THE SYSTEM OF THE WORLD,

BY

M. Le MARQUIS DE LAPLACE,

TRANSLATED FROM THE FRENCH,

AND

ELUCIDATED WITH EXPLANATORY NOTES.

BY THE

Rev. HENRY H. HARTE, F. T. C. D. M. R. I. A.

VOL. I.

DUBLIN:

PRINTED AT THE UNIVERSITY PRESS,

FOR

LONGMAN, REES, ORME, BROWN, AND GREEN,

PATERNOSTER ROW, LONDON.

MDCCCXXX.

R. Graisberry, Printer.

CONTENTS.

BOOK I.

OF THE APPARENT MOTION OF THE HEAVENLY BODIES.

BOOK II.

OF THE REAL MOTIONS OF THE HEAVENLY BODIES.

BOOK III.

OF THE LAWS OF MOTION.

PREFACE.

It has been made a matter of surprise, that notwithstanding there are many individuals in these countries perfectly competent to the task, there has not as yet appeared a translation of the works of Laplace.

That an accurate translation of the works of this great man would render them more easily apprehended, and would also contribute to their being more extensively known, cannot be questioned by any person who considers, that they are read with avidity by many persons who are frequently embarrassed as to the author's meaning, in consequence of their imperfect acquaintance with the French language. The present Translation was drawn up for the purpose of obviating these difficulties, and of rendering the work accessible to every scientific student. It is hoped that the Notes which are subjoined at the end of each volume will tend to elucidate many of the important results which are merely announced in the text. The Translator is aware, that to those readers who are conversant with the Celestial Mechanics, many, if not all, of these might be dispensed with; but when it is considered, that his object has been to render these objects accessible to the generality of readers, he trusts he will not be deemed unnecessarily diffuse, if he has insisted longer on some points than the experienced reader would think necessary.

The decimal division of the circle, and of the day, (of which the origin is fixed at midnight,) is adopted in the text. The lineal measures are referred to the metre, and all temperatures are estimated on the centigrade thermometer, the height of the barometer being supposed equal to 76 centimetres, when this thermometer points to zero at the parallel of 45°.

By means of the following table, any number of decimal degrees, minutes, and seconds, may be obtained in sexagesimal degrees, minutes, and seconds, by simple multiplication:

Decimal.	Sexagesimal.	Sexages.	Decimal.
1°	$= 54' = 324''$	1°	$= 1^\circ\ 11'\ 11''\ 11'''$, &c.
$1'$	$= 32'',4$	$1'$	$= 1', 85''\ 18'''.51$, &c.
$1''$	$= 0'',324$	$1''$	$= 3''\ \ 8'''.64$.

As it is frequently required to know the values of the corresponding quantities, according to the English standard of weights, measures, &c., the following table is subjoined, by means of which it is extremely easy to estimate the French measures in terms of the English, or *vice versa*.

1 foot = 12 inches,	= 12.785 inches.
∴ 3 feet + $1\frac{3}{12}$ of an inch,	= 3 feet, or one yard, which is the English linear standard.
The *metre* = 10,000,000 of the distance of the pole from the equator,	= 39.383 inches.
The *litre*, which is the unit of capacity, (= the cube of the tenth part of the metre,)	= 61.083 inches.
The *gramme*, which is the unit of weight, (= the weight of a cube of distilled water, of which one side is the 100th part of the metre,	= 22.966 grains.
The *are*, which is the superficial measure,	= 11.968 square yards.

The following numerical values being of frequent occurrence will likewise be useful to the practical student: l denoting the logarithm of a quantity in the Hyperbolic or Naperian system, of which the modulus = 1, and L denoting the logarithm of a quantity in the common system, of which the base = 10, we have e, the base of the Hyperbolic system = 2, 7128 18284 59045 23536, &c., the modulus in the common system $= Le =$ 0, 43429, 44819 03251 827651 11289.

The ratio of the diameter to the periphery of a circle, or π the semiperiphery of a circle, of which the rad. is unity =

$$3,\ 14159\ 26535\ 89793\ 23846\ 26433\ 83279$$
$$\text{L. } \pi = 0,\ 49714\ 98726\ 94133\ 85635\ 127$$
$$l.\ \pi = 1\ \ 14472\ 98858\ 49400\ 17414\ 342$$

In our division of the day, one second of time is the 86400th part of the mean day. In the present French division, one second is the 100,000th part of the mean day, ∴ denoting by g the force of gravity, and by λ the length of the pendulum which vibrates seconds. In the latitude of Paris we have

	In the ancient division of time.		In the present French division of time.
$g. =$	9^{m}, 808795248	$= 7^{m}$, 32214	
L. $g.$	0, 9916156690	= 2 8646381	
$\lambda. =$	0^{m}, 9938387446	$= 0^{m}$, 741887	
L. $\lambda. =$	1, 9973159236	= 1, 8703378	

THE

SYSTEM OF THE WORLD.

Me vero primum dulces ante omnia musæ
Quarum sacra fero, ingenti perculsus amore,
Accipiant, cælique vias et sidera monstrent.
VIRG. *lib.* 11, GEOR.

OF all the natural sciences, astronomy is that which presents the longest series of discoveries. The first appearance of the heavens is indeed far removed from that enlarged view, by which we comprehend at the present day, the past and future states of the system of the world. To arrive at this, it was necessary to observe the heavenly bodies during a long succession of ages, to recognize in their appearances the real motion of the earth, to develope the laws of the planetary motions, to derive from these laws the principle of universal gravitation, and finally from this principle to descend to the complete explanation of all the celestial phenomena in their minutest details. This is what the human understanding has atchieved in astronomy. The exposition of these discoveries, and of the most simple manner, in which they may arise one from the other, will have the twofold advantage of furnishing a great assemblage of important truths, and of pointing out the true method which should be followed in investigating the laws of nature. This is the object which I propose in the following work.

BOOK THE FIRST.

OF THE APPARENT MOTIONS OF THE HEAVENLY BODIES.

CHAP. I.

Of the diurnal motion of the heavens.

IF during a fine night, and in a place where the view of the horizon is uninterrupted, the appearance of the heavens be attentively observed, it will be perceived to change at every instant. The stars are either rising above or descending towards the horizon; some appear towards the east, others disappear towards the west; several, as the pole star, and the stars of the great Bear, never reach the horizon in our climates. In these various motions, the relative position of all the stars remains the same: they describe circles which diminish in proportion as they are nearer to a point which seems to be immoveable. Thus the heavens appear to revolve about two fixed points, termed from this circumstance, *poles of the world;* and in this motion they are supposed to carry with them, the entire system of the stars. The pole which is elevated above the horizon is the *north* pole. The opposite pole, which we imagine to be depressed beneath the horizon, is the *south* pole.

Already several interesting questions present themselves to be resolved. What becomes during the day of the stars which have been seen in the night? From whence do those come which begin to appear? and where are those gone which have disappeared? An attentive examination of the phenomena furnishes very simple answers to these questions. In the morning the light of the stars grows fainter, according as the dawn advances; in the evening they become more brilliant, as the twilight diminishes; it is not therefore because they cease to shine, but because they are effaced by the more vivid light of the twilight and of the sun, that we cease to perceive them.

The fortunate discovery of the telescope has furnished us with the means of verifying this explanation, for the stars seen through this instrument are visible, even when the sun is at its greatest elevation above the horizon. Those stars, which from their proximity to the pole, never reach the horizon, are perpetually visible. With respect to the stars which rise in the east and set in the west, it is natural to suppose that they complete under the horizon the circle, part of which appeared to be described above it. This truth become more obvious as we advance towards the north, more and more of the stars situated in this part of the world are extricated from beneath the horizon, till at length these stars cease to disappear at all, while the stars which are situated towards the south become entirely invisible. When we advance towards the south pole, the contrary is observed to be the case; stars which always continued above the horizon, commence to rise and set alternately, and new stars previously invisible begin to appear. It appears from these phenomena that the surface of the earth is not what it appears to be, namely, a plane on which the celestial vault is supported. This is an illusion which the first observers rectified very soon, by considerations similar to the preceding; they observed that the heavens surround the earth on all sides, and that the stars shine perpetually, describing every day their respective circles. We shall have frequent occasion to observe in the sequel, cases in which similar illusions have been dissipated, and in which even the real objects have been recognized in their erroneous appearances, by means of astronomy.

In order to form an accurate conception of the motion of the stars, we conceive an axis to pass through the centre of the earth, and the two poles of the world, on which the celestial sphere revolves. The great circle perpendicular to this axis is called the Equator, the lesser circles which the stars describe parallel to the equator, in consequence of their diurnal motion, are termed parallel circles. The *zenith* of a spectator, is that point of the heavens to which his vertical is directed. The *nadir* is the point diametrically opposite. The *meridian* (*a*) is the great circle which passes through the *zenith* and the *poles*; it divides into two equal parts the arcs described by the stars above the horizon, so that when they are on this circle, they are at their greatest or least altitude. Finally, the *horizon* is the great circle perpendicular to the vertical, or parallel to the surface of stagnant water at the place of the observer.

The elevation of the pole being an arithmetic mean between the greatest and least altitudes of the stars which never set, an easy method is suggested of determining the height of the pole. As we advance directly towards the pole, it is observed to be elevated very nearly in proportion *(b)* to the space passed over; hence it is inferred that the surface of the earth is convex, its figure differing little from that of a sphere. The curvature of the terrestrial globe is very sensible on the surface of the seas; the sailor in his approach towards the shore perceives first the most elevated points, and afterwards the lower parts, which were concealed from his view by the convexity of the earth. It is also in consequence of this curvature, that the sun at its rising gilds the summits of the mountains before he illuminates the planes.

CHAP. II.

Of the sun, and of its motions.

All the heavenly bodies participate in the diurnal motion of the celestial sphere, but several have proper motions of their own, which it is interesting to follow, because it is by means of these alone, that we can hope to arrive at the knowledge of the true system of the world. As in measuring the distance of an object, we observe it from two different positions, so in order to discover the mechanism of nature, we must consider her under different points of view, and observe the development of her laws, in the changes of appearance which she presents to us. Upon the earth, we vary the phenomena by experiments, in the heavens we carefully determine all those which the celestial motions present to us. By thus interrogating nature, and subjecting her answers to analysis, we can by a train of inductions judiciously managed, arrive at the general phenomena, from whence these particular facts arise. It is to discover these grand phenomena, and to reduce them to the least possible number, that all our efforts should be directed; for the first causes and intimate nature of beings will be for ever unknown.

The sun has a proper motion, of which the direction is contrary to the diurnal motion. This motion is recognised by the appearances which the heavens present during the nights, which appearances change and are renewed with the seasons. The stars situated in the path of the sun, and which set a little after him, are very soon lost in his light, and at length reappear before his rising; this star therefore advances towards them, from west to east. It is thus that for a long time his proper motion was traced, (which at present can be determined with great precision), by observing every day, the meridian altitude of the sun, and the interval of time which elapses between his passage, and that of the stars over the meridian. By means of these observations, we obtain the proper motions of the sun, in the direction of the meridian, and also in the direction of the parallels; the resultant of these motions is the true motion of this star about the earth. In this manner, it has been found that this star moves in an orbit, which is called the *ecliptic*, and which at the commencement of 1801, was inclined to the equator at an angle of 26°,07315.

The variety of seasons is caused by the inclination of the ecliptic to the equator. When the sun in his annual motion arrives at the equator, he describes very nearly in his diurnal motion this great circle, which being then divided into two equal parts by all the horizons, the day is equal to the night, in every part of the earth. The points of the intersection of the equator and the ecliptic, are termed *the equinoxes*, on account of this equality. In proportion as the sun, after leaving the equinox of spring, advances in his orbit, his meridian altitudes above our horizon increase, the visible arc of the parallels, which it describes every day, continually increases, and this augments the length of the days, till the sun has attained his greatest altitude. At this epoch, the days are the longest in the year, and because the variations of the meridian height of the sun, are insensible, near the *maximum*, the sun (considering only the altitude on which the duration of the day depends) appears stationary, for which reason, (*c*) this point of the *maximum* height has been termed the summer solstice. The parallel described by the sun on that day, is called the summer *tropic*. This star then descends towards the equator, which it traverses again, at the autumnal equinox, from thence it arrives at its *minimum* of altitude, or at the winter *solstice*. The parallel then described by the sun is the winter *tropic*, and the corresponding day is the shortest of the year; having attained this term, the sun again ascends and returns to the vernal equinox, to recommence the same route.

Such is the constant regular progress of the sun and of the seasons. Spring, is the interval comprised between the vernal equinox, and the summer solstice; summer is the interval from this solstice to the autumnal equinox; and the interval from the autumnal equinox to the winter solstice, constitutes the autumn; finally, winter is the interval of time from the winter solstice to the vernal equinox.

The presence of the sun above the horizon being the cause of heat, it might be supposed that the temperature should be the same in summer as in spring, and in the winter and autumn. But the temperature is not the instantaneous effect of the presence of the sun, it is rather the result of its long continued action. It does not produce its *maximum* of effect, for each day, till some time after the greatest altitude of this star above the horizon, nor does it attain its maximum effect for the year, till the greatest solstitial altitude is passed.

The different climates exhibit remarkable varieties, which we will now examine from the equator to the poles. At the equator, the horizon divides all the parallels into two equal parts; the day is therefore constantly equal to the night. In the equinoxes the sun, at mid day, passes through the zenith. The meridian altitudes of this star, at the solstices, are least, and equal to the complement of the inclination of the ecliptic to the equator. The solar shadows are then directly opposite, which is never the case in our climates, where they are always at mid-day directed towards the north.

At the equator, therefore, properly speaking, there are two summers and two winters, every year. This is also the case in all places, where the height of the pole is less than the obliquity of the ecliptic. Beyond this limit, as the sun never can be in the zenith, there is only one summer and one winter in each year; the duration of the longest day increases and that of the shortest day diminishes as we approach the pole, and at the parallel the distance of the zenith of which from the pole, is equal to the obliquity of the ecliptic, the sun never (*d*) sets on the day of the summer solstice, nor rises on the day of the winter solstice. Still nearer to the pole, the time of his presence, and of its absence, exceeds several days, and even months. Finally, under the pole, the horizon coinciding with the equator itself, the sun is always above the horizon when on the same side of the equator as the pole; it is constantly below the horizon, when it is at the other side of the equator; so that there is then but one day and one night throughout the year. (*e*)

Let us trace more particularly the path of the sun. It is at once apparent that the intervals which separate the equinoxes and the solstices are unequal, that from the vernal to the autumnal equinox, is about eight days longer than the interval between the autumnal and vernal equinoxes; the motion is consequently not uniform: by means of accurate and repeated observations, it has been ascertained that the motion is most rapid in a point of the solar orbit, which is situated near the winter solstice, and that it is slowest in the opposite point of the orbit near to the summer solstice. The sun describes in a day 1°,1327 in the first point, and only 1°,0591 in the second. Thus during the course of the year its motion varies from the greatest to the least by three hundred and thirty-six ten thousandths of its mean value. (*f*)

This variation produces, by its accumulation, a very sensible inequality in the motion of the sun. In order to determine its law, and in general to obtain the laws of all the periodical inequalities, it should be remarked that these inequalities may be properly represented by the sines and cosines of angles which become the same after the completion of every circumference. (*g*) If therefore all the inequalities of the celestial motions are expressed in this manner, the only difficulty consists in separating them from each other, and in determining the angles on which they depend. As the inequality which we are at present considering, performs the period of its changes in a revolution of the sun, it is natural to make it depend on the motion of the sun and on its multiples. In this manner, it has been found that it is expressed by means of a series of sines depending on this motion; it is reduced very nearly to two terms, of which the first is proportional to the sine of the mean angular distance of the sun, from the point in its orbit, where his velocity is greatest, and of which the second is about ninety five times less than the first, and proportional to the sine of double of this distance.

It is probable that the distance of the sun from the earth varies with its angular velocity, and this has been proved by the measures of its apparent diameter. This diameter increases and diminishes according to the same law as the velocity, but in a ratio only half as great. When the velocity is greatest, the diameter is 6035,″8 and it is observed to be only 5836,″3, when this velocity is the least; therefore its mean magnitude is about 5936,″0.

The distance of the sun from the earth being reciprocally proportional to his apparent diameter, its increase follows the same law as the diminution of this diameter. The point of the orbit in which the sun is nearest to the earth, is termed the *perigee*, and the opposite point, in which the sun is most remote, is called the *apogee*. It is in the first of these points, that the apparent diameter and also the velocity of the sun are greatest; in the second point, the apparent diameter and velocity are at their *minimum*.

It would be sufficient, in order to explain the diminution of the sun's apparent motion, to increase his distance from the earth; but if the variation of the solar motion arose from this cause only, and if the real velocity of the sun was constant, its apparent velocity would diminish in the same ratio as the apparent diameter. It diminishes in a ratio twice as great, therefore there is an actual retardation in the motion of this star, when it recedes from the earth. From the effect of this retardation, combined with the increase of distance, its angular motion diminishes as the square of the distance increases, so that its product by this square is very nearly constant. All the measures of the apparent diameter of the sun, compared with the observations of his daily motion, confirm this result.

Let us conceive a right line joining the centres of the sun and earth, which we will call the *radius vector* of the sun, it is easy to perceive that the small sector, or area described in a day by this radius about the earth, is proportional to the square of this radius into the diurnal motion (h) of the sun. This area is therefore constant, and the entire area described by the radius vector, reckoning from a given point, increases as the number of days, elapsed since the epoch at which the sun was on this radius. *Therefore the areas described by its radius vector, are proportional to the times.* This simple relation between the motion of the sun, and its distance from the focus of this motion, must be admitted as a fundamental law in its theory, at least, until observations compel us to modify it.

If from the preceding data, the position and length of the radius vector of the solar orbit be set down every day, and a curve be supposed to pass through the extremities of all those radii, it will appear that this curve will be somewhat elongated in the direction of the right line, which, passing through the centre of the earth, joins the points of the greatest and least distance of the sun. The resemblance of this curve with the ellipse, having suggested the notion of comparing them together, their identity was ascertained; from which it has been inferred, *that the solar orbit is an ellipse, of which the centre of the earth occupies one of the foci.* (i)

The ellipse is one of those curves so celebrated both in antient and modern geometry, which being formed by the intersection of a plane with the surface of a cone, have been therefore termed *conic sections*. The extremities of a thread which is stretched on a plane, being fixed on two immovable points, called foci, any point which slides along this thread describes the ellipse; it is evidently elongated in the direction of the right line which joins the foci, and which being extended on each side to meet the curve, forms the greater axis, of which the length is equal to that of the thread. The lesser axis is the right line drawn through the centre perpendicularly to the greater axis, and extended on both sides to meet the curve: the distance of the centre from one of the *foci* is the *excentricity* of the ellipse. When the two foci are united in the same point, the ellipse becomes a circle; by increasing their distance the ellipse gradually lengthens, and if the mutual distance becomes infinite, the distance of the focus from the nearest summit of the curve, remains finite, and the ellipse becomes a parabola.

The solar ellipse differs but little from a circle; for the excess of the greatest above the least distance of the sun from the earth is equal to the hundred and sixty ten thousandth part of this distance. This excess is the excentricity itself, in which observations indicate a very slow diminution, and hardly perceptible in a century.

In order to have a just conception of the elliptic motion of the sun, let us conceive a point to move uniformly on the circumference of a circle, of which the centre coincides with the centre of the earth, the radius being equal to the perigeon distance of the sun; suppose moreover that this point and the sun set off together from the perigee, and that the angular motion of the point is equal to the mean angular motion of the sun, while the radius vector of this point revolves uniformly about the earth, the radius vector of the sun moves unequally, always constituting with the distance of the perigee, and the arcs of the ellipse, sectors proportional to the times. At first, it precedes the radius vector of the point, and makes with it an angle, which after having increased to a certain limit (k), diminishes, and at length vanishes, when the sun arrives at his apogee. The two radii will then coincide with the greater axis. In the second half of the ellipse, the radius vector of the point precedes in its turn, that of the sun, and makes with it angles exactly equal to those, which it made in the first half, at the same distance from the perigee, at which point it coincides again with the radius vector of the sun, and with the greater axis of the ellipse. The angle by which the radius vector of the sun precedes that of the point, is termed *the equation of the centre.* Its *maximum* was 2°,13807 at the commencement of the present century, *i. e.* at the midnight, on which the first of January 1801 commenced. It diminishes by a quantity equal to about 53″ for every century.

From the duration of the sun's revolution in its orbit, the angular motion of the point about the earth may be inferred. The angular motion of the sun will be obtained by adding to this motion, the equation of the centre. The investigation of this equation is a very interesting problem of analysis, which can only be resolved by approximation; but the small excentricity of the solar orbit leads to very converging series, which are easily reduced to the form of tables.

The greater axis of the solar ellipse is not fixed in the heavens; it has relatively to the fixed stars an annual motion of about 36,″ in the same direction as that of the sun.

The solar orbit approaches by insensible degrees to the equator; the secular diminution of its obliquity, to the plane of this great circle, may be estimated at about 148″.

The elliptic motion of the sun does not exactly represent modern observations; their great precision has enabled us to perceive small inequalities, of which it would have been impossible to have developed the laws by observations alone. The investigation of these inequalities appertains to that branch of astronomy, which redescends from causes to the phenomena, and which will constitute the subject of the fourth book.

The distance of the sun from the earth, has at every period interested astronomers. Observers have (*l*) endeavoured to determine it, by all the means astronomy has successively furnished them with. The most natural and simple is that which

Geometers employ in measuring the distance of terrestrial objects. At the two extremities of a known base, the angles, which the visual rays of an object make with it, are observed, and by deducting their sum from two right angles, the angle will be obtained which these rays form at the point where they meet; this angle is termed the *parallax* of the object, the distance of which from the extremities of the base is easily obtained. In applying this method to the sun, the most extensive base which can be taken on the surface of the earth should be selected. Suppose two observers situated under the same meredian, and observing at noon, the distance of the centre of of the sun from the north pole; the difference of these two observed distances will be the angle, which the line joining the observers would subtend at this centre; the differences of the elevations of the pole gives this line in parts of the terrestrial radius; it will therefore be easy to infer from thence the angle under which the semidiameter of the earth would appear at the centre of the sun. This angle is the *horizontal parallax* of the sun; but it is too small to be accurately determined by this method, which only enables us to judge that the distance of this star is at least nine thousand diameters of the earth. In the sequel, it will be seen, that the discoveries in astronomy furnish other methods much more accurate for determining the parallax, which we now know to be about 27″, very nearly, at its mean distance from the earth; hence it follows that this distance is about 23984 terrestrial radii.

Black spots are observed on the surface of the sun, of an irregular and variable form. Sometimes they are very numerous, and of considerable extent; some have been observed, of which the magnitude was equal to four or five times that of the earth. At other times, though rarely, the surface of the sun has appeared pure, and without spots for several successive years. Frequently the solar spots are enveloped by penumbras, which are themselves surrounded by a more brilliant light than that of the rest of the sun, in the middle of which these spots are observed to form and to disappear. The nature of these spots is yet unknown, however they have made us acquainted with a remarkable phenomenon, namely, the rotation of the sun. Amidst all the variations which they undergo in their position and magnitude, we can discover regular motions precisely the same as those of corresponding points of the surface of the sun, if we suppose it to have a motion of rotation in the direction of its motion round the earth, on an axis almost perpendicular to the ecliptic. From a continued observation of these spots, it has been inferred that the duration of an entire revolution of the sun is about twenty-five days and a half, and that the solar equator is inclined at an angle of eight degrees and one third to the plane of the ecliptic.

The extensive spots of the sun are almost always comprised in a zone of its surface, the breadth of which, measured on the solar meridian, does not extend beyond thirty-four degrees on each side of the equator; however, spots have been observed which were forty-four degrees from this equator. There has been observed, particularly about the vernal equinox, a faint light which is visible before the rising and after the setting of the sun, to which has been given the name of *zodiacal light*. Its colour is white, and its apparent figure that of a spindle, the base of which rests on the solar equator; such as would be the appearance of an ellipsoid of revolution extremely flattened, the centre and plane of equator coinciding with those of the sun. The length of this zodiacal light appears sometimes to subtend an angle of more than one hundred degrees. The fluid which reflects this light to us, must be extremely rare, since the stars are sometimes visible through it. The most received opinion respecting its nature is, that this fluid is the atmosphere itself of the sun; but this atmosphere certainly does not extend to so great a distance.—At the conclusion of this work we will suggest what appears to us to be the cause of this light, which is unknown, and has hitherto baffled our enquiries.

CHAP. III.

Of Time, and of its measure.

Time is, relatively to us, the impression which a series of events, of which we are assured that the existence has been successive, leaves in the memory. Motion is a proper measure of time; for since a body cannot be in several places at the same time, when it moves from one place to another, it must pass successively through all the intermediate points. If it is actuated by the same force at every point of the line, which it describes, its motion is uniform, and the several portions of this line will measure the time employed to describe them. When a pendulum, at the termination of each oscillation, is in precisely the same circumstances as at the commencement of the motion, the durations of these oscillations are the same, and the time may be measured by their number. We may also employ for this measurement, the revolutions of the celestial sphere, in which the motions appear to be perfectly uniform; and mankind have universally agreed to make use of the motion of the sun for this purpose, the returns of which to the meridian, and to the same equinox or the same solstice, constitute the day and the year.

In civil life, the day is the interval of time which lapses from the rising to the setting of the sun: the night is the time, during which the sun remains below the horizon. The astronomical day comprises the entire duration of the diurnal revolution; it is the interval of time between two successive noons or midnights. It is greater than the duration of a revolution of the heavens, which constitutes the *sidereal day*; for if the sun and a star pass the meridian at the same instant, on the following day the sun will pass later, in consequence of its proper motion, by which it advances from west to east, and in the interval of a year it will pass the meridian once less than the star. It is found by assuming the mean astronomical day equal to unity, that the sidereal day is 0,99726957.

The astronomical days are not equal; their difference arises from two causes, namely, the inequality of the proper motion of the sun, and the obliquity of the ecliptic. The effect of the first cause is evident; thus, at the summer solstice, near to which the motion of the sun is slowest, the astronomical day approaches more to the sidereal day than at the winter solstice, when the motion is most rapid.

In order to conceive the effect of the second cause, it should be observed that the excess of the astronomical over the sidereal day arises solely from the proper motion of the sun reduced to the equator. If we conceive two great circles to pass through the poles of the world, and through the extremities of the small arc which the sun describes on the ecliptic each day, the arc of the equator, which they intercept, is the daily motion of the sun referred to the equator, and the time which this arc takes to pass over the meridian, is the excess of the astronomical over the sidereal day; but it is evident that in the equinoxes, the arc of the equator is less than the corresponding arc of the ecliptic, in the ratio of the cosine of the obliquity of the ecliptic to radius; in the solstices it is greater in the ratio of radius to the cosine (m) of the same obliquity; therefore the astronomical day is diminished in the first case, and increased in the second.

To obtain a mean day, independent of these causes; we imagine a second sun, which moving uniformly in the ecliptic, passes always at the same instant as the true sun the greater axis of the solar orbit; this will cause the inequality of the proper motion of the sun to disappear. The effect arising from the obliquity is then made to disappear, by imagining a third sun to pass through the equinoxes at the same moment as the second sun, and to move on the equator in such a manner, that the angular distances of these two suns from the vernal equinox, may be constantly equal to each other. The interval of time between two consecutive returns of this third sun to the meridian, consitutes the mean astronomical day. *Mean time* is measured by the number of these returns, and the *true time* is measured by the number of returns of the true sun to the meridian. The arc of the equator, intercepted between two meridians drawn through the centres of the true sun, and of the third sun, converted into time, in the proportion of the entire circumference to one day, is what is termed the (*n*) *equation of time.*

The day has been divided into twenty-four hours, and its origin has been fixed at midnight. The hour is divided into sixty minutes, the minutes into sixty seconds, the second into sixty thirds, &c. But the division of the day into ten hours, of the hours into one hundred minutes, of the minutes into one hundred seconds, will be adopted in this work, as being much more convenient for astronomical purposes.

The second sun, which we have imagined, determines by its returns to the equinoxes and the solstices, the mean equinoxes and solstices. The duration of its returns to the same equinox, or the same solstice, forms the *tropical year*, of which the actual length is about $365^{d}24226419$. Observation shews us that the sun employs a longer time to return to the same fixed stars. The *sidereal year* is the interval between two of these consecutive returns; it exceeds a tropical year by about $0,^{i}014119$. Therefore the equinoxes have a retrograde motion on the ecliptic, or contrary to the proper motion of the sun, in consequence of which they describe every year, an arc equal to the mean motion of this star, in the interval of about $0^{i},014119$, which is very nearly equal to 154″,63. This motion is not exactly the same every century, on which account, the duration of the tropical year is subject to a small inequality; it is now about 13″ shorter than in the time of Hipparchus.

It is natural that the year should be made to commence at one of the equinoxes or solstices; but if the origin of the year was placed at the summer solstice, or at the autumnal equinox, the same operations and labours would be appropriated to two different years. A like inconvenience would arise if the day was supposed to commence at noon, according to the custom of the old astronomers. It seems therefore most natural, that the year should be made to commence at the vernal equinox, at which period nature begins to revive; but it is equally natural to fix its commencement at the winter solstice, when, according to the received opinion of all antiquity, the sun begins to revive, and which is the middle of the longest night in the year under the poles.

If the length of the civil year was constantly 365 days, its commencement would always (*o*) anticipate that of the true tropical year, and it would pass through the different seasons with a retrograde motion in a period of about 1508 years. But this year (which was formerly in use in Egypt) would deprive the calendar of the advantage of attaching the months and festivals to the same seasons, and of rendering them useful epochs for the purposes of agriculture. This inestimable advantage would be secured, by considering the origin of the year as an astronomical phenomenon, which should be fixed by computation to the midnight which immediately precedes the equinox or the solstice: this has been done in France at the end of the last century. But then the bissextile years being intercalated according to a very complicated law, it would be difficult to resolve any given number of years into days, which would cause great confusion in history and chronology. Besides the origin of the year, which is always required to be known in advance, would be uncertain and arbitrary when it approached midnight, by a quantity less than the error (*p*) of the solar tables. Finally, the order of the bissextiles would be different for different meridians, which would be an obstacle to the adoption of the same calendar by all nations; indeed, when it is considered how pertinacious different nations are in reckoning geographical longitudes from their respective principal observatories, it cannot be supposed that they would all concur in making the commencement of the year to depend on the same meridian. We are therefore compelled to abandon the method pointed out by nature, and to recur to a mode of intercalating, which, though artificial, is regular and convenient. The simplest of all is that which Julius Cæsar introduced into the Roman calendar, and which consists in intercalating (*p*) one bissextile every four years. But if the short duration of life was sufficient to make the origin of the Egyptian years to deviate considerably from the solstice or the equinox, it only required a small number of centuries to produce the same displacement in the origin of the Julian year. This renders a more complicated intercalation indispensable. In the eleventh century the Persians (q) adopted one remarkable for its accuracy. It consists in rendering the fourth year bissextile seven times successively, and to defer this change on the eighth time to the fifth year. This supposes that the tropical year is $365\frac{8}{33}$, which is greater than the year as determined by observations by 0,0001823. So that a great number of centuries is requisite to produce a sensible displacement in the origin of the civil year. The mode of intercalating adopted in the Gregorian calendar is less exact, but it furnishes greater facilities in reducing the years and centuries into days, which is one of the principal objects of the calendar. It consists in intercalating a bissextile every fourth year, the bissextile at the end of each century being suppressed, to reestablish it at the end of the fourth. The length of the year which this intercalation supposes is about $365, \frac{97}{400}$ days, or about 365,242500, which is greater than the true length by about $0,^{d}0002185$. But if, according to the analogy of this mode of intercalating, a bissextile is also suppressed every four thousand years, which would reduce the number of bissextiles in this interval to 969, the length of the year would be $365^{d}.\frac{969}{4000}$; or $365^{d},2422500$, which approaches so near to 365,2422419, which is the length as determined by observation, that the difference may be rejected, particularly as there exists some slight uncertainty about the true length of the year, which besides is not rigorously constant.

The division of the year into twelve months is very ancient, and almost universal. Some nations have supposed that all the months are equal, and each to consist of thirty days, and they have completed the year by the addition of an adequate number of complementary days. Among other nations the entire year is comprized in the interval of twelve months, which are supposed to be unequal. The system of months, each consisting of thirty days, leads naturally to their subdivision into three decads. This period enables us to find out with great facility how much of the month has lapsed, but at the end of the year the complementary days would derange the order of things appropriated to the different days of the decad, which must necessarily embarras the measures of governments. This inconvenience would be obviated by making use of a short period, equally independent of months and of years, such as the week, which from the most remote antiquity in which its origin is confounded, has uninterruptedly pervaded all nations, always constituting a part of the successive calenders of different people. It is very remarkable that it is identically the same over the entire earth, as well relatively to the denomination (*r*) of its days, which has been regulated by the most ancient system of astronomy, as also with respect to their correspondence to the same physical instant. This is perhaps the most ancient and most incontrovertable monument of human attainments; it seems to indicate a common origin from which they have been derived, but the astronomical system on which they were founded is a proof of their imperfection at this commencement.

An interval of one hundred years constitutes a century, which is the longest period ever employed in the measurement of time; for the interval which separates us from the most ancient known events does not require a longer period.

CHAP. IV.

Of the motions of the moon, its phases, and eclipses.

After the sun, the moon, of all the heavenly bodies, is that which most interests us; its phases furnish a measure of time so remarkable, that it has been primitively made use of by all people. The moon, like the sun, has a proper motion from west to east; the duration of its sidereal revolution was $27^{d},321661423$, at the commencement of this century: it is not always the same, and the comparison of ancient with modern observations evinces incontrovertably an acceleration in the mean motion of the moon. This acceleration, though hardly sensible since the most ancient eclipse on record, will be developed in the progress of time. But will it go on always increasing, or will it cease to increase, and at length be changed into a retardation? This cannot be determined by observations, except after a very great number of ages. Fortunately, the discovery of its cause has anticipated them, and shewn us that it is periodical. At the commencement of this century, the mean angular distance of the moon from the vernal equinox, and reckoned from this equinox in the direction of the proper motion of this star, was 124,01321.

The moon moves in an elliptic orbit, of which the centre of the earth occupies one of the foci. Its radius vector traces about this point areas which are very nearly proportional to the times. The mean distance of this star from the earth being assumed equal to unity, the excentricity of its ellipse is 0,0548442, which gives the greatest equation of the centre equal (*s*) to 6,9854 : it appears to be invariable. The lunar perigee has a direct motion, that is to say, in the direction of the proper motion of the sun, the duration of its sidereal revolution was, at the commencement of this century, $3232^d,575343$, and its mean angular distance from the vernal equinox was 295°,68037. Its motion is not uniform; it is retarded when that of the moon is accelerated.

The laws of the elliptic motion are very far from representing the observations of the moon; it is subject to a great number of inequalities, which have an evident connection with the position of the sun. We shall indicate the three principal.

The most considerable, and that which was first recognised is, what has been termed the *evection.* This inequality, which at its *maximum* amounts to 1°,4907, is proportional to the sine of double the distance of the moon from the sun, minus the distance of the moon from its perigee. In the oppositions and conjunctions (*t*) of the moon with the sun, it is confounded with the equation of the centre, which it constantly diminishes. For this reason the ancient observers, who only determined the elements of the lunar theory, in order to be able to predict the phenomena of the eclipses, found the equation of the centre of the moon less than the true equation, by the entire quantity of the evection.

Another great inequality is also observed in the lunar motions, which disappears in the oppositions and conjunctions of the moon, and also in those points where these two stars are distant from each other by a quarter of the circumference. It arrives at its *maximum*, which is 0°,6611, when their mutual distance is fifty degrees: hence it has been inferred that it is proportional to double of the angular distance of the moon from the sun. This inequality is termed (*t*) the *variation*: as it disappears in the eclipses, it could not have been recognized by the observation of these phenomena.

Finally, the motion of the moon is accelerated, when that of the sun is retarded, and conversely; hence arises an inequality which is denominated the *annual equation*, the law of which is precisely the same as that of the equation of the centre of the sun, only affected with a contrary sign. This inequality, which at its maximum (*u*) amounts to 0°,2074, is confounded with the equation of the centre of the sun in the eclipses. In the computation of the moment at which these phenomena occur, it is indifferent whether these two equations are considered separately, or whether the annual equation of the lunar theory is suppressed, in order to increase the equation of the centre of the sun. This is the reason why the ancient astronomers assigned too great an excentricity to the orbit of the sun; while they assigned too small a one to the lunar orbit, in consequence of the *evection*.

This orbit is inclined to the ecliptic at an angle of 5°,7185: its points of intersection with the ecliptic, which are called the *nodes*, are not fixed in the heavens; they have a retrograde motion, or contrary to that of the sun; this motion is easily recognized by the succession of stars which the moon meets with when it traverses the ecliptic. The *ascending node* is that, in which the moon ascends above the ecliptic towards the north pole, and the *descending node* is that in which it descends below the ecliptic towards the south pole. The duration of a sidereal revolution of the nodes was at the commencement of this century 6793^{d},39108, and the mean (*v*) distance of the ascending node from the vernal equinox, was 15°,46117, but the motion of the nodes is retarded from one century to another.

It is subject to several inequalities, of which the greatest is proportional to the sine of double the distance of the moon from the sun, and amounts at its *maximum* to 1°8102. The inclination of the orbit is likewise variable, its greatest inequality, which amounts to 0°,1627 at its *maximum*, is proportional to the cosine of the same angle on which the inequality of the motion of the nodes depends; however the mean inclination appears to be constant in different centuries, notwithstanding the secular variations of the plane of the ecliptic.

The lunar orbit, and generally the orbits of the sun and of all the heavenly bodies, have no more a real existence than the parobolas described by projectiles at the surface of the earth. In order to represent the motion of a body in space, we conceive a line to pass through all the successive positions of its centre; this line is its orbit, of which the fixed or variable plane is that which passes through two consecutive positions of the body, and through the point about which it is supposed to move.

Instead of considering the motion of a body in this manner, we may in imagination project it on a fixed plane, and determine its curve of projection and height above this plane. This method, which is extremely simple, has been adopted by astronomers in the tables of the celestial motions.

The apparent diameter *(w)* of the moon changes in a manner analogous to the variations of the lunar motion; it is 5438″ at the greatest distance of the moon from the earth, and about 6207″ at the least distance. *(x)*

The same methods which were insufficient to determine the parallax of the sun, in consequence of its extreme smallness, have assigned 10661″ for the mean parallax of the moon; consequently at same distance at which the moon appears under an angle of 5823″, the earth would subtend an angle of 21332″; their diameters are therefore in the ratio of these numbers, or in the ratio of three to eleven, very nearly; and the volume of the lunar globe is forty-nine times less than the volume of the earth.

The phases of the moon are one of the most striking phenomena of the heavens. When it extricates itself in the evening from the rays of the sun, it appears with a feeble crescent, which increases according as it elongates itself from the sun; and it becomes a perfect circle of light when it is in opposition with this star. When it afterwards approaches this star, its phases diminish in the same proportion as they had previously increased, until in the morning it is immersed in the sun's rays. The lunar crescent being always turned towards the sun, evidently indicates that it receives its light from that body, and the law of the variation of its phases, which increase nearly as the versed sine of the angular distance of the moon (y) from the sun, proves that it is spherical.

The recurrence of the phases depends on the excess of the motion of the moon above that of the sun, which excess (z) has been termed the *synodic* motion of the moon. The duration of the synodic revolution of this star, or the period of its mean conjunctions, is now about $29^{d},530588716$: it is to the tropical year very nearly in the ratio of 19 to 235, that is to say, nineteen solar years are equivalent to about two hundred and thirty-five lunar months.

The *syrygies* are the points of the orbit in which the moon is in opposition or conjunction with the sun. In the first case the moon is said to be new, it is called full moon in the second. The *quadratures* are those points in which the moon is elongated from the sun one hundred or three hundred degrees, reckoning in the direction of its proper motion.

In those points, which are called the first and second quarters of the moon, we see the half of its illuminated hemisphere; strictly speaking, we see a little more; for when the exact half is presented to (*a*) us, the angular distance of the moon from the sun is a little less than one hundred degrees. At this instant the line which separates the illuminated from the darkened hemisphere, appears to be a right line, and the line drawn from the observer to the centre of the moon is perpendicular to the line which joins the centres of the sun and moon. Therefore in the triangle formed by the lines which join these centres and the eye of the observer, the angle at the moon is a right angle, and the angle at the observer is determined by observation, consequently the distance of the earth from the sun may be determined in parts of the distance of the earth from the moon. The difficulty of determining, with precision, the instant when the half of the disk of the moon is observed to be illuminated by the sun, renders this method not rigorously exact; we are indebted to it nevertheless, for the first just notions that have been formed (*b*) concerning the immense magnitude of the sun, and its great distance from the earth.

The explanation of the phases of the moon is connected with that of the eclipses, which, in times of ignorance, have been an object of terror to men, and of their curiosity in all ages. The moon can only be eclipsed by an opaque body, which deprives it of the light of the sun, and it is evident that this body is the earth, because an eclipse of the moon never happens except when it is in opposition, or when the earth is between this star and the sun. The terrestrial globe projects behind it a conical shadow, of which the axis is on the right line, which joins the centres of the sun and of the earth, and which terminates at the point where the apparent diameter of these two bodies, would be the same. Their diameters seen from the centre of the moon in opposition, and at its mean distance, are nearly 5920″ for the sun, and 21322′ for the earth: therefore the length of the cone of the earth's shadow is at least three times and a half greater than the distance of the moon from the earth, and its breadth at the points where it is traversed by the moon is about eight thirds of the diameter of the moon. The moon would be therefore eclipsed every time that it is in opposition to the sun, provided that the plane of its orbit coincided with the ecliptic; but in (*c*) consequence of the mutual inclination of these planes, the moon in its oppositions is frequently elevated above, or depressed below the cone of the earth's shadow, and it does not enter into it except when it is near to its nodes. If the entire disk of the moon is plunged in the shadow of the earth, the eclipse of the moon is *total*; it is said to be *partial* if only a portion of this disk is obscured; and we may easily conceive that a greater or less proximity of the moon to its nodes, at the moment of opposition, may produce all the varieties (*d*) which are observed in these eclipses.

Each point of the surface of the moon, before it is eclipsed, loses successively the light of different parts of the sun's disk. Its brightness therefore diminishes gradually, and at the moment when it penetrates into the earth's shadow it is extinguished. The interval through which this diminution has place is termed the *penumbra*, the breadth of which is equal to the apparent diameter of the sun, as seen from the centre of the moon.

The mean duration of a revolution of the sun, with respect to the node of the moon's orbit, is about $346^{d},619851$; it is to the duration of a synodic revolution of the moon, very nearly in the ratio of 223 to 19. Therefore, after a period of 223 lunar months, the sun and moon return to the same position relatively to the lunar orbit; the eclipses must consequently recur very nearly in the same order, this circumstance suggests a simple manner of predicting them, which was employed by the ancient astronomers. But the inequalities in the motions of the sun, and of the moon, ought to produce very sensible differences; besides the return of these two stars to the same position with respect to the node, in the interval of 223 months, is not rigorously exact; and the deviations which result, change at length the order of the eclipses which have been observed during one of these periods.

The circular form of the earth's shadow in the eclipses of the moon, indicated to the first astronomers that the figure of the earth was very nearly spherical; we shall see hereafter that the most exact method of determining the compression of the earth, is furnished by the great perfection to which the lunar theory has been brought.

It is solely in the conjunctions of the sun and of the moon, when this star, by being interposed between the sun and the earth, deprives us of the light of the sun, that the eclipses of the sun can be observed. Although the moon is incomparably smaller than the sun, yet on account of its proximity to the earth, its apparent diameter differs little from that of the sun; it even happens from the variations of these diameters, that they surpass each other alternately. Let us suppose the centres of the sun and moon to be on the same right line with the eye of the spectator, he will observe the sun to be eclipsed; and if the apparent diameter of the moon exceeds that of the sun, the eclipse will be total; but if this diameter be less, the observer will perceive a luminous ring, formed by that part of the sun which extends beyond the disk of the moon, and then the eclipse will be *annular*. If the centre of the moon does not exist in the right line drawn from the eye of the observer to the centre of the sun, the moon can only eclipse a part of the sun's disk, and the eclipse will be partial. Thus the changes of distance of the sun and moon from the centre of the earth, combined with the greater or less proximity of the moon to its nodes, at the moment of its conjunctions, ought to produce very sensible changes in the eclipses of the sun. To these causes may be added the elevation of the moon above the horizon, which changes the magnitude of its apparent diameter, and which, by the effect of the lunar parallax, may so increase or diminish the apparent distance of the centres of these two stars, that of two observers at some distance from each other, the one may see an eclipse of the sun which will not be visible to the other. In this respect the eclipses of the sun differ from eclipses of the moon, which are the same to all places on the earth where the two stars are elevated above the horizon.

We often see the shadow of a cloud, borne along by the winds, to pass rapidly over the hills and planes, and to deprive the spectators in those places of the view of the sun, which is visible to those who are out of the reach of its influence: this is an exact representation of a total eclipse of the sun. We may perceive then about the disk of the moon a crown of pale light, which is probably the solar atmosphere; for its extent cannot correspond to that of the moon, because it has been ascertained, by eclipses of the fixed stars and of the sun, that this last atmosphere is almost insensible.

The atmosphere which may be supposed to surround the moon, inflects the rays of light towards the centre of this star; and if, as ought to be the case, the atmospherical strata are rarer in proportion as they are farther removed from the surface, these rays, according as they penetrate farther into it, ought to be more inflected, and should consequently describe a curve which is concave to its surface. Hence it appears that a spectator on the surface of the moon, would not cease to see the star till it was depressed below the horizon by an angle equal to the *horizontal refraction.* The rays which emanate from this star seen at the horizon, after having touched the surface of the moon, continue their route, describing a curve similar to that which they described in approaching the surface. Thus, a second spectator placed behind the moon, with respect to the star, would still continue to perceive it in consequence of the inflexion of its rays in the moon's atmosphere. The diameter of the moon is (*e*) not sensibly increased by the refraction of its atmosphere; therefore a star appears to be eclipsed later than if this atmosphere did not exist, and for the same reason it ceases to be eclipsed sooner, so that the effect of the atmosphere of the moon is principally apparent in the duration of the eclipses of the sun, and of the stars, by the moon. Precise and numerous observations have enabled us with difficulty to suspect its existence; and it has been ascertained that at the surface of the moon the horizontal refraction does not exceed four seconds.

This refraction at the surface of the earth is at least one thousand times greater; therefore the lunar atmosphere, if any such exists, is of an extreme rarity, greater even than that which can be produced on the surface of the earth by the best constructed air pumps. It may be inferred from this that no terrestrial animal could live or respire at the surface of the moon, and that if the moon be inhabited, it must be by animals of another species. There is ground for supposing that all is solid at its surface, for it appears in our powerful telescopes as an arid mass, on which some have thought they could perceive the effects, and even the explosions of volcanoes.

Bouguer has found by experiment that the light of the full moon (*f*) is three hundred thousand times more feeble than that of the sun; this is the reason why this light, collected in the focus of the most powerful mirrors, produces no sensible effect on the thermometer.

We may distinguish, especially near to the new moons, that part of the disk of the moon which is not illuminated by the sun. This feeble light, which has been termed the *lumiere cendree*, is supposed to be the effect of the light which the illuminated hemisphere of the earth reflects on the (*g*) moon; and that which confirms this supposition is the circumstance of this light being most sensible near to the new moon when the greatest part of this hemisphere is directed towards this star. In fact, it is evident that the earth exhibits to a spectator at the moon, phases similar to those which the moon presents to us, but accompanied with a much stronger light, on account of the greater extent of the earth's surface.

The disk of the moon presents a great number of invariable spots, which have been carefully observed and described. They prove to us that this star always presents to us the same hemisphere; it revolves on its axis in a period equal to that of its revolution about the earth; for if a spectator be placed at the centre of the moon, supposed transparent, he will perceive the earth and his visual ray to revolve about him; and as this ray transverses always the same point of the moon's surface very nearly, it is evident that this point must revolve in the same time, and in the same direction as the earth about the spectator.

Nevertheless, a continued observation of the moon's disk indicates slight varieties in its appearances; spots are observed to approach and recede alternately from its borders; those which are very near to the borders, disappear and reappear successively, making periodical oscillations, which have been denominated *the libration of the moon.* In order to form a precise idea of the principal causes of this phenomenon, it should be considered that the disk of the moon, as seen from the centre of the earth, is terminated by the circumference of a circle of the lunar globe, which is perpendicular to its radius (h) vector, it is on the plane of this circle that the hemisphere of the moon, which is directed towards the earth, is projected, the appearances of which are connected with the motion of rotation of this star. If the moon had no motion of rotation, its radius vector would describe on its surface, in each lunar revolution, the circumference of a great circle, all the parts of which would be successively presented to us. But at the same time that the radius vector tends to describe this circumference, the lunar globe, by revolving, brings always very nearly the same point of its surface to this radius, and consequently the same hemisphere to the earth. The inequalities of the motion of the moon produce slight changes in its appearances; for as its motion of rotation does not participate in a sensible manner in these inequalities, it is variable relative to its radius vector, which thus meets its surface in different points; therefore the lunar globe makes, relatively to this radius, oscillations which correspond to the inequalities of its motion, and which alternately deprive us of and exhibit to us some parts of its surface.

Moreover, the lunar globe has another libration perpendicular to the preceding; in consequence of which the regions (*i*) which are situated near to the poles of rotation alternately disappear and reappear. In order to conceive this phenomenon, let the axis of rotation, be supposed perpendicular to the plane of the ecliptic. When the moon is in the ascending node, its two poles will be in the southern and northern extremities of the visible hemisphere. According as it ascends above the ecliptic, the north pole and those parts which are contiguous to it, disappear, whilst more and more of those parts which border on the south pole are discovered, until the moon having attained its greatest northern latitude, recommences to descend towards the ecliptic. The preceding phenomena then takes place in a reverse order; and after that the moon, having arrived at the descending node, is depressed below the ecliptic, the north pole will present precisely the same phenomena as the south pole had previously exhibited.

The axis of rotation of the moon is not exactly perpendicular to the plane of the ecliptic, and its inclination produces appearances which may be conceived by supposing the moon to move on the plane itself of the ecliptic, in such a manner that its axis of rotation remains always parallel to itself. It is manifest that then each pole will be visible during one half of the revolution of the moon about the earth, and invisible during the other half, so that those parts which are contiguous to the poles will be alternately perceived and concealed.

Finally, the observer is not at the centre, but at the surface of the earth; it is the visible ray drawn from his eye to the centre of the moon, which determines the middle of the visible hemisphere; and on account of the lunar parallax, it is evident that this radius intersects the surface of the moon in points which depend on the height of the moon above the horizon.

All these causes produce only an apparent libration in the lunar globe; they are purely optical, and do not affect the real motion of rotation. However, this motion may be subject to some small inequalities, though they are not sufficiently sensible to be discerned.

This is not the case with the variations of the plane of the lunar equator. From an attentive observation of the spots of the moon, Dominick Casini inferred that the axis of this equator is not exactly perpendicular to the plane of the ecliptic, as had been supposed previous to his time, and also that its successive positions are not exactly parallel. This celebrated astronomer was led to the following remarkable result, one of his most splendid discoveries, and which contains the entire astronomical theory of the real libration of the moon. If through the centre of this star a plane be conceived to pass perpendicular to its axis of rotation, which plane coincides with that of its equator; if moreover we conceive a second plane to pass through the same centre parallel to that of the ecliptic, and a third plane, which is the plane of the lunar orbit, abstracting from the periodic inequalities of its inclination and of the nodes, these three planes have invariably a common intersection; the second situated between the two others, makes with the first an angle of about 1°,67, and with the third an angle of 5°,7155; consequently the intersections of the lunar equator with the ecliptic, or its nodes, coincide always with the mean nodes of the lunar orbit, and like them they have a retrograde motion, of which the period is about $6793^{d},391081$. In this interval, the two poles of the equator and of the lunar orbit describe small circles parallel to the ecliptic, its pole being comprised between them, so that these three poles exist always on the same great circle of the celestial sphere.

There are mountains on the surface of the moon, which rise to a considerable height; their shadows projected on the planes, form spots which vary with the position of the sun. At the edges of the illuminated part of the lunar disk, these mountains present the form of an indented border, which extends beyond the line of light by a quantity of which the measurement proves that their height is at least three thousand metres. From the direction of these shadows it has been inferred that the surface of the moon is intersected by deep cavities, similar to the basons of our seas. Finally, this surface seems to shew traces of volcanoes; and the formation of new spots, and the sparks which are observed in its obscure part appear to indicate (*k*) volcanoes in actual operation.

CHAP. V.

Of the Planets, and in particular of Mercury and of Vènus.

In the midst of the infinite number of shining points which are spread over the celestial vault, and of which the relative position is very nearly constant, ten stars, always visible, except when they are immersed in the rays of the sun, move according to very complicated laws, the investigation of which constitutes one of the principal objects of astronomy. These stars, which have been denominated *planets*, are, Mercury, Venus, Mars, Jupiter, and Saturn, which have been known from the remotest antiquity, because they can be observed by the naked eye; and likewise Uranus, Ceres, Pallas, Juno, and Vesta, which have been recently discovered by means of the telescope. The two first planets never recede from the sun beyond certain limits; the others are elongated from it to all possible angular distances. The motions of all these bodies are comprehended in a zone of the celestial sphere, which is called the *zodiac*, the breadth of which is divided into two equal parts by the ecliptic.

The elongation of Mercury from the sun never exceeds thirty-two degrees: when it begins to appear in the evening, it is distinguished with difficulty in the rays of twilight; it extricates itself more and more on the succeeding days; and after it is elongated from the sun about twenty-five degrees, it returns towards him again. In this interval, the motion of Mercury, with respect to the stars, is direct; but when in approaching the sun, its distance from this star is about twenty degrees; it appears stationary, (*l*) and afterwards the motion becomes retrograde. Mercury still continues to approach the sun, and at length in the evening is again immersed in his rays. After continuing for some time invisible, it is again seen in the morning, emerging from the sun's rays, and receding from him. Its motion is still retrograde, as it was previous to the disappearance; but when the planet is twenty degrees elongated from the sun, it becomes again stationary, and afterwards resumes a direct motion; its elongation from the sun continues to increase till it becomes equal to twenty-five degrees, when the planet returns again, disappearing in the morning in the light of the dawn, and very soon after appearing in the evening, after which the same phenomena as before take place.

The extent of the greatest digressions of Mercury, or of his greatest deviations on each side of the sun, varies from eighteen to about thirty-two degrees. The duration of its total oscillations, (*m*) or of its return to the same position relatively to the sun, varies in like manner from one hundred and six, to one hundred and thirty days. The mean arc of retrogradation is about fifteen degrees, and its mean duration is twenty-three days; but these quantities differ considerably in different retrogradations. In general, the motion of Mercury is extremely complicated; it does not take place exactly in the plane of the ecliptic; some time this planet deviates five degrees from it.

A long series of observations was no doubt required to enable us to recognize the identity of the two stars, which were alternately observed in the morning, and in the evening, to recede from, and approach to the sun; but as the one was never seen until the other was invisible, it was at last concluded that it was the same planet which oscillated on each side of the sun.

The apparent diameter of Mercury is very variable, and its changes are evidently connected with its position relatively to the sun, and with the direction of its motion. It is a *minimum*, either when the planet in the morning is immersed in the sun's rays, or when in the evening it is disengaged from them. It is at its *maximum*, when in the evening it is immersed in these rays, or when it disengages itself from them in the morning. The mean apparent diameter is about 21″, 3.

Sometimes during the interval of its disappearing in the evening, and its re-appearing in the morning, the planet is seen projected in the form of a black spot on the disk of the sun, of which it describes a chord. It is recognized by its position, by its apparent diameter, and by its retrograde motion, being exactly those which it ought to have. These transits of Mercury are real annular eclipses of the sun, which prove to us that this planet derives its light from the sun. When seen through a powerful telescope, it exhibits phases analagous to those of the moon, and, like to them, directed towards the sun, the variable extent of which, according to the position of the planet with respect to the sun, and according to the direction of its motion, throws great light on the nature of its orbit.

The planet Venus exhibits the same phenomena as Mercury, with this difference, that its phases are much more sensible, its oscillations more extensive, and their duration more considerable. The greatest digressions of Venus vary from about fifty to fifty-three degrees; and the mean duration of its oscillations, or of its return to the same position with respect to the sun, is about five hundred and eighty-four days. The retrogradation commences, or terminates, when the planet, approaching to the sun in the evening, or receding from him in the morning, is elongated from this star about thirty-two degrees. The arc of retrogradation is eighteen degrees very nearly, and its mean duration is forty-two days. Venus does not exactly move in the plane of the ecliptic, but sometimes deviates from it several degrees.

The durations of the passages of Venus over the disk, observed at places which are at considerable distances from each other on the surface of the earth, are very sensibly different, for the same cause which (*n*) makes the durations of the same eclipse of the sun different in different places. In consequence of the parallax of this planet, different spectators refer it to different points of this disk, of which they observe it to describe chords of different lengths.

In the transit, which took place in 1769, the difference of its duration, as observed at Otaheite in the South Sea, and at Cajanibourgh in (*o*) Swedish Lapland, amounted to more than fifteen minutes. As these durations may be determined with very great exactness, their differences determine very accurately the parallax of Venus, and consequently its distance from the earth at the moment of its conjunction. A remarkable law, which we (*p*) shall explain at the end of our account of the discoveries which have made it known, connects this parallax with that of the sun and of all the planets; which circumstance renders these transits of peculiar importance in astronomy. After (*q*) succeeding each other in an interval of eight years, they do not recur again for more than a century, when they again succeed each other in the short interval of eight years, and so on continually. The two last transits happened on the fifth of June, 1761, and on the third of June 1769. Astronomers were sent to different places where the observations could be made under circumstances the most favourable for observing them, and from the result of their observations it has been concluded, that the parallax of the sun is about 26″,54 at its mean distance from the earth. The two next transits will take place on the eighth of December, 1874, and on the sixth of December, 1882.

The great variations of the apparent diameter of Venus, prove that its distance from the earth is very variable. This distance is least when it passes over the disk of the sun, and the apparent diameter is then about 189″: the mean magnitude of this diameter is, according to Arrago, about 52″,173.

From the motion of some spots observed on this planet, Dominique Cassini concluded that it revolves in an interval somewhat less than a day. Schroeter, by a continued observation of the variations of its horns, and by that of some luminous points near to the borders of those parts which are not illuminated, has confirmed this result, relative to which some doubts were entertained. He has determined the duration of its rotation to be $0^d,973$; and he has found, with Dominique Cassini, that the equator of Venus makes a considerable angle with the ecliptic. (*r*) Finally, he has inferred from his observations that mountains of a considerable height exist on its surface; and from the law of the degradation of light in the passage from the obscure to the enlightened part, he inferred that the planet is surrounded by an extensive atmosphere, of which the refracting power does not differ much from that of the earth's atmosphere. The great difficulty of observing these phenomena even in the most powerful telescopes, makes it a matter of extreme delicacy to observe them in our climate: they demand every attention from those astronomers who, from their southern situation, enjoy a more favourable climate. But it is very important, when the impressions are so feeble, to guard against the effects of imagination, which may considerably influence them; for then the interior images which it suggests, frequently modify and change those which the contemplation of objects produce.

Venus surpasses in brightness all the other stars and planets; it is sometimes so brilliant as to be seen in full daylight, and with the naked eye. This phenomenon, which depends on the return of the planet to the same position with respect to the sun, recurs in the interval of nineteen months very nearly, and its greatest brightness returns after an interval of eight years. Although it is of such frequent recurrence, it invariably excites surprise in the minds of the vulgar, who in their credulous ignorance, always suppose that it is connected with the most remarkable cotemporary events.

CHAP. VI.

Of Mars.

The two planets which we have just considered, seem to accompany the sun, like satellites; and their mean motion about the earth is the same as that of this star. The other planets recede from the sun, to all possible angular distances, but their motions are so connected with that of the sun, that there can be no doubt of his influence on these motions.

Mars appears to us to move from west to east about the earth; the mean duration of his sidereal revolution is 687 days, very nearly; that of his synodic revolution, or of his return to the same position, relatively to the sun, is about 780 days. Its motion is very unequal; when it begins to be visible in the morning, the motion is direct and most rapid; it becomes gradually *(s)* slower, and vanishes when the angular distance of the planet from the sun is about 152°; afterwards the motion becomes retrograde, increasing in velocity till the moment of opposition of Mars with the sun. This velocity having then attained its *maximum*, diminishes, and again vanishes, when Mars in approaching to the sun, is distant from it by 152°. The motion after this becomes again direct, having been retrograde for the space of seventy-three days, and in this interval the arc of retrogradation described by the planet is about eighteen degrees; continuing still to approach the sun, it finally is immersed in the evening in its rays. These remarkable phenomena are renewed at every opposition of Mars with the sun, but with a considerable difference as to the extent and duration of the retrogradations.

Mars does not move exactly in the plane of the ecliptic: it deviates sometimes several degrees from it. The variations of its apparent diameter are very great; it is about 19″,40 at the mean distance of the planet, and increases with the approach of the planet to opposition, where it amounts to 56″,43. At this time the parallax of Mars becomes sensible, and is nearly double of that of the sun. The same law which subsists between the parallaxes of the sun and of Venus, obtains also between the parallaxes of the sun and of Mars, and the observation of this last parallax had furnished a very near approximation of the solar parallax, before that it was determined with greater precision by the transits of Venus.

The disk of Mars is observed to change its form, and to become sensibly oval, according to its position relatively to the (*t*) sun. These phases shew that it receives its light from the sun. From the spots which are observed on its surface, it has been inferred that it revolves in a period of 1^d,02733, on an axis inclined to the ecliptic in an angle of 66°,33. Its diameter in the (*u*) direction of the poles, is somewhat less than the equatorial diameter. According to the measures of Arrago, these two diameters are in the ratio of 189 to 194, the preceding diameter being the mean between these two.

CHAP. VII.

Of Jupiter, and of his Satellites.

JUPITER moves from west to east in a period of $4332^d,6$ very nearly, the duration of his synodic revolution is about 399^d. It is subject to inequalities similar to those of Mars. Previous to the opposition of this planet to the sun, and when its elongation from this star is almost one hundred and twenty-eight degrees, its motion becomes retrograde, its velocity increases till the moment of opposition, it then diminishes, vanishes, and finally resumes the direct state, when the distance of the planet as it approaches the sun, is only one hundred and twenty-eight degrees. The duration of this retrograde motion is one hundred and twenty-one days, and the arc of retrogradation is about eleven degrees; but there are very perceptible differences in the extent and in the durations of the different retrogradations of Jupiter. The motion of this planet does not exactly take place in the plane of the ecliptic; it sometimes deviates from it three or four degrees.

Several obscure belts have been observed on the surface of Jupiter; they are apparently parallel to each other, and to the ecliptic; other spots have also been observed, the motion of which has indicated the rotation of this planet from west to east, on an axis very nearly perpendicular, to the plane of the ecliptic, and in a period (*v*) of 0^d,41377. From the variations of some of these spots, and from the marked differences in the durations of the rotation, as inferred from their motions, it has been supposed that these spots do not adhere to the surface of Jupiter; they appear to be clouds which the winds transport with different velocities in a very agitated atmosphere.

Jupiter is, after Venus, the most brilliant of the planets, and even sometimes surpasses it in brightness. Its apparent diameter is the greatest possible in the oppositions, when it amounts to 141″,6, its mean magnitude is 113″,4, in the direction of the equator; but it is not the same in every direction. The planet is evidently compressed at the poles of rotation, and Arrago found, by very accurate measurement, that the polar is to the equatorial diameter, in the ratio of 167 to 177 very nearly.

Four small stars are observed to revolve about Jupiter, and to accompany this planet constantly. Their relative position changes every instant; they oscillate on each side of this planet, and it is by the extent of these oscillations, that their order is determined; we term the *first* satellite, that of which the oscillation is the least. They are sometimes observed to pass over the disk of Jupiter, and to project on it their shadow, which then describes a chord of the disk. It follows from this, that Jupiter and his satellites are opaque bodies, illuminated by the sun; and when they interpose between the sun and Jupiter, they produce real eclipses of the sun, precisely similar to those which the moon causes on the earth.

The shadow which Jupiter projects behind him, with respect to the sun, enables us to explain another phenomenon which the satillites present. They are observed frequently to disappear, although at a considerable distance from the disk of the planet: the third and fourth satillites reappear sometimes at the same side of this disk. These disappearances are altogether similar to the eclipses of the moon, and indeed all doubt on this head is removed by the concomitant circumstances. The satellites are always observed to disappear on the side of the disk of Jupiter which is opposite to the sun, and consequently on the same side with that to which the shadow of the cone is projected. The eclipse takes place nearest to the disk, when the planet is nearest to its opposition; and finally, the duration of these eclipses corresponds exactly to the time which they should employ in traversing the cone of the shadow of Jupiter. Consequently these satellites move from west to east about this planet.

The observation of their eclipses furnish the most exact means of determining their motions. The durations of their periodical and synodical revolutions (*w*) about this planet are very precisely obtained, by comparing together eclipses which are separated from each other by considerable intervals, and which are observed near to the opposition of this planet. By this means it has been ascertained that the motion of the satellites of Jupiter is almost circular and uniform, because this hypothesis satisfies very nearly those eclipses in which the planet is observed in the same position, with respect to the sun; therefore the position of these satellites, as seen from the centre of Jupiter, may be always determined.

Hence results a simple and tolerably exact method of comparing together the distances of Jupiter and the sun from the earth, a method which the antient astronomers did not possess; for the parallax of Jupiter, when nearest to us, is insensible even to the precision of modern observations; they had no data from which that distance could be judged of, except the duration of their revolutions, these planets being supposed to be most remote, the durations of whose revolutions were longest.

Let us suppose that the entire duration of an eclipse of the third satellite has been observed. At the middle of the eclipse, the satellite, as seen from the centre of Jupiter, is very nearly in opposition to the sun; therefore its sidereal position such as would be observed from this centre, and which it is easy to infer from the mean motion of Jupiter and of the satellite, is then the same as that of the centre of Jupiter seen from that of the sun. The position of the earth, as seen from the centre of the sun, may be had either from direct observation, or from the known motion of this star; therefore supposing a triangle to be formed by lines joigning the centres of the earth, of the sun and of Jupiter, the angle at the sun will be obtained by what precedes; the angle at the earth will be given by direct observation; therefore at the middle of the eclipse the rectilinear distances of Jupiter from the earth and from the sun will be given in parts of the distance of the sun from the earth. It is found by this means, that when the apparent diameter of Jupiter is about 113″,4, he is at least five times more remote from us than the sun. The diameter of the earth would only appear under an angle of 10″,4, at the same distance; therefore the volume of Jupiter is at least one thousand times greater than that of the earth.

The apparent diameters of the satellites being insensible, their magnitudes cannot be measured exactly. An attempt has been made to estimate them, by the time which they take in penetrating into the shadow of the planet; but there is a great discordance in the observations which have been made to ascertain this circumstance, which arise from the different powers of the telescope, from the different degrees of perfection in the sight of the observers, from the state of the atmosphere, the heights of these satellites above the horizon, their apparent distance from Jupiter, and the change of the hemispheres which they present to us. The comparative brightness of the satellites is independent of the four first causes, which only produces a proportional change in their light; it may therefore furnish some information concerning the return of the spots, which the rotation of these bodies ought to present successively to the earth, and consequently on the rotation itself. Herschell, who has been occupied with this delicate investigation, observed that they surpass each other successively in splendor, a circumstance that enables us to judge of the *maximum* and of the *minimum* of their light; and from a comparison of these maxima and minima, with the mutual positions of these stars, he has ascertained that they revolve on themselves, like the moon, in a period equal to the duration of their revolutions round Jupiter, a result which Maraldi had concluded to obtain in the case of the fourth satellite, from the returns of the same spot observed on his disk in its passages over the planet. The great distance of the heavenly bodies renders the phenomena which their surfaces present so extremely feeble, that they are reduced to slight variations of light, which cannot be perceived at the first view, and it is only after frequent experience in this kind of observation, that they become perceptible. But this means of supplying the imperfection of our organs, over which imagination has such control, ought to be employed with the greatest circumspection, to avoid being deceived respecting the existence of those varieties, and also lest we should be bewildered as to the causes on which they depend.

CHAP. VIII.

Of Saturn, of his Satellites, and of his ring.

Saturn revolves from west to east, in a period of 10759 days: the duration of his synodical revolution is 378 days. Its motion, which is performed very nearly in the plane of the ecliptic, is subject to inequalities similar to those of the motions of Mars and of Jupiter. Its retrograde motion commences and terminates when the distance of the planet from the sun before and after opposition is 121°: the duration of this retrogradation is about one hundred and thirty-nine days, and the arc of its retrogradation is about seven degrees. At the moment of opposition, the diameter of Saturn is at its *maximum*: its mean magnitude is about 50″.

Saturn presents a phenomenon which is *unique* in the system of the world. It is frequently observed in the middle of two small bodies which seem to adhere to it, the figure and magnitude of which are very variable; sometimes they are changed into a ring, which seems to surround the planet; at other times they disappear altogether, and Saturn then appears round like the other planets. By carefully following these remarkable appearances, and by combining them with the positions of Saturn relatively to the sun and to the earth, Huygens ascertained that they are produced by a large and slender ring which surrounds the globe of Saturn, and is every where detached from it. This ring being inclined at an angle of 31°,85 to the plane of the ecliptic, always presents itself obliquely to the earth, in the form of an ellipse, of which the length when a maximum, is very nearly double the breadth. The ellipse becomes narrower in proportion as the visual ray drawn from Saturn to the earth, becomes less inclined to the plane of the ring, of which the more remote arc is at length concealed behind the planet, while the anterior arc is confounded with it; but its shadow, projected on the disk of Saturn, forms an obscure band, which being perceived in powerful telescopes, proves that Saturn and his ring are opaque bodies illuminated by the sun. We then only distinguish those parts of the rings which are extended on each side of Saturn; the breadth of these parts diminishes gradually, and they finally disappear, when the earth is in the plane of the ring, the thickness of which is imperceptible. The ring is likewise invisible when the sun being in its plane, only illuminates its thickness. It continues to be invisible as long as its plane is between the sun and earth, (*z*) and it reappears when the sun and earth are on the same side of this plane, in consequence of the respective motions of the sun and of Saturn.

As the plane of the ring meets the solar orbit at every semirevolution of Saturn; the phenomena of the disappearance and reappearance recur very nearly after the interval fifteen years, but frequently under very different circumstances: two disappearances and two reappearances may occur in the same year, but never more.

During the disappearance of the ring, its thickness reflects to us the light of the sun, but in too small a quantity to be perceptible. However it may be conceived that by increasing the power of the telescope, it might be seen; and this is in fact what Herschell experienced during the last disappearance of the ring—which continued visible to him, when it had disappeared to other observers.

The inclination of the ring to the plane of the ecliptic is measured by the greatest opening which the ellipse presents to us: the position of its nodes with the plane of the ecliptic, is easily determined from the position of Saturn, when the appearance or disappearance of the ring, depends on the meeting of its plane with the earth. Therefore all the phenomena of this kind, which determine the same sidereal position of the nodes, take place when this plane meets the earth. When this plane passes through the sun, the position of its nodes determine that of Saturn, as seen from the centre of the sun, and then the rectilinear distance of Saturn from the earth, may be determined in the same manner as the distance of Jupiter is determined from the eclipses of his satellites. In the triangle formed by the three lines which join the centres of the sun, of Saturn, and of the earth, the angles at the earth and sun are given, hence it is easy to conclude the distance of the sun from Saturn, in parts of the radius of the solar orbit. It is thus found that Saturn is about nine times and a half farther from us than the sun, when his apparent diameter is 50″.

The apparent diameter of the ring, at its mean distance from the planet is, according to the accurate measures of Arrago, equal to 118″,58; its apparent breadth is 17″,858. Its surface is not continuous; a black band, which is concentrical with it, divides it into two parts, which appear to form two distinct rings, the breadth of the exterior being less than that of the interior. From several black bands which have been observed by some astronomers, it would appear, that there is a greater number of these rings. From the observation of some luminous spots of the ring, Herschell has ascertained that it revolves from west to east in a period of 0^{d},437, about an axis which is perpendicular to its plane, and passing through the centre of Saturn.

Seven satellites have been observed to revolve round this planet from west to east, in orbits nearly circular. The six first move very nearly in the plane of the ring: the orbit of the seventh approaches more to the plane of the ecliptic. When this satellite is to the east of Saturn, its light becomes so feeble, that it is with very great difficulty perceived; this can only arise from the spots which cover the hemisphere which is presented to us. But in order that this phenomenon should occur always in the same position, it is necessary that this satellite, (in this respect similar to the moon, and to the satellites of Jupiter,) should revolve on its own axis, in a period equal to that of its revolution about Saturn. Thus an equality between the periods of rotation and revolution appears to be a general law of the motion of the satellites.

The diameters of Saturn are not equal to each other. The diameter which is perpendicular to the plane of the ring, appears less by the eleventh part at least, than that which is situated in this plane. From a comparison of this compression with that of Jupiter, it may be inferred with great probability, that Saturn revolves rapidly about the least of his diameters, and that the ring revolves in the plane of his equator; this result has been confirmed by the direct observations of Herchell, which have indicated to him that the motion of this planet, like that of the other celestial bodies, is from west to east, and that its duration is 0,428, which differs very little from the duration of Jupiter's rotation. It is remarkable that this duration is very nearly the same, and less than half a day, for the two largest planets, while the planets which are less than them, revolve on their axes in the interval of a day very nearly.

Herchell has also observed on the surface of Saturn five belts, which are nearly parallel to his equator.

CHAP. IX.

Of Uranus and of his Satellites.

THE planet Uranus escaped the observation of the ancient Astronomers on account of its minuteness. Flamstead at the end of the last century, Mayer and Le Monnier in the present, had already observed it as a small star. But it was not till 1781 that Herchell recognised its motion, and shortly after, by following this star carefully, he ascertained that it is an actual planet. Like to Mars, Jupiter and Saturn, Uranus moves from west to east about the earth. The duration of its sidereal revolution is about 30687 days; its motion, which takes place very nearly in the plane of the ecliptic, commences to be retrograde previous to its opposition, when the distance of the planet from the sun is 115°; its retrograde motion terminates, after opposition, when the elongation of the planet from the sun, as it approaches to this star, is 115°. The duration of its retrogradation is about 151 days, and the arc of retrogradation is four degrees.

If the distance of Uranus was to be estimated from the slowness of its motion, it should be on the confines of the planetary system. Its apparent diameter is very small, and hardly amounts to twelve seconds. According to Herchel six satellites revolve about this planet in orbits almost circular, and very nearly perpendicular to the plan of the ecliptic. Telescopes of a very high magnifying power are required to enable us to perceive them ; two only, the second and fourth, have been recognized by other observers. The observations which Herchell has published relative to the four others, are not sufficiently numerous to enable us to determine the elements of their orbits, or even to be assured incontrovertably of their existence (*a*).

CHAP. X.

Of the Telescopic planets, Ceres, Pallas, Juno and Vesta.

THESE four planets are so minute, that they can be only perceived by means of very powerful telescopes. The first day of the present century is remarkable for the discovery which Piazzi made at Palermo of the planet Ceres. Pallas was recognized in 1802, by Olbers; Juno was discovered in 1803 by Harding; and lastly, Vesta was perceived in 1807 by Olbers. These stars, like the other planets, move from west to east; and like to them, they are alternately direct and retrograde. But in consequence of the short time which has elapsed since their discovery, we have not been able to determine with precision, the durations of their revolutions, and the laws of their motions. We only know that the durations of their sidereal revolutions differ little from each other; and that those of the three first are about four years and two thirds: the duration of the revolution of Vesta appears to be shorter by a year. Pallas deviates considerably more from the plane of the ecliptic than the other planets, so that in order to comprize its deviations, we should enlarge considerably the breath of the zodiac (*b*).

CHAP. XI.

Of the motion of the Planets about the sun.

Had man restricted himself to a mere compilation of facts, the sciences would present nothing but a barren nomenclature, and a knowledge of the great laws of nature would never have been attained. It is from a comparison of facts with each other, by attentively considering their relations, and by this means reascending to phenomena, which are continually more and more extensive, that at length we have been enabled to discover these laws, which are continually impressed on the various effects which they produce. Then it is, that nature by revealing herself, shews how the infinite variety of phenomena which have been observed, may be traced up to a small number of causes, and thus enables us to determine antecedently those effects, which ought to be produced; and being assured that nothing will derange the connexion between causes and their effects, we can extend our thoughts forwards to the future, and the series of events which shall be developed in the course of time, will be presented to our view. It is solely in the theory of the system of the world, that the human mind has, by a long train of successful efforts, attained to this eminence. The first hypothesis which was devised to explain the phenomena of the planetary motions, could only be an imperfect sketch of this theory, but by representing these phenomena in a very ingenious manner, it furnished the means of subjecting them to the calculus; and we shall now see, that by making this hypothesis to undergo the modifications which have been successively indicated by observation, it will be changed into the true system of the world.

The most remarkable of the planetary appearances is their change from a direct to a retrograde motion, a change which can only arise from two motions alternately conspiring together, and opposing their effects. The most natural hypothesis for explaining them, was that devised by the ancient philosophers, and which consisted in making the three superior planets to move in consequentia on epicycles, of which the centres describe circles in the same direction. It is manifest, that if the planet be supposed to exist in the lowest point of the epicycle, or that which is nearest to the earth, it has in this position a motion contrary to that of the epicycle, which is always moved parallel to itself; therefore if the first of these motions be supposed to predominate over the second, the apparent motion of the planet will be retrograde, and at its maximum; on the contrary, if the planet be situated at the most elevated point of its epicycle, the two motions conspire together, and the apparent motion is direct, and the greatest possible. In proceeding from the first to the second of these positions, the apparent motion of the planet continues to be retrograde; however, it constantly diminishes, till at length it vanishes, and then changes into a direct motion. It appears from observation, that the *maximum* of the retrograde motion obtains always at the moment of the opposition of the planet with the sun; it therefore follows that each epicycle is described in the time of a revolution of this star, and that the planet is at the lowest point, when it is in opposition to the sun. Hence we may see the reason why the apparent diameter of the planet is then at its *maximum*. With respect to the two inferior planets, which never deviate from the sun beyond certain limits, their alternate retrograde and direct motions may likewise be explained, on the hypothesis that they move in consequentia on epicyles, of which the centres describe, each year, circles about the earth in the same direction; and by supposing likewise that when the planet attains the lowest point of its epicycle, it is in conjunction with the sun. The preceding is the most ancient astronomical hypothesis, which being adopted and brought to perfection by Ptolemy, has been denominated from this astronomer.

The absolute magnitudes of the circles and of the epicycles are not indicated in this hypothesis: the phenomena only assign the relative magnitudes of the radii. In like manner Ptolemy did not attempt to investigate the respective distances of the planets from the earth; he only supposed those superior planets to be farther from the earth, of which the times of revolution were the longest. He then placed the epicycle of Venus below the sun, and that of Mercury the lowest of all. In an hypothesis so indeterminate, it does not appear why the arcs of retrogradation of the superior planets are smaller, for those which are most remote; and why the moveable radii of the superior epicycles are parallel, to the radius vector of this star, and to the moveable radii of the inferior circles. This parallelism, which Kepler had already introduced into the hypothesis of Ptolemy, is clearly indicated by all observations of the motion of the planets, parallel and also in a direction perpendicular to the ecliptic. But if these epicycles and circles be supposed equal to the orbit of the sun, the cause of these phenomena become immediately apparent. It is easy to be satisfied that by such a modification of the preceding hypothesis, all the planets are made to revolve about the sun, which in his real or apparent motion about the earth carries along with it the centres of their orbits. A disposition of the planetary system so simple, leaves nothing undetermined, and clearly points out, the relations of the direct and retrograde motions of the planets, with the motion of the sun. It removes from the hypothesis of Ptolemy, the circles and epicycles which are described annually by these planets, and likewise those which he introduced in order to explain their motions perpendicular to the ecliptic. The relations which this astronomer had determined to exist between the radii of the two inferior epicycles, and the radii of the circles described by their centres, express then the mean distances of the planets from the sun in parts of the mean distance of the sun from the earth; and the same relations being reversed for the superior planets, express their mean distances from the sun or from the earth. The simplicity of this hypothesis should of itself, induce us to admit it; but the observations which have been made by means of the telescope, remove all doubts on this subject.

It has been already observed, how the distance of Jupiter from the sun may be determined by the eclipses of the satellites of this planet, from which it appears that it describes about the sun, an orbit almost circular. We have also seen, that the appearances and disappearances of the ring of Saturn determine its distance from the earth to be about nine times and a half greater than the distance of the earth from the sun; and according to the determination of Ptolemy, this is very nearly the relation which obtains between the radius of the orbit of Saturn, and the radius of its epicycle; hence it follows that this epicycle is equal to the solar orbit, and that consequently Saturn describes very nearly a circle about the sun. From the phases which have been observed in the two inferior planets, it follows that they revolve about the sun. Let us for example follow the motion of Venus, and the variations of its apparent diameter and of its phases. When in the morning it commences to extricate itself from the rays of the sun, it appears before the rising of this star, under the form of a crescent, and its apparent diameter is a *maximum;* it is then nearer to us than to the sun, and very nearly in conjunction with it. Its crescent increases, and its apparent diameter diminishes according as the planet elongates itself from the sun. When its angular distance from this star is about fifty degrees, it approaches towards it again, exhibiting to us more and more of its illuminated hemisphere: and the diminution of the apparent diameter continues to the moment, that in the morning it is immersed in the sun's rays. At this intant, Venus appears to us full, and its apparent diameter is a *minimum;* in this position it is farther from us than the sun. After continuing invisible for some time, this planet appears again in the evening, and reproduces in an inverted order, the phenomena which it exhibited previous to its disappearance. More and more of its illuminated hemisphere is averted from the earth: its phases diminish, and at the same time its apparent diameter increases with its increased elongation from the sun. When its angular distance from this star is about fifty degrees, it returns towards him: its phases continue to diminish, and its apparent diameter to increase, till it is again immersed in the rays of the sun. Sometimes in the interval between its disappearance in the evening, and its appearance in the morning, it is observed to move on the disk of the sun, in the form of a spot. It is clear from these phenomena, that the sun is very nearly in the centre of the orbit of Venus, which it carries along with it, while it revolves about the earth. As Mercury exhibits phenomena which are similar to those of Venus, it follows that the sun is likewise in the centre of its orbit.

We are therefore conducted by the phenomena of the motions and of the phases of the planets, to this general result, namely, *that all these stars revolve about the sun, which in his real or apparent revolution about the earth, appears to carry with it the foci of their orbits.* It is remarkable that this result is derived from the hypothesis of Ptolemy, by supposing the solar orbit to be equal to the circles and epicycles which are described each year, in this hypothesis, which then ceases to be purely ideal, and only proper to represent to the imagination, the celestial motions. Instead of making the planets to revolve about imaginary centres, it places in the foci of their orbits, those great bodies which by their action can retain them in these orbits, and by this means it enables us to get a glimpse of the causes of the heavenly motions.

CHAP. XII.

Of the Comets.

Stars are frequently observed, which though at first scarcely perceptible, increase in magnitude and velocity, then diminish, and finally disappear. These stars, which are called *comets*, appear almost always accompanied with a nebulosity, which increasing, terminates sometimes in a tail of considerable length, and which must be extremely rare, as the stars are seen through its immense depth. The appearance of the comets followed by these long trains of light, had for a long time terrified nations, who are always affected with extraordinary events, of which they know not the causes. The light of science has dissipated these vain terrors which comets, eclipses, and many other phenomena excited in the ages of ignorance.

The comets participate, like the other stars, in the diurnal motion of the heavens; and this, combined with the smallness of their parallax, proves that they are not meteors generated in our atmosphere. Their proper motions are extremely complicated; they have place in every direction, and are not restricted, like the planets, to a motion from west to east, and in planes very little inclined to the ecliptic.

CHAP. XIII

Of the Stars, and of their motions.

The parallax of the stars is insensible; (*c*) their disks, viewed through the most powerful telescopes, are reduced to luminous points; in this respect, these stars differ from planets, of which the apparent magnitude (*d*) is increased by the magnifying power of the telescope. The smallness of the apparent diameter of the stars is particularly evinced by their rapid disappearance in their occultations by the moon, the time of which, not amounting to a second, indicates that this diameter is less than five seconds of a degree. The vivacity of the light of the most brilliant stars compared with the smallness of their apparent disk, induces us to think that they are much farther from us than these planets, and that they do not, like them, borrow their light from the sun, but are themselves luminous; and as the smallest stars are subject to the same motions as the most brilliant, and preserve the same position relatively to each other; it is extremely probable that the nature of all these stars is the same, and that they are so many luminous bodies of different magnitudes; and situated at greater or less distances from the limits of the solar system.

Periodical variations have been observed in the intensity of the light of several stars, which have been termed on that account *changeable*. Sometimes stars have been observed to appear suddenly, and then to vanish, after having shone with the most brilliant splendor. Such was the famous star observed in 1572 in the constellation of Cassiopeia. In a short time, it surpassed the most beautiful stars, and even Jupiter himself in brilliancy. Its light afterwards grew feeble, and in sixteen months after its discovery it disappeared, without having changed its place in the heavens. Its colour experienced considerable variations : it was first of a dazzling white, afterwards of a reddish yellow, and lastly, of a lead coloured white. What is the cause of these phenomena ? The extensive spots which the stars present to us periodically, in their revolution on their axes, in the same manner very nearly as the last satellite of Saturn, and perhaps the interposition of great opaque bodies which revolve about them, are sufficient to explain the periodical variations of the changeable stars. As to those stars which suddenly shine forth with a very vivid light, and then immediately disappear, it is extremely probable that great conflagrations, produced by extraordinary causes, take place on their surface ; and this conjecture is confirmed by their change of colour, which is analogous to that which is presented to us on the earth by those bodies, which are set on fire, and then gradually extinguished.

A white light of an irregular figure, (*d*) which has been denominated the *milky way*, surrounds the heavens in the form of a zone. As a very great number of small stars has been discovered in it by means of the telescope, it is very probable that the milky way is nothing more than an assemblage of stars, which appear to us so near as to constitute an uninterrupted band of light. Small white spots, which are termed nebulæ, have also been observed in different parts of the heavens; several of which appear to be of the same nature as the milky way. When viewed through a telescope they likewise exhibit the union of a great mumber of stars; others only display a white and continuous light, perhaps on account of their great distance, which confounds the light of the stars which compose them. It is very probable that they are formed of a very rare nebulous matter, which is diffused in different masses in the heavenly regions, of which the successive condensation produces the nuclei, and all the varieties which they exhibit. The remarkable changes which have been observed in some of them, and particularly in the beautiful nebula of Orion, admit of a very easy explanation on this hypothesis, and render it extremely probable.

The immobility of the fixed stars with respect to each other, has determined astronomers to refer to them as to so many fixed points, the proper motions of the other heavenly bodies; but for this purpose it was necessary to classify them, in order that they might be recognized;

and it is with this view, that the heavens have been distributed into various groups of stars called constellations. It was likewise necessary to determine exactly the positions of the fixed stars on the celestial sphere, which has been accomplished in the following manner:

Let a great circle be conceived to pass through the two poles of the world, and through the centre of any star; this circle, which is termed the circle of declination, is perpendicular to the equator. The arc of this circle, comprised between the equator and the centre of the star, measures its declination, which is *north* or *south*, according to the denomination of the pole, to which it is nearest.

As all the stars situated in the same parallel have the same declination, it was necessary to introduce a new element in order to determine their position. The arc of the equator, comprised between the circle of declination and the vernal equinox, has been selected for this purpose. This arc, reckoned from the equinox in the direction of the proper motion of the sun, *i. e.* from west to east, is termed the *right ascension*, consequently, the position of the stars is determined by their right ascension and declination.

The distance from the equator, or the right ascension, is determined by the meridian altitude of the star compared with the height of the pole. The determinations of its right ascenstion presented greatər difficulties to the antient astrononers, on account of the impossibility of comparing directly the fixed stars with the sun. As the moon may be compared during the day with the sun, and during the night, with the fixed stars, they made use of it as an intermediate term, in order to measure the difference between the right ascension of the sun and of the fixed stars, having regard to the proper motions of the sun and moon, in the interval between the observations. The theory of the sun afterwards giving its right ascension, they inferred from it that of some of the principal stars, to which they compared the rest. It was by this means, that Hipparchus formed the first catalogue of fixed stars of which we have any knowledge. A considerable time after, this method was rendered much more precise, by employing, instead of the moon, the planet Venus, which is sometimes visible during the day, and of which during a short interval the motion is slower aud less unequal than the lunar motion. Now, that the important application of the pendulum to clocks, furnishes a very exact measure of time, we can determine directly, and with much greater precision than the ancient astronomers, the difference between the right ascenscion of the star and of the sun, by the interval of time which elapses between their transits over the meridian.

The position of the stars may be referred to the ecliptic in a similar manner, which is particularly useful in the theory of the moon and of the planets. A great circle is supposed to pass through the centre of the star, perpendicular to the plane of the ecliptic, which is called a circle of *latitude*. The arc of this circle comprised between the ecliptic and the star, measures its latitude, which is north or south, according to the denomination of the pole situated at the same side of the ecliptic. The arc of the ecliptic comprised between the circle of latitude and the vernal equinox, reckoned from this equinox, in the direction of the sun's proper motion *i, e*, from west to east, is called the *longitude* of the star, the position of which is thus determined by its longitude and latitude. It may be easily conceived that the inclination of the ecliptic to the equator being known, the longitude and latitude of a star may be deduced from its observed right ascension and declination.

An interval of only a few years, was necessary to observe the variation of the fixed stars in right ascension and declination. It was very soon remarked that while they changed their position with respect to the equator, they preserved the same latitude, from which it may be inferred that the variations in right ascension and declination, arise solely from a motion common to these stars about the poles of the ecliptic. These variations might also be represented by supposing the stars immoveable, and by making the poles of the equator to move about those of the ecliptic. In this motion the inclination of the equator to the ecliptic remains constant, and its nodes or equinoxes regrade uniformly, at the rate of 154″,63 for each year. It has been already remarked that this retrogradation of the equinoxes, renders the tropical somewhat shorter than the sidereal year. Thus the difference between the tropical and sidereal years, and the variations of the fixed stars in right ascension and declination, depend on this motion, by which the pole of the equator describes annually an arc of 154″,63 of a small circle of the celestial sphere parallel to the ecliptic. It is *(e)* in this, that the phenomenon known by the name of the precession of the equinoxes, consists.

The precision of modern astronomy, for which it is indebted to the application of telescopes, to astronomical instruments, and to that of the pendulum to clocks, has rendered perceptible, minute periodical variations in the inclination of the equator to the ecliptic, and in the precession of the equinoxes. Bradley, who discovered, and attentively followed them for several years, has observed their law, which may be geometrically represented in the following manner. Let the pole of the equator be supposed to move on the circumference of a small ellipse, a tangent to the celestial sphere, and of which the centre, which may be regarded as the mean pole of the equator, describes every year 154″,63 of the parallel to the ecliptic, on which it is situated. The greater axis of this ellipse, always in the plane of the circle of latitude, is equivalent to an arc of this great circle, equal to 59″,56 ; and the lesser axis is equivalent to an arc of this parallel, which is equal to 111″,30. The situation of the real pole of the equator on this ellipse is determined in the following manner: Suppose a small circle to be described in the plane of this ellipse, concentrical with it, and having its diameter equal to the greater axis; conceive also a radius of this circle moved uniformly with a retrograde motion, so that this radius may coincide with that half of the greater axis which is nearest to the ecliptic, every time that the ascending node of the moon's orbit, coincides with the vernal equinox; and lastly, from the extremity of this moveable radius let fall a perpendicular on the greater axis of the ellipse, the point where this perpendicular intersects the circumference of the ellipse is the place of the true pole of the equator. This motion of the pole is termed *nutation.*

The fixed stars, in consequence of the motions which we have described, preserve an invariable position relatively to each other; but the illustrious observer to whom we are indebted for the discovery of the nutation, has discovered in all the stars a general periodical motion, which produces a slight change in their respective positions. In order to represent this motion, each star is supposed to describe annually a small circumference parallel to the ecliptic, of which the centre is the mean position of the star, and of which the diameter, as seen from the earth, subtends an angle of 125″, and that it moves on this circumference like the sun in his orbit, in such a manner however, that the sun is always more advanced than the star, by one hundred degrees; this circumference, projected on the surface of the heavens, appears under the form of an ellipse more or less flattened according to the height of the star above the ecliptic; the lesser axis of the ellipse being to the greater axis, as the sine of this height is to the radius. Hence arise all the varieties of that periodical motion of the stars, which is called *abberration*.

Independently of those general motions, several stars have proper motions peculiar to themselves, very slow, but which the lapse of time has rendered sensible. They have been hitherto principally remarkable in Syrius and Arcturus, two of the most brilliant stars, but every thing induces us to think that in succeeding ages similar motions will be developed in the other stars.

CHAP. XIV.

Of the figure of the earth, of the variation of gravity at its surface, and of the decimal system of weights and measures.

Let us now descend from the heavens to the earth, and see what can be derived from observations relative to its dimensions and figure. It has been already observed that the earth is very nearly spherical: gravity being every where directed to the centre, retains bodies on its surface, although in places diametrically opposite, which are antipodes one to the other, they have directly contrary positions. The sky and the stars appear always above the earth; for elevation and depression are only relative terms with respect to the direction of gravity.

From the moment that man recognized the spherical form of the globe which he inhabits, he must have been anxious to measure its dimensions; it is therefore extremely probable that the first attempts to attain this object were made at a period long anterior to those of which history has preserved the record, and that they have been lost in the moral and physical changes which the earth has undergone. The relation which several measures of the most remote antiquity have to each other, and to the terrestrial circumference, gives countenance to this conjecture, and seems to indicate not only that this length was very exactly known at a very ancient period, but that it has also served as the base of a complete system of measures, the vestiges of which have been found in Asia and in Egypt. Be this as it may, the first precise measure of the earth, of which we have any certain knowledge, is that which Picard executed in France towards the end of the seventeenth century, and which has been repeatedly verified. It is easy to conceive this operation. As we advance towards the north, the pole seems to be elevated more and more; the meridian heights of the stars situated towards the north increases, and that of the southern stars diminishes; some of them even become invisible. The notion of the curvature of the earth was no doubt suggested by observing these phenomena, which could not fail to attract the attention of men in the first age of society, when the return of the seasons was only distinguished by the rising and setting of the principal stars, compared with that of the sun. The elevation or the depression of the stars makes known the angles, which verticals raised at the extremities of the arc of the earth, which has been passed over, make at the point where they meet; for this angle is evidently equal to the difference of the meridian heights of the same star, *minus* the angle which the arc described would subtend at the centre of the star; and we are certain that this last angle is insensible. It is then only necessary to measure this space. It would be a tedious and troublesome operation to apply our measures to so great an extent; it is much simpler to connect its extremities, by means of a series of triangles, with those of a base of twelve or fifteen thousand metres; and considering the precision with which the angles of these triangles may be determined, its length can be obtained very accurately. It is thus, that the arc of the terrestrial meridian which traverses France has been measured. The part of this arc of which the amplitude is the hundreth part of a right angle, and whose middle point corresponds to 50°, of altitude of the pole, is very nearly one hundred thousand metres.

Of all the re-entring figures, the spherical is the simplest, because it depends only on one element, namely, the magnitude of its radius. The natural inclination of the human mind to attribute that figure to objects, which it conceives with the greatest facility, disposed it to assign a spherical form to the earth. But the simplicity of nature should not be always regulated by that of our conceptions. Infinitely varied in her effects, nature is only simple in her causes, and her economy consists in producing a great number of phenomena, which are frequently very complicated, by means of a small number of general laws. The figure of the earth is the result of those laws, which modified by a thousand circumstances, might cause it to deviate sensibly from that of a sphere. Small variations, observed in the length of the degrees in France, indicate these deviations; but the inevitable errors of observation left doubts on this interesting phenomenon; and the Academy of Sciences, in which this interesting question was anxiously discussed, judged with reason, that the difference of degrees, if it really existed, would be principally evinced in a comparison of the degrees at the equator and towards the poles. And academicians were sent even to the equator itself, where they found the degree of the meridian less than the degree of France. Other academicians travelled towards the north, where the degree was observed to be greater than the degree in France. Thus the increase of the degrees of the meridian, from the equator to the poles, was proved incontrovertably by these measures, from which it was concluded that the earth was not exactly spherical.

These celebrated voyages of the French Academecians having directed the attention of astronomers towards this object, new degrees of the meridian were measured in Italy, Germany, Africa, India and Pennsylvania. All these measures concur in indicating an increase in the degrees, from the equator to the poles.

The following table exhibits the values of the extreme degrees which have been measured, and also of the mean degree between the equator and the pole. The first was measured in Peru, by Bouguer and La Condamine. The length of the second has been inferred from the great operation which was recently executed, in order to determine the amplitude of the arc, which traverses France from Dunkirk to Perpignan, and which has been extended to the south, as far as Formentera. It was joined towards the north with the meridian of Greenwich, by connecting the sides of France with those of England, by means of a series of triangles. This immense arc, which comprises the seventh part of the distance of the pole from the equator, has been determined with the greatest precision. The astronomical and geodesical observations have been made with repeating circles. Two bases, each of which is more than twelve thousand metres, have been measured, the one near Melun, the other near to Perpignan, by a new process, which is free from all uncertainty; and what confirms the accuracy of these observations is, that the base of Pepignan concluded from that of Melun, by the chain of triangles which unites them, does not differ by a third of a metre from its actual measure, although the distance between those two places is upwards of nine hundred thousand metres.

In order to render this important observation as perfect as possible, the height of the pole, and the number of ocillations performed in a day by the same pendulum, have been observed on different points of this arc; from which the variations of the degrees and of gravity have been inferred. Thus this operation, the most accurate and extensive of the kind, which has been undertaken, will remain a monument of the state of arts and sciences in this enlightened age. Lastly, the third degree was measured by M. Swanberg in Lapland.

Height of the pole.	*Length of the degree.*
0°,00	99523^{m},9.
50°,08	100004,3.
73°,71	100323,6.

The increase of the degrees of the meridian, according as the height of the pole increases, is even sensible in different parts of the great arc already mentioned. In fact let us compare its extreme points, and the Pantheon at Paris, which is one of the intermediate positions. It is found by means of observation,

	Height of the pole.	*Distances from Greenwich in the direction of the meridian.*
Greenwich	57,°19753	0^{m},0
Pantheon	54,°27431	292719,3
Formentera	42,°96178	1423636,1

The distance from Greenwich to the Pantheon, gives 100135^{m},2 for the length of the degree, of which the middle point corresponds to 55,°73592 of elevation of the pole; and from the distance of the Pantheon from Formentera, it is found that the length of a degree, the middle point of which corresponds to a latitude of 48,61804, is equal to 99970^{m},3, from which it follows that in the interval between these two points, the increment of a degree is 23^{m},167.

The ellipse being after the circle, the most simple of all the re-entring curves, the earth was considered as a solid of revolution formed by the revolution of an ellipse about its lesser axis. Its compression in the direction of the poles, is a necessary consequence of the observed increase of the meridional degrees, from the equator to the poles. The radii of these degrees being in the direction of gravity, they are by the laws of the equilibrium of fluids, perpendicular to the surface of the seas with which the earth, is in a great measure covered. They do not terminate, as in a sphere, in the centre of the ellipsoid; they have neither the same direction, nor the same magnitude, as radii drawn from the centre to the surface, and which cut it obliquely every where except at the equator and at the poles. The point where two adjoining verticals situated in the same meridian meet, is the centre of a small terestrial arc comprized between them; if this arc was a right line, these verticals would be parallel, *i. e.* they would meet at an infinite distance; but in proportion as they are curved, they meet at a distance which is proportionally less as the curvature is greater; thus the extremity of the lesser axis being the point where the ellipse approaches most to a right line, the radius of a degree of the pole, and consequently the degree itself, is of its greatest length. It is the contrary at the extremity of the greater axis of the ellipse, *i. e.* at the equator, where the curvature being the greatest, the degree in the direction of the meridian is least of all. In proceeding from the second to the first of these extremes, the degrees continually increase; and if the compression of the ellipse is inconsiderable, their increment is very nearly proportional to the square of the sine of the height of the pole above the horizon.

The excess of the equatorial axis, above that of the pole, assumed equal to unity, is termed the *compression* or *ellipticity* of the spheroid. The measure of two degrees in the direction of the meridian, is sufficient to determine it. A comparison of the arcs measured in France and Peru, which from their extent, their distance from each other, and from the accuracy and reputation of the observers, deserve the preference, makes the ellipticity of the terrestrial spheriod equal to $\frac{1}{3117}$; the semiaxis major equal to 6376606^{m}, and the semiaxis minor is equal to 635625^{m}.

If the earth was elliptical, the same compression should be nearly obtained, from a comparison, two by two, of different measures of the terrestrial degrees; but their comparison gives, on this point, differences which it is difficult to ascribe solely to the errors of observations. It therefore appears that the earth differs sensibly from the ellipsoid. This difference is even indicated by the measures of different parts of the great arc of the meridian which traverses France; for it has been observed already, that the increment of its degrees is 23^{m},167, which answers to an ellipticity of $\frac{1}{203}$, which more inconsiderable than the preceding ellipticity $\frac{1}{311,7}$; there is even reason to suppose that the two terrestrial hemispheres are not similar on each side of the equator. The degree measured by La Caille at the Cape of Good Hope, where the height of the south pole is 37°,01, is (*f*) found to be equal to $100050^m,5$; which is greater than that which was measured in Pennsylvania, where the height of the north pole is equal to 43°,56, the length of which was equal to $99789^m,1$; it even exceeds the degree which was measured in France at an elevation of the pole equal to 50°, yet the degree at the Cape ought to be less than these degrees, if the earth was a regular solid of revolution formed of two similar hemispheres; every thing therefore leads us to think that this is not the case. But the considerable errors which new measures have frequently indicated in this kind of observation, ought to make us very cautious in the conclusions which we deduce from it, and to resolve to take all possible precautions to avoid for the future similar errors. Let us see then what is the nature of the terrestrial meridians, the earth being supposed to be any figure whatever.

The plane of the celestial meridian determined by astronomical observations, passes through the axis of the world and through the zenith of the observer; because this plane bisects the arcs of all lesser circles parallel to the equator, which are described by the stars above the horizon. All places of the earth, which have their zeniths in the circumference of this meridian, form the corresponding terrestrial meridian. Considering the immense distance of the stars, verticals elevated from each of these places may be supposed parallel to the plane of the celestial meridian; the terrestrial meridian may therefore *(g)* be defined to be that curve which is formed by the junction of the bases of all the verticals parallel to the plane of the celestial meridian. This curve lies altogether in this plane, when the earth is a solid of revolution; in every other case it deviates from it, and generally it is one of those lines which geometricians term *curves of double curvature.*

The terrestrial meridian is not exactly the line which determines trigonometrical measurements in the direction of the celestial meridian. The first side of the line which is measured, is a tangent to the surface of the earth, and parallel to the plane of the celestial meridian; if this side be extended till it meets a vertical indefinitely near to it, and if then this prolongation be bent to the base of vertical, the second side of the curve will be formed; and thus with all the others. The line thus traced is the shortest which can be drawn on the surface of the earth *(h)* between any two points assumed on this line; it does not lie in the plane of the celestial, and is not confounded with the terrestrial meridian, except in the case in which the earth is a solid of revolution; but the difference between the length of this line and that of the corresponding arc of the terrestrial meridian is so small that it may be neglected without any sensible error.

The figure of the earth being extremely complicated, it is important to multiply its measures in every direction, and in as many places as possible. We may always at every point of its surface suppose an osculatory ellipse, which sensibly coincides with it for a small extent, about the point of osculation. Terrestrial arcs measured in the direction of the meridians, and of perpendiculars to the meridians, will make known the nature and position of this ellipsoid, which may not be a solid of revolution, and which varies sensibly at great distances.

Whatever be the nature of the terrestrial meridians, it is evident that as the degrees diminish from the poles to the equator, the earth is flattened in the direction (*i*) of the poles, *i. e.* that the axis of the earth is less than the diameter of the equator. In order to explain this, let us suppose that the earth is a solid of revolution; and let the radius of a degree at the north pole, and the series of those radii from the pole to the equator, which radii by hypothesis continually diminish, be supposed to be drawn, it is evident that these radii form by their consecutive intersections a curve, which at first touches the polar axis on the other side of the equator relatively to the north pole; it afterwards detaches itself from this axis, turning its convexity towards this axis, and continually raises itself towards the surface of the earth, until the radius of the meridional degree assumes a direction perpendicular to the primary direction; it is then in the plane of the equator. If the radius of the polar degree be supposed flexible, and that it involves successively the arcs of the curve which have been just described, its extremity will describe the terrestrial meridian, and the part of it which is intercepted between the meridian and the curve will be the corresponding radius of the meridional degree. This curve is what Geometricians term the *evolute* of the meridian. Let the intersection of the diameter of the equator and of the polar axis be assumed for the present to be at the centre of the earth; the sum of the two tangents to the evolute of the meridian drawn from this centre, the first in the direction of the polar axis, and the second in the direction of the diameter of the equator, will be greater than the arc of the evolute comprised between them; but the radius drawn from the centre of the earth to the north pole is equal to the radius of the polar degree *minus* the first tangent; the semidiameter of the equator is equal to the radius of the meridional degree at the equator *plus* the second tangent; therefore the excess of the semidiameter of the equator above the terrestrial radius of the pole, is equal to the sum of those tangents, minus the excess of the radius of the polar degree above the radius of the meridional degree at the equator: this last excess is the arc itself of the evolute, which arc is less than the sum of the extreme tangents; consequently the excess of the semidiameter of the equator above the radius drawn from the centre of the earth to the north pole is positive. It can be proved in the same manner, that the excess of this same semidiameter of the equator above the radius drawn to the south pole is positive, therefore the entire axis of the poles is less than the diameter of the equator, or what comes to the same thing, the earth is flattened in the direction of the poles.

Each part of the meridian being regarded as a very small arc of its osculatory circumference, it is easy to see that the radius drawn from the centre of the earth to the extremity of the arc, which is nearest to the pole, is less than the radius drawn from the same centre to the other extremity; hence it follows that the terrestrial radii continually increase from the poles to the equator, if, as all observations seem to indicate, the degrees of the meridian increase from the equator to the poles.

The difference of the radii of the meridional degrees at the poles and at the equator, is equal to the difference of the corresponding terrestrial radii *plus* the excess of (k) twice the evolute above the sum of the extreme tangents, which excess is evidently positive; thus, the degrees of the meridian increase from the equator to the poles in a greater ratio than that of the diminution of the terrestrial radii. It is evident that these demonstrations are equally applicable in the case in which the northern and southern hemispheres are not similar and equal, and it is easy to extend them to the case of the earth's not being a solid of revolution.

Curves have been constructed at the principal places in France, which lie on the meridian of the observatory, traced in the same manner as this line, with this difference, that the first side, which is always a tangent to the surface of the earth, instead of being parallel to the plane of the celestial meridian of the observatory of Paris is perpendicular to it. It is by the length of these curves, and by the distances of the observatory from the points where they meet the meridian, that the positions of these places have been determined. This operation, the most useful which has been undertaken in geography, is a model which every enlightened nation should hasten to imitate, and which will very soon be extended to all Europe.

As the respective positions of places separated by vast seas cannot be fixed by geodesical observations, we must have recourse to celestial observations, in order to determine them. The knowledge of these positions is one of the greatest advantages which astronomy has procured. In order to arrive at it, the method which was made use of to form a catalogue of the fixed stars, was followed, by conceiving circles to be drawn on the surface of the earth corresponding to those which have been imagined on the celestial surface. Thus the axis of the celestial equator intersects the surface of the earth in two points diametrically opposite, which have respectively one of the poles of the world in their zenith, and which may be considered as the poles of the earth. The intersection of the plane of the celestial equator with this surface, is a circumference which may be regarded as the terrestrial equator; the intersections of all the planes of the celestial meridians with the same surface are so many curved lines, which are reunited at the poles, and which are the corresponding terrestrial meridians, if the earth be a solid of revolution, which may be supposed in geography, without any sensible error. Finally, small circles traced on the earth parallel to the equator are terrestrial parallels; and that of any place whatever, corresponds to the celestial parallel which passes through its zenith.

The position of a place on the earth is determined by its distance from the equator, or by the arc of the terrestrial meridian comprised between its parallel and the equator, and by the angle which its meridian makes with the first meridian, of which the position is arbitrary, and to which all others are referred. Its distance from the equator depends on the angle comprized between its zenith and the celestial equator, and this angle is evidently equal to the height *(l)* of the pole above the horizon; this height is what in geography is termed *latitude*. The *longitude* is the angle which the meridian of a place makes with the first meridian; it is the arc of the equator contained between these two meridians. It is eastern or western, according as the place is to the east or west of the first meridian.

An observation of the height of the pole determines the latitude; the longitude is determined by means of a celestial phenomenon, which is observed simultaneously on the meridians of which the relative position is required. If the meridian of which the longitude is required is to the west of that from which the longitude is reckoned, the sun will arrive sooner at the celestial meridian; if, for example, the angle formed by the terrestrial meridian be a fourth part of the circumference, the difference between the instants of noon, at those meridians, will be the fourth part of the day. Suppose, therefore, that a phenomenon is observed on each of them which occurs at the same physical instant for all places on the earth, such as the commencement or termination of an eclipse of the moon or of the satellites of Jupiter, the difference of the hours which the observers will reckon at the moment of the occurrence of the phenomenon, will be to an entire day as the angle formed by the inclination of the two meridians is to the circumference. Eclipses of the sun, and the occultations of the fixed stars by the moon, furnish the most exact means of obtaining the longitude, by the precision with which the commencement and termination of these phenomena may be observed; they do not in fact occur at the same physical instant at every place on the earth, but the elements of the lunar motions are sufficiently well known to enable us to make an exact allowance of this difference.

To determine the longitude of a place, it is not necessary that the celestial phenomenon should be observed at the same time on the first meridian. It is sufficient if it be observed under a meridian of which the position with respect to the first meridian is known. It is thus that by connecting meridians with each other, the respective positions of the most distant points on the surface of the earth have been ascertained.

The longitudes and latitudes of a great number of places have been already determined by astronomical observations; considerable errors in the position and extent of countries a long time known, have been corrected: the position of those countries, which the interests of commerce, or the love of science have caused to be discovered, has been fixed; but though the voyages lately undertaken have added considerably to our geographical knowledge, much yet remains to be discovered. The interior of Africa, and that of New Holland, includes immense countries totally unknown: we have only uncertain, and frequently contradictory accounts concerning several others, of which geography hitherto abandoned to the hazard of conjecture, only waits for more accurate information from astronomy to fix and settle their position unalterably.

The longitude and latitude are not sufficient to determine the position of a place on the earth; besides these two horizontal coordinates, a vertical coordinate must be introduced, which expresses the elevation of the place above the level of the sea: this is the most useful application of the barometer; numerous and accurate observations with this instrument would throw the same light on the figure of the earth, (*m*) with respect to the comparative elevation of places, that has been already furnished by astronomy, on the other two dimensions.

It is principally to the navigator, when in the midst of the seas he has no other guide but the stars and his compass, that it is of consequence to know his position, that of the place for which he is bound, and of the shoals which he may meet in his passage. He may easily know his latitude by an observation of *(n)* the height of the stars: the fortunate inventions of the octant and of the repeating circle have rendered observations of this kind extremely accurate. But the celestial sphere, in consequence of its diurnal motion, presenting itself daily in very nearly the same manner to all the points of his parallel, it is difficult for the navigator to fix the *point* to which he corresponds. To supply the deficiency of celestial observations, he measures his velocity and the direction of his motion, thence he infers his progress in the direction of the parallels, and by a comparison of it with his observed latitude, he determines his longitude relatively to the place of his departure. The inaccuracy of this method subjects him to errors, which might become fatal when he abandons himself during the night to the winds near the shores and banks which, in his estimation, he believed himself at a considerable distance from. It is to secure him from these dangers that, as soon as the progress of arts and of astronomy led to the hope that methods might be devised to obtain the longitude at sea, commercial nations hastened to direct the views of scientific men and of artists to this important object, by powerful encouragements. Their expectations have been realised by the invention of chronometers, and by the great accuracy with which the tables of the lunar motions have been constucted; two methods, good in themselves, and which are further improved by the mutual support which they confer on each other.

A chronometer, well regulated in a port, the situation of which is known, and which preserves the same rate when carried on board a vessel, would indicate, at every instant, the time which was reckoned in this port.

This hour being compared with that observed at sea, the relation of the difference of these hours to the entire day would be, as *(o)* has been already observed, that of the corresponding difference of longitude to the circumference. But it was difficult to obtain such watches; the irregular motion of the ship, the variations of temperature, and the inevitable friction which is extremely sensible in such delicate machines, were so many obstacles, all opposed to their accuracy. These have been fortunately surmounted; chronometers are now made which *(p)* for several months preserve a rate nearly uniform, and which thus furnish the simplest means of obtaining the longitude at sea; and as this method is always more exact as the time is shorter, during which these chronometers are employed, without verifying their rate, they are particularly useful in determining the position of places very near to each other. They have even, in this respect, some advantages over astronomical observations, the accuracy of which is not increased by the proximity of the observers to each other.

The frequent recurrence of the eclipses of Jupiter's satellites would furnish an observer with an easy method of obtaining his longitude, if he could observe them at sea; but the endeavours which have been made to surmount the difficulties which the motion of the ship oppose to this kind of observations, have been hitherto fruitless; notwithstanding this, navigation and geography have derived considerable advantages from these eclipses, particularly from those of the first satellite, of which the commencement and termination can be accurately observed. The navigator employs them with success when he can land; indeed, it is necessary to know the hour at which the same eclipse which he observes would be seen upon a known meridian, since the difference of time, which is reckoned on these two meridians, gives the difference of longitudes; but from the great improvement which has been made in the tables of the first satellite in our time, the moment of the occurrence of these eclipses is given with a precision equal to that of observation itself.

The extreme difficulty of observing these eclipses at sea, has obliged us to have recourse to other celestial phenomena, among which the lunar motions are the only ones which can be made subservient to the determination of terrestrial longitudes. The position of the moon, such as it would be observed from the centre of the earth, may be easily inferred from the measure of its angular distance from the sun and fixed stars: the tables of its motion then give the hour at the principal meridian when the same phenomenon is observed, and the navigator comparing the time which he reckons on board his ship at the moment of observation, determines his longitude by the difference of time.

To appreciate the accuracy of this method, it should be considered that from the errors of observation, the place, of the moon as determined by the observer, does not exactly correspond to the hour indicated by his chronometer; and that in consequence of the errors of the tables this same place does not refer exactly to the corresponding hour which the sun indicates on the first meridian; the difference of these hours would not therefore be such as would be furnished by an observation and tables rigorously correct. Suppose that the error produced by this difference is a minute. In this interval, forty minutes of the equator is passed over the meridian; this is the corresponding error in the longitude of the vessel, and which is at the equator about forty thousand metres; but it is less on the parallels, besides it may be diminished by multiplying observations of the lunar distances from the sun and stars, and repeating them during several days, in order that the errors of observation and of the tables may be mutually compensated and destroyed. It is obvious that the error in longitude corresponding to those of observation and of the tables are so much the less considerable, as the motion of the celestial body is more rapid; thus observations made on the moon when in perigee, are in this respect, preferable to those made when the moon is in apogee. If the motion of the sun be employed, which is thirteen times slower than that of the moon, the errors in longitude will be about thirteen times as great; from hence it follows, that of all the celestial bodies the moon is the only one of which the motion is sufficiently rapid to be employed for the determination of the longitude at sea; we may consequently perceive of what great importance it is to render the tables as perfect as possible.

It is much to be desired that all the nations of Europe, instead of reckoning geographical longitudes from the meridian of their principal observatory, would concur in counting them from the same meridian, which being furnished by nature itself, might be easily found at all times. This agreement would introduce into their geography the same uniformity which their calendar and arithmetic present, a conformity which being extended to the various objects of their mutual relations, would constitute of these several nations but one immense family. Ptolemy caused his first meridian to pass through the Canaries, which were then the western limit of the known world. The reason of this selection no longer obtains, in consequence of the discovery of America. But one of these islands, presents one of the most remarkable points on the surface of the earth, in consequence of its great elevation and of its insulation, namely, the summit of the peak of Teneriffe. We might with the Hollanders assume its meridian, from which to reckon terrestrial longitudes, by determining its position relatively to the principal observatories, by means of a great number of astronomical observations. But whether we agree or not as to a common meridian, it will be extremely useful for future ages to know accurately their position, with respect to some mountains which may be always recognized by their solidity and great elevation, such as Mount Blanc, which towers over the immense and imperishable woods of the Alpine regions.

A remarkable phenomenon, the knowledge of which we owe to astronomical voyages, is the variation of gravity at the surface of the earth. This singular power acts in the same place, on all bodies proportionally to their masses, and tends to impress on them equal velocities in equal times. It is impossible by means of a balance to ascertain these variations, because they equally affect the body weighed, and the weight to which it is compared; but they can be determined by a comparison of their weight with a constant force, such as the elasticity of the air at the same temperature. *(q)* Thus, by transporting to different places, a manometer filled with a column of air, which elevates by its tension a column of mercury in an interior tube, it is evident that an equilibrum must always subsist between the weight of this column and the elasticity of the air; its elevation, when the temperature is given, will be reciprocally proportional to the force of gravity, the variations of which it consequently indicates. A very precise way of determining them is also furnished by observations of the pendulum; for it is obvious that its oscillations must be slower in those places where the gravity is less.

This instrument, the application of which to clocks is one of the principal causes of the progress of modern astronomy and geography, consists of a body suspended at the end of a thread or rod, moveable about a fixed point placed at the other extremity. The instrument is drawn a little from its vertical position, and being then remitted to the action of gravity, it makes small oscillations, which are very nearly of the same duration, notwithstanding the difference of the arcs described. This duration depends on the magnitude and figure of the suspended body, on the mass and length of the rod; but geometricians have found general rules to determine by observations of the compound pendulum, of any figure whatever, the length of a pendulum, the oscillations of which will be of a given duration, and in which the mass of the rod may be supposed nothing with respect to that of (*r*) the body, considered as an infinitely dense point. It is to this imaginary pendulum, termed the *simple pendulum*, that all the experiments of the pendulum made in different parts of the earth are referred.

Richer, sent in 1672 to Cayenne, by the Academy of Sciences, to make astronomical observations there, found that his clock regulated to mean time, at Paris, lost each day at Cayenne a perceptible quantity.

This interesting observation furnished the first direct proof of the diminution of gravity at the equator. It has been carefully repeated in a great number of places, taking into account the resistance of the air and the temperature. It follows from all the observed measures of a pendulum vibrating seconds, that it increases from the equator to the poles.

The length of the pendulum, which at the observatory of Paris makes one hundred thousand vibrations in a day, being assumed equal to unity, its length at the equator and at the level of the sea is equal to 0,99669, and in Lapland at an elevation of the pole equal to 74,22, it is observed to be 1,00137. Borda found by very exact and numerous experiments, that the length at the observatory at Paris which represented unity, was when reduced to a vacuum equal to 0,741887. From a repetition of these experiments by Biot and Mathieu, this length came out equal to 0,7419076, which differs very little from the preceding result. *(s)*

The increase in the length of the pendulum as we proceed from the equator to the poles, is even sensible on different points of the great arc of the meridian which traverses France, as will appear from an inspection of the following table, which gives the result of numerous accurate experiments made by Biot, Arrago and Mathieu.

Places.	*Height of the Pole.*	*Elevation above the sea.*	*Observed length of the pendulum which vibrates seconds.*
Fromentera	42°, 96	196^{m}	0^{m},7412061
Bourdeaux	49, 82	0	0^{m},7412615
Paris	54, 26	65	0, 7419076
Dunkirk	56, 67	0	0, 7420865

The observed lengths at Dunkirk and Bourdeaux give by the method of interpolations, 0,7416274 for the length of the pendulum which vibrates seconds on the coast of France, at the level of the sea, and at an elevation of the pole equal to fifty degrees. This length, and that of the meridional degree, the middle point of which corresponds to the same latitude, will enable us to recover our measures, if in the course of time they should be changed.

There is more regularity observed in the increase of the length of the pendulum, than in that of the meridional degrees: it deviates less from the ratio of the square of the sine of the pole's elevation; whether that its measurement being easier than that of degrees, it is less liable to error, or that the causes which disturb (*t*) the irregularity of the earth's form produce less effect on gravity. From comparison of all the observations which have been made on this subject, in different parts of the earth, it is found that if we assume for unity the length of the pendulum at the equator, its increase, as we proceed from the equator to the poles, is equal to the product of 0,005515 by the square of the sine of the latitude.

There has been likewise remarked by means of the pendulum, a small diminution of gravity on the summit of high mountains. Bouguer instituted a great number of experiments on this subject. At Peru he found that the force of gravity at the equator and at the level of the sea being expressed by unity, it is 0,999249 at Quito, which is elevated 2857^{m} above this level; and it is 998816 at Pinchincha, the elevation of which is 4744^{m}. This diminution of gravity (n) being sensible at elevations which are comparatively small with respect to the earth's radius, is a ground for supposing that it is considerable at great distances from the centre of the earth.

The observations of the pendulum furnishing a length which is invariable, and easy to be recovered at all times, has suggested the idea of employing it as an universal measure. The prodigious number of measures in use, not only among different people, but in the same nation; their whimsical divisions, inconvenient for calculation, and the difficulty of knowing and comparing them; finally, the embarrassments and frauds which they produce in commerce, cannot be observed without acknowledging that the adoption of a system of measures, of which the uniform divisions are easily subjected to calculation, and which are derived in a manner the least arbitrary, from a fundamental measure, indicated by nature itself, would be one of the most important services which any government could confer on society. A nation which would originate such a system of measures, would combine the advantage of gathering the first fruits of it with that of seeing its example followed by other nations, of which it would thus become the benefactor; for the slow but irresistible empire of reason predominates at length over all national jealousies, and surmounts all the obstacles which oppose themselves to an advantage, which would be universally felt. Such were the reasons that determined the Constituent Assembly, to charge the Academy of Sciences with this important object. The new system of weights and measures is the result of the labours of a committee appointed by them, seconded by the zeal and abilities of several members of the national representation.

The identity of the decimal calculus with that of integral numbers, leaves no doubt as to the advantages of dividing every kind of measure into decimal parts. To be convinced of this, it is only necessary to compare the difficulties of complicated divisions and multiplications, with the facility by which the same operations are performed on integral numbers, which facility may be increased by logarithms, the use of which might be rendered very popular by simple and cheap instruments. Indeed our Arithmetical scale is not divisible by three and four, two divisors which, from their great simplicity, are (*v*) of very frequent occurrence. This advantage would be secured by the addition of two new characters. But such a marked alteration would be inevitably rejected, together with the system of measures which would have been conformed to it. The duodecimal scale would be also subject to the additional inconvenience of requiring us to remember the binary products of the eleven first numbers, which surpasses the ordinary compass of the memory, to which the decimal scale is well adapted; lastly we could not retain the advantage which probably gave rise to our arithmetic, namely, that of making use of our fingers in reckoning. The academy therefore, did not hesitate in adopting the decimal division; and to render the entire system of measures uniform, it was resolved that they should all be derived from the same lineal measure, and from its decimal divisions. The question was thus reduced to the choice of this universal measure, which was denominated the *metre*.

The length of the pendulum, and that of the meridian, are the two principal means furnished by nature itself to fix the unity of linear measures. Both being independent of moral revolutions, they cannot experience a sensible alteration except by very great changes in the physical constitution of the earth. The first means, though easily applied, is notwithstanding subject to the inconvenience of making the measure of distance to depend on two elements which are heterogeneous to it, namely, gravity and time, the measure of which last is arbitrary; and as it is divided sexagesimally, it cannot be admitted as the foundation of a system of decimal measures. The second means was therefore selected, which appears to have been employed in the remotest antiquity; so natural is it for man to compare itinerary measures with the dimensions of the globe itself which he inhabits; so that in travelling he may know by the mere denomination of the space he has passed over, the relation of that space to the entire circuit of the earth. There is also the additional advantage of making nautical and celestial measures to correspond. The navigator has frequent occasion to determine the one by the other, the distance he has traversed, and the celestial arc included between the zeniths of the places of his departure and arrival; it is therefore of consequence that one of these measures should be the expression of the other, by nearly the difference of their unities. But for this purpose, the fundamental unity of linear measures should be an aliquot part of the terrestrial meridian, which corresponds to one of the divisions of the circumference. Thus the choice of the metre was reduced to that of the unity of angles.

The right angle is the limit of the inclination of a line to a plane, and of the elevation of objects above the horizon; besides it is in the first quadrant of the circumference that the sines are formed, and generally all the lines which are employed in trigonometry, of which the proportions to the radius have been reduced into tables; it was therefore natural to assume the right angle as the unity of angles, and the quarter of the circumference for the unity of their measures. It is divided into decimal parts, and in order to obtain corresponding measures on the earth, the quarter of the terrestrial meridian has been divided into the same parts, which had been done at a very ancient period; for the measure of the earth mentioned by Aristotle, the origin of which is unknown, assigns a hundred thousand stadia to the quarter of the meridian. It was then only necessary to obtain its exact length. Here two questions present themselves to be resolved. What is the proportion of an arc of the meridian measured at a given latitude, to the entire circumference? Are all the meridians similar? In the most natural hypotheses on the constitution of the terrestrial spheroid, the difference of the meridians is insensible, and the decimal degree of the middle point answering to the fiftieth degree of latitude, is the hundreth part of the quarter of the meridian. The error of these hypotheses can only influence geographical distances, where it is of no consequence. The length of the quarter of the meridian may therefore be conclnded from that of the arc which traverses France from Dunkirk to the Pyrenees, and which was measured in 1740, by the French Academicians. But as a (x) new measure of a greater arc, in which more accurate methods were employed, would excite an interest in favour of the new system of measures calculated to extend its utility, it was resolved to measure the arc of the terrestrial meridian contained between Dunkirk and Barcelona. This great arc extended as far south as Formentera, and to the north as far as the parallel of Greenwich, and of which its point of bisection, corresponds very nearly to the mean parallel between the Pole and the Equator, has given for the length of the quarter of the meridian 5130740 toises.

The ten millioneth part of this length was taken for the metre or the unity of linear measures. The decimal above this was too great, and the decimal below it was too small, and the metre, the length of which is 0,513074 toises, supplies advantageously the place of the toise and ell, which were two of our measures in most common use.

All the measures are derived from the metre, in the simplest possible manner; the linear measures are decimal multiplies and sub-multiplies of it.

The unity of the measure of capacity is the cube of the tenth of a metre; it is called *litre*. The unity of the superficial measure of land is a square, the side of which is ten metres; it is called *are*.

A *stere* is a volume of fire-wood, equal to a cubic metre.

The unity of weight, which is termed *gramme*, is the absolute weight of the cube of a millioneth part of a metre of distilled water, when at its *maximum* of density. By a remarkable peculiarity, this *maximum* does not correspond to the freezing point, but is above it by about four degrees of the thermometer. Water, as it falls below this temperature, again dilates, and thus prepares itself for that increase of volume, which it undergoes in its passage from the fluid to the solid state. Water has been selected as being one of the most homogeneous substances, and which may be easily reduced to a state of purity. Le Ferre Gineau has determined the *gramme* by a long series of delicate experiments on a hollow cylinder of brass, the volume of which he measured with extreme care; the result of these experiments is, that the livre being supposed equal to the twenty-fifth part of the pile of fifty marcs, which is preserved at the mint of Paris, is to the gramme in the ratio of 489,5058 to unity. The weight of a thousand grammes, which is denominated the *kilogramme or decimal livre*, is consequently equal to the livre, the weight of the marc multiplied by 2,04288.

In order to preserve the measures of length, and the unity of weights, standards of the metre and of the kilogramme, executed under the immediate superintendence of the committee to whom the determination of these measures was intrusted, and verified by them, were deposited in the national archives, and at the observatory of Paris. The standards of the metre do not represent it, except at a definite temperature. The temperature of melting ice was selected as being the most invariable, and independent of the modifications of the atmosphere. The standards of the kilogramme do not represent its weight, except in a vacuum, in which case the pressure of the atmosphere is insensible. In order to be able to recover the metre at all times, without having recourse to the measure of the great arc which furnished it, it was necessary to determine its relation to the length of the pendulum which vibrates seconds; this has been effected by Borda in the most accurate manner.

As there was necessarily a constant comparison of all these measures with the livre in money, it was particularly important to divide it into decimal parts. Its unity has been denominated the silver franc, its tenth part, *decime*, its hundreth *centieme*. The values of golden pieces of money, of gold and brass, have been referred to the franc.

In order to facilitate the calculation of the fine gold and silver contained in pieces of money, the alloy was fixed at the tenth part of their weight, and that of the franc has been made equal to five grammes. Thus the franc being an exact multiple of the unity of weights, it can be made use of in weighing bodies, which is extremely useful in commerce.

Finally, the uniformity of the whole system of weights and measures required that the day should be divided into ten hours, the hour into one hundred minutes, and the minute into one hundred seconds. This division of the day, which will be indispensable to astronomers, is of less consequence in civil life, where there is little occasion to imploy time as a multiplier and divisor. The difficulty of adapting it to watches and clocks, and our commercial relations with foreigners in the sale of watches, will suspend its application indefinitely. We may however be assured, that at length the decimal division of the day will supersede its present division, which differs too much from the division of the other measures not to be abandoned.

Such is the new system of weights and measures, presented by the Academy to the National Convention, which immediately adopted it. This system, founded on the measure of the terrestrial meridians, corresponds equally to all nations. It has no other relation with France than what is furnished by the arc of the meridian which traverses it. But the position of this arc is so advantageous, that if the learned of all nations had combined to fix an universal measure, they would have selected it. To multiply the advantages of this system, and to render it useful to the entire world, the French government invited foreign powers to participate in an object of such general interest: many have sent eminent men of science to Paris, who, in conjunction with the committee of the National Institute, have determined by a discussion of observations and experiments, the fundamental unites of weights and lengths; so that the determination of these unites may be considered as a work common to the learned who have assembled there, and to the people of whom they are the representatives. It is therefore permitted to hope, that one day this system, which reduces all measures and their computations to the scale, and to the simplest operations of the decimal arithmetic, will be as universally adopted as the system of numeration of which it is the completion, and which, without doubt, had to surmount the same obstacles which prejudices and long established habit oppose to the introduction of the new measures; but when once introduced, these measures will be maintained by this same power which, combined with that of reason, secures to human institutions an eternal duration.

CHAP. XV.

Of the flux and reflux of the sea, and of the daily variation of its figure.

Although the earth, and the fluids which are diffused over it, must long since have assumed the state which corresponds to the equilibrium of the forces which actuate them, nevertheless, the figure of the sea changes every instant of the day, by regular and periodical oscillations, which are denominated, *the ebbing and flowing of the sea.* It is a circumstance truly astonishing to behold, in calm serene weather, the intense agitation of this great fluid mass, of which the waves break with violence against the shores. This phenomenon gives rise to reflexions, and excites a strong desire to penetrate the cause. But in order that we may not be mislead by vague hypotheses, it is necessary previously to know the laws of this phenomenon, and to follow it in all its details. As a thousand accidental causes may alter the regularity of these phenomena, it is necessary to consider at once a great number of observations, in order that the effects of transient causes, mutually compensating each other, the mean results may only indicate the regular and constant effects. It is likewise necessary, by a judicious combination of observations, to make each of these effects which we wish to determine, as conspicuous as possible. But this is not sufficient. The results of observations being always liable to error. it is necessary to know the probability that these errors are confined within given limits. Indeed it is evident, that for the same probability, these limits are more restricted, as the observations are more numerous; and this is the cause why observers have been at all times anxious to multiply the number of experiments and observations. But the degree of accuracy of the results is not indicated by this general impression; it does not make known the number of observations necessary to obtain a determinate probability. Sometimes even, it has induced us to investigate the cause of phenomena which arose from mere chance. It is by means of the calculus of probabilities alone that we are enabled to appreciate these objects, which renders its application in physical and moral sciences of the greatest importance.

At the request of the Academy of Science, a great number of observations were made in the beginning of the last century, in our harbours: they were continued every day at Brest during six successive years. The situation of this port is peculiarly favourable to this kind of observations. It communicates with the sea by a vast and long canal, at the extremity of which this port has been constructed. The irregularities in the motion of the sea, are consequently much diminished when they arrive at this port; just as the oscillations which the motion of a vessel impresses on a column of mercury in the barometer, are considerably lessened by the contraction of the tube of this instrument. Moreover, the tides being very sensible at Brest, the accidental variations constitute but a very inconsiderable part of them; and if we particularly consider, as I have done, the excess of the high water over the preceding and subsequent low water, it will appear that the winds, which are the principal cause of the irregularities in the motion of the sea, have very little influence on the results; because if they raise the high water, they elevate very nearly as much the preceding and subsequent low water; so that a very great regularity has been observed in these results, considering the fewness of the observations which have been made. Struck by this regularity, I requested the government to order a new series of observations to be made in the harbour of Brest, during the entire period of the motion of the nodes of the lunar orbit. This has been accordingly done; they commenced in the year 1806, and have been uninterruptedly continued each successive day. All these observations being discussed, in the manner I previously made mention of, the following results have been obtained respecting which there cannot remain any doubt.

The sea rises and falls twice in the interval of time comprehended between two consecutive returns of the moon to the meridian, above the horizon. The mean interval of these returns is $1^d,035050$,; thus, the interval between two consecutive high tides is $0^{d},517525$, so that there are some solar days in which only one high tide can be observed. The moment of low water very nearly divides the extremities of this interval equally at Brest, the sea is longer rising than falling by above nine or ten minutes. Similar to (*a*) all magnitudes, which are susceptible of a *maximum* or a *minimum*, the increase or diminution of the tide near to these limits is proportional to the square of the time elapsed, since the moments of high or low water.

The elevation of the sea at high tide is not always the same; it varies every day, and its variations are evidently connected with the phases of the moon. It is greatest about the time of full or of new moon; it then diminishes and becomes least near to the time of quadrature. The highest tide at Brest does not take place exactly the day of the syzygy, but a day and a half later, so that if the syzygy happens at the moment of high tide, the greatest tide is the third that follows. In like manner, if the quadrature happens at the moment of high water, the third tide which follows will be the least. This phenomenon is observed to be very nearly the same in all the ports of France, although the hours of high and low water are very different.

The greater the elevation of the sea at high water, the more will it fall at the low water which succeeds it. A *total tide* is termed half the sum of the heights of two consecutive high waters, above the level of the intermediate low water. The mean value of this total high water at Brest, at its maximum near to the syzygies, and when the sun and moon are in the equator, and at their mean distances from the earth, is about five metres and a half. In the same circumstances it is less by one half in the quadratures.

From an attentive consideration of these results, it appears that the number of high waters being equal to the number of passages of the moon over the upper or inferior meridian, this star has the principal influence on the tides; but from the circumstance of the tides in the quadratures, being fuller than those in syzygies, it follows that the sun also influences this phenomenon, and in some measures modifies the effect of the moon's influence. It is natural to think that each of these influences, if they existed separately, would produce a system of tides, of which the period would be the same as that of the respective stars over the meridian, and that from the combination of the two systems, there should arise a compound tide, in which the lunar high water would correspond to the solar high water near to the syzygies, and to the solar low water near to the quadratures. The declinations of the sun and of the moon have a remarkable influence on the tides; they diminish the total high waters of the syzygies and of the quadratures; they increase by the same quantity the total high waters of the solstices. Thus the received opinion that the tides are greatest in the equinoctial syzygies, is confirmed by an exact discussion of a great number of observations. However, several philosophers, and especially Lalande, have questioned the truth of this observation, because that near to some solstices the sea rises to a considerable height. It is here that the calculus of probabilities is of such importance in enabling us to decide this important question in the theory of the tides. It has been found by applying this calculus to the observations, that the superiority of the syzygial equinoctial tides and of the solstitial sides in quadratures is indicated, with a probability much greater than that on which most of the facts respecting which there exists no doubt, rest.

The distance of the moon from the earth influences, in a very perceptible manner, the magnitude of the high water. All other circumstances being the same, they increase and diminish with the diameter and lunar parallax, but in a greater ratio. The variations of the distance of the sun from the earth, influences the tides in a similar manner, but in a much less degree.

It is principally near the *maxima* and *minima* of the total tides, that it is interesting to know the law of their variation. We have seen that the moment of their maximum at Brest follows the time of the occurrence of the syzygy by a day and a half. The diminution of the total tides which are near to it, is proportional to the square of the time which has elapsed from that instant, to that of the intermediate low water, to which the total tide is referred. Near the instant of the *minimum*, which follows the quadrature by a day and a half, the increment of the total tide is proportional to the square of the time which has elapsed since this instant; it is very nearly double of the diminution of the total tides near to their *maximum*.

The declinations of the sun and of the moon sensibly influence these variations; the diminution of the tides near the syzygies of the solstices is only about three fifths of the corresponding diminution near the syzygies of the equinoxes; the increment of the tides near to the quadratures is twice greater in the equinoxes than in the solstices. But the effect of the different distances of the moon from the earth is still more considerable, than that of the declinations. The diminution of the syzygial high waters is nearly three times greater near to the lunar perigee, than it is near to its apogee.

A small difference has been observed between the morning and evening tides, which must depend on the declinations of the sun and of the moon, as the differences disappear when these stars are in the equator. In order to recognize them, we should compare the tides of the first and of the second day after the syzygy or the quadrature; the tides being then very near to the maximum or the minimum, vary very little from one day to another, which enables us easily to observe the difference between two tides of the same day. It is thus found at Brest, that in the syzygies of the summer solstice, the tides of the morning of the first and second day after the syzygy are smaller than those of the evening by about a sixth of a metre very nearly; they are greater by the same quantity in the syzygies of the winter solstice. In like manner, in the quadrature of the autumnal equinox, the morning tides of the first and second day after the quadrature, surpass those of the evening by about the eighth part of a metre: they are smaller by the same quantity, in the quadratures of the vernal equinox.

Such are, in general, the phenomena which the heights of the tides present in our ports; their intervals furnish other phenomena, which we now proceed to develope.

When the high tide happens at Brest at the moment of the syzygy, it follows the instant of midnight, or that of the true mid day by $0^d,1780$, according as it happens in the morning or in the evening: this interval, which is very different in harbours extremely near to each other, is termed the hour of port, because it determines the hours of the tides relative to the phases of the moon. The high tide which takes place at Brest at the moment of the quadrature, follows the instant of midnight, or of mid day, by 0,358.

The tide which is near to the syzygy, advances or retards $270''$ for each hour by which it precedes or follows the syzygy; the tide which is near to the quadrature, advances or retards $502''$ for each hour it precedes or follows the quadrature.

The hours of the high water in the syzygies and in the quadrature, vary with the distances of the sun and of the moon from the earth, and principally with the distance of the moon. In the syzygies each minute of increase or diminution in the apparent semidiameter of the moon, advances or retards the hour of high water by 354″. This phenomenon obtains equally in the quadratures, but it is there three times less.

In like manner the declinations of the sun and of the moon influence the hours of high water in the syzygies and in the quadratures. In the solstitial syzygies, the hour of high water advances by about two minutes, and it is retarded by the same quantity in the equinoctical syzygies; on the contrary, in the equinoctial quadratures, the hour of high water advances by about eight minutes, and it is retarded by the same quantity in the solstitial quadratures.

We have seen that the retardation of the tides from one day to another is about 0,03505, in its mean state; so that if the tide happens at 0,1 after the true midnight, it will arrive on the morning after but one at 0^d,13505. But this retardation varies with the phases of the moon. It is the least possible near the syzygies, when the total tides are at their *maximum*, and then it is only 0^d,02723. When the tides are at their *minimum* or near to the quadratures, it is the greatest possible, and amounts to 0^d,05207. Thus, the difference of the hours of the corresponding high water, at the moments of the syzygy and of the quadrature, and which by what precedes is 0^d,20642, increases, for the tides which follow in the same manner these two phases, and becomes very nearly equal to a quarter of a day, relatively to the *maximum* or the *minimum* of the tides.

The variations of the distances of the sun and of the moon from the earth, and principally those of the moon, influence the retardation of the tides from one day to another. Each minute of increase or of diminution of the apparent semi-diameter of the moon, increases or diminishes this retardation by 258″ towards the syzygies. This phenomenon obtains equally in the quadratures, but it is then three times less.

The daily retardation of the tides varies also with the declination of the two stars. In the solstitial syzygies it is about one minute greater than in its mean state ; it is smaller by the same quantity in the equinoxes. On the contrary, in the equinoctial quadratures it surpasses its mean magnitude by about four minutes ; it is less by the same quantity in the solstitial quadratures.

The results which have been just detailed, were deduced from a series of observations made at Brest since the year 1807, up to the present day. It was interesting to compare them with similar results which have been deduced from observations made at the commencement of the last century. I have found that all the results accord with each other very nearly, their small differences being comprized within the limits to which the errors of observations are liable. Thus, after the interval of a century, Nature has been found agreeing with herself.

Hence it appears that the inequalities of the heights and of the intervals of the tides have very different periods, the one are equal to half a day and to an entire day, others to half a month, to a month, to half a year, and of a year; and finally others are the same as those of the revolutions of the nodes and of the perigee of the lunar orbit, the position of which influences the height of the tides by the effect of the declinations of the moon, and of its distances from the sun. These phenomena obtain indifferently in all harbours and on the shores of the sea, but local circumstances, without making any change in the laws of the tides, have a considerable influence on their height and the hour of high water for a given port.

CHAP. XVI.

Of the terrestrial atmosphere, and of astronomical refractions.

A RARE elastic and transparent fluid envelopes the earth, and extends to a considerable height. It gravitates (*a*) like all other bodies, and its weight balances that of the mercury in the barometer. At the parallel of fifty degrees, the temperature being supposed to be that of melting ice, and at the mean height of the barometer at the level of the sea, which height may be supposed to be $0^{m},76$, the weight of the air is to that of an equal volume of mercury, in the ratio of unity to 10477,9; (*b*) hence it follows that if it be then elevated, by $10^{m}4779$, the height of the barometer will be depressed very nearly one millimetre, and that if the density of the atmosphere was uniform throughout its entire extent, its height would be 7963 metres. But the air is compressible, and if its temperature be supposed constant, its density, according to a general law for gases and fluids reduced to vapours, is proportional to the weight which compresses it, and consequently to the height of the barometer. Its inferior strata being compressed by the superior ones, are consequently more dense than the latter, which become rarer according as we ascend above the earth's surface. The height of these strata being supposed to increase in arithmetic progression, their density would diminish in geometric progression, provided that the temperature of these strata was the same. In order to understand this, suppose a vertical canal to traverse two atmospherical strata indefinitely near to each. The part of the more elevated stratum through which the canal passes, will be less compressed than the corresponding part of the lower stratum, by a quantity equal to the weight of a small column of air intercepted between these two parts. The temperature being supposed to be the same, the difference of compression of these two strata, is proportional to the difference of their densities; therefore this last difference is proportional to the weight of the small column, and consequently to the product of its density by its length, at least if we abstract from the variation of gravity according as we ascend. The two strata being supposed indefinitely near to each other, the density of the column may be supposed the same as that of the inferior stratum; hence the differential variation of this last density is proportional to the product of this density by the variation of the vertical height; consequently if this height varies by equal quantities, the ratio of the differential *(c)* of the density to the density itself will be constant; which is the characteristic property of a decreasing geometric progression, all the terms of which are indefinitely near to each other.

Hence it follows that the heights of the strata increasing in arithmetical progression, their densities diminish in geometric progression, and their logarithms, whether hyperbolical or naperian, will decrease in arithmetic progression.

These data have been advantageously applied to the measurement of heights by means of the barometer. The temperature of the atmosphere being supposed to be constant throughout its entire extent, the difference of the heights of the two stations will be obtained by multiplying, by a constant cœfficient, the difference of the logarithms of the observed heights of the barometer at each station. One sole observation is sufficient to determine this coefficient. Thus we have seen that at zero of temperature, the height of the barometer being $0^m,76000$ at the inferior station, and $0^m,75999$ at the superior station, this last station was elevated $0^m,104779$ above the first; consequently the constant coefficient was equal to this quantity divided by the difference of the tabular logarithims of the numbers $0^m,76000$, $0^m,75999$, which renders this coefficient equal to 18336^m. But this rule for measuring heights by means of the barometer, requires several modifications, which we proceed to develope.

The temperature (d) of the atmosphere is not uniform; it diminishes according as we ascend. The law of this diminution changes every instant; but a mean result between several observations gives sixteen or seventeen degrees for the diminution of the temperature relative to an height of three thousand metres. Now the air, like all other bodies, expands by heat, and contracts by cold; and it has been found by very accurate experiments, that its volume being represented by unity, at the temperature of zero, it varies like that of all gazes and vapours by 0,00375 for *(f)* each degree of the thermometer; it is therefore necessary to take these variations into account in the computation of heights, for it is evident that in order to produce the same depression in the barometer, it is necessary to ascend so much the higher, as the stratum of air through which we must pass is rarer. But as it is impossible to know accurately the variation of the temperature, the simplest method of proceeding is to suppose this temperature uniform, and a mean between the temperatures of the two stations which are considered. The volume of the column of air comprised between them being increased in the ratio of this mean temperature, the height due to the observed depression of the barometer must be increased in the same ratio, which comes to multiplying the cœfficient 18336^{m}, by unity plus the fraction 0,00375, taken as often as there are degrees in the mean *(g)* temperature. As the aqueous vapours which are diffused through the atmosphere are less dense than the air at the same pressure and temperature, they diminish the density of the atmosphere, and every thing else being the same, they are more abundant when the heat is greater; this effect will be partly taken into account, by increasing a little the number 0,00375, which expresses the dilation of the air for each degree of the thermometer. It has been ascertained that the observations are sufficiently well satisfied by making this fraction equal to 0,004; we may therefore make use of this last number, at least until by a long series ofobservations on the hygrometer, we are enabled to introduce this instrument in the measurement of heights by the means of the barometer.

Hitherto the force of gravity has been supposed to be constant, but it has been already observed that it is less according as we ascend in the atmosphere; this circumstance also contributes to increase the height due to the depression of the barometer, consequently this diminution of gravity will be taken into account, if the constant factor be increased by a small quantity. From a comparison of a great number of observations of the barometer, made at the base and at the summit of several mountains, the heights of which were previously ascertained by trigonometrical means, Raymond has determined this factor to be equal to 18393^{m}. But if the *(h)* diminution of gravity be taken into account, a comparison of the same data would only give this factor equal to 18336^{m}. This last factor gives 10477,9 for the ratio of the weight of mercury to that of an equal volume of air at the parallel of fifty degrees; the temperature as indicated by the barometer being zero, and the height of the mercury in the barometer being $0^{m},76$. Biot and Arrago having carefully weighed known measures of mercury and of air, found this ratio to be 10466,6 reduced to the same parallel. But they made use of very dry air, while that of the atmosphere is always mixed with a greater or less quantity of aqueous vapour, the actual quantity of which is determined by means of the hygrometer: this vapour is lighter than the air in the ratio of ten to seventeen very nearly; consequently direct experiment ought to assign a less specific gravity to mercury than that determined by barometrical observations. These experiments reduce the factor 18336^m to 18316^m. In order that it should be supposed equal to the number 18393^m, which is given by observations of the barometer, when the variation of gravity is not taken into account, we should assign to the mean humidity of the atmosphere a value much too great; thus the diminution of gravity is even sensible in barometrical observations. The factor 18393 corrects very nearly the effect of this diminution, but another variation of gravity, namely, that which depends on the latitude, ought also to influence this factor. We have determined it for a parallel of which the latitude may without sensible error be supposed equal to 50°: it should therefore be increased at the equator where the gravity is less *(i)* than at this latitude. In fact, it is evident that it should be elevated more, in order to pass from a given pressure of the atmosphere to a pressure which is smaller by a determined quantity; because in this interval the weight of the air is less, the cœfficient 18393^m must therefore vary as the length of the pendulum which vibrates seconds, which is greater or less according as the gravity diminishes or increases. It is easy to infer from what has been previously stated relative to the variations of this length, that the product of 26^{m},164 by the cosine of twice the latitude, must be added to this coefficient.

Finally, a slight correction should be applied to the heights of the barometer, depending on the difference of temperature of the mercury of the barometer at the two stations. In order to determine this difference accurately, a small mercurial thermometer is inchased in the frame of the barometer, so that the temperature of the mercury of these two instruments may be very nearly the same. In the colder station the mercury is denser, and consequently the column of mercury of the barometer is diminished. In order to reduce it to the length which it would *(k)* have, if the temperature was the same as at the warmer station, it should be increased by its 5412^{mth} part, as often as there are degrees of difference between the temperatures of the mercury at the two stations.

Hence the following appears to be the simplest and most exact rule for measuring heights by means of the barometer. First, the height of the barometer in the colder station must be corrected in the manner just specified. Then to the factor 18393^{m}, should be added the product of 26,164 by the cosine of twice the latitude. This factor, thus corrected, should be multiplied by the tabular logarithm of the ratio of the greatest to the least corrected height of the barometer. Finally, this product must be multiplied by twice the sum of the degrees of the thermometer which indicates the temperature of the air at each station, and this product, divided by one thousand, should be added to the preceding; the sum will give very nearly the elevation of the superior station above the inferior, especially if the observations of the barometer are made at the most favourable time of the day, which appears to be at noon.

The air is invisible in small masses, but the rays of light reflected by all the strata of the terrestrial atmosphere, produce a sensible impression. They (*l*) give it a blue shade which diffuses a tint of the same colour over all objects perceived at a distance, and which forms the celestial azure. This blue vault, to which the stars appear to be attached, is therefore very near to us: it is only the terrestrial atmosphere, beyond which these bodies are placed at (*m*) immense distances. The solar rays, which its particles reflect to us in abundance before the rising and after the setting of the sun, produce the dawn and twilight, which, extending to more than twenty degrees of distance from this star, proves that the extreme particles of the atmosphere are elevated at least sixty thousand metres. If the eye could distinguish and refer to their true place, the points of the exterior surface of the atmosphere, we should see the heavens like the segment of a sphere formed by the portion of the surface which would be cut off by a plane tangent to the earth; and as the height of the atmosphere is very small relatively to the radius of the earth, the sky would appear to us under the form of a flattened vault. But although the limits of the atmosphere cannot be distinguished, yet as the rays which it transmits come from a greater depth at the horizon than at the zenith, we ought to consider it as more extended in the first direction. To this cause must be also combined the interposition of objects at the horizon, which contributes to increase the apparent distance of that part of the sky we refer to it; the sky therefore should appear to us very much flattened, like a small portion of a sphere. A star, elevated twenty-six degrees above the horizon, appears to divide into two equal parts the length of the curve which the section of the surface of the sky by a vertical plane forms from the horizon to the zenith; hence it follows that if this curve be an arc of a circle, the horizontal radius of the apparent celestial vault (*n*) is to its vertical radius very nearly as three fourths is to unity; but this ratio varies with the causes of the illusion. As the apparent magnitudes of the sun and of the moon are proportional to the angles under which they are seen, and to the apparent distance of the point of the sky to which they are referred, they *appear* greater at the horizon than at the zenith, although they subtend a smaller angle.

The rays of light do not move in a right line through the atmosphere, they are continually inflected towards the earth. As an observer beholds objects in the direction of the tangent to the curve which they describe, he sees them more elevated than they really are, so that the stars appear above the horizon when they are depressed below it. By this means the atmosphere, by inflecting the rays of the sun, lengthens the time during which he appears to us, and thus prolongs the duration of the day, which is further increased by the morning dawn and twilight. It is extremely important to astronomers, to know the laws and quantity of the refraction of light in our atmosphere, in order to be able to determine the position of the stars. But before we present the result of their researches on this subject, we shall briefly explain the principal properties of light.

A ray of light, in passing from one transparent medium into another, approaches to, or recedes from the perpendicular to the surface which separates them, in such a manner, that the sines of the two angles which its directions make with this perpendicular, the one before and the other after its entrance into the new medium, are in a constant (*o*) ratio, whatever be the magnitude of these angles. But light, when refracted, presents a remarkable phenomenon, which has led to the discovery of its nature. A ray of solar light received into a dark chamber forms, after its passage through a prism, an oblong image variously coloured; this ray is a pencil of an infinite number of rays of different colours, which are separated by the prism in consequence of their different refrangibility. The most refrangible ray is the violet, then the indigo, the blue, the green, the yellow, the orange and the red. But though we only distinguish seven species of rays, the continuity of the image proves that there exists an infinite variety of shades, which approach each other by insensible gradations of colours and refrangibility. All these rays being collected by means of a lens, reproduce the white light of the sun, which is therefore only a mixture of all the homogeneous or simple colours in determined proportions.

When a ray of an homogeneous colour is perfectly separated from the others, it does not change either its refrangibility or colour, whatever reflexions or refractions it may undergo; therefore its colour is inherent in its nature, and not a modification which light receives in the media which it traverses. However, a similitude of colour does not prove a similitude of light. If several of the differently coloured rays of the solar image, decomposed by the prism, be mixed together, a colour perfectly similar to one of the simple colours of this image will be formed; thus the mixture of the homogeneous red and yellow produces an orange similar in appearance to the homogeneous orange. But by refracting the rays of this mixture by a second prism, the component colours can be separated and made to reappear, while the rays of the homogeneous orange remain unaltered. The rays of light are reflected when they fall on a mirror, making, with the perpendicular to its surface, the angles of reflexion equal to the angles of incidence. The refractions and reflections which rays of light undergo in drops of rain, produce the rainbow, the explanation of which, founded on a rigorous computation which satisfies all the phenomena, is one of the most beautiful results of natural philosophy.

Most bodies decompose the light which they receive; they absorb one part and reflect the other in every direction; they appear blue, red, green, &c. according to the colour of the rays which they reflect. Thus the white light of the sun diffusing itself over all (*p*) natural objects, decomposes and reflects to our eyes an infinite variety of colours.

After this short digression, we return to astronomical refractions. From very accurate experiments it has been ascertained that the refraction of the air is almost independent of its temperature, and proportional to its density. In passing from a vacuo into air, of which the temperature is equal to that of melting ice, and under a pressure measured by a height of the barometer equal to 76 centimetres, a ray of light is so refracted that the sine of incidence is to the sine of refraction as 1,0002943321 to 1. Therefore in order to determine the rout of a ray of light through the atmosphere, it is sufficient to know the law of the density of its strata; but this law, which depends on their temperature, is very complicated, and varies for every instant of the day. We have seen already, that when the atmosphere is throughout at the temperature of zero, the density of the strata (q) diminish in geometric progression; and it has been found by analysis, that the height of the barometer being $0^{m},76$, the refraction is then $7391''$ at the horizon. It would be but $5630''$ if the density of the strata diminished in arithmetic progression, and vanished at the surface. The horizontal refraction which is observed is about $6500''$, a mean between these limits. Consequently the law of the diminution of the density of the atmospherical strata is very nearly a mean between these two progressions. By adopting an hypothesis which participates of the two, we are enabled to represent at once all the observations of the barometer and thermometer, according as we ascend in the atmosphere, and also the astronomical refractions, without having recourse, as some natural philosophers have done, to a particular fluid, which, being combined with the atmospheric air, refracts the light.

When the apparent altitude of the stars above the horizon exceeds eleven degrees, their refraction depends only sensibly on the state of the barometer and thermometer at the place of the observer, and it is very nearly proportional to the tangent of the apparent zenith distance of the star, diminished by three times and one fourth of the refraction corresponding to (r) this distance at the temperature of melting ice, the height of the barometer being $0^{m},76$. It follows from the preceding data that at this temperature, and when the height of the barometer is seventy-six centimetres, the coefficient, which multiplied by this tangent gives the astronomical refraction, is 187″,24; and what is very remarkable, a comparison of a great number of astronomical observations gives the same result, which must therefore be supposed extremely accurate; but it varies with the density of the air. Each degree of the thermometer increases by 0,00375 the volume of this fluid, its unity being assumed at zero of the temperature, it is therefore necessary to divide the coefficient 187″,24 by unity, plus the product of 0,00375 into the number of degrees indicated by the thermometer; moreover, the density of the air is, every thing else being the same, proportional to the height of the barometer; it is therefore necessary to multiply the preceding coefficient by the ratio of this height to $0^m,76$, the column of mercury being reduced to the zero of temperature. By means of these data a very exact table of refractions may be constructed, from eleven degrees of apparent altitude to the zenith, in which interval almost all astronomical observations are made.

