] As $D = MV$, we have:

$$\frac{1}{D}\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{VT^2}\left(\frac{M - M_D}{M}\right) + \frac{1}{VT}\left(\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}t}\right)$$

and as $TD = TMV$, if we put $TD = M$, we get $VT = 1$ and

$$\frac{\mathrm{d}\log D}{\mathrm{d}t} = V(\log M - \log M_D) + \frac{\mathrm{d}\log M}{\mathrm{d}t}\,.$$

See Allais, 1967, Vol. I, pp. 75–83.
[25] Allais also considers in this theoretical formulation that the money supply is also an invariable function of a banking psychological coefficient of economic expansion Z_B. In this model, the money supply is considered as a function

$$M(t) = B(t)\gamma(Z_B)$$

where $B(t)$ represents the monetary base. See Allais, 1953, 1954, 1955 and 1967. See also Allais, 1986, note 11, pp. 62–63. Unfortunately, no empirical result has yet been published.
[26] See Allais, 1975b and 1987.
[27] Allais dealt with this equation as early as 1972 and again in 1975. He considers it essential that this analysis be extended to all the elements of the assets; the corresponding substitutability ratio being weaker when the liquidity of the assets decreases.
[28] See especially Allais, 1987a, pp. 527–529, and 1987b.
[29] Allais has shown that the purchasing power created by the credit mechanism is equivalent to the updated value of interest corresponding to the creation of money. See Allais, 1975b, pp. 130–132, and 1987a, pp. 518–523.
[30] For the logic of such a reform, see Allais, 1977.
[31] This implies, in Allais' view, adequate measures of transition that do not raise any difficulty. See Allais, 1977, p. 319.
[32] In order to ease salary and price adjustments (avoiding the difficult sociological lowering of certain nominal incomes) Allais suggests that money supply be increased every year at a rate exceeding the gross national product rate of increase by 2%. In such a situation the expansion of the global expenditure would be regular and the velocity of money practically constant. In this perspective, indexation takes its full meaning. See Allais, 1977, pp. 118–119.

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BIOGRAPHICAL DATA

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Maurice, Félix, Charles Allais, born in Paris, May 31, 1911.

Recent Activities

- Professor of Economics at the Ecole Nationale Supérieure des Mines, Paris, since 1944;
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- Engineer General Honorary of the Mines National Corps.

Previous Activities

- Head of the Mines Agency, Nantes, and in charge of Railway Control, 1937–1943; Lieutenant (Artillery), French Alpine Army, September 1939–July 1940; Director of the Bureau of Mines Documentation and Statistics, Paris, 1943–1948.
- Director of a Research Unit, National Centre for Scientific Research, 1946–1980; Professor of Theoretical Economics at the Institute of Statistics of the University of Paris, 1947–1968; Director of the Group of Economic and Social Research, Paris, 1944–1970; 'Distinguished Visiting Scholar', Thomas Jefferson Center, University of Virgina, 1958–1959; Professor of Economics at the Graduate Institute of International Studies, Geneva, 1967–1970; Director of the Seminar of Monetary Analysis at the University of Paris X, 1970–1985.
- Member of the National Committee of the National Centre for Scientific Research, 1947–1980; Member of the Energy Commission of the French Economic and Social Council, 1960–1961; Chairman of the Commission of Experts for the Study of Options in the Transport Tariff Policy, European Economic Community, 1963–1964.

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Former student at the Ecole Polytechnique, 1931–1933, ranked first on graduation; Former student at the Ecole Nationale Supérieure des Mines, Paris, 1934–1936; Doctor-Engineer of the Faculty of Science of the University of Paris, 1949; Doctor *honoris causa* of the University of Groningen, 1964.

Scientific Associations
Fellow of the *International Econometric Society*, 1949; Member of the *International Statistical Institute*, 1951; Member of the *Editorial Board of the Revue d'Economie Politique*, 1952–1984; Fellow of the *New York Academy of Sciences*, 1956; Fellow of the *Operations Research Society of America*, 1958; *Member of the Editorial Board* of the international review *Econometrica* (together with Ragnar Frisch, Milton Friedman, Tjalling Koopmans, Wassily Leontieff, and Richard Stone), 1959–1969; Member of the *Council of the Econometric Society*, 1960–1965; Elected Chairman of the *Association Française de Science Economique*, July 1972, (resigned for reasons of health, October 1972); Honorary Member of the *American Economic Association*, 1976; Member of the *French Academy of Moral and Political Sciences* (1990).

