SELECTED PAPERS

of Charles H. Hinton

about the fourth dimension

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About the Fourth Dimension

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What Is the Fourth Dimension?

1884

Chapter I

AT THE PRESENT TIME our actions are largely influenced by our theories. We have abandoned the simple and instinctive mode of life of the earlier civilizations for one regulated by the assumptions of our knowledge and supplemented by all the devices of intelligence. In such a state it is possible to conceive that a danger may arise, not only from a want of knowledge and practical skill, but even from the very presence and possession of them in any one department, if there is a lack of information in other departments. If, for instance, with our present knowledge of physical laws and mechanical skill, we were to build houses without regard to the conditions laid down by physiology, we should probably—to suit an apparent convenience—make them perfectly draught-tight, and the best-constructed mansions would be full of suffocating chambers. The knowledge of the construction of the body and the conditions of its health prevent it from suffering injury by the development of our powers over nature.

In no dissimilar way the mental balance is saved from the dangers attending an attention concentrated on the laws of mechanical science by a just consideration of the constitution of the knowing faculty, and the conditions of knowledge. Whatever pursuit we are engaged in, we are acting consciously or unconsciously upon some theory, some view of things. And when the limits of daily routine are continually narrowed by the ever-increasing complication of our civilization, it becomes doubly important that not one only but every kind of thought should be shared in.

There are two ways of passing beyond the domain of practical certainty, and of looking into the vast range of possibility. One is by asking, “What is knowledge? What constitutes experience?” If we adopt this course we are plunged into a sea of speculation. Were it not that the highest faculties of the mind find therein so ample a range, we should return to the solid ground of facts, with simply a feeling of relief at escaping from so great a confusion and contradictoriness.

The other path which leads us beyond the horizon of actual experience is that of questioning whatever seems arbitrary and irrationally limited in the domain of knowledge. Such a questioning has often been successfully applied in the search for new facts. For a long time four gases were considered incapable of being reduced to the liquid state. It is but lately that a physicist has succeeded in showing that there is no such arbitrary distinction among gases. Recently again the question has been raised, “Is there not a fourth state of matter?” Solid, liquid, and gaseous states are known. Mr. Crookes attempts to demonstrate the existence of a state differing from all of these. It is the object of these pages to show that, by supposing away certain limitations of the fundamental conditions of existence as we know it, a state of being can be conceived with powers far transcending our own. When this is made clear it will not be out of place to investigate what relations would subsist between our mode of existence and that which will be seen to be a possible one.

In the first place, what is the limitation that we must suppose away?

An observer standing in the corner of a room has three directions naturally marked out for him; one is upwards along the line of meeting of the two walls; another is forwards where the floor meets one of the walls; a third is sideways where the floor meets the other wall. He can proceed to any part of the floor of the room by moving first the right distance along one wall, and then by turning at right angles and walking parallel to the other wall. He walks in this case first of all in the direction of one of the straight lines that meet in the corner of the floor, afterwards in the direction of the other. By going more or less in one direction or the other, he can reach any point on the floor, and any movement, however circuitous, can be resolved into simple movements in these two directions.

But by moving in these two directions he is unable to raise himself in the room. If he wished to touch a point in the ceiling, he would have to move in the direction of the line in which the two walls meet. There are three directions then, each at right angles to both the other, and entirely independent of one another. By moving in these three directions or combinations of them, it is possible to arrive at any point in a room. And if we suppose the straight lines which meet in the corner of the room to be prolonged indefinitely, it would be possible by moving in the direction of those three lines, to arrive at any point in space. Thus in space there are three independent directions, and only three; every other direction is compounded of these three. The question that comes before us then is this. "Why should there be three and only three directions?" Space, as we know it, is subject to a limitation.

In order to obtain an adequate conception of what this limitation is, it is necessary to first imagine beings existing in a space more limited than that in which we move. Thus we may conceive a being who has been throughout all the range of his experience confined to a single straight line. Such a being would know what it was to move to and fro, but no more. The whole of space would be to him but the extension in both directions of the straight line to an infinite distance. It is evident that two such creatures could never pass one another. We can conceive their coming out of the straight line and entering it again, but they having moved always in one straight line, would have no conception of any other direction of motion by which such a result could be effected. The only shape which could exist in a one-dimensional existence of this kind would be a finite straight line. There would be no difference in the shapes of figures; all that could exist would simply be longer or shorter straight lines.

Again, to go a step higher in the domain of a conceivable existence. Suppose a being confined to a plane superficies, and throughout all the range of its experience never to have moved up or down, but simply to have kept to this one plane. Suppose, that is, some figure, such as a circle or rectangle, to be endowed with the power of perception; such a being if it moves in the plane superficies in which it is drawn, will move in a multitude of directions; but, however varied they may seem to be, these directions will all be compounded of two, at right angles to each other. By no movement so long as the plane superficies remains perfectly horizontal, will this being move in the direction we call up and down. And it is important to notice that the plane would be different to a creature confined to it, from what it is to us. We think of a plane habitually as having an upper and a lower side, because it is only by the contact of solids that we realize a plane. But a creature which had been confined to a plane during its whole existence would have no idea of there being two sides to the plane he lived in. In a plane there is simply length and breadth. If a creature in it be supposed to know of an up or down he must already have gone out of the plane.

Is it possible, then, that a creature so circumstanced would arrive at the notion of there being an up and down, a direction different from those to which he had been accustomed, and having nothing in common with them? Obviously nothing in the creature's circumstances would tell him of it. It could only be by a process of reasoning on his part that he could arrive at such a conception. If he were to imagine a being confined to a single straight line, he might realize that he himself could move in two directions, while the creature in a straight line could only move in one. Having made this reflection he might ask, "But why is the number of directions limited to two? Why should there not be three?"

A creature (if such existed), which moves in a plane would be much more fortunately circumstanced than one which can only move in a straight line. For, in a plane, there is a possibility of an infinite variety of shapes, and the being we have supposed could come into contact with an indefinite number of other beings. He would not be limited, as in the case of the creature in a straight line, to one only on each side of him.

It is obvious that it would be possible to play curious tricks with a being confined to a plane. If, for instance, we suppose such a being to be inside a square, the only way out that he could conceive would be through one of the sides of the square. If the sides were impenetrable, he would be a fast prisoner, and would have no way out.

What his case would be we may understand, if we reflect what a similar case would be in our own existence. The creature is shut in in all the directions he knows of. If a man is shut in in all the directions he knows of, he must be surrounded by four walls, a roof and a floor. A two-dimensional being inside a square would be exactly in the same predicament that a man would be, if he were in a room with no opening on any side. Now it would be possible to us to take up such a being from the inside of the square, and to set him down outside it. A being to whom this had happened would find himself outside the place he had been confined in, and he would not have passed through any of the boundaries by which he was shut in. The astonishment of such a being can only be imagined by comparing it to that which a man would feel, if he were suddenly to find himself outside a room in which he had been, without having passed through the window, doors, chimney or any opening in the walls, ceiling or floor.

Another curious thing that could be effected with a two-dimensional being, is the following. Conceive two beings at a great distance from one another on a plane surface. If the plane surface is bent so that they are brought close to one another, they would have no conception of their proximity, because to each the only possible movements would seem to be movements in the surface. The two beings might be conceived as so placed, by a proper bending of the plane, that they should be absolutely in juxtaposition, and yet to all the reasoning faculties of either of them a great distance could be proved to intervene. The bending might be carried so far as to make one being suddenly appear in the plane by the side of the other. If these beings were ignorant of the existence of a third dimension, this result would be as marvellous to them, as it would be for a human being who was at a great distance—it might be at the other side of the world—to suddenly appear and really be by our side, and during the whole time he not to have left the place in which he was.

Chapter II

THE FOREGOING EXAMPLES MAKE IT CLEAR that beings can be conceived as living in a more limited space than ours. Is there a similar limitation in the space we know?

At the very threshold of arithmetic an indication of such a limitation meets us.

If there is a straight line before us two inches long, its length is expressed by the number 2. Suppose a square to be described on the line, the number of square inches in this figure is expressed by the number 4, *i.e.*, 2 x 2. This 2 x 2 is generally written 2^2, and named "2 squared."

Now, of course, the arithmetical process of multiplication is in no sense identical with that process by which a square is generated from the motion of a straight line, or a cube from the motion of a square. But it has been observed that the units resulting in each case, though different in kind, are the same in number.

If we touch two things twice over, the act of touching has been performed four times. Arithmetically, 2 x 2 = 4. If a square is generated by the motion of a line two inches in length, this square contains four square inches.

So it has come to pass that the second and third powers of numbers are called "square" and "cube."

We have now a straight line two inches long. On this a square has been constructed containing four square inches. If on the same line a cube be constructed, the number of cubic inches in the figure so made is 8, i.e., 2 x 2 x 2 or 23. Here, corresponding to the numbers 2, 2^2, 2^3, we have a series of figures. Each figure contains more units than the last, and in each the unit is of a different kind. In the first figure a straight line is the unit, viz., one linear inch; it is said to be of one dimension.

In the second a square is the unit, viz., one square inch. The square is a figure of two dimensions. In the third case a cube is the unit, and the cube is of three dimensions. The straight line is said to be of one dimension because it can be measured only in one way. Its length can be taken, but it has no breadth or thickness. The square is said to be of two dimensions because it has both length and breadth. The cube is said to have three dimensions, because it can be measured in three ways.

The question naturally occurs, looking at these numbers 2, 2^2, 2^3, by what figure shall we represent 2^4, or 2 x 2 x 2 x 2. We know that in the figure there must be sixteen units, or twice as many units as in the cube. But the unit also itself must be different. And it must not differ from a cube simply in shape. It must differ from a cube as a cube differs from a square. No number of squares will make up a cube, because each square has no thickness. In the same way, no number of cubes must be able to make up this new unit. And here, instead of trying to find something already known, to which the idea of a figure corresponding to the fourth power can be affixed, let us simply reason out what the properties of such a figure must be. In this attempt we have to rely, not on a process of touching or vision, such as informs us of the properties of bodies in the space we know, but on a process of thought. Each fact concerning this unknown figure has to be reasoned out; and it is only after a number of steps have been gone through, that any consistent familiarity with its properties is obtained. Of all applications of the reason, this exploration is perhaps the one which requires, for the simplicity of the data involved, the greatest exercise of the abstract imagination, and on this account is well worth patient attention. The first steps are very simple. We must imagine a finite straight line to generate a square by moving on the plane of the paper, and this square in its turn to generate a cube by moving vertically upwards. Figure 1 represents a straight line; figure 2 represents a square formed by the motion of that straight line; figure 3 represents perspectively a cube formed by the motion of that square ABCD upwards. It would be well, instead of using figure 3, to place a cube on the paper. Its base would be ABCD, its upper surface EFGH.

The straight line AB gives rise to the square ABCD by a movement at right angles to itself. If motion be confined to the straight line AB, a backward and forward motion is the only one possible. No sideway motion is admissible. And if we suppose a being to exist which could only move in the straight line AB, it would have no idea of any other movement than to and fro. The square ABCD is formed from the straight line

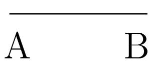

Figure 1

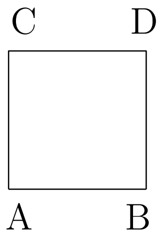

Figure 2

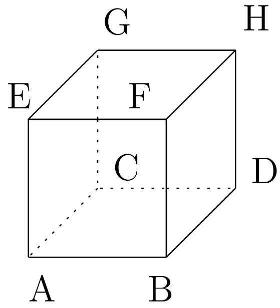

Figure 3

by a movement in a direction entirely different from the direction which exists in AB. This motion is not expressible by means of any possible motion in AB. A being which existed in AB, and whose experience was limited to what could occur in AB, would not be able to understand the instructions we should give to make AB trace out the figure ABCD.

In the figure ABCD there is a possibility of moving in a variety of directions, so long as all these directions are confined to one plane. All directions in this plane can be considered as compounded of two, from A to B, and from A to C. Out of the infinite variety of such directions there is none which tends in a direction perpendicular to figure 2; there is none which tends upwards from the plane of the paper. Conceive a being to exist in the plane, and to move only in it. In all the movements which he went through there would be none by which he could conceive the alteration of figure 2 into what figure 3 represents in perspective. For 2 to become 3 it must be supposed to move perpendicularly to its own plane. The figure it traces out is the cube ABCDEFGH.

All the directions, manifold as they are, in which a creature existing in figure 3 could move, are compounded of three directions. From A to B, from A to C, from A to E, and there are no other directions known to it.

But if we suppose something similar to be done to figure 3, something of the same kind as was done to figure 1 to turn it into figure 2, or to figure 2 to turn it into figure 3, we must suppose the whole figure as it exists to be moved in some direction entirely different from any direction within it, and not made up of any combination of the directions in it. What is this? It is the fourth direction.

We are as unable to imagine it as a creature living in the plane figure 2 would be to imagine a direction such that moving in it the square 2 would become the cube 3. The third dimension to such a creature would be as unintelligible as the fourth is to us. And at this point we have to give up the aid that is to be got from any presentable object, and we have simply to investigate what the properties of the simplest figure in four dimensions are, by pursuing further the analogy which we know to exist between the process of formation of 2 from 1 and of 3 from 2, and finally of 4 from 3. For the sake of convenience, let us call the figure we are investigating—the simplest figure in four dimensions—a four-square.

First of all we must notice, that if a cube be formed from a square by the movement of the square in a new direction, each point of the interior of the square traces out part of the cube. It is not only the bounding lines that by their motion form the cube, but each portion of the interior of the square generates a portion of the cube. So if a cube were to move in the fourth dimension so as to generate a four-square, every point in the interior of the cube would start *de novo*, and trace out a portion of the new figure uninterfered with by the other points.

Or, to look at the matter in another light, a being in three dimensions, looking down on a square, sees each part of it extended before him, and can touch each part without having to pass through the surrounding parts, for he can go from above, while the surrounding parts surround the part he touches only in one plane.

So a being in four dimensions could look at and touch every point of a solid figure. No one part would hide another, for he would look at each part from a direction which is perfectly different from any in which it is possible to pass from one part of the body to another. To pass from one part of the body to another it is necessary to move in three directions, but a creature in four dimensions would look at the solid from a direction which is none of these three.

Let us obtain a few facts about the fourth figure, proceeding according to the analogy that exists between 1, 2, 3, and 4. In the figure 1 there are two points. In 2 there are four points—the four corners of the square. In 3 there are eight points. In the next figure, proceeding according to the same law, there would be sixteen points.

In the figure 1 there is one line. In the square there are four lines. In the cube there are twelve lines. How many lines would there be in the four-square? That is to say that there are three numbers–1, 4, and 12. What is the fourth, going on accordingly to the same law?

To answer this question let us trace out in more detail how the figures change into one another. The line, to become the square, moves; it occupies first of all its original position, and last of all its final position. It starts as AB, and ends as CD; thus the line appears twice, or it is doubled. The two other lines in the square, AC, BD, are formed by the motions of the points at the extremities of the moving line. Thus, in passing from the straight line to the square the lines double themselves, and each point traces out a line. If the same procedure holds good in the case of the change of the square into the cube, we ought in the cube to have double the number of lines as in the square—that is eight–and every point in the square ought to become a line. As there are four points in the square, we should have four lines in the cube from them, that is, adding to the previous eight, there Should be twelve lines in the cube. This is obviously the case. Hence we may with confidence, to deduce the number of lines in a four-square, apply this rule. *Double the number of lines in the previous figure, and add as many lines as there are points in the previous figure*. Now in the cube there are twelve lines and eight points. Hence we get 2 x 12 + 8, or thirty-two lines in the four-square.

In the same way any other question about the four-square can be answered. We must throw aside our realizing power and answer in accordance with the analogy to be worked out from the three figures we know.

Thus, if we want to know how many plane surfaces the four-square has, we must commence with the line, which has none; the square has one; the cube has six. Here we get the three numbers, 0, 1, and 6. What is the fourth?

Consider how the planes of the cube arise. The square at the beginning of its motion determines one of the faces of the cube, at the end it is the opposite face, during the motion each of the lines of the square traces out one plane face of the cube. Thus we double the number of planes in the previous figure, and every line in the previous figure traces out a plane in the subsequent one.

Apply this rule to the formation of a square from a line. In the line there is no plane surface, and since twice nothing is nothing, we get, so far, no surface in the square; but in the straight line there is one line, namely itself, and this by its motion traces out the plane surface of the square. So in the square, as should be, the rule gives one surface.

Applying this rule to the case of the cube, we get, doubling the surfaces, 12; and adding a plane for each of the straight lines, of which there are 12, we have another 12, or 24 plane surfaces in all. Thus just as by handling or looking at it, it is possible to describe a figure in space, so by going through a process of calculation it is within our power to describe all the properties of a figure in four dimensions.