This table will be independent of all hypotheses relative to the diminution of the density of the atmospherical strata, and might as well be applied at the summit of the highest mountains as at the level of the sea. But as the gravity varies with the elevation and latitude, it is evident, that as at the same temperature, equal heights of the barometer do not indicate an equal density in the air, this density must be less in those places where the gravity is less. Thus the coefficient 187″,24, determined for the parallel of 50°, must *at the surface of the earth* (s) vary as the weight, it is therefore necessary to subtract from it the product of 0″,53 by the cosine of twice the latitude.

The table of which we have been speaking, supposes that the constitution of the atmosphere is every where and always the same, which has been proved by direct experiment. It is now ascertained that our atmosphere is not an homogeneous substance, and that in every hundred parts, it contains 79 parts of azotic gas, and 21 parts of oxygen gas, a gas remarkably respirable, which is indispensably necessary for the combustion of bodies (t) and the respiration of animals, which is in fact but a slow combustion, the principle source of animal heat; three or four parts of cartonic acid air are diffused in a thousand parts of atmospheric air. This air, taken at all seasons, in the most remote climates, on the summits of the highest mountains, and even at greater heights, has been most carefully analyzed, and it has always been found to contain the same proportions of azotic and oxygen gas. A slight envelope filled with hydrogen gas, the rarest of all elastic fluids, ascends with the bodies which are attached to it, untill it meets with a stratum of the atmosphere sufficiently rare for it to remain (u) in equilibrium. By this means, for the fortunate discovery of which we are indebted to the French philosophers, man has extended his power and sphere of action; he may launch into the air, traverse the clouds, and interrogate nature in the elevated regions of the atmosphere formerly inaccessible. The ascent from which the greatest advantages have been derived to the sciences, was that of Gay-Lussac, who ascended to a height of seven thousand and sixteen metres above the level of the sea, the greatest height to which an aeronaut has hitherto attained, and which is higher than the top of Chimboraco, one of the highest known mountains, by about five hundred metres. At this elevation, he measured the intensity of the magnetic force, the inclination of the magnetized needle, which he found to be the same as at the surface of the earth. At the instant of his departure from Paris, near to ten o'clock A. M. the height of the barometer was $0^{m},7652$, the thermometer indicated $30°,7$, and a hygrometer made of hair, $60°$. Five hours after, at the greatest height to which he ascended, the same instruments indicated respectively $0^{m},3288$;—$9°,5$ and $33°$. A balloon having been filled with the air of these elevated strata, and its contents being then carefully analyzed, the contents were found to be precisely the same as those of the lowest strata of the atmosphere.

It is not more than half a century since astronomers introduced the consideration of the heights of the barometer and thermometer, into the tables of refractions. The great precision which is now required in instruments and astronomical observations, makes it a matter of importance to ascertain whether the humidity of the atmosphere has any influence on the refracting force, and consequently to know whether it is necessary to take into account the indications of the hygrometer.

In order to supply the defect of direct experiment on this subject, let us suppose (v) that the action of water and vapour on light are proportional to their densities, which hypothesis is extremely probable from the circumstance, that changes in the constitution of bodies much more essential, than the reduction of liquids into vapours do not alter in a sensible degree the relation of their action on light, to their density. In this hypothesis, the refracting power of the aqueous vapour may be inferred from the refraction which a ray of light experiences in passing from air into water, which refraction has been exactly measured. It has been thus ascertained that this refracting power surpasses that of air reduced to the same density as the vapour; but at equal pressures, the density of the air surpasses that of vapour in very nearly the same ratio; hence it follows that the refraction due to the aqueous vapour diffused through the atmosphere, is very nearly the same as that of the air of which it occupies the place, and that consequently the effect of the humidity of the air on the refraction is insensible. Biot has confirmed this result by direct experiments, which shew moreover that the temperature does not influence the refraction, except so far as it produces a change in the density of the air. Finally, Arrogo ascertained, by an ingenious and accurate method, that the influence of the humidity of the air on its refraction is altogether insensible.

It is supposed in the preceding theory that the atmosphere is perfectly calm, so that the density is every where the same at equal heights above the level of the sea. But this hypothesis is affected by winds and inequalities of temperature, which must influence in a very sensible manner the astronomical refractions. However perfect astronomical instruments may be rendered, the effect of these perturbating causes, if it is considerable, will be always an obstacle to the extreme accuracy of observations, which should be multiplied considerably in order to annihilate them. Fortunately we are assured that this effect can never exceed a small number of seconds.

The atmosphere weakens the light of the stars, especially near the horizon, where their rays transverse through a greater extent of it. It follows, from the experiments of Bouguer, that when the height of the barometer is seventy-six centimetres, if the intensity of the light of a star at its entrance into the atmosphere be represented by unity, its intensity when it arrives at the observer, the star being supposed to be in the zenith, will be reduced 0,8123. The height of the homogeneous atmosphere, of which the temperature was zero, would in this case be 7945^{m}. (x) Now it is natural to suppose that the extinction of a ray of light which traverses the atmosphere, is the same as in this hypothesis, since it meets with the same number of aerial particles, consequently a stratum of air of the preceding density, and of which the thickness was 7945^{m}, would reduce the force of light to $0^{m},8123$. It is easy to determine from hence the diminution of light in a stratum of air of the same density, and of any given thickness; for it is evident that if the density of light is reduced to a fourth in traversing a given thickness, an equal thickness will reduce this fourth to a sixteenth of its primitive value; hence it appears that while the thickness increases in arithmetical progression, the intensity of light decreases in geometrical progression; consequently its logarithms are proportional to the thickness. Thus in order to obtain the tabular logarithm of the intensity of light after it has traversed any given thickness, it is necessary to multiply —0,0902835 (which is the tabular logarithm of 0,8123) by the ratio of this thickness to 7945^{m}; and if the density of the air is greater or less than the preceding, it is necessary to diminish this logarithm in the same ratio.

In order to determine the diminution of the light of the stars with respect to their apparent altitude, we may suppose the luminous ray to move in a canal, the air in this canal being reduced to the preceding density. The length of the column of air thus reduced, will determine the extinction of the light of the star which is considered; now we may suppose that from twelve degrees of apparent elevation to the zenith, the path of the light of the stars is rectilineal, and we can, in this interval, consider the atmospherical strata as planes parallel to each other; then the thickness of each stratum in the direction of the ray of light, is to its thickness in a vertical direction, as the secant of the apparent distance of the star from the zenith, (y) is to radius. Therefore if this secant be multiplied by —0,0902835, and by the ratio of the height of the barometer to $0^{m},76$, and if this product be then divided by unity plus 0,00375 multiplied by the number of degrees in the thermometer, we shall have the logarithm of the intensity of light of the star. This rule, which is extremely simple, will determine the extinction of the light of the stars on the summit of mountains and at the level of the sea, and may be usefully applied, both in correcting the observations of the eclipses of Jupiter's satellites, and also in estimating the intensity of solar light in the focus of burning glasses. It ought however to be observed, that vapours floating in the air influence considerably the extinction of light. The serenity of the sky and the rarity of the air make the light of the stars more brilliant on the tops of elevated mountains, and if we could transport our great instruments to the summit of the Cordilieres, there is no doubt but that we should observe several celestial phenomena, which a thicker and less transparent atmosphere renders invisible in our climates.

The intensity of the light of the stars at small altitudes like to their refraction, depends on the density of the elevated strata of the atmosphere.

If the temperature was every where the same, the logarithms of the intensities of light would be proportional to the astronomical refractions (z) divided by the cosines of the apparent heights; and then this intensity at the horizon would be reduced to about the four thousandth part of its primitive value; it is on this account that the sun, whose splendour at noon is too dazling to be borne, can be contemplated without pain at the horizon.

We can by means of these data determine the influence of our atmosphere in eclipses. As it refracts the rays of the sun which traverse it, it inflects them into the cone of the terrestrial shadow, and as the horizontal refractions surpasses the semi sum of the parallaxes of the sun and moon, the centre of the lunar disk supposed to exist on the axis of the cone, receives from the upper and lower limbs of the earth the rays which issue from the same point of the sun's surface; this centre would be therefore more illuminated than in full moon, if the atmosphere did not in a great measure extinguish the light which reaches it. If the light of this point at full moon be taken for unity, it is found by applying the analysis to the preceding data, that the light is 0,02 in the central apogean eclipses, and only 0,0036, or about six times less, in the central perigean eclipses. If it then happens by an extraordinary concurrence of circumstances, that the vapours absorb a considerable part of this feeble light, when it traverses the atmosphere (a) in passing from the sun to the moon, this last star will be altogether invisible. The history of astronomy furnishes us with examples, of rare occurrence indeed, of the total disappearance of the moon in eclipses. The red colour of the sun and moon at the horizon shews that the atmosphere gives a free passage to the rays of this colour, which, on this account, is that of the moon when eclipsed.

In eclipses of the sun, the obscurity which they produce is diminished by the light reflected by the atmosphere. Suppose in fact, the spectator to be placed in the equator, and that the centres of the sun and moon are in his zenith. If the moon was in perigee, the sun would be in the direction of the apogee; in this case the obscurity would be very nearly the most profound, and its duration would be about five minutes and a half. The diameter of the shadow projected on the earth will be the twenty-two thousandth part of that of the earth, and six times and a half less than the diameter of the section of the atmosphere by the plane of the horizon, at least if we suppose the height of the atmosphere equal to a hundreth part of the earth's radius, which is the height inferred from the duration of twilight; and it is very probable that the atmosphere reflects sensible rays from still greater heights. It appears therefore, that in eclipses, the sun illuminates the greater part of the atmosphere which is above the horizon. But it is only illuminated by a portion of the sun's disk, which increases according as the atmospherical molecules are more distant from the zenith ; in this case the solar rays traverse a greater extent of the atmosphere, in passing from the sun to these molecules, and after this in returning by reflexion to the observer, they are sufficiently diminished in intensity to enable us to perceive stars of the first and second magnitude. Their tint, participating of the blue colour of the sky, and of the red colour of twilight, diffuses over all objects a sombre colour, which combined with the sudden disappearance of the sun, fills all animals with terror.

BOOK THE SECOND.

OF THE REAL MOTIONS OF THE HEAVENLY BODIES.

Provehimur portu, terræ urbesque recedunt.

A COMPARISON of the principal appearances of the heavenly bodies, of which the exposition has been given in the preceding book, has led us to make the planets move round the sun, which in its revolution round the earth, carries along with it the foci of their orbits. But the appearances would be precisely the same, if the earth was transported, like the other planets, about the sun: then this star would be the centre of the planetory motions in place of the earth. It is consequently of the greatest importance to the progress of astronomy, to ascertain which of these two cases obtains in nature. We therefore proceed, under the guidance of induction and analogy, to determine, by a comparison of phenomena, the real motions which produce them, and thence to ascend to the laws of these motions.

CHAP. I.

Of the motion of rotation of the earth.

When we reflect on the diurnal motion to which all the heavenly bodies are subject, we evidently recognize the existence of one general cause which moves, or appears to move them about the axis of the world. If it be considered that these bodies are detached from each other, and placed at very different distances from the earth; that the sun and stars are much more removed than the moon, and that the variations of the apparent diameters of the planets indicate great changes in their distances; lastly, that the comets traverse the heavens freely in all possible directions, it will be extremely difficult to conceive that one and the same cause impresses on all these bodies, a common motion of rotation: but since the heavenly bodies present the same appearances to us, whether the firmament carries them about the earth, supposed immoveable, or whether the earth itself revolves in a contrary direction, it seems much more natural to admit this latter motion, and to regard that of the heavens as only apparent.

The earth is a globe, of which the radius is only about four thousand metres; the magnitude of the sun is, as we have seen, incomparably greater. If its centre coincided with that of the earth, its volume would embrace the orb of the moon, and extend as far again; from which we may form some judgment of its immense magnitude, besides its distance from us is about twenty-three thousand times the semidiameter of the earth. Is it not infinitely more probable to suppose that the globe which we inhabit revolves on an axis, than to imagine that a body so considerable and distant as the sun, should revolve with the rapid motion which it should have in order that it might revolve in a day, about the earth? What immense force must it not then require to keep it in its orbit, and to counterbalance its centrifugal force. Each of the stars presents similar difficulties, all of which are removed by supposing the earth to revolve on its axis.

It has been already observed, that the pole of the equator appears to move slowly about that of the ecliptic, from whence results the precession of the equinoxes. If the earth be immoveable, the pole of the equator will be equally so, because it always corresponds to the same point of the earth's surface; consequently the celestial sphere moves round the poles of the ecliptic, and in this motion, it carries along with it all the heavenly bodies. Thus the entire system, composed of so many bodies, differing from each other, in their magnitudes, their motions, and their distances, would be again subject to a general motion, which disappears, and is reduced to a mere appearance, if the axis of the earth be supposed to move round the poles of the ecliptic.

Carried along with a motion in which all the surrounding bodies participate, we are like to a mariner borne by the winds over the seas. He supposes himself to be at rest, and the shore, the hills, and all the objects situated beyond the vessel, appear to him to move. But on comparing the extent of the shore, the plains, and the height of the mountains, with the smallness of his vessel, he is enabled to distinguish the apparent motion of these objects from a real motion to which he himself is subject. The innumerable stars distributed through the celestial regions are, relatively to the earth, what the shore and mountains are with respect to the navigator; and the very same reasons which convince the navigator of the reality of his own motion, evince to us that of the earth.

These arguments are likewise confirmed by analogy. A motion of rotation has been observed in almost all the planets, the direction of which is from west to east, similar to that which the diurnal motion of the heavens seems to indicate in the earth. Jupiter, whose magnitude is considerably greater than that of the earth, revolves on an axis, in less than half a day. An observer on his surface would suppose that the heavens revolved round him in that time; yet that motion would be only apparent. Is it not therefore reasonable to suppose that it is the same with that which we observe on the earth? What confirms, in a very striking manner, this analogy is, that the earth and Jupiter are flattened at the poles. In fact, we may conceive that the centrifugal force which tends to make every particle of a body recede from its axis of rotation, should flatten it at the poles and elevate it at the equator. This force should likewise diminish that of gravity at the equator; and that this diminution does actually take place, is proved by experiments which have been made on the lengths of pendulums. Every thing therefore leads us to conclude that the earth has really a motion of rotation, and that the diurnal motion of the heavens is merely an illusion which is produced by it; an illusion similar to that which represents the heavens as a blue vault to which all the stars are attached, and the earth as a plain on which it rests. Thus astronomy has surmounted the illusions of the senses, but it was not till after they were dissipated by a great number of observations and computations, that man at last recognized the motion of the globe which he inhabits, and its true position in the universe.

CHAP. II.

Of the motion of the earth about the sun.

Since it appears from the preceding chapter that the diurnal revolution of the heavens is an illusion produced by the rotation of the earth, it is natural to think that the annual revolution of the sun, carrying with it all the planets, is also an illusion arising from the motion of translation of the earth about the sun. The following considerations remove all doubt on this subject.

The masses of the sun and of several of the planets are considerably greater than that of the earth; it is therefore much more simple to make the latter to revolve about the sun, than to put the whole solar system in motion about the earth. What a complication in the heavenly motions would the immobility of the earth suppose? What a rapid motion must be assigned to Jupiter, to Saturn, (which is nearly ten times farther from the sun than we are) and to Uranus (which is still more remote,) to make them revolve about us every year, while they move about the sun.? This complication and this rapidity disappear entirely by supposing the earth to revolve about the sun, which motion is conformable to a general law, according to which the small celestial bodies revolve about the larger ones, which are situated in their vicinity.

The analogy of the earth with the planets, confirms the supposition of the earth's motion; like Jupiter it revolves on its axis, and is accompanied by a satellite. An observer at the surface of Jupiter, would suppose that the whole solar system revolved about him, and the magnitude of the planet would render this illusion less improbable than for the earth. Is it not therefore natural to suppose that the motion of the solar system round us, is likewise only an illusion?

Let us transport ourselves in imagination to the surface of the sun, and from thence let the earth and planets be contemplated. All these bodies would appear to move from west to east; this identity in the direction is an evident proof of the motion of the earth; but what evinces it to a demonstration, is the law which exists between the times of the revolutions of the planets, and their distances from the sun. The angular motions are slower for those bodies which are more removed from the sun, and the following remarkable relation has been observed to exist between the times and the distances, namely, that the squares of the times are as the cubes of their mean distances from this star. According to this remarkable law, the duration of the revolution of the earth, supposed to move above the sun, should be exactly a sidereal year. Is not this an incontestable proof that the earth moves like the other planets, and that it is subject to the same laws? Besides, would it not be absurd to suppose that the terrestrial globe, which is hardly visible at the sun, is immoveable amidst the other planets which are revolving about this star, which would itself be carried along with them about the earth? Ought not the force which balances the contrifugal force, and retains the planets in their respective orbits, act also on the earth, and must not the earth oppose to this action the same centrifugal force? Thus the consideration of the planetary motions, as seen from the sun, removes all doubt of the real motion of the earth. But an observer placed on this body, has besides a sensible proof of this motion, in the phenomena of the aberation which is a necessary consequence of it, as we shall now explain.

About the close of the 17th century, Roemer observed that the eclipses of Jupiter's satellites happened sooner than the computed time near the oppositions of this planet with the sun, and that they occurred later towards the conjunctions; this led him to suspect that the light was not transmitted instantaneously *(a)* from these stars to the earth, and that it employed a sensible interval in passing over the diameter of the orbit of the sun. In fact, Jupiter being in the oppositions, nearer to us than in the conjunctions, by a quantity equal to this diameter, the eclipses must happen sooner in the first case than in the second, by the time which the light takes to traverse the solar orbit. The law of the retardation observed in these eclipses, corresponds so exactly to this hypothesis, that it is impossible to refuse our assent to it. It follows therefore that light employs 571″, in coming from the sun to the earth.

Now an observer at rest would see the stars in the direction of their rays, but this is not the case, on the hypothesis that he moves along with the earth. In order to reduce this case to that of a spectator at rest, it would be sufficient to transfer in a contrary direction both to the stars, to their light, and to the observer himself, the motion with which he is actuated, which does not make any change in the apparent position of the stars; for it is a general law of optics, that if a common motion be impressed on all the bodies of a system, there will not result any change in their apparent situation. Suppose then that at the instant a ray of light penetrates the atmosphere, a motion equal and contrary to that of the observer be impressed on the air and the earth; and let us consider what effects this motion ought to produce in the apparent position of the star from which the ray emanates. We may leave out of the question the consideration of the motion of rotation of the earth, which is about sixty times less at the equator itself, than that of the earth about the sun, and we may also, without sensible error, suppose that all the rays of light which each point of the star's disk transmits to us, are parallel to each other, and to the rays which would pass from the centre of the star to that of the earth, on the hypothesis that it was transparent. Thus the phenomena which these stars would present to a spectator situated at the centre of the earth, and which depend solely on the motion of light combined with that of the earth, are very nearly the same for each observer on its surface. Finally, we may neglect the small excentricity of the earth's orbit. This being premised, in the interval of 571′, that light takes to traverse the radius of the earth's orbit, the earth describes a small arc of this orbit equal to 62″5; now it follows from the composition of motions, that if through the centre of the star a small circle parallel to the ecliptic be described, the diameter of which subtends in the heavens an arc of 125″, the direction of the motion of light, when compounded with the motion of the earth applied in a contrary direction, meets this circumference at the point where it is intersected by a plane drawn through the centre of the star and of the earth tangentially to the terrestrial orbit, the star must therefore appear to move in this circumference, and to describe it in (*c*) a year, in such a manner that it is always less advanced by one hundred degrees, than the sun in his apparent orbit.

This is precisely the phenomenon which has been explained in the eleventh chapter of the first book, from the observations of Bradley, to whom we are indebted for its discovery and that of its cause. The true place of the stars is the centre of the small circumference which they appear to describe; their annual motion is only an illusion produced by the combination of the motion of light with that of the earth. From its evident relations with the position of the sun, it might be justly supposed that it was only apparent; but the preceding explanation proves it to a demonstration. It also furnishes a sensible proof of the motion of the earth about the sun, in the same manner as the increase of degrees and of the force of gravity from the equator to the poles, proves the rotation of the earth on its axis.

The aberation of light affects the positions of the sun, the planets, the satellites, and the comets, but in a different manner from the fixed stars, in consequence of their respective motions. In order to divest them of this, and to obtain the true position of the stars, we should impress at each instant, on all these bodies, a motion equal and contrary to that of the earth, which by this means becomes immoveable, and which, as has been already observed, neither changes their respective positions, nor their appearances. It is evident then that the star, at the moment that it is observed, has not the direction of the rays of light which strike our eye; it deviates from it in consequence of its real motion combined *(d)* with that of the earth, which we suppose to be impressed on it in a contrary direction. The combination of these two motions, when observed from the earth, produces the apparent, or as it is termed *the geocentrick motion*. Therefore the true position of the star will be obtained by adding to its observed geocentrick longitude or latitude, its geocentrick motion in longitude and in latitude, in the interval of time which light takes to come from the star to the earth. Thus the centre of the sun always appears to us less advanced in its orbit by 62″,5, than if the light was transmitted to us instantaneously.

The aberration changes the apparent relations of the celestial phenomena with respect both to their situation and duration. At the moment we see them they no longer exist. The satellites of Jupiter have ceased to be eclipsed twenty-five or thirty minutes, when we observe the termination of the eclipse; and the variations of the changeable stars precede by several years, the instant at which they are observed. But the cause of all these illusions being well understood, we can always refer the phenomena of the solar system, to their true place and exact epoch.

The consideration of the celestial motions leads us, then, to displace the earth from the centre of the world, where we had placed it, deceived by appearances, and by the natural propensity of man to regard himself as the principal object of nature. The globe which he inhabits is a planet in motion on itself and about the sun. When it is considered in this point of view, all the phenomena are explained in the simplest manner; the laws of the celestial motions are rendered uniform, and the analogies are all observed. Thus, like Jupiter, Saturn and Uranus, the earth is accompanied by a satellite; it revolves on an axis like Venus, Mars, Jupiter and Saturn, and probably all the other planets; like them it borrows its light from the sun, and revolves about him in the same direction, and according to the same laws. Finally, the hypothesis of the earth's motion combines in its favour simplicity, analogy, and generally every thing which characterises the true system of nature. We shall see, by following it in all its consequences, that the celestial phenomena are reduced in their minutest details to one sole law, of which they are the necessary developements. The motion of the earth will thus acquire all the certainty of which physical truths are susceptible, and which may result either from the great number and variety of phenomena which it explains, or from the simplicity of the laws on which it is made to depend. No branch of natural science combines in a higher degree these criterea than the theory of the system of the world, founded on the motion of the earth.

This motion enlarges our conceptions of the universe, by furnishing for a measure of the distance of the heavenly bodies, an (*e*) immense base, namely, the diameter of the earth's orbit; by means of this, the dimensions of the planetary orbits have been exactly determined. Thus the motion of the earth, after having, by the illusions of which it was itself the cause, retarded our knowledge of the planetary motions for a great length of time, has at last conducted us to a knowledge of them, and that in a more accurate manner than if we had been placed at the focus of these motions. Nevertheless the annual parallax of the fixed stars, or the angle which the diameter of the earth's orbit would subtend at this centre, is insensible, and does not amount to 6″, even relatively to those stars which *(f)* from their great brilliancy appear to be nearest to us; they are therefore at least two hundred thousand times farther from us than the sun. Their great brilliancy, at such an immense distance, proves to us that they do not, like the planets and satellites, borrow their light from the sun, but that they shine with their own proper light; so that they may be considered as so many suns distributed in the immensity of space, and similarly to our own, may be the foci of so many planetary systems. It would in fact be sufficient to place ourselves at the nearest of those stars, in order to see the sun as a luminous star, the diameter of which was less than the thirtieth part of a second.

It follows from the immense distances of the stars, that their motions in right ascension and declination are only apparent, and that they are produced by the motion of the earth's axis of rotation. But some stars appear to have motions proper to themselves, and it is probable that all of them are in motion as well as the sun, which carries with it in space the entire system of the planets and comets, in the same manner as each planet carries along with it, its satellites in their motions about the sun.

CHAP. III.

Of the appearances which arise from the motion of the earth.

From the point of view in which a comparison of the celestial phenomena has placed us, let us consider the stars, and shew the perfect identity of their appearances with those which we observe. Whether the heavens revolve about the axis of the world, or the earth revolves on its axis in a contrary direction to the apparent motion of the heavens, supposed to be at rest, it is clear that the appearances of the stars, on either hypothesis will be precisely the same. The only difference will be, that in the first case they will place themselves over the different terrestrial meridians, which, in the second case, will place themselves under these stars.

The motion of the earth being common to all bodies situated on its surface, and also to the fluids which cover it, their relative motions are the same as if it was immoveable. Thus, in a vessel transported with an uniform motion, every body moves as if it was in a state of rest. A projectile thrown directly upwards falls on the same spot from which it was projected; it appears to those in the vessel to describe a vertical line, but to a spectator on the shore, it will appear to move obliquely to the horizon, and to describe a curve which is sensibly parabolic. However, the real velocity which arises from the rotatory motion of the earth being somewhat less at the bottom than at the top of an elevated tower, if a body be let fall freely from this top, it is evident that in consequence of the excess of its real velocity of rotation above that of the bottom of the tower, it should not fall exactly at the point where the plumb line from the summit of the tower meets the surface of the earth, but a little to the east. In fact, it appears from analysis that its deviation from this point towards the east, is proportional to the sesquiplicate ratio of the height of the tower, and to the cosine of latitude, (a) and that at the equator it is but $21^{m},952$ for one hundred metres of height. We may therefore, by means of very accurate experiments on falling bodies, render the rotation of the earth sensible. Those which have been already instituted with this view in Germany and Italy, agree sufficiently well with the preceding results; but these experiments which require the most delicate manipulation, ought to be repeated with still greater precision. The rotation of the earth is principally indicated at its surface, by the effects of the centrifugal force, which flattens the terrestrial spheroid at the poles, and diminishes the gravity (b) at the equator, two phenomena, which the measures of the pendulum and of the degrees of the meridian, have made known to us.

In the revolution of the earth about the sun, its centre and all the points of its axis of rotation move with equal and parallel velocities; this axis therefore remains *(c)* always parallel to itself. If at every instant, a motion equal and contrary to that of the earth's centre be impressed on the heavenly bodies, and also on all the parts of the earth, this centre would remain immoveable, as also its axis of rotation; but this impressed motion does not change at all the appearances of that of the sun, it only transfers to this star, and in a contrary direction, the real motion of the earth: the appearances are consequently the same, whether the earth be supposed to be at rest, or to revolve about the sun. In order to trace more particularly the identity of these appearances, let us conceive a radius drawn from the centre of the earth to that of the sun; this radius will be perpendicular to the plane which separates the enlightened from the darkened hemisphere of the earth. The sun is vertical to the point where it intersects the surface of the earth, and all the points of the terrestrial parallel, which this ray meets successively, in consequence of the diurnal motion have this star in their zenith at noon. But whether the sun revolves about the earth, or the earth about the sun and on its own axis, as it always preserves its parallel position, it is evident that this radius will trace the same surve on the surface of the earth; in each case it intersects the same terrestrial parallels. When the apparent longitude of the sun is the same, this star will be equally elevated above the horizon, and the duration of the days will be equal. Thus, the seasons and the days are precisely the same, whether the sun be supposed to be at rest, or to revolve about (*d*) the earth; and the explanation of the seasons, which has been given in the preceding book, is equally applicable to the first hypothesis.

The planets all move in the same direction about the sun, but with different velocities; the durations of their revolutions increase in a greater ratio than their distances from this star; for instance, Jupiter employs nearly twelve years to perform its revolution, but the radius of the orbit is only five times greater than the radius of the earth's orbit; its real velocity is consequently less than that of the earth. This diminution of velocity in the planets according as they are more distant from the sun, obtains generally from Mercury, which is the nearest, to Uranus, which is the most remote from this star; and it follows, from the laws which we shall hereafter demonstrate, that the mean velocities of the planets are reciprocally as the square roots of (*e*) their mean distances from the sun.

Let us consider a planet of which the orbit is surrounded by that of the earth, and follow it from its superior to its inferior conjunction: its apparent or geocentric motion is the result of its real motion combined with that of the earth, estimated in a contrary direction. In the superior conjunction, the real motion of the planet is contrary to that of the earth; therefore its geocentrick motion is then equal to the sum of these two motions, and it has the same direction as the geocentrick motion of the sun, which results from the motion of the earth transferred to this star in a contrary direction; consequently the apparent motion of the planet is direct. In inferior conjunction, the direction of the motion of the planet is the same as that of the earth, and as it is greater, the geocentrick motion preserves the same direction, which consequently is contrary to the apparent motion of the sun; therefore the planet is then retrograde. It is easy to conceive that in the passage from the direct to the retrograde motion, it must appear without motion, or stationary, and that this will happen between the greatest elongation and inferior *(f)* conjunction, when the geocentrick motion of the planet resulting from its real motion and that of the earth, applied in a contrary direction, is in the direction of the visual ray of the planet. These phenomena are entirely conformable to the motions that are observed to take place in the planets Mercury and Venus.

The motion of the planets, whose orbits comprehend that of the earth, has the same direction in their oppositions, as the motion of the earth, but it is less, and being combined with this last motion applied in a contrary direction, the direction of the motion which it assumes is opposed to its primitive direction, therefore in this position, the geocentrick motion of these planets is retrograde, it is direct in the conjunctions like the motions of Venus and of Mercury, which are also direct in their superior conjunctions.

If the motion of the earth be transferred to the stars in a contrary direction, they must appear to describe in the interval of a year, a circumference equal and parallel to the terrestrial orbit, the diameter of which would subtend at the star, an angle equal to that under which the diameter of this orbit would appear (*g*) from their centre. This apparent motion is very similar to that which results from the combination of the motion of the earth with that of light, in consequence of which the stars appear to describe annually, a circumference parallel to the ecliptic, the diameter of which subtends an angle equal to 125″, but it differs from it in this, that in the first circumference the position of the stars is precisely the same as that of the sun, whereas in the second circumference they are less advanced than this star, by one hundred degrees. It is by means of this circumstance that we are able to distinguish between these two motions, and that we are assured that the first is at least extremely small, as the immense distance of the fixed stars renders the angle, which the diameter of the earth's orbit subtends when seen from this distance, almost insensible.

As the axis of the world is the prolongation of the axis of rotation of the earth, the motion of the poles of the celestial equator, indicated by the phenomena of precession and nutation (which have been explained in the XIIIth

Chapter of the first book), must be referred to this last axis. Therefore at the same time that the earth revolves on its axis and about the sun, its axis of rotation moves very slowly about the poles of the ecliptic, making very small oscillations, the period of which is the same as the motion of the nodes of the lunar orbit. Finally, this motion is not peculiar to the earth, for it has been observed in the IVth Chapter of the first book, that the axis of the moon moves in the same period, about the poles of the ecliptic.

CHAP. IV.

Of the laws of motion of the planets about the sun, and of the figure of their orbits.

Nothing would be more easy than to calculate from the preceding data, the position of the planets at any given moment, if their motions about the sun were circular and uniform. But they are subject to very sensible inequalities, the laws of which constitute one of the most important objects of Astronomy, and the only clew which can conduct us to a knowledge of the general principle of the heavenly motions. In order to recognize these laws in the appearances which the planets present to us, we must divest their motions of the effects of the motion of the earth, and refer to the sun, their position as observed from different points of the earth's orbit. The dimensions of this orbit must be therefore first of all determined, and the law of the motion of the earth.

It has been shewn in the second Chapter of the first book, that the apparent orbit of the sun is an ellipse of which the earth occupies one of the foci, but as the sun is really immoveable, he should be placed in the focus, and the earth in the circumference of the ellipse. The motion of the sun will be the same, and in order to obtain the position of the earth as seen from the centre of the sun, we should increase the position of that star, by two right angles.

It was also observed, that the sun appears to move in his orbit in such a manner that the radius vector, which connects its centre with that of the earth, traces about it areas proportional to the times in which they are described, but in reality these areas are traced about the sun. In general, every thing that has been stated in the chapter already cited, relative to the excentricity of the solar orbit and its variations, and respecting the position and motion of its perigee, may be also applied to the terrestrial orbit, with this sole exception, namely, that the earth's perigee is distant by two right angles from the perigee of the sun. The figure of the earth's orbit being thus known, let us examine how those of the other planets may be determined. For example, let us consider the planet Mars, which, from the great excentricity of its orbit, and its proximity to the earth, is peculiarly well adapted to make known the laws of the planetary motions.

The orbit of Mars and its motion about the sun would be known, if the angle which its radius vector makes with an invariable line passing through the centre of the sun be known at any instant, and also the length of this radius. In order to simplify the problem, we select those positions of Mars, in which one of these quantities can be found separately; and this is very nearly the case in the oppositions, when the planet is observed to correspond to the same point of the ecliptic, to which it would be referred from the centre of the sun. From the difference between the angular motions of the earth and Mars, this planet corresponds to different points of the heavens in successive oppositions, therefore by comparing together a great number of observed oppositions, we are enabled to discover the law which exists between (*a*) the time and the angular motion of Mars about the sun, which is termed his *heliocentrick motion.* The different methods which are furnished by analysis, are considerably simplified in the present case, by considering that as the principal inequalities of Mars become the same at the termination of each sidereal revolution, their sum may be (*b*) expressed by a rapidly converging series of the sines of angles which are multiples of its mean motion, the coefficients of which series may be easily determined by means of some select observations.

The law of the radius vector of Mars may afterwards be obtained, by comparing observations of this planet made near its quadratures, in which case the angle which this radius subtends is the greatest. In the triangle formed by lines which join the centres of the earth, of the sun and of Mars, the angle at the earth is determined by direct observation, the law of the heliocentrick motion (*c*) of Mars, furnishes the angle at the sun, by means of which we may determine the radius vector of Mars in parts of the radius of the earth, which is itself determined in parts of the mean distance of the earth from the sun. By comparing together a great number of radii vectores thus determined, the law of their variations corresponding to the angles which they make with an invariable right line, may be determined, by which means the figure of the orbit can be traced.

It was by a method very nearly similar, that Kepler discovered the lengthened form of the orbit of Mars; he conceived the fortunate idea of comparing its figure with that of an ellipse, the sun being in one of the foci; and the numerous observations of Tycho exactly represented in the hypothesis of an elliptic orbit, left no doubt as to the truth of this hypothesis.

The extremity of the greater axis of the orbit which is nearest to the sun, is called the *perihelion*, and the *aphelion* is the extremity which is farthest from the sun. The angular velocity of Mars about the sun is greatest at the perihelion; it diminishes according as the radius vector increases, and it is least at the aphelion. A comparison of this velocity with the powers of the radius vector, shews that it is reciprocally proportional to its square, from which it follows, that the product (d) of the daily helioecentrick motion of Mars, into the square of its radius, is constant. This product is double of the small vector, traced by its radius about the sun, therefore the area which it describes departing from an invariable line passing through the centre of the sun, increases as the number of days which have lapsed since the epoch when the planet was upon this line ; consequently the areas described by the radius of Mars are proportional to the times. These laws of the motion of Mars, which have been discovered by Kepler, are the same as those of the apparent motion of sun, which have been developed in the second Chapter of the first book, they equally obtain in the case of the earth. It was natural to extend them to the other planets; Kepler therefore established as fundamental laws of the motions of these bodies, the two following, which all subsequent observations have fully confirmed.

The orbits of the planets are ellipses, of which the centre of the sun occupies one of the foci.

The areas described about this centre by the radii vectores of the planets, are proportional to the times of their description.

These laws are sufficient to determine the motion of the planets about the sun ; but besides it is necessary to know for each of them, seven quantities, which have been called the elements of *elliptic motion.* Five of these elements respect the motion in the ellipse, and are, 1st, the duration of the sidereal revolution ; 2d, the semiaxes major of the orbit, or the mean distance of the planet from the sun; 3d, the excentricity, from which may be obtained the greatest equation of the centre; 4th, the mean longitude of the planet at a given epoch ; 5th, the longitude of the perihelion at the same epoch. The two other elements are relative to the position of the orbit itself; and are 1st, the longitude at a given epoch, of the nodes of the orbit, or of its points of intersection with a plane which is usually assumed to be that of the ecliptic. 2d, The inclination of the orbit to this plane. Therefore, for the seven planets which were known previous to the present century, there were forty-nine elements to be determined. The following table exhibits all those elements for the first instant of the present century, *i. e.* for the first of January, 1801, at midnight, according to the mean time of Paris.

The examination of this table shews that the durations of the revolutions of the planets increase with their mean distances from the sun. Kepler, for a long time, sought the relation which existed between the distances and periods; after a great number of trials, continued during sixteen years, he at length recognized that the squares *(e)* of the times of the planets' revolutions, are to each other as the cubes of the major-axes of their orbits.

Such are the fundamental laws of the planetary motion, which by exhibiting atronomy under a new aspect, have led to the discovery of universal gravitation.

The planetary ellipses are not invariable; their major axes appear to be always the same; but their excentricities, their inclinations to a fixed plane, the positions of their nodes and perihelions, are subject to variations, which hitherto appear to increase proportionally to the time. As *(f)* these variations do not become sensible until after the lapse of ages, they have been denominated *secular inequa-*

lities. There can be no doubt of their existence; but modern observations are not sufficiently removed from each other, nor are the ancient observations sufficiently exact to enable us to determine exactly their precise quantity.

There have been likewise observed *periodic* inequalities, which derange the elliptic motions of the planets. That of the earth is a little affected; for it has been before observed, that the apparent elliptic motion of the sun appears to be so. But these inequalities are principally apparent in the two larger planets, Jupiter and Saturn. From a comparison of ancient with modern observations, astronomers have inferred a diminution in the duration of Jupiter's revolution, and an increase in that of Saturn. A comparison of (g) modern observations with each other furnishes a contrary result; which seems to indicate in the motion of these planets, great inequalities of very long periods. In the preceding century, the duration of the revolutions of Saturn seemed to be different, according as the departure of the planet is supposed to take place from different points of its orbit; its returns to the vernal equinox, have been more rapid than to the autumnal. Finally, Jupiter and Saturn experience inequalities, which amount to several minutes, and which seem to depend on the situation of these planets, either among themselves, or with respect to their perihelions. Thus, every thing indicates that in the planetary system, independently of the principal cause which makes the planets to revolve in elliptic orbits about the sun; there exists several particular causes, which derange their motions, and at length change the elements of their ellipses.

TABLE OF THE ELLIPTIC MOTION OF THE PLANETS.

Durations of their sidereal revolutions.

		days.
Mercury	.	87, 96925804
Venus	.	224, 70078690
The Earth	.	365, 25638350
Mars .	.	686, 9796458
Jupiter	.	4332, 5848212
Saturn	.	10759, 2198174
Uranus	.	30686, 8208296

Semiaxes axes majores of their orbits, or their mean distances.

Mercury	.	.	0,3870981
Venus	.	.	0,7233316
The Earth	.	.	1,0000000
Mars .	.	.	1,5236923
Jupiter	.	.	5,202776
Saturn	.	.	9,5387861
Uranus	.	.	19,1823901

Ratio of the excentricities to the semiaxes majores at the commencement of 1801.

Mercury . .	0,20551494
Venus . .	0,00686074
The Earth . .	0,01685318
Mars . .	0,0933070
Jupiter . .	0,048162160
Saturn . .	0,05615051
Uranus . .	0,0466108

The mean longitude for the midnight, which separates the 31st of December, 1800, and the 1st of January, 1801, mean time at Paris.

Mercury . .	182°,15647
Venus . .	11,93259
The Earth . .	111,28179
Mars . .	71,24071
Jupiter . .	124,68251
Saturn . .	150,35354
Uranus . .	197,55589

Mean longitude of the perihelion, at the same epoch.

Mercury . .	82°,6256
Venus . .	143,0349
The Earth . .	110,5571
Mars . . .	369,3323
Jupiter . .	12,3810
Saturn . .	99,0647
Uranus . .	186,1500

The sidereal and secular motion of the perihelion, (the sign — indicates a retrogade motion.)

			″
Mercury	.	.	1801,22
Venus .	.	.	826,76
The Earth	.	.	3646,61
Mars .	.	.	4882,70
Jupiter	.	.	2054,44
Saturn	.	.	5978,67
Uranus	.	.	740,98

The inclination of the orbit to the ecliptic at the commencement of 1801.

			°
Mercury	.	.	7,78058
Venus .	.	.	3,76807
The Earth	.	.	0,00000
Mars .	.	.	2,05746
Jupiter .	.	.	1,46029
Saturn .	.	.	2,77027
Uranus	.	.	0,86063

The secular variation of the inclination to the true ecliptic, (the sign — indicates a diminution.)

				″
Mercury	.	.	.	56,12
Venus	.	.	.	14,05
The Earth		.	.	0,00
Mars	.	.	.	0,81
Jupiter	.	.	.	69,77

		″
Saturn	. . .	47,88
Uranus	. . .	9,73

The longitude of the ascending node at the commencement of 1801.

Mercury	. .	51°,0651
Venus	. . .	83,2262
The Earth	. .	0,0000
Mars	. . .	53,3344
Jupiter	. . .	109,3762
Saturn	. . .	124,3819
Uranus	. .	81,1035

The sidereal and secular motion of the node on the true ecliptic (the signs — indicates a retrograde motion,)

		″
Mercury	. .	2414,39
Venus	. . .	5775,92
The Earth	. .	0,00
Mars	. . .	7187,50
Jupiter	. .	4880,97
Saturn	. . .	5995,35
Uranus	. .	11107,43

The elements of the orbits of the four planets recently discovered cannot be yet obtained with precision, as the time during which they have been observed has been very short; besides the considerable perturbations which they experience, have not as yet been determined. Underneath are presented the elliptic elements which best satisfy the observations hitherto made, but they ought only to be considered as a first sketch of the theory of the planets.

Durations of their sidereal revolutions.

	days.
Ceres . .	1681,3931
Pallas . .	1686,5388
Juno . . .	1592,6608
Vesta . .	1325,7431

Semi axes-majores of their orbits.

Ceres . . .	2,767245
Pallas . .	2,772886
Juno . . .	2,669009
Vesta . .	2,36787

Ratio of the excentricity to the semiaxis major.

Ceres . . .	0,078439
Pallas . .	0,241648
Juno . . .	0,257848
Vesta . . .	0,089130

Mean longitude at the midnight commencing 1820.

	°
Ceres . . .	136,8461
Pallas . . .	120,3422
Juno . . .	222,3989
Vesta . . .	309,2917

Longitude of the perihelion at the same epoch.

	″
Ceres . . .	163, 4727
Pallas . . .	134, 5754
Juno . . .	59, 5142
Vesta . . .	277, 2853

Inclination of the orbit to the ecliptic.

	°
Ceres . . .	11,8044
Pallas . . .	38,4344
Juno . . .	11,5215
Vesta . . .	7,9287

Longitude of the ascending node at the commencement of 1810

Ceres . . .	87°,6557
Pallas . . .	191,8416
Juno . . .	190,1421
Vesta . . .	114,6908

CHAP. V.

Of the figure of the orbits of the comets, and of the laws of their motion about the sun.

THE sun being at the focus of the planetary orbits, it is natural to suppose that he is also in the focus of the orbits of the comets. But as these stars disappear after having been visible some months at most, their orbits, instead of being nearly circular, like those of the planets, are very excentric, and the sun is very near to that part in which they are visible. The ellipse, by means of the infinite varieties which it admits of from the circle to the parabola, may represent these different orbits. Analogy leads us then to suppose that the comets move in ellipses, of which the sun occupies one of the foci, and to consider them as moving according to the same laws as the planets, so that the areas traced by their radii vectores are equal in equal times.

It is almost impossible to know the duration of the revolution of a comet, and consequently the greater axis of its orbit, by an observation of only one of its appearances; hence the area which its radius vector describes in a given time, cannot be determined rigorously. But it should be considered that the small portion of the ellipse, described by the comet during its appearance, may be supposed to coincide with a parabola, and that thus its motion may be calculated in this interval, as if it was parabolical.

According to the laws of Kepler, the sectors, traced in equal times by the radii vectores of two planets, are to each other as the areas of *(a)* their ellipses, divided by the times of their revolutions; and the squares of these times are to each other as the cubes of their greater semiaxes. It is easy to infer from this, that if a planet be supposed to move in a circular orbit, of which the radius is equal to the perihelion distance of the comet, the sector, described by the radius vector of the comet, will be to the corresponding sector described by the radius vector of the planet, in the ratio of the square root of the aphelion *(b)* distance of the comet to the square root of the semiaxis major of its orbit, which ratio, when the ellipse changes into a parabola, becomes that of the square root of two to unity; by this means, the ratio of the sector of the comet to that of the fictitious planet may be obtained; and it is easy by what precedes to obtain the ratio of this sector to that which the radius vector of the earth traces in the same time. We can therefore determine for any instant whatever, the area traced by the radius vector of the comet, commencing with the moment of its passage through the perihelion, and fix its position in the parabola, which it is supposed to describe.

Nothing more is necessary but to determine, by means of observations, the elements of the parabolick motion, that is to say, the perihelion distance of the comet, in parts of the mean distance of the sun from the earth, the position of the perihelion, the instant of the passage through the perihelion, the inclination of the orbit to the ecliptic, and the position of its nodes. The investigation of these five elements presents greater difficulties than that of the elements of the planets, which being always visible, may be compared in positions the most favourable to the determination of these elements, while on the other hand, the comets are only visible for a short time, and almost always in circumstancs, in which their apparent motion is extremely complicated by the real motion of the earth, which we transfer to them in a contrary direction. Notwithstanding all these obstacles, we have succeeded by different methods, in determining the orbits of the comets. Three complete observations are more than sufficient for this purpose; all the others serve only to confirm the accuracy of these elements, and the truth of the theory which we have explained. More than one hundred comets, of which the numerous observations, are exactly represented by this theory, remove all doubt as to its accuracy. Thus, the comets, which for a long time were regarded as meteors, are stars similar to the planets; their motions and their returns are regulated by laws similar to those which influence the planetary motions.

Let us take notice here, how the true system of nature, according as it developes itself, receives more confirmation. The simplification of the celestial phenomena, on the hypothesis that the earth moves, compared with their great complexity, on the hypothesis of its immobility, renders the first of these hypotheses extremely probable. The laws of elliptic motion, common then to the planets and to the comets, increase this probability considerably, which becomes still greater from the consideration that the motions of the comets are subject to the same laws.

These stars do not all move in the same direction, like the planets. Some have an actual direct motion, the direction of the motion of others is retrograde; the inclinations of their orbits are not confined within a narrow zone like those of the planetary orbits; they exhibit all varieties of inclination, from the orbit situated on the plane of the ecliptic, to an orbit perpendicular to this plane.

A comet is recognized when it reappears, by the identity of the elements of its orbit with those of the orbit of a comet already observed. If the perihelion distance, the position of this perihelion and of its nodes, and the inclination of its orbit be very nearly the same, it is then extremely probable that the comet which appears is that which had been observed before, and which after having receded to such a distance that it was invisible, returns into that part of its orbit which is nearest to the sun. As the durations of the revolutions of comets are very long, and as it is not quite two centuries since these stars have been carefully observed, the period of the revolution of only one comet is known with certainty, namely, that of 1759, which had been before observed in 1682, 1607, and 1531. This comet returns to its perihelion in about seventy-six years. Therefore if the mean distance of the sun from the earth be assumed equal to unity, the greater axis of its orbit is very nearly 35,9 ; and as its perihelion distance is only 0,58, it recedes from the sun, at least thirty-five times farther than the earth, describing an extremely excentric ellipse. Its return to the perihelion was longer by thirteen months from 1531 to 1607, than from 1607 to 1682 ; and it was eighteen months shorter, from 1607 to 1682, than from 1682 to 1759. It appears therefore that causes similar to those which derange the elliptic motion of the planets, disturb also that of the comets in a much more sensible manner.

The return of some other comets has been suspected ; the most probable of these returns was that of the comet of 1532, which was supposed to be the same with that of 1661, the time of the revolution of which has been fixed at 129 years, but this comet not having appeared in 1790, as was expected, there is great reason to believe that these two comets were not the same, and we shall not be surprized at this, if we consider the inaccuracy of the observations of Appian and Frucastor, from which the elements were determined in 1532.

These observations are so rude, that according to Mechain, who has carefully examined them, they leave an uncertainty of 41°, on the position of the node, of 10°, on the inclination, of 22°, on the position of the perihelion, and of 0,255 on the perihelion distance.

The elements of the orbit of the comet observed in 1818, correspond so exactly with those of the orbit of the comet observed in 1805, that it has been inferred that these comets are the same, which would assign the short period of thirteen years for the time of revolution, provided that there was no intermediate return of the comet to its perihelion; but M. Enk has ascertained by a careful discussion of the numerous observations of this star in 1818 and 1819, that its revolution is still less by 1203^d very nearly; he concluded that it should reappear in 1822: and in order to facilitate to observers the means of finding it again, he computed the position which it aught to have on each day of its approaching appearance. From the southern declinations of the comet during the time of this appearance, it is almost impossible to observe it in Europe. Fortunately it has been observed at New Holland by M. Rumker, an expert Astronomer, who was brought there by General Brisbane, who is himself an able Astronomer, and has interested himself very much in the advancement of this science. M. Rumker observed it for each successive day, from the 2d to the 23d of June 1822, and its observed positions accord so well with those which M. Enk had previously calculated, that there cannot remain any doubt on this return of the comet, predicted by M. Enk.

The nebulosity with which the comets are almost always surrounded, seems to be formed by the vapours which the solar heat excites from their surface. In fact, the great heat which they experience near to their perihelion, may be supposed to rarify the particles which have been congealed by the excessive cold of the aphelion. This heat is most intense for those comets, whose perihelion distance is very small. The perihelion distance of the comet of 1680, was one hundred and sixty-six times less than the distance of the sun from the earth, and consequently it ought to experience a heat twenty-seven thousand five hundred times greater than that which is communicated to the earth, if, as (*d*) every thing induces us to think, the heat is proportional to the intensity of its light. This excessive heat, which is much greater than any which we could produce, would volatilise, according to all appearances, the greater number of terrestrial substances.

Whatever be the nature of heat, we know that it dilates all bodies. It changes solids into fluids, and fluids into vapours. These changes of form are indicated by certain phenomena which we will trace from ice. Let us consider a volume of snow or of pounded ice in an open vessel submitted to the action of a great heat. If the temperature of this ice be below that of melting ice, it will increase up to zero of temperature. After having attained this (*e*) point, the ice will melt by new additions of heat; but if care be taken to agitate it, until all the ice is melted, the water into which the ice is converted, will always remain at the same temperature, and the heat communicated by the vessel will not be sensible to the thermometer immersed in it, as it will be entirely occupied in converting the ice into water. After all the ice is melted, the additional heat will continually raise the temperature of the water and of the thermometer till the moment of ebullition. The thermometer will then become stationary a second time; and the heat communicated by the vessel will be entirely employed in reducing the water into steam, the temperature of which will be the same as that of boiling water. It appears from this detail, that the water produced by the melting of ice and the vapours into which boiling water is converted, absorb at the moment of their formation a considerable quantity of caloric, which reappears in the reconversion of aqueous vapours to the state of water, and of water to the state of ice; for these vapours, when condensed on a cold body, communicate much more heat to it than it would receive from an equal weight of boiling water; besides we know that water can preserve its fluidity, though its temperature may be several degrees below zero; and that in this state, if it is slightly agitated, it is converted into ice, and the thermometer, when plunged in it, ascends to zero, in consequence of the heat given out during this change. All bodies which we can make pass from a solid to a fluid (*f*) state, present similar phenomena; but the temperatures at which their fusion and ebullition commences, are very different for each of them.

The phenomenon which has been just detailed, although very universal, is only a particular case of the following general law, "*in all the changes of condition which a body undergoes from the action of caloric, a part of this caloric is employed in producing them, and becomes latent,* that is to say, insensible *to the thermometer; but it reappears when the system returns to its primitive state.*" Thus when a gas contained in a flexible envelope is dilated by an increase of temperature, the thermometer is not affected by the part of the caloric which produces this effect, but this latent part becomes sensible when the gaz is reduced by compression to its original density.

There are bodies which cannot be reduced to a state of fluidity, by the greatest heat which we can produce. There are others which the greatest cold experienced on earth is unable to reduce to a solid state: such are the fluids which compose our atmosphere, and which, notwithstanding the pressure and cold to which they have been subjected, have still maintained themselves in the state of vapours. But their analogy with aeriform fluids, to which we can reduce a great number of substances by the application of heat, and their condensation by compression and cold, leaves no doubt but that the atmospheric fluids are extremely volatile bodies, which an intense cold would reduce to a solid state. To make them assume this state, it would be sufficient to remove the earth farther from the sun, as it would be sufficient in order that water and several other bodies should enter into our atmosphere, to bring the earth nearer to the sun. These great vicissitudes take place in the comets, and principally on those which approach very near to the sun in their perihelion. The nebulosities which surround them, being the effect of the vaporisation of fluids at their surface, the cold which follows ought to moderate the excessive heat which is produced by their proximity to the sun; and the condensation of the same vaporised fluids when they recede from it, repairs in part the diminution of temperature, which this remotion ought to produce, so that the double effect of the vaporisation and condensation of fluids, makes the difference between the extreme heat and cold, which the comets experience at each revolution, much less than it would otherwise be.

When the comets are observed with very powerful telescopes, and under circumstances in which we ought only to perceive a part of the illuminated hemisphere, we are not able to discover any phases. One only, comet namely, that of 1682, presented them to Hevelius and La Hire.

We shall see in the sequel, that the masses of the comets are extremely small, the diameters of their disks must therefore be nearly insensible, and what is termed their *nucleus* is most probably made up in a great part, of the densest strata of the nebulosity which surrounds them. Thus Herschel has discovered by means of very powerful telescopes in the nucleus of the comet of 1811, a brilliant point which he judged with reason to be the disk of the comet. These strata are extremely rare, in as much as the stars have been sometimes observed through them. It appears that the tails which accompany the comets, are formed by the most volatile particles, which are excited at their surface by the heat of the sun, and which are dispersed indefinitely by the impulsion of its rays. This may be inferred from the direction of these trains of vapour, which are always beyond the comet relatively to the sun, and which continually increasing according as these stars approach to this luminary, do not attain their maximum till after these bodies have passed through the perihelion. From the extreme tenuity of the molecules, the ratio of the surfaces to the masses is increased, so that it may render sensible the impulsion of the solar rays, (*g*) which ought then to make each particle to describe an hyperbolic orbit, the sun being in the focus of the corresponding conjugate hyperbola. A succession of molecules moving on these curves from the head of the comet, form a luminous train directed from the sun, and forming a small angle with that part of the comet's orbit which it has passed over; this is in fact what observation indicates. From the quickness with which these tails increase, we may form some estimate of the rapidity of ascension of their molecules. It is possible to conceive that differences of volatility, of magnitude, and of density, in the molecules, may produce considerable differences in the curves which they describe, which must cause great varieties in the form, the length, and the magnitude of the tails of the comets. If these effects be combined with those which may arise from a rotatory motion in these stars, and from the illusions of the annual parallax, we may be able to account for the singular appearances which their nebulosities and tails exhibit to us.

Although the dimensions of the tails of the comets may be several millions of myriametres, still they do not sensibly dim the light of the stars, which are seen through them; they are therefore extremely rare, and it is probable that their masses are less than those of the smallest mountains of the earth. Consequently in the event of their meeting with this planet, they cannot produce any sensible effect. It is extremely probable that they have several times enveloped it without its being observed. The state of the atmosphere has a considerable influence on their apparent length and magnitude; between the tropics they appear much greater than in our climates. Pingre states, that he observed a star which appeared in the tail of the comet of 1769, and which receded from it in a few moments. But this appearance is only an illusion, which is produced by the clouds floating in our atmosphere, which are sufficiently dense to intercept the feeble light of this tail, at the same time that they are sufficiently rare to enable us to perceive the more vivid light of the star. It cannot (*h*) be supposed that the molecules of the vapours of which the tails are composed, make such rapid oscillations, of which the extent surpasses a million of myriametres.

If the evaporable substances of a comet diminish at each of its returns to the perihelion, they ought after several revolutions to be entirely dissipated in space, and the comet ought only to exhibit afterwards the appearance of a solid neucleus; those comets whose revolution is short, will arrive at this state sooner than others. The comet of 1682, the time of whose revolution is only seventy-six years, is the only one which has as yet exhibited appearances which correspond to this state of fixity. If the neucleus be too small to be perceived, or if the evaporable substances which remain at its surface, are in too small a quantity to constitute by their evaporation, a sensible head to the comet; the star will be for ever invisible. Perhaps this is one of the reasons, which renders the reappearances of the comets so rare; perhaps it is on this account that the comet of 1770 has totally disappeared, though during the time of its appearance it described an ellipse in a period of five years and a half; so that if it has continued to describe this curve, it must since that epoch have returned at least five times to its (*i*) perihelion. Perhaps finally this is the cause, why several comets whose routs we can trace in the heavens by means of the elements of their orbits, have disappeared sooner than might be expected.

CHAP. VI.

Of the laws of the motion of the satellites about their respective primary planets.

We have explained in the sixth chapter of the first book the laws of the motion of the satellite of the earth, it now remains to consider those of the motions of the satellites of Jupiter, of Saturn, and of Uranus.

If the semidiameter of the equator of Jupiter, which is supposed to be 56″,702 at the mean distance of Jupiter from the sun, be assumed equal to unity, the mean distances of the satellites from its centre and the durations of their sidereal revolutions will be (*a*) as follows:

	Mean distances.	Durations. days.
I. Satellite	6,04853	1,769137788148
II. Sat. . .	9,62347	3,551181017849
III. Sat. . .	15,35024	7,154552783970
IV. Sat. . .	26,99835	16,688769707084

It is easy to infer the durations of the synodic revolutions of the satellites, or the intervals between the return of their mean conjunctions with Jupiter, from the durations of their sidereal revolutions, and from that of the revolution of Jupiter. From a comparison of their mean distances with the durations of their sidereal revolutions, it appears that the same beautiful proportion which has been observed to obtain between the durations of the revolutions of the planets and their mean distances from the sun, obtains also in the case of the satellites, namely, that the squares of the times of the sidereal revolutions of the satellites are as the cubes of their mean distances from the centre of Jupiter. The frequent recurrence of the eclipses of Jupiter's satellites, has furnished astronomers with the means of tracing their motions with a precicision, which could not be obtained by observing their angular distance from Jupiter. They have enabled us to recognize the following results :

The ellipticity of the orbit of the first satellite is insensible; its plane coincides very nearly with the plane of Jupiter's equator, the inclination of which to the plane of the orbit is about 4°,4352.

The ellipticity of the orbit of the second satellite is also insensible, its inclination to Jupiter's orbit is variable, as is also the position of its nodes. All these variations may be very nearly represented, by supposing the orbit of the satellite to be inclined in an angle of *5152″* to the equator of Jupiter, and by making its nodes to move on this plane with a retrograde motion, of which the period is thirty Julian years.

A slight ellipticity has been observed in the orbit of the third satellite, the extremity of its greater axis which is nearest to Jupiter, and which has been termed its *perigove*, has a direct motion, but of a variable quantity. The excentricity of the orbit is also subject to very sensible variations near the close of the last century, the equation of the centre had attained its *maximum*, and amounted to 2458″ very nearly: it afterwards diminished, and near to 1777 it was at its *minimum*, when it amounted to 949″. The inclination of the orbit of this satellite to that of Jupiter, and the position of its nodes are variable; all these variations may be very nearly represented, by supposing the orbit to be inclined at an angle of 2284″, to the equator of Jupiter, and by assigning to its nodes a retrograde motion on the plane of this equator, in a period of 142 years. Notwithstanding this, astronomers who have determined by the eclipses of this satellite (*b*) the inclination of the equator of Jupiter on the plane of its orbit have found that it is invariably nine or ten minutes less than what is assigned by the eclipses of the first and of the second satellite. The orbit of the fourth satellite has a very visible ellipticity; its perigove moves in consequentia with an annual motion amounting to 7939″. The inclination of this orbit to that of Jupiter is about 2°,7. It is in consequence of this inclination, that the fourth satellite passes frequently behind the planet relatively to the sun without being eclipsed. From the discovery of the satellites until the year 1760, the inclination appeared to be constant, and the annual motion of the nodes on the orbit of Jupiter, has been direct and about 788″. But, since 1760, the inclination has increased, and the motion of the nodes has diminished in a very sensible manner. We shall resume the consideration of these inequalities, after their cause shall have been explained.

Independently of these variations, the satellites are subject to inequalities, which derange their elliptic motions, and render their theory extremely complicated. They are principally sensible in the three first satellites, of which the motions exhibit very remarkable relations.

It appears from a comparison of the times of their revolutions, that the period of the first satillite is only about half the duration of the period of the second, which itself is only half of that of the period of the third satellite. Thus, the mean motions of these three satellites follow very nearly a geometric progression, of which the ratio is one half. If this proportion obtained accurately, the mean motion of the first satellite, plus twice the mean motion of the third, would be precisely equal to three times the mean motion of the second. But this equality is much more accurate than the progression itself; so that we are induced to consider it as rigorously true, and to reject the very small quantities by which it deviates from it, as arising from the errors of observation; at least we can affirm that it will subsist for a long series of ages.

A result which is equally remarkable, and which is given by observation with equal precision, is, that from the discovery of the satellites, the mean longitude of the first minus three times that of the second, plus twice that of the third, does not differ from two right angles, by any perceptible quantity.

These two results also obtain, between the mean motions, and the mean synodic longitudes; for as the synodic motion of a satellite is the excess of its sidereal motion above that of the planet, if in the preceding results, the synodic motions be substituted in place of the sidereal motions, the mean motion of Jupiter disappears, and these results remain the same. It follows from this, that for a great number of years at least, the three first satellites of Jupiter cannot be eclipsed together, but in the simultaneous eclipses of the first and third, the first will be always in conjunction with Jupiter; it will be always in opposition, in the simultaneous eclipses of the sun, produced at Jupiter by the two other satellites.

The periods and the laws of the principle inequalities of these satellities are the same. The inequality of the first advances or retards its eclipses, by 223″,5 of time at its *maximum*. A comparison of its quantity, in the respective positions of the two first satellites, shews that it disappears when these satellites seen from the centre of Jupiter, are at the same time, in opposition to the sun; that it afterwards increases, and becomes the greatest possible, when the first satellite at the moment of its opposition is 50° more advanced than the second; that it vanishes again when it is more advanced by 100 than the second, and that beyond this, it is affected with a contrary sign and retards the eclipses; that it increases until the distance of the planets from each other is 150°, when it is at its negative *maximum*; that then it diminishes, and disappears when this distance is 200°; finally, in the second half of the circumference, it runs through the same series of changes as in the first. From this it has been inferred, that there exists in the motion of the first satellite about Jupiter, an inequality, which at its maximum is 5050″,6 of a degree, and proportional to the sine of double of the excess of the mean longitude of the first satellite above that of the second, which excess is equal to the difference of the mean synodic longitudes of the two satellites. The period of this inequality is only four days; but how is it transformed in the eclipses of the first satellite into a period of 437^d,6592? this is what we proceed to explain.

Suppose that the first and second satellite depart together from their mean oppositions with the sun. After the description of each circumference, the first satellite will be, in virtue of its mean synodic motion, in its mean opposition with the sun. If we suppose an imaginary star, of which the angular motion is equal to the excess of the mean synodic motion of the first satellite, above twice that of the second; then twice the difference of the mean synodic motions of the two satellites will be, in the eclipses of the first, equal to a multiple of the circumference plus the motion of the imaginary star; consequently the sine of this last motion will be proportional to the inequality of the first satellite in the eclipses, and may represent it. Its period is equal to the duration of the revolution of the imaginary star, which duration is, from the mean synodic motions of the two satellites about $437^d,6592$; it is thus determined with greater accuracy than by direct observation.

The law of the inequality of the second satellite, is precisely the same as that of the first, with this difference, that it is always of a contrary sign; at its *maximum* it advances or retards the eclipses by about $1059'',2$ of a degree; from a comparison of the respective positions of the two satellites, it appears that it vanishes when they are at the same time in opposition to the sun; that it then retards the eclipses of the second more and more, until those two satellites are at the moment of the occurrence of the phenomena, elongated from each other one hundred degrees; that this retardation diminishes and becomes nothing a second time, when the mutual distance of the two satellites is two hundred degrees; finally, that beyond this time, the eclipses advance as they had previously retarded. From these observations it has been inferred, that there exists in the motion of the second satellite, an inequality of $11920'',7$ at its *maximum*, and that it is proportional to, and affected with a contrary sign, to the sine of the mean longitude of the first satellite over that of the second, which excess is equal to the difference of the mean synodic motions of the two satellite.

If the two depart together from their mean opposition to the sun, the second will be in its mean opposition after the completion of each circumference, which it will have described in consequence of its mean synodic motion. If, as in the case of the first satellite, we conceive a star of which the angular motion is equal to the excess of the mean synodic motion of the first satellite, over twice that of the second, then the difference of the mean synodic motions of the two satellites will be, in the eclipses of the second equal to a multiple of the circumference, plus the motion of the fictitious star; consequently the inequality of the second satellite will, in its eclipses, be proportional to the sine of the motion of this imaginary star. Hence we see the reason why the period and the law of this inequality, are the same, as those of the inequality of the first satellite.

The influence of the first satellite, on the inequality of the second is very probable. But if the third produces in the motion of the second, an inequality similar to that which the second seems to produce in the motion of the first, that is to say, proportional to the sine of double of the difference of the mean longitudes of the second and third satellite; this new inequality will be confounded with that which arises from the first satellite, for in consequence of the relation which exists between the mean longitudes of the three first satellites, and what has been already explained, the difference of the mean longitudes of the two first satellites is equal to the same circumference plus, twice the difference between the mean longitudes of the second and third satellites, so that the sine of the first difference is the same as the sine of twice the second difference, only affected with a contrary sign. The inequality produced by the third satellite, in the motion of the second, would thus have the same sign, and would follow the same law as the inequality observed in this motion; it is therefore extremely probable that this inequality is the result of two inequalities depending on the first and third satellite. If in the progress of time, the preceding relation between the longitudes should cease to exist; these two inequalities which are now blended together would be separated, and we might by observation determine their respective values. But we have seen that this relation must subsist for a very long time, and in the fourth book it will appear, that this relation is rigorously true. Finally, the inequality relative to the third satellite in its eclipses, compared with the respective positions of the second and third, presents the same relations as the inequality of the second, compared with the respective positions of the two first satellites. Consequently there exists in the motion of the third satellite, an inequality proportional to the sine of the excess of the mean longitude of the second satellite above that of the third, which inequality at its *maximum* is 808″, of a degree. If we conceive a star of which the angular motion is equal to the excess of the mean synodic motion of the second satellite above twice the mean synodic motion of the third, the inequality of the third satellite in the eclipses will be proportional to the sine of the motion of the imaginary star; but in consequence of the relation which subsists between the mean longitudes of the three first satellites, the sine of this motion is with the exception of the sign, the same as that of the motion of the first imaginary star which has been considered. Thus the inequality of the third satellite in its eclipses has the same period, and follows the same laws, as the inequalities of the two first satellites.

Such are the periods of the principal inequalities of the three first satellites of Jupiter, which Bradley seems to have suspected, but which Vargenten has since detailed with the greatest accuracy. Their correspondence and that of the mean motions and mean longitudes of these satellites, appear to constitute a separate system of these three bodies, actuated according to all appearance by common forces, from which arise those relations, which they have in common.

If the apparent semidiameter of the equator of this planet, at its mean distance from the sun, which is about 25′′, be assumed as unity, the mean distances of the satellites from its centre, and the durations of their sidereal revolutions are:

	Mean distances.	Durations.
		d
I.	3,351	0,94271.
II.	4,300	1,37024.
III.	5,284	1,88780.
IV.	6,819	2,73948.
V.	9,524	4,51749.
VI.	22,081	15,94530.
VII.	64,359	79,32960.

By comparing the durations of the revolutions of the satellites, with their mean distances from the centre of Saturn, we recognize the beautiful relation discovered by Kepler relatively to the planets, and which we have already observed to exist in the system of the satellites of Jupiter, *i. e.* that the squares of the times of the revolutions of the satellites of Saturn, are as the cubes of their mean distances from the centre of this planet.

The great distance of the satellites of Saturn, combined with the difficulty of observing their position, has not enabled us to recognize the ellipticity of their orbits, and still less the inequalities of their motions. However, the ellipticity of the orbit of the sixth satellite is perceptible.

If we assume as unity, the semidiameter of Uranus, which is 6″, when seen from the mean distance of the planet from the sun; the mean distances of the satellites from its centre, and the durations of their revolutions are, according to the observations of Herchell :

	Mean distances.	Durations.
		d
I.	13,120	5,8926.
II.	17,022	8,7068.
III.	19,845	10,9611.
IV.	22,752	13,4559.
V.	45,507	38,0750.
VI.	91,008	107,6944.

These durations, with the exception of the second and fourth, have been inferred from the greatest observed elongations, and from the law according to which the squares of the periods are proportional to the cubes of their mean distances from the planet, which law is confirmed by observations made on the second and fourth satellite, the only ones which are sufficiently well known; so that it may be considered as a general law of the motion of a system of bodies which revolve about a common focus.

It may now be asked what are the principal forces which retain the planets, the satellites and the comets in their respective orbits? what particular forces derange their elliptic motions; what cause makes the equinoxes to regrade, and produces the rotation of the earth and moon about their axes; finally, by the action of what forces, are the waters of the sea raised twice each day; the supposition of one sole principle on which all these laws depend, is worthy of the majestic simplicity which pervades all nature. The generality of the laws which the celestial motions present, seems to indicate its existence; even already we may suspect that such a principle is in existence, from the connection between these phenomena and the respective positions of the bodies of the solar system. But in order that we may be able to place it in the clearest light, the laws of the motion of matter must be known.

BOOK THE THIRD.

OF THE LAWS OF MOTION.

At nunc per maria ac terras sublimaque cæli
Multa modis multis varia ratione moveri
Cernimus ante oculos.

LUCRET: lib. 1.

SURROUNDED as we are by an infinite variety of phenomena, which continually succeed each other in the heavens and on the earth, philosophers have succeeded in recognizing the small number of general laws to which matter is subject in its motions. To them, all nature is obedient; and every thing is as necessarily derived from them, as the return of the seasons; so that the curve which is described by the lightest atom that seems to be driven at random by the winds, is regulated by laws as certain as those which confine the planets to their orbits. The importance of these laws, on which we continually depend, ought to have excited the curiosity of mankind in all ages; yet by the effect of an indifference but too common to the human mind, they were utterly unknown, until the commencement of the

17th century, at which epoch Gallileo laid the first foundations of the science of motion by his beautiful discoveries on the descent of bodies. Geometricians, following up the steps of this great man, have finally reduced the whole science of mechanics to general formula, which leaves nothing to be desired but to bring the analysis to perfection.

CHAP. I.

Of forces, of their composition, and of the equilibrium of a material point.

A body appears to us to move, when it changes its situation relatively to a system of bodies which we suppose to be at rest. Thus in a vessel which moves in an uniform manner, bodies seem to us move, when they correspond successively to its different parts. This motion is only relative; for the vessel moves on the surface of the sea, which revolves round the axis of the earth, the centre of which moves about the sun, which is itself carried along in the regions of space, together with the earth and the planets. In order to conceive a term to those motions, and to arrive at length at some fixed points, from which we may reckon the absolute motion of bodies, we conceive a space without bounds, immoveable, and penetrable to matter; and it is to different parts of this space, whether real or imaginary (*a*) that we in imagination refer the position of bodies; and we conceive them to be in motion when they correspond successively to different points of this space.

The nature of that singular modification, in consequence of which a body is transported from one place to another, is, and always will be, unknown. It has been designated by the name of *Force;* but we can only determine its effects and the law of its action.

The effect of a force acting on a material point, is, if no obstacle intervenes, to put it in motion. The direction of the force is the right line, which it tends to make the point described. It is evident that if two forces act in the same direction, their effect to move the point is the sum of the forces, and that when they act in opposite directions, the point is moved by a (*b*) force represented by their difference, so that if the forces were equal, the point would remain at rest.

If the directions of the two forces make an angle with each other, a force results, the direction of which is intermediate between the directions of the composing forces, and it can be demonstrated by geometry alone, that if from the point of concourse of these forces, and in their respective directions, right lines be assumed (*c*) which represent them, and if then the parallellogram of which these lines are adjacent sides, be completed, its diagonal will represent their resultant, both in quantity and in direction. We may substitute in place of the two composing forces, their resultant, and conversely we can in place of any force whatever, substitute two others, of which it is the resultant, consequently any force may be resolved into two others parallel to two axes perpendicular to each other, and situated in a plane which passes through its direction. To effect this, it is sufficient to draw through the first extremity of the line which represents this force, two lines respectively parallel to these axes, and to form on these lines a rectangle, the diagonal of which represents the force to be decomposed. The two sides of the rectangle will represent the forces, into which the given force may be decomposed parallel to these axes.

If the force be inclined to a plane given in position, then by assuming to represent it, a line in its direction, the extremity of which is in the point where it meets the plane; the perpendicular demitted from the extremity of this line on the plane, will be the primitive force resolved perpendicularly to this plane. The right line drawn in this plane, connecting the line representing the given force and the perpendicular, will be the primitive force, decomposed parallel to the plane. This second partial force may itself be resolved into two others parallel to two axes situated in this plane, and at right angles to each other. Consequently every force may be resolved into three others, respectively parallel to three axes perpendicular to each other.

Hence arises a simple method of obtaining the resultant force of any number of forces, which act on a material point; for by resolving each of them into three others parallel to three axes given in position, and at right angles to each other, it is evident that all the forces parallel to the same axis are reducible to a single force, equal to the sum of those which act in one direction, minus the sum of those which act in a contrary direction. Consequently the point will be acted on by three forces, perpendicular to each other; if then three right lines in their respective directions be assumed to represent them, reckoning from their point of concourse, and a rectangular parallelopiped be formed on these three lines, its diagonal will represent the quantity and direction of *(d)* the force resulting from all those which act on the point.

Whatever may be the number, the magnitude, and the directions these forces, if the position of the point be varied in any manner by an indefinitely small quantity, the product of the resultant into the quantity by which the point advances in its direction, is equal to the sum of the products of each force into the corresponding quantity. The *(e)* quantity by which the point advances in the direction of any force, is the projection of the line connecting the two positions of the point, on the direction of the force; if the point advances in the opposite way from this direction, this quantity should be taken negatively.

In a state of equilibrium the resultant of all the forces vanishes, provided the point be free. If it is not, the resultant should be perpendicular to the surface, or to the curve on which the point is constrained to exist; and then, when the position of the point is changed by an indefinitely small quantity, the product of the resultant into the quantity by which the point advances in its direction, vanishes ; this product is therefore always equal to (f) nothing, whether the point be supposed to be altogether free, or whether it be constrained to exist on a curve or surface. Consequently in all cases, in which the equilibrium obtains, the sum of the products of each force, into the quantity by which it advances in its direction, when an indefinitely small change is made in its position, vanishes ; and if this condition is satisfied, the equilibrium subsists.

CHAP. II.

Of the motion of a material point.

A Point in repose cannot excite any motion in itself, because there is nothing in its nature to determine it to move in one direction in preference to another. When sollicited by any force, and then abandoned to itself, it will move constantly and uniformly in the direction of that force, if it meets with no resistance; that is to say, at every instant its force and the direction of its motion are the same. This tendency of matter, to persevere in its state of rest or of motion, is what is termed its *inertia*; it is the first law of the motion of bodies.

The direction of the motion in a right line, follows necessarily from this, that there is no reason why the body should deviate to the right, rather than to the left of its primitive direction; but the uniformity of its motion is not equally evident. The nature of the moving force being unknown, it is impossible to know *a priori*, whether this force should continue without intermission or not. Indeed, as a body is incapable of exciting any motion in itself, it seems equally incapable of producing any change in that which it has received, so that the law of inertia is at least the most simple, and the most natural that can be imagined. It is likewise confirmed by experience; in fact, we observe that motions are perpetuated on the earth, in proportion as the obstacles which oppose them are diminished; which induces us to think that if these obstacles were entirely removed, the motions would never cease. But the inertia of matter is most remarkable in the heavenly bodies, which for a great number of ages have not experienced any perceptible alteration. For these reasons, we shall consider the inertia of bodies as a law of nature; and when we observe any change in the motion of a body, we shall suppose that it arises from the action of some extraneous cause.

In uniform motions, the spaces described are proportional to the times; but the time employed in describing a given space, is longer or shorter according to the intensity of the moving force. These differences have suggested the idea of velocity, which in uniform motions is the ratio of the space to the time employed in describing it. In order to avoid the comparison of time and space which are heterogeneous quantities, we assume an interval of time, a second for example, as the unity of time, and in like manner a portion of space, such as a metre, for the unity of space. Time and space become then abstract numbers, which express how often they contain units of their species, and thus they may be compared one with another. By this means, the velocity becomes the ratio of two abstract numbers, and its unity is the velocity of a body which describes a metre in a second. By reducing in this manner, the space, time, and velocity to abstract numbers, it appears that the space is equal to the product of the velocity into the time, which latter is consequently equal to the space divided by the velocity.

Force being known to us by the space which it causes to be described in a given time, it is natural to assume this space as its measure. But this supposes that several forces, acting in the same direction, would cause to be described in a second of time, a space equal to the sum of the spaces which each would have caused to be described separately in the same time, or in other words, that the force is proportional to the velocity; but of this we cannot be assured *a priori*, (*a*) in consequence of our ignorance of the nature of the moving force. Therefore it is necessary, on this subject, also to have recourse to experiments; for whatever is not a necessary consequence of the few data which we possess on the nature of things, must be to us the result of observation.

The force may be expressed by an infinity of functions of the velocity, which do not imply a contradiction. There is none, for instance, in supposing it proportional to the square of the velocity. In this hypothesis, it is easy to determine the motion of a point solicited by any number of forces, the velocities of which are known; for if we assume on the directions of these forces, right lines representing their velocities, reckoning from their point of concourse, and if from the same point other lines be taken which are to each other as the squares of the first assumed lines, these lines will represent the forces themselves. By compounding them according to the rules already laid down, we shall obtain the direction of the resulting force, and also the right line which represents it; and which will be to the square of the corresponding velocity as the right line which represents one of the composing forces, to the square of its velocity. By this it appears how the motion of a point may be determined, whatever be the function of the velocity which expresses the force. Among all the functions mathematically possible, let us examine which is that of nature.

It is observed on the earth, that a body sollicited by any force whatever moves in the same manner, whatever be the angle which the direction of this force makes with the direction of the motion which is common to the body, and to the part of the terrestrial surface to which it corresponds. A slight change in this respect, would produce *(b)* a very sensible difference in the durations of the oscillations of a pendulum, according to the position of the vertical plane in which it it oscillates; but it appears from experiment, that in all vertical planes, this duration is exactly the same. In a ship which moves uniformly, a moveable body subjected to the action of a spring, of gravity, or of any other force, moves relatively to the parts of the ship, in the same manner, whatever may be the velocity and the direction of the vessel. It may therefore be established as a general law of terrestrial motions, that if in a system of bodies which participate in a common motion, any force be impressed on one of them, its relative or apparent motion will be the same, whatever be the general motion of the system, and the angle which its direction makes that of the impressed force.

The proportionality of the force to the velocity, results from this law supposed rigorously exact; for if we suppose two bodies moving on the same right line with equal velocities, and that by impressing on one of them a force which is added to the primitive force, its velocity relatively to the other body is the same as if the two bodies had been originally in a state of rest; and it is evident that the space described by the body in consequence of the primitive force, and of that which is added to it, is then equal to the sum of the spaces which each of them would have caused it to describe in the same time, which supposes that the force is proportional to the velocity.

And conversely, if the force be proportional to the velocity, the relative motions of a system of bodies actuated by any forces whatever, are the same whatever be their common motion; for this motion being resolved into three others, parallel to three fixed axes, only increases by the same quantity, the partial velocities of each body parallel to these axes; and since the relative velocities depend only on the difference of the partial, it is the same, whatever may be the motion common to all the bodies. It is therefore impossible to judge of the absolute motion of a system, of which we make a part, by the appearances which are observed, which circumstance characterises this law, the ignorance of which has so long retarded our knowledge of the true system of the world, by the difficulty of conceiving the relative motions of projectiles above (c) the surface of the earth, which is itself carried along by a double motion, of rotation round its own axis, and of revolution about the sun.

But considering the extreme smallness of the most considerable motions which we can impress on bodies, compared with that which they have in common with the earth, it is sufficient for the appearances of a system of bodies to be independent of the direction of this motion, that a small increase in the force by which the earth is actuated mey be to (*d*) the corresponding increase of its velocity, in the ratio of the quantities themselves. Thus our experiments only prove the reality of this proportion, which if it really obtained, whatever the velocity of the earth might be, would give the law of the velocity proportional to the force. It would likewise give this law, if the function of the velocity which expresses it was composed of only (*e*) one term. If then the velocity be not proportional to the force, we must suppose that in nature the function of the velocity which expresses the force, consists of several terms, which is very improbable. Moreover, we must suppose that the velocity of the earth is exactly such as corresponds to the preceding proportion, which is contrary to all probability. Besides, the velocity of the earth is different, in different seasons of the year; it is about one thirtieth greater in winter than in summer: this variation is still more considerable, if, as every thing appears to indicate, the solar system itself is in motion in space; for according as this progressive motion conspires with that of the earth, or is contrary to it, there should result great variations in the course of the year in the absolute motion of the earth, and this should alter the proportion of which we are speaking, and the ratio of the impressed force, to the relative velocity which results from it, if this proportion and this relation were not independent of the absolute velocity.

All the celestial phenomena serve to confirm these proofs. The velocity of light, determined by the eclipses of Jupiter's satellites, is combined with that of the earth, exactly according to the law of the proportionality of the force to the velocity; and all the motions of the solar system, computed according to this law, are entirely conformable to observations. Hence it appears that we have two laws of motion, namely, the law of inertia, and that of the force proportional to the velocity, both furnished by observation; they are the most simple and the most natural that can be imagined, and are, without doubt, derived from the nature itself of matter; but this nature being unknown, these laws are to us only observed facts, and the only ones which the science of mechanics borrows from experience.

The velocity being proportional to the force, these two quantities may be represented the one by the other; therefore, by what goes before, we can obtain the volocity of a point solicited by any number of forces, the respective directions and velocities of which are known.

If the point is solicited by a number of forces which act in a continued manner, it will describe with a motion incessantly variable, (*f*) a curve, the nature of which will depend on the forces by which the point is solicited. To determine it, we must consider the curve in its elements, examine how they arise the one from the other, and ascend from the law of the increase of the ordinates to their finite expression. This is precisely the object of the infinitesimal calculus, the fortunate discovery of which has produced so many advantages to mechanics; hence we may perceive the utility of bringing to perfection this powerful instrument of the human mind.

We have, in the case of gravity, a daily example of a force which seems to act without intermission. It is true, we cannot determine whether its successive actions are separated by intervals of time, the duration of which is insensible, but the phenomena being nearly the same, on this hypothesis and on that of a continued action; geometricians have adopted the former, as the most simple and commodious. Let us investigate the laws of these phenomena.

Gravity seems to act in the same manner on bodies, whether they are in a state of rest or of motion. In the first instant a body remitted to its (*g*) action, acquires an indefinitely small degree of velocity; in the second instant, an additional degree of velocity is added to the first, and so on successively; so that the velocity increases in the ratio of the times.

If we imagine a right angled triangle, one of the sides of which represents the time and increases with it, while the other side represents the velocity. The element of the area of this triangle, being equal to the product of the element of the time into the velocity, it will represent the element of the space which gravity causes a body to describe; this space will be therefore represented by the entire area of the triangle, which as it increases in the ratio of the squares of one of its sides, shews that in motion accelerated by the action of gravity, the velocities increase as the times, and the heights through which bodies fall from a state of rest, vary as the squares of the times, or of the last acquired velocities. Therefore if the space through which a body descends in the first second, be represented by unity, it will descend through four unities in two seconds, through nine unites in three seconds, and so on; so that in the successive seconds, it will describe spaces which increase as the odd numbers, 1, 3, 5, 7, &c.

The space which a body actuated by the velocity acquired at the end of its fall, will describe in a time equal to that of the fall, will be represented by the product of this time into its velocity; this product is double of the area of the triangle, therefore, a body moving uniformly with its last acquired velocity, will describe in a time equal to that of its fall, (h) a space double of that through which it has fallen.

The ratio of the last acquired velocity to the time, is constant for the same accelerating force; it increases or diminishes according as these forces are greater or less; it may therefore serve to express them. As the product of the time into the velocity is double of the space described, the accelerating force is equal to double of the space described divided by the square of the time; it is also equal to the square of the time divided by this double space. These three formulæ for expressing the accelerating forces (i), are useful on various occasions; they do not give the absolute values of these forces, but only their ratio to each other, which is all that is required in mechanics.

On an inclined plane, the action of gravity is decomposed into two others; the one perpendicular to the plane which is destroyed by its resistance; the other parallel to the plane, which is to the primitive force of gravity, as the height of the plane to its length. Therefore the motion on an inclined plane is uniformly accelerated; but the velocities and the spaces described, are to the velocities and spaces described in the same time, in the direction of the vertical, as the height of the plane to its length. It follows from this, that all the chords of circles, which are (*k*) terminated in one of the extremities of the vertical diameter, are described by the action of gravity, in the same time as its diameter.

A body projected in the direction of any right line whatsoever, deviates from it continually, describing a curve concave to the horizon, of which this right line is the first tangent. Its motion when referred to this right line by vertical ordinates, is uniform, but it is accelerated in the direction of the verticals, according to the laws already explained; therefore, if from (*l*) every point of the curve verticals be extended to meet the first tangent, they will be proportional to the squares of the corresponding intercepts of this tangent, which is the characteristic property of the parabola. If the force of projection is in the direction of the vertical itself, the parabola is confounded with the vertical line, and thus the formulæ for parabolic motion give those for accelerated or retarded motions, in the direction of the vertical.

Such are the laws of the descent of heavy bodies discovered by Gallileo; at the present day, it seems to require no great power of mind to have discovered them; but since they eluded the investigations of philosophers, although perpetually presented to them by the phenomena, it must no doubt have required an extraordinary genius to have developed them.

We have seen in the first book, that a material point suspended at the extremity of a straight line supposed without mass, and firmly fixed at its other extremity, constitutes the simple pendulum. This pendulum, when removed from its vertical position, tends to return by its gravity, and this tendency is very nearly proportional to this deviation, when it is not considerable. Suppose that two pendulums of the same length, depart at the same (m) instant from the vertical position, with very small velocities. In the first instant, they will describe arcs proportional to their velocities; at the commencement of the second instant, equal to the first, the velocities will be retarded proportionally to the arcs described, and consequently to the primitive velocities; therefore the arcs described in this instant will be also proportional to these velocities, and this will be likewise true for the arcs described in the third, fourth, &c. instants; thus at every instant the velocities, and the arcs measured from the vertical, will be proportional to the primitive velocities, consequently the pendulums will arrive at the state of rest, simultaneously. They will return again to the vertical with a motion accelerated, according to the same laws by which their velocites had been previously retarded, and they will reach it at the same instant, and with their primitive velocity. They will oscillate in the same manner on the other side of the vertical, and they would thus continue to oscillate for ever, but for the resistances they meet with. It is evident that the extent of their oscillations depends on their primitive velocities, but the duration of these oscillations is the same, and consequently independent of their amplitude. The force which accelerates or retards the pendulum, is not exactly proportional to the arc measured from the vertical; so that when a body moves in a circle (n) the isochronism relatively to the small oscillations of a heavy body, is only approximative. But it is rigorously exact in a curve, in which the gravity resolved parallel to the tangent, is proportional to the arc reckoned from the lowest point of the curve, which immediately gives its differential equation. Huygens, to whom we are indebted for the application of the pendulum to clocks, was the first who investigated the nature of this curve. He found that it was a cycloid, the plane of which was vertical, the vertex being the lowest point; and in order that a body suspended at the extremity of an inextensible thread, should describe this curve, it was only required that the other extremity should be fixed at the point of concourse of two cycloids equal to that to be described, and situated vertically in an opposite direction, in such a manner that the thread in its vibration might envelope alternately each of these curves. But whatever ingenuity may have been displayed in these investigations, a long experience has given the preference to the circular pendulum, as being more simple, and sufficiently accurate to be applied even to the astronomical computations. But the theory of evolutes which has been suggested by them, is become very important by its applications to the system of the world.

The duration of the very small oscillations of a circular pendulum, is to the time of a body's descent through a height equal to double of the length of the pendulum, as the semi-circumference is (*o*) to the diameter. Consequently the time of descent through a small arc terminated by the vertical diameter, is to the time of descent down the diameter, or what comes to the same thing, to the time required to describe the chord of the arc, as the fourth part of the circumference to the diameter; therefore the *right* line connecting two given points, is not the line of quickest descent from the one to the other. The investigation of this line has excited the attention of geometers, and they have (*p*) found that it is a cycloid, the origin of which coincides with the most elevated point.

The length of the simple pendulum which vibrates seconds, is to twice the height through which bodies fall by the force of gravity in the same time, as the square of the diameter to the square of the (*q*) circumference. As this length may be measured with great precision, the time which heavy bodies take to descend through a determinate space may be obtained by this theorem much more accurately than by direct experiments. It has been observed in the first book, that by means of very exact experiments, the length of the pendulum vibrating seconds at Paris, has been determined to be $0^{m}741887$, hence it follows that gravity causes bodies to fall through $3^{m},66107$, in the first second. This connection between the time of oscillation, the duration of which may be precisely observed, and the rectilinear motion of heavy bodies, is an ingenious remark, for which we are also indebted to Huygens.

The durations of very small oscillations of pendulums of different lengths, and actuated by the same force of gravity, vary as the square roots of these lengths. If the length of the pendulums be the same, but actuated by different forces, the times of their oscillations will be reciprocally as the square roots of the force of gravity. It is by means of these theorems that the variation of the force of gravity at the surface of the earth, and on the summit of mountains, has been determined. From observations made on pendulums, it has been likewise inferred, that gravity depends (*r*) neither on the figure nor on the surface of bodies; but that it penetrates their inmost parts, and tends to impress on them equal velocities in equal times. To be assured of this, Newton made several bodies of the same weight, but of different figures and matter, to oscillate, by placing them in the interior of the same surface, in order that they may experience the same resistance from the air. And though he instituted these experiments with the greatest accuracy possible, he was never able to perceive the smallest difference in the length of simple pendulums, vibrating seconds, as inferred from the durations of the oscillations of these bodies; hence it follows, that if bodies did not experience any resistance in their fall, the velocity which they would acquire by the action of gravity, would be always the same in equal times.

We have likewise in circular motion, an instance of a force which acts without intermission. The motion of matter abandoned to itself being uniform and rectilineal, it is evident that a body which moves on a curve must perpetually tend to recede from the centre in the direction of the tangent. The effort which it makes for this purpose is termed *centrifugal force;* and the force directed to the centre is called a *central* or *centripetal force.* In circular motion the central force is equal and directly contrary to the centrifugal force; it tends incessantly to draw the body from (*s*) the centre to the circumference, and in an extremely short interval of time its effect may be measured by the versed sine of the small arc described.

We are enabled by this result, to compare the force of gravity with the centrifugal force which arises from the rotatory motion of the earth. At the equator, bodies describe in consequence of this rotation in each second of time, an arc of $40'',1095$ of the periphery of the terrestrial equator. As the radius of this equator is very nearly 6376606^m, the versed sine of this arc will be $0^m,0126559$. The force of gravity causes bodies to descend through $3^m,64930$ in a second at the equator; therefore the central force necessary to retain bodies at the surface of the earth, and consequently the centrifugal force arising from the rotatory motion, is to the force of gravity at the equator, in the ratio of 1 to 288.4. As the centrifugal force acts in opposition to gravity at the equator, it diminishes it, and bodies descend to the earth by the difference only between these two forces; therefore if the entire weight which would subsist without this diminution be called *gravity*, the centrifugal force at the equator is very nearly the $\frac{1}{289}$th part of gravity. If the rotation of the earth was seventeen times more rapid, the arc described at the equator in a second, would be seventeen times greater, and its versed sine would be 289 times more considerable, consequently the centrifugal force would be equal to the force of gravity, and bodies at the equator would cease to gravitate to the earth.

In general, the expression of a constant accelerating force which acts always in the same direction, is equal to twice the space which it causes to be described, divided by the square of the time, every accelerating force, in an extremely short in-interval of time, may be considered constant and acting in the same direction; moreover, the space which the central force causes to be described in circular motion, is the versed sine of the arc described, and this versed sine is very nearly equal to the square of the arc divided by the diameter; the expression of this force is therefore the square of the arc described, divided by the square of the time, and by the radius of the circle. The arc divided by the time is the velocity itself of the body; consequently the central force, and likewise the centrifugal force, are equal to the square of the velocity divided by the radius.

A comparison of this result, with that found above, according to which the gravity is equal to the square of the acquired velocity divided by twice the space described in the direction of the (t) vertical, shews that the centrifugal force is equal to the force of gravity, when the velocity of the revolving body is the same as that acquired by a heavy body, in falling through a height equal to half the radius of the described circumference.

The velocities of several bodies moving in circles, are equal to the circumferences which they describe divided by the times of their revolutions; the circumferences being as the radii, the squares of the velocities are as the squares of the radii divided by the squares of the times. The centrifugal forces are therefore as the radii of the circumferences divided by the squares of the times of the revolutions. It follows from this, that on the different terrestrial parallels, the centrifugal force arising from the motion of rotation of the earth, is proportional to the radii of those parallels. These beautiful theorems discovered by Huygens, conducted Newton to the general theory of curvilinear motion, and thence to the law of universal gravitation.

A body which describes any curve whatever, tends to deviate from it in the direction of the tangent: now we can easily conceive a circle to pass through two consecutive elements of the curve; this circle is termed the osculating circle, or the circle of curvature; the body may be conceived in two consecutive instants to move on the circumference of the circle; its centrifugal force is therefore equal to the square of its velocity divided by the radius of the osculatory circle; but the magnitude and position of this circle are constantly varying.

If the curve be described by the action of a force directed to a fixed point; this force may be resolved into two, one in the direction of the radius of the osculating circle, the other in the direction of the element of the curve. The first is in equilibrio with the centrifugal force, the second increases (*u*) or diminishes the velocity of the body, therefore this velocity continually varies, but it is always such, *that the areas described by the radius vector about the origin of the force, are proportional to the times. Conversely, if the areas traced by the radius vector about a fixed point, increase proportionally to the times, the force which solicits the body, is constantly directed towards this point.* These fundamental propositions in the theory of the system of the world, are easily demonstrated in the following manner.

The accelerating force may be supposed to act only at the commencement of each instant, during which the motion of the body is uniform; the radius vector will thus describe a small triangle. If the force should cease to act in the following instant, the radius vector will trace in this second instant a small triangle equal to the first; because the vertices of these two triangles being at the fixed point, which is the origin of the force, their bases, which exist in the same right line, will be equal; as being described with the same velocity, during two equal and consecutive instants. But at the commencement of the second instant, the accelerating force combined with the tangential force of the curve, causes the body to describe the diagonal of a parallelogram, of which the adjacent sides represent these forces. The triangle which the radius vector describes in consequence of the action of this combined force, is equal to that which would have been described without the action of the accelerating force; for these two triangles are situated on the same base, namely, the radius vector of the end of the first instant, and their vertices exist on a right line parallel to this base; therefore the areas traced by the radius vector in two consecutive instants, are equal; and consequently the sector described by this radius increases as the number of these instants, or as the times. It is evident that this only obtains when the accelerating force is directed towards the fixed point; otherwise the triangles which we have considered will not have the same altitude. Therefore, the proportionality of the areas to the times, demonstrates that the accelerating force is constantly directed to the origin of the radius vector.

In this case, if we suppose a very small sector to be described in a very short interval of time, and if from the first extremity of the arc of this sector, a tangent to the curve be drawn, and the radius vector drawn from the origin of the force to the other extremity of the vector be prolonged to meet this tangent, it is evident that the part of this radius intercepted between the curve and the tangent, will be the space which the central force would cause the body to describe. If twice this space be divided by the square of the time, we obtain an expression for this force; but since the sector is proportional to this time, the central force is proportional to the part of the radius vector intercepted between the curve and the tangent, divided by the square of the sector. Strictly speaking, the central force in different points of the curve is not proportional to these quotients, but the accuracy is always greater according as the sectors are taken smaller, so that it is exactly proportional to the limits of these quotients. When the nature of the curve is known, this limit may be obtained in a function of the radius vector, by means of the differential analysis, and then that function of the distance to which the central (*v*) force is proportional will be determined.

If the law of the force be given, the investigation of the curve described presents greater difficulties. But whatever be the nature of the forces by which a body is actuated, the differential equations of its motion may be determined in the following manner: let us imagine three axes perpendicular to each other; the position of a body at any instant will be determined by three coordinates parallel to these axes. Each force which acts on the point being resolved into three others parallel to the same axes, the product of the resultant of all the forces, parallel to one of the coordinates, into the element of time during which it acts, will express the increment of the velocity parallel to this coordinate; but this velocity being equal to the element of the coordinate divided by the element of the time, the differential of the quotient of this division, is equal to (*x*) the preceding product. The consideration of the two other coordinates furnishes two similar equations; thus the determination of the motion of a body, becomes an investigation of pure analysis, which is reduced to the integration of these differential equations.

In general, the element of time being supposed to be constant, the second differential of each coordinate divided by the square of this element, represents a force, which being applied to the point, in an opposite direction constitutes an equilibrium with the force which solicits it in the direction of this coordinate. If the difference of these forces be multiplied by the arbitrary variation of the coordinate, the sum of the three similar products relative to the three coordinates will be equal to cypher by the condition of equilibrium. If the point be free, the variations of the three coordinates will be all arbitrary, and by putting the coefficient of each of them respectively equal to cypher, the (*y*) three differential equations relative to the motion of a point will be obtained. But if the point is not entirely free, there will be given one or two relations between the three coordinates, which will furnish a corresponding number of equations between their arbitrary variations. If then a like number of variations be eliminated by means of these relations, the coefficients of the remaining variations will be respectively equal to cypher; and the differential equations of motion will be obtained, which being combined with the relations existing between the coordinates, will determine the position of the point, for any instant.

The integration of these equations is easy when the force is directed to a fixed point, but very often it becomes impossible from the nature of the forces. Nevertheless the consideration of the differential equations leads to some interesting principles of mechanics, such as the following. The differential of the square of the velocity of a point subject to the action of accelerating forces, is equal to twice the sum (z) of the products of each force into the small space advanced by the body in the direction of this force; from which it is easy to infer, that the velocity acquired by a heavy body descending along a line or a curved surface, is the same as it would acquire in falling vertically through the same height.

Several Philosophers, struck with the order which prevails in nature, and with the fecundity of its means in the production of phenomena, have supposed that she always accomplishes her ends in the simplest manner possible. In extending this conjecture to mechanics, they have investigated what was the economy of nature in the employment of forces and of time. Ptolemy ascertained that reflected light passed from one point to another, by the shortest possible route, and consequently in the least time, the velocity of the luminous ray being supposed to be always the same. Fermat, one of the most original men that France ever produced, generalized this principle, by extending it to the refraction of light. He supposed therefore that it passes from a point assumed without a diaphonous medium to an interior point, in the shortest possible time; then supposing that the velocity is less in this medium than in a vacuo, which is extremely probable, he investigated the law of the refraction of light in these hypotheses. By applying to this problem his beautiful method *de maximis* and *de minimis*, (which should be considered as the true origin of the differential calculus,) he found agreeably to experience, that the sines of (*aa*) incidence and of refraction ought to be in a constant ratio, greater than unity. The ingenious manner in which Newton deduced this ratio from the attraction of the media which the rays traverse, indicated to Maupertius, that the velocity of light increases in diaphanous media, and that consequently it is not, as Fermat supposed, the sum of the quotients of the spaces described in a vacuo and in the medium, divided by their corresponding velocities, but the sum of the products of these quantities which should be a *mini-*

mum. Euler extended this hypothesis to motions which are every moment variable, and he demonstrated by several examples, that of all the curves that a body may describe in passing from one point to another, it always selects that *in which the integral of the product of its mass, into its velocity and the element of the curve, is a minimum.* Thus the velocity of a point which moves on a curved surface, and is not actuated by any force, being constant, it passes from one point to another by the shortest line (*bb*) which can be traced on this surface. The preceding integral has been termed *the action of the body,* and the sum of all the similar integrals relative to each body of the system, has been called the action of the system. Therefore Euler has demonstrated that this action is a *minimum,* so that the economy of nature consists in sparing this action; this is what constitutes *the principle of least action,* the discovery of which is certainly due to Euler; though Lagrange has since derived it from the primordial laws of motion. But this principle is only at bottom a remarkable result of those laws, which are, as we have seen, the most simple and the most natural that can be conceived, and which seem to be derived from the very essence of matter. All laws mathematically possible between the force and the velocity, furnish analogous results, provided that we substitute in this principle, instead of the velocity, that function of the velocity by which the force is expressed. Therefore the principle of the least action ought not to be elevated to the rank of a final cause, for so far from having given birth to the laws of motion, it has not even contributed to their discovery, without which we would still dispute about what was to be understood by the last action of nature.

CHAP. III.

Of the equilibrium of a system of bodies.

The simplest case of equilibrium between several bodies, is that of two material points meeting each other, with equal and directly contrary velocities. Their mutual impenetrability, that property of matter which prevents two bodies from occupying the same place at the same instant, evidently annihilates their velocities, and reduces them to a state of rest. But if two bodies of different masses impinge on each other, with opposite velocities, what relation exists between the velocities and the masses in the case of an equilibrium? In order to solve this problem, suppose a system of contiguous material points arranged in the same right line, and actuated by a common velocity, in the direction of this line; suppose also a second system of contiguous material points, situated on this same line and actuated also by a common velocity, but in a direction opposite to the preceding, so that the two systems, after impinging on each other, may constitute an equilibrium. It is evident, that if the first system consisted of only one material point, each point in the second system would destroy in the striking point, a part of its velocity equal to the velocity of the second system; therefore in the case of equilibrium, the velocity of the striking point should be equal to the product of the velocity of the second system into the number of material points composing it, and thus we may substitute for the first system, one sole point actuated by a velocity equal to this product. We may likewise substitute in place of the second system, a material point actuated by a velocity, equal to the product of the velocity of the first system, into the number of its material points. Thus in place of the two systems we shall have two points which will sustain each other in equilibrio with contrary velocities, of which one will be the product of the velocity of the first system into the number of its points, and of which the other will be the product arising from multiplying the velocity of the points of the second system by their number; therefore in the case of an equilibrium these products should be equal to each other. The mass of a body is the aggregate of its material points. The product of the mass by the velocity is termed *the quantity of motion*; this is also what is understood by *the force of a body*. In order that two bodies, or two systems of points, which impinge on each other in opposite directions, may be in equilibrio, the quantities of motion or the opposite forces should be equal, and consequently the velocities should be inversely as the masses.

Two material points cannot act, the one on the other, except in the direction of the right line which connects them: the action which the first exercises on the second communicates to it a certain quantity of motion; now we may conceive that previous to the action, the second body is actuated by this quantity, and by another which is equal and directly opposite to it, the action of the first body is therefore employed (*a*) in destroying this last quantity of motion, but to effect this, it must employ a quantity of motion equal and contrary to that which is to be destroyed. Hence it appears generally, that in the mutual action of bodies, the reaction is always equal and directly contrary to the action. It likewise appears, that this equality does not imply the existence of any particular force inherent in matter, but results from this, that a body cannot acquire motion from the action of another, without depriving it of a portion of its motion; in the same manner, as a vessel can only be filled at the expence of another which communicates with it.

The equality between action and reaction manifests itself in all the actions of nature; iron attracts the magnet as it is attracted by it; the same is observed in electric attractions and repulsions, and even in the developement of animal forces; for whatever be the nature of the prime motive power in man and animals, it is clear that they experience, from the reaction of matter, a force equal and contrary to that which they communicate to it, and that consequently when they are considered in this point of view, they are subject to the same laws as inanimate beings.

The reciprocity of the velocity to the mass in the case of equilibrium, enables us to determine the ratio of the masses of different bodies. Those of homogeneous bodies are proportional to their volumes, which geometry teaches us to measure; but all bodies are not homogeneous, and from the differences which exist either in their integrant molecules or in the number and magnitude of the intervals or pores which separate those molecules, there arise very considerable diversities in the masses which are contained under the same volume. Geometry then becomes inadequate to determine these masses, and we are necessarily obliged to have recourse to mechanics.

If we conceive that in two globes composed of different substances, their diameters are so varied, that they may constitute an equilibrium when they meet with equal and directly opposite velocities, we may be assured that then they contain the same number of material points, and that consequently their masses are equal. The ratio of the volumes of these substances, the masses being equal, will thus be obtained; and afterwards, we can determine by geometry, the ratio of the masses of any two volumes of the same substance. But this method would be extremely troublesome in the numerous comparisons which are continually required in the various relations of commerce. Fortunately, nature furnishes, in the weight of bodies, a simple method of comparing their masses.

It has been observed in the preceding chapter, that every material point in the same place on the earth, tends to move with the same velocity by the action of gravity. The sum of these tendencies is that which constitutes the weight of a body; therefore *(b)* the weights are proportional to the masses. It follows from this, that if two bodies suspended at the extremities of a thread, which passes over a pully, are in equilibrio when an equal portion of the thread is on each side of the pully, the masses of those bodies are equal, because tending to move with the same velocity by the action of gravity, their mutual action on each other is precisely the same, as if they impinged on each other, with equal and directly contrary velocities. Likewise if two bodies placed in a balance, of which the arms and plates are perfectly equal, be in equilibrio, we may be assured of the equality of their masses. The ratio between the masses of different bodies may thus be obtained by means of an exact and sensible balance, and of a great number of small equal weights, by determining how many of these weights are necessary to retain these masses in equilibrio.

The density of a body depends on the number of its material points, included in a given volume; it is therefore proportional to the ratio of the mass to the volume.

The density of a substance destitute of pores would be the greatest possible; and a comparison of its density with that of other bodies, would give the quantity (*c*) of matter which they contain. But as we are not acquainted with any such substance, we can only obtain the relative densities of bodies; these densities are in the proportion of the weights when the volumes are the same, for the weights are proportional to the masses: assuming therefore as unity, the density of any substance, at a constant temperature, for instance, the *maximum* of the density of distilled water, the density of a body will be the ratio of its weight to that of an equal volume of water reduced to its maximum density. This ratio is termed its *specific gravity*.

What has been said seems to suppose that matter is homogeneous, and that bodies only differ from each other in the figure and magnitude of their pores and of their integrant molecules. It is however possible that there may be essential differences in the very nature of these molecules; and it is not repugnant to the limited information which we possess of matter, to suppose the celestial regions filled with a fluid devoid of pores, and still of such a nature as not to oppose any sensible resistance to the planetary motions; we may thus reconcile the uniformity of these motions, which is (*d*) evinced by the phenomena, with the opinion of those philosophers who regard a vacuum as an impossibility; but this is of no consequence in mechanics, which takes into account no other properties of matter, but extension and motion. We may therefore, without any apprehension of error, assume the homogeneity of the elements of matter, provided that by equal masses we understand masses which being sollicited by equal and directly contrary velocities, constitute an equilibrium.

In the theory of the equilibrium, and motion of bodies, we abstract from the consideration of the number and figure of the pores which are distributed through them. But we may have regard to the differences of their respective densities, by supposing them to be constituted of material points more or less dense, which in fluids are perfectly free, and which in hard bodies are connected by inflexible straight lines, destitute of mass, and which in elastic and soft bodies, are connected by flexible and extensible lines. It is evident that in these hypotheses, bodies should present the appearances which they actually exhibit.

The conditions of the equilibrium of a system of bodies may be always determined by the law of the composition of forces, which has been explained in the first chapter of this book; for we may conceive the force by which every material point of the system is actuated, to be applied to that point of its direction where all the forces which destroy it concur, or which by combining with it, constitute a resultant, which in the case of equilibrium is destroyed by the fixed points of the system. Let us consider, for example, two material points, attached to the extremities of an inflexible lever, and suppose that the forces which sollicit them exist in the plane of the lever: these forces being supposed to meet at the point of concourse of their directions, their resultant should, in order to constitute an equilibrium, pass through the fulcrum, which can alone destroy it; (*e*) and according to the law of the composition of forces, the two composing forces should be reciprocally proportional to perpendiculars demitted from the fulcrum or point of support, on their directions.

If we suppose two heavy bodies to be attached to the extremities of a rectilinear inflexible lever, of which the mass is indefinitely small, relatively to the masses of these bodies, the directions respectively parallel to that of the gravity, may be conceived to meet at an infinite distance. In this case, the forces by which each body is actuated, or what is the same thing, their weights must be in the case of equilibrium reciprocally proportional to perpendiculars let fall from the fulcrum on the directions of these forces; these perpendiculars are proportional to the arms of the levers, consequently the weights of two bodies are, in the case of equilibrium, reciprocally proportional to the arms of the lever to which they are attached.

A very small weight may therefore sustain a very considerable one in equilibrio, and in this manner we can raise an enormous weight by a very slight effort; but for this purpose the arm of the lever to which the power is attached, must be very long with respect to that which elevates the weight, so that the power must describe a great space to elevate the weight a small height.

Consequently what is gained in force, is lost in time, and this is the case universally in all (f) machines. But we may almost always dispose of time at pleasure, when we can only employ a very limited force. In other cases where it is required to produce a great velocity, it may be effected by applying the force to the shorter arm of the lever. It is in this possibility of augmenting, according to circumstances, the mass or the velocity of the bodies to be moved, that the principal advantage of machinery consists.

From a consideration of the lever has been suggested the notion of moments. By the *moment* of a force to make a system turn about a point, is understood the product of this force, into the distance of the point from its direction. Therefore in the case of the equilibrium of a lever, to the extremities of which two forces are applied, the moments of these forces with respect (g) to the fulcrum or point on which it turns, must be equal and contrary, or what comes to the same thing, the sum of the moments relatively to this point must be equal to cypher.

The projection of a force on a plane drawn through a fixed point, multiplied into the distance of the point from this projection, is termed the moment of the force to make the system to revolve about an axis which passes through the fixed point, and is perpendicular to the plane.

The moment of the resultant of any number of forces with respect to a point, or any axis, is equal to the sum of the corresponding moments of the composing forces.

Parallel forces may be supposed to meet at an infinite distance, they are therefore reducible to an unique force, equal and parallel to their sum; therefore if each force be resolved into two, one of which exists on a given plane, the other being perpendicular to this plane, all the forces situated in the plane are reducible to a unique force, as likewise all the forces which are perpendicular to this plane. There exists always a plane passing through the fixed point, such that the resultant of the forces which are perpendicular to it, either vanishes or passes through this point: in these two cases the (*h*) moment of this resultant vanishes relatively to the axes which have this point for the origin, and the moment of the forces of the system, with respect to these axes is reduced to the moment of the resultant situated in the plane in question. The axis about which this moment is a *maximum*, is that which is perpendicular to this plane, and the moment of the forces relative to an axis, which passing through the fixed point makes any angle with the axis of greatest moment, is equal to the greatest moment of the system, multiplied into the cosine of this angle; so that this moment vanishes for all axes situated in the plane to which the axis of the greatest moment is perpendicular.

The sum of the squares of the cosines of the angles made by the axis of greatest moment, with any three axes perpendicular to each (*i*) other and passing through the fixed point being equal to unity; the squares of the three sums of the moments of the forccs, with respect to these axes, are equal to the square of the greatest moment.

In order that a system of bodies connected in an invariable manner, and which revolves about a fixed point, may be in equilibrio, the sum of the moments of the forces must vanish with respect to any axis passing through this point. It follows from what goes before, that this will always be the case if the preceding sum be equal to cypher, relatively to three fixed axes, perpendicular to each other. If there is no fixed point in the system, it is required in addition to the preceding conditions, in order to insure an equilibrium, that the three sums of forces resolved parallel to these axes, be *respectively* (*k*) equal to cypher.

Let us considsr a system of ponderable points firmly connected, referred to three planes at right angles to each other, and connected with the system. The action of gravity being resolved parallel to the intersections of these planes, all the forces parallel to the same plane may be reduced to an unique resultant parallel to this plane and equal to their sum. The three resultants relative to the three planes must concur in the same point; for the action of gravity on the several points of the system being parallel, they have an unique resultant, which is obtained by first combining two of these forces, and afterwards their resultant with the third force; the resultant of the three forces with a fourth, and so on. The situation of this point of concourse with respect to the system, is independent of the inclination of the planes to the direction of gravity; for a greater or less inclination can only change the (*l*) values of the three partial resultants, without altering their position with respect to the planes; therefore this point being supposed fixed, all the efforts of the weights of the system will be annihilated in all the positions which it can assume in revolving about this point, which for this reason has been termed *the centre of gravitg of the system.* Let us conceive the position of this centre, and that of the different points of the system to be determined by coordinates parallel to three axes at right angles to each other. The actions of gravity being equal and parallel, and the resultant of those actions passing in all positions of the system through its centre of gravity; if this resultant be supposed to be successively parallel to each of the three axes, the equality of the moment of the resultant to the sum of the moments of the composing forces gives any one of these coordinates, multiplied by the entire mass of the system, equal to the sum of the products of the mass of each point into its corresponding coordinate. Thus the determination of this centre, of which gravity first suggested the idea, is independent of it. The consideration of this centre extended to a system of bodies ponderable or not, free or connected in any manner whatever, is extremely useful in mechanics.

The theorem which was given at the close of the first chapter on the equilibrium of a point, when generalized, leads to the following theorem, which contains, in the most general manner, the conditions of the equilibrium of a system of material points actuated by any forces whatever.

If an indefinitely (*m*) small change be made in the position of the system, in a manner compatible with the connection of its parts, each material point will advance in the direction of the force which sollicits it, by a quantity equal to the part of this direction, comprised between the first position of the point and the perpendicular let fall from the second position of the point on this direction; this being premised, in the case of equilibrium, *the sum of the products of each force into the quantity by which the point to which it is applied advances in its direction, is equal to cypher; and conversely if this sum is equal to cypher, whatever may be the variation of the system, it is in equilibrio.* It is in this that the principle of virtual velocity consists, for which we indebted to John Bernoulli, but in applying it, it should be observed, that those products must be taken negatively, of which the points in the change of position of the system, advance in a direction contrary to that of their forces: it should be likewise recollected, that the force is the product of the mass of a material point, into the velocity with which it would move, if entirely free.

If we conceive the position of each point of the system, to be determined by three rectangular coordinates, the sum of the products of each force into the quantity advanced in its direction by the point which it sollicits, when an indefinitely small change is made in the system, will be expressed by a linear function of the variation of the coordinates of its several points; these variations have with each other relations, which depend on the manner in which the parts of the system are connected together, therefore in reducing the arbitrary variations to the least possible number by means of these relations in the preceding sum which should be equal to cypher, in the case of equilibrium; it is necessary, in order that the equilibrium may take place in every direction, to make the coefficient of each of the remaining variations separately equal to cypher, which will furnish us with as many equations as there are arbitrary variations. These equations, combined (n) with those which are furnished by the connection of the parts of the system, will contain all the conditions of its equilibrium.

There are two states of equilibrium, which are essentially different. In one, if the equilibrium be a little deranged, all the bodies of the system only make small oscillations about their primitive position; and then the equilibrium is *firm* or *stable*. This stability is absolute, if it obtains whatever may be the oscillations of the system; it is only relative, if it only obtains with respect to oscillations of a certain species. In the other state of equilibrium, when the system is disturbed, the bodies deviate more and more from their primitive position. We may form a just notion of these two states, by considering an ellipse situated vertically on a horizontal plane. If the ellipse be in equilibrio on its lesser axis, it is clear that by making it to deviate a little from this situation by a slight *(o)* motion on itself, it tends to revert, making oscillations which will be soon annihilated by the friction and resistance of the air. But if the ellipse be in equilibrio on its greater axis; when it once deviates from this situation, it continually deflects from it more and more, and is at length upset on its lesser axis. Consequently the stability of the equilibrium depends on the nature of the small oscillations, which the system, when deranged in any manner, makes about this state. In order to determine generally in what manner the different states of stable and tottering equilibrium succeed each other, let us consider a curve returning into itself, situated vertically in a position of stable equilibrium. When it is a little deranged from this state, it tends to revert to it; this tendency varies as the deviation increases, and when it vanishes, the curve is found in a new position of equilibrium, but which is not stable, for the curve previous to its arrival tended to revert to its primitive position. Beyond this last position, the tendency to the first state, and consequently to the second, becomes negative, until it vanishes a second time, and then the curve is in a position of stable equilibrium. By pursuing this illustration; it appears that the states of stable and tottering equilibrium succeed each other alternately, like the maxima and minima of the ordinates of curves. The same reasoning may be easily extended to the different states of equilibrium of a system of bodies.

CHAP. IV.

Of the equilibrium of fluids.

THE characteristic property of fluids, whether elastic or incompressible, is the extreme facility with which each of their molecules yields to the slightest pressure which it experiences on one side, rather than on the other. We proceed therefore to establish on this property, the laws of the equilibrium of fluids, by considering them as constituted of molecules perfectly moveable among each other.

It follows immediately from this mobility, that the force by which a molecule of the free surface of a fluid, is actuated, must be perpendicular to this surface, for if it was inclined to it, by resolving the force into two others, one perpendicular, and the other parallel to this surface, the molecule would glide on the surface *(a)* in consequence of this last force. Gravity is consequently perpendicular to the surface of stagnant waters, which is on this account horizontal; for the same reason, the pressure which each fluid molecule exerts against a surface, must be perpendicular to it.

Each molecule in the interior of a fluid mass, experiences a pressure, which in the atmosphere is measured by the height of the barometer, and which may be estimated in a similar manner for every other fluid. By considering each molecule as an (*b*) indefinitely small rectangular prism, the pressure of the ambient fluid will be perpendicular to the faces of this prism, which will consequently tend to move perpendicularly to each face, by virtue of the difference of pressures, which the fluid exerts on two opposite faces. From these different pressures arise three forces perpendicular to each other, which must be combined with the other forces which sollicit the molecule. It is easy to shew from this, that in the state of equilibrium the differential of the pressure is equal to the density (*e*) of the fluid molecule multiplied into the sum of the products of each force by the element of its direction; therefore if the fluid be incompressible and homogeneous, this sum will be an exact differential, this important result was first announced by Clairaut, in his beautiful treatise on the figure of the earth.

When the forces arise from attractions, which are always a function of the distance from the attracting centres, the product of each force into the element of its direction is an exact differential; therefore the density of the fluid molecule must be a function of the pressure, for the differential of the (*d*) pressure divided by this density is equal to an exact differential. Consequently all the strata of the fluid mass, in which the pressure is constant, are of the same density throughout their entire extent. The resultant of all the forces which actuate each molecule at the surface of these strata, is perpendicular to this surface, on which the molecule would glide if this resultant was inclined to it. In consequence of this property these strata have been termed *strata of level.*

The density of a molecule of atmospheric air, is a function of the pressure and of the temperature; its gravity is very nearly a function of its height above the surface of the earth. If its temperature was likewise a function of this height, the equation of the equilibrium of the atmosphere, would be a differential equation between the pressure and the elevation, and consequently the equilibrium *(d)* would be always possible. But in nature, the temperature of the different regions of the atmosphere depends also on the latitude, on the presence of the sun, and on a thousand variable or constant causes which ought to produce in this great fluid mass, motions often very considerable. In consequence of the mobility of its molecules, a heavy fluid may produce a pressure much more considerable than its weight. For example, a small column of water, terminated by a large horizontal surface, presses the base on which it is incumbent, as much as a cylinder of water of the same base and height. In order to evince the truth of this paradox, suppose a fixed cylindrical (*e*) vase, of which the horizontal base is moveable; and let this vase be filled with water, its base is sustained in equilibrio by a force equal and contrary to the pressure which it experiences. It is evident that the equilibrium would still obtain, in the case in which a part of the water was to consolidate and unite itself with the sides of the vessel; for the equilibrium of a system of bodies, is not deranged by supposing that in this state, several of them unite or become attached to fixed points. We may in this manner form an infinity of vessels of different figures, having all the same height and base as the cylindrical vessel, and in which the water will exert the same pressure on the moveable base.

In general, when a fluid acts only by its weight, the pressure which it exerts against a surface, is equivalent to the weight of a prism of this fluid, of which the base is equal to the pressed surface, (*f*) and of which the height is equal to the distance of the centre of gravity of this surface, from the plane of the level of the fluid.

A body plunged in a fluid, loses a part of its weight equal to the weight of a volume of the displaced fluid; for before the immersion, the surrounding fluid was in equilibrio with the weight of this volume of the fluid, which may be supposed, without deranging the equilibrium, to have formed itself into a solid mass, the resulting force of all the actions of the fluid on this mass must therefore be in equilibrio with its weight, and pass through its centre of gravity; now it is clear that (*g*) the same actions are exerted on a body which occupies its place; consequently the action of the fluid destroys a part of the weight of this body, equal to the weight of the volume of the displaced fluid. Hence it follows that bodies weigh less in air than in a vacuo; the difference, though for the most part hardly perceptible, should not be neglected in very delicate experiments.

By means of a balance, which carries at the extremity of one of its arms a body which can be plunged in a fluid, we can estimate exactly the diminution of weight which the body experiences in this immersion, and determine its *specific gravity*, or its density relative to that of fluid. This gravity is the ratio of the weight of the body in a vacuo, to its loss of weight, when it is entirely immersed in the fluid. It is thus that the specific gravities of bodies have been determined, by comparing them with distilled water at its *maximum* density.

In order that a body which is lighter than a fluid may be in equilibrio at its surface, its weight must be equal to the volume of the displaced fluid. It is moreover necessary, that the centres of gravity of this portion of the fluid and of the body should exist in the same vertical line; for the resultant of the actions of gravity on all the molecules of the body, passes through its centre of gravity, and the resultant of all the actions of the fluid on this body passes (h) through the centre of gravity of the volume of the displaced fluid; and as these resultants must be on the same right line in order to destroy each others effect, the centres of gravity must exist in the same vertical.

But in order to secure the *stability* of the equilibrium, it is necessary that other conditions, besides the two preceding, should be satisfied. It may be always determined by the following rule.

If through the centre of gravity of the section of a floating body on a level with the water, we conceive a horizontal axis, such that the sum of the products of each element of the section, into the square of its distance from this axis be less than a similar sum relatively to any other horizontal axis drawn through the same centre, the equilibrium will be stable in every direction, when this sum is greater than the product of the volume of the displaced fluid, into (*i*) the height of the centre of gravity of the body, above the centre of gravity of this volume. This rule is principally useful, in the construction of vessels to which it is necessary to give sufficient stability, in order to enable them to resist the efforts of storms and waters which tend to submerge them. In a ship the axis drawn from the stern to the prow is the line, relatively to which, the above mentioned sum is a *minimum*; it is therefore easy by means of the preceding rule, to determine the stability.

Two fluids contained in a vessel, dispose themselves in such a manner that the heaviest occupies (*k*) the lowest part of the vessel, and the surface which separates them is horizontal.

If two fluids communicate with each other by means of a bent tube, the surface which separates them in a state of equilibrium is nearly horizontal, when the tube is very large; their heights above this surface, are reciprocally proportional to their specific gravities. The entire atmosphere being therefore supposed to be of a uniform density, equal to that of the air at the temperature of melting ice; its height will be 7963^{m}, when compressed by a column of mercury of seventy-six centimetres; but because the density of the atmospheric strata diminishes, as they are more elevated above the level of the seas, the height of the atmosphere is much greater.

CHAP. V.

Of the motion of a system of bodies.

Let us consider first, the action of two material points of different masses, which moving in the same right line impinge on each other. We may conceive immediately before to the shock, their motions to be decomposed in such a manner, that they may have a common velocity, and two opposite velocities, such that if they were actuated by these alone they would have remained in equilibrio. The velocity common to the two points, is not affected by their mutual action, and therefore it will subsist alone after the shock. To determine it we shall observe, that the quantity of motion of the two points arising from this common velocity, plus the sum of the quantities of motion which are due to the velocities, which are destroyed, represent the sum of the quantities of motion previous to the shock, provided that the quantities of motion arising from the opposite velocities, be taken with contrary signs; but (*a*) by the conditions of equilibrium, the sum of the quantities of motion produced by the destroyed velocity vanishes; hence, the quantity of motion arising from the common velocity, is equal to that which existed in the two points previous to the impact; and consequently this velocity is equal to the sum of the quantities of motion, divided by the sum of the masses.

The impact of two material points is purely ideal, but it is easy to reduce to it that of any two bodies, by observing that if these bodies impinge in the direction of a right line passing through their centres of gravity, and perpendicular to their surfaces of contact, they will act on each other as if their masses were condensed into these centres; therefore motion is communicated between them, as between two material points, of which the masses are respectively equal to these bodies.

The preceding demonstration supposes, that after the shock, the two bodies must have the same velocity. We may readily suppose that this must be the case for soft bodies, in which the communication of motion is made successively, and by insensible gradations; for it is evident, that from the instant when the struck body has the same velocity as the striking body, all velocity between them ceases. But between two bodies of absolute hardness, the shock is instantaneous, and it does not appear to be necessary that their velocities should be (*b*) afterwards the same; their mutual impenetrability solely requires that the velocity of the striking body should be less; in other respects it is indeterminate. This indetermination demonstrates the absurdity of an absolute hardness. In fact, in nature the hardest bodies, if they are not elastic, have an imperceptible softness, which renders their mutual action successive, although its duration is insensible.

Where bodies are perfectly elastic, it is necessary, in order to obtain their velocity after the shock, to add or subtract from the common velocity which they would have, if they were destitute of elasticity, the velocity which they would gain or lose in this hypothesis; for the perfect elasticity doubles these effects, by the restitution of the springs which were compressed by the shock; therefore the velocity of each body after the shock will be obtained by subtracting its velocity before the shock, from twice this common velocity.

Hence it is easy to infer, that the sum of the products of each mass by the square of its velocity, is the same before and after the shock of the two bodies; which obtains universally in the impact of any number of perfectly elastic bodies, however they may be supposed to act on each other.

Such are the laws of the communication of motion by impulse, laws which have been confirmed by experience, and which may be mathematically deduced from the two fundamental laws of motion, explained in the second chapter of this book. Several philosophers have endeavoured to determine them from the consideration of final causes. Descartes, supposing that the quantity of motion in the universe should always remain the same without any regard to its direction, has deduced from this false hypothesis erroneous laws of the communication of motion, which furnish a remarkable example of the errors to which we are liable, when we endeavour to develope the laws of nature, by attributing to her, particular views.

When a body receives an impulsion, in a direction which passes through its centre of gravity, all its parts move with an equal velocity. If this direction is at one side of this point, the velocities of different parts of this body will be unequal, and from this inequality arises a motion of rotation of the body about its centre of gravity, at the same time that this centre is carried forward with the velocity with which it would have moved if the direction of the impulsion had passed through this point. This (*c*) case is that of the earth, and of the planets. Thus to explain the double motion of rotation and of translation of the earth, it is sufficient to suppose that in the beginning, it received an impulse of which the direction was at a small distance from its centre of gravity, and supposing this planet to be homogeneous, this distance is very nearly the hundreth and sixtieth part of its radius. It is extremely improbable that the primitive direction of the planets, the satellites and comets, should pass exactly through their centres of gravity; all these bodies should therefore revolve round their axes.

For the same reason the sun, which revolves on an axis, must have received an impulsion, of which the direction not passing accurately through its centre of gravity, carries it along in space with the planetary system, unless an impulse in a contrary direction should have destroyed this motion, which (*d*) is not at all probable.

The impulsion given to an homogeneous sphere, in a direction which does not pass through its centre, causes it to revolve constantly round a diameter perpendicular to a plane passing through its centre, and through the direction of the impressed force. New forces which sollicit all its points, and of which the resulting force passes through its centre, do not alter the parallelism of the axis of rotation. It is thus that we explain how the axis of the earth, remains always very nearly parallel to itself in its revolution about the sun, without assuming with Copernicus, an annual motion of the poles of the earth about those of the ecliptic. If the body be of any figure whatever, its axis of rotation may vary at every instant: the investigation of these variations, whatever be the forces which act on the body, is one of the most interesting problems in the science of mechanics which relates to hard bodies, in consequence of its connexion with the procession of the equinoxes and the libration of the moon. Its solution has led to this curious and useful result, namely, that in every body there exist three axes, perpendicular to each other, about which it may revolve (*e*) uniformly, when it is not sollicited by any external force. These axes have on this account been termed the *principal axes of rotation*. They possess this remarkable property, that the sum of the products of each molecule of the body, into the square of its distance from the axis, is a *maximum* with respect to two of these axis, and a *minimum* with respect to the third. If we suppose the body to revolve round an axis which is inclined in a very small angle to either of the two first, the instantaneous axis of rotation will always deviate from either of them by an indefinitely small quantity; therefore the rotation is stable relatively to the two first axes; it is not so with respect to the third principal axis, and if the instantaneous axis deviates from it, by ever so small (*f*) a quantity, this deviation will increase and become continually greater and greater.

A body, or a system of bodies of any figure whatever, oscillating about a fixed horizontal axis, constitutes the compound pendulum. These are the only species of pendulums which really exist in nature, and the simple pendulums, which have been noticed in the second chapter, are purely geometrical conceptions which have been (*g*) devised in order to simplify the subject. It is easy to reduce to them the compound pendulums, of which all the points are firmly connected together. If the length of the simple pendulum, the oscillations of which are of the same duration as those of the compound pendulum, be multiplied by the mass of this last pendulum, and by the distance of its centre of gravity from the axis of oscillation, the product will be equal to the sum of the products of each molecule of the compound pendulum, into the square of its distance from the same axis. It is by means of this rule, which was discovered by Huygens, that experiments on compound pendulums make known the length of the simple pendulum which vibrates seconds.

Conceive a pendulum to make very small oscillations, all of which exist in the same plane, and suppose that at the moment of its greatest deviation from the vertical, a small force is impressed on it, perpendicular to the plane of its motion; it will describe an ellipse about the vertical. In order to represent this motion, we may conceive a fictitious pendulum which continues to vibrate as the real pendulum would do, if the new force had not been impressed on it; while the real pendulum, in virtue of the impressed force vibrates at each side of the ideal pendulum, as if this fictitious pendulum had been immoveable and vertical. Thus it appears, that the motion of the real pendulum (h) is the result of two simple oscillations co-existing and perpendicular to each other.

This manner of considering the small oscillations of bodies, may be extended to any system whatever. If we suppose the system to be deranged from its state of equilibrium by very small impulsions, and that afterwards new ones are impressed on it, it will oscillate relatively to the successive states which it would have assumed in virtue of the first impulsions, in the same manner as would vibrate with respect to its state of equilibrium, if the new impulsions had been solely impressed in this state. Therefore the very small oscillations of a system of bodies, however complicated, may be considered as made up of simple oscillations, perfectly similar to those of the (i) pendulum. In fact, if we conceive the system to be primitively in repose, and then very little disturbed from its state of equilibrium, so that the force which sollicits each body may tend to reduce it to this state, and may moreover be proportional to the distance of the body from this point, it is evident that this will be the case during the oscillation of the system, and that at each instant the velocity of the different points will be proportional to their distance from the position of equilibrium. They will therefore attain this position at the same instant, and they will vibrate in the same manner as the simple pendulum. But the state of derangement which we have assigned to the system, is not unique. If one of the bodies be elongated from the position of equilibrium, and if then the situations of the other bodies which satisfy the preceding conditions be investigated, we arrive at an equation of a degree equal to the number of the bodies of the system, which are moveable between themselves; which furnishes for each body, as many species of simple oscillations, as there are bodies. Let us conceive that the first species of oscillations exists in the system; and at any given instant, let all the bodies be supposed to be elongated from their position, proportionally to the quantities which are relative to the second species of oscillations. In virtue of the coexistence of the oscillations, the system will oscillate with respect to the consecutive states, which it would have assumed in consequence of the first species of oscillation, as it would have oscillated about its state of equilibrium, if the second species had been solely impressed on it; its motion will therefore be made up of the two first species of oscillation: we may in like manner combine with this motion, the third species of oscillations, and so by proceeding in this manner combine all these species in the most general manner; we can thus synthetically compound all possible motions, which may be impressed on a system, provided that they be very small, and conversely we may by analysing these motions, resolve them into simple oscillations. Hence arises an easy method of recognizing the absolute stability of the equilibrium of a system of bodies.

If in all positions relative to each species of oscillations, the forces tend to reduce the bodies to a state of equilibrium, this state will be stable; this will not be the case, or the stability will be only relative, if in any one (k) of these positions, the forces tend to encrease the distance of the bodies from the position of equilibrium.

It is evident that this manner of viewing the very small oscillations of a system of bodies, may be extended to fluids themselves, of which the oscillations are the result of simple oscillations existing simultaneously, and frequently of an infinite number.

We have a very sensible example of the existence of very small oscillations, in the case of waves, when a point of the surface of stagnant water is slightly agitated; circular waves are observed to form and to extend themselves about it. If the surface be agitated at a second point, new waves are observed to arise, and mix themselves with the former; they are superimposed over the surface agitated by the first waves, as they would be (l) dispersed on this surface, if it had remained tranquil, so that they are perfectly distinct in their commingling. What is observed by the eye to be the case with respect to waves, the ear perceives with respect to sounds or the vibrations of the air, which are propagated simultaneously without any alteration, and make very distinct impressions.

The principle of the coexistence of simple oscillations, for which we are indebted to Daniel Bernoulli, is one of these general results which assists the imagination, by the facility with which it enables us to exhibit phenomena and their successive changes.

It may be easily deduced from the analytical theory of the small oscillations of a system of bodies. These oscillations depend on linear differential equations, of which the complete integrals, are the sum of the (m) particular integrals. Thus the simple oscillations are disposed one on the other, to form the motion of the system, as the particular integrals which represent them, are combined together to constitute the complete integrals. It is interesting to trace in this manner, the intellectual truths of analysis in the phenomena of nature. This correspondence, of which the system of the world furnishes us with numerous examples, constitutes one of the great charms of mathematical speculations.

It is natural to reduce the laws of the motion of bodies to a general principle, in the same manner as the laws of their equilibrium have been reduced to the sole principle of virtual velocities. To effect this, let us consider the motions of a system of bodies acting the one on the other, without being sollicited by accelerating forces. Their velocities change at every instant, but we may conceive each velocity at any instant to be compounded of the velocity which it would have at the following instant, and of another velocity which ought to be destroyed at the commencement of this new instant. If the velocity destroyed be known, it would be easy, by the law of the resolution of forces, to determine the velocity of the body at the second instant; now it is evident, that if the bodies were only actuated by the velocities which are destroyed, they would mutually constitute an equilibrium; thus the laws of equilibrium will give the relations of the velocities which are destroyed, and it will be easy to determine from thence the velocities which remain, and their (n) directions. Therefore by means of the infinitismal analysis we shall have the successive variations of the motion of the system, and its position at every instant. It is evident that if the bodies are actuated by accelerating forces, the same resolution of velocities may be employed, but then, the equilibrium ought to obtain between the velocities destroyed and these forces.

This method of reducing the laws of motion to those of equilibrium, for which we are principally indebted to d'Alembert, is very luminous and universally applicable. It would be a matter of surprise that it had escaped the notice of geometers, who had occupied themselves with the principles of dynamics previously to its discovery, if we did not know that the simplest ideas are almost always those which are the last suggested to the human mind.

It still remained to combine the principle which has been just explained, with that of virtual velocities, in order to give to the science of mechanics all the perfection of which it appears to be susceptible. This is what Lagrange has atchieved, and by this means has reduced the investigation of the motion of any system of bodies, to the integration of differential equations. The object of mechanics is by this means accomplished, and it is the province of pure analysis to complete the solution of problems. The following is the simplest manner of forming the differential equations of the motion of any system whatever. If we imagine three fixed *(o)* axes perpendicular to each other, and that at the end of any instant the velocity of each material point of a system of bodies is resolved into three others parallel to those axes; we may consider each partial velocity as being uniform during this instant; we can then suppose that at the end of this instant, the point is actuated parallel to one of these axes by three velocities, namely, by its velocity during this instant, by the small variation which it receives in the following instant, and by this same variation applied in a contrary direction. The two first of these velocities exist in the following instant; the third must therefore be destroyed by the forces which sollicit the point, and by the action of the other points of the system. Consequently, if the instantaneous variations of the partial velocities of each point of the system, be applied to this point in a contrary direction, the system should be in equilibrio, in consequence of all these variations, and of the forces which actuate it. The equations of this equilibrium will be obtained by means of the principle of virtual velocities; and by combining them with those which arise from the connection of the parts of the system, the differential equations of the motion of each of these points will be obtained.

It is evident that we can in the same manner, reduce the laws of the motion of fluids to those of their equilibrium. In this case, the conditions relative to the connection of the parts of the system are reducible to this, namely, that the volume of any molecule of the fluid remains always the same, if the fluid be incompressible, and that it depends on the pressure exerted according to a *(n)* given law, if the fluid be elastic and compressible. The equations which express these conditions, and the variations of the motion of the fluid, contain the partial differences of the coordinates of the molecule, taken either relatively to the time, or with respect to the primitive coordinates. The integration of this species of equations presents great difficulties, and we have as yet been only able to succeed in some particular cases, relative to the motions of ponderable fluids in vases, to the theory of sound, and to the oscillations of the sea and of the atmosphere.

The consideration of the differential equations of the motion of a system of bodies, has led to the discovery of several very general and useful principles of mechanics, which are an extension of those already announced in the second chapter of this book, relative to the motion of a point.

A material point moves uniformly in a right line, if it is not subjected to the action of extraneous causes. In a system of bodies which act on each other without being subjected to the action of exterior causes, the common centre of gravity moves uniformly in a right line, and its motion is the same, as if all the bodies were united in this point, all the forces which actuate them being immediately applied (q) to it; so that the direction and the quantity of their resultant, remain constantly the same.

We have seen that the radius vector of a body, sollicited by a force, which is directed to a fixed point, describes areas which are proportional to the times. If we suppose a system of bodies acting on each other, in any manner, and sollicited by a force directed to a fixed point; and if from this point, radii vectores drawn to each of them, be projected on an invariable plane passing through this point, the sum of the products of the mass of each body into the area which the projection of its (r) radius vector traces, is proportional to the time. It is in this that the *principle of the conservation of areas consists.* If there is no fixed point, towards which the system is attracted, and if it be only subjected to the mutual action of its parts, we may then assume any point whatever, for the origin of the radii vectores.

The product of the mass of the body into the area described by the projection of its radius vector in an unit of time, is equal to the projection of the entire force of this body multiplied into the perpendicular let fall from the fixed point, on the direction of the force thus projected ; this last product is the moment of the force to make the system revolve about an axis passing through the fixed point, and perpendicular to the plane of projection ; the principle of the conservation of areas is therefore reduced to this, namely, that the sum of the moments of the finite forces to make the system revolve about any axis passing through the fixed point, which sum vanishes in the state of equilibrium, is constant in the state of motion. When it is announced in this manner, this principle is applicable in all possible laws between the force and velocity.

The *vis viva* of a system, is the sum of the products of the mass of each body, by the square of its velocity. When the body moves on a curve or on a surface, without being subject to a foreign action, its *vis viva* is always the same, because its velocity is constant; if the bodies of the system experience no other action, but such as arise from their mutual tractions and pressures, either directly or by the intervention of rods and inextensible and unelastic threads, the *vis viva* of the system remains constant, even though several of the bodies should be constrained to move on curved lines or surfaces. This (*s*) principle, which has been termed the *principle of the conservation of living forces*, is applicable to all possible laws between the force and the velocity, provided that by the *vis viva* or living force of a body, is understood twice the integral of the product of its velocity, into the differential of the finite force by which it is actuated.

In the motion of a body sollicited by any forces whatever, the variation of the *vis viva* is equal to twice the product of the mass of the body, by the sum of the accelerating forces multiplied respectively by the elementary quantities, by which the body advances towards their origins. In the motion of a system of bodies, twice the sum of all these products, is the variation of the living force of the system. Let us conceive that in the motion of the system, all the bodies arrive at the same instant in the position, in which it would be in equilibrio in consequence of the accelerating forces which sollicit it: by the principle of virtual velocities the variation of the living force vanishes; therefore the *vis viva* will then have attained its *maximum* or *minimum*. If the system be moved by one sole species of simple oscillations, the bodies after departing from the position of equilibrium will tend to revert to it, if the equilibrium be stable; therefore, their velocities diminish in proportion as their distance from this position is increased, and consequently in this position, the *vis viva* will be a *maximum*. But if the equilibrium be not stable, the bodies in proportion as their distance from this position is increased will tend to deviate more from it, and their velocities will continue to increase, consequently their *vis viva* will be in this case a *minimum*. Hence we may infer, that if the *vis viva* be constantly a *maximum*, when the bodies simultaneously attain the position of equilibrium, whatever that velocity may be, the equilibrium will be stable, and on the contrary, the stability will be neither absolute or relative, if the *vis viva* in this position of the system, be constantly a *minimum*.

Finally, we have seen in the second chapter, that the sum of the integrals of the product of each finite force of the system, by the element of its direction, which sum vanishes in the state of equilibrium, becomes a minimum in the state of motion. It is in this (t) that the principle of the least action consists, which principle differs from those of the uniform motion of the centre of gravity, of the conservation of areas and of living forces, in this, that these principles are the real integrals of the differential equations of the motion of bodies; whereas that of the least action is only a remarkable combination of these same equations.

The finite force of a body, being the product of its mass into its velocity, and the velocity multiplied into the space described in an element of time, being equal to the product of this element by the square of the velocity, the principle of the least action may be announced in the following manner: the integral of the *vis viva* of a system, multiplied by the element of the time, is a *minimum;* so that the true economy of nature is that of the *vis viva.* To produce this economy ought to be our object in the construction of machines, which are more perfect in proportion as less *vis viva* is required, to produce a given effect. If the bodies are not sollicited by any accelerating forces, the *vis viva* of the system is constant; consequently the system passes from one point to another in the shortest time.

Another important remark remains to be made relative to the extent of these different principles. That of the uniform motion of the centre of gravity, and the principle of the conservation of areas, subsist even when by the (u) mutual action of the bodies of the system they undergo sudden changes in their motions, which renders these principles extremely useful in several circumstances; but the principle of the conservation of *the vis viva* and of the least action require, that the variations of the motions of the system be made by insensible gradations.

When the system undergoes sudden changes, either from the mutual action the of bodies of the system, or from meeting with obstacles, the *visviva* experiencesat each of these changes, a diminution equal to the sum of the products of each body into the square of the velocity destroyed, conceiving the velocity previous to the change to be resolved into two, of which one subsists after the shock, the other being annihilated, the square of which is evidently equal to the sum of the squares of the variations which the change makes the decomposed velocity to experience, parallel to any three coordinate axes. All these principles would still obtain, regard being had to the (v) relative motion of the bodies of the system, if it was carried along by a general motion common to the foci of the forces, which we have supposed to be fixed. They obtain likewise in the relative motions of bodies on the earth, for it is impossible, as has been already observed, to judge of the absolute motion of a system of bodies, by the sole appearances of its relative motion.

Whatever be the motion of the system and the variation which it experiences from the mutual action of its parts, the sum of the products of each body, by the area which its projection traces about the common centre of gravity, on a plane which passing through this point remains always parallel to itself, is constant. The plane on which this sum is a maximum, preserves its relative position (x) during the motion of the system, the same sum vanishes for every plane which passing through the centre of gravity, is perpendicular to that just mentioned; and the squares of the three similar sums relative to any three planes drawn through the centre of gravity, and perpendicular to each other, are equal to the square of the sum which is a *maximum*. The plane which corresponds to this sum, posesses also the following remarkable property, namely, that the sum of the projections of the areas traced by bodies about each other, and multiplied respectively by the product of the masses of the two bodies which are connected by each radius vector, is a *maximum* on this plane, and on all planes which are parallel to it. We may therefore find at all times a plane which passing through any one of the points of the system preserves always a parallel situation; and as by referring the motion of the bodies of the system to it, two of the constant arbitrary quantities of this motion disappear, it is as natural to select this plane for that of the coordinates, as it is to fixtheir origin, at the centre of gravity of the system.

NOTES.

(*a*) The meridian is therefore a secondary both to the equator and to the horizon; and as from Napier's rules the sine of the elevation of any point of the equator above the horizon is equal to the sine of the inclination of the equator to the horizon multiplied into the sine of the arc of the equator, intercepted between the given point and the horizon; it follows, that as the inclination of the equator is constant in the same place, the elevation of the point is greatest when it is 90° from the horizon, *i. e.* when it is on the meridian; in which case also the sine of the greatest elevation of the equator, (which is equal to the complement of latitude,) is equal to the sine of the inclination of the equator to the horizon, and as the most elevated point of the equator exists on the meridian, the most elevated points of all parallels to the equator exist also on the meridian.

As the star is always at the same distance from the pole, when it is on the meridian, it is as much below the pole in one observation as it is above it in the other; hence, the three elevations constitute an arithmetic progression. This observation gives us at the same time the declination, for this last quantity is equal to 90°, minus half the difference between the greatest and least heights;

however, this method requires some corrections for refraction, &c. as will be hereafter specified; indeed it has been employed to determine the *quantity of refraction* when the latitude is known from other considerations, (see Brinkley's Astronomy, Chap. 4.); the nearer the star is to the pole the less will be the error from the hypothesis, that there is no refraction; those stars never set, of which the distance from the pole is equal to the complement of the elevation of the pole above the horizon.—*See* Note (*d*), Chap. 2.

(*b*) The actual magnitude of the earth, considered as spherical, may be determined from this circumstance, for if we proceed north or south until the pole is elevated or depressed a degree, we know that we must have travelled over a degree on the earth's surface, the number of miles in which being measured and multiplied by 360, gives the number of miles in the earth's circumference, by means of which it is easy to determine the number of miles in the earth's radius; what is stated in the text shews, that the earth is convex at the place of the spectator; the circumnavigation of the globe in *various* directions proves, that it is a curved surface returning into itself, and likewise the circumstance of the boundary of the earth's shadow in a lunar eclipse being always circular, proves that it is globular or round.

(*c*) The sun's motion is always performed in the same plane; for the sine of right ascension bears to the tangent of declination an invariable ratio, it follows consequently that the plane passingthrough the sun and the vernal equinox must always make the same angle with the equator, the radius being to the tangent of this angle in the given invariable ratio; it is also observed, that the difference between the right ascensions of those stars, which are near to the sun at the commencement of spring and at the commencement of autumn, is 180, hence it follows, that the two intersections of the equator and ecliptic are 180° distant; and if the points of the horizon, when the sun sets in the beginning of summer and winter, be accurately marked, it will be found that they are equally distant from the east and west, hence, and as all the points of the orbit are always in the same plane, it follows, that the ecliptic is a *great* circle.

(*d*) If l denotes the latitude, d the declination, and h the horary angle from noon, we have, when the sun is rising or setting, $\cos. h = \text{tang.}\, l.\, \text{tang.}\, d$; when the height of the pole and d are of the same denomination $\cos. h$ is negative, and $\therefore h >$ than 90, $\therefore$ the day is longer than the night; when l or d, or both, vanish, $h = 90°$, therefore, the day is always equal to the night; when $\text{tang.}\, l = \cot.\, d$, or *vice versâ*, $h = 0$, $\therefore$ the sun does not set.

(*e*) The horizon of spectators situated at the equator passes through the poles, hence the horizon, being in this case a secondary to the equator must pass through the centres of all circles parallel to the equator, and bisect them all at right angles; hence, as also appears from the preceding note, the days are always equal to the night; such a position of the sphere is called a right sphere. To a spectator situated at the pole, the equator and horizon coincide, consequently the planes of all the diurnal circles are parallel to the plane of the horizon, so that when the sun is at the northern side of the equator, he does not set for six months; this position of the sphere is called a parallel sphere. In all places intermediate between the equator and poles, the length of the day is different at different periods of the year. Such positions of the sphere are called oblique spheres; what is stated here is immediately apparent from the preceding part of this note (*d*).

If by means of the observed declinations and right ascensions of the sun, the daily increments of longitude be computed, it will be found that they are not proportional to the intervals of time which separate the consecutive passages of the sun over the meridian; the greatest difference exists in two points of the ecliptic, of which one is situated near to the summer, and the other near to the winter solstice; these two points are in the same line, though situated on opposite sides of the equator, and their right ascensions differ by 180.

(*f*) Consequently the mean velocity between these two extremes is $\frac{1°,1327 + 1°,0591}{2} = 1°,0959$.

(*g*) The angles being supposed to increase proportionably to the times, their sines will be periodical; for the sine, which at the commencement is cypher, increases with the arc and becomes equal to radius when the arc $= 90°$, it then decreases and finally becomes = to cypher when the arc becomes equal to 180; the sine then passing to the other side of the diameter changes its sign, and runs through the same series of changes in this semicircumference. It may be remarked here, that it appears from analysis that all the inequalities of the heavenly bodies may be expressed by the sines and cosines of angles, which increase proportionally to the time. No other function of the circle occurs in the expressions for these inequalities. *See* Vol. 2, Book 6, Chap. 2.

(*h*) In fact, as it is a matter of observation that the angular motion of the sun varies as the square of the apparent diameter, it follows, as a general law, that the angle described each day by the sun multiplied by the square of the distance is constant, *i. e.* if r and r' represent the distances, and dv, dv' the angles described by the sun at the two different epochs, we have $dv.r^2 = dv'.r'^2$; but the areas described at these points are respectively $= \frac{dv.r^2}{2}$, $\frac{dv'.r'^2}{2}$, hence it follows, that equal areas are described in equal times.

Otherwise thus, let v and v' represent the angular motions of the sun in two different points of the orbit, a and a' the small diurnal arcs described by the sun at these points, r and

$r+\delta r$, the corresponding distances, and d, d' the corresponding apparent diameters, the small sectors described by the sun are equal respectively to $\frac{r.a}{2}$ and $\frac{(r+\delta r).a'}{2}$, or as $a=vr$, $a'=v'.(r+\delta r)$, these sectors are equal to $\frac{r^2.v}{2}$, $\frac{(r+\delta r)^2.v'}{2}$; now by means of very exact measurements of the apparent diameter of the sun made with a micrometer, it is found that the apparent angular motions vary as the squares of the apparent diameters, i. e. $v:v'::d^2:d'^2$, or $v:v'::(r+\delta r)^2:r^2$, $\therefore vr^2=v'.(r+\delta r)^2$, hence the small sectors are always proportional to the times.

(*i*) In fact, suppose lines to be drawn in a plane passing through a given point, (which represents the common centre of the earth and of the celestial sphere,) so that their angular distances may be equal to the diurnal motion. These lines will represent the visual rays, which are drawn to the sun each successive day. Lay off from the fixed point in the direction of these rays the corresponding distances of the sun from the earth, (which may be estimated from the diurnal motion, one of these distances being assumed equal to unity,) the points which are determined in this manner will indicate the place of the sun for each day, and the curve which is traced by uniting these points will be similar to the sun's orbit. It is evident, that if the angles described by the sun each succsssive day be determined by means of its observed longitudes, the ratio of the distances will be obtained; for, from the equation $vr^2=\text{A}$, it follows, that these distances are reciprocally as the square roots of the angular motions. But in order to ascertain whether the curve indicated by the observations of the sun is an *exact* ellipse, we should assume the indeterminate equation of any ellipse, and make it to satisfy some of these observations; and when the elements have been determined by this condition, we can investigate and try whether it equally represents the other observations, *i. e.* if it assigns for the distances of the sun from the earth in different longitudes, values equal to those which have been deduced from observation.

We might have inferred from an observation of the sun's apparent diameter that his apparent orbit is an ellipse, for if m be his mean, and $m-n$ his least apparent diameter, then this diameter at any other point is observed to be equal to $m-n.\cos.v$, v representing his angular distance from the point where his diameter is least; now, as the distance varies inversely as the apparent diameter, $r=\frac{B}{m-n.\cos.e}$, which is an equation of the same form as $r=\frac{a.(1-e^2)}{1+e.\cos.v}$.

Or thus, let D, D′ represent the greatest and least diameters of the sun, which have been already given in numbers in the text; it is found that if d denote any other diameter, and v the angular distance of the sun when the diameter is d, from the point in the ellipse where the diameter is D, we have $D-D' : D-d ::$ $1-\cos.180$ (*i. e.* 2) : $1-\cos.v$, $\therefore (D-D')(1-\cos.v)=$ $2.(D-d)$ and $d=D-\frac{(D-D')}{2}.(1-\cos.v)=\frac{D+D'}{2}+$ $\frac{D-D'}{2}.\cos.v$, $\therefore \frac{1}{r}=\frac{1}{a.(1-e^2)}+\frac{e}{a.(1-e^2)}.\cos.v=$ $\frac{1+e.\cos.v}{a(1-e^2)}$, this is the equation of an ellipse whose major axis passes through the points where the apparent diameter is greatest and least.

(*k*) This point may be easily determined in the case of any elliptic orbit. About the focus of the ellipse, as centre, describe a circle, of which the radius is a mean proportional between the semiaxes of the ellipse; this circle is equal to the ellipse, and if a body be conceived to revolve in this circle with the mean angular motion of the sun, its periodic time will be equal to the periodic time of the sun. Conceive this imaginary body to set off from the same radius as the sun, at the same time that the sun begins to move from the apogee. As the sun's velocity in this point is less than his mean angular velocity, the fictitious body will precede the sun, and it will continue to precede this star by greater quantities perpetually, till the angular motion of the sun becomes equal to the angular motion of this body, afterwards the angular motion of the sun becoming greater than the mean angular motion, the sun will begin to gain on the body, and will overtake it, when it arrives at perigee; hence it is evident, that the body precedes the sun by the greatest quantity, when its angular motion is equal to the mean angular motion; now it appears from the equation $vr^2 = A$, that the angular motions vary as the synchronous areas directly, and inversely as the squares of the distance, but the synchronous areas are equal in the ellipse and circle, for they are as the whole areas divided by the respective periodic times, *i. e.* in a ratio of equality, hence, the angular motions are equal when the distances are equal, *i. e.* when the distance of the sun from the focus is a mean proportional between the semiaxes.

The radius of the circle whose area is equal to that of the ellipse $= a. \sqrt{a^2 - e^2}$.

(*l*) This parallax is given with great accuracy by theory, as we shall see in the sequel, (see Book 4, Chap. 4,) the reason why it is so particularly interesting to determine the parallax is, because our knowledge of the absolute magnitude of the solar system depends on it.

If the exact time when the spots describe right lines was known, the longitude of the sun or earth at this instant would determine the place of the nodes. However, this place is best determined by means of corresponding observations, made before and after the passage through the nodes when the openings of the ellipse is the same, but in opposite directions.

Calling l, x the heliocentric longitudes of the earth and spot, y the heliocentric latitude, and B the geocentric latitude, Δ the sun's semi-diameter, r the distance of spot from centre of the sun, and R the distance of spot from centre of the earth, which is very nearly equal to the distance of the centre of the sun from earth, we have $r : R :: \sin. B : \sin. y$, $\therefore$ as $\sin. \Delta = \frac{r}{R}$, we have $\sin. y = \frac{R}{r} \cdot \sin. B = \frac{\sin. B}{\sin. \Delta}$, likewise $r. \cos. y : R. \cos. B$ expresses the ratio of the curtate distances of the spot from the centres of the sun and earth, which is also expressed by that of $\sin. E : \sin. (l - x)$, E being equal to the difference between the geocentric longitudes of the centre of the sun and spot, $\therefore$ we have $r. \cos. y : R. \cos. B :: \sin. E : \sin. (l - x)$, hence $\sin. (l - x) = \frac{\sin. E. \cos. B}{\cos. y} \cdot \frac{R}{r} = \frac{\sin. E. \cos. B}{\cos. y. \sin. \Delta} =$ (by substituting for $\cos. y$ its value $\frac{\sqrt{\sin.^2 \Delta - \sin.^2 B}}{\sin. \Delta}$) $\frac{\sin. E. \cos. B}{\sqrt{\sin.^2 \Delta - \sin.^2 B}}$, hence we can determine x.

Observing three positions of the same spot, we are given by what precedes their distances, l, l', l'', from the pole of the ecliptic or their co-latitudes. We can also, by what precedes, determine their differences of longitude; hence in the three spherical triangles, which are formed by drawing lines from the pole of the ecliptic to the three observed positions of the spot, we have in each of them, respectively, two sides and the included angle, which enables us to determine the remaining sides, A, A$'$, A$''$, (or the arcs connecting the three positions of the spot), and also the base angles, and consequently their sum; now as the spot moves parallel to the solar equator, its distances from the pole of this equator are the same, consequently a perpendicular from this pole bisects the arcs connecting the different positions of the spots; and from the consideration of these triangles it is evident, that we are given the ÷ of the sum of the cosines of the angles, which an arc from the pole of the sun's equator makes with the connecting arcs, to the difference of the cosines of these angles, *i. e.* we are given the ratio of the cotangents of half the angle made by connecting arcs to the tangent of half the difference of the preceding angles; having determined this difference we can obtain the angle which the arc from the pole of the equator makes with connecting arc, and hence we obtain, by subtraction, the angle formed by arcs drawn from a given position of the spot to the poles of equator and ecliptic, and as we have these arcs we can obtain the third side, which measures the inclination of the equator to ecliptic; and as we also know a, the angle formed by l, l', and the time, t, in which it is described, we can obtain the time of revolution for $t : T :: a : 360°$.