National Orders
Officier dans l'Order des Palmes Académiques (December 20, 1949); Chevalier dans l'Ordre de l'Economie Nationale (March 5, 1962); Commandeur dans l'Ordre de la Légion d'Honneur (March 26, 1989); Grand-Officier dans l'Ordre National du Mérite (March 8, 1994).

SCIENTIFIC PRIZES

1933–1987

- Prix Laplace 1933 and Prix Rivot 1933 of the Academy of Sciences for his rank (first) on graduation from the Ecole Polytechnique;
- Prix Charles Dupin 1954 of the Academy of Moral and Political Sciences for his book *A la Recherche d'une Discipline Economique* (In Quest of an Ecomonic Discipline) of 1943;
- The Lanchester Prize 1958 of the Johns Hopkins University and the Operations Research Society of America for the outstanding paper on operations research published in 1957, *Method of Appraising*

Economic Prospects of Mining Exploration over Large Territories – Algerian Sahara Case Study;

- Prix Joseph Dutens 1959 of the Academy of Moral and Political Sciences for his work *La Gestion des Houillères Nationalisées et la Théorie Economique* (The Management of Nationalized Coal Mines and Economic Theory) of 1953;
- Prix Galabert 1959 of the French Astronautical Society for his research on gravity and the movement of the paraconical pendulum;
- Laureate of the 'Gravity Research Foundation', 1959, U.S.A., for his memoir *New Theoretical and Experimental Research Work on Gravity* of 1959;
- Grand Prix de la Communauté Atlantique 1960 of the French Association for the Atlantic Community for his book *L'Europe Unie, Route de la Prospérité* (United Europe and the Road of Prosperity) of 1959;
- Grand Prix André Arnoux 1968 of the Association for Economic Liberty and Social Progress for his whole work;
- Gold Medal 1970 of the Society for the Promotion of National Industry for his whole work;
- Prix Robert Blanché 1983 of the Academy of Moral and Political Sciences for his memoir *Fréquence, Probabilité et Hasard* (Frequency, Probability and Chance) of 1982;
- Grand Prix Zerilli Marimò of the Academy of Moral and Political Sciences for his whole work;
- Prix Spécial du Jury, on the occasion of the creation of the 'Prix Dupuit-de-Lesseps, 1987', for his whole work on the economics of transportations infrastructures;
- Nobel Prize in Economic Sciences, 1988;
- Médaille de l'Université de Paris X, April 27, 1989;
- Grande Médaille de la Ville de Paris, June 1, 1989.

BIBLIOGRAPHY 1943–1991

MAURICE ALLAIS: MAIN WORKS AND MEMOIRS, 1943–1991
SUMMARY

I. Theory of Economic Evolution and General Equilibrium, of Maximum Efficiency, and of the Foundations of Economic Calculus

II. Theory of Intertemporal Processes and of Optimal Capitalistic Structure

III. Theory of Choice under Uncertainty and the Criteria for Rational Economic Decisions

IV. Theory of Money, Credit, and Monetary Dynamics

V. Probability Theory, Analysis of Time Series and Their Exogenous Components

VI. Economic Policy
- Management and Economic Development
- Income Distribution and Taxation
- Monetary Policy

VII. International Economics
- Comparative Analysis of the Standards of Living and of Productivities
- International Trade and Economic Growth
- International Monetary System
- Economic Unions and Federal Unions

VIII. Sectoral Economics
- The Economics of Energy
- The Economics of Transports
- The Economics of Mining Research

IX. History of Economic Thought

X. Scientific Research, Methodology and Education

XI. Sociology and Political Science

XII. Overview on Maurice Allais' Scientific Contributions

B. Munier (ed.), Markets, Risk and Money, 329–371.

Note[1]

This Bibliography may be considered as a complete list, at the beginning of 1992, of the main works of Maurice Allais in *Economics with a scientific scope*. The works and papers relating to contemporaneous history and policy are not mentioned in this Bibliography, neither are significant works in experimental and theoretical physics.