There is another characteristic so remarkable as to need a special statement. In the case of a finite straight line, the boundaries are points. If we deal with one dimension only, the figure 1, that of a segment of a straight line, is cut out of and separated from the rest of an imaginary infinitely long straight line by the two points at its extremities. In this simple case the two points correspond to the bounding surface of the cube. In the case of a two-dimensional figure an infinite plane represents the whole of space. The square is separated off by four straight lines, and it is impossible for an entry to be made into the interior of the square, except by passing through the straight lines. Now, in these cases, it is evident that the boundaries of the figure are of one dimension less than the figure itself. Points bound lines, lines bound plane figures, planes bound solid figures. Solids then must bound four dimensional figures. The four-square will be bounded in the following manner. First of all there is the cube which, by its motion in the fourth direction, generates the figure. This, in its initial position, forms the base of the four-square. In its final position it forms the opposite end. During the motion each of the faces of the cube give rise to another cube. The direction in which the cube moves is such that of all the six sides none is in the least inclined in that direction. It is at right angles to all of them. The base of the cube, the top of the cube, and the four sides of the cube, each and all of them form cubes. Thus the four-square is bounded by eight cubes. Summing up, the four-square would have 16 points, 32 lines, 24 surfaces, and it would be bounded by 8 cubes.

If a four-square were to rest in space it would seem to us like a cube.

To justify this conclusion we have but to think of how a cube would appear to a two-dimensional being. To come within the scope of his faculties at all, it must come into contact with the plane in which he moves. If it is brought into as close a contact with this plane as possible, it rests on it by one of its faces. This face is a square, and the most a two-dimensional being could get acquainted with of a cube would be a square.

Having thus seen how it is possible to describe the properties of the simplest shape in four dimensions, it is evident that the mental construction of more elaborate figures is simply a matter of time and patience.

In the study of the form and development of the chick in the egg, it is impossible to detect the features that are sought to be observed, except by the use of the microscope. The specimens are accordingly hardened by a peculiar treatment and cut into thin sections. The investigator going over each of these sections, noticing all their peculiarities, constructs in his mind the shape as it originally existed from the record afforded by an indefinite number of slices. So, to form an idea of a four-dimensional figure, a series of solid shapes bounded on every side differing gradually from one another, proceeding, it may be, to the most diverse forms, has to be mentally grasped and fused into a unitary conception.

If, for instance, a small sphere were to appear, this to be replaced by a larger one, and so on, and then, when the largest had appeared, smaller and smaller ones to make their appearance, what would be witnessed would be a series of sections of a four-dimensional sphere. Each section in space being a sphere.

Again, just as solid figures can be represented on paper by perspective, four-dimensional figures can be represented perspectively by solids. If there are two squares, one lying over the other, and the underneath one be pushed away, its sides remaining parallel with the one that was over it, then if each point of the one be joined to the corresponding point of the other, we have a fair representation on paper of a cube. Figure 3 may be considered to be such a representation if the square CDGH be considered to be the one that has been pushed away from lying originally under the square ABEF. Each of the planes which bound the cube is represented on the paper. The only thing that is wanting is the three-dimensional content of the cube. So if two cubes be placed with their sides parallel, but one somewhat diagonally with regard to the other, and all their corresponding points be supposed joined, there will be found a set of solid figures, each representing (though of course distortedly) the bounding cubes of the four-dimensional figure, and every plane and line in the four-dimensional figure will be found to be represented in a kind of solid perspective. What is wanting is of course the four-dimensional content.

Chapter III

HAVING NOW PASSED IN REVIEW SOME OF THE PROPERTIES of four-dimensional figures, it remains to ask what relations beings in four dimensions, if they did exist, would have with us.

And in the first place, a being in four dimensions would have to us exactly the appearance of a being in space. A being in a plane would only know solid objects as two-dimensional figures—the shapes namely in which they intersected his plane. So if there were four-dimensional objects, we should only know them as solids—the solids, namely, in which they intersect our space. Why, then, should not the four-dimensional beings be ourselves, and our successive states the passing of them through the three-dimensional space to which our consciousness is confined?

Let us consider the question in more detail. And for the sake of simplicity transfer the problem to the case of three and two dimensions instead of four and three.

Suppose a thread to be passed through a thin sheet of wax placed horizontally. It can be passed through in two ways. Either it can be pulled through, or it can be held at both ends, and moved downwards as a whole. Suppose a thread to be grasped at both ends, and the hands to be moved downwards perpendicularly to the sheet of wax. If the thread happens to be perpendicular to the sheet it simply passes through it, but if the thread be held, stretched slantingwise to the sheet, and the hands are moved perpendicularly downwards, the thread will, if it be strong enough, make a slit in the sheet.

If now the sheet of wax were to have the faculty of closing up behind the thread, what would appear in the sheet would be a moving hole.

Suppose that instead of a sheet and a thread, there were a straight line and a plane. If the straight line were placed slantingwise in reference to the plane and moved downwards, it would always cut the plane in a point, but that point of section would move on. If the plane were of such a nature as to close up behind the line, if it were of the nature of a fluid, what would be observed would be a moving point. If now there were a whole system of lines sloping in different directions, but all connected together, and held absolutely still by one framework, and if this framework with its system of lines were as a whole to pass slowly through the fluid plane at right angles to it, there would then be the appearance of a multitude of moving points in the plane, equal in number to the number of straight lines in the system. The lines in the framework will all be moving at the same rate—namely, at the rate of the framework in which they are fixed. But the points in the plane will have different velocities. They will move slower or faster, according as the lines which give rise to them are more or less inclined to the plane. A straight line perpendicular to the plane will, on passing through, give rise to a stationary point. A straight line that slopes very much inclined to the plane will give rise to a point moving with great swiftness. The motions and paths of the points would be determined by the arrangement of the lines in the system. It is obvious that if two straight lines were placed lying across one another like the letter X, and if this figure were to be stood upright and passed through the plane, what would appear would be at first two points. These two points would approach one another. When the part where the two strokes of the X meet came into the plane, the two points would become one. As the upper part of the figure passed through, the two points would recede from one another.

If the line be supposed to be affixed to all parts of the framework, and to loop over one another, and support one another[1], it is obvious that they could assume all sorts of figures, and that the points on the plane would move in very complicated paths. Figure 4 represents a section of such a framework. Two lines XX and YY are shown, but there must be supposed to be a great number of others sloping backwards and forwards as well as sideways.

Let us now assume that instead of lines, very thin threads were attached to the framework: they on passing through the fluid plane would give rise to very small spots. Let us call the spots atoms, and I regard them as constituting a material system in the plane. There are four conditions which must be satisfied by these spots if they are to be admitted as forming a material system such as ours. For the ultimate properties

[1] ABCD framework, X and Y two lines interlinked

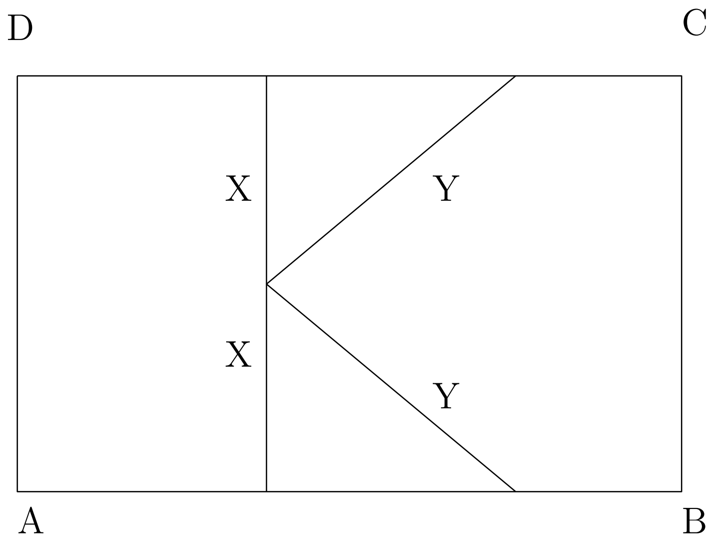

Figure 4

of matter (if we eliminate attractive and repulsive forces, which may be caused by the motions of the smallest particles), are—1, Permanence; 2, Impenetrability; 3, Inertia; 4, Conservation of energy.

According to the first condition, or that of permanence, no one of these spots must suddenly cease to exist. That is, the thread which by sharing in the general motion of the system gives rise to the moving point, must not break off before the rest of them. If all the lines suddenly ended this would correspond to a ceasing of matter.

2. Impenetrability.—One spot must not pass through another. This condition is obviously satisfied. If the threads do not coincide at any point, the moving spots they give rise to cannot.

3. Inertia.—A spot must not cease to move or cease to remain at rest without coming into collision with another point. This condition gives the obvious condition with regard to the threads, that they, between the points where they come into contact with one another, must be straight. A thread which was curved would, passing through the plane, give rise to a point which altered in velocity spontaneously. This the particles of matter never do.

4. Conservation of energy.—The energy of a material system is never lost; it is only transferred from one form to another, however it may seem to cease. If we suppose each of the moving spots on the plane

to be the unit of mass, the principle of the conservation of energy demands that when any two meet, the sum of the squares of their several velocities before meeting shall be the same as the sum of the squares of their velocities after meeting. Now we have seen that any statement about the velocities of the spots in the plane is really a statement about the inclinations of the threads to the plane. Thus the principle of the conservation of energy gives a condition which must be satisfied by the inclinations of the threads of the plane. Translating this statement, we get in mathematical language the assertion that the sum of the squares of the tangents of the angles the threads make with the normal to the plane remains constant.

Hence, all complexities and changes of a material system made up of similar atoms in a plane could result from the uniform motion as a whole of a system of threads.

We can imagine these threads as weaving together to form connected shapes, each complete in itself, and these shapes as they pass through the fluid plane give rise to a series of moving points. Yet, inasmuch as the threads are supposed to form consistent shapes, the motion of the points would not be wholly random, but numbers of them would present the semblance of moving figures. Suppose, for instance, a number of threads to be so grouped as to form a cylinder for some distance, but after a while to be pulled apart by other threads with which they interlink. While the cylinder was passing through the plane, we should have in the plane a number of points in a circle. When the part where the threads deviated came to the plane, the circle would break up by the points moving away. These moving figures in the plane are but the traces of the shapes of threads as those shapes pass on. These moving figures may be conceived to have a life and a consciousness of their own.

Or, if it be irrational to suppose them to have a consciousness when the shapes of which they are momentary traces have none, we may well suppose that the shapes of threads have consciousness, and that the moving figures share this consciousness, only that in their case it is limited to those parts of the shapes that simultaneously pass through the plane. In the plane, then, we may conceive bodies with all the properties of a material system, moving and changing, possessing consciousness. After a while it may well be that one of them becomes so disassociated that it appears no longer as a unit, and its consciousness as such may be lost. But the threads of existence of such a figure are not broken, nor is the shape which gave it origin altered in any way. It has simply passed on to a distance from the plane. Thus nothing which existed in the conscious life on the plane would cease. There would in such an existence be no cause and effect, but simply the gradual realization in a superficies of an already existent whole. There would be no progress, unless we were to suppose the threads as they pass to interweave themselves in more complex shapes.

Can a representation, such as the preceding, be applied to the case of the existence in space with which we have to do? Is it possible to suppose that the movements and changes of material objects are the intersections with a three-dimensional space of a four-dimensional existence? Can our consciousness be supposed to deal with a spatial profile of some higher actuality?

It is needless to say that all the considerations that have been brought forward in regard to the possibility of the production of a system satisfying the conditions of materiality by the passing of threads through a fluid plane, holds good with regard to a four-dimensional existence passing through a three-dimensional space. Each part of the ampler existence which passed through our space would seem perfectly limited to us. We should have no indication of the permanence of its existence. Were such a thought adopted, we should have to imagine some stupendous whole, wherein all that has ever come into being or will come co-exists, which passing slowly on, leaves in this flickering consciousness of ours, limited to a narrow space and a single moment, a tumultuous record of changes and vicissitudes that are but to us. Change and movement seem as if they were all that existed. But the appearance of them would be due merely to the momentary passing through our consciousness of ever existing realities.

In thinking of these matters it is hard to divest ourselves of the habit of visual or tangible illustration. If we think of a man as existing in four dimensions, it is hard to prevent ourselves from conceiving him prolonged in an already known dimension. The image we form resembles somewhat those solemn Egyptian statues which in front represent well enough some dignified sitting figure, but which are immersed to their ears in a smooth mass of stone which fits their contour exactly.

No material image will serve. Organized beings seem to us so complete that any addition to them would deface their beauty. Yet were we creatures confined to a plane, the outline of a Corinthian column would probably seem to be of a beauty unimprovable in its kind. We should be unable to conceive any addition to it, simply for the reason that any addition we could conceive would be of the nature of affixing an unsightly extension to some part of the contour. Yet, moving as we do in space of three dimensions, we see that the beauty of the stately column far surpasses that of any single outline. So all that we can do is to deny our faculty of judging of the ideal completeness of shapes in four dimensions.

Chapter IV

LET US NOW LEAVE THIS SUPPOSITION of framework and threads. Let us investigate the conception of a four-dimensional existence in a simpler and more natural manner in the same way that a two-dimensional being should think about us, not as infinite in the third dimension, but limited in three dimensions as he is in two. A being existing in four dimensions must then be thought to be as completely bounded in all four directions as we are in three. All that we can say in regard to the possibility of such beings is, that we have no experience of motion in four directions. The powers of such beings and their experience would be ampler, but there would be no fundamental difference in the laws of force and motion.

Such a being would be able to make but a part of himself visible to us, for a cube would be apprehended by a two-dimensional being as the square in which it stood. Thus a four-dimensional being would suddenly appear as a complete and finite body, and as suddenly disappear, leaving no trace of himself, in space, in the same way that anything lying on a flat surface, would, on being lifted, suddenly vanish out of the cognizance of beings, whose consciousness was confined to the plane. The object would not vanish by moving in any direction, but disappear instantly as a whole. There would be no barrier, no confinement of our devising that would not be perfectly open to him. He would come and go at pleasure; he would be able to perform feats of the most surprising kind. It would be possible by an infinite plane extending in all directions to divide our space into two portions absolutely separated from one another; but a four-dimensional being would slip round this plane with the greatest ease.

To see this clearly, let us first take the analogous case in three dimensions. Suppose a piece of paper to represent a plane. If it is infinitely extended in every direction, it will represent an infinite plane. It can be divided into two parts by an infinite straight line. A being confined to this plane could not get from one part of it to the other without passing through the line. But suppose another piece of paper laid on the first and extended infinitely, it will represent another infinite plane. If the being moves from the first plane by a motion in the third dimension, it will move into this new plane. And in it it finds no line. Let it move to such a position that when it goes back to the first plane it will be on the other side of the line. Then let it go back to the first plane. It has appeared now on the other side of the line which divides the infinite plane into two parts.

Take now the case of four dimensions. Instead of bringing before the mind a sheet of paper conceive a solid of three dimensions. If this solid were to become infinite it would fill up the whole of three-dimensional space. But it would not fill up the whole of four-dimensional space. It would be to four-dimensional space what an infinite plane is to three-dimensional space. There could be in four-dimensional space an infinite number of such solids, just as in three-dimensional space there could be an infinite number of infinite planes.

Thus, lying alongside our space, there can be conceived a space also infinite in all three directions. To pass from one to the other a movement has to be made in the fourth dimension, just as to pass from one infinite plane to another a motion has to be made in the third dimension.

Conceive, then, corresponding to the first sheet of paper mentioned above, a solid, and as the sheet of paper was supposed to be infinitely extended in two dimensions, suppose the solid to be infinitely extended in its three dimensions, so that it fills the whole of space as we know it.

Now divide this infinite solid in two parts by an infinite plane, as the infinite plane of paper was divided in two parts by an infinite line. A being cannot pass from one part of this infinite solid to another, on the other side of this infinite plane, without going through the infinite plane, *so long as he keeps within the infinite solid.*

But suppose beside this infinite solid a second infinite solid, lying next to it in the fourth dimension, as the second infinite plane of paper was next to the first infinite plane in the third dimension. Let now the being that wants to get on the other side of the dividing plane move off in the fourth dimension, and enter the second infinite solid. In this second solid there is no dividing plane. Let him now move, so that coming back to the first infinite solid he shall be on the other side of the infinite plane that divides it into two portions. If this is done, he will now be on the other side of the infinite plane, without having gone through it.

In a similar way a being, able to move in four dimensions, could get out of a closed box without going through the sides, for he could move off in the fourth dimension, and then move about, so that when he came back he would be outside the box.

Is there anything in the world as we know it, which would indicate the possibility of there being an existence in four dimensions? No definite answer can be returned to this question. But it may be of some interest to point out that there are certain facts which might be read by the light of the fourth-dimensional theory.

To make this clear, let us suppose that space is really four dimensional, and that the three-dimensional space we know is, in this ampler space, like a surface is in our space.

