ON THE EQUATION $\nabla \times \boldsymbol{a} = \kappa \boldsymbol{a}$

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Abstract

We show that when correctly formulated the equation $\nabla \times \boldsymbol{a} = \kappa \boldsymbol{a}$ does not exhibit some inconsistencies atributed to it, so that its solutions can represent physical fields.

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Let us consider the *free* Maxwell equations:

$$\nabla \cdot \vec{E} = 0, \quad \nabla \cdot \vec{B} = 0, \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}. \tag{2}$$

We want to look for solutions of Maxwell equations which describe *stationary* electromagnetic configurations – in the sense that the energy of the field does not propagate. In order to obtain one such stationary solution it is sufficient to find solutions of the vector equation

$$\nabla \times \vec{a} = \kappa \vec{a}, \quad \kappa \quad \text{constant.} \tag{3}$$

In fact, if we are looking for stationary solutions then in the rest frame we can make the following *ansatz*:

$$\vec{E} = \vec{a} \sin \kappa t, \quad \vec{B} = \vec{a} \cos \kappa t. \tag{4}$$

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All Maxwell equations are automatically satisfied within this *ansatz* for $\vec{a}$ satisfying the vector equation (3). The solution is obviously stationary since the Poynting vector $\vec{S} = \vec{E} \times \vec{B} = 0$. It also follows that $\vec{E}$ and $\vec{B}$ satisfy the same equation:

$$\nabla \times \vec{E} = \kappa \vec{E}, \quad \nabla \times \vec{B} = \kappa \vec{B}. \tag{5}$$

The vector equation $\nabla \times \vec{B} = \kappa \vec{B}$ is very important in plasma physics and astrophysics, and can also be used as a model for force-free electromagnetic waves [1].

The identification of solutions of the eq.(3) with physical fields (as in eq.(4) above) has been criticized by Salingaros [2]. In particular, he discussed the question of violation of gauge invariance and of parity invariance. The inconsistencies have been identified in [2] with the lack of covariance of the eq.(3) with respect to transformations. Our proposal in this letter is to show that there is *no* violation of gauge invariance and of parity invariance.

The argument leading to the lack of gauge invariance [2] runs as follows. From $\nabla \times \vec{B} = \kappa \vec{B}$ we have, since $\vec{B} = \nabla \times \vec{A}$, that $\nabla \times \vec{B} = \kappa \nabla \times \vec{A}$, and then $\vec{B} = \kappa \vec{A} + \nabla \phi$. Now, in [2] it was argued that for $\vec{B}' = \kappa \vec{A}' + \nabla \phi = \kappa(\vec{A} + \nabla \lambda) + \nabla \phi = \vec{B} + \kappa \nabla \lambda$, that is, gauge invariance requires $\kappa = 0$ or the specific gauge $\lambda = 0$. The mistake in this argument is easily identified since for $\vec{B}' = \nabla \times \vec{A}'$ we have $\vec{B} = \kappa \vec{A}' + \nabla \psi$, where the arbitrary function ψ must *not* be identified *a priori* with ϕ. In this case $\vec{B}' = \kappa(\vec{A} + \nabla \lambda) + \nabla \psi = \kappa \vec{A} + \kappa \nabla(\lambda + \psi) = \kappa \vec{A} + \kappa \nabla \phi = \vec{B}$.

The argument used in [2] leading to the lack of parity invariance is that since $\vec{B}$ is a parity eigenvector of even parity [3] and since under upon reflection we have $\nabla \mapsto -\nabla$ then $\kappa \vec{B} = \nabla \times \vec{B} \mapsto \kappa \vec{B} = -\nabla \times \vec{B}$, $\vec{B} = -\vec{B} = 0$, which means that solutions of $\nabla \times \vec{B} = \kappa \vec{B}$ must necessarily *not* be a parity eigenvector, and then they cannot be associated with neither $\vec{E}$ nor $\vec{B}$ since both fields have definite parity. The origin of the mistake in this case is not trivial, and requires a detailed explanation.

The problem in the above argument is essentially due to the definition of the vector product $\times$ in the usual Gibbs-Heaviside vector algebra. The usual definition of the vector product $\vec{v} \times \vec{u}$ as

$$(v_1, v_2, v_3) \times (u_1, u_2, u_3) = (v_2 u_3 - v_3 u_2, v_3 u_1 - v_1 u_3, v_1 u_2 - v_2 u_1) \tag{6}$$

is a *nonsense* since it equals a pseudo-vector (L.H.S.) and a vector (R.H.S.). This nonsense is therefore also expected in the definition of $\nabla \times \vec{v}$:

$$\nabla \times \vec{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_2}{\partial x_2} \right). \tag{7}$$

In other words, in the Gibbs-Heaviside vector algebra the vector product of two vector $\vec{v}, \vec{u} \in V \simeq \mathbb{R}^3$ is the mapping $\times : (\vec{v}, \vec{u}) \mapsto \vec{w}$. Obviously $\vec{w}$ cannot belong to the same space V where $\vec{v}$ and $\vec{u}$ live because $\vec{w}$ is a pseudo-vector. So, let us call this new vector space $V^\times$. We also have the vector product of vectors and pseudo-vectors, $\times : V \times V^\times \to V$ and $\times : V^\times \times V \to V$. The non-specification of these two spaces V and $V^\times$ in the usual presentation produces nonsense. If we usually identify V and $V^\times$ as in eq.(6) and consider the sum $\vec{v} + \vec{v}^\times = \vec{z}$, then under reflection is $\vec{z}$ a vector or a pseudo-vector? Obviously this means that the usual vector product is a nonsense.