CHAPTER III.

This position, with respect to the equality which subsists between the duration of each oscillation of a pendulum, is, in fact, the principle of sufficient reason which was first propounded as a general axiom by Leibnitz, though it was long before *virtually* assumed by Archimedes in demonstrating some of the first principles of mechanics.

The sun in the course of the year passes the meridian once less than the star, because the sum of all the retardations in that time is equal to 360°, being equal to the sum of the arcs described by the sun in the year, *i. e.* to 360.

It may be remarked here, that in consequence of the precession of the equinoxes, the star takes a longer time to return to the meridian than the revolution of the earth on its axis; however, the difference is not appreciable, for supposing that the annual precession in right ascension is $50''1$, which it is very nearly for stars near the equator, this converted into time gives 3,3 seconds, by which the star passes the meridian later at the end of a year, which being distributed over the entire year is altogether insensible.

(m) Let I be the obliquity of the ecliptic, l the longitude of the sun, and A the right ascension; then if $\cos. I = s$, $\text{tang}. l = x$, we have, by Napier's rules, $sx = \text{tang}. A$, $\therefore$ $s.dx$, i. e. $dl.(1+x^2).s = dA(1+\text{tang}.^2 A)$, or $sdx = s.dl.\sec.^2 l = dA.\sec.^2 A$, that is, $\frac{dl.s}{\cos.^2 l} = \frac{dA}{\cos.^2 A}$, and since $\cos. l = \cos. A.\cos. D$, (D being equal to the declination,) we obtain $\frac{dl.s}{\cos.^2 A.\cos.^2 D} = \frac{dA}{\cos.^2 A}$; therefore, dA (which converted into time determines the variation of the astronomical day,) is equal to $d.l.\sec.^2 D$; and as dl and s are constant, dA varies as $\sec.^2 D$, and therefore it is greatest at the solstice, and least at the equinox; for $dA = dl.s$ at the equinox, and at the solstice $dA = \frac{dl}{s}$, $\therefore$ dA at the equinox is to dA at the solstice as $s^2 : 1$, consequently dl is a mean proportional between the increment in the equinoxes and in the solstices; l is evidently equal to the right ascension of the fictitious sun s'', which is supposed to move in the equator with a motion equal to the sun's mean motion in the ecliptic; $\therefore$ $l - A$ is equal to the separation of s'' from s', and $\tan.(l-A) = \frac{\tan. l - \tan. A}{1 + \tan. l.\tan. A} = \frac{x - sx}{1 + sx^2} = (1-s).\frac{x}{1+sx^2}$, $\therefore$ $d.(l-A) = (1-s)\,dx.\frac{(1-sx^2)}{(1+sx^2)^2}$; which is a *maximum* when $1 = sx^2$, that is, $\tan. l = \frac{1}{\sqrt{\cos. I}}$, and consequently $\sqrt{\cos. I} = \tan. A$.

hence $A=43°, 43', 56''$, and $l=46°, 14'$, and $l-A$ when a *maximum* $= 2°, 28', 20''$; it appears from this that the greatest separation of s'' from s' is greater than the greatest separation of s' from s on the equator, corresponding to the greatest equation of the centre, for the latter is only $2°, 6'$, when the greatest equation is $1°, 55' 33''$; besides, this greatest separation happens about the 8th of May, which is later than when the radius vector of the solar orbit is a mean proportional between the semiaxes, that is, when the equation of the centre is *maximum.*—*See* Note (k), page 302.

(n) Hence, as the second and third sun's depart from the equinox together, the one describing the equator, and the other the ecliptic, with the same uniform motion; the distance of the latter (which is equal to the mean longitude of the true sun) will be equal to the right ascension of the third sun. Hence the equation of time may be defined to be the *difference between the true sun's right ascension and his mean longitude, corrected by the equation of the equinoxes in right ascension*; therefore, naming e the equation of the centre, μ, ν the increments in longitude and right ascension which result from the nutation, r the reduction to the equator, or the difference between the longitude and right ascension, λ', λ the true and mean longitudes of the sun, ρ' ρ the true and mean right ascensions, and p the effect produced by the perturbations of the planets, we have $\lambda'=\lambda+e+p+\mu$, $\rho=\lambda+v$, $\rho'=\lambda'+\rho=\lambda+e+p+\mu+\rho$, $\therefore \rho'-\rho=e+p+r+\mu-v$; we will see hereafter that $v=\mu.\cos.\varepsilon$ (ε being the obliquity) and $\mu=18''.\sin.\zeta$, (ζ depending on the : situation of the lunar orbit), therefore x, the equation of time, $=\dfrac{e+p+r}{15}+\dfrac{18.\sin.\zeta}{15}\cdot(1-\cos.e)$; since, therefore, both e and r are variable in this expression, the equation must (without taking into account the disturbing force μ or p) be variable from these two causes; and as e and r are not the same on corresponding days of two different years, in consequence of the secular disturbances, it follows, that the equation of time is continually varying.

There are four times in the year in which the equation of time vanishes, for denoting the true sun, the sun which moves with a mean motion in the ecliptic, and the sun which moves with a mean motion in the equator by s, s', s'' respectively. As s' precedes s from apogee to perigee, and s'' precedes s' from the autumnal equinox to the solstice, the order of the sun's near the winter solstice is s, s', s''; at the solstice s' coinciding with s'' the order is $s, \left\{\frac{s'}{s''}\right\}$ immediately after s' passes s'', (as appears from what has been established above respecting the increments of dA, at the equinox and at the solstice), ∴ after the solstice the order is s, s'', s'; at the perigee, which is very little beyond the solstice, s coincides with s'; ∴ it must have passed s'' in order to effect this, for s'' does not overtake s' till their arrival at the vernal equinox; hence, at the moment when s passed s'', the equation of time vanished. After the perigee the order of the sun's is $s''s's$, which continues to the vernal equinox, therefore in that interval the equation of time does *not* vanish; after the equinox s'' begins to precede s', and the order becomes $s's''s$; very near this point the distance from the focus of the solar ellipse is a mean proportional between the semiaxes, *i. e.* the true angular motion is equal to the mean angular motion, and therefore s' is at the greatest distance from s. But the greatest separation of s'' from s' is subsequent to this, and as it is greater in quantity than the deviation of s from s', it follows, that previous to the greatest separation of s'' from s', the order of the sun's is not $s's''s$ but $s'ss''$; therefore s'' must have passed s, consequently the equation of time must have vanished; but at the summer solstice s'' joins s', and as s' does not join s till after the time of the solstice, when the sun is in the apogee, this junction of s'' with s' must have been effected by s'' repassing s, this caused the equation of time again to vanish, previous to the time of the solstice; after this takes place the order of the sun is $s's''s$, at the solstices s' coincides with s'', and after this the order becomes $s''s's$ till the sun arrives at apogee. Immediately after s' moving with a greater angular motion than s, the order becomes $s''ss'$; now as s cannot overtake s' till it arrives at perigee, whereas s'' reaches s' at the equinox, it follows, that previous to this s'' must have passed s, and at the instant of passing, the equation of time vanishes. If the apogee and perigee coincided with the solstices, the equation of time would vanish in these points, which was the case in the year 1250; but as the apsides continually prograde, the points, at which the equation of time vanishes, continually vary. As the moments when the equation from each cause separately is a maximum, do not coincide, the greatest equation can never be equal to the sum of the two equations arising from each cause separately; when the equation of time is a *maximum*, its increment is cypher, *i. e.* the mean and true day have the same length, when the equation of time vanishes, their difference is the greatest possible.

(*o*) The reason why the day was divided into 24 hours, and the hours into 60 minutes, and the minutes into 60 seconds, was, because these numbers admitted many different divisors.

If the year was exactly $=365+\frac{1}{4}$, in four years the commencement of the year would have regraded an entire day, and in 1460 *Julian* years the commencement would have regraded an entire Julian year, for dividing 1460 by by 4, the quote will be 365, ∴ 1460 Julian are equivalent to 1461 Egyptian years, but as the year is accurately only $=365,2422640$, in order that the difference between this and 365 may produce a tropical year, it is necessary that 1508 years should be accomplished; this period of 1460 is called the *sothiac* period. The Egyptians supposed all their months to consist of 30 days, and they added at the end of the year five days, which were called ἐπαγομενα. *See* Vol. 2, Book 4, Chap. 3.

(*p*) Suppose that the moment of the solstice or equinox preceded midnight by a quantity less than the errors of the tables, then according to the tables the moment would happen *after* midnight, and as the commencement of the year is reckoned from *the midnight* which precedes the solstice *as* determined by the *tables*, this origin would differ nearly by an entire day from the true origin.

(*p*) In the *Julian* arrangement of the year, it is supposed that $365+\frac{1}{4}=R$, a revolution of the sun; consequently, though there is not an integral number of days in *one* revolution, still four years may be made equal to four revolutions of the sun, and $4R=4.365+1=3.365+366$; now as the true length of the year is not 365.25, but 365.242264, which is less than the former by $11', 15''$; before a new year has commenced, the sun has passed the point in the ecliptic where the last year began, by a small fraction $= 11', 15'' \times 59', 8''$; therefore, the Julian reckoning and the course of the seasons fall behind the sun, and in 132 years this difference is very nearly a day, hence in 3.132 or 396, which is nearly equal to four centuries, their loss would be three days; this is the reason why Gregory proposed to omit the intercallary day at the commencement of three successive centuries, which would be, in the Julian arrangement, intercallary years, and to retain it on the fourth century, and two hours fifteen minutes is all that remains uncorrected; for $11', 15''$, the annual error $=$ .007736, which in a century is .7736, and in four centuries it is equal to 3.0944, of which the decimal part .0944, which is not corrected, $= 2^h, 15'$; now this part becomes equal to an entire day in 4237 years, and therefore it would be corrected nearly by omitting, as is suggested in the text, a bissextile every four thousand years.

(q) The Persian intercalation was more correct than the Julian, for Omar proposed to delay to the 33d year the intercalation which ought regularly to take place on the 32d, and by this means the Julian intercalation would be altogether omitted in the 128th year; but it has been before observed, that the Julian intercalation is too much by one day in 132 years, this method is therefore more exact than that proposed by Gregory, for it differs from the truth only by one minute in 120 years; (in fact, if we determine the series of continued fractions which express the ratio between $5^h\,48'\,49''$ and 24^h, the first terms of the series are $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{31}{128}$, and among the terms of this series the ratio, which would exist according to the intercalation proposed by Gregory, does not occur; $\frac{1}{4}$ is greater than the true difference, and $\frac{7}{29}$ less, therefore, as the fractions converge towards the true value, the correction proposed by Omar is more accurate than Cæsar's or Gregory's).—*See* Vol. 2, Book 6, page 220.

(r) The order of the planets, according to the ancients, is Saturn, Jupiter, Mars, the Sun, Venus, Mercury, Moon; now the names of the planets are imposed on the days *δια τεσσαρων*, *i. e.* as the sun is the fourth from Saturn inclusively, he denominates the first day of the week; the moon being the fourth from the sun, denominates the second day of the week, and so on. An astrological reason has also been assigned; for as the planets were supposed to preside over *each* hour of the day, and as the planet gave its name to that day, over the first hour of which it presided, if the sun would have the first hour it would have also the 8th, the 15th, and in general all those of the form $1+7n$; Venus would preside over the second hour, and in general all those of the form $7n+2$; Mercury over all those of the form $7n+3$; the moon over those of the form $7n+4$; Saturn over those of the form $7n+5$; Jupiter over those of the form $7n+6$; Mars over those of the form $7n+7=7.(n+1)$. The general formula is $7n+a=24m+1$, therefore $a-1=24m-7n$ if $7n+a=1$; $n=0$; $a=1$; the first day belongs to the sun; $7n+a=25$, n must be equal to 3; therefore $24m-7n=4$, therefore the moon presides over the second day; if $7n+a=73$; $a=73-7.10$, therefore $a=3$, and the fourth day will be that of Mercury; and after the seven planets are exhausted, the days will return in the same order as before; for let $7n+a=169$, therefore $a=169-7.24=1$, therefore the eighth day belongs also to the sun as well as the first.

CHAPTER IV.

Calling M the diurnal motion of the moon, μ the diurnal motion of the sun, $(M-\mu)$ will be the relative motion with which the moon regains the sun, ∴ we shall have $M-\mu : 1^\circ :: 360 : L = \frac{360^\circ}{M-\mu}$, as we know very nearly the duration of a synodic revolution, we know the number of synodic revolutions in a given interval N; hence, if n represent this number, we have $n.\left(\frac{360}{M-\mu}\right) = N$, therefore $\left(\frac{n}{N}\right)\cdot 360^\circ = M-\mu$, which is the relative motion, hence we can determine M and $\left(\frac{360}{M-\mu}\right)$; now if m represent the mean motion of the sun, $n.360+m : 360 :: N : \frac{N.360}{n.360+m} =$

$$\frac{\left(\frac{N}{n}\right)}{1+\frac{m}{n.360^\circ}} = \left(\frac{N}{n}\right).\left(1-\frac{m}{n.360^\circ}+\left(\frac{m}{n.360^\circ}\right)^2-\&c.\right) = P =$$

the revolution in longitude of the moon, in order to determine P′, the sidereal revolution, we have $360° - p$: $360° :: P : P' = \frac{P}{1-\frac{p}{360°}} = P.\left(1+\frac{p}{360}+\left(\frac{p}{360}\right)^2+\&c.\right)$, ($p$ being the precession in a tropical month), if M represent the motion of the apsis during P, we have $360° - M ::$ $360 :: P : P''$ the anomalistic revolution, $= \frac{P}{1-\frac{M}{360}} =$ $P\left(1+\frac{M}{360}+\left(\frac{M}{360}\right)^2-\&c.\right)$ Let M′ represent the motion of the node, and as it regrades we have $360 + M' : 360 ::$ $P : P''' = P.\left(1-\frac{M'}{360}+\left(\frac{M}{360}\right)^2-\&c.\right)$ hence it appears how all these different periods may be readily inferred from the synodic revolution, which may be accurately determined by means of two eclipses separated by a considerable interval from each other.

The orbit of the moon may be proved to be elliptical in the same manner as the sun's orbit was shewn to be elliptical.

(*s*) Let D, D′ represent the greatest and least apparent diameters of the moon, the eccentricity $=\frac{D-D'}{D+D'}$, and if p be the horizontal parallax when the moon's apparent diameter is d, the parallax at the least distance $=\frac{p.D}{d}$, and at the greatest distance $=\frac{p.D'}{d}$, therefore the least distance $=\frac{r.d}{p.D}$, and the greatest distance $=\frac{r.d}{p.D'}$, consequently the mean distance on the hypothesis that the orbit is elliptical $=\frac{r}{2}.\frac{d}{p}.\left(\frac{1}{D}+\frac{1}{D'}\right)$; r represents the radius of the earth, supposed spherical; but on the supposition that the earth is an ellipsoid, r is the radius corresponding to the latitude, of which the square of the sine $=\frac{1}{3}$. *See* Chap. 4.

From the eccentricity, the equation of the centre may be inferred by means of the formula

$$(2e-\tfrac{1}{4}e^3+\tfrac{5}{96}.e^5).\sin.nt+(\tfrac{5}{4}e^2-\tfrac{11}{24}e^4+\tfrac{11}{192}.e^6).\sin.2nt+(\tfrac{13}{12}e^3-\tfrac{43}{64}.e^5).\sin.3nt+\&c.$$

See Notes to Book 2, Chap. 3. Hence it is easy to shew when it will be a maximum.

(t) If ☾ be the mean longitude of the moon, and ☉ that of the sun, the mean anomaly being nt, as before, this inequality is

$$(1°.21, 5''.5)\sin.2(☾-☉)-nt));$$

hence, as ☾ — ☉ = 0, or 180°, in the oppositions or conjunctions of the moon with the sun, the argument in these positions is $-nt$, which renders the evection negative, if nt is < 180, and positive if nt is > 180, contrary to what happens in the equation of the centre, as is evident from an inspection of its value; therefore, in both cases it is diminished. The period of the evection may be inferred from the rate of increase of its argument, which is 11ᵈ.3166 per day, its period therefore is $\frac{360}{11.3166}$, or 31°.8119. The evection may be considered as an inequality in the equation of the centre, arising from an increase of the eccentricity at the quadratures, and a diminution of it at the syzygies; it appears from its argument that it depends on the position of the axis major of the moon's orbit, with respect to the line connecting the sun and earth.—*See* Princip. Math. Lib. I., Prop. 66, Cor. 9.

(t) This inequality may therefore be represented by the formula $(35' 42'').\sin.2(☾-☉)$. Its period is evidently equal to 14ᵈ.7655, or half a lunar month. Mayer has added to the preceding value of the variation two other terms, which are respectively proportional to sin. 3 (☾ —☉), sin. 4 (☾ —☉).—*See* Notes to Vol. II. Chap. 4.

(*u*) The argument of this inequality is sin. mean. Anom. ☉. Hence, in the eclipses it is confounded with the equation of the centre of the sun. This equation arises from the variation of the sun's distance from the earth.—*See* Notes to Vol. II. Chap. 4.

(*v*) During *each* revolution of the moon the nodes advance, and regrade alternately; but the quantity of the regress exceeding that of the advance, the nodes during a revolution may be said on the whole to regrade, as the excess of the arc of regression above the arc, during the description of which the nodes advance, is twice the distance of the node from syzygy, the regress of the nodes will increase in the passage of the nodes from syzygy to quadrature, and again decrease in the passage from quadrature to syzygy.

Let l, l' represent two latitudes of the moon on successive days, before and after passing the node, λ, λ' the corresponding longitudes, and n the longitude of the node, and I the inclination, we have $l+l' : l : \lambda-\lambda' : n-\lambda = \frac{l.(\lambda-\lambda')}{l+l'}$, hence we get n, and as by Napier's rules $\sin.(\lambda-n)=\tan.l.\cot.\mathrm{I}$, we obtain $\cot.\mathrm{I}=\sin.(\lambda-n).\cot.l$, I might also be obtained by observing the moon's latitude on several days near to its maximum, for the greatest latitude is evidently $=\mathrm{I}$.

Since the argnment of the greatest inequality is proportional to the sine of double the distance of the sun from the ascending node of the lunar orbit, its period must be equal to a semi-revolution of the sun with respect to the nodes of the moon.

(*w*) It was from the variation of the moon's apparent diameter that Newton inferred that the areas were proportional to the times.—*See* Princip. Lib. 3, Prop. 3.

(*x*) The lunar inequalities have been distinguished into three classes, namely, those which affect the longitude, those which affect the latitude, and those which affect the radius vector of the moon. The reason why it was so easy to discover them, was because their periods were of such different durations. With respect to the inequalities which affect the longitude of the moon, three, namely, the evection, variation, and annual equation, have been known to the ancient astronomers; but there are several others, the existence and form of which have been indicated by theory, and which may be considered as so many corrections to be applied to the above mentioned inequalities, in order to determine the position of the moon with the accuracy required by the precision of modern observations. It is the same with with respect to the inequalities which affect the latitude and the radius vector of the moon. The forms of the inequalities are determined by physical astronomy; the coefficients are determined by observing when they attain their greatest values, for then the angular functions into which they are multiplied are equal to unity.

(*y*) The greatest breadth of the illuminated part of the moon's surface is observed to vary as the versed sine of the moon's elongation; but if the moon was spherical, the illuminated part would vary as the versed sine of the exterior angle at the moon, which differs very little from the angle of elongation. Strictly speaking, the illuminated portion varies as the versed sine of the exterior angle at the moon $=$E, (the angle of elongation) $+$(S) angle at the sun; this last quantity, or its sine, which is nearly the same thing, $=\sin. E \times$ into the $\div$ of the moon's distance into the sun's distance from the earth, $=\frac{60.r}{d}\cdot\sin. E$, where r represents the rad. of the earth, d the distance of sun from earth, but $\frac{r}{d}=8''.47$ the sun's parallax, therefore $S=8'.47.\sin. E$ and $P=\Delta$ versed sine $(E+8'.47.\sin. E)$; now from Napier's rules cos. E=cos. l. cos. (☽—☉); l and ☽, ☉, representing the same as in the preceding notes, in conjunction ☽—☉=0 and E=l; therefore P=Δ vers. sine (l+8′.47. sin. l), ∴ unless the moon is in its node, the illuminated part does not vanish; when E+8′.47. sin. E=90°, P=Δ½, ∴ half the disk is illuminated; in this case E is less than 90°, as stated in the text, when ☽—☉=90, E=90°, and P is greater than Δ; when ☽—☉=180, P=Δ. versed sin. (180°—l—8′47. sin. l); and when l vanishes, *i. e.* at the node, P=2Δ; calling 2Δ′ the apparent disk of the earth as seen from the moon, 180°—E is the exterior angle at the earth, and P′ the illuminated part =Δ′. ver. sin. (180—E)=Δ′. (1+cos. E), but P=Δ. ver. sin. (S+E)=Δ. (1—cos. E), nearly; hence, when E=0, P=0, and P′=2Δ′, and when E=180°, P=2Δ, P′=0.

If the angular motion of the moon was exactly equal to that of the sun, the lines drawn from the earth to the sun and moon would preserve the same relative position, and the moon would invariably present the same aspect, the quantity of the illuminated surface being always the same.

(*z*) It appears, therefore, that the period of the phases is the time required to describe four right angles with an angular motion, equal to the difference between the angular motion of the sun and of the moon, it is consequently greater than the time of tropical revolution.

(*a*) This is the method employed by Aristarchus to determine the distance of the sun from the earth, and is the first attempt on record to determine this distance.

(*c*) Half the angle of this cone is equal to semid. ☉—parallax ☉; therefore, if r be the radius of the earth, s the apparent semidiameter, and p the horizontal parallax of the sun, the height of this shadow reckoned from the earth's centre $=\frac{r}{\sin.(s-P)}$, and the semiangle of the section of the shadow $=P+p-s$; P representing the horizontal parallax of the moon.

(*d*) The ecliptic limits, or the greatest distance from the node at which an eclipse can happen, is determined by computing the moon's distance from the node, when she just touches the earth's shadow; we might by a similar manner compute the limits of a *total* eclipse.

When the angle at the moon is 90, the moon must be dichotomized, and therefore the boundary of the illuminated part is a right line; and conversely when the boundary is a right line, the angle at the moon is a right angle, therefore in this case the sun's distance from earth is to moon's distance from earth, *i. e.* moon's parallax : sun's parallax as 1 : cos. elongation.

The rad. of the penumbra $P+p+s$, therefore we might compute the time of the moon's entering and emerging from the penumbra. As the earth's atmosphere intercepts some of the rays of light coming from the sun, it causes the shadow of the earth to appear somewhat greater than it would be if there was no atmosphere, the parallax of the moon ought, according to Mayer, to be increased its sixtieth part.

The ecliptic limits for the sun may be computed in a manner similar to that for computing the ecliptic limits of the moon, and as they are greater than those of the moon, there are more solar eclipses than lunar in a year, though more lunar eclipses are visible at any given place.

(*e*) The ray of light at its entrance into the lunar atmosphere is inflected towards the perpendicular, and it suffers an equal deflection from the perpendicular at its egress; each of these deviations is equal to the horizontal refraction of the lunar atmosphere, so that the entire inflection of the ray equals very nearly twice the horizontal refraction. Hence the star continues visible some time after the moon has been actually interposed between the star and observer; and it is also, for the same reason, seen some time before it ought to be visible, from which it follows, that the duration of an occultation of a fixed star by the moon is less than if there was no lunar atmosphere; however, as the entire duration is never lessened eight seconds of time, the beginning of the occultation will not be retarded, nor the end of it accelerated by four seconds of time; if the retardation was four seconds of time, the horizontal refraction would be two seconds of space, for the moon moves over $2''$ of space in $4''$ of time; therefore as the densities are proportional to the horizontal refractions, the density of the lunar atmosphere is 1000 times less than the density of the terrestrial atmosphere, which is a density much less than what can produced in the best constructed air pumps. And as without the pressure of the terrestrial atmosphere, all the liquids which at present exist on its surface would be dissipated into vapours, (see Chap. 16, Book 1,): the pressure of the lunar atmosphere being so very inconsiderable, it follows, that if there was any large collection of water on its surface, it would long since have been dissipated. Besides, if there was a quantity of water spread over the lunar surface, whenever the circle of light and darkness passed through it, it would exhibit a regular curve.

(*f*) Bouguer found that if the light of the sun, when elevated 31° above the horizon, and introduced into a darkened chamber, be made to pass through a concave mirror, it would be dilated into a space of 108 lines of diameter, or weakened 11664 times, and in this state it was equivalent to the light of a candle 16 inches distant. The light of the moon when full, and at the same elevation above the horizon, was found to be dilated into a space of eight lines of diameter, or weakened 64 times, which is equivalent to the light of the same candle when it is distant fifty feet. Thus the light of the sun when enfeebled 11664 times, was still 443 times stronger than the light of the moon when rendered weaker only 64 times. Hence the ratio of the one one to the other is about that of 1 to 268000. Other observations made the ratio that of 1 to 300,000, which is very nearly the mean of several observations. A different estimation is given in Smith's Optics.—*See* Young's Analysis, p. 305.

(*g*) The part of the moon in which this light is visible corresponds exactly to the part of the moon which is not illuminated by the sun; which is exactly equal to the part of the earth which would appear to a spectator on the moon illuminated by the sun.

(*h*) If the axis of rotation of the moon was in the plane of the moon's orbit, every part of the moon would be successively presented to the earth, though the moon revolved on her axis in the time of her revolution about the earth; so that the perpendicularity of the axis of rotation to the plane of the orbit is a condition, which must be combined with the equality of the times of rotation and revolution, in order that the same face may be always presented to us. If the axis of rotation was exactly perpendicular to the plane of the moon's orbit, the libration in longitude would be a *maximum* at the point where the equation of the centre was greatest, (see page 303). From apogee to this point parts of the western edge of the moon come into view, and from this point to perigee these parts are gradually restored; the contrary takes place in the other half of the orbit. The libration of a spot towards the centre of the lunar disk, is much more sensible than the libration of a spot near to the border.

It appears from what is stated in the text, that there are four kinds of librations of the moon; three apparent, one real.

(*i*) The axis of rotation remains parallel to itself, making with the plane of the ecliptic an angle of 88½°, and therefore with the plane of its orbit, which is inclined to that of the ecliptic in an angle of 5°, 10′, an angle of 83°

at its greatest latitude. The descending node of the lunar orbit coincides with the ascending node of the moon's equator. The axis of the earth being inclined to the plane of the ecliptic at an angle of 66°, 23′, the earth must exhibit to a spectator at the sun, appearances similar to those which the moon presents to us, *i. e.* at the time of the summer solstice a portion of its disk about the north pole of 23°, 28′ extent, would be visible, which would contract according as the earth approached to the equinox, after which a like extent of its southern disk would be successively developed till the moment of the winter solstice. This spectator would therefore suppose that there existed in the earth a motion of libration.

(*k*) It may be objected to this explanation, that in consequence of the great rarity of the lunar atmosphere, no explosion would be visible; but in answer it is sufficient to observe, that there are several substances which develope during their ignition the oxygen gas, which is required in order that they may burn.

CHAPTER V.

(*l*) l and L denoting the heliocentric longitudes of the planet and earth, λ the geocentric longitude of the planet, we have

$$r.\cos.l-\rho.\cos.\lambda=\cos.L, \quad \text{and} \quad r.\sin.l-\rho.\sin.\lambda=\sin.L,$$

therefore,

$$\tan.\lambda = \frac{r.\sin.l - \sin.L}{r.\cos.l - \cos.L}, \text{ and } \frac{d\lambda}{\cos.^2\lambda} =$$

$$(r.\cos.l - \cos.L)(r.\cos.l\,dl - \cos.L\,dL) + (r.\sin.l - \sin.L)$$
$$(r.\sin.l.\,dl - \sin.L.\,dL) \div (r.\cos.L - \cos.L)^2$$

equal by concinnating to

$$\frac{\{r^2 - r.\cos.(L-l)\}.dl + \{1 - r.\cos.(L-l)\}.dL}{(r.\cos.l - \cos.L)^2};$$

but as the mean motions which are proportional to dL, dl, are inversely as the periodic times, we have $dL : dl :: r^{\frac{3}{2}} : 1$, unity denoting the radius of the earth's orbit, therefore $dL = dl.r^{\frac{3}{2}}$, $\therefore$ if $\frac{1}{r.\cos.l - \cos.L}$ be put equal to $\frac{P}{\cos.\lambda}$, we shall have $d\lambda = P^2.(r^2 + r^{\frac{3}{2}}) - (r + r^{\frac{3}{2}}).\cos.(L-l).\,dl$; in inferior conjunction or opposition $L - l = 0$, therefore $d\lambda = P^2.r.(r + r^{\frac{1}{2}} - 1 - r^{\frac{3}{2}}).\,dl = P^2.r.(r-1)(1 - r^{\frac{1}{2}}).\,dl$, which is always negative, hence the motion of the planet is always retrograde, in superior conjunction $L - l = 180$, therefore $\cos.L - l = -1$, hence λ must be positive, therefore the motion is direct; when $d\lambda = 0$, the planet appears stationary from the earth, and then we have $\cos.L - l = \frac{r + r^{\frac{1}{2}}}{1 + r^{\frac{3}{2}}}$. If m, m' represent the daily motions in longitude, we have $L = mt$, $l = m't$, and $L - l = (m - m')t$, t being the time when the longitude was the same, *i. e.* the time of syzygy, therefore as t in this case $= \frac{L-l}{m-m'}$ the planet will be retrograde while it describes $2m.\frac{L-l}{m-m'}$, and direct while it describes $360° - 2m.\frac{L-l}{m-m'}$, hence it appears, that the greater the difference between m and m', the less the arc of retrogradation. The preceding investigation goes on the supposition that the orbits are circular, which is not the case, therefore it is that the arc of regression, and also the duration, are not always of the same magnitude.

(m) The illuminated portion of a planet varies as the versed sine of the exterior at the planet, *i. e.* as $1+\cos.\phi$, where ϕ is the angle at the planet, when ϕ is a maximum, *i. e.* when $\sin.\phi=\frac{1}{r}$ the planet is most gibbous, which is evidently in quadrature for a superior planet, in superior conjunction and opposition $\phi=0$, therefore the whole disk is illuminated; for an inferior planet, $\phi=180$ in inferior conjunction, hence in this position $1+\cos.\phi=0$, and the disk is invisible.

(m) M and m representing the angular velocities of the earth, t the time between two conjunctions, we have $t.(m-\mathrm{M})=360$, $m=\frac{360}{p}$, and $\mathrm{M}=\frac{360}{\mathrm{P}}$, therefore $\left(\frac{1}{p}\pm\frac{1}{\mathrm{P}}\right).t=1$, and $t=\frac{\mathrm{P}p}{\mathrm{P}\pm p}$.

(n) It is the parallax of Venus which is obtained by this method; however as its ratio to the parallax of the sun is known from having the ratio of the distances, which latter is given from the observed periods of the sun and Venus, we obtain the parallax of the sun; the transit of an inferior planet over the disk of the sun is a phenomenon of exactly the same kind as that of a solar eclipse, and may be calculated in precisely the same way. The parallax of the sun may be also inferred from theory.—*See* Book 4, Chap. 5, Vol. 2.

(o) It was originally proposed to observe the difference between the times of total ingress of Venus, as seen from two different places on the earth; this requires that the difference of longitudes of the two places should be known accurately; besides it supposes that the spectators are either accurately, or very nearly in the plane of the orbit of Venus; to avoid this it was suggested, that by comparing the difference of duration of the transits, as seen from the different places, we might determine the parallax. From an approximate knowledge of the sun's parallax, we can compute the difference of duration at any place, compared with what it would be as seen from the centre of the earth. Hence, comparing the difference of duration at two distant places, at one of which the duration is shortened, and at the other lengthened, we get a double effect of parallax. It is, therefore, a matter of considerable importance to select places where the effects of the increase of the duration, or of its diminution, is greatest, and it is clear that with respect to the first, the duration is most lengthened when the commencement is near sun-set, and the end near to sun-rise; but in order to secure this it is evident, that the place must have a very considerable northern latitude; the duration would be evidently most shortened when the commencement was near sun-rise, and the termination near sun-set; hence, as the duration is only six hours, and as the time of the occurrence of the last transit was in June, it was necessary that the place should be to the south of the equator, where the days were then shorter than the nights; in places where the complement of latitude was less than the sun's declination, the sun would not set, consequently in such places the entire transit is visible, and the sun's elevation being then inconsiderable, the effect of the parallax would be very great; and also as Venus is depressed, the duration is increased.

(p) The law here adverted to is that which connects the periods and distances, namely, that the squares of the periods are as the cubes of the distances.

(q) Calling x the number of revolutions made by the earth, and y the number made by Venus in the interval between two conjunctions, we must have $xP=yP'$, P, P' being the periods of the earth and planet, hence we must have $\frac{x}{y}=\frac{P'}{P}$, and by substituting for P′, P, we find, by means of the principle of continued fractions, that the numbers expressing this ratio are $\frac{8}{13}$, $\frac{235}{382}$, $\frac{717}{1159}$, &c. It does not necessarily follow that a transit will happen at these intervals, for it is likewise requisite that the least distance of the sun and Venus must be less than the sum of their semi-diameters, and as the nodes of Venus's orbit regrade, we cannot be ascertained of this without computation.

(*r*) The rotation of Mercury is not stated in the text; however Schroeter thought that certain periodical inequalities observed near the *horns* of his disk seemed to indicate a revolution in 24^h, 5′, 30″ on an axis, which coincided very nearly with the plane of his orbit. It was by a continued observation of the *horns* of Venus that he ascertained its rotation. The asperities of this planet, and the different situations of the shades which they project from the side opposed to the sun, change the form of the horns in the course of 23^h, 21,′ 29″; this can only be explained by the circumstance of its rotation, and that the horns resume always the same form at the end of a revolution. The compressions of Venus and Mercury ought not, if the time of their rotation be nearly the same as that of the earth, sensibly to differ from that of the earth, however the observed compression of Venus is nearly insensible.

Schroeter observed when the planet was dichotomized, that a bright spot moved very nearly in the line of the horns, hence he inferred, that the motion was very nearly perpendicular to the ecliptic; however, some uncertainty rests on this matter. Hence it appears, that since the mean length of a revolution is nearly the same for Mercury, Venus, and the Earth, there must be a much greater variation in the length of the days, and also in the seasons for Venus and

Mercury, as the inclination of their equator to their ecliptic, is considerably greater; indeed their torrid zone must embrace very nearly 150 or 180°; in fact, as the sun ranges to within 15° of one pole, the cold and darkness experienced at the other must be very great. It was to a mountain, situated near to the southern horn of Venus, that Schroeter directed his observations; strictly speaking, the line of the horns should be always a diameter, and those of a crescent should be very pointed; however, Schroeter remarked, that this was not always the case with respect to Venus, the horn of the northern extremity was always pointed, but the southern horn appeared sometimes obtuse, or blunted, which indicates the existence of a mountain, which covers a part with its shade.

To find the position of a planet when brightest, let k, r, and ρ denote the distances of the sun and earth, the sun and planet, the earth and planet, and χ the angle at the planet, the quantity of light received at the earth will vary as $\frac{1 + \cos.\chi}{\rho^2}$, $\cos.\chi = \frac{\rho^2 + r^2 - k^2}{2r\rho}$, and $1 + \cos.\chi = \frac{(r+\rho)^2 - k^2}{2\rho r}$, consequently the quantity of light will vary as $\frac{(r+\rho)^2 - k^2}{\rho^3}$, therefore differentiating this we obtain $0 = \rho^2 + 4\rho r - (3k^2 - r^2)$; and $\rho = -2r \pm \sqrt{3k^2 + r^2}$, hence we obtain the value of $\cos.\chi$.

CHAPTER VI.

(s) The motions of Mars are subject to more variations than those of any other planet, which circumstance induced Kepler to direct his observations more particularly to this planet. The position when stationary, and also the duration of his direct and retrograde motion, is computed in the same manner as for an inferior planet. The cause of the differences which are observed in the quantity and duration of the retrogrations, arises from the ellipticity of the orbit.

(*t*) The brilliancy of a fixed star when approaching this planet was observed to become sensibly faint, hence it was inferred, that Mars was environed by a dense atmosphere, which was the cause of this faintness. Besides, from a continued observation of the spots, particularly two, which are near to the poles, there was observed a periodical increase and diminution, according as they are exposed to the action of the sun's rays in a more or less oblique manner; from this circumstance it has been conjectured, that they are like the collections about our polar regions of the earth.

(*u*) The inclination being very nearly the same as the inclination of the earth's axis to the ecliptic, the variations of seasons must be also nearly the same.

CHAPTER VII.

(*v*) The duration of Jupiter's rotation is the shortest, and his magnitude and mass are the greatest of any of the planets. This great rapidity of rotation may compensate for the greater weight which bodies experience at the surface of this planet, (*see* Vol. 2, Chap. 8, page 143); in fact, a point on the surface of Jupiter moves twenty-six times faster than a point on the earth's surface.

In consequence of the inclination being so inconsiderable, it follows, that there is no great variety in the seasons.

(*w*) If the commencement and termination of an eclipse be accurately observed, then the middle of the eclipse is found, which is nearly the time when Jupiter is in opposition with respect to the satellite; let the time of another opposition, separated by a considerable interval from the first, be found in the same manner, calling τ this interval, and n the number of oppositions which have occurred in τ, we have T the time of a synodic revolution $=\frac{\tau}{n}$, hence, if P′ be the periodic time of Jupiter, we shall have P, the period of the satellite, $=\frac{P'T}{P'+T}$. *See* Notes to Chap. 9, Book 2.

It has been also inferred, from the circumstance of the greatest elongations of the satellites, when measured with a micrometer at their mean distances from the earth, being always the same, that the orbits are Q. P. circular, and it is in this manner that the distances are found in terms of the radius of Jupiter's equator; however, as in a comparison of a great number of observations, we must modify a little the laws of circular motion for the orbit of the third satellite; it follows, that this orbit is elliptical.—*See* Chap. 10, Book 2.

Calling Jupiter's geocentric longitude λ, l, the longitude of the satellite, as seen from Jupiter, and θ the longitude of the sun, the angle at the earth is equal to $\lambda - \odot$, that at Jupiter, $= l - \lambda$, and $r : 1 :: \sin.(\lambda - \odot) : \sin.(l - \lambda)$.

(*y*) From this circumstance of their alternately surpassing each other in splendour, it is probable that certain parts of their surface reflect more light than others, and then the epochs of the maximum or minimum of illumination ought to happen when the very same parts of the satellites are turned towards us; from a comparison of these returns with the positions of the satellites relatively to Jupiter he ascertained that they always present the same face to this planet, hence he inferred, that they revolve on their axes in the time of their revolution about Jupiter.

Naming t, T the durations of the longest and shortest eclipse of the same satellite, and r the radius of Jupiter's equator, we have $T : t :: r : \frac{r.t}{T} = c$, half the chord of the arc described in the shortest eclipse, consequently d its distance from the centre of Jupiter $= r.\sqrt{1 - \frac{t^2}{T^2}}$; but $r : 360 :: T : L$, (a synodic revolution of the satellite,) $\therefore r = \frac{360°.T}{L}$, hence $d = \frac{T}{L} \cdot 360°.\sqrt{1 - \frac{t^2}{T^2}}$, and calling n the longitude of the satellite, and l that of Jupiter, we have in the right angled spherical triangle, of which the hypothenuse is $n-l$, and d a side about the right angle $\sin. d = \left(\sin. \frac{360° \sqrt{T^2 - t^2}}{L}\right) = \sin. (n-l). \sin. N$, (N being the inclination which consequently can be found). n is found by observing the position of Jupiter when the duration of the eclipse is the greatest possible, for the heliocentric longitude of Jupiter and of his node are in this case precisely the same.

CHAPTER VIII.

(z) The circular ring must always appear as an ellipse, as the eye of the spectator invariably looks at it obliquely, being never raised 90° above the plane of the ring, therefore the major axis of the ellipse is to the minor as radius to the sine of the angle ϕ, at which the line drawn from the earth to the centre of the ring, is inclined to its plane, consequently if λ represent the geocentric longitude of Saturn, the longitude of the earth as seen from Saturn $=180+\lambda$, and if the longitude of the ring's node $=n$, B being the geocentric latitude of Saturn, and consequently $-$B the latitude of the earth as seen from the planet, we can, from knowing $180+\lambda-n$, B, and also v the inclination of the plane of the ring to the ecliptic, compute ϕ, and thus obtain the ratio of the axes of the ellipse $=\sin.\phi=\sin.\nu.\cos.$ B. $\sin.(n-\lambda)+\sin.$ B. $\cos.v$, if $\sin.\phi=0$, i. e. if the earth is in the plane of the ring, we shall have $\sin.(n-\lambda)=\tan.$ B. $\cot.\nu$, in this case the thickness of the ring is turned towards us, which being inconsiderable is therefore invisible; this occurs twice during each revolution of Saturn, *i. e.* every fifteen years; if the plane of the ring passes through the sun it will disappear, because its thickness is then only illuminated; naming ϕ' the elevation of the sun above the plane of the ring, H h the heliocentric longitude and latitude of Saturn, we have $\sin.v'=\sin.\phi'.\cos.h.\sin.(\text{H}-\text{N})-\cos.v.$ $\sin.h$, and therefore $\sin(\text{H}-\text{N})=\cot.v.\tan.h$, when $\phi'=0$, *i. e.* when the ring disappears. When ϕ, ϕ' have the same sign, the earth will see the illuminated part, and the ring will be visible; when they have contrary signs the ring will be invisible, for the ring will turn one of its faces towards the earth, and the other towards the sun; but as λ and H never differ by 5°, which is described by Saturn in five months nearly, this difference of sign cannot last longer; it is in this interval that the phenomena of the appearances and disappearances occur, Saturn being near to his nodes, similar phenomena occur at the following node; if the ring disappears a short time before Saturn becomes stationary, the earth will meet it soon again, since Saturn becomes retrograde after the second occurrence, the ring will again become visible as $\sin.\phi$ and $\sin.\phi'$ then have the same sign, shortly after $\sin.\phi'$ vanishes, the plane of the ring passing through the sun, and as afterwards $\sin.\phi'$ changes its sign, when the ring continues to be invisible until a short time after the planet becomes direct, when sin. ϕ vanishes, and consequently the plane of the ring passes through the earth, afterwards as sin. ϕ changes its sign, it will be the same as sin. ϕ', and consequently the ring will be visible for fifteen years; if when the plane of the ring passes through the sun, the angular distance of the earth from the ascending node of the ring, as seen from the sun, is greater than 90, and less than 180, there will be only one disappearance, which commences when the sun passes through the plane of the ring, and ends when the earth meets it, consequently it will last less than three months; if the preceding angle is > 180 and < 270, the earth meets the plane shortly before this plane passes through the sun, after this, sin. ϕ, sin. ϕ' will have the same sign, consequently the ring will be visible, consequently in this case as well as the preceding, the invisibility lasts three months; if this angle be > 270 and < 360, there will be two disappearances, namely, when the earth meets the plane of the ring a little before it passes through the sun, after this the earth again meets the plane of the ring, consequently there will be a second disappearance. If this angle be between 0 and 90, there are two disappearauces also, namely, when the earth meets the ring before opposition; secondly, when the ring passes through the sun after opposition, after the second reappearance, the ring bocomes visible for fifteen years. The most favourable circumstances for seeing the ring are when the plane passes through the sun and earth at the same time, the earth being in conjunction; then the earth is always on the illuminated side of the ring, which only ceases to be visible in consequence of the plane passing through the sun, if the plane passes through the earth and sun at the same time, the planet being in opposition, the circumstances for seeing the ring are the most unfavourable, in this case the ring is invisible nine months, four months before the passage of the plane through the sun, and five months after. If sin. ϕ be positive, we see the northern face of the ring; the semi-ellipse, which will be visible, will be below the centre of Saturn, and the other half will be behind the planet; if sin. ϕ be negative we see the half above the centre. The inclination of the ring to the ecliptic, or the angle v = the angle at the earth + the angle which a visual ray from the earth makes with the border of the ring, this last angle = 30°, the first angle = the geocentric latitude of Saturn + the angle which the minor semi-axis subtends.

Sin. $\phi=\frac{b}{a}$, therefore according as the earth ascends above the plane of the ring, the ellipse increases, when $b=a$ sin. $\phi=\frac{1}{2}$ diameter of Saturn, its extremities coincides with those of its disk, in this case evidently sin. $\phi=\frac{3}{7}$, if $n—\lambda=90$, $\frac{b}{a}=\sin.\phi=\sin.v.\cos.g-\cos.v.\sin.g=\sin.(v-g)$, therefore $\phi=v-g$, and since $n=90+\lambda$, we have the place of the ascending node. As the phenomena of the disappearances recur after a complete revolution of Saturn, it follows, that these two positions of the ring always correspond to the same points of the orbit of Saturn, and consequently the plane remains always parallel to itself, therefore its inclination to the ecliptic is invariable, or if we substitute for tan. h its value tan. I′. sin. (H—N′), I′ and N′ being the inclination and longitude of the node of the orbit of Saturn, we have

$$\sin.(H-N)=\cot.v.\tan.v'.\sin.(H-N'),$$

therefore,

$$\tan.v'.\cot.v=\frac{\sin.(H-N)}{\sin.(H-N')}=\frac{\sin.H.\cos.N-\cos.H.\sin.N}{\sin.H.\cos.N'-\cos.H.\sin.N'}$$

$$=\frac{\tan.H.\cos.N-\sin.N}{\tan.H.\cos.N'-\sin.N'},$$

therefore,

$$\tan. H = \frac{\tan. v'. \cot. v. \sin. N' - \sin. N}{\tan. v'. \cot. v. \cos. N' - \cos. N};$$

hence we find H, which is very nearly constant. When the plane of the ring passes through the sun, the heliocentric longitude of Saturn on the orbit $= N$, ∴ the place of the nodes of the ring on the orbit is determined, or *vice versa*, which is found to be the same always; let $N_{,}$ be this longitude reduced to the ecliptic, the angle at the sun between rad. of earth and curtate distance of Saturn $= N_{,} + 180 - \odot$, the angle at the earth subtended by curtate distance $= \odot - z$, rad. of earth $= a$, we have curtate distance $= \frac{a. \sin. (\odot - z)}{\sin. (z - N_{,})}$, and radius vector of Saturn $= \frac{\text{custate distance}}{\cos. \phi}$. ($z =$ the geocentric longitude of Saturn).

The apparent headth of the ring is equal to the distance of its interior border from the surface of Saturn, as is indeed evident from what has been already observed, and it revolves in a time equal to the periodic time of a satellite whose distance from Saturn would be the same as that of the ring.

CHAPTER IX.

(*a*) In determining the elements of the planetary orbits, a great number of observations is supposed to be made about the time of opposition or conjunction, and also the periodic times of the planets are supposed to be known; but as this last element is most accurately determined by means of a great number of complete revolutions, and as the motion of this planet is so slow as to preclude the possibility of observing more than one opposition in eighty years, a considerable time must elapse before the elements of Uranus can be known with the same accuracy as those of the other planets. However, as will be shewn in the third Chapter of the second Book, the very extreme slowness of the observations enables us to make a tolerably accurate approximation to a knowledge of the elements.

In the consequence of the *q. p.* perpendicularity of the planes of the orbits of the satellites of Uranus to that of their primary, they must experience considerable disturbance from the action of the sun; indeed the investigation of the sun's action would be a new case in the problem of the three bodies, for in general the inclinations are assumed to be inconsiderable.—*See* Vol. ii. p. 51.

CHAPTER X.

(*b*) These planets are so small that they belong to that class of stars which are termed telescopic, the volume of all the four taken together does not surpass the magnitude of the moon, therefore, though nature has elevated them above the rank of satellites, as far as their magnitude is concerned, they are below these bodies. These circumstances of their extreme smallness, and of their being at the same distance very nearly from the sun, have induced philosophers to think that they are the fragments of one planet divided into parts; indeed an explosion with a velocity twenty times greater than that of a cannon ball, would be sufficient to make these detached fragments describe orbits similar to those described by these planets; such an hypothesis explains why the excentricities and inclinations of these planets are so considerable, and also why they are moved in such various directions, and with such different velocities.—*See* Vol. ii. p. 250.

The elements of these planets cannot be known with the same precision as those of the other planets, for as not more than five revolutions of them have been observed, their periodic time, a most important element, cannot be determined with great accuracy. The best method for determining the elements of these stars is that given by M. Gauss in his *Theoria motus corporum cœlestium;* but when their proximity to Jupiter, the perturbations which result from their mutual attractions, and their great eccentricities and inclinations, are taken into account we cannot expect to have as yet a very accurate knowledge of these elements.

CHAPTER XI.

An epicycle is a curve produced by the combination of two circular motions. The circles described by the centres were called deferents. The epicycle of a superior planet was supposed to be described in the time between two conjunctions or oppositions. The epicycle of an inferior was described in the time between two inferior conjunctions. The deferent of the superior planets were supposed to be described in the time of a planet's revolution about the sun; those of inferior planets in the time of the earth's revolution. In a position of an inferior planet on the Ptolemaic system, if lines be drawn from the planet to the earth and centre of the deferent, the angle at this centre will be equal to the angle which an inferior planet has gained on the earth since last inferior conjunction, hence if the rad. of the deferent is to the rad. of the epicycle, as the distance of the sun from earth, to the distance of the sun from planet, the angle at the earth is equal to the angle of elongation in the true system, and if the rad. of deferent be assumed equal to the distance of the sun from the earth, (which we are permitted to do on Ptolomy's system), we have then the inferior planets moving about the sun, which is itself carried in a year about the earth; in like manner, if lines be drawn from the place of superior planet to earth, and to centre of deferent, the angle at the centre will correspond to the angle gained by the earth on the planet, and if those distances are proportional to the distances of the earth and planet from the sun, the angle made by lines drawn from earth to sun and planet, will be equal to the elongation, for the rad. of the epicycle may be shewn to be parallel to the moveable rad. of the sun; it is evident also, that if the rad. of the deferent be equal to the distance of the planet from sun, the rad. of the epicycle is equal the distance of the sun from the earth.

CHAPTER XIII.

(*a*) Considering the great perfection of Astronomical instruments, and the precision with which observations have been made, it is supposed that if the parallax was equal to 3″ of the decimal division of the circle, or 1″ of the sexagesimal, it might be observed; if the parallax was equal to 9′ 1″, the diameter of the earth's orbit would hardly subtend an angle equal to the thickness of a spider's thread at the star. Various methods have been suggested for determining the distances of the fixed stars, of which the most successful appears to be that which was first suggested by Galileo, and subsequently improved on by Herschell, of which the principle consists in determining the angle, or variations in the angle, which two stars very near to each other appear to subtend at opposite seasons of the year.—*See* Philosophical Transactions for the year 1794.

Another method was from the consideration of the quantity of light in the stars compared with the light of the sun; in this way M. Mitchell concluded, that the parallax of a star of the second magnitude is not more than the $\frac{1}{5}$th of a second, and of a star of the fifth or sixth magnitude not more than $\frac{1}{20}$ or $\frac{1}{30}$th of a second. The attempts to discover the parallax of the stars by direct observations, have not been attended with any success previously to the time of Doctor Brinkley, Professor in Trinity College, Dublin. His observations which were made with the greatest care, seem to indicate the existence of parallax in a Lyræ amounting to 2″, 52. *See* Philosophical Transactions for the years 1812 and 1813.

(*b*) On the hypothesis, that Sirius was of the same magnitude as the sun, Huyghens found by diminishing the aperture of a telescope, so that the sun when seen through it might appear of the same apparent magnitude as *Sirius*, that the diameter of the sun was diminished in the ratio of 1 : 27664, hence *Sirius* is 27664 times more distant than the sun.

The smallest apparent diameter of an opaque body which is visible is about 40″, but if the body be luminous *per se*, the limit of visibility will be so much less as the light of the body is stronger, and as the stars with a diameter less than 1″ have a splendour so great that some of them are visible immediately after sun-set, there cannot be any doubt but that they have a light of their own like the sun; their extreme smallness is proved from their scintillating, which shews that the least molecule floating in the air is sufficient to intercept their light.

When a fixed star is eclipsed by the moon it ought to disappear by degrees, if it had a sensible apparent diameter, conformably to the moon's mean motion, which is such that it describes its apparent diameter, which is about 30′ in an hour, consequently in two seconds of time it ought to describe one second of space.

(*c*) A third explanation of these phenomena has been suggested; this supposes that the figures of these stars is very compressed, which makes them to appear much less flattened in some aspects than in others.

(*d*) The milky way environs the sphere very nearly in the plane of a great circle, which by half of its breadth intersects the equator at the 100th and 277th degree, its inclination to the equator is equal to 60°, the breadth is from 9 to 18 degrees; it is narrowest near the poles of the equator, between the constellations Cassiopea and Perseus, and its greatest breadth is in the plane of the equator. The milky way is divided into several departments, by a space void of stars, in the middle of the breadth, chiefly from 254° of right ascension, and 40 of south declination to 310 of right ascension, and 45° of northern declination. Hershel could distinguish the stars of which this milky way was composed, which were so near to each other, that in telescopes of inferior magnifying power their light was confounded; according as the direction of the telescope deviated from the milky way, the number of these stars diminished. Having counted the stars in different parts of this way, he found that on a medium estimate, a segment 15° long, and two degrees wide, contained 50,000 stars, of sufficient magnitude to be distinguished through his powerful telescope; ∴ on the supposition that the breadth of the milky way is 14°; it follows, that it contains more than eight millions of stars, without reckoning those, which even with this great telescope cannot be distinguished; with respect to the arrangement and nature of the stars which constitute the milky way, some observations will be suggested in the Notes to Chap. VI. of the 2nd Volume.

(*e*) The declination of an object is best obtained by observing its distance when on the meridian from the horizon or zenith, for this distance added or subtracted from the distance of the zenith from the pole, gives the distance of the object from the pole, and consequently the declination; if the object has apparent magnitude, the altitudes or zenith distances of its upper and lower limbs should be observed, and then half their sum should be taken as the altitude of the centre; this method requires that the exact zenith distance, corrected by refraction and parallax should be known, which is best obtained in the manner indicated in the notes to the first chapter.—*See* also Notes, to Chap. XIV. If the star does not exist in the meridian, then in order to determine the declination, it is necessary to know the zenith distance of the star, that of the pole and either the azimuth or hour angle from noon. Indeed, of the five preceding quantites any three being given, the other two may be found by the solution of a spherical triangle. This general problem contains twenty different cases, of which the most useful are given in the Treatises of Astronomy. However, there is an obvious advantage in determining the declination by means of an observation made in the meridian, for in this case parallax and refraction only affect the declination, but do not at all alter the right ascension.

With respect to the right ascension, its determination is more difficult than the declination, as the first point of Aries, from which it is reckoned, is not fixed in the heavens. The difference of the right ascension of two stars is obtained by observing the time intervening between their passages over the meridian; this converted into time at the rate of 15° for an hour, gives the difference. Hence, as this difference is easily observed, if we had the right ascension of some one star, that of all others might be determined; the method which Flamstead proposed was as follows:

He noted when the sun had equal declinations, some time after the vernal and before the autumnal equinox; in these positions the distances of the sun from the respective equinoxes must be the same, call this distance E, and let D, $D'+p$, represent the differences of right ascensions of the sun and some star in these two positions, then we have $D+D'+p+2E=180$, hence we obtain E, and consequently the right ascension of the star; p is the correction to be made to the right ascension, in consequence of precession and displacement of ecliptic, which will be afterwards noted.

It is easy to compute the angular distance of two stars, of which we know the right ascensions and declinations, for if d, d', represent the polar distances of the stars, and Δ the angle made by d, d' at the poles, which is measured by their difference of right ascension, and D the arc of a great circle which measures their angular distance, we have by the formulæ of spherical trigonometry, $\cos. D=\sin. d. \sin. d'. \cos. \Delta+\cos. d. \cos. d'$. Let λ represent the longitude, β the latitude, ρ the right ascension, δ the declination of a star; p its angle of position; π the arc of a great circle intercepted between the star and equinoxial point; ϕ the angle contained between this arc and equator, and ε the obliquity of ecliptic; then $\cos. \phi=\tan. \rho. \cot^{nt}. \pi$; $\cos. (\phi-\varepsilon)=\text{tang}. \lambda \cot^{nt}. \pi$; $\sin. \delta=\sin. \phi. \sin \pi$; $\sin. \beta=\sin. (\phi-\varepsilon). \sin. \pi$. Hence we obtain $\tan. \lambda=\frac{\cos. (\phi-\varepsilon). \tan. \rho}{\cos. \phi}$ and sin. $\beta=$ sin. $\frac{(\phi-\varepsilon)}{\text{sin. }\phi}$ sin. δ; hence we can obtain λ and β when ϕ is known.

The reverse formulæ for finding ρ and δ from knowing λ and β, may be obtained by merely changing β into δ, and λ into ρ, and by making ε negative; θ representing the arc of the circle of declination passing through the star, intercepted between equator and ecliptic; we have sin $\theta=$ tan. ρ. cot$^{\text{nt}}$. ν, and cos. $(\delta-\theta)=$ cot. p. cot ν, $\therefore$ cot$^{\text{nt}}$. $p=\frac{\text{cos. }(\delta-\theta)\text{ tan. }\rho}{\text{sin. }\theta}$ (note ν = the angle at which circle of declination passing through the star, is inclined to ecliptic) cos. δ. cos. ρ. = cos. λ. cos. β; tan. ρ = $\frac{\text{cos. }(\phi+\varepsilon)\text{. tan }\lambda}{\text{cos. }\phi\text{.}}$ and sin. $\delta=\frac{\text{sin. }\beta\text{. sin. }(\phi+\varepsilon)}{\text{sin. }\phi}$; tan. ρ, may become negative in several cases, in the first quadrant, if sin. λ is less than tan. ε. tan. β, for by substituting for ϕ, the preceding value of tan. $\rho=\frac{\text{cos. }\varepsilon\text{. sin. }\lambda}{\text{cos. }\lambda}-\frac{\text{sin. }\varepsilon\text{. tan. }\beta}{\text{cos. }\lambda}$ this may happen when λ is small and β great, *i. e.* if the star is in the circle of latitude near to the pole of the ecliptic; in the 2nd quadrant tan. ρ is negative, unless that sin. λ is less than tan. ε. tan. β; in the 3rd quadrant tan. ρ is positive, and in the 4th quadrant — $\therefore$, except when sin. λ is less than tan. ε. tan. β; ρ is always in the same quadrant as λ; ρ is negative or in the 4th quarant if $\lambda=0$; unless $\beta=$ either 0, or is —; in the first case $\rho=0$, *i. e.* the star is in the equinoxial point; in the second case it is in the first quadrant; if $\lambda=90$, and tan. $\beta >$ than sin λ. cot. $\varepsilon=$ cot$^{\text{nt}}$. ε, *i. e.* if β is greater than 66° 32′, in this case tan. $\rho=-\infty$. and the star is in the solstitial colure between the two poles.

(*e*) As the distance between the pole of the equator and the pole of the ecliptic = the obliquity of the ecliptic which is very nearly constant, it follows that the axis of the equator describes, in consequence of this precession, a cone about the pole of the ecliptic. In order to obtain the variation in right ascension and declination; supposing β and ε constant, we shall have by differentiating $\frac{d.\,\delta}{d.\,\lambda}=\frac{\sin.\,\varepsilon.\,\cos.\,\beta.\,\cos.\,\lambda}{\cos.\,\delta.}$ and $\frac{d\,\rho}{d\,\lambda}=\cos.^2\rho\left(\frac{\cos.\,\varepsilon-\sin.\,\varepsilon.\,\tan.\,\beta.\,\sin.\,\lambda}{\cos.^2\lambda}\right)$ and as we have from comparing the two values of p obtained by substituting,

$$\cos.\,\rho\left(\frac{\cos.\,\varepsilon-\sin.\,\varepsilon.\,\tan.\,\beta.\,\sin.\,\lambda}{\cos.\,\lambda}\right)$$
$$=\frac{\cos.\,\varepsilon\cos.\,\delta+\sin.\,\varepsilon\,\sin.\,\delta.\,\sin.\,\rho}{\cos.\,\beta}$$

by substituting this value and from the equation $\cos.\,\rho.\,\cos.\,\lambda=\cos.\,\lambda.\,\cos.\,\beta$, these will assume the form. $d.\,\delta=d\lambda.\,\sin\,\varepsilon.\,\cos\,\rho$; $d\rho=d\lambda\,(\cos.\,\varepsilon+\sin.\,\varepsilon.\,\tan.\,\delta.\,\sin.\,\rho.)$ Note, as the equinoxes regrade uniformly, $d\,\lambda$ is constant, and it appears that the right ascension cannot diminish except when the star is in the southern hemisphere, or when it is in the 3rd or 4th quadrants, $\tan.\,\delta.\,\sin.\,\rho.$ being greater than $\cot.\,\varepsilon$; and as near to the pole $\tan.\,\delta$ approaches to ∞, the variation of right ascension may become then indefinitely great.

The preceding formulæ are sufficiently exact, when the effects of precession are computed for an interval which is near to the epoch, for which we have determined the arguments ε, δ, ρ. But as ε changes continually within certain limits as shall be observed in Chapter XIII, Vol ii, and as $d\lambda$ likewise, is not always the same, the preceding expressions are only correct for a short interval of time.

Bradley, in his endeavour to ascertain whether the parallax of the fixed stars was of a sensible quantity, observed that for the space of nine years the declination of the stars increased, and that it diminished by the same quantity the nine following years; so that all was re-established after eighteen years. He likewise observed, that the greatest difference of declination was 18″, and that the latitude was not affected; hence he inferred, that the pole of the equator approached the pole of the ecliptic for the first nine years, and that it receded from it by the same quantity the following nine years. He observed, likewise, that this motion was connected with an irregularity of the precession of the equinoxes, which obeyed precisely the same period; hence it follows that the motion of the poles of the equator, does not take place in the solstitial colure, or in other words, that the poles neither describe right lines nor the arc of a great circle of the sphere, but a curve or small circle intersecting the solstitial colure; ∵ as the true motion of the pole takes place in the periphery of an ellipse of which the centre retrogrades on the periphery of the circle described by the mean place of the pole, its locus will be a species of epicycle. In the superior part the direction of the motion of the pole is the same as that of the epicycle, ∵ the actual motion being quicker than the mean motion, the true pole precedes the mean; it is the contrary in the lower part of the ellipse, and as the mean motion is considerably greater than the motion in the ellipse, it predominates over it; ∵ the motion in the epicycle is still retrograde. From a comparison of observations of the nutation with the nodes of the moon, it appears that the right ascension of the true pole, reckoned from the mean pole precedes by 90°, the longitude of the ascending node of the moon; *i. e.* $\rho = 90° + \Omega$. then $d\varepsilon$ the variation of obliquity $= q. p$, the cosine of ☊ to a radius $= 9''$, 65, *i. e.* $d\varepsilon = 9''$, 65. cos. ☊. To determine the variation in longitude, it is to be remarked that the angle formed by lines drawn to the true and mean poles of the equator, from the pole of the ecliptic = distance of true pole from the axis major, divided by sine of the distance between poles of the equator and ecliptic $= \frac{7'', 17, \sin. \Omega}{\sin. \varepsilon}$, see notes to Chap. XIII, Vol, 2, in order to determine the effects of nutation on the right ascension and declination, naming δ' and ρ' the right ascensions and declination when the longitude becomes $\lambda + d\lambda$, and the obliquity becomes $\varepsilon + d\varepsilon$, then by formula given in page 341; we have,

$$\sin.\ \delta' = \sin.\ (\varepsilon + d\varepsilon).\ \cos.\ \beta \sin.\ (\lambda + d\lambda) + \cos.\ (\varepsilon + d\varepsilon) \sin.\beta$$

$$\tan.\ \rho' = -\tan.\ \beta.\frac{\sin.\ (\varepsilon + d\,\varepsilon) + \sin.\ (\lambda + d\,\lambda).\ \cos.\ (\varepsilon + d\varepsilon),}{\cos.\ (\lambda + d\,\lambda)}$$

then by omitting all terms after the first we obtain, $\delta' = d + d\,\lambda.\ \sin.\ \varepsilon.\ \cos.\ \rho + d\varepsilon\,.\ \sin.\ \rho$; $\rho' = \rho + d\,\lambda.\ (\cos.\ \varepsilon + \sin.\ \varepsilon.\ \sin.\ \rho.\ \tan.\ \delta) - d\,\varepsilon.\ \cos.\ \rho.\ \tan.\ \delta$; substituting for $d\,\lambda$, $d\,\varepsilon$, their values previously found, and making $9''$, $65 = h$; $18''$, 0 3. $\sin.\ \varepsilon = g$, we obtain the nutation in right ascension or the value of $\rho' - \rho'$, which is the same thing, $= -g.\ \sin.\ \Omega\ \cot^{nt}.\ \varepsilon - \tan.\ \delta.\ (h.\ \cos.\ \Omega.\ \cos.\rho + g\ \sin.\ \Omega\ \sin.\ \rho)$ the first term being independent of the stars place, is the same for them all; assuming $h.\ \tan.\ \delta.\cos.\ \rho = g.\ (\cot.\ \varepsilon + \sin\ \rho.\ \tan.\ \delta).\ \tan.\ B'$, then the nutation in right ascension $= -g.\ (\cot.\ \varepsilon + \sin.\rho.\ \tan.\ \delta.\ \sin.\ (B' + \Omega,)$ hence it is easy to perceive that for the same star the nutation in right ascension is a max^{m}, when $\Omega + B' = 90$; by making the substitutions already indicated, the nutation in declination or north polar distance, which is the same thing, becomes $-g.\ \cos.\ \rho.\ (\sin.\ \Omega - \frac{h}{g}\,.\ \tan.\ \rho.\ \cos.\ \Omega)$, let $-\frac{h}{g}\,.\ \tan.\ \rho = \tan.\ B$, and then $-\frac{g.\ \cos.\ \rho}{\cos.\ B}\,.\ \sin.\ (B + \Omega)$ = the nutation in north polar distance; it is easy to perceive that this becomes a max^{m}, when $\Omega + B = 90^{\circ}$.

Beside the nutation just examined, the pole of the equator is subject to a similar inequality arising from the disturbing action of the sun, it is much feebler than that of the moon, however, it is not altogether insensible, and is always introduced in the tables. In consequence of this action the true pole describes a circle about the mean pole *according* to the order of signs; its period is half a year, and the true pole is always 90 before the sun; indeed, if extreme accuracy was required, it is theoretically true that in the course of half a month, the pole is disturbed from the inequality in the moon's action; however, this last is altogether insensible; now as the true pole would, in consequence of *each* of these actions, if they obtained separately, combined with the motion of the pole arising from precession, describe an epicycloid, the curve actually described, will be that which results from the combined action of all these motions; however, as they are separately extremely small, if we estimate the effect of each by itself, and then take the sum, the total effect may be considered very nearly as = to the sum of all the partial results.

With respect to the aberration of light, which is the third correction to be applied in order to obtain the true place of a star, *see* Notes to Chapter II. Book II.

Besides the three *apparent* motions of the fixed stars, which are adverted to in this chapter, namely the precession, the nutation, and the abberration, there is a fourth, which though obscurely indicated by observation, is completely established by theory, namely the diminution of the obliquity of the ecliptic. *See* Notes to Chapter XIII. Volume 2nd.

CHAPTER XIII.

(f) In order to a clearer understanding of the articles treated of in the text, it will be necessary to establish a few principles relating to the radius of curvature and expressions for a degree of the meridian, &c., for this

purpose let a, b denote the semi-axes of an ellipse, p the principal semi-parameter $= \frac{b^2}{a}$, n the normal and ρ the radius of meridional curvature, λ the latitude, $x\ y$ the co-ordinates of any point, and s its subnormal; then as $n^2 = y^2 + s^2$, and as $x = \frac{a^2}{b^2} \cdot s$, and as $y^2 = \frac{b^2}{a^2} \cdot (a^2 - x^2)$, we shall obtain by observing that $n \sin. \lambda = y$, $n. \cos \lambda = s$, $n.^2 \sin.^2 \lambda = \frac{b^2}{a^2} \cdot \left(a^2 - \frac{a^4}{b^4} \cdot n^2 \cos.^2 \lambda\right)$ ∴ by concinating, n^2.

$$(b^2. \sin.^2 \lambda + a^2. \cos.^2 \lambda) = b^4, \text{ and}$$

$$n = \frac{b^2}{\sqrt{a^2. \cos.^2 \lambda + b^2. \sin.^2 \lambda}}; \text{ and as } \rho = \frac{n^3}{p^2} = \frac{n^3. a^2}{b^4}$$

by substituting we obtain $\rho = \frac{a^2. b.^2}{(a^2. \cos.^2 \lambda + b^2. \sin.^2 \lambda)^{\frac{3}{2}}}$

hence we obtain $D = \frac{a^2. b^2.}{m\,(a^2. \cos.^2 \lambda + b^2. \sin.^2 \lambda)^{\frac{3}{2}}}$ (m expressing the number of degrees in an arc = to the radius.) Making $b = a - c$, and neglecting the square and higher powers of c; we obtain $m\ D = (a^4 - 2\,a^3.\ c). (a^2 - 2\ a\ c. \sin.^2 \lambda)^{-\frac{3}{2}} = a\ \left(1 - \frac{2c}{a} + \frac{3c}{a}.\ \sin.^2 \lambda\right) = a\left(1 - \frac{c}{2\,a} - \frac{3\,c}{2\,a}. \cos.\ 2\,\lambda\right)$; ∵ at the equator $m\ D = a - 2\,c$, and at the pole $m\,D = a + c$, at the parallel 45, $m\ D$ is an arithmetic mean between $m\ D$ at the equator, and $m\,D$ at the poles, for at 45° $m\ D = a - \frac{c}{2}$; If D' be a degree to the latitude

λ', we have $m\,D' = a - \frac{c}{2} - \frac{3\ c}{2}. \cos.\ 2\,\lambda'$; hence

$$c = \frac{2\,m\,(D' - D)}{3. (\cos. 2\,\lambda - \cos. 2\,\lambda')} \text{ and } \frac{c}{a} =$$

$\frac{2\,(D' - D)}{3\ D. \cos.\ 2\ \lambda - \cos.\ 2\ \lambda')}$; these expressions may be reduced respectively into $c = \frac{2\ m\ (D' - D)}{3. \sin.\ (\lambda + \lambda') \sin.\ (\lambda - \lambda)}$; $\frac{c}{a}$

$=\frac{2\,(D'-D)}{3\,D.\ \sin.\ (\lambda+\lambda)\ \sin.\ (\lambda'-\lambda)}$, ∵ at the equator c $=\frac{m\,(D'-D)}{3.\ \sin.\ ^2\lambda}$ hence the increment of the degree at any latitude λ', above the degree at the equator is as $\sin\ ^2\lambda'$; likewise as D′—D ∽ sin. $(\lambda'+\lambda)$ sin. $(\lambda'-\lambda)$ if D′ D are two contiguous degrees, so that, $\lambda'=\lambda+1°$; then D′—D$=\frac{3\,c}{m}$ sin. $(2\,\lambda+1°)$. sin. $1°$; ∵ as the difference of contiguous degrees is ∷ l to sin. $(2\,\lambda+1)$ it is a maximum when $2\,\lambda+1=90$, *i. e.* when the middle latitude is 45°. The semi-diameter² to any latitude $\lambda=r^2=x^2+y^2=n^2\ \sin.\ ^2\lambda+\frac{a^4}{b^4}.\ n^2\cos.^2\lambda\ \therefore\ r=n.\ \frac{\sqrt{b^4.\ \sin.\ ^2\lambda+a^4\ \cos.^2\lambda}}{b^2}=$ $n.\ \frac{\sqrt{a^4-4\,a^3.c\sin.^2\lambda,}}{a^2-2\,a\,c}$ and by expanding this expression and neglecting c^2, we obtain $r=a\,(1-\frac{c}{a}\sin.^2\lambda.)$ The circumference of the elliptic meridian may be found by multiplying the mean degree, *i. e.* the degree in the parallel of 45° by 360°. By the series expressing the rectification of the ellipse, it may be found still more accurately.

In an ellipsoid of revolution, the normal terminated in the minor axis is equal to ρ', the rad. of curvature of a degree perpendicular to the meridian, for as in this hypothesis the direction of gravity always passes through the axis af the earth, the direction of a plumb line which is perpendicular to the meridian, and indefinitely near to it on the east and west sides, will intersect the axis in the same point, which point is ∵ the centre of curvature of the arc; as this normal is greater than the rad. of meridional curvature, a degree perpendicular to the meridian is greater than a degree of the meridian; $\rho'=$

$$\frac{a^2}{\sqrt{a^2\ \cos.^2\lambda+b^2\ \sin.^2\lambda}}$$

Now,

$$x = \frac{a^2 \cos. \lambda}{\sqrt{a^2. \cos.^2 \lambda + b^2. \sin.^2 \lambda}} \; y = \frac{b^2. \sin. \lambda}{\sqrt{a^2. \cos.^2 \lambda + b^2. \sin.^2 \lambda}}$$

$\therefore$ as $r^2 = x^2 + y^2$ we obtain by substituting e^2 for $\frac{a^2 - b^2}{a^2}$,

$r^2 = a \left(1 \quad \frac{-e^2. (1-e^2). \sin.^2 \lambda}{1-e^2. \sin.^2 \lambda.}\right)^{\frac{1}{2}}$; naming h the angle at the centre between r and a, we have $\tan. h = \frac{b^2}{a^2}. \tan. \lambda$, if l represent the angle between a and a line drawn to the extremity of the produced ordinate, we have

$$\tan. l = \frac{b}{a}. \tan. \lambda, \because \sin.^2 l = \frac{\frac{b^2}{a^2}. \tan.^2 \lambda}{1 + \frac{b^2}{a^2}. \tan.^2 \lambda},$$

hence by putting $1-e^2$ for $\frac{b^2}{a^2}$ we obtain $\frac{(1-e^2). \sin.^2 \lambda}{1-e^2. \sin.^2 \lambda}$ $= \sin.^2 l$, and $\therefore r = (1-e^2 \sin.^2 l)$, hence as l differs very little from λ, it follows as before that the increments of the rad. are very nearly as the squares of the sines of λ. π the angle between n and $r = \lambda - h$, $\therefore$ substituting for $\tan. h$ its value $\frac{b^2}{a^2} \tan. \lambda$, we obtain $\tan. \pi = \frac{\tan. \lambda - \frac{b^2}{a^2} \tan. \lambda}{1 + \frac{b^2}{a^2} \tan.^2 \lambda}$

$= \frac{(a^2 - b^2) \tan. \lambda}{(a^2 + b^2) \tan. \lambda}$; likewise it follows that

as we have always $\tan. \lambda + \tan. h : \tan. \lambda - \tan. h :: \sin. (\lambda + h) : \sin. (\lambda - h) :: a^2 + b^2 : a^2 - b^2$; it may be shewn that $\pi = \lambda - h$ is a max[m] when $(\lambda + h) = 90°$; it is evident also from other considerations that the point where the angle between the r and n is a max[m], must be at the extremity of the equal conjugate diameters: if the value of x, which is given above, be differentiated, we obtain after all reductions; $dx = = -a \frac{(1-e^2) \sin. \lambda \, d. \lambda}{(1-e^2 \sin.^2 \lambda)^{\frac{3}{2}}}$;

now $ds = -dx \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = -dx. \cos^{nt} \lambda$

$\because = a \frac{(1-e^2) \, d \lambda}{(1-e^2 \sin.^2 \lambda)^{\frac{3}{2}}}$; hence we derive $\rho = \frac{ds}{d\lambda}$, as

$$m\Delta = \frac{a^2}{\sqrt{a^2 \cos.^2 \lambda + b^2 \sin.^2 \lambda}} = a^2. (a^2 - 2 \, a \, c. \sin.^2 \lambda) - \frac{1}{2}$$

$= a + c \sin.^2 \lambda$; $m. (\Delta - D) = 2 \, c. \cos.^2 \lambda$; hence we can determine c, a, &c. Δ representing the same as before, a degree of longitude $= \Delta \cos. \lambda$. If R denotes the rad. of curvature of any sect., perpendicular to the tangent plane, at the earth's surface, it would not be difficult to show that it was equal to $\frac{\rho \, \rho'}{\rho. \sin.^2 \theta + \rho' \cos.^2 \theta}$, θ being the angle which the cutting plane makes with the meridian; hence it follows that when θ is 45°, R is an harmonic mean between ρ and ρ'.—See *Puissant*, tom. I. p. 288.

It follows from the expression cos. ρ, that a degree of longitude at the equator is the first of two mean proportionals between first and last degrees of latitude, for D at the equator is to D at the poles, as $a^3 : b^3$, the general ratio being that of

$$(a^2 \sin.^2 \lambda + b^2 \cos.^2 \lambda)^{\frac{3}{2}} : (a^2 \sin.^2 \lambda' + b^2. \cos.^2 \lambda')^{\frac{3}{2}}$$

The compression is obtained more accurately by comparing a meridional degree with a degree of a perpendicular to the meridian, than from a comparison of two meridional degrees with each other.—See *Puissant*.

The following is a brief outline of the method for determining the length of any arch of the meridian: two points are assumed nearly at the distance of the required arch, these two points are then connected by a series of triangles, the angles of which are determined by means of stations taken on the tops of hills, or other elevated positions; the angles of the triangles and also the azimuths of the sides, at the points where the series commences and ends, are to be measured. By this means the species of all these triangles are given, and also the bearings of their sides, with respect to the meridian of the first station. The lengths of the sides of the triangles are known by measuring a base on a level ground, and connecting it with the sides of one of the triangles. In these computations the process is on the supposition that the triangles are plane; however the error from this hypothesis is corrected by knowing the spherical excess which is given from knowing the area.—See *Puissant's Geodesique*, tom. I. p. 223.

(*g*) In like manner the terrestrial equator may be defined to be the plane, formed by all the points of the terrestrial surface, the verticals of which are parallel to the plane of the celestial equator, or which is the same thing, which are perpendicular to the axis of rotation of the heavenly sphere; consequently unless the earth be a solid of revolution, the terrestrial equator will be a curve of double curvature; if it be a solid of revolution, the terrestrial equator is a great circle of the sphere.—(*see* p. 102.) In like manner, the poles of the earth are those points of its surface, whose verticals are parallel to the axis of rotation; so that these points are not necessarily diametrically opposed to each other, except the earth be a solid of revolution. However, though when the earth is not a solid of revolution, neither the equator nor meridians are plane curves, still the corresponding celestial equator and celestial meridians may be considered as great circles, for the verticals when indefinitely prolonged may be conceived as terminating in the celestial sphere, in different points of the same great circle.—See *Puissannt*, tom. II. Book 6th.

Conformably to the above definitions, the terrestrial parallels will be formed by points, of which the verticals meet the celestial sphere under the same parallel, so that all points of the same parallel will have the same stars in the zenith; however, unless the earth be a solid of revolution these points will not form a circle, or even exist in the same plane. The latitude of all the points of these parallels is the same.—(*See* p. 111.) N. B. It is evident that the length of degrees of the terrestrial parallels decrease in proceeding from the equator to the pole, in the ratio of the cosine of latitude. From some measurements made by Biot and Arrago, it would appear that the parallel to the equator at the southern extremity of the meridian measured by them, is sensibly elliptic.

(*h*) *ds* denoting the first side of this line, *ds'* the second side, &c. These sides may be considered as equal, at least if quantities indefinitely small of the third and higher orders be neglected, for let *i* denote the angle which the prolongation af the first side (which is evidently equal to the first side) makes with the second, (*i* being a quantity indefinitely small of the first order,) then as the prolongation of the first side is evidently equal to it, we have $ds' = ds. \cos. i = ds - \frac{ds. i^2}{2}$, $= ds$, as ds is of the same order as i; hence it follows that in a geodesique line its differential is constant, likewise the normal comprised between the prolongation of the first side and the terrestrial surface is of the second order, for it $= ds. \sin. i$, or simply $i. ds.$ and since this geodesic line is equal to the right line, it necessarily follows, that it is the shortest which can be traced on the earth between any two points, it therefore measures the itinerary distances of places; its curvature likewise exists in a plane at right angles to the horizon, as is evident from the manner in which it has been traced. It is evident from what precedes that the difference between the length of this line and that of the corresponding arc of the terrestrial meridian may be neglected. Another property of the geodesic line is, that the sines of the angles made by the perpendicular with the respective meridians are inversely as the ordinates of the point of concourse.

It is clear that when the earth is a solid of revolution, all the normals to the surface of this solid meet the axis of rotation, consequently those which pass through the points of the generating curve are necessarily in the plane of this curve, and ∵ in that of the celestial meridian.

(*i*) Calling a b the equatorial and polar semidiameters, ρ ρ' the corresponding radii, t t' the two tangents, &c. c the arc of the evolute, then $a = \rho + t$, $b = \rho' - t'$, $\therefore a - b = \rho - \rho' + t + t'$.

(*k*) In determining the position of places in a region of considerable extent, it is necessary first to traverse it with a meridian line, from one extremity to the other, on this a certain number of points are selected, through which perpendiculars to the meridian are drawn. The meridian and its perpendiculars in this manner constitute a system of cervilinear coordinates, to which the different points of the earth's surface may be transferred. The great advantage of this method is, that when the extent of the region is not very considerable, these perpendiculars may be considered as great circles, and distances measured on them are the shortest between two given points.

(*l*) The method indicated in the notes to page 212 is perhaps the best and simplest of all, however it cannot be always applied; in that case, other methods have been devised, all of which may be reduced to the solution of certain cases of obtuse angled spherical triangles. Such as from having two altitudes of the sun, and the time between, or from observing the zenith distances of a heavenly body when near the zenith, the latitude is determined; the method which employs two altitudes of the sun has the advantage of enabling us continually to approximate to the true value.

(*m*) In fact the longitude and latitude only give the projection of a place on the earth's surface, but do not define its position in space; in order to determine this we must know the elevation of the place about the level of the sea. A determination of the heights of the most remarkable places in Europe would, combined with a knowledge of their longitude and latitude, be a more complete way of levelling than by trigonometrical operations, and would perfectly point out the directions of chains of mountains, and also the falls of rivers, &c., and thus give a most accurate notion of the form of the earth. As illustrative of the utility of these kind of observations, it may be remarked that a comparison of the heights of the barometer in the Euxine and Caspian Seas, evince that the level of the latter is considerably lower than the former.

(*n*) The repeating circle is an invaluable instrument to the practical astronomer, it supplies the place of a mural quadrant, and also of a transit instrument; besides it is capable of almost indefinite exactness, and from the smallness of its size it may easily be transported from one place to another.

(*o*) In general the retardation of time is proportional to the angle contained between the meridians of the two places, hence appears the reason of what has been already adverted to, namely, that if while one observer be fixed, another proceeds round the earth, he will on his arrival at the place from whence he set out, have either gained or lost a day, according as he went, eastward or westward.

(*p*) The chronometers now in use, being furnished with compensators, which secure them from the effects arising from changes of temperature, and also from the inevitable effects of the agitation which they experience during a long voyage, give the time with extreme accuracy.

The *true* time H at the place of observation is easily obtained when the latitude of the place or vessel, Z the zenith distance or altitude of the star, and *d*, its declination are given, for it is easy to show that

E E

$$\text{Sin. } \frac{H}{2} = \sqrt{\frac{\text{Sin. } \frac{(Z+P-D.)}{2} \text{ Sin. } \frac{(Z+D-P.)}{2}}{\text{Sin. } P. \text{ Sin. } D.}}$$

See Notes, p 304, 292. But as the chronometer indicates *mean* time we must apply the equation of time in order to obtain the mean time at the place of observation. This method assumes that the time indicated by the chronometer is exact, which is not the case; however its rate of going and small inequalities may be ascertained by comparing it with the time pointed out by observing the altitudes of the sun or stars as often as possible.

As lunar eclipses are of comparatively rare occurrence, they are not of very great use in finding the longitude at sea; this objection does not apply to eclipses of Jupiter's satellites, as eclipses of the first satellite recur every third hour; however the difficulty of rightly adjusting a telescope on board a ship is such, that it is now very rarely used, except when the observer can land.

The problem for determining the true distances of the centres of the sun and moon, from knowing the observed values of the heights of the sun and moon, and from having the observed distances of the centres, is one which has occupied astronomers who applied themselves to the perfecting nautical instruments; the best methods are those given by Maskeylyne and Borda.—*See* Nautical Almanack.

Besides the methods suggested in the text, it has been proposed to determine the difference of longitudes of two places, by means of signals, such as an explosion, which may be seen at the same time from the two places; and if the places are too distant to observe the same signal, a series of such signals are made, and noted in places intermediate between those whose difference of longitude is required.—*See* Lardner's Trigonometry, 189.

When the difference of longitude of two places, and their respective latitudes are known, their distance in an arc of a great circle, is easily determined, for calling λ, λ'

the respective latitudes, and D the difference of longitudes, cos. a the mutual distance = cos. λ. cos. λ′. sin. D + sin. λ. sin. λ′. This is on the hypothesis that the earth is *q. p.* circular; if it be supposed to be an ellipsoid of revolution, the direction of verticals from the two places do not meet in the same point of the axis, and ∵ do not make a solid angle; in that case we deduce the angles which rad. from centre of ellipsoid to the two places make with the axis, and the inclinations of the planes of these angles to each other is also given, hence the angle which the rad. vectors make with each other may be determined, and hence the mutual distance of the places, the distance of each place from the centre being known.

(*q*) This instrument is a common barometer, except that the open branch, which communicates with the external air in the barometer, communicates with a closed vessel in which the gas or vapour is placed, of which the elastic force is required. As the height of the mercury in the barometer, of which the open branch communicates with the atmosphere, gives a measure of the elastic force of the air at the point where the fluid is in contact with the mercury, the same will be true when the aperture is closed, for it is evident that the state of the air is not affected by this circumstance; hence if g represents the force of gravity, ρ the density of the mercury in the barometer, and h the difference of heights of the mercury in the two tubes, we have an equilibrium between $g\rho h$ and the elastic force of the air, which we will denominate by E; now as E is always the same when the density and temperature of the air are the same, if the manometer be transported from one place to another, taking care that the state of the air contained in it does not undergo any change, $g\rho h$ must also remain unchanged; hence if g varies, h must vary in the inverse ratio, provided that ρ is constant.

(*r*) The length of the ideal pendulum, which is isochronous with the observed pendulum, = the distance between the point of suspension and a point in it called the centre of oscillation.

(*s*) *See* Notes to Chap. II. Book IV. Naming l the length of the pendulum, t the time of vibration, and g the force of gravity, it will be proved in the 4th Book, Chap. II. that $t=\pi.\sqrt{\frac{l}{g}}$ when the arch of vibration is very small, hence as t increases towards the equator, g must diminish, for if the time of vibration increases, the number of vibrations performed in T must diminish, and consequently the clock must lose for $t=\frac{\mathrm{T}}{n}$. What is advanced in this Note suffices to show that the gravity decreases as we approach the equator. A fuller investigation of this subject will be given in the Notes to Chap. II. Book IV. of this volume, and in Notes to Chap. VI. Book I. of next volume.

(*t*) Indeed it is natural to suppose that the intensity of gravity is less affected by local variations than its direction, for the inequalities on the surface of the earth, and the very irregular manner in which the rocks are distributed, necessarily cause considerable deviations in the directions of the plumb line, and are most probably the causes of the discrepancies which are observed in the measurement of contiguous arcs of the meridian, which are extremely near to each other, which must consequently cause the results as to the ellipticity, &c. of the earth, to differ considerably from each other.

(*u*) If (g) be the intensity of gravity at the level of the sea, and g the intensity at the top of the mountain, whose height is h, r being the radius of the earth, $\frac{(g)}{g}=\frac{(r+h)^2}{r^2}$ $=1+\frac{2h}{r}$ neglecting the square of h, $\therefore$ if l' be the length of the pendulum on the top of the mountain, l the length at the level of tha sea $= l' + \frac{2hl'}{r}$. *See* Notes to Chap. III. Vol. II.

(x) It does not appear that the new system of weights and measures explained in the text, has been adopted with that generality which was anticipated by the illustrious author; on the contrary, a Committee of the House of Commons, which was appointed to revise and examine the standard weights and measures of Great Britain, appeared to think the only practical advantage of having a quantity commensurate to any original quantity existing, or which might be supposed to exist in nature, consisted in its affording some little encouragement to its universal adoption by other nations; but this advantage would by no means compensate for the great inconveniencies which must necessarily result from a departure from a universally established standard; nor would the adoption of the decimal scale in weights and measures have any very marked advantages over the present subdivisions; on the contrary, as the standard measure consisted of twelve inches, we can express a greater number of subdivisions of it without fractions, than in any other scale.—*See* Note in next page; and as to the weights and the measurement of capacities, the continual division by two, enable us to make up any given quantity with the smallest number of standard weights, and ∴ in this respect has an advantage over the decimal scale.—*See* Notes to next page.

The Committee above mentioned suggested that the standard measure should be the standard executed by Bird in 1760, which is in the custody of the clerk of the House of Commons; likewise in the event of its being lost, its length could be easily ascertained, as they have declared its proportion to that of a pendulum vibrating seconds of mean time at the latitude of London, in a vacuo, and at the level of the sea to be that of 36 to 39,

1393. They have also declared that a brass weight equal to half the brass weight of two pounds gravitating in air, at the temperature of 62, the barometer being 30, which is kept in the House of Commons, should be the imperial standard troy pound, or the unit of weight; if lost they have also determined its relation to a cubic inch of distilled water weighed by brass weights in a vacuum at the temperature of 62 of Fahrenheit, to be as 5760 to 252,724. The standard measure of capacity for liquids and dry goods not heaped, is a gallon containing ten pounds avoirdupois weight of distilled water weighed in air at the temperature of 62°, and the standard measure for goods sold by heaped measure shall be a bushel containing eighty pounds avoirdupois of water as aforesaid.

(*y*) With respect to the different scales of notation, it is plain that if mere simplicity of arithmetical operations be considered, the number 2 is preferable to any other; but there is always another point to be considered, namely, the facility and ease of arithmetical expressions, and in this point of view the binary scale would be extremely embarrassing, as it requires such a multiplicity of figures to express any considerable number. The senary, at the same time that it would secure most of the advantages of the Binary scale, would not be liable to this last objection, at least in so great a degree, it has this peculiar advantage, that there would be a considerably greater number of finite fractions in this scale than in the denary; however as the operations proceed rather slow it was never brought into use. The duodenary combines all the advantages of the senary scale, and is free from this objection; the only inconvenience attending it, is the trouble of requiring us to remember two additional characters; but though it is stated in the text that this is a great objection to its use, in point of fact it is not considered so, as we find by experience that our multiplication table is carried on as far as

12 multiplied by 12, though, strictly speaking, it ought to terminate with the product of 9 into 9.

In fine the great objection against the French system is, that it depends upon an accurate measure of a quadrant of the meridian, at the same time that no such measure has hitherto been obtained, besides the meridians differ so widely among themselves that it is likely that no accurate mean length of the pendulum will ever be obtained.

The idea of verifying a standard by some other means than by a comparison with some actually existing standard, though suggested a great while ago, was never completely acted on, until the new system of weights and measures was introduced into France.

CHAPTER XIV.

(*a*) Conceive a vertical to be elevated from the level of low water by a quantity equal to the height of the high water, and if a circle be described on this line, the tide will rise or fall through equal arches on equal times; hence if we assume any arc, reckoning from the lowest point, to represent the interval from the instant of low water, the versed sine of the arc will represent the height to which the water will have risen; hence it is evident that near the high or low water, the differences of depths from those of high or low water, are as squares of the times.

The causes which produce a difference in the height of the tides, arise either from the circumstances of the sun's action sometimes conspiring with, and at other times opposing the moon's action, from the variations in the respective distances of these luminaries, and also from the declinations not being always the same; the effects arising from these causes influence the interval between two successive high waters, as well as the heights.—*See* Mechanique Celeste, Tome 2, Chap. 3, and also the Notes to Chap. 4, Vol. 2, of this work.

CHAPTER XV.

(*a*) A bottle when filled with air is heavier than after the air is extracted; the pressure of the atmosphere on every square inch of the earth's surface is 14 lbs., for a cubic inch of mercury is nearly 8 ounces, ∵ ,76 + 8,238 ounces = 15 lbs. nearly; it appears from this, that the pressure to which the bodies of animals and vegetables are subjected is very considerable, and could not in fact be sustained but for the elasticity of the air, which being always ∷ *l* to the compressing force, enables the small quantity of air contained in their bodies to counteract the violent pressure of the atmosphere; hence it might easily be shewn that the pressure on the entire convex surface of the earth = 10,686,000,000 hundreds of millions of pounds.

(*b*) If *g* represent the force of gravity, *h* the vertical height of the barometer above the surface of the mercury, which is exposed to the external air, ρ the density of the mercury, the pressure on the exterior surface of the mercury, and ∵ the = and contrary pressure of the air = $g. \rho. h.$

The numbers mentioned in page 136 exhibit the ratio of the specific gravity of air to that of mercury; which numbers also indicate the ∷ of the height of the homogeneous atmosphere to the height of the mercury in the barometer; for let h' represent this height, m' representing the specific gravity or density of the air, we have $mh = m'h'$; consequently, as we ascend from the surface of the earth, h' and $\because h$ diminishes; the height of the homogeneous atmosphere, *i. e.* of an atmosphere which is the same density as the air at any elevation above the earth's surface, is a constant quantity, if the effects arising from the action of heat and cold are not taken into account, for $h' = h\frac{m}{m'}$, but as $\frac{m}{m'}$ measures the air's density and pressure, it will vary as $\frac{1}{h}$, $\because \frac{m}{m'}$ is constant, hence h at any station is not affected by any difference in the weight of the air.

(*c*) Let z represent the vertical height, m' the density, g' the gravity, p the pressure or elastic force of the air, x the temperature, we have $adp = m'g'dz$, $\therefore -\frac{dp'}{p} \backsim -dz'$, $\because$ log. $p \backsim$ as z, and if z be taken in arithmetical progression, the Naperian logarithm of $\frac{1}{p}$ is in arithmetical progression, and $\because \frac{1}{p}$ is in geometric progression, and as the densities are as the compressing forces, *i. e*, as the heights of the mercury in the barometer, in the same circumstances these heights will decrease in geometrical progression (*a* expresses the ratio of the elastic force to the density, when the temperature is zero, and is evidently the same for the same elastic fluid, but is different for each) and $\because$ if h, h' are the columns of the mercury at the surface, and at any elevation z from the surface, K representing the constant coefficient to be determined by experiment, we have $z = K.(\log. h - \log. h')$ $= K. \log. \frac{h}{h'}$, hence K will be had if z is determined trigonometrically in any case, where h, h' are previously ascertained.

(f) The intervals between which it has been ascertained from experiment that this = dilatation obtains, is from zero to 100° of the centigrade thermometer, and it is even true for those aeriform substances which are produced by vaporization, provided that they are not charged with any liquidity, hence 00375 being represented by a, and the increase of temperature by x, we have $p = am'.(1+ax)$.

(g) The aqueous vapours are necessarily less dense than the air in which they float, and from their being mixed in the air, it is enabled to sustain with a less density a column of mercury of the same height, ∴ this vapour weighs less than dry air, perfectly free from humidity, of the same elastic force. *See* Note (v) of this Chapter.

(h) From these weights the ratio of the specific gravity, and ∴ the constant coefficient may be deduced, which ought to agree with the coefficient deduced *a priori* from a comparison of the same height as furnished by barometrical and trigonometrical observations, but these disagree; and as this disagreement cannot be accounted for by introducing the consideration of humidity, the variation of the force of gravity as we ascend from the earth must be taken into account; this diminution of the force of gravity will be taken into account, if in the equation $adp = m'.g'.dz$, we substitute $\frac{gr^2}{(r+z)^2}$ for g', then we have $\frac{dp}{p} = -\frac{gr^2}{a.(1+ax)} \cdot \frac{dz}{(r+z)^2}$, which gives by integrating, $\log. p = \frac{K.gr^2}{a.(1+ax)} \times \frac{1}{r+z} + C$, x is supposed to be constant, and ∴ if π be the value of p, when $z = 0$; $\log. \frac{\pi}{p} = \frac{K.gr}{a.(1+ax)} \times \frac{z}{r+z}$, hence when z is known, and the heights in the barometer observed, we can determine

K; by means of this equation, combined with the value of p, given in Note (c), we can determine the laws of density and elastic force of the air for a given state of equilibrium of the atmosphere. Now in order to apply this formula to the mensuration of heights we have $\pi = mgh$, $p = mg'h'\left(1 + \frac{T-T'}{5412}\right)$, T, T', being the temperatures of the mercury at the two stations; in order to abbreviate, let h' represent the height of the barometer at the second place of observation multiplied into $1 + \frac{T-T'}{5412}$, then we have $\frac{\pi}{p} = \frac{h}{h'} \cdot \frac{(r+z)^2}{r^2}$; $\because \log. \frac{\pi}{p} = \log. \frac{h}{h'} + 2 \log. \left(1 + \frac{z}{r}\right)$; let t, t' be the respective temperatures of the air, which differ from T, T', as a given difference of temperature, is not so rapidly communicated to the mercury as to the external air, $x = \frac{t+t'}{2}$, $a = ,004 = \frac{1}{250}$, $\because ax = 2\frac{(t+t')}{1000}$; hence substituting these values we get $z = \frac{a}{Kg} \cdot \left(1 + \frac{2.(t+t')}{1000}\right) \cdot \left(\log. \frac{h}{h'} + 2 \log. \left(1 + \frac{z}{r}\right)\right)$; $\frac{a}{Kg}$ is the coefficient 18336, mentioned in the text, it is obtained by an equation of condition which is given from knowing z, h, h', t, t', and r the radius of the earth; this value is for the latitude 45, for any other $\frac{a}{Kg} = 18336\,(1,002837 \cos. 2\psi)$. In the determination of z, as $\frac{z}{r}$ occurs in the second member, where it is an extremely small fraction, we 1st compute z on the supposition that this fraction is wanting, we then substitute the value of $\frac{z}{r}$ determined in this supposition, and as it is extremely small, the result differs inconsiderably from the truth. Besides the corrections mentioned in the text, when extreme accuracy is required we must take into account the convexity of the mercury in the upper part of the tube, and also the effect of capillary attraction. With respect to the cause of the variation of the length in the barometric column, various theories have been suggested, none however completely satisfactory. In the Notes to Chapter X. Vol. II. we shall enter into some details respecting the periods, &c. of these variations.

By very precise experiments made with the hygrometer, it has been ascertained that the power of the air to retain moisture is doubled at every increase of temperature of the centrigrade thermometer by 15 degrees, or in other words, while the temperature increases in an arithmetical progression, the quantity of moisture which the air is capable of holding in solution increases in a geometrical ratio; these indications of the hygrometer do not point out the absolute degrees, but only its relative dryness with respect to the ball of the hygrometer.

It has been computed, that if the atmosphere would pass from its point of saturation in dampness, to a state in which the air would be completely destitute of humidity, the whole quantity of water discharged would not constitute a sheet of water five inches in depth.

(*i*) The natural colour of the air is blue; but in order to be apparent, the depth of the air should be considerable. This is the reason why the colours of very distant objects are always tinged with the blue of the intermediate atmosphere. In fact, as the particles which compose the air are extremely small, and at a distance from each other, they could not be perceived unless they were united in a mass; conformably to this, it is found that according as we ascend in the atmosphere, the blue colour becomes less brilliant, for the brightness diminishes with the density af the air which reflects it, so that on the summit of a high mountain, or to an aeronaut, the sky appears black.

As no coloured substance discloses its inherent colour, but by separating the rays of light, in order that its real colour should be exhibited the particles of light must penetrate the atmosphere, and after undergoing some change be again emitted. In the atmosphere, besides the internal dispersion of the blue rays, the white light is reflected in various quantities without any change, as is evident from the phenomena of polarization. And as the white light, in its transit through the air, continually loses more and more of the blue rays, it must, according as it advances, assume the complimentary colours of the spectrum, and ∵ become successively yellow, orange, red and crimson.

(*k*) It is the reflective power of the atmosphere, which makes objects to appear uniformly enlightened in *every* direction; if it had not this power, the bright sides of objects would be only visible, and their shadows would be, in all probability, insensible. The evening twilight is longer than the morning, because the atmosphere is then more dilated by heat.

The last ray which comes to the spectators eye touches the earth when it is first emitted from the sun; and secondly, when it reaches the spectator after being reflected at the extreme verge of the atmosphere.

In this method allowance should be made for the inflection of the ray, or for its deviation from a rectilinear course by the action of the continually denser strata. For the greatest height of the atmosphere at the equator, *see* Vol. II. Chapter XIII.

If the density of the air decreased in geometric progression at fifteen miles elevation, the height of the barometer would be only one inch; ∵ the greatest part of the atmosphere is always within fifteen, or at farthest twenty miles of the earth, and ∵ though from the refraction of the sun's light, and from the duration of twilight, it has been inferred that the height is from forty to forty-five miles; Wollaston thinks that it is limited to the former height; in fact, the force of gravity on a single particle is then equal to the resistance which arises from the repelling force of the particles of air; hence he likewise infers, that there is a limit to the divisibility of matter.—*See* Philosophical Transactions, 1822.

On the contrary, a stratum of air at five and a half miles depth from the surface, would have such a density that it would never rise to the surface; ∵ as the mean depth of the sea, as given by the theory of the tides, see Vol. II. Chap. XII., is twice that quantity, the conjecture of some philosophers may be true, that the bed of the ocean rolls on this subaqueous air, which, though it never rises to the surface, may support the combustion which we know goes on below the surface of the earth.

It is easy to compute the duration of twilight, when the latitude and declination are known,; for as it appears from repeated observations, that it lasts until the sun is 18° below the horizon, if h', h, represent the hour angles at the termination and beginning of twilight, we have $\cos. h = -\tan. l. \tan. \delta$, $\cos. h' = -\dfrac{\sin. 18°}{\cos. l. \cos. \delta} - \tan. l. \tan. \delta$, $\because \cos. \frac{1}{2}(h'-h) = -\dfrac{\sin. 18°}{2 \sin. \frac{1}{2}(h'+h). \cos. h. \tan. l}$, ∵ it is shortest when l and $\delta = 0$, as the greatest depression of the sun $= 90 - (l + \delta)$, if this quantity is less than 18, or $72 \angle l + \delta$, twilight will last all night, or rather the morning twilight will immediately succeed the evening. Cos. h' is always > 90 until l and δ are of opposite affections, and $\sin. l. \sin. > \sin. 18°$, or $\sin. l > \dfrac{\sin. 18°}{\sin. 23°, 28'}$, $\because l > 50°, 54$; hence all parts of the earth, of which the latitude exceeds 51°, have the days longer than the nights in consequence of this power of the air to reflect light, and at the poles it lasts until the sun is 18° at the other side the equator, so that the two twilights, before and after the commencement of summer, last fifteen weeks; if $\cos.(h'-h)=-1$, twilight lasts all night; in this case, $\sin. 18=\cos.(l+\delta)$, $\because \delta=72-l$, hence, the part of the year during which twilights can last all night increases with l, and its least value is 48°, 32′. To determine the day in which, in a given place, the duration of twilight may be given quantity. Let $h'-h=\gamma$ then we have $\cos. h=-\frac{\sin. l. \sin.\delta}{\cos. l. \cos.\delta}$, and $\cos.(h+\gamma)$ $=\cos. h'=-\frac{\sin. 18°+\sin. l. \sin.\delta}{\cos. l. \cos.\delta}$, *i. e.* $\cos. h. \cos.\gamma. -\sin. h. \sin.\gamma=\cos. h-\frac{\sin. 18°}{\cos. l. \cos.\delta}$, $\because \sin. h=$ $\frac{\sin. 18°+\sin. l. \sin.\delta.(1-\cos.\gamma)}{\sin.\gamma. \cos. l. \cos.\delta}=\frac{\sqrt{\cos.^2 l-\sin.^2\delta}}{\cos. l. \cos.\delta}$; $\because$ by squaring and concinnating we obtain $(\sin.^2\delta.(2\sin.^2 l.(1-\cos.\gamma)+\cos.^2 l. \sin.^2\gamma)+2\sin.\delta. \sin. 18° \sin. l.(1-\cos.\gamma)+\sin.^2 18-\cos.^2 l. \sin.^2\gamma=0$; the solution of this equation gives two different values of δ, and as the sun has the same declination twice every year, there are four different days in which the duration of twilight is the same. To find the shortest twilight, we have by differentiating the preceding values of $\cos. h$, and $\cos.(h+\gamma)$ supposing γ and δ to vary, $dh=\frac{d\delta. \sin. l}{\cos. l. \cos.^2\delta. \sin. h}$, and $dh+d\gamma=\frac{d\delta.(\sin. l+\sin. 18°. \sin.\delta)}{\cos. l. \cos.^2\delta. \sin.(h+\gamma)}$, $\because$ as γ is supposed to be a minimum, $d\gamma=0$, $\because \frac{\sin.(h+\gamma)}{\sin. h}=$ $\frac{\sin. l+\sin. 18°. \sin.\delta}{\sin. l}$, but $\sin. h=\frac{\sqrt{\cos.^2 l-\sin.^2\delta}}{\cos. l. \cos.\delta}$, and $\sin.(h+\gamma)=$

$$\frac{\sqrt{\cos.^2 l-\sin.^2\delta-2\sin. 18°. \sin. l. \sin.\delta-\sin.^2 18}}{\cos. l. \cos.\delta},$$

$$\therefore \frac{\sin.(h+\gamma)}{\sin. h} =$$

$$\frac{\sqrt{\cos.^2 l - \sin.^2\delta - 2\sin. 18^\circ. \sin. l. \sin.\delta - \sin.^2 18}}{\cos.^2 l - \cos.^2\delta},$$

$\therefore$ equalling these two values of $\frac{\sin.(h+\gamma)}{\sin. h}$, squaring and dividing by $\sin.^2 18^\circ. \cos.^2\delta$ we obtain $\sin.^2\delta + \frac{2\sin. l. \sin.\delta}{\sin. 18^\circ} + \sin.^2 l = 0$, $\therefore \sin.\delta = -\frac{\sin. l.(1 \mp \cos. 18^\circ)}{\sin. 18^\circ}$ $=$ either $-\sin. l. \tan. 9^\circ$, or $-\sin. l. \cot. 9^\circ$, $\therefore$ the shortest twilight occurs four times in the year, and always in winter time, for δ is negative; but as δ cannot exceed 23,28, in the second value of δ, if $\sin. l$ is $>r$ than $\tan. 9^\circ. \sin. 23,28$ it is impossible, $\therefore$ this solution only obtains for latitudes less than $3^\circ,37'$, but the first is true for all latitudes for its maximum value, *i. e.* when $l = 90$, is $\sin.\delta = \tan. 9^\circ$; this would appear therefore to determine δ for the shortest twilight under the pole; however this problem is not applicable to the pole, as we can have but one day, and consequently but one twilight under the pole during the entire year; in general that several twilights may occur successively, it is necessary that $180 - l + \delta > 108$, *i. e.* that $l < 72 + \delta$; $\therefore$ conformably to this condition, it results from the first value of δ, that $\sin.\delta \angle$ than $\tan. 9^\circ. \sin.(72+\delta)$ or $\tan.\delta \angle \frac{\tan. 9. \cos. 18}{1 - \tan. 9. \sin. 18}$, or $\tan\delta \angle \tan. 9$; $\therefore l$ is less than $72+9$, or $81 \because l+\delta \angle 90^\circ$; this shews that the firs root is not applicable to all the earth, for all places whose latitude is $>$ than 80°, the sun does not set for the day of shortest twilight; it is evident that if $l=0, \delta=0$, $\therefore$ the shortest twilight at the equator is when the sun is in the equator. To find the duration of the shortest twilight, let the angles formed by the vertical and circle of declination at the sun set and at the end of twilight $=$ s and S respectively, then we have $\cos. s = \frac{\sin. l}{\cos.\delta}$, $\cos. S =$

$\frac{\sin. l + \sin. 18. \sin. \delta}{\cos. 18. \cos. \delta}$, substituting for sin. δ its value $-\tan. 9. \sin. l$, we have $\cos. S = \frac{\sin. l.(1 - 2. \sin.^2 9^\circ)}{\cos. 18. \cos. \delta} =$ $\frac{\sin. l. \cos. 18^\circ}{\cos. \delta. \cos. 18} = \cos. s$, $\because s = S$, $\because$ in the vertical passing through the sun, if an arc $= 18^\circ$ be taken, it is easy to prove that the zenith distance is equal the arc of a great circle, formed by lines from the pole to the extremity of this arc, and that the angle between them $= \gamma$, $\because$ in this isoceles triangle we have $\cos. \gamma = \frac{\cos. 18 - \sin.^2 l}{\cos.^2 l}$

$\therefore 1 - \cos. \gamma = 2 \sin.^2 \tfrac{1}{2}\gamma = \frac{1 - \cos. 18}{\cos.^2 l} = \frac{2 \sin.^2 9}{\cos.^2 l}$,

$\because \sin. \tfrac{1}{2}\gamma = \frac{\sin. 9}{\cos. l}$.

(*o*) A ray of light is made to pass through a prism, out of which the air is supposed to be completely excluded, and if the sides of the prism be perfectly parallel, the deviation which the ray experiences must arise from the refraction of the external air; and from knowing this deviation, and also the refracting angle of the prism, the ratio of the sine of incidence to the sine of refraction can be determined for gases or liquids.

There is however this difference, that in case of gaseous substances the refracting angle of the prism may be considerably greater than for liquids; in the latter it cannot exceed a certain limit, which is thus determined, sine of half the angle of prism is to radius as sine of incidence to sine of refraction from the liquid into air.

It is easy to shew that for any ray refracted by the prism, the sine of the deviation of the ray is to the sine of refracting angle of the prism, as sine of incidence is to the difference between the sine of incidence and the sine of refraction from the prism into air. It is in this manner that the ratio of the sine of incidence to the sine of refraction is determined in page 145 of the text.

As it is nearly impossible to procure a perfect vacuum, the height of the mercury in the gage must be observed, and account made of it in the calculus. If we wished to obtain the refraction of the air at different densities it would be only necessary to note the height of the mercury in the gage at the respective densities; or if the refractions of other gases were required, we should exhaust the prism as far as possible, and then after noting the height of the mercury in the gage, introduce the gas. Caustic potash is generally introduced to absorb the aqueous vapours which exist in the air, when its density is so reduced; on the contrary, if the refractions of aqueous vapours were required, we should charge the atmosphere with them, by means of vessels of water and of moistened towels. See *Biot's Physique*, tom. 3.

(*p*) The diversity of colours arises from the particular disposition of bodies to reflect some rays rather than others. When this disposition is such that the body reflects every kind of ray in the mixed state in which it receives them, that body appears white; ∵ white is not a colour, but rather the assemblage of all colours; if a body has a disposition to reflect one sort of rays more than others, by absorbing all the others, it will appear of the colour belonging to that species of rays. As different bodies are fitted to reflect different kinds of rays, they must appear of different colours; when a body absorbs all the light which reaches it, it appears black, as it transmits so few reflected rays that it is scarcely perceivable.

(*q*) The density of the atmospherical strata decrease in arithmetrical progression, when the temperature diminishes in arithmetrical progression; for the density m being equal to Q the quantity of matter divided by the volume, if 1 represent the volume previously to x the increase of temperature, we have $m = \frac{Q}{1+ax}$, (a representing 0,375) $= Q.(1-ax)$ nearly; ∵ when the increments of the temperature are given, the densities decrease by an arithmetical progression.

There are two causes of the decrease of heat, according as we ascend in the atmosphere, namely, our receding from the earth, the principal source of heat, and also from the circumstance of the air being less compressed, which makes its absorbing power greater. But though the heat thus decreases in a less ratio than the distances increase, still the rate of decrease is nearly uniform when the height is inconsiderable.

When the altitude exceeds eleven degrees the inclination of a ray of light to the atmospheric strata is less oblique, consequently the curvature of the portion of the trajectory to be described by the star is less, and according as the altitude increases, it approaches, more and more to the rectilinear direction; now if the strata of the atmosphere were parallel to each other, and to the earth, considered as a plane, the refraction would be what would take place if the ray passed from a vacuo into air of the same density as that at the earth's surface; the error, ∵ arises from the earth being supposed to be a plane, when it is in point of fact spheroidical, which shape is communicated to the atmospherical strata. In the former case, the refraction would depend on the total increase of density of the atmosphere, *i. e.* on the pressure and temperature which are indicated by the barometer and thermometer.

It may likewise be observed here, that when the elevation is greater than eleven degrees, the differential equation of the trajectory described by the ray of light, namely $dr = dv\sqrt{Q}$ (where r is the radius from the centre to any point of the trajectory, and v the angle between r and a vertical at this point, Q a function of r depending on the law of the decrease of densities) may be expressed in a very convergent series; but when the trajectory is horizontal, dr and ∴ $\sqrt{Q}=0$; ∴ if it is near to a horizontal state, Q is inconsiderable, ∴ $\sqrt{Q}$ cannot be developed in a convergent series, because the several terms which compose it have a finite ÷ to each other; but when the point is at a considerable distance from its *minimum* state, some of the terms composing Q are considerably greater than others, ∴ the expansion of $\sqrt{Q}$ into a series is possible, and ∴ the equation of the trajectory may be obtained by approximation. —*See* Mechanique Celeste, tom. 4, livre 10.

(s) If $n:1$ be the ratio of sin. I to sin. R from a vacuum into air, we have, if ir be the angles of incidence and refraction, z the zenith distance, a the radius of the earth, and h the height of the homogeneous atmosphere, $a+l : a ::$ sin. $z :$ sin r; $1 : m$: sin. r : sin. i, ∴ sin. $i = \frac{m.a. \sin. z}{a+l} = m. \sin. z. \left(1-\frac{l}{a}\right)$; sin. $r =$ sin. $z\left(1-\frac{l}{a}\right)$; $i = r + R$; ∴ sin. $(r+R) =$ sin. i; and sin. r + cos. r. sin. $R = m.$ sin. r, ∴ $(m-1)$. tan. $r =$ sin. R, or R; hence substituting for sin. r, and sin. i, and also for cos. $r = \sqrt{1-\sin.^2 r}$, $=$

$$\sqrt{1-\sin.^2 z\left(1-\frac{l}{a}\right)^2} = \sqrt{\cos.^2 z + \frac{2l \sin.^2 z}{a}} = \cos. z.$$

$$\left(1+\frac{l}{a}. \tan.^2 z,\right) ∴ R = \frac{\sin. i - \sin. r}{\sin. 1''. \cos. R}$$

$$= \frac{(m-1). \sin. z.\left(1-\frac{l}{a}\right)}{\sin. 1''. \cos. z. \left(1+\frac{l}{a}. \tan.^2 z\right)} = \frac{(m-1). \tan. z}{\sin. 1''}$$

$-\frac{(m-1). l \tan.^2 z}{a. \sin. 1''. \cos.^2 z}$. If $z = 80°$, $l = 5$, $a = 4000$ miles, the second term will not exceed $10''$; this is what arises from the spherical shape of the earth; if a was infinite, *i. e*, if the earth was a plane, it would vanish; ∴ as far as 80° of zenith distance, the error from the supposition that the density of the atmosphere is uniform, and the earth a plane, must be less than 10″. Now from the ratio of sin. i : sin. r from a vacuum into air, m the coefficient of the refraction may be determined (p. 369). This coefficient is as the refractive force of the air, *i. e*, as its density, or as $\frac{Q}{M}$; ∴ to reduce the coefficient to a given temperature and pressure, it must be divided by $1+ax$. (*see* page 370), and then multiplied by the direct ratio of the pressures, ∴ the true coefficient $= \frac{p}{,0,76(1+at)}$; but if these quantities are determined for the latitude of 45, they should be multiplied by cos. 2ψ for any other latitude ψ.—(*See* p. 341.)

(t) It may be doubted whether the analysis given in the text is complete, for a recomposition of these materials will not give air of the same nature as the atmosphere, ∴ some of the elements or constituents must have escaped during the decomposition, which is indeed probable, as the air is charged with emanations from the various substances with which it comes in contact; we are certain, as was before observed, that the quantity of aqueous vapour is not always the same; it appears from this that, if its chief constituents are always in the same ÷n, the purity or insalubrity of the air must depend on somethiug besides this proportion. It is conjectured with some degree of probability, that the higher regions consist ofi nflammable materials, which is the cause of those appearances which it frequently exhibits, namely, of shooting stars, fire-balls, and luminous arches; these materials arise from the numerous emanations from volcanoes, &c. &c., and as hydrogen is lighter than common air, and has very little affinity for its constituents, it ascends upwards from its greater levity, and from the extent and celerity of these phenomena they must necessarily take place in the most elevated regions of the atmosphere: this conjecture is confirmed by astronomical refractions, for the refraction in these elevated regions is greater than what computation assigns to them, but on the supposition that hydrogen gas is one of their chief constituents, this discrepancy disappears, for the refraction of this gas is greater than that of other substances in proportion to its density, while oxygen gas is the least refractive of the gases.

Chemists are not agreed as to the manner in which the constituents of the atmosphere exist in it; some suppose that they are chemically united, chiefly from the uniform manner in which they are always combined, and because they are not arranged according to their respective specific gravities; others think that the particles of the gases which compose the atmosphere neither attract nor repel one another, and that the weight on any one particle of the atmosphere arises solely from particles of its own kind. —*See* Manchester Memoirs, p. 538.

(*u*) It is easy to find the stratum of air, of which the density is such as is described in the text, for let c^3 be the capacity of the balloon, y the specific gravity of the stratum of air in which the balloon floats in equilibrio, since a cubic foot of water weighs 62,48lbs, $c^3. (62.48)y$ is the weight of the displaced air, and the whole weight is $w + (62.48).\ c^3.\ \frac{y}{13}$, when these quantities are $=$, we can determine y, and $\because$ the density of the stratum, and consequently the height, from knowing the density of the air and height of the mercury at the earth's surface.

(Note. w is the weight with which the balloon is loaded, and the hydrogen gas which is generally used is only six times lighter than common air.)

Besides the circumstances mentioned in page 151, it was ascertained, as mentioned above, that the elasticity of the upper regions of the atmosphere was greater than near the earth's surface, also the diminution in the temperature was less than what was experienced in corresponding heights on the earth's surface, and the indications of the hygronometer shewed that the atmosphere became dryer according as we ascended; but indeed this might arise from the increased attraction of the air for moisture in consequence of its less density.

(*v*) Knowing the refractive power of water, from note page 372, we can determine it for water reduced to vapour of the same density as the air, for these refractions are ∻l to their densities; now the density of this vapour would give its refraction greater than that of air; but as the density of the vapours which float in the air are less than that of air, the refraction of the vapour must be diminished, and by nearly the quantity by which it was greater than that of air. Biot made his direct experiments on the refraction of air saturated with humidity, and at high temperatures. Note, there are some exceptions to the position that the refractions are ∻l to the densities, for it is not true for the class of inflammable substances.

Suppose for instance, as stated in page 153, that a wind blew for a long time in the same direction, the curvature of the inferior strata would be necessarily affected by it, and ∵ the refractions computed from it would be very unequal. The temperature may produce equally anomalous effects, as for instance, if from the greater heat of the surface of the earth, the density of the lower strata was less than of those more elevated, as is the case in the phenomena observed frequently in Egypt, which are called *mirages*.

The effects of diurnal parallax and refraction are very different, and may easily be distinguished one from the other; as refraction elevates, and parallax depresses;—the first increases and the second diminishes the duration of the visibility of the stars above the horizon. Each is greatest at the horizon, but as the refraction varies nearly as the tangent of the zenith distance, near to the horison it varies very irregularly and with great rapidity, and near the zenith slowly and regularly; on the contrary, near the zenith the variation of the parallax is quickest, and slowest near to the horizon; as the refraction of the sun is greater than his parallax, we enjoy his light longer than if these effects did not exist, on the contrary, the parallax of the moon being greater than the refraction, we enjoy the light of the moon for a shorter time than without these effects. At the horizon refraction diminishes the vertical and horizontal diameters of the sun and moon; the diminution of the latter is insensible, but that of the former is more than 4′; both are nearly insensible when the altitudes are more than ten degrees. Parallax increases both diameters, at the horizon however the quantity is insensible; on the contrary, at the zenith the vertical diameter of the moon is increased a sixtieth part. From the horizontal refraction of the sun being greater than the corresponding diameter, we see the entire disk when it is beneath the horizon, and a spectator at the poles will see the sun two days sooner than if it did not exist.

(x) Let X, X', X'', &c. represent the light in the 1st, 2d, 3d, &c. strata of air, as the same quantity, namely its $\frac{1}{t}$ part, is supposed to be lost in each of those = strata, we have $X-\frac{X}{t}=X'$, $X'-\frac{X'}{t}=X''$, $\because \frac{(t-1)}{i}. X=X'$, $\frac{(t-1)^2}{t}. X=X''$, &c.; hence the logarithms of the intensity of light are $\div$1 to the thickness of the stratum; in fact, ε denoting the intensity of light at any stratum we have, $d\varepsilon=-A\varepsilon.m.\sqrt{dr^2+r^2.dv^2}$, where m denotes the density of the stratum r its radius, and v the zenith distance, $\because \frac{d\varepsilon}{\varepsilon}=-A\rho.\sqrt{dr^2+r^2dv^2}$; now rdv is of the form $\frac{1}{\cos. v}$; when the altitude of the star is greater than $12°$; $\frac{d\epsilon}{\epsilon} = -\frac{A\rho.dr}{\cos. v}$, and $\log. \epsilon = -\int \frac{A\rho.dr}{\cos. v}$. Let E be the value of ϵ in the zenith where $\cos. v = 1$; then we have $\log. \epsilon = \frac{\log. E}{\cos. v}$; $\log. E \backsim (\rho). l$, *i. e.* it is ∷ to the height of the barometer.

From some observations, founded on the preceding analysis, it was inferred, that at the altitude of 25° when the sky is most serene, the sun loses $\frac{1}{2}$ of its light, and at an elevation of 15° it loses $\frac{1}{3}$ of its light.

The continual agitation of the atmosphere produces momentary condensations and dilatations in the particles composing it, which causes the direction of the luminous rays to vary continually from the diversity of refractions which they occasion.—*See* Notes to page 362.

(*a*) In note (*c*) to page 317, it was stated that the height of the shadow was $= \frac{r}{\sin. (s-p)}$; but if the effect of refraction be taken into account, this expression should be $\frac{r}{\sin. (s+2R-p)}$; in like manner, the semidiameter of the section of the shadow $= p + P - s - 2R$; in the first expression, if s denote the distance of the centre of the sun from any point in its disk, it will determine the distance at which this point commences to be seen; if $s = 0$, we have the distance at which the centre of the sun becomes visible by the refraction at the earth's surface, or if s becomes negative, we have the distance at which points of the disk at the other side of the centre become visible; in like manner, by determining the value of s from the equation $p + P - s - 2R = 0$, we could determine the quantity of the sun's disk visible by refraction to a spectator at the moon, for any given distance from the earth by computing P for this distance, and then determining s from the equation $P + p - s - 2R = 0$; from a computation made under the most unfavourable circumstances, it might be shewn that $\frac{3}{4}$ of the solar disk is visible by means of the earth's atmosphere.

Another effect of refraction was, that in consequence of it, the sun and moon were both so elevated in a total eclipse, as to be both visible at the same time.

BOOK THE SECOND.

CHAPTER I.

The arguments for the earth's rotation, which are detailed in this and the third chapter, may be reduced to the five following:—1st, the internal probability; 2d, the impossibility of the contrary; 3d, the analogy of the other planets; 4th, the compression of the earth, and the diminished lengths of isochronous pendulums as we approach the equator (which may be termed the physical proofs of this motion); 5th, the deviation of falling bodies to the east of the tower from which they are let fall. As shewing the far greater probability of the earth's rotation than that of the celestial bodies in a contrary direction, let us investigate the relative velocities of the earth and fixed star in the two hypotheses; the distance of the nearest fixed star is at least 200,000 radii of the earth's orbit (*see* Notes, page 337), its circumference, which is at least six times greater, is described in twenty-four hours; hence, it is easy to shew that its velocity is at least 6570 times greater than that of light; ∵ the star describes more than 270 milions of leagues, or more than twice the diameter of the earth's orbit in a second; and this velocity must be still greater, for the more distant stars, such as those which compose the milky way; on the conrary, supposing the earth to revolve, a point on the equator describes 5400 leagues in 24 hours, or in one second the sixtieth part of a league, which is a velocity a little greater than that of sound, and at least 4600 millions of times less than the preceding. Besides the motion to which, on the hypothesis of the earth's immobility, all celestial bodies must be subjected in order to explain the precession of the equinoxes, they must be in like manner subjected to another, in order to account for the nutation. Likwise, as all actions are accompanied with a contrary reaction, if the earth exerts a force to retain the celestial bodies in their diurnal paths, an = and contrary force must be exerted by them on the earth. And as the circles described by the stars are not concentric, but rather have their centres all existing in the axis of the earth, the central force should be different for each body; and as they all revolve in the same time, the force, whatever it is, should be greater for the more remote objects, contrary to what is observed in other cases of nature.

As an inhabitant of Jupiter would suppose the heavens to revolve in the time of Jupiter's rotation, so likewise an inhabitant of Saturn would come to the same conclusion for his planet, but one is inconsistent with the other. It is evident from the measurement of degrees, which was explained in the XIV. Chapter, that the earth is flattened at the poles; for a greater space must be traversed in the direction of the meridian near the poles than at the equator, in order to have the same inclination of two plummets.

If the earth be considered an ellipsoid, it is easy to prove that the attraction, or weight of a body, increases as we proceed from the equator to the poles, proportionally to the square of the sine of the latitudes (*see* Vol. II. Chapter VIII.); and if the earth revolves on its lesser axis, the centrifugal force, which is always perpendicular to this axis, makes an angle which is continually more oblique, with the direction of gravity; and it is easy to shew that the part of this force which is efficacious varies very nearly as the square of the cosine of latitude; ∵ the difference between the centrifugal force in equator and any parallel is ∻l to the square of the sine of the latitude; ∵ in consequence of those two causes, the increase of weight from the equator to the poles must be ∻l to the square of the sine of the latitude; and the acceleration of falling bodies must increase in the same proportion, which is confirmed by experiments made with pendulums.—*See* Notes to Chapter II. Book III.

CHAPTER II.

(*a*) If light was progressive and not instantaneous, the last ray which issues from the satellite, at the commencement of the eclipse, or the first which we see at the termination of an eclipse, should strike our eye sooner in opposition, and later in conjunction, than if the eclipse occurred when the planet was at its mean distance from us. If the earth was in repose, a spectator on its surface would see a star in the direction of a ray of light issuing from the star; but if the earth be in motion, it is clear that in order to see the star, his telescope must be inclined to the direction of the first ray of light. If the ray and spectator were in motion in the respective directions of the light coming from the star, and of the direction of the earth's motion, the sensation or impression on the eye will be the same, as if the spectator was supposed to be at rest, and there was impressed on the ray, besides its own motion, that with which the spectator is actuated in a contrary direction, he would then see the star in the direction of the diagonal of a parallelogram, of which the two previously mentioned motions constituted the sides, and the angle which this makes with the primitive direction of the ray of light is the aberration; ∵ if $1 : \mu$ be the ratio of the velocity of the earth to that of light, π the angle of aberration, ϕ the angle of the earth's way, we have $\sin. \pi = \frac{\sin. \phi}{\mu}$, μ is determined by the eclipses of Jupiter's satellites, and consequently for reflected light; however we shall see hereafter in Vol. II. Chapter XII., that the value is precisely the same for the direct light of the stars. Light traverses the diameter of the earth's orbit in 16′, 26″, 4‴; in this time the earth describes an arch = 40″, 5, ∵ velocity of light is to that of the earth as the diameter of a circle to an arc of 40″, 5, or as 2 to that number which expresses 40″, 5, in parts of the radius, ∵ $\frac{1}{\mu} = \frac{\sin. 40'', 5}{2} = \sin. 20''.25$, and $\pi = 20'', 25. \sin. \phi$, ∵ it is a maximum when ϕ is 90 or 270. As the diameter of the earth is 23000 less than that of its orbit, a point on the equator describes in a day a circle whose radius = 1; and in 365,25 days it describes a circle 23000 times greater, ∵ as the velocities are directly as the spaces and inversely as the times, the velocity of the annual motion is $\frac{2300000}{36525}$, or 63 times greater than that of the diurnal motion, and the diurnal aberration at the equator and at its maximum is $\frac{20''}{63}$ at most, *i. e.* less than a third of a second; and for any parallel of latitude χ, the coefficient $\frac{20''}{63}$, must be multiplied by $\cos. \chi$.

(*c*) The aberration of a fixed star takes place in a plane which passes through the star and the tangent to the earth's orbit, and is always in the direction of those parts towards which the earth moves, ∵ if the angle of the earth's way be acute, the star will appear elevated.

In the quadratures of the stars with the sun, relatively to the earth, the aberration is made entirely in the plane of a circle of latitude passing through the star, so that the longitude is not at all affected; in the first quadrature the apparent latitude is β—20″. sin. β, in the last quadrature it is $\beta+20''$. sin. β, and their difference is 40″. sin. β; in the syzygies on the contrary, the plane of the circle of aberration is at right angles to the plane of the circle of latitude, and ∵ the latitude is not at all affected, whereas the longitude is most affected in those cases; hence it appears that the phenomena of aberration do not arise from the annual parallax.—*See* Notes to page 234. If a plane be conceived to pass through the star parallel to the plane of the earth's orbit, and if a line be drawn from the star parallel to a tangent at the earth, which may be to the stars' distance as the velocity of the earth to that of light, the star will always appear at the extremity of such line, and it will appear to trace the curve described by the extremity of this line, but as this line is ∻1 to the velocity of the earth, and ∵ to the perpendicular let fall from empty focus on a tangent to the earth's orbit, it will appear to describe a curve similar to that traced by the intersection of the perpendicular with tangent, which curve is known to be a circle, ∵ a star viewed directly, or in pole of ecliptic, will describe a circle; between the pole and plane of ecliptic it describes an ellipse; and when in plane of ecliptic it describes an arc of a circle; the true place of the star divides the diameter of the circle, as the diameter of earth's orbit is divided by the sun. As the axes majores of the ellipses which the stars appear to describe are the same for them all; the velocity of the light as it emanates from them must be the same.

If λ be the longitude of a star, β its latitude, $\odot$ the longitude of the sun, the aberration in longitude is $-\frac{a. \cos. (\odot-\lambda)}{\cos. \beta}$, and the aberration in latitude $= a.$ $\sin. (\odot-\lambda). \sin. \beta$; the aberration in right ascension =

$-\frac{b.\cos.(\rho-\lambda)-c.\cos.(\rho+\lambda)}{\cos.\delta}$, the aberration in declination $=\sin.\delta.\ b.\sin.(\rho-\lambda)-c.\sin.(\rho+\lambda)-8''.\cos.\beta.\cos.\delta$ (*see* Cagnoli, 1529;) hence we see that the aberration in longitude for a given star is a maximum when $\odot-\lambda$ is 0, or 180°, in which case the aberration in latitude vanishes, ∴ it cannot arise from the parallax of the annual orb. In general the longitudes increase if $\odot-\lambda$ is between 90° and 270, and diminish in the first and last quadrants; the latitudes, whether northern or southern, diminish or increase according as $\odot-\lambda$ is ∠ or > than 180. The greatest difference between the latitudes of a star arising from aberration $=2a.\sin.\beta$, the greatest difference of longitude $=2a.\sec.\beta$, this increases to infinity for stars situated near the pole of the ecliptic.

The coefficient of aberration might be determined, *a priori*, suppose the change of declination in a star existing in the solstitial colure produced by aberration, be observed; in this case $\sin.\rho=1$, $\cos.\rho=0$, $\odot=0$ at the vernal and 180 at the autumnal equinox, ∴ the aberration at the vernal equinox $=a.\sin.(\delta-\varepsilon)$, and at the autumnal, the aberration $=-a.\sin.(\delta-\varepsilon)$, ∴ the entire difference $D=2a.\sin.(\delta-\varepsilon)$, and $a=\frac{D}{2.\sin.(\delta-\varepsilon)}$.

With respect to the coefficient a, as the motion of the ray of light is accelerated by the action of the transparent bodies, namely, the atmosphere, the object glass of the telescope and humours of the eye, which it must traverse before it reaches the retina, it follows that the value of a is not the velocity of the ray when it enters our atmosphere, but rather the velocity of the ray when it reaches the retina. However, be the quantity of this acceleration ever so great, since from the most accurate observation it appears that the *quantity* of aberration is not increased in consequence of the increased velocity of the ray, it follows that these bodies must also impart to light a velocity in the direction of the earth's motion $\div$1 to the increase of velocity which they produce.—*See* note (*b*), Chap. II. Book I.

The motion of the planet about the earth in the time in which light comes from the planet to the earth is the whole aberration; $\because$ if 1 : *r* represent the ratio of the sun's distance from the earth to the planet's distance from earth, we have $8', 7''. r$ for the time light takes to come from planet to earth, and if *m* be the diurnal motion of the planet we have the aberration of the planet $= \frac{8', 7''. r.m}{24^h}$; for the sun the aberration is nearly constant, in order to get the true place we should add $20'', 25$ to the place, as given in the tables.

As it is very probable that our planetary system has a motion in space, there must result from it an aberration in the fixed stars, which depends on their situation with respect to the path described by the system; however as the direction of this translation, and also its velocity are unknown, the aberration which results from it is confounded with that arising from the *proper motions* of the stars, so that the coefficient *a* does not arise *solely* from the velocity of light, combined with the motion of the earth.

Since the distances and magnitudes of all the bodies composing the planetary system are determined relatively to the distance of the earth from the sun, it is of the last consequence that this base should be determined as accurately as possible; this is the reason why the problem of finding the distance of the sun from the earth has occupied so much of the attention of astronomers.

If the annual parallax amounted to $6''$, in a triangle of which the vertix is the angle at the star = to $3''$, and whose subtense is half the diameter of the earth's orbit, the distance of the star will be expressed by 212,207, the radius of the earth's orbit being unity; and as the radius is 24,096 times the semidiameter of the earth, the distance of the star from the earth = 5113339872 terrestrial radii, *i. e.* more than five trillions of leagues.

CHAPTER III.

(*a*) As the top and bottom of the tower are supposed to describe, during the fall, similar arcs, and as the body when it arrives at the ground is as far from its first position, as the top of the tower is from its first position. (If the experiment be supposed to be instituted at the equator, and in a vacuo) we have from similar triangles, dividendo, the deviation to the east = to the height of the tower multiplied into the arc described by the bottom, and divided by the radius of the equator, but as the earth revolves uniformly, the arc described varies as the time, *i. e.* as the square root of the height, $\because$ the deviation varies as $h \times h^{\frac{1}{2}}$, *i. e.* as $h^{\frac{3}{2}}$, in any latitude ψ the arc described is to the arc described at the equator as cos. ψ : 1 —Mechanique Celeste livre 10, chap. 5.

(*b*) See Notes to Chapter I.

(*c*) This is called the motion of translation; it supposes that each element of the earth has a motion = and parallel to that of the centre, and consequently that the resultant of all the motions is equal to the sum of the motions of the elements. And as all the particles or elements are equally affected by this motion of translation, it cannot affect the rotation of the whole about an axis. The double motion of the earth may result from one sole impulse. The axis of the earth's rotation is not strictly speaking always parallel to itself, for the phenomena of precession and nutation arise from slow motions in the equator, which necessarily implies a motion in the axis.—*See* Note (*d*) to page 280.

(*d*) On the supposition that the earth was immoveable, the change of seasons and different lengths of days were produced by the sun, ascending or descending from one tropic to another; on the hypothesis of the earth revolving on its axis, it presents itself to the sun under different aspects in different parts of its orbit; in both cases, the different lengths of the day and of the seasons, depend on the latitude of the place and declination of either the sun or earth; one of those being = and of a contrary denomination with the other.

(*e*) The orbits being supposed to be circular, or the velocity being that of a planet at its mean distance, we have $\frac{r}{p^2} = \frac{v^2}{r}$, but p^2 is as r^3, $\therefore v^2 \backsim \frac{1}{r}$, or $v \backsim \frac{1}{\sqrt{r}}$.

(*f*) For in this case the motion being directed either from or towards the earth, it is evident the planet will appear relatively to the earth to be stationary.

(*g*) If lines be supposed to be drawn from different points of the earth's orbit to a star situated in the pole of the ecliptic, they will constitute a conical surface, of which the summit is the star, and the base the orbit of the earth, and the production of this surface beyond the summit, will form another cone opposite to the first, the intersection of which with the celestial sphere will be an ellipse, in the circumference of which the star will always appear diametrically opposite to the earth, in the continuation of a ray drawn from it to the summit of the cone; this circumstance sufficiently distinguishes the effect of annual parallax from that of aberration, which affects the apparent position of the star perpendicularly to the radius of the earth's orbit and not in its direction; the centre of the ellipse is the true place of the star, its greater axis = the parallax, and the minor = the parallax × into the sine

sine of the stars latitude, and it exists in the plane of a circle of latitude passing through the pole; this ellipse is therefore different from that which is described in consequence of aberration; however though the two causes act at once, it would not be difficult to prove that a star under the influence of both would still appear to describe an ellipse about its true place.

Let $c =$ rad. of the earth's orbit, b the distance of star from plane of the ecliptic; $a, a' \doteq$ the curate distances of the star from the sun, and earth; β, β', the heliocentric and geocentric latitudes of the star, α the distance of earth from syzygies, e distance of star from sun; tan. $\beta = \frac{b}{a} = m$, and $a' = \overline{a^2 + c^2 + 2ac. \cos. \alpha}|^{\frac{1}{2}}$, let $\frac{c}{a} = n$, $\frac{c}{e} = p$,

then tan, $\beta' = \frac{b}{a'} = \frac{m}{\sqrt{1 + 2n. \cos. \alpha + n^2}} =$ (neglecting n^2 which is inconsiderable) $m.(1 - n. \cos. \alpha)$, $\because$ tan. $(\beta - \beta')$ $= \frac{mn. \cos. \alpha}{1 + m^2.(1 - n. \cos. \alpha)}$, *i. e*, $\beta - \beta' = \frac{m.n. \cos. \alpha}{1 + m^2} = n.$ $\cos. \alpha. \sin. \beta. \cos. \beta$, $\because$ as $\cos. \beta = \frac{a}{e}$, and $p = n. \cos. \beta$, $\beta - \beta'$ the parallax in latitude $= p. \cos. \alpha. \sin. \beta$; note p is the semidiameter of the orbit of the earth as seen from the star, and $\because$ it is = to the annual parallax. The tangent of the angle formed by lines drawn from projection of star on the plane of the ecliptic to sun and earth, or the parallax in longitude $= \frac{c. \sin. \alpha}{a + c. \cos. \alpha} =$ $\frac{p. \sin. \alpha}{\cos. \beta + p. \cos. \alpha}$, *i. e.* the parallax in longitude $= \lambda' =$ $p. \sin. \alpha. \sec. \beta.$ very nearly; consequently, $\lambda' : \beta - \beta' ::$ tan. α : sin. β. cos. β :: 2 tan. α : sin. 2β, hence we can determine the one from the other; $\beta - \beta'$ vanishes in the quadratures, *i. e.* when a $\alpha = 90$ or 270; it is a maximum in the syzygies; in this particular it differs from the aberration; its maximum being p. sin. β, it is greatest near to the pole of the ecliptic, $\beta - \beta$, is positive, or the apparent latitude is less than the true from the last quadrature through conjunction to the first quadrature; in the other half of the orbit it is negative, or the apparent latitude is greater than true, λ' vanishes in the syzygies, and it is a maximum and $= p$. sec. β, in the quadratures, it $\because$ increases with the latitude, and from conjunction to opposition it is positive, or the apparent latitude is greater than true, and from opposition to conjunction it is less than true; the apparent latitude in opposition $= m.(1 - n$ cos. $a)$, and is a minimum; it is $= m.(1 + n.$ cos. $a)$ in conjunction, when it is a maximum.

If $\triangle$ be the difference between the longitude of a star in the 90th and 270th degrees of distance from conjunction, we have $\triangle = 2p$. sec. β, $\because p = \frac{\triangle}{2}$. cos. β.

If $p = 20'' = a$, the same tables would serve for parallax and aberration, if they are computed for the aberration it is only necessary to add 90° to the sun's place.

CHAPTER IV.

(*a*) The locus of a planet and consequently its orbit, which is composed of all its points, is determined by the magnitude of the radius vector and by the angle which it makes with some line fixed in space, such is that drawn to the first point of Aries. With respect to the *direction* of the radius vector, this is found by observing the planet in opposition or conjunction; for in this case, on account of the irrationality which exists between the period of the earth and planet, they occur in different points of the orbit, consequently, we can by means of oppositions and conjunctions, find all the points of the orbit, and also the epoch, when the planets are in those positions; hence, may be obtained the law which exists between the heliocentrick longitude and time, from which may be derived the true longitude; and as the principal inequalities are destroyed at the termination of each revolution, this law may be developed in a series, proceeding according to the sines of angles ∻ to the time and their multiples; the coefficients of this series may be determined by observations made under the most favourable circumstances.

(*b*) See Notes to page 12.

(*c*) In order to determine the magnitude of the radius vector, the observations made at quadratures are the most useful, for the radius being then perpendicular to the visible ray, it appears under the greatest angle; and as the quadratures occur in every point of the orbit, the law between the time and radius vector, and ∴ between this last and the longitude can be determined, ∴ the orbit can be completely constructed; in case of an inferior planet, the greatest elongations are employed in place of the quadratures to determine the radii vectores.

(*d*) Let ψ be the arc described about the sun, r the distance of planet from sun, then the angular velocity $= \frac{d\psi}{r}$ is observed to be equal to $\frac{A}{r^2}$, $\therefore r.d\psi =$ twice the sector described in an indefinitely short period of time $\doteq$ A.

To completely determine the orbit of a planet, 1st, the plane in which it moves—2dly, the nature of the curve described—3dly, the position of this curve in the plane of its orbit, aad 4thly, the law according to which this curve is described, must be determined; the law is given by the application of Kepler's 2d law, the position by that of its greater axis; the species of the curve by Kepler's 1st law; the particular form by the excentricity, the *magnitude* of the axis, by the revolution or mean motion—which last, as determined by a comparison of ancient and modern observations is the best known of all the elements.

(*f*) The period may be found by noting the time between two returns of the planet to the same node, and this interval being divided by the number of revolutions, will give the period with respect to the node; but as this node regrades, the period thus deduced will be less than the true period; however P may be easily computed from knowing the quantity of regression a, for if n be the number of revolutions, we have $n.360—a : 360 ::$ observed time : P. The period may be also found from the formula given in Notes, page 323, for $P = \frac{p.t}{p+t}$; (t being the time between two conjunctions and oppositions). The axis major or mean distance can be determined by means of Kepler's third law; the earth's orbit and period being accurately known already. To determine the excentricity, let the heliocentric positions of the planet, when the equation of the centre is observed to be a maximum, *i. e.* when the planet is moving with its mean angular velocity, be determined; the mean places of the planet at these epochs can be determined, and they always lie between the perihelion and the true places; ∵ the angle at the sun formed by lines drawn to the true places are given by observation, and the time between the two observations gives the angle at the sun formed by lines drawn to the mean places; the difference between these angles = twice the greatest equation; as the points when the true and mean motions are the same, are not exactly known, among a great number of observations, those two should be selected which give the difference between the preceding angles the greatest possible, we may then assume their difference equal to twice the greatest equation, as near to the maximum, this variation is inconsiderable.

The excentricity is given, from the greatest equation, by means of the series—(*see* Notes to page 14.)

$e = \frac{1}{2} h - \frac{11}{768} h^3 - \frac{587}{983040} h^3$, &c. e represents the excentricity; $h = \frac{g}{57.29578}$ g expressing the greatest equation of centre.

If the planet be observed near the aphelion, the difference between angle proportional to the interval from the planets being in the point where the equation of the centre is a maximum, to the time when the planet is in aphelion, and the angle between axis major and line drawn to this point, should be = to the greatest equation, as it is next to impossible that this should be accurately the case, let is be less by a small angle c, and make a second observation when it is greater by an angle c'; now as the longitudes of the planet when observed at each side of the aphelion, and ∵ their difference e are known, and also t the interval between the observations, we have when the angles are very small, $c + c' : e :: c$ to the angular distance of the first assumed point which is known, from the aphelion, we have also $q.p : c+c' : c :: t$: to the *time* from this point to aphelion, which ∵ determines its epoch; ∵ we can obtain the longitude at any epoch, or *vice versa*.

Let L, l represent the heliocentric longitudes of the sun and node, S the angle at the sun subtended by the earth and planet = L—l, E the elongation of planet from sun = difference between the geocentric longitudes of sun and planet; then r the planet's distance from sun : R the earth's distance sin. E; sin. (S + E), ∵ r. sin. (S + E) = R. sin. E; *i. e*, r. sin. (E+L—l) = R. sin. E; let E′, R′, L′, be the values of E, L, R, when the planet returns again to the node, then r. sin. (E′+L′—l') = R′. sin. E′,

$$\therefore \frac{\sin.(E' + L' - l') + \sin.(E + L - l)}{\sin.(E' + L' - l') - \sin.(E + L - l)} = \frac{(R'.\sin.E' + R.\sin.E)}{R'.\sin.E' - R.\sin.E}, \therefore$$

$$\frac{\tan.(\frac{1}{2}(E' + E + L' + L) - l)}{\tan.\frac{1}{2}(E' - E + L' - L)} = \frac{R'.\sin.E' + R.\sin E}{R'.\sin.E' - R.\sin.E};$$

hence as R′, R, E′, E, L′, L, can be determined, we can find l the longitude of the node; this method supposes the planet to be in its node, if not, let β, β', be the geocentric latitudes of the planet before and after its passage through the node, t the interval between the observations, then $\beta + \beta' : \beta' : t$ to the interval between the first observation and the time when the planet is in the node; hence we can find E and L when the planet is in the node. This method supposes also that the node is stationary, which is not the case, (*see* Chapter III. Vol. II.) However a determination of the node in this manner will give the motion of the node, by means of which l can be determined accurately; the inclination i is easily determined, for we have $\sin. E = \tan. \beta. \cot. i$.

The preceding methods not being rigorously exact, the elements determined by means of them will be found to differ somewhat from the truth; their values should be corrected by the formation of equations of condition, of which the number is indeed indeterminate; it is only necessary to have as many of them as there are unknown quantities to be determined.

(f) On the secular inequalities *see* Notes to Chapter II. Vol. II.

(g) The reader is likewise referred to Chapter II. for an explanation of the variation in Jupiter's and Saturn's motions.

An inspection of the axes majores, or mean distances of the ancient planets, shews that, with one exception, their distances are embraced in the formula $4 + 3.2^{n-2}$, (n being the place occupied by the planet; commencing with mercury, however a blank occurred between Mars and Jupiter; and in order to have the preceding law exact, a planet should exist at the distance where the four new ones have been observed. The circumstance of there being four instead of one planet at this distance, does not militate against the preceding law, as from some circumstances connected with them it has been conjectured that these might originally have constituted but one planet.—*See* Notes, page 333, and Vol. II. Chapter II.

More particularly, the causes which disturb the motions of the four new planets arise from their orbits mutually intersecting each other, from their comparatively great excentricities, and from the proximity of Jupiter, the greatest of all planets.

CHAPTER V.

(*a*) Let a, b, represent the major and minor semiaxes of the ellipse A, P the periodic time, s the sector described in any time t, a', b', A′, P′, s', corresponding quantities for another ellipse, then since the areas are ÷l to the times, we have $s = \frac{A.t}{P}$, and $s : s' :: \frac{A}{P} : \frac{A'}{P'}$ but $A = ab$, $A' = a'.\, b'$, and $P = Ka^{\frac{3}{2}}$, $\therefore s : s' :: \frac{a.b}{K.a^{\frac{3}{2}}}$ $\frac{a'.\, b'}{K.a'^{\frac{3}{2}}}$.

(*b*) Let x be the perihelion distance, and we have $b^2 = x.(2a - x)$, $\therefore s : s' :: \frac{\sqrt{x.(2a-x)}}{\sqrt{a}} : \sqrt{x}$; the ellipse A′

being supposed to become a circle of which the rad. $=x$; $s : s' :: \sqrt{2a-x} : \sqrt{a}$, which when the ellipse A becomes a parabola, in which case, x vanishes relatively to a, the proportion becomes that of $\sqrt{2} : 1$; the ratio of the sector described by the fictitious planet to the synchronous sector described by the earth at a distance from the sun equal to r, is that of $\sqrt{x} : \sqrt{r}$; ∵ we can determine for any instant whatever the area traced by the radius vector of the comet, commencing with the instant of its passage through the perihelion.

The time t of describing a sector $s = \frac{P.s}{A} \backsim \frac{a^{\frac{3}{2}}.s}{a.b} \backsim \frac{s}{\sqrt{p}}$, p being the parameter; hence it appears that the times in different sectors, are as the sectors described divided by the square root of the parameters.

(*c*) In orbits of great excentricity, such as the comets, the equation $r = \frac{a.(1-e^2)}{1+e.\cos. v}$, may, by substituting $1-\alpha$ for e, be made to assume the form

$$\frac{D}{\cos.^2\frac{1}{2}v.\left(1+\frac{\alpha}{2-\alpha}.\tan.^2\frac{1}{2}v.\right)},$$

(D being the perihelion distance.) For as $\cos.^2\frac{1}{2}v + \sin.\frac{1}{2}v = 1$, and as $\cos. v = \cos.^2\frac{1}{2}v - \sin.^2\frac{1}{2}v$, we have $r =$

$$\frac{a.\alpha.(2-\alpha)}{\cos.^2\frac{1}{2}v + \sin.^2\frac{1}{2}v + (1-\alpha).(\cos.^2\frac{1}{2}v - \sin.^2\frac{1}{2}v)}$$

$=$ by concinnating and dividing by $(2-\alpha)$,

$$\frac{a.\alpha}{\cos.^2\frac{1}{2}v \frac{+\alpha}{2-\alpha} \sin.^2\frac{1}{2}.v.}$$, and as $D = a.(1-e)$

$=a.\alpha$, we ∵ obtain $r = \frac{D}{\cos.^2\frac{1}{2}v.\left(1+\frac{\alpha}{2-\alpha}.\tan.^2\frac{1}{2}v\right)}$, by expanding this expression into a series we obtain r to any degree of accuracy; if a vanished the expression would become $\frac{D}{\cos.^2\frac{1}{2}v}$; the time corresponding to the true anomaly in an orbit, such as the preceding, may be likewise found, for as $u = 2\tan.\frac{1}{2}u.(1-\frac{1}{3}.\tan.^2\frac{1}{2}u+\frac{1}{5}.\tan.^4\frac{1}{2}u$, &c.) and as $\tan.\frac{1}{2}u = \frac{\sqrt{1-e}}{\sqrt{1+e}}.\tan.\frac{1}{2}v = \frac{\sqrt{a}}{\sqrt{2-a}}.\tan.\frac{1}{2}v$, by substituting we obtain $u = \frac{\sqrt{a}}{\sqrt{2-a}}.\tan.\frac{1}{2}v.\left(1-\frac{1}{3}\left(\frac{a}{2-a}\right).\tan.^2\frac{1}{2}v+\frac{1}{5}.\left(\frac{a}{2-a}\right)^2.\tan.\frac{1}{2}v-\&c.\right)$; but

$$\sin.u = \frac{2\tan.\frac{1}{2}u}{1+\tan.^2\frac{1}{2}u} = 2\tan.\frac{1}{2}u.(1-\tan.^2\frac{1}{2}u+\tan.^4\frac{1}{2}u,\&c.),$$

$$e.\sin.u = 2(1-a).\frac{\sqrt{a}}{\sqrt{2-a}}.\tan.\frac{1}{2}v.\left(1-\frac{a}{2-a}.\tan.^2\frac{1}{2}v+\left(\frac{a}{2-a}\right)^2.\tan.^4\frac{1}{2}v,\ \&c.\right).$$

∵ in the equation $nt = u - e.\sin.u$, the substitution of these values of u and of $e.\sin.u$ will give t in a very converging series, in a function of the anomaly v, and $= \frac{1}{n}.\tan.\frac{1}{2}v.\left(1+\frac{(\frac{2}{3}-a)}{2-a}.\tan.^2\frac{1}{2}v-\frac{\frac{4}{5}-a}{(2-a)^2}.a.\tan.^4\frac{1}{2}v+\&c.\right)$, which when $a = 0$, $\frac{1}{n}.(\tan.\frac{1}{2}v+\frac{1}{3}.\tan.^3\frac{1}{2}v.)$

If $v = 90$, then $\tan.\frac{1}{2}v+\frac{1}{3}.\tan.^3\frac{1}{2}v = \frac{4}{3}$; and t' the time corresponding to this anomaly $= \frac{4}{3.n} = 109^d, 6154$, when $D=1$; ∵ a comet, of which the perihelion distance $=1$, will describe in this time a sector of which the anomaly is 90, *i. e.* it will reach the parameter in that time, ∵ for any other anomaly u, we can obtain the corresponding time; the determination of the anomaly from knowing the time is more difficult than the reverse problem, for u must be determined by an equation of the third degree.

Note.—This is called the comet of 109 days, and for any time t' we have $\tan.\ \frac{1}{2}v + 3.\ \tan.\ ^{3}\frac{1}{2}v = \frac{t'}{27{,}40385}$, and for that of which the perihelion distance $= x$, $\tan.\ \frac{1}{2}v + 3.\ \tan.\ ^{3}\frac{1}{2}v = \frac{t}{27{,}40385x^{\frac{3}{2}}}$; hence, if the comets move in parabolas, their anomalies depend only on their perihelion distance.

The formula for determining the time of describing *any* arc intercepted between the radii vectoris r, r'. is $T = \frac{T}{12\pi}.\ ((r + r' + c)^{\frac{3}{2}} \pm (r + r' - c)^{\frac{3}{2}})$.—*See* Celestial Mechanics, Book II. Chapter IV.

(*c*) In consequence of the smallness of the diameter of a comet, and the feebleness of its light, it does not become visible until it approaches very near to the sun, so that the greater number of comets which have been observed, appear nearer than Mercury, shortly after their distances become so great that they cease to be seen ; ∵ their orbits are extremely excentric ellipses, in which particular they differ from the planetary orbits, and likewise in the circumstance that they are inclined at every species of angle to the ecliptic, from which it follows, that their motions are sometimes retrograde; though they receive their light from the sun, their disk is not so accurately terminated as the planetary disks, nor are there any apparent phases; indeed the side averted from the sun appears to be luminous likewise.

The great inclination to the ecliptic is not a distinguishing property of comets, neither is the feebleness of their light or the smallness of their masses, as in all these particulars they do not differ from the planets recently discovered.

The method of determining the elements of the planetary orbits is not applicable to comets which are visible only in a small portion of its orbits; ∵ the most important elements, namely, the mean distance and the mear motion cannot be thus determined, it is necessary, in order to obtain them, to avail ourselves of Kepler's laws.

In the methods made use of for determining the planetary orbits, it is assumed that the planet has been observed more than once in the same point of its orbit, from which the periodic term and distance from the sun can be determined. The sun being assumed to be in the focus of the ellipse or parabola, which the comet is supposed to describe, if the comet be observed in three different positions from three corresponding points of the earth, in the triangle formed by lines joining the sun, earth, and comet; we only know the angle of elongation at the earth, and the distance of the earth from the sun, which is not enough; however, in the two triangles formed by lines drawn from the sun to the observed places of the comet, we have not only the *ratio* of their areas from knowing the times between the respective observations, but also the areas themselves, the conic section described being supposed to be known, and by combining these data we can determine the orbit.

In this determination an indirect method is generally employed as less complicated, and as more exact than the direct determination of the elements, on account of the errors of observation. In this way two of the unknown quantities are assumed arbitrarily, by combining them so as to satisfy one of the observations, with those elements, the other observations are calculated hypothetically, and then a comparison of the computation with the observations will indicate the correction required for the elements. Now, as the great excentricity of the orbit justifies us in assuming that the orbit is q. p. parabolic; there is also this peculiar advantage in assuming them to be such, namely, that the ::nality of the areas to the times is reduced to the quadrature of the curve, which, in the case of the parabola is extremely simple; besides as all parabolas are similar curves, we can compute a general table for all orbits.

Several indirect methods have been proposed for determining the cometary orbits on the parabolic hypothesis, and they only differ from each other in the elements which are supposed to be known. The following is a brief outline of the method which supposes the angle at the sun to be known.

