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The Geometrization of the Seven Musical Scales and its Mathematical Implications

by

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Abstract

The seven possible types of musical scales contain 14 different notes (7 notes and the 7 intervals, or tonal "complements," between them and the octave). The 91 intervals between these notes are found to consist of 40 Pythagorean intervals (notes found in the Pythagorean musical scale) and 51 non-Pythagorean intervals. As a sequence of monotonically increasing tone ratios, they group into 65 intervals up to the 7th note and 26 larger intervals that are complements of some of these intervals. This shows how the Divine Name ADONAI with gematraic number value 65 and the Godname YAHWEH with number value 26 prescribe the composition of the 91 intervals between the basic set of 14 notes. The Godname EHYEH with number value 21 prescribes the 21 intervals that are not notes of the seven scales because they are intervals between notes belonging to different scales. EHYEH also prescribes all 91 intervals because 91 is the sum of the 21 odd integers making up the squares of the first 6 integers. There are 25 pairs of notes and their complements. The Godname ELOHIM with number value 50 prescribes these 50 intervals. The Godname ELOHA with number value 36 prescribes the 36 intervals between the eight notes of each scale. The Godname YAH with number 15 prescribes the 15 intervals that have no complements. The Godname YAHWEH ELOHIM with number value 76 prescribes the number of remaining intervals that do have complements. There are 24 pairs of intervals other than 1 and the octave. EHYEH prescribes the 21 pairs that are notes, as well as the 21 types of intervals found in them. The 24 pairs of intervals are symbolized by the 24 pairs of vertices and their mirror images outside the shared root edge of the first (6+6) enfolded polygons of the inner Tree of Life. They are also symbolized by the 24 vertices that are above or below the equator of the 120 Polyhedron, its 12 vertices representing the 12 basic notes between the tonic and octave found in the 7 musical scales. The 8 basic intervals and their 8 complements found in the set of 90 intervals below the octave are analogous to the 8 simple roots of E_8 and the 8 simple roots of E_8' appearing in $E_8 \times E_8'$ heterotic superstring theory. There are also 8 triplets of notes with tone ratios in the proportion $1:T:T^2$, where T (= 9/8) is the tone ratio of the Pythagorean tone interval. As four triplets of intervals and four triplets of their complements, they are the counterpart of the four trigrams of the I Ching and their four polar opposites with Yang and Yin lines interchanged. They are also the counterpart of the 8 unit octonions. The geometrical realisation of the 26 unpaired intervals and the 24 pairs of intervals is the 144 Polyhedron, 26 vertices of which belong to its underlying disdyakis dodecahedron, the remaining 24 diametrically opposite pairs pointing outward from the 24 pairs of faces of this polyhedron. These pairs of intervals spanning the octave constitute the source of the 7 musical scales. They group into 8 sets of 3 intervals and their 3 complements with tone ratios in the proportions $1:T:T^2$: They correspond to the 8 bundles of six 'beams of energy' that emanate from 48 vertices of the 144 Polyhedron. Each musical scale is the joining of two tetrachords of 4 notes. This is why the beams focus on 8 of the 12 vertices of the icosahedron in the 120 Polyhedron representing the 12 notes between the tonic and octave. The 90 edges in one half of the 120 Polyhedron represent the 90 rising intervals below the octave. The 90 edges in the other half represent the 90 falling intervals. The 6 edges and their mirror images in the equator represent the six rising intervals of a perfect fifth and the six falling intervals of a perfect fifth. The 168 edges outside the equator and the 168 intervals other than these 12 intervals are both analogous to, if not actual manifestations of, the 168 symmetries of the group PSL(2,7), whose centre SZ(3,2) is isomorphic to the 3rd roots of 1: 1, r & r^2, where r = $\exp(2\pi i/3)$. The 120° rotational symmetry in their locations in the Argand diagram is displayed in the Y-shaped cross-section of each beam.

Table 1. Tone ratios of the notes in the seven musical scales.

Musical scale	Tone ratio							
B scale	1	256/243	32/27	4/3	1024/729	128/81	16/9	2
A scale	1	9/8	32/27	4/3	3/2	128/81	16/9	2
G scale	1	9/8	81/64	4/3	3/2	27/16	16/9	2
F scale	1	9/8	81/64	729/512	3/2	27/16	243/128	2
E scale	1	256/243	32/27	4/3	3/2	128/81	16//9	2
D scale	1	9/8	32/27	4/3	3/2	27/16	16/9	2
C scale	1	9/8	81/64	4/3	3/2	27/16	243/128	2

(*Tone ratios belonging to the Pythagorean scale are written in black and non-Pythagorean tone ratios are written in red*).

The seven species of musical octaves[1] comprise 14 different notes (Table 1). In order of increasing tone ratios, they are:

1 256/243 9/8 32/27 81/64 4/3 1024/729 | 729/512 3/2 128/81 27/16 16/9 243/128 2

They form seven pairs of notes x and their complements y, where xy = 2:

1.		1	2	T^5L^2	$1\times 2 = 2$
2.	L	256/243	243/128	T^5L	$256/243\times 243/128 = 2$
3.	T	9/8	16/9	T^4L^2	$9/8\times 16/9 = 2$
4.	TL	32/27	27/16	T^4L	$32/27\times 27/16 = 2$
5.	T^2	81/64	128/81	T^3L^2	$81/64\times 128/81 = 2$
6.	T^2L	4/3	3/2	T^3L	$4/3\times 3/2 = 2$
7.	T^2L^2	1024/729	729/512	T^3	$1024/729\times 729/512 = 2$

(T = 9/8 is the Pythagorean tone interval and L = 256/243 is the Pythagorean leimma). Let $X = (x_1,x_2,x_3, \ldots x_7)$ be the set of the first seven notes ($x_m>x_n$ for m>n) and $Y = (y_1,y_2,y_3, \ldots y_7)$ be the set of their complements ($x_7<y_n<y_m$ for m>n), where $x_ny_{8-n} = 2$. There are (${}^{14}C_2 = 91$) intervals between the 14 notes. The largest of these is the octave, so that 90 intervals are below it. Their explicit values can be calculated in three steps:

1. Work out the (${}^{7}C_2 = 21$) rising intervals X_{nm} between the notes x_n and x_m in X (m>n), where $X_{nm} \equiv x_m/x_n$. By definition, $x_n = X_{1n}$;
2. Work out the 21 rising intervals Y_{nm} between the notes y_n and y_m in Y (m>n), where $Y_{nm} \equiv y_m/y_n$. As $y_m = 2/x_{8-m}$ and $y_n = 2/x_{8-n}$, $Y_{nm} = x_{8-n}/x_{8-m} = X_{(8-m)(8-n)}$.
3. Work out the (7×7=49) rising intervals Z_{nm} between the notes x_n and y_m, where $Z_{nm} \equiv y_m/x_n = 2/x_{8-m}x_n$. By definition, $y_n = Z_{1n}$, so that the octave y_7 is Z_{17}.

Tables 2, 3 and 4 display the magnitudes of the 90 rising intervals below the octave.

Table 2. Intervals X_{nm}.

n	m	1	2	3	4	5	6	7
		1	256/243	9/8	32/27	81/64	4/3	1024/729
1	1	1	256/243	9/8	32/27	81/64	4/3	1024/729
2	256/243		1	2187/2048	9/8	19683/16384	81/64	4/3
3	9/8			1	256/243	9/8	32/27	8192/6561
4	32/27				1	2187/2048	9/8	32/27
5	81/64					1	256/243	65536/59049
6	4/3						1	256/243
7	1024/729							1

(Cells highlighted in turquoise are the tone ratios of the first seven notes. Cells for the falling intervals are left blank).