Moreover, some communications to important scientific congresses are not indicated for brevity, when their essential content has been published elsewhere.

Some publications, on the other hand, are mentioned in this Bibliography either because of their originality at the time of their publication, or because of the light they throw on the sequence of the works, or because of the value of the summary they give of the analysis of some questions.

Each reference is given with the indications of *two dates*, that of its writing (in the margin) and that of its publication (according to the indication of the publisher). In this way, we think, it is possible to give a better understanding of the evolution of the economic thought of Maurice Allais.

I. THEORY OF ECONOMIC EVOLUTION AND GENERAL EQUILIBRIUM, OF MAXIMUM EFFICIENCY, AND OF THE FOUNDATIONS OF ECONOMIC CALCULUS

1943 *A la Recherche d'une Discipline Economique – Première Partie – l'Economie Pure* (In Quest of an Economic Discipline – Part One – Pure Economics)
First edition, Ateliers Industria, Paris, 1943, 852 pp. and Appendices, 68 pp.
Second edition, published with the help of the *Centre National de la Recherche Scientifique*, under the title: *Traité d'Economie Pure* (Treatise on Pure Economics), Imprimerie Nationale, 5 Vols., in 4°, Paris, 1952, 984pp. This second edition differs only from the first in that a 63-page introduction has been added.

[1] I want here to express my gratitude to Professor Allais as well as to several of his coworkers at the Ecole Nationale Supérieure des Mines de Paris. Without their contribution, this bibliography would not have been possible. All errors and omissions remain entirely mine (Ed.)

Third edition, Montchrestien, Paris, 1994.

1945 *Economie Pure et Rendement Social* (Pure Economics and Social Efficiency)
Sirey, Paris, 1945, 72 pp.
Second edition, Montchrestien, 1989.

1947 *Economie et Intérêt* (Economy and Interest)
See Section II below.

1948 *Le Concept d'Equilibre en Economie Politique* (The Concept of Equilibrium in Economics)
Travaux du Congrès des Economistes de Langue Française, 1948, Editions Domat-Montchrestien, 1949, pp. 66–70.

1967 *Les Conditions de l'Efficacité dans l'Economie* (The Conditions of Efficiency in the Economy)
Report presented at the Fourth International Seminar organized by the *Centro Studi e Ricerche su Problemi Economico-Sociali* of Milan, September 12–14, 1967, 145 pp.
Published in Italian under the title *Le Condizioni dell'Efficienza nell'Economia* in *Programmazione e Progresso Economico*, Centro Studi e Ricerche su Problemi Economico-Sociali, Franco Angeli Editore, Milano, 1969, pp. 13–153, 169–181, 207–208 and 221–242.
Republished by the Ecole Nationale Supérieure des Mines de Paris (E.N.S.M.P.), 1968.

1967 *Les Fondements du Calcul Economique* (The Foundations of Economic Calculus)
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1968 *The Conditions of Efficiency in the Economy*
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1971 *Les Théories de l'Equilibre Economique Général et de l'Efficacité Maximale – Impasses Récentes et Nouvelles Perspectives* (Theo-

ries of General Economic Equilibrium and Maximum Efficiency – Recent Blind Alleys and New Perspectives)
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Revue d'Economie Politique, May–June 1971, no. 3, pp. 331–409.
Spanish translation: *Las Teorias del Equilibrio Economico General y de la Eficacia Maxima – Recientes Callejones sin Salida y Nuevas Perspectivas, El Trimestre Economico* (Mexico), July–September 1972, Vol. XXXIX (3), No. 155, pp. 557–633.

1973 *La Théorie Générale des Surplus et l'Apport Fondamental de Vilfredo Pareto*
Revue d'Economie Politique, November–December 1973, No. 6, 83è année, pp. 1044–1097.

1973 *The General Theory of Surplus and Pareto's Fundamental Contribution*
Convegno Internationale Vilfredo Pareto (Roma, 25–27 October 1973), Roma, Academia Nazionale dei Lincei, 1975, pp. 109–163. (English translation of Allais, 1973).