By a comparison of two geocentrick positions reduced to the ecliptic, and by assuming the corresponding angles of commutation arbitrarily, we can compute by means of *these* angles, and of the given elongations and distances of the earth from the sun at the times of the two observations, the curtate distances of the comet from sun at these times, and also the heliocentric movement on the ecliptic, or the angle contained between these distances; from knowing the angles at the earth and sun, and also the geocentric latitudes, we can determine the heliocentric latitudes, and also the true distances of comet from sun at the times of observation. With the heliocentric latitudes and longitudes we can determine the inclination, the position of the node, and the longitudes on the orbit; ∵ we have two radii vectores, and the angle contained between them; hence we can determine from the nature of the parabola, the perihelion distance, the longitude of the perihelion, the area of the sector contained between the radii, the time employed in moving from perihelion to the observed places, from which we can determine the instant of the passage through the perihelion; if the time *computed* for passing from one observed position to the other, does not agree with the time elapsed between the two observations, the *assumed* angles of commutation do not take place simultaneously; ∵ one should be changed until the computed time agrees with the observed, the other remaining the same; now all the elements of the orbit being determined, we can calculate for the time of the third observation, the true anomaly conformably to the parabolic hypothesis, and consequently the longitude on the orbit and the distance from the sun at this time, then from knowing the position of the node, and the inclination, the heliocentric longitude and latitude, and also the curtate distance of the third point from the sun may be determined; the longitude of this point and of the earth at the time of the third observation, will make known the angle of commutation at this time. Knowing this angle, and the distances of the earth and comet from the sun, we can compute the angle of elongation, which ought to be equal to the observed angle; likewise the first angle of commutation is also erroneous, ∵ by assigning another value to it, the second commutation will be changed until the first and second observations agree with the computation; we should operate on the third observation in the preceding manner, and if it does not agree with the computation, the first angle of commutation should be again changed. After thus making two hypotheses for the first angle of commutation, their errors will indicate by the method of interpolations the correction to be applied to this angle, in order that the hypothesis should satisfy the three observations. With those elements we can reduce any observation to its heliocentric position, from which it is easy to calculate with the true anomaly the time of any observation, which enables us to verify the elements by all the observations which have been made, and to correct them by taking the mean.

The element which in the case of the planets is the first and easiest to be determined, namely, the periodic time, is in the case of the comets the last and most difficult, and cannot be found except by a computation on the hypothesis that the orbit is elliptical.—*See* Celestial Mechanics, Book II. Chapter IV. and Delambre, tom. III. Chapter XXXIII.

If, as stated in page 197, the elements of a comet nearly agree with those of a comet formerly observed, we can apply the calculus of probabilities to determine to what degree of probability we can be sure that they are exactly the same.

(*d*) The heat of the sun is as the density of his rays, *i. e.* inversely as the square of the distance; now the heat of boiling water is three times greater than that produced by the action of the sun in summer on the earth; and iron heated to a red heat is four times greater than that of boiling water, therefore the heat which a body of the same density as our earth would acquire at the perihelion distance of the comet, is at least 2000 times greater than that of iron heated to a red heat; and it is quite evident that with such a heat, all vaporous exhalations, and in fact every species of volatile matter ought immediately to be dissipated; the preceding is Newton's estimation, *see* Princip. Math. Book III. page 509; he assumes that the comets are compact solid substances like the planets; this he infers from their passing so near to the sun in their perihelion without being dissipated into space.

Heat expands all bodies, but = additions of caloric do not produce equal increments of magnitude, for as it acts by diminishing the cohesive tendency, the greater that tendency the less will be its effect; on the contrary, in the case of gases, as no such tendency exists = increments of heat must necessarily produce equal augmentations of bulk. In general, when the density of bodies is increased they must give out caloric. The quantity given out by water when freezing is 140°, its capacity is by this increased one-ninth; from this it has been inferred, that the zero of temperature is 1260 degrees below the freezing point; but there are great discrepancies in the results from different liquids.

The latent heat of the vapours of fluids, though cor-stant for vapour of the same kind and of a given elasticity, still varies in different vapours; thus, according to a recent investigation, the vapour of water at its boiling point = 967°. However, though this heat is different in different fluids, still the point at which all solid bodies, and all those liquids which are susceptible of ignition, *i. e.* of becoming heated so as to be luminous *per se*, is nearly the same for all, and about 840° of Fahrenheit.

In permanently elastic fluids, the caloric is held so forcibly that no diminution of temperature can separate it from them.

The comet of 1770 is the only one which cannot be computed on the hypothesis that it moves in a parabola.—*See* Vol. II. Chap. IV. Notes.

The nebulosity which environs the comet is its atmosphere, which extends farther than our atmosphere; it increases according as it approaches the sun. The parts which are volatilized become so very light, that the attraction of the comet on them is nearly insensible, so that they yield without difficulty to the impulsion of the solar rays; the orbit described by each particle must be an hyperbola, for previously to the impulsion, as it described a parabola, its velocity is to the velocity in a circle at the same distance as $\sqrt{2} : 1$, and the impulsion of the solar rays increasing this velocity, it will be to the velocity in a circle in a greater ratio than that of $\sqrt{2} : 1$, it must consequently describe an hyperbola.

The tail is generally behind the comet; this is the cause of the curvature which has been observed in it, and also of the deflection towards that part from which the comet is moving.

It has been supposed that the loss sustained by the evaporation near the perihelion may be repaired by new substances which it meets with in its route.—*See* Chapter VI. Book V. Vol. II. Notes.

CHAPTER VI.

The elements of the orbits of the satellites in the order in which they are derived, the one from the other, are the periodic time, or mean motion, the distance from the primary, the inequalities and true motion, the inclination, and nodes, and magnitude.

In determining the period from the interval between two consecutive conjunctions, we obtain it as affected by all the inequalities in the motions of the satellites; but when it is obtained from two conjunctions, separated by a considerable interval from each other, these inequalities are in a great measure compensated. Observations with the micrometer give, as was stated in page 96, the angle which the radius of the orbit subtends at the earth, it must change with the distance of Jupiter from the earth; but as the apparent diameter of Jupiter varies in the same ratio, it is only necessary to measure this diameter at the same time, in order to have the diameter of the orbit relatively to that of Jupiter; and as a comparison of these diameters at different times gives this ratio always the same, it follows that the orbit is q. p. circular. The distances of the satellites might also be inferred from the greatest durations of the eclipses, and *vice versa*. Some of the observed inequalities are only apparent, others are real; if there is a difference in the *periodic* revolutions of the satellites, it must arise from a real inequality in the motion of the satellite; but as the synodic revolution depends on the motion of Jupiter, there may be a difference in the observed synodic revolutions, without there being any inequality in the satellite from which it may have originated. When the computed time of an eclipse is corrected for the inequalities in the motion of Jupiter, and also for the velocity of light, &c., then a comparison of this time, with that furnished by observation, will enable us to discover the real inequalities.

The cause of the deviations from mean motion arise either from the excentricity of the orbits, or from the disturbing action of Jupiter combined with that of the sun: the disturbing action of the satellites on each other depends on their relative positions; its period therefore will be the time at the end of which the satellites return to the same relative position, with respect to the sun; and as the eclipses are the most important observations, and those most commonly made, it is therefore the period in which each satellite makes a complete number of revolutions; but a comparison of the values given in page 208, shews that the shortest period which satisfies these conditions for the *three* first satellites is 437 days. This period is less exact with respect to the fourth satellite, as it performs in 435 days 26 revolutions; however as its actions are less than that of the other satellites, on account both of its greater distance and smaller mass, and as the difference does not exceed one day and a half, it is assumed that even with respect to it, the period is 437 days. Astronomers made use of this period to form empirical equations, for which those founded on the theory of universal gravitation have been substituted. Their arguments are composed of the position of each satellite with respect to the others, the apsides of the third and fourth, and the nodes of their orbits.

The orbits are unquestionably elliptic, however the ellipticity of the two first satellites cannot be observed. The eclipses will be observed sooner when the planet is in its perijove, and later in the apogove, than the computed time, which will enable us to determine the position of the apsides. If there was no *penumbra*, and if the diameter of the satellite was insensible, the duration of the computed and observed eclipses would be the same; but as these causes affect the observed time of commencement, it is evident that it depends on the eye of the spectator, and also on the goodness of the telescope.

The tables are so constructed as to give the eclipses in the mean state of the atmosphere, mean power of the telescope, and mean accuracy of vision; besides what is mentioned in page 155, the proximity of the star to the horizon, its proximity to Jupiter, or Jupiter's too great proximity to the sun; all, or any of these circumstances affect the observations. In order that the results given by stationary observers should agree with those given by voyagers, we should employ only telescopes of a medium magnifying power.

At the extreme distance from the node at which an eclipse can happen, the duration of an eclipse is the *least* possible, and would be always the same if the inclination was constant; but as this duration is variable, for the 1st, 2d, and 3d satellites particularly, it follows that the inclination is likewise variable.

The position of the node will be given from knowing the duration of the longest eclipse; the shortest observed eclipses are at the limit, and will give the inclination; knowing the position of the node and inclination we can compute antecedently the duration of any eclipse.

Calling M, M′, M″, the mean motions of the three first satellites, and l, l', l'', their mean longitudes; we have also $M+2M''=3M'$, and $l+2l''=3l'+180°$; these equations are so exact, that the deviations from them, which are observed, must arise from errors of observations, or from the small oscillations which they make about these mean values, *see* Vol. II. Chap. V.

It follows from this, that these three satellites cannot be simultaneously eclipsed, for then we would have $l=l'=l''$; or $l+2l''=3l'$, which, in consequence of the second equation, is impossible; and it appears from the first equation, that if the first is true once, it will be always so; it likewise follows, that the real inequalities of those three satellites must have precisely the same laws and periods.

The method alluded to in page 328 would evidently give a diameter, as seen from Jupiter, smaller than the actual magnitude. It has been suggested, that if in geocentric conjunctions of the satellites with Jupiter, the instants of interior and exterior contact with Jupiter were observed at immersion and emersion, we would have the time which the planet takes to describe a chord equal to its diameter; this will give the ratio of the diameter of the satellite to that of Jupiter, if that observation in which the ratio of the duration of the passage to that of the immersion is the greatest possible, be observed.

BOOK THE THIRD.

CHAPTER I.

(*a*) THAT which admits of the introduction of a finite body has been called *space*; it is said to be pure if it be totally devoid of matter. Whether there be such a thing as any space absolutely pure has been disputed, but that such a space is *possible*, admits of no dispute; for if any body be annihilated, and all surrounding bodies kept from rushing into the space which this body occupied, that portion of space, with respect to matter, would be pure space. Pure space is therefore conceivable, and it is conceived as having length, breadth, and depth. In the notion of motion, as announced in the text, the author assumes that there would be motion even though all the other bodies in the converse were annihilated, but this position is not acceded to by all philosophers. Berkeley, for instance, thought that all motion was relative; however, though with respect to the origin of our ideas of motion, his account is unanswerable; nevertheless it must be admitted, that a body might spontaneously produce motion in itself; still we may venture to affirm with him, that as long as the body would remain in absolute solitude it would not acquire the idea of motion; but if other bodies be called into existence, while the body is under the influence of its own spontaneous energy, it certainly would then acquire the idea of motion, from perceiving its change of place with respect to those bodies; but as this creation of bodies at a distance could produce no real alteration in the condition of a body which existed before them, if the body *now* perceives itself to be moving, we may conclude that it was moving previously to the existence of those bodies, and that its motion was *absolute.*

(*b*) All cases of the equilibrium of forces acting on a material point, may ultimately be reduced to that of two equal and opposite forces, as when any number of forces acting on the same point constitute an equilibrium, all of them but one may be reduced to a force equal and contrary to this one, so that these forces are always as the sides of a polygon, having the same number of sides drawn parallel to their directions. (Note, the sides of the polygon are not necessarily in the same plane.)

If three forces acting on a material point constitute an equilibrium, they must exist in the *same* plane; four forces acting in *different* planes constitute an equilibrium, when they are as the three sides and diagonal of a parallelopiped respectively parallel to their directions. If two equal and parallel forces act in opposite directions, an equilibrium between them cannot be effected by the introduction of any third force.

(*c*) It is evident from this, that in the composition of forces, force is expended—in the resolution force is gained. The two given forces into which the given one is resolved are reciprocally as perpendiculars from the given force on the directions of its components. The less the angle made by the components, the greater will be the resultant, therefore it is a maximum when this angle $= 0$, *i. e.* when the components are parallel; in this case it is easy to prove that the resultant $=$ the sum of the components, and that its point of application divides the line connecting them inversely as the forces.

(*d*) Any force being resolved into three others, at right angles to each other, as stated in page 225, the line representing it will be the diagonal of a rectangular parallelopiped, of which the composing forces represent the sides, $\therefore$ A, B, C, representing the composing forces, $\sqrt{A^2+B^2+C^2}$ will represent the resultant or diagonal; and

$$\frac{A}{\sqrt{A^2+B^2+C^2}}, \frac{B}{\sqrt{A^2+B^2+C^2}}, \frac{C}{\sqrt{A^2+B^2+C^2}}, =$$

the cosines of the angles which A, B, C respectively make with $\sqrt{A^2+B^2+C^2}$ it is also evident that the sum of their squares $=1$; if A', B', C' be the components of a second force parallel to the same rectangular coordinates, the coordinates of the resultant of $\sqrt{A^2+B^2+C^2}=S$, and of $\sqrt{A'^2+B'^2+C'^2}=S'$, are $A+A'$, $B+B'$, $C+C'$, respectively, therefore as these are the coordinates of the diagonal of a parallelogram whose sides $=\sqrt{A^2+B^2+C^2}$ $\sqrt{A'^2+B'^2+C'^2}$, this diagonal must be the resultant of the given forces S, S', and if the angle between their directions$=\Delta$, we have $S^2+S'^2-2\,S.S'.\cos.\,\Delta=(S.\cos.a-S'\cos.b)^2+(S.\cos.a'-S'.\cos.b')^2+(S.\cos.a''-S'.\cos.b'')^2$ $=S^2+S'^2-2SS'.(\cos.a.\cos.b+\cos.a'.\cos.b'+\cos.a''\cos.b'')$, therefore $\cos.\,\Delta=\cos.a.\cos.b+\cos.a'.\cos.b'+\cos.a''.\cos.b''$.

Note a, a', a'', b, b', b'', are the angles made by S, S' with the rectangular coordinates. The value of $\cos.\,\Delta=0$, when S, S' are at right angles to each other; as $A+A'$, $B+B'$, $C+C'$, are the coordinates of the resultant of S and S', $A+A'+A''$, $B+B'+B''$, $C+C'+C''$, are the coordinates of V, the resultant of S'', and this last resultant, $\therefore$ V will be as stated in the text, the diagonal of a parallelopiped, whose sides are $A+A'+A''+$ &c.; $B+B'+B''+$ &c., $C+C'+C''$ &c.; $V^2=(A+A'+A''+\text{\&c.})^2+(B+B'+B''\text{\&c.})^2+(C+C'+C''$

$+$ &c.)^2 and if $m\ n\ p$ be the angles which V makes with the axes, we have $\cos. m = \frac{A+A'+A''+\ \&c.}{V}$, $\cos. n = \frac{B+B'+B''+\&c.}{V}$, $\cos. p = \frac{C+C'+C''+\&c.}{V}$, $\because$ we have both the quantity and direction of the resultant.

The coordinates of the origin of the force S being supposed to be A B C, if $x\ y\ z$ be the coordinates of its point of application to the given point, the distance of the point of application from the origin, $= s =$ $\sqrt{(x-A)^2+(y-B)^2+(z-C)^2}$ $\because$ the force resolved parallel to the coordinates $= S\frac{(x-A)}{s}$, $S\frac{(y-B)}{s}$, $S.\frac{(z-C)}{s}$ $= \left(\text{as } \delta x = \frac{\delta s}{\delta x}.\, \delta x + \frac{\delta s}{\delta y}.\ \delta y + \frac{\delta s}{\delta z}.\ \delta z.\right)$ $S.\frac{\delta s}{\delta x}$, $S.\frac{\delta s}{\delta y}$, $S.\frac{\delta s}{\delta z}$, respectively, in like manner for a second or third force S′, S″, $S'.\frac{\delta s'}{\delta x}$ $S'.\frac{\delta y'}{\delta y}$, or $S.''\frac{\delta s''}{\delta x}$, $S.''\frac{\delta s''}{\delta y}$ &c. are the forces S′ S,″ parallel to x, y &c. $\because \Sigma. S. \frac{\delta s}{\delta x}$ is the sum of all the forces S S′ S″, resolved parallel to x; now if u be the distance of V the resultant of all the forces S, S′, S″, &c. from the given point, $V\frac{\delta u}{\delta x}$ will express the resultant resolved parallel to x, and as by what has been already established, this is equal to the sum of the composing forces parallel to x, we have $V.\frac{\delta u}{\delta x} = \Sigma. S. \frac{\delta s}{\delta x}$; $V.\frac{\delta u}{\delta y} = \Sigma. S. \frac{\delta s}{\delta y}$; $V.\frac{\delta u}{\delta z} = \Sigma. S. \frac{\delta s}{\delta z}$; multiplying these equations by $\delta x\ \delta y\ \delta z$ respectively, we obtain by adding them together $V.\ \delta u = \Sigma.\ S.\ \delta s$. If S, S′, S″, &c. are Algebraic functions of S, S′, S″, &c. then $\Sigma.\ S.\ \delta s$ is an exact variation.

(*e*) The quantity advanced in the direction of the force is termed its virtual velocity, in the direction of that force. *See* Note (*m*) page 265.

In the state of equilibrium $V=o$, ∵ $\Sigma. S. \delta s=o$, ∵ when a point acted on by any number of forces is in equilibrio, the sum of the products of each force by the quantity advanced in its direction is equal to cypher. In this case S one of the forces is = and directly contrary to the resultant V' of all the rest S', S'', S''', &c. for from what has been already stated, we have $V'. \cos. a = S'. \cos. b + S.'' \cos. c +$ &c. but since $S. \cos. a + S.' \cos. b + S.'' \cos. c +$&c. $=o$, we have $V'. \cos. a = - S. \cos. a$; in like manner it may be shewn that $V'. \cos. l = - S. \cos. a'$, $V' \cos. o = -S. \cos. a''$ ∵ $V'^2 = S^2$; and $a = 180 - a$, $l = 180 - a'$ &c.

(*f*) If the resultant was not perpendicular to the surface it might be resolved into two forces, one perpendicular to the surface, which would be destroyed by the reaction of the surface, and the other parallel to this surface, which, as it is not counteracted, would cause the point to move on the surface, contrary to the hypothesis. The re-action which the body experiences from the curve or surface is = and directly contrary to the force with which the point presses it; ∵ if R denote this reaction, r being a perpendicular from the point of application to the surface, we must have $o = \Sigma S \delta s + R \delta r$, instead of the equation $o = \Sigma S \delta s$. If we suppose δx, δy, δz, which are arbitrary, to belong to the surface on which the point is subjected to exist, we have $\delta r = o$; for r is by hypothesis perpendicular to the surface, ∵ $R \delta r$ vanishes from the preceding equation, consequently the position of the text is true, or in other words, in the case of the equilibrium of a point, the sum of the forces which solicit it, each multiplied by the space through which the point moves in its direction, is equal to nothing; it ought however to be remarked, that when the point exists on a surface, the equation $o = \Sigma S \delta s$ is not equivalent to three *distinct* equations, but only to two; for as the variations δx, δy,

δz, belong to the curved surface, one of them may be eliminated by means of the equation of the surface. Laplace substitutes for δr its value N δu, u being the equation of the surface, and N being a function of x, y, z, such that $\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2 = \frac{1}{N^2}$; then if λ be supposed = to N. R, the equation of equilibrium becomes $o = \Sigma. S\, \delta s + \lambda\, \delta u$; in this case we may put each of the coefficients of δx δy δz $= o$, but still they are only equivalent to two distinct equations, on account of the indeterminate quantity λ; the advantage of this expression is, that by means of it we can determine λ, and ∵ — R, the pressure. The equations of the equilibrium of a material point being independent, two or more of them may obtain without the others having place; this is an advantage connected with the resolution of the forces parallel to three rectangular coordinates.—*See* Notes to page 249.

CHAPTER II.

(*a*) Let v be the velocity common to all bodies on the earth's surface, and f the force with which a given body M is actuated in consequence of this velocity, and let the body be sollicited by any new force f', a b c being the components of f resolved parallel to three rectangular axes, and a' b' c' the components of f' resolved parallel to the same, by the notes to preceding chapter F, the resultant of f, $f' = \sqrt{(a+a')^2+(b+b')^2+(c+c')^2}$.

(*b*) If $v = f\phi(f)$, $v' = f'\phi(f')$, $V = F\phi(F)$; the *relative* velocity of the body resolved parallel to the axis of $a = \frac{(a+a').V}{F} - \frac{av}{f} = (a+a')\ \phi\ (F) - a.\ \phi\ (f)$; but as f' is very small relatively to f, we have by neglecting indefinitely small quantities of the second and higher orders, $F = f + \frac{aa' + bb' + cc'}{f}$ and $\phi\ (F) = \phi\ (f). + \frac{aa' + bb' + cc'}{f}.\ \phi'\ (f)$ ∵ by substituting, the relative velocity of the body parallel to $a = a'\ \phi\ f + \frac{a}{f}(aa' + bb' + cc').\ \phi'\ (f)$, parallel to $b = b'\ \phi\ (f,) + \frac{b}{f}\ (aa' + bb' + cc').\phi'(f)$, parallel to $c = c'\ \phi\ (f) + \frac{c}{f}.\ (aa' + bb' + cc').\ \phi'(f)$; if the direction of the impressed motion coincided with a, then the preceding expressions would become $a'\ (\phi f + \frac{a^2}{f}.\ \phi^1\ (f))$; $\frac{ab}{f}.\ a'\ \phi'\ (f)$; $\frac{ac}{f}.\ a'\ \phi'\ (f)$.

(c) If $\phi'\ (f)$ does not vanish, the body, in consequence of the impressed force a' will have a relative velocity perpendicular to the direction of a, if b and c do not vanish, *i. e.* if the direction of a does not coincide with that of the motion of the earth; but as in all cases, those perpendicular velocities vanish; it follows, that $\phi'\ (f)$ vanishes and therefore $\phi\ (f)$ is constant, consequently the function of the velocity which expresses the force is f.

(d) If $\phi\ (f)$ consisted of several terms, $\phi'(f)$ could never be = to cypher, if f was not = to cypher; if ∵ v was not ∻l to f; $\phi\ (f)$ consists of several tems, and also the velocity of the earth must be such as to render $\phi'\ (f) = o$; which cannot be reconciled with the known fact, that the velocity of the earth is different at different seasons of the same year and at corresponding seasons of different years.

(e) Some philosophers hold that this discussion, as to the ∻nality of the force to the velocity, is altogether superfluous, as we are not sure that forces such as we *conceive* them, exist without our conceptions; for what is termed force is only an abstraction, which we make use of to enable us to subject the laws of motion to the calculus; the *true law of nature* is that discovered by Newton, namely, that the velocity communicated by the sun in an instant to the planets, is in the inverse ratio of the square of the distances, and all his physical discoveries might be deduced without using the term force instead of velocity; it follows from this law, that whatever is ∻1 to the velocity follows necessarily the same ∻, so that if Newton assumed that the velocity $f \backsim v^2$, he would have obtained the same results, but then he should say, not that f but that $\sqrt{f}$ varied as $\frac{1}{d^2}$.

(f) If the spaces successively described in = times, constitute an increasing series, the motion of the body is said to be accelerated; if they constitute a decreasing series the motion is retarded; in these cases the measure of the velocity is obtained by determining the space which would be described in a given time, if all causes of acceleration or retardation were to cease after the point attains that position; now as the change in the velocity may be diminished indefinitely by diminishing the space, and ∵ the time in which it is described, if dv ds dt be the indefinitely small increments or decrements of v, s, t, &c. the spaces described in the times dt, immediately preceding and subsequent to the time in which the velocity is required to be estimated, are $(v \pm dv).\ dt$; but as one of those spaces is described with a greater and the other with a less velocity than that with which ds is described, we have $(v+dv).\ dt > ds > (v-dv).\ dt$; but when dv and ∵ dt are indefinitely diminished, the extreme quantities approach within any assignable difference, ∵ $v.\ dt$ and ds, which always exist between them, must differ by a quantity less than any assignable difference ∵ $v = \frac{ds}{dt}$, whatever be the nature of the force; hence if on an assumed line = portions be taken representing the = intervals of time, and if at these points of equal section perpendiculars to the assumed line be drawn, representing the velocities acquired at the corresponding moments, the areas formed by connecting the extremities of the perpendiculars will represent the spaces, this area will be made up of a series of trapezia, if the velocity increases per saltum; if however the intervals of time be increased indefinitely, the velocity will continually approach to that in which the variation is continued, and the figure will be a nearer representation of the space actually described: its limit is a curvilinear area, on the base of which the elements of time are taken the ordinates being ∻l to the velocities; this limit differs from the figure of which it is the limit, by a triangle under one of the equal subdivisions of the base, which are supposed to represent *dt* the element of time, and the difference between the extreme ordinates, hence when *dt* is indefinitely small this difference vanishes.

(*g*) If the velocity receives = increments in = times, *i. e.* if it be uniformly increased, the velocity is as the number of = increments, or as the number of = portions of time from the commencement of the motion, *i. e.* as the times, ∵ in this case, if on the line representing the time, ordinates be erected, they will be as the corresponding abscissæ, the velocity being supposed = to *o*, when the time = *o*, and the locus of the extremities of these ordinates will be a right line diverging from the given line at the point where velocity and time = *o*, and the area of this triangle at the end of any time will represent the space described, and as the triangles representing the spaces described in the given intervals of time are always similar, the spaces described are as the squares of the times of their description, or of the last acquired velocities, ∵ the spaces described in $1''$, $2''$, $3''$, &c. are as 1, 4, 9, &c. and the spaces described in the 1st, 2nd, 3rd, &c. = moments are as the difference of the squares of these moments *i, e,* as 1 3 5 7 9, &c.

(*h*) If a body at the commencement is actuated by any finite velocity, then the space described is geometrically represented by a trapezium, one of whose sides is the initial velocity, and the other an ordinate, = to the sum of this ordinate and of the ordinate which would express the velocity of the body, had it fallen freely in the same time; if v' be the initial velocity, $v = v' \pm ft \therefore vt = v't \pm \frac{ft^2}{2} \therefore s$ the space described $= v't \pm \frac{ft^2}{2}$; if the body moved with a uniform velocity v during t, s, the space described $= vt$. If it acquired the velocity v in the time t, by being urged by an uniform force from a state of rest, s' the space described would be $\frac{v.t}{2} \therefore s : s' :: 2 : 1$.

(*i*) Let $v\ t\ s$ represent the velocity, time, and space, and f the accelerating force $= \frac{v}{t}, = \frac{vt}{t^2} = \frac{v^2}{vt}$ *i, e,* $= \frac{2s}{t^2} = \frac{v^2}{2s}$, f denoting the unit of velocity or the velocity generated in a unit of time, $\therefore s = \frac{ft^2}{2}$; $v^2 = 2f.s.$

(*k*) The force acting parallel to the inclined plane being to the force of gravity which is constant, as h the height of the plane to l the height, *i. e.* in a constant ratio, a body moving down an inclined plane has its motion uniformly accelerated, $\therefore$ if v' s' represent the spaces described by a body descending down an inclined plane in any time t, and v' the acquired velocity, f' the accelerating force, we have $f' = \frac{h}{l}f$, $v' = \frac{h}{l}.ft$; $s' = \frac{h}{l}.\frac{ft^2}{2}$ $v'^2 = 2s'.\frac{h}{l}.f$; $\therefore v' : v :: h : l$; $s' : s :: h : l$; if $s' = l$ then we have $l = \frac{h}{l}.\frac{ft^2}{2} \therefore t = l.\sqrt{\frac{2}{fh}}$; as $\frac{2h}{f}$. expresses the square of the time acquired in falling down the vertical, and as we have $t^2 =$

$\frac{2ls}{fh}$; when this is = the square of the time acquired in falling down the vertical, we have $ls = h^2$, ∴ $s = \frac{h^2}{l}$, ∵ if a perpendicular be let fall from the right angle on the plane, it will cut off a portion of the plane, which will be described in the same time as the perpendicular height; and if a circle be described on this height as diameter, it is evident from what has been just established, that all chords drawn from *its* extremity to the circumference, are described in the same time as the diameter, ∵ in = times.

(*l*) Let v' the velocity of projection be resolved into two, of which one is vertical and the other parallel to the horizon, and let *e* be the elevation of the line of direction, we have $v'.\sin. e$, $v'.\cos. e$, for the velocity of projection estimated in the direction of x and y respectively; $v'.\cos. e$ is the motion parallel to the horizon, $v'.\sin. e - ft$ is the vertical motion of the projectile, ∵ if in the equations given in page 416, we make $v' = o$, we shall have for the height of ascent $s = \frac{v'^2.\sin. e}{2f}$, and $t = \frac{v'.\sin. e}{f}$, for the time of ascent, ∵ $\frac{2v'.\sin. e}{f}$ for the time of flight; to find the horizontal range, the velocity $v'.\cos. e$, must be multiplied into $2t$ or its equivalent $2\frac{v'.\sin. e}{f}$, it ∵ is equal to $\frac{v'^2.\sin. 2e}{f}$; therefore it is a maximum when $e = 45$, and for any elevations which are complements of each other, the horizontal ranges are =, the coordinates of the place of the body for any time t, are $x = v'.\ t\cos. e$, $y = v'.\ t\sin. e.-\frac{ft^2}{2}$, ∵ as t is the same in these two equations we obtain by eliminating it and substituting $2fh$ for v'^2, $y = x\tan. e - \frac{x^2}{4h\cos.^2 e}$ which is the equation of a parabola, $4\ h\cos.^2 e$

is the principal parameter, and $4\,h$ the parameter of the diameter passing through the point of projection, hence being given of $x\,y\,e\,h$, any three, the fourth may be found.

(*m*) Let the arc described, reckoning from the lowest point $= s$, the ordinate $= y$, and the vertical abscissa $= x$, the origin of the coordinates being at the lowest point, if $b =$ the value of x at the commencement of the motion; v the velocity at the end of any time t, is the same as would be acquired by falling through the vertical height, $b - x$, *i. e.* $v = \sqrt{2g.(b-x)} = -\frac{ds}{dt}$; *see* Note($x$) $\because dt = -\frac{ds}{\sqrt{2g(b-x)}}$, the negative sign being taken, because s diminishes according as t increases; but as $ds = \sqrt{dx^2 + dy^2}$, $y^2 = 2rx - x^2$, we obtain by substituting, $ds = \frac{rdx}{\sqrt{2rx - x^2}} \because dt = -\frac{rdx}{\sqrt{(2rx - x^2)2g(b-x)}}$, if the oscillations are very small, x may be neglected relatively to r, then the value of dt becomes $\frac{-r.\,dx}{\sqrt{2rx\,(2g(b-x)}}$,

$$= \frac{1}{2}.\sqrt{\frac{r}{g}} \times \frac{-dx}{\sqrt{bx - x^2}}$$

the integral of the variable factor $=$ arc $\left(\cos. = \frac{2x-b}{b}\right) = \pi$, when we integrate from $x = b$ to $x = o$, $\because$ the time of a semioscillation

$$= \tfrac{1}{2}.\pi.\sqrt{\frac{r}{g}};$$

(*n*) Hence it follows, that provided the amplitudes be inconsiderable, the time of oscillation is always the *same*, when r and g are given, when these quantities vary the time varies as $\sqrt{\frac{r}{g}}$ *i. e.* directly as the square roots of the lengths of the pendulums, and inversely as the square root of the force of gravity.—*See* Note (s) page 355. As $3\,\pi$, $5\,\pi$, &c. and in general any odd multiple of π satisfies the preceding integral, of $\frac{rdx}{\sqrt{bx-bx^2}}$; it is evident that the body arrives at the lowest point an indefinite number of times, which are separated from each other by the time $\pi\sqrt{\frac{r}{g}}$; hence it follows, that if all obstacles were removed, the number of oscillations would be infinite and the time of each $=$.

The value of dt may be made to assume the form $\frac{1}{2}\sqrt{\frac{r}{g}}.\frac{-dx}{\sqrt{bx-x^2}}\ \frac{1}{\sqrt{(1-\frac{x}{2r})}}$ = (by developing the factor

$$\left(1-\frac{x}{2r}\right)^{-\frac{1}{2}} \text{ in a series) } \frac{1}{2}\sqrt{\frac{r}{g}}.\ \frac{-dx}{\sqrt{bx-x^2}}.$$

$\left(1+\frac{1}{2}.\frac{x}{2r}+\frac{1.3}{2.4}.\frac{x^2}{4r^2}+\text{\&c.}\right)$, if $\frac{dx}{\sqrt{bx-x^2}}$ be multiplied by each term of this series the resulting terms will be of the form $\frac{-x^m dx}{\sqrt{bx-x^2}}$, of which the integral when taken between the limits $x=o$, $x=b$, is

$\frac{(\frac{1}{2}b)^m.\ \pi.\ 1.\ 3,\ 5,\ \text{\&c.}\ (2m-3).\ (2m-1)}{1.\ 2.\ 3\ ..\ m}$ from which if m be made successively $= 0$ 1, 2, &c. the value of t becomes $T=\frac{1}{2}\pi.\sqrt{\frac{r}{g}}.\left(1+(\frac{1}{2})^2.\frac{b}{2.r}+...\left(\frac{1.3(...)(2m-1)}{2.4-2m}\right)^2.\right.$ $\left.\frac{b^m.}{2^m.r^m}+\text{\&c.}\right)$; b is the versed sine of the arc described, which when it is inconsiderable may evidently be neglected, in this case the value of t is the same as was obtained in the preceding page; when great accuracy is required, the two first terms of the series are retained, in that case the aberration from isochronism varies as the square of the sine of half of the amplitude of the arc described.

(*o*) As τ the time of fall in the vertical through a space = to half the length of the pendulum $= \sqrt{\frac{r}{g}}$, we have $\text{T} : \tau :: \pi.\sqrt{\frac{r}{g}} : \sqrt{\frac{r}{g}} :: \pi : 1$; if t' be the time employed to describe the chord of the indefinitely small arc, as this time = the time of falling vertically through the diameter $2r$, *see* preceding Note ∵ it is $= \sqrt{\frac{4r}{g}}$, ∴ $\frac{\text{T}}{2} : t' :: \frac{\pi}{2}.\sqrt{\frac{r}{g}} : \sqrt{\frac{4r}{g}}$, *i. e.* $\pi : 4$ or as the periphery of a circle to four times the diameter; hence it is evident that the chord is not the line of swiftest descent, *see* Note (*p*).

Naming $\tilde{\omega}$ = the angular velocity, we have $v = r.\tilde{\omega}$: ∴ $\tilde{\omega} = \frac{v}{r} = \frac{\sqrt{2g.(b-x)}}{r}$ but if a be the angular distance from the vertical at the commencement of the motion, and θ the angular distance at the end of any time t, we have $b = r.\cos. a$, $x = r.\cos.\theta$, ∴ $\tilde{\omega} = \sqrt{\left(\frac{2g}{r}(\cos.\theta - \cos. a)\right.}$. The acccelerating force in any point, = the force of gravity resolved in the direction of the tangent; ∴ if any vertical line be assumed to represent the force of gravity, the accelerating or tangential force = this line multiplied into the sine of the angular distance from the lowest point. If the body, instead of falling freely, had a velocity at the commencement of the motion due to the height h, then the velocity at any point of which the height $= x$, is $\sqrt{2g(h+b-x)}$ and $= o$, when $x = h + b$, ∴ when the body attains a height $= h + b$, it ceases to rise; v will never vanish when $h + b$ is > than the diameter which is the greatest value of x, ∴ the body will gyrate for ever with a variable velocity, the greatest being when at the lowest, and the least at the highest extremity of the vertical diameter. When a body M attached to a string describes an arc of a curve, the tension at the point to which the string is attached, arises from the centrifugal force and the force of gravity resolved in the direction of the string; if the arc described be the arc of a circle, the part of the force of gravity which acts in the direction of the string $= g.\left(\frac{r-x}{r}\right)$, r being the length of the string, and x the distance above the lowest point; the centrifugal force $=\frac{v^2}{r}= 2g.\left(\frac{b-x}{r}\right)$; this always acts from the centre; $\because$ the whole tension $=$ M$g.$ $\left(\frac{r+2b-3x}{r}\right)$; if M falls from an horizontal diameter, $r=b$, and the tension at any point $=$ 3M$g.\left(\frac{r-x}{r}\right)$, *i. e.* three times the effect of the weight resolved in the direction of the radius vector. If the pendulum fell from the vertical position freely, then $b=2r$ and $\because$ the tension $=$ M$g\left(\frac{5r-3x}{r}\right)$, and when $x = o$, it is equal 5Mg. or five times the weight; making M$g\left(\frac{r+2b-3x}{r}\right)=$ Mg we obtain $x=\frac{2}{3}b$, the value of x when the tension $=$ the weight; when $x = \frac{r+2b}{3}$ the tension $= o$; but as x can never exceed either b or $2\,r$; when it is respectively $=$ these quantities, we have $b=r$, $b=\frac{5r}{2}$, if $b \angle r$ then the force of gravity resolved in the direction of the string is directed from the centre, $\therefore$ this point then suffers a tension from both causes; if $b > \frac{5r}{2}$, the centrifugal force is throughout $>$ than weight, $\because$ the whole tension can never vanish, but if b is not $\angle r$ or $> \frac{5r}{2}$ the tension may vanish; at this point the body will quit the circle, and as its direction will be that of a tangent to this circle it will describe a parabola. In a cycloid if a body falls freely from the extremity of the base, the pressure arising from the weight resolved in the direction of the string $= g.\frac{\sqrt{a-x}}{\sqrt{a}}$, and likewise that produced by the centrifugal force $= \frac{2g(a-x)}{2.\sqrt{a(a-x)}} = g.\frac{\sqrt{a-x}}{\sqrt{a}}$, hence at the lowest point, the entire tension = twice the weight; in any other point the entire tension is to weight, as twice the cosine of the inclination of the tangent to the horizon to radius; hence, when the body falls from the horizontal base, they are equal at the point of the cycloidal arc where the tangent is inclined at an angle of 60° to the horizon.

(*q*) Calling this space x, we have $2x :: r :: \pi^2.\frac{r}{g} : \frac{r}{g} :: \pi^2 : 1$; the equation $T = \pi.\sqrt{\frac{r}{g}}$ gives likewise a very exact measure of g, for if l be the length of this pendulum vibrating seconds, we get $g = \pi^2.l$, which expresses the velocity generated in one second by the space which would be described with that velocity continued uniformly for that time, the space described by a body falling from rest in a second is one half of this, or $\pi^2.\frac{l}{2}$; substituting for π, l their numerical values given in the text we obtain $3^{m},66107$ for the space described in the first second.

As the sine of the angle which the tangent at any point of a vertical curve makes with the horizon, $= \frac{dx}{ds}$, the accelerating force along the tangent $= g.\frac{dx}{ds} = \frac{g}{2a}.s$; (when the curve described is a cycloid, in consequence of the equation of the cycloid $s^2=4ax$), the preceding is the expression for the accelerating force, in any curve whatever which renders it tautochronous, $\because$ this force is at each instant ÷ 1 to the length of the arc to be described, in order to arrive at the lowest point of the curve; and conversely if $\frac{dx}{ds} = As$, it is easy to shew that when the curve is one of *single curvature existing in a vertical plane*, its equation is that of a cycloid, for by integrating the preceding equation, and then eliminating s between the integral $x = \frac{1}{2} As^2$, and $\frac{dx}{ds} = As$, we obtain $\frac{1}{2A} \cdot \frac{dx^2}{x} = ds^2 = \overline{dy^2 + dx^{2\prime} + dz^2}$; $\therefore \frac{1}{A} x = s^2$, if the curve is one of single curvature inclined to the horizon at an angle $= \theta$, then if y' x' be the coordinates in that plane, we have $y' = y$, $x = x'$. sin. θ, consequently the equation of the curve is $\frac{1}{2A} . x'$. sin. $\theta = s^2$; note the relation $\frac{1}{A} \frac{dx^2}{x} = ds^2 =$ generally $dx^2 + dy^2 + dz^2$, and as $\frac{1}{2A} . x = s^2$ is independent of z y, these quantities may vary according to any law whatever, which satisfies the equation $ds^2 = dx^2 + dy^2 + dz^2$; $\therefore$ any curve of double curvature which arises from wrapping a cycloid around a vertical cylinder of which the base is a continuous curve, will satisfy the preceding conditions, and $\therefore$ be tauto chronous; and conversely such a curve so unfolded as that it might entirely exist in the same plane would continue to possess this property, and $\therefore$ from what has been stated above, would necessarily be a cycloid. We might investigate a priori, the time necessary for a body to describe any portion of a cycloidal arc on the hypothesis, that it moves with an initial velocity represented by $\sqrt{2gh}$, for let h' represent the vertical ordinate at the commencement of the motion, the origin being as before at the lowest point, and x the ordinate after any time t, we have $v^2 = 2g(h + h' - x)$; $\therefore dt = \frac{-ds}{\sqrt{2g(h + h' - x)}}$ but as $ds = dx . \sqrt{\frac{a}{x}}$ by substituting we have $dt = -\sqrt{\frac{a}{2g}} . \frac{dx}{(x(h + h' - x))^{\frac{1}{2}}}$ $\therefore$ integrating we obtain $t =$

$$\left(\frac{a}{2g}\right)^{\frac{1}{2}} \text{arc}\left(\cos. = \frac{x - \frac{(h+h')}{2}}{\frac{(h+h')}{2}}\right) + \text{C}; \text{ as } t = o \text{ when } x = h',$$

$\text{C} = -\left(\frac{a}{2g}\right)^{\frac{1}{2}}. \text{ arc}\left(\cos. = \frac{h'-h}{h'+h}\right)$; and when $x = o$, *i e*, at the lowest point $t = \left(\frac{a}{2g}\right)^{\frac{1}{2}}. \left(\pi - \text{arc}\left(\cos. = \frac{h'-h}{h+h}\right)\right)$; if $h = o$ *i.e.* if the initial velocity vanishes $t = \pi.\left(\frac{a}{2g}\right)^{\frac{1}{2}}$, ∵ as h does not occur in this expression, the time is independent of the amplitude of the arc described; it appears from a comparison of this value of t with that given in page 418, that the oscillations in a cycloid are isochronous with the indefinitely small vibrations in a circle, of which the radius is equal to twice the axis of the cycloid.

Huygen's contrivance depended on the known property of cyloids, namely, that their evolute was a curve = and similar to the given cycloid, hence it follows, that if two metallic curves, each consisting of an inverted semi cycloid with an horizontal base touched at their upper extremities; and if at their point of contact, the thread of the pendulum was attached (its length being equal to either of the semi cycloids,) when it is enveloped on the curves, its other extremity will trace a curve = and similar to the given curve, having its axis however in an opposite direction.

(r) From the times of vibration and lengths of these pendulums being the same, the times of falling down the = axes are the same, ∵ all bodies falling freely are equally accelerated by the force of gravity.

It is easy to shew that the time of describing the chord of a semi cycloidal arc is to the time of describing the arc, as the chord to half the base of the cycloid, which is evidently a ratio of major inequality.—*See* Note (p), page 425.

(p) In investigating the nature of the curve of swiftest descent in a *vacuo*, it is easy to shew that if the entire line be supposed to be described in the shortest possible time, so any portion of this line intercepted between two assumed points is described in a less time than any other curve joining these two points; hence if $x\,y$ be the vertical and horizontal coordinates of any point, reckoning from the point whence the body has commenced to move, s the corresponding arc of the curve, the time of describing $ds = \frac{ds}{\sqrt{2gx}}$, in like manner if a point indefinitely near to the first point be taken whose coordinates are $x'y'$ and the corresponding arc described from commencement $= s'$, we have $x' = x + dx$, $s' = s + ds$, and the time of describing $ds' = \frac{ds'}{\sqrt{2gx'}}$, $\because$ the time of describing the entire arc made up of $ds' + ds = \frac{ds}{\sqrt{2gx}} + \frac{ds'}{\sqrt{2gx'}}$, therefore we have $o = \delta\left(\frac{ds}{\sqrt{2gx}} + \frac{ds'}{\sqrt{2gx'}}\right)$, but from the conditions of the problem $x\,x'$ are independent of these variations $\because \delta x, \delta x' = o$, and consequently $\frac{\delta.\,ds}{\sqrt{x.}} + \frac{\delta.\,ds'}{\sqrt{x'}} = o$; and as $dx\,dx'$ have no variations, $\delta.\,ds = \delta.\,d\,\sqrt{dy^2 + dx^2} = \frac{dy.\,\delta\,dy}{ds}$, &c. $\because$ by substituting we have $\frac{dy.\,\delta\,dy}{ds\sqrt{x}} + \frac{dy'.\,\delta\,dy'}{ds'\sqrt{x'}} = o$, but $dy + dy'$ is constant, therefore $\delta\,dy = -\,\delta\,dy'$, consequently $\frac{dy}{ds.\sqrt{x}} - \frac{dy'}{ds.\sqrt{x'}} = o$, *i. e.* $d.\left(\frac{dy}{ds\sqrt{x}}\right) = o$, (for the two points $x\,y$, $x'\,y'$, are continuous,) and $\frac{dy}{ds.\sqrt{x}} = \text{C}$; now as $\frac{dy}{ds}$ is the sine of the angle which the tangent makes with the axis of x, when the arc is horizontal, this angle is right, and $\therefore \frac{1}{\sqrt{a}} = C$, ($a$ being the value of y at this point,) $\therefore \frac{dy}{ds} = \sqrt{\frac{x}{a}}$ by squaring and substituting we get $dy^2 . \left(1 - \frac{x}{a}\right) = dx^2 \frac{x}{a} \therefore dy = dx . \frac{x^{\frac{1}{2}}}{\sqrt{a-x}}$ and $ds = dx \sqrt{\frac{a}{a-x}} \therefore s = -2\sqrt{a(a-x)} + C'$; but when $s=o$, $x=o$, $\therefore C' = 2a$, and $s = 2a - 2\sqrt{a(a-x)}$, which is the equation of a cycloid, of which the axis is a, the arc being measured from the horizontal base.

If the curve is not required to pass between two given points, but between two given curves, then it would not be difficult to shew that the required curve is a cycloid meeting the two given curves at right angles.

(s) In an indefinitely small portion of time, the quantity by which the body is deflected from the tangent to the circle, which measures the centripetal and consequently the centrifugal force, is the versed sine of the arc described; and as this is the space which the central force causes a body to describe, the force of gravity will be to the centrifugal force as the space described, in consequence of the action of gravity in this time, to this versed sine.

(t) Calling f the accelerating force, we have $f = \frac{2\,dr}{dt^2}$; $dr = \frac{ds^2}{2.r}$, $\therefore f = \frac{ds^2}{dt^2 r}$ but $\frac{ds}{dt} = v \therefore f = \frac{v^2}{r}$; the curve described being a circle in which the deflection from the tangent is always the same, the force acting on the point is a constant accelerating force; hence as v^2 always $=2gh$, we have $\frac{v^2}{r} = f = \frac{2gh}{r}$ and $\frac{f}{g} = \frac{2h}{r}$ which gives generally the relation between the centrifugal force in a circle and the force of gravity, and they are $=$ when $h = \frac{r}{2}$; *i. e.* the body must fall through half the radius in order to acquire the velocity which renders the centrifugal force equal to the gravity; if P = the time of revolution we have $v = \frac{2\pi r}{P}$ $\because f = \frac{4\pi^2 r}{P^2}$, this expression gives $\frac{11154}{32 \cdot 1250} = \frac{1}{288}$ for the ratio of the centrifugal force to the force of gravity at the equator, and because when r is given, f varies inversely as P^2, if P′ be the time of the earth's rotation when the centrifugal force = the force of gravity, we have $P^2 : P'^2 :: 289 : 1$ therefore $P' = \frac{P}{17}$, hence if the earth revolved on its axis in the 17th part of a day, *i, e*, in $1^h, 24' \; 28\frac{1}{2}''$ the centrifugal force would be equal to the gravity. *See* Notes to Chapter VIII. Vol. II.

It follows from the expression $f = \frac{4\pi^2 r}{P^2}$, that the centrifugal force on the earth's surface is greatest at the equator, and that it decreases as the cosine of latitude; however as its direction is inclined to the direction of gravity it is not entirely efficacious at any *parallel*, and by a resolution of forces it may be shewn that the efficacious part is to the whole centrifugal force at the parallel, as the cosine of the latitude λ to the radius, and therefore to the centrifugal force at the equator as $\cos. \lambda^2 : 1$; the part of the resolved force which acts perpendicularly to the direction of gravity, and is therefore inefficacious, varies as $\sin. \lambda. \cos. \lambda$.

(*u*) The force which is in equilibrio with the centrifugal force is $\because$ the measure of the pressure arising from the tendency of the body to recede in the direction of the tangent; hence, by note (*t*) it is $\frac{2dr}{dt^2} = \frac{v^2}{r}$; (*r being the radius of curvature,*) the effect of the part of the force resolved in the direction of *dr* is therefore to produce a continued change in the direction of the motion; and the effect of the other part is evidently to *accelerate* or *retard* the motion of the body, its variation = $\frac{1}{p^2 . c} . \sqrt{1 - \frac{p^2}{\rho^2}}$; ρ being the radius vector.

(v) Calling $d\rho$ the part of the radius vector intercepted betwen the curve and the tangent, ds the arc and c the chord of curvature, we have $f = \frac{2d\rho}{dt^2}$, but $d\rho = \frac{ds^2}{c}$ ∴ $f = \frac{ds^2}{dt^2 . c} = \frac{v^2}{c}$, this expression is general, and true independently of the equal description of areas; on the hypothesis that the areas are ÷1 to the times, $v \backsim \frac{1}{p}$, p being a perpendicular let fall from the centre of force on tangent, and ∴ $f \backsim \frac{1}{p^2 . c}$ which is one of Newton's expressions.

Let x be the space through which the body should fall to acquire the velocity in the curve, the velocity acquired in falling through dc is to the velocity with which the arc is described, as $2dc : ds$; and $dc : x$ ∷ as the square of the velocity acquired in falling through dc to the square of the velocity with which dc is described, ∴ $dc : x :: 4dc^2 : ds^2$ ∴ $x = \frac{ds^2}{4dc} = \frac{c}{4}$, *i. e.* a body falls through one-fourth of the chord of curvature to acquire the velocity in the curve.

(v) It is by taking the function of the radius vector, which is equal to this limit, that Newton determines the expression for force in conic section, spiral, &c., *see* Princip. Math. sec. 2 and 3. It would not be difficult to shew by reasoning precisely similar to that in pages 249, 250, that if a body is attracted to two fixed points which are not in the same plane as that in which it moves, the body will describe = solids in equal times about the line connecting the attracting points.

The proposition established in page 246 may be thus proved, by what is stated in page 249, $X = \frac{d^2x}{dt^2}$, $Y = \frac{d^2.y}{dt^2}$ $Z = \frac{d^2z}{dt^2}$; multiplying the first equation by y and z, the second by x and z, and the third by x and y, we obtain by subtracting, $\frac{d^2y}{dt^2}.x - \frac{d^2x}{dt^2}.y = Y.x - X.y$, $\frac{d^2y}{dt^2}.z - \frac{d^2z}{dt^2}.y$

$= \text{Y}.z - \text{Z}.y$, $\frac{d^2z}{dt^2}.x - \frac{d^2x}{dt^2}.z = \text{Z}.x - \text{X}z$, by integrating we obtain $\frac{dy.x - dx.y}{dt} = \text{C} + \int(\text{Y}x - \text{X}y)\,dt : \frac{dy}{dt}.z - \frac{dz}{dt}.y = \text{C}' + \int(\text{Y}z - \text{Z}y)\,dt ; \frac{dzx}{dt}. - \frac{dxz}{dt} = \text{C}'' + \int(\text{Z}x - \text{X}z)dt$; but when the force is directed to a fixed point, which is the origin of $x\ y\ z$, $(\text{Y}x - \text{X}y)$, $(\text{Y}z - \text{Z}y)$, $(\text{Z}x - \text{X}z)$, are respectively $= o$, *see* Chapter IV. Note (h), $\because dyx - dxy = \text{C}.dt$, a constant quantity, but this quantity is evidently $=$ to the projection of the element of the area on the plane $x\ y$, for let ρ be the projection of the radius vector, ψ the angle which it makes with x and y, we have $x = \rho \cos.\psi$, $y = \rho \sin.\psi$, $\because xdy - ydx = \rho^2.\,d\psi$, which is the element of the area. The quantities C C′ C″ depend on the nature of the curve described. In the case of a conic section, origin being in the focus, they are respectively $\div 1$ to the cosines of the inclinations of the planes xy, xz, yz, to the plane in which the body moves, multiplied by the square root of the parameter.

Multiplying each of the preceding equations by the variable which does not occur in it, and then adding them together we obtain the equation $o = \text{C}z + \text{C}'y + \text{C}''x$, which shews that when a body is acted on by a force directed to a fixed point, it will describe a curve of single curvature.

(x) By referring the position of a point in space to rectangular coordinates, every species of curvilinear motion may be reduced to *two* or *three* rectilinear motions, according as the curve described is of single or double curvature, for the position of a point in space is completely determined when we can determine the position of its projections on three rectangular axes, each coordinate is the rectilinear space described by the point parallel to the axis to which it is referred, it will $\because$ be some given function of the time; if we could determine these functions for the *three* coordinates, the species of the curve described would be given, by eliminating the time by means of the three equations between the coordinates and the time. The space s being considered a function of the time t it is easy to shew that the velocity is $=\frac{ds}{dt}$, and f the force is $\div$ 1 to $\frac{d^2s}{dt^2}$, for t receiving the increment dt, then $s = \phi\ (t)$ becomes $s' = \phi(t+dt)$ and $s'-s = \frac{ds}{dt}.\ dt + \frac{d^2s}{dt^2}\ dt^2 + \frac{d^3s}{dt^3}.\ dt^3 + \&c.$; if dt be considered as indefinitely small, in which case we can consider the velocity as uniform and the force as constant, $\frac{ds}{dt}$ being the coefficient of dt expresses the velocity, and $\frac{d^2s}{dt^2}$ being the coefficient of dt^2, it is $\div$ 1 to the force; ∵ if the action of the forces soliciting the point should cease suddenly $\frac{d^2s}{dt^2}$ would vanish, and the point would move with an uniform velocity, if instead of vanishing $\frac{d^2s}{dt^2}$ became constant, then $\frac{d^3s}{dt^3}$ and all subsequent coefficients would vanish, and the motion of the point would be composed of a uniform motion and of a motion uniformly accelerated, both commencing at the same instant; now if f represents the force, it is evident that $f.\ dt = dv, = d.\frac{ds}{dt} = \frac{d^2s}{dt}.$

(y) Let P Q R represent the resultants of all the forces which act on the point parallel to $x\ y\ z$ respectively, we have $\frac{d^2x}{dt^2} = \mathrm{P}, \frac{d^2y}{dt^2} = \mathrm{Q}, \frac{d^2z}{dt^2} = \mathrm{R}$, consequently if the point was actuated by the forces

$$- d.\frac{dx}{dt} + \mathrm{P}; - d.\frac{dy}{dt} + \mathrm{Q}; - \frac{dz.}{dt.} + \mathrm{R}$$

they would keep it in an equilibrium; hence from what has been already established in Notes, page 354, we have

$\left(d.\frac{dx}{dt} - \text{P}\right).\ \delta x + \left(d.\frac{dy}{dt} - \text{Q}\right).\ \delta y + \left(d.\frac{dz}{dt} - \text{R}\right).\ \delta z = o;$ if the point be free we shall have, as is stated in the text, the coefficients of δx, δy, δz, separately $= o$; *i. e.* $\frac{d^2x}{dt^2} = \text{P}$; $\frac{d^2y}{dt^2} = \text{Q}$, $\frac{d^2z}{dt^2} = \text{R}$; but if the point is constrained to move on a curve or surface, by means of the equations to this curve or surface, we can eliminate as many of the variations $\delta x\ \delta y\ \delta z$ as there are equations; the coefficients of the remainder may be put = to cypher; it appears from this process, which is that made use of by Laplace in his Celestial Mechanics, how the laws of the motion of a point may be deduced from those of their equilibrium: we shall see in the sixth chapter that the laws of the motion of any system of bodies may be reduced to those of their equilibrium; if P Q R are given in functions of the coordinates, then by integrating twice we obtain $x\ y\ z$ in a function of the time; two constant arbitrary quantities are introduced by these integrations; the first depends on the velocity of the point at a given instant, the second depends on the position of the point at the same instant: if $x\ y\ z$ came out respectively $= a.f(t)$, $b.f(t)$, $c.f(t)$, the point will move in a right line, the cosines of the angles which it makes with $x\ y\ z =$

$$\frac{a}{\sqrt{a^2+b^2+c^2}} \quad \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad \frac{c}{\sqrt{a^2+b^2+c^2}};$$ the constant quantities $a\ b\ c$ depend on the nature of the function $f(t)$, if $f(t) = t$ then $a\ b\ c$ represent the uniform velocities parallel to $x\ y\ z$, and the uniform velocity of the point $= \sqrt{a^2+b^2+c^2}$; if $f(t) = t^2$; $a\ b\ c$ are proportional to the accelerating forces parallel to $a\ b\ c$, and the point will move with an uniformly accelerated motion represented by $\sqrt{a^2+b^2+c^2}$; if $x = a'.f(t) + b'\,\text{F}(t)$; $y = cf(t) + d.\,\text{F}(t)$, $z = ef(t) + g.\,\text{F}(t)$, the path of the point will be a curve, however it will be of single curva-

tion; for by eliminating t we obtain an equation of the form $Ax + By + Cz = o$, which is that of a plane; the simplest case of this form is $x=a't+b't^2$, $y=c't+d't^2$, $z=e't+g't^2$; eliminating t between the two first equations we shall obtain an equation of the second order between x and y, which is evidently a parabola from the relation which exists between the coefficients of the three first terms. If $x = f(t)$, $y = F(t)$ $z = \phi(t)$, all the points in the curve will not exist in the same plane. The law of the force being given, the investigation of the curve which this force causes to be described, is more difficult than the reverse problem of determining the force, velocity, &c. the nature of the curve being given, as the integrations which are required in the first case are much more difficult than the differentiations which determine the force and velocity in the second. It may be remarked here, that the number of the equations of condition of the motion of a material point is necessarily *less than three*; for if there were three equations of condition between the coordinates $x\ y\ z$, it is evident that if these equations were independent of the time, their resolution would give particular values for each of the coordinates, ∴ the point could not move; and if the equations contained the time the values of $x\ y\ z$ are given in a function of the time, so that the motion of the point being determined *a priori* by the equations of condition, it cannot be modified by any accelerating force; if there were more than three equations of condition their simultaneous existence would imply a contradiction.

(z) As $\delta x\ \delta y\ \delta z$ are arbitrary they may be assumed = to $dx\ dy\ dz$ respectively, in which case we have

$$d.\frac{dx}{dt}\,dx + d.\frac{dy}{dt}\,dy + d.\frac{dz}{dt}dz = P\,dx + Q\,dy + R\,dz;$$

∴ by integrating $\frac{dx^2+dy^2+dz^2}{dt^2} = C + 2\int(Pdx+Qdy+Rdz)$; if this integral $= f(x\,y\,z)$, then $v^2 = C + f(x\,y\,z)$

let A be the velocity corresponding to the coordinates $a\ b\ c$, then $A^2 = C + 2f(a\ b\ c)$, $\because v^2 - A^2 = 2f(x\ y\ z) - 2f(a\ b\ c)$, *i. e.* the difference of the squares of the velocities depends on the coordinates of the extreme points of the line described, $\because$ is independent of the line described; so that when the point describes a curve, the pressure of the moving point on the curve does not affect the velocity. The constant quantity C depends on the values of v and of $x\ y\ z$ at any given instant; when the moving point describes a curve returning into itself, the velocity is always the same at the same point, and if the velocities of two points of which one describes a curve while the other describes a right line, are equal at = distances from the centre of force at any given instant, they will be equal at all other = distances; if the force varies as the n^{th} power of s the distance from the centre, then $f(x\,y\,z) = s^{n+1}$, $\because v^2 - A^2 = s^{n+1} - a^{n+1}$, and $2dv.\ v = (n+1).\ s^n\,ds$, $\because$ by erecting in the line drawn from the centre ordinates $\doteqdot 1$ to s^n, the resulting figure will represent the square of the velocity, when n is positive, this figure is of the parabolic species, when it is negative it will be of the hyperbolic species; if $Pdx + Qdy + Rdz$ be an exact differential, then $\frac{dP}{dy} = \frac{dQ}{dx}$, $\frac{dP}{dz} = \frac{dR}{dx}$ &c. and P Q R must be functions of $x\ y\ z$ independently of the time; now if the centres to which the forces were directed had a motion in space, the time would be involved, and $\because Pdx + Qdy + Rdx$ would not be an exact differential; if P Q R arose from friction or the resistance of a fluid, the equation $Pdx + Qdy + Rdz$ would not satisfy the preceding conditions of integrability, for as in such cases P Q R depend on the velocities $\frac{dx}{dt}\frac{dy}{dt}\frac{dz}{dt}$, $Pdx + Qdy + Rdz$ cannot be an exact differential of $x\ y\ z$ considered as independent variables, consequently in order to integrate, we should in the expression $Pdx + Qdy + Rdz$ substitute for these variables and their differentials, their values in a function of the time, which supposes that the problem is already solved, ∵ when the point to which the force is directed is in motion, or when the force arises from friction or resistance, the velocity involves the time and $Pdx + Qdy + Rdz$ is not an exact differential. When a point moves in a right line, the velocity is = to the element of the space ÷ded by the element of the time, *i. e.* $v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt}$ this is also true for curvilinear motion, for if P Q R should suddenly cease, the velocity in the direction of each coordinate is uniform and $= \frac{dx}{dt} \frac{dy}{dt} \frac{dz}{dt}$ respectively; therefore v the velocity of the point will be uniform and its direction rectilinear, *i. e.* $v = \frac{ds}{dt} = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt}$; the rectilinear direction is that of the tangent, for if A, B, C, denote the angles which this direction makes with $x\ y\ z$, we have

$v.$ cos. A $= \frac{ds}{dt}.$ cos. A $= \frac{dx}{dt}$; and v cos. B $= \frac{ds}{dt}.$ cos. B $= \frac{dy}{dt}$, $v.$ cos. C $= \frac{ds}{dt}.$cos. C $= \frac{dz}{dt}$ ∵ cos. A $= \frac{dx}{ds}$, cos. B $= \frac{dy}{ds}$, cos. C $= \frac{dz}{ds}$ which are the expressions for the angles, which any tangent makes with the coordinates, ∵ the tangent coincides with the line along which the point would move if P Q R should suddenly cease.

If the point moves on any curve whatever, the centrifugal force $= \frac{v^2}{r}$, *see* Notes, page 428, and as this force acts in the direction of a normal to the curve, if all the accelerating forces which act on the point be resolved to two, of which one acts perpendicularly to the trajectory, and the other in the direction of the tangent, the resultant of the first of these forces and of $\frac{v^2}{r}$, is the entire pressure of the point on the curve, and the resistance of the curve is an accelerating force = and contrary to this resultant, denoting the normal force by L, if A′, B′, C′, be the angles which it makes with $x\ y\ z$ respectively; by the Notes to page 431, we have $\frac{d^2x}{dt^2} = \text{P} + \text{L cos. A}'$, $\frac{d^2y}{dt^2} = \text{Q} + \text{L cos. B}'^2$, $\frac{d^2z}{dt^2} = \text{R} + \text{L cos. C}'$; but since the normal is perpendicular to the tangent we have $\frac{dx}{ds}$ cos. A′ + $\frac{dy}{ds}$ cos. B′ + $\frac{dz}{ds}$. cos. C′ = *o*, we have also cos. ^{2}A′ + cos. ^{2}B′ + cos.^{2}C′ = 1, ∴ between these five equations we can eliminate A′ B′ C′ L, and the resulting equation, which as of the second order being combined with the equations of the trajectory, which are given in each particular case, will determine the coordinates $x\ y\ z$ in a function of the time; if the three preceding equations be multiplied by $dx\ dy\ dz$ respectively, and then added together, we obtain

$$\frac{d^2x\,dx + d^2y\,dy + d^2z\,dz}{dt^2} = \text{P}dx + \text{Q}dy + \text{R}dz + \text{L (cos. A}'$$

dx + cos. B′. dy + cos. C′. dz) as the latter part of the second member = *o*, we have, by substituting for the first member its value, $\frac{(d^2s.\ ds)}{dt^2} = \text{P}.dx + \text{Q}.dy + \text{R}.dz$; ∴

$\frac{d^2s}{dt^2} = \text{P}.\frac{dx}{ds} + \text{Q}.\frac{dy}{ds} + \text{R}.\frac{dz}{ds}$ *i. e.* the accelerating force resolved in the direction of the tangent, is equal to the second differential coefficient of the arc considered as a function of the time, which is an extension of what has been established in Notes, page 430; it likewise appears that the force in the direction of the tangent is totally independent of L; and also that when there is no accelerating force, $\frac{d^2 s}{dt^2} = o$. It appears from what has been just established, that when the equations of condition of the motion of the material point are independent of the time, the resultant of the forces which are equivalent to the equations of condition is normal to the curve described by the point, for in that case $P'dx + Q'dy + R'dz = o$; P' Q' R' being the resultant of these forces resolved parallel to x y z respectively; but if these equations are functions of the time $P'dx + Q'dy + R'dz$ is not $= o$. If V denotes the resultant of all the accelerating forces which act on the point, and the θ angle which this resultant makes with the normal, V cos. θ expresses the resultant resolved in the direction of the normal, and when the curve described is of *single* curvature, $\frac{v^2}{r} + \text{V cos. } \theta$. expresses the entire pressure $= \text{L}; = \frac{v^2}{r} + \text{P}.\frac{dy}{ds} + \text{Q}.\frac{dx}{ds}$, ∴ if the equation of the trajectory be given, and also the values of P, Q, in terms of x y, we can determine v, and ∴ L, and substituting for L this value, in the expressions for $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ &c. we might by integrating, determine the position of the point at any given moment, and also its velocity. As the coordinates are arbitrary, if we make one of them to coincide with the normal to the curve, denoting by A′, B′, the angles which the radius of the osculating circle makes with the normal and with the coordinate, which is in the plane of the tangent, and by m, n, l, the angles which V, the resultant of all the forces, makes with the three coordinates, the force expressed by $\frac{v^2}{r}$ resolved parallel to these coordinates $= \frac{v^2}{r}. \text{cos. A}, \frac{v^2}{r}. \text{cos. B}, \frac{v^2}{r}. \text{cos. } 90°$; and V resolved parallel to these coordinates $= \text{V cos. } m$, $\text{V cos. } n$, $\text{V cos. } l$, and as A m denote the inclination of the radius of curvature, and of V to the normal, $\frac{v^2}{r}$. cos. A + V cos. m expresses the pressure of the point on the surface, V cos. n $+\frac{v^2}{r}$. cos. 90° expresses the force by which the point is moved; ∵ V cos.l + $\frac{v^2}{r}$. cos. B = the motion perpendicular to the tangent $=o$; hence, if V l v and r were given, we might determine B and ∵ the inclination of the plane of the osculating circle to the tangent plane, and when there is no accelerating force, $\frac{v^2}{r}$. cos. B = o, *i. e.* B=90, or the plane of the osculating circle is at right angles to the surface $\frac{v^2}{r}$. cos. B=$\frac{v^2}{r}$. sine of the inclination of plane of osculating circle to the plane which touches the surface.

(*aa*) Let the perpendicular distances of the given points from the plane which separates the two media = a, a', if through these two points a plane be conceived to pass perpendicular to the plane surface which separates the media, and if the line described be supposed to be projected on this plane, then, since the extreme points of this line are given, a a' the perpendicular distances of these points from the separating plane will also be given; and also c the intercept between these perpendiculars reckoned on this plane, let x, x', denote the angles which the projection of the line on the perpendicular plane makes with the perpendicular to the separating plane at the point, where the projection of the line described meets the separating plane; then we have evidently $c = a$ tang. $x + a'$ tang. x', if z denotes the perpendicular distance of the point where the ray of light meets the separating plane from its projection on the perpendicular plane, and y y' the distances of the given points from this plane, we have evidently $y=\sqrt{z^2+\frac{a^2}{\cos.^2 x}}$, $y'=\sqrt{\left(z^2+\frac{a'^2}{\cos.^2 x'}\right)}$; but as the density of the two media through which the light passes, though different, from one to the other, is uniform for each of them respectively; $n\ n'$ the velocities in those media will be uniform; $\because \int v\,d\,s = n\,y$ is the part of the integral of $v\,d\,s$ which appertains to the first medium, and $n'\,y'$ the part of this integral which appertains to the second, consequently by Note (*bb*) $ny + n'\,y' = \int v\,d\,s$ is a minimum, *i. e.* $n.\sqrt{z^2 + \frac{a^2}{\cos.^2 x}} + n'\sqrt{z^2 + \frac{a'^2}{\cos.^2 x'}}$ is a minimum with respect to z, x, x', of these $x\ x'$ are connected by the equation $c = a.\tan.x + a'.\tan.x'$; $\therefore$ in the first place the differential of the preceding function with respect to $z = o$, *i. e.* $n.\frac{dy}{dz} + n'.\frac{dy'}{dz} = o$, but $\frac{dy}{dz} = \frac{z}{y}$, $\frac{dy'}{dz} = \frac{z}{y'}$, $\therefore \frac{nz}{y} + \frac{n'z}{y'} = o$, but as this equation cannot be satisfied unless $z = o$, it follows, that the track of the luminous ray coincides with the plane perpendicular to the plane separating the surfaces, and passing through the two given points; therefore

$$ny + n'y' = \frac{a\,n}{\cos.x} + \frac{a'\,n'}{\cos.x'}$$ which as it is a minimum,

$$\frac{a\,n.\sin.x.}{\cos.^2 x}dx + \frac{a'n'.\sin.x'}{\cos.^2 x'}dx' = o,$$ but differentiating the equation $c = a.\tan.x + a'.\tan.x'$ we obtain

$$\frac{a.\,dx}{\cos.^2 x} + \frac{a'.\,dx'}{\cos.^2 x'} = o,$$ hence eliminating $\frac{dx'}{dx}$ between these two equations we find $n.\sin.x = n'\sin.x'$; but x is the angle of incidence, and x' the angle of refraction, *whose sines are therefore in a given ratio.* If the ray of light instead of penetrating the second medium is reflected back, then the velocity remains the same during the entire route, and $\int v\,d\,s$ becomes $v\int d\,s$, which is by hypothesis a minimum: therefore the track of the ray is the shortest possible, consequently it makes = angles with the reflecting surface, $\because$ the angles of incidence and reflexion are =.

(*bb*) $v^2 = C + 2.\int(Pdx + Qdy + Rdz)$ *see* page 432 $\because v\delta v = P\delta x + Q\delta y + R\delta z$, $\because$ substituting in the equation of page 431; we have $\delta x\,d.\frac{dx}{dt} + \delta y\,d.\frac{dy}{dt} + \delta z\,d.\frac{dz}{dt}. = v.\,dt.\,\delta v =$

$ds. \delta v$, now as $ds = \sqrt{dx^2 + dy^2 + dz^2}$, $\frac{ds}{dt}. \delta ds =$

$\frac{dx}{dt} \delta dx + \frac{dy}{dt}. \delta dy + \frac{dz}{dt}. \delta dz$, and as it is indifferent which of the characteristics d or δ precedes the other; we have

$$\frac{ds}{dt}. \delta ds = v. \delta ds = d. \left(\frac{dx\ \delta x + dy\ \delta y + dz.\ \delta z}{dt}\right) - \delta x\ d. \frac{dx}{dt} -$$

$$\delta y.\ d\frac{dy}{dt} - \delta z\ d\frac{dz}{dt} \therefore v. \delta ds + \delta v. ds = \delta (vds) =$$

$d. \left(\frac{dx.\ \delta z + dy\ \delta y + dz.\ \delta z}{dt}\right)$; integrating with respect to d, we have $\delta \int (v\ d\ s) = C' +$

$\frac{dx\ \delta x + dy\ \delta y + dz\ \delta z}{dt}$; when the extreme points of the line described by the point are fixed, $\delta x\ \delta y\ \delta z$ are $=$ to cypher at these points; $\therefore \delta \int (v. ds) = o$ for C' evidently vanishes; $\therefore \int (v\ d\ s)$ is either a maximum or minimum: but it is evident from the nature of the function $\int (v\ d\ s)$ that it is not a maximum; hence of all curves which a point sollicited by the forces P Q R, describes in its passage from one given point to another, it describes that in which $\delta. (v\ d\ s) = o$, consequently that in which $v. ds$ is a minimum; if there are no accelerating forces v is constant, and $\int (v. ds)$ becomes $v. \int d\ s$, $\therefore$ in this case the curve described by the moving point is the shortest, and in consequence of the uniformity of the motion the time will also be a *minimum*: since $\delta \int v. ds = o$, is true in all cases in which $Pdx + Qdy + Rdz$ is an exact differential, it is true for all curves described by the actions of forces directed to fixed centres, the forces being $\div$l to functions of distances from them; and if the form of these functions was given, we could determine the species of the curve described, by substituting for v its value in terms of the force, and then investigating by the calculus of variations the relation between the ordinates of the curve, which satisfies $\delta(vds)=o$. If the force varied as $\frac{1}{s^2}$ it would be easy to shew that the curve was a conic section origin in the focus, if the force varied as s the distance from centre, the curve described would be also a conic section, origin being in the centre.

CHAPTER III.

(a) In fact let p denote the action which m the first exerts on m' the second, if previous to the impact, m' is actuated by p and $-p$; the first m is employed in destroying $-p$, and to effect this it must employ a force $=$ and directly contrary to $-p$, and therefore it will lose a force $=$ to p;

(b) g the gravity must, however, be distinguished from w the weight, for g denotes the intensity of the power as it exists in nature without any reference to the quantity of matter put in motion; w denotes the force of gravity applied to the particular body under consideration, which depends not only on the intensity of the gravity, but also on the mass of the body on which it is exerted, so that w is the resultant of all the forces of gravity acting on each molecule. w is $\div$l to m, the quantity of matter, at a given place, but to determine the value of w in different latitudes, we must take into account the intensity of gravity, which varies from one place to another, $\therefore w = mg$ and as $m = v\,d$, $w = v.\,d.\,g.$ v being the volume and d the density.

(c) The reason why distilled water was selected as the term of comparison was, that it was one of the most homogeneous substances, and the maximum of its condensation was easily ascertained, as it always obtained about 4° above the freezing point the centigrade thermometer.

(*d*) What is here stated does not in the least tend to establish the exploded position of Des Cartes, that all space was equally full of matter, for according to him, all matter was homogeneous, and the subtle ether which was diffused through the planetary regions was of the same nature with other matter.

(*e*) Since perpendiculars from any point in the direction of the resultant of two forces, on the directions of the forces, are inversely as the forces, it follows that as in this case the resultant passes through the fulcrum, perpendiculars from fulcrum on the directions of the composing forces, are inversely as the forces.

(*f*) In general it may be remarked that the *whole* force necessary to perform any work is not diminished by the application of the mechanic powers, their use is either to diminish the force applied at once by lengthening the time, or to shorten the time, by increasing the force applied at once.

(*g*) This will immediately appear from Notes to Chapter II, for V the resultant resolved parallel to the axis of $x = V. \left(\frac{x-A}{u}\right)$, $=$ (as $x = \rho.\cos.\psi$, ρ being the projection u on the plane of xy) $V.\left(\frac{\rho.\cos.\psi - A}{u}\right)$ and this force resolved in the direction perpendicular to ρ *i. e.* in the direction of

$$\rho\,\delta\,\psi = \frac{V.}{u}(\rho.\cos.\psi - A).\frac{y}{\rho} = \frac{V.}{u}(\rho.\cos.\psi - A).\sin.\psi,$$

in like manner V when resolved parallel to the axis of y, and then perpendicular to ρ or in the direction of $\rho\,d\,\psi$ $= \frac{V.}{u}(\rho.\sin.\psi - B).\cos.\psi$, $\because$ the efficient part of V resolved in the direction of the element $\rho\,\delta\,\psi =$ $\frac{V}{u}.((\rho.\sin.\psi - B).\cos.\psi - (\rho.\cos.\psi - A)\sin.\psi)$, which as u^2

$=(\rho.\cos.\psi-A)^2+(\rho.\sin.\psi-B)^2+(z-C)^2$, and $\because u\left(\frac{\delta u}{\delta\psi}\right)$ $=-\rho.\sin.\psi\ (\rho.\cos.\psi-A)+\rho.\cos.\psi\ (\rho.\sin.\psi-B)$ is $=$ to $\frac{V}{\rho}.\left(\frac{\delta u}{\delta\psi}\right)=\frac{p\,V'}{\rho}$; V' being the projection of the given force on the plane xy, and p a perpendicular from the axis of z on the direction of V', and $\because \frac{p.V'}{\rho}$, the projected force resolved in a direction perpendicular to ρ, therefore we have $V.\left(\frac{\delta u}{\delta\psi}\right)=p\ V'=$ the moment of the projection of V with respect to the origin, but $V.\left(\frac{\delta u}{\delta\psi}\right)=\Sigma.S.\left(\frac{\delta s}{\delta\psi}\right)=$ the sum of moments of the composing forces, *see* page 410.

(*g*) It appears from the expression $p.V'$ that the moment of a force may be geometrically represented by means of a triangle, whose vertex is at the point, and whose base represents the intensity of the force; and if X, Y indicate the force V, resolved parallel to the axes of x,y respectively, $X=V_{,}.\left(\frac{x-A}{u}\right)$, $Y=V_{,}.\left(\frac{y-B}{u}\right)$, and these forces resolved respectively perpendicular to ρ, are $V_{,}.\left(\frac{x-A}{u}\right).\frac{y}{\rho}$, $V_{,}.\left(\frac{y-B}{u}\right).\frac{x}{\rho}$; their difference $=\frac{Yx-Xy}{\rho}=\frac{p.V'}{\rho}$.

(*h*) Hence if either p or V', vanish the moment is $=$ to o, and as the projection of the area of a plane curve on another plane, is equal to this area multiplied by the cosine of the angle contained between the two planes, it follows that the moment of the forces relative to any axis inclined to the greatest moment is equal to the greatest moment multiplied into the cosine of this inclination.

(*i*) If $s=$ the inclination of two planes of the moments H and V_0, or which is the same thing, the inclination of two perpendiculars to these planes; and if a, a', a'', b, b', b'', represent the angles which these perpendiculars make respectively with three rectangular axes, $\cos. s = \cos. a. \cos. b + \cos. a'. \cos. b' + \cos. a'' \cos. b''$; ∵ when $s = 90$, this function $= o$; we have also $\cos.^2 a + \cos.^2 a' + ^2 a'' = 1$; $\cos.^2 b + \cos.^2 b' + \cos.^2 b'' = 1$; ∵ if $V_{,}$ $V_{,,}$ $V_{,,,}$ represent the projections of the given moment H on three rectangular planes, xy, $x z$, $y z$, we have $V_{,} = H. \cos. a$, $V_{,,} = H. \cos. a'$, $V_{,,,} = H. \cos. a''$; in like manner we have $V_o = H. \cos. s = H. \cos. a. \cos. b + H \cos. a'. \cos. b' + H. \cos. a''. \cos. b'' = V_{,}. \cos. b + V_{,,}. \cos. b' + V_{,,,}. \cos. b''$; ∵ if we know the projection of the greatest moment on any three rectangular planes, we have its projection on any plane whose inclination to those is given; in like manner, if $V_{o,}$ $V_{o,,}$ represent the projections of H on two planes rectangular to each other and to the plane of projection of V_o, $b_{,}$ $b_{,}'$ $b_{,}''$ $b_{,,}$ $b_{,,}'$ $b_{,,}''$ being the angles which perpendiculars to these planes make respectively with $x\, y\, z$, we have $V_{o,} = V_{,}. \cos. b_{,} + V_{,,}. \cos. b_{,}' + V_{,,,}. \cos. b_{,}'''$, $V_{o,,} = V_{,}. \cos. b_{,,} + V_{,}. \cos. b_{,,}' + V_{,} \cos. b_{,,}''$, ∵ it follows, that $V_{,}^2 + V_{,,}^2 + V_{,,,}^2 = V_o^2 + V_{o,}^2 + V_{o,,}^2$; hence it appears that $V_{,}^2 + V_{,,}^2 + V_{,,,}^2$ is independent of the direction of the three perpendicular planes of projection, and $V_o = \sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2 - V_{o,}^2 - V_{o,,}^2}$; ∵ V_o is a maximum and $=$ H *i.e.* $\sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2}$ when $V_{o,} = o$, $V_{o,,} = o$, ∵ this constant quantity is the value of the maximum moment, and $V_{,} = V_o. \cos. a$, $V_{,,} = V_o \cos. a'$, $V_{,,,} = V_o. \cos. a''$, ∵ $\cos. a =$

$$\frac{V_{,}}{\sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2}},\ \cos. a' = \sqrt{\frac{V_{,,}}{V_{,,,}^2 + V_{,,}^2 + V_{,,,}^2}},\ \cos. a'' =$$

$$\frac{V_{,,,}}{\sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2}}$$; ∵ if we know the moments with respect to three rectangular planes arbitrarily selected, we have the value of the principal moment, and also its position; and if on perpendiculars to each of these three planes, lines be assumed respectively ∻l to the projections of the moments on these planes, the diagonal of the parallelopiped, of which, these three lines are the sides, represents the maximum moment in quantity and direction.

(*k*) Therefore it appears from Note (*i*) there will be an equilibrium, if the principal moment and the resultant of all the forces = cypher respectively; if there is a fixed point in the system, the resultant of all the forces is destroyed by its reaction; if there is no fixed point the resultant V must vanish, but this cannot be the case, unless each of the forces X Y Z respectively vanish; as $Xy - Yx = V_{,}$, so it might be shewn that $Zx - Xz = V_{,,}$; $Yz - Zy = V_{,,,}$; but as these three equations obtain at the same time, we have by multiplying the first by Z, (*see* Celestial Mechanics, page 89,) the second by Y, and the third by X, and then adding them together $V_{,}Z + V_{,,}Y + V_{,,,}X = o$, this is the =n of condition, which must be satisfied when the forces have an unique resultant; if X Y Z are = respectively to cypher, then the forces are reducible to two respectively =, but not directly opposed to each other.

(*l*) It is evident from what has been established in Notes (*g*) (*h*) of this Chapter, that generally the sum of the three composing forces, parallel to the three rectangular coordinates, are $\Sigma.\, m\, S.\, \left(\frac{\delta s}{\delta x}\right)$, $\Sigma.\, m.\, S.\, \left(\frac{\delta s}{dy}\right)$, $\Sigma\, m\, S \left(\frac{\delta s}{\delta z}\right)$; and the sum of the moments projected on the three planes may be expressed thus:

$$\Sigma\, m\, S \left(y.\left(\frac{\delta s}{\delta x}\right) - x.\left(\frac{\delta s}{\delta y}\right)\right);\ \Sigma.\, m.\, S. \left\{ z.\left(\frac{\delta s}{dx}\right) - x\left(\frac{\delta s}{\delta z}\right)\right\};$$

$$\Sigma\, m\, S \left\{ y.\left(\frac{\delta s}{\delta z}\right) - z.\left(\frac{\delta s}{\delta y}\right)\right\},$$ in the case of equilibrium, and that the point is free, these quantities are respectively = to cypher; if the forces acting on the system be those of gravity, $S = S', = S''$, &c. $\frac{\delta s}{\delta x} = \frac{\delta s}{\delta x'} = \frac{\delta s}{\delta x''}$, &c. the first three equations become $S.\left(\frac{\delta s}{\delta x}\right).\, \Sigma\, m$, $S.\left(\frac{\delta s}{\delta y}\right)\Sigma\, m$, $S.\left(\frac{\delta s}{\delta z}\right)\Sigma\, m$, and the last three become $S.\left(\frac{\delta s}{dx}\right).\, \Sigma\, m\, y - S.\left(\frac{\delta s}{\delta y}\right).\, \Sigma\, m\, x$;

$S.\left(\frac{\delta s}{\delta x}\right).\ \Sigma\, m\, z - S.\left(\frac{\delta s}{\delta z}\right).\ \Sigma\, m\, x$; $S.\left(\frac{\delta s}{\delta z}\right).\Sigma\, m\, y - S.\frac{\delta s}{\delta y}.$ $\Sigma\, m.\, z$; and the three first compound a unique force $= S.\,\Sigma\, m$ *i.e.* the weight of the system, which is destroyed by the reaction of the origin when it is fixed. If the origin of the coordinates be a given point different from the centre of gravity, and if C B A be the coordinates of the centre with respect to this point; then $\Sigma m.(x-A)=o$; $\Sigma m.(y-B)=o$; $\Sigma m.(z-C)=o$ when the origin is *fixed*; ∵ we have $A.\Sigma m = \Sigma mx$; $B.\ \Sigma m = \Sigma my$; $C.\ \Sigma m = \Sigma mz$; hence knowing the positions of the several bodies of the system with respect to the axes of x, y, z, we can determine the coordinates of the centre of gravity with respect to the same axes. As $(\Sigma(mx))^2 = \Sigma(m^2x^2) + 2\ \Sigma(mm', xx')$; and $\Sigma mm'(x-x')^2 = mm'x^2 + m\,m'x'^2 + m\,m''x^2 + m\,m''^2 + m'm''x''^2 + \&c. - 2mm''\ xx'' - 2m'\ m''\ x'x'' - \&c. = \Sigma(m\,m'x^2) - 2\Sigma(mm'\ xx')$; and as $\Sigma(mx^2).\Sigma m = \Sigma(m^2x^2) + \Sigma(m\,m'\ x^2)$, ∵ $\Sigma(mx)^2 = \Sigma(mx^2)\Sigma m - \Sigma m\,m'x^2 - \Sigma mm'(x-x')^2 + \Sigma(mm'x^2)$, ∵ $A^2 =$

$$\frac{(\Sigma mx)^2}{\Sigma m^2} = \frac{(\Sigma mx^2)}{\Sigma m} - \frac{\Sigma m\,m'(x-x)^2}{(\Sigma m)^2},$$

we might obtain corresponding values for B^2 and C^2, hence it is evident that $A^2+B^2+C^2 =$

$$\frac{\Sigma m.(x^2+y^2+z^2)}{\Sigma m} - \frac{\Sigma mm'((x'-x)^2+(y'-y)^2+(z'-z)^2}{(\Sigma m^2)}$$

consequently if we have the distances of the several bodies of a system from a given point and also their mutual distances from each other, we have the distance of the centre of gravity of those bodies from the same point, and if the same be given for three fixed points, the position of the centre of gravity in space will be obtained. If the expression $\Sigma((x-A)^2+(y-B)^2+(z+C)^2)$ be differentiated with respect to $x\,y\,z$ respectively, and the differential coefficient be then put $= o$, we shall have $\Sigma(x-A)=o$, $\Sigma(y-B) = o$, &c. this implies that the sum of the squares of the distances of the molecules from the point ABC is a minimum, and if these molecules are all equal to each other, and represented by m, we have $\Sigma m(x-A)=o$, $\Sigma m(y-B)=o$, &c.

$$\because A = \frac{\Sigma mx}{\Sigma m},\ B = \frac{\Sigma my}{\Sigma m},\ C = \frac{\Sigma mz}{\Sigma m};$$

consequently the centre of gravity of a system possesses this property, namely, that the sum of the squares of the distances of the points of the system from it, is less than for any other point whatever. If several forces concurring in a point constitute an equilibrium, and if at the extremities of lines ∻l to and in the directions of these forces, be placed the centres of gravity of = bodies, the common centre of gravity of these bodies will be the point where the forces concur; for as the forces are represented by lines taken in their direction and concurring in one point, if this point be made the origin of the coordinates, the sum of the forces parallel to the axes of x y z are $\Sigma(x)$, $\Sigma(y)$, $\Sigma(z)$, and by hypothesis they are = to cypher, $\because \Sigma(x), =o$, $\Sigma(y)$ $\Sigma(z)=o$, *i. e.* since the bodies are equal $\Sigma(mx)$, $\Sigma(my)$, $\Sigma(mz)$, are = to cypher, consequently the origin is in the centre of gravity of a system of bodies of which each is equal to m, ∵ if to all the points of any body, forces be applied directed towards the centre of gravity, and ∻l to the distances between these points and the centre of gravity, these forces constitute an equilibrium; it likewise appears that when several forces constitute an equilibrium, the sum of the squares of the distances of the point of concourse of these forces from the extremities of lines ∻l to these forces, is a minimum.

(*m*) This principle was established first by a copious induction of particular cases; it may be thus analytically announced, if S, S′, S″, &c. represent the forces actuating the several points of the system and δs, $\delta s'$, $\delta s''$, &c. the spaces moved over in the respective directions of these forces, we have $m\ S\ \delta s + m'\ S'\ \delta s' + m''\ S''\ \delta s'' +$&c. $= o$ in the case of the equilibrium of the system, the variations being subjected to the condition of the connexion of the parts of the system.—(*See* page 411, and also Celestial Mechanics, page 82.) It is also evident, that if the preceding equation obtains the system is in equilibrio; for suppose that while the preceding equation obtains the points m, m', m'', &c. are actuated by the velocities v, v', v'', in consequence of the action of the forces m S, m'S$'$, m''S$''$, &c. which are applied to them, the system will evidently be in equilibrio, in consequence of the action of these forces, and of $m\,v$, $m'\,v'$, $m''\,v''$, &c. applied in a contrary direction, ∴ δv, $\delta v'$, $\delta v''$, &c. denoting the variations of the directions of the new forces, we shall have from the preceding principle, $m\mathrm{S}.\delta s + m'\mathrm{S}\;\delta s' + m''\;\mathrm{S}''\;\delta s'' +$ &c. $- mv\delta v - m'v'\delta v' - m''v''\delta v''$, &c. $=$ o, but the positive part of this equation vanishes by hypothesis, ∴ $mv\delta v + m'v'\delta v' + m''v''\delta v''$, &c. $= o$, if we assume $\delta v = vdt$, $\delta v' = v'dt$, $\delta v'' = v''\delta t$, &c. as we are permitted to do, we shall have $mv^2 + m'v'^2 + m''\,v''^2 +$ &c. $= o$; ∴ $v = o$, $v' = o$, $v'' = o$, &c.; *i. e.* the system is in equilibrio when $m\,\mathrm{S}\delta s + m'\;\mathrm{S}'\,\delta s' + m''\;\mathrm{S}''\delta s''$, &c. $= o$;

The condition of the connexion of the parts of the system may be reduced to equations between the coordinates of the several bodies, if $u' = o$, $u'' = o$, &c. be these different equations, we should add, as in page 412, $\lambda\;\delta u$, $\lambda'\;\delta u'$, &c. to the function $\Sigma m.\;\mathrm{S}\,\delta s$; λ, λ', &c. being indeterminate functions which should be determined in the manner suggested in page 412, the equation given above then becomes $o = \Sigma m\;\mathrm{S}\delta s + \Sigma\lambda\;\delta u$; in this case we may treat the variations of the coordinates as arbitrary, and ∴ put their respective coefficients $= o$; which will give as many distinct equations, and thus enable us to determine $\lambda\;\lambda'$ &c. and therefore R R$'$ &c.

The six equations of equilibrium which were given in page 422 may be deduced from the equation $o = \Sigma m\mathrm{S}ds$, for if the bodies of the system are firmly united to each other, their mutual distances $d\;d'\;d''$ &c. are invariable, and

$$\therefore \sqrt{(x'-x)+(y'-y)^2+(z'-z)^2}$$
$$\sqrt{(x''-x)^2+(y''-y)^2+(z''-z')^2}$$
$$\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}$$

are constant, and therefore their variations $= o$; $\therefore$ whatever suppositions satisfy these conditions, will also obtain for the equation $\Sigma m.\ S\,\delta s$, but these variations $= o$, from either of two suppositions, namely, either from making $\delta x = \delta x' = \delta x''$, &c. $\delta y = \delta y' = \delta y''$, &c. $\delta z = \delta z' = \delta z''$, &c. or from making $\delta x = y\delta\tilde{\omega}$, $\delta y = -x\delta\tilde{\omega}$ $\delta x' = y'\ \delta\tilde{\omega}, \delta y' = -x'\delta\tilde{\omega}$, &c. $\delta\tilde{\omega}$ being any variation whatever, as is evident from making these suppositions in the variations of d, d', d'', &c. in the first case it is evident that these substitutions make

$$m\ \mathrm{S}.\frac{\delta s}{\delta x}+m'\mathrm{S}'\frac{\delta s'}{\delta x}+m''\mathrm{S}''\frac{\delta s''}{\delta x}+\&c. = o;\quad i.\ e.$$

$\Sigma m\,S\dfrac{\delta s}{\delta x} = o$, $\Sigma m\mathrm{S}\dfrac{\delta s}{\delta y} = o$; $\Sigma m\ \mathrm{S}.\ \dfrac{\delta s}{\delta z} = o$; which imply that in the case of the equilibrium of a system of bodies, the sum of the forces of the system resolved parallel to the axes of $x\ y\ z$ are $=$ to cypher. By substituting the other values of $\delta x\ \delta y$ &c. which satisfy the condition of the invariability of the distances of the bodies of the system, in the equation $o = \Sigma m.\ \mathrm{S}\delta s$, we obtain $o = \Sigma m\mathrm{S}.\left(y\dfrac{\delta s}{\delta x} - x\dfrac{\delta s}{\delta y}\right)$, and by changing the coordinates x, x', x'', &c. or y, y', y'', &c. into z, z', z'', &c. we shall obtain $o = \Sigma m\mathrm{S}\left\{y\left(\dfrac{\delta s}{\delta z}\right) - z\left(\dfrac{\delta s}{\delta y}\right)\right\}$; $o = \Sigma m\mathrm{S}.\ \left\{z\left(\dfrac{\delta s}{\delta x}\right) - x\left(\dfrac{\delta s}{\delta z}\right)\right\}$; which are the three other equations of the equilibrium of a system, indicating that the sum of the moments of the forces, parallel to any two axes, which would cause the system to revolve about the remaining axis, are respectively equal to cypher; if the origin of the coordinates was fixed and attached invariably to the system, the forces parallel to the three axes will be destroyed by the reaction of the fixed point, so that the three last equations are those which remain to be satisfied ; in this case the resultant of all the forces which act on the body, passes through the fixed point, and therefore is destroyed by its reaction. If there are two points fixed in the system there is only one equation of equilibrium, namely, that which expresses that the sum of the moments of the forces, which would make the system revolve about the line connecting the fixed points as an axis, is equal to cypher; in general the number of equations of equilibrium is equal to the number of possible motions which can be impressed on the system, or to the least number of indeterminate quantities.

(*o*) Perhaps it would be more accurate to say, that there were three kinds of equilibrium, namely, stable, unstable, and neutral, in the last the body is in a state of indifference, and has no tendency either to recover its primary position, or to deviate more from it; it is evident that it only obtains when the equilibrium exists under a continued change of position, a homogeneous sphere, or a cylinder whose axis is horizontal, floating in a fluid, are instances of this species of equilibrium, for they have no tendency to maintain one position in preference to another.—*See* Note (*h*) Chapter IV, and Notes (*f*) (*k*) Chapter V.

CHAPTER IV.

(*a*) By Notes, page 411, it appears that when the fluid is at rest, and ∵ each molecule at rest, the resultant of all the forces, by which it is actuated, must be at right angles to the surface on which it exists; for it follows from the perfect mobility of the particles of fluids, that when a fluid mass is in equilibrio, each constituent molecule of the fluid must also be in equilibrio. When a molecule exists on the surface of the fluid, the resultant of all the forces by which it is actuated, must from what is already established, be perpendicular to that surface; a molecule in the interior of the fluid mass is subjected to two distinct actions, one arising from the forces which solicit it, and the other from the pressure produced by the surrounding particles; and the entire pressure at any point arises from the combined action of the two. If a fluid mass, of which the molecules are solicited by any accelerating forces whatever, be in equilibrio, when contained in a vessel, which is closed on *every* side; and if the equilibrium would cease to exist if an aperture was made in the side of the vessel, it is necessary in this case, that the pressure exerted on the sides, should be perpendicular to them, as otherwise the resistance of the surface would not destroy the pressure; the intensity of this pressure in general varies from one point to another, and depends on the accelerating forces and on the position of the point; with respect to those fluids, which are termed elastic, they may indeed press on the sides of the vessel in which they are enclosed, though no motive forces act on the particles, or without any pressure urging the surface of the fluid; for as they perpetually endeavour to dilate themselves in consequence of their elasticity, this gives rise to a pressure on the sides of the vessel, which is always constant for the same fluid, and depends in general on the matter of the fluid, its density and temperature.

(*b*) By considering each molecule as an indefinitely small rectangular parallelopiped, we are permitted to suppose that the pressure accelerating forces and density of each point of any one of its surfaces are the same; we also can thus express the fact of the equality of pressure which is the fundamental principle from which the whole theory of their equilibrium may be deduced; let p denote the mean of all the pressures on any side, p' the corresponding pressure on the opposite side, ρ the density, P, Q, R, the accelerating forces which solicit the molecule, resolved parallel to the three coordinates of the angle of the parallelopiped next to the origin; then $dx\ dy\ dz$ represent the dimensions of the parallelopiped, and p being a function of $x\,y\,z$, we have $\delta p = \frac{\delta p}{\delta x}. \delta x + \frac{\delta p}{\delta y}. \delta y + \frac{\delta p}{\delta z}. \delta z$; now the parallelopiped, in consequence of the *pressure* to which it is subjected, will be urged in the direction of x by the force $(p'-p).\ dy.\ dz$, but as $p'-p$ is the differential of p, taken on the hypothesis that x only is variable, we have $p' - p = \frac{dp}{dx}dx = \frac{\delta p}{\delta x}.\ dx$; $\therefore (p - 'p).\ dy.\ dz = -\frac{dp}{dx}\ dx.\ dy.\ dz$, (we have taken $\frac{dp}{dx}$, &c. negatively, because they diminish the coordinates); but ρ being the density of the molecule, its quantity of matter $= \rho.\ dx.\ dy.\ dz.$ and its *motive force* arising from P, $= \rho$ P. $dx\ dy\ dz$; the force with which the molecule is solicited in consequence of the action of this force and of the pressure, both of which act on the molecules $= \left\{\rho \text{ P} - \left(\frac{dp}{dx}\right)\right\} dx.\ dy.\ dz$; similar equations may be obtained for the forces parallel to y and z.

(*c*) By hypothesis the molecule is in equilibrio, therefore in consequence of what is established in Notes, page 430, we have

$$o = \left\{\rho\, P - \left(\frac{dp}{dx}\right)\right\}. \delta x + \left\{\rho\, Q - \left(\frac{dp}{dy}\right)\right\}. \delta y +$$

$\left\{\rho\, R - \left(\frac{dp}{dz}\right)\right\}. \delta z$, *i. e.* $\delta p = \rho(P.\delta x + Q.\delta y + R.\delta z)$; as δp is an exact variation, the second member must be so likewise, therefore

$$\frac{d.\rho P}{dy} = \frac{d.\rho.Q}{dx};\ \frac{d.\rho.P}{dz} = \frac{d.\rho.R}{dx};\ \frac{d.\rho.R}{dy} = \frac{d.\rho.Q}{dz}$$

and multiplying the first of these equations by R, the second by — Q, and the third by P, we obtain by expanding,

$$\frac{\rho.R.dP}{dy} + \frac{R.P.d\rho}{dy} = \frac{R\rho.\ dQ}{dx} + \frac{RQ.\ d\rho}{dx};\ -\frac{\rho.Q.dP}{dz} -$$

$$\frac{Q.P.d\rho}{dz} = -\frac{\rho Q.dR}{dx} - \frac{RQ.d\rho}{dx};\ \frac{\rho P.d.Q}{dz} + \frac{P.Q.\delta\rho}{dz} = \frac{\rho P.dR}{dy}$$

$+\frac{RP.\ d\rho}{dy}$ by reducing all the terms of which $\delta\rho$ is a factor to one side, and adding them together, we obtain

$$\rho.\left\{\frac{R.dP}{dy} - \frac{R.\ dQ}{dx} - \frac{Q.\ dP}{dz} + \frac{Q.\ dR}{dx} + \frac{P.\ dQ}{dz} - \frac{P.\ dR}{dy}\right\}$$

$$= \left(-\frac{RP}{dy} + \frac{RQ}{dx} + \frac{QP}{dz} - \frac{R\,Q}{dx} - \frac{P\,Q}{dz} + \frac{RP}{dy}\right)\delta\rho = o;$$

therefore by concinnating

$$P.\frac{dQ}{dz} - Q.\frac{dP}{dz} + R\frac{dP}{dy} - P.\frac{dR}{dy} + Q.\frac{dR}{dx} - R.\frac{dQ}{dx} = o;$$

when this equation is satisfied, the equilibrium obtains though ρ remains undetermined. But if the relation indicated by this equation does not obtain between the forces P, Q, R, the fluid will be in a perpetual state of agitation whatever figure it may be made to assume; and if P Q R be functions of the coordinates, $\rho(P\delta x + Q\delta y +$

$R\delta z$) can be integrated by the method of quadratures, by means of which we can find the value of p for any given point, and ∴ the force with which any side of the vessel is pressed; but though the equilibrium be impossible when the equation of condition is not satisfied, it does not follow that when it is satisfied that the equilibrium will obtain; for in most cases this fluid must assume a determined figure depending on the nature of P Q R. Likewise though in the state of equilibrium all the molecules in the same strata have necessarily the same density, and experience the same pressure, the converse is not true, for in homogeneous incompressible fluids ρ is constant, in those sections of the fluid in which neither $\delta\phi$ nor $\delta p = o$;

δp never $= o$, if the fluid be elastic, because ρ being a function of p, if the density has a finite value, p can never vanish.

(d) If the fluid be free at its surface, $p = o$, ∴ if δx δy δz belong to the surface we have $o = P\delta x + Q\delta y + R\delta z = \lambda du$, u being the equation of the surface, and λ a function of $x\,y\,z$; therefore the resultant of P Q R must be perpendicular to those parts of the surface in which the fluid is free, for in this case

$$\frac{P}{\sqrt{P^2+Q^2+R^2}} \quad \frac{Q}{\sqrt{P^2+Q^2+R^2}}, \quad \frac{R}{\sqrt{P^2+Q^2+R^2}}$$

are the cosines of the angles which the resultant of P Q R makes with the axes of x y z, but as $P\delta x + Q\delta y + R\delta z = \lambda\delta u$ they express also the cosines of the angles which the same axes make with the normal, ∴ the normal coincides with the resultant; this coincidence of the normal with the resultant is a condition which must also be satisfied to insure the equilibrium; by means of it we can determine in each particular case, the figure corresponding to the equilibrium of the fluid; if for instance there is only one attractive force directed towards a fixed point, the form of the surface will be spherical, the fixed point being the centre of the sphere; (if this point be at an infinite distance the sphere degenerates into a plane,) for in this case $\frac{dp}{\rho}=\frac{F}{r}\{(x-a)\,dx + (y-b)dy + (z-c).dz\}=F.\,dr$; $a\ b\ c$ are the coordinates of the centre, to which F the force is directed, and if the origin of the coordinates be in the centre, we have $\frac{dp}{\rho}=\frac{F}{r}(xdx+ydy+zdz)$, if the density ρ is constant, or a function of p, the equation of each stratum of level becomes $x^2+y^2+z^2=C$, which belongs to a sphere of which the centre coincides with the point of common reunion of the directions of all the forces.

When $P\delta x+Qdy+R\delta z$ arise from attractive forces, as is stated in the text, it must be an exact variation $=\delta\phi$, $\because$ we have $\delta p=\rho.\,\delta\phi$, consequently as $\frac{\delta p}{\rho}=\delta\phi$, ρ must be a function of p, therefore p will be a function of ρ, and they will be same for all those molecules in which the value of ϕ is given, *i. e.* for molecules of the *same strata af level*; and for a fluid in which ϕ varies, an equilibrium cannot take place unless each respective stratum be homogeneous, for in this case ρ and $\because$ p is constant; for surfaces in which ρ is constant $\delta p=o$, therefore for such surfaces $P.\,\delta x+Q.\,\delta y+R.\,\delta z=o$, and the resultant coincides with the normal. The integral of $o=\rho\delta\phi$ is a constant arbitrary quantity, which indicates that the given equation $\phi=C$ appertains to an indefinite number of surfaces differing from each other by the value which is assigned to this quantity; in the equation $\phi=C$, $d\phi$ evidently $=o$, $\because$ ϕ is either a *maximum* or *minimum*, and generally when $P\delta x+Q\delta y+R\delta z$ is an exact variation, ρ is a function of ϕ, $\because$ the equation $\delta p-\rho.\,\delta\phi=o$, indicates that in the state of equilibrium, there is a function of p and $x\,y\,z$, which is either a maximum or minimum. If this quantity be increased by insensible gradations, we shall have an indefinite number of surfaces, distributing the entire mass into an indefinite series of strata constituting between any two strata what have been termed *strata of level;* the law of the variation of density ρ in passing from one stratum to the consecutive one, is altogether arbitrary, as it depends on what function of ϕ, ρ is, but this is undetermined. It appears, therefore, from what precedes, that there are two cases in which $\delta p = o$, when it is at the free surface, in which case p vanishes of itself, and also when p is constant, *i. e.* for all surfaces of the same level; ∵ when the fluid is homogeneous, the strata to which the resultant of the forces is perpendicular, are necessarily of the same density; when the fluid is contained in a vessel closed on every side, it is only necessary that all strata of the same level should have the same density; in elastic fluids it never could happen that p should vanish, or that $P\delta x + Q\delta y + R\delta z = o$, therefore unless the fluid extends indefinitely into space, so that ρ may be altogether insensible, it cannot be in equilibrio except in a vessel closed on every side.

(*d*) In the case of our atmosphere ρ is observed to be ÷l to p *i. e.* $p = k\rho$, k depends on the temperature and matter of the fluid, by substituting for ρ in the equation $\delta p = \rho\delta\phi$ we obtain $\delta p = \frac{p}{k}.\delta\phi$ ∵ by integrating $\log. p + C = \frac{\phi}{k}$, because when the temperature and matter is given, k is constant, by making $C =$ to $\log. E$, we obtain $p = Ec^{\frac{\phi}{k}}$, since ∵ p and ρ are functions of ϕ, they will be constant for each stratum of level, but the law of the variation of density is not arbitrary as in the case of incompressible fluids, for the equation $\rho = \frac{p}{k} = \frac{E}{k}.c^{\frac{\phi}{k}}$ determines the law; if the matter of the fluid remaining homogeneous the temperature should vary, k will be a function of the variable temperature, but in order that $\frac{\delta\phi}{k} = \frac{\delta p}{k}$ should be an exact variation, k and consequently the temperature should be functions of ϕ; these functions are arbitrary, hence when the fluid is in a state of equilibrium, the temperature is arbitrary though uniform for each stratum; if this law was given we could integrate $\frac{\delta\phi}{k}$, from which we could infer the law of the densities and pressures by means of the equations

$$p = \mathrm{E}\,e^{\int\frac{\delta\phi}{k}};\ \rho = \frac{\mathrm{E}}{k}.e^{\int\frac{\delta\phi}{k}}\ \&c.$$

(e) Let the horizontal surface of the quiescent fluid be the plane of the coordinates of x, y, the axis of z is in this case vertical, which is also the direction of g the accelerating force of gravity; hence $x\ y$ are $= o$, and $\mathrm{R} = g$ in the equation given in page 452, and then δp becomes $= \rho.\ g\delta z$, ∴ $p = \rho g z$, since $p = o$ when at the surface of the water where $z = o$, there is no constant; calling h the height of any level above the pressed surface, and A the area of the pressed surface, since all the points are equally pressed, and the pressure on each unit of the surface $= p$, Π the pressure on the entire base $= \mathrm{A}.p = \rho.gh\ \mathrm{A}$, $h\mathrm{A} =$ the volume of a cylinder whose base $=$ A and height that of the level of the water, and $\rho gh.$ A is the weight of a corresponding cylinder of water, ∴ whatever be the shape of the vessel, provided the base and height of the water above the base remain the same, the pressure which the base experiences from the incumbent fluid, remains the same, we suppose in this case that the fluid is in a vacuo, otherwise p does not vanish when $z = o$, and we must have at the surface $p' = \rho.ga$, this is the pressure due to the atmosphere, or to any force which acts equally on all the points of the horizontal surface.

(f) The pressure on each point $\rho g z . \pi = p \pi$, π expressing one of the equal elements of the base, into which the pressed surface is divided, and as the pressure of all the elements are parallel to each other, their resultant is obtained by taking the integral of $\rho g z . \pi$ extended to the entire area, this integral is = to $A z_{,}$, A denoting the area of the pressed surface, and $z_{,}$ the distance of the centre of gravity of this surface from the plane of the level of the fluid; from this it appears that the pressure depends only on the extent of the pressed surface, and on the depth of its centre of gravity below the level of the fluid, therefore if this surface was supposed to revolve about its centre of gravity, the pressure which it experiences will remain the same.

It is easy to estimate the lateral pressure of a fluid in a vessel whose sides are perpendicular to the base, for as the pressure is propagated equally in every direction, the pressure of each molecule is ∻ to its distance from the horizontal surface of level, hence it is easy to shew that the entire lateral pressure in such a vessel is equal to the weight of a triangular prism of water, whose altitude is that of the fluid, and whose base is a parallelogram, one side of which is equal to the altitude of the vessel, and the other side to its perimeter.

What precedes suggests a method of finding the centre of pressure of a fluid, this centre is that point to which if a force equal to the whole pressure were applied, but in a contrary direction, it would keep the surface at rest, it is therefore the point where the resultant of the pressures of all the elements of the surface meets it, and as the pressures of the elements are parallel forces, the point of applica- of this resultant must be determined by the theory of the moments of these forces, ∵ as $\rho g z \pi$ denotes the pressure for each element, $\int \rho g z^2 \pi$ expresses the sum of the moments of these elements with respect to the surface of the fluid, which is consequently $= A z_{,} z_{,,}$, $z_{,,}$ being = the distance of the centre of pressure from the surface, $\therefore z_{\prime\prime} = \frac{\int \rho g z^2 \pi}{A z_{\prime}}$, which shews that this centre coincides with that of percussion, hence if a plane surface which is pressed by a liquid be produced to the surface of the liquid, and their common intersection be made the axis of suspension the centre of percussion will be the centre of pressure :—*see* Note (g), Chapter V. This centre of pressure always lower than he centre of gravity except all the points of the surface are equally pressed, in which case they coincide.

(g) Let, as in page 352, P represent the weight of a body in a vacuo, P′ its weight in any fluid, V its volume, D its density, Π the weight of the displaced fluid, ρ its density, and g the accelerating force of gravity, we have $P = V.Dg$, $\Pi = V\rho G$, $P - \Pi = P'$, eliminating V and Π, we obtain $\frac{P}{P-P'} = \frac{D}{\rho}$; which equation gives the specific gravity of the body with respect to the specific gravity of the fluid it follows from what is stated in the text, that two bodies which balance in air, are not necessarily of equal weight, unless they are constituted of the same materials; it follows likewise from this, that as $gVD - gV\rho =$ the motive force of a body existing in air, by dividing this expression by V.D, the mass, the quote $= g. \left\{1 - \frac{\rho}{D}\right\}$ (ρ being the density of the air) represents the accelerating force of a body in the air; hence it appears that the air retards the motion of bodies, both because it diminishes its accelerating force, and also because it produces a retarding force depending on the velocity and figure of the moving body. When a body floats on the water, it actually exists in two different fluids, part being in the air and the other part in the water; hence the common rule for determining the specific gravities of bodies is incorrect; to correct the result we should subtract the number expressing the specific gravity of the air, from the two numbers expressing the specific gravities of the body and fluid on which it floats.

(h) A body, whether it floats on a fluid, or whether it is entirely submerged, will be *in equilibrio* when it satisfies the two following conditions, namely, when the centres of gravity of the body and that of the part immersed, or of that of the displaced fluid exist in the same vertical; secondly, when the weight of this portion = that of the body itself; if the body is homogeneous and entirely submerged, the two centres of gravity coincide, and if its density is the same with that of the fluid, it will remain suspended.

(i) The body being supposed to be in equilibrio in a fluid, the plane of its intersection with the fluid, which is termed the plane of flotation, will be horizontal, if it then be raised or depressed in a vertical line, and then inclined by an indefinitely small quantity θ to the horizontal position, and a plane parallel to the horizon being supposed to be drawn through the centre of gravity of the first plane, if ζ be the distance of this plane from the present plane of flotation of the fluid, the stability or instability of the fluid depends on the circumstance of ζ θ, which at the commencement are supposed to be very small, remaining always so.

u being the variable velocity of any element dm of the mass of the body, $\int u^2\, dm = C + 2\phi$ expresses the sum of the living forces, where ϕ depends on the forces of gravity, and on the vertical pressures which the fluid exerts on all the points of the surface of the body which are submerged under the water; but as the resultant of the motive forces, which are equal for each molecule, to the weight of an equal molecule of the water, is the same as that of the vertical pressures of the fluid, if dv be an element of the volume of the body, corresponding to dm, an element of its mass; the entire motive force of dm when immersed in the fluid $= gdm - g.\rho.dv$, and therefore from what is established in page 452, $\phi = \int zg.dm - \int zg\rho.dv$; $\therefore$ if $z_{,}$ represent the distance of the centre of gravity of the entire mass M, from the horizontal plane, we shall have $\int zg.dm = g.\int z.dm = gMz_{,}$; $\int zg\rho dv$ consists of two distinct parts, one relative to the volume V, the part of M which is beneath the original section of the body in its second position, it $\therefore = gV\rho z_{,,}$, $z_{,,}$ being the variable distance of the centre of gravity of V from the plane of flotation; $\therefore$ if K represent the value of $\int zdv$ taken in the limits of V, so that $g.\rho K$ may be the second part of $g.\int z\rho dv$, we shall have $\phi = gMz_{,}, - g\rho Vz_{,,}, - g\rho K$; but if a be the distance between the centres of gravity of M and V, in the second position of the body, this distance reduced to the vertical $= a.\cos.\theta$, $\therefore$ as the difference between $z_{,}$ and $z_{,,}$ is always $=$ to this reduced distance, we have $z_{,} = z_{,,} \pm a.\cos.\theta$, and $\therefore$ by substituting $\phi = \pm g\rho Va.\cos.\theta - g\rho K$. Now it is not difficult to prove by decomposing the area of the original section into an infinite number of elements, and then projecting them on the plane of flotation, (quantities of the third and higher orders being neglected,) that the value of $K = q.p.$ to $\int zdv = \frac{1}{2}b\zeta.^2\cos.\theta + \frac{1}{2}\gamma\sin.^2\theta.\cos.\theta$ where $b =$ the area of the original section, and $\gamma = \int l^2 d\lambda$, l being a perpendicular from any point in the original section on the intersection of the original section with the horizontal one drawn through the centre of gravity, and $d\lambda$ an element of the original section; hence we obtain the value of $\phi = \pm g.\rho Va.\cos.\theta - \frac{1}{2}g\rho b\zeta.^2\cos.\theta - \frac{1}{2}g\rho\gamma\sin.^2\theta.\cos.\theta =$ (neglecting quantities of the third and higher orders) $\pm g\rho Va \mp g\rho Va\theta^2 - \frac{1}{2}g\rho b\zeta^2 - \frac{1}{2}g\rho\gamma.\theta^2$ $\therefore \int u.^2dm + g\rho(\gamma.\pm Va).\theta^2 + g\rho b\zeta^2 = C$; $2g\rho Va$ being contained in the value of C; as θ^2, ζ^2 are positive, it may be shewn as in Notes to page 381 that if $\gamma \pm Va$ is positive, the constant quantity will always remain so; and the value of C depends on the values of $u\,\theta\,\zeta$ at the commencement of the motion, it therefore is a very small quantity;

$$\because \theta^2 \text{ is always} \angle \frac{C}{g\theta(\gamma \pm Va)};\ \zeta \angle \frac{C}{g\rho\, b};$$

the stability of the equilibrium depends on the sign of $\gamma \pm Va$, and it will be always stable when the coefficient is + at the origin and during the entire duration of the motion; as $\int l^2 d\lambda$ is necessarily +, if Va is also positive, the coefficient $Va + \int l^2 d\lambda$ is +, and the equilibrium is stable, but from what has been established already Va is +, when the centre of gravity of the entire mass is lower than that of the volume of the displaced water; but if this latter centre be the lower, then Va must be taken with a negative sign, and in order that $\gamma - Va$ may be + it is necessary that γ should be $> Va$; but γ varies with the position of the intersection of the horizontal plane with the original plane, which passes through the centre of gravity of the original plane, therefore in its revolution about this centre it must assume different values, and if in that part of the revolution in which γ is a minimum its value predominates over Va, it must do so in all other positions, and $\because \gamma - Va$ will be always positive; *e.g* in a ship the line, relatively to which γ or $\int l^2 d\lambda$ is a minimum, is evidently the line drawn from the prow to the stern; and if the area of its plane of flotation be divided into an indefinite number of elements, and if the sum of all these multiplied into the square of their respective distances from this line be greater than the product of the volume of displaced water multiplied into the distance of its centre of gravity from that of the vessel, the equilibrium will be stable relatively to all the small oscillations of the vessel.

(*k*) When fluids communicate by means of a level tube, the pressure of each is equal to a cylinder of the fluid whose base A = the common horizontal surface, and whose altitude = the vertical height of the upper surface of the respective fluids above the surface of contact; hence if s s' denote the specific gravities of two fluids and h h' their respective heights, we have $sh\text{A}=s'h'\text{A}$; hence as we know s', the specific gravity of the air at the earth's surface relatively to s that of the mercury in the barometer the ratio of s to s' gives the ratio of h to h' (the height of the homogemogeneous atmosphere.) It likewise appears that all barometers, whatever may be the diameters of their bores, stand at the same height.

CHAPTER V.

(a) Let the masses of the two bodies A A′ be represented by m and m', their velocities by v v', and let u be the common velocity after the shock $\int v-u$ will be the velocity lost by A the body, whose velocity is the greater of the two, and $u-v'$ will be the velocity gained by A′; by hypothesis $(m+m')u+m(v-u)+m'.(u-v')$ represents the sum of the quantities of motion previous to the shock, but in consequence of the conditions of equilibrium $m.(v-u)=m'(u-v')$, $\because$ $(m+m')u$ is what existed previously to the shock; and it is evident from the preceding equation that the common velocity $u=\frac{mv+m'v'}{m+m'}$; if, however A A′ moved in opposite directions with the velocities v v', then we would have $m(v-u)=m'(u+v')$ and therefore $u=\frac{mv-m'v'}{m+m'}$, but this value may be comprised in the first by attending to the signs of the velocities; this effect of the mutual shock of the two bodies is the same as if the forces F F′, which separately actuated m, m' to make them acquire the velocities v v', were impressed on m and m' simultaneously, for the velocity communicated by F to $m+m=$ $\frac{mv}{m+m'}$ and that communicated by $F'=\frac{m'v'}{m+m'}$, and $\because$ the velocity of $m+m'$ arising from the combined action of F and $F'=\frac{mv}{m+m'}\pm\frac{m'v'}{m+m'}$, the sign being + or — according as F, F′ act in the same or in contrary directions; if m' has no motion previous to the shock $v'=o$ and $u=\frac{mv}{m+m'}$; $\because$ if m' be very great relatively to m, this quantity vanishes. This is the case with respect to all bodies which impinge on the earth, and all points which are immoveable may be considered as bodies whose masses are infinite relatively to the striking bodies; as in this case, $mv=(m+m')u$, m loses by the shock a quantity of motion $=m'.\,u$, which is that gained by m', *see* Notes, page 440; multiplying the equation $(m+m')\,u=mv\pm m'v'$, by $2u$ and then subtracting from both sides $mv^2+m'v'^2+(m+m')u^2$ we obtain $mv^2+m'v'^2-(m+m')u^2=mv^2+m'v'^2-2u(mv\pm m'v')+(m+m')u^2$ *i. e.* $mv^2+m'v'^2-m'u^2-mu^2=m(v-u)^2+(u\mp v')^2$. $\because$ if the motion of a system of bodies experience a sudden change, there results a diminution in the sum of the living forces of all the bodies = to the sum of the living forces which would arise from the velocities lost or gained by the bodies.—*See* Notes (s) (t) of this Chapter.

(b) In fact if the principle of D. Alembert be applied to determine the circumstances of the impact of two bodies of any form whatever, this principle furnishes us in general with but twelve equations between the unknown quantities of the problem, which in the most general case of it, are thirteen in number, the percussion which the bodies experience at the instant of the shock being considered as one of them; ∵ there are not a sufficient number of equations to determine these unknown quantities; but the consideration of the compressibility of the two bodies furnishes an additional equation, and thus renders the problem completely determinate.

In order to prove what is asserted in page 279, let, as in the case of non-elastic bodies, $v\ v'$ be the velocities of $m\ m'$ previously to impact, they may be assumed respectively $=u+(v-u)$, $u-(u-v')$, let V, V′ be the velocities of $m\ m'$ after the impact, those being considered as positive which move in the direction of m before the shock, and those as negative which move in an opposite direction, ∵ v will be always positive, but $v'\ u$ V V′ may be either positive or negative; let u be determined from the equation $m(v-u)=m'(u-v')$, $u-v$, $u-v'$ will be destroyed by the impact, but in consequence of the perfect elasticity of m, m' they will be reflected back with those destroyed velocities; ∵ V the *entire* velocity of m after the shock $=u-(v-u)$ and V′ that of m', $=u+(u-v')$, ∵ substituting for u its value

$$\frac{mv+m'v'}{m+m'},\text{ we obtain } V=\frac{(m-m')\,v+2m'v'}{m+m'},$$

$$V'=\frac{(m'-m)\,v'+2mv}{m+m'};\ \because V-V'=v-v',\text{ and if } m=m';\ V=v';$$

$V'=v$; we have also by conscinnating $mV^2+m'V'^2=4u^2(m+m')-4u(mv+m'v')+m\,v^2+m'v'^2.=mv^2+m'v'^2.$

(*c*) In general when a body receives an impulsion in any direction, it acquires two different motions, namely, a motion of rotation, and a motion of translation, which are respectively determined by the equations given in page 444, when the three first equations vanish, the forces are reducible to two = and opposite forces acting in parallel directions, when the rotatory motion does not exist the instantaneous forces have an unique resultant passing through the centre of gravity; when the molecules of the body are solicited by accelerating forces, these in general modify the two motions which have been produced by an initial impulse; if, however, the resultant of the accelerating forces passes through the centre of gravity of the body, the rotatory motion is not affected by them, as is the case with respect to a sphere and planets supposed to be spherical, but in consequence of the oblateness of these bodies the direction of the accelerating force does not pass accurately through the centre; ∴ the axis of rotation does not remain accurately parallel to itself, however the velocity of rotation is not sensibly altered.—*See* Vol. II. Chapter VI.

(*c*) In order to determine the position, &c. of these axes, it is necessary to determine the pressure on a fixed axis which arises from a body revolving about this axis in consequence of a primitive impulse; for this purpose, if this fixed axis be the axis of z, $x\ y\ z$ being the coordinates of dm, $\bar{\omega}$ the angular velocity, and r the distance of dm from the axis of z; the centrifugal force of the element dm is ∺l to $r\bar{\omega}^2$; for it is $=$ to $\frac{r}{T^2}$ and T varying as $\frac{1}{\bar{\omega}}$, it is proportional to $r\bar{\omega}^2$; the fixed axis is therefore urged perpendicularly to its length by the force $r\bar{\omega}^2dm$, and the resultant of all such forces for the sum of the elements dm, or their two resultants, when they are not reducible to one sole force, expresses the entire pressure which the axis experiences during the motion of the body, and as $\frac{x}{r}, \frac{y}{r}$, are the cosines of the angles, which the direction of the force $r\bar{\omega}^2.dm$ makes with the axes of x and of y, $x\bar{\omega}^2dm$, $y\bar{\omega}^2.dm$ represent the components of this force resolved parallel to these axes, ∴ $\int x\bar{\omega}^2dm = \bar{\omega}^2\int xdm$ is the resultant of all the forces parallel to the axis of x, which integral is $= \bar{\omega}^2 Mx_{\prime}$ M being the mass of the body, and $x_{\prime}$ the coordinate of the centre of gravity parallel to the axis of x; and $\bar{\omega}^2 My_{\prime}$ expresses the resultant of the forces $\bar{\omega}^2 \int y\, dm$ parallel to the axis of y; if z' z'' represent the respective distances of the resultants $\bar{\omega}^2 Mx_{,}$, $\bar{\omega}^2 My_{,}$, from the plane of x, y, by what is established in Notes page 428, we shall have $Mx_{,}z_{,} = \int xz\, dm$, $My_{,}z_{,,} = \int yz\, dm$, by means of these equations we can determine $z_{,}$ $z_{,,}$ and the intensities of the forces which acting in the planes xz, yz urge the fixed axis perpendicularly to its length, if $z_{,} = z_{,,}$ the forces $\bar{\omega}^2 My_{,}$ $\bar{\omega}^2 Mx_{,}$, are applied to the same point, and are ∵ reducible to one force, of which the intensity $= \bar{\omega}^2 M(x^2_{,} + y_{,}^2)$ which therefore expresses the pressure on the axis of the body; now if the axis of z be supposed to be *entirely* free, *i. e.* if the body is supposed to revolve in such a manner that the centrifugal forces of the several points do not exercise any pressure on the axis of rotation, and so that this axis has in itself no motion of rotation, neither is it subjected to any pressure; then, not only the moments of the forces which would cause them to revolve about the axis of z, but also the pressures on this axis are = to cypher, *i. e.* $\int xz\, dm = o$, $\int yz\, dm = o$, $\int x\, dm = o$, $\int y\, dm = o$; from the two last it follows, that $x_{,}$ $y_{,}$ are = respectively to cypher, therefore each free axis must pass through the centre of gravity; however, it is evident from the two first, that an axis may pass through the centre of gravity without being free, for $\bar{\omega}^2 \int x dm = \bar{\omega}^2 Mx_{,}$ this quantity $= o$, when the origin is in the centre of gravity; but $z_{,}$ the coordinate of its point of application $= \dfrac{\int xz\, dm}{Mx_{,}}$ is infinite, unless $\int xz\, dm$ is at the same time equal to cypher.

In order to determine the position of the principal axis of rotation, conceive a plane to pass through this axis perpendicular to the plane xy, and let θ equal the angle formed by the intersection of this plane with this principal axis, and ψ the angle between the axis of x, and this intersection; now, if the position of an element dm be referred to three coordinates x' y' z' of which the first is parallel to the intersection above mentioned, the second is perpendicular to this intersection, and the third z' is parallel to the axis of z, then it is evident from the transformation of coordinates that we have $x' = x.\cos.\psi + y.\sin.\psi$, $y' = y\cos.\psi - x.\sin.\psi$, $z' = z$, and consequently $x'' = x.\cos.\psi\cos.\theta + y.\sin.\psi.\cos.\theta + z\sin.\theta$; $y'' = z.\cos.\theta - x.\cos.\psi.\sin.\theta - y.\sin.\psi.\sin.\theta$, $z'' = y.\cos.\psi - x.\sin.\psi$, x'', y'', z'' being the coordinates of which the free axis is one; therefore $x''y'' = -\frac{1}{2}x^2.\cos.^2\psi.\sin.2\theta - \frac{1}{2}y^2\sin.^2\psi.\sin 2\theta + \frac{1}{2}z.^2\sin.2\theta - xy.\sin.\psi.\cos.\psi.\sin.2\theta + xz\cos.\psi.\cos.2\theta + yz\sin.\psi.\cos.2\theta$; $x''z'' = -x^2.\sin.\psi.\cos.\psi.\cos.\theta + y^2\sin.\psi.\cos.\psi.\cos.\theta + xy.\cos.\theta.(\cos.^2\psi - \sin.^2\psi) - xz\sin.\psi.\sin.\theta + yz\cos.\psi.\sin.\theta$; ∵ substituting these values in the equation $\frac{\int x''y''.dm}{\cos.2\theta} = o$; $\frac{\int x''z''.dm}{\cos.\theta} = o$; by assuming $\int x^2dm = A$, $\int yd^2m = B$, $\int z^2dm = C$; $\int xydm = D$, $\int xzdm = E$, $\int yzdm = F$, we shall obtain the following equations, $o = -\frac{1}{2}.\tan.2\theta\,(A\cos.^2\psi + B.\sin.^2\psi - C) - D\sin.\psi.\cos.\psi.\tan.2\theta + E.\cos.\psi + F.\sin.\psi.$; $o = (B - A)\sin.\psi.\cos.\psi + D(\cos.^2\psi - \sin.^2\psi) - E\sin.\psi.\tan.\theta + F.\cos.\psi.\tan.\theta$; by assuming $\tan.\psi = t$ the last equation gives us

$$\tan.\theta = \frac{(A-B).t + D(t^2-1)}{(F-Et).\sec.\psi};$$

$$\because \tan.2\theta = \frac{2(F-Et)\{Dt^2+(A-B)t-D\}\sec.\psi}{(F-Et)^2.\sec.^2\psi - \{Dt^2+(A-B)t-D\}^2};$$

but by means of the first equation we obtain

$$\tan.2\theta = \frac{2(E+Ft).\sec.\psi}{(B-C)t^2\ 2Dt + A - C};$$

by equating these two values we derive an equation of the fifth degree and divisible by $\sec.^2\psi = 1 + t^2$; ∵ t will be finally given by an equation of the third degree; therefore for every body, there is either one or three real values of $t = \tan.\psi.$ and consequently of ψ, and as

$$\tan.\theta = \frac{(A-B)\sin.2\psi - 2D\cos.2\psi}{2(F\cos.\psi - E\sin.\psi)}$$

it is evident that we have always a real value for θ, and thus by means of the angles ψ and θ we can determine the position of a free axis for every body; but in point of fact, the three roots of the cubic equation are real, and therefore there are three free axes belonging to every body; for if the axis of x be the free axis determined by that one of the three roots which is given to be real, and if the axis of x'' of which the position is determined by ψ, θ, was also free, when these angles are always positive, the three roots are real, and ∵ then besides the axis of x'' the axis of x is also free; x'', y'', z'', x, y, z, representing the same as before, if x be a free axis we have $\int x\,y\,dm = D = 0$, $\int x\,z\,dm = E = 0$; consequently the axis of x' will be also a free axis, if the equations $\int xydm$ $\int xzdm$ respectively $= 0$, after substituting for them $D = 0$, $E = 0$, give real values for ψ and θ, but these equations then become, $o = F.\sin.\psi - \frac{1}{2}.\tan.2\,\theta.(A\cos.^2\psi + B.\sin.^2\psi - C)$, $o = F.\cos.\psi.\tan.\theta - (A - B)\sin.\psi.\cos.\psi$; these two equations of condition must be satisfied when the axes of x and x' are simultaneously free axes, but the last may be satisfied by supposing $\cos.\psi = 0$, or $\psi = 90°$, and $\sin.\psi = 1$. By substituting this value in the first equation we obtain $\tan.2\theta = \frac{2F}{B - C}$, which is satisfied by θ or $\theta' = \theta + 90$; since $\psi = 90$, and the *planes* xx', $x'x''$ are by hypothesis perpendicular to each other, the angle between the axis of x'' and the axis of x is also 90°, and therefore the axis of x is perpendicular to the two others, and as $\theta' - \theta = 90$, it follows that every body has at least three axes intersecting each other perpendicularly in their centre of gravity. It is evident from the preceding analysis that the values of θ ψ are deduced by assuming $\int xydm = o$; which it appears from the reality of the roots may obtain without the equations $\int x\,dm = o$, $\int y\,dm = o$, having place, if these equations do not vanish neither x, nor y, will vanish, but as by hypothesis, $\int xzdm, \int yzdm = 0$ respectively, z' z'' will vanish; and the axis of rotation will experience a pressure represented by $\bar{\omega}^2 M \sqrt{x_{\prime}^2 + y_{\prime}^2}$ applied at the origin of the coordinates, *see* page 466; in this case therefore if the origin be fixed, the pressure arising from the centrifugal forces will be destroyed; hence if the axis of rotation for which $\int y\, xdm = o$, $\int z\, xdm = o$, be at liberty to turn about the fixed point, the body will revolve about this axis, as if it was fixed; consequently it appears that a *fixed* point being given in a body of any figure whatever, there always exist three axes passing through this point, about which the body may revolve uniformly, without any displacement of these axes, and just as if these axes were altogether immoveable, and these are the only axes which possess this property, for supposing the body to revolve about any other axis passing through the fixed point, this axis would evidently experience a pressure, which would not pass through the fixed point; since then z' and z'' would no longer vanish, if the axis was suddenly remitted to revolve freely about the fixed point the pressure will no longer be destroyed, therefore this force would displace the axis of rotation and the motion would be deranged; it appears from this that if a body impinge on another retained by one sole point, its motion will be continued uniformly, if it commences to revolve about one of the principal axes which intersect in this fixed point, in the same manner as if the axis were fixed, in this case it is necessary that the percussion on the axis of rotation at the commencement of the motion be reduced to one sole force perpendicular to this axis and passing through the fixed point, for it is then counteracted by the resistance of the fixed point.

This sum of the products of each molecule of the body into the square of its distance from an axis is termed the moment of inertia of the body with respect to that axis.

If r, r', r'', denote the distances of the element dm from the axis of x, y, z, respectively, we have $r = \sqrt{y^2+z^2}$, $r' = \sqrt{x^2+z^2}$, $r'' = \sqrt{x^2+y^2}$ the moments of inertia relatively to x, y, z, are $\int r^2dm = (B+C)$, $\int r'^2dm = (A+C)$, $\int r''^2dm = A+B$ respectively, which are severally $= Ma^2$, Mb^2, Mc^2, a, b, c, being the coordinates of the centre of gravity, and besides these three equations between A, B, C, we have also $D=o$, $E=o$, $F=o$; x, y, z being free axes; relatively to any other axis x', of which the position, with respect to the plane xy, is determined by θ and ψ, we have the moment of inertia $x'' = \int(y''^2+z'^2)dm = A.(\sin.^2\psi+\cos.^2\psi \sin.^2\theta)+B(\cos.^2\psi+\sin.^2\psi. \sin.^2\theta)+C\cos.^2\theta = Mk^2$; for each axis situated in the plane of xy, we have $\theta=o$, and the moment of inertia with respect to such an axis $= A \sin.^2\psi + B.\cos.^2\psi + C = Mm^2$, $\therefore Ma^2 - Mm^2 = (B-A).\sin.^2\psi$, $Mb^2 - Mm^2 = (A-B).\cos.^2\psi$, $Mc^2 - Mm^2 = (A\cos.^2\psi + B\sin.^2\psi - C$, $Mk^2 - Mm^2 = (A\cos.^2\psi + B\sin.^2\psi - C).\sin.^2\theta$; whatever, therefore, be the value of ψ, or the *position* of the plane passing through the axis of x at right angles to the plane x, y, the difference between the moments of inertia relatively to a line which coincides with the intersection of these planes and to any axis x'' existing in the perpendicular plane $(=Mk^2-Mm^2,)$ will be a *maximum*, when $\theta=o$, in which case the axis of x'' coincides with z, a free axis of rotation, *i. e.* the difference between the moments of inertia with respect to the free axis of z, and an axis perpendicular to it existing in the plane of x, y which is also a free axis, is $>$ than the difference between the moments of inertia relatively to the first of these axes and any other axis, therefore the moment of inertia relatively to the first axis is $>$ or $<$ than with respect to any other axis according as it is $>$ or $\angle$ than relatively to the axis in the plane $x\,y$, $\therefore$ the moment of inertia with respect to the the first axis is either a *maximum* or *minimum*. As the same may be shewn to be true with respect to the other two principal axes at right angles to this first axis, it follows in general that in every body, the moment of inertia with respect to each of the three free axes is a maximum or minimum. But as from the nature of moments of inertia in general, they can neither become negative or infinite, it follows, that these three moments are alternately maxima or minima. If $\psi=o$, the intersection of the two planes coincides with the axis of x, and the moment with respect to the axis of x'' being by what goes before $>$ or $\angle$ than that relatively to the axis of x, the moment relatively to the first axis will be so of course, $\because$ if the moments relatively to the axis of x and the first axis are $=$, the moments relatively to all axes existing in the plane of these axes will necessarily be $=$; this is equally true for the other two pair of axes. In every case one of the three free axes may be considered as that to which the greatest moment appertains, and the other as that to which the least appertains, so that $Ma^2 > Mb^2$ and $Mb^2 > Mc^2$, $\because$ if $a^2=c^2+\alpha^2$, $b^2=c^2+\beta^2$; $c^2=a^2-\alpha^2$, $b^2=a^2-\gamma^2$, $\alpha\ \beta\ \gamma$ vanish if two or three of the moments are equal; from what precedes $A=\frac{1}{2}M(b^2+c^2-a^2)$, $B=\frac{1}{2}M(a^2+c^2-b^2)$, $C=\frac{1}{2}M(a^2+b^2-c^2)$, this being substituted in the value of $\int(y''^2+x''^2).dm$, given in preceding page, will give $Mk^2=Ma^2.\cos.^2\psi.\cos.^2\theta+Mb^2.\sin.^2\psi\cos.^2\theta.+Mc^2.\sin.^2\theta.=$ by substituting for $a^2\ b^2$ their values $c^2+\alpha^2$, $c^2+\beta^2$, $Mc^2+M\alpha^2.\cos.^2\psi.\cos.^2\theta+M\beta^2\sin.^2\psi.\cos.^2\theta$, (or by substituting $a^2-\gamma^2$ for b^2, and $a^2-\alpha^2$ for c^2)$=Ma^2-M\gamma^2.\sin.^2\psi.\cos.^2\theta-M\alpha^2.\sin.^2\theta$, $\because$ all moments Mk^2 are less than the greatest and greater than the least of those which belong to the three free axes, one of these last is an absolute maximum, and the other an absolute minimum. If Ma^2, Mb^2 Mc^2 the three principal moments are equal; α, β, γ are$=$to cypher, and $Mk^2=Ma^2=Mc^2=Mb^2$, $\because$ all the moments of the body are $=$. If only two of the moments Ma^2 and Mb^2 are equal, $\gamma=o$, and $Mk^2=Ma^2-M\alpha^2.\sin.^2\theta=Ma^2.\cos.^2\theta+Mc^2.\sin.^2\theta$, and by making $\theta=o$, $Mk^2=$

$Ma^2 = Mb^2$, ∵ all moments relatively to axes in the plane of $x\,y$ are $=$, which is indeed evident of itself, for if Mk^2 was $>$ or $\angle$ Ma^2, the difference between Ma^2 and Mb^2 would be still greater. It is on account of these remarkable properties that these three axes have been termed principle axes, since Mk^2 is $\angle M.a^2$ and $> Mc^2$, $k \angle a$ and $> c$, ∵ $b \angle a$ and $> c$, if $B = C$, then $Mb^2 = A + C = Mc^2 = A + B$, ∵ if $\psi = o$, we shall have because $D = E = F = o$, $\tan. 2\theta = \frac{0}{0}$; therefore the angle θ is indeterminate, ∵ all axes which exist in the plane x, y, satisfy the conditions of free axes, and are therefore principal axes; if $A = B = C$ the three moments are then equal, namely, $Ma^2 = Mb^2 = Mc^2$, ∵ sin. $\psi = \frac{0}{0}$ *i. e.* the two angles θ and ψ are indeterminate.

(f) If the body is not actuated by any extraneous forces, the equations of its motion of rotation are

$$\frac{dp}{dt} + \frac{C^2 - B^2}{C^2}.\ qr. = 0;\quad \frac{dq}{dt} + \frac{A^2 - C^2}{B^2}.pr = 0,$$

$$\frac{dr}{dt} + \frac{B^2 - A^2}{C^2}.pq = 0.$$

where $p = \bar{\omega}.$ cos. a, $q = \bar{\omega}.$ cos. β, $r = \bar{\omega}.$ cos. γ, cos. a being supposed $=$ cos. ψ. cos. θ, cos. $\beta =$ cos. ψ. sin. θ, cos. $\gamma =$ sin. θ, and ψ, θ denote the same as in page 467, and as cos. $^2 a +$ cos. $^2\beta +$ cos. $^2\gamma = 1$, we have $\bar{\omega}$ the angular velocity $= \sqrt{p^2 + q^2 + r^2}$. (Celestial Mechanics, p. 207). In order that the motion should be uniform about an invariable axis, it is necessary that the quantities p, q, r, should be constant; hence we have the three following equations of condition:

$$(C^2 - B^2).\ qr = 0.\ (A^2 - C^2).\ pr = 0.\ (B^2 - A^2).\ pq = 0;$$

we satisfy these conditions by assuming $A = B = C$, *i. e.* if all the moments of inertia are equal, and consequently all the diameters of the revolving body principal axes, the simplest case of this is that of an homogeneous sphere; it is also satisfied if two of the quantities p, q, r, vanish, which is evidently the case when the body revolves about a principal axis; in fact, if it be the axis of z, then $\alpha = \beta = 90$, and $\gamma = 0$; hence, it follows, that $p = 0$, $q = 0$, and $r = \tilde{\omega}$. The rotation is therefore uniform and invariable if the solid revolves about a principal axis, and conversely, the rotation cannot be uniform except about a principal axis; in fact, either the three moments of inertia A, B, C, are unequal, or only two of them A, B, are $=$, or all the three are equal; in the first case, pq, pr, qr, and therefore two of the quantities p, q, r, must vanish, or which is the same thing, two of the angles α, β, γ must be right angles, and the third $= 0$, in order that the motion may be uniform; hence it follows, that the axis of rotation is a principal axis; in the second case $pr = qr$, therefore $p = q = 0$, or $r = 0$, the first supposition makes $\alpha = \beta = 90$, and $\gamma = 0$, because $\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 0$, therefore the axis of rotation is a principal axis.

The second supposition $r = 0$, gives $\gamma = 90$, therefore the axis of rotation coincides with the plane of x, y, in which all the diameters are principal axes, because the two moments A, B, are equal. In the third case, $A = B = C$, the quantities p, q, r are undetermined, and as all diameters which pass through the centre of gravity are in this case principal axes, the axis of rotation will be one also. In all cases in which the axis of rotation is not a principal axis, whether the solid be free or solicited by extraneous forces, the velocity of the rotation as well as the position of the axis will be liable to changes, which depend on the conformation of the solid, *i. e.* on the quantities A, B, C, and on the position of the axis of rotation with respect to the principal axes, *i. e.* on p, q, r. If the axis of rotation is inclined to the principal axis in a very small angle, this axis being that of z, the quantities p, p are so very small, that we may neglect their product, therefore $dr = 0$, and $\because r = h$ a constant quantity. In

U U

this case also the velocity of rotation $= \tilde{\omega}$ is nearly constant; hence the two other equations become $0 = dp + \frac{C^2 - B^2}{A^2}. hq.dt$, $0 = dq + \frac{A^2 - C^2}{B^2}. hp.dt$; and it is evident that the integral of these equations must assume the form $p = m. \sin.(nt + e)$, $q = m' \cos. (nt + e)$, n, m', n, e being constant quantities, it is easy to prove by substitution that these circular functions satisfy the preceding differential equations, and therefore may be assumed as the values of p and q. *See* Celestial Mechanics, page 208. We may deduce from them $0 = mn + \frac{C^2 - B^2}{A^2}. hm'$; $0 = -m'n + \frac{A^2 - C^2}{B^2}. hm$, hence we have $n = \frac{B}{AB}. \sqrt{(A^2 - C^2).(B^2 - C^2)}$, and $m' = \frac{A}{B}. m. \sqrt{\frac{A^2 - C^2}{B^2 - C^2}}$. If $(A^2 - C^2). (B^2 - C^2)$ is positive, *i. e.* if the moment of inertia with respect to the axis of rotation Mc^2, is greater or less than Ma^2 and Mb^2, and consequently the greatest or least of all the moments, n, m, m' are real quantities, and p, q are expressed by real sines, therefore the variations of rotation being periodic, and confined within very narrow limits, the axis of rotation will make small oscillations about its primitive state, the magnitude of which may be determined by the equations $p = m. \sin. e$, $q = m'. \cos. e$. As by hypothesis, p, q, were at the commencement of the motion extremely small, m, m' are extremely small, and thus p, q will always differ very little from cypher. The state of rotation is therefore stable, if the body commenced to move about an axis inclined in a very small angle to one of the two principal axes, of which the moments of inertia are the greatest or least of all; the velocity will then experience only insensible and periodic oscillations, and the axis of rotation will make slight excursions about the principal axis, the rotation always returning to its primitive state; but if the solid revolves nearly about the principal axis, of which the moment of inertia Mc^2 exists between the two others, Ma^2 and Mb^2, the rotation will be subject to changes, which instead of being periodic may increase indefinitely; for n being then imaginary, the sine and cosine of $nt+e$, will be changed into exponentials, which are susceptible of continual increase; in this case, therefore, the motion of rotation is not stable, and the slightest derangement may cause the changes to be indefinitely great. And as observation proves that the rotation of the sun, planets, and satellites, (which are observed) is in a stable state, it appears certain that all the celestial bodies revolve very nearly about a principal axis, with respect to which the moment of inertia is the greatest or least, most probably the first, for on account of the compression of the earth arising from the rotation, the axis is smaller than the diameter of the equator, and therefore its moment of inertia is greater. *See* Tom. II. Chap. VI.

(g) Suppose the axis of x' to be this horizontal axis, and if the axis of y' be also horizontal, the axis of z' will be vertical, let the plane which passes through the axis of y' and z', pass through the centre of gravity, and let ϕ be the angle which the axis of z' makes with an axis passing through the centre of gravity and the origin of the coordinates; if y'', z'' be the coordinates referred to this new axis, then $y' = y''.\cos.\phi + z''.\sin.\phi$; $z' = z''.\cos.\phi - y''.\sin.\phi$; now as the coordinates y'', z'' are constantly the same for the same body, and only vary in passing from one molecule to another, by taking the differential of y' and z' with respect to the time, we obtain

$$\int.\frac{(x'dy'-y'dx')}{dt}.\ dm = -\frac{d\theta}{dt}.\int.dm.(y''^2+z''^2),$$

$\int.dm.(y''^2+z''^2)$ is the moment of inertia of the body with respect to the axis of x'; if this moment $=C'$, then from what is stated in page 443, multiplying by dm, and extending the expression to all the molecules, we obtain $\int . \frac{y'.dz'-z'.dy'}{dt} . dm = V = - C'. \frac{d\theta}{dt}$, and as C′ is constant we have $-C' \frac{d^2\theta}{dt^2} = \frac{dV}{dt}$, if the only force actuating the body be that of gravity, then the values of P, Q, which are supposed to act horizontally, will vanish, and R, which acts vertically, will be constant; hence, we obtain

$$\frac{dV}{dt} = \int .Ry'dm = R.\cos.\theta.\int .y''.dm + R.\sin.\theta \int z''.dm,$$

since the axis of z'' passes through the centre of gravity of the body, $\int y''.dm = 0$, and if h be the distance of the centre of gravity of the body from the axis of x', $\int .z''.dm = Mh$, M being the entire mass of the body, therefore

$$\frac{dV}{dt} = Mh.\ R.\sin.\theta, \text{ and consequently } \frac{d^2\theta}{dt} = -\frac{Mh.\ R.\sin.\theta}{C'};$$

suppose a second body, all whose molecules are condensed into one point, of which the distance from the axis of x' is $=$ to l, we shall have for this body $C' = M'l^2$, M′ expressing the mass, for as all the molecules are condensed into one point

$$h = l, \text{ and } \frac{d^2\theta}{dt^2} = -\frac{M'l}{M'l^2}.\ R.\sin.\theta = -\frac{R}{l}.\sin.\theta,$$

moreover $\frac{d^2\theta}{dt^2} = -\frac{R}{l}.\sin.\theta$, hence the two bodies will have the same oscillatory motion, if their initial angular velocities are the same, when their centre of gravity exists in the same vertical, and when $l = \frac{C'}{Mh}$, which is equivalent to the rule given in the text. Multiplying both sides of the equation $\frac{d^2\theta}{dt^2} = -\frac{R.\sin.\theta}{l}$ by $2d\theta$, then integrating we obtain $\frac{d\theta^2}{dt^2}=\frac{2R}{l}.\cos.\theta+B'$, the constant quantity B' depends on the angular velocity and value of θ at the commencement of the motion. From the equation $l=\frac{C'}{Mh}$, it appears that when the axis of rotation passes through the centre of gravity, $h=0$, and l is infinite, therefore the time of oscillation is infinite; in fact, in this case the action of gravity being destroyed the primitive impulse will communicate a motion of rotation, which will be perpetuated for ever if the resistance of extraneous causes be removed. The point which is distant from the axis of rotation by a quantity equal to l is termed the centre of oscillation of the body; and if the axis of rotation passed through this point, the centre of oscillation, with respect to the new axis, will be in the former axis of rotation; for the moment of inertia, with respect to the centre of gravity, being equal to $C'-Mh^2$, the moment of inertia with respect to the new axis will be $C'+Ml^2-2Mlh$, therefore the value of l for the new axis $=\frac{C+Ml^2-2Mlh}{Ml-Mh}$, but $C'=Mlh$, therefore the value of l for the new axis $=\frac{Ml^2-Mlh}{Ml-Mh}=l$.

Let $C'=A.\sin.^2\theta.\sin.^2\phi+B.\sin.^2\theta.\cos.^2\phi+C.\cos.^2\theta+Mh^2$, A, B, C, being the moments of inertia relatively to the principal axes passing through the centre of gravity, *see* page 467, we shall have $l=$

$$\frac{Mh^2+A.\sin.^2\theta.\sin.^2\phi+B.\sin.^2\theta.\cos.^2\phi+C.\cos.^2\theta}{Mh},$$

therefore l will be a minimum when the quantity C' becomes the least of the three principal moments of inertia, for in that case the two other moments must vanish; let A be the least of these moments, then $l=\frac{Mh^2+A}{Mh}$, for sin. θ. cos. $\phi = 0$, cos. $\theta = 0$, and to determine when l is a minimum, $d\theta = \frac{2M^2.h^2 - M^2h^2 - MA}{M^2h^2}$. $dh = 0$, hence $h = \sqrt{\frac{A}{M}}$, therefore l, and consequently the time of rotation will be a minimum when the axis of rotation is that principal axis relatively to which the moment of inertia is a minimum, and at a distance from the centre of gravity by a quantity $= \sqrt{\frac{A}{M}}$; lh is constant and $= \frac{C'}{M}$, which is equal to the square of the distance of a point called the centre of gyration from the axis of rotation, *i. e.* that point where if all the matter contained in the revolving body was collected, any point to which a given force is applied to communicate motion would be accelerated in the same manner as when the parts of the system revolve in their respective places, and consequently the same angular velocity is generated in both cases; therefore this distance is a geometric mean proportion between the distances of the centres of gravity and oscillation from the axis of rotation, and from what precedes it appears that when the time of vibration is a *minimum*, the distance of the centre of gyration from the axis of rotation is equal to the distance of the centre of gravity from the same, and the distance of the centre of oscillation from the same axis $= 2.\sqrt{\frac{A}{M}}$, in this case the centre of gyration is termed the principal centre of gyration.

If, as in the case of the planets the rotatory motion arises from a primitive impulse, of which the direction does not pass through the centre of gravity, then, in consequence of what is stated in notes (c) and (u), it follows this centre will move in the same manner as if the impulse was applied immediately to it, and the rotatory motion about this centre will be the same as if it was fixed; the sum of the areas described about this point by the radius vector of each molecule projected on the plane passing through the centre of gravity, and the direction of the impulsion multiplied respectively by these molecules will be proportional to the moment of the primitive force projected on the same plane; but this moment is evidently the greatest possible, for the plane which passes through its direction and through the centre of gravity, therefore this is the invariable plane. (*see* note (x) of this chapter.) If f be the distance of the primitive impulse from the centre of gravity, and v the velocity impressed on this point, Mfv will be the moment of the impulsion, and being multiplied by $\frac{1}{2}t$, the product will be equal to the sum of the areas described in t; but, as will be seen hereafter, this sum is equal to

$$\sqrt{C^2p^2+A^2q^2+B^2.r^2},$$

$$\because mfv=\sqrt{C^2p^2+A^2q^2+B^2r^2},$$

and if at the commencement of the motion we know the position of the principal axis with respect to the invariable plane, *i. e.* the angles θ and ϕ, we shall have the values Cp, Aq, Br, at the commencement, and therefore at any subsequent instant. Now, if the moving body was a sphere, of which the radius $=R$, and if U be the angular velocity with which it revolves about the sun, the distance from the sun being $=d$, $v=dU$, and as the planet is put in motion by a primitive impulse, the axis of rotation will be perpendicular to the invariable plane; and on the hypothesis that this axis coincides with the third principal axis, $\theta=0$, therefore $Aq=0$, $Br=0$; consequently $Cp=mfv=mfdU$; in a sphere $C=\frac{2}{5}.\ mR^2$, therefore $f=\frac{2}{5}.\frac{R^2}{d}.\frac{p}{U}$, by means of which we can determine the distance of the direction of the force which causes a planet to revolve about the sun with a velocity of rotation $= p$, and a velocity of revolution $=$ U. (*See* note (*c*).

(*h*) If a body describes an ellipse, the centre of the ellipse being the point to which the force is directed, the force will vary as the distance from the centre, and *vice versa* if the force vary as the distance, the curve describes an ellipse, the point to which the force is directed being in the centre, but it is evident that in cases of small impulsions made on the vibrating body, the force varies very nearly at the distance. The time of the revolution is twice the duration of the vibration of a pendulum whose length is the distance of the plane of the ellipse described from the point of suspension.

The general solution of the problem of the very small oscillations of a system of bodies about their points of equilibrium, is very complicated. However, the following may be considered as a *precis* of the method of Lagrange: he assumes that the coordinates of the several bodies may be expressed by the coordinates which appertain to the body in a state of equilibrium, increased by the very small variables which vanish in the state of equilibrium; this is always possible when the equations of condition, reduced into a series, contain the first powers of the variables, which are assumed to be extremely small; as for instance, if a, b, c, be the coordinates of a body in a state of equilibrium, when it deviates very little from this state, let the coordinate $x = a + \alpha$, $y = b + \beta$, $z = c + \gamma$, α, β, γ, being so extremely small that powers of them higher than the first may be neglected; then if the equations of condition $\mathrm{L} = 0$, $\mathrm{M} = 0$, &c., are in any position algebraic functions of x, y, z, x', &c.; as the position of equilibrium is one of the positions of the system, it is evident that the equations $\mathrm{L} = 0$, $\mathrm{M} = 0$, &c. must still subsist; x, y, z, x', &c. being supposed to become a, b, c, a', &c. hence, it is evident that these equations do not involve the time t, let A, B, &c. be what L, M, &c. become when x, y, z, x', become a, b, c, a', &c. it is evident that by substituting for x, y, z, x', &c. their values $a+\alpha$, $b+\beta$, $c+\gamma$, &c.

$$L = A + \frac{dA}{da}.\ \alpha + \frac{dA}{db}.\ \beta + \frac{dA}{dc}.\ \gamma + \frac{dA}{da'}.\ \alpha' + \text{\&c.};$$

$$M = B + \frac{dB}{da}.\ \alpha + \frac{dB}{db}.\ \beta + \frac{dB}{dc}.\ \gamma + \frac{d\beta}{da'}.\ \alpha', \text{\&c.}$$

∵ as relatively to the state of equilibrium $A = 0$, $B = 0$, The values $L - A$, $M - B$, are respectively equal to cypher, which will give the relation which ought to subsist between α, β, γ, α', &c. and by neglecting very small quantities of the second and higher orders, we will obtain linear equations by means of which we can determine the values of some of these variables in terms of the others, then by means of these first values, we shall find others more exact, taking into account the second, and even higher powers, as we wish.

(k) In general assuming $x = a + a_{,}\xi + a_{,,}\psi + a_{,,,}\phi$, &c. $y = b + b_{,}\xi + b_{,,}\psi + b_{,,,}\phi$, &c. $z = c + c_{,}\xi + c_{,,}\psi + b_{,,,}\phi$ +&c., where a, $a_{,}'$, $a_{,,}$, &c. b, $b_{,}$, $b_{,,}$, &c., c, $c_{,}$, $c_{,,,}$, &c. are constant, and ξ, ψ, ϕ, &c. are very small variable quantities, which are = to cypher in the case of equilibrium; when the variables ξ, ψ, ϕ, &c. are supposed to be in a constant ratio to each other, then in the expression for the sum of the living forces, and for its variation, we would arrive at an equation of the form $\frac{d^2\zeta}{dt^2} + k\xi = 0$, of which the integral is $\zeta = E.\ \sin.\ t.\sqrt{k+e}$, where k has as many values as there are unknown quantities; it is evident, that this expression represents the very small isochronous oscillations of a simple pendulum, the length of which is equal to $\frac{g}{k}$, g representing the force of gravity, therefore the oscillations of the different bodies of the system may be considered as made up of small oscillations analogous to those of pendulums, the lengths of which are $\frac{g}{k}, \frac{g}{k'}, \frac{g}{k''}$.