We should then be in this ampler space like beings confined to the surface of a plane would be in ours. Let us suppose that just as in our space there are centers of attraction whose influence radiates out in every direction, so in this ampler space there are centers of attraction whose influence radiates out in every direction. Is there anything to be observed in nature which would correspond to the effect of a center of attraction lying out of our space, and acting on all the matter in it? The effect of such a center of attraction would not. be to produce motion in any known direction, because it does not lie off in any known direction.

Let us pass to the corresponding case in three and two dimensions, instead of four and three. Let us imagine a plane lying horizontally, and in it some creatures whose experience was confined to it. If now some water or other liquid were poured on to the plane, the creatures, becoming aware of its presence, would find that it had a tendency to spread out all over the plane. In fact it would not be to them as a liquid is to us—it would rather correspond to a gas. For a gas, as we know it, tends to expand in every direction, and gradually increase so as to fill the whole of space. It exercises a pressure on the walls of any vessel in which we confine it.

The liquid on the plane expands in all the dimensions which the two-dimensional creatures on the plane know, and at the same time becomes smaller in the third dimension, its absolute quantity remaining unchanged. In like manner we might suppose that gases (which by expansion become larger in the dimensions that we know) become smaller in the fourth dimension.

The cause in this case would have to be sought for in an attractive force, acting with regard to our space as the force of gravity acts with regard to a horizontal plane.

Can we suppose that there is a center of attraction somewhere off in the fourth dimension, and that the gases, which we know are simply more mobile liquids, expanding out in every direction under its influence. This view receives a certain amount of support from the fact proved experimentally that there is no absolute line of demarcation between a liquid and a gas. The one can be made to pass into the other with no moment intervening in which it can be said that now a change of state has taken place.

We might then suppose that the matter we know extending in three dimensions has also a small thickness in the fourth dimension; that solids are rigid in the fourth as in the other three dimensions; that liquids are too coherent to admit of their spreading out in space, and becoming thinner in the fourth dimension, under the influence of an attractive center lying outside of our space; but that gases, owing to the greater mobility of their particles, are subject to its action, and spread out in space under its influence, in the same manner that liquids, under the influence of gravity, spread out on a plane.

Then the density of a gas would be a measure of the relative thickness of it in the fourth dimension: and the diminution of the density would correspond to a diminution of the thickness in the fourth dimension. Could this supposition be tested in any way?

Suppose a being confined to a plane; if the plane is moved far off from the center of attraction lying outside it, he would find that liquids had less tendency to spread out than before.

Or suppose he moves to a distant part of the plane so that the line from his position to the center of attraction lies obliquely to the plane; he would find that in this position a liquid would show a tendency 40 spread out more in one direction than another.

Now our space considered as lying in four-dimensional space, as a plane does in three-dimensional space, may be shifted. And the expansive force of gases might be found to be different at different ages. Or, shifting as we do our position in space during the course of the earth's path round the sun, there might arise a sufficient difference in our position in space, with regard to the attractive center, to make the expansive force of gases different at different times of the year, or to cause them to manifest a greater expansive force in one dire(Sion than in another.

But although this supposition might be worked out at some length, it is hard to suppose that it could afford any definite test of the physical existence of a fourth dimension. No test has been discovered which is decisive. And, indeed, before searching for tests, a theoretical point of the utmost importance has to be settled. In discussing the geometrical properties of straight lines and planes, we suppose them to be respectively of one and two dimensions, and by so doing deny them any real existence. A plane and a line are mere abstractions. Every portion of matter is of three dimensions. If we consider beings on a plane not as mere idealities, we must suppose them to be of some thickness. If their experience is to be limited to a plane this thickness must be very small compared to their other dimensions. Transferring our reasoning to the case of four dimensions, we come to a curious result.

If a fourth dimension exists there are two possible alternatives.

One is, that there being four dimensions, we have a three-dimensional existence only. The other is that we really have a four-dimensional existence, but are not conscious of it. If we are in three dimensions only, while there are really four dimensions, then we must be relatively to those beings who exist in four dimensions, as lines and planes are in relation to us. That is, we must be mere abstractions. In this case we must exist only in the mind of the being that conceives us, and our experience must be merely the thoughts of his mind–a result which has apparently been arrived at, on independent grounds, by an idealist philosopher.

The other alternative is that we have a four-dimensional existence. In this case our proportions in it must be infinitely minute, or we should be conscious of them. If such be the case, it would probably be in the ultimate particles of matter, that we should discover the fourth dimension, for in the ultimate particles the sizes in the three dimensions are very minute, and the magnitudes in all four dimensions would be comparable.

The preceding two alternative suppositions are based on the hypothesis of the reality of four-dimensional existence, and must be conceived to hold good only on that hypothesis.

It is somewhat curious to notice that we can thus conceive of an existence relative to which that which we enjoy must exist as a mere abstraction.

Apart from the interest of speculations of this kind they have considerable value; for they enable us to express in intelligible terms things of which we can form no image. They supply us, as it were, with scaffolding, which the mind can make use of in building up its conceptions. And the additional gain to our power of representation is very great.

Many philosophical ideas and doctrines are almost unintelligible because there is no physical illustration which will serve to express them. In the imaginary physical existence which we have traced out, much that philosophers have written finds adequate representation. Much of Spinoza's Ethics, for example, could be symbolized from the preceding pages.

Thus we may discuss and draw perfectly legitimate conclusions with regard to unimaginable things.

It is, of course, evident that these speculations present no point of direct contact with fact. But this is no reason why they should be abandoned. The course of knowledge is like the flow of some mighty river, which, passing through the rich lowlands, gathers into itself the contributions from every valley. Such a river may well be joined by a mountain stream, which, passing with difficulty along the barren highlands, flings itself into the greater river down some precipitous descent, exhibiting at the moment of its union the spectacle of the utmost beauty of which the river system is capable. And such a stream is no inapt symbol of a line of mathematical thought, which, passing through difficult and abstract regions, sacrifices for the sake of its crystalline clearness the richness that comes to the more concrete studies. Such a course may end fruitlessly, for it may never join the main course of observation and experiment. But, if it gains its way to the great stream of knowledge, it affords at the moment of its union the spectacle of the greatest intellectual beauty, and adds somewhat of force and mysterious capability to the onward current.

Many Dimensions

1885

IN CONNECTION WITH THE SUBJECT OF HIGHER SPACE there is a remark which is sometimes made, a question which is put–"If there are four dimensions, then there may be five and six, and so on up to any number?"

This question is one, I own, which it would never have occurred to me to ask. Still it often happens that a line of thought which is most foreign and unattractive does repay investigation. And so let us follow the ready algebraist, to whom it is as easy to write down five as four, and n as five. Let us see what it is reasonable to think on the subject.

If we take four-dimensional shapes and examine them, we find that there is in them a peculiarity of the same kind which led us to be sure of the reality of a four-dimensional existence from the inspection of these dimensional shapes. In four dimensions we can have two figures which are precisely similar in all their parts, and which yet will not move so that one shall occupy the place of the other.

And the same observation can be made with regard to five-dimensional figures.

Hence it would seem that there is an indication of a higher and higher reality. And if we suppose that the same fact of absolute similarity, without the possibility of superposition, were found again and a gain, then we should be compelled to recognize the existence of higher and still higher space, and we should have to admit the existence of an indefinite number of dimensions.

But let us turn away from this direct inquiry. Let us ask what the phrase "an infinite number of dimensions" denotes.

The question reminds me so forcibly of an Eastern story that I must digress fur a moment.

For it is said that once, in the cool of the morning, beneath the spreading branches of a great palm the master stood. And round him were gathered three or four with whom he spent the hours of his quiet life.

And not for long had they gathered together.

One was a warrior, and long ago he had come to the master, asking him what he should do, and had received for answer—"Go back and serve your commander. The day will come when you will have fulfilled your life, and the voice within you will speak clearly."

And the soldier had returned to the life of camps and marches and combats, till at length, at the close of a hard-fought day, he threw down his weapons, and passing through the enemy's land, came to where the master taught.

And his comrades, seeking long for their leader, at last buried with honor a corpse unrecognizable for wounds.

He now sat on a bare stone listening. Beside him stood a younger man. He had been a merchant, travelling over the whole earth in search of gain, and in restlessness of curiosity. And when in wonder he had begged the master what he should do, he had been told—"Wander over the earth, and visit every part; when thy eagerness for change is satisfied, an inward voice will lead thee."

And he had travelled far, till, even, in the course of his wanderings, he had come to the most distant lands, and gained great riches by what he bought and sold.

But when his stores were full, and his possessions had increased beyond his dreams, he left them all, and, seeking that hillside, lived obediently to the master's words.

Half lying upon the ground was one whose countenance hardly bespoke him—fitting companion for the others. And, indeed, he had been that one, whose life had afforded the master the most interest of all of them.

For he had not, like the others, been immersed in an active and adventurous life, but had been a slave to the wants of his own body. And seeing amidst his vices that the master had words for others, he had besought him to tell him too what to do.

And the master had told him first one thing and then another, but always he fell back, unable to withdraw himself, even for a short while, from his bodily cravings, hut, gratifying them with drink and Sloth, he passed his days in brutishness.

Then at length the master, hailing him as a friend, had said to him—I will not seek to withdraw you any longer, for is not your body like the rain-clouds, and the sky a part of the changing show that hangs before our faces? Gaze, therefore, earnestly on your body, attend to it the more intently, for this is your vocation; and when you see the flimsy veil it is, come to me.

And this man had sat for ten years contemplating the middle portion of his body, till his frame had grown so cramped that he could not rise. At last he had bidden his fellows carry him to the master; and now he too listened to the words that fell on welcome ears.

And many days they had spoken together, and retiring each to his hut of reeds at nightfall, had pondered over the master's words. And on each of them had come a change.

Into the soldier's face, hard and stern set, had come the dawn of gentleness. The quick, observant gaze of the traveller now at times changed almost into such an expression as one would wear who looks at the wide fields that lie above the countries of the earth. And in the dull, inexpressive countenance of him who had sat absorbed in the contemplation of his body, had come the kindling light of intelligence.

And on this day the master opened his lips, and began to instruct them about the universe.

He told them much that made them wonder. He told them of the mysterious currents of life that passed away from the bodies and frames which they could see, and that, spreading into the minutest particles of the earth, collected again, and eddying back through seed and leaf and fruit, participated anew with the soul, which also in its turn had gone through many vicissitudes, in that mingling ground of various principles which we call a human life.

And seeing their wonder and interest, and feeling that they were desirous to know, and since, moreover, he saw no harm in gratifying their wish, he began to explain to them the deepest facts of their physical being. And talking of the universe, which contained all that they saw and knew, from the beneficent stars to the humblest blade of grass, he said—"The world rests upon an elephant." And then he paused.

The warrior did not speak. He who had been absorbed in the contemplation of his body did not open his lips—or if he had it would not have mattered; for with the instinctive and right attitude of the half-cultured mind to the proximate object which is the last to come before its intelligence, he would have said if he had spoken, "worship the elephant;" and the master would have greeted this remark with a kindly smile, and proceeded with his discourse.

But just as he was about to take up the thread of his speech, there came from the traveller, who had been listening eagerly, a hurried question.

For, alas, in his wanderings, this one had traversed the greater part of the globe, and in the course of them had come to the West, where even at this early period a habit of mind reigned, very unlike that which characterized the calm, deep, contemplative souls of the East.

Moved by this restless and questioning spirit, he cried out—"And on what does the elephant rest?"

"Upon a tortoise," the holy man replied. And had he not been beyond all human passions, his tone would have been one of mockery.

He taught them no more. Why should he tell them of these things? Was it not better rather to dwell in the daily perfectionment of brotherly love, and in the ministering offices of devoted lives?

And yet one cannot help wishing that unlucky question had not been put. If only the unfortunate disciple had but said, "Let us investigate the elephant," or, better still, had said nothing—what should we not have known now!

And if then such a question sealed the fount of sacred wisdom at that remote epoch, what must not the effect of our modern mind be?

For now such a disciple would not simply ask, "Upon what does the elephant rest?" but he would have glibly asked, all in one breath—"Upon what does the elephant rest, and upon what does the support of the elephant rest, and on what the support of that? and so on, ad infinitum; do tell me."

And so too, even on the rivulet from the fount of wisdom that trickles sparingly through our own minds, is there not a checking effect coming from this mental attitude of ever asking what is behind and behind and behind, seeking formal causes always, instead of living apprehension of the proximate?

Indeed, that question was a misfortune if the possession of fact knowledge is a boon. For what could have been a more apt description of this all-supporting elastic solid ether than the broad arching back of the largest animal known on earth—the created being that could bear the most, and of all not-human creatures, the most intelligent and responsive?

The master knew how all the worlds were held together—and how much more!

And, indeed, does not this feeling come upon us strongly with regard to those of the Eastern world, with whom we have the privilege of talking?

For my own part, however much I have learnt in the intervals of my speaking with them, there they still hover on the weather-bow of my knowledge—they, or those from whom they learn, are in the possession of knowledge of which all my powers are but secondary instances or applications.

What it is I know not, nor do they ever approach to tell me. Yet with them I feel an inward sympathy, for I too, as they, have an inward communion and delight, with a source lying above all points and turns and proofs—an inward companion, whose presence in my mind for one half-hour is worth more to me than all the cosmogonies that I have ever read of, and of which all the thoughts I have ever thought are but minutest fragments, mixed up with ignorance and error. What their secret is I know not, mine is humble enough—the inward apprehension of space.

And I have often thought, travelling by railway, when between the dark underground stations the lads and errand boys bend over the scraps of badly printed paper, reading fearful tales—I have often thought how much better it would be if they were doing that which I may call "communing with space." 'Twould be of infinite delight, romance, and interest; far more than are those creased tawdry papers, with no form in themselves or in their contents.

And yet, looking at the same printed papers, being curious, and looking deeper and deeper into them with a microscope, I have seen that in splodgy ink stroke and dull fibrous texture, each part was definite, exact, absolutely so far and no farther, punctiliously correct; and deeper and deeper lying a wealth of form, a rich variety and amplitude of shapes, that in a moment leapt higher than my wildest dreams could conceive.

And then I have felt as one would do if the dark waters of a manufacturing town were suddenly to part, and from them, in them, and through them, were to uprise Aphrodite, radiant, undimmed, flashing her way to the blue beyond the smoke; for there, in these crabbed marks and crumpled paper, there, if you but look, is space herself, in all her infinite determinations of form.

Thus the reverent and true attitude is, not to put formal questions, but to press that which we know of into living contact with our minds.

And so the next step, when we would pass beyond the knowledge of the things about us in the world, is to acquire a sense and living apprehension of four-dimensional space.

But the question does come to many minds. "What lies beyond?" And, although our knowledge is not ripe enough to answer this question, still, hurrying on before, we may ask—not what does lie beyond, but what is it natural for us in our present state of knowledge to think about the many dimensions of space?

Let us drop for a moment into the most common sense mode of looking at it. Why do we think of space at all? To explain what goes on. If everything followed uniformly, we should not need to think of three, or even two dimensions—one would do. But problems come up, practical problems, which need to be reconciled. Things get "behind" one another, are hidden, and disappear. So we find that one variable will not suffice. If we were in a line looking at only one thing, its gradual changes of distance from us would be all our experience. We should not call this "distance"; it would be the one fact of our experience; and if we treated it mathematically, we should express it as the variation of one variable. So we may consider as identical, one-dimensional space, and the variation of one variable. Now plane space requires two variables. May not plane space then be defined as our knowledge of the variation of two variables? The being in plane space requires two variables to account for his experience. He lives, we say, in a space of two dimensions.

Now why should we not identify these, and say that that which he calls space is the organized mass of knowledge of the relations of two variables that has grown up in his mind?

We talk of distance and size as if each were something known in itself. But suppose a percipient soul subjected to a series of changes depending on two independent causes, which always operated together, and which were each of them continuous in their increase and diminution, would not this percipient soul form an idea of space of two dimensions? Would he not say that he lived in a space of two dimensions? His apprehension of the number of variables by which he was able to account for his experience would project itself into a feeling of being in space; and the kind of space would depend on the number of variables he habitually worked with.

Now we have become habituated to use, for practical thought, three variables; these explain the greater part of our daily life. Is that which we call space simply the organized knowledge of the relations of these variables? Without pledging ourselves to this view, let us adopt it and note its consequences.

Then it is evident that as we come into the presence of more and more independent causes—I mean, as we find that these are in nature working independently of one or more in number than three—we shall have to study the general aspect of events which turn up from the combinations in varying intensity of these four or more principles, or causes of our sensation. Then we shall get a mental organization capable of dealing readily and rapidly with the combinations of these causes. And this mental organization will be indicated in our consciousness by the feeling of being in four-dimensional (or more-dimensional) space.

It seems strange to talk of there being three independent causes, or of some such limited number, for in the events that happen around us we see a vast variety of causes. There is the tendency to fall, there is the motion of the wind, there are the actions of human beings, each of them producing effects, and besides these many other causes.

But if we look at them, we find that they are not all independent one of the other, but may be different forms of the same cause.