The 21 intervals X_{nm} consist of 3 Pythagorean notes, 5 Pythagorean intervals, 3 non-Pythagorean notes and 10 non-Pythagorean intervals.

Table 3. Intervals Y_{nm}.

n	m	7	6	5	4	3	2	1
		2	243/128	16/9	27/16	128/81	3/2	729/512
7	2	1	256/243	9/8	32/27	81/64	4/3	1024/729
6	243/128		1	2187/2048	9/8	19683/16384	81/64	4/3
5	16/9			1	256/243	9/8	32/27	8192/6561
4	27/16				1	2187/2048	9/8	32/27
3	128/81					1	256/243	65536/59049
2	3/2						1	256/243
1	729/512							1

(The 7 complements are tabulated in order of decreasing tone ratio in order to demonstrate that the set of 21 intervals Y_{nm} is identical to the set X_{nm}).

The 21 intervals Y_{nm} comprise 8 Pythagorean intervals and 13 non-Pythagorean intervals.

Table 4. Intervals Z_{nm}.

n	m	1	2	3	4	5	6	7
		729/512	3/2	128/81	27/16	16/9	243/128	2
1	1	729/512	3/2	128/81	27/16	16/9	243/128	2
2	256/243	177147/131072	729/512	3/2	6561/4096	27/16	59049/32768	243/128
3	9/8	81/64	4/3	1024/729	3/2	128/81	27/16	16/9
4	32/27	19683/16384	81/64	4/3	729/512	3/2	6561/4096	27/16
5	81/64	9/8	32/27	8192/6561	4/3	1024/729	3/2	128/81
6	4/3	2187/2048	9/8	32/27	81/64	4/3	729/512	3/2
7	1024/729	531441/524288	2187/2048	9/8	19683/16384	81/64	177147/131072	729/512

(The cell with tone ratio 2 is coloured black to indicate that it does not belong to the set of 90 intervals).

The 48 intervals Z_{nm} below the octave consist of 3 Pythagorean notes, 20 Pythagorean intervals, 3 non-Pythagorean notes and 22 non-Pythagorean intervals, that is, 23 Pythagorean intervals and 25 non-Pythagorean intervals, regarding notes as intervals.

The 91 intervals consist of 7 Pythagorean notes, 6 non-Pythagorean notes, 33 Pythagorean intervals and 45 non-Pythagorean intervals. In increasing order of size, their tones ratios are:

	tone ratio	number
TL^{-2}	531441/524288*	1
L	256/243	8
TL^{-1}	2187/2048*	6
L^2	65536/59049*	2
T	9/8	11
TL	32/27	8
T^2L^{-1}	19683/16384*	4
TL^2	8192/6561*	3

	tone ratio	number
	2	1
T^5L	243/128	2
T^5	59049/32768*	1
T^4L^2	16/9	2
T^4L	27/16	4
T^4	6561/4096*	2

Table 5. Gematraic number values of the ten Sephiroth in the four Worlds.

	SEPHIRAH	GODNAME	ARCHANGEL	ORDER OF ANGELS	MUNDANE CHAKRA
1	Kether (Crown) **620**	EHYEH (I am) **21**	Metatron (Angel of the Presence) **314**	Chaioth ha Qadesh (Holy Living Creatures) **833**	Rashith ha Gilgalim First Swirlings. (Primum Mobile) **636**
2	Chokmah (Wisdom) **73**	YAHVEH, YAH (The Lord) **26, 15**	Ratziel (Herald of the Deity) **331**	Auphanim (Wheels) **187**	Masloth (The Sphere of the Zodiac) **140**
3	Binah (Understanding) **67**	ELOHIM (God in multiplicity) **50**	Tzaphkiel (Contemplation of God) **311**	Aralim (Thrones) **282**	Shabathai Rest. (Saturn) **317**
	Daath (Knowledge) **474**				
4	Chesed (Mercy) **72**	EL (God) **31**	Tzadkiel (Benevolence of God) **62**	Chasmalim (Shining Ones) **428**	Tzadekh Righteousness. (Jupiter) **194**
5	Geburah (Severity) **216**	ELOHA (The Almighty) **36**	Samael (Severity of God) **131**	Seraphim (Fiery Serpents) **630**	Madim Vehement Strength. (Mars) **95**
6	Tiphareth (Beauty) **1081**	YAHVEH ELOHIM (God the Creator) **76**	Michael (Like unto God) **101**	Malachim (Kings) **140**	Shemesh The Solar Light. (Sun) **640**
7	Netzach (Victory) **148**	YAHVEH SABAOTH (Lord of Hosts) **129**	Haniel (Grace of God) **97**	Tarshishim or Elohim **1260**	Nogah Glittering Splendour. (Venus) **64**
8	Hod (Glory) **15**	ELOHIM SABAOTH (God of Hosts) **153**	Raphael (Divine Physician) **311**	Beni Elohim (Sons of God) **112**	Kokab The Stellar Light. (Mercury) **48**
9	Yesod (Foundation) **80**	SHADDAI EL CHAI (Almighty Living God) **49, 363**	Gabriel (Strong Man of God) **246**	Cherubim (The Strong) **272**	Levanah The Lunar Flame. (Moon) **87**
10	Malkuth (Kingdom) **496**	ADONAI MELEKH (The Lord and King) **65, 155**	Sandalphon (Manifest Messiah) **280**	Ashim (Souls of Fire) **351**	Cholem Yesodeth The Breaker of the Foundations. The Elements. (Earth) **168**

The Sephiroth exist in the four Worlds of Atziluth, Beriah, Yetzirah and Assiyah. Corresponding to them are the Godnames, Archangels, Order of Angels and Mundane Chakras (their physical manifestation). This table gives their number values obtained by the ancient practice of gematria, wherein a number is assigned to each letter of the alphabet, thereby giving a number value to a word that is the sum of the numbers of its letters.

T^2	81/64	8	T^3L^2	128/81	3
T^2L	4/3	8	T^3L	3/2	6
T^3L^{-1}	177147/131072*	2			
T^2L^2	1024/729	4	T^3	729/512	5
	Total = 65			Total = 26	

(The 21 starred intervals are not notes of the seven musical scales). In total, there are 40 Pythagorean intervals and 51 non-Pythagorean intervals. The Godname ELOHIM (Table 5) with number value 50 prescribes the latter because 51 is the 50th integer after 1. There are 65 intervals up the seventh and last note with tone ratio 1024/729 before the crossover to notes that are complements of the first seven notes. The Godname ADONAI with number value 65 prescribes how many *independent* intervals there are between the 14 notes making up the seven musical scales. They are independent in the sense that all other larger intervals complete the octave as their complements and so are *determined* by them. The Godname YAHWEH with number value 26 prescribes the number of these complementary intervals. The Godname EHYEH with number value 21 prescribes the 21 asterisked intervals (eight types) between notes in *different* scales.

As 65 is the sum of the first 10 integers after 1:

$$
65 = \begin{array}{ccccccc} & & & 2 & & & \\ & & 3 & & 4 & & \\ & 5 & & 6 & & 7 & \\ 8 & & 9 & & 10 & & 11 \end{array},
$$

we see how the Decad, given the title "All Perfect" by the ancient Pythagoreans, defines the number of independent intervals between the 14 different notes in the seven scales.