(*m*) As the coefficients E, E′, E″, &c. depend solely on the initial state of the system, we may always suppose this state to be such that all the coefficients E′, E″, E‴, &c. except one vanishes, then all the bodies of the system make simple oscillations analogous to those of the same pendulum, and it thus appears that the same system is susceptible of as many different simple oscillations, as there are moveable bodies; therefore, generally speaking, the oscillations of the system, of what kind soever they are, will only be made up of those simple oscillations, which from the nature of the system may have place; consequently however irregular the small oscillations which are observed in nature appear to be, they may be always reduced to simple oscillations, the number of which is equal to the number of vibrating bodies in the same system; this immediately follows from the linear equations, by which the motions of the body which compose any system are expressed, when those motions are very small. The system can never resume its original position when $\sqrt{k'}$, $\sqrt{k''}$, $\sqrt{k'''}$, &c. are incommensurable, for in that case the times of the oscillations are incommensurable:—if they are commensurate the system will return to the same position at the end of the time $T = \frac{2\pi}{\mu}$, where $\pi = 180$, and $\mu =$ the greatest common measure of $\sqrt{k'}$, $\sqrt{k''}$, $\sqrt{k'''}$, &c. θ will then be equal to the time of the compound oscillation of the system.

(*n*) This principle is called the principle of D'Alembert, as it was first announced by that philosopher, by means of it the laws of the motion of a system are reducible to one sole principle, in the same manner as the laws of the equilibrium of bodies have been reduced to the principle of virtual velocities.

In consequence of the mutual connexion which subsists between the several bodies of the system, the effect which the forces immediately applied to the several parts of the system would produce, is modified so that their velocities, and the directions of their motions are different from what would take place if the bodies composing the system were altogether free; therefore if at any instant if we compute the motions which the bodies would have at the subsequent instant, if they were not subject to their mutual action, and if we also compute the motions which they have, in the subsequent instant, in consequence of their mutual actions, the motions which must be compounded with the first of these in order to produce the second, are such, as if they acted on the system alone, would constitute an equilibrium between the bodies of the system, for if not, the second of the above-mentioned motions are not those which have actually place, contrary to hypothesis, *i. e.* if v, v', v'', &c. be the velocities which the bodies m, m', m'', &c. composing the system would have if each of them was isolated, and if u, u', u'', &c. are the unknown velocities with which the bodies are actuated in directions equally unknown, in consequence of the mutual connexion of the parts of the system; and if p, p', p'', be the velocities which must be compounded with u, u', u'', &c. acting in a contrary direction, in order to produce v, v', v'', &c. respectively, then there is evidently an equilibrium between mp, $m'p'$, $m''p''$, &c., the quantities of motion lost or gained; otherwise, u, u', u'', would not be the velocities which have actually place; as mp is the resultant of mu and of mv, taken in a direction the contrary to its motion, by substituting for mp, $m'p'$, $m''p''$, &c. their components, we may announce the principle by stating that there is an equilibrium in the system between the quantities of motion mv, $m'v'$, $m''v''$, &c. impressed on the bodies, and the quantities mu, $m'u'$, $m''u''$,

&c. which actually obtain; these latter being taken in a contrary direction, by announcing the principle in this way we avoid complicated and embarrassing resolutions, and we need not consider the quantities of motion lost or gained, besides we are enabled by it to establish directly equations of equilibrium between the *given* velocities v, v', v'', &c., and the unknown velocities u, u', u'', &c., which can therefore be determined by means of these equations. However it must be observed, that the above equation is not sufficient of itself to determine u, u', u'', &c. we must in addition, obtain another to be determined by the nature of the system.

If the bodies are actuated by accelerating forces, then if those resolved parallel to the coordinates x, y, z, be P, Q, R, for m, P′, Q′, R′, for m', &c., mP, mQ, mR, mP′, &c., will represent the motions parallel to the three axes which the bodies would have if they were altogether free, and

$$m.\frac{d^2x}{dt^2},\ m.\frac{d^2y}{dt^2},\ m.\frac{d^2z}{dt^2},$$

represent the motions parallel to the same, which the bodies actually have at the commencement of the second instant, which since they are to be taken in a direction opposite to their true one, must be affected with contrary signs to mP, mQ, mR. *See* page 432.

$$\because -m.\left(d.\frac{dx}{dt}+\text{P}.dt\right);\quad -m.\left(d.\frac{dy}{dt}+\text{Q}.dt\right);$$

$$-m.\left(d.\frac{dz}{dt}+\text{R}.dt\right),\ \&\text{c.}$$

will be destroyed.

$$i.\ e.\ \text{if}\ m.\frac{dz}{dt},\ m.\frac{dy}{dt},\ m.\frac{dz}{dt},$$

represent the partial forces of the body m at any instant, resolved parallel to the three axes, in the subsequent instant they will become

$$m.\frac{dx}{dt}+m.d.\frac{dx}{dt}-m.d.\frac{dx}{dt}+m.\,\mathrm{P}dt.$$

$$m.\frac{dy}{dt}+m.d.\frac{dy}{dt}-m.d.\frac{dy}{dt}+m.\,\mathrm{Q}.dt+\ \&c.$$

and as $m.\frac{dx}{dt}+m.d.\frac{dx}{dt},\ m.\frac{dy}{dt}+m.d.\frac{dy}{dt}$, &c.

only remain in the subsequent instant,

$$-m.d.\frac{dx}{dt}+m.\mathrm{P}.dt,\ -m.d.\frac{dy}{dt}+m.\,\mathrm{Q}.dt+\ \&c.$$

will be destroyed; by distinguishing in this expression the characters in x, y, z, P, Q, R, by one, two, &c., marks we shall have an expression for the velocities destroyed in m', m'', &c., and multiplying these forces by $\delta x, \delta y, \delta z$, &c., the respective variations of their directions, by means of the principle of virtual velocities, the following equation will be obtained,

$$0=m\delta x.\left(\frac{d^2x}{dt}-\mathrm{P}.\right)+m.\delta y.\left(\frac{d^2y}{dt^2}-\mathrm{Q}\right)$$

$$+m.\delta z.\left(\frac{d^2z}{dt^2}-\mathrm{R}\right)+m'\delta x'.\left(\frac{d^2x'}{dt^2}-\mathrm{P}'\right)+\ \&c.$$

if we eliminate by means of the particular conditions of the parts of the system as many variations as there are conditions, and then make the coefficients of the remaining variations separately equal to cypher, we shall obtain all the equations necessary for determining the motions of the bodies of the system.

As $\frac{d^2x}{dt^2}$ is made to express the increase of the velocity, the changes in the motion of m are made by insensible degrees. The preceding equation consists of two parts, entirely distinct, namely,

$$\Sigma.m.(\mathrm{P}.\delta x+\mathrm{Q}.\delta y+\mathrm{R}.\delta z),$$

and $\Sigma m\left(\frac{d^2x}{dt^2}.\,\delta x+\frac{d^2y}{dt^2}.\,\delta y+\frac{d^2z}{dt^2}.\,\delta z\right)$, &c.

the first member would be equal to cypher if P, Q, R, P′, &c., which are applied to the several bodies of the system constituted an equilibrium; the other part arises from the motion which is produced by the forces P, Q, R, P′, &c., when they do not constitute an equilibrium, and the equation in page 431 is only a particular case of this; the second member is totally independent of the position of the axis of the coordinates, for substituting

$$\text{for } x,\ ax' + by' + cz',\ \text{for } y,\ a'x' + b'y' + c'z',$$
$$\text{for } z,\ a''x' + b''y' + c''z' + \&c.$$

and substituting also

for d^2x, d^2y, d^2z, δx, δy, δz, &c., their values in terms of these quantities, (a, b, c, a', &c. being supposed to be constant,) we obtain an equation of the same form as the preceding, for

$a^2 + a'^2 + a''^2 = 1$, $ab + ac + bc = 0$, &c. see page 409, the same substitutions being made in the expressions of the mutual distances, the coefficients a, b, c', a', &c., will disappear for the same reasons. The principle of D'Alembert by itself, without introducing the consideration of virtual velocities, would enable us to infer several important results; but it is its combination with that of virtual velocities which has contributed so much to the improvement of rational mechanics, as by means of it all mechanical problems are reducible to one sole principle, namely, that of virtual velocities; and thus every problem of dynamics may be reduced to the integrations of differential equations, so that as it belongs to pure analysis alone to effect the integration, the only obstacle to the perfect solution of every problem of dynamics arises from the imperfection of our analysis.

(p) In order to determine the condition of a fluid mass at each instant, we must know the direction of the motion of a molecule, its velocity, its pressure p, and the density ρ, but if we know the three partial velocities, parallel to the three ordinates, we shall have the entire velocity, and also the direction, for the partial velocities divided by the entire velocities, express the cosines of the angles which the coordinates make with its direction; hence we have five unknown quantities. Now, in the general equation of equilibrium furnished in Notes, page 452, namely, $\delta p = P.\delta x + Q.\delta y + R.\delta z$, the characteristic δ is independent of the time; but when the fluid is in motion we must, by what has been just established, substitute

$$P - \frac{d^2x}{dt^2} \text{ for P, } Q - \frac{d^2y}{dt^2} \text{ for Q, } R - \frac{d^2z}{dt^2} \text{ for R,}$$

and after the substitution, if we concinnate, and assume that

$$P.\delta x + Q.\delta y + R.\delta z = \delta V,$$

then we shall have

$$\frac{\delta p}{\rho} = \delta V - \frac{d^2x}{dt^2}.\,\delta x - \frac{d^2y}{dt^2}.\,\delta y - \frac{d^2z}{dt^2}.\,\delta z;$$

since the variations δx, δy, δz, are independent, this equation is equivalent to three distinct equations; besides these, we obtain another from the circumstance of the continuity of the fluid, for though each indefinitely small portion of the fluid changes its form, and if it is compressible, its volume likewise, during the motion, still as the mass must be constant, the product of the volume into the density must be the same as at the commencement; and by equating the two values of the mass we obtain the equation relative to the continuity of the fluid.

Neglecting quantities indefinitely small of the fifth order, the volume of the element at the end of the time

$$t + dt \text{ is } dx.dy.dz.\left(1 + \frac{du}{dx}.\,dt + \frac{dv}{dy}.\,dt + \frac{dw}{dz}.\,dt\right),$$

and the density at the same epoch becomes

$$\rho + \frac{d\rho}{dt}.\,dt + \frac{d\rho}{dx}.\,udt + \frac{d\rho}{dy}.\,vdt + \frac{d\rho}{dz}.\,wdt,$$

multiplying this expression by the corresponding volume, the product expresses the mass at the end of $t+dt$, from which subtracting $\rho.dx.dy.dz$, the remainder will be the variation of this mass, which should be $=0$; hence, we obtain by suppressing common factors, and neglecting dt^2

$$\frac{d\rho}{dt}+\frac{d\rho}{dx}.u+\frac{d\rho}{dy}.v+\frac{d\rho}{dz}w+\rho.\frac{du}{dz}+\rho.\frac{dv}{dy}$$
$$+\rho.\frac{dw}{dz}=0;\ i.\ e.\ \frac{d\rho}{dt}+\frac{d.\rho u}{dx}+\frac{d.\rho v}{dy}+\frac{d.\rho z}{dz}=0;$$

if the fluid is incompressible this equation is resolvable into two, for both the mass and also the density remain the same. The two into which it is resolvable are

$$\frac{d\rho}{dt}+\frac{d\rho}{dx}.u+\frac{d\rho}{dy}.u+\frac{d\rho}{dz}w=0,$$
$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0;$$

and these combined with the three equations already mentioned, will be sufficient to determine p, ρ, u, v, w, in a function of x, y, z, t; the first of these equations becomes a purely identical one when ρ is constant, but in this case we have only four unknown quantities. With respect to elastic fluids we have also only four different equations; however, it is to be remarked, that in this species of fluid, the density is always in a given ratio to the pressure p, therefore they are reduced to one unknown quantity, provided that the temperature is given; and even if it is not, if it varies according to a given law, so that that the temperature may be assumed a given function of x, y, z, and t, the coefficient which expresses the ratio of ρ to p, will be a given function of these variables; consequently, whether the motion to be determined be that of an incompressible fluid, or of one, of which the temperature is constant, or variable according to a given law, we shall in all cases have as many differential equations as unknown quantities; as these equations are those of partial differences between four independent variables,

x, y, z, and t, their perfect integrations cannot be effected by the ordinary methods, except that by means of some hypothesis they are simplified; and even in such a case, we should determine by means of the state of the fluid at the commencement of the motion, the arbitrary functions which their integrals contain.

(q) If in the expressions of p. 485, we suppose the origin of the coordinates to be in a point x, y, z, then in the values of $\delta f, \delta f', \delta f''$, &c., we have evidently $\delta x' = \delta x + \delta x_{,}'$, $\delta y = \delta y + \delta y_{,}'$, $\delta z' = \delta z + \delta z_{,}' +$&c., therefore if in the values of δf, $\delta f'$, $\delta f''$, &c., of the variations of the mutual distances given in page 448, we substitute these values for $\delta x'$, $\delta y'$, $\delta z'$, &c., the variations δx, δy, δz, &c. will disappear from these expressions; consequently, by substituting these values for $\delta x'$, $\delta y'$, $\delta z'$, &c. in the equation given in page 485, we obtain

$$0 = m.\delta x. \left(\frac{d^2x}{dt^2} - P\right) + m.\delta y \left(\frac{d^2y}{dt^2} - Q\right) + m.\delta z \left(\frac{d^2z}{dt^2} - R\right)$$
$$+ m'\delta x. \left(\frac{d^2x'}{dt^2} - P'\right) + m'.\delta x_{,}'. \left(\frac{d^2x'}{dt^2} - P'\right) + m'.\delta y.$$
$$\left(\frac{d^2y'}{dt^2} - Q'.\right) + m'\delta y_{,}' \left(\frac{d^2y'}{dt^2} - Q'.\right) + \&c.$$

the terms in the expression which are multiplied by δx, δy, δz, respectively, are, by adding them together

$$\Sigma m. \left(\frac{d^2x}{dt^2} - P\right); \quad \Sigma m. \left(\frac{d^2y}{dt^2} - Q\right); \quad \Sigma m. \left(\frac{d^2z}{dt^2} - R\right)$$

Consequently, if, as is supposed, the system be free, the conditions relative to the mutual connexion of the bodies will only depend on their mutual distances, hence the variations of δx, δy, δz, are independent of these conditions, and therefore the preceding expressions by which they are respectively multiplied, must be put severally equal to cypher; and as from what is laid down in page 446,

$$A = \frac{\Sigma mx}{\Sigma m}; \; B = \frac{\Sigma my}{\Sigma m}; \; C = \frac{\Sigma mz}{\Sigma m},$$

we have

$$\frac{d^2A}{dt^2} = \frac{\Sigma m.\frac{d^2x}{dt^2}}{\Sigma m} = \frac{\Sigma m.P}{\Sigma m},$$

we obtain in the same manner

$$\frac{d^2B}{dt^2} = \frac{\Sigma m.Q}{\Sigma m}; \quad \frac{d^2C}{dt^2} = \frac{\Sigma m.R}{\Sigma m};$$

therefore if all the bodies of the system were united in the centre of gravity, and the forces which are applied to them, when separate, were simultaneously impressed on them; the motion of such a body is the same as that of the centre of gravity; if the system was only subject to the mutual actions p, p', &c., of the bodies composing it, and to their reciprocal attractions; then since f, f', f'', &c. the distances of the bodies are

$$= \sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2},$$
$$\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2} + \&c.$$

in consequence of the sole action p, we have

$$mP = p.\frac{(x-x')}{f}, \; mQ = p.\frac{(y-y')}{f}, \; mR = p.\frac{(z-z')}{f};$$

$$m'P' = p.\frac{(x'-x)}{f}, \; m'Q' = p.\frac{(y'-y)}{f}, \; m'R' = p.\frac{(z'-z)}{f}.$$

$$\because mP + m'P' = 0; \; mQ + m'Q' = 0; \; mR + m'R' = 0; \; \&c.$$

and a similar proof may be shewn for the bodies in the case of their mutual attractions. As action is equal to reaction, though its direction be contrary, when two bodies impinging on each other exercise a *finite* action in an instant, their reciprocal action will disappear in the expressions $\Sigma.mP$, $\Sigma.mQ$, &c.; in fact, as we can always suppose the action of the bodies to be effected by means of a spring interposed between them, which endeavours to restore itself after the shock, the effect of the shock will be produced by forces of the same nature with p, which, as we have seen, disappear in the expressions $\Sigma.mP$, $\Sigma.mQ$,

&c. By integrating $\frac{d^2A}{dt^2}=0$, we obtain $\frac{dA}{dt}=b$, and $A=bt+a$; a is the value of A at the commencement of the motion, and b is the uniform velocity of the centre of gravity resolved parallel to A. In like manner the invariability of the motion of the centre of gravity of a system of bodies, notwithstanding their mutual action, subsists even in the case in which some of the bodies lose in an instant, by this action a finite quantity of motion; for since $d.\frac{A}{dt}.\ \Sigma m=\Sigma m.\frac{dx}{dt}=$ the quantity of motion, and since by the principle of D'Alembert the quantity of motion $\Sigma m.\frac{dx}{dt}$ before and after impact, should be equal to cypher, *i. e.* such as would cause an equilibrium in the system, it follows that $\frac{dA}{dt}.\ \Sigma m$, before and after impact should be equal to nothing, *i. e.* as Σm is given, $\frac{dA}{dt}$, the velocity of the centre of gravity in the direction of the axis of x, is not affected by the impact. We can therefore always determine the motion and direction of the centre of gravity of a system, by the law of the composition of forces, for it moves in the same manner as a body equal to the sum of the bodies would move, provided that the same momenta are communicated to it as are impressed on the respective bodies of the system; and if the several bodies of the system were only subject to their mutual action, then they would meet in the centre of gravity, for they must meet, and the centre of gravity remains at rest.

(r) We may make the variation δx disappear from the expressions for $\delta f, \delta f'$, &c., by another supposition beside that of page 489; for if we assume

$$\delta x'=y'\frac{\delta x}{y}+\delta x'_{,};\ dx''=\frac{y''\delta x}{y}+\delta x_{,}'';\ \delta y=-\frac{x\delta x}{y}.$$
$$+\delta y_{,},\ \&c.$$

then if in the expression for δf, $\delta f'$, &c, =

$$\frac{\sqrt{(x'-x).(\delta x'-\delta x)+(y'-y).(\delta y'-\delta y)+(z'-z(.(\delta z'-\delta z)}}{f},$$

&c. we substitute for $\delta x'$, $\delta x''$, $\delta y,'$ $\delta y''$, &c. their preceding values, they become

$$(x'-x).\left(\frac{y'\delta x}{y}+\delta x_{,}'-\delta x\right)+(y'-y)\left(-\frac{x'\delta x}{y}+\delta y_{,}'+\frac{x\delta x}{y}-\delta y_{,,}\right) \text{ divided by } f, =$$

by omitting quantities which destroy each other

$$(x'-x),\ \delta x_{,}'+(y'-y).\ (\delta y_{,}'-\delta y_{,}),\ \&c.$$

hence, making those substitutions, the variation δx disappears from the expressions for δf, $\delta f'$, &c., and it is easy to perceive, if the preceding values be substituted for δx, δy, $\delta x''$, &c., in the equation given in page 485, that the coefficient of δx will be

$$\Sigma.m.\frac{(xd^2y-yd^2x)}{dt^2}+\Sigma m.\ (\mathrm{P}y-\mathrm{Q}x),$$

which is, by what precedes, equal to cypher, therefore its integral with respect to t is

$$c=\Sigma m.\ \frac{(xdy-ydx)}{dt}+\Sigma\ \textstyle\int.m(\mathrm{P}y-\mathrm{Q}x).\ dt,$$

if in place of the forces Q, Q′, &c., parallel to the axis of y, we substitute the forces R, R′, &c., parallel to the axis of z, or in this last if we substitute Q, Q′, for P, P′, &c., we shall obtain the corresponding equations

$$c'=\Sigma m.\ \frac{(xdz-zdx)}{dt}+\Sigma\textstyle\int m.(\mathrm{P}z-\mathrm{R}x).dt,$$

$$c''=\Sigma m.\ \frac{(ydz-zdy)}{dt}+\Sigma\textstyle\int m.(\mathrm{Q}z-\mathrm{R}y).\ dt;$$

it is evident, from what precedes, if the bodies of the system are only subjected to the action of forces arising from their mutual action, and of forces directed to a fixed point, that then

$\Sigma m.(\mathrm{P}y - \mathrm{Q}x)$, $\Sigma m.(\mathrm{P}z - \mathrm{R}x) +$ &c. are respectively $= 0$;

$$\because c = \Sigma m. \frac{xdy - ydx)}{dt};\ c' = \Sigma m.\ \left(\frac{xdz - zdx}{dt}\right)\text{\&c. but}$$

from what has been stated in pages 390, 429, $\frac{xdy - ydx}{2}$, = the area traced by the radius vector of m in dt, hence then appears the truth of what is asserted in the text, that when the bodies composing the system are only subject to their mutual actions, and to attractions directed towards a fixed point, the sum of the areas multiplied respectively by the masses of the bodies is proportional to the time. The constant quantities c, c', c'', may be determined at any instant, when the velocities and coordinates of the bodies are given at that instant. There are three cases in which this principle of the conservation of areas obtains, when the forces are only the result of the mutual action of the bodies composing the system, when the forces pass through the origin of the coordinates, when the system is moved by an initial impulse; in the first and last cases the origin of the coordinates may be any point whatever; if there is a fixed point in the system, as by what is stated in page 442, the principle of the conservation of areas may be reduced to that of moments, the principle obtains when this point is made the origin of the coordinates; for in that case, $\mathrm{P}y - \mathrm{Q}x$, which is the moment with respect to the origin, will disappear, *see* Notes, page 442; if there are two *fixed* points in the system, only one of the three equations obtains, namely, that which contains those coordinates, the plane of which is perpendicular to the line joining the given points. If all the bodies of the system are equal, the theorem comes to this, that the sum of the areas traced by the radii vectores about the focus is proportional to the times.

(t) If the variations δx, δy, dz, $\delta x'$, &c., be supposed equal to dx, dy, dz, dx', &c., which supposition we are permitted to make, the equation given in page 481, becomes

$$0 = mdx.\left(\frac{d^2x}{dt^2} - \text{P.}\right) + mdy.\left(\frac{d^2y}{dt^2} - \text{Q.}\right) + m.dz.\left(\frac{d^2z}{dt^2} - \text{R}\right)$$

$$+ m'.dx'.\left(\frac{dx'^2}{dt^2} - \text{P.}\right) + m'dy'.\left(\frac{d^2y'}{dt^2} - \text{Q}'\right) + m'.dz'.\left(\frac{d^2z}{dt^2} - \text{R}'.\right)$$

+ &c., of which the integral is

$$\Sigma m.\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = c_i' + \Sigma.\int m.(\text{P}dz + \text{Q}dy + \text{R}dz),$$

this last term is an exact integral, if the forces P, Q, R, P′, &c. are the results of attractive forces directed towards fixed centres and of a mutual attraction between the bodies, which is some function of the distance; if we suppose it = to ϕ, the preceding equation will become $\Sigma mv^2 = c + 2\phi$, see page 432; hence, if the bodies composing the system are not solicited by any forces, ϕ vanishes, and $\Sigma mv^2 = c$, *i. e.* the sum of the living forces is constant, and if it does not vanish, the sum of the increments of the living forces is the same, whatever be the nature of the curves described, provided that their points of departure and arrival are the same. What has been stated respecting the mutual attraction of the bodies of the system, is equally true respecting repulsive forces, which vary as some function of the distance; it is also true, when the repulsions are produced by the action of springs interposed between the bodies, for the force of the spring must vary as some function of the distance between the points; hence in the impact of perfectly elastic bodies, though the quantity of motion communicated may be *increased indefinitely*, still the *vis viva* after the impact remains the same as before; indeed, *during* the impact, the *vis viva* varies as the coordinates of the respective points vary, but after the restitution of the bodies, from their perfect elasticity they resume their original position, and therefore the value of the *vis viva* remains the same as before; but if the elasticity be not perfect, in order to have the *vis viva* at any instant, we should have the relation which exists between the compressive and restitutive force. The *vis viva* of a system is evidently diminished when the motion is modified by friction, or the resistance of a medium, for in that case $(\mathrm{P}dx+\mathrm{Q}dy+\mathrm{R}dz)$ is not a perfect integral.

It is evident from the manner in which the principle of the *vis viva* was deduced, that it only obtains when the motions of the bodies change by imperceptible gradations, if these motions undergo abrupt changes, the living force is diminished by a quantity which is thus determined, let $\Delta.\frac{dx}{dt}.\ \Delta.\frac{dy}{dt}$, &c. denote the differences of $\frac{dx}{dt}, \frac{dy}{dt}$, &c., from one instant to another, and from the principle of D'Alembert, as $\Delta.\frac{dx}{dt}$ is the variation of the velocity on the supposition that the body is entirely free, and $\mathrm{P}.dt$, the variation which actually takes place, in consequence of the actions of the bodies of the system, we may apply the reasoning of page 483 to this case; therefore the following equations obtain

$$\Sigma m.\left(\Delta.\frac{dx}{dt}\cdot\frac{\delta x}{dt}+\Delta.\frac{dy}{dt}\cdot\frac{\delta y}{dt}+\Delta.\frac{dz}{dt}\cdot\frac{\delta z}{dt}+\&c.\right)$$
$$-\Sigma m.(\mathrm{P}\delta x+\mathrm{Q}\delta y+\mathrm{R}\delta z)=0;$$

now as dx, dy, dz become in the subsequent instants

$$dx+\Delta.dx,\ dy+\Delta.dy,\ dz+\Delta.dz,\ \&c.$$

if we assume $\delta x, \delta y, \delta z$, &c.= to these quantities, we evidently satisfy the condition of the connexion of the parts of the system, therefore substituting these quantities for δx, δy, δz, &c., the preceding equation becomes

$$\Sigma m.\left\{\left(\frac{dx}{dt}+\Delta.\frac{dx}{dt}.\right)\Delta.\frac{dx}{dt}+\left(\frac{dy}{dt}+\Delta.\frac{dy}{dt}\right)\Delta.\frac{dy}{dt}+\right.$$

$$\left(\frac{dz}{dt} + \Delta.\frac{dz}{dt}\right)\Delta.\frac{dz}{dt}\Big\} + \&c.$$

$$-\Sigma m.\mathrm{P}(dx + \Delta\, dx) + \mathrm{Q}.(dy + \Delta.dy) + \mathrm{R}.(dz + \Delta.dz)\&c.,$$

the integral of $m\mathrm{P}.(dx + \Delta.dx)$ is evidently equal to $\int m.\mathrm{P}.dx$, &c., and the integral of

$$m.\frac{dx}{dt}.\ \Delta.\frac{dx}{dt} = m.\frac{dx^2}{dt^2},$$

for $\Delta.(x^2) = 2xh + h^2$, and if h be made equal to Δx, it becomes

$2x\Delta.x + (\Delta x^2)$, $\therefore 2\mathrm{S}.(x\Delta x + (\Delta x)^2) = \mathrm{S}.(2x\Delta x + (\Delta x)^2$ $+ \mathrm{S}.(\Delta x)^2 = x^2 + \mathrm{S}.(\Delta x)^2$; *see* Lacroix, tom. 3, No. 344, therefore if we multiply the preceding equation by 2, and substitute dx, in place of x, we shall obtain after concinnating

$$\Sigma m.\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) - 2\int m.(\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz)$$

$$+\mathrm{S}.\Sigma m.\left\{\left(\Delta.\frac{dx^2}{dt^2}\right) + \left(\Delta.\frac{dy^2}{dt^2}\right) + \left(\Delta.\frac{dz^2}{dt^2}\right)\right\};$$

i. e. if v, v', v'', &c., denote the velocities of the several bodies m, m', m'', &c., we have

$$\Sigma.mv^2 = \mathrm{C}. + 2\Sigma\int m.(\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz)$$

$$-\mathrm{S}.\Sigma m.\left\{\left(\Delta.\frac{dx}{dt}\right)^2 + \Delta.\left(\frac{dy}{dt}\right)^2 + \left(\Delta.\frac{dz}{dt}\right)^2\right\},$$

as the quantity under the sign S is always positive, the living force of the system is diminished by the mutual action of the bodies as often as $\Delta.\dfrac{dx}{dt}$ is finite, as

$\dfrac{dx^2 + dy^2 + dz^2}{dt^2}$ expresses the square of the velocity of

m before the shock, and

$$\frac{(dx + \Delta.dx)^2 + (dy + \Delta.dy)^2 + (dz + \Delta dz)^2}{dt^2},$$

the square of the velocity of m after the shock; and since from the principle of D'Alembert,

$$\Sigma m.(2dx \triangle.dx + 2(\triangle dx)^2 + 2dy.\ \triangle.dy + 2(\triangle.dy)^2 + 2dz.\triangle.dz + 2(\triangle.dz)^2) = 0,$$

If we subtract this from the preceding expression the difference becomes

$$= \Sigma m.\frac{dx^2 + dy^2 + dz^2}{dt^2} - \Sigma m.\frac{(\triangle.dx)^2 + (\triangle.dy)^2 + (\triangle.dz)^2}{dt^2},$$

$$\text{and as } \Sigma m.\ \frac{(dx^2 + dy^2 + dz^2)}{dt^2} = \Sigma mv^2,$$

the living force of the system before the shock,

$$\frac{(\triangle.dx)^2 + (\triangle.dy)^2 + (\triangle.dz)^2}{dt^2} = V^2 =$$

the square of the velocity lost by the shock, and $\Sigma.mV^2$(= the loss which the *vis viva* sustains by the shock) is equal to the sum of the living forces which would belong to the system, if each body was solely actuated by that which is lost by the shock. This theorem was first announced by Carnot.

(s) The variation of the *vis viva* of the system is equal to

$$2\Sigma m.(P.dx + Q.dy + Rdz) = d.(\Sigma m.v^2),$$

therefore when this expression vanishes, Σmv^2 is either a maximum or a minimum; but from the principle of virtual velocities it appears that when P, Q, R, P′, &c., constitute an equilibrium

$$P.\delta x + Q.\delta y + R.\delta z + P'.\delta x' + \&c. = 0\,;$$

and when δx, δy, δz, &c., are subjected to the conditions of the connexion of the parts of the system, we may substitute dx, dy, dz for these variations; consequently, we have

$$\Sigma m.(P.dx + Q.dy + R.dz),$$

the variation of the *vis viva* equal to nothing in this case, and therefore the *vis viva* is either a maximum or minimum. If the system was slightly disturbed from the position of equilibrium, expressing P, Q, R, &c., in terms of the coordinates and expanding the resulting expressions into a series proceeding according to the variations of the coordinates; the first term of the series will be the value of ϕ when the system is in equilibrio; and since it is given, it may be made to coalesce with $c_{\prime}'$ in the expression given in page 494; the second term vanishes by the conditions of the problem; and when Σmv^2 is a maximum, the theory of maxima and minima shews that the third term of the expansion may be made to assume the form of a sum of squares affected with a negative sign, *see* Lacroix, No. 134, the number of terms in this sum being equal to the number of variations or independent variables.

The terms whose squares we have assumed, are linear functions of the variations of the coordinates, and vanish at the same time with them; and they are greater than the sum of all the remaining terms of the expansion. The constant quantity being equal to $c_{\prime}'$ + the value of Σmv^2, when P, Q, R, P′, &c., constitute an equilibrium, it is necessarily positive, and may be rendered as small as we please by diminishing the velocities; but it always exceeds the greatest of the quantities whose squares have been substituted in place of the variations of the coordinates; for if it were less, this negative quantity would exceed the constant quantity, and therefore render the value of Σmv^2, negative; consequently, these squares and the variations of the coordinates, of which they are linear functions, always remain very small, therefore the system will always oscillate about the position of equilibrium, and hence this equilibrium will be one of stability. But in the case of ϕ being a minimum, it is not requisite that the variations should be always constrained to be very small in order to satisfy the equation of living forces; this indeed does not prove that there is no limit then to those variations, which should be done in order to prove the equilibrium to be instable; in order to demonstrate this, we should substitute for these variations their values in a function of the time, and then shew from the form of those functions, that they increase indefinitely with the time, however small the primitive velocities may be.

Let P, P′, P″, &c., denote the weights of any number of bodies in equilibrio, and z, z', z'', &c., their coordinates with respect to an horizontal plane; then if the position of the system be disturbed by any quantity, however small, we have

$$R.\delta z + R'.\delta z' + R''.\delta z'' + \&c. = 0; \therefore \frac{Rz + Rz' + R''z''}{\Sigma.R} + \&c.$$

(which is equal to $z_{/}$ the distance of the centre of gravity of all the bodies of the system from the horizontal plane) is either a maximum or a minimum; and the sum of the living forces is a maximum when the centre ceases to descend, and commences to ascend, for

$$\Sigma m.(P.dx + Q.dy + R.dz),$$

in this case becomes $\Sigma m.R.dz$; and therefore by substitution we have $\Sigma m.v^2 = c_{/}' + z_{/}R.\Sigma m$, consequently Σmv^2 is a maximum or minimum, according as $z_{/}$ is a maximum or minimum; when Σmv^2 is a maximum the equilibrium is stable, when a minimum the equilibrium is instable. For from the definition of stability, it appears that then the bodies tend to revert to the position of equilibrium, therefore the velocities will diminish according as the system deviates more from the position of equilibrium, consequently the sign of the second differential of ϕ will be negative; hence Σmv^2 will be a maximum in this case, and in the contrary it will be evidently a minimum.

Let, as in page 413, F the force be ∻l to $\phi(v)$, then this force resolved parallel to the axes of x, y, z, becomes respectively

$$= \phi.(v).\ \frac{dx}{ds},\ \phi.(v).\left(\frac{dy}{ds}\right),\ \phi.(v).\left(\frac{dz}{ds}\right);$$

moreover, the forces at the subsequent instant are

$$\phi.(v).\ \frac{dx}{ds} + d.\left(\phi.(v).\ \frac{dx}{ds}\right),\ \phi.(v).\ \frac{dy}{ds},$$

$$+ d.\left(\phi.(v).\frac{dy}{ds}\right),\ \ \phi.(v).\frac{dz}{ds} + d.\left(\phi.(v).\ \frac{dz}{ds}\right) + \&c.$$

if P, Q, R, P′, &c., denote the same quantities as before, the system will, by what is established in page 485, be in equilibrio in consequence of these forces and the differentials

$$d.\left(\frac{dx}{dt}.\ \frac{\phi.(v)}{v}\right),\ d.\left(\frac{dy}{dt}.\ \frac{\phi.(v)}{v}\right),\ \ d.\left(\frac{dz}{dt}.\ \frac{\phi.(v)}{v}\right),$$

taken with a contrary sign, therefore in place of the equation given in page 485, we shall have the following

$$0 = \Sigma m.\left(\delta x.d.\left(\frac{dx}{dt}.\ \frac{\phi.(v)}{v}\right) - \text{P}.dt.\right)$$

$$+ \delta yd.\left(\frac{dy}{dt}.\ \frac{\phi.(v)}{v}\right) - \text{Q}.dt.\Big) + \delta z.d.\left(\frac{dz}{dt}.\frac{\phi.(v)}{v}\right) - \text{R}.dt,\Big),$$ it

differs from that equation in this respect, that $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are multiplied by the function $\frac{\phi\,(v)}{v}$, which in the case of the force ∻1 to the velocity is = to unity; this difference renders the solution of problems extremely difficult; however we may obtain from the preceding equation principles analogous to those of the conservation of living force, of areas, and of the motion of the centre of gravity. For instance, the preceding expression, by changing δx, δy, δz, &c., into dx, dy, dz, &c., becomes

$$\Sigma m.\left(dx.d.\left(\frac{dx}{ds}.\ \phi.\ (v).\right) + dy.d.\left(\frac{dy}{ds}.\ \phi.(v).\right)\right.$$

$$\left. + dz.d.\left(\frac{dz}{ds}.\ \phi.)v)\right)\right)$$

i. e. by expanding the expression

$$= \Sigma m. \frac{(dx.d^2x + dy.d^2y + dz.dz^2)}{ds}. \ \phi.(v)$$

$$- \Sigma m. \frac{(dx^2 + dy^2 + dz^2)}{ds^2}. \ d^2s. \ \phi.(v) +$$

$$\Sigma m. \left(\frac{dx^2 + dy^2 + dz}{ds}. \ d.\phi.(v) = \Sigma m.d^2s.\phi.(v) - \Sigma m.d^2s.\phi.(v)\right.$$

$$+ \Sigma m.ds.d.\phi(v).$$

and this last quantity is equal by substitution to

$$\Sigma m.vdt.dv.\phi'.(v),$$

therefore we have

$$\Sigma f mv.dv.\phi'.(v) = c_{\prime}' + \Sigma f m.(\text{P}.dx + \text{Q}.dy + \text{R}.dz);$$

if this last term is an exact differential equal to $d\lambda$, we shall have

$$\Sigma.f mvdv.\phi'.(v) = c_{\prime}' + \lambda;$$

an equation which establishes what is stated in page 292. If as in page 489, we make

$$\delta x = \delta x + \delta x_{\prime}', \quad \delta y = \delta y + \delta y_{\prime}', \quad \delta z = \delta z + \delta z_{\prime}',$$

and make, as in that page, the coefficients of δx, δy, δz, respectively equal to cypher, we shall have

$$0 = \Sigma m. \left(d. \left(\frac{dx}{dt}. \ \frac{\phi.(v)}{v}\right) - \text{P}.dt.\right)$$

$$0 = \Sigma m. \left(d. \left(\frac{dy}{dt}. \ \frac{\phi.(v)}{v}\right) - \text{Q}.dt.\right);$$

$$0 = \Sigma m. \left(d. \left(\frac{dz}{dt}. \ \frac{\phi.(v)}{v}\right) - \text{R}.dt.\right),$$

which are analogous to those of page 464, from which the conservation of the motion of the centre of gravity was inferred, when the system is only subject to the mutual action and reciprocal attraction of the bodies composing it, in which case $\Sigma m\text{P}$, $\Sigma m\text{Q}$, $\Sigma m\text{R}$ are respectively equal to cypher; we can infer from the preceding equation

$$\text{C} = \Sigma m. \frac{dx}{dt}. \ \frac{\phi.(v)}{v}; \ \text{C}' = \Sigma m. \frac{dy}{dt}. \ \frac{\phi.(v)}{v};$$

$C'' = \Sigma.m.\frac{dz}{dt}.\frac{\phi.(v)}{v}$; but $m.\frac{dz}{dt}.\frac{\phi.(v)}{v} = m.\phi.(v).\frac{dz}{ds} =$ the finite force of the body resolved parallel to the axis of z; provided that we understand by the force of a body, the product of the mass into that function of the velocity which expresses it; consequently in the preceding case the sum of the finite forces of the bodies composing the system is constant, whatever may be the nature of ϕ; but unless $\frac{\phi.(v)}{v} = 1$, the motion of the centre of gravity will not be uniform and rectilinear, for it is only in that case that we could prove from the expression $C = \Sigma m.\frac{dx}{dt}.\frac{\phi.(v)}{v}$, that dA, the differential of the coordinate of the centre of gravity, was constant.

Making the substitutions indicated in page 491, and afterwards putting the coefficient of $\delta x = 0$, we obtain, when the system is not actuated by extraneous forces,

$$0 = \Sigma m.\left(x.d.\left(\frac{dy}{dt}.\frac{\phi.(v)}{v}\right) - y.d.\left(\frac{dx}{dt}.\frac{\phi.(v)}{v}\right)\right) + \Sigma m.(Py - Qx).dt,$$

and by integrating

$$c = \Sigma m.\left(\frac{x\,dy - ydx}{dt}\right).\frac{\phi.(v)}{v} + \Sigma.\int m.(Py - Qx).dt,$$

and in like manner

$$c' = \Sigma m.\left(\frac{x.dz - zdx}{dt}\right).\frac{\phi.(v)}{v} + \Sigma\int m.(Pz - Rx).dt;$$

$$c'' = \Sigma m.\frac{ydz - zdy}{dt}.\frac{\phi.(v)}{v} + \Sigma\int m.(Qz - Ry).dt,$$

and since, by what has been stated above, $m.\left(x.\frac{dy}{dt} - y.\frac{dx}{dt}\right).\frac{\phi.(v)}{v}$ is the moment of the finite force by which the body is actuated, $\Sigma m. \left(\frac{xdy-ydx}{dt}\right). \frac{\phi.(v)}{v}$ expresses the sum of the moments of all the finite forces of the bodies of the system to make it revolve about the axis of z, which, when $Py - Qx = 0$, is constant, and it evidently vanishes in the case of equilibrium.

(t) If the equation

$$\Sigma m.v^2 = C + 2\Sigma m.(P.dx + Q.dy + R.dz)$$

be differentiated with respect to the characteristic δ we shall have

$$\Sigma m.v\delta v = \Sigma m.(P.\delta x + Q.\delta y + R.\delta z),$$

and the equation given in page 485 then becomes

$$0 = \Sigma m. \left(\delta x.d.\frac{dx}{dt} + \delta y.d.\frac{dy}{dt} + \delta z.d.\frac{dz}{dt}\right)$$

$-\Sigma.m.dt.vdv$; and as $vdt = ds$, $v'dt = ds'$, &c. we can obtain by the same process as in page 439,

$$\Sigma m.\delta.(vds) = \Sigma m.\ d.\left(\frac{dx.\delta x + dy.\delta y + dz.\delta z}{dt}\right),$$

integrating with respect to d, and extending the integrals to the entire curves described by the bodies m, m', &c. we shall have

$$\Sigma.\delta.\int m.vds = C + \Sigma m.\left(\frac{dx.\delta x + dy.\delta y + dz.\delta z}{dt}\right),$$

C, and also the variations $\delta x.\ \delta y$, δz, $\delta x'$, &c. refer to the extreme points of the curves described, and when these are invariable, we have $0 = \Sigma.\delta.\int mv.ds$, therefore $\Sigma.\int m.vds$ is a *minimum*. This expression becomes, by substituting for ds, ds', &c. $v.dt$, $v'.dt$, &c. $= \Sigma\int mv^2.dt =$ the sum of the living forces of the bodies composing the system, consequently, the principal of the least action, in fact, indicates that the sum of the living forces of the bodies composing the system is, in its transit from one position to another, a minimum; and when the bodies are not actuated by any accelerating forces, the velocities v, v', &c.

and the sum of the living forces are constant at each instant, see page 439;

$$\because \Sigma \int mv^2.dt = \Sigma mv^2. \int dt,$$

and the sum of the living forces for any interval of time is $\div$l to this time, consequently in this case the body passes from one position to another in the shortest possible time. As $\Sigma \int m.vds = \Sigma \int m.v^2.dt$, La Grange proposed to alter the denomination of the principle of least action, and to term it the principle of the greatest or least living force. The advantage from this mode of expression would be, that it is equally applicable to a state of equilibrium and motion, since, in the state of equilibrium, it has been already shewn to be either a maximum or minimum.

(w) This is evident from what goes before, for from the principle of action and reaction the expressions

$$\Sigma.mP, \ \&c. \ \Sigma m.(Py - Qx)\&c. = 0,$$

whatever changes are produced by the mutual actions of the bodies. Let X, Y, Z represent the coordinates of the moveable origin of the coordinates,

$$x = X + x_{,}; \ y = Y + y_{,}; \ z = Z + z_{,}; \ x' = X + x_{,}'; \ \&c.$$

If the origin moves with a uniform rectilinear motion

$$d^2X, = 0, \ d^2Y = 0, \ \&c.$$

therefore substituting for d^2x, we have, when the system is free, by the nature of the centre of gravity,

$$\Sigma m.(d^2X + d^2x_{,}) - \Sigma m.P.dt^2 = 0,$$

$$\Sigma m.(d^2Q + d^2y) - \Sigma m.Q.dt^2) = 0, \ \&c.$$

by substituting

$$\delta X + \delta x_{,}, \ \delta Y + \delta y_{,}, \ \&c.$$

in place of δx, δy, &c. in the equation of page 485, we shall have

$$0 = \Sigma m.\delta x_{,}.\left(\frac{d^2x_{,}}{dt^2} - P.\right) + \Sigma m.\delta y_{,}.\left(\frac{d^2y_{,}}{dt^2} - Q.\right)$$

$$+ \Sigma m.\delta z_{,}.\left(\frac{d^2z_{,}}{dt^2} - R.\right),$$

which is precisely of the same form as the equations given in page 485, and the same consequences may evidently be derived from them; if X, Y, Z denote the coordinates of the centre of gravity, by the nature of it we have

$$\Sigma m x_{,} = 0, \ \Sigma m y_{,} = 0, \ \Sigma m z_{,} = 0,$$

$$\therefore \Sigma m. \left(\frac{xdy - ydx}{dt}\right) = \frac{Xdy - Ydx}{dt}. \ \Sigma m +$$

$$\Sigma m. \frac{(x_{,}dy_{,} - y_{,}dx_{,})}{dt},$$

in like manner

$$\Sigma m. \left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = \frac{dX^2 + dY^2 + dZ^2}{dt^2}. \ \Sigma m +$$

$$\Sigma m. \left(\frac{dx_{,}^2 + dy_{,}^2 + dz_{,}^2}{dt^2}\right),$$

for $\Sigma m.dx^2 = \Sigma m.dX^2 + 2\Sigma m.dx_{,}.\ dX + \Sigma m.dx_{,}^2$,

and as $2dX.\Sigma m.dx_{,} = 0$, we have

$$\Sigma m.dx^2 = dX^2.\Sigma m + \Sigma m.\ dx_{,}^2.$$

therefore it appears, that if the origin be transferred from another point to the centre of gravity, the quantities which result are composed of two different expressions, namely of those which would obtain if all the bodies of the system were concentrated in the centre of gravity; and secondly, of quantities relative to the centre of gravity supposed fixed; and since the first described quantities are constant, the reason why the principles in question obtain, with respect to the centre of gravity is evident; also if the origin of the coordinates be supposed in this point, the plane which passes through it, and relatively to which $\Sigma m. \left(\frac{xdy - ydx}{dt}\right)$ is a maximum, remains always parallel to itself during the motion of the system, and the same function relatively to every other plane perpendicular to it, vanishes, see note (x), and page 509.

(x) There exists a plane with respect to which c' and c'', in page 492 vanish, which is thus determined, let θ represent the inclination of the required plane formed by two of the new axes x'', y'' with the plane of x, y, and let ψ, represent the angle between the axis of x and the intersection of x'', y'' with x, y, and ϕ the angle between x'', and the intersection of x, y, x'', y'', then by substituting it would be easy to shew that

$$x'' = x.(\cos.\,\theta.\,\sin.\,\psi.\,\sin.\,\phi + \cos.\,\psi.\,\cos.\,\phi) +$$
$$y.(\cos.\,\theta.\,\cos.\,\psi.\,\sin.\,\phi - \sin.\,\psi.\,\cos.\,\phi) - z.\,\sin.\,\theta.\,\sin.\,\phi;$$
$$y'' = x.(\cos.\,\theta.\,\sin.\,\psi.\,\cos.\,\phi - \cos.\,\psi.\,\sin.\,\phi) +$$
$$y.(\cos.\,\theta.\,\cos.\,\psi.\,\cos.\,\phi + \sin.\,\psi.\,\sin.\,\phi) - z\,\sin.\,\theta.\,\cos.\,\phi.$$
$$z'' = x.\,\sin.\,\theta.\,\sin.\,\psi + y.\,\sin.\,\theta.\,\cos.\,\psi + z.\,\cos.\,\theta;$$

if we take the expressions $x''dy'' - y''dx''$, by substituting for $y''dx''$, $x''dy''$, &c. their values, neglecting quantities which destroy each other, and observing that $xdy - ydx = c$, $xdz - zdx = c'$, &c. we shall obtain after all substitutions

$$\Sigma m.\left(\frac{x''.dy'' - y''.dx''}{dt}\right) = c.\,\cos.\,\theta - c'.\,\sin.\,\theta.\,\cos.\,\psi + c''.\,\sin.\,\theta.\,\sin.\,\psi\,;\quad \Sigma m.\left(\frac{x''.dz'' - z''.dx''}{dt}\right) = c.\,\sin.\,\theta.\,\cos.\,\phi.$$
$$+ c'.(\sin.\,\psi\,\sin.\,\phi + \cos.\,\theta.\,\cos.\,\psi.\,\cos.\,\phi) + c''.(\cos.\,\psi.\,\sin.\,\phi - \cos.\,\theta.\,\sin\,\psi.\,\cos.\,\phi),\quad \Sigma m.\left(\frac{y''dz'' - z''dy''}{dt}\right) = -c.\,\sin.\,\theta.\,\sin.\,\phi + c'.(\sin.\,\psi.\,\cos.\,\phi - \cos.\,\theta.\,\cos.\,\psi.\,\sin.\,\phi) + c''.\,\cos.\,\psi.\,\cos.\,\phi + \cos.\,\theta.\,\sin.\,\psi.\,\sin.\,\phi),$$

if θ and ψ are so determined that $\sin.\,\theta.\,\sin.\,\psi =$

$$\frac{c''}{\sqrt{c^2 + c'^2 + c''^2}};\quad \sin.\,\theta.\,\cos.\,\psi = \frac{-c'}{\sqrt{c^2 + c'^2 + c''^2}};$$

and therefore $\cos.\,\theta = \dfrac{c}{\sqrt{c^2 + c'^2 + c''^2}}$, we shall have

$$\Sigma m.\left(\frac{x''.dy'' - y''.dx''}{dt}\right) = \sqrt{c^2 + c'^2 + c''^2}$$

$$\Sigma m. \left(\frac{x''.dz''-z''.dx''}{dt}\right) = 0; \quad \Sigma m. \left(\frac{y''.dz''-z''.dy''}{dt}\right) = 0;$$

therefore with respect to a plane determined in this manner, c', c'' vanish; there exists only one plane which possesses this property, for supposing it to be the plane of x, y, then

$$\Sigma m. \left(\frac{x''.dz''-z''.dx''}{dt}\right) = c. \sin. \theta. \cos. \phi;$$

$$\Sigma m. \left(\frac{y''.dz'-z''.dy''}{dt}\right) = -c. \sin. \theta. \sin. \phi;$$

if these two functions be put = to cypher, we shall have $\sin. \theta = 0$; therefore the plane x'', y'', coincides with the plane x, y; since whatever has been the direction of the original plane x, y, the value of

$$\Sigma m. \left(\frac{x''.dy''-y''.dx''}{dt}\right) \text{ is } \sqrt{c^2+c'^2+c''^2},$$

it follows that $c^2+c'^2+c''^2$ is constant, and that the plane of x'', y'', determined by what precedes, is that with respect to which $\Sigma m. \left(\frac{x''.dy''-y''.dx''}{dt}\right)$ is a maximum: this plane therefore possesses these remarkable properties, namely, that the sum of the areas traced by the projections of the radii vectores of the several bodies on it, and multiplied by their masses, is the greatest possible, and that the same sum vanishes for every plane which is perpendicular to it; by means of these properties we can always find its position, whatever variations may be induced in the respective positions of the bodies in consequence of their mutual action; as $\cos \theta$, $\sin. \theta. \cos. \psi$, $\sin. \theta. \sin. \psi$ represent the cosines of the angles which the plane x'', y'' makes with the plane x, y; z, x; y, z, it follows that where we have the projections c, c', c'' of any area on three coordinate planes, we have its projection $\Sigma m. \left(\frac{x''.dy''-y''.dx''}{dt}\right)$ on the plane of x'', y'', the position of which, with respect to the three planes xy, xz, yz, is given;

it also appears from the expression of $\Sigma m.\left(\frac{x''.dy''-y''.dx''}{dt}\right)$, that for all planes equally inclined to the plane, on which the projection is a maximum, the values of the projection of the area are equal; c, c', c'', being constant, and ∻l to the cosines of the angles, which the plane on which the projection of the area is a maximum, makes with xy, xz, yz, the position of this plane is necessarily fixed and invariable; and as c, c', c'' depend on the coordinates of the bodies at any instant, and on the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c. when these quantities are given, we can determine the position of this plane, which may be called invariable because it depends on c, c', c'', which are constant when the bodies are only subject to their mutual action, and to the action of forces directed towards a fixed point. Since the plane x, y, is undetermined in the text, we infer that the sum of the squares of the projections of any areas existing in the invariable plane, on any three coordinate planes existing in the same point of space is constant, therefore if on the axes to the coordinate planes xy, xz, yz, lines be assumed ∻l to c, c', c'', then the diagonal of the parallelopiped whose sides were ∻l to these lines, will represent the quantity and direction of the greatest moment, and this direction is the same whatever three coordinate planes be assumed, but the *position* in absolute space is undetermined, for the projections on all parallel planes are evidently the same. The conclusions to which we have arrived respecting the projections of areas, are evidently applicable to the projections of moments, since, as has been remarked in page 442, these moments may be geometrically represented by triangles, of which the bases represent the projected force, the altitudes being equal to perpendiculars let fall from the point to which the moments are referred, on the directions of the bases; therefore when the forces applied to the several points of the system have an unique resultant V, since the sum of the moments of any forces projected on a plane is equal to the moment of the projection of their resultant, it follows that the unique resultant V, and the point to which the moments are applied, must exist in the invariable plane, therefore the axis of this plane must be at right angles to this resultant; and as $\frac{P}{V}$, $\frac{Q}{V}$, $\frac{R}{V}$, (see page 443) are equal to the cosines of the angles, which V makes with the coordinates; and as

$$\frac{c}{\sqrt{c^2+c'^2+c''^2}}, \quad \frac{c'}{\sqrt{c^2+c'^2+c''^2}}, \quad \frac{c''}{\sqrt{c^2+c'^2+c''^2}}$$

are equal to the cosines of the angles, which the axis to the invariable plane makes with the same coordinates, we have $\frac{cP+c'Q+c''R}{\sqrt{c^2+c'^2+c^{2''}}}=0$, (see page 409).

The practical rule for the determination of the plane of greatest projection is given in Chapter II. Vol. II. From what has been established in notes, page 505, it appears that for all points in which $\frac{AdX-BdY}{dt}\cdot\Sigma m=0$, the value of c remains constantly the same; but it is evident that this equation will be satisfied, if the locus of the origin of the coordinates be either the right line described by the centre of gravity or any line parallel to this line; therefore for all such positions the invariable plane remains constantly parallel to itself, however, though for all points of the *same* parallel the direction of the invariable plane remains constantly parallel to itself, still in the passage from one parallel to another, the direction of this plane changes. (A, B, C, are the coordinates of the new origin. *See* Celestial Mechanics, page 145.)

When the forces are reducible to an unique resultant, if the origin of the coordinates be any point in it, the quantities c, c', c'', and therefore the plane, with respect to which the projection of the areas is a maximum, vanishes; and if the locus of the origin be any line parallel to this line, the value of the projection of the area on the plane x, y, with respect to this line $= \frac{A.dX - B.dY}{dt}. \Sigma m$, for c in this case vanishes, if the locus of the origin be a right line diverging from this resultant, the expression $\frac{AdX - BdY}{dt}. \Sigma m$, is susceptible of continual increase. The plane, with respect to which the value of $\sqrt{c^2 + c'^2 + c''^2}$ is the minimum maximorum is perpendicular to the direction of the general resultant, or of the common motion with which the system is actuated, its axis is a perpendicular to this plane, erected at the origin, which may be any point in the direction; for all equidistant origins existing in a perpendicular plane, the maximum areas will have the same values, and their planes will be normal to the different generatrices of an hyperboloid of revolution described about this central axis; although the value of the maximum area should be given, still if the origin be not also given, its plane cannot be distinguished from an infinity of others perpendicular to the generatrices of an hyperboloid of revolution; but if with the preceding we combine the condition that the areas should be the *minimum* of the maxima areas, relatively to different origins in space, the plane sought may be easily found, inasmuch as it enjoys not only an exclusive property with respect to those which pass through the same origin, but likewise another exclusive property with respect to those which have the first property common with it.

In the system of the world, as we do not know any fixed point to which the different heavenly bodies may be referred, and as we are also ignorant of the direction and force with which this system moves in space, neither the plane nor the value of the area which is the minimum maximorum can be determined, we can solely select the plane of the maximum area with respect to *any point* which moves with the velocity of the common centre of gravity of the system in a right line; therefore the origin may be assumed at the common centre of gravity, which, during the entire motion possesses the property of moving in a right line.

The principle of the conservation of areas, and also that of living forces, may be reduced to certain relations between the coordinates of the mutual distances of the bodies composing the system; for if the origin of the coordinates be supposed to be at the centre of gravity, the equation given in page 507 may be made to assume the form

$$c.\Sigma m = \Sigma mm'.\left(\frac{(x'-x).(dy'-dy)-(y-y).dx'-dx)}{dt}\right),$$

$$c'.\Sigma m = \Sigma mm'.\left(\frac{(x'-x).(dz'-dz)-(z'-z).(dx'-dx)}{dt}\right),$$

$$c''.\Sigma m = \Sigma mm'.\left(\frac{(y'-y).(dz'-dz)-(z'-z).(dy'-dy)}{dt}\right),$$

(for the verification of those formula see Celestial Mechanics, page 145), the second members of these equations multiplied by dt, express the sum of the projections of the elementary areas traced by each line which joins the two bodies of the system, of which one is supposed to move round the other considered as immoveable, each area being multiplied by the product of the two masses, which are connected by the same right line. It might be made appear, as in page 508, that the plane passing through any of the bodies of the system, and with respect to which the preceding function is a maximum, remains always parallel to itself, during the motion of the system, and that this plane is parallel to the plane passing through the centre of gravity, relatively to which the function $\Sigma m.\left(\frac{xdy-ydx}{dt}\right)$ is a maximum, &c. Also the second members of the preceding equations vanish with respect to all planes passing through the same body, and perpendicular to the plane in question.

In like manner the equation given in page 494 may be made to assume the form

$$\Sigma mm'.\left(\frac{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2}{dt^2}\right)=$$
$$C''-2\Sigma m.\ \Sigma.\int mm'.\ F.df,$$

which only respects the coordinates of the mutual distances of the bodies, in which the first member expresses the sum of the squares of the relative velocities of the bodies of the system about each other, considering them two by two, and supposing, at the same time, that one of them is immoveable, each square being multiplied by the product of the two masses which are considered. *See* Celestial Mechanics, page 148.

It may be remarked, with respect to the preceding conclusions about the invariable plane, that in any system of solid or fluid molecules actuated primitively by any forces, and subjected to their mutual action, if it happens that after a great number of oscillations these molecules are arranged in a permanent state of rotation about an invariable axis passing through their common centre of gravity, (which is most probably the case with respect to the celestial bodies), then their equator will be parallel to that plane which would furnish the maximum of areas with respect to the centre of gravity. See Vol. II. Chap. IX. page 121.

It may be likewise remarked here, that planes are not the sole surfaces on which the areas remain constant without undergoing any change during the motion of the system; the same property appertains to every circular conic surface, of which the summit is the origin of the radii vectores, but it is necessary that these radii should be projected on the cone by lines parallel to its axis, the areas described on the surfaces of different cones having the same axis and summit, (see page 507), are inversely as the sines of the angles of the cones, therefore the area will be least, which is projected on the right cone. If the angle of the cone is given but the axes different, there is only one on the surface of which the area traced by the radius vector will be a maximum; also among all those which assign the same value to the maximum areas relatively to different origins in space, there is only one which will give the least of these maxima areas. The axes of these remarkable cones are the same as the axes of the moments or areas which possess the same properties.

END OF THE FIRST VOLUME.

ERRATA.

Page 15, line 12, *from bottom, for* becomes *read* becoming.
— 21, — 5, *from bottom, for* and *read* but.
— 7, — 3, *for* plan *read* plane.
— 96, — 4, *from bottom, after* which *read* is.
— 114, — 7, *from bottom, after* from *read* a.
— 129, — 13, *for* fuller *read* feebler.
— 135, — 4, *dele* of.
— ib. — 9, *for* sun *read* earth.
— 137, — 8, *after* each *read* other.
— 150, — 19, *for* cartonic *read* carbonic.
— 151, — 8, *from bottom, for* the contents *read* they.
— 153, — 20, *for* transverse *read* traverse.
— 191, — 6, *for* the *read* these.
— 197, — 9, *for* comets *read* earth.
— ib. — 19, *for* on *read* in.
— 209, — 5, *after* the *read* periods of the.
— 210, — 17, *for* on *read* to.
— 212, — 15, *for* first *read* second.
— 226, — 14, *after* directions *read* of.
— 237, — 17, *for* time *read* velocity.
— 278, — 22, *after* all *for* velocity *read* action.

THE SYSTEM OF THE WORLD,

BY

M. LE MARQUIS DE LAPLACE,

TRANSLATED FROM THE FRENCH,

AND

ELUCIDATED WITH EXPLANATORY NOTES.

BY THE

REV. HENRY H. HARTE, F. T. C. D. M. R. I. A.

VOL. II.

DUBLIN:

PRINTED AT THE UNIVERSITY PRESS,

FOR

LONGMAN, REES, ORME, BROWN, AND GREEN,

PATERNOSTER ROW, LONDON.

MDCCCXXX.

R. Graisberry, Printer.

CONTENTS.

BOOK IV.

OF THE THEORY OF UNIVERSAL GRAVITATION.

BOOK V.

SUMMARY OF THE HISTORY OF ASTRONOMY.

THE

SYSTEM OF THE WORLD.

BOOK IV.

OF THE THEORY OF UNIVERSAL GRAVITATION.

Opinionum commenta delet dies, naturæ judicia confirmat.

CIC. DE NAT. DEOR.

HAVING, in the preceding Books, explained the laws of the celestial motions, and those of the action of forces producing motion, it remains to compare them together, to determine what forces animate the solar system, and to ascend without the assistance of any hypothesis, but by strict geometrical reasoning, to the principle of universal gravitation, from which they are derived. It is in the celestial regions, that the laws of mechanics are observed with the greatest precision; on the earth so many causes tend to complicate their results, that it is very difficult to unravel them, and still more difficult to submit them to calculation.

But the bodies of the solar system, separate by immense distances and subject to the action of a principal force, whose effect is easily calculated, are not disturbed in their respective motions, by forces sufficiently considerable, to prevent us from including under general formulæ, all the changes which a succession of ages has produced, or may hereafter produce in the system. There is no question here of vague causes, which cannot be submitted to analysis, and which the imagination modifies at pleasure, to accommodate them to the phenomena. The law of universal gravitation has this inestimable advantage, that it may be reduced to calculation, and by a comparison of its results with observation, it furnishes the most certain means of verifying its existence. We shall see that this great law of nature, represents all the celestial phenomena even in their minutest details, that there is not one single inequality of their motions, which is not derived from it, with the most admirable precision, and that it has frequently anticipated observations by revealing the cause of several singular motions, just perceived by astronomers, and which were either too complicated or too slow to be determined by observation alone, except after a lapse of ages. By means of it, empericism has been entirely banished from astronomy, which is now a great problem of mechanics, of which the elements of the motions of the stars, their figures, and masses are the arbitrary quantities, and these are the only indispensable data, which this science must derive from observation. The most profound geometry was required to establish these theories: I have collected them in *my Treatise of Celestial Mechanics.* I shall confine myself here to detail the principal results of this work, indicating the steps that lead to them, and explaining the reasons, as far as can be done, without the assistance of analysis.

CHAP. I.

Of the Principle of Universal Gravitation.

Of all the phenomena of the solar system, the elliptic motion of the planets and of the comets, seems the most proper to conduct us to the general law of the forces by which they are actuated. Observation has shewn that the areas described by the radii vectores of the planets and comets about the Sun, are proportional to the times. Now we have seen in the preceding Book, that for this to take place, the force which deflects the path of these bodies from a right line, must constantly be directed towards the origin of the radii vectores. The tendency of the planets and comets to the Sun, is therefore a necessary consequence of the proportionality of these areas to the times in which they are described.

To determine the law of this tendency, let us suppose that the planets move in circular orbits, which supposition does not greatly differ from the truth. The squares of their real velocities will then be proportional to the squares of the radii of these orbits, divided by the squares of the times of their revolutions. But by the laws of Kepler, the squares of these times are to each other as the cubes of the same radii. The squares of the velocities are therefore reciprocally as these radii. It has been already shewn that the (*a*) central forces of several bodies moving in circular orbits, are as the squares of the velocities, divided by the radii of the circumferences described; the tendencies therefore of the planets to the Sun are reciprocally, as the squares of the radii of their orbits supposed circular. This hypothesis, it is true, is not rigorously exact, but the constant relation of the squares of the times to the cubes of the greater axes of their orbits, being independent of their excentricities, it is natural to think it would subsist also in the case of the orbits being circular. Thus, the law of gravity towards the Sun, varying reciprocally as the square of the distance, is clearly indicated by this relation. Analogy leads us to suppose that this law, which extends from one planet to another, subsists equally for the same planet, at its different distances from the Sun, and its elliptic motion confirms this beyond a doubt. To comprehend this, let us follow this motion from the departure of the planet from its perihelion: its velocity is then at its maximum, (*b*) and its tendency to recede from the Sun, surpassing its gravity towards it, its radius vector augments and forms an obtuse angle with the direction of its motion. The force of gravity towards the Sun, decomposed according to this direction, continually diminishes the velocity, till it arrives at the aphelion; at this point, the radius vector becoming perpendicular to the curve, its velocity is a minimum, and its tendency to recede from the Sun, being less than its gravity towards it, the planet will approach it describing the second part of its ellipse. In this part, the gravity towards the Sun, increases its velocity in the same manner as it before diminished it, and the planet will arrive at its perihelion with its primitive velocity, and recommence a new revolution similar to the first. Now, the curvature of the ellipse at the aphelion and perihelion being the same, the radii of curvature are the same, and consequently the centrifugal forces of these two points are as the squares of the velocities. The sectors described in the same time being equal, the aphelion and perihelion velocities are reciprocally as the corresponding distances of the planet from the Sun; the squares of these velocities are therefore reciprocally as the squares of these same distances; but at the perihelion and aphelion the centrifugal (*c*) forces in the osculatory circumferences are evidently equal to the gravity of the planet towards the Sun, which is therefore in the inverse proportion of the squares of the distances from this star. Thus the theorems of Huygens on the centrifugal force, were sufficient to demonstrate the tendency of the planets towards the Sun: for it is highly probable that this law, which extends from one planet to another, and which is verified in the same planet, at its aphelion and perihelion, extends also to every part of the planetary orbit, and generally to all distances from the Sun. But to establish it in an incontestable manner, it was requisite to determine the general expression of the force which, directed towards the focus of an ellipse, makes a projectile to describe that curve. And it was Newton who demonstrated that this force was reciprocally as the square of the (*d*) radius vector. It was essential also to demonstrate rigorously that the force of gravity, towards the Sun, only varies from one planet to another, in consequence of their different distances from this star.

This great geometrician shewed, that this followed necessarily from the law of the squares of the periodic (*e*) times being reciprocally as the cubes of the greater axes of the orbits. Supposing, therefore, all the planets in repose at the same distance from the Sun, and abandoned to their gravity towards its centre, they would descend from the same height in equal times; this result should likewise be extended to the comets, notwithstanding the greater axes of their orbits are unknown, for we have seen in the second Book, chap. 6, that the magnitude of the areas described by their radii (*f*) vectores, supposes the law of the squares of their periodic times, proportional to the cubes of these axes.

An analysis, which in all its generalities, embraces every possible result from a given law, shews us that not only an ellipse, but any other conic section, may be described by virtue of the force, which retains the planets in their orbits; a comet may therefore move in an hyperbola, but then it would only be once visible, and would after its apparition recede from the limits of the solar system to approach other suns, which it would again abandon, thus visiting the different systems that are distributed through the immensity of the heavens. It is probable, considering the infinite variety of nature, that such bodies exist. Their apparition should be a very rare occurrence; the comets we usually observe, are these which, having reentrant orbits, return at the end of intervals more or less considerable, into the regions of space which are in the vicinity of the Sun. The satellites tend also, as well as the planets, perpetually to the Sun. If the Moon was not subject to its action, instead of describing an orbit almost circular round the earth, it would very soon abandon it; and if this satellite and those of Jupiter were not sollicited towards the Sun, according to the same law as the planets, sensible inequalities would result in their motions, which have not been recognized by observation. The planets, comets, and satellites are therefore subject to the same law of gravity towards the Sun. At the same time that the satellites move round their respective primary planets, the whole system of the planet and its satellites is carried by a common motion in space, and retained by the same force, round the Sun. Thus the relative motion of the planet and its satellites, is nearly the same as if the planet was at rest, and not acted on by any external force.

We are thus conducted without the aid of hypothesis, by a necessary consequence of the laws of the celestial motions, to regard the Sun as the centre of a force, which, extending indefinitely into space, diminishes as the square of the distance increases, and which attracts all bodies similarly. Every one of the laws of Kepler indicates a property of this attractive force. The law of the areas proportional to the times, shews us that it is constantly directed towards the centre of the Sun; the elliptic orbits of the planets shew that this force diminishes as the square of the distance increases; finally, the law of the squares of the periodic times proportional to the cubes of the distance, demonstrates that the gravity of all the planets towards the Sun is the same at equal distances; we shall call this gravity *the solar attraction*, for without knowing the cause, we may by one of those conceptions, common to geometricians, suppose an attractive power to exist in the Sun.

The errors to which observations are liable, and the small alterations in the elliptic motion of the planets, leave a little uncertainty in the results which we have just deduced from the laws of motion; and it may be doubted whether the solar gravity diminishes exactly in the inverse ratio of the square of the distance. But a very small variation in this law, would produce a very sensible difference in (*g*) the motions of the perihelia of the planetary orbits. The perihelion of the terrestrial orbit, would have an annual motion of 200″, if we only increased by one ten-thousandth part, the power of the distance to which the solar gravity is reciprocally proportional; this motion is only 36″4, according to observation, and of this we shall hereafter see the cause. The law of the gravity inversely as the square of the distance, is then at least, extremely near; and its extreme simplicity should induce us to adopt it, as long as observations do not compel us to abandon it. However we must not estimate the simplicity of the laws of nature, by our facility of conception; but when those which appear to us the most simple, accord perfectly with all the phenomena, we are justified in supposing them rigorously exact.

The gravity of the satellites towards the centre of their primary planet, is the necessary consequence of the proportionality of the areas described by their radii vectores to the times, and the law of the diminution of this force, according to the square of the distance, is indicated by the ellipticity of their orbits. But this can hardly be perceived in the orbits of the satellites of Jupiter, Saturn, and Uranus, which renders the law of the diminution of the force difficult to ascertain by the motion of any one single satellite; but the constant ratio of the squares of the times of their revolutions, to the cubes of their distances, indicates it beyond a doubt, by demonstrating, that from one satellite to another, the gravity towards the planet is reciprocally as the square of the distance from its centre.

This proof is wanting for the earth, which has but one satellite, but it may be supplied by the following considerations.

The force of gravity extends to the summits of the highest mountains, and the small diminution which it there experiences, does not permit us to doubt, but that at still greater altitudes it would also be sensible. Is it not natural to extend this to the Moon, and to suppose that this star is retained in its orbit by its gravity towards the earth, in the same manner as the solar gravity retains the planets in their orbits round the Sun? For in fact these two forces seem to be of the same nature: they both of them penetrate the most intimate parts of matter, animating them with the same velocities; for we have seen that the solar gravity sollicits equally all bodies placed at equal distances from the Sun, just as the terrestrial gravity causes all bodies to fall in a vacuo, through the same height in equal times.

A heavy body forcibly projected horizontally from a great height, falls on the earth at a considerable distance, describing a curve which is sensibly parabolic, it will fall still farther if the force is greater; and if the velocity of projection was about seven thousand metres in a second, it would not fall to the Earth, but would setting aside the resistance of the air, circulate round it like a satellite, its centrifugal force being then equal to its gravity. To form a moon of this projectile, it must be taken to the height (h) of that body, and there receive the same motion of projection.

But what completes the demonstration of the identity of the moon's tendency towards the earth with gravity, is that, to obtain this tendency, it is sufficient to diminish the terrestrial gravity according to the general law of the variation of the attractive force of the celestial bodies. Let us enter into the details suitable to the importance of this subject.

The force which at every instant deflects the Moon from the tangent of her orbit, causes it to move over, in one second, a space equal to the versed sine of the arc which it describes in that time; since this sine is the quantity by which the Moon, at the end of a second, deviates from the direction it had in the beginning. This quantity may be determined by the distance of the Earth, inferred from the lunar parallax, in parts of the terrestrial radius; but to obtain a result independent of the inequalities of the Moon, we must take (*i*) for the mean parallax, that part of it which is independent of these inequalites, and which corresponds to the semiaxis major of the lunar ellipse. Burgh determined, by a comparison of a great number of observations, the lunar parallax and it results that the part of which we have been speaking, is about 10541″, at the parallel of which the square of the sine of the latitude is equal to $\frac{1}{3}$. We select this parallel, because the attraction of the Earth, on the corresponding points of its surface is, as at the distance of the Moon, very nearly equal to the mass of the Earth, divided by the square of the distance from its centre of gravity. The radius drawn from a point of this parallel to the centre of gravity of the Earth is 6369809 metres, from whence it may be computed that the force which sollicits the Moon towards the Earth, causes it to fall $0^{me}.00101728$ in one second of time. It will be shewn hereafter, that the action of the Sun diminishes the lunar gravity by a $\frac{1}{358}$th part. The preceding height must therefore be augmented a $\frac{1}{358}$th part, to render it independent of the action of the Sun; it then becomes $0^{me}.00102012$. But in its relative motion round the Earth, the Moon is sollicited by a force equal to the sum of the masses of the Earth and Moon, divided by the square of their mutual distance; therefore to obtain the height which the Moon would fall through in one second by the action of the Earth alone, the preceding space must be diminished in the ratio of the mass of the Earth to the sum of the masses of the Earth and Moon. But a great number of phenomena depending on the action of the Moon, have given the mass of the Moon equal to $\frac{1}{75}$th of that of the earth, multiplying therefore this space by $\frac{75}{76}$, we have $0^{me}.0010067$ for the height which the Moon falls through in one second, by the action of the Earth.

Let us now compare this height, with that which results from observations made on the pendulum. At the parallel above mentioned, the length of the pendulum vibrating seconds is (by Chapter XIV, Book I.) equal to $3^{me}.65631$. But on this parallel, the attraction of the Earth is less than the force of gravity, by $\frac{2}{3}$ of the centrifugal force due (h) to the motion of rotation of the Earth at the equator; and this force is the $\frac{1}{288}$th part of that of gravity; the preceding space must therefore be augmented a $\frac{1}{432}$d part, to have the space due to the action of terrestrial gravity alone, which on this parallel is equal to the mass divided by the square of the terrestrial radius, we shall therefore have $3^{me}.66477$ for this space. At the distance of the Moon, it should be diminished in the ratio of the square of the radius of the terrestrial spheriod to the square of the distance of the Moon: for this it is sufficient to multiply it by the square of the sine of the lunar parallax, or by 10541″, this will give $0^{me}.00100464$ for the height which the Moon should fall through in one second by the attraction of the Earth. This quantity derived from experiments on the pendulum, differs very little from that which results from direct observation of the lunar parallax; to make them coincide, it is sufficient to diminish by about 2″ the preceding value. This small difference being within the limits of the errors of observation, and of the elements employed in the calculation, it is certain, that the principal force which retains the Moon in its orbit is the terrestrial gravity diminished in the ratio of the square of the distance. Thus the law of the diminution of gravity, which, in planets accompanied by several satellites, is proved by a comparison of their periodic times with their distances, is demonstrated for the Moon, by comparing its motion with that of projectiles at the surface of the Earth.

The observations of the pendulum made on the summits of mountains, had already indicated this diminution of the terrestrial gravity; but they were insufficient to discover the law, because of the small height of the most elevated mountains, compared with the radius of the Earth: it was requisite to find a body very remote from us, as the Moon, to render the law perceptible, and to convince us that the force of gravity on the Earth, is only a particular case of a force which pervades the whole universe.

Every successive phenomenon elucidates and confirms the laws of nature. It is thus that the comparison of experiments on gravity, with the lunar motion, shews us, that the origin of the distances of the Sun and of the planets in the calculation of their attractive forces, should be placed in their centres of gravity; for it is evident that this takes place for the Earth, whose attractive force is of the same nature as that of the Sun and planets.

The striking similarity between the Sun and the planets which are attended by satellites, and those which have none, should induce us to extend to them this attractive force. The spherical figure common to all these bodies, indicates that their particles are united round their centers of gravity, by a force which, at equal distances, equally sollicits them towards these points; this force is also indicated by the perturbations which planetary motions experience; but the following considerations leave no doubt on this subject.

We have seen that if the planets and the comets were placed at the same distance from the Sun, their gravity towards it would be in proportion to their masses: now it is a general law in nature, that action and reaction are equal and contrary, all these bodies therefore react on the Sun, and attract it in proportion to their masses; they are therefore endowed with an attractive force proportional to their masses, and inversely as the square of the distances. By the same principle, the satellites attract the planets and the Sun according to the same law. This attractive property then is common to all the celestial bodies: it does not disturb the elliptic motion round the Sun, when we consider only their mutual action; for the relative motion of the bodies of a system, are not changed by giving them a common velocity: by impressing therefore, in a contrary direction to (*l*) the Sun and to the planet, the motion of the first of these two bodies, and the action which it experiences on the part of the second, the Sun may be considered as immoveable; but the planet will be sollicited towards it, with a force reciprocally as the squares of the distance, and proportional to the sum of the masses: its motion round the Sun will therefore be elliptic. And we see by the same reasoning, that it would be so if the planet and Sun were carried through space, with a motion common to each of them. It is equally evident that the elliptic motion of a satellite is not disturbed by the motion of translation of its planet, nor would it be by the action of the Sun, if it was always exactly the same on the satellite and planet. Nevertheless, the action of a planet on the Sun influences the length of its revolution, which is diminished as the mass of the planet is more considerable, so that the relation of the square of its periodic time to the cube of the major axis of its orbit, is proportional to the sum of the masses of the Sun and planet. But since this relation is nearly the same for all the planets, their masses must evidently be very small compared with that of the Sun, which is equally true for the satellites with respect to their respective primary planets. This is what is confirmed by the volumes of these different bodies.

The attractive property of the heavenly bodies, does not only appertain to them in the aggregate, but likewise belongs to each of their particles. If the Sun only acted on the centre of the Earth, without attracting in particular every one of its particles, there would arise in the ocean oscillations incomparably more considerable, and very different from those which we observe. The gravity of the Earth therefore to the Sun is the result of the gravity of all its particles which consequently, attract the Sun in proportion to their respective masses; besides each body on the earth, tends towards its centre proportionally to its mass, it (*m*) reacts therefore on it, and attracts it in the same ratio. If that was not the case, and if any part of the Earth, however small, attracted another part without being attracted by it, the centre of gravity of the earth would move in space in virtue of the force of gravity, which is inadmissable.

The celestial phenomena compared with the laws of motion, conduct us, therefore, to this great principle of nature, namely, *that all the particles of matter mutually attract each other, in the ratio of their masses, divided by the squares of their distances.*

Already we may perceive in this universal gravitation, the cause of the perturbations to which the heavenly bodies are subject; for as the planets and comets are subject to the action of each other, they must deviate a little from the laws of elliptic motion, which they would otherwise exactly follow, if they only obeyed the action of the Sun. The satellites also, deranged in their motions round their planets, by their mutual action and that of the Sun, deviate a little from these laws.

We perceive, then, that the particles of the heavenly bodies, united by their attraction, should form a mass nearly spherical; and that the result of their action at the surface of the body, should produce all the phenomena of gravitation. We see, moreover, that the motion of rotation of the celestial bodies should slightly alter their spherical figure, and flatten them at the poles: and then the resulting force of all their mutual actions not passing through their centres of gravity, should produce in their axes of rotation motions similar to those discovered by observation. Finally, we may perceive why the particles of the ocean, unequally acted on by the Sun and Moon, should have oscillations similar to the ebbing and flowing of the tides. But these different effects of the principle of gravitation, must be particularly developed, to give it all the certainty of which physical truth is susceptible.

CHAP. II.

Of the Perturbations of the Elliptic Motion of the Planets.

If the planets only obeyed the action of the Sun, they would revolve round it in elliptic orbits, but they act mutually upon each other and upon the Sun, and from these various attractions, there result perturbations in their elliptic motions, which are to a certain degree perceived by observation, and which it is necessary to determine to have exact tables of the planetary motions. The rigorous solution of this problem, surpasses the actual powers of analysis, and we are obliged to have recourse to approximations. Fortunately, the smallness of the masses compared to that of the Sun, and the smallness of the excentricity and mutual inclination of their orbits, afford considerable facilities for this object. It is still, however, sufficiently complicated, (*a*) and the most delicate and intricate analysis is requisite to detect among the infinite number of inequalities to which the planets are subject, those which are sensible to observation, and to assign their values.

The perturbations of the elliptic motion of the planets may be divided into two distinct classes. Those of the first class affect the elements of the elliptic motion of the planets, they increase with extreme slowness, and are called *secular inequalities.* The other class depends on the configurations of the planets, both with respect to each other and to their nodes and perihelia, and being re-established every time these configurations become the same, they havebeen termed *periodical inequalities* to distinguish them from the secular inequalities, which are equally periodic, but whose periods are much longer, and independent of the mutual configurations of the planets.

The most simple manner of considering these various perturbations, consists in imagining a planet to move according to the laws of elliptic motion, upon an ellipse, whose elements vary by imperceptible gradations, and conceiving at the same time the true planet to oscillate round the imaginary planet in a (*b*) small orbit, the nature of which must depend on its periodic inequalities.

Let us first consider those secular inequalities which, by developing themselves in the course of ages, should change at length, both the form and position of the planetary orbits. The most important of these inequalities is that which may affect the mean motion of the planets. By comparing together, the observations which have been made since the restoration of astronomy, the motion of Jupiter appears to be quicker and that of Saturn slower, than by a comparison of the same observations, with those of the ancient astronomers: from which astronomers have concluded that the first of these motions has accelerated, while the second has been retarded from one century to another. And to take into account these variations, astronomers have introduced into the tables of those planets, two secular equations increasing with the squares of the times, one additive to the mean motion of Jupiter, the other subtractive from that of Saturn. According to Halley, the secular equation of Jupiter is 106″ for the first century reckoned from 1700, the corresponding equation of Saturn is 256″94. It was natural to look for the cause of these equations, in the mutual actions of these planets, the most considerable of our system. Euler, who first directed his attention to this problem, found a secular equation, equal for both the planets, and additive to their mean motions, which is inconsistent with observation. Lagrange obtained a result which accorded more nearly with them. Other geometricians obtained other equations. Struck with this difference, I examined again this subject, and by applying the greatest possible care to the investigation, I arrived at the true analytical expression for the secular inequality of the planets. In substituting the numerical values, relative to Jupiter and Saturn, I was surprised to find that it became equal to nothing. I suspected that this was not peculiar to these planets, and that if this expression was put in the most simple form of which it was susceptible, (by reducing to the least possible number the different quantities which it contains by means of the relations which subsist between them), all its terms would destroy each other. Calculation confirmed this supposition, and shewed me that, in general, the mean motions of the planets and their mean distances from the Sun are invariable; at least when we neglect *(c)* the fourth powers of the excentricities and of the inclinations of the orbits, and the squares of the perturbating masses, which is more than sufficient for the actual purposes of astronomy. Lagrange has since confirmed this result, and shewn, by a beautiful method, that it is even true, when the powers and products of any order whatever, of the excentricities and inclinations, are taken into the calculation. M. Poisson has shewn by an ingenious analysis, that the same result subsists even when the approximations are extended to the squares and products of the masses of the planets. Thus the variations of the mean motions of Jupiter and Saturn, do not depend on their secular inequalities.

The permanency of the mean motions of the planets and of the greater axes of their orbits, is one of the most remarkable phenomena in the system of the world. All the other elements of the planetary ellipses are variable, these ellipses approach to and depart insensibly from the circular form; their inclination to a fixed plane or to the ecliptic augments and diminishes, and their perihelia and nodes are continually changing their places. These variations, produced by the mutual actions of the planets on each other, are performed with such extreme slowness, that for a number of centuries they are nearly proportional to the times. They have already become apparent by observation; we have seen, in the first Book, that the perihelion of the Earth's orbit has a direct annual motion of 36″, and that its inclination to the equator diminishes every century 148″. It was Euler who first investigated the cause of this diminution, which all the planets contribute to produce, by the respective situation of the planes of their orbits. In consequence of these variations of the orbit of the earth, the perigee of the Sun coincided with the equinox of spring at an epoch to which we can ascend by analysis, which is anterior to our æra by about 4089 years. It is remarkable that this astronomical epoch is nearly that at which chronologists have fixed the creation of the world. The ancient observations are not exact enough, and the modern are too near each other to fix the exact quantity of these great changes of the planetary orbits, nevertheless they combine to prove their existence, and to shew that their progress is the same as would result from the law of gravitation. If we knew exactly the masses of the planets, future observations might be anticipated, and the true values assigned to the secular inequalities of the planets; and one of the surest means of determining them, will be the developement of these inequalities in the progress of time. We may then in imagination look back to the successive changes which the planetary system has undergone, and foretell those which future ages will offer to astronomers, and the geometrician will at once comprehend in his formulæ both the past and future states of the world.

Many interesting questions here present themselves to our notice. Have the planetary ellipses always been, and will they always be nearly circular. Among the number of the planets have any of them ever been comets whose orbits have gradually approached to the circular form, by the mutual attractions of the other planets? Will the obliquity of the ecliptic continually diminish till at length it coincides with the equator, and the days and nights become equal on the earth, throughout the year? Analysis answers these questions, in a most satisfactory manner. I have succeeded in demonstrating that whatever be the masses of the planets, in as much as they all move in the same direction, in orbits of small excentricity, and little inclined to each other; (*c*) their secular inequalities will be periodic, and contained within narrow limits, so that the planetary system will only oscillate about a mean state, from which it will deviate but by a very small quantity; the planetary ellipses therefore always have been, and always will be nearly circular, from whence it follows that no planet has ever been a comet, at least if we only take into account the mutual action of of the bodies of the planetary system. The ecliptic will never coincide with the equator, and the whole extent of its variations will not exceed three degrees.

The motions of the planetary orbits and of the stars will one day embarrass astronomers, when they attempt to compare precise observations separated by long intervals of time; already this difficulty begins to be apparent; it would be interesting therefore to find some plane that should remain invariable, that is, constantly parallel to itself. We have given at the end of the preceding book, a simple means of determining a similar plane, in the motion of a system of bodies which are only subject to their mutual action; this method when applied to the solar system, gives the following rule. If at any instant of time (*h*) whatever, and upon any plane passing through the centre of the Sun, we draw from this point straight lines to the ascending nodes of the planetary orbits referred to this plane, and if we take on these lines, reckoning from the centre of the Sun, lines equal to the tangents of the inclinations of these orbits to this plane, and if at the extremities of these lines, we suppose masses equal to the masses of the planets multiplied respectively into the square roots of the parameters of the orbits, and by the cosines of their inclinations; and lastly, if we determine the centre of gravity of this new system of masses, then the line drawn from the centre of the Sun to this point will be the tangent of the inclination of the invariable plane, to the assumed plane; and continuing this line to the heavens, it will there mark its ascending node.

Whatever changes the succession of ages may produce in the planetary orbits, and whatever be the plane to which they are referred, the plane determined by this rule, will always be parallel to itself. It is true, its position depends on the masses of the planets; but these will soon be sufficiently known to determine it with exactness. In adopting the values of these masses which will be given in the following chapter, we find that the longitude of the ascending node of the invariable plane was 114°,7008 at the commencement of the nineteenth century, and at the same epoch its inclination to the ecliptic was 1°,7565. In this computation we have neglected the comets, which nevertheless ought to enter into the determination of the invariable plane, since they constitute a part of the solar system. It would be easy to include them in the preceding rule, if their masses and the elements of their orbits were known. But in our present ignorance of the nature of these objects, we suppose their masses too small to influence the planetary system, and this is the more probable, since the theory of the mutual attraction of the planets, suffices to explain all the inequalities observed in their motions. But if the action of the comets should become sensible in the progress of time, it should principally affect the position of the plane, which we suppose invariable, and in this new point of view the consideration of this plane will still be useful, if the variations of this plane could be recognised, which would be attended with great difficulties.

The theory of the secular and periodic inequalities of the motions of the planets, founded on the law of universal gravitation, has been confirmed by its agreement with all observations ancient and modern. It is particularly in the motions of Jupiter and Saturn, that these inequalities are most sensible, but they present themselves under a form so complicated, and the length of their periods is so considerable, that it would have required several ages to have determined their law by observations alone, which has in this instance been anticipated by theory.

After having established the invariability of the mean motions of the planets, I suspected that the alterations observed in the mean motions of Jupiter and Saturn, proceeded from the action of comets. Lalande had remarked in the motion of Saturn, irregularities which did not appear to depend on the action of Jupiter: he found its returns to the vernal exquinox, more rapid than its returns to the autumnal equinox, although the positions of Jupiter and Saturn, both with respect to each other, and to their aphelia, were nearly the same. Lambert likewise observed that the mean motion of Saturn, which seemed to diminish from century to century by the comparison of ancient with modern observations, appeared on the contrary, to accelerate by the comparison of modern observations with each other, at the same time that Jupiter presented phenomena exactly contrary. All this seemed to indicate that causes independent of the action of Jupiter and Saturn on each other, had altered their motions. But on mature reflection, the order of the variations observed in the mean motions of these planets, appeared to me to agree so well with the theory of their mutual attraction, that I did not hesitate to reject the hypothesis of a foreign cause.

It is a remarkable result of the mutual action of the planets on each other, that if we only consider (*i*) the inequalities which have very long periods, the sum of the masses of every planet, divided respectively by the greater axes of their orbits considered as variable ellipses, is always pretty nearly constant. From this it follows, that the squares of the mean motions, being reciprocally as the cubes of these axes, if the motion of Saturn is retarded by the action of Jupiter, that of Jupiter should be accelerated by the action of Saturn, which is conformable to observation. I perceived, moreover, that the law of these variations was the same as corresponded to the preceding theory. In supposing with Halley the retardation of Saturn to be 256″94 for the first century, reckoned from 1700, the corresponding acceleration of Jupiter should be 109″80, and Halley found it to be 106″02 by observation. It was therefore very probable that the variations observed in the mean motions of Jupiter and Saturn, were the effects of their mutual action; and since it is certain that this action cannot produce any inequality either constantly increasing or periodic, but of a period independent of the configuration of these planets, and that it cannot effect in it any irregularities but what are relative to this configuration, it was natural to think that there existed in their theory a considerable inequality of this kind, of a very long period, and which was the cause of these variations.

The inequalities of this kind, although very small and almost insensible in differential equations, augment considerably in the integrations, and may acquire very great values in the expressions of the longitudes of the planets. (k) I easily recognized the existence of similar inequalities, in the differential equations of the motions of Jupiter and Saturn. These motions are very nearly commensurable; so that five times the mean motion of Saturn differs very little from twice that of Jupiter: from which I concluded that the terms which have for their argument five times the mean longitude of Saturn, minus twice that of Jupiter, might by integration become very sensible, although multiplied by the cubes and products of three dimensions of the excentricities and inclinations of the orbits. I considered therefore that these terms were the probable cause of the variations observed in the mean motions of these planets. The probability of this cause, and the importance of the object, determined me to undertake the laborious calculation, necessary to determine this question. The result of this calculation fully confirmed my conjecture; and it appeared, that in the first place there exists in the theory of Saturn a great inequality of 8895″7 at its maximum, of which the period is 929 years; and which ought to be applied to the mean motion of this planet; and secondly, that the motion of Jupiter is subject to a similar inequality, whose period and law are the same, but affected with a contrary sign, its amount is only 3662″41. The magnitude of the coefficients of these inequalities and the duration of their period are not always the same, they participate in the secular variations of the elements of the orbits on which they depend, I have determined with especial care, those coefficients and their secular diminution. It is to these two inequalities, formerly unknown, that that we must attribute the apparent retardation of Saturn, and the apparent acceleration of Jupiter. These phenomena attained their maximum about the year 1560; since this epoch, their mean apparent motions have approximated to their true mean motions, and they were equal in 1790. This explains the reason why Halley, in comparing the ancient with modern observations, found the mean motion of Saturn slower, and that of Jupiter more rapid than by the comparison of modern observations with each other, instead of which these last indicated to Lambert an acceleration in the motion of Saturn, and a retardation in that of Jupiter. And it is very remarkable that the quantities of these phenomena, deduced from observation alone by Halley and Lambert, are very nearly the same as result from the two great inequalities which I have just mentioned. If astronomy had been revived four centuries and a half later, observations would have presented the direct contrary phenomena. The mean motions which the astronomy of any people have assigned to Jupiter and Saturn, should afford us information concerning the time of its foundation. Thus it appears that the Indian astronomers determined the mean motions of these planets, in that part of the period of the preceding inequalities, when the motion of Saturn was the slowest, and that of Jupiter the most rapid. Two of their principal astronomical epochs, the one 3102 A. C. the other 1491 A. C. answer nearly to this condition. The nearly commensurable relation that exists in the motions of Jupiter and Saturn, occasions other very perceptible inequalities, the most considerable of which affects the motion of Saturn; it would be entirely confounded in the equation of the centre, if twice the mean motion of Jupiter was exactly equal to five times that of Saturn. The difference observed in the last century in the intervals of the returns of Saturn to the equinoxes of spring and autumn, arises principally from this cause.

In general, when I had recognised these various inequalities, and examined more carefully than had been done before, those which had been submitted to calculation, I found that all the observed phenomena of the motions of these two planets adapted themselves naturally to the theory; before they seemed to form an exception to the law of universal gravitation; they are now become one of the most striking examples of its truth. Such has been the fate of this brilliant discovery of Newton, that every difficulty which has arisen, has only furnished a new subject of triumph for it, which is the most indubitable characteristic of the true system of nature.

The formulæ which I have obtained for representing the motions of Jupiter and Saturn, satisfy with remarkable precision the last oppositions of these two planets, which have been observed by the most skilful astronomers with the best meridian telescopes and the greatest quadrants of circles, the error never amounted to 40″; and twenty years ago the errors of the best tables sometimes surpassed four thousand seconds. These formulæ also represent with the same accuracy as observations themselves, the observations of Flamstead, those of the Arabians, and the observations cited by Ptolomy. This great precision with which the two largest planets of our system, have obeyed from the most remote period, the laws of their mutual attraction, evinces the stability of this system, since Saturn, of which the attraction to the Sun is about an hundred times less than the attraction of the earth to the same star, has not since the æra of Hipparchus to the present day, experienced any sensible derangement from the action of extraneous causes.

I cannot in this place, refrain from making a comparison of the real effects of this relation between the mean motions of Jupiter and Saturn, with those which astrology had attributed to it. In consequence of this relation, the mutual conjunctions of these two planets are renewed after an interval of twenty years, but the point of the heavens to which they arrive, retrogrades by about a third of the zodiac, so that if the conjunction of the two planets arrives in the first point of Aries, it will in twenty years afterwards take place in Sagittarius, and in twenty years afterwards in Leo, to return then to the sign of the ram at ten degrees from its original position. It will continue to take place in these three signs, for nearly two hundred years. In the same manner, in the next two hundred years, it will go through the signs Taurus, Capricornus, and Virgo. In the next two hundred years, it will proceed through the signs Gemini, Aquarius, and Libra; and finally, in the last two hundred years, it will describe the remaining signs, Cancer, Pisces, and Scorpio; after which it will again begin with the sign Aries as before. From hence arises a great year, each season of which is equal to two centuries. They attributed different temperatures to the different seasons of this year, as likewise to the signs which belonged to them. The assemblage of these three signs was called a *trigon*. The first trigon was that of Fire, the second of Earth, the third of Air, and the fourth of Water.—We may easily imagine that astrology made great use of these trigons, which even Kepler himself describes with great exactness, in several of his works: but it is very remarkable that sound astronomy, while it dissipated the imaginary influence that was supposed to attend this relation in the motion of the two planets, should have recognised in this relation, the source of the greatest perturbations of the planetary system.

The planet Uranus, though lately discovered, offers already incontestable indications of the perturbations which it experiences from the action of Jupiter and Saturn. The laws of elliptic motion do not exactly satisfy its observed positions, and to represent them, its perturbations must be considered. Their theory, by a very remarkable coincidence, places it in the years 1769, 1756, and 1690, in the same points of the heavens, where Monnier, Mayer, and Flamstead, had determined the position of three stars, which cannot be found at present: this leaves no doubt of the identity of these stars with the new planet.

The small planets which have been discovered, are subject to very great inequalities, which will throw new light on the theory of the attractions of the heavenly bodies, and will enable us to render it perfect; but hitherto we have been unable to recognize these inequalities by means of observations. It is only three centuries since Copernicus first introduced into the astronomical tables the motion of the planets about the Sun: about a century after, Kepler took into account the laws of elliptic motion, which had been discovered by means of the observations of Tycho Brahe; this led Newton to the discovery of universal gravitation. Since these three epochs, which will be always memorable in the history of the sciences, the improvements in the infinitesimal calculus have enabled us to subject to computation the numerous inequalities of the planets which arise from their mutual attraction, and by this means the tables have acquired a degree of precision which could never have been anticipated; formerly their errors amounted to several minutes, they are now reduced to a small number of seconds, and very often, it is probable, that their apparent deviations arise from the inevitable errors of the observations.

CHAP. III.

Of the Masses of the Planets, and of the Gravity at their Surface.

The ratio of the mass of a planet to the mass of the Sun, being the principal element of the theory of the perturbations which it produces, the comparison of this theory with a great number of very precise observations, ought to give its value so much the more accurately, as the perturbations of which it is the cause are more considerable. It is in this manner (*a*) that the following values of the masses of Venus, of Mars, of Jupiter and of Saturn, have been determined. The masses of Jupiter, of Saturn, and of those planets which have satellites, may be determined in the following manner.

It follows from the theorems on centrifugal force, given in the preceding book, that the gravity of a satellite towards its primary is to the gravity of the Earth towards the Sun, as the mean radius of the orbit of the satellite divided by the square of the time of its sidereal revolution, is to the mean distance of the Earth (*b*) from the Sun, divided by the square of a sidereal year. To reduce these gravities, to the same distance from the bodies which produce them, they must be multiplied respectively by the squares of the radii of the orbits which they describe. And as at equal distances, the masses are proportional to their attractions, the mass of the planet is to that of the Sun, as the cube of the mean radius of the orbit of the satellite, divided by the square of the time of its sidereal revolution, is to the cube of the mean distance of the Earth from the Sun, divided by the square of the sidereal year. This result supposes that the mass of the satellite relatively to that of the planet has been neglected, and also the mass of the planet with respect to that of the Sun, which may be done without any sensible error, it will become more exact if we substitute in place of the mass of the planet, the sum of the masses of the planet and of its satellite, and instead of the mass of the Sun, the sum of the masses of the Sun and planet, since the force which retains a body in its relative orbit, about that which attracts it, depends on the sum of their masses. Let this result be applied to Jupiter; the mean radius of the orbit of the fourth satellite, such as it has been given in the second book, seen at the mean distance of the Earth from the Sun, would appear under an angle of 7964″75; the radius of the circle contains 636619″8: thus the mean radii of the orbits of the fourth satellite, and of the terrestrial orbit, are in the proportion of these two last numbers. The duration of the sidereal revolution of the fourth satellite (*c*) s 16^d 6890, and the sidereal year is 365^d 2564. Setting out from these data, the mass of Jupiter is found to be $\frac{1}{1067,09}$, that of the Sun being represented by unity. To obtain greater exactness, it is necessary to diminish by unity the denominator of this fraction; the mass of this planet then is $\frac{1}{1066,09}$. I have determined by the same method, the masses of Saturn and of Uranus, equal respectively to $\frac{1}{3359,4}$, $\frac{1}{19504}$.

The perturbations which these three large planets experience from their reciprocal attractions, furnish an accurate method of obtaining the values of their masses. Mr. Bouvard, from a comparison of the formulæ which are given in the Celestial Mechanics, with a great number of observations carefully discussed, constructed new tables of Jupiter, of Saturn, and of Uranus. He has formed for this important object equations of condition, in which he left as indeterminate, the masses of these planets, and from a resolution of these equations, he obtained the following numerical values for these masses, $\frac{1}{1070,5}$, $\frac{1}{3512}$; $\frac{1}{17918}$. If we consider the great difficulty of measuring the elongations of the satellites of Saturn and Uranus, and our ignorance of the ellipticity of the orbits of these satellites; the little difference which exists between the values inferred from these elongations, and those which result from the perturbations, is really astonishing. These last values include for each planet, its mass and those of its satellites, to which it is necessary to add, in the case of Saturn (d), that of its ring. But every thing induces us to think that the mass of the planet is far superior to that of the bodies which surround it; at least this is certainly the case for the earth and Jupiter. But by applying the theory of probabilities to the equations of condition of M. Bouvard, it has been found, that it is a million to one, that the value given above for the mass, does not differ by a hundreth part from its true value. There is eleven thousand to one, that this is the case with respect to the mass of Saturn. Since the perturbations which Uranus produces in the motion of Saturn are inconsiderable, a great number of observations is required to obtain its mass with the same probability, but in the actual state of the case, it is 2500 to 1, that the preceding result does not differ from its true value by a fourth part. The perturbations which the earth experiences from the attractions of Venus and Mars, are sufficiently sensible to indicate the masses of these planets. M. Buckhardt, to whom we are indebted for excellent tables of the Sun, founded on four thousand observations, has concluded that the values of these masses are respectively $\frac{1}{405871}$ and $\frac{1}{2546320}$.

We may obtain in the following manner, the mass of the earth. If the mean distance of the Earth from the Sun be assumed equal to unity, the arc described by the Earth in a second of time, will be the proportion of the circumference to radius, divided by the number of seconds in the sidereal year, or by 36525636″1; dividing the square of this arc by the diameter, we shall get

$\frac{479565}{10^{20}}$ for its versed sine, it is the quantity which the Earth falls towards the Sun, during one second, in consequence of its relative motion round it. It has been seen, in the preceding chapter, that upon the terrestrial parallel, of which the square of the sine of the latitude is $\frac{1}{3}$, the attraction of the Earth causes bodies to fall through $3^{me.}$ 66477 in one second. To reduce this attraction to the mean distance of the Earth from the Sun, it must be multiplied by the square of the sine of the solar parallax, and then the product should be divided by the number of metres contained in this distance. Now the terrestrial radius, at the parallel we are considering, is 6369809 metres; dividing this number, therefore, by the sine of the solar parallax, or by 26″54, we shall get the mean radius of the terrestrial orbit, expressed in metres. It follows from hence, that the effect of the Earth's attraction, at the mean distance of this planet from the Sun, is equal to the product of the fraction $\frac{3,66477}{6369809}$ by the cube of the sine of 26″54, it is consequently equal to $\frac{4,16856}{10^{20}}$: taking this fraction from $\frac{1479565}{10^{20}}$ we shall have $\frac{1479560}{10^{20}}$ for the effect of the Sun's attraction at the same distance (*e*). The masses of the Sun and Earth are therefore in the proportion of the numbers 1479560,8 and 4,16856; from whence it follows that the mass of the Earth is $\frac{1}{354936}$. If the parallax of the Sun is a little different from what we have supposed, the value of the mass of the Earth should vary as the cube of this parallax compared to that of 26″54.

The mass of Mercury has been determined by its volume, supposing the densities of this planet and of the Earth, inversely as their mean distances from the Sun. An hypothesis indeed very precarious, but which corresponds with sufficient exactness to the respective densities of the Earth, Jupiter and Saturn. It will be necessary to rectify all these values, when in the progress of time the secular variations of the celestial motions shall be determined more correctly.

Masses of the Planets, that of the Sun being taken as unity.

Mercury	$\frac{1}{2025810}$
Venus	$\frac{1}{405871}$
The Earth	$\frac{1}{354936}$
Mars	$\frac{1}{2546320}$
Jupiter	$\frac{1}{1070.5}$
Saturn	$\frac{1}{3512}$
Uranus	$\frac{1}{17918}$.

The densities of bodies are proportional to their masses divided by their volumes, and when they are nearly spherical, their volumes are as the cubes of their radii. The densities therefore are as the masses divided by the cubes of the radii; but to obtain greater accuracy, that radius of a planet must be taken, which corresponds to the parallel, the square of the sine of whose latitude is $\frac{1}{3}$. It was stated in the first book, that the semidiameter of the Sun seen at its mean distance from the earth, subtends an angle of 2966″, and at the same distance the, radius of the earth would appear under an angle of 26″,54. It is easy to infer from this that the mean density of the solar globe being assumed equal to unity, (*f*) that of the earth is 3,9326. This value is independent of the parallax of the Sun; for the volume and the mass of the earth increase respectively, as the cube of this parallax. The semidiameter of the equator of Jupiter seen at its mean distance from the Sun, is according to the accurate measures of Arrago, equal to 56″,7″02; the semiaxis passing through the poles is 53,497, therefore the radius of the spheroid of Jupiter, corresponding to the parallel of which the square of the sine of the latitude is $\frac{1}{3}$, will subtend at the same distance, an angle of 55,″967, and seen at the mean distance of the Earth from the Sun, it will be 291″,185. Hence it is easy to infer that the density of Jupiter is equal to 0,99239. The density of the other planets may be determined in the same manner, but the errors of which the measures of their apparent diameters, and the estimation of their masses are also susceptible, will cause considerable uncertainty in the results of the computation; if the apparent diameter of Saturn seen at its mean distance from the Sun be supposed equal to 50″, its density will be equal to 0,55, that of the Sun being unity. A comparison of the respective densities of the earth, of Jupiter and of Saturn, indicates that they are smaller for the more distant planets; Kepler was led to the same result from his notions of suitableness and harmony, and he supposed the density of the planets to be reciprocally proportional to the square roots of their distances. But he concluded from the same considerations that the Sun was the densest of all the stars, which is not the case. The planet Uranus, of which the density appears to surpass that of Saturn, is an exception to the preceding rule. In consequence of the uncertainty which hangs over the measures of his apparent diameter, and the measures of his greatest elongations, we cannot pronounce with certainty on this subject.

To obtain the intensity of gravitation at the surface of the Sun and planets, it has been proved that if Jupiter and the Earth were exactly spherical, and deprived of their rotatory motion, gravity at their equators would be proportional to the masses of these bodies, divided by the squares of their diameters; now at the mean distance of the Sun from the Earth, Jupiter's apparent semi-diameter is 291″,185, and that of the Earth's equator is 26″,541; representing then the weight of a body at the terrestrial equator by unity, the weight of this body transported to the equator of Jupiter would be 2,716, but this weight must be diminished by about a ninth part (*g*) from the effcts of the centrifugal force, due to the rotation of these planets. The same body would weigh 27,9 at the equator of the Sun, and falling bodies would describe one hundred and two metres in the first second of their descent. The immense interval which separates us from these great bodies, seemed for ever to debar us from obtaining a knowledge of the effects of gravity at their surface: but the connexion of truths leads to results which appear inaccessible, when the principle on which they depend is unknown. It is thus that the measure of the intensity of gravity at the surface of the Sun and of the planets, is rendered possible by the discovery of universal gravitation.

CHAP. IV.

....

Of the Perturbations of the Elliptic Motion of Comets.

The action of the planets produces in the motion of comets, inequalities which are principally sensible in the intervals of their returns to the perihelion. Halley having remarked that the elements of the orbits of the comets observed in 1531, 1607, and 1682, were nearly the same, concluded that they belonged to the same comet which in the space of 151 years, had made two revolutions. It is true, that the period of its revolution is thirteen months longer from 1537 to 1607, than from 1607 to 1682. But this great astronomer thought, and with reason, that the attraction of the planets, particularly of Jupiter and Saturn, might have occasioned this difference, and after a vague estimation of this action during the course of the following period, he judged that it should retard the return of the comet, and he fixed it for the end of 1758, or the commencement of 1759. This prediction was too important in itself, and too intimately connected with the theory of universal gravitation, not to excite the curiosity of all those who were interested in the progress of the sciences; and in particular of a theory which already accorded with a great number of phenomena. Astronomers, uncertain of the epoch at which it ought to return, sought it about the year 1757; and Clairaut, who had been one of the first to solve the problem of the three bodies, applied his solution to the determination of the inequalities which the comet had sustained by the action of Jupiter and Saturn. On the 14th November, 1758, he announced in the academy of sciences, (*a*) that the interval of the return of the comet to its perihelion, would be 618 days longer in the present period than in the former one, and consequently, the comet would pass its perihelion, about the middle of April 1759. He observed, at the same time, that the small quantities neglected in this approximate calculation, might advance or retard this term, a month. He remarked also that a body which passes into regions so remote, and which escapes our sight during such long intervals, may be subject to the action of forces entirely unknown, as the attraction of other comets, or even of some planet, whose distance is too great to be ever visible to us. This philosopher had the satisfaction of seeing his prediction accomplished; the comet passed its perihelion the 12th March 1759, within the limits of the errors of which he thought his results susceptible. After a new revision of his calculations, Clairaut fixed this passage at the 4th of April, and he would have brought it to the 24th

March, if he had employed the mass of Saturn, such as is given in chap. III.; that is, within thirteen days of the actual observation. This difference will appear very small, if we consider the great number of quantities neglected, and the influence which the planet Uranus might produce, whose existence was at that time unknown.

Let us remark, for the honour of the human understanding, that this comet, which in the last century only excited the most lively interest among astronomers and mathematicians, had been regarded in a very different manner, four revolutions before, when it appeared in 1456. Its long tail spread consternation over all Europe, already terrified by the rapid success of the Turkish arms, which had just destroyed the great empire. Pope Callixtus, on this occasion, ordered a prayer, in which both the comet and the Turks were denounced in the same anathema.

In those times of ignorance, mankind were far from thinking that nature obeyed immutable laws, and according as phenomena succeeded with regularity or without apparent order, they were supposed to depend either on final causes or on chance; so that whenever any thing happened which seemed out of the natural order, they were considered as so many signs of the anger of heaven.

To the terrors which the apparition of comets then inspired, succeeded the apprehension, that of the great number which traverse the planetary system in all directions, one of them might overturn the earth.

They pass so rapidly by us, that the effects of their attraction are not to be apprehended. It is only by striking the earth that they can produce any disastrous consequences. But this circumstance, though possible, is so little probable in the course of a century, and it would require such an extraordinary combination of circumstances for two bodies, so small in comparison with the immense space they move in, (*b*) to strike each other, that no reasonable apprehension can be entertained of such an event.

Nevertheless, the small probability of this circumstance may, by accumulating during a long succesion of ages, become very great. It is easy to represent the effect of such a shock upon the earth: the axis and motion of rotation would be changed, the waters abandoning their antient position, would precipitate themselves towards the new equator; the greater part of men and animals drowned in a universal deluge, or destroyed by the violence of the shock given to the terrestrial globe; whole species destroyed; all the monuments of human industry reversed: such are the disasters which the shock of a comet ought to produce, if its mass was comparable to the mass of the earth.

We see then why the ocean has covered the highest mountains, on which it has left incontestible marks of its former abode: we see why the animals and plants of the south may have existed in the climates of the north, where their relics and impressions are still to be (*c*) found: *lastly*, this explains the short period of the existence of the moral world, whose earliest monuments do not go much farther back than five thousand years. The human race reduced to a small number of individuals, in the most deplorable state, occupied only with the immediate care for their subsistence, must necessarily have lost the remembrance of all sciences and of every art; and when the progress of civilization had created new wants, every thing was to be invented again, as if mankind had been just placed upon the earth. But whatever may be the cause assigned by philosophers to these phenomena, I repeat it, we may be perfectly at ease with respect to such a catastrophe during the short period of human life, especially since it appears that the masses of the comets are extremely small, and therefore their shock ought only to produce local changes.

But man is so disposed to yield to the dictates of fear, that the greatest consternation was excited at Paris, and thence communicated to all France in 1773, by a memoir of Lalande, in which he determined, of those comets which had been observed, the orbits that most nearly approached the earth; so true it is, that error, superstition, vain terrors, and all the evils of ignorance, are ever ready to start up, when the light of science is unfortunately extinguished.

The observations of the comet which was first perceived in 1770, have conducted astronomers to a very remarkable result. After having in vain attempted to subject these observations to the laws of parabolic motion, which have hitherto represented the motions of the comets with sufficient accuracy, they at length recognized that it described during its appearance, an ellipse the duration of whose revolution did not surpass six years. Lexel, who first made this curious remark, satisfied on this hypothesis, a great number of observations of the comet. But so very short a duration could not be admitted, (*d*) except after incontrovertible proofs, founded on a new and profound discussion of the observations of the comet, and of the positions of the fixed stars to which it was compared. The Institute therefore proposed this discussion for the subject of a prize, which Buckhardt gained, and his investigations has conducted us to very nearly the same result, as Lexel, on which there ought not now to remain any doubt. A comet, of which the period was so short ought frequently to appear, notwithstanding which it was not observed previously to 1770, nor has it been seen again, since that period. To account for this twofold phenomenon, Lexel remarked that in 1767 and 1779, this comet was very near to Jupiter, of which the powerful attraction diminished in 1767, the perehelion distance of its orbit, so as to render this star visible in 1770, which was before invisible, and then in 1779 it increased this same distance, so as to render this comet perpetually invisible. But it was necessary to demonstrate the possibility of the two effects which have been ascribed to the attraction of Jupiter, by shewing that the elements of the ellipse described by the comet, ought to satisfy them. This I have accomplished by subjecting this question to analysis, and by this means the preceding explanation has been rendered very probable. Of all the comets, this approached the nearest to the earth, consequently it ought to experience a sensible action from it, if its mass was comparable to that of the earth.

These two masses being supposed to be equal, the sidereal year would have been increased 11612″, by the action of the comet. By a computation of a great number of observations which Delambre and Buckhardt made in order to construct the tables of the Sun, we may be assured that since 1770, the sidereal year has not increased 3″, consequently the mass of the comet is (*e*) not the $\frac{1}{5000}$ part of the mass of the earth; and if we consider that in 1767 and 1779 this star traversed the system of the satellites of Jupiter without producing the slightest derangement, it will be evident that it must be even less. The smallness of the masses of the comets is in general indicated by their insensible influence on the motions of the planetary system. These motions are represented by the sole action of the bodies of the system, with such remarkable precision, that the small abberrations of our best tables may be ascribed to the sole errors of approximations and of observations. But very exact observations continued for a great number of years, and compared with the theory, can alone throw light on this important point in the system of the world.

CHAP. V.

Of the Perturbations of the Motion of the Moon.

The Moon is attracted at the same time by the Sun and by the Earth; but its motion round the Earth is only disturbed by the difference of the actions of the Sun, upon these two bodies: if the Sun was at an infinite distance, it would act equally upon them, and in the direction of parallel lines; their relative motion, therefore, would not be affected by an action which was common to both; but its distance, though (*a*) very great compared with that of the Moon, cannot be considered as infinite: the Moon is alternately nearer and farther from the Sun than the Earth, and the straight line joining the centres of the Sun and Moon, forms angles more or less acute with the terrestrial radius vector. Thus the Sun acts *unequally* and in *different* directions on the Earth and Moon; and from this diversity of action, inequalities must necessarily arise in the lunar motion, depending on the respective positions of the Moon and Sun. This constitutes the famous problem of the three bodies, the exact solution of which surpasses the powers of analysis, (*b*)but from the proximity of the Moon, compared with its distance from the Sun, and from the comparative smallness of its mass, an approximation may be obtained extremely near the truth. Nevertheless, the most delicate analysis is necessary to extricate all the terms, whose influence becomes sensible.

Their discussion is the most important point of this analysis, when it is proposed to perfect the lunar theory, which indeed ought to be its principal object; there are various ways of reducing this problem of the three bodies to an equation; but its principal difficulty consists in discriminating in the differential equations, and determining exactly, the terms which, though extremely small in themselves, acquire by successive integrations a sensible value; this requires a judicious selection of coordinates, delicate considerations on the nature of the integrals, approximations accurately conducted, computations carefully made and frequently verified. I have endeavoured to fulfil these conditions in the theory of the Moon, which has been explained in the Celestial Mechanics, and I have the satisfaction of seeing my results coincide with those found by Mason and Burgh from a comparison of near five thousand observations of Bradley and Maskeline, and which have given to the lunar tables a precision which it will be difficult to surpass, and to which geography, and nautical astronomy are principally indebted for their progress. It is due to Mayer, one of the greatest astronomers that ever lived, to observe that he was the first who brought the tables to the degree of perfection which is necessary for this important object. Mason and Burg have adopted the form which he gave to them; they have corrected the coefficients of his inequalities, and have added to them some others indicated by his theory. Mayer moreover, by the invention of the repeating circle, which has been cansiderably improved by Borda, has brought observations made at sea, to the same accuracy, to which he has reduced the lunar tables. Finally, M. Burkhardt has rendered the lunar tables nearly perfect, by assigning to their arguments a simple and more commodious form, and by determining their coefficients from a great collection of modern observations. The object of my theory has been to shew, in the sole law of universal gravitation, the source of all the inequalities of the lunar motion, and to make use of this law, to perfect the lunar tables, and to infer from them, several important elements in the system of the world, such as the secular equations of the Moon, its parallax, that of the Sun and the compression of the earth. Fortunately, while I was occupied in these investigations, Burg on his part was endeavouring to perfect the lunar tables. My analysis indicated to him several new and extremely sensible equations; and from a comparison of them with a great number of observations, he has ascertained their existence, and thrown great light on the elements, of which I have been speaking. The motions of the nodes, and of the lunar perigee, are the principal effects of the perturbations which this satellite experiences. A first aproximation had only given to geometers, half of the second of these motions; from (c) which Clairaut concluded that the law of attraction was not quite so simple as had been imagined; and he supposed it to consist of two parts, of which one varying inversely as the square of the distances, is sensible only at the great distances of the planets from Sun, and that the other, increasing in a greater ratio as the distance diminished, became sensible at the distance of the Moon from the Earth. This conclusion was vehemently opposed by Buffon: he maintained that since the primordial laws of nature should be the most simple possible, they could only depend on one *modulus*, and their expression, therefore, must consist of one single term. This consideration should no doubt lead us not to complicate the law of attraction, except in case of extreme necessity; at the same time our ignorance respecting the nature of this force, does not permit us to pronounce with certainty as to the simplicity of its expression. However this may be, the metaphysician was in the right, this time, in his contest with the mathematician, who retracted his error on making this important discovery, that by carrying on the approximation farther than had been done at first, the law of attraction, reciprocally as the squares of the distances, gave the motion of the lunar perigee, exactly conformable to observation, which has since been confirmed by all those who have occupied themselves with this subject. The motion which I inferred from my theory, differs from the true motion by its four hundreth and fortieth part, and this difference is only the three hundredth and fiftieth part, with respect to the motion of the nodes. Although analysis may be indispensible to make known how all the inequalities of the Moon, result from the action of the Sun combined with that of the earth, it is possible, nevertheless, without analysis, to explain the cause of the annual equation of the Moon, and its secular equation. I shall the more willingly stop to explain the causes of these equations, because it will be seen that from them are derived the greatest inequalities of the Moon, which the course of ages may develope to observers, but which up to the present period have been almost insensible.

In its conjunctions with the Sun, the Moon is nearer to it than the Earth, and experiences from it a more considerable action: the difference of the attractions of the Sun upon these two bodies, tends to diminish the gravity of the Moon towards the Earth. In a similar manner, in the oppositions of the Moon to the Sun, this satellite being farther (*d*) from the Sun than the Earth, is more weakly attracted: thus the difference of the actions of the Sun, tends also in this case to diminish the gravity of the Moon to the Earth. In each case, the diminution is very nearly the same, and equal to twice the product of the mass of the Sun, by the quotient of the radius of the lunar orbit, divided by the cube of the distance of the Sun from the Earth. In the quadratures, the action of the

Sun upon the Moon, decomposed in the direction of the lunar orbit, tends to augment the gravity of the Moon to the Earth: but this augmentation is only half the value of the diminution which it experienced in the syzygies. Thus, from all the actions of the Sun upon the Moon in the course of a synodical revolution, there results a mean force in the direction of the lunar radius vector, which diminishes the gravity of this satellite; and it is equal to half of (*e*) the product of the mass of the Sun, by the quotient of the radius of the lunar orbit, divided by the cube of the distance of the Sun from the Earth.

To find the ratio which this product bears to the gravity of the Moon, we may observe, that this force which retains it in its orbit, is nearly equal to the sum of the masses of the Earth and Moon, divided by the square of their mutual distance; and the force which retains the Earth in its orbit, is very nearly equal to the mass of the Sun divided by the square of its distance from the Earth. According to the theory of central forces, explained in the third Book, these two forces are as the radii of the orbits of the Sun and of the Moon, divided respectively by the squares of the times of their revolutions. Hence it follows that the preceding product is to the gravity of the Moon, as the square of the time of the sidereal revolution of the Moon, is to the square of the time of the sidereal revolution of the Earth. This product therefore is very nearly the $\frac{1}{179}$th of the lunar gravity, which by the mean action of the Sun is thus diminished by its 358th part.

In consequence of this diminution, the Moon is sustained at a greater distance from the Earth, than if it was abandoned (*f*) entirely to the action of its own force of gravity. The sector described by its radius vector is not altered, since the force which produces it, is in the direction of this radius, but its real velocity and angular motion are diminished; and it is easy to see, that by placing the Moon at a greater distance, so that its centrifugal force might equal its gravity, diminished by the action of the Sun, and that its radius vector might describe the same sector, that it would have described without this action; this radius would be augmented by its 358th part, and its angular motion diminished by a 179th part.

These quantities vary reciprocally as the cubes of the distances of the Sun from the Earth. When the Sun is in perigee, its action being most powerful, dilates the lunar orbit, but this orbit contracts again, as the Sun approaches its apogee; thus the Moon describes in space, a series of epicycloids whose centres are on the terrestrial orbit, and which dilate and contract as the Earth approaches to or recedes from the Sun. From hence an inequality (*g*) arises in the angular motion, very similar to the equation of the centre of the Sun, with this difference, that it retards this motion when that of the Sun augments, and that it accelerates it, when the motion of the Sun diminishes. These two equations are therefore always affected with contrary signs. The angular motion of the Sun is, as we have shewn in the first Book, reciprocally as the square of its distance, at the perigee, this distance being $\frac{1}{60}$th less than the mean distance, its angular velocity is augmented $\frac{1}{30}$th; the diminution of $\frac{1}{179}$th produced by the action of the Sun in the lunar motion, is then greater by a twentieth; the increase of this diminution is therefore the 3580th part of this motion. Hence (*h*) it follows that the equation of the centre of the Sun, is to the annual equation of the Moon, as a thirtieth of the solar motion is to the 3580th of the lunar motion, which gives 2398″ for the annual equation. It is about an eighth part less according to observation; this difference depends on the quantities that have been neglected in this first calculation.

The secular equation of the Moon, is produced by a cause similar to that of the annual equation. Halley first remarked this equation, which Dunthorn and Mayer have confirmed by a profound discussion of the observations. These two learned astronomers have proved that the mean motion of the Moon cannot be reconciled with modern observations, and with the eclipses observed by the Chaldeans and Arabians. They have attempted to represent them, by adding to the mean longitudes of this satellite a quantity proportional to the square of the (*i*) number of centuries before or after the year 1700. According to Dunthorn, this quantity is 30″9, for the first century: Mayer made it 21″6, in his first tables, which he increased to 27″8, in his last. And since that time, Lalande, after a new investigation of the subject, was led nearly to the same result as Dunthorn. The Arabian observations which have been chiefly made use of, are two eclipses of the Sun and one of the Moon, observed by Ibn Junis, near Cairo, towards the end of the tenth century, and extracted some time ago, from a manuscript of this astronomer's existing in the library at Leyden. Doubts have arisen concerning the reality of these eclipses; but the translation which M. Caussin has lately made of the part of this valuable manuscript, which contains the observations, has dissipated these doubts; it has moreover made us acquainted with twenty-five other eclipses observed by the Arabians, and which confirm the acceleration of the mean motion of the Moon. Besides, our modern observations compared with those of the Grecians and of the Chaldeans, are sufficient to establish the existence of the secular equation of the Moon. In fact, Delambre and Bouvard and Burg, have determined, by means of a great number of observations of the two preceding centuries, the actual secular motion, with a precision that leaves a very slight uncertainty: they found it six or seven hundred seconds greater than what is given by a comparison of antient and modern observations. The lunar motion is therefore accelerated since the time of the Chaldeans, and the Arabian observations being made in the interval that separates them and confirming this supposition, it is impossible any longer to question the truth of it.

Now, what is the cause of this phenomenon? Does the theory of universal gravitation, which has so well explained the numerous inequalities of the Moon, account likewise for its secular variation? These questions are the more interesting to resolve, because if we succeed, we shall obtain the law of the secular variations of the motion of the Moon, for it is evident that the hypothesis of an acceleration proportional to the time, as admitted by astronomers, is only on approximation, and cannot extend to an indefinite period.

This object has greatly occupied the attention of geometricians, but their researches were for a long time fruitless, having discovered nothing either in the action of the Sun or planets on the Moon, nor in the figures not exactly spherical of of this satellite and the Earth, that could change the mean motion of the Moon, some rejected the secular equation altogether, others to explain it, had recourse to various hypotheses, such as the action of comets, the resistance of an ether, and the successive transmission of gravity. Yet the correspondence of the other celestial phenomena with the theory of gravitation is so perfect, that we could not observe without regret, that the secular variation of the Moon appeared to refuse to submit to it, and continued the only exception to a general and simple law, whose discovery, by the grandeur and variety of the objects which it embraces, does so much honour to the human understanding. This consideration having determined me to reconsider this phenomenon, after several attempts I was at last so fortunate as to discover its cause. *The secular equation of the Moon arises from the action of the Sun upon this satellite, combined with the variation of the excentricity of the terrestrial orbit.* To form a just idea of this cause, we must recollect that the elements of the orbit of the Earth, are subject to alterations from the action of the planets; its greater axis remains always the same, but its excentricity, its inclination to a fixed plane, and the position of its nodes and of its perihelion, are incessantly changing. It must also be considered, that the action of the Sun upon the Moon diminishes by $\frac{1}{179}$, its angular velocity, and that its numerical co-efficient varies reciprocally as the cube of the distance of the Earth from the Sun. Now in expanding the inverse third power of this distance, into a series arranged according to the sines and cosines of the mean motions of the Moon, (k) and of their multiples, taking for unity the semi-major axis of the terrestrial orbit; it is found that this series contains a term equal to three times the half of the square of the excentricity of this orbit; the diminution of the angular velocity of the Moon, contains therefore a term equal to the 179th part of this velocity, multiplied by this term. If the excentricity of the terrestrial orbit was constant, this term would be confounded with the mean angular velocity of the Moon; but its variation, though very small, has nevertheless in progress of time a sensible influence on the motion of the Moon. It is evident that this motion will be accelerated, when the excentricity diminishes, which has been (*l*) the case ever since the most ancient observations to the present time, this acceleration will be changed into a retardation, when the excentricity having arrived at its *minimum*, will cease to decrease, and begin to augment.

In the interval from 1750 to 1850, the square of the excentricity of the terrestrial orbit diminishes 0.00000140595, the corresponding increase in the angular velocity of the moon is therefore 0.0000000117821 of this velocity: this increase taking place successively and proportionally to the time, its effect on the Moon's motion is only half what it would be, if during the whole course of the century, it was the same as at the end. To determine therefore this effect, or the secular equation of the Moon at the end of a century, reckoning from 1700, we must multiply the secular motion of the Moon by the half of the very small increase in its angular velocity; but in a century, the motion of the Moon is 5347405406″, which gives 31″,5017 for its secular equation.

As long as the diminution of the square of the excentricity of the terrestrial orbit may be supposed proportional to the time, the secular equation of the Moon will increase sensibly as the square of the time; it would be sufficient therefore to multiply 31″,5017 by the square of the number of centuries contained between the time for which the calculation is made, and the commencement of the nineteenth century. But I have found that in going back to the observations of the Chaldeans, the term proportional to the cube of the times, in the expression in a series, of the secular equation of the Moon, becomes sensible, this term is equal to $0'',057214$ for the first century; it should be multiplied by the cube of the number of centuries reckoned from 1801, the product being taken as negative for the centuries anterior to this epoch.

The mean action of the Sun upon the Moon depends also on the inclination of the lunar orbit to the ecliptic, and we might suppose that the position of the ecliptic being variable, there should result inequalities in the motion of this satellite, similar to those produced by the diminution of the excentricity of the terrestrial orbit; but I have recognised by analysis, that the lunar orbit is constantly brought back by the action of the Sun, to the same inclination to that of the Earth, so that the greatest and least declinations of the Moon are, in consequence of the secular variations in the obliquity of the eliptic, subject to the same changes as the declinations of the Sun.

This constancy in the inclination of the lunar orbit, is confirmed by all observations both ancient and modern.

The excentricity of this orbit experiences in like manner only an insensible alteration, from the change of the excentricity of the terrestrial orbit.

It is not thus with the variations of the motion of the nodes and perigee, to which it is indepensably necessary to pay attention in investigations, the object of which is to perfect the lunar tables. In submitting these variations to analysis, I have found that the influence of the terms depending on the square of the perturbating force, and which, as we have seen, double the mean motion of the perigee, is yet greater on the variation of this motion. The result of this intricate analysis, has given me a secular equation, triple of the secular equation of the mean motion of the Moon, to be subtracted from the mean longitude of the perigee, so that the mean motion of the perigee is retarded, when that of the Moon is accelerated. I have found likewise in the motion of the nodes of the lunar orbit upon the true ecliptic, a secular equation to be added to their mean longitude, and equal to 735 thousandths of the secular equation of the mean motion. Thus the motion of the nodes is retarded, like that of the perigee, when that of the Moon augments, and the secular equations of these three motions, are constantly in the proportion of the numbers 0,735, 3, 1. It is easy to infer from this, that the three motions of the moon, with respect to the sun, to its perigee and its nodes, continually increase, and that their secular equations are as the three numbers 1, 4, 0,265.

Future ages will develope these great inequalities, which will produce one day variations at least equal to a fortieth of the circumference, in the secular motion of the Moon, and to a thirteenth of the circumference in that of its perigee.

These inequalities do not always continue increasing; they are periodical, like those of the excentricity of the terrestrial orbit on which they depend, and do not re-establish themselves till after millions of years.

They must at length, alter the periods which have been devised for the purpose of comprehending complete numbers of revolutions of the Moon, relatively to its nodes, to its perigee, and to the Sun, periods which differ sensibly in different parts of the immense period of the secular equation.

The luni-solar period *(m)* of six hundred years, has been rigorously exact at a certain epoch, which it would be easy to find by analysis, if the masses of the planets were accurately determined; but this determination, so desirable for the perfection of our astronomical theories, is yet wanting. Fortunately Jupiter, whose mass we know exactly, is the planet which has the greatest influence on the secular equation of the Moon, and the values of the other planetary masses, are sufficiently accurate, for us to be certain that there cannot exist a sensible error in the magnitude of this equation.

Already ancient observations, notwithstanding their imperfection, confirm these inequalities, and we may trace their progress, either in these ancient observations, or in the astronomical tables which have succeeded them to the present time. We have seen that the ancient eclipses, made known the acceleration of the Moon's motion, before the theory of gravity had developed the cause.

In comparing modern observations, and the eclipses observed by the Arabians, Greeks, and Chaldeans, with this theory, we find an agreement between them that appears surprising, when we consider the imperfection of ancient observations, the vague manner in which they have been transmitted to us, and the uncertainty which still exists concerning the variations of the excentricity of the earth's orbit, and from the obviously imperfect manner in which the masses of Venus and Mars have been determined. The developement of the secular equations of the Moon, will be one of the most proper data to determine these masses.

It was particularly interesting to verify the theory of gravity, relatively to the secular equation of the motions of the perigee of the lunar orbit or to that of the anomaly, four times greater than the secular equation of the mean motion. From its discovery I have inferred that the actual motion of the perigee, made use of by astronomers, and which they inferred from a comparison of ancient and modern observations, must be diminished by from fifteen to sixteen minutes. In fact, when they did not take into account its secular equation, they should find this motion too rapid in the same manner, as they assumed too small a mean motion to the Moon, when they did not take its secular equation into account; this is what Bouvard and Burgh have confirmed by determining the actual secular equation of the lunar perigee, by means of a great number of modern observations; moreover Bouvard has found the same motion, by means of the most ancient observations, and by those of the Arabians, if its secular equation be taken into account of which the existence is by this means incontestably established.

The mean motions and the epochs of the tables of the Almageste and of the Arabians, indicate evidently these three secular equations of the lunar motion. The tables of Ptolemy are the result of immense calculations made by this astronomer and by Hipparchus; the work of Hipparchus has not come down to us: we only know from the evidence of Ptolemy, that he had taken the greatest care to select eclipses the most advantageous for the determination of the elements of which he was in search. Ptolemy, after two centuries and a half of new observations, found very little to change in these elements; there is therefore reason to believe that those which he made use of in his tables, have been determined by a great number of eclipses, of which he only preserved those that appeared to him to coincide most with the mean results which had been obtained by Hipparchus and himself. Eclipses only make known correctly the mean synodical motion of the Moon, and its distances from its nodes and its perigee: we can only then depend upon these elements in the tables of the Almageste: now in going back to the first epoch of these tables, by means of motions determined only by modern observations, we do not find the mean distances of the Moon from its nodes, its perigee, and from the Sun, that are given in these tables at this epoch. The quantities which must be added to these distances, are very nearly those which result from the secular equations; therefore, the elements of these tables at the same time, confirm the existence of these equations, and the values which I have assigned to them.

The motions of the Moon relative to its nodes, to its perigee, and to the Sun, being slower in the tables of the Almageste, than in our days, indicate also in these motions an acceleration, equally indicated both by the corrections that Albatenius, eight centuries after Ptolemy, made to the elements of these tables, and also by the epochs of the tables which Ibn Junis constructed about the year one thousand, from the collection of the Chaldean, Greek and Arabian observations.

It is remarkable that the diminution of the excentricity of the terrestrial orbit should be much more sensible, in the lunar motion than in itself. This diminution which, since the most ancient eclipse we are acquainted with, has not altered the equation of the Sun's centre 15′, has produced a variation of two degrees in the Moon's longitude, and nearly a variation of eight degrees in its mean anomaly; we could hardly suspect it from the observations of Hipparchus and Ptolemy. Those of the Arabians indicated it with much probability; but the ancient eclipses, compared with the theory of gravitation, leave no doubt on this subject. This reflexion, if I may make use of the term, of the secular variations of the orbit of the Earth on the lunar motion, in consequence of the action of the Sun, has place also for the periodic inequalities. It is in this manner that the equation of the centre of the Earth's orbit, reappears in the lunar motion with a contrary sign, and reduced to about a tenth of its value ; in like manner the inequality produced in the motion of the Earth by the action of the Moon, is reproduced in the motion of the Moon, but diminished in the ratio of five to nine. Finally, the action of the Sun in transmitting to the Moon, the inequalities which the planets produce in the motion of the Earth, renders this indirect action of the planets on the moon, more considerable than their direct action on this satellite.

Here we see an example of the manner in which phenomena as they are developed, lead us to the knowledge of their true causes. When only the acceleration of the mean motion of the Moon was known, it might be attributed to the resistance of ether, or to the successive transmission of gravity. But analysis proves that these two causes cannot produce any sensible alteration in the mean motion of the nodes and of the lunar perigee, and that alone would suffice to exclude them, even when the true cause of the variations observed in these motions was unknown.

The agreement of theory with observations, proves that if the mean motions of the Moon are altered by causes foreign to the principle of universal gravitation, their influence is very small, and hitherto insensible.

This agreement evinces in a decisive manner, the invariability of the duration of the day, which is an essential element in astronomical theories. If this duration was greater now by the hundredth part of a second, than in the time of Hipparchus, the duration of the present century would be greater than at that time, by 365″25; in this interval the Moon describes an arch of 534″,6; therefore the actual secular mean motion of the Moon would appear to be increased by this interval, which would increase its secular equation by 13″,51 for the first century, commencing from 1801, and which by what goes before is 31″,5017. Observations do not permit us to suppose so considerable an increase. We may therefore be assured that since the time of Hipparchus, the duration of the day has not varied by the hundredth part of a second.

One of the most important equations in the lunar theory, in as much as it depends on the compression of the earth, is relative to the motion of the Moon in latitude. This inequality is proportional to the (n) sine of the true longitude of this satellite. It arises from a nutation in the lunar orbit produced by the action of the terrestrial spheroid, and corresponding to that which the Moon produces in our equator, so that one of these nutations is the reaction of the other; and if all the molecules of the Earth and Moon were connected firmly together by inflexible lines, void of mass, the entire system would be in equilibrio about the centre of gravity of the earth, in consequence of the forces which produce these two nutations; the force which actuates the Moon, compensating its smallness, by the length of the lever to which it is attached. This inequality in latitude may be represented, by conceiving that the orbit of the Moon, instead of moving uniformly on the ecliptic with a constant inclination, moves according to the same conditions on a plane a little inclined to the ecliptic, and passing always through the equinoxes, between the ecliptic and equator; this phenomenon is produced in a more sensible manner in the motions of the satellites of Jupiter, in consequence of the very great compression of that planet. Thus, this inequality diminishes the inclination of the orbit of the Moon to the ecliptic, when its ascending node coincides with the equinox of spring, it increases it when the node coincides with the equinox of autumn, which being the case in 1755, renders the inclination too great, which Mason determined by the observations of Bradley, made between the intervals of 1750 and 1760. In fact Burg, who determined it by observations made during a much longer interval, and by taking into account the preceding inequality, found the inclination to be smaller by about $11'',\frac{1}{2}$; this astronomer undertook at my request to determine the coefficient of this inequality, and from a great number of observations he found it equal to—$24'',6914$. Burchardt, by employing for this purpose a still greater number of observations, arrived at the very same result, which gives the compression of the earth equal to $\frac{1}{304,6}$.

This compression may also be determined by means of an inequality in the Moon's motion in longitude, which depends on the longitude of the Moon's node. Observation indicated it to Mayer, and Mason fixed its quantity at 23,″765; but as it did not appear to result from the theory of gravity, the greater number of astronomers neglected it. This theory pointed out to me that its cause existed in the compression of the earth. Burgh and Burchardt from a great number of observations, fixed it at 20″,987, which answers to a compression of $\frac{1}{305,05}$, very nearly the same as is given by the preceding inequality of the motion in latitude. Thus the Moon by the observation of its motion, renders sensible to astronomy when brought to a state of perfection, the ellipticity of the earth, the round form of which it first made known to astronomers, by its eclipses.

The two preceding inequalities demand the greatest attention of observers. They have an advantage over geodesical observations, in as much as they give the compression of the earth, in a manner less dependant on the inequalities of the surface of the earth. If the earth was homogeneous, they would be much greater than what observation determines them to be, consequently the earth is not homogeneous. It follows also from this that the attraction of the Moon towards the Earth arises from the attractions of all the molecules of this planet; which is a new proof of the mutual gravitation of all the parts of matter.

Theory combined with the experiments of the pendulum and the measures of degrees on the earth, assigns to the parallax, as we have seen in the first chapter of this book, a quantity very nearly conformable (*o*) to observations, so that conversely, the magnitude of the earth might be inferred from these observations.

Finally, the parallax of the Sun might be inferred with accuracy from a lunar equation which depends on the simple angular distance of the Moon from the Sun. For this purpose, I have computed with great care, the coefficient of this equation, and by putting it equal to that, which Burgh and Burchardt concluded from a long series of observations, I concluded that the mean parallax of the Sun was 28″,56 the same which several astronomers deduced from the last transit of Venus.

It is worthy of remark that an astronomer without leaving his observatory, by merely comparing observations with analysis, can determine exactly the magnitude and compression of the earth, and its distance from the Sun and from the Moon, which elements have been determined by long and troublesome voyages in the two hemispheres. The agreement of the results obtained by these two different methods is one of the most striking proofs of the theory of universal gravitation.

The numerous comparisons which Bouvard and Burgh made of the lunar tables, with the observations of the end of the seventeenth century by La Hire and Flamstead; of the middle of the eighteenth century by Bradley, and with the uninterrupted series of observations of Maskeyline from Bradley to this day, furnish a result which we would be far from anticipating. The observations of La Hire and of Flamstead, compared with those of Bradley, indicate a secular sidereal motion of the Moon, greater at least by one hundred and thirty seconds, than what results from the observations of Bradley compared with the last of Maskeline; and the observations made during the last twenty years proves that the diminution of the secular motion of the Moon has been greater still during this interval; so that the existence of an anomaly in the mean motion of the Moon is at least very probable. Hence arises the necessity of perpetually retouching the epochs of the tables, until we can determine the cause or the law of this remarkable anomaly. It is evidently connected with one or more unknown inequalities, with long periods, of which theory alone can indicate the laws.

The best lunar tables are founded on theory and observation combined. They borrow from theory the arguments of the inequalities which it would have been difficult to know by means of observation alone. I have determined in my Treatise of Celestial Mechanics, the coefficients of these arguments in a very approximate manner, but in consequence of the slowness of the convergance of these approximations, combined with the difficulty of extricating from among the immense number of terms which the analysis developes, those which can acquire from integration a sensible value, this investigation is extremely troublesome. Nature itself furnishes in the collections of observations, the results of those integrations so difficult to obtain by analysis. Messrs. Buckhardt and Burgh have employed for their determination several thousand observations, and by this means have rendered their lunar tables extremely accurate. Being anxious to banish all empericism, and that other geometers should discuss several intricate points of the theory to which I first arrived, such as the secular equations of the motions of the Moon; I induced the Academy to propose for the subject of its mathematical prize for the year 1820, the formation by theory alone of lunar tables equally perfect with those which have been inferred from theory and observation combined. Two pieces were crowned by the Academy, the author of one of them, M. Damoiseau, accompanied it with tables which compared with observations, have represented them with the accuracy of our best tables. The authors of the two pieces agree on the periodical and secular inequalities of the motions of the Moon. They differ a little from my result on the secular equation of mean motion; but instead of the numbers 1; 4; 0,265 by which I represented the ratios of the secular inequalities of the motion of the Moon relatively to the Sun, to the perigee of the lunar orbit and to its nodes, they have found the numbers 1; 46776; 0,391. M. Domoiseau in his essay has made the second of these numbers, very nearly equal to 4; but after mature reflection on his analysis, he has arrived at the same result as

Messrs. Plana and Carlini, the authors of the other essay. As they extended their approximations a considerable way, their numbers appear preferable to those which I have determined. Finally, from those approximations the mean motions of the perigee and of the nodes of the orbit, have been inferred exactly conformable to observations.

It follows indubitably from what we have seen, that the law of universal gravitation is the sole cause of all the inequalities of the Moon, and if we consider the great number and the extent of these inequalities, and the proximity of this satellite to the earth, it will be agreed on, that of all the heavenly bodies, it is the best adapted to establish this great law of nature, and the power of analysis, of that wonderful instrument, without which it had been impossible for the human mind to penetrate into a theory so complicated, and which may be employed as a means of discovery, equally certain with observation itself.

Some partizans of final causes, have imagined that the Moon was given to the Earth, to afford it light during the night. But in this case, nature would not have attained the end proposed, since we are often deprived at the same time of the light of each of them. To have accomplished this end, it (*p*) would have been sufficient to have placed the Moon at first in opposition to the Sun and in the plane of the ecliptic, at a distance from the Earth equal to the one hundredth part of the distance of the Earth from the Sun, and to have impressed on the Earth and Moon, parallel velocities proportional to their distances from the Sun. In this case, the Moon being constantly in opposition to the Sun, would have described round it an ellipse similar to that of the Earth. These two stars would then constantly succeed each other, and as at this distance the Moon could not be eclipsed, its light would always replace that of the Earth. Other philosophers struck with the singular opinion of the Arcadians that they were older than the Moon, thought that this planet was originally a comet which, passing near to the Earth, was forced by its attraction to accompany it. But by reascending by means of analysis to the most remote periods, we shall find that the Moon always moved in an orbit nearly circular about the Earth, as the planets move about the Sun, therefore neither the Moon nor any satellite was originally a comet.

As the gravity at the surface of the Moon, is much less than at the surface of the Earth, and as this star has no atmosphere which can oppose a sensible resistance to the motion of projectiles, we may conceive that a body projected with a great force, by the explosion of a lunar volcanoe, may attain and pass the limit, where the attraction of the Earth commences to predominate over that of the Moon. For this purpose it is sufficient that its central velocity in the direction of the vertical may be 2500 metres in a second; then in place of falling back on the Moon, it becomes a satellite of the Earth, and describes about it an orbit more or less elongated.

The direction of its primitive impulsion may be such as to make it move directly towards the atmosphere of the earth ; or it may not attain it, till after several and even a great number of revolutions, for it is evident that the action of the Sun, which changes in a sensible manner the distances of the Moon from the Earth, ought to produce in the radius vector of a satellite which moves in a very excentrick orbit, much more considerable variations, and thus at length so diminish the perigean distance of the satellite, as to make it penetrate our atmosphere. This body traversing it with a very great velocity, and experiencing a very sensible resistance, might at length precipitate itself on the Earth : the friction of the air against its surface would be sufficient to enflame it, and make it detonate, provided that it contained ingredients proper to produce these effects, and then it would present to us all those phenomena which meteoric stones exhibit. If it was satisfactorily proved that they are not produced by volcanoes, or generated in our atmosphere, and that their cause must be sought beyond it, in the regions of the heavens, the preceding hypothesis, which likewise explains the identity of composition observed in meteoric stones, by an identity of origin, will not be devoid of probability.

CHAP. VI.

Of the perturbations of the satellites of Jupiter.

Of all the satellites, the most interesting, after that of the earth, are the satellites of Jupiter. The observations of these stars, the first which the telescope discovered in the heavens, are not older than two centuries, and it is only about a century and a half since their eclipses have been observed. But in this short interval they have presented, by the quickness of their revolutions, all the great changes which time would not develop except with great slowness in the planetary system, of which that of the satellites is only an epitome. The inequalities produced by their mutual attraction, do not differ materially from those of the planets and of the Moon: however the relations which exist between the mean motions of the three first satellites give rise to some inequalities of considerable magnitudes, which have a great influence on their theory. We have seen in the second book, that the differences between the mean motions of the first and second is very nearly twice the difference between the mean motions of the second and third, and that they are subject to very sensible inequalities, of which the periods though different one from the other, are in the eclipses transformed into one sole period of 437^{d},659.

The first inequalities which observation discovered in the motion of these bodies, are also the first which are derived from the theory of universal gravitation of the satellites. This theory not only determines these inequalities, but it shews us also, what observation seemed to indicate with great probability, namely, that the inequality of the second sateltite is the result of two inequalities, of which one being caused by the action of the first satellite, varies as the sine of the excess of the longitude of the first satellite (*a*) above that of the second; and of which the other, produced by the action of the third, varies as the sine of double the excess of the longitude of the second satellite above that of the third. Thus the second satellite experiences a perturbation from the action of the first, similar to that which itself causes in the third; and it experiences from the third a similar perturbation to that which itself causes in the first.

These two inequalities are combined into one in consequence of the relation which exists between (*b*) the mean motions and the mean longitudes of the three first satellites, according to which the mean motion of the first satellite *plus* twice that of the third, is equal to three times that of the second; and the mean longitude of the first satellite, *minus* three times that of the second, *plus* twice that of the third, is constantly equal to a semi-circumference: but will these relations always exist, or are they only approximate, and will the two inequalities of the second satellite which at present are combined, be separated in the course of time? It is to theory that we must apply for a solution to this question.

The approximate manner with which the tables furnished the preceding relations, made me suppose that they were rigorously exact, and that the small quantities by which they still differed, depended on the errors to which they were liable; for it was against all probability that chance should have originally placed the three first satellites at the precise distances and positions suitable to the above relation: it was therefore extremely probable that it arose from some particular cause; I looked therefore for this cause in the mutual action of the satellites. A scrupulous investigation of this action, has shewn me that it has rendered these relations rigorously exact: from whence I concluded, that in determining again by the examination of a great many distant observations, the mean motions and the mean longitudes of the three first satellites, it would be found that they would approximate still more to these relations, to which the tables should be made exactly to agree. I had the satisfaction of seeing this consequence of the theory confirmed, with remarkable precision, by the researches which Delambre has made concerning the satellites of Jupiter. It is not necessary that these relations should have taken place exactly at their origin, it was enough that they did not greatly differ, then the mutual actions of the satellites upon each other were sufficient to subject them to this law, and to maintain it unaltered; but the little difference between this and the primitive relation, has given rise to a small inequality of an arbitrary extent, which is distributed among the three satellites, and which I have designated by the name of *libration*. The two constant arbitrary quantities of this inequality, replace whatever arbitrary quantity is made to disappear by the two preceding relations, in the mean motions and in the epochs of the mean longitudes of the three first satellites; for the number of arbitrary (*c*) quantities included in the theory of a system of bodies is necessarily sextuple the number of bodies: as observation does not indicate this inequality, it must evidently be very small, and even insensible.

The preceding relations would still subsist, even if the mean motions of the satellites were subject to secular variations analogous to that of the motion of the Moon. They would subsist also in the case of these motions being altered by the resistance of a medium, or by other causes, of which the effects would not be perceived until after a long time. In all these cases, the secular equations so arrange themselves by the reciprocal action of the satellites, that the secular equation of the first plus twice that of third is equal to three times that of the second: even their inequalities, which increase with extreme slowness, approach so much the more to coordinate themselves thus, as their periods are more considerable. This libration, in consequence of which the motions of the three first satellites are balanced in space according to the laws which we have just announced, extends also to their motions of rotation, if as all observations appear to indicate these motions are equal to those of revolution. The attraction of Jupiter must then maintain this inequality, by impressing on the motions of rotation, the same secular equations as affect the motions of revolution. Thus, the three first satellites of Jupiter consitute a system of bodies connected together by the preceding inequalities and relations, which their mutual action will maintain uninterruptedly, unless some extraneous cause should abruptly derange their respective motions and positions.

Such would be the effect of a comet, which traversing this system, as the first comet of 1770 appears to have done, would impinge on one of these bodies. It is probable that such rencounters have taken place, in the immensity of ages, which have lapsed, since the commencement of the planetary system. The shock of a comet, of which the mass was only the hundreth millioneth part of that of the earth, would be sufficient to render the libration of the satellites sensible. As this inequality has not been recognised, notwithstanding all the care of Delambre to detect it in his observations, we ought to conclude that the masses of any of the comets which might have (*d*) impinged on the three satellites of Jupiter must have been extremely small; which confirms what has been already observed on the smallness of the masses of the comets.

The orbits of the satellites experience changes analogous to the great variations which the planetary orbits undergo; their motions are in like manner subjected to secular equations similar to those of the Moon. The developement of all these inequalities in the progress of time, will furnish the most advantageous data for determining the masses of the satellites and the compression of Jupiter. The great influence which this last element has on the motions of the nodes, determines its value with more accuracy than direct measurement. By this means it is found that the ratio (*e*) of the lesser axis of Jupiter to the diameter of his equator, is equal to 0,9368, which differs very little from the ratio, sixteen to seventeen, which is given by a mean of the most accurate measures of the compression of this planet. This agreement is a new proof that the gravity of the satellites towards the primary planet, arises from the attractions of all its molecules.

One of the most remarkable consequences of the theory of the satellites of Jupiter is the knowledge of their masses, which would appear to be interdicted by their extreme smallness and by the impossibility of measuring their diameters. I have selected for this purpose, the data which in the actual state of astronomy have appeared to me the most advantageous, and I apprehend that the following values, which I have inferred from them, are very accurate.

Masses of the satellites of Jupiter, that of the planet being assumed equal to unity.

I Satellite	0,0000173281.
II Satellite	0,0000232355.
III Satellite	0,0000884972.
IV Satellite	0,0000426591.

These values should be corrected, when in the progress of time, we become better acquainted with the secular variations of the orbits.

Whatever be the perfection of the theory, an immense labour is reserved for the astronomer to convert the analytical formulæ into tables. These formulæ contain thirty-one constant arbitrary quantities, namely, the twenty-four arbitrary (f) quantities of the twelve differential equations of the motion of the satellites, the masses of these stars, the compression of Jupiter, the inclination of his equator, and the position of his nodes. In order to obtain the values of all these unknown quantities, we should discuss a very great number of eclipses of each satellite, and combine them in the manner best adapted to make each element arise. Delambre has performed this important work with the greatest success; and his tables, which represent observations with the accuracy of the observations themselves, afford to the navigator a sure and easy means of obtaining immediately by the eclipses of the satellites, especially by those of the first, the longitude of the places at which he can land. The following are the principal elements of the theory of each satellite, which result from a comparison which was made by Delambre of my formulæ, with observations.

The orbit of the first satellite moves uniformly with a constant inclination on a fixed plane, which passes constantly between the equator and the orbit of Jupiter, through the mutual intersection of these two last planes, of which the respective inclination is according to observations, equal to 3,°4352. The inclination of this fixed plane with the equator of Jupiter, is only 20″ by theory; it is consequently insensible. The inclination of the orbit of the satellite on this plane is in like manner insensible to observations; thus the first satellite may be supposed to move in the plane of the equator of Jupiter. An excentricity peculiar to this orbit has not been recognised, which only participates a little in the excentricities of the orbits of the third and fourth satellites, for in virtue of the mutual action of all these bodies, the excentricity proper to each orbit is diffused over the others, but more feebly as they are more distant. The sole inequality of this satellite which is sensible is that of which the argument, is double of the excess of the mean longitude of the first satellite above that of the second, and which produces in the recurrence of the eclipses, an inequality of $437^d,659$; it is one of the data which I have made use of to obtain the masses of the satellites, and as it arises from the action of the second alone, it determines its mass with great accuracy.

The eclipses of the first satellite of Jupiter, gave rise to the (*g*) discovery of the successive transmission of light, which the penomenon of aberration has ascertained with still greater precision. It appeared to me that as the theory of the motion of this satellite is now better known, and as the observations of its eclipses are become more numerous, their discussion should give the quantity of aberration more exactly than direct observation. Delambre, who undertook this investigation at my request, found the entire quantity of aberration $62''5$, which is exactly that which Dr. Bradley derived from his observations. It is very curious to observe such a perfect agreement in results which have been obtained by such very different methods.

It follows from this agreement, that the velocity of light is uniform (*h*) through the whole space comprehended by the terrestrial orbit. In fact, the velocity of light given by the aberration is that which subsists at the circumference of the terrestrial orbit, and which, being combined with the motion of the Earth produces this phenomenon. The velocity of light, as given by the eclipses of the satellites of Jupiter, is determined by the time which light employs to traverse the terrestrial orbit; these two velocities being the same, their velocity is uniform through the whole length of the diameter of the terrestrial orbit. It results also from these eclipses, that the velocity of light is uniform through the whole diameter of the orbit of Jupiter; for, from the excentricity of this orbit, the effect of the variation of the radii vectores, is very sensible in the eclipses of the satellites; and these exactly correspond to the hypothesis of a uniform velocity, in the motion of light.

If light is an emanation from luminous bodies, the uniformity of the velocity of its rays requires *(i)* that they should be projected from each of them with the same force, and that their motion should not be sensibly retarded by the attraction which they experience on the part of foreign bodies. If we suppose light to consist in the vibrations of an elastic fluid, the uniformity of its velocity requires that the density of this fluid, throughout the whole extent of the planetary system, should be proportional to its elasticity. But the great simplicity with which the aberration of the stars, and the phenomena of the refraction of light in passing from one medium to another, are explained on the hypothesis that light is an emanation from a luminous body, renders this hypothesis very probable.