Indeed, if we suppose that we live in three-dimensional space, and that every change and occurrence is the result of the movements of the small particles of matter, there would ultimately be only three independent causes—the three independent movements, namely, which a particle could go through.

Thus it would appear that, since no one would deny that there are an infinite number of perfectly independent causes in nature, the formation of a sense of higher and higher kinds of space was simply necessary as, our knowledge becoming deeper, we came into contact with more and more of these causes.

It might be said that these causes might be very diverse from each other; one might be apprehended as love, another as color, another as distance. But this view is hardly tenable, for to apprehend a cause it must be congruent with the others which we already apprehend. If it is known at all it must work uniformly in with the rest of our experience. No doubt there are an infinite number of causes, which give that richness to experience of which the intellect can take hold only by a small part. But when the intellect does take hold of a part, it takes hold of it by seeing how it comes in, modifying each of the already existing possibilities and producing a new variety, out of which the actual experience is a selection. Thus, if a being having an experience derived from two causes, and so living in a space of two dimensions, were to be affected by a third cause, he would first of all find that there were many things which he would say could not be explained by space relations. Then he would gradually arrive at the idea of a three-dimensional space. Space being due then not to anything in the nature of the causes themselves, but to the number of them.

Then, to us, when mentally we come into the comprehension of any new independent cause, we must acquire the sense of a new dimension, and the question of space and space relation is altogether independent of the nature of these causes—the real and systematic apprehension of them necessitating an enlargement of our sense of space. Now the unknown comes to us generally in the properties of the minute particles of matter which make the different "kinds." Hence as we study matter closer and closer we shall find that we need more and more dimensions. And the molecular forces in one kind of space will be the physical forces of the next higher.

That is to say, when in our space we have explained all that we can explain by the supposition of particles moving in our space, we shall find that there is a residuum, and this residuum will be explained by the four-dimensional movement of the minutest particles. The large movements are simply movements in three-dimensional space, but to explain the residual phenomenon a higher kind of space will be requisite.

Still, this all seems to me a barren view, and I am convinced that it is far truer to think of space, as indeed we can hardly help doing, as a beneficent being, supporting us all looking at us in every lovely leafy bough, and bending towards us in the forms of those we know.

And, moreover, there is one very valid objection to the conclusion that we have explained anything, or made any step by using the word "variable."

It will be found that such a notion as a continuously varying quantity is a mere verbal expression. All that we can conceive or understand are definite steps, definite units. We can conceive a great many definite magnitudes, but not continuous magnitude. The idea of continuity is one which we use and apply; but to think men have explained anything by speaking of continuous variables, is really to lose ourselves in words.

But, although we dismiss the previous supposition, still we see that, even if it were true, the practical thing to do is to acquire the sense of a higher dimensional space.

And, indeed, what a field is here! Take a single example. The idea of magnitude is one dimensional simply adding and adding on in a straight line.

The idea of rotation, or twisting, in its very nature involves the idea of two dimensions—for it is the passage from one dimension to another—it is an idea which, in its essence, has two dimensions.

If we think of a twist, it is the change from one direction to another. It cannot be thought without the two directions being present to the mind—the direction from which and the direction to which the change takes place.

In our space we have nothing more than this rotation. If a ball is twisting, and a blow is given to it, which tends to set up a twist in a new direction, the old twist and the new one combine together into a single twist about a new axis.

But in four-dimensional space there is such a thing as a twist of a twist—a rotation of a rotation—bearing to a simple rotation the same relation that an area does to a line. Perfectly independent rotations may exist in a four-dimensional body.

And again, evidently if there is an idea which in its essence involves two dimensions, may there not be an idea which, of its very nature, includes three dimensions?

What that idea is, we do not know now; but some time, when the knowledge of space is more highly developed, that idea will become as familiar to us as the idea of a twist is now.

And, indeed, space is wonderful. We all know that space is infinite in magnitude—stretching on endlessly.

And when we look quietly at space, she shows us at once that she has infinite dimensions.

And yet, both in magnitudes and dimensions there is something artificial.

To measure, we must begin somewhere, but in space there is no "somewhere" marked out for us to begin at. This measuring is something, after all, foreign to space, introduced by us for our convenience.

And as to dimensions, in order to enumerate and realize the different dimensions, we must fix on a particular line to begin with, and then draw other lines at right angles to this one.

But the first straight line we take can be drawn in an infinite number of directions. Why should we take any particular one?

If we take any particular line, we do something arbitrary, of our own will and decision, not given to us naturally by space.

No wonder then that if we take such a course we are committed to an endless task.

We feel that all these efforts, necessary as they are to us to apprehend space, have nothing to do with space herself. We introduce something of our own, and are lost in the complexities which this brings about.

May we not compare ourselves to those Egyptian priests who, worshipping a veiled divinity, laid on her and wrapped her about ever with richer garments, and decked her with fairer raiment.

So we wrap round space our garments of magnitude and vesture of many dimensions.

Till suddenly, to us as to them, as with a forward tilt of the shoulders, the divinity moves, and the raiment and robes fall to the ground, leaving the divinity herself, revealed, but invisible; not seen, but somehow felt to be there.

And these are not empty words. For the one space which is not this form or that form, not this figure or that figure, but which is to be known by us whenever we regard the least details of the visible world—this space can be apprehended. It is not the shapes and things we know, but space is to be apprehended in them.

The true apprehension and worship of space lies in the grasp of varied details of shape and form, all of which, in their exactness and precision, pass into the one great apprehension.

And we must remember that this apprehension does not lie in the talking about it. It cannot be conveyed in description.

We must beware of the attitude of standing open-mouthed just because there is so much mechanics which we do not understand. Surely there is no mechanics which we do not understand, but geometry and mathematics only spring up there where we, in our imperfect way, introducing our own limitations, tend towards the knowledge of inscrutable nature.

If we want to pass on and on till magnitude and dimensions disappear, is it not done for us already? That reality, where magnitudes and dimensions are not, is simple and about us. For passing thus on and on we lose ourselves, but find the clue again in the apprehension of the simplest acts of human goodness, in the most rudimentary recognition of another human soul wherein is neither magnitude nor dimension, and yet all is real.

The answer to this is twofold. In order to live, self knowledge is necessary. That knowledge of self which is distinctly a matter of ethical inquiry, is altogether foreign to these pages.

But there is a no less important branch of self knowledge which seems altogether like a research into the external world. In this we pass into a closer and closer contemplation of material things and relations, till suddenly we find that what we thought was certain and solid thought is really a vast and over-arching crust, whose limitlessness to us was but our conformity to its limits a shell out of which and beyond which we may at any time pass.

But if we do so pass, we do not leave behind us the idea of matter. All that we thus attain is a different material conception of ourselves.

In ancient times there was no well-defined line between physics and metaphysics. And our present physical notions are derived from amongst the mass of metaphysical notions. Metaphysics is so uncertain, because when any one of its doctrines becomes certain, it takes a place in physics.

And the exploration of the facts of higher space is the practical execution of the great vision of Kant. He turned thought in an entirely new direction. And where he turned, all seemed blank—all positive assertions fell away, as he looked into the blackness of pure thought.

But out of this absence can come any amount of physical knowledge. It is like an invisible stuff out of which visible garments can be woven.

But, indeed, many would say: What is the use of these speculations?

Does not the contemplation of space leave the mind cold, the heart untouched? Not altogether.

Is not our life very much a matter of fact, concerned with events? All our feelings are bound up with things which we do or suffer.

And thus a right conception of the possibilities of action in our world, and in i higher world, must have some influence on ourselves.

Then also there is a path through which we can pass, leading from the most complete materialism to something very different from the first form in which it presents itself.

Any one, who will try, can find that, by passing deeper and deeper into absolute observation of matter, and familiarity with it, that which he first felt as real passes away—though still there, it passes away, and becomes but the outward sign of realities infinitely greater.

Thus there springs before the mind an idealism which is more real than matter; a glimpse of a higher world, which is no abstraction, or fancy, or thought, but of which our realities are the appearances.

And with this there comes overpoweringly upon the mind of one, who thinks on higher space, the certainty that all we think, or do, or imagine, lies open.

In that large world our secrets lie as clear as the secrets of a plane being lie to an eye above the plane. For howsoever closely a being living on a plane may hide from his fellows, he has nothing secret from an eye that gazes down upon his plane.

The very idea that he can put forward to such a one any false pretenses, is absurd.

And so we lie palpable, open. There is no such thing as secrecy.

And as I have said before, the difference between the moral life and the animal life, in a world of any dimensions, lies in this—that the animal life consists of actions which are those natural to the possibilities of space of that world; the moral life (viewed as exhibited in physical arrangements) lies in the striving, by modification and restraint of the natural actions, towards those actions and modes of existence which are natural in a higher space world.

It has been shown how plane beings could only pass each other by courtesy and mutual forbearance. And the great effort wherein the higher spirit most plainly shows itself, apart from convenience, or profit, or any obvious physical good, is in one very simple and obvious tendency towards a higher-dimensional existence. For, as to a higher space being no secrets of ours are hidden—nothing is unknown, so, in making towards one another our limited lives open and manifest, we treat each other in the service of truth, as if we were each members of that higher world.

It is often said and felt, that all our actions do in the course of time impress their effect on the world. Nothing is lost. And if we, being limited, know that this is so, how much the more apparent is it when we realize our higher being. We know that, as animal frames moving and acting in the world, the effect of every movement passes on and on.

And with this effort corruption and evil fall. Space is so large that no interior can be hidden from the vivifying breath of the universe; no part can be cut off, however foul, from direct contact with the purifying winds which traverse space higher than itself.

As conscious minds, we realize the oneness of past and future in our open communication one with another. We attain a mental consciousness of the higher fact. Whether we represent it to ourselves as a day wherein all that ever has been done will be told, or as an omnipresent and all-knowing mind, it is the same.

Truth is nothing but an aspiration to our higher being. And the first sign of love towards individuals, as towards the world as distinguished from the easy and yielding good nature which always tries to please that which is nearest at the moment—is veracity. This is the secret of the mysterious effect of science on our emotions—the simple description of fact, apart from our own conditions and prejudices. And also in the material world around us, this is the secret of the beauty of the crystal and of still water. For in them the near and the far are brought together; in their translucency they give an emblem of the one vision wherein a higher being grasps every part of the solid matter, of which we can only see outside and surface.

The acceptance of the rule of the great master of empiric religion, Comte—"Live openly"—is really to imitate in our world, and make ourselves conscious of our true existence in a higher world.

There are two sides of religion—the inductive and the deductive. To the realm of deduction belongs theology, with its central assertion and its manifold consequences. But inductive religion consists in grasping, amidst the puzzling facts of life, those greater existences in which the individual organizations are bound up, and which they serve, passing, as in every science, from the details to the whole. And the connecting link between materialism and the conduct of life, lies in the doctrine of the limited nature of our present space perceptions. For, with the elevation of our notion of space to its true place, the antagonism between our present materialistic and our present idealistic views of life falls away.

The Fourth Dimension

1904

Chapter I
Four-Dimensional Space

There is nothing more indefinite, and at the same time more real, than that which we indicate when we speak of the "higher." In our social life we see it evidenced in a greater complexity of relations. But this complexity is not all. There is, at the same time, a contact with, an apprehension of, something more fundamental, more real.

With the greater development of man there comes a consciousness of something more than all the forms in which it shows itself. There is a readiness to give up all the visible and tangible for the sake of those principles and values of which the visible and tangible are the representation. The physical life of civilized man and of a mere savage are practically the same, but the civilized man has discovered a depth in his existence, which makes him feel that that which appears all to the savage is a mere externality and appurtenage to his true being.

Now, this higher—how shall we apprehend it? It is generally embraced by our religious faculties, by our idealizing tendency. But the higher existence has two sides. It has a being as well as qualities. And in trying to realize it through our emotions we are always taking the subjective view. Our attention is always fixed on what we feel, what we think. Is there any way of apprehending the higher after the purely objective method of a natural science? I think that there is.

Plato, in a wonderful allegory, speaks of some men living in such a condition that they were practically reduced to be the denizens of a shadow world. They were chained, and perceived but the shadows of themselves and all real objects projected on a wall, towards which their faces were turned. All movements to them were but movements on the surface, all shapes but the shapes of outlines with no substantiality.

Plato uses this illustration to portray the relation between true being and the illusions of the sense world. He says that just as a man liberated from his chains could learn and discover that the world was solid and real, and could go back and tell his bound companions of this greater higher reality, so the philosopher who has been liberated, who has gone into the thought of the ideal world, into the world of ideas greater and more real than the things of sense, can come and tell his fellow men of that which is more true than the visible sun—more noble than Athens, the visible state.

Now, I take Plato's suggestion; but literally, not metaphorically. He imagines a world which is lower than this world, in that shadow figures and shadow motions are its constituents; and to it he contrasts the real world. As the real world is to this shadow world, so is the higher world to our world. I accept his analogy. As our world in three dimensions is to a shadow or plane world, so is the higher world to our three-dimensional world. That is, the higher world is four-dimensional; the higher being is, so far as its existence is concerned apart from its qualities, to be sought through the conception of an actual existence spatially higher than that which we realize with our senses.

Here you will observe I necessarily leave out all that gives its charm and interest to Plato's writings. All those conceptions of the beautiful and good which live immortally in his pages.

All that I keep from his great storehouse of wealth is this one thing simply—a world spatially higher than this world, a world which can only be approached through the stocks and stones of it, a world which must be apprehended laboriously, patiently, through the material things of it, the shapes, the movements, the figures of it.

We must learn to realize the shapes of objects in this world of the higher man; we must become familiar with the movements that objects make in his world, so that we can learn something about his daily experience, his thoughts of material objects, his machinery.

The means for the prosecution of this enquiry are given in the conception of space itself.

It often happens that that which we consider to be unique and unrelated gives us, within itself, those relations by means of which we are able to see it as related to others, determining and determined by them.

Thus, on the earth is given that phenomenon of weight by means of which Newton brought the earth into its true relation to the sun and other planets. Our terrestrial globe was determined in regard to other bodies of the solar system by means of a relation which subsisted on the earth itself.

And so space itself bears within it relations of which we can determine it as related to other space. For within space are given the conceptions of point and line, line and plane, which really involve the relation of space to a higher space.

Where one segment of a straight line leaves off and another begins is a point, and the straight line itself can be generated by the motion of the point.

One portion of a plane is bounded from another by a straight line, and the plane itself can be generated by the straight line moving in a direction not contained in itself.

Again, two portions of solid space are limited with regard to each other by a plane; and the plane, moving in a direction not contained in itself, can generate solid space.

Thus, going on, we may say that space is that which limits two portions of higher space from each other, and that our space will generate the higher space by moving in a direction not contained in itself.

Another indication of the nature of four-dimensional space can be gained by considering the problem of the arrangement of objects.

If I have a number of swords of varying degrees of brightness, I can represent them in respect of this quality by points arranged along a straight line.

If I place a sword at A, figure 1, and regard it as having a certain brightness, then the other swords can be arranged in a series along the line, as at A, B, C, etc., according to their degrees of brightness.

If now I take account of another quality, say length, they can be arranged in a plane. Starting from A, B, C, I can find points to represent different

Figure 1

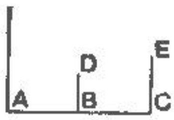

Figure 2

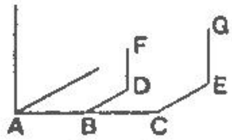

Figure 3

degrees of length along such lines as AF, BD, CE, drawn from A and B and C (see fig. 2). Points on these lines represent different degrees of length with the same degree of brightness. Thus the whole plane is occupied by points representing all conceivable varieties of brightness and length.

Bringing in a third quality, say sharpness, I can draw, as in figure 3, any number of upright lines. Let distances along these upright lines represent degrees of sharpness, thus the points F and G will represent swords of certain definite degrees of the three qualities mentioned, and the whole of space will serve to represent all conceivable degrees of these three qualities.

If now I bring in a fourth quality, such as weight, and try to find a means of representing it as I did the other three qualities, I find a difficulty. Every point in space is taken up by some conceivable combination of the three qualities already taken.

To represent four qualities in the same way as that in which I have represented three, I should need another dimension of space.

Thus we may indicate the nature of four-dimensional space by saying that it is a kind of space which would give positions representative of four qualities, as three-dimensional space gives positions representative of three qualities.

Chapter II
A Chapter in the History of Four Space

PARMENIDES, AND THE ASIATIC THINKERS with whom he is in close affinity, propound a theory of existence which is in close accord with a conception of a possible relation between a higher and a lower dimensional space. This theory, prior and in marked contrast to the main stream of thought, which we shall afterwards describe, forms a closed circle by itself. It is one which in all ages has had a strong attraction for pure intellect, and is the natural mode of thought for those who refrain from projecting their own volition into nature under the guise of causality.