1974 *Theories of General Economic Equilibrium and Maximum Efficiency*
In the volume: *Equilibrium and Disequilibrium in Economic Theory*, D. Reidel Publishing Company, Dordrecht, 1977, pp. 129–201. (English translation of Allais, 1971. The English version only differs from the French one in a few marginal additions, particularly in note 18).

1975 *Les Implications des Rendements Croissants et Décroissants sur les Conditions de l'Equilibre Economique Général et d'une Efficacité Maximale* (The Implications of Scale Economies and Diseconomies in the Production Functions for the Conditions for General Economic Equilibrium and Maximum Efficiency)
In the volume: *Hommage à François Perroux*, Presses Universitaires de Grenoble, 1978, pp. 605–674.

1975 *De Quelques Contributions Significatives à la Théorie des Surplus de David Ricardo à John Hicks* (Some Significant Contributions

to the Theory of Surpluses from David Ricardo to John Hicks)
Mimeo, Centre d'Analyse Economique, E.N.S.M.P., 1975, 70 pp.

1978 *La Théorie Générale des Surplus* (The General Theory of Surpluses)
Economies et Sociétés, Paris, January–May 1981, Institut de Sciences Mathématiques et Economiques, 2 Vols., 718 pp.
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1987 *The Equimarginal Principle, Meaning, Limits and Generalization*
Rivista Internazionale di Scienze Economiche e Commerciali, Vol. 34, No. 8, 1987, pp. 689–750.

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1960 *Influence du Coefficient Capitalistique sur le Revenu Réel par Tête* (The Influence of the Capital Output Ratio on Real per Capital Income)
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1961 *The Influence of the Capital Output Ratio on Real National Income*
Walras Bowley Lecture, American Meeting of the Econometric Society, New York, December 28, 1961, *Econometrica*, Vol. 30,

No. 4, October 1962.
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1963 *Some Analytical and Practical Aspects of the Theory of Capital*
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Published in: *Activity Analysis in the Theory of Growth and Planning*, E. Malinvaud and M. Bacharach (Eds.), MacMillan, London and St. Martin's Press, New York, 1967, pp. 64–107.

1963 *The Role of Capital in Economic Development*
Memoir presented at the *Semaine d'Etude sur le Rôle de l'Analyse Econométrique dans la Formulation de Plans de Développement*, organized by the Pontifical Adademy of Social Sciences (Rome, 7–13 October, 1963).
Published in the volume *Le Rôle de l'Analyse Econométrique dans la Formulation de Plans de Développement*, Pontificiae Academiae Scientiarum Scripta Varia, 28, Pontifica Academia Scientiarium, 1965, Vol. II, pp. 697–1002.
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1966 *Produit National Brut Réel, Revenu National Réel, et Fonction de Cobb–Douglas* (Real Gross National Product, Real National Income, and the Cobb–Douglas Function)
Memoir presented to the *Symposium International sur la Statistique en tant que Méthodologie des Sciences Sociales*, University of Rome, March 13–15, 1966.
Paper published in the volume: *La Statistica Come Methodologia Delle Scienze Sociali*, in honor of Professor Gini, Statistical Institute, Rome 1966. And also: *Metron*, Vol. XXV, No. 1–4, 31 XII 1966, pp. 310–355.

1966 *L'Impôt sur le Capital* (Tax on Capital)
Droit Social, No. 29, special issue, September–October 1966, pp. 465–544. Librairie Sociale et Economique, 3 rue Soufflot,

Paris, 5e.

1968 *A Theorem about the Optimum Accumulation of Capital*
Centre d'Analyse Economique, E.N.S.M.P., 1968, 27 pp.

1968 *Growth and Inflation*
American Bankers Association, Conference of University Monetary Economists, Ditchley Park, September 10–13, 1968.
Journal of Money, Credit and Banking, August 1969, pp. 355–426 and 427–462.