Excluding the octave leaves 25 complements. The 65:25 division of intervals below the octave between the 14 notes is represented in the Lambda Tetractys (Fig. 1). The sum of the four numbers forming its base is 65 and the sum of the six remaining numbers is

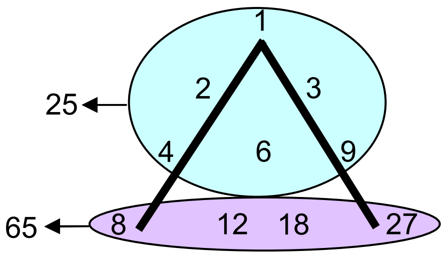


Figure 1. The sum of the four numbers in the base of the Lambda Tetractys is the number of independent intervals up to the crossover in the octave between notes and their complements. The sum of the remaining six numbers is the number of intervals past the crossover.

25. That this is no coincidence is the fact that the 65 intervals are made up of 27 Pythagorean intervals, 18 that are not notes in the seven scales, 12 non-Pythagorean intervals and eight leimmas of 256/243, all of which are the numbers forming the base of the Lambda Tetractys. Indeed, the central number 6 denotes the number of perfect fifths, the number 4 denotes the number of the note A with tone ratio 27/16, the number 2 is the number of the note B with tone ratio 243/128, the number 1 denotes the *largest* tone interval 59049/32768 not belonging to the seven scales, the number 3 is the number of intervals 128/81 and the number 9 is the number of intervals 729/512, 6561/4096 and 16/9. In other words, *every* number in the Lambda Tetractys denotes the number of different subsets of intervals in the set of 90 intervals between the notes of the seven musical scales. This reveals the amazing, archetypal quality of the Lambda Tetractys in quantifying such holistic systems, as well as in defining the tone ratios themselves as ratios of its numbers.

The same 65:26 division as that exhibited by the intervals between the 14 different notes of the seven scales is expressed arithmetically by the fact that 91 is the sum of the squares of the first six integers:

$$
\begin{array}{rc}
1^2 = & 1 \\
2^2 = & 1 + 3 \\
3^2 = & 1 + 3 + 5 \\
4^2 = & 1 + 3 + 5 + 7 \\
5^2 = & 1 + 3 + 5 + 7 + 9 \\
6^2 = & 1 + 3 + 5 + 7 + 9 + 11
\end{array}
\quad = 65 + 26.
$$

91 is the sum of 21 odd integers, showing how this number is prescribed by the Godname EHYEH with number value 21 (the sum of the first six integers). The sum of the six integers within the central blue triangles is 26, which is the number of intervals with tone ratios that takes them past the crossover between notes and their complements. The sum of the 15 integers on its boundary is 65, which is the number of intervals that are not notes and which are not paired with their complements (see below). 15 is the number value of YAH, the older version of the Godname YAHWEH.

EHYEH determines the 21 intervals that are not notes of the seven scales. This leaves 40 Pythagorean intervals and 30 non-Pythagorean intervals that are notes, that is, 70 intervals. YAHWEH determines the 27 Pythagorean intervals before the crossover into complementary notes because 27 is the 26th integer after 1.

Let us now examine those intervals that do or do not have complements. There are 15 intervals without complements, none of which are found as notes in the seven musical scales. They are of the type TL^{-2}, TL^{-1}, T^2L^{-1} and T^3L^{-1}. This means that there are (91-15=76) intervals, some of which are paired as an interval and its complement. This shows how the Godname YAHWEH ELOHIM with number value 76 prescribes these intervals. Six of these are not notes of the scales, leaving 70 intervals that are notes. Some of them, however, cannot be paired with their complements because the number of complements for a given interval is not always equal to the number of intervals of that type.

Tabulated below in order of increasing tone ratio are the number of intervals of each type that are left after the 24 pairs of intervals and their complements are subtracted from the complete set of 91 intervals between the 14 notes of the seven scales:

	number		**number**		**number**
				T^5L^2	1
TL^{-2}*	1				
		L	6		
TL^{-1}*	6				
L^2*	1				
		T	9		
		TL	4		
T^2L^{-1}*	4				
TL^2*	1				
		T^2	5		
		T^2L	2		
T^3L^{-1}*	2				
				T^3	1
Total = 15		= 26		= 2	

The Godname YAH with number value 15 prescribes the number of unpaired intervals that are not notes and the full Godname YAHWEH with number value 26 prescribes the number of unpaired notes before the crossover point. There is one note T^3 and the octave T^5L^2 *after* the crossover point. There are 24 pairs of notes and their complements (see below), so that the set of 76 intervals consists of 26 unpaired notes before the crossover point and 50 other intervals. This reflects the number values 26 and 50 of the words YAHWEH and ELOHIM in the Godname YAHWEH ELOHIM.

Listed below are those intervals between the tonic and octave and their numbers that *do* form pairs of intervals and their complements:

	tone ratio		tone ratio	number of pairs
L	256/243	T^5L	243/128	2
L^2*	6536/59049*	T^5*	59049/32768*	1
T	9/8	T^4L^2	16/9	2
TL	32/27	T^4L	27/16	4
TL^2*	8192/6561*	T^4*	6561/4096*	2
T^2	81/64	T^3L^2	128/81	3
T^2L	4/3	T^3L	3/2	6
T^2L^2	1024/729	T^3	729/512	4
				Total = 24

There are 48 intervals forming 24 pairs. Including the tonic and octave, there are 25 pairs, i.e., 50 intervals. The Godname ELOHIM with number value 50 prescribes how many of the intervals between notes in the seven scales actually group together as complementary pairs. Including the tonic and octave, there are 25 Pythagorean intervals and 25 non-Pythagorean intervals. The 25:25 split exists not only for the intervals and their complements but also for Pythagorean and non-Pythagorean intervals! There are 49 intervals above the tonic that form pairs, showing how EL CHAI, the Godname of Yesod with number value 49, prescribes the spectrum of intervals between the 13 notes above the tonic. There are (49-13=36) intervals that are not notes

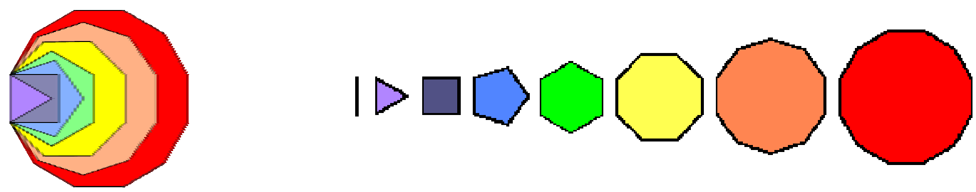


Figure 1. The 50 corners of the seven separate, regular polygons and root edge correspond to the 50 intervals that form complementary pairs. The endpoints of the root edge symbolize the tonic and the octave. The 36 corners of the first six separate symbolize the 36 intervals other than notes that form complementary pairs. The 36 corners of the enfolded polygons denote the 28 intervals between the eight notes of each scale, including the 8 intervals 1 between themselves.

(i.e., 18 pairs), showing how ELOHA, Godname of Geburah with number value 36, prescribes these intervals. The 50 intervals therefore become 36 intervals. This illustrates how the musical potential defined by ELOHIM, Godname of Binah, becomes restricted by ELOHA, the Godname of the Sephirah *below* Binah on the Pillar of Severity.

This 50→36 reduction is geometrically represented in the inner form of the Tree of Life (Fig. 1). The seven separate polygons have 48 corners symbolizing the 48 intervals that can form complementary pairs. The two endpoints of the root edge, which formally are corners, symbolize the unit interval and the octave. Together, they constitute 50 corners. The 12 notes in the seven scales other than the octave are symbolized by the 12 corners of the dodecagon. The 36 corners of the first six separate polygons symbolize the intervals forming pairs that are not notes. These extra musical intervals are symbolized by the 36 corners of the seven enfolded polygons.