The orbit of the second satellite moves uniformly with a constant inclination, on a fixed plane, which passes constantly between the equator and orbit of Jupiter through their mutual intersection, of which the inclination to this equator is 201″. The orbit of the satellite is inclined by 5152″ to its fixed plane, and its nodes have on this plane a retrogade tropical motion, of which the period is 29^{yrs},9142 : this period is one of the data which I have made use of in determining the masses of the satellites. Observation has not made known the excentricity peculiar to this orbit; but it participates a little in the excentricities of the orbits of the third and fourth satellite. The two principal inequalities of the second satellite depend on the actions of the first and of the third satellite. The ratio existing between the longitudes of the three first satellites always combines those inequalities (*l*) into one sole, of which the period in the recurrence of the eclipses is 437^{d},659, and of which the value is the third quantity which I made use of in determining the masses. The orbit of the third satellite moves uniformly with a constant inclination, on a fixed plane, which passes constantly between the equator and the orbit of Jupiter, through their mutual intersection, and of which the inclination on this equator is 931″. The orbit of the satellite is inclined by 2284″ to its fixed plane, and its nodes have on this plane a retrograde tropical motion, of which the period is 141^{yrs},739. Astronomers supposed the orbits of the three first satellites to move in the plane of the equator itself of Jupiter, but they deduced from the eclipses of the third satellite, a smaller inclination of this equator to the orbit of the planet, than what was collected from those of the two others. This difference, of which they did not know the cause, arose from this, that the orbits of the satellites do not move with a constant inclination to the equator, but on different planes, of which the inclination is greater for those satellites which are more distant. Our moon presents a similar result, as we have observed in the preceding chapter ; it is on this, that the lunar inequality in latitude depends, from which the compression of the earth has been inferred, perhaps with more accuracy than from the measures of the degrees of the meridian.

The excentricity of the orbit of the third satellites exhibits singular anomalies, of which theory has indicated the cause. They depend on two distinct equations of the centre. The one peculiar to this orbit respects a perijove, of which the annual sidereal motion is about 29010″. The other, which may be regarded as an emanation from the equation of the centre of the fourth satellite, respects the perijove of this last body. It is one of the data from which I have determined the masses. These two equations form by their combination a variable equation of the centre respecting a perijove, of which the motion is not uniform. They coincided and combined their effects in 1682 and their sum amounted to 2458 ;″ in 1777 the effect of one was taken from that of the other, and the difference amounted to 949″. Wargentin endeavoured to represent these variations by means of two equations of the centre, but as he did not refer one of them to the perijove of the fourth satellite, he was obliged, by observations, to abandon his hypothesis, and he had recourse to that of one variable equation of the centre, of which he determined the changes by observations, which conducted him very nearly to what we have indicated. Finally, the orbit of the fourth satellite moves uniformly with a constant inclination on a fixed plane, inclined by 4457″ to the equator of Jupiter, and which passes through the line of the nodes of this equator, between this last plane and that of the orbit of the planet; the inclination of the orbit of the planet to its fixed plane is 2772″, and its nodes have on this plane a retrograde tropical motion, of which the period is 531 years. In consequence of this motion the inclination of the orbit of the fourth satellite on the orbit of Jupiter varies continually. Having attained its *minimum* towards the middle of the last century, it has been nearly stationary, and about 2°,7 from 1680 to 1760. In this interval its nodes have a direct motion in a year of 8′ very nearly. This circumstance, which observation indicated, was for a long time made use of by astronomers, who were employed in the tables of these satellites; it is a consequence of the theory which gives the inclination and the motion of the nodes very nearly the same, as Astronomers found them by a discussion of the eclipses. But in these last years the inclination of the orbit has undergone a considerable increase, of which it was difficult to know the law, without the aid of analysis. It is curious to see these remarkable phenomena, which observation indicated, resulting from the analytical formulæ ; but which arising from the combination of several simple inequalities are too complicated for Astronomers to discover their laws. The excentricity of the orbit of the fourth satellite is much greater than those of the other orbits, its perijove has a direct annual motion of 7959″ ; it is the fifth data which I employed in determining the masses. Each orbit participates a little in the motion of the others. The fixed planes to which we have referred them are not strictly speaking fixed ; they move very slowly with the equator and orbit of Jupiter, always passing through the mutual intersection of those last planes, and preserving on the equator of Jupiter inclinations which, though variable, have to each other, and to the inclination of the orbit of the planet on its equator, a constant ratio.

Such are the principal results of the theory of the satellites of Jupiter compared with numerous observations of their eclipses. Observations of the ingress and egress of their shadows on the disk of Jupiter would throw considerable light on several elements of their theory. This kind of observations, hitherto too much neglected by Astronomers, ought, as it appears to me, to attract their attention, for it seems that the interior contact of the shadows would determine the time of conjunctiom more accurately than eclipses. The theory of the satellites is now so far advanced, that whatever deficiency is required to complete this theory, can only be determined by the most exact observations; it is therefore necessary to try new modes of observations, or at least to be certain that those which we make use of, deserve the preference.

CHAP. VII.

Of the Satellites of Saturn and of Uranus.

THE extreme difficulty of observing the satellites of Saturn renders their theory so imperfect, that we hardly know with any precision their revolutions and their mean distances from the centre of this planet; it is therefore as yet unnecessary to consider their perturbations. But the position of their orbits presents a phenomenon worthy of the attention of Geometers and Astronomers. The orbits of the six first satellites appear to be in the plane of the ring, while the orbit of the seventh satellite deviates from it sensibly. It is natural to think that this depends on the action of Saturn, which, in consequence of his compression, retains the first six orbits and its rings in the plane of its equator. (*a*) The action of the sun tends to make them deviate from it, but this deviation increasing very rapidly and very nearly as the fifth power of the radius of the orbit, it only becomes sensible for the last satellite. The orbits of the satellites of Saturn, like those of Jupiter, move in planes, which pass constantly between the equator and orbit of the planet, through their mutual intersection, and which are always more inclined to this equator according as the satellites are farther from Saturn. This inclination is considerable relatively to the last satellite, and about 24°,0, if we refer to observations already made; the orbit of the satellite is inclined by 16°,96 to this plane, and the annual motion of its nodes on the same plane is 940″. But as these observations are extremely uncertain, the preceding results can only be considered as a very imperfect approximation.

We are even less informed with respect to the satellites of Uranus. It solely appears from the observations of Herschel, that they move in the same plane, almost perpendicular to that of the orbit of the planet; which evidently indicates a similar position in the plane of its equator. Analysis shews that the ellipticity of the planet, combined with the action of the satellites, can very nearly maintain their different orbits, in the same plane. This is all which can be affirmed of these stars, which in consequence of their distance and inconsiderable magnitude, will be for a long time inaccessible to the most extended researches.

CHAP. VIII.

Of the Figure of the Earth and Planets, and of the Law of Gravity at their Surface.

We have detailed in the First Book, what has been indicated by observations on the figure of the Earth, and of the planets: let us compare these results with those of universal gravitation.

The force of gravity towards the planets, is composed of the (*a*) attractions of all their particles. If their masses were in a state of fluidity, and without motion, their figure and those of the different strata would be spherical, those nearer the centre being more dense. The force of gravity at their exterior surface, and at any distance whatever, without the sphere, would be exactly the same, as if the whole mass of the planet was conndensed into the centre of gravity. It is in consequence of this remarkable property, that the Sun, the planets, comets, and satellites, act upon each other, very nearly (*b*) as if they were so many material points. At very great distances the attraction of the particles of a body of any figure, which are the most remote, and those which are nearest the particle attracted, (*c*) compensate each other in such a manner, that their total attraction is very nearly the same as if they were united in the centre of gravity; and if the ratio of the dimensions of the body to its distance from the attracted point, be considered as a very small quantity of the first order, this result will be exact to quantities of the second order. But in a sphere, it is rigorously true, and in a spheroid differing but little from a sphere, the error is of the same order as the product of its excentricity, by the square of the ratio of its radius, to its distance from the point attracted. This property of the sphere, of attracting as if its mass was concentered in its centre, contributes greatly to the simplicity of the motions of the heavenly bodies. It does not belong exclusively to the law of nature, it equally appertains to the law of attraction varying proportionably to (d) the simple distance, and cannot belong to any other law but those formed by the addition of these two. And of all the laws which render the force of gravity nothing at an infinite distance, that of nature is the only one in which the sphere possesses this property.

According to this law, a body placed within a spherical stratum of uniform thickness, is equally attracted by all its parts, so as to remain at rest in the midst of the various attractions which act upon it. The same circumstance takes place in an elliptic stratum, when the exterior and interior surfaces are similar and similarly situated. Supposing therefore the planets to be homogeneous spheres, the force of gravity in their interior, must diminish as the distance from the centre; for the exterior part, relatively to the attracted particle, contributes nothing to its gravity, which is only produced by the attraction of the internal sphere, whose radius is equal to the distance of this point from the centre. But this attraction is equal to the mass of the sphere, divided by the square of the radius, and the mass, is as the cube of this same radius. The force of gravity on the attracted particle, is therefore proportional to the radius. But if, (as is probably the case) the strata are more dense as they are nearer to the centre, the force of gravity will diminish in a less ratio, than in the case of homogeneity. The rotatory motion of the (*e*) planets causes them to deviate a little from the spherical figure. The centrifugal force arising from this motion, causing the particles situated at the equator to recede from the centre, and thus to produce a flattening of the poles.

Let us consider first the effects of this compression in the simplest case, namely, that in which the Earth is considered as an homogeneous fluid, the gravity residing in its centre and varying reciprocally as the square of the distance from this point. It is then easy to prove that the terrestrial spheroid is an ellipsoid of revolution; for if we conceive two columns of fluids, communicating with each other at the centre, and terminating, the one at the pole, the other at any point on the surface, these two columns ought to be in equilibrio. The centrifugal force does not alter the weight of the column directed to the pole, but it diminishes the weight of the other column. This force is nothing at the centre of the Earth, and at the surface it is proportional to the radius of the terrestrial parallel, or very nearly, to the cosine of the latitude; but the whole of this force is not entirely employed in diminishing the force of gravity; for these two forces making an angle with each other, *(f)* equal to the latitude, the centrifugal force, decomposed according to the direction of gravity, is weakened in the ratio of the cosine of this angle to radius. Thus, at the surface of the Earth, the centrifugal force diminishes the force of gravity, by the product of the centrifugal force at the equator, by the square of the cosine of the latitude; therefore the mean value of this diminution in the length of a fluid column, is the half of this product, and since the centrifugal force is $\frac{1}{289}$ of the force of gravity at the equator, this value is the $\frac{1}{578}$th part of the force of gravity, multiplied by the square of the cosine of the latitude. And since it is necessary, for the maintenance of the equilibrium, that the column by its length should compensate the diminution of its weight, it ought to surpass the polar column by a $\frac{1}{578}$th of its length, multiplied by the square of the above cosine. Thus the augmentation of the radii, from the pole to the equator, is proportional to the squares of these cosines, from which it is easy to conclude, that the Earth is an ellipsoid of revolution, the equatorial and polar axis of which are in the proportion of 578 to 577.

It is evident that the equilibrium of the fluid mass would still subsist, even if a part should be supposed to consolidate itself in the interior, provided the force of gravity remains the same.

To determine the law of gravity at the surface of the Earth, we may observe that the force of gravity to any point on this surface, is less than that at the pole, from its being situated farther from the centre. This diminution is nearly equal to double the augmentation of the terrestrial radius; it is equal therefore to the product of the $\frac{1}{289}$th part of the force of gravity by the square of the cosine of the latitude. The centrifugal force diminishes likewise the force of gravity by the same quantity; thus by the union of these two causes, the diminution of gravity from the pole to the equator, is $=$ 0,00694, multiplied by the square of the cosine of the latitude, the force of gravity at the equator being taken as unity.

It has been shewn in the First Book, that the measures of meridional degrees, assign to the Earth an ellipticity greater than $\frac{1}{378}$, and that the measures of the pendulum indicate a diminution in the force of gravity, from the poles to the equator, less than 0,00694, and equal to 0,00567. The measures of the degrees and of the pendulum concur, therefore, to prove that the force of gravity is not directed to a single point, which confirms *a posteriori* what has been antecedently demonstrated, namely, that the gravity is composed of the attractions of all the particles of the Earth.

This being the case, the law of gravity depends on the figure of the terrestrial spheroid, which depends itself on the law of gravity. It is this mutual dependance of the two unknown quantities on each other, that renders the investigation of the figure of the earth so extremely difficult. Fortunately, however, the elliptic figure, the most simple of all the re-entering figures next to the sphere, satisfies the condition of the equilibrium of a fluid mass, subject to a motion of rotation, and of which all the particles attract each other reciprocally, as the squares of the distances. Newton, upon this hypothesis, and supposing the earth a homogeneous fluid, found the ratio of the equatorial to the polar axis, to be 230 to 229.

It is easy to determine the law of the variation of the force of gravity on the earth upon this hypothesis. For this purpose let us consider two different points situated on the same radius, drawn from the centre to the surface of an homogeneous fluid, in equilibrio. All the similar elliptic strata, which cover any one amongst them, contribute nothing to its gravity. The resulting force of all the attractions which act on it, is derived entirely from the attraction of the interior spheroid, similar to the entire spheroid, and whose surface passes through the point in question. The similar and similarly situated particles of these two spheroids, attract the interior (g) point, and the corresponding point of the exterior surface, proportionally to their masses, divided by the squares of their distances. These masses are in the two spheriods, as the cubes of their similar dimensions, and the squares of their distances, are as the squares of these dimensions. The attractions on similar particles, are proportional therefore to these dimensions; from which it follows, that the entire attractions of the two spheroids, are in the same ratio, and their directions are parallel. The centrifugal forces of the two points, now under consideration, are likewise proportional to the same dimensions. Therefore the force of gravity in each of them, being the result of these two forces, will likewise be proportional to their distances from the centre of the fluid mass.

Now, if we conceive two fluid columns directed as before, to the centre of the spheroid, one from the pole, and the other from any point on the surface, it is evident, if the ellipticity of the spheroid is very small, that is, if it differs but little from a sphere, that the force of gravity, decomposed according to the directions of these columns, will be nearly the same as the total gravity. Dividing, therefore, the length of these columns into an equal number of parts, infinitely small and proportional to their lengths, the weights of the corresponding parts will be to each other as the products of the lengths of the columns, by the force of gravity at the points of the surface where they terminate. The whole weight of the columns will therefore be to each other in this ratio; and as these weights must be equal, to be in equilibrio, the force of gravity at their surface must consequently be reciprocally, as the length of these columns. Thus the length of the radius of the equator, surpassing the radius at the pole a 230th part, the force of gravity at the pole should likewise exceed that at the equator a 230th part.

This supposes the elliptic figure sufficient for the equilibrium of a homogeneous fluid mass. Maclaurin has demonstrated this in (*h*) a beautiful manner, from which it results, that the equilibrium is rigorously possible ; and that, if the ellipsoid differs little from a sphere, the ellipticity will be equal to $\frac{5}{4}$ of the quantity, which expresses the proportion of the centrifugal force, to that of gravity under the equator.

Two different figures of equilibrium may correspond to the same motion of rotation. But the equilibrium cannot exist with every motion of rotation. The shortest period of rotation of an homogeneous fluid in equilibrio, of the same density as the earth, is 0.1009 of a day, and this limit varies reciprocally, as the (*i*) square root of the density. When the motion of rotation increases in rapidity, the fluid mass becoming more flattened at the poles, its period of rotation becomes less, and ultimately falls within the limits suitable to a state of equilibrium. After a great many oscillations, the fluid, in consequence of the friction and resistances which it experiences, fixes itself at last in that state which is *unique*, and determined by the primitive motion ; and whatever may have been the primitive forces, the axis drawn through the centre of gravity of the fluid mass, and relative to which the moment of the forces was a maximum, at the origin, becomes the axis of rotation.

The preceding results furnish an easy method of verifying the hypothesis of the homogeneity of the earth. The irregularity of the measured degrees, may be supposed to leave too much uncertainty on the ellipticity of the earth to enable us to decide, if it is really such as the above hypothesis requires. But the regular increase of the force of gravity, from the equator to the pole, is sufficient to throw great light upon this subject.

By taking as unity the force of gravity at the equator, its increase at the pole, according to the hypothesis of homogeneity, should be equal to 0.00435. But by observations on the pendulum, this increase is 0.0054 : the earth therefore is not homogeneous. And indeed it is natural to suppose, that the density of the strata increases as they approach the centre. It is even necessary, for the stability of the equilibrium of the waters of the ocean, that their density should be less than the mean density of the earth; otherwise, when agitated by the winds and other causes, they would overflow their limits, and inundate the adjoining continents.

The homogeneity of the earth being thus excluded by observation, we must, to determine its figure, suppose the sea covering a nucleus, composed of different strata, diminishing in density from the centre to the surface. (*k*) Clairaut has demonstrated, in his beautiful work, that the equilibrium is still possible, on the supposition that the surface, and strata of the interior nucleus, have an elliptic figure. In the most probable hypothesis, relative to the law of the densities and ellipticities of these strata, the ellipticity of the earth is less than in the case of homogeneity, and greater than if the force of gravity was directed to a single central point. The increase of gravity from the equator to the poles is greater than in the first case, and less than in the second. But there exists between the total increase of the force of gravity, taken as unity at the equator, and the ellipticity of the earth, this remarkable analogy, that in all the hypotheses relative to the constitution of the internal nucleus, which the sea incloses, the ellipticity of the earth is just so much less than that which would take place in the case of homogeneity, as the increase of the force of gravity exceeds that which should exist, according the same supposition, and reciprocally, so as that the fractions expressing the sum of the ellipticity and of the increment, make a constant quantity equal to five times the half of the ratio of the centrifugal force, to the force of gravity at the equator, which, for the earth is $\frac{1}{115.2}$.

In attributing an elliptic figure to the strata of the terrestrial spheroid, the increase of its radii, and of the force of gravity, and the diminution of the degrees, from the pole to the equator, will vary as the squares of the cosines of the latitude, and these *(l)* are connected with the ellipticity of the earth, in such a manner, that the total increase of the radii is equal to the ellipticity. The total diminution of the degree, is equal to the ellipticity, multiplied by three times the degree at the equator; and the total increase of the force of gravity, is equal to the force of gravity at the equator, multiplied by the excess of $\frac{1}{115.2}$, above the ellipticity.

Thus the ellipticity of the Earth may be determined, either by direct measurement of degrees, or by observations on the length of the pendulum.

A consideration of a great number of observations of the pendulum give 0,00561, for the increase of the force of gravity, which taken from $\frac{1}{115.2}$ gives $\frac{1}{34.8}$, for the ellipticity of the Earth. If the hypothesis of the ellipse be conformable to nature, this ellipticity should agree with the measures of degrees; but it implies errors that are altogether improbable: and this circumstance, joined to the difficulty of reconciling all these measures to the same elliptic meridian, proves that the figure of the earth is much more complicated than had been supposed. This will not appear surprising, if we consider the different depths of the sea, the elevation of the continents, and islands above its level, the heights of mountains, and the unequal density of the water, and different substances which are at the surface of this planet.

To embrace, in the most general manner possible, the theory of the figure of the Earth and planets, it is necessary to determine the attraction of spheroids, differing little from spheres, and formed of strata, variable both in figure and density, according to any law whatever.

It will be also necessary to determine the figure which is suitable to the equilibrium of a fluid, expanded over its surface, for we must imagine the planets covered with a fluid in equilibrio similar to the case of the Earth, or their form would be entirely arbitrary. Dalembert has given, for this purpose, an ingenious method, which extends to a great number of cases, but which is deficient in that simplicity so desirable, in such complicated investigations, and which constitutes their principal merit.

A remarkable equation of partial differences relative to the (*l*) attraction of spheroids, led me, without the aid of integrations, and by differential methods only, to general expressions, for the radii of the spheroids; for the attractions upon any points whatever, either within the surfaces, or without them; for the condition of equilibrium of the fluids that surround them; for the law of gravity, and for the variation of the degrees at the surface.

All these quantities are connected with each other, by analogies extremely simple, from which results an easy method of verifying all the hypotheses that may be formed to represent either the variation of the force of gravity, or that of the values of different degrees of the meridian.

Thus Bouguer, with a view of reconciling the degrees measured at the equator, in France and in Lapland, supposed the Earth to be a spheroid of revolution, in which the increase of the degrees, from the equator to the pole, was proportional to the fourth power of the sine of the latitude. It is found that this hypothesis does not satisfy the increase of the force of gravity from the equator to Pello.—An increase, which according to observation, is equal to forty-five ten millionths of the whole gravity, and which would be only twenty-seven ten millionths on this hypothesis.

The above mentioned expressions give a direct and general solution of the problem, the object of which is to determine the figure of a fluid mass in equilibrio, supposing it subject to a motion of rotation, and composed of an infinity of fluids, of different densities, whose particles attract each other directly as their masses, and inversely as the squares of their distances.

Legendre had already solved this problem by a very ingenious analysis, which supposes the mass homogeneous. In the general case, the fluid necessarily takes the form of an ellipsoid of revolution, of which all the strata are elliptic, whose densities diminish at the same time that their ellipticities increase, from the centre to the surface.

The limits of compression of the whole ellipsoid, are $\frac{5}{4}$ and $\frac{1}{2}$ of the ratio of the centrifugal force, to the force of gravity at the equator. The first limit is relative to the hypothesis of homogeneity, and the second, to the supposition of the strata, indefinitely near to the centre, (m) being infinitely dense, and consequently the whole mass of the spheroid acting as if concentered in that point. In the latter case, the force of gravity being directed to a single point, and varying inversely as the square of the distance, the figure of the Earth would be such as has been above determined; but in the general hypothesis, the line which determines the direction of the force of gravity from the centre to the surface of the spheroid, is a curve, every element of which is perpendicular to the stratum through which it passes.

The analysis to which I have adverted, supposes that the terrestrial spheroid is entirely covered by the sea; but as this fluid leaves a considerable part of this spheroid uncovered; the analysis, notwithstanding its generality, does not represent nature exactly, and it is necessary to modify the results obtained on the hypothesis of a general inundation. Indeed the mathematical theory of the figure of the Earth presents on this supposition greater difficulties; but the progress of analysis particularly in this department, furnishes us with the means of surmounting them, and of considering the seas and continents such as they appear to observers. By thus adhering to nature, we get glimpses of several phenomena which natural history and geography present; which may thus diffuse great light on these two sciences, by connecting them with the theory of the system of the world. These are the principal results of my analypsis. One of the most interesting is the following theorem, which incontestably establishes the heterge-neity of the terrestrial strata.

"If to the length of a pendulum vibrating seconds at any point of the surface of the terrestrial spheroid, be added the product of this length into half the height of this point above the level of the sea, determined by observations made on the barometer, and divided by the semiaxis of the pole; the increase of this length thus corrected will be, from the equator to the poles on the hypothesis that the density of the earth to an inconsiderable depth is constant, the product of this length at the equator, into the square of the sine of the latitude, and by five fourths of the ratio of the centrifugal force to the gravity at (n) the equator, or by 43 ten thousaandths."

This theorem, to which I was conducted by a differential equation of the first order, which belongs to the surface of homogeneous spheroids, differing little from spheres, is generally true whatever may be the density of the sea and the manner in which it covers part of the earth. It is remarkable, in as much as it does not suppose a knowledge of the figure of the terrestrial spheroid, nor of that of the sea, figures which it would be impossible to obtain.

Experiments on the pendulum made in the two hemispheres, agree in giving to the square of the sine of latitude a coefficient greater than 43 ten thousandths, and very nearly equal to 54 ten thousands of the length of the pendulum at the equator. It is therefore satisfactorily proved by experiments, that the earth is not homogeneous in its interior. It appears moreover, by comparing them with analysis, that the densities of the terrestrial strata continually increase from the surface to the centre.

The regularity with which the observed variation of the lengths of the pendulum vibrating seconds, follows the law of the square of the sine of the latitude, proves that these strata are regalarly arranged about the centre of gravity of the earth, and that their form is very nearly an ellipse of revolution.

The ellipticity of the terrestrial spheroïd, may be determined by measures of degrees of the meridian. The different measures which have been made, compared two by two, give ellipticities which are sensibly different, so that the variation of degrees does not follow as exactly as gravity, the law of the square of the sine of latitude. This depends on the second differentials of the terrestrial radius, which the expressions of the degrees of the meridian and of the osculating circle contain, while the expression for the gravity contains only the first differentials of this radius, of which the small deviations from the elliptic radius, increase by successive differentiations. But if degrees at a considerable distance from each other be compared, such as those of France and the equator, their anomalies must be insensible on their difference; and it is found by this comparison that the ellipticity of the terrestrial spheroid is $\frac{1}{308}$.

But a more certain means of obtaining this ellipticity, consists, as has been already observed, in comparing with a great number of observations, the two lunar inequalities which are due to the compression of the earth, the one in longitude and the other in latitude. They agree in making the compression of the terrestrial spheroid very nearly equal to $\frac{1}{305}$, and what is very worthy of remark, each of the two inequalities leads to this result, which as we have seen differs very little from that furnished by a comparison of degrees in France and at the equator.

As the density of the sea is only the fifth part of the mean density of the earth; this fluid ought to have very little influence on the variations of degrees, and of gravity, and on the two inequalities of which we have spoken. Its influence is still more diminished by the smallness of its mean depth, which is thus proved. Conceive the terrestrial spheroid to be deprived of the ocean, and suppose that in this state the surface became fluid and was in equilibrio; we shall have its ellipticity by subtracting from five times the half of the ratio of the centrifugal force to the gravity at the equator, the coefficient assigned by experiments to the square of the sine of the latitude in the expression of the length of the pendulum which vibrates seconds; this length at the equator being assumed equal (n) to unity. By this means it is found that the compression of the terrestrial spheroid is $\frac{1}{304,8}$, the trifling influence which the action of the sea has on the variation of the gravity being neglected. The little difference which exists between this compression, and those furnished by the measures of terrestrial degrees and of the lunar inequalities, proves that the surface of this spheroid would be very nearly one of equilibrium, if it became fluid. On this account, and because the sea leaves vast continents uncovered, it is inferred that its depth is inconsiderable, and that its mean depth is of the same order as the mean height of continents and isles above the level of the sea, which height does not surpass a thousand metres. This depth is therefore a small fraction of the excess of radius of the equator above that of the pole, which excess does not surpass twenty thousand metres. But as high mountains are spread over some parts of the continents, so there may be great cavities in the bottom of the seas. However, it is natural to suppose that their depth is less than the elevation of high mountains: as the depositions of rivers and the remains of marine animals carried along by currents, must at length fill these cavities.

This is an important result for natural history and geology. There can be not the least doubt but that the sea covered a great part of our continents on which it has left incontestable proofs of its existence. The successive subsidence of isles, and of a part of the continents, followed by extended subsidences of the bason of the sea which have uncovered parts previously submerged, appear to be indicated by the different phenomena which the surface and strata of the existing continent present to us. In order to explain these subsidences, it is sufficient to assign more energy to causes, similar to those which have produced the subsidences of which history has preserved the record. The subsidence of one part of the bason of the sea, renders visible another part, so much the more extensive as the sea is less profound. Thus great continents might emerge from the ocean without producing great changes in the figure of the terrestrial spheroid. The property, which this figure possesses, of differing little from that, which its surface would assume if it became fluid, requires that the depression of the level of the sea, should be only a small fraction of the difference of the two axes of the pole and of the equator. Every hypothesis founded on a considerable displacement of the poles on the surface of the earth, must be rejected as incompatible with the property of which I have been speaking. Such a displacement has been suggested, in order to explain the existence of elephants, of which fossel remains are found in such great abundance in northern climates, where living elephants cannot exist. But an elephant, which is with great probability supposed to be cotemporaneous with the last flood, was found in a mass of ice well preserved with its skin, and as the hide was covered with a great quantity of hair, this species of elephant was guarantied by this means, from the cold of the northern climates, which it might inhabit and even select as a place of residence. The discovery of this animal has therefore confirmed what the mathematical theory of the earth had shewn us, namely, that in the revolutions which have changed the surface of the globe and destroyed several species of animals and vegetables, the figure of the terrestrial spheroid, and the position of its axis of rotation on its surface, have undergone only slight alterations.

Now what is the cause which has given to the strata of the earth forms very nearly elliptical, with densities increasing from the surface to the centre, which has arranged them regularly about their common centre of gravity, and which has rendered its surface very little different from what it would be, if it had been primitively in a fluid state? If the different substances which compose the earth had been primitively, by the effect of great heat in a fluid state, the most dense must have been carried towards the centre: all would have assumed elliptic forms, and the surface would have been in equilibrio. These strata in consolidating having changed their figure very little, the earth should at present exhibit the phenomena of which I have been speaking. This case has been amply discussed by geometers. But if the earth was homogeneous in the chymical sense, *i. e.* if it was composed of one sole substance in its interior, it might also exhibit these phenomena. In fact, we may conceive that the immense weight of the superior strata, should increase considerably the density of the inferior strata. Hitherto geometers have not taken into account in their investigations on the figure of the earth, the compression of the substances of which it was composed; although Daniel Bernoulli in his essay on the tides had already pointed out the cause of the increase of density of the strata of the terrestrial spheroid. From the analysis which I have applied in the eleventh book of the celestial mechanics, it appears that it is possible to satisfy all the observed phenomena, on the hypothesis of the earth being composed of one sole substance in its interior. The law of the densities which the compression of the earth assigns to the strata of this substance not being known, we can only make suppositions on this subject.

It is known (*o*) that the density of gases increases proportionally to their compression, when the temperature remains the same. But this law does not appear to agree to liquid and solid bodies; it is natural to think that these bodies resist the compression, so much the more as they are more compressed. This is in fact confirmed by experiment, so that the ratio of the differential of the pressure to the differential of the density, instead of being constant as in the case of gas, increases with the density. The simplest expression of this ratio, supposed variable, is the product of the density by a constant quantity. This is the law which I have adopted, since it combines to the advantage of representing in the simplest possible manner, what we know respecting the compression of bodies, that of adapting itself easily to the calculus in the investigation of the figure of the earth; my object in this investigation being only to shew that this manner of considering the interior constitution of the earth, may be reconciled with all the phenomena, which depend on this constitution, at least if the terrestrial spheroid had been primitively fluid. In the solid state, the adherence of the molecules, diminishes extremely their mutual compression, and it prevents the entire mass from assuming the regular figure which it would have in the fluid state, if it had primitively deviated from it.

Therefore in this very hypothesis on the constitution of the earth, as in all others, the primitive fluidity of the earth appears to me to be indicated by the regularity of gravity and by the figure at its surface.

All astronomers have assumed the invariability of the axis of rotation of the earth, and the uniformity of this rotation. The duration of a revolution of the earth about its axis is the standard of time; it is therefore of great importance to appreciate the influence of all the causes which may alter this element. The axis of the earth moves about the poles of the ecliptic, but since the epoch, at which the application of the telescope to philosophical instruments furnished the means of observing terrestrial latitudes with precision, no variation has been recognized in these latitudes, but what may have arisen from the errors of observation, which proves that since that epoch, the axis of rotation has existed very nearly on the same point of the terrestrial surface; it therefore appears that this axis is invariable. The existence (*p*) of a similar axis on solid bodies has been known for a long time. We know that each of these bodies has three principal rectangular axes, about which it may revolve uniformly; the axis of rotation remaining invariable. But does this remarkable property appertain to bodies, which, like the earth, are partly covered with a fluid? The condition of the equilibrium of the fluid must be then combined with the conditions of the principal axes: it changes the figure of the surface, when the axis of rotation is changed. It is therefore interesting to know whether among all the possible changes there is one, in which the axis of rotation and the equilibrium of the fluid remain invariable. Analysis proves that if we make to pass very near to that centre of gravity of the terrestrial spheroid a fixed axis about which it may revolve freely, the sea may always assume on the surface of the spheroid a constant state of equilibrium. I have given in the eleventh book already cited, in order to determine this state, a method of approximation arranged according to the powers of the ratio of the density of the sea to the mean density of the earth, and as this ratio is only $\frac{1}{5}$, the approximation is extremely converging. The irregularity of the depth of the sea, and of its contour, does not permit us to obtain this approximation. But it is sufficient to recognize the possibility of this circumstance, in order to be assured of the existence of a state of equilibrium of the sea. The position of the fixed axis of rotation being arbitrary, it is natural to think, that among all the positions which this axis may be made to undergo, there is one in which the axis passes through the common centre of gravity of the sea and of the spheroid which it covers, so that this fluid being in equilibrio, and congealed in that state, this axis should be a principal axis of rotation of the terrestrial spheroid and of the sea, considered as one body; it is evident that if its fluidity be restored to the congealed mass, the axis will be always an invariable axis for the entire earth; I have shewn that such an axis is always possible, and I have given the equations which determine its position. By applying these equations to the case in which the sea covers the entire spheroid, I have arrived at the following theorem.

"If the density of each stratum be supposed "to be diminished by the density of the sea; and "if through the centre of gravity of this imaginary spheroid, we conceive a principal axis of "the spheroid to be drawn, the earth being "made to revolve about this axis, if the sea be "in equilibrio, this axis will be the principal axis "of the entire earth, of which the centre of "gravity will be that of the imaginary spheroid."

Thus the sea which partly covers the terrestrial spheriod not only does not render impossible the existence of a principal axis of rotation, but it even by its mobility, and by the resistances which its oscillations experience, would restore to the earth a permanent state of equilibrium, if any causes should derange it.

If the sea was sufficiently profound to cover the surface of the terrestrial spheroid, supposing it to turn successively round the three principal axes of the terrestrial spheroid, each of these axes would be a principal axis for the entire earth. But the stability of the axis of rotation has not place, as in the case of a solid body, but relatively to the two principal axes, for which the moment of inertia is a *maximum* or a *minimum*. However there is this difference between the earth and a solid body, that in the case of the solid body, if the axis of rotation be changed, the figure of the solid body will not be changed, whereas in the case of the earth the surface of the sea assumes another figure altogether. The three figures, which this surface assumes in revolving, successively with the same angular velocity of rotation, about each of the three axes of rotation of the imaginary spheroid, have very simple relations which, I have determined; and it follows from my analysis, that the mean radius between the radii of the three surfaces of the (q) sea, corresponding to the same point of the surface of the terrestrial spheroid, is equal to the radius of the surface of the sea in equilibrio on this spheroid, and deprived of its motion of rotation.

In the fifth book of the Celestial Mechanics, I have discussed the influence of interior causes, such as volcanoes, earthquakes, winds, currents of the sea, &c. on the duration of the rotation of the earth, and I have shewn by means of the principle of (r) areas, that this influence is insensible, and that in consequence of these causes, it is necessary in order that a sensible effect might be produced, that considerable masses should be transported considerable distances; which has not been the case since the periods of which history has preserved the records, but there exists an interior cause of alteration of the day, which has not been yet considered, and which, considering the importance of this element, deserves a particular discussion. This cause is the heat of the terrestrial spheroid. If, as every thing induces us to think, the earth had been primitively fluid, its dimensions have diminished successively with its temperature; its angular velocity of rotation has increased gradually, and it will continues to increase until the earth arrives at the constant state of the mean temperature of the space through which it moves. In order to form a just conception of this movement of angular velocity, suppose in a space of a given temperature, a globe (r) of homogeneous matter to revolve on its axis in a day. If this globe be transported into a space of which the temperature is less by the hundredth part of a degree, and if we suppose that the rotation is not altered, either by the resistance of the medium, or by friction; its dimensions will diminish with the diminution of temperature; and when at length it shall have assumed the temperature of the new space, its radius will be diminished by a quantity, which I shall suppose the hundred thousandth part, which is the case very nearly for a globe of glass, and which may be admitted for the earth. The weight of the heat is inappreciable to all experiments which have been made to (*s*) measure it; it appears therefore like to light to produce no sensible variation in the mass of bodies, consequently, in the new space two things may be supposed the same as in the first, namely, the mass of the globe and the sum of the areas described in a given time, by each of its molecules referred to the plane of its equator. The molecules approach to the centre of the globe by a hundreth thousand part of their distance from this point. The areas which they describe on the plane of the equator (*t*) being proportional to the square of this distance, will diminish therefore very nearly by a fifty thousandth part, if the angular velocity of rotation does not increase; hence it follows, that in order that the sum of the areas described in a given time may be constant, the increment of this velocity, and consequently the diminution of the duration of rotation, ought to be a fiftieth thousandth part; such is therefore the final diminution of this duration. But previous to its attaining this final state, the temperature of the globe continually diminishes, and more slowly at the centre than at the surface, so that from observation of this diminution, compared with the theory of heat, we can determine the epoch when the globe was transported into the new space. The earth appears to be in a similar state. This follows from thermometrical observations made in profound mines, and which indicate a very sensible increase of heat, according as we penetrate into the interior of the earth. The mean of the observed increments appears to be a centesimal degree for a depth of 32 metres, but a very great number of observations will make its value known very accurately, which cannot be the same for all climates. It was necessary, in order to obtain the increment of the earth's rotation, to know the law of the diminution of heat from the centre to the surface. This I have investigated in the eleventh book of the Celestial Mechanics, for a globe primitively warmed in any manner, and besides subjected to the heating action of an exterior cause. The law in question, which I published in 1819, in the Connaisance des temps, and which M. Poisson has since confirmed by a learned analysis, is represented by an infinite series of terms, which have for factors constant quantities, which are always less than unity, and of which the exponents increase proportionably to the time. The length of the time makes these terms to disappear the one after the other; so that before the establishment of the final temperature, only one of those terms which produces the increase of temperature in the interior of the globe, is sensible. I have supposed the earth to have attained this state, from which it is perhaps still far removed. But as I only wish to give here a general idea of the influence of the diminution of the interior heat on the duration of the day I have adopted this hypothesis, and I have inferred from it, the increase of the velocity of revolution.

It is necessary in order to reduce this increment to numbers, to determine numerically the two constant arbitrary quantities of which one depends on the conducting power of the earth with respect to heat, and the other on the elevation of temperature of its superficial stratum above the temperature of the ambient space. I have determined the first constant by means of the variations of the annual heat at different depths, and for this purpose I have made use of the experiments of M. Saussure, which this philosopher has cited in No. 1422 of his voyage to the Alps. In these experiments, the annual variation of the heat at the surface has been reduced to a twelfth part at the depth of $9^{m},6$. I have afterwards supposed that in our mines, the increase of heat is a centesimal degree for a depth of 32 metres, and that the linear dilatation of the earth's strata is a hundred thousand part of each degree of temperature. I have found by means of these data, that the duration of the day has not increased by half a hundredth of a centesimal second for the last two thousand years, which is chiefly owing to the magnitude of the earth's radius. Indeed I have supposed that the earth is homogeneous, and it is certain that the densities of its strata increase from the surface to the centre. But it should be observed here that the quantity of heat and its internal motion would be the same in a heterogeneous substance, if in the corresponding parts of the two bodies, the heat and the property of conducting it were the same. The matter may here be considered as a vehicle of heat, which may be the same in substances of different densities. This is not the case for dynamical properties, which depend (u) on the mass of the molecules. Thus we can in this conception of the effects of terrestrial heat on the duration of the day extend to the earth, considered as heterogeneous, the data relative to heat considered as homogeneous. In this manner it is found, that the increment of the density of the strata of the terrestrial spheroid diminishes the effect of heat on the duration of the day, which effect since the time of Hipparchus has not increased this duration by $\frac{1''}{300}$.

The term on which the increment of the interior heat of the earth depends, does not now add the fifth of a degree to the mean temperature of its surface. Its annihilation, which a very long series of ages ought to produce, will not consequently cause any species of organized beings actually existing to disappear, at least as long as the proper heat of the sun, and its distance from the earth, do not experience any sensible alteration.

In fine, I am far from thinking that the preceding suppositions obtain in nature ; besides, the observed values of the two constants of which I have spoken, depend on the nature of the soil which in different countries has not the same qualities with respect to heat. But the sketch which I have given, suffices to shew that the phenomena which have been observed on the heat of the earth, may be reconciled with the result which I have deduced from a comparison of the theory of the secular inequalities of the moon, and observations of ancient eclipses, namely, that since the time of Hipparchus, the duration of the day has not varied by the hundredth part of a second.

But what is the ratio of the mean density of the earth, to that of a known substance at its surface? The effect of the attractions of mountains, on the oscillations of the pendulum, and on the direction of the plumb line, ought to conduct us to the solution of this interesting problem.

It is true, that the highest mountains are always very small, in proportion to the Earth; but we may approach very near to the centre of their action, and this joined to the precision of modern observations, ought to render their effects perceptible.

The mountains of Peru, (*v*) the highest in the world, seemed the most proper for this object. Bouguer did not neglect so important an observation in the journey which he undertook, for the measure of the meridional degrees at the equator.

But these great bodies being volcanic and hollow in their interior, the effect of their attraction was found to be much less than might be expected from their size. However it was perceptible; the diminution of the force of gravity at the summit of Pichincha, would have been 0,00149, without the attraction of the mountain, and it was observed to be 0.00118. The effect of the deviation of the plumb-line, from the action of another mountain, surpassed 20″. Dr. Maskelyne has since measured, with great care, a similar effect produced by the action of a mountain in Scotland: the result was, that the mean density of the Earth, is double that of the mountain, and four or five times greater than that of common water. This curious observation deserves to be repeated several times on different mountains, whose interior constitution is well known. Cavendish determined this density by the attraction of two metallick globes of a great diameter, and he succeeded in rendering it sensible by a very ingenious process. It follows from these experiments that the mean density of the earth, is to that of water, very nearly in the ratio of eleven to two, which agrees with the preceding ratio as well as could be expected from such delicate observations and experiments.

I proceed here to present some considerations on the level of the sea, and on the reductions to this level. Conceive an extremely rare fluid of a uniform density throughout, and of an inconsiderable elevation, to surround the earth; let it, however, embrace the highest mountains; such would be very nearly our atmosphere if reduced to its mean density. Analysis shews that the corresponding points of the two surfaces, of the sea, and of this level, are separated by the same interval. If we conceive the surface of the sea to be prolonged below the continents and the surface of the fluid, so that the two surfaces may be always separated by this interval, this will be what is termed *the level of the sea*,. It is the ellipticity of those two surfaces, that is determined by the measures of degrees; it is also the variation of gravity at the surface of the supposed fluid, which added to the ellipticity of this surface, gives a constant sum equal to $\frac{5}{2}$ of the ratio of the centrifugal force to the gravity at the equator. It is therefore to this surface or to the surface of the sea prolonged in the manner above specified, that it is necessary to refer the measures of degrees, and of the pendulum observed on the continents. But it is easily proved that the gravity does not vary from a point on the continent to the corresponding point of the surface of the fluid, but in consequence of the distance of those two points, when the slope to the sea is inconsiderable. Therefore in the reduction of the length of the pendulum to the level of the sea, we ought only to consider the height above this level such as we have defined it. In order to render this sensible by the results of the calculus in a case which I have subjected to analysis, conceive that the earth is an ellipsoid of revolution partly covered by the sea, of which we shall suppose the density to be very small relatively to the mean density of the earth. If the ellipticity of the terrestrial spheroid be less than that which corresponds to the equilibrium of the surface of the supposed fluid, the sea will cover the terrestrial equator to a certain latitude. The degrees measured on the continents, and increased in the ratio of their distance from the surface of the supposed fluid, (the radius of the earth being assumed equal to unity), will be those which are measured on this

K 2

surface. The length of the pendulum which vibrates seconds diminished by twice this ratio, will be that which is observed on this surface; and the ellipticity determined by the measures of degrees, will be the same as would be obtained by subtracting from $\frac{5}{2}$ of the ratio of the centrifugal force to the gravity at the equator, the excess of the polar over the equatorial gravity being assumed equal to unity.

Let us apply the preceding theory to Jupiter.

The centrifugal force due to the motion of rotation of this planet, is nearly $\frac{1}{12}$ of the force of gravity at its equator; at least, if the distance of the fourth satellite from its centre, as given in the second Book, be adopted.

If Jupiter was homogeneous, (x) the diameter of its equator might be obtained, by adding five-fourths of the preceding fraction to its shorter axis taken as unity, these two axes would, therefore, be in the proportion of 10 to 9,06. According to observation, their proportion is that of 10 to 9,43. Jupiter, therefore, is not homogeneous. Supposing it to consist of strata, of which the densities diminish from the centre to the surface, its ellipticity should be included between $\frac{1}{24}$ and $\frac{5}{48}$, the observed ellipticity being within these limits, proves the heterogenity of its strata, and by analogy that of the strata of the terrestrial spheroid, already rendered very probable from the measures of the pendulum, and which have been confirmed by the inequalities of the Moon depending on the ellipticity of the Earth.

CHAP. IX.

On the Figure of the Ring of Saturn.

It was shewn in the first book, that the ring of Saturn consisted of two concentric rings of very small thickness. By what mechanism do these rings sustain themselves about the planet? It is not probable that this should take place from the simple adhesion of their particles. Since, were this the case, the parts nearest to Saturn, sollicited by the constantly renewed action of gravity, would be at length detached from the rings, which would, by an insensible diminution, finally disappear, like all those works of nature which have not had sufficient force to resist the action of external causes. These rings support themselves then without effort, and by the sole laws of equilibrium. But for this it is requisite to suppose them endowed with a rotary motion about an axis perpendicular to their plane, and passing through the centre of Saturn, so that their gravitation towards the planet, may be balanced by the centrifugal force due to this motion.

Let us imagine a homogeneous fluid spread about Saturn in the form of a ring, and let us see what ought to be its figure, for it to remain in equilibrio, in consequence of the mutual attraction of its particles, of their gravitation towards Saturn, and their centrifugal force. If, through the centre of the planet, a plane is imagined to pass, perpendicular to the surface of the ring, the section of the ring by this plane, is what I shall call the *generating curve.* Analysis proves that if the magnitude of the ring is small in (*a*) proportion to its distance from the centre of Saturn, the equilibrium of the fluid is possible, when the generating curve is an ellipse of which the greater axis is directed towards the centre of the planet. The duration of the rotation of the ring, is nearly the same as that of the revolution of a satellite, moved circularly at the distance of the centre of the generating ellipse, and this duration is about four hours and a third, for the interior ring. Herschel has confirmed by observation this result, to which I had been conducted by the theory of gravitation.

The equilibrium of the fluid would also exist, supposing the generating ellipse variable in size and position, within the extent of the circumference of the ring; provided that these variations are sensible only at a much greater distance, than the axis of the generating section. Thus, the ring may be supposed of an unequal breadth in its different parts, it may even be supposed of double curvature. These inequalities are indicated by the appearances and disappearances of Saturn's ring, in which the two arms of the ring have presented different phenomena. They are even necessary to maintain the ring in equilibrio about the planet, since if it was perfectly similar in all its parts, its equilibrium would be deranged by the slightest force, such as the attraction of a satellite, and the ring would finally precipitate itself upon the planet.

The rings by which Saturn is surrounded, are consequently irregular solids, of unequal breadth in the different points of their circumference, so that their centres of gravity do not coincide with their centres of figure. These centres of gravity may be considered as so many satellites, moving about the centre of Saturn, at distances dependant on the inequalities of the rings, and with angular velocities equal to the velocities of rotation of their respective rings.

We may conceive, that these rings, sollicited by their mutual action, by that of the Sun, and of the satellites of Saturn, ought to oscillate about the centre of this planet, and thus produce the phenomena of light, of which the period comprises several years. It might likewise be supposed, that sollicited by different forces, they should cease to exist in the same plane; but Saturn having a rapid rotatory motion, and the plane of its equator being the same with that of its ring, and of its six first satellites, its action retains the system of these different bodies in the same plane. The action of the Sun, and of the seventh satellite, only changes the position of the plane of Saturn's equator, which in this motion carries with it the ring, and the orbits of the six first satellites.

CHAP. X.

On the Atmosphere of the Celestial Bodies.

The thin, transparent, compressible, and elastic fluid which surrounds a body, and rests upon it, is called its *atmosphere*. We conceive, with great appearance of probability, that a similar atmosphere surrounds every celestial body; and the existence (*a*) of such a fluid, relatively to the Sun and Jupiter, is indicated by observations. In proportion as the atmospherical fluid is elevated above the surface of a body, it becomes thinner, in consequence of its elasticity, which dilates it so much the more, as it is less compressed. And if the particles of its exterior surface were (*b*) perfectly elastic, it would extend itself indefinitely, and would eventually dissipate itself in space.

It is then requisite that the elasticity of the atmospherical fluid should diminish in a greater proportion than the weight which compresses it; in order that there may exist a state of rarity, in which it may be without elasticity. It should be in this state at the surface of the atmosphere.

All the atmospheric strata should acquire, after a time, the rotatory motion, common to the body which they surround. For the friction of these strata against each other, and against the surface of the body, should accelerate the slowest motions, and retard the most rapid, till a perfect equality is established among them. In these changes, and generally in all those, which the atmosphere undergoes, (c) the sum of the products of the particles of the body, and of its atmosphere, multiplied respectively by the areas, which their radii vectores projected on the plane of the equator, describe round their common centre of gravity, are always equal in the same times.

Supposing then, that by any cause whatever, the atmosphere should contract itself, or that a part should condense itself on the surface of the body, the rotatory motion of the body, and of its atmosphere, would be accelerated, because the radii vectores of the areas, decribed by the particles of the primitive atmosphere becoming smaller, the sum of the products of all the particles, by the corresponding areas, could not remain the same, unless the velocity of rotation is increased.

At its surface the atmosphere is only retained by its weight, and the form of this surface is such, that the force which results from the centrifugal and attractive forces of the body (*d*), is perpendicular to it. The atmosphere is flattened towards the poles, and distended at its equator, but this ellipticity has limits, and in the case where it is the greatest, the proportion of the axis of the pole to that of the equator is as two to three.

The atmosphere can only extend itself at the equator, to that point where the centrifugal force exactly balances the force of gravity, for it is evident that beyond this limit, the fluid would dissipate itself. Relatively to the Sun, this point is distant from its centre by the length of the radius of the orbit of a planet, the period of whose revolution it equal to that of the Sun's rotation.

The Sun's atmosphere then does not extend so far as Mercury, and consequently does not produce the zodiacal light, which appears to extend beyond even the terrestrial orbit. Besides, this atmosphere, the axis of whose poles should be at least two-thirds of that of the equator, is very far from having the lenticular form which observation assigns to the zodiacal light.

The point where the centrifugal force balances gravity, is so much nearer to the body, in proportion as its rotatory motion is more rapid. Supposing that the atmosphere extends itself as far as this limit, and that afterwards it contracts and condenses itself from the effect of cold at the surface of the body, (*e*) the rotatory motion would become more and more rapid, and the farthest limit of the atmosphere would approach continually to its centre: it will then abandon successively in the plane of its equator, fluid zones, which will continue to circulate about the body, because their centrifugal force is equal to their gravity. But this equality, not existing relatively to those particles of the atmosphere, distant from the equator, they will continue to adhere to it. It is probable that the rings of Saturn are similar zones, abandoned by its atmosphere.

If other bodies circulate round that which has been considered, or if it circulates itself round another body, the limit of its atmosphere (*f*) is that point where its centrifugal force, *plus* the attraction of the extraneous bodies, balances exactly its gravity. Thus the limit of the Moon's atmosphere, is the point where the centrifugal force due to its rotatory motion, *plus* the attractive force of the Earth, is in equilibrio with the attraction of this satellite. The mass of the moon being $\frac{1}{75}$ of that of the earth, this point is therefore distant from the centre of the Moon, about the ninth part of the distance from the Moon to the Earth. If, at this distance, the primitive atmosphere of the Moon had not been deprived of its elasticity, it would have been carried towards the Earth which might have retained it. This is perhaps the cause why this atmosphere is so little perceptible.

CHAP. XI.

Of the Tides.

It was Newton, who first gave the true explanation of the tides, by shewing that they arose from the great principle of universal gravitation. Kepler had recognised the tendency of the waters of the sea towards the centres of the sun and moon; but being ignorant of the law of this tendency, and of the methods necessary to subject it to computation, he could only assign a very probable conjecture on this object. Galileo in his dialogues on the system of the world, expresses his astonishment and regret, that this conjecture, which appeared to bring back into natural philosophy the occult qualities of the ancients, had been suggested by such a man as Kepler. He explained the ebbing and flowing of the sea, by the diurnal changes which the rotation of the earth, combined with its revolution about the sun, ought to produce in the absolute motion of each molecule of the sea. This explanation appeared to him so incontestable, that he gave it, as one of the principal proofs of the Copernican system, for the defence of which he was afterwards so persecuted. Further discoveries have confirmed the conjecture of Kepler, and overturned the explication of Galileo, which is inconsistent with the laws of the equilibrium and motion of fluids.

The theory of Newton appeared in 1687, in his Treatise on the Mathematical Principles of Natural Philosophy. He there considered the sea as a fluid of the same density as the earth which it entirely covers, and he supposed that it assumed at each instant, the figure in which it would be in equilibrio under the action of the sun. If then this figure be supposed to be that of an ellipsoid (*a*) of revolution, of which the greater axis is directed towards the sun; he determined the ratio of the two axes, in the same way as he determined the ratio of the two axes of the earth, compressed by the centrifugal force of its motion of rotation. The greater axis of the aqueous ellipsoid being constantly directed towards the sun, the greatest height of the sea in each port, ought to happen when the sun is on the equator at midday and midnight, and the greatest depression ought to be at the rising and setting of this star.

Let us consider the manner in which the sun acts on the sea, when it deranges its equilibrium. It is evident, that if the Sun acted on the centre of gravity of the Earth, and of every particle of the ocean, by exerting equal and parallel forces, (*b*) the whole system of the terrestrial spheroid would obey these forces by a common motion, and the equilibrium of the waters would not be at all altered. This equilibrium then, is only deranged by the difference of these forces, and by the inequality of their directions. A particle of the ocean, placed directly under the Sun, is more attracted than the centre of the Earth. It tends therefore, to separate itself from it, but it is retained by its gravity, which this tendency diminishes. Half a day afterwards, this particle is opposite to the Sun, which attracts it less forcibly than it does the centre of the Earth; the surface of the terrestrial globe therefore tends to separate itself from it, but the gravity of the particles retains it. This force is therefore diminished also in this case by the solar attraction. But since the distance of the Sun is very great, compared with the radius of the Earth, it is easy to see that the diminution of gravity in each case is very nearly the same. A simple decomposition of the action of the Sun upon the particles of the ocean, is sufficient to shew, that in any position of this body, relatively to these particles, its action in disturbing their equilibrium, becomes the same after half a day.

The law according to which the water rises and falls, may be thus determined. Let us conceive a vertical circle, whose circumference represents half a day, and whose diameter is equal to the whole tide, or to the difference between the height of high and low water, and let the arcs of this circumference, (*c*) reckoning from the lowest point, express the time elapsed since low water, the versed sines of these arcs will express the heights of the water, corresponding to these times. Thus, the ocean in rising, covers in equal times, equal arcs of this circumference.

The greater the extent of the surface of the water, the more perceptible are the phenomena of the tides. In a fluid mass, the impressions which a fluid particle receives, are communicated to the whole. It is thus that the action of the Sun, which is insensible on an insulated particle, produces on the ocean such remarkable effects. Let us imagine, at the bottom of the sea, a curved canal, terminated at one of its extremities by a vertical tube, rising above the surface of the water, and which, if prolonged, would pass through the centre of the Sun.

The water will rise in (*d*) this tube by the direct action of the Sun, which diminishes the gravity of its particles, and particularly by the pressure of the particles enclosed in the canal, which all make an effort to unite themselves beneath the Sun. The elevation of the water in the tube, above the natural level of the sea, is the integral of all these infinitely small efforts. If the length of this canal is increased, this integral also becomes greater, because it extends over a larger space, and because there will be a greater difference in the quantity and direction of the forces, by which the extreme particles are sollicited.

By this example we see the influence which the extent of the sea has upon the phenomena of the tides, and the reason why they are insensible in the seas of inconsiderable extent, as the Euxine and the Caspian. The magnitude of the tides depends also much on local circumstances. The oscillations of the ocean, when confined in a narrow channel, may become extremely great, and these may be augmented by the reflection of the waters from the opposite shore. It is thus, that the tides, very small in the South Sea islands, are very considerable in our harbours.

If the ocean covered a spheroid of revolution, and experienced no resistance to its motion, the instant of high water would be that of the passage of the Sun over the superior or inferior meridian; but it is not thus in nature; local circumstances produce great variations in the times of high water, even in harbours that are very near each other. To have a just idea of these variations, we may suppose a large canal communicating with the sea, and extending into the land; it is evident that the undulations which take place at its entrance, will be propagated successively through its whole length, so that the figure of its surface will be formed by the undulations of large waves in motion, which will be incessantly renewed, and will describe their length in the interval of half a day. These waves will produce at every point of the canal, a flux and reflux, which will follow the preceding laws, but the hours of the flowing will be retarded, in proportion as the points are farther from the entrance of the canal. What we have here said of a canal, may be applied to rivers whose surfaces rise and fall by similar waves, notwithstanding the contrary motion of their streams. These waves are observed in all rivers near to their entrance; they extend to considerable distances in great rivers, and at the straights of Pauxis in the river of the Amazons, they are as sensible at the distance of eighty myriameters from the sea.

The action of the Moon on the sea produces an ellipsoid similar to that produced by the action of the Sun, but it is more elongated, because the lunar action is more powerful than that of the Sun. In consequence of the inconsiderable excentricity of these ellipsoids, we may conceive (*e*) them to be placed the one over the other, so that the radius of the surface of the sea is half the sum of the corresponding radii of their surfaces.

From hence arise the principal varieties of the tides. In the syzygies, the greater axes coincide, and the greatest elevation happens at the instant of mid-day and mid-night, and the greatest depression at the rising and setting of these stars. In the quadratures, the greater axis of the lunar ellipsoid and the lesser axis of the solar ellipsoid coincide; the full tide happens therefore at the rising and setting of these stars, and it is the least high water: the low water happens at the instants of mid-day and mid-night, and it is the greatest of low waters. If therefore the action of each star be expressed by the difference of the semiaxes of its ellipsoid, which is evidently proportional to it, when the place is situated at the equator, the excess of the greatest syzygial tide over the low water in syzygies will express the sum of the solar and lunar actions, and the excess of the least high water, which is in quadrature, over the greatest low water, which is likewise, (as we have seen in quadrature), will express the difference of these actions. If the harbour be not in the equator, this excess should be multiplied by the square of the cosine (f) of latitude. Therefore the ratio of the action of the Moon to that of the Sun may be determined by observing the heights of the tides in syzygies and in quadratures. Newton inferred from some observations made near Bristol, that this ratio is that of four and a half to unity. The distances of those stars from the centre of the earth influence all these effects; the action of each star being reciprocally as the cube of the distance.

As to the intervals between high water from one day to another, Newton observed that it is least in syzygies, and that it increases from syzygy to the following quadrature, that at the first octant it is equal to a lunar day, and that it attains its *maximum* at the quadrature; that it afterwards diminishes, becoming equal to a lunar day at the subsequent octant, and that it finally resumes its *minimum* at the syzygy. Its mean value being a lunar day, there are as many high waters as there are passages of the Moon over the superior or inferior meridian.

Such would be, according to the theory of Newton, the phenomena of the tides, if the sun and moon moved in the plane of the equator. But it appears from observation, that the highest tides do not arrive at the very moment of the syzygy, but a day and a half later. Newton ascribed this retardation to the oscillatory motion of the sea, which remains some time after the sun and moon cease to act. The exact theory of the undulations of the sea, produced by this action, shews that, without the accessory circumstances, the highest tides would coincide with the syzygies, and the lowest would coincide with the quadratures. Consequently their retardation at the moments of these phases cannot be attributed to the cause assigned by Newton, it therefore must depend, as also the hour of high water, in each harbour, on accessary circumstances. This example shews that we ought to distrust the most specious conjectures, when they are not confirmed by a rigorous analysis.

However the consideration of two ellipses, superimposed the one over the other, may also represent the tides, provided that the greater axis of this ellipsoid be conceived to be directed towards a fictitious sun, always equally elongated from the true sun. The axis of the lunar ellipsoid should be likewise always directed towards an maginary moon equally elongated from the true, but at such a distance that the conjunction of the two imaginary stars, does not arrive until a day and a half after the syzygy.

This consideration of the two ellipsoids, extended to the case, in which the stars move in orbits inclined to the equator, cannot be reconciled with observations. If the harbour be situated in the equator it gives near the *maximum* of the tides, the two high waters in the morning and in the evening, very nearly equal, whatever may be the declinations of these stars; only the action of each star is diminished in the ratio of the square of the cosine of its declination (*g*) to unity. But if the place is not on the equator, these two high waters may be extremely different, and when the declination of the stars is equal to the obliquity of the ecliptic, the evening tide at Brest should be eight times greater than that of the morning. However it appears from numerous observations made at this port, that these two tides are very nearly equal, and their greatest difference is not the thirtieth part of their sum. Newton ascribed the smallness of this difference to the same cause, by means of which, he explained the retardation of the high water beyond the moment of the syzygy, namely to a motion of oscillation in the sea, which, according to him, bringing back a great part of the evening tide on the subsequent morning tide, renders these tides very nearly equal. But the theory of the undulations of the sea shews that this explanation is not exact, and that without accessary circumstances the two consecutive tides would not be equal, unless the sea had every where the same depth.

In 1738, the Academy of Sciences proposed the cause of the ebbing and flowing of the sea, as the subject of the mathematical prize, which it decided in 1740. Four essays were crowned, the three first, founded on the principle of universal gravitation, were those of Daniel Bernouilli, of Euler, and Maclaurin. The Jesuit Cavalleri, the author of the fourth, adopted the system of vortices. This was the last honour paid to this system by the Academy, which was then composed of many geometers, whose successful labours contributed so powerfully to the advancement of the celestial mechanics.

The three essays which were founded on the law of universal gravitation, are developements of the theory of Newton. They depend not only on this law, but also on the hypothesis adopted by this great geometer, namely, that the sea assumes at each instant the figure in which it would be in equilibrio, under the star which attracts it.

The essay of Bernouilli contains the most extensive developements. He, like Newton, ascribed the retardation of the *maxima* and of the *minima* of the tides, after the instants of the occurrence of the syzygies and the quadratures, to the motion of the waters of the ocean ; and he adds, perhaps, a part of this retardation is owing to the time the action of the moon takes to arrive at the earth. But I have ascertained that, between the heavenly bodies all attractions are transmitted with a velocity, which, if it be not infinite, surpasses several thousand times the velocity of light; and we know that the light (*h*) of the moon reaches the earth in less than two seconds.

D'Alembert, in his treatise on the general course of the winds, which bore away the prize, proposed on the subject by the Academy of Sciences in Prussia, considered the oscillations of the atmosphere produced by the attractions of the sun and moon. And on the hypothesis that the earth is deprived of its motion of rotation, the consideration of which he judged to be totally useless in his investigations, and supposing the atmosphere every where equally dense, and acted on by a star at rest, he determined the oscillations of this fluid. But when he wished to consider the case of a star in motion, the difficulty of the problem obliged him to have recourse, in order to simplify his results, to a precarious hypothesis, and even with such restrictions the results cannot even be considered as approximations. His formulæ gave a constant wind blowing from east to west, of which the expression depends on the initial state of the atmosphere; now the quantities depending on this state ought long since to have disappeared, in consequence of all the causes which would reestablish the equilibrium of the atmosphere, if the action of the stars should cease; consequently we cannot thus explain the trade winds. The treatise of D'Alembert is particularly remarkable for the solutions of some problems on the integral calculus of partial differences, which solutions he successfully applied a year afterwards, to explain the motion of vibrating chords.

The motion of the fluids which cover the planets was a subject almost entirely new, when I undertook in 1772 to discuss it. Assisted by the discoveries made in the calculus of partial differences, and in the theory of the motion of fluids discovered in a great measure by D'Alembert, I published in the Memoirs of the Academy of Sciences for the year 1775, the differential equations of the motions of the fluids which being spread over the earth, are attracted by the Sun and Moon. I first applied these equations to the problem which D'Alembert in vain essayed to resolve, namely, that of the oscillations of a fluid spread over the entire earth, supposed spherical, and without rotation, the attracting star being supposed to be in motion about this planet. I gave the general solution of this problem, whatever might be the density of the fluid and its initial state, supposing that each fluid molecule experiences a resistance proportional to its velocity, which shews that the primitive conditions of motion are at length annihilated by the friction and the small viscosity of the fluid. But an inspection of the differential equations shewed me very soon, that I ought to take into account the rotatory motion of the earth. I therefore considered this motion, and I applied myself particularly to the determination of the oscillations of the fluid, which are independent of its initial state, and the only ones which are permanent. These oscillations are of three kinds. Those of the first kind are independent of the motion of rotation of the earth, and their determination presents few difficulties. The oscillations depending on the motion of rotation of the earth, and of which the period is about a day, constitute the second species; finally, the third species is composed of oscillations, of which the period is very nearly half a day. They surpass the others considerably in our harbours. I have accurately determined those different oscillations, in the case in which it can be determined, and by very convergent approximations, in the other cases. The excess of two consecutive high waters, one over the other in the solstices, depends on the oscillations of the second species. This excess, which is hardly sensible at Brest, ought, according to the theory of Newton, to be very considerable. This great geometer and his successors attributed, as I have already stated, this difference between the formulæ and observations, to the inertia of the waters of the ocean. But analysis shews that it depends on the law of the depth of the sea. I therefore investigated the law which would render this excess nothing, and I found that the depth of the sea ought to be every where constant. The figure of the earth being then supposed to be elliptical, which would render to the sea an elliptic figure of equilibrium, I have given the general expression of the inequalities of the second species: and I have deduced this remarkable proposition, namely, that the motions of the earth's axis are exactly the same as if the sea consituted a solid mass with the earth, which was contrary to the opinion of geometers, and particularly of D'Alembert, who in his *(i)* important Treatise on the precession of the Equinoxes, asserted that, in consequence of the fludity of the sea, it had no influence on this phenomenon. My analysis also indicated to me the general condition of the stability of the equilibrium of the sea. The geometers who considered the equilibrium of a fluid spread over an elliptic spheroid, remarked that if its figure be a little compressed, it does not tend to revert to its first state, except in the case in which the ratio of its density to that of the spheroid, was below $\frac{3}{5}$; and they have inferred from this condition, that of the stability of the equilibrium of the fluid. But in this investigation, it is not sufficient to consider a state of quiescence of the fluid, very near to the state of equilibrium, it is necessary to assign to this fluid some initial motion very small, and then to determine the condition necessary, in order that this motion may be always confined within very narrow limits. By considering the problem in this general point of view I have found, that if the mean density of the earth surpass that of the sea, this fluid, when deranged by any causes from its state of equilibrium, will never deviate from it, except by small quantities; but that the durations may be very considerable, if this condition be not satisfied. Finally, I have determined the oscillations of the atmosphere, on the ocean which it covers, and I have found that the attractions of the Sun and Moon cannot produce the constant motion from east to west, which is observed under the name of *trade winds.* The oscillations of the atmosphere produce *(k)* in the height of the barometer, small oscillations, of which the extent at the equator being only half a millimetre, demands the utmost attention of observers. The preceding observations, though extremely general, are still far from representing accurately the tides, which have been observed in our harbours. They suppose that the surface of the terrestrial spheroid is entirely covered by the sea; now it is evident that the great irregularities of its surface ought to modify considerably the motion of the waters, with which it is only partly covered. Experience shews in fact, that accessary circumstances produce considerable varieties in the heights, and in the hours of high water in the harbours, which are very near to each other. It is impossible to subject these varieties to the calculus, since the circumstances on which they depend are not known, and even if they were, we would not be able to solve the problem, in consequence of its extreme difficulty. However, in the midst of the numerous modifications of the motion of the sea, arising from these circumstances, this motion preserves, with the forces which produce it, relations which are proper to indicate the nature of those forces, and to verify the law of the attractions of the Sun and Moon on the sea. The investigations of these relations between causes and their effects, is not less useful in natural philosophy than the direct solution of problems, as well in verifying the existence of these causes, as also in determining the laws of their effects: we can frequently apply it, and it is like the calculus of probabilities a fortunate supplement to the ignorance and imperfection of the human mind.

In the present question, I make use of the following principle, which may be useful on various occasions. "The state of a system of bodies, in "which the primitive conditions of motion have "disappeared in consequence of the resistances "which this motion experiences, is periodic, like "the forces which actuate the system."

From this I have inferred, that if the sea is sollicited by a periodic force expressed by the cosine of an angle which increases proportionally to the time; there will result from it a partial tide expressed by the cosine of an angle increasing in the same manner, but of which the constant contained under the sign *Cosine*, and the coefficient of this cosine, may be in consequence of accessory circumstances, very different from the same constant quantities in the expression of the force, so that they can be determined by observation only. The expressions of the actions of the Sun and Moon on the sea may be developed into a convergent series of similar cosines. Hence arise so many partial tides, which in consequence of the coexistence of the small oscillations, combine together to form the total tide which is observed in any harbour. It is in this point of view that I have considered the tides in the fourth book of the Celestial Mechanics. In order to connect together the different constants of the partial tides, I have considered each tide as produced by the action of a star, which moves uniformly in the plane of the equator; the tides, of which the period is about half a day, arise from the action of stars, of which the proper motion is very slow, with respect to the rotatory motion of the earth; and as the angle of the cosine, which expresses the action of one of these stars, is a multiple of the rotation of the earth, plus or minus a multiple of the proper motion of the star, and since, besides the constants of the cosines, which express the tides of the two stars, would have the same ratio to the constants of the cosines which express their actions, if the proper motions were equal; I have assumed that the ratio varies from one star to another, proportionally to the difference of the proper motions. The error of this hypothesis, if there be any such, has no sensible influence on the principal results of my computations.

The greatest variations of the height of the tides in our harbours, arise from the action of the sun and moon, being supposed to move uniformly in the plane of the equator. But in order to have the law of these variations, it is necessary to combine the observations in such a manner, that all the other variations may disappear from their result. This is obtained by considering the height of high waters, above the neighbouring very low waters, in the syzygies, and the quadratures, assuming an equal number, near to each equinox and solstice. By this means the tides, independent of the rotation of the earth, and those of which the period is about a day, disappear, and likewise the tides produced by the variation of the distance of the sun from the earth. By considering three consecutive syzygies or solstices, and by doubling the intermediate, the tide produced by the variation of the distance of the moon from the earth is made to disappear ; since if this star be in perigee at one of her phases, it is very nearly in apogee at the other corresponding phase, and the compensation is the more exact, according as a greater number of observations is employed. By this process the influence of the winds on the result of observation becomes very nearly nothing, for if the wind raises the height of one high water, it elevates very nearly by the same quantity the neighbouring low water, and its effect disappears in the difference of those two heights. It is thus that by combining the observations in such a manner that their sum may present only one element, we are enabled to determine successively all the elements of the phenomena. The analysis of probabilities furnishes for the determination of these elements (*l*) a method still more certain, and which may be termed *the most advantageous method.* It consists in forming between these elements, as many equations of condition as there are observations. By the rules of this method, the number of these equations is reduced to that of the elements, which are determined by resolving the equations thus reduced. It is by this process that M. Bouvard has constructed his excellent tables of Jupiter, Saturn and Uranus. But observations relative to the tides are far from having the same accuracy as astronomical observations; the very great number of those which it is necessary to employ, in order, that the errors may compensate each other, does not permit us to apply to them *the most advantageous method.* At the suggestion of the Academy of Sciences, observations on the tides were made in the harbour of Brest, during the space of six consecutive years. It is to those observations published by Lalande, that I have compared my formulæ, in the book already cited. The situation of this harbour is very favourable to this kind of observation. It communicates with the sea by a vast canal, at the extremity of which this port has been constructed. Therefore the irregularities in the motion of the sea, when they arrive at this harbour, are very much diminished, just as the oscillations, which the the irregular motion of a ship produces in a barometer, are lessened by a contraction made in the tube of this instrument. Besides, the tides at Brest being considerable, the accidental variations are only an inconsiderable part of them. Thus, considering the fewness of the observations relative to the tides, a great regularity is observed, which are not affected by the little river which empties itself into the immense road of this harbour. Struck with this regularity, I suggested to government to order a new series of obsvervations relative to the tides to, be made at Brest, which might be continued during the period of the motions of the nodes of the moon. This has been undertaken. These new observations commenced on the first of June, 1806, and they have been continued uninterruptedly since that period. We have examined those of 1807, and of the fifteen following years. The immense computations which the comparison of my analysis with observations required, are due to the indefatigable zeal of M. Bouvard, for every thing which concerns astronomy, near six thousand observations are employed, in order (*m*) to obtain the height of the high waters and their variation, which, near to the *maximum*, is proportional to the square of the time. I have considered near to each equinox and solstice, three consecutive syzygies, between which the equinox or the solstice were included; and the results of the intermediate syzygy, were doubled in order to destroy the effects of the lunar parallax, at the occurrence of each syzygy, the height of the evening high water above the low water of the morning, was taken, on the day which preceded the syzygy, on the very day of the syzygy, and on the four succeeding days; because the *maximum* of the tides occurs very nearly in the middle of this interval. The observations of these heights, made during the day, became more certain and exact. During each of the sixteen years, the sum of the heights of the corresponding days of the equinoctial syzygies has been taken, and a like sum relative to the solstitial syzygies, and from hence the *maxima* of the heights of the high waters, near to the equinoctial and solstitial syzygies has been inferred, and the variations of these heights near to their *maxima*. From an inspection of these heights, and of their variations, the regularity of this kind of observation on the harbour of Brest is immediately apparent.

In the quadratures a similar process has been pursued, with the sole difference that the excess of the height of the morning over the low water of the evening, has been taken on the day of the quadrature, and on the three succeeding days. The increment of the tides at the quadratures departing from their *minimum*, being much more rapid, than the diminution of the syzygial tides in departing from their *maximum;* the law of the variation proportional to the square of the time, ought to be restricted to a shorter interval.

All those heights evidently indicate the influence of the declinations of the Sun and Moon, not only on the absolute heights of the tides, but also on their variations. Several philosophers, and particularly La Lande, has questioned this influence, because instead of considering a great number of observations, they attended only to isolated observations in which the sea, by the effect of accidental causes, was elevated to a great height in the solstices. But the simplest application of the calculus of probabilities, to the results of Mr. Bouvard, is sufficient to shew that the probability of the influence of the declination of the stars is very great, and far superior to that of a great number of facts respecting which there does not exist any doubt.

From the variations of the high waters near to their maxima and minima, the interval at which these maxima and minima follow the syzygies and quadratures, has been inferred, and this interval has been found to be a day and a half very nearly, which perfectly accords with what I deduced from ancient observations in the fourth book of the Celestial Mechanics. The same agreement obtains relative to the magnitude of these *maxima* and *minima*, and with respect to the variations of the heights of the tides, in departing from these points, so that nature after the lapse of a century is found agreeing with herself. The interval to which I allude, depends on the constant quantities involved under the signs of *Cosines*, in the expressions of the two principal tides due to the actions of the Sun and Moon. The corresponding constant quantities of the expressions of the forces are differently modified by accessory circumstances; at the moment of the syzygy, the lunar tide precedes the solar tide, and it is not till a day and a half after that, (*m*) (the lunar tide retarding each day on the solar tide,) these two tidies coincide and thus produce the *maximum* high water. We shall have an adequate conception of the retardation of the highest tides at the instant of syzygy, if we conceive in the plane of a meridian, a canal at the mouth of which the highest tide arrives at the moment of the occurrence of the syzygy, and that it employs a day and a half to arrive at the port situated at the extremity of this canal. A similar modification obtains in the constant quantities, which multiply the cosines, and there results from it an increase in the action of the stars on the sea. I have given in the fourth book of the Celestial Mechanics the means of recognizing this increment, which by the ancient observations I have found to be a tenth part; but although the observations of the tides in the quadratures accord with the observations of the syzygial tides on this subject, I have stated that an element so delicate as this, requires a much greater number of observations. The computations of Mr. Bouvard have confirmed the existence of this increment, and made it very nearly equal to a fourth part, in the case of the Moon. The determination of this relation is necessary to enable us to infer from the observations of the tides, the true relations between the actions of the Sun and of the Moon, on which the phenomena of the precession of the equinoxes and of the nutation of the earth's axis depend. The actions of the stars on the sea being corrected by the increments due (*n*) to accessory circumstances, the nutation is found equal to 9″,4 in sexagesimal minutes; the lunar equation of the tables of the Sun is found equal to 6″8, and the mass of the Moon comes out to be a 75th of the mass of the earth.

These are very nearly the results furnished by astronomical observations. The agreement of values obtained from such different sources is extremely remarkable. It is from a comparison of my formulæ with the *maxima* and *minima* of the observed heights of the seas, that the actions of the Sun and Moon and their increments have been determined. The variations of the heights of the tides near to these points, is a necessary consequence of them; therefore by substituting the values of these actions in my formulæ, we ought to find very nearly the observed variations. This is in fact the case. This agreement is a striking confirmation of the law of universal gravitation. It receives an additional confirmation from observations of the syzygial tides near to the apogee and perigee of the Moon. In the work cited, I have only considered the difference of the heights of the tides in those two positions of the Moon. Here I have moreover considered the variations of these heights in departing from their *maxima*, and on these two points my formulæ coincide with the observations.

The times of high water, and the retardations of the tides from one day to another, present the same varieties as their heights. M. Bouvard constructed tables of them for the tides, which he employed in the determination of the heights. The influence of the declinations of the stars, and of the lunar parallax, are very evident in them. These observations, compared with my formulæ, exhibit the same agreement as the observations relative to the heights. The small anomalies which observations still present may be made to disappear, by a suitable determination of the constant quantities of each partial tide; the principle by which these various constant quantities have been connected together cannot be rigorously exact. Perhaps also, the quantities which have been neglected in adopting the principle of the coexistence of oscillations, become sensible in the great tides. I have barely adverted to those slight inaccuracies, in order to direct those who might wish to resume the computations when observations of the tides, which are making at Brest, and which are deposited at the Royal Observatory, will be sufficiently numerous to enable us to determine with certainty whether these anomalies arise from the errors of observations. But previously to making any modification in the principles which I have employed, it will be necessary to extend farther our analytical approximations. Finally, I have considered the tide, of which the period is about half a day. From a comparison of the differences of the two high and the two consecutive low waters, among a great number of syzigeal solstices, the magnitude of this tide and the hour of its *maximum*, in the harbour of Brest, have been determined. Although its magnitude is not the thirtieth part of the magnitude of the semidiurnal tide, still the forces which produce these two tides are very nearly equal, which shows how differently accessory circumstances affect the magnitude of the tides. We shall not be surprized at this, if we consider that even in the case in which the surface of the earth was regular, and entirely covered by the sea, the daily tide would disappear if the depth of the sea was constant.

The accessory circumstances may also cause the semidiurnal inequalities to disappear, and render the diurnal inequalities very sensible. Then we shall have on each day, but one tide, which disappears when the stars *(o)* are in the equator. This is what takes place at Batsham, a harbour in the kingdom of Tonquin, and in some islands of the south sea.

With respect to those circumstances, it may be observed that the one appertains to the entire sea, and refers to causes operating at a considerable distance from the harbour where they are observed, for instance, there can be no doubt but that the oscillations of the Atlantic ocean, and of the south sea, being reflected by the eastern side of America, which extends almost from one pole to another, has a considerable influence on the tides at the harbour of Brest. It is chiefly on these circumstances that the phenomena depend which are nearly the same in our harbours. Such appears to be the retardation of the highest tides at the moment of syzygy. Other circumstances more nearly connected with the ports, such as the shores or neighbouring straits, may produce the differences, which are observed between the heights and hour of port in harbours which are very near to each other. Hence it follows, that the partial tide has not with the latitude of the harbour, (p), the relation indicated by the force which produces it; since it depends on similar tides corresponding to very distant latitudes, and even to another hemisphere. Therefore the sign and magnitude of the tide can be determined by observation alone.

The phenomena of the tides which I have considered depend on terms arising from the expansion of the action of these stars, divided by the cubes of their distance, which are the only ones that have been hitherto considered. But the Moon is sufficiently near to the Earth to have the terms of the expression of its action divided by the fourth power of its distance, sensible in the results of a great number of observations; for we know from the theory of probabilities, that the number of observations compensates for their want of accuracy, and includes inequalities much less than the errors, of which each observation is susceptible. We can even by this theory, assign the number of observations necessary to acquire a great probability, when the error of the result which has been obtained, is contained within narrow limits. It therefore occurred to me that the influence of the terms of the Moon's action, divided by the fourth power of the distance, might be apparent in the collection of the numerous observations which have been discussed by M. Bouvard. The tides, which correspond to the terms divided by the cube of the distance, do not assign any difference between the high waters of full moon and those of new moon. But those of which the devisor is the fourth power, produce some difference between these tides. They produce a tide, of which the period is about the third part of a day, and observations discussed under this point of view, indicate with a great degree of probability the existence of this partial tide. They also unquestionably prove that the action of the Moon to raise the sea at Brest, is greater when its declination is southern, than when it is northern, which can only arise from the terms of the lunar action, divided by the fourth power of the distance.

It appears from the preceding *expose*, that the investigation of the general relations between the phenomena of the seas, and the actions of the Sun and Moon on the ocean, most fortunately supplies the impossibility of integrating the differential equations of this motion, and our ignorance of the data necessary to determine the arbitrary functions which occur in their integrals; it also follows, that these phenomena have one sole cause, namely, the attraction of these two stars conformably to the law of universal gravitation.

If the earth had no satellite, and if its orbit was circular and situated in the plane of the equator, we should only have in order to recognize the action of the Sun on the ocean, the hour of high water, (which would be always the same,) and the law according to which the sea rises. But the action of the Moon by combining with that of the

Sun, produces in the tides, varieties relative to its phases, the agreement of which with observations, renders the theory of universal gravitation extremely probable. From all the inequalities in the motion, in the declination, and in the distance of these two stars, there arise the phenomena indicated by observation, which places this theory beyond all doubt ; it is thus that varieties in the actions of causes establish their existence.

The action of the Sun and Moon on the Earth, a necessary consequence of the universal attraction, demonstrated by all the celestial phenomena, being directly confirmed by the phenomena of the tides, ought to leave no uncertainty on the subject. It is indeed brought now to such a degree of perfection, that not the least difference of opinion exists upon the subject, among men sufficiently learned in the science of geometry and mechanics, to comprehend its relation with the law of universal gravitation.

A long series of observations, more precise than have hitherto been made, and continued during the period of the revolution of the nodes, will rectify the elements already known, fix the value of those which are uncertain ; and develope phenomena which before were obscured in the errors of observation. The tides are not less interesting to understand than the inequalities of the heavenly bodies, and equally merit the attention of observers. We have hitherto neglected to follow them with sufficient precision, because of the irregularities they present. But I can assert, after a careful investigation, that these irregularities disappear by multiplying the observations; nor is it necessary that their number should be extremely great, particularly at Brest, of which the situation is very favourable to this species of observation.

I have now only to speak of the method of determining the time of high water, on any day whatever. We should recollect, that each of our ports may be considered as the extremity of a canal, at whose *embouchure* the partial tides happen at the moment of the passage of the Sun and Moon over the meridian, and that they employ a day and a half to arrive at its extremity, supposed eastward of its embouchure, by a certain number of hours. This number is what I call the *fundamental hour* of the port. It may easily be computed from the hour of the establishment of the port, by consider- the former as the hour of the full tide, when it coincides with the syzygy. The retardation of the tides, from one day to another, being then 2705″, it will be 3951″ for one day and a half, which quantity is to be added to the hour of the establishment, to have the fundamental hour. Now, if we augment the hours of the tides at the *embouchure* by the fifteen hours, *plus* the fundamental hour, we shall have the hours of the corresponding tides in our ports. Thus, the problem consists in finding the hours of the tides in a place whose longitude is known, on the supposition that the partial tides happen at the instant of the passage of the Sun and Moon over the meridian. For this purpose analysis affords very simple formulæ, which are easily reduced to tables, and very useful to be inserted in the ephemerides that are destined for the use of navigators.

The great tides have frequently produced in harbours, and near shores, disastrous effects which might have been foreseen, if we were previously apprised of the height of these tides. The winds may have on this phenomenon an influence which it is impossible to anticipate. But we can predict with certainty the influence of the Sun and Moon, and this is sufficient most frequently to secure us from the accidents which high tides may occasion, when the direction and force of the wind is combined with the action of regular causes. In order that the maritime departments may participate in the advantage produced by the sciences, the Bureau of longitude publishes each year in its Ephemerides, the table of the syzygial tides, the mean height in the syzygies of the equinoxes being assumed equal to unity.

I have dwelt more particularly on the theory of the tides, because of all the effects of the attraction of the heavenly bodies, it is the most obvious, and most within our reach; besides it appeared of consequence to shew, how by means of a great number of observations, although inaccurate, we can recognize and determine the laws and the causes of the phenomena, the analytical expressions of which it is impossible to determine by the formation and integration of their differential equations. Such are the effects of the solar heat on the atmosphere, in the production of the trade winds and monsoons, and in the regular variations both annual and diurnal, of the barometer and thermometer.

CHAPTER XII.

Of the Oscillations of the Atmosphere.

THE action of the Sun and Moon on the ocean must previously traverse the atmosphere, which must necessarily be subject to their influence, and experience motions similar to those of the sea. Hence result periodic variations in the height of the barometer, and of the winds, the direction and intensity of which are also periodic. These winds are inconsiderable, and nearly insensible in an atmosphere already very much agitated from other causes: the extent of the oscillations of the barometer is not a millimeter at the equator itself, where it is greatest.

In the fourth book of the Celestial Mechanics I have given the theory of all these variations, and I have directed the attention of observers to this subject. It is at the equator that observations on the variations in the height of the barometer ought to be made, not only because they are greater than in any parallel, but also because the changes arising from irregular causes are smaller there. However, as local circumstances considerably increase the heights of the tides in our harbours, they may produce a similar effect in the oscillations of the atmosphere, and also in the corresponding variations of the barometer, it is therefore of importance to be assured of them by observations.

The atmospheric tide is produced by the three following causes; the first is the direct action of the sun and moon on the atmosphere; the second is the periodic elevation and depression of the ocean, which is the moveable base of the atmosphere; finally, the third is the attraction of this fluid by the sea, the figure of which varies periodically. These three causes arise from the same attractive forces of the sun and moon; they have, like their effects, the same periods as these forces, (*a*) conformably to the principle on which I have founded my theory of the tides. The atmospheric tide is therefore subject to the same laws as the tides of the ocean; it is, like to the latter, the combination of two partial tides produced, the one by the action of the Sun, the other by the action of the Moon. The period of the atmospheric solar tide is half of a solar day; and that of the lunar tide is half of a lunar day. The action of the Moon on the sea at Brest being triple (*b*) of the action of the sun, the atmospheric lunar tide is at least double of the solar tide. These observations should guide us in the selection of observations proper to determine such small quantities, and also in the modes of combining them together, so as to abstract as much as possible from the influence of causes which produce great variations in the height of the barometer. For several years the heights of the barometer and thermometer have been observed at nine o'clock A. M., at mid-day, at three o'clock P. M. and at nine o'clock P. M. These observations being made with the same instrument, and almost by the same observer, are from their precision, and their great number, very proper to indicate an atmospheric tide, if it be sensible. In the results of these observations, a diurnal variation of the barometer is indicated very plainly: one month only is sufficient to manifest it. The excess of the greatest observed height of the barometer, which occurs at nine o'clock, A. M., over the least height, which happens at three o'clock P. M., is at Paris eight-tenths of a millemetre, according to the mean result of observations made each successive day during six consecutive years. As the height of the barometer due to the solar tide, becomes the same at the same hour of each day; this tide is confounded with the diurnal variation, which it modifies, so that it cannot be distinguished by observations made at the Royal Observatory. This is not the case with respect to the barometric heights due to the lunar tide, and which regulating itself by lunar hours, does not become the same at the same solar hours, until after the lapse of half of a month. The observations of which I have spoken being compared from one half month to another, are arranged in the most advantageous manner for indicating the lunar tide. If, for example, the *maximum* of this tide occurs at one o'clock A. M. on the day of thc syzygy, its *minimum* will happen towards three o'clock P. M. The contrary will be the case on the day of the quadrature. This tide will therefore increase the daily variation of the first of these days ; it will diminish the daily variation of the second ; and the difference of these variations (*e*) will be twice the height of the atmospheric lunar tide. But as the *maximum* of this tide does not take place at nine o'clock A. M. in the syzygy, it is necessary, in order to determine its magnitude, and the hour it happens, to employ barometrical observations made at nine o'clock A. M., at midday, and at three o'clock P. M. for each day, both of the syzygy and of the quadrature : We may likewise make use of observations made on the days which precede or which follow those phases by the same number of days, and make all the observations of the year concur in the determination of these delicate elements.

An important observation may be made here, without which it would have been impossible to recognise so inconsiderable a quantity as the lunar tide, in the midst of the great variations of the barometer. The more the observations approach to each other, the less sensible will be the effects of these variations; it is almost nothing on a result inferred (*d*) from observations made on the same day, and in the short interval of six hours. The barometer varies sufficiently slow, as not to derange in a sensible manner the effects of regular causes. This is the reason why the mean result of the daily variation of each respective year is always very nearly the same, although differences to the amount of several millemetres may exist in the mean absolute barometrical heights of different years: so that if the mean height of nine o'clock A. M. of one year, be compared with the mean height of three o'clock P. M. of another year; a diurnal variation will result frequently very erroneous, and even sometimes affected with a sign the contrary of the true sign. It is therefore of importance, in order to determine such very small quantities, to deduce them from observations made on the same day, and to take the mean between a great number of observations thus obtained. Consequently we cannot determine the lunar tide, except by a system of observations made on each day, at three different hours at least according to the method followed at the observatory.

M. Bouvard wished to insert in his registers, barometric observations made on the respective days of each quadrature and syzygy, and also on the day which precedes those phases, and on the first and second days which follow it. They embrace the eight years which have lapsed from the first of October, 1815, to the first of October,

1823. I have made use of the observations of nine o'clock in the morning, of midday, and of three o'clock P. M. However I did not take into account observations made at nine o'clock P. M., in order to diminish as much as possible, the interval at which observations are made. Besides those of the three first hours, which have been specified, were made more exactly at the time pointed out, than those made at nine o'clock P. M., and moreover the barometer being illuminated by the light of the day at the three first hours, the difference (*e*) which may arise from the different manner in which the instruments are illuminated, disappears. From a comparison of these numerous results (which embrace an interval of 1584 days,) with my observations, I have found that the magnitude of the lunar atmospheric tide is an eighteenth part of a millimetre, and the time of its maximum, on the evening of the day of the syzygy, is three hours and a quarter.

It is here particularly that the necessity is apparent of employing a great number of observations, of combining them in the most advantageous manner, and of having a method for determining the probability that the errors of the results, which (*f*) are obtained, are confined within narrow limits, without which we would be liable to present as laws of nature the effects of irregular causes, which is frequently the case in meteorology. I have given this method in my analytical theory of probabilities. And in the application of it to observations, I have determined the law of the anomalies of the diurnal variation of the barometer, and I have ascertained that we cannot without every appearance of improbability, attribute the preceding results to these anomalies solely: it is probable that the lunar atmospheric tide diminishes the diurnal variation in the syzygies, and that it increases it in the quadratures, but it is so inconsiderable that in the limits, this tide does not produce a variation in the height of a barometer of an eighteenth of a millimetre, more or less; which shews, that the action of the Moon on the atmosphere, is nearly insensible at Paris. Although these results have been obtained from 4752 observations, the method already adverted to, shews that in order to secure the requisite probability, and to obtain with sufficient accuracy such a small element as the lunar atmospheric tide, it is necessary to employ at least forty thousand observations. One of the principal advantages of this method is, that it indicates to what extent it is necessary to multiply observations, in order that no reasonable doubt may rest on their results.

It follows from the laws of the anomalies of the diurnal variation of the barometer, which I obtained, that there is a probability of $\frac{1}{2}$, or of one to one, that the daily variation from 9 o'clock A. M. to three o'clock P. M. will be constantly positive in its mean result for each month of 30 days, during 75 consecutive months. I have requested M. Bouvard to examine whether this is the case for each of the 72 months of the six years which have lapsed from the first of January 1817 to the first of January 1823, from which he inferred that the mean diurnal variation was equal to 0^{m}.,^{m}801. A comparison of his observations has given the most probable result, namely, that the mean diurnal variation of each month has been always positive.

What is the respective influence on the lunar tide, of the three causes already cited of the atmospherical tide; it is difficult to give an answer to this question. However the little density of the sea comparatively to the mean density of the earth, does not permit us to ascribe a sensible effect to the periodic change of its figure. Without local circumstances, the direct effect of the action of the Moon would be insensible in our latitudes. These circumstances have indeed a great influence on the height of the tides in our harbours; but as the atmospheric fluid is diffused about the earth, much less irregularly than the sea, their influence on the atmospheric tide must be much less than on the tide of the ocean. From these considerations I am induced to (*f*) consider the periodic elevation or depression of the sea, as the principal cause of the lunar atmospheric tide in our climates. Barometric observations made every day in the harbours, where the sea ascends to a considerable height, would throw considerable light on this curious point of meteorolgy.

It may be remarked here, that the attraction of the Sun and Moon, does not produce either in the sea or in the atmosphere, any constant motion from east to west; that which is observed between the tropics, under the name of *trade* *winds*, must therefore arise from some other cause, the following appears to be the most probable.

The Sun, which for greater simplicity, we shall suppose in the plane (*h*) of the equator, rarifies by its heat the strata of the air, and makes them to ascend above their true level; they must therefore in consequence of their greater weight subside, and move towards the poles in the higher regions of the atmosphere; but at the same time a fresh current of air must arrive in the lower regions from the poles, in order to supply that which has been rarified at the equator. There is thus established two currents of air, blowing in opposite directions, the one in the inferior, and the other in the higher region of the atmosphere; but the actual velocity of the air, arising from the rotation of the earth, is always less according as it is nearer to the pole; it must therefore, as it approaches towards the equator, revolve slower than the corresponding parts of the earth, and bodies placed on the surface of the earth, must strike it with the excess of their velocity, and thus experience from its reaction, a resistance contrary to their motion of rotation. Therefore to an observer, who considers himself as immoveable, the air appears to blow in a direction opposite to that of the earth's rotation, *i. e.* from east to west; this is in fact the direction of the trade winds.

If we consider all the causes which derange the equilibrium of the atmosphere, its great mobility arising from its elasticity and mobility, the influence of heat and cold on its elasticity, the immense quantity of vapours with which it is alternately charged and unloaded, finally, the changes which the rotation of the earth produces in the relative velocity of its molecules, from this alone that they are displaced in the direction of the meridians, we will not be astonished at the variety of its motions, which it will be extremely difficult to subject to certain laws.

CHAP. XIII.

Of the Precession of the Equinoxes, and of the Nutation of the Axis of the Earth.

Every part of nature is linked together, and its general laws connect phenomena with each other, which appear to be altogether distinct. Thus, the rotation of the terrestrial spheroid compresses the poles, and this compression, combined with the action of the Sun and Moon, produces the precession of the equinoxes, which, before the discovery of universal gravitation, did not appear to have any connection with the diurnal motion of the Earth.

Let us suppose this planet to be an homogeneous speroid, protuberant at the equator, it may then be considered as composed of a sphere of a diameter equal to the axis of the poles, and of a meniscus surrounding the sphere, of which the greatest thickness corresponds with the equator of the spheroid. The particles of this meniscus may be considered as so many small moons adhering together, and making their revolutions in a period equal to the revolution of the Earth on its axis.

The nodes of all their orbits should therefore have a retrograde motion, arising from the action of the Sun, in the same manner as the nodes of the lunar orbit; and from the connection of these bodies together, there should arise a motion of the whole meniscus which would make its points of intersection with the ecliptic to retrograde, but this meniscus imparts to the sphere to which it is attached, its retrograde motion, which, for this reason, becomes slower; the intersection of the equator and the ecliptic, that is to say, the equinoctial points, should consequently have a retrograde motion. Let us endeavour to investigate both the law and the cause of this phenomenon.

And first let us consider the action of the Sun upon a ring, situated in the plane of the equator. If we conceive the mass of the Sun to be distributed uniformly over the circumference of its orbit, (supposed circular) it is evident that the action of this solid orbit will represent the mean action of the Sun. This action, on every one of the points of the ring above the ecliptic, being decomposed into two, one in the plane of the ring, and the other perpendicular to it, (*a*) it follows that the resulting force, arising from these last actions, on all the particles of the ring, is perpendicular to its plane, and situated on that diameter of the ring, which is perpendicular to the line of its nodes. The action of the solar orbit, on the part of the ring below the ecliptic, produces also a resulting force, perpendicular to the plane of the ring, and situated in the inferior part of the same diameter. These two resulting forces combine to draw the ring towards the ecliptic, by giving it a motion round the line of nodes; its inclination, therefore, to the ecliptic, would be diminished by the mean action of the Sun, the nodes all the time continuing stationary; and this would be the case but for the motion of the ring, which we now suppose to revolve in the same time as the Earth. In consequence of this motion, the ring is enabled to preserve a constant inclination to the ecliptic, and to change the effect of the action of the Sun, into a retrograde motion of the nodes. It gives to the nodes a variation, which otherwise would be in the inclination, and it gives to the (*b*) inclination a permanency, which otherwise would rest with the nodes. To conceive the reason of this singular effect, let us suppose the situation of the ring varied by an infinitely small quantity, in such a manner, that the planes of its two positions may intersect each other, in a line perpendicular to the line of the nodes.

At the end of any instant whatever, we may decompose the motion of each of its points into two, one of which should subsist alone in the following instant, the other being perpendicular to the plane of the ring, and which should therefore be destroyed. It is evident that the resulting force of these second motions relative to all the points of the upper part of the ring, will be perpendicular to its plane, and placed on the diameter which we just now considered, and this is equally true for the lower part of the ring. That this resulting force may be destroyed by the action of the solar orbit, and that the ring, by virtue of these forces, may remain in equilibrio on its centre, it is requisite that these forces should be contrary to each other, and their moments, relatively to this point, equal. The first of these conditions requires that the change of position, supposed to be given to the ring, be retrograde; the second condition determines the quantity of this change, and consequently the velocity of the retrograde (*c*) motion of the nodes. And it is easily demonstrated, that this velocity is proportional to the mass of the Sun, divided by thecube of its distance from the Earth, and multiplied by the cosine of the obliquity of the ecliptic.

Since the planes of the ring, in its two consecutive positions, intersect each other in a diameter perpendicular to the line of its nodes, it follows that the inclination of these two planes to the ecliptic is constant; therefore the inclination of the ring does not vary in consequence of the mean action of the Sun.

That which has been explained relatively to a ring, may be demonstrated by analysis, to hold true in the case of a spheroid, differing but little from a sphere. The mean action of the Sun produces in the equinoxes a motion proportional to its mass, divided by the cube of its distance, and multiplied by the cosine of the inclination to the ecliptic. This motion is retrograde when the spheroid is flattened at the poles; its velocity depends on the compression of the spheroid, but the inclination of the equator to the ecliptic always remains the same.

The action of the Moon produces likewise a similar retrogradation of the nodes of the terrestrial equator in the plane of its orbit; but the position of this plane and its inclination to the equator incessantly varying, by the action of the Sun, and as the retrograde motion of the nodes of the equator on the lunar orbit, produced by the action of the Moon, is proportional to the cosine *(d)* of this inclination, this motion is consequently variable.

Besides, even supposing it uniform, it would, according to the position of the lunar orbit, cause a variation both in the retrograde motion of the equinoxes, and in the inclination of the equator to the ecliptic. A calculation, by no means difficult, is sufficient to show, that the action of the Moon, combined with the motion of the plane of its orbit, produces. 1st A *(e)* mean motion in the equinoxes, equal to that which it would produce if it moved in the plane of the ecliptic. 2dly. An inequality *subtractive*, from this retrograde motion, and proportional to the sine of the longitude of the ascending node of the lunar orbit. 3dly. A diminution in the obliquity of the ecliptic, proportional to the cosine of this same angle. These two inequalities are represented at once by the motion of the extremity of the terrestrial axis (prolonged to the heavens) round a small elipse, conformably to the laws explained in Chap. XII. of Book I. The greater axis of this ellipse is to the lesser, as the cosine of the obliquity of the ecliptic is to the cosine of double this obliquity. We may comprehend from what has been said, the cause of the precession of the equinoxes, and of the nutation of the Earth's axis, but a rigorous calculation, and a comparison of its results with observation, is the best test of the truth of a theory. That of universal gravitation is indebted to d'Alembert, for the advantage of having been thus verified in the case of the two preceding phenomena. This great mathematician first determined, by a beautiful analysis, the motions of the axis of the Earth, on the supposition that the strata of the terrestrial spheroid were of any density or figure whatever, and he not only found his results exactly conformable to observation, but obtained an accurate determination of the dimensions of the small ellipse described by the pole of the Earth, with respect to which the observations of Bradley had left some little doubt.

The influence of a heavenly body, either upon the motion of the axis of the Earth, or upon the ocean, is always proportional to the mass of that body, divided by the cube of its distance from the Earth. The nutation of the Earth's axis being due to the action of the Moon alone, while the mean precession of the equinoxes arises from the combined actions of the Sun and Moon, it follows that the observed values of these two phenomena, should give the ratio of their respective actions (*f*). If we suppose, with Bradley, the annual precession of the equinoxes to be 154″4, and the entire extent of the nutation equal to 55″6, the action of the Moon would be found to be double that of the Sun. But a very small difference in the extent of the nutation, produces a very considerable one in the ratio of the actions of these two bodies. The most accurate observations give for this extent 58″,02 hence it results that $\frac{1}{75}$ expresses the ratio of the mass of the Moon to that of the Earth.

The phenomena of the precession and of the nutation, throw a new light on the constitution of the terrestrial spheroid. They give a limit to the compression of the earth supposed elliptic, for it appears from them that this compression does not exceed $\frac{1}{247,7}$, which accords with the experiments that have been made on pendulums. We have seen in Chap. VII. that there exists in the expression of the radius vector of the terrestrial spheriod, terms, which, but little sensible in themselves, and on the length of the pendulum, cause the degrees of the meridian to deviate considerably from the elliptic figure. These terms disappear entirely in the values of the precession and nutation, and for this reason, these phenomena agree with the experiments on pendulums. The existence, of these terms, therefore reconciles the observations of the lunar parallax, those of the pendulums and degrees of the meridian, and the phenomena of precession and nutation.

Whatever figure and density we may suppose in the strata of the Earth, whether or not it be a solid of revolution, provided it differs little from a sphere, we can always assign an elliptic solid of revolution, with which the precession and nutation will always be the same. Thus in the hypothesis of Bouguer, of which we have spoken in Chap. VII, and according to which the increase of the degrees varies as the fourth power of the sine of the latitude, these phenomena are exactly the same as if the Earth was an ellipsoid, whose ellipticity was $\frac{1}{183}$, but we have seen that observations do not permit us to suppose a greater ellipticity than $\frac{1}{247,7}$, so that these observations, and the experiments on pendulums, combine to disprove the hypothesis of Bouguer.

We have hitherto supposed the Earth entirely solid, but this planet being covered in a great part by the waters of the ocean, ought not their action to change the phenomena of the precession and nutation? It is of importance to consider this question.

The ocean, in consequence of its fluidity, is obedient to the action of the Sun and of the Moon. It seems at first sight that their re-action should not affect the axis of the Earth. D'Alembert and every subsequent mathematician, who has investigated these motions, have entirely neglected it, they have even commenced from that point, to reconcile the observed quantity of the precession and nutation, with the measures of the terrestrial degrees. Nevertheless, a more profound examination of this question has shewn us, that the fluidity of the waters of the sea is not a sufficient reason why their effect in the precession of the equinoxes should be neglected; for if on one hand, they obey the action of the Sun and Moon, on the other, the force of gravity tends to bring them back without ceasing, to a state of equilibrium, and consequently permits them to make but small oscillations; it is therefore possible, that by their attraction and pression on the spheriod which they cover, they may communicate, at least in part, the same motion to the axis of the Earth, which they would, if they could possibly become solid. Besides, we may, by simple reasoning, be convinced that their action is of the same order as action of the Sun and Moon, on the solid part of the Earth.

Let us suppose this planet to be homogeneous and of the same density as the ocean, and moreover, that the waters assume at every instant the figure that is requisite for the equilibrium of the forces that animate them. If in these hypotheses the Earth should suddenly become entirely fluid, it would preserve the same figure, all its parts would remain in equilibrio, and the axis of the Earth would have no tendency to move; now it is evident that the same should be the case, if a part of this mass formed by becoming solid, the spheroid which the ocean covers. The preceding hypotheses serve as a foundation to the theories of Newton, relatively to the figure of the Earth (*g*) and of the tides.

It is remarkable, that among the infinite number of those which may be chosen on this subject, this great geometrician has selected two which neither give the precession nor the nutation; the re-action of the waters destroying the effect of the action of the Sun and Moon upon the terrestrial nucleus, whatever may be its figure. It is true that these two hypotheses, particularly the last, are not conformable to nature, but we may see, *à priori*, that the effect of the re-action of the waters, although different from that which takes place in the hypothesis of Newton, is nevertheless of the same order.

The investigations which I have made on the oscillations of the ocean, have enabled me to determine this effect of the re-action of the waters in the true hypotheses of nature, and have led to this remarkable theorem.

Whatever may be the law of the depth of the ocean, and whatever be the figure of the spheroid which it covers, the phenomena of the precession and nutation will be the same as if the ocean formed a solid mass with this spheroid.

If the Sun and Moon acted only on the Earth, the mean inclination of the equator to the ecliptic would be constant, but we have seen that the action of the planets continually changes the position of the terrestrial orbit, and produces a diminution of its obliquity to the equator, which is fully confirmed by observations ancient and modern, the same cause gives to the equinoxes, a direct annual motion of 0″9659; thus the annual precession produced by the action of the Sun and Moon, is diminished by this quantity in consequence of the action of the planets; without this action it would be 155″5927. These effects of the action of the planets are independent of the compression of the terrestrial spheroid, but the action of the Sun and Moon upon this spheroid, modifies these effects and changes their laws.

If we refer to a fixed plane, the position of the orbit of the Earth, and the motion of its axis of rotation, it will appear, that the action of the Sun in consequence of the variations of the ecliptic, will produce in this axis an oscillatory motion similar to the nutation, but with this difference, that the period of these variations being incomparably longer than that of the variations of the plane of the lunar orbit, the extent of the corresponding oscillation in the axis of the Earth, is much greater than in the nutation. The action of the Moon produces in this same axis a similar oscillation, because the mean inclination of its orbit to that of the Earth, is constant. The displacement of the ecliptic, by being combined with the action of the Sun and Moon upon the Earth, produces upon its obliquity to the equator, a very different variation from that which would arise from this change of position only: the entire extent of this variation would be, by this alteration of the ecliptic, about twelve degrees, however in consequence of the action of the Sun and Moon, it is reduced to about three degrees.

The variation in the motion of the equinoxes, produced by these same causes, changes the duration of the (*h*) tropical year in different cenutries, The duration diminishes as this motion augments, which is the case at present, so that the actual length of the year is now shorter by about 13′, than in the time of Hipparchus. But this variation in the length of the year has its limits, which are also restricted by the action of the Sun and Moon, upon the terrestrial spheroid. The extent of these limits which would be about 500″, in consequence of the alteration in the position of the ecliptic, is reduced to 120″ by this action.

Lastly, the day itself, such as we have defined it in the First Book, is subject by the displacement of the ecliptic, combined with the action of the Sun and Moon, to very small variations, which though indicated by the theory, are quite insensible to observation. According to this theory, the rotation of the Earth is uniform, and the mean length of the day may be supposed constant, an important result for astronomy, as it is the measure of time, and of the revolutions of the heavenly bodies. If it should undergo any change, it would be recognized by the durations of these revolutions, which would be proportionably increased or diminished, but the action of the heavenly bodies does not produce any sensible alteration.

Nevertheless, it might be imagined that the trade winds which blow constantly from east to west between the tropics, would diminish the velocity of the rotation of the Earth, by their action on the continents and mountains. It is impossible to submit this action to analysis; fortunately it may be demonstrated that this action on the rotation of the Earth is nothing, by means of the principle of the conservation of areas, which we have explained in the Third Book. According to this principle, (*i*) the sum of all the particles of the Earth, the ocean and the atmosphere, multiplied respectively by the areas which their radii vectores describe round the centre of gravity of the Earth, projected on the plane of the equator, is constant in a given time.

The heat of the Sun can produce no effect, because it dilates bodies equally in every direction; now it is evident, that if the rotation of the Earth should diminish, this sum would be less. Therefore the trade winds, which are produced by the heat of the Sun, cannot alter the rotation of the Earth. The same reasoning shews us that the currents of the sea ought not to produce any sensible change. To produce any perceptible alteration in its period, some great change must take place in the parts of the terrestrial spheroid: thus a great mass taken from the poles to the equator, would make this rotation longer, it would become shorter if the denser materials were to (*k*) approach the centre or axis of the Earth; but we see no cause that can displace such great masses to distances considerable enough to produce any variation in the length of the day, which may be regarded as one of the most constant elements in the system of the world. This is likewise the case with respect to the points where the axis of rotation meets the surface. If the Earth revolved successively about different diameters, making with each other considerable angles, the equator and the poles would change their positions on the Earth; and the ocean, flowing continually towards the new equator, would alternately overwhelm and then abandon the highest mountains: but all the investigations which I have made upon this change of position in the poles, have convinced me that it is insensible.

CHAP. XIV.

On the Libration of the Moon.

We have now only to explain the cause of the libration of the Moon, and of the motion of the nodes of its equator.

The Moon, in virtue of its motion of rotation, is a little flattened at its poles; but the attraction of the Earth must have lengthened a little that axis which is turned towards it. If the Moon was homogeneous and fluid, it would (to be in equilibrio) assume the form of an ellipsoid, of which the lesser axis passed through the poles of rotation; (*a*) the greater axis would be directed to the Earth, and in the plane of the lunar equator, and the mean axis would be situated in the same plane, perpendicular to the other two. The excess of the greatest above the least axis would be quadruple the excess of the mean above the least, and nearly equal $\frac{1}{27640}$, the least axis being taken as unity.

We may easily conceive that if the greater axis of the Moon deviates a little from the direction of the radius vector, which joins its centre with that of the Earth, the terrestrial attraction will tend to bring it down to this radius, in the same manner as gravity brings a pendulum towards the vertical. If the primitive motion of rotation of this satellite had been sufficiently rapid to have overcome this tendency, the period of its rotation would not have been perfectly equal to that of its revolution, and the difference would have discovered to us (*b*) successively every point in its surface. But at their origin the angular motions of rotation and revolution having differed but little, the force by which the greater axis of the Moon tended to deviate from the radius vector, was not sufficient to overcome the tendency of this same axis towards the radius, due to the terrestrial gravity, which by this means has rendered their motions rigorously equal, and in the same manner as a pendulum, drawn aside from the vertical by a very small force, continually returns, making small vibrations on each side of it, so the greater axis of the lunar spheroid ought to oscillate on each side of the mean radius vector of its orbit. Hence would arise a motion of libration, of which the extent would depend on the primitive difference between the angular motions of rotation and revolution of the Moon. This difference must have been very small, since it has not been perceived by observation.

Hence we see that the theory of gravitation explains in a sufficiently satisfactory manner, the rigorous equality of these two mean motions of rotatiou and revolution of the Moon. It would be against all probability to suppose that these two motions had been at their origin perfectly equal, but for the explanation of this phenomenon, it is enough to assume that their primitive difference was but small, and then the attraction of the Earth would establish the equality which at present subsists.

The mean motion of the Moon being subject to great secular inequalities, which amount to several circumferences, it is evident that if its mean motion of rotation was perfectly uniform, this satellite would, by virtue of these inequalities, present successively to the Earth every point on its surface, and its apparent disk would change by imperceptible degrees, in proportion as these inequalities were developed; the same observers would see pretty nearly the same hemisphere, and there would be no considerable difference, except to observers separated by an interval of several ages. But the cause which has thus established an equality between the mean motions of revolution and rotation, must take away all hope from the inhabitants of the Earth, of seeing the opposite side of the lunar hemisphere. The terrestrial attraction, by continually drawing towards us the greater axis of the Moon, causes its motion of rotation to participate in the secular inequalities of its motion of revolution, and the same hemisphere to be constantly directed towards the Earth.

The same theory ought to be extended to all the satellites, in which an equality between their motions of rotation and of revolution round their primary, has been observed.

The singular phenomenon of the coincidence of the nodes of the equator of the Moon, with those of its orbit, is another consequence of the terrestrial attraction. This was first demonstrated by Lagrange, who by a beautiful analysis was conducted to a complete explanation (*c*) of all the observed phenomena of the lunar spheroid. The planes of the equator and of the orbit of the Moon, and the plane passing through its centre parallel to the ecliptic, have always very nearly the same intersection; the secular motions of the ecliptic neither alter the coincidence of the nodes of these three planes, nor their mean inclination, which the attraction of the Earth constantly maintains the same.

We may observe here, that the preceding phenomena cannot subsist with the hypothesis in which the Moon, originally fluid and formed of strata of different densities, should have taken the figure suited to their equilibrium. They indicate between the axes of the Moon, a greater inequality than would take place in this hypothesis. The high mountains which we observe at the surface of the Moon, have without doubt a sensible influence on these phenomena, and so much the greater as its ellipticity is very small, and its mass inconsiderable.

Whenever nature subjects the mean motions of the celestial bodies to determinate conditions, they are always accompanied by oscillations, whose extent is arbitrary. Thus the equality of the mean motions of revolution and rotation produces a real libration in this satellite. In like manner the coincidence of the mean nodes of the equator and lunar orbit, is accompanied by a libration of the nodes of this equator round those of the orbit, a libration so small as hitherto to have escaped observation. We have seen that the real libration of the greater lunar axis is insensible, and it has been observed, (Chap. VI.) that the libration of the three satellites of Jupiter is also insensible. It is remarkable, that these librations, whose extent is arbitrary, and which might have been considerable, should nevertheless be so very small; we must attribute this to the same causes which originally established the conditions on which they depend.

But relatively to the arbitrary quantities, which relate to the initial motion of the rotation of the celestial bodies, it is natural to think that without foreign attractions, all their parts, in consequence of the friction and resistance which is opposed to their reciprocal motion, would, in process of time, acquire a permanent state of equilibrium, which cannot exist but with an uniform motion of rotation round an invariable axis; so that observation should no longer indicate in this motion, any other inequalities than those derived from these attractions. The most exact observations show that this is the case with the Earth, the same result extends to the Moon, and probably to the other celestial bodies.

If the Moon had encountered a comet (which according to the theory of chances ought to happen in the immensity of time), their masses must have been very minute ; for the impact of a comet, which would only be the hundredth millioneth part of that of the earth, would be sufficient to render the real libration of this satellite sensible, which however is not perceived by observations. This consideration, combined with those which we (*d*) have presented in the fourth chapter, ought to satisfy those astronomers who apprehend that the elements of their tables may be deranged by the action of these bodies.

The equality of the motions of rotation, and of revolution, furnishes the astronomer, who may wish to describe its surface, a universal meredian, (*e*) suggested by nature, and easy to be found at all times, an advantage which geography has not in the description of the earth. This meridian is that which passes through the Poles of the Moon, and through the extremity of its greater axis, always very nearly directed towards us. Although this extremity is not distinguished by any spot, still its position at each instant may be fixed, by considering that it coincides with the line of the mean nodes of the lunar orbit, when the line itself coincides with the mean place of the Moon. The situation of different spots of the Moon have been thus determined as exactly as that of many of the most remarkable places on the earth .

CHAP. XV.

Of the proper motions of the fixed stars.

After having considered the motions of the bodies composing the solar system, it remains to examine those of the stars, all of which ought in consequence of the universal gravitation of matter, to tend towards each other, and describe immense orbits. Already observations have indicated (*a*) these great motions, which probably in part arise from the motion of translation of the solar system, which motion, according to the laws of optics, is transferred in a contrary direction to the stars. When a great number of them are considered together, as their real motions have place in every direction, they ought to disappear in the expression of the motion of the Sun, which is inferred from a consideration of their proper observed motions taken collectively. By this means we have recognised that the system of the Sun, and of every thing which surrounds it, is carried towards the constellation Hercules, with a velocity at least equal to that (*b*) of the earth in its orbit. But very exact and multiplied observations, made for the interval of one or two centuries, will determine exactly this important and delicate point of the system of the world.

Besides these great motions of the Sun and of the Stars, we observe particular motions in several stars, which are called *double.* Thus two stars are termed, which being very near, appear to constitute but one, in telescopes whose magnifying power is inconsiderable. Their apparent proximity may arise from their being very nearly in the same visual ray. But a similar disposition is itself an index of their real proximity ; and if moreover their proper motions are considerable, and differ little in right ascension and declination, it becomes extremely probable that they constitute a system of two bodies very near to each other, and that the small differences of their proper motions arise from a motion of revolution of each of them, about their common centre of gravity: without this, the simultaneous existence of these three circumstances, (*c*) namely the apparent proximity of these two stars, and their motions both in right ascension and declination being nearly equal, would be altogether improbable.

The 61[me] of the swan and the star next to it, combine these three conditions in a remarkable manner: the interval which separates them is only 60″; their proper annual motions from the time of Bradley to the present day, have been 15″75, and 16″,03 in right ascension, and 10″,24 and 9″,56 in declination: it is therefore very probable that these two stars are very (*d*) near to each other, and that they revolve about their common centre of gravity in the period of several ages. The direction of their proper motions being almost contrary to that of the motion of the solar system, seems to indicate that they are at least in a great part an optical illusion due to this last motion; and as they are very considerable, the annual parallax of those two stars ought to be one of the greatest. If we could succeed in determining it, we would obtain by the time of their revolution, the one about the other, the sum of their (*e*) masses relatively to those of the Sun and of the Earth.

The contemplation of the heavens exhibits also several groups of brilliant stars comprised in a very small space; such is that of the Pleades. A like disposition indicates, with much probability, that the stars of each group are very near, relatively to the distance which separates them from the other stars, and that they have about their common centres of gravity, motions which the progress of time will make known.

BOOK THE FIFTH.

SUMMARY OF THE HISTORY OF ASTRONOMY.

Multi pertransibunt et augebitur scientia.

BACON.

THE principal phenomena of the system of the world have been detailed in the preceding books, according to the simplest and most direct analytical order. The appearances of the celestial motions were first considered, and then their mutual comparison conducted us to the discovery of the real motions which produced them. In order to arrive at the principal regulator of those motions, it was necessary to know the laws of the motion of matter, and accordingly, these have been developed in all their detail. By applying them to the bodies and motions observed in the solar system, it was ascertained that there exists not only between these bodies, but also between their smallest molecules, an attraction which varies as the respective masses divided (*a*) by the square of their mutual distance. Finally, proceeding in a reverse order, from this universal force to its effects, it was shewn that not only all the known phenomena, and also those *merely* perceived by astronomers, but likewise a great number of others *entirely* new, which subsequent observation has verified, arise from this source. This indeed is not the order according to which those results were discovered. The preceding method supposes that we have exhibited before our view the entire series of ancient and modern observations, and that in comparing them together, and in deducing from them, the laws of the heavenly motions, and the causes of their inequalities, we have employed all the resources which are now furnished by analysis and mechanics. But as our knowledge in these two departments of science has advanced concurrently with the improvements made in Astronomy, their condition at its various epochs, must necessarily have influenced our astronomical theories. Several hypotheses have been successively adopted, although directly contrary to the known laws of mechanics ; but of many of those laws, even to this very day we are ignorant, so that it should not be a matter of surprize if, in consequence of this ignorance, difficulties have been raised against the true system of the world, interspersed as it is on all sides with such complicated phenomena. Hence the progress of our astronomical knowledge has been frequently embarrassed, and the evidence of our acquirement in this science has been rendered doubtful, from the truths with which it was enriched, being combined with errors, which nothing but time, observation, and the progress of the other sciences could separate from it. We proceed to give, in the following book, a summary of its history, and in this account we shall have occasion to observe how, after remaining for a long series of years, in its infancy, it sprung up and flourished in the Alexandrian school; that then it remained stationary, until the time of the Arabs, who improved and advanced it by their observations; and that, finally passing from Asia and Arabia, where it originated, it settled in Europe, where in less than three centuries it has obtained the eminence which it now holds among the sciences. This detail of the most sublime of the natural sciences will furnish the best excuse for the aberrations of the human mind in the invention of Astrology, which from the remotest antiquity has every where occupied the attention of ignorant and timid man, but which the improvements in this science have for ever dissipated.

CHAP. I.

Of the Astronomy of the Ancients, till the Foundation of the Alexandrian School.

The view of the firmament must at all times have arrested the attention of mankind, and more particularly in those happy climates, where the serenity of the air invited them to observe the stars. Agriculture required, that the seasons should be distinguished and their returns known. It could not be long before it was discovered that the rising and setting of the principal stars, when they are immersed in the Sun's rays, or when they are again extricated from his light, might answer this purpose. Hence we find that among most nations, this species of observations may be traced back to such early times, that their origin is lost. But some rude remarks on the rising and setting of the stars, could not constitute a science. Astronomy did not commence till anterior observations being registered and compared, and the celestial motions examined with greater care, some attempt was made to explain their motions and their laws.

The motion of the Sun in an orbit inclined to the equator; the motion of the Moon, the cause of its phases and eclipses, the knowledge of the planets and their revolutions, and the sphericity of the Earth, were probably the objects of this ancient astronomy; but the few monuments, that remain of it, are insufficient to determine either its epoch or its extent. We can only judge of its great antiquity, by the astronomical periods which have come down to us, and which suppose a series of observations so much the longer, as they were more imperfect. Such has been the vicissitude of human affairs, that printing, the art, by which alone the events of past ages can be transmitted in a durable manner, being of modern invention, the remembrance of the first inventors in the arts and sciences has been entirely effaced. Great nations, whose names are hardly known in history, have disappeared, without leaving in their transit any traces of their existence.

The most celebrated cities of antiquity have perished with their annals, and the language itself which the inhabitants spoke; with difficulty can the scite of Babylon be recognised. Of so many monuments of the arts and of industry, which adorned their cities and passed for the wonder of the world, there only remains a confused tradition, and some scattered wrecks, of which the origin is for the most part uncertain, but of which notwithstanding the magnitude attests the power of the people who have elevated these monuments.

It appears that the practical astronomy of these early ages was confined to observations of the rising and setting of the principal stars, with their occultations by the Moon and planets, and of eclipses. The path of the Sun was followed, by means of the stars, the light of which was obscured by the twilights, and perhaps by the variations in the meridian shadow of the gnomon. The motion of the planets was determined by the stars which they came nearest to, in their course. To recognize all these stars and their various motions, the heaven was divided into constellations; and that celestial zone from which the Sun, Moon and planets were never seen to deviate, was called the Zodiac. It was divided into the twelve following constellations: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricornus Aquarius and Pisces. These were called *signs*, because they served to distinguish the seasons. Thus the entrance of the Sun into Aries, in the time of Hipparchus, marked the commencement of the spring, after which it described the other signs, Taurus, Gemini, Cancer, &c. but the retrograde motion of the equinoxes changed, though slowly, the coincidence of the constellations with the seasons of the year, and at the æra of this great astronomer it was already very different from what it was at the origin of the zodiac; nevertheless since astronomy, according as it became more perfect, had need of signs to indicate the motion of the stars, they still continued as in the time of Hipparchus to denote the commencement of the spring by the entrance of the Sun into sign of Aries. Afterwards they distinguished the signs of the zodiac from the constellations, the first being ideal, and serving only to designate the course of the heavenly bodies. Now that we endeavour to refer our ideas to the most simple expressions, we no longer use the signs of the zodiac, but mark the positions of the heavenly bodies on the ecliptic, according to their distance from the equinoctial point.

The names given to the constellations of the zodiac were not assigned to them fortuitously; for they express relations which were the object of a great number of investigations and of systems. Some of these names appear to relate to the motion of the Sun. Cancer and Capricorn, for example, seem to indicate the retrogradation of this body from the solstices, and Libra denotes the equality of the day and night. The other names seem to refer to the climate and agriculture of those nations to whom the zodiac owes its origin. Capricorn, or the constellation of the goat, appears to be more properly placed at the highest than at the lowest point of the Sun's course. In this position, which goes backward fifteen *(b)* thousand years, the balance was at the equinox of spring; and the constellations of the zodiac had striking relations with the climate of Egypt and with its agriculture. All these relations would also subsist if the constellations of the zodiac, instead of being named from their rising with the Sun, or the commencement of the day, had been denominated from their setting, at the beginning of night; if, for example, the setting of libra had at this moment indicated the commencement of spring. The origin of the zodiac, which would not then go farther back than two thousand years before our æra, agrees much better than the preceding, with the little data which we possess of the antiquity of the sciences, and particularly of astronomy.

The Chinese are, of all people, those who furnish the most ancient astronomical observations. The first eclipses of which mention is made cannot be made use of in chronology, in consequence of the indeterminate manner in which they are detailed; notwithstanding this, these eclipses evince that when the Emperor Yao lived, which was more than two thousand years before our æra, astronomy was cultivated in China as the basis of their ceremonies. The calender, and the announcement of eclipses, were important objects for which a mathematical tribunal was created. At that period the length of the meridian shadows of the gnomon, at the time of the solstices, and the passage of the stars over the meridian, were measured; time was measured by means of clepsydræ, and the position of the Moon, with respect to the stars at the eclipses, was determined, which would give the sidereal positions of the Sun and of the solstices. They also constructed instruments for measuring the angular distances of the stars. From a combination of these means, the Chinese ascertained that the duration of the solar year exceeded, by a quarter of a day very nearly, three hundred and sixty-five days, and they fixed its commencement at the winter solstice. Their civil year was lunar, and in order to reduce it to the solar year, they made use of a period (*c*) of nineteen solar years, corresponding to two hundred, and thirty five lunations, which is exactly the same period as that which Calippus, sixteen centuries afterwards, introduced into the Grecian calendar. Their months consisted alternately of twenty-nine and thirty days, and their lunar year consisted of three hundred and fifty-four days; it was consequently shorter than their solar year by eleven days and a quarter; but in the year when the sum of these differences exceeded a lunation, they intercalated one month. They divided the equator into twelve immoveable signs, and into eighteen constellations, in which they carefully determined the position of the solstices. The Chinese, instead of a century, made use of a period of sixty years; and instead of a week, a period of sixty days; but this short cycle of seven days, which was in use throughout the entire east, was known to them from the most remote periods. The division of the circumference was always in China, subordinated to that of the length of the year, so that the Sun described exactly a degree every day; but the subdivisions of the degree, of the day, of weights, and of every kind of lunar measure, were decimal; and this precedent, furnished for upwards of four thousand years at least, by the most populous nation on the earth, evinces that these divisions, which besides offer so many advantages, may at length by use become extremely popular. The first observations which were useful to astronomy are those of Tcheou-Kong, whose memory is still held in the highest veneration in China, as one of the best princes who ever swayed the sceptre. Being brother of Ou Ouang, who founded the dynasty of Tcheou, he governed the empire after his death, during the minority of his nephew, from the year 1104 to the year 1098 before our æra. Confucius, addressed in the Chou-King, the book held in the highest veneration by the Chinese, through this great prince, to his pupil, the wisest maxims of government and morality. Tcheou-Kong himself, with his astronomers, made a great number of observations, three of which have fortunately come down to us, and they are of inestimable value, from their great antiquity. Two of them are about the meridian lengths of the gnomon, which were observed with the greatest care at the summer and winter solstice, in the town of Loyang; they assign an obliquity to the ecliptic, at this remote period, which perfectly corresponds to the the theory of universal gravitation. The other observation is relative to the position of the winter solstice in the heavens at the same epoch. It likewise answers to the theory, as far as can be expected from the means employed, to determine such a delicate element. This remarkable agreement does not permit us to doubt of the authenticity of these observations.

The burning of the Chinese books, commanded by the emperor Chi Hoanti, about the year 213, before our æra, destroyed all vestiges of the ancient methods of computing eclipses, and several interesting observations, so that in order to discover those which may be useful to the Atronomer, it is necessary to descend to four centuries after Tscheou-Kong, and to pass over to Chaldea. Ptolemy has transmitted several to us; the most ancient are three eclipses of the moon observed at Babylon in the years 719 and 720 before our æra, and which he made use of in determining the motions of the moon. Unquestionably, Hipparchus and he were not in possession of the most ancient, which were sufficiently accurate to be employed in these determinations, as their precision is always proportional to the interval which separates the extreme observations. This consideration should diminish our regrets on account of the loss of the Chaldean observations, which Aristotle, according to Porphyry, as cited by Simplicius, caused to be communicated by the interference of Callisthenes, and which went back to nineteen centuries before Alexander. But the Chaldeans could not discover, except after a long series of observations, the period of 6585 days and $\frac{1}{3}$, during which the moon makes 223 revolutions with respect to the sun, 239 anomalistic revolutions, and 241 revolutions with respect to its nodes. They added $\frac{4}{135}$ of the circumference, in order to obtain the sidereal revolution of the sun in this interval, which supposes that the length of the sidereal year is 365 days and $\frac{1}{4}$. Ptolemy, in recording this period, attributes it to the most ancient mathematicians; but the astronomer Geminus, who was cotemporary with Sylla, affirms that the Chaldeans discovered this period, and he explains the manner, in which they deduced from it, the diurnal motion of the moon, and the method by which they computed the lunar anomaly. His testimony should remove every doubt on the subject, if it be considered that the Chaldean *saros*, consisting of 223 lunar months, which brings back the moon to the same position with respect to the nodes, its perigee, and the sun, makes a part of the preceding period. Thus, the eclipses observed during one period, furnish a simple means of predicting those which ought to occur in subsequent periods. This period, and the ingenious manner in which they computed the principal lunar inequality, required a great number of observations, skilfully discussed; it is the most remarkable astronomical monument before the foundation of the Alexandrian school. The preceding is all we know with certainty respecting the Astronomy of a people whom all antiquity consider as the most advanced in the science of the heavens. The opinions of the Chaldeans respecting the system of the world have been various, as must necessarily be the case, concerning objects respecting which observation and theory had previously furnished so little information. However, some of their philosophers, more fortunate than others, or guided by juster views of the order and immensity of the universe, have thought that the comets were, like the planets, subject to motions regulated by immutable laws.

We have very little positive information respecting the Astronomy of the Egyptians. The exact direction of the faces (*d*) of their pyramids towards the four cardinal points, gives us a favourable notion of their mode of observing; but none of their observations have reached us. It is surprising that the astronomers of Alexandria were obliged to make use of the Chaldean observations, either because the record of the Egyptian observations had been lost, or that the Egyptians did not wish to communicate them, from a feeling of jealousy, which might excite the favour of the kings for the school which they had founded. Previously to this epoch the reputation of their priests had attracted to Egypt, the first philosophers of Greece. Thales, Pythagoras, Eudoxus and Plato, journeyed thither to acquire from them the knowledge with which they enriched their own country; and it is extremely probable that the school of Pythagoras is indebted to them for the sound notions which they taught respecting the constitution of the world. Macrobius expressly attributes to them the suggestion of the motions of Mercury and Venus about the sun. Their civil year consisted of three hundred and sixty five days, and they added at the end of each year five complimentary days called επαγομενα. But according to the ingenious remark of M. Fourrier, the observation of the heliacal rising of Sirius, the most brilliant of all the stars, might have taught them that the return of these risings would then be retarded each year by a fourth part of a day; and on this remark they founded (*e*) the Sothiac period of 1461 years, which would very nearly reduce their months and fetes to the same seasons. This period is renewed in the year 139 of our æra. If it had been preceded by a similar period, as every thing induces us to suppose, the origin of this anterior period would go back to an epoch when we may, with great probability, suppose that the Egyptians gave names to the signs of the zodiack, and when consequently their Astronomy was founded. They had observed, that in twenty-five of their years there were three hundred and nine returns of the moon to the sun, which assigns a very accurate value to the length of the month. Finally we may perceive, from what remains of their zodiacks, that they observed with great care the position of the solstices in the zodiacal constellations. According to Dion Cassius the week is due to the Egyptians. This period is founded on the most ancient system of Astronomy, which placed the Sun, the Moon, and the Planets in the following order of distances from the earth, commencing with the greatest; Saturn, Jupiter, Mars, the Sun, Venus, Mercury, the Moon: the successive parts of the series of days, divided respectively into twenty-four parts, were consecrated in the same order to these stars. Each day took the name of the star corresponding to its first part; the week is found in India among the Bramins with our denominations; and I am satisfied that the days denominated by them and by us in the same manner, correspond to the same physical instant. This period, which was made use of by the Arabians, by the Jews, the Assyrians, and throughout the entire East, is uninterruptedly renewed, and always the same, pervading all nations and changes of empires. It is impossible, among such a variety of nations, to ascertain which was its inventor; we can only affirm that it is the most ancient monument of astronomical knowledge. The civil year of the Egyptians consisted of 365 days; it is easy to perceive that if the name of its first day was assigned (*d*) to each year; the names of these years would be invariably those of the days of the week. It is thus that weeks of years might be formed, which was in use among the Hebrews, but which evidently belonged to a nation whose year was solar and consisting of 365 days.

The knowledge of astronomy appears to have constituted the basis of all the theogonies, the origin of which is thus explained in the simplest possible manner. In Chaldea and ancient Egypt, astronomy was only cultivated in their temples, and by priests, who made no other use of their knowledge than to consolidate the empire of superstition, of which they were the ministers. They carefully disguised it under emblems, which presented to credulous ignorance, heroes and gods, whose actions were only allegories of celestial phenomena, and of the operations of nature; allegories which the power of imitation, one of the chief springs of the moral world, has perpetuated to our own days, and mingled with our religious institutions. The better to enslave the people, they profited by their natural desire of penetrating into futurity, and invented astrology. Man being induced, by the illusions of his senses, to consider himself as ths centre of the universe, it was easy to persuade him, that the stars influenced the events of his life, and could prognosticate to him his future destiny. This error, dear to his self-love, and necessary to his restless curiosity, seems to have been co-eval with astronomy. It has maintained itself through a very long period, and it is only since the end of the last century, that our knowledge of our true relations with nature, has caused them to disappear.

In Persia and India, the commencement of astronomy is lost in the darkness which envelopes the origin of these people.

The Indian tables indicate a knowledge of astronomy cosiderably advanced, but every thing shews that it is not of an extremely remote antiquity. And here, with regret, I differ in opinion from a learned and illustrious astronomer, whose fate is a terrible proof of the inconstancy of popular favour, who, after having honoured his career by labours useful both to science and humanity, perished a victim to the most sanguinary tyranny, opposing the calmness and dignity of virtue, to the revilings of an infatuated people, of whom he had been once the idol.

The Indian tables have two principal epochs, which go back, one to the year 3102, the other to the year 1491 before our æra. These epochs are connected with the mean motions of the Sun, Moon, and planets, in such a manner, that setting out from the position which the Indian tables assign to all the stars at this second epoch, and reascending to the first by means of these tables, the general conjunction which they suppose at this primitive epoch, is found. Baillie, the celebrated astronomer, already alluded to, endeavours, in his Indian astronomy, to prove, that the first of these epochs is founded on observation. Notwithstanding all the arguments are brought forward, with that perspicuity he so well knew how to bestow on subjects the most absract, I am still of opinion, that this period was invented for the purpose of giving a common origin to all the motions of the heavenly bodies in the zodiac. Our last astronomical tables being rendered more perfect by the comparison of theory with a great number of observations, do not permit us to admit the conjunction supposed in the Indian tables; in this respect indeed they made much greater differences than the errors of which they are still susceptible, but it must be admitted that some elements in the Indian astronomy have not the magnitude which they assigned to them, until long before our æra; for example, it is necessary to ascend 6000 years back to find the equation of the centre of the Sun. But, independently of the errors to which the Indian observations are liable, it may be observed, that they only considered the inequalities of the Sun and Moon relative to eclipses, in which the annual equation of the Moon is added to the equation of the centre of the Sun, and augments it by a quantity which is very nearly the difference between its true value and that of the Indians. Many elements, such as the equations of the centre of Jupiter and Mars, are very different in the Indian tables from what they must have been at their first epoch.

A consideration of all these tables, and particularly the impossibility of the conjunction, at the epoch they suppose, prove, on the contrary, that they have been constructed, or at least rectified in modern times. This also may be inferred from the mean motions which they assign to the Moon, with respect to its perigee, its nodes, and the Sun, which being more rapid than according to Ptolemy indicate that they are posterior to this astronomer, for we know, by the theory of universal gravitation, that these three motions have accelerated for a great number of ages. Thus this result of a theory so important for lunar astronomy, throws great light on chronology. Nevertheless, the ancient reputation of the Indians does not permit us to doubt, but that they have always cultivated astronomy.

When the Greeks and Arabs began to devote themselves to sciences, they drew their first elements from India. It is there that the ingenious manner of expressing all numbers in ten characters originated, by assigning to them at once an absolute and a local value, a subtle and important conception, of which the simplicity is such that we can with difficulty, appreciate its merit. But this very simplicity and the great facility with which we are enabled to perform our arithmetical computations place it in the very first rank of useful inventions; the difficulty of inventing it will be better appreciated if we consider that it escaped the genius of Archimedes and Appollonius, two of the greatest men of antiquity.

The Greeks did not begin to cultivate astronomy till a long time after the Egyptians, of whom they were the disciples.

It is extremely difficult to ascertain the exact state of their astronomical knowledge, amidst the (e) variety of fable which fills the early part of their history. Their numberless schools for philosophy produced not one single observer, before the foundation of the Alexandrian school. They treated astronomy as a science purely speculative, often indulging in the most frivolous conjectures.

It is singular, that at the sight of so many contending systems, which taught nothing, the simple reflection, that the only method of comprehending nature is to interrogate her by experiment, never occurred to one of these philosophers, though so many were endowed with an admirable genius. But we must reflect, that as the first observations only presented insulated facts, little suited to attract the imagination, impatient to ascend to causes, they must have succeeded each other with extreme slowness. It required a long succession of ages to accumulate a sufficient number, to discover, among the various phenomena, such relations, which by extending themselves should unite with the interest of truth, that of such general speculations as the human understanding delights to indulge in.

Nevertheless, in the philosophic dreams of Greece, we trace some sound ideas, which their astronomers collected in their travels, and afterwards improved. Thales, born at Miletus, 640 years before our æra, went to Egypt for instruction: on his return to Greece he founded the Ionian school, and there taught the sphericity of the Earth, the obliquity of the ecliptic, and the true causes of the eclipses of the Sun and Moon; he even went so far as to predict them, employing no doubt the periods which had been communicated to him by the priests of Egypt.

Thales had for his successors—Anaximander, Anaximenes, and Anaxagoras; to the first is attributed the invention of the gnomon and geographical charts, which the Egyptians appear to have been already acquainted with.

Anaxagoras was persecuted by the Athenians for having taught these truths of the Ionian school. They reproached him with having destroyed the influence of the gods on nature, by endeavouring to reduce all phenomena to immutable laws. Proscribed with his children, he only owed his life to the protection of Pericles, his disciple and his friend, who succeeded in procuring a mitigation of his sentence, from death to banishment. Thus, truth, to establish itself on earth, has almost always had to combat established prejudices, and has more than once been fatal to those who have discovered it. From the Ionian school arose the chief of one more celebrated. Pythagoras, born at Samos, about 590 years before Christ, was at first the disciple of Thales. This philosopher advised him to travel into Egypt, where he consented to be initiated into the mysteries of the priests, that he might obtain a knowledge of all their doctrines. The Brachmans having then attracted his curiosity, he went to visit them, as far as the shores of the Ganges. On his return to his own country, the despotism under which it groaned, obliged him again to quit it, and he retired to Italy, where he founded his school. All the astronomical truths of the Ionian school, were taught on a more extended scale in that of Pythagoras; but what principally distinguished it, was the knowledge of the two motions of the earth, on its axis, and about the Sun. Pythagoras carefully concealed this from the vulgar, in imitation of the Egyptian priests, from whom, most probably, he derived his knowledge; but his system was more fully explained, and more openly avowed by his disciple Philolaus.

According to the Pythagoricians, not only the planets, but the comets themselves, are in motion round the Sun. These are not fleeting meteors formed in the atmosphere, but the eternal works of nature. These opinions, so perfectly correct, on the system of the universe, have been admitted and inculcated by Seneca, with the enthusiasm which a great idea, on a subject the most vast of human contemplation, ought naturally to excite in the soul of a philosopher.

" Let us not wonder," says he " that we are " still ignorant of the law of the motion of comets, " whose appearance is so rare, that we can nei- " ther tell the beginning nor the end of the revo- " lution of these bodies, which descend to us from " an immense distance. It is not fifteen hundred " years since the stars have been numbered in " Greece, and names given to the constellations. " The day will come, when, by the continued " study of succesive ages, things which are now " hid, will appear with certainty, and posterity " will wonder that they have escaped our notice."

In the same school, they taught that the planets were inhabited, and that the stars were suns, distributed in space, being themselves centres of planetary systems. These philosophic views ought from their grandeur and justness, to have obtained the suffrages of antiquity ; but having been taught combined with systematic opinions, such as the harmony of the heavenly spheres, and wanting, moreover, that proof which has since been obtained, by the agreement with observations, it is not surprising that their truth, when opposed to the illu sions ofthe senses, should not have been admitted.

The only observation which the history of Grecian Astronomy furnishes us with, previously to the foundation of the school of Alexandria, is that of the solstice of the summer of the year 432, before our æra, by Meton and Euclemon. The former of these Astronomers is celebrated for the cycle of nineteen years, which he introduced into the calendar, corresponding to the two hundred and thirty-five lunations already mentioned. The simplest method of measuring time, is that which makes use of solar revolutions, but in the infancy of society, the phases of the moon presented to their ignorance so natural a division of time, that it was universally adopted. They regulated their fetes and games by the return of those phases, and when the necessities of agriculture compelled them to have recourse to the sun, in order to distinguish the seasons, they did not give up the old custom of measuring time by the revolutions of the moon, the age of which may be thus determined by the days of the month. They endeavoured to establish between the revolutions of this star and those of the sun, an agreement depending on the number of periods, which contain entire numbers of these revolutions. The simplest is that of nineteen years. Meton therefore established this cycle of nineteen years, of which twelve were common, or consisting of twelve months, the seven remaining consisted of thirteen. These months were unequal, and so constituted, that in two hundred and thirty-five months of this cycle, one hundred and ten contained twenty-nine days, and one hundred and twenty-five thirty days. This arrangement was proposed by Meton to the

Greeks assembled to celebrate the Olympic games, and was unanimously adopted. But it was not difficult to perceive that at the end of each period, the new calendar retarded about the fourth part of the day on the new moon. Calippus proposed to quadruple the cycle of nineteen years, and to form a period consisting of seventy-six years, at the termination of which one day was to be subtracted. This period was denominated the Calippean, from the name of its inventor; and although not so ancient as the *Saros* of the Chaldeans, it is inferior to it in accuracy. About the time of Alexander, Pythias rendered Marseilles, his country, celebrated by his works as an Astronomical Geographer. We are indebted to him for an observation on the meridian length of the gnomon in this town, at the summer solstice; it is the most ancient observation of this kind after that of Tscheou-Kong. And it is extremely important, in as much as it confirms the continued diminution of the obliquity of the ecliptic. It is to be regretted that the ancient Astronomers did not make a greater use of the gnomon, which produces much more accuracy than their armillæ. By taking some easy precautions to level the surface on which the shade is projected, they might have left us observations on the declinations of the sun and moon, which would be at this day extremely useful.

CHAP. II.

Of Astronomy, from the Foundation of the Alexandrian School to the Time of the Arabs.

Hitherto the practical astronomy of different people has only offered us some rude observations relative to the seasons and eclipses; objects of their necessities or their terrors. Their theoretical astronomy consisted in the knowledge of some periods, founded on very long intervals of time, and of some fortunate conjectures, relative to the constitution of the universe, but mixed with considerable error. We see, for the first time, in the school of Alexandria, a connected series of observations; angular distances were made with instruments suitable to the purpose, and these were calculated by trigonometrical methods. Astronomy then assumed a new form, which the following ages have adopted and brought to perfection. The positions of the fixed stars were determined with more accuracy than before, the paths of the planets were carefully traced, the inequalities of the Sun and Moon were better known, and, finally, it was the school of Alexandria that gave birth to the first system of astronomy that ever comprehended an entire series of celestial pheuomena. This system was, it must be allowed, very inferior to that of the school of Pythagoras, but being founded on a comparison of observations, it afforded, by this very comparison, the means of rectifying itself, and of ascending to the true system of nature, of which it was an imperfect sketch.

After the death of Alexander, his principal generals having divided his empire among themselves, Ptolemy Soter received Egypt for his share. His munificence, and love of the sciences, attracted to Alexandria, the capital of his kingdom, a great number of the most learned men of Greece. Ptolemy Philadelphus, who inherited, with the kingdom, his father's love of the sciences, established them there under his own particular protection. A vast edifice, in which they were lodged, contained both an observatory and that magnificent library, which Demetrius Phalereus had collected with such trouble and expence. Being supplied with whatever books and instruments were necessary to their pursuits, they devoted themselves without distraction to their studies; and their emulation was excited by the presence of a prince, who often came amongst them to participate in their conversation and their labours. The impulse given to the sciences by this school, and the great men which it produced, or which were cotemporary with them, constitutes the epoch of the Ptolemies one of the most memorable in the history of the human mind.

Arystillus and Thimocares were the first observers of the Alexandrian school; they flourished about the year 300 before the Christian æra. Their observations of the principal stars of the zodiac enabled Hipparchus to discover the precession of the equinoxes, and served as the basis of a theory which Ptolemy gave of this phenomenon.

The next astronomer which the school of Alexandria produced, was Aristarchus of Samos. The most delicate elements of astronomy were the subjects of his investigation, unhappily they have not come down to us. The only one of his works which remains is his Treatise *on the magnitudes and distances of the Sun and of the Moon*, where he gives an account of the ingenious manner in which he endeavoured to determine the ratio of these distances. Aristarchus measured the angle contained between the Sun and the Moon, at the moment he judged half of the lunar disk to be illuminated by the Sun, at this instant the visual ray drawn from the eye of the observer to the centre of the Moon is perpendicular to the line which joins the centre of the Moon and Sun, and having found the angle of the observer smaller than a right angle by about the thirtieth part of this angle, he concluded that the Sun was nineteen times farther from us than the Moon. Notwithstanding the inaccuracy of this result, it extended the boundaries of the universe much farther than had been done before. In this treatise Aristarchus supposes the apparent diameters of the Sun and Moon, equal to each other, and to the 180th part of the circumference, which value is much too great; but he afterwards corrected this error, as we learn from Archimede that he made the diameter of the Sun equal to about the 720th part of the zodiac, which is a mean between the limits which Archimede himself, a few years afterwards, assigned by a very ingenious process to this diameter. This correction was unknown to Pappus, a celebrated geometer of Alexandria, who lived about the fourth century, and commented on the treatise of Aristarchus. This induces us to apprehend that the burning of a considerable part of the library of Alexandria during the siege which Cesar sustained in this city, had already distroyed the greater part of the writings of Aristarchus, and also a number of other works equally precious. Aristarchus revived the opinion of the Pythagoricians, relative to the motion of the Earth. But as his writings have not been transmitted to us, we are ignorant to what extent he carried this theory in his explanation of the celestial phenomena. We only know, that this judicious astronomer, from the consideration that the motion of the Earth produced no change in the apparent position of the stars, placed them at a distance incomparably greater than the Sun. Thus it appears, that of all the ancient astronomers, Aristarchus had formed the most just notions of the magnitude of the universe. They have been transmitted to us by Archimede in his Treatise on the *Arenarea*. This great geometer had discovered the means of expressing all numbers, by conceiving them formed of successive periods of myriads of myriads, the units of the first being simple units; those of the second being myriads of myriads, and so on. He denoted the parts of each period by the same characters as the Greeks employed, as far as an hundred millions. In order to evince the advantage of this method, Archimede proposed to express the number of grains of sand which the celestial sphere could contain, a problem of which he increased the difficulty by selecting the hypothesis which assigns to this sphere the greatest extent: it is with this view, that he adduces the opinion of Aristarchus.

The celebrity of his successor, Eratosthenes, is principally due to his measure of the Earth, and of the obliquity of the ecliptic. It is probable that the measurement of the earth was undertaken a long time before, but there only remained of these observations some evaluations of the terrestrial circumference, which it was sought by some approximations, more ingenious than certain, to reduce to the same value, very nearly agreeing with the result of modern observations. Having, at the summer solstice, remarked a deep well, whose whole depth, was illuminated by the Sun, at Syene, in Upper Egypt, he compared this with the altitude of the Sun, observed at the same solstice at Alexandria. He found the celestial arc, contained between the zeniths of these two places, equal to the fiftieth part of the whole circumference; and as their distance was estimated at five hundred stadia, he fixed at two hundred and fifty thousand stadia, the length of the whole terrestrial circumference. It is not at all probable that for such an important result, this astronomer would be content with the rough observation of a well illuminated by the Sun. This consideration, and the account given by Cleomedes, authorises us to suppose that he made use of observations of the meridian lengths of the gnomons at the summer and winter solstices at Syene and Alexandria. This is the reason why the celestial arc between these two places, as determined by him, differs little from the results of modern observations. Eratosthenes erred in supposing that Syene and Alexandria existed under the same meridian; he also erroneously supposed that the distance between these two cities was only five thousand stadia, if the stadium which he most probably employed contained three hundred cubits of the nilometer of Elephantinus. Then the two errors of Eratosthenes would be very nearly compensated, which would lead us to conclude that this astronomer only employed a measure of the earth, formerly executed with great care, the origin of which was lost.

The observation of Erastothenes on the obliquity of the ecliptic, is very valuable, inasmuch as it confirms the diminution of it, determined *à priori*, by the theory of gravitation. He found the distance between the tropics equal to eleven parts of the circumference, divided into eighty-three parts. Hipparchus and Ptolemy found no reason to alter this result by new observations. It is remarkable, that if we suppose, with the Alexandrian astronomers, the latitude of this city equal to thirty-one sexagesimal degrees; this measure of the obliquity places Syene exactly under the tropic, agreeably to the opinion of antiquity.

But of all the astronomers of aniquity, the science is most indebted to Hipparchus of Nice, in Bithynia, for the great number and extent of his observations, and by the important results he ohtained, from a comparison of them with those that had been formerly made by others; and for the excellent method which he pursued in his researches. He flourished at Alexandria in the second century before our æra. Ptolemy, to whom we are principally indebted for a knowledge of his work, and who recurs always to his observations and his theorems, pronounces him, with justice, an astronomer of great skill, of rare sagacity, and a sincere friend of truth. Not content with what had already been done, Hipparchus determined to recommence every thing, and not to admit any results but those founded on a new examination of former observations, or on new observations, more exact than those of his predecessors.

Nothing affords a stronger proof of the uncertainty of the Egyptian and Chaldean observations on the Sun and stars, than the circumstance of his being compelled to recur to the observations of the Alexandrian school, to establish his theories of the Sun, and of the precession of the equinoxes. He determined the length of the tropical year, by comparing one of his observations of the summer solstice with one made by Aristarchus of Samos, 381 years before our æra. This duration appeared to him less than the year of 365¼ days, which had been hitherto adopted, and he found that at the end of three centuries we should subtract one day. But he remarks himself on the little reliance that can be placed on a determination from solstitial observations, and on the advantage of employing observations of the equinoxes. Those which he made in an interval of nearly thirty-three years led him to the same result very nearly. Hipparchus recognized also that the two intervals from one equator to another, were unequally divided by the solstices, so that 94 days and a half elapse from the vernal equinox to the summer solstice, and 92 days and a half from this solstice to the autumnal equinox.

To explain these differences, Hipparchus supposed the Sun to move uniformly in a circular orbit; but, instead of placing the Earth in the centre he supposed it removed to the twenty-fourth part of the radius from the centre, and fixed the apogee at the sixth degree of Gemini. From these data he formed the first solar tables to be found in the History of Astronomy. The equation of the centre, which they suppose, was too great; and it is very probable, that a comparison of the eclipses, in which this equation is augmented by the annual equation of the Moon, confirmed Hipparchus in his error, or perhaps even led him into it. For this error, which surpasses a sixth of the entire value of the equation, is reduced to one sixteenth of this value in the computation of these phenomena. He was mistaken also in supposing the orbit of the Sun, which is really elliptical, to be circular, and that the real velocity of this body was constantly uniform. The contrary is now demonstrated by direct measures of the Sun's apparent diameter; but such observations were impossible at the time of Hipparchus, whose solar tables, with all their imperfections, are a lasting monument of his genius and which Ptolemy so respected, that he subjected own observations to them.

This great Astronomer next considered the motions of the moon. He determined, by a comparison of a great number of eclipses, selected in the most favourable circumstances, the durations of their revolutions relatively to the stars, to the sun, to its nodes, and to its apogee. He found that an interval of $126007^{d}\frac{1}{24}$ contained 4267 months, 4573 returns of the anomaly, 4612 sidereal revolutions of the moon minus $\frac{15}{720}$ of the circumference. He found moreover, that in 5458 months, the moon returns 5923 times to the same node of its orbit. These results are perhaps the most precious of ancient astronomy from their accuracy, and becaus they represent at this epoch the perpetually variable durations of its revolutions (Note IV). Hipparchus determined also the excentricity of the lunar orbit and its inclination to the ecliptic, and he found them very nearly the same as those which have now place in eclipses, in which we know that the one and the other of these elements are diminished by the evection, and the great inequalities of the motion of the moon in latitude. This constancy of the inclination of the lunar orbit to the plane of the ecliptic, notwithstanding the variations which this plane experiences relatively to the stars, and which by the ancient observations are sensible on its obliquity to the equator, is, as we have seen in the fourth book, a result of universal gravitation which the observations of Hipparchus confirm. Finally, from the determination of the parallax of the moon, he endeavoured to conclude that of the Sun, by the breadth of the cone of the terrestrial shadow, (*a*) in an eclipse at the moment it was traversed by the Moon, which led him nearly to the same result as had been obtained by Aristarchus. He made a great number of observations on the planets, but too much the friend of truth to explain their motions by uncertain theories, he left the task of this investigation to his successors. A new star which appeared in his time induced him to undertake a catalogue of the fixed stars, to enable posterity to recognize any changes that might take place in the appearances of the heavens. He was sensible also of the importance of such a catalogue for the observations of the Moon and the planets. The method he employed was that of Arystillus and Timochares, which we have already explained in the third chapter of the First Book. The reward of this long and laborious task, was the important discovery of the precession of the equinoxes; in comparing his observations with those astronomers, he discovered that the stars had changed their situation with respect to the equator, but had preserved the same latitude with respect to the ecliptic ; he at first supposed that this was only true for the stars situated in the zodiack, but having observed that they all preserve the same relative position, he concluded that this phenomenon was general. To explain these different changes, he assigned a direct motion to the celestial sphere round the poles of the ecliptic, which produces a retrograde motion in longitude of the equinoxes with respect to the stars, which appeared to him to be for each century the three hundred and sixtieth part of the zodiack. But he announced his discovery with some reserve, being doubtful of the accuracy of the observations of Arystillus and Timochares. Geography is indebted to Hipparchus for the method of determining places on the Earth, by their latitude and longitude, for which he first employed the eclipses of the Moon. Spherical trigonometry, also, owes its origin to Hipparchus, who applied it to the numberless calculations which these investigations required. His principal works have not been transmitted to us, and we are only acquainted with them through the Almagest of Ptolemy, who has transmitted to us the principal elements of the theories of this great Astronomer, and some of his observations. Their comparison with modern observations having shewn their accuracy, and their use even to astronomers at the present day, makes us regret others, and particularly those which he made on the planets, of which there remains very few ancient observations. The only work of Hipparchus which has come down to us is a critical commentary on the sphere of Eudoxus, described in a poem of Aratus; it is anterior to the discovery of the precession of the equinoxes. The positions assigned to the stars on this sphere are so erroneous, and thay gave for the epoch of its origin such different results, that it is astonishing to see Newton establish on these imperfect positions a system of chronology, which besides deviates considerably from dates assigned with much probability to several ancient events. The interval of near three centuries, which separated these two astronomers, presents to us Geminus and Cleomedes, whose works have come down to us; and some observers, as Agrippa, Menelaus and Theon of Smyrna. We may also notice in this interval the reformation of the Roman calendar by Julius Cæsar, for which purpose he made Sosthenes come to Alexandria, and the precise knowledge of the ebbing and flowing of the sea. Possidonius observed the law of this phenomenon, which appertains to astronomy by its evident relation to the motion of the Sun and Moon, and of which Pliny the naturalist has given a description remarkable for its exactness.

Ptolemy, born at Ptolemais in Egypt, flourished at Alexandria about the year 130 of our æra. Hipparchus had given, by his numerous works, a new face to Astronomy, but he left to his successors the care of rectifying his theorems by new observations, and of establishing those which were deficient. Ptolemy continued this labour, and has given a treatise on this science in his great work entitled the Almagest.

His most important discovery is that of the evection of the Moon. Astronomers previously had only considered the motion of this body relatively to eclipses; in which it was solely sufficient to have regard to the equation of the centre, especially if we suppose with this astronomer that the equation of the centre of the Sun is greater than its true value, which in part replaces the annual equation of the Moon. It appears that Hipparchus had recognized that this did not represent the motion of the moon in its quadratures, and that observations presented great anomalies in this respect. Ptolemy carefully followed these anomalies, determined its law and fixed its value with great accuracy. In order to represent it, he supposed the moon to move on an epicycle carried by a moveable excentrick, of which the centre revolved about the earth in a contrary direction to the motion of the epicycle.