According to Parmenides of the school of Flea, the all is one, unmoving and unchanging. The permanent amid the transient—that foothold for thought, that solid ground for feeling, on the discovery of which depends all our life—is no phantom; it is the image amidst deception of true being, the eternal, the unmoved, the one. Thus says Parmenides.

But how explain the shifting scene, these mutations of things!

"Illusion," answers Parmenides. Distinguishing between truth and error, he tells of the true doctrine of the one—the false opinion of a changing world. He is no less memorable for the manner of his advocacy than for the cause he advocates. It is as if from his firm foothold of being he could play with the thoughts under the burden of which others labored, for from him springs that fluency of supposition and hypothesis which forms the texture of Plato's dialectic.

Can the mind conceive a more delightful intellectual picture than that of Parmenides, pointing to the one, the true, the unchanging, and yet on the other hand ready to discuss all manner of false opinion, forming a cosmogony too, false "but mine own" after the fashion of the time?

In support of the true opinion he proceeded by the negative way of showing the self-contradictions in the ideas of change and motion. It is doubtful if his criticism, save in minor points, has ever been successfully refuted. To express his doctrine in the ponderous modern way we must make the statement that motion is phenomenal not real.

Let us represent his doctrine.

Imagine a sheet of still water into which a slanting stick is being lowered with a motion vertically downwards. Let 1, 2, 3 (fig. 4), be three consecutive positions of the stick. A, B, C, will be three consecutive positions of the meeting of the stick with the surface of the water. As the stick passes down, the meeting will move from A on to B and C.

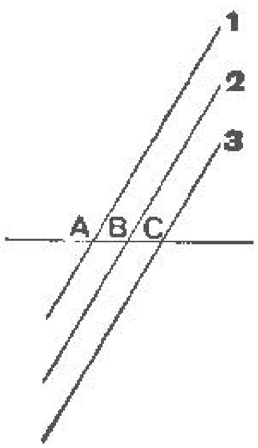

Figure 4

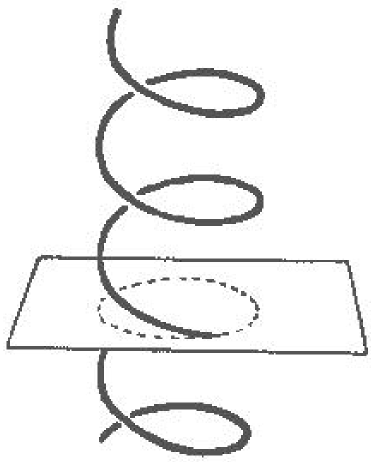

Figure 5

Suppose now all the water to be removed except a film. At the meeting of the film and the stick there will be an interruption of the film. If we suppose the film to have a property, like that of a soap bubble, of closing up round any penetrating object then as the stick goes vertically downwards the interruption in the film will move on.

If we pass a spiral through the film, the intersection will give a point moving in a circle shown by the dotted lines (fig. 5). Suppose now the spiral to be still and the film to move vertically upwards the whole spiral will be represented in the film of the consecutive positions of the point of intersection. In the film the permanent existence of the spiral is experienced as a time series—the record of traversing the spiral is a point moving in a circle. If now we suppose a consciousness connected with the film in such a way that the intersection of the spiral with the film gives rise to a conscious experience, we see that we shall have in the film a point moving in a circle, conscious of its motion, knowing nothing of that real spiral the record of the successive intersections of which by the film is the motion of the point.

It is easy to imagine complicated structures of the nature of the spiral, structures consisting of filaments, and to suppose also that these structures are distinguishable from each other at every section. If we consider the intersections of these filaments with the film as it passes to be the atoms constituting a filmar universe, we shall have in the film a world of apparent motion; we shall have bodies corresponding to the filamentary structure, and the positions of these structures with regard to one another will give rise to bodies in the film moving amongst one another. This mutual motion is apparent merely. The reality is of permanent structures stationary, and all the relative motions accounted for by one steady movement of the film as a whole.

Thus we can imagine a plane world, in which all the variety of motion is the phenomenon of structures consisting of filamentary atoms traversed by a plane of consciousness. Passing to four dimensions and our space, we can conceive that all things and movements in our world are the reading off of a permanent reality by a space of consciousness. Each atom at every moment is not what it was, but a new part of that endless line which is itself. And all this system successively revealed in the time which is but the succession of consciousness, separate as it is in parts, in its entirety is one vast unity. Representing Parmenides' doctrine thus, we gain a firmer hold on it than if we merely let his words rest, grand and massive, in our minds. And we have gained the means also of representing phases of that Eastern thought to which Parmenides was no stranger. Modifying his uncompromising doctrine, let us suppose, to go back to the plane of consciousness and the structure of filamentary atoms, that these structures are themselves moving—are acting, living. Then, in the transverse motion of the film, there would be two phenomena of motion, one due to the reading off in the film of the permanent existences as they are in themselves, and another phenomenon of motion due to the modification of the record of the things themselves, by their proper motion during the process of traversing them.

Thus a conscious being in the plane would have, as it were, a twofold experience. In the complete traversing of the structure, the Intersection of which with the film gives his conscious all, the main and principal movements and actions which he went through would be the record of his higher self as it existed unmoved and unacting. Slight modifications and deviations from these movements and actions would represent the activity and self-determination of the complete being, of his higher self.

It is admissible to suppose that the consciousness in the plane has a share in that volition by which the complete existence determines itself. Thus the motive and will, the initiative and life, of the higher being, would be represented in the case of the being in the film by an initiative and a will capable, not of determining any great things or important movements in his existence, but only of small and relatively insignificant activities. In all the main features of his life his experience would be representative of one state of the higher being whose existence determines his as the film passes on. But in his minute and apparently unimportant actions he would share in that will and determination by which the whole of the being he really is acts and lives.

An alteration of the higher being would correspond to a different life history for him. Let us now make the supposition that film after film traverses these higher structures, that the life of the real being is read off again and again in successive waves of consciousness. There would be a succession of lives in the different advancing planes of consciousness, each differing from the preceding, and differing in virtue of that will and activity which in the preceding had not been devoted to the greater and apparently most significant things in life, but the minute and apparently unimportant. In all great things the being of the film shares in the existence of his higher self as it is at any one time. In the small things he shares in that volition by which the higher being alters and changes, acts and lives.

Thus we gain the conception of a life changing and developing as a whole, a life in which our separation and cessation and fugitiveness are merely apparent, but which in its events and course alters, changes, develops; and the power of altering and changing this whole neélies in the will and power the limited being has of directing, guiding, altering himself in the minute things of his existence.

Transferring our conceptions to those of an existence in a higher dimensionality traversed by a space of consciousness, we have an illustration of a thought which has found frequent and varied expression. When, however, we ask ourselves what degree of truth there lies in it, we must admit that, as far as we can see, it is merely symbolical. The true path in the investigation of a higher dimensionality lies in another direction.

The significance of the Parmenidean doctrine lies in this: that here, as again and again, we find that those conceptions which man introduces of himself, which he does not derive from the mere record of his outward experience, have a striking and significant correspondence to the conception of a physical existence in a world of a higher space. How close we come to Parmenides' thought by this manner of representation it is impossible to say. What I want to point out is the adequateness of the illustration, not only to give a static model of his doctrine, but one capable as it were, of a plastic modification into a correspondence into kindred forms of thought. Either one of two things must be true-that four-dimensional conceptions give a wonderful power of representing the thought of the East, or that the thinkers of the East must have been looking at and regarding four-dimensional existence.

And from the numerical idealism of Pythagoras there is but a step to the more rich and full idealism of Plato. That which is apprehended by the sense of touch we put as primary and real, and the other senses we say are merely concerned with appearances. But Plato took them all as valid, as giving qualities of existence. That the qualities were not permanent in the world as given to the senses forced him to attribute to them a different kind of permanence. He formed the conception of a world of ideas, in which all that really is, all that affects us and gives the rich and wonderful wealth of our experience, is not fleeting and transitory, but eternal. And of this real and eternal we see in the things about us the fleeting and transient images.

And this world of ideas was no exclusive one, wherein was no place for the innermost convictions of the soul and its most authoritative assertions. Therein existed justice beauty-the one, the good, all that the soul demanded to be. The world of ideas, Plato's wonderful creation preserved for man, for his deliberate investigation and their sure development, all that the rude incomprehensible changes of a harsh experience scatters and destroys.

Plato believed in the reality of ideas. He meets us fairly and squarely. Divide a line into two parts, he says (fig. 6); one to represent the real objects in the world, the other to represent the transitory appearances, such as the image in still water, the glitter of the sun on a bright surface, the shadows on the clouds.

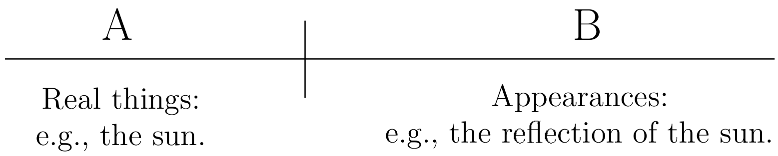

Figure 6

Take another line and divide it into two parts (fig. 7), one representing our ideas, the ordinary occupants of our minds, such as whiteness, equality, and the other representing our true knowledge, which is of eternal principles, such as beauty, goodness.

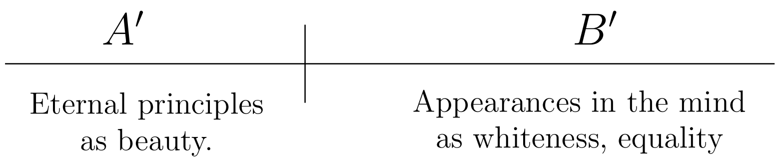

Figure 7

Then as A is to B, so is A' to B'.

That is, the soul can proceed, going away from real things to a region of perfect certainty, where it beholds what is, not the scattered reflections; beholds the sun, not the glitter on the sands; true being, not chance opinion.

Now, this is to us, as it was to Aristotle, absolutely inconceivable from a scientific point of view. We can understand that a being is known in the fullness of his relations; it is in his relations to his circumstances that a man's character is known; it is in his acts under his conditions that his character exists. We cannot grasp or conceive any principle of individuation apart from the fullness of the relations to the surroundings.

But suppose now that Plato is talking about the higher man—the four-dimensional being that is limited in our external experience to a three-dimensional world. Do not his words begin to have a meaning? Such a being would have a consciousness of motion which is not as the motion he can see with the eyes of the body. He, in his own being, knows a reality to which the outward matter of this too solid earth is flimsy superficiality. He too knows a mode of being, the fullness of relations, in which can only be represented in the limited world of sense, as the painter unsubstantially portrays the depths of woodland, plains, and air. Thinking of such a being in man, was not Plato's line well divided?

It is noteworthy that, if Plato omitted his doctrine of the independent origin of ideas, he would present exactly the four-dimensional argument; a real thing as we think it is an idea. A plane being's idea of a square object is the idea of an abstraction, namely, a geometrical square. Similarly our idea of a solid thing is an abstraction, for in our idea there is not the four-dimensional thickness which is necessary, however slight, to give reality. The argument would then run, as a shadow is to a solid object, so is the solid object to the reality. Thus A and B' would be identified.

In the allegory which I have already alluded to, Plato in almost as many words shows forth the relation between existence in a superficies and in solid space. And he uses this relation to point to the conditions of a higher being.

He imagines a number of men prisoners, chained so that they look at the wall of a cavern in which they are confined, with their backs to the road and the light. Over the road pass men and women, figures and processions, but of all this pageant all that the prisoners behold is the shadow of it on the wall whereon they gaze. Their own shadows and the shadows of the things in the world are all that they see, and identifying themselves with their shadows related as shadows to a world of shadows, they live in a kind of dream.

Plato imagines one of their number to pass out from amongst them into the real space world, and then returning to tell them of their condition.

Here he presents most plainly the relation between existence in a plane world and existence in a three-dimensional world. And he uses this illustration as a type of the manner in which we are to proceed to a higher state from the three-dimensional life we know.

It must have hung upon the weight of a shadow which path he took! Whether the one we shall follow toward the higher solid and the four-dimensional existence, or the one which makes ideas the higher realities, and the direct perception of them the contact with the truer world.

Chapter III
Metageometry

THE THEORIES WHICH ARE GENERALLY CONNECTED with the names of Lobatchewsky and Bolyai bear a singular and curious relation to the subject of higher space.

In order to show what this relation is, I must ask the reader to be at the pains to count carefully the sets of points by which I shall estimate the volumes of certain figures.

No mathematical processes beyond this simple one of counting will be necessary.

Let us suppose we have before us in figure 8 a plane covered with points at regular intervals, so placed that every four determine a square.

Now it is evident that as four points determine a square, so four squares meet in a point.

Thus, considering a point inside a square as belonging to it, we may say that a point on the corner of a square belongs to it and to four others equally: belongs a quarter of it to each square.

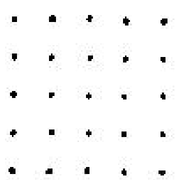

Figure 8

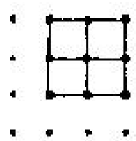

Figure 9

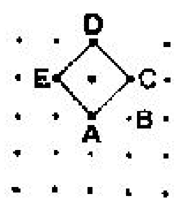

Figure 10

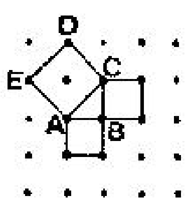

Figure 11

Thus the square ACDE (fig. 10) contains one point, and has four points at the four corners. Since one-fourth of each of these four belongs to the square, the four together count as one point, and the point value of the square is two points—the one inside and the four at the corner make two points belonging to it exclusively.

Now the area of this square is two unit squares, as can be seen by drawing two diagonals in figure 11.

We also notice that the square in question is equal to the sum of the squares on the sides AB, BC, of the right-angled triangle ABC. Thus we recognize the proposition that the square on the hypotenuse is equal to the sum of the squares on the two sides of a right-angled triangle.

Now suppose we set ourselves the question of determining whereabouts, in the ordered system of points, the end of a line would come when it turned about a point keeping one extremity fixed at the point.

We can solve this problem in a particular case. If we can find a square lying slantwise amongst the dots which is equal to one which goes regularly, we shall know that the two sides are equal, and that the slanting side is equal to the straight-way side. Thus the volume and shape of a figure remaining unchanged will be the test of its having rotated about the point, so that we can say that its side in its first position would turn into its side in the second position.

Now, such a square can be found in the one whose side is five units in length.

In figure 12, in the square on AB, there are

9 points interior	9
4 at the corners	1
4 sides with 3 on each side, considered as $1\frac{1}{2}$, on each side, because belonging equally to two squares	6

The total is 16. There are 9 points in the square on BC. In the square on AC there are—

24 points inside	24
4 at the corners	1

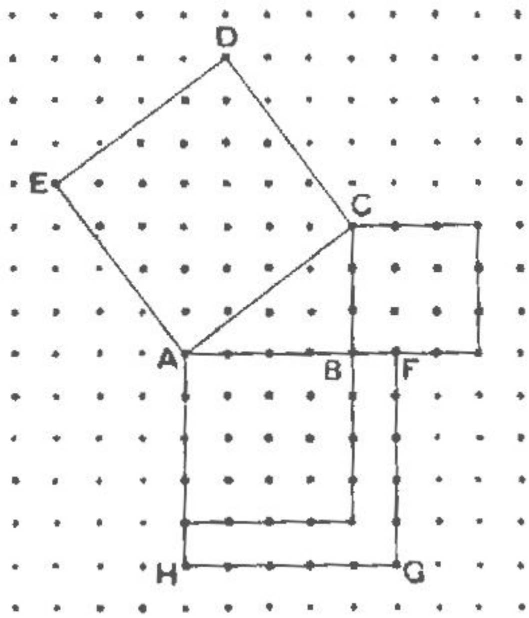

Figure 12

or 25 altogether.

Hence we see again that the square on the hypotenuse is equal to the squares on the sides.

Now take the square AFHC, which is larger than the square on AB. It contains 25 points.

16 inside	16
16 on the sides, counting as	8
4 on the corners	1

making 25 altogether.

If two squares are equal we conclude the sides are equal. Hence, the line AF turning round A would move so that it would after a certain turning coincide with AC.

This is preliminary, but it involves all the mathematical difficulties that will present themselves.

There are two alterations of a body by which its volume is not changed.

One is the one we have just considered, rotation, the other is what is called shear.

Consider a book, or heap of loose pages. They can be slid so that each one slips over the preceding one, and the whole assumes the shape b in figure 13.

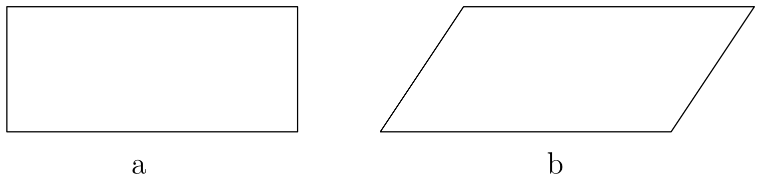

Figure 13.