The intervals in three pairs are not notes of the seven scales, leaving 21 pairs that *are* such notes. EHYEH prescribes those pairs of intervals and their complements that are notes of the scales. There are (8+8=16) types of intervals, 12 of which are notes of the seven scales and four of which are not. Taking into account the four types of intervals that have no complements, the 14 notes of the seven scales have (16+4+1=21) types of intervals. EHYEH prescribes how many kinds of intervals there are in the 91 intervals between the 14 notes.

There are 16 types of rising intervals below the octave (6 Pythagorean, 10 non-Pythagorean). Including the octave, there are 17 types (7 Pythagorean, 10 non-Pythagorean). Similarly, there are 16 types of falling intervals with tone ratios that are the reciprocal of those of the rising intervals. Including the interval 1, there are (16+1+16=33) rising and falling types of intervals between the 24 pairs of intervals. 33 = 1! + 2! + 3! + 4! and 24 = 1×2×3×4. This demonstrates how the Pythagorean integers 1, 2, 3, 4, which are symbolized by the tetractys and whose ratios define the octave, perfect fifth and perfect fourth, express the number of pairs of intervals and the number of types of intervals in them.

Including the unit interval and octave, the (9+9=18) types of intervals form 25 pairs:

1	2/1	(×1)
L	2/L	(×2)
L^2*	$2/L^2$*	(×1)
T	2/T	(×2)
TL	2/TL	(×4)
TL^2*	$2/TL^2$ *	(×2)
T^2	$2/T^2$	(×3)
T^2L	$2/T^2L$	(×6)
T^2L^2	$2/T^2L^2$	(×4)

(As before, the tone ratios of intervals written in red are not those of notes in the Pythagorean scale, and asterisked intervals are not notes of the seven musical scales).

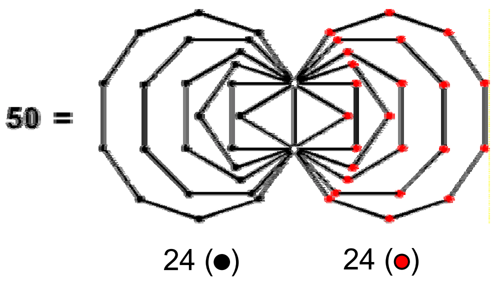


Figure 2. The 50 corners of the (6+6) enfolded polygons symbolize the 50 intervals between notes of the seven musical scales that form complementary pairs. The two endpoints of the shared edge denote the unit interval and the octave. The 24 corners outside this edge of one set of polygons denote the 24 intervals and their 24 mirror-image corners denote their 24 complements.

Figure 2 shows how they constitute a Tree of Life pattern. The first (6+6) enfolded polygons are a subset of the (7+7) enfolded polygons that constitute such a pattern in themselves because they, too, are prescribed by the Godnames of the ten Sephiroth.[2]

The 50 intervals are symbolized by the 50 corners of the first (6+6) enfolded polygons (Fig. 2). The unit interval and the octave are denoted by the two endpoints of the shared root edge. The 24 intervals and their complements are symbolized by the 24 corners on each side of this edge. The mirror symmetry of the two sets of polygons is the geometrical counterpart of the complementarity between certain pair of notes. The detailed correspondence between intervals and corners is set out below:

	Interval	**Complement**
Corner of triangle	$1{\times}L^2$*	$1{\times}2/L^2$*
Two corners of square	$2{\times}TL^2$*	$2{\times}2/TL^2$*
Three corners of pentagon	$3{\times}T^2$	$3{\times}2/T^2$
Four corners of hexagon	$2{\times}L + 2{\times}T$	$2{\times}1/L + 2{\times}2/T$
Six corners of octagon	$6{\times}T^2L$	$6{\times}2/T^2L$
Eight corners of decagon	$4{\times}TL + 4{\times}T^2L^2$	$4{\times}1/TL + 4{\times}2/T^2L^2$

The three intervals that are not notes of the seven scales are symbolized by the corners of the triangle and square. This means that the 21 intervals that *are* notes are naturally symbolised by the 21 corners of the next four polygons.

The eight kinds of intervals between the notes of the seven scales that form pairs correspond to the eight trigrams of the Taoist I Ching:

☰ ☱ ☲ ☳ ☴ ☵ ☶ ☷

This is another example of the eight-fold way discussed in Article 19[3] They divide into two sets of four trigrams that express the two Yang/Yin halves of a cycle. A musical octave is such a cycle and its eight notes, symbolised by the eight trigrams, are created

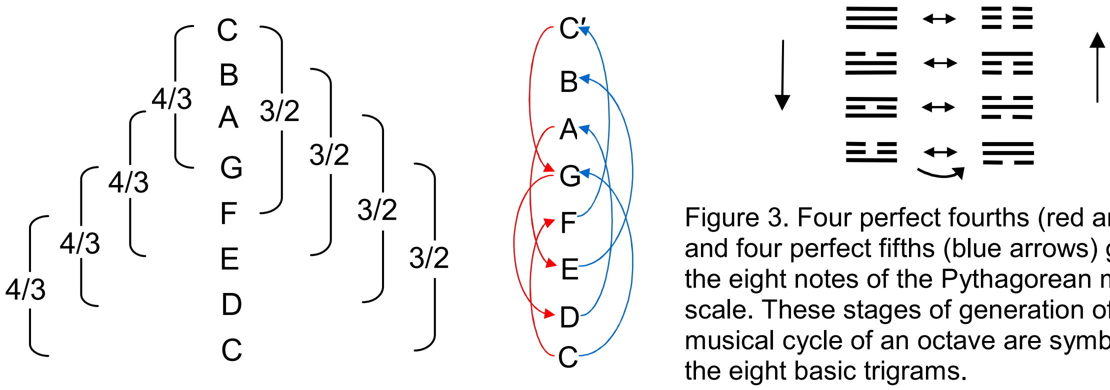


Figure 3. Four perfect fourths (red arrows) and four perfect fifths (blue arrows) generate the eight notes of the Pythagorean musical scale. These stages of generation of the musical cycle of an octave are symbolized by the eight basic trigrams.

by leaps of four perfect fifths and four perfect fourths (Fig. 3). The ancient Greeks regarded the eight-note musical scale as two joined tetrachords, or groups of four notes. The fact that eight-fold cyclical systems divide into two sets of four phases raises the question of whether the eight types of intervals naturally split into two quartets. We pointed out in Article 32 that the 12 notes between the tonic and octave that create the seven musical scales form two triplets: (T, T^2, T^3) and (T^2L^2, T^3L^2, T^4L^2), whose tone ratios are in the proportions $1{:}T{:}T^2$, and two triplets (L, TL, T^2L) and (T^3L, T^4L, T^5L),

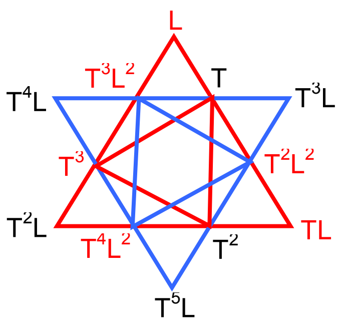


Figure 4. The 12 notes between the tonic and octave of the seven types of musical scales form two triplets and two ‘antitriplets’ represented by pairs of inverted triangles in two nested Stars of David. The note at any point of a Star of David is the complement of that at its opposite point.