It was a general opinion of the ancients, that the uniform circular motion being the most simple and natural, was necessarily that of the heavenly bodies. This error maintained its ground till the time of Kepler, and for a long time impeded him in his researches. Ptolemy adopted it, and, placing the Earth in the centre of the celestial motions, he endeavoured to represent their inequalities in this false hypothesis. Conceive to move on a circumference, of which the Earth occupies the centre, that of another circumference, on which moves that of a third, and so on, up to the last circumference, on which the body is supposed to move uniformly. If the radius of one of these circles surpasses the sum of the others, the apparent motion of the body round the Earth, will be composed of a mean uniform motion, and of several inequalities depending on the proportions these several radii, the motions of their centres, and of the Star, have to each other. By increasing their number, and giving them suitable dimensions, we may represent the inequalities of this apparent motion. Such is the most general manner of considering the hypothesis of epicycles and excentrics. For an excentric may be considered as a circle of which the centre moves about the earth with a greater or less velocity, and which vanishes if it is immoveable. The Geometers who preceded Ptolemy were occupied with the appearances of the motions of the planets on this hypothesis, and it appears in the Almagest that the great geometer Appollonius had already resolved the problem of their stations and retrogradations. Ptolemy supposed the Sun, Moon, and planets in motion round the Earth in this order of distances—the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn; each of the planets superior to the Sun, was moved on an epicycle, of which the centre described an excentrick about the earth, in a time equal to that of the revolution of a planet. The period of the motion of the star on the epicycle was that of the solar revolution; and it was always found in opposition to the Sun, when it attained the point of the epicycle which was nearest to the earth. Nothing in this system determines the actual magnitude of the circles and of the epicyles. Ptolemy had only occasion to know the ratio which the radius of each epicycle had to that of the circle described by its centre. In like manner he made each inferior planet, to move on an epicycle of which the centre described an excentrick about the earth; but the motion of this point was equal to the solar motion, and the planet described its epicycle in a time which in modern astronomy is that of the revolution of the Sun: the planet was always in conjunction with it when it arrived at the lowest point of its epicycle. Here also nothing determines the absolute magnitude of the circles and of the epicycles. Astronomers anterior to Ptolemy were divided in their opinions as to the position of Mercury and Venus; Ptolemy followed the most ancient opinion, and placed them below the Sun; others placed them above, and finally, the Egyptians made them move round it. It is singular, that Ptolemy does not mention this hypothesis, which is equivalent to making the excentrics of those two planets equal to the solar orbit. If moreover he supposed the epicycles of the superior planets equal and parallel to this orbit, his system would make all the planets to move about the Sun, while this star revolves about the earth, and then there was but one step to make, in order to arrive at the true system of the world. This man-

R 2

ner of determining the arbitrary quantities in Ptolemy's system, by supposing the circles and epicycles described in an annual motion equal to the solar orbit, renders the agreement of this motion with that of the Sun evident. By thus modifying this system, he can exhibit the mean distances of the planets from this star, in parts of its distance from the earth; for those distances are the ratios of the radii of the excentricks to those of the epicycles for the superior planets, and of the radii of the epicycles, to the radii of the excentricks for the two inferior. Such a simple and natural modification of the system of Ptolemy escaped all astronomers till the time of Copernicus. None of them appeared to have sufficiently considered the relations which subsist between the geocentrick motion of the planets and that of the Sun, to have investigated its cause; none of them were curious to know their respective distances from the Sun and Earth, they were content with connecting by new observations the elements determined by Ptolemy, without making any change in his hypothesis. But even, if by means of epicycles we could represent the inequalities of the motions of the heavenly bodies, still it would be impossible to represent the variations in their distances. In the time of Ptolemy, these variations were almost insensible in the planets, whose apparent diameters could not then be measured. But his observations on the Moon should have taught him that his hypothesis was erroneous, according to which the perigean diameter of the Moon in the quadratures, should be double of the apogean diameter in the sysigies. Besides every new inequality which the improvements in the art of observing discovered, incumbered this system with an additional epicycle, which, instead of being confirmed by the progress of the science, has only grown more and more complicated; and this should convince us, that it is not that of nature. But in considering it as a method of adapting the celestial motions to calculation, this first attempt of the human understanding towards an object so very complicated, does great honour to the sagacity of its author. Such is the weakness of the human understanding, that it frequently requires the aid of hypotheses to connect phenomena together, and to determine their laws; and if hypotheses are restricted to this use, by avoiding to ascribe any reality to them, and by restifying them perpetually by new observations, we arrive finally at the true causes, or at least we can supply them, and conclude from the observed phenomena those which given circumstances ought to develope. The history of philosophy furnishes us with several examples which hypotheses may procure in this point of view, and of the errors to which it is exposed when they are realized.

Ptolemy confirmed the motion of the equinoxes discovered by Hipparchus, by comparing his observations with those of this great astronomer. He established the respective immobility of the Stars, their invariable latitude to the ecliptic, and their motion in longitude, which he found conformable to what Hipparchus had suspected.

We now know that this motion is much more considerable; which circumstance, considering the interval between the observations of Ptolemy and Hipparchus, implies an error of more than one degree in their observations. Notwithstanding the difficulty which attended the determination of the longitude of the Stars, when observers had no exact measure of time, we are surprised that so great an error should have been committed, particularly when we observe the agreement of the observations with each other, which Ptolemy cites as a proof of the accuracy of his result. He has been reproached with having altered them, but this reproach is not well founded; his error, in the determination of the motion of the equinoxes, seems to have been derived from too great confidence in the result of Hipparchus, relative to the length of the tropical year. In fact, Ptolemy determined the longitudes of the stars, by comparing them either with the Sun, or with the Moon, which was equivalent to a comparison with the Sun, since the synodical revolution of the Moon was well known by the means of eclipses. Now, Hipparchus having supposed the year too long, and consequently the motion of the Sun, with respect to the equinoxes, too slow, it is clear that this error diminished the longitudes of the Sun employed by Ptolemy. The motion in longitude, which he attributed to the Stars, ought to be increased by the arc described by the Sun in the time, equal to the error of Hipparchus in the length of the year, and then it comes out very nearly what it ought to be. The sidereal year being the tropical year increased by the time necessary for the Sun to describe an arc equal to the annual motion of the equinoxes, it is evident that the sidereal year of Hipparchus and of Ptolemy ought to differ from the true year; in fact the difference is only the tenth part of that which exists between their tropical year and ours.

This remark has led to the examination of another question. It has been generally believed, that the catalogue of Ptolemy, was that of Hipparchus, reduced to his time by means of a precession of one day in ninety years. This opinion is founded on the circumstance, that the constant error in longitude of his Stars, disappears when reduced to the time of Hipparchus. But the explanation which we have given of the cause of this error, justifies Ptolemy from the reproach which has been imputed to him, of having taken the merit of Hipparchus to himself; and it seems fair to believe him, when he asserts that he has observed all the Stars of his own catalogue, even to the stars of the sixth magnitude. He adds, at the same time, that he found very nearly the same position of the Stars, relatively to the ecliptic, as Hipparchus, and we are always more induced to think so, as Ptolemy continually endeavours to make his results approximate to those of this great astronomer, who was in fact a much more accurate observer.

Ptolemy inscribed on the temple of Serapis at Canoeum the principal elements of his system; this astronomical edifice subsisted near fourteen centuries, and now that it is entirely destroyed, his Almagest considered as a depositary of ancient observations, is one of the most precious monuments of antiquity. Unfortunately it contains but a small number of the observations anterior to his æra. The author only related those which were necessary to explain his theory. The astronomical tables being once formed, he judged it useless to transmit with them to posterity the observations which Hipparchus and he employed for this purpose, and his example has been followed by the Arabs and the Persians. The great collections of precious observations collected solely for themselves, and without any application to theories, belong to modern astronomy, and is one of the fittest means of rendering it perfect. Ptolemy has not rendered less service to geography, in collecting all the known longitudes and latitudes of different places, and laying the foundation of the method of projections, for the construction of geographical charts. He composed a great treatise on optics, which has not been preserved, in which he explained the astronomical refractions: he likewise wrote treatises on the several sciences of chronology, music, gnomonics, and mechanics. So many labours, and on such a variety of subjects, manifest a very superior genius, and will ever obtain him a distinguished rank in the history of science. On the revival of astronomy, when his system gave way to that of nature, mankind avenged themselves on him for the despotism he had so long maintained; and they accused Ptolemy of having appropriated to himself the discoveries of his predecessors. But the honourable mention which he makes of Hipparchus, whom he frequently cites to support his theories, fully justifies him from this charge. At the revival of letters among the Arabs, and in Europe, his hypotheses combining the attraction of novelty with the authority of antiquity, were generally adopted by minds desirous of knowledge, and who were anxious at once to obtain possession of that which antiquity had acquired after long labour. Their gratitude elevated Ptolemy too high, whom they afterwards too much depressed. The fame of Ptolemy has met with the same fate as that of Aristotle and Descartes. Their errors were no sooner recognized, than a blind admiration gave way to an unjust contempt, for even in science itself, the most useful revolutions are not always exempt from passion and prejudice.

CHAP. III.

Of Astronomy from the time of Ptolemy to the period of its restoration in Europe.

The progress of astronomy in the school of Alexandria terminated with the labours of Ptolemy. This school continued to exist for five centuries, but the successors of Ptolemy and Hipparchus contented themselves with commenting on their works without adding to their discoveries. The phenomena of the heavens continued unobserved during a period of more than six hundred years. Rome, for a long time, the seat of valour, glory, and learning, did nothing useful to science. The consideration that was always attached by the republic to eloquence and military talents, attracted all talents to those pursuits: and science, offering no advantage, was necessarily neglected in the midst of conquests undertaken by ambition, and of internal commotions, in which liberty expired, and gave way to the despotism of the emperors. The division of the empire, the necessary consequence of its vast extent, brought on its fall, and the light of science, extinguished by the barbarians, was only again revived among the Arabians.

This people, actuated by a wild spirit of fanaticism, after having extended their religion and arms over a great part of the Earth, had no sooner reposed in peace, than they devoted themselves with ardour to letters and science.

It, however, was but a short time before that they destroyed their most beautiful ornament, by burning the famous library of Alexandria.

In vain the philosopher Philoponus exerted himself for its preservation. If these books, replied Omar, are conformable to the alcoran, they are useless; if they are contrary to it, they are detestable. Thus (*a*) perished this immense treasure of erudition and genius. Repentance and regret soon followed this barbarous execution, for the Arabians were not long before they perceived their irreperable loss, and that they had deprived themselves of the most precious fruits of their conquests.

About the middle of the eighth century, the caliph Almansor gave great encouragement to astronomy; but among the Arabian princes who distinguished themselves for their love of the sciences, the most celebrated in history was Almamoun, of the family of the Abassides and son of the famous Aaron-al-Rashid, so celebrated throughout Asia. Almamoun reigned in Bagdat in 814; having conquered the Greek emperor

Michael III., he imposed on him, as one condition of peace, that he should have delivered to him the best books of Greece;—the Almagest was among the number; he caused it to be translated into the Arabian language, and thus diffused the astronomical knowledge which had formerly acquired so much celebrity for the Alexandrian school. Not content with encouraging learned men by his liberality, he was himself an observer, and determined the obliquity of the ecliptic; he likewise caused a degree of the meridian to be measured on the vast plain of Mesopotamia. He did more still, he wished to render the science more perfect, and for this purpose he collected together several distinguished astronomers, who after making a great number of observations, published new tables of the Sun and Moon, more accurate than those of Ptolemy, and for a long time celebrated in the East, under the name of the *verified tables*. In this table, the solar perigee has the position which it ought to have, the equation of the centre of the Sun, which, according to Hipparchus, is considerably greater, is reduced to its true value; but this precision became then a source of error in the computation of the eclipses, in which the annual equation of the Moon, partly corrected the inaccuracy in the equation of the centre of the Sun, which was adopted by this astronomer. The duration of the tropical year is much more exact than that of Hipparchus, it is however too short by almost two minutes, but this error arises from this; that the authors of the verified table compared their observations with those of Ptolemy; it would have been nearly nothing if they had employed the observations of Hipparchus. This is also the reason why they supposed the precession of the equinoxes too great.

Almamon caused also to be measured, with great care, in the extensive plane of Mesopotamia, a terrestrial degree which he found equal to two hundred thousand five hundred cubits. This measurement exhibits the same uncertainty as that of Eratosthenes, relatively to the length of the modulus made use of. These measures cannot now interest us, unless their modulus is made known; but the errors to which these observations were then liable do not permit us to draw from thence the advantage, which can only result from the accuracy of modern operations, by means of which we can always find our measures if in the course of time their standards should alter.

The encouragement given to astronomy by this prince and his successors, produced a great number of astronomers, among whom Albategnius deserves to be placed the first. His Treatise on *The Science of the Stars* contains several interesting observations, and the principal elements of the theory of the Sun and Moon; they differ little from those of the astronomers of Almamon. His work being for a long time the only known treatise of Arabian astronomy, the advantageous changes which were made in the tables of Ptolemy have been attributed to him. But a precious fragment, extracted from the astronomy of Ebn. Junis, and translated by Caussin, evinces that these changes are due to the authors of the verified tables. Besides it has furnished us with precise and very accurate notions of the Arabian astronomy. Ebn. Junis, astronomer of Hakenn, caliph of Egypt, observed at Cairo about the year one thousand. He arranged a great treatise of astronomy, and constructed tables of the celestial motions, which were celebrated through the East for their accuracy, and which appear to have served as the foundation of tables formed afterwards by the Arabians and the Persians. We perceive in this fragment, from the age of Almanon to the time of Ebn. Junis, a long series of observations of eclipses, of equinoxes, of solstices, of conjunctions of planets, and of occultations of stars; observations important for the perfection of astronomical theories, inasmuch as they have enabled us to recognise the secular equation of the Moon, and have thrown considerable light on the great variations of the system of the world. (Note 5). These observations are still only a small part of those of the Arabian astronomers, of which the number has been prodigious. They perceived the inaccuracy of the observations of Ptolemy on the equinoxes, and by comparing their observations either together, or with those of Hipparchus, they determined very exactly the true length of the year; that of Ebn. Junis only exceeds ours by thirteen seconds, and it ought to exceed it by five seconds.

It appears by this work, and by the tables of several manuscripts existing in our libraries, that the Arabians were particularly occupied with the perfection of astronomical instruments, the treatises which they left on this subject shew the importance which they attached to it, which importance is confirmed by the accuracy of their observations. They also paid particular attention to the measure of time by clepsydræ, by immense solar dials, and also by the vibrations of the pendulum. Notwithstanding this, their observations of the eclipses exhibit the same uncertainty as those of the Greeks and of the Chaldeans; and their observations on the Sun and Moon are far from having over those of Hipparchus that superiority, which can compensate the advantage of the distance which separates us from this great astronomer. The activity of the Arabian astronomers is confined to observations: it is not extended to the investigation of new inequalities, and in this point of view they have added nothing to the hypotheses of Ptolemy. That lively curiosity which attaches us to phenomena till their laws and cause are perfectly known, is what characterises the learned of modern Europe. (Note *5.*)

The Persians, after having for a long time submitted to the same sovereigns as the Arabians, and professing the same religion, about the middle of the eleventh century shook off the yoke of the Caliphs. About this time their calendar received a new form, by the care of the astronomer Omar Cheyam; it was founded on an ingenious intercalation, which consists in making in every thirty-three years, eight of them bissextile; Dominick Cassini, at the end of the seventeenth century, suggested the adoption of this intercalation as more exact and simple than the Gregorean: not knowing that the Persians had for a long time employed it. In the thirteenth century Holagu Ilecoukan, one of their last sovereigns, assembled the most learned astronomers at Maragha, where he constructed a magnificent observatory, the direction of which he entrusted to Nassireddin. But no prince of this nation distinguished himself more for his zeal for Astronomy than Ulugh-Beigh, whom we ought to place in the first rank of great observers. He himself formed at Samarcand, the capital of his states, a new catalogue of the stars, and of the best astronomical tables which we had before Tycho Brahe. He measured in 1437, with a great instrument, the obliquity of the ecliptic, and his results, when corrected by refraction and the erroneous parallax which he employed, gives this obliquity greater by seven minutes than at the commencment of this century, which confirms its successive diminution.

The annals of China furnish us with the most ancient astronomical observations. They present to us also twenty-four centuries after, the most accurate observations which have been made previously to the restoration of Astronomy, and even before the application of the telescope to the quadrant of the circle. We have seen that the astronomical year of the Chinese commenced about the winter solstice, and that to fix its origin, they repeatedly observed the meridian shades of the gnomon near the solstices. Gaubil, one of the most learned and judicious Jesuit missionaries sent to this empire, has made us acquainted with a series of observations of this kind, which extend from the year 1100 before our æra, to 1280 years after. These indicate with great clearness the diminution of the obliquity of the ecliptic, which in this long interval has been the thousandth part of the circumference. Tsou-tchong, one the most skilful astronomers of China, by a comparison of the observations made at Nankin in 461, with those which were made at Loyang in the year 173, determined the magnitude of the tropical year more exactly than the Greeks had done, or even the astronomers of Almamon. He found it $365^{\text{days}}24282$, the same very nearly with that of Copernicus. While Holagu Ilecoukan made astronomy to flourish in Persia, his brother Cobelai, who in 1271 founded the dynasty of Yuun, granted the same protection to it in China: he named Cocheou King, the first of the Chinese Astronomers, chief of the tribunal of mathematicians. This great observer constructed instruments much more exact than those hitherto made use of; the most valuable of all being a gnomon of forty Chinese feet, terminated by a plate of brass which was vertical, and pierced by a hole of the diameter of a needle. It is from the centre of this opening that Cocheou King reckoned the height of the gnomon; he measured the shade to the centre of the image of the Sun. "Hitherto," says he, "the higher limb of the Sun has been observed, and the extremity of the shade can with difficulty be distinguished; besides the gnomon of eight feet, which has been constantly made use of, is too short. These reasons have induced me to use the gnomon of forty feet, and to take the centre of the image." Gaubel, from whom we have these details, has recommended to us several observations made from 1277 to 1286, and they are precious for their accuracy, and prove unquestionably the diminutions of the obliquity of the ecliptic, and of the excentricity of the earth's orbit, from that epoch to our days. Cocheou King determined with remarkable precision the position of the lunar solstice with respect to the Stars in 1280, he made it to coincide with the apogee of the Sun, which took place thirty years before. The length which he assigned to the year is exactly that of our Gregorian year.

The Chinese methods for the computations of eclipses are inferior to those of the Arabians and of the Persians; the Chinese have not profited by the knowledge acquired by these people, notwithstanding their frequent communications with them; they have extended to Astronomy itself their constant attachment for their ancient customs.

The history of America, before its conquest by the Spaniards, exhibits some traces of astronomy; for the most elementary notions of this science have been amongst all nations the first fruits of their civilization. The Mexicans had, instead of the week, a short period of five days. Their months were each twenty days, and eighteen of these months constituted a year, which commenced at the winter solstice, and to which they added five complementary days. There is reason to suppose they composed from the combination of one hundred and four years, a great cycle, in which they intercalated twenty-five days. This supposes a duration of the tropical year more exact than that of Hipparchus; and what is very remarkable, it is nearly the same as that of the astronomers of Almamon. The Peruvians and the Mexicans carefully observed the shades of the gnomon at the solstices and at the equinoxes; they had even elevated for this purpose columns and pyramids. However, when we consider the difficulty of obtaining such an exact determination of the length of the year, we are induced to think that it was not accomplished by them, and that they obtained it from the ancient continent. But from what people or by what means did they receive it? Wherefore, if they received it from the north of Asia, have they a division of time so different from those in use in this part of the world? These are questions which it appears impossible to determine. There exist in the numerous manuscripts which our libraries contain, several ancient observations yet unknown, which would throw great light on astronomy, and especially on the secular inequalities of the heavenly motions. To their discussion, the attention of the learned skilled in the eastern languages ought to be directed; for the great variations which have taken place in the system of the world are not less interesting to man than the revolutions of empires. Posterity, which can compare a long series of very exact observations with the theory of universal gravitation, will much more enjoy the agreeement of these results than we, to whom antiquity has left observations for the most part very uncertain. But those observations, subjected to a sound discussion, can at least, in part, compensate for the errors to which they are liable, and supply the place of exact observations; as in geography, in order to fix the position of places, we compensate for the want of astronomical observations, by comparing together the different relations of travellers. Thus, though the account which the series of observations from the ancient times to the present day presents to us, be very imperfect; still we may perceive, in a sensible manner, the variations of the excentricity of the orbit of the Earth, and of the position of its perigee; those of the secular motions of the Moon, with respect to its nodes, to its perigee, and to the Sun; finally, the variations of the elements of the orbits of the planets. The successive diminution of the obliquity of the ecliptic during a period of nearly three thousand years, is particularly remarkable in the comparison of the observations of Tcheou Kong, of Pytheas, of Ebn Junis, of Cocheou King, of Uleugh-Beigh, and of the moderns.

CHAP. IV.

Of Astronomy in modern Europe.

It is to the Arabians that modern Europe is indebted for the first rays of light that dissipated the darkness in which it was enveloped during twelve centuries. They have transmitted to us the treasure of knowledge which they received from the Greeks, who were themselves disciples of the Egyptians; but by a deplorable fatality the arts and sciences have disappeared among all these nations, almost as soon as they had communicated them.

Despotism has for a long period extended its barbarism over those beautiful countries where science first had its origin, so that those names which formerly rendered them so celebrated, are now utterly unknown to them.

Alphonso, king of Castille, was one of the first sovereigns who encouraged the revival of astronomy in Europe, but he was ill seconded by the astronomers, whom he had assembled at a considerable expense, and the tables which they published did not answer to the great cost they had occasioned.

Endowed with a correct judgment, Alphonso was shocked at the confusion of the circles and epicycles, in which the celestial bodies were supposed to move; he felt that the expedients employed by nature ought to be more simple. "*If the Deity*," said he, "*had asked my advice, these things would have been better arranged.*" By these words, in which he was charged with impiety, he meant to express that mankind were still far from knowing the true mechanism of the universe.

In the time of Alphonso, Europe was indebted to the encouragement of Frederic II. Emperor of Germany, for the first Latin translation of the Almagest of Ptolemy, which was made from the Arabic version.

We are now arrived at that celebrated epoch when astronomy, emancipating itself from the narrow sphere in which it was hitherto confined, advanced by a rapid and continued progress to its present exalted eminence. Purbach, Regiomontanus, and Walterus, prepared the way to these prosperous days of the science, and Copernicus gave them birth by the fortunate explanation of the celestial phenomena, by means of the motion of the Earth on its axis, and round the Sun.

Shocked, like Alphonso, at the extreme complication of the system of Ptolemy, he tried to find among the ancient philosophers a more simple arrangement of the universe. He found that many of them had supposed Venus and Mercury to move round the Sun: that Nicetas, according to Cicero, made the Earth revolve on its axis, and by this means freed the celestial sphere from that inconceivable velocity which must be attributed to it to accomplish its diurnal revolution. He learnt from Aristotle and Plutarch that the Pythagoreans had made the Earth and planets move round the Sun, which they placed in the centre of the universe. These luminous ideas struck him; he applied them to the astronomical observations which time had multiplied, and had the satisfaction to see them yield, without difficulty, to the theory of the motion of the Earth. The diurnal revolution of the heavens was only an illusion due to the rotation of the Earth, and the precession of the equinoxes is reduced to a slight motion of the terrestrial axis. The circles, imagined by Ptolemy, to explain the alternate direct, and retrograde motions of the planets, disappeared. Copernicus only saw in these singular phenomena, the appearances produced by the combination of the motion of the Earth round the Sun, with that of the planets; and he concluded, from hence, the respective dimensions of their orbits, which, till then, were unknown. Finally, every thing in this system announced that beautiful simplicity in the expedients of nature, which delights so much when we are fortunate enough to discover them. Copernicus published it in his work, *On the Celestial Revolutions*; not to shock received prejudices, he presented it under the form of an hypothesis. "Astronomers," said he, in his dedication to Paul III., "being permitted to "imagine circles, to explain the motions of the "stars, I thought myself equally entitled to ex- "amine if the supposition of the motion of the "Earth would render the theory of these appear- "ances more exact and simple."

This great man did not witness the success of his work. He died suddenly, by the rupture of a blood vessel, at the age of seventy-one years, a few days after receiving the first proof. He was born at Thorn, in Polish Prussia, the 19th of February, 1473. After learning the Greek and Latin languages, he went to continue his studies at Cracovia. Afterwards, induced by his taste for astronomy, and by the reputation which Regiomontanus had acquired, he undertook a journey to Italy, where this science was taught with success: being greatly desirous to render himself illustrious in the same career, he attended the lectures of Dominique Maria, at Bologna. When arrived at Rome, his talents obtained him the place of professor, where he made several observations: he afterwards quitted this city, to establish himself at Fravenberg, where his uncle, then Bishop of Warmia, made him a canon. It was in this tranquil abode that, by thirty-six years of observation and meditation, he established his theory of the motion of the Earth. At his death he was buried in the cathedral of Fravenberg, without any pomp or epitaph; but his memory will exist as long as the great truths which he taught with a clearness that eventually dissipated the illusions of the senses, and surmounted the difficulties which ignorance of the laws of mechanics had opposed to them.

These truths had yet to vanquish obstacles of another kind, and which, arising from a respected source, would have extinguished them altogether, if the rapid progress of all the mathematical sciences had not concurred to support them.

Religion was invoked to destroy an astronomical system, and one of its defenders, whose discoveries did honor to Italy, was harassed by repeated prosecutions. Rethicus, the disciple of Copernicus, was the first who adopted his ideas; but they were not in great estimation till towards the beginning of the seventeenth century, and then they owed it principally to the labours and misfortunes of Galileo.

A fortunate accident had made known the most wonderful instrument ever discovered by human ingenuity, and which, by giving to astronomical observations a precision and extent hitherto unhoped for, displayed in the heavens new inequalities, and new worlds. Galileo hardly knew of the first trials of the telescope, before he bent his mind to bring it to perfection. Directing it towards the stars, he discovered the four satellites of Jupiter, which shewed a new analogy between the Earth and planets; he afterwards observed the phases of Venus, and from that moment he no longer doubted of its motion round the Sun. The milky way displayed to him an infinite number of small stars, which the irradiation blends to the naked eye, into a white and continued light; the luminous points which he perceived beyond the line which separated the light part of the Moon from the dark, made him acquainted with the existence and height of its mountains. Finally he observed the singular appearances occasioned by Saturn's ring, and by the spots and rotation of the Sun. In publishing these discoveries, he showed that they proved incontestibly the motion of the Earth; but the idea of this motion was declared heretical by a congregation of cardinals; and Galileo, its most celebrated defender, was cited to the tribunal of the inquisition, and compelled to retract this theory, to escape a rigorous prison.

One of the strongest passions in a man of genius, is the love of truth. Full of the enthusiasm which a great discovery inspires, he burns with ardour to disseminate it, and the obstacles which ignorance and superstition, armed with power, oppose to it, only stimulate and increase his energy; besides, the subject is of the highest importance to us, from the rank which it assigns to the globe which we inhabit. Galileo, more and more convinced by his own observations of the motion of the Earth, had long meditated a new work, in which he proposed to develope the proofs of it. But to shelter himself from the persecution of which he had escaped being the victim, he proposed to present them under the form of dialogues between three interlocutors, one of whom defended the system of Copernicus, combated by a Peripatetician. It is obvious that the advantage would rest with the defender of this system;

but, as Galileo did not decide between them, and as he gave as much weight as possible to the objections of the partisans of Ptolemy, he had a right to expect that tranquillity which his age and labours merited.

The success of these dialogues, and the triumphant manner with which all the difficulties against the motion of the Earth were resolved, roused the inquisition. Galileo, at the age of seventy, was again cited before this tribunal. The protection of the Grand Duke of Tuscany could not prevent his appearance. He was confined in a prison, where they required of him a second disavowal of his sentiments, threatening him with the punishment incurred by contumacy, if he continued to teach the system of Copernicus.

He was compelled to sign this formula of abjuration:

" *I Galileo, in the seventieth year of my age,* " *brought personally to justice, being on my knees,* " *and having before my eyes the holy evangelists,* " *which I touch with my own hands; with a sin-* " *cere heart and faith, I abjure, curse, and detest,* " *the error, and heresy, of the motion of the Earth,*" " &c.

What a spectacle! A venerable old man, rendered illustrious by a long life, consecrated to the study of nature, abjuring on his knees, against the testimony of his own conscience, the truth which he had so evidently proved. A decree of the inquisition condemned him to a perpetual prison. He was released after a year, at the solicitations of the grand duke; but, to prevent his withdrawing himself from the power of the inquisition, he was forbidden to leave the territory of Florence.

Born at Pisa, in 1564, he gave early indications of those talents which were afterwards developed. Mechanics owe to him many discoveries, of which the most important is the theory of falling bodies, the most splendid discovery of his genius.

Galileo was occupied with the libration of the Moon, when he lost his sight; he died three years afterwards, at Arcetre, in 1642, regretted by all Europe, which he left enlightened by his labours, and indignant at the judgment passed against so great a man, by an odious tribunal.

While this passed in Italy, Kepler, in Germany, developed the laws of the planetary motions. But, previous to the account of his discoveries, it is necessary to look back and to describe the progress of astronomy in the north of Europe, after the death of Copernicus.

The history of this science presents at this epoch a great number of excellent observers. One of the most illustrious was William IV., Landgrave of Hesse-Cassel. He had an observatory built at Cassel, which he furnished with instruments, constructed with care, and with which he observed a long time. He procured two celebrated astronomers, Rothman and Juste Byrge; and Tycho was indebted to his pressing solicitations for the favours which were conferred on him by Frederic King of Denmark.

Tycho Brahe, who was one of the greatest observers that ever existed, was born at Knucksturp, in Norway. His taste for astronomy was manifested at the age of fourteen years, on the occasion of an eclipse of the Sun, which happened in 1560. At this age, when reflection is so rare, the justice of the calculation which announced this phenomenon, inspired him with an anxious desire to know its principles; and this desire was still further increased by the opposition of his preceptor and family. He travelled to Germany, where he formed connexions of correspondence and friendship with the most distinguished persons, who pursued astronomy either as a profession, or amusement, and particularly with the Landgrave of Hesse-Cassel, who received him in the most flattering manner.

On his return to his own country, he was fixed there by his sovereign, Frederic, who gave him the little island of Huene, at the entrance of the Baltic. Tycho built a celebrated observatory there, which was called Uranibourg. There, during an abode of twenty-one years, he made a prodigious number of observations, and many important discoveries. At the death of Frederic, envy, then unrestrained, compelled Tycho to leave his retreat. His return to Copenhagen did not appease the rage of his persecutors; the Minister, Walchendorp, (whose name, like that of all men who have abused the power intrusted to them, ought to be handed down to the execration of posterity,) forbad him to continue his observations. Fortunately, Tycho found a powerful protector in the

Emperor Rodolph II. who settled on him a considerable pension, and lodged him commodiously at Prague. He died suddenly at this city, on the 24th of October, 1601, in the midst of his labours, and at an age when astronomy might have expected great services from him.

The invention of new instruments, and great improvements made in the old ones, a much greater precision in observations; a catalogue of stars much more accurate than those of Hipparchus, and Ulugh Beigh; the discovery of that inequality of the Moon, which is called variation; that of the inequalities of the motion of the nodes, and of the inclination of the lunar orbit; the interesting remark, that the comets are beyond this orbit; a more perfect knowledge of astronomical refractions; finally, very numerous observations of the planets, which have served as the basis of the discoveries of Kepler, are among the principal services which Tycho Brahe has rendered astronomy. The accuracy of his observations, to which he was indebted for his discoveries on the lunar motion, shewed him also, that the equation of time with respect to the Sun and the planets is not applicable to the moon, and that the part depending on the anomaly of the Sun, and even a quantity much greater must be deducted from it. Kepler, carried away by his imagination to investigate the relations and the cause of those phenomena, thought that the moving virtue of the Sun caused the Earth to move more rapidly on itself in its perihelion than in its appelion. The effect of this variation of diurnal motion could not be recognized by the observations of Tycho, except in the motions of the Moon, where it is thirteen times more considerable than in that of the Sun. But clocks brought to perfection by the application of the pendulum, shew that this effect vanishes in this last motion, and that the rotation of the earth is uniform. Flamstead transferred to the Moon itself the inequality depending on the anomaly of the Sun, which he had regarded as only apparent. This inequality, which Tycho first perceived, is that which is termed the *annual equation.* By this example we may perceive how observations, in becoming more perfect, discover to us inequalities till then enveloped in errors. The researches of Kepler furnishes us also with a still more remarkable example. Having shewn in his commentary on Mars, that the hypothesis of Ptolemy necessarily differs from the observations of Tycho by eight sexagesimal minutes, he adds; "This difference is less than the uncertainty of the "observations of Ptolemy, which uncertainty, by "the confession of this Astronomer, is less than "ten minutes, but the divine goodness giving "us Tycho Brahe, a very exact observer, it is "meet to be grateful for the kindness of the divi-"nity, and to render him thanks for it. Being "now convinced of the error of the hypothesis "which we made use of, we ought to direct all "our efforts to the true laws of the heavenly mo-"tions. These eight minutes which I am no longer "permitted to neglect have enabled me to reform "all Astronomy, and indeed constitute the mate-"rials of the greatest part of this work." Struck with the objections which the adversaries of Copernicus made to the motion of the Earth, and perhaps influenced by the vanity of wishing to give his name to an astronomical system, he mistook that of nature. According to him the Earth is immoveable in the centre of the universe; all the Stars move every day round the axis of the world; and the Sun, in its annual revolution, carries with it the planets. In this system, which ought, in the natural order of things, to precede that of Copernicus, the appearances are the same as in the theory of the motion of the Earth. For we may, in general, consider any point we choose, for example, the centre of the Moon as immoveable, provided that we assign the motion with which it is animated, to all the stars in a contrary direction.

But, is it not physically absurd to suppose the Earth immoveable in space, while the Sun carries along the planets in the midst of which the Earth is included? How could the distance from the Earth to the Sun, which agrees so well with the duration of its revolution in the hypothesis of the motion of the Earth, leave any doubt of the truth of this hypothesis in a mind constituted to feel the force of analogy? Ought we not to confess with Kepler, that nature proclaims with a loud voice the truth of this hypothesis. Indeed it must be admitted, that Tycho, though a great observer, was not fortunate in his research after causes; his unphilosophical mind had even imbibed the prejudices of astrology, which he tried to defend.

It would be, however, unjust to judge him with the same rigor, as one who should refuse at present to believe the motion of the Earth, confirmed by the numerous discoveries made in astronomy since that period.

The difficulties which the illusions of the senses opposed to this theory, were not then completely removed. The apparent diameter of the fixed stars, greater than their annual parallax, gave to these stars in this theory, a real diameter, greater than that of the terrestrial orbit. The telescope, by reducing them to luminous points, made this improbable magnitude disappear. It could not be conceived how these bodies, detached from the Earth, could follow its motion. The laws of mechanics have explained these appearances; they have proved, what Tycho, deceived by an erroneous experiment, had refused to admit, that a body, falling from a considerable height, and abandoned to the action of gravity alone, ought to fall very nearly in a vertical line, only deviating towards the east by a quantity difficult to estimate accurately by observation, from its minuteness, so that at present there is as much difficulty in proving the motion of the Earth by the direct experiment of a falling body, as formerly existed to prove that it should be insensible.

The reformation of the Julian calendar is also to be traced up to the æra of Tycho Brahe. It is convenient that the months and festivals should be attached to the same seasons, to make them to be remarkable epochs for agriculture. But to secure this inestimable advantage to the inhabitants of a country, it was necessary by the regular intercalation of a day, to compensate the excess of the solar year above the common year of three hundred and sixty-five days. The simplest mode of intercalation was that employed by Julius Cæsar; it consisted in making a bissextile to succeed three common years. But as the length of the year, which this intercalation supposes, was too great, the vernal equinox always preceded it, so that after the interval of fifteen centuries, it had approached to the commencement of the year by above eleven days and a half. In order to remedy this inconvenience, Pope Gregory decreed in 1582, that the month of October of that year should only consist of twenty-one days; that the year 1600 should be bissextile; and that henceforth for years which terminate each century, only three of them should be bissextile in four centuries. Even this intercalation assigns too great a length to the year, so that the equinox would anticipate it by about a day in four thousand years; but if the bissextile which terminates this interval is considered as a common year, the Gregorian intercalation would be very nearly correct. In other respects the Julian calendar has not been altered. It would then have been easy to fix the origin of the year, at the winter solstice, and to render the length of the months more uniform, by assigning thirty-one days to the first, and twenty-nine days to the second month in common years, and thirty days in bissextile years, and by making the other months alternately thirty and thirty-one days; it would have been convenient also to denote them by their numerical order, which would have done away with the improper denominations of the four last months of the year. If then this intercalation adopted by Gregory was corrected in the manner specified above, the Gregorian calendar would be as perfect as could be desired. But is it requisite to give it all this perfection? When it is considered that this calendar has been now adopted by almost every nation in Europe and America, and that it required two centuries, and all the influence of religion, to secure to it this advantage, it will be immediately apparent that it ought to be retained, notwithstanding some imperfections which attach to it, and which, it may be observed besides, are comparatively of trifling importance. For the principal object of a calendar is, to connect by a simple mode of intercalation events to the series of days, and to make the seasons for a great number of years to correspond to the same months of the year, which conditions are sufficiently well secured in the Gregorian calendar. As the part of this calendar which refers to the fixing of Easter is foreigu to the science of Astronomy, I have not adverted to it here.

Towards the close of his life, Tycho Brahe had Kepler for a disciple and assistant. He was born in 1571, at Viel, in the duchy of Wirtemberg, and was one of these extraordinary men whom nature grants now and then to the sciences, to bring to light those grand theories which have been prepared by the labour of many centuries.

The career of the sciences did not appear to him adequate to satisfy the ambition he felt of rendering himself illustrious; but the ascendancy of his genius, and the exhortations of Mæstlin, led him to astronomy; and he entered into the pursuit with all the avidity of a mind passionately desirous of glory.

The philosopher, endowed with a lively imagination, and impatient to know the causes of the phenomena which he sees, often obtains a glimpse of them, before observation can conduct him to them. Doubtless he might, with greater certainty, ascertain the cause from the phenomena; but the history of science proves to us, that this slow progress has not always been that of inventors.

What rocks has he to fear, who takes his imagination for his guide!

Prepossessed with the cause which it presents to him, instead of rejecting it when contradicted by facts, he alters them to make them agree with his hypotheses; he mutilates, if I may be allowed the expression, the work of nature, to make it resemble his imagination, without reflecting that time destroys with one hand these vain phantoms, and with the other confirms the results of calculation and experience.

The philosopher who is really useful to the cause of science, is he, who, uniting to a fertile imagination, a rigid severity in investigation and observation, is at once tormented by the desire of ascertaining the cause of the phenomena, and by the fear of deceiving himself in that which he assigns.

Kepler owed the first of these advantages to nature, and the second to Tycho Brahe, who gave him useful advice, from which he too frequently deviated, but which he followed in all cases where he could compare his hypotheses with observations, which, by the method of exclusion, conducted him from hypothesis to hypothesis to the laws of the planetary motions. This great observer, whom he went to see at Prague, and who had discovered the genius of Kepler in his earliest works, notwithstanding the mysterious analogies of numbers and figures with which they were filled, exhorted him to devote his time to observation, and procured him the title of imperial mathematician.

The death of Tycho, which happened a few years afterwards, put Kepler in possession of his valuable collection of observations, of which he made a most noble use, founding on them three of the most important discoveries that have been made in natural philosophy.

It was an opposition of Mars which determined Kepler to employ himself to examine, in preference, the motions of this planet. His choice was fortunate in this circumstance, that the orbit of Mars, being one of the most eccentric of the planetary system, the inequalities of his motion were more perceptible, and therefore led to the discovery of their laws with greater facility and precision. Though the theory of the motion of the Earth had made the greater part of those circles with which Ptolemy had embarrassed astronomy disappear, yet Copernicus left several to remain, in order to explain the real inequalities of the celestial bodies.

Kepler, deceived like him, by the opinion that their motions ought to be circular and uniform, tried a long time to represent those of Mars, in this hypothesis. Finally, after a great number of trials, which he has related in detail in his famous work *de Stella Martis*, he surmounted the obstacle, which an error, supported by the suffrage of every period, had opposed to him; he discovered that the orbit of Mars is an ellipse, of which the Sun occupies one of the foci, and that the motion of the planet is such, that the radins vector, drawn from its centre to that of the Sun, describes equal areas in equal times. Kepler extended these results to all the planets, and published from this theory, in 1626, the Rudolphine tables, for ever memorable in astronomy, as being the first founded on the true laws of the planetary motions, and freed from all the circles with which anterior tables were encumbered.

If we separate the astronomical investigations of Kepler from the chimerical ideas with which they were frequently accompanied, we will perceive that he arrived at those laws in the following manner: He first ascertained that the equality of the angular motion of Mars had sensibly place about a point situated beyond the centre of his orbit, with respect to the Sun. He recognised the same thing for the Earth, by comparing together select observations of Mars, of which the orbit, by the magnitude of its annual parallax, is proper to make known the respective dimensions of the orbit of the Earth. Kepler, from these results, concluded that the real motion of the planets were variable, and that at the points of the greatest and least velocities, the areas described in a day, by the radius vector of a planet, are the same. He extended this equality of areas to all the points of the orbit, which gives him the law of the areas, proportional to the times. Afterwards, the observations of Mars, near his quadratures, showed him that the orbit of this planet is an oval, elongated in the direction of the diameter which joins the points of the extreme velocities. Finally, he concluded from this the elliptic motion.

Without the speculations of the Greeks, on the curves formed from the section of a cone by a plane, these beautiful laws might have been still unknown. The ellipse being one of these curves, its oblong figure gave rise, in the mind of Kepler, to the idea of supposing the planet Mars, whose orbit he had discovered to be oval, to move on it, and soon, by means of the numerous properties which the ancient geometricians had found in the conic sections, he became convinced of the truth of this hypothesis. The history of the sciences offers us many examples of these applications of of pure geometry, and of its advantages; for every thing is connected in the immense chain of truths, and often a single observation has been sufficient to show the connection between a proposition apparently the most sterile, and the phenomena of nature, which are only mathematical results of a small number of immutable laws.

The perception of this truth probably gave birth to the mysterious analogies of the Pythagoricians: they had seduced Kepler, and he owed to them one of his most beautiful discoveries. Persuaded that the mean distances of the planets from the Sun, ought to be regulated conformably to these analogies, he compared them a long time, both with the regular geometrical solids, and with the intervals of tones. At length, after seventeen years of meditations and calculation, conceiving the idea of comparing the powers of the numbers which expressed them, he found that the squares of the times of the planetary revolutions, are to each other as the cubes of the major axes of their orbits; a most important law, which he had the advantage of observing in the system of satellites of Jupiter, and which extends to all the systems of satellites.

After having determined the curves which the planets describe about the Sun, and discovered the laws of their motions, Kepler was too near the principle whence those laws are derived, not to anticipate it. The investigation of this principle frequently exercised his active imagination; but the moment of making this last step was not yet arrived, which supposes the invention of dynamics, and of the infinitesmal analysis. Far from approaching this end, Kepler deviated from it by vain speculations on the moving cause of the planets. He supposed in the Sun a motion of rotation on an axis perpendicular to the ecliptic; immaterial species emanating from this star, in the plane of its equator, and endowed with an activity decreasing in the ratio of the distances, and preserving their primitive motion of rotation, cause each planet to participate in this circular motion. At the same time the planet, by a sort of instinct or magnetism, approaches and recedes alternately from the Sun, elevates itself above the solar equator, or is depressed below it, so as to describe an ellipse always situated in the same plane, passing through the centre of the Sun. In the midst of those numerous errors, Kepler was nevertheless led to sound views on universal gravitation in the introduction of the work *de Stella Martis*, in which he presents his principal discoveries.

"Gravity," says he, "is only a mutual corpo-"real affection between bodies, by which they tend "to unite. The gravity of bodies is not directed "towards the centre of the world, but towards that "of the bodies of which they make a part; and if "the Earth was not spherical, heavy bodies placed "on different parts of its surface would not fall "towards the same centre. Two isolated bodies "are carried towards each other, as two magnets, "in running to join, describe spaces inversely as "their masses. If the Earth and Moon were not "retained at the distance which separates them "by an animal force, or by some other equivalent force, they would fall towards each "other; the Moon would fall $\frac{53}{54}$ of the way, and "the Earth would describe the rest, supposing "them to be equally dense. If the Earth ceased "to attract the waters of the ocean, they would "flow towards the Moon, in virtue of the attractive force of this star. This force, which extends to the Earth, produces there the phenomena of the tides." Thus the important work which we have cited contains the first germs of the celestial mechanics which Newton and his successors have so happily developed.

We may be astonished that Kepler should not have applied the general laws of elliptic motion to comets. But, misled by an ardent imagination, he lost the clue of the analogy, which should have conducted him to this great discovery. The comets, according to him, being only meteors, engendered in ether, he neglected to study their motions, and thus stopped in the middle of the career which was open to him, abandoning to his successors a part of the glory which he might yet have acquired. In his time, the world had just begun to get a glimpse of the proper method of proceeding in the search of truth, at which genius only arrived by instinct, frequently connecting errors with its discoveries. Instead of passing slowly by a succession of inductions, from insulated phenomena, to others more extended, and from these to the general law of nature, it was more easy and more agreeable, to subject all the phenomena to the relations of suitableness and harmony, which the imagination could create and modify at pleasure.

Thus, Kepler explained the disposition of the solar system by the laws of musical harmony. It is a humiliating sight for the human mind to behold this great man, even in his latest works, amusing himself with these chimerical speculations, even so far as to regard them as the "*life and soul*" of astronomy. These being blended with his true discoveries was unquestionably the cause why the astronomers of his age, Des Cartes himself and Galileo, who might have drawn the most advantageous consequences from his laws, do not appear to have perceived their importance. They were not generally admitted till after that Newton made them the base of his theory of the system of the world.

Astronomy likewise owes to Kepler many useful works. His treaties on optics are full of new and interesting matter; he brought the telescopic theory to perfection; he there explains the mechanism of vision, which was unknown before him. He assgned the true cause of the *lumiere cendrée* of the Moon; but he gave the honour of this discovery to his master, Maestlin, entitled to notice from this discovery, and from having recalled Kepler to astronomy, and converted Galileo to the system of Copernicus.

Finally, Kepler, in his work entitled *Stereometria Doliorum,* has presented some conceptions on infinity, which have influenced the revolution experienced by geometry towards the end of the last century; and Fermat, whom we ought to regard as the true inventor of the differential calculus, has founded on them his beautiful method *de maximis et de minimis.*

With so many claims to admiration this great man lived in misery, while judicial astrology, every where honoured, was magnificently recompensed.

Fortunately the enjoyment which a man of genius receives from the truths which he discovers, and the prospect of a just and grateful posterity, console him for the ingratitude of his contemporaries.

Kepler had obtained pensions which were always ill paid: going to the diet of Ratisbon to solicit his arrears, he died in that city the 15th of November 1630. He had in his latter years the advantage of seeing the discovery of logarithms, and making use of them. This was due to Nepier, a Scottish baron; it is an admirable contrivance, an improvement on the ingenious algorithm of the Indians, and which, by reducing to a few days the labour of many months, we may almost say doubles the life of astronomers, and spares them the errors and disgusts inseparable from long calculations;—an invention so much the more gratifying to the human mind, as it is entirely due to its own powers: in the arts, man makes use of the materials and forces of nature to increase his powers, but here all is his own work.

The labours of Huygens followed soon after those of Kepler and Galileo. Very few men have deserved so well of the sciences, by the importance and sublimity of their researches. The application of the pendulum to clocks is one of the most beautiful acquisitions which astronomy and geography have made, and to which fortunate invention, and to that of the telescope, the theory and practice of which Huygens considerably improved, they owe their rapid progress.

He discovered, by means of excellent object-glasses which he succeeded in constructing, that the singular appearances of Saturn were produced by a very thin ring with which this planet is surrounded: his assiduity in observing enabled him also to discover one of the satellites of Saturn. He published these two discoveries in his *Systema Saturneum*, a work which contains some traces of the Pythoragean notions which Kepler had so much abused, but which the genuine spirit of science, which in this celebrated age has made so much progress, has for ever effaced. This satellite of Saturn rendered the number of satellites equal to that of the planets then known. Huygens judging this equality necessary for the harmony of the system of the world, dared to affirm, that there were [no more satellites to discover; and Cassini, a few years afterwards discovered four new satellites belonging to the same satellite.

He made numerous discoveries in geometry, mechanics and optics; and if this extraordinary genius had conceived the idea of combining his theorems on centrifugal forces with his beautiful investigation on involutes, and with the laws of Kepler, he would have preceded Newton in his theory of curvilinear motion, and in that of universal gravitation. But it is not such approximations that constitute invention.

Towards the same time, Hevelius rendered himself useful to astronomy by his immense labours. Few such indefatigable observers have existed; it is to be regretted that he would not adopt the application of telescopes to quadrants, an invention for which we are indebted to Picard, which gives to observations a precision previously unknown to astronomy, and has rendered the greater number of those of Hevelius useless.

At this epoch astronomy received a new impulse from the establishment of learned societies.

Nature is so various in her productions and phenomena, that it is extremely difficult to ascertain their causes, hence it is requisite for a great number of men to unite their intellect and exertions in order to comprehend and develope her laws. This union is particularly requisite when the progress of the sciences multiplying their points of contact, and not permitting one individual to penetrate them all; they can only receive from several learned men the mutual support which they require.

It is then that the natural philosopher has recourse to geometry, to arrive at the general causes of the phenomena which he observes, and the geometrician in his turn interrogates the philosopher, in order to render his own investigation useful, by applying them to experience, and to open in these applications a new road in the analysis. But the principal advantage of learned societies is the philosophical feeling on every subject which is introduced into them, and from thence diffuses itself over the whole nation. The insulated philosopher may resign himself without fear to the spirit of system; he only hears contradiction at a distance; but in a learned society the shock of systematic opinions at length destroys them, and the desire of mutually convincing each other, establishes between the members an agreement only to admit the results of observation and calculation. Hence experience has proved that since the origin of these establishments, true philosophy has been generally extended.

By setting the example of submitting every opinion to the test of severe scrutiny, they have detroyed prejudices which had so long reigned among the sciences, and in which the highest intellects of the preceding age had participated. Their useful influence on opinion has dissipated the errors accumulated in our own time, with an enthusiasm which at other periods would have perpetuated them. Equally removed from the credulity which admits every thing, and the prejudices which would induce us to reject every thing which was at variance with preconceived notions, they have always in difficult questions and in extraordinary phenomena, sagely waited for the answers of observation and of experience, exciting them by prizes, and by their proper works, regulating their estimation as much by the greatness and difficulty of a discovery, as by its immediate utility; and convinced, by several instances, that the most barren in appearance may have one day the most important consequences. They have encouraged the investigation of truth on all subjects, only excluding those which, by the limits of the extent of human understanding, will be for ever inaccessible. Finally, it is among them that those grand theories have been formed which are placed above the reach of the vulgar by their comprehensiveness; and which, extending themselves by the numerous occasions in which they are applicable, to nature and to the arts, are inexhaustible sources of delight and intelligence. Wise governments, convinced of the utility of learned societies, and viewing them as one of the principal foundations of the glory and of the prosperity of empires, have instituted and placed them near themselves, to illuminate by their information, from which they have frequently derived great advantages.

Of all the learned societies, the two most celebrated for the number and importance of their discoveries in astronomy, are the Academy of Sciences at Paris, and the Royal Society in London.

The first was founded in 1666, by Louis XIV. who foresaw the lustre which the arts and sciences were to diffuse over his reign. This monarch, worthily seconded by Colbert, invited many learned strangers to fix themselves in his capital. Huygens availed himself of this flattering invitation; he published his admirable work, *De horologio oscillatorio*, in the midst of the academy, of which he was one of the first members. He would have finished his days in this country, had it not been for the disastrous edict which, towards the end of the last century, widowed France of so many valuable citizens. Huygens, departing from a country in which the religion of his ancestors was proscribed, retired to the Hague, where he was born the 14th of April, 1629, and died there the 15th of June, 1695.

Dominic Cassini was likewise induced to go to Paris, by the advantages held out by Louis XIV. During forty years of useful labours, he enriched astronomy with a crowd of discoveries: such are the theory of the satellites of Jupiter, the motions of which he determined from observations of their eclipses; the discovery of the four satellites of

Saturn, that of the rotation of Jupiter, of the belts parallel to his equator, of the rotation of Mars, of the zodiacal light, a very approximate knowledge of the Sun's parallax, a very exact table of refractions, and, above all, a complete theory of the libration of the Moon.

Galileo had only considered the libration in latitude. Hevelius explained the libration in longitude, by supposing that the Moon always presents the same face to the centre of the lunar orbit, of which the Earth occupies one of the foci. Newton, in a letter addressed to Mercator in 1675, rendered the explanation of Hevelius perfect, by reducing it to the simple conception of a uniform rotation of the Moon on itself, while it moves unequally about the Earth. But he supposed with Hevelius the axis of rotation always perpendicular to the plane of the ecliptic. Cassini, by his own observations, recognized that it was inclined to the plane of the ecliptic at a small angle of an invariable magnitude; and to satisfy the condition already observed by Hevelius, according to which all the inequalities of libration are re-established after each revolution of the nodes of the lunar orbit, he made the nodes of the lunar equator to coincide constantly with them. Such has been the progress of opinions on one of the most curious points of the system of the world.

The great number of astronomical academicians of extraordinary merit, and the limits of this historical abridgment, do not permit me to give an account of their labours; I shall content myself with observing, that the application of the telescope to the quadrant, the invention of the micrometer and heliometer, the successive propagation of light, the magnitude of the Earth, its ellipticity, and the diminution of gravity at the equator, are all discoveries due to the Academy of Sciences.

Astronomy does not owe less to the Royal Society of London, the origin of which is a few years anterior to that of the Academy of Sciences. Among the astronomers which it has produced, I shall cite Flamstead, one of the greatest observers that has ever appeared; Halley, rendered illustrious by his travels, undertaken for the advantage of science, by his beautiful investigations concerning comets, which enabled him to discover the return of the comet in 1759; and by the ingenious idea of employing the transit of Venus over the Sun, in order to determine its parallax: I shall mention, lastly, Bradley, the model for observers, and who will be for ever celebrated for two of the most beautiful discoveries that have been made in astronomy, namely, the aberration of the fixed stars, and the nutation of the axis of the Earth.

When the application of the pendulum to clocks, and of telescopes to quadrants, had rendered the slightest changes in the position of the celestial bodies perceptible to observers, they endeavoured to determine the annual parallax of the fixed stars; for it was natural to suppose, that so great an extent as the diameter of the terrestrial orbit, would be sensible even at the distance of these stars. Observing them carefully, at every season of the year, there appeared slight variations; sometimes favourable, but more frequently contrary to the effects of parallax. To determine the law of these variations, an instrument of great radius, and divided with extreme precision, was requisite. The artist who executed it, deserves to partake of the glory of the astronomer who owed his discovery to him. Graham, a famous English watch-maker, constructed a great sector, with which Bradley discovered the aberration of the fixed stars, in the year 1727. To explain it, this great astronomer conceived the fortunate idea of combining the motion of the earth with that of light, which Roemer had discovered at the end of the last century, by means of the eclipses of Jupiter's satellites. We should be surprised that in the interval of half a century, which intervened between this discovery and that of Bradley, none of the distinguished philosophers who then existed, and who knew the motion of light, should have paid any attention to the very simple effects which result from it, in the apparent position of the fixed stars. But, the human mind, so active in the formation of systems, has almost always waited till observation and experience have acquainted it with important truths, which its powers of reasoning alone might have discovered.

It is thus that the invention of telescopes has followed by more than three centuries that of lenses, and even then it was solely due to accident.

In 1745, Bradley discovered by observation, the nutation of the terrestrial axis and its laws. In all the apparent variations of the fixed stars, observed with extraordinary care, he perceived nothing which indicated a perceptible parallax. We are also indebted to this great man for the first sketch of the principal inequalities of the satellites of Jupiter, which was soon afterwards extended. Finally, he left an immense number of observations of all the phenomena which the heavens presented towards the middle of the last century, for more than ten consecutive years. The great number of these observations and the accuracy which distinguishes them, form by their collection one of the principal foundations of modern Astronomy, and the epoch whence we ought to set out in all the delicate investigations of this science. He has given us a model for several similar collections, which being rendered successively more perfect by the progress of the arts, are so many signs placed in the path of the heavenly bodies to denote their periodic and secular changes.

At the same epoch Lacalle flourished in France and Tobias Mayer in Germany, both of them indefatigable observers and laborious compilers; they have rendered the tables and astronomical theories perfect, and from their own observations have formed catalogues of the stars, which, compared wiih those of Bradley determine with great precision the state of the heavens in the middle of the last century.

The measures of the degrees of the terrestrial meridian, and of the pendulum, (repeated in different parts of the globe, of which France gave the example, by measuring the whole arc of the meridian, which crosses it, and by sending academicians to the north and to the equator, to observe the magnitude of these degrees, and the intensity of the force of gravity ;) the arc of the meridian, comprised between Dunkirk and Barcelona, determined by very precise observations, and forming the base of the most natural and simple system of measures; the voyages undertaken to observe the two transits of Venus over the Sun's disk, in 1761 and 1769, and the exact knowledge of the dimensions of the solar system, which has been derived from these voyages; the discovery of achromatic telescopes, of chronometers, of the sextant, and repeating circle, invented by Mayer, and brought to perfection by Borda; the formation by Mayer of lunar tables sufficiently exact for the determination of the longitude at sea; the discovery of the planet Uranus, by Herschel, in 1781; that of its satellites, and of the two new satellites of Saturn, due to the same observer; these, with the discoveries of Bradley, are the principal obligations which astronomy owes to our century, which, with the preceding, will always be considered as the most glorious epoch of the science.

The present age has commenced under the most favourable auspices for astronomy: its first day is remarkable for the discovery of the planet Ceres, made by Piazzi, at Palermo; and this discovery was soon followed by those of the two planets, Pallas and Vesta, by Olbers, and of the planet Juno, by Harding.

CHAP. V.

Of the Discovery of universal Gravitation.

After having shewn by what successive efforts the human mind has attained the knowledge of the laws of the celestial motions, it only remains to consider the means by which it has arrived at the general principle, on which these laws depend. Descartes was the first who endeavoured to reduce the motions of the heavenly bodies to some mechanical principle. He imagined vortices of subtle matter, in the centre of which he placed these bodies. The vortex of the Sun forced the planet, its satellites, and their vortices, into motion; that of the planet, in the same manner, forced its satellite to revolve round it. The motion of comets traversing the heavens in all directions, destroyed these vortices, as they had before destroyed the solid heavens, and the whole apparatus of circles imagined by the ancient astronomers. Thus, Descartes was no happier in his mechanical, than Ptolemy in his astronomical theory. But their labours have not been useless to science. Ptolemy has transmitted to us, through fourteen centuries of ignorance, the few astronomical truths which the ancients had discovered, and which he had also increased. When Descartes appeared on the stage, the impulse given by the invention of printing, by the discovery of the New World, by the Reformation, and by the Copernican system, rendered people eager for new discoveries. But this philosopher, by substituting in the place of ancient errors, others most seducing, and resting on the authority of his geometrical discoveries, was enabled to destroy the empire of Aristotle, which might have stood the attack of a more careful philosopher; and his system of vortices, at first received with enthusiasm, being founded on the motion of the planets and Earth about the Sun, contributed to make these notions be generally adopted; but by establishing as a principle, that we should begin by doubting of every thing, he himself warned us to adopt his own system with great caution, and his astronomical system was very soon overturned by later discoveries, in which his own, combined with those of Kepler and Galileo; and also, with the just philosophical notions then entertained on all subjects, has rendered his age, already illustrious for so many chefs d'œuvre in literature and the fine arts, likewise the most remarkable epoch in the history of the human mind. It was reserved for Newton to make known the general principles of the heavenly motions. Nature not only endowed him with a profound genius, but placed his existence in a most fortunate period. Descartes had changed the face of the mathematical sciences, by the application of algebra to the theory of curves and variable functions. Fermat had laid the foundation of the geometry of infinites, by his beautiful method *de maximis* and *de minimis*, and of tangents. Wallis, Wren, and Huygens, had discovered the laws of motion; the discoveries of Galileo, on falling bodies, and of Huygens on evolutes and on the centrifugal force, led to the theory of motion in curves; Kepler had determined those described by the planets, and had formed a remote conception of universal gravitation; and finally, Hook had distinctly perceived that their motion was the result of a projectile force, combined with the attractive force of the Sun. The science of celestial mechanics wanted nothing more to bring it to light, but the genius of a man, who, by generalizing these discoveries, should be capable of deducing from them the law of gravitation: it is this which Newton accomplished in his immortal work on the mathematical principles of natural philosophy. This philosopher, so deservedly celebrated, was born at Woolstrop, in England, in the latter end of the year 1642, the year in which Galileo died. His first success in mathematics announced his future reputation; a cursory perusal of elementary books, was sufficient for him to comprehend them; he next read the geometry of Descartes, the optics of Kepler, and the arithmetic of infinites, by Wallis; but soon aspiring to new inventions, he was, before the age of twenty-seven, in possession of his method of fluxions, and of his theory of light. Anxious for repose, and averse to literary controversy, which he had better avoided by sooner making known his discoveries, he delayed publishing his works. His friend and preceptor, Dr. Barrow, exerted himself in his favour, and obtained for him the situation of professor of mathematics in the university of Cambridge; it was during this period, that, yielding to the request of Halley, and the solicitations of the Royal Society, he published his *Principia*. The university of Cambridge, whose privileges he strenuously defended when attacked by James II., chose him for their representative, in the conventional parliament of 1688, and for that which was convened in 1701. He was knighted and appointed director of the mint, by Queen Anne: he was elected president of the Royal Society in 1703, which dignity he enjoyed till his death, in 1727. During the whole of his life he obtained the most distinguished consideration, and the nation to whose glory he had so much contributed, decreed him at his death, public funeral honours.

In 1666, Newton, retired into the country, for the first time, directed his thoughts to the system of the world. The descent of heavy bodies, which appears nearly the same at the summit of the highest mountains as at the surface of the Earth, suggested to him the idea, that gravity might extend to the Moon, and that being combined with some motion of projection, it might cause it to describe its elliptic orbit round the

Earth. To verify this conjecture, it was necessary to know the law of the diminution of gravity. Newton considered, that if the Moon was retained in its orbit by the gravity of the Earth, the planets should also be retained in their orbits by their gravity towards the Sun, and demonstrated this by the law of the areas being proportional to the times. Now it results from the relation of the squares of the times to the cubes of the greater axes of their orbits, discovered by Kepler, that their centrifugal force, and consequently their tendency to the Sun, diminishes inversely as the squares of the distances from this body. Newton, therefore, transferred to the Earth this law of the diminution of the force of gravity, and reasoning from the experiments of falling bodies, he determined the height which the Moon, abandoned to itself, would fall in a short interval of time. This height is the versed sine of the arc which it describes in the same interval; and this quantity the lunar parallax gives in parts of the radius of the Earth, so that, to compare the law of gravitation with observation, it was necessary to know the magnitude of this radius; but Newton having, at that time, an erroneous estimate of the terrestrial meridian, obtained a different result from what he expected, and suspecting that some unknown forces operated concurrently with the gravity of the Moon, he abandoned his original idea. Some years afterwards, a letter from Dr. Hook induced him to investigate the nature of the curve described by projectiles round the centre of the Earth. Picard had lately finished the measure of a degree in France, and Newton found, by this measure, that the Moon was retained in its orbit by the force of gravity alone, supposed to vary inversely as the square of the distance. From the action of this law he found that bodies in their fall, describe ellipses, of which the centre of the Earth occupies one of the foci, and then, considering that the planetary orbits are likewise ellipses, having the Sun in one of their foci, he had the satisfaction to see, that the solution which he had undertaken from curiosity, could be applied to the greatest objects in nature. He arranged the several propositions relative to the elliptic motions of the planets, and Dr. Halley having induced him to publish them, he composed his grand work, the *Principia*, which appeared in 1687. These details, which have been transmitted to us by his friend and cotemporary, Dr. Pemberton, prove that this great philosopher had, so early as 1666, discovered the principal theorems on centrifugal force, which Huygens published sixteen years afterwards, at the end of his work De Horologio Oscillatorio; indeed it is extremely probable that the author of the method of fluxions, who seems then to have been at that time in possession of it, should easily have discovered these theorems. Newton arrived at the law of the diminution of gravity, by the relation which subsists between the squares of the periodic times of the planets, and the cubes of the greater axes of their orbits, supposed circular. He demonstrated that this relation exists in elliptic orbits generally, and that it indicates an equal gravity of the planets towards the Sun, supposing them at an equal distance from its centre. The same equality of gravity towards the principal planet, exists likewise in all the systems of satellites, and Newton verified it on terrestrial bodies, by very accurate experiments. Whence it results, that the development of gases, of electricity, of heat, and of affinities, in the mixture of several substances contained in a closed vessel, do not alter the weight of the system, neither during, nor after the mixture.

This great geometrician, by considering the question generally, demonstrated that a projectile can move in any conic-section whatever, in consequence of a force directed towards its centre, and varying reciprocally as the square of the distances. He investigated the different properties of motion in this species of curves; he determined the conditions requisite for the section to be a circle, an ellipse, a parabola, or an hyperbola, which conditions depend entirely on the velocity and primitive position of the body.

Any velocity, position, and initial direction of a body being given, Newton assigned the conic section which the body should describe, and in which it ought consequently to move, which refutes the objectiou advanced by John Bernouilli against him of not having demonstrated, that the conic sections are the *only* curves which a body, solicited by a force varying reciprocally as the squares of the distance, can describe. These investigations, applied to the motion of comets, informed him that these bodies move round the Sun, according to the same laws as the planets, with the difference only of their ellipses being very excentric ; and he indicated the means of determining by observations the elements of these ellipses.

He learned from the comparison of the distances and durations of the revolutions of the satellites, with those of the planets, the respective densities and masses of the Sun, and of planets accompanied by satellites, and the intensity of the force of gravity at their surface.

By considering that the satellites move round their planets very nearly, as if the planets were immoveable, he discovered that all these bodies obey the same force of gravity towards the Sun.

The equality of action and reaction, did not permit him to doubt, but that the Sun gravitated towards the planets, and these towards their satellites ; and even that the Earth is attracted by all the bodies that gravitate on it. He extended this proposition afterwards to all the celestial bodies, and established as a principle, *that each particle of matter attracts all others directly as its mass, and inversely as the square of its distance from the attracted particle.*

Arrived at this principle, Newton saw that the great phenomena of the system of the world might be deduced from it. By considering the gravity at the surfaces of the celestial bodies, as the result of the attractions of all their particles, he ascertained this remarkable and characteristic property of the law of attraction varying inversely as the square of the distance, namely that two spheres composed of concentrical strata and of densities varying according to any given law, attract each other mutually as if their masses were united in their centres ; hence the bodies of the solar system act on each other, and likewise on the bodies placed at their surfaces, as so many attracting points, which result contributes to the regularity of their motions, and enabled this great geometer to recognise the terrestrial gravity in the force which retains the moon in its orbit.

He proved that the motion of rotation of the Earth ought to have flattened it in the direction of the poles, and he determined the law of the variation of the degrees and of gravity at its surface.

He demonstrated that the action of the Sun and Moon on the terrestial spheroid combined with its rotatory motion, ought to produce the retrograde motion of the equinoxes, to elevate the waters of the ocean, and to produce in this great fluid mass the oscillations which are observed under the name of tides.

Lastly, he was convinced that the lunar inequalities and the retrograde motion of the nodes were produced by the combined action of the Sun and Earth on this satellite. This principle is not simply a hypothesis that satisfies phenomena, which may be otherwise explained, as the equations of an indeterminate problem may be satisfied in different ways. Here the problem is determined by laws observed in the celestial motions, of which this principle is a necessary result. The gravity of the planets towards the Sun is demonstrated by the proportionality of the areas to the times; the diminution in the inverse ratio of the squares of the distances is proved by the ellipticity of the planetary orbits; and the law of the squares of the times of the revolutions, proportional to the cubes of the greater axes, demonstrates that the solar gravity would act equally on all bodies supposed at the same distance from the Sun, of which the weight would therefore be proportional to the masses. The equality of action to reaction shews that the Sun gravitates in its turn towards the planets, proportionably to their masses divided by the squares of their distances from this star; the motions of the satellites prove that they gravitate at the same time to the Sun and to the planets, which reciprocally gravitates towards them, so that there exists between all the bodies of the solar system a mutual attraction directly proportional to the masses, and inversely as the squares of the distances. Finally their spherical figure, and the phenomena of gravity at the surface of the earth, do not permit us to suppose that this attraction appertains to the bodies considered in mass, but belongs to each of their particles.

Considering then the elevation of the earth at the equator as a system of satellites adhering to its surface, he found that the combined actions of the sun and moon tended to make the nodes of the circles which they describe about the axis of the earth, to retrograde, and that all these tendencies being communicated to the entire mass of the planet ought to produce, in the intersection of the ecliptic and equator, that slow retrogradation which is termed *the precession of the equinoxes.*

Thus the cause of this phenomenon depends on the compression of the earth, and on the retrograde motion which the action of the Sun communicates to the nodes, two facts which Newton first announced, and which could not before his time be suspected. Kepler himself, carried along by an ardent imagination to explain every thing by hypotheses, was obliged to admit the inutility of his efforts on this subject. But, with the exception of what concerns the elliptic motion of the planets and comets, the attraction of spherical bodies, and the intensity of gravity at the surface of the Sun, and of those planets that are accompanied by satellites, all these discoveries were only sketched by Newton. His theory of the figure of the planets is limited by the supposition of their homogeneity: his solution of the problem of the precession of the equinoxes, though very ingenious, is, notwithstanding the apparent agreement of his result with observation, in many respects defective; among the great number of the perturbations of the celestial motions, he has only considered those of the lunar motion, of which the most considerable, the evection, escaped his investigation. He perfectly established the existence of the principle which he discovered, but the developement of its consequences and its advantages, has been the work of the successors of this great geometrician. The state of imperfection in which the infinitesmal calculus must have been in the hands of its inventor, did not permit him to resolve completely the difficult problems which the theory of the system of the world presents; and he has been often obliged to give conjectures, always uncertain, till they have been verified by a rigorous calculation. Notwithstanding these inevitable defects, the importance and extent of his discoveries, the great number of original and profound conceptions, which have been the germ of the most brilliant theories of the geometricians of this century, and arranged with much elegance, insures to his *Principia* a pre-eminence over all other productions of the human intellect.

The case is not the same with the sciences as with general literature: this has limits which a man of genius may reach, when he employs a language brought to perfection; he is read with the same interest in all ages; and time only adds to his reputation by the vain efforts of those who try to equal him.

The sciences, on the contrary, like nature herself, without bounds, indefinitely increase by the labours of successive generations, the most perfect work; and thus by raising them to a height from which they can never again descend, gives birth to new discoveries, which produce in their turn new works, which efface those from which they

x 2

originated. Others will present in a point of view more general and more simple, the theories detailed in the *Principia*, and all the truths which it has brought to light; but it will ever remain as an eternal monument of the profundity of that genius which has revealed to us the greatest law of the universe.

This work, and the equally original treatise by the same author on optics, have still the merit of being the best models which can be proposed in the sciences, and in the delicate art of making experiments and submitting them to calculation. We there see the most beautiful applications of the method, which consists in tracing the principal phenomena to their causes, by a succession of inductions, and afterwards in redescending from these causes, to all the details of the phenomena.

General laws are impressed in all individual cases, but they are complicated with so many extraneous circumstances, that the greatest address is often necessary to develope them. The phenomena most proper for this object must be chosen, and these must be multiplied by varying the attendant circumstances, so that whatever they have in common may be observed.

We thus ascend successively to relations more and more extended, until we arrive at length at general laws, which are verified either by proofs or by direct experiment, if that is possible, or by examining if they satisfy all the known phenomena.

This is the most certain method by which we can be guided in the search of truth. No philosopher has adhered more closely to this method than Newton; none ever possessed, in a higher degree, that felicitous tact of discerning in objects the general principles involved in them, and which enabled him to recognise in the fall of bodies, the principle of universal gravitation. Other philosophers in England, cotemporaries of Newton, adopted the method of inductions by his example, which thus became the basis of a great number of excellent works in physics and analysis.

The philosophers of antiquity following a contrary path, and considering themselves as the source of every thing, imagined general causes to explain them.

Their method, which was only productive of vain systems, had not greater success in the hands of Descartes. In the time of Newton, Leibnitz, Malebranche and other philosophers employed it with as little advantage.

At length the inutility of the hypotheses to which it led its followers, and the progress for which the sciences are indebted to the method of inductions, have recalled all philosophers to this last method, which was explained by Chancellor Bacon, with the whole force of reason and eloquence, and which Newton has yet more strongly recommended by his discoveries.

At the period of their appearance, Descartes had substituted for the occult qualities of the Peripatetics, the intelligible ideas of motion, of impulse, and centrifugal force. His ingenious system of vortices, founded on these ideas, were received with avidity by the learned, who rejected the obscure and trifling doctrines of the school; and they thought that they perceived to arise in the doctrine of universal gravitation, those occult qualities which the French school had so justly proscribed. It was not till after the vagueness of Descartes' explanation was recognised, that attraction was considered as it ought to be, *i. e.* as a general fact, to which Newton was led by a series of inductions, and from which he descended again to explain the heavenly motions. This great man would justly have merited the reproach of re-establishing the occult qualities, if he was content to ascribe to universal attraction, the elliptic motion of the planets and of the comets, the inequalities of the motion of the Moon, those of terrestrial degrees and of gravity, the precession of the equinoxes and the ebbing and flowing of the sea, without demonstrating the connection of his principle with the phenomena. But as the Geometers who corrected and generalized these his demonstrations, and compared all the observations to the same principle, found the most perfect agreement between them and the results of analysis; they therefore have unanimously adopted his theory of the system of the world, which has thus become, by their researches, the basis of all Astronomy.

This analytical connection of particular with general facts, is what constitutes a theory. It is thus that having deduced, by a rigorous calculus, all the effects of capillary action, from the sole principle of a mutual attraction between the particles of matter, which is only sensible at imperceptible distances, we may presume that we have found out the true theory of these phenomena. Some philosophers, struck with the advantages which the admission of unknown causes have produced in several branches of the natural sciences, have brought back the occult qualities of the ancients, and their trifling explanations. Viewing the Newtonian philosophy under the same point of view which made it reject the Cartesians, they have assimilated their doctrines to it; which, however, have nothing common in the most essential circumstance, namely, the rigorous agreement of the results with the phenomena.

It is by means of synthesis that this great geometrician has explained his theory of the system of the world. It appears, however, that he discovered the greater part of his theorems by analysis, the limits of which he has considerably extended, and to which he allows himself to have owed his general results on the quadratures of curves.

But his great predilection for synthesis, and his esteem for the geometry of the ancients, has induced him to represent his theorems, and even his method of fluxions, under a synthetic form. And it is evident, by the rules and examples which he has given of these transformations in many works, how much importance he attached to it. We may regret with the geometricians of his time, that he has not followed in the exposition of his discoveries, the path by which he arrived at them; and that he has suppressed the demonstration of many results, such as the equation of the solid of least resistance, preferring the pleasure of leaving it to be divined, to that of enlightening his readers.

The knowledge of the method which has guided a man of genius is not less serviceable to the progress of the sciences, and even to his own glory, than his discoveries. This method is frequently the most interesting part; and if Newton, instead of merely announcing the differential equation of the solid of least resistance, had, at the same time, furnished the analysis of it, he would have the honour of giving the first essay on the method of variations, one of the most fruitful branches of modern analysis; and his example has perhaps prevented his countrymen, from contributing as much as they might to the advancement, which astronomy has made, from the application of analysis to the principle of universal gravitation.

The preference of Newton for the synthetical method, may be explained by the elegance with which he connected his theory of curvilinear motion with the investigations of the ancients on the conic sections, and the beautiful discoveries which Huygens had published according to this method. Geometrical synthesis has besides the property of never losing sight of its object, and of enlightening the whole path which leads from the first axioms to their last consequences, while algebraic analysis soon makes us forget the principal object, to occupy ourselves with abstract combinations, and it is only at the end that it brings us back to it. But though it thus separates itself from the object of investigation, after having assumed what is indispensably necessary to arrive at the required result; still by directing our attention to the operations of analysis, and reserving all our forces to overcome the difficulties which present themselves, we are conducted by the universality of this method, and by the inestimable advantage of thus transferring the train of reasoning into mechanical processes, to results often inaccessible to synthesis. Such is the fecundity of analysis, that if we translate particular truths into this universal language, we shall find a number of new and unexpected truths arise merely from the form of expression. No language is so susceptible of the elegance which arises from the developement of a long train of expressions connected with each other, and all flowing from the same fundamental idea. Analysis unites to all these advantages, that of always being able to conduct us to the most simple methods. Nothing more is requisite than to apply it in a convenient manner by a judicious selection of unknown quantities, and by giving to the results the form most easily reducible to geometrical construction, or to numerical calculation. Newton himself furnishes many examples in his Universal Arithmetic. The geometricians of this century, convinced of its superiority, have principally applied themselves to extend its domain, and enlarge its boundaries.

However, geometrical considerations ought not to be abandoned; they (*a*) are of the greatest utility in the arts. Besides, it is curious to show how the different results of analysis may be represented in space; and reciprocally, to read all the affections of lines and surfaces, and all the variations in the motions of bodies, in the equations which express them. This connection of geometry and analysis, diffuses a new light over the sciences; the intellectual operations of the latter, rendered perceptible by the images of the former, are more easy to comprehend, and more interesting to pursue; and when observations realizes, and transforms these geometrical results into laws of nature, and when these, embracing the whole universe, display to our view its present and future state, the view of this sublime spectacle presents to us one of the most noble pleasures reserved for mankind.

About fifty years passed after the discovery of the theory of gravitation, without any remarkable addition to it. All this time had been requisite for this great truth to be generally understood, and to surmount the obstacles opposed to it by the system of vortices, and the authority of geometricians contemporary with Newton, who combated it perhaps from vanity, but who have nevertheless accelerated its progress by their labours on the infinitesmal analysis.

Among the contemporaries of Newton, Huygens, who appears more than any other to have appreciated the value of this discovery, admits the gravitation of the heavenly bodies towards each other in the inverse ratio of the squares of the distances, and all the results which Newton deduced relative to the elliptic motion of the planets, of the satellites and comets, and relative to the gravity at the surfaces of planets, which are accompanied by satellites. On these points he rendered to Newton all the justice to which he was entitled. But his erroneous notions respecting the cause of gravity, made him to reject the mutual attraction of molecules, and the theories of the figure of the planets and of the variation of gravity at their surface, which depends on it. It must however be observed, that the law of universal gravitation had not, for Newton himself and his cotemporaries, all the certainty which the subsequent progress of observations and of mathematical sciences has secured to it. Euler and Clairault, who first with D'Alembert applied analysis to the perturbations of the celestial motions, did not deem it sufficiently established, to attribute to the inaccuracies of approximations and computations, the differences which were found to exist between observation and their results, on the motions of Saturn and the Lunar Perigee. But these three great Geometers having rectified these results, perfected the methods, and carried the approximation as far as is necessary, succeeded in explaining by the sole law of universal gravitation all the phenomena of the system of the world, and have thus assigned to the tables and astronomical theories a precision which could not be anticipated.

It is about three centuries since Copernicus first introduced into his tables the motions of the Earth, and of the planets round the Sun; about a century after Kepler introduced the laws of elliptic motion, which depend on the solar attraction alone; now they contain the numerous inequalities, which arise from the mutual attraction of all the bodies of the solar system, so that all empiricism is banished, and they only borrow from observations indispensable data.

It is principally in the application of analysis that the power of this wonderful instrument is evinced, without which it would be impossible to penetrate a mechanism so complicated in its effects, at the same time that it is so simple in its cause. The Geometer now embraces in his formulæ the entire of the solar system and its successive variations; he can ascend in imagination to the various changes which this system has undergone at the remotest periods, and he can redescend to all those which time will reveal to observers. He perceives those great changes, of which the entire developement requires millions of years, to be repeated in a few centuries, in the system of the satellites of Jupiter, by the quickness of their revolutions, and thus to produce those remarkable phenomena, just conjectured by Astronomers, but which were too complicated or too slow to enable them to determine the laws. The theory of gravity becomes, by so many applications, a means of discovering, as certain as observation itself; it has made known a great number of new inequalities, of which the most remarkable are the inequalities of Jupiter and Saturn, and the secular inequalities of the Moon with respect to its nodes, to its perigee and the Sun. By this means the Geometer has known to derive from his observations, as from a fruitful source, the most important elements of the system of the world, which would remain for ever concealed, without the aid of analysis. He has determined the respective values of the masses of the Sun, of the planets, and of the satellites, by the revolutions of these different bodies, and by the developement of their periodic and secular inequalities: the velocity of light, and the ellipticity of Jupiter have been made known to him by the eclipses of the satellites with more precision than by direct observation: he has inferred the rotation of Uranus, of Saturn, and of its ring, and the ellipticity of these two planets, from the respective position of the orbits of their satellites; the parallaxes of the Sun and of the Moon, and the ellipticity itself of the earth, are indicated in the lunar inequalities; as we have already seen that the moon by its motion reveals to astronomy, when brought to perfection, the compression of the earth, of which it made known the round form to the first observers, by its eclipses. Finally, by a fortunate combination of analysis with observations of the Moon, which seems to have been given to the Earth to illuminate it in the night, it has become the surest guide to the mariner, which guards him from the dangers to which he has for a long time been exexposed by the errors of his reckoning.

The perfection of the lunar theory, to which this inestimable advantage is owing, and that of determining accurately the position of the places where he lands, is the fruit of the labours of Geometers for the last half century, so that during this short interval, geography, by the use of the lunar tables and of the chronometer, has made more progress than in all preceding ages. These sublime theories thus combine every thing which can give importance to discovery: the greatness and utility of the object, the fruitfulness of the result, and the merit of surmounted difficulties.

It has been necessary, for this object, to bring to perfection at the same time mechanics, optics, and analysis, which principally owe their rapid improvements to their being necessary to the purposes of physical astronomy. It might be rendered yet more correct and simple, but future ages will no doubt see with gratitude that the geometricians of this century have transmitted no astronomical phenomenon to posterity, of which they have not determined the cause and the law.

Justice to France requires us to observe, that if England had the advantage of giving birth to the discovery of universal gravitation, it is principally to the French geometricians, and to the encouragements of the Academy of Sciences, that the numerous developments of this discovery are due, and the revolution which it has produced in astronomy. (*b*)

The attraction which regulates the motions and the figures of the heavenly bodies, is not the only force which exists between their molecules; they are also subject to attractive forces, which depend on the internal constitution of these bodies, and which are only sensible at distances inappreciable to our senses. Newton gave the first example of the calculation of this species of force by demonstrating that in the passage of light from one transparent medium to another, the attraction of the media refracts it so, that the sines of refraction and incidence are always in a constant ratio: experience had previously made known this. He has moreover conjectured that cohesion, all chemical affinities and capillary phenomena arise from the action of similar forces; but the explanations which he gave of them are not satisfactory, and the complete mathematical theory of these phenomena is the work of his successors.

Is the principle of universal gravitation a primordial law of nature, or is it a general effect from an unknown cause? Can we not reduce to this principle, the explanation of affinities? Newton, more wary than his successors, has not ventured to pronounce on these questions, which, in our present ignorance respecting the intimate properties of matter, we cannot answer satisfactorily. Instead of forming hypothesis, we shall restrict ourselves to some reflections on this principle, and on the manner in which it has been employed by geometers.

Newton inferred from the equality between action and reaction, that each molecule of a body should attract as it is attracted, and that consequently the gravity is the resultant of the attractions of all the molecules of the attracting body. The principle of action being equal to reaction, is embarrassing, when the mode of action of the forces is unknown. Thus Huygens, who had founded on this principle his investigations on the collisions of elastic bodies, did not find it sufficient to establish the mutual attraction of each molecule. It was therefore necessary to confirm this attraction by observation, in order to remove every doubt on this important point of the Newtonian theory. The celestial phenomena may be divided into three classes. The first comprehends all those which depend on the mutual tendency of the heavenly bodies towards each other; such are the elliptic motions of the planets and satellites, and their reciprocal perturbations, which are independent of their figures. Under the second class are contained those phenomena which are produced by the tendency of the molecules of the attracted body towards the centres of the attracting bodies; such are the ebbing and flowing of the tide, the precession of the equinoxes, and the libration of the Moon. Finally, I have arranged under the third class, the phenomena which depend on the action of the molecules of the attracting bodies, on the centres of those which are attracted, and on their own molecules. The two lunar inequalities which arise from the compression of the Earth, the motion of the orbits of the satellites of Jupiter and Saturn, the figure of the Earth and the variation of gravity at its surface, are phenomena of this kind. The Geometers who, in order to explain the cause of gravity, surround each of the heavenly bodies with a vortex, ought to admit the Newtonian theories relative to the phenomena of the two first classes; but they ought to reject, as Huygens did, the theories of the phenomena of the third class, founded on the action of the molecules of the attracting bodies. The perfect agreement of these theories with all observations, ought now to remove every doubt of the mutual attraction of the molecules. The law of attraction, inversely as the square of the distance, is that of emanations which proceed from a centre. It appears to be the law of all forces, of which the action is sensible at a distance, as has been recognised in electrical and magnetic forces. Hence, as this law corresponds exactly to all the phenomena, it should be regarded from its simplicity and generality, as rigorously true. One of its remarkable properties is, that if the dimensions of all the bodies in the universe, their mutual distances and velocities, increase or diminish proportionably, they would describe curves entirely similar to those which they at present describe; so that if the universe be successively reduced to the smallest imaginable space, it would always present the same appearances to observers. These appearances are consequently independent of the dimensions of the universe, as they are also, in consequence of the law of the proportionality of the force to the velocity, independent of the absolute motion which it may have in space. The simplicity of the laws of nature therefore only permits us to observe the relative dimensions of the universe. (*b*)

In the law of attraction, the heavenly bodies attract each other very nearly as if their masses were united in their centres of gravity; their surfaces and orbits also assume in this law the elliptical form, which is the simplest after the spherical and circular, which last the ancients deemed to be essential to the stars and their motions.

Is the attraction communicated instantaneously from one body to another? The time of its transmission, if it was sensible to us, would be particularly evinced in a secular acceleration of the Moon's motion. I suggested this as a means of explaining the acceleration which is observed in this motion; and I have found, that in order to satisfy observations, we must ascribe to the force of gravity, a velocity seven million of times greater than that of a ray of light. As the cause of the secular equation of the Moon (*c*) is now well ascertained, we may affirm that the attraction is transmitted fifty millions of times more rapidly than light. We can therefore assume, without any apprehension of error, that its transmission is instantaneous.

The attraction may also produce, and continually maintain the motion in a system of bodies which were primitively in repose; for it is not true, as some philosophers have asserted, that it must at length reunite them all about their common centre of gravity. The only elements which must always remain equal to nothing, are the motion of this centre, and the sum of the areas described about it, in a given time, by all the molecules of the system projected on any plane whatever.

CHAP. VI.

Considerations on the system of the World, and future progress of Astronomy.

The preceding summary of the history of Astronomy presents three distinct periods, which referring to the phenomena, to the laws which govern them, and to the forces on which these laws depend, point out the career of this science during its progress, and which consequently ought to be pursued in the cultivation of other csiences. The first period embraces the observations made by Astronomers antecedently to Copernicus, on the appearances of the celestial motions, and the hypotheses which were devised to explain those appearances, aud to subject them to computation. In the second period, Copernicus deduced from these appearances, the motions of the Earth on its axis and about the Sun, and Kepler discovered the laws of the planetary motions. Finally in the third period, Newton, assuming the existence of these laws, established the principle of universal gravitation; and subsequent

Geometers, by applying analysis to this principle, have derived from it all the observed phenomena, and the various inequalities in the motion of the planets, the satellites, and the comets. Astronomy thus becomes the solution of a great problem of mechanics, the constant arbitraries of which are the elements of the heavenly motions. It has all the certainty which can result from the immense number and variety of phenomena, which it rigorously explains, and from the simplicity of the principle which serves to explain them. Far from being apprehensive that the discovery of a new star will falsify this principle, we may be antecedently certain that its motion will be conformable to it; indeed this is what we ourselves have experienced with respect to Uranus and the four telescopic stars recently discovered, and every new comet which appears, furnishes us with an additional proof.

Such is unquestionably the constitution of the solar system. The immense globe of the Sun, the focus of these motions, revolves upon its axis in twenty-five days and a half. Its surface is covered with an ocean of luminous matter. Beyond it the planets, with their satellites, move, in orbits nearly circular, and in planes little inclined to the ecliptic. Innumerable comets, after having approached the Sun, recede to distances, which evince that his empire extends beyond the known limits of the planetary system. This luminary not only acts by its attraction upon all these globes, and compels them to move around him, but imparts to them both light and heat; his benign influence gives birth to the animals and plants which cover the surface of the Earth, and analogy induces us to believe, that he produces similar effects on the planets; for, it is not natural to suppose that matter, of which we see the fecundity develope itself in such various ways, should be sterile upon a planet so large as Jupiter, which, like the Earth, has its days, its nights, and its years, and on which observation discovers changes that indicate very active forces. Man, formed for the temperature which he enjoys upon the Earth, could not, according to all appearance, live upon the other planets; but ought there not to be a diversity of organization suited to the various temperatures of the globes of this universe? If the difference of elements and climates alone causes such variety in the productions of the Earth, how infinitely diversified must be the productions of the planets and their satellites? The most active imagination cannot form any just idea of them, but still their existence is, at least, extremely probable.

However arbitrary the elements of the system of the planets may be, there exists between them some very remarkable relations, which may throw light on their origin. Considering it with attention, we are astonished to see all the planets move round the Sun from west to east, and nearly in the same plane, all the satellites moving round their respective planets in the same direction, and nearly in the same plane with the planets. Lastly, the Sun, the planets, and those satellites in which a motion of rotation have been observed, turn on their own axes, in the same direction, and nearly in the same plane as their motion of projection.

The satellites exhibit in this respect a remarkable peculiarity. Their motion of rotation is exactly equal to their motion of revolution; so that they always present the same hemisphere to their primary. At least, this has been observed for the Moon, for the four satellites of Jupiter, and for the last satellite of Saturn, the only satellites whose rotation has been hitherto recognized.

Phenomena so extraordinary, are not the effect of irregular causes. By subjecting their probability to computation, it is found that (*a*) there is more than two thousand to one against the hypothesis that they are the effect of chance, which is a probability much greater than that on which most of the events of history, respecting which there does not exist a doubt, depends. We ought therefore to be assured with the same confidence, that a primitive cause has directed the planetary motions.

Another phenomenon of the solar system equally remarkable, is the small excentricity of the orbits of the planets and their satellites, while those of comets are very much extended. The orbits of this system present no intermediate shades between a great and small excentricity. We are here again compelled to acknowledge the effect of a regular cause; chance alone could not have given a form nearly circular to the orbits of all the planets. It is therefore necessary that the cause which determined the motions of these bodies, rendered them also nearly circular. This cause then must also have influenced the great excentricity of the orbits of comets, and their motion in every direction; for, considering the or bits of retrograde comets, as being inclined more than one hundred degrees to the ecliptic, we find that the mean inclination of the orbits of all the observed comets, approaches near to one hundred degrees, which would be the case if the bodies had been projected at random. (*b*)

What is this primitive cause? In the concluding note of this work I will suggest an hypothesis which appears to me to result with a great degree of probability, from the preceding phenomena, which however I present with that diffidence, which ought always to attach to whatever is not the result of observation and computation.

Whatever be the true cause, it is certain that the elements of the planetary system are so arranged as to enjoy the greatest possible stability, unless it is deranged by the intervention of foreign causes. From the sole circumstance that the motions of the planets and satellites are performed in orbits nearly circular, in the same direction, and in planes which are inconsiderably inclined to each other, the system will always oscillate about a mean state, from which (*c*) it will deviate but by very small quantities. The mean motions of rotation and of revolution of these different bodies are uniform, and their mean distances from the foci of the principal forces which actuate them are constant; all the secular inequalities are periodic.

The most considerable are those which affect the motions of the Moon, with respect to its perigee, to its nodes and the Sun; they amount to several circumferences, but after a great number of centuries they are reestablished. In this long interval all the parts of the lunar surface would be successively presented to the earth, if the attraction of the terrestrial spheroid, which causes the rotation of the Moon to participate in these great inequalities, did not continually bring back the same hemisphere of this satellite to us, and thus render the other hemisphere (*d*) for ever invisible. It is thus that the primitive attraction of the three first satellites of Jupiter originally established, and maintains the relation which is observed between their mean motions, and which consists in this, that the mean longitude of the first satellite minus three times that of the second, plus twice that of the third is equal to two right angles. In consequence of the celestial attractions the duration of the revolution of each planet is always very nearly the same. The change of inclination of its orbit to that of its equator being confined within narrow limits, only produces slight changes in the seasons. It seems that nature has arranged every thing in the heavens, to secure the continuation of the planetary system, by views similar to those which she appears to follow so admirably on the earth, for the preservation of individuals and the perpetuity of the species. It is principally to the attraction of the great bodies which are placed in the centre of the system of the planets, and the system of the satellites, that the stability of these systems is due, which the mutual action of all the bodies of the system, and extraneous attractions tend to derange. If the action of Jupiter ceased; his satellites, which now appear to move with such admirable regularity, would be immediately disturbed, and each would describe about the Sun a very excentric ellipse; (*e*) others would recede indefinitely in hyperbolic orbits. Thus an attentive inspection of the solar system evinces the necessity of some paramount central force, in order to maintain the entire system together, and secure the regularity of its motions.

These considerations of themselves will be sufficient to explain the disposition of this system, unless the Geometer extends his view farther, and seeks, in the primordial laws of nature, the cause of the most remarkable phenomena of the universe. Some have been already reduced to these laws. Thus the stability of the poles of the Earth, and that of the equilibrium of the seas, which are both necessary for the preservation of organised beings, are simple consequences of the rotation of the earth and of universal gravitation. By its rotation the earth has been compressed, and its axis of revolution is become one of the principal axes about which the motion of rotation is invariable. In consequence of this gravity the denser strata are nearer to the centre of the earth, of which the mean density thus surpasses that of the waters which surround it, which is sufficient to secure the stability of the equilibrium of the seas, and to put a check to the fury of the waves; in fine, if the conjectures which I have proposed on the origin of the planetary system have any foundation, the stability of this system is also a consequence of the laws of motion. (*f*) These phenomena, and some others which are explained in a similar manner, induce us to think that every thing depends on these laws by relations more or less concealed; but of which it is wiser to avow our ignorance than to substitute imaginary causes, for the sole purpose of dissipating our anxiety. I must here remark how Newton has erred on this point, from the method which he has otherwise so happily applied. Subsequently to the publication of his discoveries on the system of the world and on light, this great philosopher abandoned himself to speculations of another kind, and inquired what motives induced the author of nature to give to the solar system its present observed constitution. After detailing in the scholium which terminates the principles of natural philosophy, the remarkable phenomenon of the motions of the planets and of the satellites in the same direction, very nearly in the same plane, and in orbits Q. P. circular, he adds, all these motions, so very regular, do not arise from mechanical causes, because the comets move in all regions of the heavens, and in orbits very excentric. (*g*) "This ad-"mirable arrangement of the Sun, of the planets,

" and of the comets, can only be the work of an " intelligent and most powerful being." At the end of his optics he suggests the same thought, in which he would be still more confirmed, if he had known that all the conditions of the arrangement of the planets and of the satellites are precisely those which secure their stability. " A blind " fate," says he, " could never make all the pla- " nets to move thus, with some irregularities " hardly perceivable, which may arise from the " mutual action of the planets and of the comets, " and which, probably, in the course of time will " become greater, till in fine the system may re- " quire to be restored by its author." But could not this arrangement of the planets be itself an effect of the laws of motion; and could not the supreme intelligence which Newton makes to interfere, make it to depend on a more general phenomenon? such as, according to us, a nebulous matter distributed in various masses throughout the immensity of the heavens. Can one even affirm that the preservation of the planetary system entered into the views of the Author of Nature? The mutual attraction of the bodies of this system cannot alter its stability, as Newton supposes; but may there not be in the heavenly regions another fluid besides light? Its resistance, and the diminution which its emission produces in the mass of the Sun, ought at length to destroy the arrangement of the planets, so that to maintain this, a renovation would become evidently necessary. And do not all those species of animals which are extinct, but whose existence Cuvier has ascertained with such singular sagacity, and also the organization in the numerous fossil bones which he has described, indicate a tendency to change in things, which are apparently the most permanent in their nature? The magnitude and importance of the solar system ought not to except it from this general law; for they are relative to our smallness, and this system, extensive as it appears to be, is but an insensisble point in the universe. If we trace the history of the progress of the human mind, and of its errors, we shall observe final causes perpetually receding, according as the boundaries of our knowledge are extended. These causes, which Newton transported to the limits of the solar system, were, in his time, placed in the atmosphere in order to explain the cause of meteors: in the view of the philosopher, they are therefore only an expression of our ignorance of the true causes.

Leibnitz, in his controversy with Newton, relative to the invention of the infinitesmal calculus, attacks him with great force on account of his introducing the divinity to restore order into the solar system. "It is," says he, "to have too "confined notions of the wisdom and power of "the Deity." Newton rejoined by an equally severe critique on the preestablished harmony of Leibnitz, which he denominated a continual miracle. Subsequent ages have not admitted these vain hypotheses; they have, however, rendered the most ample justice to the mathematical labours of these two great men; the discovery of universal gravitation, and the efforts of its author to explain all the heavenly phenomena by means of it, will for ever secure to him the admiration and gratitude of posterity.

Let us now pass in imagination beyond the solar system to the innumerable suns distributed in the immensity of space, at such a distance from us, that the entire diameter of the terrestrial orbit, observed from their centre, would be insensible. Several stars experience in their colour and splendour remarkable periodical changes, which indicate the existence, at the surface of these stars, of great spots, which their motion of rotation alternately presents and removes from our view. Other stars, on the contrary, have suddenly appeared, and then disappeared, after having shone for several months with the most brilliant splendour. Such was the star observed by Tycho Brahe in the year 1572, in the constellation Cassiopeia. In a short time it surpassed the most brilliant stars, and even Jupiter himself. Its light then waned away, and finally disappeared sixteen months after its discovery. Its colour underwent several changes; it was at first of a brilliant white, then of a reddish yellow, and finally of a lead coloured white like to Saturn. What great changes must take place on these great bodies, in order that they may be perceptible at the distance which intervenes between them and us? How much must they surpass those which are observed on the surface of the Sun, and convince us that nature is far from being always every where the same. All these stars, after they become invisible, do not change their place during their appearance. Therefore there exists, in the immensity of space, opaque bodies as considerable in magnitude, and perhaps equally numerous as the stars.

It appears that far from being distributed at distances which are nearly equal, the stars are disposed in groups, some of which contain thousands of these objects. Our Sun, and the most brilliant stars, probably constitute part of one of those groups, which, seen from the earth, appear to surround the earth, and form the milky way. The great number of stars, which are seen in the field of a powerful telescope, directed towards this way, evinces its immense distance, which is a thousand times greater than the distance of Syrius from the earth, so that it is probable that rays emanating from these stars have employed several centuries to reach the earth. To a spectator at an immense distance from the milky way, it would present the appearance of an uninterrupted band of white light, having a very inconsiderable diameter, for the irradiation which subsists even in our best constructed telescopes, would not cover the interval between the stars. It is therefore probable, that amongst the nebulæ several consist of groups of a great number of stars, which, viewed from their interior, appear similar to the milky way. If now we reflect on the profusion of stars and nebulæ distributed through the heavenly regions, and on the immense intervals between them, the imagination, struck with astonishment at the magnitude of the universe, will find it difficult to assign any limits to it.

Herschel, while observing the nebulæ by means of his powerful telescopes, traced the progress of their condensation, not on one only, as their progress does not become sensible until after the lapse of ages, but on the whole of them, as in a vast forest we trace the growth of trees, in the individuals of different ages which it contains. He first observed the nebulous matter diffused in several masses, through various parts of the heavens, of which it occupied a great extent. In some of these masses he observed that this matter was fully condensed about one or more nuclei, a little more brilliant. In other nebulæ, these nuclei shine brighter, relatively to the nebulosity which environs them. As the atmosphere of each nucleus separates itself by an ulterior condensation, there result several nebulæ constituted of brilliant nulcei very near to each other, and each surrounded by its respective atmosphere; sometimes the nebulous matter being condensed in a uniform manner, produces the nebulæ which are termed *planetary*. Finally, a greater degree of condensation transforms all these nebulæ into stars. The nebulæ, classed in a philosophic manner, indicate, with a great degree of probability, their future tranformation into stars, and the anterior state of the nebulosity of existing stars. Thus, by tracing the progress of condensation of the nebulous matter, we descend to the consideration of the Sun, formerly surrounded by an immense atmosphere, to which consideration we can also arrive, from an examination of the phenomena of the solar system, as we shall see in our last note. Such a marked coincidence, arrived at by such different means, renders the existence of this anterior state of the Sun extremely probable.

Connecting the formation of comets with that of nebulæ, they may be considered as small nebulæ, wandering from one solar system to another, and formed by the condensation of the nebulous matter which is so profusely distributed throughout the universe. The comets will be thus, relatively to our system, what the meteoric stones appear to be relatively to the earth, to which they do not appear to have originally belonged. (*e*) When these stars first become visible, they present an appearance perfectly similar to the nebulæ; so much so, that they are frequently mistaken for them, and it is only by their motion, or by our knowing all the nebulæ contained in our part of the heavens, that we are able to distinguish one from the other. This hypothesis explains, in a satisfactory manner, the increase of the heads and tails of the comets, according as they approach the sun, and the extreme rarity of their tails, (which, notwithstanding their great depth, do not sensibly diminish the brilliancy of the stars seen through them;) the motions of the comets, which are performed in every direction, and the great excentricity of their orbits.

From the preceding considerations, which are founded on telescopic observations, it follows, that the motion of the solar system is extremely complicated. The Moon describes an orbit nearly circular about the earth, but seen from the Sun, it appears to describe a series of epicycles, of which the centres exist on the terrestrial orbit. In like manner, the earth describes a series of epicycles, of which the centres lie on the curve, which the Sun describes about the common centre of gravity of the group of stars, of which it makes a part. Finally, the Sun himself describes a series of epicycles, of which the centres lie on the curve described by the centre of gravity of this group, about that of the universe. Astronomy has already made an important step, in making us acquainted with the motion of the earth, and the epicycles which the Moon and the satellites describe on the orbits of their respective primary planets. But if ages were necessary in order to know the motions of the planetary system, what a great length of time must be required for the determination of the motions of the Sun and the stars; notwithstanding this, such motions appear to be already indicated by observations. From all of them considered together, it has been inferred, that the bodies of the solar system are in motion towards the constellation Hercules; but however they at the same time seem to prove that the apparent motions of the stars result from a combination of their proper motions with that of the Sun. *(f)*

There are also remarked some singular motions in the *double* stars: we have denominated such, those stars which, when seen through a telescope, appear to be composed of two neighbouring stars. These two stars revolve about one another in a manner sufficiently sensible to enable us to determine for some of them, by means of a few years observation, the duration of their revolutions.

All these motions of the stars, their parallaxes, the periodic variations of the light of the changeable stars, and the durations of their motions of rotation; a catalogue of those stars which just appear and then disappear, and their position at the instant of their transient passage; finally, the successive changes in the figure of those nebulæ which are already sensible in some of them, and particularly in the beautiful nebula of Orion, will be, relatively to the stars, the principal objects of Astronomy in subsequent ages. Its progress depends on these three things: the measure of time, that of angles, and the perfection of optical instruments. The two first are nearly as perfect as we could wish; it is therefore to the improvement of the latter that our attention should be directed, for there can be no doubt but that if we succeeded in enlarging the apertures of our achromatic telescopes, they would enable us to discover in the heavens, phenomena which have been hitherto invisible, especially if we were able to remove them to the pure and rare atmosphere of the high mountains of the equator.

There are also numerous discoveries to be made in our own system. The planet Uranus and its satellites, which have been lately discovered, give grounds to suppose that other planets, as yet not observed, exist. It has been even conjectured that one must exist between Jupiter and Mars, in order to satisfy the double progression, which obtains (*g*) very nearly, between the intervals of the planetary orbits, to that of Mercury. This conjecture has been confirmed by the discovery of four small planets, whose distances from the Sun differ little from that, which this double progression assigns to a planet intermediate between Jupiter and Mars. The action of Jupiter on these planets increased by the magnitude of the excentricities and of the inclinations of the intersecting orbits, produces considerable inequalities in their motions, which throw new light on the theory of the celestial attractions, and will enable us to render them more perfect. The arbitrary elements of this theory, and the convergence of its approximations, depend on the precision of observations and on the progress of analysis, and this should thereby acquire every day more and more accuracy. The great secular inequalities of the heavenly bodies, which is a consequence of their mutual attractions, and which has been already indicated by observation, will be developed in the course of ages. By means of observations on the satellites, made with powerful telescopes, we shall be able to render their theory more perfect, and perhaps to discover new satellites. By accurate and repeated measures of the Earth, all the inequalities of the figure of the Earth, of gravity at its surface, will be determined, and in a short time all Europe will be covered with a chain of triangles, which will accurately determine the position, the curvature, and the magnitude of all its parts. The phenomena of the tides, and their remarkable varieties in the two hemispheres, will be determined by a long series of observations, compared with the theory of gravity. We will ascertain whether the motions of rotation and revolution of the Earth are sensibly changed by the changes which it experiences at its surface, and by the impact of meteoric stones, which, according to all probability, come from the depths of the heavenly regions. The new comets which will appear ; those which, moving in hyperbolic orbits, wander from one system to another ; the returns of those which move in elliptic orbits, and the changes in the form and intensity of light, which they undergo at each appearance, will be observed ; and also the peturbations which all those stars produce in the planetary motions, those which they experience themselves, and which, at their approach to a large planet, may entirely derange their motions ; finally, the changes which the motions and the orbits of the planets and satellites experience from the action of the stars, and perhaps likewise from the resistance of ethereal media ; such are the principal objects which the solar system offers to the investigations of future Astronomers and Mathematicians.

Astronomy, from the dignity of the subject, and the perfection of its theories, is the most beautiful monument of the human mind—the noblest record of its intelligence. Seduced by the illusion of the senses, and of self-love, man considered himself, for a long time, as the centre about which the celestial bodies revolved, and his pride was justly punished by the vain terrors they inspired. The labour of many ages has at length withdrawn the veil which covered the system. And man now appears, upon a small planet, almost imperceptible in the vast extent of the solar system, itself only an insensible point in the immensity of space. The sublime results to which this discovery has led, may console him for the limited place assigned to the Earth, by showing him his proper magnitude, in the extreme smallness of the base which he made use of to measure the heavens. Let us carefully preserve, and even augment the number of these sublime discoveries, which constitute the delight of thinking beings.

These indeed have rendered important services to navigation and astronomy ; but their great benefit consists in their having dissipated the alarms occasioned by extraordinary celestial phenomena, and thus exterminating the errors arising from the ignorance of our true relation with nature ; errors and apprehensions which would speedily spring up again, if the light of the sciences was extinguished.

NOTES.

NOTE I.

A SEPARATE history of the Chinese Astronomy was published by the Jesuit Gaubel, who appears to have been particularly well acquainted with the subject. He discussed again in much detail, in the 26th letter of the Instructive Letters, the ancient part of this history. I published in the Connaisance des Tems for the year 1809, an invaluable manuscript of the same Jesuit, on the solstices and meridian shadows of the gnomon observed at China. From these treatises it appears that Tcheou Kong observed the meridian shadows of a gnomon eight Chinese feet long, at the solstices in the city of Loyang, now called Honan Fou, in the province of Honan. He carefully traced the meridian, and levelled the earth on which the shadow was projected. He found the length of the meridian shadow to be one foot and a half at the summer solstice, and thirteen feet at the winter solstice. In order to infer from these observations the obliquity of the ecliptic, he applied several corrections to them; the most considerable is that of the Sun's semi-diameter, for it is evident that the extremity of the shadow of the gnomon indicates the height of the upper limb of this star; it is therefore necessary to subtract from this height the apparent semidiameter of the Sun, in order to obtain the height of its centre. It is strange, that so simple and essential a correction should have escaped the observation of all the old Astronomers of the Alexandrian school; it must have caused their geographical latitudes to have erred by a quantity very nearly equal to this semidiameter. A second correction respects the astronomical refraction, which not being observed, may without sensible error be supposed such as would correspond to a temperature of ten degrees, and to a height of the barometer equal to 0,76. Finally, a third correction respects the parallax of the Sun, and reduces these corrections to the centre of the Earth. By applying these three corrections, to the preceding observations, the height of the centre of the Sun, referred to the centre of the Earth, is found to be equal to 87°,9049 at the summer solstice, and to 34°,7924 at the winter solstice. These heights assign 38°,6513 for the height of the pole at Layang, which result differs very little from the mean between all the observations of the Jesuit missionary on the latitude of this city: they make the obliquity of the ecliptic at the epoch of Tcheou-Kong to be about 26,6563. This epoch may without sensible error be fixed at the year 1100 before our æra. If by means of the formula given in the sixth book of the treatise on Celestial Mechanics, we go back to this epoch, we shall find that the obliquity ought then to be equal to 26,5161. The difference 402″ will appear very inconsiderable, if we consider the uncertainty which exists relative to the masses of the planets, and that which the observations on the gnomon present, especially on account of the penumbra, which renders the umbra itself very indistinctly terminated.

Tcheou-Kong also observed the position of the winter solstice, with respect to the stars, and he fixed it at two Chinese degrees of *Nu*, a Chinese constellation, which commences with ϵ of Aquarius. In China, the division of the circumference was always regulated by the length of the year, so that the Sun described a degree every day; and the year at the epoch of Tcheau-Kong being supposed equal to $365^{d}\frac{1}{4}$; two degrees correspond to 2°,1905 of the decimal division of the quadrant of the circle. The stars having been at the same epoch referred to the equator, the right ascension of the star was, according to this observation, about 297°,8096. By the formulæ of the celestial mechanics it ought to be 298°,7265, in the year 1100 before our æra. In order to get rid of the difference 9169″, it is sufficient to go back fifty-four years beyond this, which is inconsiderable if we consider the uncertainty of the precise epoch at which this great prince made his observations, and particularly that of the observations themselves. There also exists an observation on the instant of the solstice, but the greatest error to apprehend is in the manner of referring the solstice to the star ε of Aquarius; whether Tcheou-Kong made use of the difference in time, between the passages of the star and Sun over the meridian, or whether he measured the distance of the Moon from this star, at the moment of the occurrence of a lunar eclipse, two means employed by the Chinese Astronomers.

NOTE II.

By means of a long series of observations, the Chaldeans recognised that in 19756 days, the Moon made 669 revolutions with respect to the Sun, 717 anomalistic revolutions, *i. e.* with respect to the points of its greatest velocity, and 726 revolutions with respect to its nodes. They added $\frac{4}{45}$ of a revolution to the position of those two stars, in order to obtain in this interval 723 sidereal revolutions of the Moon, and 54 of the Sun. Ptolemy, in explaining this period, attributes it to the ancient Astronomers, without specifying the Chaldeans; but Geminus, a cotemporary of Sylla's, whose treatise on Astronomy has come down to us, removes all doubt on this head, for he not only attributes this period to the Chaldeans, but he even gives their method for computing the anomaly of the Moon. They supposed that from the least to the greatest velocity of the Moon, its angular motion accelerated by a third of a degree every day, during one half of the anomalistic revolution, and that it retarded by the same quantity during the other half. He is mistaken in supposing that the increments, which are proportional to the cosines of the distance of the Moon from its perigee, are constant. Notwithstanding this error, the preceding method is creditable to the sagacity of the Chaldean Astronomers ; it is the only monument of this kind which remains previously to the foundation of the Alexandrian school. The period of which I have spoken supposes that the sidereal year is very nearly equal to $365\frac{1}{4}$; that of 365,2576, which Albaterius ascribed to the Chaldeans, cannot only belong to times posterior to Hipparcus.

NOTE III.

In the second book of his Geography, chap. iv., Strato states, that according to Hipparcus, the proportion of the shadow at Byzantium to the gnomon, is the same which Pythias asserts that he observed it to be at Marseilles ; and in the 5th chapter of the same book he quotes from Hipparchus, that at Byzantium at the summer solstice, the proportion of the shadow to the gnomon is that of 42 minus $\frac{1}{5}$ to 120. It is unquestionable from this observation, that Ptolemy, in the 6th chapter of the second book of the Almagest, makes the parallel on which the duration of the longest day of the year is, five-eighths of the astronomical day, to pass through Marseilles ; which supposes that the proportion of the meridian shadow to the gnomon at the summer solstice, is that of 42 minus $\frac{1}{5}$ to 120. Pytheas was at the latest, a contemporary of Aristotle ; therefore we may without sensible error refer his observation to the year 350 before our æra. By correcting it for the refraction, the parallax of the Sun, and its semidiameter, it makes the zenith solstitial distance of the centre of the Sun from the zenith of Marseilles, equal to 21,6386. The latitude of the Observatory of this city is 48,1077. If the preceding distance be subtracted from it, the obliquity of the ecliptic at the time of Pytheas comes out equal to 26,4691. This obliquity, when compared with that given in the time of Tcheou-Kong, indicates already a diminution in this element. From the formula given in the Celestial Mechanics, the obliquity of the ecliptic 350 years before our æra comes out equal to 26,4095; the difference 596″ between this result and that of the observation of Pytheas, is within the limits of the errors of this kind of observation.

NOTE IV.

Hipparcus found, from comparing together a great number of eclipses of the Moon, 1st, that in the interval of 126007^d plus $\frac{1}{24}$ of a day, the Moon performs 4267 revolutions with respect to the Sun, 4573 revolutions relatively to its perigee, and 4612 revolutions relatively to the fixed stars, minus eight degrees and one-third; 2dly, that during 5458 synodic months it performs 5923 revolutions relatively to its nodes. According therefore to this result, the motions of the Moon in the interval of $126007^d\frac{1}{24}$ are

with respect to the Sun 1706800°
with respect to the perigee 1829200°
with respect to the node 1852212°, 89368.

A comparison of these motions with those which have been determined by combining together all the modern observations, should render their acceleration, which is indicated by the theory of universal gravitation, very sensible. In fact, those who have thus determined it for the commencement of this century, assign, for the same interval of time, the preceding quantities increased respectively by + 2657″,0; + 10981″,9; + 432″,8. The acceleration of these three motions from the time of Hipparchus to the present, is evident: we see, moreover, that the acceleration of the motion of the Moon with respect to the Sun, is about four times less than that of its motion with respect to the perigee, whilst it surpassses considerably the acceleration of its motion with respect to the node. This is very nearly conformable to the theory of gravity, according to which, these accelerations are Q. P. in the ratio of the numbers 1; 4,70197; 0,38795. Hipparchus supposed that Babylon was more eastward than Alexandria, by 3472″ of time. According to the observations of Beauchamp it was still more eastward by 557″. This ought to increase a little the mean lunar motions, which Hipparchus inferred from a comparison of his observations, with those of the Chaldeans.

Ptolemy has not transmitted to us the epochs of the lunar motions of Hipparchus; but from the slight changes which he made in these motions, and from his always endeavouring to make his results approximate to those of this great Astronomer, we are justified in supposing that the epochs of Hipparchus differ very little from those of the tables of Ptolomy, which assign at the epoch of Nobonassar, *i. e.* the 26th of February of the year 746 before our æra, at mid-day, mean time of Alexandria,

distances from the Moon	to the Sun	78°, 4630
	to the perigee	98 , 6852
	to the ascending node	93 , 6111

If we go back to this epoch, by means of the mean motions determined for the commencement of this century, from the comparison of modern observations solely; if, moreover, we suppose, agreeably to the latest observations, that Alexandria is more eastward than Paris by 7731″,48 of time, we shall find the distances less than the preceding by the respective quantities — 1°,6316; — 7°,6569; — 0°,8205. These differences, which are much too great to be ascribed to the errors of either ancient or modern determinations, evince incontrovertibly the acceleration of the lunar motions, and the necessity of admitting the secular equations. The secular equation of the distance of the Sun from the Moon, which equation is the same as that of the mean motion of the Moon, since that of the Sun is uniform, becomes at the epoch of Nabonassar, 2°, 0480. In order to obtain those of the distance of the Moon from its perigee and its ascending node at the same epoch, it is necessary to multiply the preceding, by the numbers 4,70197 and 0,38795 respectively. Therefore the three secular equations will be 2,°480; 9°,6299; 0°,7949. By adding them to the three preceding differences, they are reduced to the three following +4164″;+19730″;—260″. These differences thus reduced may depend on the errors of ancient and modern observations, for the secular mean motion of the node being determined, for example, by a comparison of the observations of Bradley with those made since his time, *i. e.* by the observations made in the last half century; there may exist in its value, an uncertainty of half a minute at least.

NOTE V.

The Astronomers of Almamon found, by their observations, the greatest equation of the centre of the Sun equal to 2°,2037, greater than ours by 635″. Albatenius, Ebn Junis and a great number of other Arabian astronomers, make very slight changes in this result, which evinces incontrovertably, the diminution of the excentricity of the terrestrial orbit from their time to the present. The same astronomers found the longitude of the apogee of the Sun to be 830, equal to 91,°8333; which corresponds very nearly with the theory of gravity, according to which the longitude at the same epoch ought to be 92°,047. This theory assigns 36,″44 for the annual motion of this apogee with respect to the fixed stars; and the preceding observation gives the same motion to within a few seconds. Finally, from a comparison of their observations of the equinoxes with those of Ptolemy, they found the duration of the period of the tropical year to be 365^d,240706. About the year 803, which is more than twenty-five years before the formation of the verified table, the Arabian astronomer found by comparing his observations with those of Hipparchus, a much more exact duration of the year; he determined it to be 365,242181. Almost all the Arabian astronomers supposed that the obliquity of the ecliptic was about 26,2037; but it seems that this result is influeneed by the erroneous parallax which they assigned to the Sun; at least it is certainly the case with respect to the observations of Ebn Junis, which when corrected for this erroneous parallax, and for the refraction, make this obliquity 26,1932 for the year 1000. Theory makes it at this epoch, 26,°2009, the difference —77″ is within the limits of the errors of the Arabian observations. The epochs of the astronomical tables of Ebn Junis, confirm the secular equations of the motions of the Moon; the great inequalities of of Jupiter and Saturn are likewise confirmed by these epochs, and by the conjunction of these two planets, observed at Cairo by this astronomer. This observation, one of the most important in Arabian astronomy, was made on the 31st of October, 1007, at 0^d,16 of mean time at Paris. Ebn Junis found the excess of the geocentric longitude of Saturn above that of Jupiter, equal to 4444″. The tables constructed by M. Bouvard, according to my theory, and from a comparison of all the observations made by Bradley, Maskeline, and at the Royal Observatory, make this excess 5191″; the difference 747″ is less than the error of which this observation is susceptible.

NOTE VI.

The observations of the meridian shadows of the gnomon, made by Cocheou-King, and inserted in the Connaissance des Tems of the year 1809, assign 2°,1759 as the greatest equation of the Sun for the year 1280, which exceeds its actual value by about 377″. They likewise make the obliquity of the ecliptic at the same epoch, about 26°,1489, which is greater by 757″, than the actual obliquity. Hence it appears that the diminution of these two elements is demonstrated by these observations.

An observation of the obliquity of the ecliptic by Ulug-Beigh, when corrected for refraction and parallax, makes the obliquity in 1437 equal to 26°,1444; it is smaller than the preceding, as it ought to be, on account of the interval of 157 years, which separates the corresponding epochs. The following table clearly points out the successive diminution of this element in an interval of 2900 years.

Tcheou King, 1100 years before our æra - -	26°,5563	402″
Pythias, 350 years before our æra - - -	26°,4691	596″
Ebn Junis, the year one thousand - - -	26°,1932	—77
Cocheou-King, 1280 -	26°,1489	—62
Ulug-Beigh, 1437 -	26°,1444	130
In 1801 - - -	26°,0732.	

The second row of numbers indicates the excess of this obliquity over the results of the formulæ given in the Celestial Mechanics.

NOTE VII. AND LAST.

From the preceding chapter it appears, that we have the five following phenomena to assist us in investigating the cause of the primitive motions of the planetary system. The motions of the planets in the same direction, and very nearly in the same plane; the motions of the satellites in the same direction as those of the planets; the motions of rotation of these different bodies and also of the Sun, in the same direction as their motions of projection, and in planes very little inclined to each other; the small eccentricity of the orbits of the planets and satellites; finally, the great eccentricity of the orbits of the comets, their inclinations being at the same time entirely indeterminate.

Buffon is the only individual that I know of, who, since the discovery of the true system of the world, endeavoured to investigate the origin of the planets and satellites. He supposed that a comet, by impinging on the Sun, carried away a torrent of matter, which was reunited far off, into globes of different magnitudes and at different distances from this star. These globes, when they cool and become hardened, are the planets and their satellites. This hypothesis satisfies the first of the five preceding phenomena; for it is evident that all bodies thus formed should move very nearly in the plane which passes through the centre of the Sun, and through the direction of the torrent of matter which has produced them: but the four remaining phenomena appear to me inexplicable on this supposition. Indeed the absolute motion of the molecules of a planet ought to be in the same direction as the motion of its centre of gravity; but it by no means follows from this, that the motion of rotation of a planet should be also in the same direction. Thus the Earth may revolve from east to west, and yet the absolute motion of each of its molecules may be directed from west to east. This observation applies also to the revolution of the satellites, of which the direction in the same hypothesis, is not necessarily the same as that of the motion of projection of the planets.

The small eccentricity of the planetary orbits is a phenomenon, not only difficult to explain on this hypothesis, but altogether inconsistent with it. We know from the theory of central forces, that if a body which moves in a re-entrant orbit about the Sun, passes very near the body of the Sun, it will return constantly to it, at the end of each revolution. Hence it follows that if the planets were originally detached from the Sun, they would touch it, at each return to this star; and their orbits, instead of being nearly circular, would be very eccentric. Indeed it must be admitted that a torrent of matter detached from the Sun, cannot be compared to a globe which just skims by its surface: from the impulsions which the parts of this torrent receive from each other, combined with their mutual attraction, they may, by changing the direction of their motions, increase the distances of their perihelions from the Sun. But their orbits should be extremely eccentric, or at least all the orbits would not be *q. p.* circular, except by the most extraordinary chance. Finally, no reason can be assigned on the hypothesis of Buffon, why the orbits of more than one hundred comets, which have been already observed, should be all very eccentric. This hypothesis, therefore, is far from satisfying the preceding phenomena. Let us consider whether we can assign the true cause.

Whatever may be its nature, since it has produced or influenced the direction of the planetary motions, it must have embraced them all within the sphere of its action; and considering the immense distance which intervenes between them, nothing could have effected this but a fluid of almost indefinite extent. In order to have impressed on them all a motion *q. p.* circular and in the same direction about the Sun, this fluid must environ this star, like an atmosphere. From a consideration of the planetary motions, we are therefore brought to the conclusion, that in consequence of an excessive heat, the solar atmosphere originally extended beyond the orbits of all the planets, and that it has successively contracted itself within its present limits.

In the primitive state in which we have supposed the Sun to be, it resembles those substances which are termed nebulæ, which, when seen through telescopes, appear to be composed of a nucleus, more or less brilliant, surrounded by a nebulosity, which, by condensing on its surface, transforms it into a star. If all the stars are conceived to be similarly formed, we can suppose their anterior state of nebulosity to be preceded by other states, in which the nebulous matter was more or less diffuse, the nucleus being at the same time more or less brilliant. By going back in this manner, we shall arrive at a state of nebulosity so diffuse, that its existence can with difficulty be conceived.

For a considerable time back, the particular arrangement of some stars visible to the naked eye, has engaged the attention of philosophers. Mitchel remarked long since how extremely improbable it was that the stars composing the constellation called the Pleiades, for example, should be confined within the narrow space which contains them, by the sole chance of hazard; from which he inferred that this group of stars, and the similar groups which the heavens present to us, are the effects of a primitive cause, or of a primitive law of nature. These groups are a general result of the condensation of nebulæ of several nuclei; for it is evident that the nebulous matter being perpetually attracted by these different nuclei, ought at length to form a group of stars, like to that of the Pleiades. The condensation of nebulæ consisting of two nuclei, will in like manner form stars very near to each other, revolving the one about the other like to the double stars, whose respective motions have been already recognized.

But in what manner has the solar atmosphere determined the motions of rotation and revolution of the planets and satellites? If these bodies had penetrated deeply into this atmosphere, its resistance would cause them to fall on the Sun. We may therefore suppose that the planets were formed at its successive limits, by the condensation of zones of vapours, which it must, while it was cooling, have abandoned in the plane of its equator.

Let us resume the results which we have given in the tenth chapter of the preceding book. The Sun's atmosphere cannot extend indefinitely; its limit is the point where the centrifugal force arising from the motion of rotation balances the gravity; but according as the cooling contracts the atmosphere, and condenses the molecules which are near to it, on the surface of the star, the motion of rotation increases; for in virtue of the principle of areas, the sum of the areas described by the radius vector of each particle of the Sun and of its atmosphere, and projected on the plane of its equator, is always the same. Consequently the rotation ought to be quicker, when these particles approach to the centre of the Sun. The centrifugal force arising from this motion becoming thus greater; the point where the gravity is equal to it, is nearer to the centre of the Sun. Supposing therefore, what is natural to admit, that the atmosphere extended at any epoch as far as this limit, it ought, according as it cooled, to abandon the molecules, which are situated at this limit, and at the successive limits produced by the increased rotation of the Sun. These particles, after being abandoned, have continued to circulate about this star, because their centrifugal force was balanced by their gravity. But as this equality does not obtain for those molecules of the atmosphere which are situated on the parallels to the Sun's equator, these have come nearer by their gravity to the atmosphere according as it condensed, and they have not ceased to belong to it, inasmuch as by this motion, they have approached to the plane of this equator.

Let us now consider the zones of vapours, which have been successively abandoned. These zones ought, according to all probability, to form by their condensation, and by the mutual attraction of their particles, several concentrical rings of vapours circulating about the Sun. The mutual friction of the molecules of each ring ought to accelerate some and retard others, until they all had acquired the same angular motion. Consequently the real velocities of the molecules which are farther from the Sun, ought to be greatest. The following cause ought likewise to contribute to this difference of velocities : The most distant particles of the Sun, and which, by the effects of cooling and of condensation, have collected so as to constitute the superior part of the ring, have always described areas proportional to the times, because the central force by which they are actuated has been constantly directed to this star ; but this constancy of areas requires an increase of velocity, according as they approach more to each other. It appears that the same cause ought to diminish the velocity of the particles, which, situated near the ring, constitute its inferior part.

If all the particles of a ring of vapours continued to condense without separating, they would at length constitute a solid or a liquid ring. But the regularity which this formation requires in all the parts of the ring, and in their cooling, ought to make this phenomenon very rare. Thus the solar system presents but one example of it ; that of the rings of Saturn. Almost always each ring of vapours ought to be divided into several masses, which, being moved with velocities which differ little from each other, should continue to revolve at the same distance about the Sun. These masses should assume a spheroidical form, with a rotatory motion in the direction of that of their revolution, because their inferior particles have a less real velocity than the superior ; they have therefore constituted so many planets in a state of vapour. But if one of them was sufficiently powerful, to unite successively by its attraction, all the others about its centre, the ring of vapours would be changed into one sole spheroidical mass, circulating about the Sun, with a motion of rotation in the same direction with that of revolution. This last case has been the most common ;

however, the solar system presents to us the first case, in the four small planets which revolve between Mars and Jupiter, at least unless we suppose with Olbers, that they originally formed one planet only, which was divided by an explosion into several parts, and actuated by different velocities. Now if we trace the changes which a farther cooling ought to produce in the planets formed of vapours, and of which we have suggested the formation, we shall see to arise in the centre of each of them, a nucleus increasing continually, by the condensation of the atmosphere which environs it. In this state, the planet resembles the Sun in the nebulous state, in which we have first supposed it to be; the cooling should therefore produce at the different limits of its atmosphere, phenomena similar to those which have been described, namely, rings and satellites circulating about its centre in the direction of its motion of rotation, and revolving in the same direction on their axes. The regular distribution of the mass of rings of Saturn about its centre and in the plane of its equator, results naturally from this hypothesis, and, without it, is inexplicable. Those rings appear to me to be existing proofs of the primitive extension of the atmosphere of Saturn, and of its successive condensations. Thus the singular phenomena of the small eccentricities of the orbits of the planets and satellites, of the small inclination of these orbits to the solar equator, and of the identity in the direction of the motions of rotation and revolution of all those bodies with that of the rotation of the

Sun, follow from the hypothesis which has been suggested, and render it extremely probable. If the solar system was formed with perfect regularity, the orbits of the bodies which compose it would be circles, of which the planes, as well as those of the various equators and rings, would coincide with the plane of the solar equator. But we may suppose that the innumerable varieties which must necessarily exist in the temperature and density of different parts of these great masses, ought to produce the eccentricities of their orbits, and the deviations of their motions, from the plane of this equator.

In the preceding hypothesis, the comets do not belong to the solar system. If they be considered, as we have done, as small nubulæ, wandering from one solar system to another, and formed by the condensation of the nebulous matter, which is diffused so profusely throughout the universe, we may conceive that when they arrive in that part of space where the attraction of the Sun predominates, it should force them to describe elliptic or hyperbolic orbits. But as their velocities are equally possible in every direction, they must move indifferently in all directions, and at every possible inclination to the ecliptic ; which is conformable to observation. Thus the condensation of the nebulous matter, which explains the motions of rotation and revolution of the planets and satellites in the same direction, and in orbits very little inclined to each other, likewise explains why the motions of the comets deviate from this general law.

The great eccentricity of the orbits of the comets, is also a result of our hypothesis. If those orbits are elliptic, they are very elongated, since their greater axes are at least equal to the radius of the sphere of activity of the Sun. But these orbits may be hyperbolic; and if the axes of these hyperbolæ are not very great with respect to the mean distance of the Sun from the Earth, the motion of the comets which describe them will appear to be sensibly hyperbolic. However, with respect to the hundred comets, of which the elements are known, not one appears to move in a hyperbola; hence the chances which assign a sensible hyperbola, are extremely rare relatively to the contrary chances. The comets are so small, that they only become sensible when their perihelion distance is inconsiderable. Hitherto this distance has not surpassed twice the diameter of the Earth's orbit, and most frequently, it has been less than the radius of this orbit. We may conceive, that in order to approach so near to the Sun, their velocity at the moment of their ingress within its sphere of activity, must have an intensity and direction confined within very narrow limits. If we determine by the analysis of probabilities, the ratio of the chances which in these limits, assign a sensible hyperbola to the chances which assign an orbit, which may without sensible error be confounded with a parabola, it will be found that there is at least six thousand to unity that a nebula which penetrates within the sphere of the Sun's activity so as to be observed, will either describe a very elongated ellipse, or an hyperbola, which, in consequence of the magnitude of its axis will be as to sense confounded with a parabola in the part of its orbit which is observed. It is not therefore surprising that hitherto no hyperbolic motions have been recognised.

The attraction of the planets, and perhaps also the resistance of the ethereal media, ought to change several cometary orbits into ellipses, of which the greater axes are much less than the radius of the sphere of the solar activity. It is probable that such a change was produced in the orbit of the comet of 1759, the greater axis of which was not more than thirty-five times the distance of the Sun from the Earth. A still greater change was produced in the orbits of the comets of 1770 and of 1805.

If any comets have penetrated the atmospheres of the Sun and planets at the moment of their formation, they must have described spirals, and consequently fallen on these bodies, and in consequence of their fall, caused the planes of the orbits and of the equators of the planets to deviate from the plane of the solar equator.

If in the zones abandoned by the atmosphere of the Sun, there are any molecules too volatile to be united to each other, or to the planets, they ought in their circulation about this star to exhibit all the appearances of the zodiacal light, without opposing any sensible resistance to the different bodies of the planetary system, both on account of their great rarity, and also because their motion is very nearly the same as that of the planets which they meet.

An attentive examination of all the circumstances of this system renders our hypothesis still more probable. The primitive fluidity of the planets is clearly indicated by the compression of their figure, conformably to the laws of the mutual attraction of their molecules; it is moreover demonstrated by the regular diminution of gravity, as we proceed from the equator to the poles. This state of primitive fluidity to which we are conducted by astronomical phenomena, is also apparent from those which natural history points out. But in order fully to estimate them, we should take into account the immense variety of combinations formed by all the terrestrial substances which were mixed together in a state of vapour, when the depression of their temperature enabled their elements to unite; it is necessary likewise to consider the wonderful changes which this depression ought to cause in the interior and at the surface of the earth, in all its productions, in the constitution and pressure of the atmosphere, in the ocean, and in all substances which it held in a state of solution. Finally, we should take into account the sudden changes, such as great volcanic eruptions, which must at different epochs have deranged the regularity of these changes. Geology, thus studied under the point of view which connects it with astronomy, may, with respect to several objects, acquire both precision and certainty.

One of the most remarkable phenomena of the solar system is the rigorous equality which is observed to subsist between the angular motions of rotation and revolution of each satellite. It is infinity to unity that this is not the effect of hazard. The theory of universal gravitation makes infinity to disappear from this improbability, by shewing that it is sufficient for the existence of this phenomenon, that at the commencement these motions did not differ much. Then, the attraction of the planet would establish between them a perfect equality; but at the same time it has given rise to a periodic oscillation in the axis of the satellite directed to the planet, of which oscillation the extent depends on the primitive difference between these motions. As the observations of Mayer on the libration of the Moon, and those which Bouvard and Nicollet made for the same purpose, at my request, did not enable us to recognize this oscillation; the difference on which it depends must be extremely small, which indicates with every appearance of probability the existence of a particular cause, which has confined this difference within very narrow limits, in which the attraction of the planet might establish between the mean motions of rotation and revolution a rigid equality, which at length terminated by annihilating the oscillation which arose from this equality. Both these effects result from our hypothesis; for we may conceive that the Moon, in a state of vapour, assumed in consequence of the powerful attraction of the earth the form of an elongated spheroid, of which the greater axis would be constantly directed towards this planet, from the facility with which the vapours yield to the slightest force impressed upon them. The terrestrial attraction continuing to act in the same manner, while the Moon is in a state of fluidity, ought at length, by making the two motions of this satellite to approach each other, to cause their difference to fall within the limits, at which their rigorous equality commences to establish itself. Then this attraction should annihilate, by little and little, the oscillation which this equality produced on the greater axis of the spheroid directed towards the earth. It is in this manner that the fluids which cover this planet, have destroyed by their friction and resistance the primitive oscillations of its axis of rotation, which is only now subject to the nutation resulting from the actions of the Sun and Moon. It is easy to be assured that the equality of the motions of rotation and revolution of the satellites ought to oppose the formation of rings and secondary satellites, by the atmospheres of of these bodies. Consequently observation has not hitherto indicated the existence of any such. The motions of the three first satellites of Jupiter present a phenomenon still more extraordinary than the preceding; which consists in this, that the mean longitude of the first, minus three times that of the second, plus twice that of the third, is constantly equal to two right angles. There is the ratio of infinity to one, that this equality is not the effect of chance. But we have seen, that in order to produce it, it is sufficient, if at the commencement, the mean motions of these three bodies approached very near to the relation which renders the mean motion of the first, minus three times that of the second, plus twice that of the third, equal to nothing. Then their mutual attraction rendered this ratio rigorously exact, and it has moreover made the mean longitude of the first minus three times that of the second, plus twice that of the third, equal to a semicircumference. At the same time, it gave rise to a periodic inequality, which depends on the small quantity, by which the mean motions originally deviated from the relation which we have just announced. Notwithstanding all the care Delambre took in his observations, he could not recognise this inequality, which, while it evinces its extreme smallness, also indicates, with a high degree of probability, the existence of a cause which makes it to disappear. In our hypothesis, the satellites of Jupiter, immediately after their formation, did not move in a perfect vacuo ; the less condensible molecules of the primitive atmospheres of the Sun and planet would then constitute a rare medium, the resistance of which being different for each of the stars, might make the mean motions to approach by degrees to the ratio in question ; and when these movements had thus attained the conditions requisite, in order that the mutual attraction of the three satellites might render this relation accurately true, it perpetually diminished the inequality which this relation originated, and eventually rendered it insensible. We cannot better illustrate these effects than by comparing them to the motion of a pendulum, which, actuated by a great velocity, moves in a medium, the resistance of which is inconsiderable. It will first describe a great number of circumferences ; but at length its motion of circulation perpetually decreasing, it will be converted into an oscillatory motion, which itself diminishing more and more, by the resistance of the medium, will eventually be totally destroyed, and then the pendulum, having attained a state of repose, will remain at rest for ever. contra medium, the resistance of which is inconsiderable, it will first describe a great number of circular vibrations; but at length its motion, or circular being perpetually decreasing, it will be converted into an oscillatory motion, which then diminishing more and more, by the resistance of the medium, will eventually be totally destroyed, and then the pendulum, having attained a state of repose, will remain at rest for ever.

NOTES TO CHAPTER I.

(*a*) The uniform velocities are proportional to the circumferences of the circles described, divided by the periodic times, or times of their description, *i. e.* $v = \frac{2 r \pi}{P}$; and as by hypothesis $P^2 \backsim r^3 \therefore v^2 \backsim \frac{r^2}{r^3} \backsim \frac{1}{r}$, hence F which is :: l to $\frac{v^2}{r}$ varies as $\frac{1}{r^2}$; this, however, only proves, that if the orbits of the planets were circular, the forces by which they are retained in their respective circumferences vary inversely as the squares of their distances from the sun.

(*b*) The areas being proportional to the times, the bases described in the interval *dt* are inversely as the altitudes or perpendiculars let fall from the centre of force on the tangents to the curve described; but as *dt* is assumed indefinitely small, the velocities with which the bases are described may be considered as uniform, and therefore proportional to the bases, consequently they are reciprocally, as the perpendiculars from the centre of forces; hence, as at the perihelion, the distance is least, the velocity at this point must be a maximum. As the body is supposed to describe an ellipse, its tendency to recede from the sun at the perihelion must be greater than its gravity towards it, for, otherwise, if they were equal, the body would describe a circle about the sun; and if this tendency was less at the perihelion than the gravity towards it, the body would fall within this circle, which is contrary to the hypothesis; this is also evident from the ratio which the centripetal bears to the centrifugal force at the same distance; it is likewise apparent from the circumstance of the velocity in the ellipse decreasing in a greater ratio than the inverse subduplicate ratio of the distance, and therefore in a greater ratio than the velocities of bodies moving in circles at the same distance (*see* note (*d*) of this chapter;) therefore, as the velocities decrease in a greater ratio than the velocities of bodies moving in circles at the same distance, the velocity of the body moving in the ellipse, continually approaches to the velocity of a body moving in a circle at the same distance; there is a certain point in the curve where the velocity becomes equal to the velocity in a circle at the same distance; this is at the mean distance of the body from focus; but though the velocities are in this case the same, the curves described will not coincide, for as the angle of projection in the ellipse is obtuse, the body will continue to recede from the centre, until it arrives at the point where the direction of its motion is at right angles to the radius vector, and as at this point the velocity is less than in a circle at the same distance, the path described by the body will fall within the circle, and the body will return to the perihelion, tracing a curve precisely equal and similar to that by which it arrived at aphelion.

(*c*) *See* note (*b*) page 256 of first volume. Let, as in notes page 248 of first volume, x and y represent the rectangular coordinates of the planets, the origin being in the sun to which the force soliciting the bodies is directed, (it is not necessary to introduce a third coordinate, because the areas being proportional to the times, the curve described is of single curvature,) by what is stated in the notes already adverted to, $o = \frac{d^2x}{dt^2} + P$; $o = \frac{d^2y}{dt^2} + Q$, multiplying the first equation by $-y$, and the second by x, and then adding them together, we have $d\left(\frac{xdy-ydx}{dt}\right) + xQ - yP = o$; but the first member being the differential of $\frac{xdy-ydx}{dt}$, which is constant and $=$ to cdt, $xQ-yP=0$; $\because x : y :: P : Q$, and the force is directed to the origin of the coordinates; if the first of the preceding equations be multiplied by dx, and the second by dy, and then added together, we obtain $\frac{dx.d^2x + dy\, d^2y}{dt^2} + Pdx + Qdy = 0$, and therefore (by including the constant arbitrary under the sign $\int$) $\frac{dx^2 + dy^2}{dt^2} + 2\int(P\,dx + Q\,dy) = 0$, substituting for dt its value $\frac{xdy-ydx}{c}$, this equation becomes

$$\frac{c^2\,(dx^2+dy^2)}{(xdy-ydx)^2} + 2\int(P\,dx+Q\,dy),$$

but if r be the radius vector, and v the angle which r makes with the axis of x, we have $x=r.\cos.v$, $y=r.\sin.v$, and therefore $dx^2+dy^2=r^2.dv^2+dr^2$, $x.dy-y.dx = r^2dv$, $P=\phi\cos.v$, $Q=\phi\sin.v$, ϕ being $= \sqrt{P^2+Q^2}$, $\therefore$ by substituting we obtain

$$c^2.\frac{(r^2.dv^2+dr^2)}{r^4.dv^2} + 2\int\phi\,dr, \text{ and therefore}$$

$$dv = \frac{c\,dr}{r\sqrt{-c^2-2r^2.\int\phi\,dr}},$$ when ϕ is given in terms of r, we can obtain v in a function of r, by the method of quadratures; but if ϕ be unknown, and the nature of the curve described be given, we obtain (by differentiating the preceding expression)

$$\phi = \frac{c^2}{r^3} - \frac{c^2}{2}.d.\frac{\frac{dr^2}{r^4.dv^2}}{dr},$$ now, as the planetary orbits are

ellipses, $\frac{1}{r} = \frac{1 + e.\cos.(v - \bar{\omega})}{a(1 - e^2)}$, ∴

$$\frac{dr^2}{r^4.dv^2} = \frac{2}{ar(1-e^2)} - \frac{1}{r^2} - \frac{1}{a^2(1-e^2)}$$, consequently

$\phi = \frac{c^2}{a(1-e^2)} \cdot \frac{1}{r^2}$; and conversely, if ϕ varies as $\frac{h}{r^2}$, the preceding equation will satisfy the differential equation, which expresses the value of ϕ, for then $h = \frac{c^2}{a(1-e^2)}$, is an equation of condition between a and e, and therefore the three quantities a e $\bar{\omega}$ are reduced to two distinct quantities, which is enough, as the differential equation between r and v is only of the second order. The coefficient $\frac{c^2}{a(1-e^2)}$ determines the *intensity* of the force ϕ for *each* planet and comet; but it is easy to show that this is the same in passing from one planet to another, for from the proportionality of the areas to the times of their description, we have $\frac{cdt}{2} : \pi a^2 . \sqrt{1-e^2}$ (which expresses the area of the ellipse) as dt : P the periodic time,

$\therefore c = \frac{2\pi a^2 . \sqrt{1-e^2}}{P}$; but since the squares of the periodic times are as the cubes of the greater axes of the ellipses, we have $P^2 = k^2.a^3$, k being the same for all the planets, and therefore we have by substituting,

$c = \frac{2\pi . \sqrt{a(1-e^2)}}{k}$; ∴ as $2a(1-e^2)$ expresses the principal parameter of the orbits traced by the planets, c which is ∻ to the areas traced in equal times, varies as the square root of the parameters. In the case of the comets, as their orbits are parabolic, the preceding value of c becomes $\frac{2\pi\sqrt{2D}}{k}$, D being the perihelion distance; in this case the preceding proportion becomes

$\frac{2\pi.\sqrt{2D}}{2k}.dt : \pi a^{\frac{3}{2}}\sqrt{2D} :: dt : P$, and $\therefore P^2 = k^2 a^3$, therefore when P is known we can determine a. The value of c gives

$$\frac{c^2}{a(1-e^2)} = \frac{4\pi^2}{k^2} \text{ and } \therefore \phi = \frac{4\pi^2}{k^2}.\frac{1}{r^2};$$ which shows that ϕ varies from one planet to another, only in consequence of the change of distance, and therefore at equal distances from the sun the accelerating force of all the planets is the same, and the moving force varies as their masses; therefore, if all the planets fell at the same instant from different points of the same spheric surface towards the sun, they would reach it in the same time, just as all bodies near to the surface of our earth are equally accelerated by the force of terrestrial gravity, and the weight of the planets to the sun is proportional to their masses divided by the squares of their distance from the sun. The greatest and least values of r in the ellipse, correspond to $r-\tilde{\omega}=\pi$, $r-\tilde{\omega}=0$, therefore they are respectively $a(1+e)$, $a(1-e)$, consequently they lie in directum, hence it follows that when ϕ varies as $\frac{1}{r^2}$, the apsides are 180° distant, and *vice versa* if the apsides are 180 distant, the force varies as $\frac{1}{r^2}$. *See* Principia Math. book 1, prop. 45; and note (*i*) chapter 5. From the equation $c^2 = h.a(1-e^2)$, it follows that the synchronous areas vary generally as the square root of the absolute forces into the square roots of the parameters of the orbits described, and therefore if the absolute forces be different, we have

$$\sqrt{h.a(1-e^2)}.\,dt : \pi a^2.\sqrt{1-e^2} :: dt : P, \text{ and } \therefore P^2 = \frac{\pi.a^3}{h}$$

i. e. the square of the periodic time varies as the cube of the distance divided by the absolute force. Generally speaking, the quantity c, which results from the integration of the equation $d.\left(\frac{x\,dy - y\,dx}{dt}\right) = o$, is common to all laws of central forces, therefore it does not depend on the law of the attractive force, but on its absolute quantity, and it will serve to determine the ratio of the central force of the sun, to every other; v the velocity

$$= \frac{\sqrt{dr^2 + r^2 dv^2}}{dt} = \frac{\sqrt{dr^2}}{dt} + \frac{c^2}{r}$$

$= (\text{as } d.(r^2.dv) = 0,)\ 2\mu.\left(\frac{2}{r} - \frac{1}{a}\right)$, (Celestial Mechanics, Nos. 18, 26.), therefore v is a maximum when r is a minimum, and *vice versa;* if U denotes the velocity which the body would have if it described a circle about the sun at the unit of distance, then

$$r = a = 1, \text{ and } \therefore U^2 = \mu, \therefore v^2 = U^2\left(\frac{2}{r} - \frac{1}{a}\right),$$

hence given the velocity of projection and distance, we can determine the axis major a, as a is positive in the ellipse, infinite in the parabola, and negative in the hyperbola, the section described will be an ellipse, a parabola, or hyperbola, according as

$$v \text{ is } \angle = \text{ or } > \text{ than } U.\sqrt{\frac{2}{r}},$$

it is remarkable that the direction of projection does not influence the *species* of conic section, for $\frac{1}{a} = \frac{2}{r} - \frac{v^2}{U^2}$, therefore, when r and v are given, a and therefore P remain the same; as $U\sqrt{\frac{1}{r}}$ = the velocity in a circle at the distance of r from the sun, in the ellipse the velocity at any point is to that in a circle at the same distance, in a less ratio than that of $\sqrt{2} : 1$; in the parabola this ratio is that of $\sqrt{2} : 1$; in the hyperbola the ratio is greater than that of $\sqrt{2} : 1$; in the ellipse, when v diminishes r increases, and when $v = o$, $r = 2a$, in which case $e = 1$; in the hyperbola, when r is infinite, the limit of the velocity is $U^2.\frac{1}{a}$ = the square of the velocity in a circle at the distance of a from the focus, when $r=a$, $v=U\sqrt{\frac{1}{r}}$ = the velocity in a circle at the same distance, and in general

$$v : U.\sqrt{\frac{1}{r}} :: \sqrt{2a-r} : \sqrt{a};$$

hence we see the truth of what is stated in notes page 372. As $\frac{dr}{dt}$ expresses the velocity resolved in the direction of the radius, it is = to v. cos. ε, ε. being the angle which the radius vector makes with the tangent, therefore

$$\frac{dr^2}{dt^2} = \mu.\left(\frac{2}{r}-\frac{1}{a}\right).\ \cos.^2\varepsilon, \text{ but as } \frac{\mu}{a}-\frac{2\mu}{r}$$

$$+\frac{dx^2+dy^2}{dt^2}=0,$$

$\mu.a\,(1-e^2) = 2\mu r - \frac{\mu r^2}{a} - \frac{r^2\,dr^2}{dt^2}$, and substituting for $\frac{dr^2}{dt^2}$, its value, we obtain

$a(1-e^2) = r^2 \sin.^2\varepsilon\left(\frac{2}{r}-\frac{1}{a}\right)$, $a\,(1-e^2)$ expresses the parameter, which when r and a are given, varies as the square of the sine of projection ; ∴ the parameter, when every thing else remains the same, depends on that part of the velocity which acts perpendicularly to the radius vector, it is termed the paracentric velocity, and is evidently a maximum at the extremity of the focal ordinate.

From the expression $a\,(1-e^2)=r^2.\ \sin.^2\varepsilon.\left(\frac{2}{r}-\frac{1}{a}\right)$, it follows that sin. ²ε varies inversely, as $r.\left(\frac{2a-r}{a}\right)$, but as

$r + 2a - r$ is given, their product is a maximum, and ∴ the sine of projection the least possible when $r = a$, *i.e.* at mean distance.

(*f*) It appears from what has been established with respect to the relation which exists between the velocity in a circle and the velocity in a conic section at the same distance, that the hyperbola and ellipse are equally possible, with this sole difference, that the hyperbola supposes a greater velocity than the ellipse; the parabola is infinitely less probable than the two other conic sections, since it supposes an *unique* case, the circle likewise requires a *perfect equality*. The parabolas may be considered as the asymptotes to which very excentric ellipses perpetually approach in the perihelion. It is on this supposition that the investigation of the cometary motions is founded. See Vol. I., page 394.

As $r^2 dv$ expresses the elementary area, it follows that $dv = \frac{cdt}{r^2}$, *i. e.* the angular velocity, varies as the square root of the parameter or of the sychronous areas divided by the square of the distance, therefore the angular velocity in a conic section is to that in a circle at the same distance r, as $c : \sqrt{r}$, and they are equal at the extremity of the focal ordinate, as

$$c = \frac{2\pi a^2 . \sqrt{1 - e^2}}{P}, \; \frac{dv}{dt} = \frac{2\pi a^2 \sqrt{1 - e^2}}{P . r^2};$$

if a circle is described at the unity of distance in a time equal to P, we have $\frac{2\pi}{P}$ = the mean angular velocity in the ellipse, therefore when the angular velocity in the ellipse is equal to the mean angular velocity, we have

$\frac{2\pi}{P} = \frac{2\pi a^2 \sqrt{1 - e^2}}{P . r^2}$, and ∴ $r = a(1 - e^2)^{\frac{1}{4}}$ = a mean proportional between the semiaxes; in this position the equation of the centre is a *maximum*.

(*g*) See Celestial Mechanics, No. 58, and also Princip. Math. Section 9. If X represents the quantity by which the force deviates from the inverse ratio of the square of the distance, then the distance between the apsides

$$= 180. \frac{\sqrt{1+X}}{\sqrt{1+3X}} = 180(1-X),$$

the square of X being neglected.

If a represent the mean distance of the satellite from the centre of Jupiter, P' its period, expressed in seconds, $\frac{2a\pi}{P'}$ will represent the arc described in a second; and $\frac{2a\pi^2}{P'^2}$ which is equal to the versed sine of the arc described, is the space through which the attractive force of the planet causes the body to descend in a second; and if a' P'', &c. represent the same quantities for another satellite, the ratio of $\frac{2a\pi^2}{P'^2}$ to $\frac{2a'\pi^2}{P''^2}$, expresses the ratio of ϕ to ϕ', the attractive forces of Jupiter at the distances a, a'; but as by observation

$P'^2 : P''^2 :: a^3 : a'^3$, we have $\phi : \phi' :: \frac{1}{a^2} : \frac{1}{a'^2}$. See page 10 of the text.

In note (*u*), page 356 of the first volume, we showed how the number of oscillations performed by a pendulum in a given time indicated the diminution of gravity.

(*h*) In note (*t*), page 426, we gave the method of determining the velocity which should be impressed on a projectile, in order that, setting aside the resistance of the air, it might perpetually revolve about the earth.

It may be shown, by a comparison of the apparent angular motion of the moon with her apparent diameter, that she describes equal areas in equal times about the earth; and therefore that the force by which she is retained in her orbit is directed to the earth. See notes page 315, Vol. I. Indeed, as will be shown in the fifth chapter, if great accuracy be required, the observations ought to be made in the syzygies and quadratures; for in the other points of the orbit the disturbing action of the sun is not directed to the centre of the earth. Newton shows, from the small quantity by which the apsides are observed to prograde, that the force must be nearly inversely as the square of the distance; for if the orbit was elliptical, the earth being in one of the foci, the distance between the apsides would be 180°, and the force by which the moon would be retained in her orbit would vary as $\frac{1}{d^2}$. See notes, page 375. Now the apsides are observed to prograde 3°, 3′ every month, and the law of the force which would produce such a progression must vary inversely as some power of the distance, intermediate between the square and the cube, but which is nearly sixty times nearer to the square; consequently, on the hypothesis that the progression is produced by a deviation from the law of elliptical motion, it must be nearly in the inverse ratio of the squares of the distance; but as Newton proves this motion of the apsides to arise from the disturbing action of the sun, it follows that the force varies accurately as $\frac{1}{d^2}$. See Luby's Physical Astronomy, page 197.

(*i*) He computes the space through which the moon would fall in a second, in consequence of the action of the force by which she is retained in her orbit; which force, in consequence of the proportionality of the areas to the times, is directed towards the centre of gravity of the earth; and assuming that the force decreases in the inverse ratio of the square of the distance, he determines, from knowing the space described by a body falling near the earth's surface in a second, the space through which, in consequence of the action of the same force diminished in the inverse ratio of the square of the distance of the moon from the earth, a body would fall at this distance; and as this space comes out equal to that by which the moon is deflected from the tangent to the orbit, he justly concludes, that this force is the terrestrial gravity diminished in the ratio of the square of the distance.

In consequence of the disturbing action of the sun, the moon's distance and motion are subject to several inequalities, which are detailed in Chapter IV. of the First Volume. The particular explanation of the most remarkable of them will be given in notes to Chapter V. of this Volume.

Knowing the parallax and radius of the earth, it is easy to obtain the distance.

In determining the space through which the moon falls in a second, in consequence of the force which sollicits it, there are two corrections applied; one arising from the disturbing action of the sun, which, taking into account the entire orbit, diminishes the lunar gravity $\frac{1}{358}$th part, see note (*f*) Chapter V.; and in consequence of this the result obtained should be increased a $\frac{1}{358}$th part. The other correction arises from this, that in the relative motion of the moon about the earth, the point about which it really revolves is the common centre of gravity of the earth and moon, and the central force which should exist in the centre of the earth, which would cause the moon to revolve about this centre in the same time in which she actually revolves about the common centre of gravity of the earth and moon, should be equal to $m+m'$, the sum of the masses of the earth and moon; for if a be the distance of the earth from the moon, y the distance at which the moon would revolve about the earth by itself, considered as quiescent,

$= \frac{a.m^{\frac{1}{3}}}{(m+m')^{\frac{1}{3}}}$, (Princip. Math. prop. 59, book 1.), ∴

$P^2 = \frac{y^3}{m^3} = \frac{a^3}{m+m'}$, *i. e.* if a be the distance, the central force should be $m+m'$, ∴ as the versed sine of the arc described in a second is the space through which the moon descends in consequence of the combined actions of the earth and moon, it must be multiplied by

$\frac{m}{m+m'}$, *i. e.* $\frac{75}{76}$, to obtain the space described by the sole action of m.

(k) Let a b represent the major and minor semiaxes of the terrestrial spheroid, its solid content $= \frac{4\pi\, a^2 b}{3}$, and if r be the radius of the equicapacious sphere, its content

$= \frac{4\pi\, r^3}{3}$, ∴ $a^2 b = r^3$, ∴ as $r = \frac{b}{\overline{1-e^2.\cos.^2\lambda}|^{\frac{1}{2}}}$ we have

$a^{\frac{4}{3}}.b^{\frac{2}{3}} = \frac{b^2}{1-e^2.\cos.^2\lambda}$, *i. e.* $a^{\frac{4}{3}} = b^{\frac{4}{3}}.(1+e^2.\cos.^2\lambda)$ very nearly; and if $a = b(1+\varepsilon)$, ε being a very small quantity of which the square may be neglected, then

$$1+\frac{4}{3}.\varepsilon = 1+2\varepsilon.\cos.^2\lambda, \text{ and } \therefore \cos.^2\lambda = \frac{2}{3};$$

now, as the efficient part of the centrifugal force at any parallel of latitude λ diminishes as cos. $^2\lambda$, and as the centrifugal force at the equator is the $\frac{1}{288}$th part of gravity, the centrifugal force at the parallel in question $= \frac{2}{3}.\frac{1}{288} = \frac{1}{432}$th part of gravity, ∴ if p represent the lunar parallax, v the versed sine of the arc described by the moon, and s the space fallen through near the surface of the earth in the same time,

$v = \frac{2a\pi^2}{p^2}$, $a = \frac{6369809}{\sin. p}$, ∴ applying the corrections specified above,

$v = \frac{2.63.69809^m}{\sin. p. P^2} \cdot \frac{369}{368} \cdot \frac{75}{76}$, and s when diminished in the ratio of the square of the distance

$= 3^m.65631 \left(1 + \frac{1}{432}.\right). \sin.^2 p$; now these two expressions come out $q.p$ equal, ∴ it follows that the force varies as $\frac{1}{d^2}$. In the Celestial Mechanics, the identity of terrestrial gravity with the force deflecting the moon, is proved from the equality of p, the lunar parallax, as determined by observation, and from the preceding equation.