1978 *Les Incidences quant à l'Efficacité sur l'Economie d'un Impôt Général sur le Capital* (Incidences on Economic Efficiency of a General Tax on Capital)
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III. THEORY OF CHOICE UNDER UNCERTAINTY AND THE CRITERIA FOR RATIONAL ECONOMIC DECISIONS

1952 *Fondements d'une Théorie Positive des Choix Comportant un Risque* (The Foundations of a Positive Theory of Choices Involving Risk)
Memoir presented at the Paris International Symposium of the Centre National de la Recherche Scientifique on the *Fondements et Applications de la Théorie du Risque en Econométrie* (Foundations and Applications of the Theory of Risk in Econometrics), Paris, May 12–17, 1952.
This Memoir was published in the volume: *Econométrie, Collection des Colloques Internationaux du Centre National de la Recherche Scientifique*, Vol. XL, Paris, 1953, pp. 127–140.

1952 *La Généralisation des Théories de l'Equilibre Economique Général et du Rendement Social au Cas du Risque* (Generalization

of the Theories of General Economic Equilibrium and Economic Efficiency to the Case of Risk)

Memoir presented at the Paris International Symposium of the Centre National de la Recherche Scientifique on the *Fondements et Applications de la Théorie du Risque en Econométrie* (Foundations and Applications of the Theory of Risk in Econometrics), Paris, May 12–17, 1952.

This memoir was published in the volume: *Econométrie, Collection des Colloques Internationaux du Centre National de la Recherche Scientifique*, Vol. XL, Paris, 1953, pp. 81–120.

1952 *Fondements d'une Théorie Positive des Choix Comportant un Risque et Critique des Postulats et Axiomes de l'Ecole Américaine* (The Foundations of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School)

Comprehensive memoir written further to the debates of the Paris International Symposium.

Actes du Colloque (Proceedings of the Symposium), *ibid.* pp. 257–332. This memoir was republished in 1955 with a few minor additions in a special issue of the *Annales des Mines*, and as a separate volume by the Imprimerie Nationale (56 pp. in 4°).

English translation: see below, 1979.

1952 *La Psychologie de l'Homme Rationnel devant le Risque. La Théorie et l'Expérience* (The Psychology of Rational Behavior in the Presence of Risk – Theory and Experience)

Journal de la Société de Statistique de Paris, January–March 1953, pp. 47–73.

1952 *L'Extension des Théories de l'Equilibre Economique Général et du Rendement Social au Cas du Risque* (Extension of the Theories of General Economic Equilibrium and Economic Efficiency to the Case of Risk)

Econometrica, Vol. 21, No. 2, April 1953, pp. 269–290.

1952 *Le Comportement de l'Homme Rationnel devant le Risque. Critique des Postulats et Axiomes de l'Ecole Américaine* (Rational

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1954 *Evaluation des Perspectives Economiques de la Recherche Minière sur de Grands Espaces – Applications au Sahara Algérien*
Study conducted for the Bureau de Recherche Minières de l'Algérie (1953–1954).
Memoir presented at the Petropolis (Brazil) Conference of the International Statistical Institute and at the Centenary Conference of Mineral Industry, Paris, June 1955.
Bulletin de l'Institut International de Statistique, Tome XXXV, 4 Rio de Janeiro, 1957, pp. 89–140, and:
Revue de l'Industrie Minérale, Paris, January 1956, pp. 329–383.
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Study made for the Bureau de Recherches Minières de l'Algérie.
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Revue d'Economie Politique, 1953, No. 3, pp. 381–390.

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Maurice Allais and Ole Hagen (Eds.), D. Reidel Publishing Company, Dordrecht, 1979, 714 pp.

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2. 'The Foundations of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School' (pp. 25–145).
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3. 'The So-Called Allais Paradox and Rational Decisions under Uncertainty' (pp. 434–698)

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In: *Progress in Decision Theory*, O. Hagen and F. Wenstøp (Eds.), D. Reidel Publishing Company, Dordrecht, 1984, pp. 3–131.

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1984 *Determination of Cardinal Utility According to an Intrinsic Invariant Model*
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In: *Risk, Decision and Rationality*, Bertrand Munier (Ed.), D. Reidel Publishing Company, Dordrecht, 1988, pp. 231–289.

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