whose tone ratios are in the same proportions. There are therefore four triplets with the same proportions of their tone ratios. In each pair of triplets, one triplet contains notes that are the complement of their corresponding notes in the other triplet. These double and triple relationships can be represented by two Stars of David (Fig. 4), one nested inside the other. The three points of one red or blue triangle denote a triplet of notes and the three points of the inverted blue or red triangle denote the 'antitriplet' of its complementary notes. The tonic and octave may be thought of as the centre of the star nest. There are two triplets of intervals and two antitriplets of their corresponding complements. Adding the two intervals L^2* and TL^2* that do not belong to any scale to the former and their complements to the latter will create two quartets of intervals, so that the eight basic intervals can be divided into two halves, thus upholding the ancient view of the number 8 as "twice 4."[4]

According to Tables 2, 3 & 4, L^2* = 6536/59049* appears twice either as X_{57} = 1024/729 ÷81/64 or as Y_{31} = 128/81÷729/512. In either case, the pair of tone ratios does not appear within the same scale. TL^2* = 8192/6561 appears three times either as X_{37} = 1024/729÷9/8, Y_{51} = 16/9÷729/512 or as Z_{53} = 128/81÷81/64. In all three cases, the two tone ratios do not appear in the same scale. This means that the extra two intervals L^2* and TL^2* and their six complements added to the six intervals and their complements are between two notes in *different* scales. In other words, they do not appear when music is played in any one scale, only if the available notes are all 14 notes.

The eight basic intervals L, L^2*, T, TL, TL^2*, T^2, T^2L & T^2L^2 and their eight complements have their counterpart in superstring theory as the eight roots of E_8 and the eight roots of E_8'. Musically speaking, the division of the octave into notes and their complements corresponds to the distinction in the $E_8 \times E_8$ heterotic superstring theory between superstrings of ordinary matter governed by E_8 and superstrings of shadow matter governed by E_8'. In music, the distinction between notes and their complements is the manifestation in tones of the duality of Yang and Yin. The same can be said for the fundamental difference between ordinary and shadow matter. The musical counterpart of the group distinction between E_8 and its exceptional subgroup E_6 with six roots is the difference between the eight distinct intervals, of which six are actual notes. It may not be coincidental that the dimension 78 of E_6 is the number of intervals between the 13 notes of the seven musical scales above the tonic: ${}^{13}C_2 = 78$.

The sequence of nine basic intervals[1]:

$$1, L, L^2, T, TL, TL^2, T^2, T^2L, T^2L^2$$

can be written

$$(1, L, L^2) \qquad T(1, L, L^2) \qquad T^2(1, L, L^2),$$

Successive triplets of intervals have the same proportion $1{:}L{:}L^2$ in the tone ratios of the members of each triplet. We discussed earlier that triplets of notes in the seven scales can be found that have the same proportion of $1{:}T{:}T^2$ of the first three notes C, D & E of the Pythagorean scale (C scale). Let us therefore carry out an exhaustive analysis of triplets of intervals drawn from the complete set of 18 intervals that exhibit proportions of the form $1{:}X{:}X^2$, where X = L, T, TL, T^2 or T^2L (the only possible values, because the largest interval is $T^5L^2 = 2$).

X = L.

1. ×1:	$(1, L, L^2)$	(T^5, T^5L, T^5L^2)
2. ×T:	(T, TL, TL^2)	(T^4, T^4L, T^4L^2)
3. ×T^2:	(T^2, T^2L, T^2L^2)	(T^3, T^3L, T^3L^2)
4. ×T^3:	(T^3, T^3L, T^3L^2)	(T^2, T^2L, T^2L^2)

[1] The asterisk and red lettering for non-Pythagorean intervals are dropped from now on.

5. $\times T^4$:	(T^4, T^4L, T^4L^2)	(T, TL, TL^2)
6. $\times T^5$:	(T^5, T^5L, T^5L^2)	$(1, L, L^2)$

As (1) is the same as (6), (5) is identical to (2) and (4) is the same as (3), there are three different triplets: (1), (2) & (3).

X = T.

1. $\times 1$:	$(1, T, T^2)$	(T^3L^2, T^4L^2, T^5L^2)
2. $\times L$:	(L, TL, T^2L)	(T^3L, T^4L, T^5L)
3. $\times L^2$:	(L^2, TL^2, T^2L^2)	(T^3, T^4, T^5)
4. $\times T$:	(T, T^2, T^3)	(T^2L^2, T^3L^2, T^4L^2)
5. $\times TL$:	(TL, T^2L, T^3L)	(T^2L, T^3L, T^4L)
6. $\times TL^2$:	(TL^2, T^2L^2, T^3L^2)	(T^2, T^3, T^4)
7. $\times T^2$:	(T^2, T^3, T^4)	(TL^2, T^2L^2, T^3L^2)
8. $\times T^2L$:	(T^2L, T^3L, T^4L)	(TL, T^2L, T^3L)
9. $\times T^2L^2$:	(T^2L^2, T^3L^2, T^4L^2)	(T, T^2, T^3)

Multiplying by the remaining intervals just replicates the pairs above because they are the complements of the first eight intervals. As (7) is the same as (6), (8) is the same as (5) and (9) is identical to (4), there are six different pairs of triplets: (1)-(6).

X = TL.

1. $\times 1$:	$(1, TL, T^2L^2)$	(T^3, T^4L, T^5L^2)
2. $\times T$:	(T, T^2L, T^3L^2)	(T^2, T^3L, T^4L^2)
3. $\times T^2$:	(T^2, T^3L, T^4L^2)	(T, T^2L, T^3L^2)
4. $\times T^3$:	(T^3, T^4L, T^5L^2)	$(1, TL, T^2L^2)$

As (1) & (4) are the same and as (2) and (3) are the same, there are two different triplets: (1) & (2).

X = T².

1. $\times 1$:	$(1, T^2, T^4)$	(TL^2, T^3L^2, T^5L^2)
2. $\times L$:	(L, T^2L, T^4L)	(TL, T^3L, T^5L)
3. $\times L^2$:	(L^2, T^2L^2, T^4L^2)	(T, T^3, T^5)
4. $\times T$:	(T, T^3, T^5)	(L^2, T^2L^2, T^4L^2)
5. $\times TL$:	(TL, T^3L, T^5L)	(L, T^2L, T^4L)
6. $\times TL^2$:	(TL^2, T^3L^2, T^5L^2)	$(1, T^2, T^4)$

There are three different triplets: (1), (2) & (3). For $X = T^2L$, there is only the triplet: (T, T^3L, T^5L^2). Hence, this case is of no interest.

There are therefore 14 independent triplets and their complements:

1.	$(1, L, L^2)$	$(T^5. T^5L, T^5L^2)$	
2.	(T, TL, TL^2)	(T^4, T^4L, T^4L^2)	X = L
3.	(T^2, T^2L, T^2L^2)	(T^3, T^3L, T^3L^2)	
4.	$(1, T, T^2)$	(T^3L^2, T^4L^2, T^5L^2)	
5.	(L, TL, T^2L)	(T^3L, T^4L, T^5L)	
6.	(L^2, TL^2, T^2L^2)	(T^3, T^4, T^5)	X = T
7.	(T, T^2, T^3)	(T^2L^2, T^3L^2, T^4L^2)	
8	(TL, T^2L, T^3L)	(T^2L, T^3L, T^4L)	
9.	(TL^2, T^2L^2, T^3L^2)	(T^2, T^3, T^4)	
10.	$(1, TL, T^2L^2)$	$(T^3, T^4L, T5L^2)$	X = TL
11.	(T, T^2L, T^3L^2)	(T^2, T^3L, T^4L^2)	
12.	$(1, T^2, T^4)$	(TL^2, T^3L^2, T^5L^2)	
13.	(L, T^2L, T^4L)	(TL, T^3L, T^5L)	$X = T^2$
14.	(L^2, T^2L^2, T^4L^2)	(T, T^3, T^5)	