The deformation is not shear alone, but shear accompanied by rotation.

Shear can be considered as produced in another way.

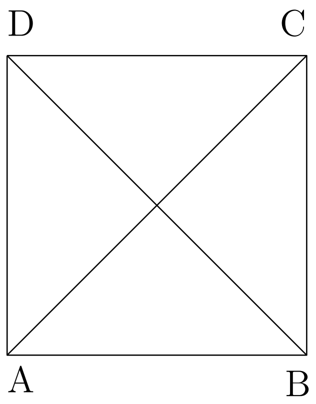

Figure 14

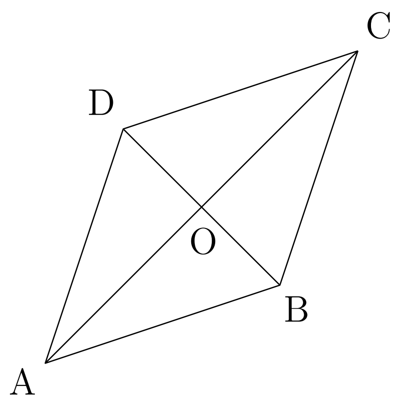

Figure 15

Take the square ABCD (fig. 14), and suppose that it is pulled out from along one of its diagonals both ways, and proportionately compressed along the other diagonal. It will assume the shape in figure 15.

This compression and expansion along two lines at right angles is what is called shear; it is equivalent to the sliding illustrated above combined with a turning round.

In pure shear a body is compressed and extended in two directions at right angles to each other, so that its volume remains unchanged.

Now we know that our material bodies resist shear—shear does violence to the internal arrangement of their particles, but they turn as wholes without such internal resistance.

But there is an exception. In a liquid shear and rotation take place equally easily, there is no more resistance against a shear than there is against a rotation.

Now, suppose all bodies were to be reduced to the liquid state, in which they yield to shear and to rotation equally easily, and then were to be reconstructed as solids, but in such a way that shear and rotation had interchanged places.

That is to say, let us suppose that when they had become solids again they would shear without offering any internal resistance, but a rotation would do violence to their internal arrangement.

That is, we should have a world in which shear would have taken the place of rotation.

A shear does not alter the volume of a body: thus an inhabitant living in such a world would look on a body sheared as we look on a body rotated. He would say that it was of the same shape, but had turned a bit round.

Let us imagine a Pythagoras in this world going to work to investigate, as is his wont.

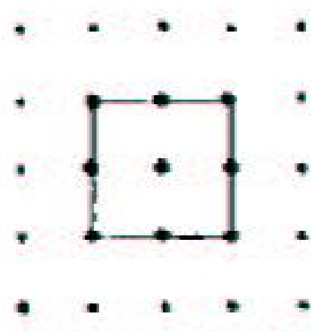

Figure 16

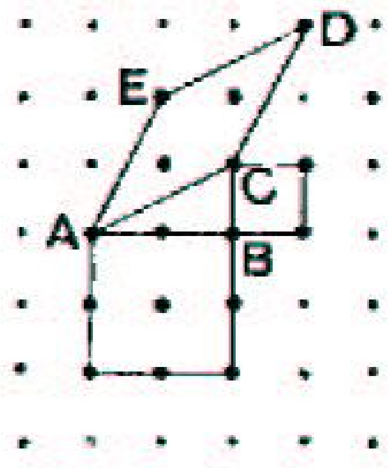

Figure 17

Figure 16 represents a square unsheared. Figure 17 represents a square sheared. It is not the figure into which the square in figure 16 would turn, but the result of shear on some square not drawn. It is a simple slanting placed figure, taken now as we took a simple slanting placed square before. Now, since bodies in this world of shear offer no internal resistance to shearing, and keep their volume when sheared, an inhabitant accustomed to them would not consider that they altered their shape under shear. He would call ACDE as much a square as the square in figure 16. We will call such figures shear squares. Counting the dots in ACDF, we find

2 inside	2
4 at corners	1

or a total of 3.

Now, the square on the side AB has 4 points, that on BC has 1 point. Here the shear square on the hypotenuse has not 5 points but 3; it is not the sum of the squares on the sides, but the difference.

This relation always holds. Look at figure 18.

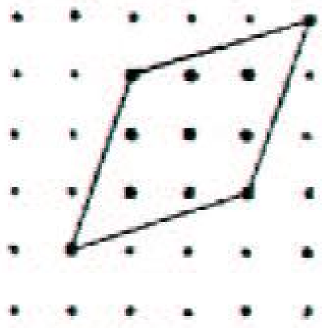

Figure 18

Shear square on hypotenuse

7 internal	7
4 at corners	1
	8

Square on one side—which the reader can draw for himself—

4 internal	4
8 on sides	4
4 at corners	1
	9

The square on the other side is 1. Hence in this case again the difference is equal to the shear square on the hypotenuse, 9 - 1 = 8.

Thus in a world of shear the square on the hypotenuse would be equal to the difference of the squares on the sides of a right-angled triangle.

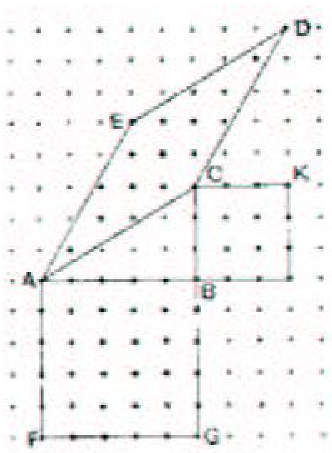

Figure 19

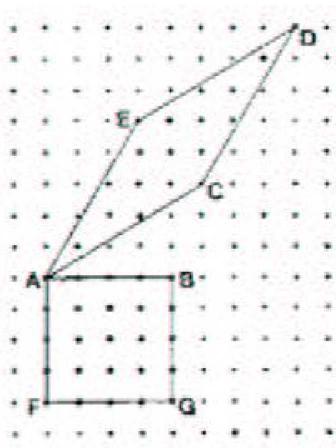

Figure 20

In figure 19 another shear square is drawn on which the above relation can be tested.

What now would be the position a line on turning by shear would take up?

We must settle this in the same way as previously with our turning.

Since a body sheared remains the same, we must find two equal bodies, one in the straight way, one in the slanting way, which have the same volume. Then the side of one will by turning become the side of the other, for the two figures are each what the other becomes by a shear turning.

We can solve the problem in a particular case—

In the figure ACDE (fig. 20) there are

15 inside	15
4 at the corners	1
a total of	16

Now in the square ABCF, there are 16—

9 inside	9
12 on sides	6
4 at corners	1
	16

Hence the square on AB would, by the shear turning, become the shear square ACDE.

And hence the inhabitant of this world would say that the line AB turned into the line AC. These two lines would be to him two lines of equal length, one turned a little way round from the other.

That is, putting shear in place of rotation, we get a different kind of figure, as the result of the shear rotation, from what we got with our ordinary rotation. And as a consequence we get a position for the end of a line of invariable length when it turns by the shear rotation, different from the position which it would assume on turning by our rotation.

A real material rod in the shear world would, on turning about A, pass from the position AB to the position AC. We say that its length alters when it becomes AC, but this transformation of AB would seem to an inhabitant of the shear world like a turning of AB without altering in length.

If now we suppose a communication of ideas that takes place between one of ourselves and an inhabitant of the shear world, there would evidently be a difference between his views of distance and ours.

We should say that his line AB increased in length in turning to AC. He would say that our line AF (fig. 33) decreased in length in turning to AC. He would think that what we called an equal line was in reality a shorter one.

We should say that a rod turning round would have its extremities in the positions we call at equal distances. So would he—but the positions would be different. He could, like us, appeal to the properties of matter. His rod to him alters as little as ours does to us.

Now, is there any standard to which we could appeal, to say which of the two is right in this argument? There is no standard.

We should say that, with a change of position, the configuration and shape of his objects altered. He would say that the configuration and shape of our objects altered in what we called merely a change of position. Hence distance independent of position is inconceivable, or practically, distance is solely a property of matter.

There is no principle to which either party in this controversy could appeal. There is nothing to connect the definition of distance with our ideas rather than with his, except the behavior of an actual piece of matter. For the study of the processes which go on in our world the definition of distance given by taking the sum of the squares is of paramount importance to us. But as a question of pure space without making any unnecessary assumptions, the shear world is just as possible and just as interesting as our world.

It was the geometry of such conceivable worlds that Lobatchewsky and Bolyai studied.

This kind of geometry has evidently nothing to do directly with four-dimensional space.

But a connection arises in this way. It is evident that, instead of taking a simple shear as I have done, and defining it as that change of the arrangement of the particles of a solid which they will undergo without offering any resistance due to their mutual action, I might take a complex motion, composed of a shear and a rotation together, or some other kind of deformation.

Let us suppose such an alteration picked out and defined as the one which means simple rotation; then the type, according to which all bodies will alter by this rotation, is fixed.

Looking at the movements of this kind, we should say that the objects were altering their shape as well as rotating. But to the inhabitants of that world they would seem to be unaltered, and our figures in their motions would seem to them to alter.

In such a world the features of geometry are different. We have seen one such difference in the case of our illustration of the world of shear, where the square on the hypotenuse was equal to the difference, not the sum, of the squares on the sides.

In our illustration we have the same laws of parallel lines as in our ordinary rotation world, but in general the laws of parallel lines are different.

In one of these worlds of a different constitution of matter, through one point there can be two parallels to a given line, in another of them there can be none; that is, although a line be drawn parallel to another it will meet it after a time.

Now it was precisely in this respect of parallels that Lobatchewsky and Bolyai discovered these different worlds. They did not think of them as worlds of matter, but they discovered that space did not necessarily mean that our law of parallels is true. They made the distinction between laws of space and laws of matter, although that is not the form in which they stated their results.

The way in which they were led to these results was the following. Euclid had stated the existence of parallel lines as a postulate—putting frankly this unproved proposition—that one line and only one parallel to a given straight line can be drawn, as a demand, as something that must be assumed. The words of his ninth postulate are these: "if a straight line meeting two other straight lines makes the interior angles on the same side of it equal to two right angles, the two straight lines will never meet."

The mathematicians of later ages did not like this bald assumption, and not being able to prove the proposition they called it an axiom—the eleventh axiom.

Many attempts were made to prove the axiom; no one doubted of its truth, but no means could be found to demonstrate it. At last an Italian, Sacchieri, unable to find a proof, said: "Let us suppose it not true." He deduced the results of there being possibly two parallels to one given line through a given point, but feeling the waters too deep for the human reason, he devoted the latter half of his book to disproving what he had assumed in the first part.

Then Bolyai and Lobatchewsky with firm step entered on the forbidden path. There can be no greater evidence of the indomitable nature of the human spirit, or of its manifest destiny to conquer all those limitations which bind it down within the sphere of sense than this grand assertion of Bolyai and Lobatchewsky.

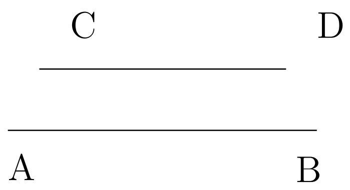

Figure 21

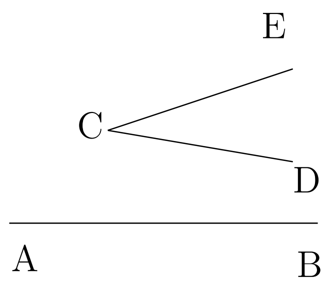

Figure 22

Take a line AB and a point C. We say and see and know that through C can only be drawn one line parallel to AB.

But Bolyai said: "I will draw two." Let CD be parallel to AB, that is, not meet AB however far produced, and let lines beyond CD also not meet AB; let there be a certain region between CD and CE, in which no line drawn meets AB. CE and CD produced backwards through C will give a similar region on the other side of C.

Nothing so triumphantly, one may almost say so insolently, ignoring of sense had ever been written before. Men had struggled against the limitations of the body, fought them, despised them, conquered them. But no one had ever thought simply as if the body, the bodily eyes, the organs of vision, all this vast experience of space, had never existed. The age-long contest of the soul with the body, the struggle for mastery, had come to a culmination. Bolyai and Lobatchewsky simply thought as if the body was not. The struggle for dominion, the strife and combat of the soul were over; they had mastered, and the Hungarian drew his line.

Can we point out any connection, as in the case of Parmenides, between these speculations and higher space? Can we suppose it was any inner perception by the soul of a motion not known to the senses, which resulted in this theory so free from the bonds of sense? No such supposition appears to be possible.

Practically, however, metageometry had a great influence in bringing the higher space to the front as a working hypothesis. This can be traced to the tendency the mind has to move in the direction of least resistance. The results of the new geometry could not be neglected, the problem of parallels had occupied a place too prominent in the development of mathematical thought for its final solution to be neglected. But this utter independence of all mechanical considerations, this perfect cutting loose from the familiar intuitions, was so difficult that almost any other hypothesis was more easy of acceptance, and when Beltrami showed that the geometry of Lobatchewsky and Bolyai was the geometry of shortest lines drawn on certain curved surfaces, the ordinary definitions of measurement being retained, attention was drawn to the theory of a higher space. An illustration of Beltrami's theory is furnished by the simple consideration of hypothetical beings living on a spherical surface (fig. 23).

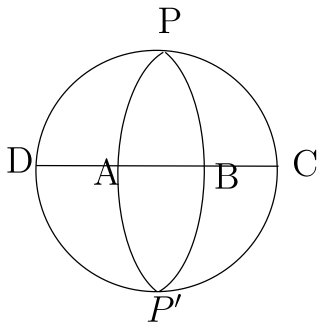

Figure 23

Let ABCD be the equator of a globe, and AP, BP, meridian lines drawn to the pole, P. The lines AB, AP, BP would seem to be perfectly straight to a person moving on the surface of the sphere, and unconscious of its curvature. Now AP and BP both make right angles with AB. Hence they satisfy the definition of parallels. Yet they meet in P. Hence a being living on a spherical surface, and unconscious of its curvature, would find that parallel lines would meet. He would also find that the angles in a triangle were greater than two right angles. In the triangle PAB, for instance, the angles at A and B are right angles, so the three angles of the triangle PAB are greater than two right angles.

Now in one of the systems of metageometry (for after Lobatchewsky had shown the way it was found that other systems were possible besides his), the angles of a triangle are greater than two right angles.

Thus a being on a sphere would form conclusions about his space which are the same as he would form if he lived on a plane, the matter in which had such properties as are presupposed by one of these systems of geometry. Beltrami also discovered a certain surface on which there could be drawn more than one "straight" line through a point which would not meet another given line. I use the word straight as equivalent to the line having the property of giving the shortest path between any two points on it. Hence, without giving up the ordinary methods of measurement, it was possible to find conditions in which a plane being would necessarily have an experience corresponding to Lobatchewsky's geometry. And by the consideration of a higher space, and a solid curved in such a higher space, it was possible to account for a similar experience in a space of three dimensions.

Now, it is far more easy to conceive of a higher dimensionality to space than to imagine that a rod in rotating does not move so that its end describes a circle. Hence, a logical conception having been found harder than that of a four-dimensional space, thought turned to the latter as a simple explanation of the possibilities to which Lobatchewsky had awakened it. Thinkers became accustomed to deal with the geometry of higher space—it was Kant, says Veronese, who first used the expression of "different spaces"—and with familiarity the inevitableness of the conception made itself felt.

From this point it is but a small step to adapt the ordinary mechanical conceptions to a higher spatial existence, and then the recognition of its objective existence could be delayed no longer. Here, too, as in so many cases, it turns out that the order and connection of our ideas is the order and connection of things.

What is the significance of Lobatchewsky's and Bolyai's work?

It must be recognized as something totally different from the conception of a higher space; it is applicable to spaces of any number of dimensions. By immersing the conception of distance in matter to which it properly belongs, it promises to be of the greatest aid in analysis; for the effective distance of any two particles is the product of complex material conditions and cannot be measured by hard and fast rules. Its ultimate significance is altogether unknown. It is a cutting loose from the bonds of sense, not coincident with the recognition of a higher dimensionality, but indirectly contributory thereto.

Thus, finally, we have come to accept what Plato held in the hollow of his hand; what Aristotle's doctrine of the relativity of substance implies. The vast universe, too, has its higher, and in recognizing it we find that the directing being within us no longer stands inevitably outside our systematic knowledge.

The Recognition of the Fourth Dimension

1904

THERE are two directions of inquiry in which the research for the physical reality of a fourth dimension can be prosecuted. One is the investigation of the infinitely great, the other is the investigation of the infinitely small.

By the measurement of the angles of vast triangles whose sides are the distances between the stars, astronomers have sought to determine if there is any deviation from the values given by geometrical deduction. If the angles of a celestial triangle do not together equal two right angles, there would be an evidence for the physical reality of a fourth dimension.