One formalism we can use which is free from the above inconsistency is the one of *differential forms* [4], or the Cartan calculus. Given the 1-forms $\{dx^i\}$ $(i = 1, 2, 3)$ and the vector fields $\{\partial_j = \partial/\partial x^j\}$ $(j = 1, 2, 3)$ such that

$$\partial_j \lrcorner\, dx^i = dx^i(\partial_j) = \delta^i_j, \tag{8}$$

we can construct 1-forms $\mathbf{v}$ and $\mathbf{u}$ as

$$\mathbf{v} = v_i dx^i, \quad \mathbf{u} = u_i dx^i. \tag{9}$$

The exterior product gives the 2-form

$$\mathbf{v} \wedge \mathbf{u} = (v_1 u_2 - v_2 u_1) dx^1 \wedge dx^2 + (v_2 u_3 - v_3 u_2) dx^2 \wedge dx^3 + (v_1 u_3 - v_3 u_1) dx^1 \wedge dx^3. \tag{10}$$

In order to relate this expression with the vector product we need the so called Hodge operator $\star$ [4]. If we denote the volume element by τ,

$$\tau = dx^1 \wedge dx^2 \wedge dx^3 \tag{11}$$

then we have that

$$\star\,(\mathbf{v}\wedge\mathbf{u}\wedge\cdots\wedge\mathbf{w}) = \vec{w}\,\lrcorner\,(\cdots\lrcorner\,(\vec{u}\,\lrcorner\,(\vec{v}\,\lrcorner\,\tau))\cdots), \tag{12}$$

where $\vec{v}=\varphi(\mathbf{v})$, etc., and φ is the isomorphism given by

$$\varphi(dx^i) = \partial_i. \tag{13}$$

Explicitly we have

$$\star dx^1 = dx^2\wedge dx^3, \quad \star dx^2 = dx^3\wedge dx^1, \quad \star dx^3 = dx^1\wedge dx^2, \tag{14}$$

$$\star(dx^2\wedge dx^3) = dx^1, \quad \star(dx^3\wedge dx^1) = dx^2, \quad \star(dx^1\wedge dx^2) = dx^3. \tag{15}$$

It follows that $\star(\mathbf{v}\wedge\mathbf{u})$ is the 1-form

$$\star\,(\mathbf{v}\wedge\mathbf{u}) = (v_2u_3 - v_3u_2)dx^1 + (v_3u_1 - v_1u_3)dx^2 + (v_1u_2 - v_2u_1)dx^3, \tag{16}$$

which we recognize as the counterpart of the vector product. If we work with $\star(\mathbf{v}\wedge\mathbf{u})$ then if we take $dx^i\mapsto -dx^i$ we have $\star(\mathbf{v}\wedge\mathbf{u})\mapsto -\star(\mathbf{v}\wedge\mathbf{u})$ while $\mathbf{v}\wedge\mathbf{u}\mapsto\mathbf{v}\wedge\mathbf{u}$. This is because the volume element τ used in the definition of $\star$ also changes sign, $\tau\mapsto-\tau$.

Now, the electric field is represented by a 1-form $\mathbf{E}$ given by

$$\mathbf{E} = E_1dx^1 + E_2dx^2 + E_3dx^3, \tag{17}$$

but the magnetic field is represented by a 2-form $\mathbf{B}$

$$\mathbf{B} = B_1dx^2\wedge dx^3 + B_2dx^3\wedge dx^1 + B_3dx^1\wedge dx^2. \tag{18}$$

The fact that the magnetic field *must* be represented by a 2-form follows from Faraday law of induction [5]. Note that for $dx^i\mapsto -dx^i$ we have $\mathbf{E}\mapsto-\mathbf{E}$ and $\mathbf{B}\mapsto\mathbf{B}$. Note also that we can define a 1-form $\mathbf{b}$ by

$$\mathbf{b} = \star\mathbf{B} = B_1dx^1 + B_2dx^2 + B_3dx^3, \tag{19}$$

and in this case $\mathbf{b}\mapsto-\mathbf{b}$ for $dx^i\mapsto -dx^i$.