Seven triplets [(1)-(6) & (10)] have intervals in the first half of the octave (they are all *notes* in four of them). There are seven triplets of intervals [(2), (3) & (5)-(9)] other than 1 with X = L or T. There are also seven such triplets with X = T or T^2. Of these, five [(5)-(9)] show the proportions 1:T:T^2 and two [(13) & (14)] show the proportions 1:T^2:T^4. They are shown below:

1.	(L, TL, T^2L)	(T^3L, T^4L, T^5L)	
2.	(L^2, TL^2, T^2L^2)	(T^3, T^4, T^5)	
3.	(T, T^2, T^3)	(T^2L^2, T^3L^2, T^4L^2)	X = T
4.	(TL, T^2L, T^3L)	(T^2L, T^3L, T^4L)	
5.	(TL^2, T^2L^2, T^3L^2)	(T^2, T^3, T^4)	
6.	(L, T^2L, T^4L)	(TL, T^3L, T^5L)	X = T^2
7.	(L^2, T^2L^2, T^4L^2)	(T, T^3, T^5)	

They contain the intervals L^2, TL^2 and their complements. These are not notes of the seven scales, merely intervals between notes in *different* scales. There are six triplets [(4), (5), (7), (8), (12) & (13)] with X = T or T^2 whose intervals are all notes. There is one triplet (3) with X = L whose intervals are notes. Hence, there are seven triplets all of whose intervals are notes with X = L, T or T^2.

However the seven triplets be defined, they bear a striking correspondence to the seven 3-tuples of octonions, as now explained. The octonions are the numbers of the

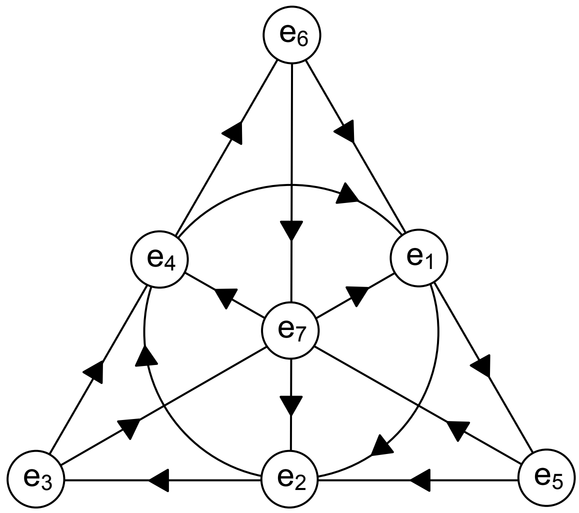


Figure 5. The Fano plane representation of the seven unit imaginary octonions.

fourth and last class of division algebras. They are linear combinations of the eight unit octonions e_i (i = 0, 1, 2, … 8) that consist of the real unit octonion $e_0 = 1$ and seven unit imaginary octonions e_j (j = 1-7) whose multiplication is non-associative and non-commutative:

$$e_ie_j = -\delta_{ij}e_0 + \sum f_{ijk}ek \qquad (i,j,k = 1, 2,\ldots.7)$$

where f_{ijk} is antisymmetric with respect to the indices i, j, k and has values 1, 0, & -1. The seven unit imaginary octonions form seven 3-tuples (e_i, e_{i+1}, e_{i+3}) with the cyclic property of multiplication

$$e_ie_{i+1} = e_{i+3}.$$

Their explicit forms are listed below:

(e_1, e_2, e_4)
(e_2, e_3, e_5)
(e_3, e_4, e_6)
(e_4, e_5, e_7)

(e_5, e_6, e_1)
(e_6, e_7, e_2)
(e_7, e_1, e_3)

Their multiplication is geometrically represented by the Fano plane, which is the simplest projective plane. A projective plane of order n consists of $(1+n+n^2)$ points and $(1+n+n^2)$ lines. The Fano plane is of order n = 2 because it comprises seven points and seven lines. The eight notes of the Pythagorean scale are analogous to the eight unit octonions. As a tone, the tonic can have any pitch, being simply the base with respect to which the tone ratios of the other notes are measured. It corresponds to $e_0 = 1$, the base of the real numbers. The seven rising intervals n_i above the tonic correspond to the seven unit imaginary octonions, their falling intervals (their reciprocals $1/n_i$) corresponding to the conjugates of the imaginary octonions $e_i^* = -e_i$, so that $e_ie_i^* = 1 = n_i\times 1/n_i$. Alternatively, the counterpart of conjugate octonions may be thought of as the complement m of a note n, where nm = 2.

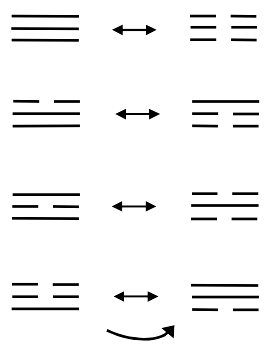
The counterparts of the seven 3-tuples are the seven musical scales. Table 1 shows the tone ratios of their notes. Table 5 shows their composition in terms of the T and L.

Table 6. Intervallic composition of the notes of the seven musical scales.

	C scale	D scale	E scale	F scale	G scale	A scale	B scale
1	1	1	1	1	1	1	1
2	T	T	L	T	T	T	L
3	T^2	TL	TL	T^2	T^2	TL	TL
4	T^2L	T^2L	T^2L	T^3	T^2L	T^2L	T^2L
5	T^3L	T^3L	T^3L	T^3L	T^3L	T^3L	T^2L^2
6	T^4L	T^4L	T^3L^2	T^4L	T^4L	T^3L^2	T^3L^2
7	T^5L	T^4L^2	T^4L^2	T^5L	T^4L^2	T^4L^2	T^4L^2
8	T^5L^2	T^5L^2	T^5L^2	T^5L^2	T^5L^2	T^5L^2	T^5L^2
Number	3	2	2	3	2	2	3

The 17 triplets that show a $1:T:T^2$ scaling of their tone ratios are not all different. Including the triplet $(1, T, T^2)$, there are eight distinct triplets (four triplets of intervals and four triplets of their complements):

$$\begin{pmatrix}1\\T\\T^2\end{pmatrix} \quad \begin{pmatrix}T^5L^2\\T^4L^2\\T^3L^2\end{pmatrix}$$
$$\begin{pmatrix}L\\TL\\T^2L\end{pmatrix} \quad \begin{pmatrix}T^5L\\T^4L\\T^3L\end{pmatrix}$$
$$\begin{pmatrix}T\\T^2\\T^3\end{pmatrix} \quad \begin{pmatrix}T^4L^2\\T^3L^2\\T^2L^2\end{pmatrix}$$
$$\begin{pmatrix}TL\\T^2L\\T^3L\end{pmatrix} \quad \begin{pmatrix}T^4L\\T^3L\\T^2L\end{pmatrix}$$



This is another musical counterpart of the eight trigrams, the Yang/Yin polarities of the lines and broken lines in each one corresponding to the intervals and their complements. It is also the musical counterpart of the eight unit octonions, with $(1, T, T^2)$ being equivalent to the real unit octonion e_0 and the seven other triplets being equivalent to the seven imaginary octonions.