This conclusion deserves a word of explanation. If space is really four dimensional, certain conclusions follow which must be brought clearly into evidence if we are to frame the questions definitely which we put to Nature. If space is four dimensional, there must be a solid material sheet against which we move. This sheet must stretch alongside every object in every direction in which it visibly moves. Every material body must slip or slide along this sheet, not deviating from contact with it in any motion which we can observe.

The necessity for this assumption is clearly apparent if we consider the analogous case of a suppositionary plane world. If there were any creatures whose experience were confined to a plane, we must account for their limitation. If they were free to move in every space direction they would have a three-dimensional motion; hence they must be physically limited, and the only way in which we can conceive such a limitation to exist is by means of a material surface against which they slide. The existence of this surface could only be known to them indirectly. It does not lie in any direction from them in which the kinds of motion they know of leads them. If it were perfectly smooth and always in contact with every material object, there would be no difference in their relations to it which would direct their attention to it.

But if this surface were curved–if it were, say, in the form of a vast sphere–the triangles they drew would really be triangles of a sphere, and when these triangles are large enough the angles diverge from the magnitudes they would have for the same lengths of sides if the surface were plane. Hence by the measurement of triangles of very great magnitude, a plane being might detect a difference from the laws of a plane world in his physical world, and so be led to the conclusion that there was in reality another dimension to space a third dimension as well as the two which his ordinary experience made him familiar with.

Now, astronomers have thought it worthwhile to examine the measurements of vast triangles drawn from one celestial body to another with a view to determine if there is anything like a curvature in our space–that is to say, they have tried astronomical measurements to find out if the vast solid sheet against which, on the supposition of a fourth dimension, everything slides is curved or not. These results have been negative. The solid sheet, if it exists, is not curved or, being curved, has not a sufficient curvature to cause any observable deviation from the theoretical value of the angles calculated.

Hence the examination of the infinitely great leads to no decisive criterion. It neither proves nor disproves the existence of a fourth dimension.

Coming now to the prosecution of the inquiry in the direction of the infinitely small, we have to state the question thus: Our laws of movement are derived from the examination of bodies which move in three-dimensional space. All our conceptions are founded on the supposition of a space which is represented analytically by three independent axes and variations along them–that is, it is a space in which there are three independent movements. Any motion possible in it can be compounded out of these three movements, which we may call: up, right, away.

To examine the actions of the very small portions of matter with the view of ascertaining if there is any evidence in the phenomena for the supposition of a fourth dimension of space, we must commence by clearly defining what the laws of mechanics would be on the supposition of a fourth dimension. It is no use asking if the phenomena of the smallest particles of matter are like–we do not know what. We must have a definite conception of what the laws of motion would be on the supposition of the fourth dimension, and then inquire if the phenomena of the activity of the smaller particles of matter resemble the conceptions which we have elaborated.

Now, the task of forming these conceptions is by no means one to be lightly dismissed. Movement in space has many features which differ entirely from movement on a plane; and when we set about to form the conception of motion in four dimensions, we find that there is at least as great a step as from the plane to three-dimensional space.

I do not say that the step is difficult, but I want to point out that it must be taken. When we have formed the conception of four-dimensional motion, we can ask a rational question of Nature. Before we have elaborated our conceptions we are asking if an unknown is like an unknown–a futile inquiry.

As a matter of fact, four-dimensional movements are in every way simple and more easy to calculate than three-dimensional movements, for four-dimensional movements are simply two sets of plane movements put together. It appears to me one of the most marvelous characteristics of the power of the intellect to find how, without any of the familiarity derived from experience, it is possible to grasp the facts of four-dimensional movement and apprehend the consequences of these conceptions.

Without the formation of an experience of four-dimensional bodies, their shapes and motions, the subject can be but formal–logically conclusive, not intuitively evident. It is to this logical apprehension that I must appeal.

It is perfectly simple to form an experiential familiarity with the facts of four-dimensional movement. The method is analogous to that which a plane being would have to adopt to form an experiential familiarity with three-dimensional movements, and may be briefly summed up as the formation of a compound sense by means of which duration is regarded as equivalent to extension.

Consider a being confined to a plane. A square enclosed by four lines will be to him a solid, the interior of which can only be examined by breaking through the lines. If such a square were to pass transverse to his plane, it would immediately disappear. It would vanish, going in no direction to which he could point.

If, now, a cube be placed in contact with his plane, its surface of y contact would appear like the square which we have just mentioned. But if it were to pass transverse to his plane, breaking through it, it would appear as a lasting square. The three-dimensional matter will give a lasting appearance in circumstances under which two-dimensional matter will at once disappear.

Similarly, a four-dimensional cube, or, as we may call it, a tesseract, which is generated from a cube by a movement of every part of the cube in a fourth direction at right angles to each of the three visible directions in the cube, if it moved transverse to our space, would appear as a lasting cube.

A cube of three-dimensional matter, since it extends to no distance at all in the fourth dimension, could instantly disappear if subjected to a motion transverse to our space. It would disappear and be gone without it being possible to point to any direction in which it had moved. All attempts to visualize a fourth dimension are futile. If must be 18 connected with a time experience in three space.

The most difficult notion for a plane being to acquire would be that of rotation about a line. Consider a plane being facing a square: If he were told that rotation about a line were possible, he would move his square this way and that. A square in a plane can rotate about a point, but to rotate about a line would seem to the plane being perfectly impossible. How could those parts of his square which were on one side of an edge come to the other side without the edge moving? He could understand their reflection in the edge. He could form an idea of the looking-glass image of his square lying on the opposite side of the line of an edge, but by no motion that he knows of can he make the actual square assume that position. The result of the rotation would be like reflection in the edge, but it would be a physical impossibility to produce it in the plane.

The demonstration of rotation about a line must be to him purely formal. If he conceived the notion of a cube stretching out in an unknown direction away from his plane, then he can see the base of it, his square in the plane, rotating round a point. He can likewise apprehend that every parallel section taken at successive intervals in the unknown direction rotates in like manner round a point. Thus he would come to conclude that the whole body rotates round a line–the line consisting of the succession of points round which the plane sections rotate. Thus, given three axes, x, y, z; if x rotates to take the place of y, and y turns so as to point to negative x – then the third axis remaining unaffected by this turning is the axis about which the rotation takes place. This, then, would have to be his criterion of the axis of a rotation–that which remains unchanged when a rotation of every plane section of a body takes place.

There is another way in which a plane being can think about three-dimensional movements; and, as it affords the type by which we can most conveniently think about four-dimensional movements, it will be no loss of time to consider it in detail.

We can represent the plane being and his object by figures cut out of paper, which slip on a smooth surface. The thickness of these bodies must be taken as so minute that their extension in the third dimension escapes the observation of the plane being, and he thinks about them as if they were mathematical plane figures in a plane instead of being material bodies capable of moving on a plane surface. Let Ax, Ay be two axes and ABCD a square (fig. 45). As far as movements in the plane are concerned, the square can rotate about a point, A, for example. It cannot rotate about a side such as AC.

But if the plane being is aware of the existence of a third dimension he can study the movements possible in the ample space, taking his figure portion by portion.

His plane can only hold two axes. But, since it can hold two, he is able to represent a turning into the third dimension if he neglect one of his axes and represent the third axis as lying in his plane. He can make a drawing in his plane of what stands up perpendicularly from his plane. Let Az be the axis, which stands perpendicular to his plane at A. He can draw in his plane two lines to represent the two axes, Ax and Az. Let figure 46 be this drawing. Here the z axis has taken the place of the y axis, and the plane of A x A z is represented in his plane. In this figure all that exists of the square ABCD will be the line AB.

The square extends from this line in the V direction, but more of that direction is represented in figure 46. The plane being can study the turning of the line AB in this diagram. It is simply a case of plane turning around the point A. The line AB occupies intermediate portions like AB' and after half a revolution will lie on Ax produced through A.

Now, in the same way, the plane being can take another point, A', and another line, A'B", in his square. He can make the drawing of the two directions at A’, one along A’B", the other perpendicular to his plane. He will obtain a figure precisely similar to figure 46, and will see that, as AB can turn around A, so A’B" around A’.

In this turning AB and A’B" would not interface with each other, as they would if they moved in the plane around the separate points A and A’.

Hence the plane being would conclude that a rotation round a line was possible. He could see his square as it began to make this turning. He could see it half way round when it came to lie on the opposite side of the line AC. But in intermediate portions he could not see it, for it runs out of the plane.

Coming now to the question of a four-dimensional body, let us conceive of it as a series of cubic sections, the first in our space, the rest at intervals, stretching away from our space in the unknown direction.

We must not think of a four-dimensional body as formed by moving a three-dimensional body in any direction which we can see.

Refer for a moment to figure 47. The point A, moving to the right, traces out the line AC. The line AC, moving away in a new direction, traces out the square ACEG at the base of the cube. The square AEGC, moving in a new direction, will trace out the cube ACEGBDHF. The vertical direction of this last motion is not identical with any motion possible in the plane of the base of the cube. It is an entirely new direction, at right angles to every line that can be drawn in the base. To trace out a tesseract the cube must move in a new direction–a direction at right angles to any and every line that can be drawn in the space of the cube.

The cubic sections of the tesseract are related to the cube we see, as the square sections of the cube are related to the square of its base which a plane being sees.

Let us imagine the cube in our space, which is the base of a tesseract, to turn about one of its edges. The rotation will carry the whole body with it, and each of the cubic sections will rotate. The axis we see in our space will remain unchanged, and likewise the series of axes parallel to it, about which each of the parallel cubic sections rotates. The assemblage of all of these is a plane.

Hence in four dimensions a body rotates about a plane. There is no such thing as rotation round an axis.

We may regard the rotation from a different point of view. Consider four independent axes each at right angles to all the others, drawn in a four-dimensional body. Of these four axes we can see any three. The fourth extends normal to our space.

Rotation is the turning of one axis into a second, and the second turning to take the place of the negative of the first. It involves two axes. Thus, in this rotation of a four-dimensional body, two axes change and two remain at rest. Four-dimensional rotation is therefore a turning about a plane.

As in the case of a plane being the result of rotation about a line could appear as the production of a looking-glass image of the original object on the other side of the line, so to us the result of a four-dimensional rotation would appear like the production of a looking-glass image of a body on the other side of a plane. The plane would be the axis of the rotation, and the path of the body between its two appearances would be unimaginable in three-dimensional space.

Let us now apply the method by which a plane being could examine the nature of rotation about a line in our examination of rotation about a plane. Figure 47 represents a cube in our space, the three axes x, y, z, denoting its three dimensions. Let w represent the fourth dimension. Now, since in our space we can represent any three dimensions, we can, if we choose, make a representation of what is in the space determined by three axes x, z, w. This is a three-dimensional space determined by two of the axes we have drawn, x and z, and in place of y the fourth axis, w. We cannot, keeping x and z, have both V and w in our space; so we will let y go and draw w in its place. What will be our view of the cube?

Evidently we shall have simply the square that is in the plane of xz, the square ACDB. The rest of the cube stretches in the y direction, and, as we have none of the space so determined, we have only the face of the cube. This a represented in figure 48.

Now, suppose the whole cube to be turned from the x to the w direction. Conformably with our method, we will not take the whole of the cube into consideration at once, but will begin with the face ABCD.

Let this face begin to turn. Figure 49 represents one of the positions it will occupy; the line AB remains on the z axis. The rest of the face extends between the x and the w direction.

Now, since we can take any three axes, let us look at what lies in the space of zyw, and examine the turning there. We must now let the z axis disappear and let the w axis run in the direction in which z ran.

Making this representation, what do we see of the cube? Obviously we see only the lower face. The rest of the cube lies in the space of xyz. In the space of xyw we have merely the base of the cube lying in the plane of xy, as shown in figure 50.

Now let the x to w turning take place. The square ACEG will turn about the line AE (fig. 51). This edge will remain along they axis and will be stationary, however far the square turns.

Thus, if the tube be turned by an x to w turning, both the edge AB and the edge AC remain stationary; hence the whole face ABEF in the yz plane remains fixed. The turning has taken place about the face ABEF.

Suppose this turning to continue till AC runs to the left from A. The cube will occupy the position shown in figure 52. This is the looking-glass image of the cube in figure 47. By no rotation in three-dimensional space can the cube be brought from the position in figure 47 to that shown in figure 52.

We can think of this turning as a turning of the face ABCD about AB, and a turning of each section parallel to ABCD round the vertical line in which it intersects the face ABEF, the space in which the turning takes place being a different one from that in which the cube lies.

One of the conditions, then, of our inquiry in the direction of the infinitely small is that we form the conception of a rotation about a plane. The production of a body in a state in which it presents the appearance of a looking-glass image of its former state is the criterion for a four-dimensional rotation.

There is some evidence for the occurrence of such transformations of bodies in the change of bodies from those which produce a right-handed polarization of light to those which produce a left-handed polarization; but this is not a point to which any very great importance can be attached.

Still, in this connection, let me quote a remark from Prof. John G. MeKendrick's address on physiology before the British Association at Glasgow. Discussing the possibility of the hereditary production of characteristics through the material structure of the ovum, he estimates that in it there exist 12,000,000,000 biophors, or ultimate particles of living matter, a sufficient number to account for hereditary transmission, and observes: "Thus it is conceivable that vital activities may also be determined by the kind of motion that takes place in the molecules of that which we speak of as living matter. It may be different in kind from some of the motions known to physicists, and it is conceivable that life may be the transmission to dead matter, the molecules of which have already a special kind of motion of a form of motion sui generis.

Now, in the realm of organic beings symmetrical structures–those with a right and left symmetry–are everywhere in evidence. Granted that four dimensions exist, the simplest turning produces the mirror-image form, and by a folding over, structures could be produced, duplicated right and left, just as in the case of a plane. A symmetrical and lifelike contour is created by the child's amusement of folding an ink-spattered paper along the line of blots.

Whether four-dimensional motions correspond to the physiologist's demand for a special kind of motion or not, I do not know. Our business is with the evidence for its existence in physics. For this purpose it is necessary to examine into the significance of rotation round a plane in the case of extensible and of fluid matter.

Let us dwell a moment longer on the rotation of a rigid body. Looking at the cube in figure 47, which turns about the face of ABFE, we see that any line in the face can take the place of the vertical and horizontal lines we have examined. Take the diagonal line AF and the section through it to GH. The portions of matter which were on one side of AF in this section in figure 47 are on the opposite side of it in figure 52. They have gone round the line AF. Thus the rotation round a face can be considered as a number of rotations of sections round parallel lines in it.

The turning about two different lines is impossible in three-dimensional space. To take another illustration (fig. 53), suppose A and B are two parallel lines in the xy plane, and let CD and EF be two rods crossing them. Now, in the space of xcz if the rods turn round the lines A and B in the same direction they will make two independent circles.

When the end F is going down the end C will be coming up. They will meet and conflict.

But if we rotate the rods about the plane of AB by the z to w rotation (fig. 54), these movements will not conflict. Suppose all the figure removed with the exception of the plane xz, and from this plane draw the axis of w, so that we are looking at the space of xzw.

Here, figure 54, we cannot see the lines A and B. We see the points G and H, in which A and B intercept the x axis, but we cannot see the lines themselves, for they run in the y direction, and that is not in our drawing.

Now, if the rods move with the z to w rotation, they will turn in parallel planes, keeping their relative positions. The point D, for instance, will describe a circle. At one time it will be above the line A at another time below it. Hence it rotates round A.

Now only two rods, but any number of rods crossing the plane will move round it harmoniously. We can think of this rotation by supposing the rods standing up from one line to move round that line and remembering that it is not inconsistent with this rotation for the rods standing up along another line also to move round it, the relative positions of all the rods being preserved. Now, if the rods are thick together, they may represent a disk of matter, and we see that a disk of matter can rotate round a central plane.

Rotation round a plane is exactly analogous to rotation round an axis in three dimensions. If we want a rod to turn round, the ends must be free; so if we want a disk of matter to turn round its central plane by a four-dimensional turning, all the contour must be free. The whole contour corresponds to the ends of the rod. Each point of the contour can be looked on as the extremity of an axis in the body, round each point of which there is a rotation of the matter in the disk.

If the one end of a rod be clamped, we can twist the rod, but not turn it round; so if any part of the contour of a disk is clamped we can impart a twist to the disk, but not turn it round its central plane. In the case of extensible materials a long, thin rod will twist round its axis, even when the axis is curved; as, for instance, in the case of a ring of India rubber.

In an analogous manner, in four dimensions we can have rotation round a curved plane, if I may use the expression. A sphere can be turned inside out in four dimensions.

Let figure 55 represent a spherical surface on each side of which a layer of matter exists. The thickness of the matter is represented by the rods CD and EF, extending equally without and within.

Now, take the section of the sphere by the yz plane; we have a circle figure 56. Now, let the w axis be drawn in place of the x axis so that we have the space of yzw represented. In this space all that there will be seen of the sphere is the circle drawn.