Consider the differential operator d, which can be defined by

$$d\mathbf{v} = \partial_i v_j dx^i \wedge dx^j, \quad d(\mathbf{v} \wedge \mathbf{u}) = (d\mathbf{v}) \wedge \mathbf{u} - \mathbf{v} \wedge (d\mathbf{u}). \tag{20}$$

The codifferential operator δ is defined as

$$\delta = \star d \star . \tag{21}$$

We can easily verify the relations

$$\begin{aligned} \nabla \times \vec{E} &\leftrightarrow \star d\mathbf{E}, \\ \nabla \cdot \vec{E} &\leftrightarrow \delta\mathbf{E}, \\ \nabla \times \vec{B} &\leftrightarrow \delta\mathbf{B}, \\ \nabla \cdot \vec{B} &\leftrightarrow \star d\mathbf{B}. \end{aligned} \tag{22}$$

The vector equation $\nabla \times \vec{B} = \kappa \vec{B}$ must be written as

$$\delta\mathbf{B} = \kappa \star \mathbf{B}. \tag{23}$$

The operators d and δ are such that $d \mapsto -d$ and $\delta \mapsto -\delta$ for $dx^i \mapsto -dx^i$. Then we have that

$$\delta\mathbf{B} = \kappa \star \mathbf{B} \mapsto (-\delta)(\mathbf{B}) = \kappa(-\star)(\mathbf{B}), \tag{24}$$

and no problem appears within the parity of $\mathbf{B}$. The same holds for the equation $\nabla \times \vec{E} = \kappa \vec{E}$ which reads $d\mathbf{E} = \kappa \star \mathbf{E}$, and transforms as

$$d\mathbf{E} = \kappa \star \mathbf{E} \mapsto (-d)(-\mathbf{E}) = \kappa(-\star)(-\mathbf{E}). \tag{25}$$

In summary, when correctly formulated in terms of differential forms, that is, the electric field being represented by a 1-form and the magnetic field being represented by a 2-form, the vector equation $\nabla \times \vec{a} = \kappa \vec{a}$ does not show any problem related to violation of parity invariance.

Moreover, since the calculus with differential forms is *intrinsic* [4], it does *not* depend on our coordinate system choice. We remember, however, that the vector equations $\nabla\times\vec{E}=\kappa\vec{E}$ and $\nabla\times\vec{B}=\kappa\vec{B}$ emerged from a separation of variables which is expected to hold only in the rest frame.

In conclusion, when correctly formulated, the vector equation $\nabla\times\vec{a}=\kappa\vec{a}$ does not deserve any of Salingaros' criticisms [2].

Before we end we recall that being $\langle x^\mu\rangle$ $(\mu=0,1,2,3)$ Lorentz coordinates of Minkowski spacetime, the Maxwell equations can be written as

$$d\mathbf{F}=0, \quad \delta\mathbf{F}=-\mathbf{J}, \tag{26}$$

where $\mathbf{F}=(1/2)F_{\mu\nu}dx^\mu\wedge dx^\nu$ and $\mathbf{J}=J_\mu dx^\mu$, with

$$F_{\mu\nu}=\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \quad J_\mu=(\rho,-j_1,-j_2,-j_3)\,. \tag{27}$$

The force-free equation appears, e.g., in the tentative to construct purely electromagnetic particles (PEP), as done, for example, in [6,7]. Following Einstein [8], Poincaré [9] and Ehrenfest [10] a PEP must be free of self-force. Then the current vector field $\underline{J}=J^\mu\partial_\mu$ must satisfy

$$\underline{J}\,\lrcorner\,\mathbf{F}=0, \tag{28}$$

or in vector notation,

$$\rho\vec{E}=0, \quad \vec{j}\cdot\vec{E}=0, \quad \vec{j}\times\vec{B}=0. \tag{29}$$

From eq.(29) Einstein concluded that the only possible solution of eq.(26) with the condition given by eq.(28) is that $\underline{J}=0$. However, this conclusion only holds if we assume that $\underline{J}$ is time-like. If we assume that $\underline{J}$ may be space-like (as, for example, in London's theory of superconductivity) then there exists a reference frame where $\rho = 0$, and a possible solution of eq.(28) is

$$\rho = 0, \quad \vec{E} \cdot \vec{B} = 0, \quad \vec{j} = kC\vec{B}, \tag{30}$$

where $k = \pm 1$ is called the chirality of the solution and C is a real constant. In [6,7] stationary solutions of eq.(26) with the condition (28) are exhibited with $\vec{E} = 0$. In this case we verify that

$$\nabla \times \vec{B} = kC\vec{B}. \tag{31}$$

What is interesting to observe is that from the solutions of eq.(31) found in [6,7] we can obtain solutions of the free Maxwell equations. Indeed, it is enough to put $\vec{E}' = \vec{B} \cos \Omega t$ and $\vec{B}' = \vec{B} \sin \Omega t$, as discussed in the beginning. In [11] we found also stationary solutions of Maxwell equations. Other solutions can be found with the methods described in [12].

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