Each musical scale is unchanged under interchange of each note and its complement. Similarly, the Fano plane is invariant under interchange of its points and lines and the eight trigrams remain the same set when their Yang and Yin lines are interchanged. The seven scales have 168 rising and falling intervals that are repetitions of the basic set of 12 notes between the tonic and octave. In the 64 hexagrams of the I Ching table, there are 28 pairings of different trigrams with 168 Yang/Yin lines. The Fano plane has 168 symmetries described by SL(3,2), the special linear group of 3×3 matrices with unit determinant over the field of complex numbers. The trigrams are the expression of the 3×3 matrices and their pairing is the counterpart of this field of order 2. SZ(3,2), the centre of SL(3,2), is the set of scalar matrices with unit determinant and zero trace. It is isomorphic to the third roots of 1. The three roots are 1, $\exp(2\pi i/3)$ and $\exp(4\pi i/3)$. Plotted in the Argand diagram, they are located at the three corners of an equilateral triangle. The cyclic group of order 3 is $C_3 = (1, r, r^2)$, where the generator $r = \exp(2\pi i/3)$ is the primitive third root of 1. It is the counterpart of the generation of the nine basic types of intervals in the seven scales:

$$(1+T+T^2)(1+L+L^2) = 1 + L + L^2 + T + TL + TL^2 + T^2 + T^2L + T^2L^2.$$

It is known that $1 + X + X^2$ is the only irreducible polynomial of degree 2 on the finite field of order 2. This plus the fact that the algebra of the octonions can be represented by the Fano plane of order 2, which is the simplest of the projective planes of order n that have $(1+n+n^2)$ points and lines, is strong evidence that the mathematical analogy between the octonions and the notes of the seven musical scales is significant. It exists because the Pythagorean mathematics of music and the mathematics of octonions are parallel manifestations of a universal paradigm.

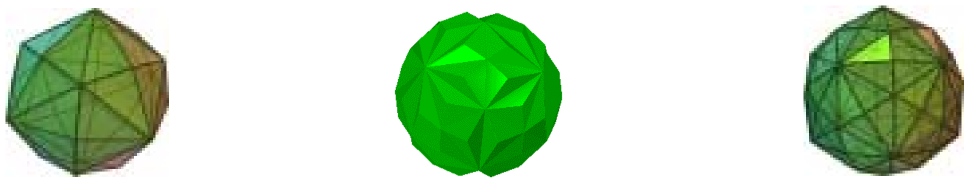
The 144 & 120 Polyhedra Geometrize the Seven Musical Scales

It was found earlier that the maximum number of intervals between the 14 basic notes of the seven scales that have complements is 24, leaving 26 unpaired intervals before the crossover into the complements of these intervals. There are therefore 74 such intervals. The 144 Polyhedron, which is the progenitor of the 120 Polyhedron (Fig. 6)

Disdyakis dodecahedron

144 Polyhedron

120 Polyhedron



The 26 vertices denote the 26 intervals without complements

The 48 additional vertices denote 24 pairs of intervals and their complements.

(90+90) edges denote 90 rising and 90 falling intervals between basic set of 14 notes in 7 musical scales.

Figure 6. The (48+26) vertices of the 144 Polyhedron denote the 48 intervals between the 14 notes of the 7 musical scales that have complements and the 26 intervals that have no complements. The (90+90) edges of the 120 Polyhedron denote the 90 rising and 90 falling intervals below the octave.

has 74 vertices, 26 of them belonging to the disdyakis dodecahedron underlying it. 48 “beams of energy” emanate from the 48 other vertices and are focussed on passage through the so-called “focussing sphere” into eight bundles of six beams, each converging on eight of the 12 vertices of the icosahedron making up the 120

Polyhedron. The 26 vertices of the 144 Polyhedron that do not emit beams correspond to musical intervals that are leftover, so to speak, unable to form complementary pairs, and therefore not actively participating in the embodiment of the seven musical scales in the 120 Polyhedron. The maximal set of 24 pairs that can do so correspond to the 48 remaining vertices. These play a dynamic, generative role because creation is a cyclic interplay of Yang and Yin represented by intervals and their complements making up the octave cycle and no more than 24 tonal intervals (Yang) have their complementary opposites (Yin). Indeed, we are countenancing here the pattern-determining character of the number 24, as explained in other contexts in Article 37.[5] The 144 Polyhedron supplies the musical potential in terms of intervals belonging to the seven scales. The 120 Polyhedron organises them into the patterns recognisable as the seven musical scales - the very basis of music itself. The beams *have* to focus into the icosahedron because it has 12 vertices representing the 12 notes within an octave of the seven scales. Eight of those notes are Pythagorean, forming the Pythagorean scale (C scale) and six are non-Pythagorean. As the Pythagorean scale is the most perfect of the scales and the generator of the other six scales, one may suppose that the 48 beams bundle into eight groups of six in order to focus on the eight vertices signifying notes of the Pythagorean scale from which all the other scales are formed by starting with successive notes as the tonic. The 14 types of notes thus generated have 90 rising intervals represented by the 90 edges in one half of the 120 Polyhedron and 90 falling intervals denoted by their 90 mirror-image edges in the other half. The 12 edges along

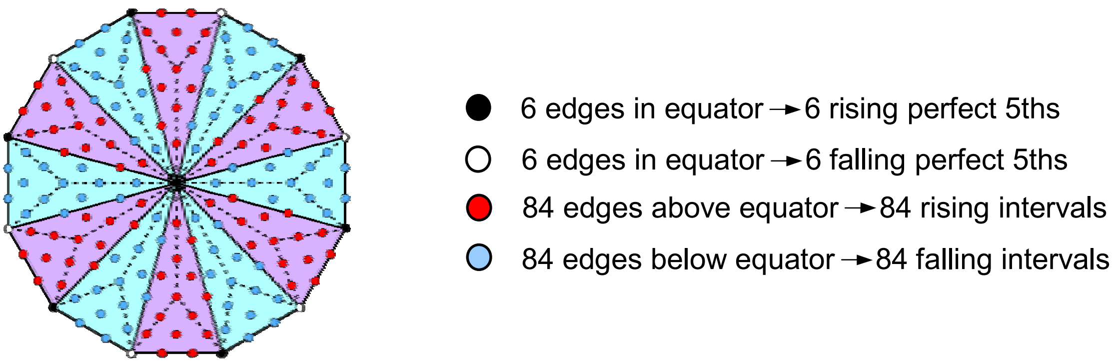


Figure 7. The 180 yods surrounding the centre of the dodecagon — the last of the regular polygons in the inner Tree of Life — symbolize the 180 edges of the 120 Polyhedron and the 180 rising and falling intervals of the 14 notes in the seven musical scales. Its 12 vertices denote the 12 vertices on the equator and the 6 rising and 6 falling perfect 5ths. The 84 yods in a set of six sectors denote the 84 edges above or below the equator and the 84 rising or falling intervals.

its equator denote the six rising and six falling perfect fifths found in the 24 intervals and their complements (the only type of interval to have six copies — see the list on page 7). This is the geometrical manifestation of the mathematical archetype represented by Plato's Lambda Tetractys:

$$90 = \begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 3 & & \\ & 4 & & 6 & & 9 & \\ 8 & & 12 & & 18 & & 27 \end{array} = 84 + 6.$$

Its ten integers add up to 90, that is, the nine integers surrounding the central integer 6 add up to 84. The 168 remaining intervals (84 rising and 84 falling) correspond to the 84 edges above the equator and the 84 edges below the equator. The 24 vertices above the equator symbolize the maximal set of 24 intervals, which are matched by their complementary intervals denoted by the 24 vertices below the equator. The two remaining vertices (the poles of the 120 Polyhedron) denote the tonic and the octave — the beginning and the end of the musical scale.