Here we see that there is no obstacle to prevent the rods turning round. If the matter is so elastic that it will give enough for the particles at E and C to be separated as they are at F and D, they can rotate round to the position D and F, and a similar motion is possible for all other particles. There is no matter or obstacle to prevent them from moving out in the w direction, and then on round the circumference as an axis. Now, what will hold for one section will hold for all, as the fourth dimension is at right angles to all the sections which can be made of the sphere.

We have supposed the matter of which the sphere is composed to be three dimensional. If the matter had a small thickness in the fourth dimension, there would be a slight thickness in figure 56 above the plane of the paper a thickness equal to the thickness of the matter in the fourth dimension. The rods would have to be replaced by thin slabs. But this would make no difference as to the possibility of the rotation. This motion is discussed by Newcomb in the first volume of the American Journal of Mathematics.

Let us now consider, not a merely extensible body, but a liquid one. A mass of rotating liquid, a whirl, eddy, or vortex, has many remarkable properties. On first consideration we should expect a rotating mass of liquid immediately to spread off and lose itself in the surrounding liquid. The water flies off a wheel whirled round, and we should expect the rotating liquid to be dispersed. But we see the eddies in a river strangely persistent. The rings that occur in puffs of smoke and last so long are whirls or vortices curved round so that their opposite ends join together. A cyclone will travel over great distances.

Helmholtz was the first to investigate the properties of vortices. He studied them as they would occur in a perfect fluid–that is, one without friction of one moving portion or another. In such a medium vortices would be indestructible. They would go on forever, altering their shape, but consisting always of the same portion of the fluid. But a straight vortex could not exist surrounded entirely by the fluid. The ends of a vortex must reach to some boundary inside or outside the fluid.

A vortex which is bent round so that its opposite ends join is capable of existing, but no vortex has a free end in the fluid. The fluid round a vortex is always in motion, and one produces a definite movement in another.

Lord Kelvin has proposed the hypothesis that portions of a fluid segregated in vortices account for the origin of matter. The properties of the aether in respect of its capacity of propagating disturbances can be explained by the assumption of vortices in it instead of by a property of rigidity. It is difficult to conceive, however, of any arrangement of the vortex rings and endless vortex filaments in the aether.

Now, the further consideration of four-dimensional rotations shows the existence of a kind of vortex which would make an aether filled with a homogeneous vortex motion easily thinkable.

To understand the nature of this vortex, we must go on and take a step by which we accept the full significance of the four-dimensional hypothesis. Granted four-dimensional axes, we have seen that a rotation of one into another leaves two unaltered, and these two form the axial plane about which the rotation takes place. But what about these two? Do they necessarily remain motionless? There is nothing to prevent a rotation of these two, one into the other, taking place concurrently with the first rotation. This possibility of a double rotation deserves the most careful attention, for it is the kind of movement which is distinctively typical of four dimensions.

Rotation round a plane is analogous to rotation round an axis. But in three-dimensional space there is no motion analogous to the double rotation, in which, while axis 1 changes into axis 2, axis 3 changes into axis 4.

Consider a four-dimensional body, with four independent axes, x,y,z,w. A point in it can move in only one direction at a given moment. If the body has a velocity of rotation by which the x axis changes into the y axis and all parallel sections move in a similar manner, then the point will describe a circle. If, now, in addition to the rotation by which the x axis changes into they axis the body has a rotation by which the z axis turns into the w axis, the point in question will have a double motion in consequence of the two turnings. The motions will compound, and the point will describe a circle, but not the same circle which it would describe in virtue of either rotation separately.

We know that if a body in three-dimensional space is given two movements of rotation, they will combine into a single movement of rotation round a definite axis. It is in no different condition from that in which it is subjected to one movement of rotation. The direction of the axis changes; that is all. The same is not true about a four-dimensional body.

The two rotations x to y and z to w are independent. A body subject to the two is in a totally different condition to that which it is in when subject to one only. When subject to a rotation such as that of x to y, a whole plane in the body, as we have seen, is stationary. When subject to the double rotation no part of the body is stationary except the point common to the two planes of rotation.

If the two rotations are equal in velocity, every point in the body describes a circle. All points equally distant from the stationary point describe circles of equal size.

We can represent a four-dimensional sphere by means of two diagrams, in one of which we take the three axes x, y, and z; in the other the axes x,w, and z. In figure 57 we have the view of a four-dimensional sphere in the space of xyz. Figure 57 shows all that we can see of the four sphere in the space of xyz, for it represents all the points in that space, which are at an equal distance from the center.

Let us now take the xz section, and let the axis of w take the place of the y axis. Here, in figure 58, we have the space of xzw. In this space we have to take all the points which are at the same distance from the center, consequently we have another sphere. If we had a three-dimensional sphere, as has been shown before, we should have merely a circle in the xzw space, the xz circle seen in the space of xzw. But now, taking the view in the space of xzw, we have a sphere in that space also. In a similar manner, whichever set of three axes we take, we obtain a sphere.

In figure 57, let us imagine the rotation in the direction xy to be taking place. The point x will turn to y, and p to p'. The axis remains stationary, and this axis is all of the plane zw which we can see in the space section exhibited in the figure.

In figure 58, imagine the rotation from z to w to be taking place. The w axis now occupies the position previously occupied by the y axis. This does not mean that the w axis can coincide with the y axis. It indicates that we are looking at the four-dimensional sphere from a different point of view. Any three-space view will show us three axes, and in figure 58 we are looking at xzw.

The only part that is identical in the two diagrams is the circle of the x and z axes, which axes are contained in both diagrams. Thus the plane z, x, z' is the same in both, and the point p represents the same point in both diagrams. Now, in figure 58 let the zw rotation take place, the z axis will turn toward the point w of the w axis, and the point p will move in a circle about the point x.

Thus in figure 57 the point p moves in a circle parallel to the xy plane; in figure 58 it moves in a circle parallel to the zw plane, indicated by the arrow.

Now, suppose both of these independent rotations compounded; the point p will move in a circle, but this circle will coincide with neither of the circles in which either one of the rotations will take it. The circle the point p will move in will depend on its position on the surface of the four sphere.

In this double rotation, possible in four-dimensional space, there is a kind of movement totally unlike any with which we are familiar in three-dimensional space. It is a requisite preliminary to the discussion of the behavior of the small particles of matter, with a view to determining whether they show the characteristics of four-dimensional movements, to become familiar with the main characteristics of this double rotation. And here I must rely on a formal and logical assent rather than on the intuitive apprehension which can only be obtained by a more detailed study.

In the first place this double rotation consists in two varieties or kinds, which we will call the A and B kinds. Consider four axes, x, y, z, a'. The rotation of x to y can be accompanied with the rotation of z to w. Call this the A kind.

But also the rotation of x to y can be accompanied by the rotation of not z to w, but w to z. Call this the B kind.

They differ in only one of the component rotations. One is not the negative of the other. It is the semi-negative. The opposite of an x to y, z to w rotation would be y to x, w to z. The semi-negative is x to y and w to z.

If four dimensions exist and we cannot perceive them because the extension of matter is so small in the fourth dimension that all movements are withheld from direct observation except those which are three dimensional, we should not observe these double rotations, but only the effects of them in three-dimensional movements of the type with which we are familiar.

If matter in its small particles is four dimensional we should expect this double rotation to be a universal characteristic of the atoms and molecules, for no portion of matter is at rest. The consequences of this corpuscular motion can be perceived, but only under the form of ordinary rotation or displacement. Thus if the theory of four dimensions is true we have in the corpuscles of matter a whole world of movement which we can never study directly, but only by means of inference.

The rotation A, as I have defined it, consists of two equal rotations–one about the plane of zw, the other about the plane of xy. It is evident that these rotations are not necessarily equal. A body may be moving with a double rotation in which these two independent components are not equal; but in such a case we can consider the body to be moving with a composite rotation–a rotation of the A or B kind and, in addition, a rotation about a plane.

If we combine an A and a B movement, we obtain a rotation about a plane; for, the first being x to y and z to w, and the second being x to y and w to z, when they are put together the z to w and w to z rotations neutralize each other, and we obtain an x to y rotation only, which is a rotation about the plane of zw. Similarly. if we take a B rotation, y to x and z to w, we get, on combining this with the A rotation, a rotation of z to w about the xy plane. In this case the plane of rotation is in the three-dimensional space of xyz, and we have–what has been described before–a twisting about a plane in our space.

Consider now a portion of a perfect liquid having an A motion. It can be proven that it possesses the properties of a vortex. It forms a permanent individuality–a separated-out portion of the liquid–accompanied by a motion of the surrounding liquid. It has properties analogous to those of a vortex filament. But it is not necessary for its existence that its ends should reach the boundary of the liquid. It is self-contained and, unless disturbed, is circular in every section.

If we suppose the aether to have its properties of transmitting vibration given it by such vortices, we must inquire how they lie together in four-dimensional space. Placing a circular disk on a plane and surrounding it by six others (fig. 59), we find that if the central one is given a motion of rotation, it imparts to the others a rotation which is antagonistic in every two adjacent ones. If A goes round as shown by the arrow, B and C will be moving in opposite ways, and each tends to destroy the motion of the other.

Now, if we suppose spheres to be arranged in a corresponding manner in three-dimensional space, they will be grouped in figures which are for three-dimensional space what hexagons are for plane space. If a number of spheres of soft clay be pressed together, so as to fill up the interstices, each will assume the form of a 14-sided figure, called a tetrakaidecagon.

Now, assuming space to be filled with such tetrakaidecagons and placing a sphere in each, it will be found that one sphere is touched by six others. The remaining eight spheres of the fourteen which surround the central one will not touch it, but will touch three of those in contact with it. Hence if the central sphere rotates it will not necessarily drive those around it so that their motions will be antagonistic to each other, but the velocities will not arrange themselves in a systematic manner.

In four-dimensional space the figure which forms the next term of the series hexagon, tetrakaidecagon, is a thirty-sided figure. It has for its faces ten solid tetrtkaidecagons and twenty hexagonal prisms. Such figures will exactly fill four-dimensional space, five of them meeting at every point. If, now, in each of these figures we suppose a solid four-dimensional sphere to be placed, any one sphere is surrounded by thirty others. Of these it touches ten, and, if it rotates, it drives the rest by the means of these. Now, it we imagine the central sphere to be given an A or a B rotation, it will turn the whole mass of spheres round in a systematic manner. Suppose four-dimensional space to be filled with such spheres, each rotating with a double rotation, the whole mass would form one consistent system of motion, in which each one drove every other one, with no friction or lagging behind.

Every sphere would have the same kind of rotation. In three-dimensional space, if one body drives another round, the second body rotates with the opposite kind of rotation; but in four-dimensional space these four-dimensional spheres would each have the double negative of the rotation of the one next it, and we have seen that the double negative of an A or B rotation is still an A or B rotation. Thus four-dimensional space could be filled with a system of self-preservative living energy. If we imagine the four-dimensional spheres to be of liquid and not of solid matter, then, even if the liquid were not quite perfect and there were a slight retarding effect of one vortex on another, the system would still maintain itself.

In this hypothesis we must look on the aether as possessing energy, and its transmission of vibrations, not as the conveying of a motion imparted from without, but as a modification of its own motion.

We are now in possession of some of the conceptions of four-dimensional mechanics, and will turn aside from the line of their development to inquire if there is any evidence of their applicability to the processes of nature.

Is there any mode of motion in the region of the minute which, giving three-dimensional movements for its effect, still in itself escapes the grasp of our mechanical theories? I would point to electricity. Through the labors of Faraday and Maxwell we are convinced that the phenomena of electricity are of the nature of the stress and strain of a medium; but there is still a gap to be bridged over in their explanation–the laws of elasticity, which Maxwell assumes, are not those of ordinary matter. And, to take another instance: a magnetic pole in the neighborhood of a current tends to move. Maxwell has shown that the pressures on it are analogous to the velocities in a liquid which would exist if a vortex took the place of the electric current; but we cannot point out the definite mechanical explanation of these pressures. There must be some mode of motion of a body or of the medium in virtue of which a body is said to be electrified.

Take the ions which convey charges of electricity 500 times greater in proportion to their mass than are carried by the molecules of hydrogen in electrolysis. In respect of what motion can these ions be said to be electrified? It can be shown that the energy they possess is not energy of rotation. Think of a short rod rotating. If it is turned over it is found to be rotating in the opposite direction. Now, if rotation in one direction corresponds to positive electricity, rotation in the opposite direction corresponds to negative electricity, and the smallest electrified particles would have their charges reversed by being turned over–an absurd supposition.

If we fix on a mode of motion as a definition of electricity, we must have two varieties of it, one for positive and one for negative; and a body possessing the one kind must not become possessed of the other by any change in its position.

All three-dimensional motions are compounded of rotations and translations, and none of them satisfy this first condition for serving as a definition of electricity.

But consider the double rotation of the A and B kinds. A body rotating with the A motion cannot have its motion transformed into the B kind by being turned over in any way. Suppose a body has the rotation x to y and z to w. Turning it about the xy plane, we reverse the direction of the motion x to y. But we also reverse the z to w motion, for the point at the extremity of the positive z axis is now at the extremity of the negative z axis, and since we have not interfered with its motion, it goes in the direction of position w. Hence we have y to x and w to z, which is the same as x to y and z to w. Thus both components are reversed, and there is the A motion over again. The B kind is the semi-negative, with only one component reversed.

Hence a system of molecules with the A motion would not destroy it in one another, and would impart it to a body in contact with them. Thus A and B motions possess the first requisite which must be demanded in any mode of motion representative of electricity.

Let us trace out the consequences of defining positive electricity as an A motion and negative electricity as a B motion. The combination of positive and negative electricity produces a current. Imagine a vortex in the aether of the A kind and unite with this one of the B kind. An A motion and a B motion produce rotation round a plane, which is in the aether a vortex round an axial surface. It is a vortex of the kind we represent as a part of a sphere turning inside out. Now, such a vortex must have its rim on a boundary of the aether–on a body in the aether.

Let us suppose that a conductor is a body which has the property of serving as the terminal abutment of such a vortex. Then the conception we must form of a closed current is of a vortex sheet having its edge along the circuit of the conducting wire. The whole wire will then be like the centers on which a spindle turns in three-dimensional space, and any interruption of the continuity of the wire will produce a tension in place of a continuous revolution.

As the direction of the rotation of the vortex is from a three-space direction into the fourth dimension and back again, there will be no direction of flow to the current; but it will have two sides, according to whether z goes to w or z goes to negative w.

We can draw any line from one part of the circuit to another; then the aether along that line is rotating round its points.

This geometric image corresponds to the definition of an electric circuit. It is known that the action does not lie in the wire, but in the medium, and it is known that there is no direction of flow in the wire.

No explanation has been offered in three-dimensional mechanics of how an action can be impressed throughout a region and yet necessarily run itself out along a closed boundary, as is the case in an electric current.

But this phenomenon corresponds exactly to the definition of a four-dimensional vortex.

If we take a very long magnet, so long that one of its poles is practically isolated, and put this pole in the vicinity of an electric circuit, we find that it moves.

Now, assuming for the sake of simplicity that the wire which determines the current is in the form of a circle, if we take a number of small magnets and place them all pointing in the same direction normal to the plane of the circle, so that they fill it and the wire binds them round, we find that this sheet of magnets has the same effect on the magnetic pole that the current has. The sheet of magnets may be curved, but the edge of it must coincide with the wire. The collection of magnets is then equivalent to the vortex sheet and an elementary magnet to a part of it. Thus, we must think of a magnet as conditioning a rotation in the aether round the plane which bisects at right angles the line joining its poles.

If a current is started in a circuit, we must imagine vortices like bowls turning themselves inside out, starting from the contour. In reaching a parallel circuit, if the vortex sheet were interrupted and joined momentarily to the second circuit by a free rim, the axis plane would lie between the two circuits, and a point on the second circuit opposite a point on the first would correspond to a point opposite to it on the first; hence we should expect a current in the opposite direction in the second circuit. Thus the phenomena of induction are not inconsistent with the hypothesis of a vortex about an axial plane.

In four-dimensional space in which all four dimensions were commensurable, the intensity of the action transmitted by the medium would vary inversely as the cube of the distance. Now, the action of a current on a magnetic pole varies inversely as the square of the distance; hence over measurable distances the extension of the aether in the fourth dimension cannot be assumed as other than small in comparison with those distances. This extension being small, the effect of a vortex sheet would be equivalent to a number of jets on one side and suctions on the other.

Such an arrangement in the case of a liquid would produce velocities in the liquid which coincide in direction with the tendency of motion of a magnetic pole. But analogies of this kind leave out of sight the fact that the action is a reciprocal one. Non-magnetic matter shows no tendency to move. To arrive at a definite conclusion it will be necessary to investigate the resultant pressures which accompany the collocation of solid vortices with surface ones.

To recapitulate: The movements and mechanics of four-dimensional space are definite and intelligible. A vortex with a surface as its axis affords a geometric image of a closed circuit, and there are rotations which by their polarity afford a possible definition of statical electricity.