Figure 7 shows how this information is embodied in the dodecagon — the last of the regular polygons enfolded in the inner Tree of Life. When its sectors are divided into three tetractyses, there are 180 yods surrounding its centre. They symbolize the 180 edges of the 120 Polyhedron — the polyhedral realisation of the inner Tree of Life. They also denote the 180 rising and falling intervals below the 14 notes of the seven musical scales. The 12 vertices of the 36 tetractyses making up the dodecagon correspond to the 12 edges in the equator of the 120 Polyhedron and, in the musical context of the intervals, the six rising perfect fifths (3/2) and the six falling perfect fifths (2/3). The 84 remaining yods in six sectors are the counterpart of the 84 edges above the equator and the 84 rising intervals. The 84 yods in the other six sectors are the counterpart of the 84 edges below the equator and the 84 falling intervals. The central yod signifies the tonic as the starting note. Its counterpart in the 120 Polyhedron is the imaginary, *internal* line joining two diametrically opposite A vertices — the axis of the polyhedron.

Of the (24+24) intervals, there are 21 notes and 21 complements with tone ratios of notes in the seven scales. Interestingly, Table 1 indicates that there are actually just 21 notes with these tone ratios! This demonstrates how the Godname EHYEH with number value 21 prescribes the composition of the 91 intervals between the 14 different notes

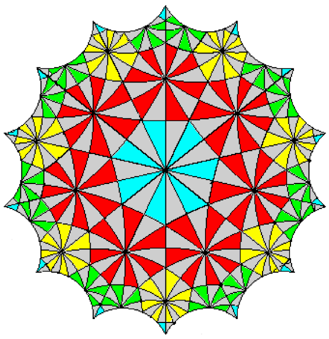


Figure 8. The Klein Configuration is the {7,3} hyperbolic tiling of the 168 symmetries of the Klein quartic into 24 heptagons. Each of its 7 sectors contains 24 hyperbolic triangles. Of these, one (coloured cyan) belongs to the central heptagon and two (coloured cyan as well) are at the two corners of a half-sector and are sectors of two heptagons. The 24 heptagons are divided into three heptagons whose 21 triangles form the corners of these seven half-sectors and 21 other heptagons.

of the seven scales. The three remaining intervals (one L^{2*} and two TL^{2*}) are not notes. This 3:21 differentiation was found in Article 21[6] in the context of the 24 lines and broken lines making up the eight trigrams. The positive and negatives lines of each trigram denote the positive and negative directions with respect to a rectangular coordinate system of the three perpendicular faces of a cube whose intersection is one of its eight corners. If such cubes are stacked together, any one corner of a cube coincides with the corners of seven other cubes (three on the same level and four either above or below it). This means that a cubic lattice point can be defined by the intersections of three faces belonging to eight cubes, three belonging to the cube itself and 21 belonging to the seven cubes that surround it.

The same 3:21 division appears in the Klein Configuration.[7] This is the hyperbolic mapping of the 168 automorphisms of the equation well known to mathematicians called the “Klein quartic”:

$$x^3y + y^3z + z^3x = 0.$$

These symmetries of its Riemann surface can be mapped onto the hyperbolic surface of a 3-torus in a number of different ways. Figure 8 shows the {7,3) tiling that requires 24 heptagons divided into 168 coloured triangles. It also has 168 anti-automorphisms represented by the 168 grey triangles of 24 other heptagons. These two sets of 24 heptagons are the counterpart of the 24 intervals and their 24 complements. The three intervals and their complements that are not notes of the seven scales correspond, respectively, to the three cyan triangles in Fig. 8 at the corners of a half-sector and to the three grey triangles at the corresponding corners of the other half-sector. Notice that the one L^2* and the two TL^2* intervals in the set of 24 match, respectively, the innermost triangle and the two outermost triangles in a half-sector. They correspond in the 3×3×3 array of cubes displaying an isomorphism with the Klein configuration to the three faces of the central cube intersecting at one corner.[8] The 168 automorphisms of the Klein quartic correspond to the 168 rising and falling intervals other than the six perfect fifths and to the 168 edges above and below the equator of the 120 Polyhedron,[9] its six edges and their inverted images corresponding, respectively, to the six rising perfect fifths and to the six falling perfect fifths in the 91 intervals between the 14 notes of the seven scales. Both are the manifestation of the projective, special linear group PSL(2,7), which is the quotient group SL(2,7)/{$\underline{1}$,-$\underline{1}$}, where $\underline{1}$ is the identity matrix, and SL(2,7) consists of all 2×2 matrices with unit determinant over F_7, the finite field with 7 elements. These elements can be the seven types of intervals between notes of the seven musical scales and the seven unit imaginary octonions e_i, whose algebra is represented by the Fano plane with the symmetry group SL(3,2) that is isomorphic to PSL(2,7). Their seven conjugates $e_i^* = -e_i$, where $e_ie_i^* = 1$, correspond to the complements y_i of the seven notes x_i, where $x_iy_i = 2$, whilst their seven 3-tuplets (e_i, e_{i+1}, e_{i+3}) and the seven 3-tuplets of their conjugates (e_i^*, e_{i+1}^*, e_{i+3}^*) correspond, respectively, to the seven triplets of intervals and to the seven triplets of their complements that display the same relative proportions $1:T:T^2$. of their tone ratios.

Whether the 168 rising and falling intervals are *actual* elements of PSL(2,7) is irrelevant except to one who cannot see the larger picture. Anyone who demands a formal proof before he takes the analogy seriously is missing the vital point. Such proof is necessary only if one makes the stronger claim that the intervals *are* such elements. However, judging the similarity to be significant evidence of a universal principle because it cannot be merely coincidental does not require the stronger claim to be made. What is sufficient is to demonstrate that 1. the mathematical properties of the two sets of seven basic intervals found in the seven musical scales are at least analogous to the properties of PSL(2,7) in too many ways for this to be plausibly coincidental, and 2. these properties can be represented by the polygonal and polyhedral forms of the outer and inner Trees of Life in too much detail and in too natural a way either for the matching to be contrived or for it to indicate anything other than that PSL(2,7) and the musical intervals between the notes in the seven scales embody the same, essential Tree of Life pattern. If it cannot plausibly be attributed to coincidence because there is too much matching (as is the case here), two systems can be analogous only because they are both holistic in nature and therefore manifest physically or conceptually the same, universal paradigm in their own way. The mathematical patterns in a system and in some symmetry group need only be similar in appearance. The former does not necessarily have to amount formally to a group symmetry that is isomorphic to the latter in order to constitute evidence of such a paradigm. The fact that such extensive analogy exists between topics as diverse as octonions, the eight simple roots of E_8, musical scales and acupuncture meridians, as demonstrated in this and previous articles, is not an illusion due to contrived selection of features that fit and ignoring those that do not fit. The remarkable, independent appearance of at least eight Godname numbers to prescribe the properties of the 90 intervals totally discredits such a suggestion and confirms the status of the seven musical scales as a holistic system musical scales as a holistic system that embodies the Tree of Life pattern. It indicates that a universal principle connects these topics as different facets of a pervading Unity within diversity.

References

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[3] Phillips, Stephen M. Article 19: “I Ching and the Eight-fold Way,” http://smphillips.8m.com.

[4] Ibid, p. 6.

[5] Phillips, Stephen M. Article 37: “The Seven Octaves of the Seven Musical Scales are a Tree of Life Pattern Mirrored in the 120 Polyhedron,” http://www.smphillips.8m.com.

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[7] Phillips, Stephen M. Article 15: “The Mathematical Connection Between Superstrings and Their Micro-psi Description: a Pointer Towards M-theory,” http://www.smphillips.8m.com, pp. 24–28; also ref. 4, pp. 29–30.

[8] Ref. 5, p. 8.

[9] Ibid, Fig. 35, p. 34.