With respect to the diminution of the force of gravity on the summits of the highest mountains, *see* note (v) Chapter VIII.

(l) In consequence of the equality of action and reaction, whatever motive force is produced in the planet by the action of the sun, an equal and contrary force is produced in the sun by the planet's reaction; now, if M m represent the respective masses of the sun and planet, and d their mutual distance, the motive force of any planet is ∻l to $\frac{M.m}{d^2}$; ∴ at equal distances from the sun, the motive force towards that body is proportional to the masses of the planets; and, therefore, as the accelerating force of the planets is the same at equal distances from the sun, it follows that the moving force is ∻l to the mass; and the same is true for bodies near the earth's surface, as is evident from experiments made with pendulums. The accelerating force of the planet is expressed by

$\frac{M}{d^2}$, and the accelerating force of the sun $= \frac{m}{d^2}$; now,

if we impress on the sun and planet in a contrary direction to the motion of the sun, an accelerating force equal

to $\frac{m}{d^2}$, the sun will be at rest, their relative motion is evi-

dently not affected, and the planet will be actuated by the accelerating force

$\frac{M}{d^2}$, and also by $\frac{m}{d^2}$, *i. e.* by $\frac{M+m}{d^2}$, and as P the periodic

time $= \frac{d^{\frac{3}{2}}}{\sqrt{M+m}}$, it will be less than if the sun was im-

moveable in the ratio of $\sqrt{M+m}$, to $\sqrt{M}$; $\therefore$ the ratio of P^2 to d^3, is, strictly speaking, different for each planet; however, as in point of fact, this ratio is nearly the same for all the planets, it follows that the masses of the planets must be very small compared with that of the sun. This comparative smallness is also evinced by the circumstance of Kepler being able to announce his laws, for from the universality of gravitation each body is attracted by every other body, therefore those laws do not accurately obtain; still their effect must be *q.p* small, as the elliptic orbit satisfies the observations.

In addition to what is stated in page 15, it is to be remarked, that every computation founded on this hypothesis, if it satisfies all the observed phenomena, furnishes an additional proof of the truth of the theory of universal gravitation; and in this way, all physical astronomy, and in particular the theory of perturbations, by means of which the modern tables accord so perfectly with observation, is one of its most satisfactory confirmations.

We shall, in the subsequent Chapter, find this fact of the universality of attraction, confirmed by numerous astronomical observations, the precession of the equinoxes, the nutation of the earth's axis, and the compression of the planets, have been computed on this hypothesis, and its truth is evinced by the accurate agreement of the results of the computation with actual observation. *See* Chapters V. and VI. of this Volume, pages 48 and 55.

(*m*) For suppose the attracted body to approach towards the earth until it came in contact with it, if the reaction was not exactly equal to action, the two bodies would move with a common velocity in the direction of the pressure which predominates, therefore the centre of gravity of the two bodies would have a rectilinear motion in some direction, in consequence of the force of gravity, which is contrary to what is established in notes page 440, Vol. I.

This principle of reaction is of the greatest consequence in physics. Let, as before, M *m* represent the masses of two attracting bodies, V v the velocities which they communicate to each other, A B the intensities of their forces, *d* their mutual distance, we shall have, in consequence of reaction,

$$MV = mv \therefore V : v :: m : M, \text{ i.e. as}$$

$$V = \frac{B}{d^2}, v = \frac{A}{d^2}; B. : A :: V. : v : B : A :: m : M; \text{ i.e.}$$

the intensity of the forces is ∻l to the masses, and the velocities communicated are inversely as the masses. It also appears that if the bodies do not receive an initial impulse, they will approach each other in a right line, and meet in their common centre of gravity.

NOTES TO CHAPTER II.

(*a*) The formulæ which analysis has furnished for the determination of the perturbations, are composed of two different descriptions of terms. The first are proportional to the sines or cosines of certain angles; the second are proportional to the angles themselves. The first description of terms have periods, at the term of which they attain their greatest or least values, without ever passing these limits, so that they can never accumulate, and thus, at the end of millions of years they will not be more considerable than they are at present. As very accurate observations are required to detect them, it is only recently that they have been observed, with the exception of the inequalities of the Moon, and one relating to the motion of Jupiter and Saturn, which have been known a considerable time. The other terms, which are not proportional to the sines, but to the arches, or to the time in which these arches are described, have no period, and, therefore, continually accummulate. When the interval between the time of making observations is not considerable, these progressive perturbations are confounded with those which are periodical; but they at length become more and more detached from them, so that eventually they are so different, that it is impossible to mistake the one for the other. Like the mean motions of the planets, they may be determined by means of observations separated by considerable intervals from each other, though not with the same accuracy. Such, in particular, is the manner in which the motion of the nodes and of the apsides has been determined, which, in one hundred years, amounts to more than one degree. The observations of several ages would, in this way, indicate whether the change of the longitudes of the aphelia and of the nodes corresponds to the retrogradation of the equinoctial points, or if the apsides have a proper motion of their own; by means of such observations, we might determine with great accuracy the *progressive* perturbations, and then theory has only to account for them; they are useful in this respect, in the investigation of the *periodical* perturbations, inasmuch, as being easily observed, they enable us to determine the constant coefficients, masses, &c. But although the *coefficients* are furnished in this way by observation, the *form* of the equations which indicate the periodic pertubations can only be determined by theory; and this form, in the case in which the coefficients are unknown, is of the greatest importance in the *empirical* determination of the equations; for, by pointing out the arguments on which they depend, they suggest a mode according to which the observations should be made. They also indicate the period of each inequality, and consequently the epoch and situation in which it is at its maximum, and therefore easiest to be observed, and thus furnish a means of separating one perturbation from the other, and of determining them separately by observation. As the progressive perturbations, or those which are proportional to the times or the arcs, are commonly so small, that they do not become apparent until after the lapse of one or of several ages; their value is generally indicated for every hundred years; hence it is that they have been denominated *secular* inequalities. The following is a brief outline of the method by which the different perturbations of the planetary system are investigated. The differential equations of the second order, which are given in page 420, Volume I., furnish at once both the elliptic motion and the perturbations: the first integration of these equations, which is not difficult, consists of differential equations of the first order, and determines the *variations* of all the elements. The second integration, if it is expressed by polar coordinates, would determine the longitude and latitude, and the other elements; but it has hitherto baffled the skill of mathematicians, who have had recourse to various artifices to effect it. They have only ascertained, that if even the exact integral could be obtained, it would be so complicated, that in order to render the result applicable, it should be developed into a series. Thus, the only practicable method is to develope the result in an *approximative* manner, in which the quantities, which are extremely small with respect to others, are neglected; and the integral should be also exhibited in a series, the form of which is indicated by that of the differential equation, and its coefficients are determined by comparing the differential of the supposed integral with the given differentials. Therefore, the solution of the problem consists in expressing the integral by a convergent series, which necessarily supposes that the mass or distance, or, in short, that the attraction of one of the bodies is inconsiderable relatively to that of the other, the last body being termed the *central*, and the first the *disturbing* body. In such a case the disturbed orbit will deviate very little from the laws of Kepler. It can be considered as a variable ellipse, subject to these laws, as Lagrange has proved. Indeed, if the disturbed orbit deviated considerably from an ellipse, which, for instance, would be the case with the moon if she was four times farther from the earth than she is, (in which case the sun might be regarded as the central body equally as the earth,) the computation of its orbit would surpass the powers of our analysis. Fortunately the solar system is so arranged that we may assume the elliptic motion as the basis of each disturbed orbit.

The constant arbitrary quantities which are introduced at each integration, are the elements of the planetary ellipses. They are data which cannot be determined by theory, but solely from observation. They do not affect the differential equations, or general laws of motion, but solely the arbitrary modifications of the elliptic orbits, which, for each planet that moves according to the laws of Kepler, may be indefinitely varied. There are, in general, six constant arbitrary quantities independent of each other. See Book 3, Vol. I., page 185. For, as is mentioned in page 428, each body being referred to three rectangular coordinates, and then putting the second differentials of the coordinates, divided by the square of the element of the time, equal to the attractions which the body experiences from the other bodies, we shall have the three differential equations of the second order which determine the motion of the body. As each body of the system furnishes three similar equations, the entire number of these equations is triple that of the bodies; therefore their *complete* integrals contain six times as many arbitrary quantities as there are bodies. These constant quantities are determined by the initial coordinates of each body, and by the initial velocities resolved in the direction of the coordinates. The bodies of the system are almost always referred to one principal one; and this is done by subtracting the differential equations of its motion resolved in the direction of each coordinate from the corresponding differential equations of the motions of the other bodies; by this means the differential equations relative to their motions about the principal body will be obtained. See Celestial Mechanics, Vol. II. page 259. By means of these differential equations, there have been obtained seven integrals relatively to the motion of a system; three of these refer to the motion of the centre of gravity of the bodies of a system which are not acted on by any extraneous forces. The four remaining integrals, which are furnished by the principles of the conservation of areas and of living forces, are differentials of the *first order*. They are, as has been noticed in the Fourth Book, the generalization of the law of the areas, proportional to the times, and of the expression for the square of the velocity, which Newton announced in the motion of the system of two bodies. The determination of the motion in the case of two bodies, is reduced to the integration of differential equations of the first order, which is easily effected; but when there is a greater number of bodies the problem becomes extremely complicated, and we are obliged to have recourse to approximations.

(*b*) If, as is stated in the text, a body A be supposed to describe about the sun an ellipse, the elements of which vary by insensible gradations, and if the planet B be supposed to describe an epicyle about it, as the satellites do about their respective primaries, the motion of A would represent the primitive orbit changed imperceptibly by the secular inequalities, while the motion of B in the epicyle would represent the periodic inequalities.

(*c*) Calling a the mean distance, nt $n't$ the mean motions, the value of $\frac{1}{a} = \frac{2im'n\text{K}}{\mu(i'n'-in)} \cdot \cos.\ (in't - int + \text{A})$,

and ζ the mean motion

$= \frac{3im'an'\text{K}}{\mu(i'n'-in)^2} \cdot \sin.\ (i'n't - int)$. In this case the disturbing action of the planet m' on m is solely considered; and as it appears that $i'n' - in$ does not vanish, the quantities a and ζ only contain periodical inequalities, the approximation being continued as far as the first power of the disturbing force; and as i i' are integral numbers (See No. 54, Celestial Mechanics,) the equation $i'n' - in = 0$, cannot have place when the mean motions of m and m' are incommensurate, which is the case of the planets. Here the disturbing action of m' on m was only considered; but if the disturbing actions of all the planets m' m'' m''', &c. were taken into account, we should have, instead of $i'n'-in=0$, $in+in'$ $i'n''$ &c. $=0$, which is still more improbable than the equation $i'n'-in=0$. In the Supplement to the third volume, Laplace extended this conclusion in the manner mentioned in the text; however, what is stated here may suffice to point out the principle of the method. It follows, consequently, from this, that the greater axes of the orbits of the planets and their mean motions are only subject to periodical inequalities, depending on their mutual configuration, and therefore if these are neglected, the greater axes are invariable, and the mean motions are uniform. It is not, perhaps, generally known, that Mr. Simpson made this observation respecting the inequalities of the planets. See Miscellaneous Tracts, 179.

(*c*) Calling $e\, e'\, e''$, &c. the eccentricities of the orbits of $m\, m'\, m''$, &c., it is proved, in No. 57, Book 2, of the Celestial Mechanics, that

$$0 = ede.m\sqrt{a} + e'de'.m'\sqrt{a'} + e''de''.m''\sqrt{a''} + \&c.;$$

now, as $a\, a'\, a''$, &c. have been shown to be constant, if we integrate this expression, we shall have

$$e^2m\sqrt{a} + e'^2m'\sqrt{a'} + e''^2.m''\sqrt{a''} + \&c. = C$$ a constant quantity, and as the planets all revolve in the same direction, the signs of $\sqrt{a}$, $\sqrt{a'}$, $\sqrt{a''}$, &c. must be the same; therefore each of the terms of the first member of the preceding equation is positive, and consequently less than C; hence, if at any given epoch the eccentricities $e\, e'\, e''$, &c. are very small, the constant quantity C will be very small; therefore each of the terms of the equations will *always* remain very small; consequently the orbits will always remain $q.p$: circular. See Celestial Mechanics, p.

332. The eccentricities of the planetary orbits are therefore subject to this condition, scilicet, that the sum of their squares, multiplied respectively by their masses, and by the square roots of their greater axes, is constantly the same. By a similar analysis, if ϕ, ϕ', ϕ'', &c. represent the inclinations of the orbits of m, m', m'', &c. to a fixed plane, we can obtain the equation

$C' = \tan.^2\phi.m\sqrt{a} + \tan.^2\phi'.m'\sqrt{a'} + \tan.^2\phi''m''\sqrt{a''} +$ &c.; and as, by hypothesis, the orbit is inclined at a very small angle to the fixed plane, it may be shown that its inclination to this plane will be always inconsiderable, and consequently the system is always stable, for the inclinations, as well as for the eccentricities.

(h) It is evident, from what has been stated in page 501, Vol. I., that if γ be the inclination of the invariable plane to the plane xy and ϖ, the longitude of its ascending node $\tan.\gamma \sin.\varpi$,

$= \frac{c''}{c'}$, but we have $\frac{x\,dy - y\,dx}{dt} = \sqrt{a(1-e^2)}$, = the square root of the parameter. See page 374. But if the area be referred to a fixed plane, it should be multiplied by the cosine of its inclination to this plane,

$$\therefore \frac{x\,dy - y\,dx}{dt} = \cos.\phi.\sqrt{a(1-e^2)} = \frac{\sqrt{a(1-e^2)}}{\sqrt{1+\tan.^2\phi}}$$

in like manner,

$$\frac{x'dy' - y'dx'}{dt} = \frac{\sqrt{a'.(1-e'^2)}}{\sqrt{1+\tan.^2\phi'}}, + \text{&c.}$$

$\therefore$ neglecting quantities of the order mm', &c.

$$c = m.\sqrt{\frac{a(1-e^2)}{(1+\tan.^2\phi)}} + m'.\sqrt{\frac{a'(1-e'^2)}{1+\tan.^2\phi'}} + \text{&c.};$$

hence, if $p = \tan.\phi.\sin.\theta$, $q = \tan.\phi.\cos.\theta$, we have

$$c' = mq.\sqrt{\frac{a(1-e^2)}{1+\tan.^2\phi}} + m'q'\sqrt{\frac{a(1-e'^2)}{1+\tan.^2\phi'}} + \text{&c.};$$

$$c'' = mp\sqrt{\frac{a(1-e^2)}{1+\tan.^2\phi}} + m'p'\sqrt{\frac{a'(1-e'^2)}{1+\tan.^2\phi'}} + \&c.,$$

$\therefore$ substituting for $\frac{c''}{c'}$, these values, and concinnating, we may arrive at the expression given in the text for the tangent of $\tilde{\omega}$.

(i) It is shown in No. 9 of the Celestial Mechanics, if the central body of the system, which in this case is the sun, be considered as unity, that h a constant quantity

$$= \Sigma\, m.\, \frac{dx^2 + dy^2 dz^2}{dt^2} - \frac{2.\Sigma m}{r},$$

we have also from No. 18,

$\Sigma\,\frac{m}{a} = \Sigma\,\frac{2m}{r} - \Sigma\, m\,\frac{dx^2+dy^2+dz^2}{dt^2}$, $\therefore$ multiplying by Σm, and neglecting quantities of the order m^2, &c. we obtain

$$h = \Sigma\,\frac{m}{a},\ i.\ e. = \frac{m}{a} + \frac{m'}{a'} + \frac{m''}{a''} + \&c.$$

(k) These inequalities which have very long periods, are expressed in this form

$$\zeta = \frac{3\,im'}{\mu}.\textstyle\int\int a\,kn^2\,dt^2.\ \sin.\,(i'n't - int + \mathrm{A}),$$

which double integration gives a term

$= -\frac{3\,im'.\,an^2k}{\mu\,(i'n'-in)^2}$. sin. $(i'n't - int + \mathrm{A})$, in which the denominator is very small, when $nt : n't$ very nearly in the ratio of $i' : i$; from which there result inequalities in ζ which increase very slowly, and which, on that account, might induce us to to suppose that the mean motions of m and m' were not uniform; $n\ n'$ the mean motions of Jupiter and Saturn, are such that $5n'$ = very nearly $2n$, $\therefore$ the term of the expression for ζ, which depends on $5n't-$

$2nt$, though of the third order, becomes extremely sensible.

Besides the stability which is secured to our system by the law of gravity varying as $\frac{1}{d^2}$, it is likewise a peculiarity of that law, that the orbits of the heavenly bodies, their distances, &c. are independent of their *dimensions* and *absolute motion* in space; for if M represent the mass of a central body, d its distance, if the dimensions are changed in any ratio of 1 to $\frac{1}{n}$, every line such as d becomes $\frac{d}{n}$, and every mass m becomes $\frac{m}{n^3}$, if $\phi(d)$ be the function of the distance, which determines the law of attraction, so that $\frac{m}{\phi(d)}$ determines the action of m on a body revolving about m, the new force at the distance $\frac{d}{n}$ will be $\frac{m}{n^3\,\phi.\left(\frac{d}{n}\right)}$, but the orbits being supposed to be similar, the force must change in the ratio of 1 to $\frac{1}{n}$, because the lines which the forces cause the bodies to describe are $\div$l to the forces,

$\therefore \frac{m}{n^3.\,\phi.\left(\frac{d}{n}\right)} = \frac{1}{n}.\,\frac{m}{\phi(d)}$ or $n^2.\,\phi.\left(\frac{d}{n}\right) = \phi(d)$, *i.e.* $\phi(d)$ must be such a function of d, that if $\frac{d}{n}$ be substituted in place of d, the function after being multiplied by n^2 has the same value as before, *i. e.* $= \phi(d)$, $\therefore$ all the terms in which n occurs must respectively vanish, this only obtains when $\phi(d) = Cd^2$, for if $\phi(d) = Ad^m$, $\phi\left(\frac{d}{n}\right) = A.\,\frac{d^m}{n^m}$ and

$n^2 . \phi\left(\frac{d}{n}\right) = Ad^m . n^{2-m}$, this must be $= \phi\left(d\right) = Ad^m$, $\therefore$

$2 = m$, $\therefore$ the force with which m acts on a body at a distance $= d$

$= \frac{m}{\phi(d)} = \frac{m}{Cd^2}$. This law also gives the *simplest* possible expression; for spherical bodies made up of particles attracting according to this law, attract each other according to the same, which would not be the case if the attraction *decreased* according to any other law; likewise $\frac{m}{d^2}$, the expression for this force, is of one dimension, which should be the expression of a force reduced to its utmost simplicity; and the lines described by two bodies acting according to this law are always of the second order; therefore, as no other law could secure the same stability, neither could any other give the same simplicity.

NOTES TO CHAPTER III.

(*a*) In fact there are, as appears from the details of this Chapter, two ways of determining the ratio of the mass of a planet to that of the sun; either from a consideration of the inequalities and perturbations which the actions of these bodies produce, or from knowing accurately the periods and distances of the primary planets, and of the satellites which accompany them. In the former case, we may make use of either the periodic or secular inequalities; the latter, if accurately determined, would obviously give the most accurate results; but as these are as yet not sufficiently well determined, we are obliged to make use of those periodic inequalities, which are determined by a great number of exact observations.

(*b*) If μ μ' represent the sum of the masses of the sun and earth, and planet and satellite, A a the respective distances of the earth and satellite from the sun and primary, and P p their respective periods, we have, by what is stated in page 384,

$$P^2 \backsim \frac{A^3}{\mu}, \text{ and } p^2 \backsim \text{ as } \frac{a^3}{\mu'} \therefore \frac{\mu'}{\mu}, \ i.e. \ \frac{m+m'}{M+m} = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2};$$

if m' be very small relatively to m, and m very small with respect to M, we have $\frac{m}{M} = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$; this would be accurately true, if we had $M : m :: m : m'$, rejecting m' and retaining m, we have $\frac{m}{M+m} = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$, *i.e.* $\frac{m}{M} - \frac{m^2}{M^2}$ $q.p = \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$, but $m = M\frac{a^3}{A^3} \cdot \frac{P^2}{p^2}$ very nearly, $\therefore \frac{m}{M}$ $= \frac{a^3}{A^3} \cdot \frac{P^2}{p^2} \cdot \left(1 + \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}\right)$, now if δ be the sine of the angle under which m appears at the mean distance A from the sun, $\frac{m}{M} = \delta^3 \frac{P^2}{p^2}\left(1 + \frac{a^3}{A^3}\frac{P^2}{p^2}\right)$, relatively to the sun and earth, if π Π denote the horizontal parallaxes of the sun and moon, $\frac{a^3}{A^3} = \frac{\pi^3}{\Pi^3}$ $\therefore \frac{m}{M} = \frac{P^2}{p^2} \cdot \frac{\pi^3}{\Pi^3}$ $\therefore \frac{m}{M} \backsim$ as $\frac{\pi^3}{\Pi^3}$, hence any error in the parallax produces an error three times as great in $\frac{m}{M}$.

(*c*) The duration of the several revolutions has been inferred by means of the formula $\frac{PT}{P+T}$, given in page 328 of the First Volume, corrected for the inequalities of light and motion of the apsides, &c.

(*d*) The action of the satellites, and also of the ring of Saturn, contribute to induce perturbations in the system, ::lly to the quantity of matter which they contain. See page 17.

(*e*) Let $s =$ the arc described in $1''$, and z its versed sine, the mean distance of the sun from earth being assumed, $= 1$, $z = \frac{s^2}{2}$, but $s = \frac{2 \times 3.14159}{36225636'',1}$ $\therefore \frac{s^2}{2} = \frac{1479565}{10^{20}}$, the space described at the latitude, the square of the sine of which $= \frac{1}{3}$, relatively to r the radius of this parallel, is $\frac{3.^{me}66477}{6369809}$, and relatively to R the radius of the earth's orbit it is $= \frac{3.^{me}66477}{6369809} \cdot \frac{r}{R} = \frac{3.^{me}66477}{6369809} \cdot \sin. \pi$,

$$\therefore \frac{3.^{me}66477}{6369809} \cdot \sin. \pi \frac{r^2}{R^2} \cdot = 3.^{me}66477. \frac{\sin. \pi. \sin. ^2\pi}{6369809} = g,$$

is the earth's attraction reduced to the mean distance of the sun from the earth, for the number of metres in this distance : 6369809 :: 1 : sin. π;

but $\frac{M}{m} = \frac{s-g}{g}$, (see page 17.) and $\frac{m}{M} = g. \frac{1}{1-\frac{g}{s}}$.

As the cube of the parallax is involved in this expression of the ratio of the masses, it follows that it is of the greatest consequence to obtain this quantity, or, in fact, the dimensions of the solar system, as accurately as possible; we shall see, in the sequel, that the perturbations of the moon furnish perhaps the most accurate means of obtaining it. See note (o) in Chapter V.

(f) Let $\tilde{\omega} = 2966''$ the apparent diameter of the sun as seen from the earth, Δ $\triangle$ the respective densities of the sun and earth, V v their respective volumes, we have

$$\frac{\Delta}{\triangle} = \frac{M}{V} \cdot \frac{v}{m} \cdot = \frac{A^3}{a^3} \cdot \frac{p^2}{P^2} \cdot \frac{\pi^3}{\tilde{\omega}^3} = \frac{P^2}{p^2} \cdot \frac{1}{\tilde{\omega}^3},$$

hence it appears that the density is independent of the parallax of the sun, or of the magnitude of the solar system. This would not be the case if the law of attraction was different from that of nature. See last note of preceding Chapter.

(g) If g' represent the space described by a body in a second, at the latitude on the earth of which the square of the sine is $\frac{1}{3}$, the acceleration h, or the space which a body would describe at the distance r from the centre of m another planet, equal to the terrestrial radius is equal to $m.g'$, $\therefore$ G the space described at a distance, $= k$ the radius of the planet at the latitude of which the square of the sine is $\frac{1}{3} = \frac{hr^2}{k^2}$, but $k = \frac{\bar{\omega}.r}{\pi}$, $\therefore \mathrm{G} = \frac{hr^2}{k^2} = \frac{\mathrm{M}g'r^2}{k^2} = \mathrm{M}.g'.\frac{\pi^2}{\bar{\omega}^2}$, $\bar{\omega}$ is the apparent diameter of m, seen from the earth.

In the numerical expressions for the masses, volumes, and densities of the planets, the only *absolute* quantity is the fall of heavy bodies at the surface of the planets. For, in the expressions for the masses and densities, their ratios to the mass and density of the earth is all that is given. In the Sixth Chapter there are several methods given of determining the mean density of the earth, which can be had relatively to that of water; but as the absolute density of water, or of any substance, is not given, this itself is only a ratio. See Book 5, p. 259, Vol. I. It is worth remarking, that the density of the sun, according to the preceding computation, does not differ much from that of water, and is considerably less than the mean density of the earth. The methods given in the text are not applicable to the moon. For obtaining the density and quantity of matter of this body, see Chapter VIII., note (g).

NOTES TO CHAPTER IV.

(*a*) WHAT constitutes the difficulty in investigating the motions of the comets, arises from the eccentricities of their orbits, and the inclinations to the ecliptic being so considerable, that the formulæ which furnished expressions for the disturbed orbit of the planet, are not at all applicable in this case; therefore, in the present state of our analysis, we cannot express these perturbations by analytical formulæ, which embrace, as in the case of the planets, an *indefinite* number of revolutions.

The following is the method which Laplace made use of to compute the perturbations which the comet of 1759 experienced in its successive revolutions, which he determined so exactly, that he was enabled to predict its next return to the perihelion, to within thirteen days of its actual appearance. From a careful discussion of the observations of this comet in 1682 and 1759, the elements of the orbit at these two epochs, on the hypothesis that it is an ellipse, of which the greater axis answers to the duration of a revolution from 1682 to 1759, were computed. Then, assuming the elements of 1682 as strictly accurate, he determines, by what is established in the Ninth Book, the changes which have taken taken place in the elements, and in the mean anomaly, in the three first quadrants of the excentric anomaly, *i. e.* from $u = 0$, to $u = 270°$. See Volume I., page 395. In order to determine the changes in the last 90°, it is better to go back from 1759 to the extremity of this quadrant, which is the same thing as if we fixed the origin of this angle at the perihelion of 1759, and then went back to 1682, making u negative, and commencing with the elements and epoch observed in 1759. As the comet is nearer to the disturbing planets, particularly to Jupiter, in the first and last quarters of the ellipse, than in the second and fourth, it is necessary to have as exactly as possible its position and distance from these planets, the attractions of which may make a change of a considerable number of degrees in the elongation of the comet. To obtain still greater accuracy, the changes in the elements and mean anomaly from 1682 should be computed, making use of the greater axis corresponding to this epoch, which is given by the preceding approximation; then, as far as 25° of excentric anomaly, we can employ the elements of the new ellipse, which answer to this anomaly; and afterwards, in this ellipse, thus rectified, compute the perturbations from 22½ to 45. The fundamental ellipse is rectified in the same manner, from 90 to 180; and the perturbations up to 270 of eccentric anomaly are then determined. The alterations in the last quadrant of the eccentric anomaly are then obtained by rectifying the ellipse to −22½—45 and −90; by this means the perturbations of the comet from 1682 to 1759 will be given much more accurately by a second approximation. Similar operations may be performed from 1759 to the next perihelion; but as the moment of the passage through this last point is unknown, when we arrive at 270, the ellipse is rectified for every 22½ up to 360. These computations, when carefully performed, ought to give, within a very few days, the instant of the passage of the comet through the next perihelion; the only uncertainty is with respect to the mass of Uranus, the determination of which is perhaps best determined by observing this passage.

(*b*) Lalande computed whether, among the sixty comets whose orbits and returns had been observed and discussed when he wrote, any of them had their nodes near to the circumference of the earth's orbit; and he found, that there were only eight of the sixty whose distance from the sun when at their node, did not differ much from that of the earth from the sun. In fact, the question comes to this, to determine whether, among the sixty known comets, it ever happens, that at the time their distance from the sun is equal to the distance of the earth from the sun, they are also in their node, and consequently in the plane of the ecliptic? for in that case it might happen, among an infinite number of revolutions, that the earth might be at that very part of its orbit at the moment of the comet's passing through the node; in which case there would be necessarily a collision between the two bodies. If even the distances were not precisely the same, still if the difference was not very great, the mutual attractions of the earth and comet, and also the actions of the other planets, might cause them to be exactly equal; and consequently produce an impact between them; and we know, from experience, that very considerable changes are frequently produced in the cometary orbits. If a comet, equal to the earth, approached three times nearer to it than the moon, the effects which it would produce in elevating the waters of the ocean would be such as entirely to inundate the earth; but then, when the rapid motion is taken into account, and also the inertia of the waters of the sea, it will be apparent that they would soon be beyond the effect of the earth's attraction. From all these circumstances taken into account, it appears there are the following conditions to be satisfied, 1st, That the exact coincidence of the node with the orbit of the earth, which is itself instantaneous, should occur at the very time that the comet passes through it: 2dly, Granting this coincidence, it is necessary that these two bodies of which the orbits accurately intersect, should meet at the same time in the very point of their intersection. The probability of this last might be estimated in the following manner, as the diameter of the earth seen from the sun is only 17″, it does not occupy more than the 76th thousand part of the circumference of its orbit; therefore, on the hypothesis that the comet traverses accurately the orbit of the earth, there is, at the instant it is in the node, 76 thousand to one that the earth is not in that point of its orbit where it can be struck. Besides, the passages through the nodes are of rare occurrence, since each revolution requires a considerable time, and thus thousands of revolutions may be performed without the nodes being accurately on the circumference of the earth's orbit.

With respect to the effect of the attraction of comets, which, though they do not actually impinge, approach to the earth so as to effect an elevation of its waters, Sejour shows, that in consequence of the inertia of the waters, if even the sea was diffused over the whole earth, it would take $10^h\ 52'$ to produce its entire effect; but then the true circumstances of the problem are not so favourable to those great perturbations; for, 1st, The comet is not always perpendicular to the same point of the earth, in consequence not only of the rotatory motion of the earth, but also on account of the very rapid motion of the comet itself; besides, the waters are not diffused over the entire earth, which necessarily diminishes the effect of the earth's attraction; and, 3dly, there is only a very short time (less considerably than $10^h\ 52'$) during which the comet is at the distance at which its effect might raise the waters of the sea. The velocity with which the comet moves at the distance of the earth from the sun is easily obtained, on the hypothesis that the orbit is parabolic, for this velocity is to that of the earth as $\sqrt{2}$ to 1.

(*c*) See note (*n*) to Chapter VIII. of this Volume. According to Cuvier, the appearances exhibited by various strata on the lowest parts of the earth, and also on the tops of mountains, where shells and various marine productions have been dug up, may be adduced as decisive proofs of a number of revolutions having taken place on the surface of our globe. That these revolutions have been very sudden, he infers, from the circumstance of the carcasses of some large quadrupeds having been arrested and preserved entire, with their skin, hair, and flesh, which could not be the case unless they were frozen almost as soon as they were killed. See note (*n*) Chapter VIII., where the real cause of these productions being found in these regions is assigned.

(*d*) From knowing the place of the ascending node, the inclination of the orbit of 1770, &c., Laplace determines the epoch at which the comet passes out from the sphere of Jupiter's attraction; and then, by means of these data, he computes the elements of the relative orbit of the comet about the sun, from which the elements of the ellipse at its entrance into the sphere of Jupiter's attraction are computed; and the value of the axis major $=13,293$, and of the perihelion distance $=5,0826$, shows that the comet is perpetually invisible. Buckardt, in like manner, determined the effect of the action of Jupiter on the comet, which appeared in 1779, at the moment it entered within the sphere of its attraction; and an investigation of its elements at that instant showed that they differed very little from the preceding; and then, by computing the effect of Jupiter's action, he found that the axis major became 6,388, and the perihelion distance 3,3346, at which distance the comet also remains invisible. Hence we see how the action of Jupiter may have rendered (Mechanique Celeste, Tom. II., p. 226.) this star visible in 1770, which was previously invisible, and render it then invisible from the year 1779. The changes produced in the orbit of this comet are the *greatest* instance of perturbation observed among the bodies of our system.

(*e*) $n\ n'$ being the mean motions of the earth and comet, $a'\ a$ the greater axes of their orbits, by assuming a' the radius of the earth's orbit = to unity, we have

$$\frac{\delta n}{n} = 104.791.m'.a, \text{ i. e. as } \frac{n^2}{n'^2} = \frac{a'^3}{a^3}, \ \frac{\delta n}{n} = 104.791.m'.$$

$\frac{n'^{\frac{3}{2}}}{n^{\frac{3}{2}}}$, if T represents the duration of the comet's revolution,

T′ that of the earth, and δ T a variation corresponding to δn, we shall have $n\,T = 2\pi$

$$= (n + \delta n).(T + \delta T), \ \therefore \ \frac{\delta n}{n} = -\frac{\delta T}{T}, \text{ but } \frac{n}{n'} = \frac{T'}{T},$$

$\therefore \delta T = -104{,}791.m' \left(\frac{T}{T'}\right)^{\frac{3}{2}}.T$; hence substituting for T′

and m', their numerical values, we find $\delta T = -2^d{,}046$; this is the quantity by which the action of the earth diminishes the period of the comet; and as, by No. 65 of the First Book of Celestial Mechanics,

$$\delta n' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}.\delta n, \text{ and } \therefore \frac{\delta n'}{n'} = -\frac{m\sqrt{a}}{m'\sqrt{a'}}.\frac{n}{n'}.$$

$\frac{\delta n}{n}$, *i.e.* as $\frac{n'^{\frac{1}{3}}}{n^{\frac{1}{3}}} = \sqrt{\frac{a}{a'}}$, and as $\frac{\delta n}{n} = 104{,}791.\frac{m'.n'^{\frac{2}{3}}}{n^{\frac{2}{3}}}$,

$\frac{\delta n'}{n'} = -104{,}791.m$, and $\therefore \delta T' = 104{,}791\,mT'$, hence if

$m'=m$, we shall have $\delta T' = 0{,}11612$; and as accurate observations prove that $\delta T'$ does not exceed $2''{,}5$, it follows that m is not the $\frac{1}{5000}$th part of m.

NOTES TO CHAPTER V.

(*a*) Let the distance of the sun from the earth (which for the present we shall suppose to be constant) $= a$, and let y, z be the distances of the moon from the earth and sun, ϕ the elongation of the moon from from the sun, $\frac{\mu}{a^2} = F =$ the force with which the earth is solicited towards the sun, $\frac{1}{y^2} (= P,) \frac{\mu}{z^2}$ are the forces with which the moon is sollicited to the earth and sun, (μ being the mass of the sun relatively to the earth represented by unity,) and it is evident, that if the sun was infinitely distant, $\frac{\mu}{a^2}$ and $\frac{\mu}{z^2}$ might be considered as $=$, and acting in parallel directions. If $\frac{\mu}{z^2}$ be resolved into two forces, one in the direction of y, and the other in the direction of a, they will be respectively $\frac{\mu y}{z^3} = Q$, $\frac{\mu a}{z^3} = R$, if $R = F$, the relative motion of the moon about the earth would not be changed, the only effect would be to make the sun approach the common centre of gravity of these two bodies. Hence it follows, that the moon is drawn from the earth parallel to a, in virtue of the force $\mu\, a.\left(\frac{1}{z^3}-\frac{1}{a^3}\right)=\mathrm{S}$; now as the greatest value of $z=a+y$, and its least value $=a-y$, and consequently the mean value $=a$, it might seem that in general this force cannot alter the mean distance of the moon from the earth; but this is not the case. See note (g) of this Chapter. From what precedes, it appears, therefore, that the forces which affect the relative motion of the moon about the earth, are P+Q in the direction of y the radius of its orbit, S in a direction parallel to a, the line connecting the earth and sun; therefore, if in a line drawn from the place of the moon in its orbit, parallel to a, a portion be assumed = S, and if this force be resolved into two, one in the direction of y, and the other in the direction of a tangent to the moon's orbit, we have N the whole central force of the moon $=\frac{1}{y^2}+\frac{\mu y}{z^3}-\mu.\frac{(a^3-z^3)}{a^2\,z^3}.\cos.\phi$, and M the force in the direction of the tangent $=\mu\,\frac{(a^3-z^3)}{a^2\,z^3}.\sin.\phi$.

(b) The manner in which the perturbations of the moon are investigated, is not essentially different from the manner employed to determine those of the planets; however, they are not exactly the same, for though the fundamental equations have the same form, still we cannot assume, as in the case of the planets, that the mass of the disturbing body is inconsiderable relatively to that of the central, for the sun is in this case the disturbing body, whose mass is 354790 times that of the earth; this quantity is indeed multiplied by the ratio of the radius of the lunar to that of the terrestrial orbit, *i. e.* by $\frac{1}{64000000}$, still, however, the product is too considerable to permit us to neglect its square, which we can do for the most part in case of the planets. Notwithstanding the facilities mentioned in the text, the theory of the moon is embarrassed with peculiar difficulty, owing in a great measure to the magnitude of its numerous inequalities, and also very much to the little convergence of the series which furnish them; hence results the necessity of a judicious selection of ordinates, &c., insisted on in the text. The solution of the problem of the three bodies which is given by Laplace, is called the direct method, in contradistinction to the more simple and indirect method which Newton followed, which is by no means so accurate as the first, in which the terms of the series by which the motions of the moon are expressed may be computed to an indefinite extent, or at least until the quantities omited are too small to affect observations, whereas, in the method of Newton, we cannot go farther than the first, or at most a *few of the leading* terms of each series.

(*c*) Though it is difficult to determine the precise quantity by which the apsides advance, it is easy to show that, in consequence of the disturbing force of the sun, they *must prograde;* for, as is evident from the nature of an elliptic orbit, if a body leaves an apsis, it will arrive at the other apsis after describing 180°; (see notes, page 375.;) but as the mean disturbing force in the direction of the radius vector tends, *on the whole,* to diminish the gravitation of the moon to the earth, (see note (*f*),) the portion of her path described in any instant will be less deflected from the tangent than if this disturbing force did not exist; therefore the actual path of the moon will be less incurvated than the elliptic orbit, which would be described if the moon was influenced solely by the force of gravity, and consequently it will not be brought to intersect the radius vector at right angles, until it has moved over a *greater arc than* 180°; therefore, in consequence of the action of the solar force, the apsides advance. The term which Clairault proposed to add to $\frac{1}{r^2}$, was one of which the effect could be only sensible in the motion of the apogee, and not on the other lunar inequalities. However, when he proceeded to fix the form of the quantity which would make this new supposition represent the motion of the apsides, he found it necessary to perform the computations more rigorously, and to extend his approximation farther than before, so as to include terms which he had previously neglected; and when these terms were taken into account, he found that it was not necessary to make any correction to the Newtonian theory; consequently it followed that no such term as $\frac{B}{r^m}$ was required, or, in other words, that a force varying inversely as the square of the distance was sufficient to explain the motion of the apsides. For an able and satisfactory analytical statement of the process by which Clairault arrived at the true expression for the motion of the lunar apsides, see Woodhouse's Astronomy, Vol. II., Chap. 13.

According to Buffon, it is more philosophical to suppose that every primordial law of nature depends on one sole function of the distance; for if two powers of the distance were introduced, the function expressing the law of attraction would necessarily contain one constant arbitrary quantity at least, so that the ratio of the attractive forces at two different distances of the attracting body would not depend solely on two different distances, but would also involve some parameter, which would modify and complicate this ratio; thus the attraction would depend not only *on the distance*, but also on this parameter, for the introduction of which there does not appear to be a sufficient reason.

As the method by which Clairault determined the precise motion of the lunar apsis was analytical, it was supposed by several mathematicians that the direct synthetic method of Newton was inadequate to the determination of its exact quantity, and it is thought that Newton himself was conscious of the insufficiency of his method, from the circumstance of his omitting, in the later editions of the Principia, any mention of the motion of the apogee. However, though Newton certainly did not mention the quantity of this progress, still there can be no doubt but that his method is fully equal to determining its exact quantity. See Stewart's Mathematical and Philosophical Tracts.

If ☾ ψ represent the relative motions of the moon and apsis, then it is easy to show that $d\psi$, the differential of the motion of the apsis $= \frac{3\mu y^3}{4a^3}(d ☾ - 3nd\phi . \cos. 2\phi)$, when d ☾ is the periodic, and $d\phi$ the synodic motion of the moon, and $n : 1$ the ratio of $d\psi$ to $d\phi$,

$\therefore \psi = \frac{3\mu y^3}{4a^3} . \left(☾ - \frac{3}{2} n . \sin. 2\phi \right)$; $\frac{\mu y^3}{a^3}$ 270 is the mean motion of the apsides in a *periodic* month. The second term $-\frac{9\mu y^3}{8a^3} . n . \sin. 2\phi$, gives the libration of the apsides; it evidently vanishes in syzygies and quadratures, is a positive *maximum* in the octants which precede the quadratures; and a negative maximum in the octants which precede the syzygies. This equation of the apsides produces a correction to be applied to the equation of the centre; the argument for which depends on the distance of the sun from the moon's apogee.

(*d*) In the conjunctions N, given in note (*a*), becomes, (as $\phi = 0$, and

$z = a - y) = \frac{1}{y^2} + \frac{\mu.(y-a)}{(a-y)^3} + \frac{\mu}{a^2} = \frac{1}{y^2} - \frac{\mu(2ay - y^2)}{a^2(a-y)^2}$,

i. e. as y is very small relatively to $a = \frac{1}{y^2} - \frac{2\mu y}{a^3}$, in opposition $N = \frac{1}{y^2} + \frac{\mu(y+a)}{(y+a)^3} - \frac{\mu}{a^2} = \frac{1}{y^2} - \frac{2\mu y}{a^3}$, in the quadratures, cos. $\phi = 0$, and $z = a$, $\therefore$ N $= \frac{1}{y^2} + \frac{\mu y}{a^3}$, hence in the syzygies the central force $\frac{1}{y^2}$, experiences the greatest diminution, and in the quadratures the greatest increase; and it is evident that the diminution in syzygies is twice the increase in quadratures. Conceive a perpendicular from the place of the moon on a, the part of a intercepted between the perpendicular and the centre of the sun is nearly

$=$ to z, $\therefore z = a - y.\cos.\phi$, and $z^3 = a^3 - 3a^2y.\cos.\phi$, hence, by substituting this value of z^3 in the expression for M, given in note (a), page 324, we obtain

$$M = \frac{3\mu y.\sin.\phi.\cos.\phi}{z^3} = \frac{3\mu y.\sin.2\phi}{2a^3},$$

$$N = \frac{1}{y^2} + \mu y\frac{(1 - 3.\cos.^2\phi)}{z^3} = \frac{1}{y^2} - \frac{\mu y(1 + 3.\cos.2\phi)}{2a^3},$$

hence it appears that M attains its greatest *positive* value in the octants which precede the syzygies, and its greatest negative value at the octants which precede the quadrature; and that the central force varies as $\frac{1}{y^2}$, when $1 + 3.\cos.2\phi = 0$, *i. e.* when $\cos.2\phi = -\frac{1}{3}$, and $\therefore \phi = 54°, 44'$; $\phi = 125°, 16'$; $\phi = 135°, 16'$; $\phi = 234°, 44'$; $\therefore$ in these four points, two of which are nearly at 10° before the octants which precede the syzygies, and two at 10° after the octants which precedes the quadratures, the central force varies inversely at the square of the distance.

It is easy to show, that in the quadratures the force varies in a ratio less than that of the inverse square of the distance, for in that case N $= \frac{a^3 + \mu y^3}{a^2 y^2}$, and if y' differs very little from y, N′ being the central force at the distance y', then

$$N : N' :: \frac{1}{y^2}\left(1+\frac{\mu y^3}{a^3}\right) : \frac{1}{y'^2}\left(1+\frac{\mu y'^3}{a^3}\right), \textit{ i. e.} \text{ if}$$

$a\ a' = \frac{\mu y^3}{a^3}, \frac{\mu y'^3}{a^3}$ respectively, $N : N' :: \frac{1+a}{y^2} : \frac{1+a'}{y'^2}$, now, as $y = 60$ radii of the earth, $q.p$ all powers of y, whether integral or fractional, whose index is positive, are necessarily greater than unity; therefore we may assume $1+a = y^\gamma$ $1+a' = y'^\gamma$, γ 1 being a small positive fraction, and consequently

$N : N' :: \frac{y^\gamma}{y^2} : \frac{y'^\gamma}{y'^2} :: \frac{1}{y^{2-\gamma}} : \frac{1}{y'^{2-\gamma}} :: \frac{1}{y^n} : \frac{1}{y'^n}$, $\therefore$ as γ is a positive fraction near to the quadratures, the force varies in a less ratio than the square of the distance; in like manner, at the syzygies

$$N : N' :: \frac{a^3-2\mu y^3}{y^2} : \frac{a^3-2\mu y'^3}{y'^2} :: \frac{1-\beta}{y^2} : \frac{1-\beta'}{y'^2}$$

$:: \frac{y^{-\lambda}}{y^2} : \frac{y'^{-\lambda}}{y'^2} :: \frac{1}{y^{2+\lambda}} : \frac{1}{y'^{2+\lambda}}$, when $\beta =$ a very small fraction, $\therefore$ as $1+a = y^\gamma$, and $1-\beta = y^{-\lambda}$, we have $q.p$ $\beta = 2a \therefore \lambda = 2\gamma$, $\therefore$ if $N = \frac{1}{y^m}$, m is $>$ than 2, and $m-2 = 2(2-n)$, therefore in the syzygies the central force varies in a greater ratio than the inverse square of the distance; $\frac{1}{y^2} - N$ is a maximum, and $= \frac{2\mu y}{a^3}$ in the syzygies; but its actual value depends on the situation of the apsides; as N : N′ near to the syzygies are as $\frac{a^3-2\mu y^3}{y^2} : \frac{a^3-2\mu y'^3}{y'^2}$, which would be the inverse :: of the squares of the distances, if the numerators were $=$, $\therefore$

it will be the more deranged, according as y y' differ more from each other; but the difference is evidently a maximum, when the line of apsides coincides with that of the syzygies; therefore the ellipse suffers the greatest derangement in this case; from the first and last quadratures to the syzygies, M increases the velocity of the moon, and from the syzygies to quadrature, M retards by the same quantity the velocity of the moon; hence the velocity is a maximum in the syzygies, and a minimum in the quadratures, and at its mean value in the octants, since the velocity is a maximum, and the central force a minimum in the syzygies, the moon in these points deviates less from the tangent; therefore the moon approaches the earth least in syzygies, and most in quadratures; and as the orbit is a curve returning into itself, it follows that, as the moon commences to recede from the earth at the syzygies, and to approach the earth in the quadratures, that the distance of the moon from the earth in an orbit which, without the disturbing force action of the sun, would be circular, is greatest in quadratures and least in syzygies. If the orbit be an ellipse, *i. e.* if the force varied as $\frac{1}{y^2}$, then if, in going from apogee to perigee, the force increases in a greater ratio (see page 341), the *true* orbit will fall within the ellipse, and the perigean distance will be less than for the ellipse, consequently the eccentricity will increase so much the more as the axis major diminishes; for a like reason, if the moon departs from the perigee, and the force decreases in a greater ÷ than the inverse square of the distance, the moon, when in the apogee, will have receded farther from the earth than if the orbit described was an ellipse; therefore, in the other half of the orbit the eccentricity will be also increased, and the contrary to this will obtain if the force varies in a less ratio than the inverse square of the distance. Now, as the force varies in a greater or less ratio than $\frac{1}{y^2}$, according as the apsides coincide with the syzygies or the quadrature, it follows that the eccentricity is a maximum in the former, and a minimum in the latter case; *i.e.* when the greatest equation of the centre coincides with the quadratures the eccentricity is a maximum, when this equation occurs near to the syzygies this eccentricity is a minimum, and generally in the progress of the apsides from the syzygies to quadratures the eccentricity diminishes, and from quadratures to syzygies the eccentricity increases. This is the explanation of the phenomenon known by the name of evection. See note (*t*) Chapter IV., Vol. I.; and Princip. Math. Prop. 66., Cor. 9.

(*e*) In order to compute the *mean* quantity of the force $\frac{\mu y}{a^3}.(1-3\cos.^2\phi)$, which is continually directed to or from the centre of the earth, if we multiply it by $d\,\phi$, the differential of the arc of elongation, we have $\frac{\mu y}{a^3}.(d\,\phi -$ $3\,d\,\phi\cos.^2\phi)$, the integral of which $=\frac{my}{a^3}.\left(-\phi+\frac{3}{2}.\sin.\phi.\cos.\phi\right)$, which, when extended to the whole orbit, *i.e.* when ϕ is four right angles, becomes $\frac{\mu y}{a^3}\times-\frac{\pi}{2}$, which therefore expresses the sum of the forces for an entire revolution; and ∴ when divided by π gives the mean force $\frac{-\mu y}{2a^3}$, which being negative, shows that the mean effect of the solar force is to diminish the gravitation of the moon to the earth.

Calling $\tau\,\tau'$ the periods of the earth and moon, F the gravity of the moon, we have

$$\frac{\mu}{a^2} : F :: \frac{a}{\tau^2} : \frac{y}{\tau'^2}, \therefore \frac{\mu y}{a^3} = \frac{F.\tau'^2}{\tau^2} = \frac{F}{179}, \text{ for } \frac{\tau'^2}{\tau^2} = \frac{1}{179},$$

and $\therefore \frac{-\mu y}{2a^3} = \frac{-F}{358}$.

(*f*) Though the mean area is therefore not altered, since the radius vector is increased $\frac{1}{358}$th part, inasmuch as the angular velocity in *this case* is inversely as the square of the distance, its diminution will be *q.p* half the increase of the radius vector, and $\therefore \frac{1}{179}$th part.

(*g*) It is evident, from what precedes, that every thing else being the same, the numerical coefficient $\frac{1}{179}$ varies inversely as the cube of the distance; therefore, making $a=1+\varepsilon$, we shall have the mean disturbing force in the direction of $y = -\frac{F}{358} \cdot (1+\varepsilon)^{-3}$; now, as the magnitude of the area is the same, we have

$$\delta.(y.^2\delta\phi) = 0, \; i.e. \; 2\delta y\delta\phi + y\delta^2\phi = 0; \therefore$$

$\frac{\delta^2\phi}{\delta\phi} = -\frac{2\delta y}{y} = (1+\varepsilon).^{-3} \frac{2}{358} = \frac{(1+\varepsilon)^{-3}}{179}$, $\therefore$ on the hypothesis that the moon's orbit is circular, and

$$\therefore \delta\phi = n\delta t, \text{ we have } \delta^2\phi = -\frac{n\,dt}{179}(1+\varepsilon)^{-3},$$

hence, if e be the eccentricity of the lunar orbit, and l the mean anomaly, we have

$$a = 1+\varepsilon = 1 + \frac{e^2}{2} - e.\ \cos.\ l - \frac{e^2}{2}.\cos.\ 2l + \text{, \&c.}$$

See Celestial Mechanics, page 146.

$\therefore (1+\varepsilon)^{-3} = 1-3\varepsilon+6\varepsilon^2$, &c. $= 1+3e(\cos.\ l - \frac{e}{2} + \frac{e}{2}.$ $\cos.\ 2l) + 6e^2 \cos.\ ^2l$, &c. $= 1+\frac{3}{2}e^2+3e.\ \cos.\ l + \frac{9}{2}e^2 \cos.$

$2l$; if m represent the mean anomalistic motion of the sun, so that $\delta a = m\delta t$, we shall have $\delta^2\phi$

$$= \frac{-n\delta t}{179}.\left(1+\frac{3}{2}e^2\right) - \frac{3en\delta a}{179m}\cdot\left(\cos.\ l + \frac{3}{2}e\cos.\ 2l\right),$$

$$\therefore \int\delta^2\phi = \frac{-nt}{179} - \frac{3n}{358}\int e^2\delta t - \frac{3en}{179m}\left(\sin.\ l + \frac{3}{4}e.\ \sin.\ 2l\right).$$

The first term of this expression is included under the mean motion of the moon, namely nt; in fact, it is the mean diminution of the lunar motion, constituting, as appears, the 179th part of the primitive motion or $\frac{-nt}{179}$; the second term would also belong to the mean motion, if e was constant; but as the eccentricity of the earth is changed in consequence of the action of the planets, there results from it a *secular equation* of the mean motion of the moon $= -\frac{3n}{358}\int e^2\delta t$; the third term gives the *annual* equation of the moon; and if their values 13,37, and $\frac{1}{60}$ be substituted for $\frac{n}{m}$ and e, we shall obtain the value of the greatest equation given in Chapter IV., Vol. I., note (u); and it is evident, that is, if the same form, only affected with an opposite sign, as the equation of the centre of the sun. See note (u), Chapter IV., Vol. I.

(h) If the mean distance $=1$, e equal $\frac{1}{60}$, and A, A$+a$ are the respective angular velocities, we have

$A : (A+a) : 1 : \left(1+\frac{1}{60}\right)^2 :: 1 : 1+\frac{1}{30}$, $\left(\frac{1}{60}\right)^2$ being neglected, $\therefore a = \frac{A}{30}$; in like manner the mean value of the diminution of the mean motion $= \frac{-nt}{179}$, and as this varies inversely as the cube of the distance, at perigee it is $= \frac{-nt}{179} \cdot (1+e)^{-3}$, ∴ neglecting e^2 and e^3, and substituting its numerical value for e, the increase of the diminution $= \frac{nt}{20.179} = \frac{nt}{3580}$, ∴ as the arguments of the equation of the centre and of the annual equation are the same, they are as their coefficients, *i. e.* as $\frac{mt}{30}$ to $\frac{nt}{3580}$.

(*i*) If the mean motion of the moon, as determined by observations made at considerable intervals from each other, varies, its correction a, as it depends on the time, must be given by a series of the form $at+bt^2+ct^3+$, &c.; but all terms of the form at, being already included in the mean motion, we shall have $a=bt^2+ct^3+$, &c.; therefore the secular equation, or rather the most considerable of its terms is proportional to the square of the time, which is indeed otherwise evident from the following consideration, the force, whatever it be, which accelerates the motion of the moon, must be considered as constant, otherwise it could not produce a *real secular* equation; therefore δv, the increment of velocity communicated at *each instant*, may be considered as constant; hence $\delta v = \gamma \delta t$, and $v = c + \gamma t$, ∴ if s be the mean motion in the time t, we shall have $\delta s = v\delta t = c\delta t + \gamma t \delta t$, and $s = ct + \frac{\gamma t^2}{2}$, cv being the mean motion, therefore the acceleration or the secular equation is proportional to the square of the time, hence appears the reason of what is stated in the text, page 65, that as the increase takes place successively, and proportionally to the time, the effect on the moon's motion is half what it would be, if, during the entire century it was the same as at the end, and also the reason why the secular equation may be considered as increasing proportionally to the square of the time, as long as the diminution of the square of the eccentricity of the earth's orbit, is supposed to be proportional to the time.

If X be the mean motion of the moon between two observations, at the interval of n years, then $\frac{X}{n}$ and $\frac{100X}{n} = x$ will be the annual and secular motions. If y be the number of seconds by which the mean motion of the moon is greater than x in the following century, and less than x in the preceding, the correct mean motion in the following century will $x+y$, and in m centuries $= mx+m^2y$; hence, if C be the mean longitude at the commencement of the epoch, the mean longitude m centuries *after* the epoch $= C+mx+m^2y$, and for m centuries *before* the epoch the mean longitude $= C-mx+m^2y$, for in the last case m must be taken negatively, and $(-m)^2=m^2$; hence in *both* cases the *secular* equation is *additive.*

(*k*) The second term of the value of $\int \delta^2\phi = -\frac{3n}{358}.\int e^2\delta t = -\frac{3n}{179}\int\frac{e^2}{2}\delta t$, hence, as has been remarked already in page 416, since e is variable, this expression will not remain always the same. If ε be value of e at the commencement of the present century, then the value of e at any subsequent time $t = \varepsilon+t.\frac{\delta\varepsilon}{\delta t} + \frac{t^2}{1.2}.\frac{\delta^2\varepsilon}{\delta t^2}$ and $e^2 = \varepsilon^2 + 2t\ \varepsilon.\frac{\delta\varepsilon}{\delta t}+t^2\left(\frac{\delta e^2}{\delta t^2} + \varepsilon.\frac{\delta^2 e}{\delta t^2}\right)$, by means of this equation we can deduce the value of $\frac{\delta^2 e}{\delta t^2}$ in terms of ε and $\frac{\delta e}{\delta t}$, which are respectively given, $\frac{\delta e}{\delta t}$ being the variation of the eccentricity, and ∴ known; consequently, by substituting their numerical values, we obtain

$$e^2 = \varepsilon^2 - 2t.\,0.''14671 - t_{,}.^2 0.''0006027, \text{ and } \therefore \frac{3n}{358}.\int e^2 \delta t =$$

$$\frac{-3\varepsilon^2}{358}.\,nt + \frac{3n}{358}.\,t_{,}^2 0.''14671 + \frac{n}{358} t'^3.\,0.''006027,$$

i. e. substituting for n its value, and neglecting the first term of this expression, which is included under the mean motion, we have the part of the mean motion which depends on the variation of the eccentricity

$$= t.^2 10,''18 + t.^3 0,''018538,$$

hence appears the reason why, as stated in page 66, he adds a term proportional to the cube of the time.

(*l*) As $\frac{3n}{358}\int e^2 \delta t$ is affected with a negative sign, it is evident that the motion will be accelerated when e^2, or the eccentricity of the earth's orbit is diminished, and that it will be retarded when the quantity to be subtracted increases with the increase of e^2, or the eccentricity of the earth's orbit.

(*m*) In reference to what is stated in page 67, it may be observed, that after a most complicated investigation, and by substituting their numerical values, the secular equation of the perigee comes out = to $3{,}03.\,\frac{3}{2} m^2 \int e.^2 n \delta t$, and has a contrary sign to the secular equation of the longitude; the secular equation of the motion of the nodes comes out = to $0{,}735452.\,\frac{3}{2}\, m^2 \int e^2\; n.\delta t$, and as in the same circumstances, the secular equation of the mean motion $= -\frac{3}{2} m^2, \int e^2 n \delta t$; it follows that when the two first are accelerated, the last is retarded, and their ratio in numbers is that given in the text.

(*n*) In consequence of the attraction of the terrestrial spheroid, a nutation arises in the orbit of the moon, corresponding to that which the attraction of the moon produces in our equator, so that one of these nutations may be shown to be the reaction of the other; (see Chapter XIII. notes;) as the extent of this oscillation or nutation depends on the *compression* of the earth, it can throw considerable light on this important element. The existence of this inequality in the latitude of the moon, was indicated by observations long before the law which it observed was discovered. It can be perfectly represented by the expression 8″ sin. L, if the compression of the earth was assumed $= \frac{1}{334}$, whereas, if the compression was that which resulted from the hypothesis of homogeneity, namely, $\frac{1}{230}$, (see Chapter VIII., page 108,) the expression of this inequality would be —13.5. sin. L. The manner in which the quantity of the compression is determined is as follows: the *theoretic* expression for this inequality, which involves the compression of the earth, is compared with the value furnished from a careful discussion of a number of observations, and then substituting numerical values for sin. L, the value of the compression is thence deduced. In like manner, the inequality of the motion of the moon in longitude, which depends on the compression of the earth, is = 6″,8 cos. L (L expressing, as before, the longitude of the ascending node) on the hypothesis of a compression $= \frac{1}{334}$, which is exactly conformable to observation; whereas, on the hypothesis of homogeneity, this inequality would be = 11.″5. sin. L, contrary to observation. There is, according to theory, a certain given relation between the coefficient of the lunar inequality in longitude and latitude, from which the value of the coefficient of the inequality in longitude may be determined; and on the supposition that the coefficient of that in latitude is 8″, namely, that which results from the hypothesis of a compression of the earth $= \frac{1}{334}$, the coefficient of the inequality in longitude comes out 6,846, very nearly that which is given by observation. A phenomenon analogous to the preceding, and arising from the same cause, is produced in the orbit of Jupiter's satellites. See Chapter VI. Besides these inequalities depending on the compression of the earth, Laplace also investigated whether the difference which is known to exist in the quantity of land distributed over the northern and southern hemispheres, had any sensible influence; but a careful discussion showed that this effect was altogether inappreciable.

(*o*) In an inequality depending on the true distance of the moon from the sun, which Laplace terms the parallactic inequality, (see Mechanique Celest. Tom. 7., page 281,) the argument is $v—mv$; it depends on the ratio of the moon's distance from the earth to the sun's distance from the same, *i. e.* on the ratio of their horizontal parallaxes, which, as that of the moon is determined in the Second Chapter, it is easy to find; the result ought to be considered as extremely accurate, inasmuch as the approximation is extended to quantities of the fifth order inclusively. The only point in the theory of the moon's motion which remains to be cleared up after the delicate investigations of Messrs. Plana, &c. and Damoiseau, is a small change which astronomers have thought they discovered in the mean motion of the moon. However, as the existence of this change, though extremely probable, is not incontrovertably established, a greater number of accurate observations, made in the most favourable circumstances, is required before there will be occasion to ascertain its cause. The imperfect manner in which accurate observations have been made and transmitted to us, may also explain why we have not been hitherto able to appreciate the changes produced in the motions of the planets and satellites, in consequence of the attraction of comets, and also of the impact of meteoric stones, which are observed sometimes to impinge on our earth, and which appear to come from the depths of celestial space. The only thing which can throw light on this subject is a series of accurate observations.

(*p*) Laplace, in the Sixth Chapter of the Second Part of his Tenth Book, investigates in what cases we can rigorously obtain the motion of a *system* of bodies, which mutually attract each; and as, in order that this may be secured, it is necessary that the resultants of the forces by which each of the bodies of the system is actuated should pass through their common centre of gravity, and be proportional to the respective distances of the bodies from that point, he shows, that if the position of the bodies of the system be such, that the lines connecting them constitutes a polygon, existing in the same plane, which remains always similar to that formed by joining the bodies at the commencement of their motion, then (the law of attraction being proportional to any power of the distance between the bodies) the resultants of the forces by which the bodies are actuated must pass always through the common centre of gravity; but it is evident that these resultants, at the commencement, being supposed to pass through the centre of gravity, and to be proportional to the distances from that centre, they will always remain so, if, on the several bodies of the system, velocities be impressed in directions equally inclined to those distances, and respectively proportional to them; then the polygons formed at each instant by lines connecting the bodies will be *always* in the same plane, and similar, and the curves which the bodies will describe about their common centre, and about each other, will be similar to each other, and they will be of the same species with that which a body, actuated by the same law of force, would describe about a fixed point. See Princip. Math., Sect. 11, Prop. 58. Let the preceding conclusions be applied to the case of three bodies, $m\, m'\, m''$, acting on each other. If s be the distance between m and m', s' the distance of m from m'', and s'' the distance between m' and m'', it is easy to perceive that the force by which m is actuated parallel to the axis of x, is

$$m' \frac{\phi(s)}{s}(x-x') + m'' \frac{\phi(s')}{s'} \cdot (x-x''),$$

(the attractive force being proportional to $\phi(s)$.) and parallel to the axis of y will be

$$m' \frac{\phi(s)}{s} \cdot (y-y') + m'' \cdot \frac{\phi(s')}{s'} \cdot (y-y'');$$

similar expressions may be obtained for the forces parallel to these axis, acting on m' and m''; now, as by hypothesis, the resultant of the two forces parallel to the axes of x and y, which act on m, passes through the centre of gravity, we have

$$\frac{m \cdot \phi(s)}{s} \cdot (x-x') + m'' \frac{\phi(s')}{s'} \cdot (x-x'')$$

$$= \mathrm{K}x, \frac{m\,\phi(s).}{s}(y-y') + m'' \frac{\phi(s').}{s'}(y-y'') = \mathrm{K}y;$$

therefore the force which solicits m towards the centre of gravity is $\mathrm{K}.\sqrt{x^2+y^2}$; in like manner, it might be shown, that the force soliciting m' towards this point is $\mathrm{K}'\sqrt{x'^2+y'^2}$, and as, by hypothesis, the forces are as the distances, $\mathrm{K}=\mathrm{K}'$; therefore, for the forces acting on $m\, m'\, m''$ parallel to the axis of x, we have the three following equations:

$$m'. \frac{\phi(s)}{s} \cdot (x-x') + m'' \frac{\phi(s')}{s'} \cdot (x-x'') = \mathrm{K}x$$

$$m. \frac{\phi(s)}{s} \cdot (x'-x) + m''. \frac{\phi(s'')}{s''} \cdot (x'-x'') = \mathrm{K}x' \qquad (a)$$

$$m.\frac{\phi(s')}{s'}.(x''-x)+m'.\frac{\phi(s'')}{s''}.(x''-x')=\mathrm{K}x''.$$

Similar equations may be obtained for the forces parallel to y, changing $x\ x'\ x''$ into y, y', y'', &c.

Multiplying the preceding equations by $m\ m'\ m''$ respectively, and then adding them together, we obtain

$$0=mx+m'x'+m''x'';$$

which shows that the point to which the forces are directed must be the centre of gravity; by combining this equation with the first of the equations (a), we obtain

$$x\left(m.'\frac{\phi(s)}{s}+(m+m'')\frac{\phi(s')}{s'}\right)+m''x'\left(\frac{\phi(s')}{s'}-\frac{\phi(s)}{s}\right)=$$

Kx, which, if $s=s'$, gives $\mathrm{K}=(m+m'+m'').\frac{\phi(s)}{s}$, the same value of K will be obtained, if we suppose $s=s''$, ∴ if $s=s'=s''$, this expression satisfies the equations (a), and the corresponding ones in $y\,y'\,y''$, ∴ if on this supposition $r\ r'\ r''$ represent the respective distances of m, m', m'', from the centre of gravity of the system, the forces which solicit m, m', m'', towards this point are Kr, Kr', Kr''; and if, on these bodies, velocities be impressed proportional to r, r', r'', respectively, and in directions equally inclined to $r\ r'\ r''$, we will have, during the motion $s=s'=s''$, *i. e.* the three bodies will always exist in the vertices of an equilateral triangle formed by connecting them, and they will describe similar curves about each other, and about their common centre of gravity; Princip. Math. Prop. 52, Sect. 11. He next proceeds to determine the expression for K in a function of r, which will evidently determine the law of force Kr in a function of r. For this purpose, let the origin of the coordinates be any point different from X, Y, the centre of gravity; as, for instance, the centre of m, then $x\,y$ are = to cypher, $\mathrm{X}^2+\mathrm{Y}^2=r^2=$ generally

$$\Sigma m.\frac{(x^2+y^2)}{\Sigma m}-\Sigma m m'\frac{(x-x')^2+(y'-y)^2}{(\Sigma m)^2}, \textit{ i. e. as}$$

$xy=0$; and $s^2=s'^2$, *i. e.* $(x'-x)^2+(y'-y)^2=$

$$(x''-x)^2+(y''-y)^2,\ r^2=\frac{(m'+m'').s^2}{m+m'+m''}$$

$$-\frac{(mm'+mm''+m'm'')s^2}{(m+m'+m'')^2},\ \therefore s=\frac{(m+m'+m'').r}{\sqrt{m'^2+m'm''+m''^2}},$$

$\therefore$ as $K=(m+m'+m'').\frac{\phi(s)}{s}$, we have

$$Kr=\sqrt{m'^2+m'm''+m''^2}.\phi\left(\frac{m+m'+m''.r}{\sqrt{m^2+m'm''+m'^2}}\right),$$

hence, as we have the expression for the law of force, we can, by what is stated in notes page 373, determine the nature of the curve described, when the form of ϕ is given; if $\phi(s)=\frac{1}{s^2}$, then the force which solicits the body m towards the centre of gravity

$$=\frac{\overline{m'^2+m'm''+m''^2}|^{\frac{3}{2}}}{(m+m'+m'').^2r^2},$$ $\therefore$ the three bodies will describe similar conic sections about the centre of gravity of the system, the lines connecting them constituting always an equilateral triangle, the sides of which continually vary, and become infinite, if the section be a parabola or hyperbola, which circumstance depends on the initial velocity. But if $s\,s'\,s''$ are not $=$, then we have

$$x.\left(m.'\frac{\phi(s)}{s}+(m+m'').\frac{\phi(s')}{s'}\right)+m''x'\left(\frac{\phi(s')}{s'}-\frac{\phi(s)}{s}\right)=$$

Kx. As a similar equation obtains between y and y', we have $x:x'::y:y'$, $\therefore m\,m'$ exist in the same right line with the centre of gravity; therefore $m\,m'\,m''$ are in the same right line. Suppose that this right line is the axis of the abscissæ, and that the bodies are ranged in the order $m\ m'\ m''$, their common centre being between m and m', let $x' = -\mu x$, $x'' = -\mathrm{V}x$, then if $\phi\ (s) = s^n$, as $s = (1+\mu).x$, $s' = (1+\mathrm{V})x$, from this and the equations (a) we obtain $\mathrm{K} = x^{n-1}.\ \{m'(1+\mu)^n + m''(1+\mathrm{V})^n\}$, $\mu\{m'(1+\mu)^n + m''(1+\mathrm{V})^n\} = m(1+\mu)^n - m^n.\ (\mathrm{V}-\mu)^n$; let $\mathrm{V}-\mu = (1+\mu).z$, then $1+\mathrm{V} = (1+\mu).(1+z)$; consequently, $\mu\{(m'+m'').(1+z)^n\} = m - m''z^n$, and as the equation $mx + m'x' + m''x'' = 0$, gives $m - m'\mu - m''\mathrm{V} = 0$;

$\therefore \mu = \dfrac{m - m''z}{m' + m''(1+z)}$; therefore we have

$(m - m''z).\ \{m' + m''(1+z)^n = (m'+m'').(1+z)\}.(m' - m''z^n)$; when $n = -2$, this equation becomes

$$mz^2.\ \{(1+z)^3 - 1\} - m'\{(1+z).^2(1-z^3)\} - m''\{(1+z)^3 - z^3\} = 0,$$

which is of the fifth degree, therefore it has one real root, which is necessarily positive, for when $z=0$, the first member of this equation is negative, and when $z=\infty$, this first member is positive. If m be the sun, m' the earth, and m'' the moon, then, as z and m', m'' are very small quantities relatively to m, we have $3m\,z^3 = m'+m''$ very nearly, and $z = \sqrt[3]{\dfrac{m'+m''}{3m}}$, which, by substituting their values, gives $z = \frac{1}{100}$ q.p; $\therefore$ if, as is stated in the text, the earth and moon were placed in the same right line, at distances from the sun proportional to 1 and $1 + \frac{1}{100}$, and if velocities were impressed on these bodies in parallel directions, and proportional to their distances from the sun, the moon would be always in opposition, and these two luminaries would succeed each other alternately; and as the extent of the earth's shadow ranges from 213 to 220 semidiameters of the earth, and therefore is much less than the $\frac{1}{100}$th part of the earth's distance from the sun, the moon would be never eclipsed, consequently, during the night, its light would succeed that of the sun; it is assumed here that the sole use of the moon was to afford light in the absence of the sun, but though this may be one use, there are others equally important, such as to elevate the waters of the ocean and air, and thus produce a continual circulation of the sea and of the atmosphere, &c.

This opinion of the Arcadians, mentioned in page 80, that their ancestors inhabited the earth before the moon was a satellite, has been transmitted to us by Ovid and other authors. In his Fasti, speaking of Arcadia, Ovid has these remarkable words:

> " Orta prior Luna, de se si creditur ipsi
> A magno Tellus Arcade nomen habet.
> Ante jovem genitum terras habuisse feruntur
> Arcades et Luna gens prior illa fuit."

And in Apollonius Rhodius we have

> *αρκάδἒς οἱ και πρόσθε σεληναιης υδεονται.*

It was in consequence of these authorities, combined with the appearance which the moon presents through a telescope, and its almost total absence of atmosphere, that some philosophers fancied they perceived on the surface of the moon, vestiges of a body burned up by the sun, and this led them to think that the moon might one time or other have been a comet, which passing very near the earth after the perihelion, was forced by the attraction of the earth to become its satellite. However, it is easy to show, by means of what has been established in notes to Chapter I. of this Volume, that no comet moving in a parabolic or in a hyperbolic orbit can become a satellite of the earth. If a comet, moving in an elliptic orbit becomes a satellite, it must, at the moment it enters within the sphere of the earth's attraction, be at right angles to the extremity of the upper apsis of the ellipse which the comet describes about the sun; for if, instead of being perpendicular, it made an acute angle with it, it is evident, that however small the velocity of the comet at this point, the comet cannot remain always within the sphere of the earth's attraction; for when, by the nature of conic sections, the comet arrives at a distance from the earth equal to that which it had when it commenced to be subject to the action of the earth, the radius vector of the comet is situated relatively to the axis of the conic section, in a manner similar to the radius vector by which the comet entered within the sphere of attraction, and the tangential velocities will be equal; but in the first case the direction of the motion makes an acute, and in the second case an obtuse angle with the radius vector, and therefore will cause the comet to move out of the sphere of the earth's attraction; consequently the comet must, at the time of its entrance within the sphere of the earth's attraction, be at its highest apsid; hence it appears how extremely improbable it is, that a comet, moving in an elliptic orbit, can ever become a satellite of the earth; but with respect to our moon the improbability is still greater, as it is considerably within the sphere of the earth's attraction; besides, it appears to be firmly connected with the earth: its motions, rigorously computed and estimated, by going back to the remotest periods, do not present any circumstances from which we can infer that it could be in a condition to cease to revolve about the earth.

Knowing the quantity of matter and magnitude of the earth and moon, it is easy to estimate the point of equal attraction. If these two bodies were at rest, a body projected from the surface of the moon, with the velocity of 12,000 feet in a second, would be carried beyond the point of equal attraction, if the moon's mass was $\frac{1}{39.78}$, which was Newton's estimation; but this estimation is now admitted to be too great: see notes to Chapter X., where its true value $\frac{1}{75}$, is assigned, therefore a force a little more than half of the above power would be sufficient to produce that effect, *i. e.* a force capable of projecting a body with a velocity a little more than a mile and a half in a second; but cannon balls have been propelled with a velocity of 2500 per second, which is upwards of half a mile; and in the experiments of Perkins, the balls were driven by the force of steam with a still greater velocity; therefore a projectile force, causing a velocity three times greater than that with which a cannon ball is projected, would move a body beyond the point of equal attraction, and cause it to reach the earth; and there can be no doubt but a force equal to that is exerted by volcanoes on the earth, and also by the steam produced by subterraneous heat, for huge masses of rock, many times larger than cannon balls, are thrown much higher; and a like cause of motion exists in all probability in the moon, as well as in the earth, and that it is even in a superior degree, is probable from the circumstance that there is no sensible atmosphere to resist or retard the motion of bodies, as at the surface of the earth; and besides, the appearances observed in the moon indicate traces of more powerful and extensive volcanoes than on the surface of the earth. After the body passes the point of equal attraction, the *path* which it describes in approaching the earth must in a great measure depend, as is stated in the text, on the direction of the primitive impulsion; for as, besides this impulsion, it also participates in the absolute motion of the moon, it must, when it reaches the point of equal attraction, be actuated by the tangential velocity of the moon, combined with the force drawing it to the centre of the earth; these two would cause it to describe an ellipse about the earth, and the sun's action would disturb its motion in the same manner as the moon's motion is disturbed; when the body reaches our atmosphere it has not lost much of its heat, inasmuch as the space which it traversed being comparatively a vacuum, it enters the upper regions of the atmosphere with little diminution of its original temperature, from which circumstance, combined with its very great velocity, which is *then* more than ten times greater than that of a cannon ball, and passing through a part of the atmosphere consisting chiefly of inflammable gas, (see notes, page 373, Volume I.,) it is easy to conceive how the body will be suddenly ignited.

These stones consist always of the same ingredients, namely, silex, magnesia, sulphur, iron in a *metallic* state, nickel, and a small quantity of chromium. As these are invariably the constituents of these stones, it has been justly concluded that they have a *common origin*, besides, iron is never found in a metallic state in terrestrial bodies; even what is found in volcanic eruptions is always oxidized; nickel is likewise very seldom met with, and never on the surface of the earth; and chromium is rarer still. At the period when they burst forth, they are a considerable height about the earth's surface, as appears from estimating their parallaxes, by means of simultaneous observations, made at the instant of their explosion. Beside the threefold opinion of their origin, given in the text, namely, a lunar, a volcanic, and atmospheric, some philosophers have supposed that they were small planets, or fragments of planets, like those lately discoved, revolving in space, and which, meeting with the earth's atmosphere, are ignited by the friction which they experience in the earth's atmosphere.

NOTES TO CHAPTER VI.

As Jupiter and his satellites are considerably more distant than the earth from the sun, and as the mass of Jupiter is much greater than that of the earth, the part of the sun's force which disturbs the motions of the satellites is much less than the corresponding part of the sun's force in case of the earth; therefore the principal cause of the inequalities in the motions of the satellites arises from their mutual attraction.

(*a*) It is impossible to give a perfect explanation of the different inequalities of these satellites without discussing the theory of these bodies more in detail than the limits which these notes admit of. The reader will find them satisfactorily accounted for in the Sixth Chapter of the Second Book, and in the Eighth Book of the Celestial Mechanics.

$m\, m'\, m''\, m'''$ being the masses of the satellites, in the order of their distances from the sun, $n\, n'\, n''\, n'''$, &c. their respective mean motions, $r\, r'\, r''$, &c. their radii vectores, and $v\, v'\, v''$ their longitudes, then, as has been observed in notes, page 405 of the First Volume, since the mean motions of the three first satellites constitute very nearly a duplicate progression, $n-2n'$, $n'-2''n$ must be very small fractions of n, and their difference $n-2n'-(n'-2n'')$, or $n-3n'+2n''$, is incomparably less than either of them. It is easy to prove that the action of m' on m produces in the radius vector r, and the longitude v, a very sensible inequality, depending on the argument $2(n't-nt+\varepsilon'-\varepsilon)$; the terms relative to this inequality have for a divisor

$$4(n'-n)^2-n^2=(n-2n')\,.\,(3n-2n'),$$

which, in consequence of the smallness of the factor $n-2n'$, is very sensible. It appears also, from a consideration of the same expressions, that the action of m on m' produces in r' and v' an inequality depending on the argument

$$n't-nt+\varepsilon'-\varepsilon,$$

which, as its divisor is

$$(n'-n)^2-n^2, \text{ or } n(n-2n'),$$

is also extremely sensible. In like manner, the action of m'' on m', and *vice versa*, of m' on m'', produces respectively, in their respective longitudes and radii vectores, inequalities depending on the arguments $2(n''t-n't+\varepsilon''-\varepsilon')$, and $n''t-n't+\varepsilon''-\varepsilon'$; (see as above;) therefore the value of $\delta v'=m''\mathrm{F}''.\sin.2(n''t-n't+\varepsilon''-\varepsilon')+m\mathrm{H}.\sin.(n't-nt+\varepsilon'-\varepsilon)$.

(*b*) By hypothesis we have $2n''t+2\varepsilon''-2n't-2\varepsilon'=\pi+n't-nt+\varepsilon'-\varepsilon$, $\therefore$ $m''\mathrm{F}'.\sin.2(n''t-n't+\varepsilon''-\varepsilon')=-m''\mathrm{F}.''\sin.(n't-nt+\varepsilon'-\varepsilon)$, consequently the value of $\delta v'$ becomes $(m\mathrm{H}-m''\mathrm{F}'')\sin.(n't-nt+\varepsilon'-\varepsilon)$, in which we see how the two inequalities are made to coalesce into one.

The manner in which it may be shown that the mutual action of the satellites rendered this proportion, which was originally only approximate, accurately true, is as follows: assuming $\mathrm{V}=nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$, it is easy to prove that $\frac{d^2\mathrm{V}}{dt^2}=\mathrm{C}n.^2\sin.\mathrm{V}$; C being a constant coefficient depending on the masses of $m\,m'\,m''$, n being likewise supposed constant, and by integrating we have

$$dt = \pm \frac{dV}{\sqrt{c - 2Cn^2 \cos. V}}; \text{ now, as } \left(\frac{dV}{dt}\right)^2 =$$

$$(n - 3n' + 2n'')^2,\ c - 2\,Cn.^2 \cos. V = \left(\frac{dV}{dt}\right)^2 =$$

$(n-3n'+2n'')^2$; and if this last quantity be greater than $\mp 2Cn^2.(1-\cos. V)$, c must be positive and $>$ than $2Cn^2$, in which case, as the radical can never vanish, V, or its equivalent, should increase continually, and become $= 2\pi, 4\pi, 6\pi$, &c.; but this is not the case, for let $\bar{\omega} = \pi - V$, and we have

$$dt = \frac{d\bar{\omega}}{\sqrt{c + 2Cn^2. \cos. \bar{\omega}}}$$; and when c is not less than $2Cn^2$,

$\sqrt{c + 2Cn^2 \cos. \bar{\omega}}$ is $>$ than $\sqrt{2Cn^2}$ from $\bar{\omega} = 0$ to $\bar{\omega} = \frac{\pi}{2}$;

therefore t the time in which the angle $\bar{\omega}$ passes from 0 to a right angle, is $\angle$ than $\frac{\pi}{2n\sqrt{2C}}$; and this last angle, by substituting for C and n comes out $\angle$ than two years; but as $\bar{\omega}$ has always remained insensible, this last case, namely that of c, not less than $2Cn^2$, is not the case of Jupiter and his satellites. If c is $\angle$ than $\mp 2Cn^2$, V will oscillate about a mean state either of two right angles, if C be positive, in which case $\sqrt{c - 2Cn^2 \cos. V}$ becomes imaginary, when $V = 0$, or 2π, 4π, &c., $\therefore\ \bar{\omega}$ can never become equal to cypher, its value is therefore periodic, oscillating about a mean state $= \pi$. If, in the same circumstances, C be negative, then the radical is imaginary, when $V = \pi, 3\pi, 5\pi$, therefore $\bar{\omega}$ can never reach π, and its mean value is cypher. Now all observations of Jupiter and its satellites give a positive value to C; therefore in the case of Jupiter, V oscillates about a mean state $= \pi$. From the equation

$$n't-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''=V=\pi-\tilde{\omega},$$

we deduce, by putting the parts which are not periodic separately = to cypher, $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''=\pi$, $\therefore n-3n'+2n''=0$; therefore the mean motion of the first satellite + twice that of the third, minus three times that of the third = 0, and $\varepsilon-3\varepsilon'+2\varepsilon''=\pi$, *i.e.* the mean longitude of the first + twice that of the third, minus three times that of the second = π; and since according to observation the angle $\tilde{\omega}$ in the equation

$dt=\dfrac{d\tilde{\omega}}{\sqrt{c+2Cn^2\cos.\tilde{\omega}}}$ must be always very small, we

can assume $\cos.\tilde{\omega}=1-\dfrac{\tilde{\omega}^2}{2}$, and the preceding equation will give by integrating $\tilde{\omega}=\lambda.\ \sin.\ (nt\sqrt{C}+\gamma)$, where λ and γ are two constant arbitrary quantities.

(*c*) As three differential equations of the second order are necessary to determine the motion of each body of the system, and as the integration of each equation involves two constant arbitrary quantities, there are in the determination of the motion of each body six constant arbitraries; therefore, in general, as is stated in the text, the number of arbitrary quantities is sextuple of the number of bodies; consequently, in the case of the four satellites of Jupiter, there are twenty-four arbitrary quantities, which are reduced to twenty-two, in consequence of the two relations between the epochs of mean longitudes and also the mean motions of the three first satellites, which is established by the two preceding theorems; but these two are supplied by the new arbitraries, which the expression of $\tilde{\omega}$ contains.

If the satellites were affected either by a secular inequality analogous to that of the moon, as stated in the text, or by one arising from the resistance of a medium, it would be necessary to add to $\dfrac{d^2V}{dt^2}$ a quantity of the form $\dfrac{d^2\psi}{dt^2}$, which can only become sensible by integrations; therefore if $V=\pi-\tilde{\omega}$, when $\tilde{\omega}$ is very small, the differential equation in V will become of the form

$$0=\frac{d^2\tilde{\omega}}{dt^2}+Cn^2\tilde{\omega}+\frac{d^2\psi}{dt^2},$$

as the period of the angle $nt\sqrt{C}$ embraces but five years, while the quantities contained in $\frac{d^2\psi}{dt^2}$ are either constant, or extend to several centuries, we shall obtain, very nearly, by integrating,

$$\tilde{\omega}=\lambda.\ \sin.(nt\sqrt{C}+\gamma)-\frac{d^2\psi}{Cn^2.\,dt^2},$$

therefore the value of $\tilde{\omega}$ will be always very small, and the secular equations of the mean motions of the three first satellites will be so coordinated by their mutual action, that the secular equation of the first plus twice that of the third, is equal to three times that of the second.

(*d*) It has been already stated, in notes, page 405, that from the circumstance of the length of the year not having been altered 2″,8 by the action of the comet of 1770, its mass is not the $\frac{1}{5000}$th part of that of the earth; and if, as is stated in the text, in the lapse of ages these bodies have more than once impinged on the satellites, the effect would be particularly perceptible in a real libration of these satellites, and also of the moon; for, as will be stated hereafter, in notes to Chapter XIV., it is by no means probable that the equality which obtains between the motion of rotation and revolution subsisted at the very origin of the planetary system.

(*e*) The compression of Jupiter has also a considerable influence on the motion of the apsides of the satellites, as well as on the motion of the nodes; and it is from the accuracy with which these quantities have been determined, that we are able to deduce such an exact expression for the compression of Jupiter.

The masses of the satellites, and the value of the compression, are determined relatively to the mass and equatorial diameter of Jupiter. In order to determine these five unknown quantities, it is necessary to have five data furnished by observation. Those are selected in which the quantity required to be known has most influence. As the ratio of Jupiter's mass to the earth's is given in page 42, we can have the mass of the satellites relatively to that of the earth, and by substituting their numerical values, it is found that the mass of the third satellite, which is nearly double of the fourth, is also double of the moon's mass.

(*f*) Three differential equations of the second order are necessary to determine the circumstances of the motions of each satellite; and for the integrations of each of them, two constant arbitrary quantities are introduced; this gives twenty-four constant arbitrary quantities, in addition to which, the masses of the four satellites, the compression of Jupiter, the inclination of his equator, and the position of his nodes, furnish seven more, which make thirty-one in all. See note (*c*.)

(*g*) See notes to Chapter II., Book 2., Volume I.

(*h*) Assuming that the velocity of the light which comes from the stars was such as is given from a comparison of the eclipses of Jupiter's satellites, the quantity of the aberration comes out exactly equal to what is deduced from actual observation, which shows that the velocity of the light which comes from the stars is equal to that which is *reflected* from Jupiter's satellites. The uniformity of the velocity through the diameter of the earth's orbit might be evinced by taking into account the effect of the ellipticity of the orbit of the earth. It also follows, that the velocity of a ray of light which emanates from stars of *different* magnitudes, and at *different* distances, is uniform and the same through the diameter of the earth's orbit.

(*i*) It is easy to show that the velocities of pulses which are propagated in any elastic medium, are in the direct subduplicate ratio of F the elasticity, and the inverse subduplicate ratio of d the density of the medium; consequently when F varies as d, the velocity must be uniform. (See Princip. Math. Prop. 48., Book 2.) It is to be observed here, that on the hypothesis of the materiality of light, it is supposed that light is emitted from stars of different magnitudes with the same velocity.

(*l*) See note (*b*) of this Chapter.

NOTES TO CHAPTER VII.

(*a*) The expression for the part of Saturn's compression which disturbs the satellite at the distance a from Saturn,

$$= \frac{\rho - \frac{\phi}{2}}{a^2},$$ where ϕ is the centrifugal force at the equator of Saturn, and $\therefore = \frac{T^2}{t^2\,a^2}$, t being the time of Saturn's rotation, and ρ the compression; and as the compression of the earth is to $\frac{1}{289} :: \rho : \phi$, we have

$$\rho - \frac{\phi}{2} = \frac{m.\,T^2}{t^2.a^3}, \therefore \frac{\rho - \frac{\phi}{2}}{a^2} = \frac{m.T^2}{t^2} \cdot \frac{1}{a^5}.$$ as is stated in the text; therefore, the more distant the satellite from Saturn the less will be this quantity; and for the last satellite, it is so small, that the disturbing action of the sun predominates over it, causing the satellite to deviate from the plane of Saturn's equator.

Laplace also infers from this deviation of the last satellite, that its mass must be inconsiderable, for otherwise its action on the last but one would cause it to deviate from the plane of Saturn's equator, in which, however, it accurately moves.

The preceding formula is not applicable to the case of Uranus's satellites; for we do not know the precise amount of t, the rotation of the satellite. If the orbits are perpendicular to the plane of Uranus's orbit, the theoretic discussion of the disturbing action of the sun on these satellites would require formulæ different from those used to investigate and express the disturbances produced in the planets, moon, and other bodies of our system.

NOTES TO CHAPTER VIII.

(*a*) See note (*d*) page 453 of Volume I., where it is proved, that the equation of the figure which the fluid affects in the hypothesis of the text, is that of a sphere; and as the density must be a function of the pressure, the fluid will be arranged in strata of equal pressure, and the same density, having the strata which are nearer to the centre denser than those which are more remote.

If the earth was fluid, and at rest, it would necessarily assume a spherical shape; for the mutual attraction of the particles would so collect them together, that if any particles were more protuberant than the others, the direction of gravity would not be perpendicular to its surface, and as it would not remain in such a form, the projecting parts would flow down; in consequence of the centrifugal force induced by the motion of rotation, all the particles have a tendency to recede from the axis of rotation, which, as it is greater near the equator, will enlarge the earth more than near the poles, and thus make the earth assume a spheroidical shape.

(*b*) A particle placed without a sphere of which each of the particles attracts in the inverse ratio of the square of the distance, is urged to the *centre* of the sphere with a force which varies in the inverse ratio of the square of the distance from the centre; (see Princip. Math. Book I., Section 12., Prop. 74.;) therefore, if f denotes the distance of the particle from the centre of the sphere, the attraction of the sphere upon the particle will be expressed by $\frac{A}{f^2}$; (A being a constant quantity, which can be determined by the actual attractive force, at any determinate distance from the centre;) if r denote the radius of the sphere, and M its mass, since no part of the matter of the sphere is nearer the attracted particle than $f-r$, and none more remote than $f+r$, the attraction of the sphere on the particle will be > than

$\frac{M}{(f+r)^2}$, and < than $\frac{M}{(f-r)^2}$, $\therefore \frac{A}{f^2}$ being always contained between those limits, A must be = to M, for if A > M, such values of f could be found as would make $\frac{A}{f^2}$ = or > than $\frac{M}{(f-r)^2}$, and if A is < than M, such values of f could be found as would render $\frac{A}{f^2}$ = or < than $\frac{M}{(f+r)^2}$, which can never be the case; $\therefore$ A = M, and the attraction of the sphere $= \frac{M}{f^2}$, or the same as if all the matter were collected in the centre. Now, if ρ denotes the density of the matter contained in the sphere, we have $M = \frac{4\pi r^3 \rho}{3}$, $\therefore$ the attraction at the distance $f = \frac{4\pi r^3 \rho}{3.f^2}$, and at the surface, where $r=f$ the attraction $= \frac{4\pi r.\rho}{3}$.

(*c*) Suppose a system of bodies $m\ m'\ m''$, &c., whose mutual distances from each other are inconsiderable relatively to their respective distances from the attracted point, let the origin of the coordinates be in the attracted point, $x\,y\,z$, $x'\,y'\,z'$, x'', &c. being the coordinates of the $m\,m'\,m''$, &c. X Y Z, those of the centre of gravity, we have

$$x = X + x_{,},\ y = Y + y_{,},\ z = Z + z_{,},\ x' = X + x'_{,},\ \&c.$$

$x_{,}\,y_{,}\,z_{,}$ being the coordinates of m with respect to the centre of gravity; if $r\ r'$, &c. denote the distances of the attracted point from $m\ m'$, &c. and R the distance of the centre of gravity from the same, the action of m on the attracted point resolved parallel to $x = \frac{m\,x}{r^3}$, ∴ the sum of the attractions of $m\,m'\,m''$, &c. resolved parallel to $x = \Sigma.\frac{m\,x}{r^3}$,

but $\frac{x}{r^3} = \frac{X + x_{,}}{\left((X+x_{,})^2 + (Y+y_{,})^2 + (Z+z_{,})^2\right)^{\frac{3}{2}}} =$

neglecting very small quantities of the second order, as is stated in the text, namely, the products and squares of $x_{,}\,y_{,}\,z_{,}\,x'$, &c.

$$X.(X^2 + 2Xx_{,} + Y^2 + 2Yy_{,} + Z^2 + 2Zz_{,})^{-\frac{3}{2}}$$

$$+\ x_{,}.\,(X^2 + Y^2 + Z^2)^{-\frac{3}{2}} = X.(X^2 + Y^2 + Z^2)^{-\frac{3}{2}} - \frac{3}{2}.$$

$$X(2Xx_{,} + 2Yy_{,} + 2Zz_{,})R^{-5} + x_{,}.\,(X^2 + Y^2 + Z^2)^{-\frac{3}{2}} =$$

$\frac{X + x_{,}}{R^3} - 3.X\frac{(Xx_{,} + Yy_{,} + Zz_{,})}{R^5}$, and as $\Sigma m x_{,} = 0$, $\Sigma m y_{,} = 0$; $\Sigma m z_{,} = 0$; $\Sigma\frac{m\,x}{r^3} = \Big(\frac{X.\Sigma m}{R^3} + \frac{\Sigma m x_{,}}{R^3} -$

$3\,X.\frac{(X.\Sigma m x_{,} + Y.\Sigma m y_{,} + Z.\Sigma m z_{,})}{R^5}\Big), = \frac{X.\,\Sigma m}{R^3}$, which is the same as if the bodies were united in the centre of gravity. Now, if m, m', m'', &c., are so near as to be parts of the same attracting body, this will be more accurately true, as in the case of spheroids differing little from a sphere; then, as in a sphere, an exterior point is attracted, as if the whole mass was united in the centre; in a spheroid differing little from a sphere, the error or the difference from what would be the case if the body was a sphere, is of the same order as the eccentricity for all points contiguous to the surface; for if a sphere be supposed to be described concentrical with the given spheroid whose radius is equal to the distance of any assumed point on its surface from centre of spheroid, then this sphere attracts as if the entire matter was collected in the centre; therefore the difference between its attractions and that of the spheroid must be of the same order as the eccentricity; and as for very distant particles, in estimating the effect of the attraction of a body of any figure whatever on them, in showing that its action is nearly the same as if the entire mass was collected in the centre of gravity; by what has been just established, the ratio of the quantities which are neglected to those which are retained is that of the square of the radius to the square of the distance of the point attracted; the error for a spheroid in the case of very distant points, must be the product of the excentricity into the square of this ratio. It would not be difficult to show, that if the force varied directly as the distance, then a point outside a body of any figure *whatever*, is attracted as if the entire mass was condensed into the centre of gravity.

(*d*) In order to demonstrate this property, it may be observed, that if a homogeneous sphere attracts a point placed without it, as if all the matter was united in its centre, the same result will have place for a spherical stratum of a uniform thickness; for if we take away from the sphere a spherical stratum of a uniform thickness, we shall obtain a new sphere of a smaller radius, which will possess the property equally with the first sphere of attracting, as if the entire mass was united in its centre; but it is evident, that if this property belongs to the two spheres, it belongs also to their difference; therefore the problem reduces itself to determine the laws of attraction, according to which a spherical stratum of a uniform and indefinitely small thickness attracts an exterior point, as if the entire mass was collected in its centre. Let r represent the distance of the attracted point from the centre of the stratum, u the radius of the stratum, θ the angle contained between r and u, $\bar{\omega}$ the angle which the plane passing through r and u makes with a fixed plane, then it is easy to prove that $u^2du.d\bar{\omega}.d\theta.\ \sin.\ \theta$ is $=$ to the element of the spherical stratum; and if f be the distance of this element from the attracted point, $f^2 = r^2 - 2ru.\ \cos.\ \theta + u^2$, $\therefore \frac{df}{dr} = \frac{r - u.\ \cos.\ \theta}{f}$; $d\theta.\ \sin.\ \theta = \frac{fdf}{r.u}$; if $\phi\,(f)$ expresses the law of attraction, then the action of the element resolved parallel to r, and directed towards the centre $=$ $u^2du.d\bar{\omega}.d\theta.\ \sin.\ \theta.\ \frac{(r - u.\ \cos.\ \theta)}{f}\ .\ \phi\,(f)$, which $\left(\text{since } \frac{r - u.\ \cos.\ \theta}{f} = \left(\frac{df}{dr}\right)\right)$ assumes the form $u^2.du.d\bar{\omega}.d\theta.\ \sin.\ \theta.\left(\frac{df}{dr}\right).\ \phi\,(f)$, *i. e.* if we denote $\int d\,(f).\ \phi.\ (f)$ by $\phi_{,}.(f)$, we obtain the entire action of the spherical stratum, by means of the integral $u^2du.\int d\bar{\omega}.d\theta.\ \sin.\ \theta.\ \phi_{,}.(f)$, differenced with respect to r, and divided by $dr.$; relatively to $\bar{\omega}$, the preceding integral, should be taken from $\bar{\omega}=0$, to $\bar{\omega}=2\pi$, the circumference, *i. e.* the preceding integral, becomes $2\pi.u^2du\int d\theta.\ \sin.\ \theta.\ \phi_{,}f=$ (by substituting for $d\theta.\sin.\ \theta,$) $2\pi.\ \frac{udu}{r}\ .\int fdf.\ \phi_{,}(f)$. Now, as relatively to θ, the integral should be taken from $\theta=0$ to $\theta=\pi$, which corresponds to $f=r-u$, $f=r+u$, when $\frac{fdf}{du}$ is substituted for $d\theta.\ \sin.\ \theta$, the integral relatively to f must be taken from $f=r-u$ to $f=r+u$; $\therefore$ if we make $\int f df.\phi_{\prime}(f)=\psi(f)$, we shall have

$$\frac{2\pi udu}{r}\int f df.\phi_{\prime}(f)=\frac{2\pi udu}{r}\left(\psi(r+u)-\psi(r-u)\right); \text{ now}$$

if $\phi(f)=\frac{1}{f^2}, \int df.\phi(f)=\phi_{\prime}(f)=-\frac{1}{f}$, and $\int df.f.\phi_{\prime}(f)$ $=\psi(f)=-f=$ at the limits $-r-u$, $r-u$, $\therefore \psi(r+u)-\psi(r-u)=-2u$, $\therefore$ the differential coefficient of the second member of this equation with respect to r, (which, as has been observed, gives the attraction of the spherical stratum,) $=\frac{-2\pi.udu}{r^2}.(-2u)=\frac{4\pi u^2 du}{r^2}$, *i. e.* as $\pi u^2=$ the area of a circle whose radius is u, $4\pi u^2=$ the *surface* of the spherical stratum, and $4\pi u^2 du=$ the mass of the stratum whose thickness $=du$; $\therefore$ when $\phi(f)=\frac{1}{f^2}$, an exterior point is attracted, as if the whole mass was united in its centre; but to determine $\phi(f)$ *generally*, when the attraction of the stratum is the same as if the mass was collected in its centre; in that case the attraction would be $=4\pi.u^2du.\phi(r)$ which, by hypothesis,

$$=2\pi u du.\left(\frac{d.\left(\frac{1}{r}[\psi(r+u)-\psi(r-u)]\right)}{dr}\right),$$

integrating with respect to r, and dividing by $2\pi udu$, we shall have

$$\psi(r+u)-\psi(r-u)=2ru.\int dr.\phi(r)+r\text{U},$$

U being a function of u, and of constant quantities, let $\psi(r+u)-\psi(r-u)=\text{R}$, and then differentiating twice with respect to r, we obtain

$$\frac{d^2\text{R}}{dr^2}=4u.\phi(r)+2ru.\frac{d.\phi(r)}{dr},$$

and differentiating twice with respect to u, we obtain

$\frac{d^2 R}{du^2} = r . \frac{d^2 U}{du^2}$, but from the nature of the function R we have

$$\frac{d^2 R}{dr^2} = \frac{d^2 R}{du^2} ; \therefore 2u . \left(2\phi r + \frac{r . d . \phi(r)}{dr}\right) = r . \left(\frac{d^2 U}{du^2}\right),$$

$$i.e. \ \frac{2\phi(r)}{r} + \frac{d\phi(r)}{dr} = \frac{1}{2u} . \frac{d^2 U}{du^2},$$

(see Celestial Mechanics, Book 2, page 68,) therefore as the first member is independent of u, and the second member is independent of r, they must each be equal to a constant quantity, which denoting by 3A, we have $\frac{2\phi(r)}{r} + \frac{d\phi(r)}{dr} = 3A$, which, by multiplying both sides by $r^2 dr$, gives

$$2r . dr . \phi(r) + r^2 . d\phi(r) = 3Ar^2 dr, \therefore r^2 \phi(r) = Ar^3 + B, \therefore$$

$\phi(r) = Ar + \frac{B}{r^2}$; $\therefore$ as is stated in the text, all the laws in which the sphere acts on an exterior point, as if the whole mass was condensed in the centre, are comprised in the general formula $Ar + \frac{B}{r^2}$; in fact, this value of $\phi(r)$ will satisfy the equation given in the preceding page. If the point be situated within a spherical stratum of uniform thickness, then since u is $> r$, the expression for the attraction of the stratum whose thickness $= du$ is $2\pi u du . \left(\frac{d . \frac{1}{r}\left(\psi(u+r) - \psi(u-r)\right)}{dr}\right)$. In order that this function should vanish, we should have $\psi(u+r) - \psi(u-r) = rU$, and it is evidently the case when $\phi(f) = \frac{B}{f^2}$, but to show that this vanishes *only* when $\phi(f) = \frac{1}{f^2}$, if

$$\psi'(f) = \frac{d . \psi(f)}{df}, \ \psi''(f) = \frac{d . \psi'(f)}{df}, \text{ \&c. \&c.}$$

$$\frac{d\psi(u+r)-d\psi(u-r)}{dr}=\psi'(u+r)-\psi'(u-r)=\text{U},$$

$$\frac{d^2\psi(u+r)-d^2\psi(u-r)}{dr^2}=\psi''(u+r)-\psi''(u-r)=\frac{d\text{U}}{dr}=0,$$

$\therefore$ as $\psi''(u+r)$ is always $=\psi''(u-r)$, each of them must be equal to a constant quantity, $\therefore \psi''(f)=$ a constant quantity, and $\psi'''(f)=0$; but as $\psi'(f)=f\phi_{,}(f)$, $\psi'''(f)=2\phi(f)+f.\phi'(f)$, *i. e.* $0=2\phi(f)+f.\phi'(f)$, or $\psi(f)=\int f.df\phi_{,}(f)$, $\psi'(f)=f.\phi_{,}(f)$, and $\psi''(f)=\phi_{,}(f)+f.\phi(f)$, and $\psi'''(f)=\phi(f)+\phi(f)+f\phi'(f)=0$; multiplying by fdf we obtain $2f\phi(f).df+f^2\phi'f.df=0$, $\therefore f^2\phi(f)=\text{B}$, and $\phi(f)=\frac{\text{B}}{f^2}$; $\therefore$ a point situated within the interior of a spherical stratum is equally attracted in every direction when the force of gravity is inversely as the square of the distance; the same is true, for a spheroid in the circumstances specified in the text, for any common chord to the two spheroids drawn through the interior point, has the portions of it which are intercepted between the two spheroidical surfaces equal; therefore, if the point in the interior be conceived to be the vertix of two *similar* pyramids, whose common axis is the chord, its gravitation to the pyramid whose axis is the distance of the point from the exterior surface, is equal and *opposite* to the gravitation to the matter contained in the *frustrum*, of the similar cone, whose axis being the part of the chord intercepted between the two spheroids at the other side, is equal to the axis of the first cone, and as this is true *whatever be* the direction of the chord drawn through the given point, this point is in equilibrio in every direction. From the expression $\frac{4\pi r\rho}{3}$ given in page 441, it follows that when the density ρ is given, the force of gravity is proportional to r; but if the strata nearer the centre are denser, then this force varies evidently in a less ratio.

(*e*) As the centrifugal force at the equator is greatest, the weight of a column of water extending from the surface to the centre, must be less than the weight of an equal column at the poles, reaching from surface to centre; therefore to compensate for the loss of weight produced by the centrifugal force, the equatorial columns should be longer than the polar.

(*f*) What is here stated may be analytically expressed in the following manner: calling m the equatorial, and n the polar diameter, r the radius belonging to any parallel λ, then if A and F denote the gravity and centrifugal force at the equator, the gravity at $\lambda = \frac{A.m^2}{r^2}$, and the efficient part of the centrifugal force $= \frac{Fr. \cos.^2\lambda}{m}$, (see notes Volume I., page 427.,) therefore the gravity diminished by the centrifugal force $= \frac{Am^2}{r^2} - \frac{F.r.\cos.^2\lambda}{m}$, and conceiving a canal, of an indefinitely small thickness, to extend from the surface at λ to the centre, the weight of an element at the surface $= Am^2. \frac{dr}{r^2} - \frac{Frdr.\cos.^2\lambda}{m}$, therefore, by integrating, the weight of the entire canal $= -\frac{Am^2.}{r} - \frac{Fr^2.\cos.^2\lambda}{2m}$, at the equator and at the poles these quantities become respectively

$-Am - \frac{Fm}{2}, -\frac{A.m^2}{n}$, and as in the case of equilibrium, these quantities must be equal, we have

$\frac{n}{m} = \frac{2A}{2A+F} = \left(\text{as } \frac{F}{A} = \frac{1}{289.}\right.$ (see notes, Vol. I., p. 427.)$\left.\right)$ $\frac{578}{579}$; and in general as $\frac{Am^2}{n}$ = (if $m=n+e$) $An + A2e$

and as $\left(A + \frac{F}{2}\right).m = \left(A + \frac{F}{2}\right)(n + e)$, we have $\Big($since $\left(A + \frac{F}{2}\right).m = A.(n+2e),\Big)$ $A : \frac{F}{2} :: m : e$; now if $n + dn = r$, G the gravity at the pole is to G′ the gravity at any parallel $:: (n+dn)^2 : n^2$, *i. e.* $n+2dn : n$, hence it appears that the diminution of gravity is nearly twice the increase of the terrestrial radius; and as the centrifugal force *at the surface* is equal to the same quantity, we can obtain *the whole* diminution which arises from the two causes of decrease.

(*g*) As the spheroids are by hypothesis similar, if they be supposed to be divided into an indefinite number of similar and similarly situated particles, the attractions are as the quantities of matter, and inversely as the squares of the distances, *i. e.* directly as their similar dimensions, which vary by hypothesis as the distances from the centre. In like manner, the centrifugal forces are as the radii of the respective circles described by the two particles, which are by hypothesis as their distances from the centre. Hence as the two forces which affect the particle, namely, the centrifugal and the attractive forces are respectively in the same ratio, the directions in which the two particles will have a tendency to move will be parallel.

(*h*) It is to be remarked here, that Newton did not prove that if the earth revolved on an axis, it would necessarily have the figure of an ellipsoid of revolution. He assumed it was an elliptic spheroid, differing little from a sphere; and he then estimated, in a manner similar to that indicated in the text, the weight of a column extending from the pole to the centre, and the weight of another column extending from the equator to the centre, and as the equatorial column must compensate for its loss of weight, by its greater length, by making the difference of the weights of the two columns equal to the sum of the centrifugal forces of the parts of the equatorial column, he has at once the ratio of the axes. (As for every particle in a column which reaches from the surface at the equator to the centre, the diminution is proportional to the distance from centre; the whole diminution must be the same as if each particle of the column lost half as much as the outermost particle loses.)

Since the weights of the corresponding parts are to each other as the magnitudes of the parts into the force of gravity at the points where they terminate, they are (as the parts are proportional to the whole lengths) as the whole lengths into the force of gravity.

M'Laurin's proof consists in the demonstration of the three following particulars: 1*st*, That the direction of gravity affected by the centrifugal force of rotation is every where normal to the surface of the spheroid, for otherwise the fluid would flow off towards that quarter to which the gravity inclines; 2*dly*, That all canals from the centre to the surface, must balance at the centre, otherwise the preponderating column would subside, and pressing up the other, would produce a change in the surface; 3*dly*, Any particle of the whole mass must be in equilibrio, being equally pressed in every direction; and he shows that these conditions will be *secured in an homogeneous spheroid revolving on its axis, if the gravity at the pole be to the equatorial gravity diminished by the centrifugal force arising from the rotation, as the radius of the equator to the semiaxis*; which was the conclusion Newton arrived at. It would be impossible, in these notes, to demonstrate these points in all their details; we shall only advert to some inferences which M'Laurin makes from establishing the first condition; namely, that the *sensible* gravity of any particle at the surface, is to the polar gravity, as the part of the normal *terminated by the axis* to the radius of meridional curvature at pole; and it is to the equatorial gravity as part of the normal, *terminating in the equator* to the radius of meridional curvature at the equator; from which it follows, that the sensible gravity is every where inversely as perpendicular from centre on tangent, and the gravity estimated in the direction of a radius to the centre is inversely as the distance from the centre. Hence it follows, from what has been established in Volume I., page 348, with respect to the decrements of the radii vectores, that the sensible increment of gravity, which, as we have seen, varies as the decrement of distance, is proportional to the square of the sine of the latitude. It is to be observed here, that M'Laurin, no more than Newton, does not prove that a fluid sphere, revolving on an axis, *must* assume the form of an elliptical spheroid, but only that it is a possible form. In fact, all that M'Laurin demonstrates is, that whatever be the proportion between the axes of an oblate spheroid, there is a certain velocity of rotation which will induce such a relation between the diminished equatorial and polar gravity, as is required in order to satisfy the three conditions of equilibrium above adverted to. Clairault, indeed, requires other conditions to be satisfied, such as: 1*st*, That a canal of any *form whatever*, must be every where in equilibrio; 2*ndly*, That such a canal, reaching from any one part to the other, shall exert no force at its extremities; 3*dly*, That a canal of any form, returning into itself, shall be in equilibrio through its whole extent. But it is not difficult to show, as Professor Robinson remarks, that these conditions are contained in the three previous ones.

What has been just established, only proves, as has been remarked, the competency of an elliptical spheroid for the rotation of the earth; another point remains to be determined, namely, given the velocity of rotation to determine the corresponding proportion which exists between the diameters. In order to determine this, M'Laurin investigated the gravitation of a particle at the pole of a spheroid, to the matter which is redundant over the inscribed sphere, and of a point in the equator to the excess of matter by which the circumscribed sphere exceeds the spheroid. The former comes out ∷ 1 to $\frac{8}{15} \cdot e$, e being $=$ the difference between the equatorial and polar diameters, and the latter is half this quantity, and very nearly $= \frac{4}{15} \cdot e$; now, the gravity at the pole to the inscribed sphere is ∷ 1 to $\frac{2}{3} \pi . (r-x)$, add to this $\frac{8}{15} \cdot \pi e$, the gravitation to the redundant matter, and the sum is $=$ $\frac{2}{3} \pi r - \frac{2}{15} \pi . e$, and the gravitation of a particle at the equator to a sphere, whose radius $= r = \frac{2}{3} \pi r$, from this substract $\frac{4}{15} \cdot \pi e$, the deficiency of gravitation, and the *undiminished* equatorial gravity comes out $= \frac{2}{3} \pi r - \frac{4}{15} . \pi e$, therefore, dividing by $\frac{2}{3} \pi r$, the ratio of E, the equatorial to P, the polar gravity comes out that of $r - \frac{1}{5} . e$ to $r - \frac{2e}{5}$, *i. e.* as e is very small relatively to r, as $r : r - \frac{e}{5}$, or *q.p*, as $r + \frac{e}{5} : r \therefore \mathrm{P} = \mathrm{E} + \frac{\mathrm{E}e}{5r}, \therefore \mathrm{E} - c : \mathrm{E} + \frac{\mathrm{E}e}{5r} :: r : r+e$, *i. e.* $\mathrm{E} : \mathrm{E} + \frac{\mathrm{E}e}{5r} + c :: r : r+e, \therefore \mathrm{E} : \frac{\mathrm{E}e}{5r} + c : r : e, \therefore \mathrm{E}e = \frac{\mathrm{E}e}{5} + rc$, and $\frac{4\,\mathrm{E}e}{5} = rc$; $\therefore e = \frac{5rc}{4\,\mathrm{E}}$ and $\frac{e}{r}$, or the ellipticity $= \frac{5c}{4\mathrm{E}}$, as is stated in the text; hence, when c and E are given, we can determine $\frac{e}{r}$; $\therefore$ as $c = \frac{r}{\mathrm{T}^2}$ and $\mathrm{E} = r . \rho$, $\frac{e}{r} = \frac{5}{4} \cdot \frac{1}{\mathrm{T}^2 \rho}$. (See Princip. Math., Book 3, Prop. 19.)

(i) In No. 18, Book 3, Laplace investigates the figure which satisfies the equilibrium of an homogeneous fluid mass endowed with a motion of rotation; and assuming that the figure is an ellipsoid of revolution, if the forces which result from this hypothesis, when substituted in the equation $0 = \mathrm{P}da + \mathrm{Q}db + \mathrm{R}dc$, which is the equation of equilibrium at the free surface of a fluid, give the differential equation of the surface of an ellipsoid, the elliptic figure satisfies the equilibrium of a fluid mass endowed with a rotatory motion, then by substituting for P Q R, their values in the case of an ellipsoid of revolution affected with a rotatory motion, he obtains the following equation, $\frac{(9+2q\lambda^2).\lambda}{3(3+\lambda^2)} - \text{arc tan.}\ \lambda = 0$, where $q = \frac{g}{\frac{4}{3}\pi\rho}$, and $\lambda^2 = \frac{1-m}{m}$, m being the coefficient of y in the equation of an ellipsoid $x^2 + my^2 + nz^2 = k^2$.

If this equation $\frac{9\lambda + 2\lambda^3}{9+3\lambda^2} - \text{arc tan.}\ \lambda = 0, = \phi$, is susceptible of several real roots, then several figures of equilibrium may correspond to the same motion of rotation, it is evident that if $\lambda = 0$ it is satisfied, but this is not the case of nature; if λ is very small, then arc tan. $\lambda = a$, becomes $= \lambda$; therefore in this case, ϕ is positive; if a is $= \frac{\pi}{4}$, $\lambda = 1$, which renders ϕ negative; now suppose a curve of which λ is the abscissa, and ϕ the ordinate, this curve intersects the axis when $\lambda = 0$, the ordinates will then be positive, and increasing until they attain their maximum, after which they will diminish; and when the abscissa has that value of λ which renders $\phi = 0$, *i. e.* which corresponds to the state of equilibrium of the fluid, the curve will cut the axis a second time; the ordinates will afterwards become negative; and since they are positive when $\lambda = \infty$, it is necessary that the curve should meet the axis a third time, which intersection determines a second value of λ, which renders $\phi=0$, or satisfies the condition of equilibrium, $\therefore$ for the same value of q, or for a given rotatory motion, there are several figures with which the equilibrium is possible; in order to determine the number of figures, we should determine how many maxima exist between the roots, and by taking the derivitive function of ϕ, we find only two maximum ordinates on the positive side, and the same number on the side of the negative abscissæ; therefore, on this side, the curve intersects its axis in three points only, of which the origin is one, consequently there are only *two* figures which satisfy the equilibrium; for as even powers of λ only occur in the determination of these figures, those furnished by the positive and negative abscissæ are identically the same. If q is very small, as in the case of the earth, the equation $\phi=0$, may be satisfied either by making λ^2 very small, or very great; in the first case, as has been already remarked, we have $\lambda^2 = q(2,5 + 5,35'7.q + 23q^2 + 123,q^3 +$, &c.) which, by substituting the value of q, gives $1 + \lambda^2 =$ 1,008746123; therefore we can obtain λ, and consequently the ratio of the axes which comes out $\frac{230}{231}\cdot q.p$; in the second case, arc tan. λ is nearly $= \frac{\pi}{2}$; $\therefore \lambda = \frac{\pi}{2} - a$, where a is very small, so that its tangent is $= q.p\,\frac{1}{\lambda}$, $\therefore a = \frac{1}{\lambda} = \frac{1}{3\lambda^3} + \frac{1}{5\lambda^5} -$ &c.; $\therefore$ arc tan. $\lambda = \frac{\pi}{2} -$ $\frac{1}{\lambda} + \frac{1}{3\lambda^2} -$ &c. $= \frac{9\lambda + 2q\lambda^3}{9 + 3\lambda^2}$, and by reversion of series we can obtain λ, which, by substituting for q its value, give the ratio of the axes $= 680$.

If two of the roots of $\phi=0$ were equal, then we would have evidently $\phi=0$; $d\phi=0$; in which case the curve will touch the axis at the origin; therefore the value of ϕ can never become negative at the side of the *positive* abscissæ; consequently the value of q, determined by these two equations $\phi=0$, $d\phi=0$, will be the limit of those with which the equilibrium can subsist; and if q has a greater value, the equilibrium would be impossible; for in that case the curve would not meet the line of the abscissæ. It follows, from what precedes, that there is only one value of q which satisfies the equations $\phi=0$, $d\phi=0$, these equations give the following values,

$$q=\frac{6\lambda^2}{(1+\lambda^2)(9+\lambda^2)};\ 0=\frac{7\lambda^5+3\lambda^3+27\lambda}{(1+\lambda^2).(3+\lambda^2).(9+\lambda^2)}-$$

arc tan. λ; the value of λ which satisfies the last equation, is $\lambda=2{,}5292$, $\therefore q=0{,}337007$, and $\sqrt{1+\lambda^2}=2{,}7197$, as in case of the earth $q=,00344957$, this value corresponds to a rotation $=0,^d99727$, but $q \backsim \frac{1}{T^2.\rho}$, note ($h$) page 453, $\therefore$ for a mass of the same density as the earth, T the time of rotation, which corresponds to $0{,}337007 = 0{,}10090$, which is the limit; for if the time of rotation be less than this, the equilibrium is impossible; if it be greater, there are two figures which satisfy the equilibrium. It appears also, as is stated in page 106, that the time of rotation varies generally inversely as the square root of the density.

If the rotatory motion should increase so as to be greater than that which answers to the limit of q, it does not necessarily follow that the fluid cannot be in equilibrio with an elliptic figure; for we may suppose, that according as the compression increases, the motion of rotation will become less rapid; therefore, if there exists between the molecules of the fluid mass a force of tenacity, this mass, after a great number of oscillations, may at length arrive at a motion of rotation, comprised within the limits of equilibrium, and fix itself in that state. In fact, when the rotation is increased, the spheroid becomes more oblate, and the fluids having less velocity of rotation than the equator, accumulate about that circle, and retard the motion; this goes on for some time, until the true shape is overpassed, and then the accumulation relaxes. Now, the motion is too slow for the accumulation, and the waters flow back towards the poles; in this way an oscillation is produced, which, however, in consequence of the mutual tenacity of the particles of the fluid, gradually subsides, and the appropriate form is eventually assumed.

In order to determine the ellipticity of Jupiter, resuming the equation $0 = \frac{9\lambda + 2\,q\lambda^3}{9 + 3\lambda^2} = \text{arc tan. } \lambda$, $\sqrt{1+\lambda^2}$ is the ratio of the equatorial to the polar diameter, and k the axis; ∴ to determine λ we must have q; now if D be the distance, and P the periodic time of the fourth satellite, t the time of Jupiter's rotation, M the mass of Jupiter, and F the centrifugal force, we have F to the force retaining the satellite in its orbit, *i. e.* $\frac{M}{D^2}$ as $\frac{1}{t^2} : \frac{D}{P^2}$, $\therefore F = \frac{MP^2}{t^2.D^3}$, $M = \frac{4}{3}\pi.k^3.(1+\lambda^2)$, ∴ as $D = 26.63$ of the radius of Jupiter's equator, $\therefore \frac{k.\sqrt{(1+\lambda^2)}}{D} = \frac{1}{26,63}$; and as $t = 0,41377$, $P = 16,.^d68902$, $\frac{F}{\frac{4}{3}\pi} = q = \frac{k^3.(1+\lambda^2).P^2}{t^2.D^3} = 0,0861450\,.(1+\lambda^2)^{-\frac{1}{2}}$, hence the preceding equation in λ, becomes

$$0 = 9\,\lambda. + \frac{0,172290.\lambda^3}{\sqrt{1+\lambda^2}} - (9+3\lambda^2).\text{ arc tan. }\lambda.\text{ and } \therefore \lambda =$$

0,481, and if the polar diameter be unity, axis of equator = 1,10967. This is the ratio of the axes on the hypothesis of homogeneity; but the observed proportion being 1,0769, as deduced from actual observation, and also from the motion of the nodes of the satellites, (see page 452,) it follows that Jupiter is not homogeneous, but his density increases as we approach towards the centre. The limits of the ellipticity of the planets, if they were primitively fluid, are $\frac{5c}{4.g}$, $\frac{c}{2g}$; (see pages 107 and 459;) and as the observed ellipticity is within these limits, it follows that the density increases towards the centre.

From the expressions given in page 454, we can compare the ellipticities of Jupiter and the earth on the hypothesis of homogeneity; or even when the densities at distances proportional to their diameters are in a given ratio. In the first case, $\frac{e}{r} = \frac{5c}{4g} = \frac{5}{4} \cdot \frac{1}{\rho . t^2}$, for g is proportional to ρr, and $c = \frac{r}{t^2}$; in the other case, $\frac{e}{r} = \frac{5c}{2g} \cdot \frac{n}{5n. - 3.f}$. See note ($x$) of this Chapter.

And from what has been established in the notes, page 493., Volume I., it appears, that whatever be the manner in which the fluid particles act on each other, whether by their tenacity, their mutual attraction, or even by impinging on one another, in which case they experience finite changes of motion; if through the centre of gravity of this fluid, supposed immoveable, we conceive a plane to pass, with respect to which the sum of the areas described on this plane by each molecule, and multiplied respectively by their corresponding molecules, is at the origin a maximum, this plane will always possess the property; therefore, when, after a great number of oscillations, the fluid mass assumes a uniform motion of rotation about a fixed axis, this axis will be perpendicular to the preceding plane, which will be, from what is stated in notes page 512., Volume I., the plane of the equator, and the motion of rotation will be such, that the sum of the areas described in the instant dt by the molecules projected on this plane, will be the same as at the commencement of the motion; and the axis in question is that with respect to which the *sum of the moments of the primitive forces of the system is a maximum*; it evidently preserves this property during the motion of the system, and finally becomes the axis of rotation. The actual velocity of rotation, as well as the axis of the ellipsoid of revolution which the fluid assumes, are determined by this *maximum;* and from what has been established in page 454, there is evidently only one possible figure of equilibrium.

(k) As it would be impossible, in the limits of these notes, to give the complete investigation of the figure of the earth, when the density increases towards the centre, we shall confine ourselves to pointing out some remarkable consequences which follow from the results, as given by Clairault and others. If n denotes the density of the nucleus, and f that of the rarer fluid which is spread over it, the value of $\frac{e}{r}$, is $\frac{5c}{2g} \cdot \frac{n}{5n-3f}$; if the density of the interior part be infinitely greater than that of the ambient fluid, (which is the hypothesis of page 101.,) $\frac{e}{r} = \frac{c}{2g}$; if $n = f$ then $\frac{e}{r} = \frac{5c}{4g}$, as we have before deduced; therefore the ratio of the ellipticity when the spheroid is homogeneous, to the ellipticity when the nucleus is infinitely denser than the fluid, is that of 5 : 2. These, as we shall see presently, are the extreme cases; likewise it appears, from the above expression, that according as f becomes less with respect to n, the ellipticity diminishes, and conversely. In the preceding hypothesis, it is easy to show that the expression of the increase of the force of gravity from the equator to the poles is expressed by the fraction $\frac{5c}{2g} \cdot \frac{4n-3f}{5n-3f}$; the sum of this and of $\frac{5c}{2g} \cdot \frac{n}{5n-3f} = \frac{5c}{2g}$, which is double the ratio of the centrifugal force at the equator to the force of gravity at the equator; or of the ellipticity of a homogeneous spheroid; for, in the case of homogeneity, the ellipticity and increase of the force of gravity are both expressed by the same fraction.

(*l*) Hence, if we can find the ratio of the equatorial and polar gravities, the ellipticity will be had by substracting the fraction expressing this ratio from twice the ellipticity of a homogeneous spheroid; and as a mean of a great number of observations made with the pendulum, gives 0,00561 for the increase of the force of gravity, $\frac{1}{115,2}$ — ,00561 $= \frac{1}{34,8}$, which shows that the *mean density* of the interior parts of the earth is > than the exterior; therefore, if l be the length a pendulum vibrating seconds at the equator, and $l+d$ the length of an isochronous pendulum at the pole, which is easily determined from the length of an isochronous pendulum at any latitude λ, and from knowing that the increments of the lengths are as sin. ${}^2\lambda$, then $\frac{d}{l} = \frac{5c}{2g} \cdot \frac{4n-3f}{5n-3f}$, $\therefore \frac{4n-3f}{5n-3f} = \frac{2gd}{5cl}$, and $\frac{n}{f} = \frac{15cl-6gd}{20cl-10gd}$, &c. See notes, page 347, Volume I.

The inequalities observed in the measurement of contiguous arcs of the meridian, which, according to Laplace, are to be attributed to the earth's not being spheroidical, arise in some measure also as well from the unequal distribution of the rocks which compose it, as from inequalities in the surface of the earth; which, according to Playfair, account for the discrepancy observed between the preceding ellipticity and that deduced from the measurement of degrees of the meridian. As a remarkable instance of this discrepancy, the spheroid which best agrees with the degrees measured in France, is one of which the ellipticity $= \frac{1}{152}$, which is very nearly double of what may be reckoned the mean ellipticity.

The equation here adverted to, in (*l*) page 110, is that which is given in No. 11 of the Second Book, and is detailed at greater length in the Second Chapter of the Third Book of the Celestial Mechanics. It is of the following form:

$$\left(\frac{d^2V}{dx^2}\right) + \left(\frac{d^2V}{dy^2}\right) + \left(\frac{d^2V}{dz^2}\right) = 0;$$

and may be generally announced in the following manner; that the sum of the three partial differences of the second order of the function V, which expresses the sum of the attracting molecules of a spheroid, divided respectively by their distances from the attracted point, (of which function the partial differences with respect to any line, is the resultant of its attractions decomposed according to this line,) is constantly equal to cypher. By combining this fundamental equation with a differential equation of the first order, which the preceding function must satisfy when the attracted point is at the surface of a homogeneous spheroid, which differs little from a sphere, Laplace obtained by developing, the attraction of a spheroid composed of fluid or solid strata of any density whatever, and endowed with a motion of rotation; the molecules being supposed to attract each other inversely as the square of the distance. The general and simple relations between the attractions and the figure of the spheroids, which are furnished by this expression, enabled Laplace *directly* to determine the figure of the fluid strata in the case of equilibrium, and the law of gravity at their surface. From the fecundity of the fundamental equation, which is the basis of his analysis, and is reproduced in the theory of the fluids, and in that of heat, Laplace was induced to think that the formulæ which he obtained were the simplest and most general which could be obtained.

(*m*) See note (*k*) of this Chapter.

In the general hypothesis, the strata increase in density and diminish in ellipticity from the surface; therefore, if a line be conceived to be drawn from the surface to the centre, the tangents drawn to the strata at the intersection of this line with them, will not be parallel, and consequently the perpendicular to these tangents which indicate the direction of gravity will not be parallel; now, if the number of these strata be increased indefinitely, these perpendiculars will form a curve. In fact, the direction of gravity being a curve line, all those elements are perpendicular to the strata of level which it traverses; this curve is the trajectory which intersects at right angles all ellipses which, by their revolution, form these strata.

The following is the analytical expression of what is stated in (n) p. 117, $p'' = \mathrm{P}.\left(1 - \frac{1}{2}\,a\,(l - y'') + \frac{5}{4}\,a\,\phi.\mu^2\right)$, when p'' is the force of gravity at the surface of the spheroid, P the force of gravity at the surface of the sea and at the equator, a a constant coefficient, so small that its square and higher powers are neglected, and $a(l-y'') =$ the depth of the sea, and $a\phi =$ the ratio of the centrifugal force to the force of gravity at the equator; p'', P are determined by means of isochronous pendulums, and ay' is the elevation of points of the surface of the spheroid above the surface of the sea, $a\,y''$ is the elevation of corresponding points of the atmosphere, $a\,l = a\,y' - a\,y''$, and $a\,l$ and $a\,y''$ can be always determined by means of barometrical measurement, μ^2 is the square of the sine of the latitude, and $\therefore \frac{5}{4}\,a\phi\mu^2$ is the increase of gravity at any latitude μ; now, from what has been established in page 107., it appears, that $\frac{5}{4}\,\phi = 0{,}004325$, $\therefore$ the increase is $0{,}004325.\mathrm{P}\mu^2$, which being less than the observed increment, it follows that the earth is not homogeneous.

In the Eleventh Book, besides the causes mentioned in the text, and which are detailed at length in the Third Book of the Celestial Mechanics, another source of deviation from the law of the square of the sine of the latitude, arises from the errors to which the observations of the amplitudes of the measured arcs are liable, which are, relatively to the measured arc much more considerable than the errors of the pendulum; the reason of which appears to be, that the intensity of gravity is much less affected by local variations than its direction; for the inequalities on the earth's surface, and unequal distribution of the rocks which compose it, must produce great local irregularities in the direction of the plumb line, which, in all probability, are the causes of the inequalities observed in the measurement of contiguous arches of the meridian, reduced to the level of the sea.

It may not be unnecessary to mention, that in general there are three modes of determining the ellipticity of the earth given in the text, either by observing the lengths of isochronous pendulums, or by measuring the arcs of degrees; or, thirdly, by means of some lunar inequalities. By means of the observed quantity of the precession of the equinoxes, Laplace shows, in the Eleventh Book, that D, the mean density of the earth $= 1,587(\rho)$ where (ρ) denotes the density at the surface; and as this density is, by the experiments of Maskeylyne and Cavendish, which will be detailed in note (r) of this Chapter, three times that of water, we have $D=4,761$; that of water being unity, which agrees very well with the conclusions of Maskeylene.

(n) In the Eleventh Book the author shows, that the radius of the terrestrial spheroid $= 1 + a\bar{h}\left(\mu^2 - \frac{1}{3}\right) + ax$, where ax is a very small quantity with respect to $a\bar{h}$, and of the same order as the mean elevation of the continents; in like manner, the expression for the radius of the surface of the sea is $al - a(h'+h)\cdot\left(\mu^2 - \frac{1}{3}\right) + ax'$, where al is a constant quantity and ax' of the same order as ax.

The depth of the sea is very nearly = the difference of these radii, and $\therefore = al - ah'\left(\mu^2 - \frac{1}{3}\right) + ax' - ax$; at the equator the continents occupy a great extent, for which this expression becomes negative; but the sea occupies a still greater extent, for which this expression is positive. In the first case, $al + \frac{1}{3}ah'$ is $<$; and in the second case it is $>$ than $ax - ax'$; consequently $al + \frac{1}{3}ah'$ is of the same order as ax; very near to the north pole, where $\mu = 1$, the sea covers part of the terrestrial spheroid, and leaves another part uncovered; in the first case, $al - \frac{2\,ah'}{3}$ is $>$, and in the second case it $<$ than the value of $ax - ax'$ corresponding to $\mu = 1$. $\therefore$ as $al + \frac{ah'}{3}$, $al - \frac{2ah'}{3}$ are respectively of the order ax, their difference ah' and also the constant quantity al are of the same order; consequently the depth of the sea must be inconsiderable, and of the same order as the elevations of continents above the level of the sea; but as there are mountains which rise very high above the level of the adjacent continents, so there may be some parts of the sea of very considerable depths. Hence it follows, that the surface of the terrestrial spheroid is *q.p* elliptic, for, by what precedes, the equation of the equilibrium of the surface of the sea, would become that of the equilibrium of the surface of the terrestrial spheroid supposed fluid, if the sea was to disappear. It is generally admitted, that at least two-thirds of the surface of the earth is at the present time fluid, and from this circumstance, combined with what is stated in the text, it would seem to follow that the earth was primitively *fluid*.

(*o*) If Π represents the pressure, and ρ the density, the equation adverted to in the text may be expressed as follows, $\frac{d\Pi}{d\rho} = 2k\rho$, $2k$ being constant, $\therefore \Pi = k.(\rho^2 - (\rho^2))$ (ρ) being the density at the surface, where $\Pi = 0$; and as it is proved in No. 30, of the Third Book of the Celestial Mechanics, that $\frac{d\Pi}{\rho} = -4\pi . \frac{da}{a^2} . \int \rho a^2 da$, where a denotes the radius of the stratum of which the pressure $= \Pi$, we have $\frac{d\rho}{da} = -\frac{n^2}{a^2} . \int \rho a^2 da$, where $n^2 = \frac{2\pi}{k}$, if $\rho' = ar$, then $a^2 d\rho = ad\rho' - \rho.' da$, $\therefore \frac{ad\rho'}{da} - \rho' = -n^2 . \int \rho' a da$, $\therefore$ by differentiating $\frac{d^2\rho'}{da^2} + n^2\rho' = 0$; and the integral of this equation is $\rho' = \text{A} \sin. an + \text{B}. \cos. an$, A and B being constant arbitrary quantities, therefore $\rho = \frac{\text{A}}{a} \sin. an + \frac{\text{B}}{a} . \cos. an$; as ρ is not = to infinity at the centre where a vanishes, B must be = 0, and $\therefore \rho = \frac{\text{A}}{a} . \sin. an$. This is the law of the density of the strata of the terrestrial spheroid, on the hypothesis that $\frac{d\Pi}{d\rho} = -2k\rho$, at the surface $a = 1$ and $\rho = (\rho)$, $\therefore (\rho) = \text{A}. \sin. n$; $\frac{-\left(\frac{d\rho}{da}\right)}{(\rho)} = 1 - \frac{n}{\tan. n.}$, and as D is the mean density of the earth, $\int \rho a.^2 da = \text{D}. \int a^2 da = \frac{\text{D}}{3}$; but at the surface the equation $\frac{d\rho}{da} = \frac{-n^2}{a^2} . \int \rho a^2 da$, becomes

$$(\rho) . \left(1 - \frac{n}{\tan. n.}\right) = n^2 . \int \rho a^2 da; \therefore \frac{\text{D}}{(\rho)} =$$

$\frac{3}{n^2} \cdot \Big)1 - \frac{n}{\tan. n}\Big) = \frac{3q}{n^2}$, q being $= 1 - \frac{n}{\tan. n}$, $\frac{D}{(\rho)}$ the ratio of the mean density of the earth to the density of its surface, and $n^2 = \frac{2\pi}{k}$, hence we can determine k and $\therefore n^2$, when D is known, and *vice versa*. From these results, Laplace obtains expressions for the gravity, ellipticity, &c. which accord sufficiently well with observation; from whence he infers, that it is extremely probable, the internal constitution of the earth is conformable to the preceding hypothesis. It is worthy of remark here, though Laplace infers, in page 120, that the primitive fluidity of the earth is clearly indicated by the regularity of gravity, and by the figure at its surface, Playfair, in his Outlines, asserts the express contrary: he states, that the approximation, which, notwithstanding the irregularities in the measured degrees, the figure of the earth has made to the spheroid of equilibrium, cannot, in consistency with other appearances, be ascribed to its having been once in a fluid state, for though the action of water may be evidently traced in the formation of those stratified rocks which constitute a large proportion of the earth's surface, it is of water depositing the *detritus* of solid bodies: with respect to those rocks which contain no such detritus, but have the character of crystallization in a greater or less degree, it is not evident that they are of aqueous formation. Indeed, the only action of water of which we have any distinct evidence in the natural history of the globe, is partial and local, and therefore insufficient to account for the spheroidical figure of the earth.

(p) See note (c) page 465, Volume I.; and No. 27 of the First Book of the Celestial Mechanics.

(q) It follows from the theorem announced in the text, that the three principal axes of rotation of the imaginary spheroid, are the principal axes of the earth.

The expressions for these radii, according as the earth revolves about the first, second, or third principal axes, are,

$$1+al+au-\frac{5}{2}.\frac{a\phi.\left(\mu^2-\frac{1}{3}\right).\int\rho\, d.a^3}{5.\int\rho d.a^3-3},$$

$$1+al+au+\frac{5}{4}.\frac{a\phi.\left(\left(\mu^2-\frac{1}{3}\right)-(1-\mu^2).\cos.(2\bar{\omega}'-2\Pi)\right)}{5.\int\rho.d.a^3-3.},$$

$$1+a.l+a.u+\frac{5}{4}.\frac{a\phi\left(\left(\mu^2-\frac{1}{3}\right)+(1-\mu^2).\cos.(2\bar{\omega}'-2\Pi)\right)}{5.\int\rho.d.a^3-3},$$

therefore, if these be added together, their mean value is $1+a.l+au$, so that it is independent of the centrifugal force $a.\phi$, as is stated in the text.

(*r*) The principal of areas in reference to the present subject, may be announced in the following manner: if we project, on a fixed plane, each molecule of a system of bodies which react on each other, and if, moreover, we draw from these projections to a fixed point assumed on this plane, lines which we shall term radii vectores, the sum of the products of each molecule by the area which its radius vector describes in a given time, is proportional to the time; so that if A denotes this sum, and t the time, we shall have $A=ht$, h being constant. Now, in the case of earthquakes, volcanoes, &c. it is easy to show, that while these phenomena diminish the motion of the earth in one way, there exist simultaneous causes which produce the contrary effect, so that the value of A remains the same: but if, as is stated in the text, considerable masses are brought from the poles to the equator, the radii vectores *increase;* therefore, in order that the value of A may remain unvaried, the other factor must diminish, consequently the rotatory motion of the earth must diminish.

(*s*) If we counterpoise a quantity of ice in a delicate balance, and then leave it to melt, the equilibrium will not be in the slightest degree disturbed; or if we substitute for the ice, boiling-water or red-hot iron, and leave them to cool, the result will be precisely the same; and if a pound of mercury be placed in one scale, and a pound of water in the other, and if they then be heated or cooled through the same number of degrees, although thirty times more heat either enters or leaves the water than the mercury, in consequence of its different capacity for heat, they will still balance each other; likewise if a beam of solar light, be condensed by means of a burning glass, and then made to fall upon the scale of a delicate balance, it will not depress the scale, as would be the case if the beam of light had the least inertia or weight.

(*t*) If a be the arc described in a given time, r the radius, and V the angle, we have $V = \frac{a}{r}$, but the area $= a.r = V.r^2$, therefore if the angular velocity of rotation does not increase, the areas described in the plane of the equator are proportional to r^2; and as the decrement of r is the 100,000th part, the decrement of r^2 will be very nearly double of this, or the 50,000th part.

If this diminution of r arose from a decrease of temperature equal to one degree; and if the duration of the earth's rotation be 100,000 *decimal* seconds, the duration of rotation will be diminished 2″ in this hypothesis; now, as it appears from the comparison of observations with the theory of the secular equation of the moon, that the duration of rotation since the time of Hipparchus has not varied $\frac{1}{100}$th of a second, the variation of the internal heat of the earth since that time is insensible. Indeed, the dilatation, specific heat, and greater or less permeability to heat, which are all unknown, may not be the same in the earth and the glass globe, in which a diminution of $\frac{1}{100}$″ in a day corresponds to a diminution $\frac{1}{200}$ of a degree in temperature. But still this difference can never increase from $\frac{1}{200}$ of a degree to $\frac{1}{10}$ of a degree, the loss of terrestrial heat corresponding to a diminution of $\frac{1}{100}$ of a second in the duration of a day; but a diminution of $\frac{1}{100}$ of a degree near the surface, supposes a much greater diminution in the temperature of the inferior strata; for, eventually, the temperature of all the strata diminishes in a geometric progression, so that the diminution of a degree near the surface, implies a much greater diminution in the strata which are nearer to the centre; therefore the dimensions and moment of inertia of the earth diminish more than in the case of the sphere of glass. From what precedes, it follows, that if, in the progress of time, any change is observed in the mean height of a thermometer placed at the bottom of a deep cavern, it must be ascribed to a change in the climate of the place, and not to a variation in the mean temperature of the earth. It is worthy of remark, that the discovery of the true cause of the secular equation of the moon, makes known at the same time the invariability of the duration of the day, and of the mean temperature of the earth. Connaissance des Temps, 1813, compared with text.

According to M. Fourier, who has discussed the subject of the interior temperature of the globe, the heat distributed within the earth is susceptible of three distinct modifications, arising, 1*st*, from the rays of the sun, which penetrating the globe, cause diurnal and annual variations in its temperature. These periodical variations cease to be perceptible at a certain distance beneath the surface. Beyond that depth, and even to the greatest accessible excavations, the temperature due to the sun has long since become fixed and stationary; the whole quantity of solar heat which regulates the periodical variations, oscillates in the exterior shell of the earth, descending further within the surface during one portion of the year, and rising up to be dissipated into space during the opposite or the winter season. Secondly, the temperature for deep excavations, which, though constant for any one place, varies for localities more or less distant from the equator; so that the *solar heat* penetrates farther at the equinoctial zones, to reascend and be dissipated at the polar regions. But besides the external focus of heat, there is also to be considered the proper or intrinsic heat of the earth; and if, as the experiments mentioned in p. 471 seem to prove, the temperature of the deep recesses of the earth becomes perceptibly greater according as we penetrate farther into the interior, it is impossible to ascribe this increase to the heat of the sun; it can only arise from a primitive heat, with which the earth was endowed at its origin, and which may diminish with greater or less celerity, by diffusion from its surface; it is evident that the increase will not be always the same in amount as at present, it will diminish progressively; but a number of ages must elapse before it is reduced to half of its present value; in general the extent of this diffusion will be proportional to its primitive intensity, and to the conducting quality of the surrounding materials.

If V be the heat of a molecule at the surface, it is proved by analysis, that the increment of heat at the depth z' relatively to r the radius of the earth, is equal to the product of this depth by the elevation of the temperature of the surface of the earth above the mean state of temperature, *i. e.* $= z'\left(\frac{dV}{dr}\right)$ which becomes $= f z' V$, when we only consider in V the part of the heat, which is independent of the action of the heating causes at the exterior.

It is to be remarked here, that at all distances to which we can penetrate, the temperature of the *sea decreases*, and at the equator at a depth of 600 metres, the temperature of the water was $7°,5$ of the centrigrade thermometer, while that at the surface was $30°$; but this is not inconsistent with what is stated in the text, as this decrease of temperature is owing to currents of water coming from the poles to the equator.

It is the opinion of geologists, that originally there existed in the interior of the crust of the earth, a great magazine of fire, which, according to them, was the cause of the deluge, and the numerous catastrophies to which an accurate examination of the various appearances of the internal constitution of the earth proves that our globe has experienced, previous to the deluge, particularly the alternation of marine and fresh-water products. According to them, this heat was much more intense formerly than at present; and as in consequence of the fluidity of the earth in its primeval state, very little heat was lost in its transmission from the interior to the surface, any warmth imparted to the bottom of the ocean would be transmitted without sensible loss to the surface. In this order of things, a genial climate would exist over the whole surface of the earth, from one pole to another; and, in like manner, this intrinsic source of heat would, when its diffusive energy was thus slightly obstructed, predominate over the solar, so that the position of the sun with respect to the equator would act a comparatively subordinate part in modifying climate; therefore, as in this case, the difference of the temperature at the pole and equator would be comparatively small, a considerable uniformity of temperature would thus obtain over the whole earth; and this may explain why animals and plants which are now peculiar to the tropical regions, might have formerly existed as far north as the arctic and antartic circles; (see page 88.)

According as the deposits after *each successive catastrophe to which the earth was subjected*, thickened, there was a progressive interception from the ocean of the subjacent heat; but besides the thickenings of the deposits of the ocean, a great mechanical change took place on the terraqueous constitution, the influence of which, in refrigerating climates, is considerable; for, at every catastrophe, the area of the land in proportion to the sea would be diminished, and that of the sea increased, with a proportionate diminution of depth, *i. e.* the cooling surface would be increased, and the ocean would rest on a cooler bed, because it is more distant from the central heat of the earth ; and besides these two, there is a third cause of the decrease of heat, namely, that which arises from its diffusion into the ambient space.

The elephant mentioned in p. 117, whatever its hide may have been, required necessarily for its subsistance an enormous supply of vegetable food, which necessarily implied a luxurious herbage in the northern regions; and the freshness of his carcass proves that the animal perished at once, with its kindred, in a sudden revolution, accompanied by a *sudden* change of climate, which prevented the decomposition of *its* flesh, and of the bones of its kindred, which are found in great abundance on the banks of the Tanais.

Suppose that three thousand metres beneath an extensive plane, there existed a vast reservoir of water, produced by rain water; at this depth it would acquire, from the heat of the earth, a temperature very nearly equal to that of boiling-water ; and if now, in consequence of the pressure of the adjacent columns of water, or from the action of vapours, which ascend in the reservoir, these waters ascend to the height of the inferior part of the surface from which they had flowed down, they will constitute a source of warm water, impregnated with such substances of the strata through which it flowed, as were soluble by it. This furnishes an extremely probable explanation of the natural tepid waters which are found in different parts of the earth.

(v) It is easy to estimate what would be the effect of the attraction of a *spherical* and *homogeneous* mass near to which a plumb-line is suspended, for if g denote the force of gravity, and x the angle which the direction of the plumb-line makes with the vertical, g sin. x expresses the force of gravity resolved perpendicular to the direction of the plumb-line ; and if y denote the distance of the centre of gravity of the attracted body from the centre of the homogeneous sphere, μ the mass of the attracting body, and f the intensity of the attractive force at the unity of distance, and for the unity of mass; $\frac{\mu f}{y^2}$ = the attraction of the spherical mass on the suspended body; now, if a denote the distance of the point of suspension from the centre of the sphere, and α the angle which this line makes with the verticle, $\alpha - x$ is the angle, which a makes with the plumb-line; the cosine of the angle which a perpendicular to the direction of the plumb-line makes with y, = sine of angle which y makes with plumb-line = $\frac{a.\ \sin.\ (\alpha - x)}{y}$, $\therefore \frac{\mu f}{y^2}$ resolved in the direction of this perpendicular = $\frac{\mu f\ a.\ \sin.\ (\alpha - x)}{y^3}$, which, when the plumb-line is at rest = $g.\ \sin.\ x$; if y be supposed = a, we have $\frac{\mu f}{g a^2} = \frac{\sin.\ x}{\sin.\ (\alpha - x)}$, by means of this equation, when μ, f and a are given, we can determine the deviation x. Now, if m denote the mass of the earth, ρ its mean density, and r its radius, and ρ', r' the density and radius of the attracting body, $\mu : m :: r'^3 \rho' : r^3.\rho$, and $\frac{\mu}{m} = \frac{\rho' r'^3}{\rho. r^3}$, and $mf = g.r^2$, $\therefore \frac{\mu f}{g} = \frac{\rho' r'^3}{\rho. r}$, $\therefore$ from what precedes we have $\frac{\rho' r'^3}{\rho r a^2} = \frac{\sin.\ x}{\sin.\ (\alpha - x)}$. The value of x will be so much the greater as a diminishes, and as α approaches to a right angle; and as the least value of a is r', it follows that the deviation from the vertical will be a *maximum* when $a = r'$ and $\alpha = 90°$, in which case we have tan. $x = \frac{\rho' r'}{\rho. r}$, $\therefore$ if ρ', ρ r', r be given, we can determine the deviation and conversely, it is easy to show that if $\rho'=\rho$ the radius of the sphere, which would cause a deviation $= 1°$, should be $= 30^m,866$. But in nature it is not easy to determine the ratio of ρ to ρ', &c. The manner in which Maskelyne determined the value of x is as follows, by observation of the zenith distances of the stars on the north and south sides of Schehallien, he determined the difference of the latitudes of two stations; from a trigonometrical survey of the mountain, the distance between the same two points was ascertained; and thence, from the known length of a degree of the meridian at that parallel, the difference of the latitudes of the two stations was again inferred, and it was found less by $11''6$, than by astronomical observations. This could only arise from the zeniths of the two places being separated from each other by the attraction of the mountain on the plummets. From the quantity of this change of direction, the ratio of the attraction of the mountain to that of the earth was concluded to be that of 1 to 17804; and from the magnitude and figure of the mountain, which was given by the survey, it was inferred that ρ' was to $\rho :: 5 : 9$, $\therefore \rho$ is nearly double of ρ' the density of the rocks which compose the mountain; indeed these last appear to be considerably more dense than the mean of those which compose the exterior crust of the earth, and at least two or three times more dense than water, $\therefore \rho$ is four or five times more dense than water; hence, as the earth is four times denser than the sun, it follows that the density of the sun is nearly = that of water. See notes, page 399.

However, as was already remarked in page 399, it is to be observed here, that the density obtained is only relative, as we do not know the absolute density of water; and indeed, we are so far from knowing the actual mean density of the earth, that there are considerable discrepancies in the results which determine the ratio of its mean density to that of water.

Though the Cordillieries exhibit evident traces of their being volcanic, and therefore hollow in their interior, there is no reason to suppose that Schehallien is of that nature; on the contrary, it is very probable that it is an extremely dense mountain. The universality of the attraction of every particle of matter is clearly established by this deflection; also it follows, that the force of gravity varies inversely as the square of the distance, for if the attraction of the hill was to that of the earth only as their respective masses, the effect of its attraction would be altogether insensible, in consequence of the comparative smallness of its mass.

We might, by means of the oscillations of the pendulum, determine the length of the pendulum, which vibrates seconds at the level of the sea, for if l be the length of a pendulum vibrating seconds at the height h of the Cordellieries, the length of an isochronous pendulum at the level of the sea $= l.\frac{(r+h)^2}{r^2} = l + \frac{2hl}{r}$, omitting $\frac{lh^2}{r^2}$ as a very small fraction; now, it is observed, that the value of the correction of the length of the pendulum, determined by observation, is less than what theory assigns to it from a diminution of distance, which can only arise from the action of the mountain itself making the difference less than it ought to be, from its increased distance from the centre of the earth. The ratio deduced from the effect of the mountain in deflecting the plummet, is considerably less than the estimation by means of the attraction of two leaden balls, the effect of which Cavendish rendered sensible by the balance of torsion, an instrument by means of which we can determine very small, and apparently inappreciable forces; it consists of a very delicate metallic thread, attached to a fixed point, at the extremity of which is suspended an horizontal lever; while the thread is not twisted, the lever quiesces in a certain position, termed the line of repose; according as it deviates from this position the thread becomes twisted, and this torsion tends to cause the lever to revert to the line of repose; therefore, in order to retain it in this position, it is necessary to apply to its extremities equal and contrary forces, existing in the horizontal plane, and acting perpendicular to its length; the common value of these forces will be the measure of the force of torsion, which, when the thread remains the same, is proportional to the angle through which the lever is deflected. Now, if two leaden balls be brought near to the opposite extremities of this line, their attraction will cause the lever to deviate from the line of repose, and according as the deviation increases, the force of torsion increases, and there exists a position in which this force constitutes an equilibrium with the attraction of the two spheres; but as the lever attains this position with an accelerated velocity, it will pass beyond it, and will perform oscillations on each side, like a *horizontal* pendulum; from observing the relation between the length of this pendulum and that of an ordinary isochronous pendulum, we can infer the ratio of the attraction of each sphere to that gravity, and consequently the proportion of the mass of this sphere to that of the earth. As it would be impossible here to enter into all the details of this experiment, we shall give the resulting equation, *i. e.* $\frac{m}{\mu} = \frac{l'.r^2.a.\sin.\alpha}{l c^3 b}$. Where m, μ denote the masses of the earth and sphere, $l\,l'$ the lengths of the lever and isochronous pendulum, a the distance of the centre of the attracting body from the point of bisection of the lever, α the angle which c the line from the centre of attracting body to quiescent extremity of the lever subtends at its point of bisection, and b a constant arbitrary quantity; as all the terms of the second member are given, we can determine the ratio of m to μ, and as we know the magnitudes, we have the ratio of the densities.

(x) If $t\,t'$ denote the times of the earth's and Jupiter's rotation, $c\,c'$ the respective centrifugal forces at their equators, of which the radii are $r\,r'$, and $g\,g'$ their gravities, we have, on the hypothesis of homogeneity,

$e : e' :: \frac{c}{g} :: \frac{c'}{g'} :: \frac{r}{g.t^2} : \frac{r'}{g'\,t'^2}$, (see page 452,) *i. e.* because g varies as $\rho.r$, $e : e' :: \frac{1}{g.\rho} : \frac{1}{g'.\rho'}$; if C′ be the centrifugal force of the fourth satellite, whose distance from the centre of Jupiter is given in terms of r', $c' : C' :: \frac{1}{t^2} : \frac{D}{T^2}$ (T being the period of the satellite), but $C' = \frac{M}{D^2}$, M being the mass of Jupiter, $\therefore c' = \frac{MT^2}{t^2.D^3}$; therefore knowing the ratio of the centrifugal force to that of gravity, we can obtain the proportion of the axes, on the hypothesis of homogeneity, in the manner indicated in the text; which proportion not agreeing with that furnished by accurate observations, it follows that Jupiter is not homogeneous; this also follows from the proportion $e : e' :: \frac{1}{g\rho} : \frac{1}{g'.\rho'}$ for the ellipticity deduced from the preceding proportion, by substituting for e, g, g', ρ, ρ', does not agree with observation.

NOTES TO CHAPTER IX.

(*a*) SUPPOSE a to be the distance of the centre of this ellipse from that of Saturn, which by hypothesis is very great relatively to the dimensions of the ellipse, and let c represent the centrifugal force due to the motion of rotation at the distance of unity from the axis of rotation, then if the coordinates of a molecule of the ring referred to its centre as origin be u, z, the centrifugal force of this molecule, multiplied by the element of its direction, will be equal to $(a+u).cdu$, and the attraction of Saturn on the same molecule is $\frac{S}{(a+u)^2+z^2}$, (S being the mass of Saturn, supposed to be condensed into its centre,) and as the element of its direction is $-d.\sqrt{(a+u)^2+z^2}, = \frac{-(du.a+du.u+dzz)}{\sqrt{(a+u)^2+z^2}}$ if the squares of z and u be neglected, by multiplying by $\frac{S}{a^2+2au}$, we obtain $-\frac{S\,du}{a^2}+\frac{2Sudu}{a^3}-\frac{S.zdz}{a^3}$, the attractions which the same molecule experiences from the ring itself, when multiplied by the element, $-du$, $-dz$ are given by the expressions $-\frac{4\pi udu}{\lambda+1}$, $-\frac{4\pi\lambda zdz}{\lambda+1}$, the equation of the generating ellipse being $u^2+\lambda^2z^2=k^2$; and therefore its differential equation is $0=udu+\lambda^2zdz$, which being compared with the preceding, gives the two following,

$c = \frac{S}{a^3}$; $\frac{4\pi\lambda}{\lambda+1} + \frac{S}{a^3}$ divided by $\frac{4\pi}{\lambda+1} - \frac{3S}{a^3} = \lambda^2$; the first equation determines the rotatory motion of the ring; the second determines the ellipticity of its generating figure, making $e = \frac{S}{4\pi . a^3}$, we obtain, by means of the second equation, $e = \frac{\lambda(\lambda-1)}{(\lambda+1).(3\lambda^2+1)}$, and since e is positive, λ must be greater than unity; as the axis of the ellipse directed towards Saturn, which measures the breadth of the ring, is $= 2k$, the axis which is perpendicular to it $= \frac{2k}{\lambda}$, and as it measures the thickness of the ring, it must be less than its breadth; as $e=0$, when $\lambda=0$, and also when $\lambda=\infty$, it follows that for the same value of e, there are two different values of λ; but we should select the greatest, which gives the most compressed form to the ring; when, therefore, e is a maximum, $\lambda=2,594$, in which case $e=$ 0,0543026., and as $S = \frac{4}{3}\pi . \rho . R^3$, ρ being the density, and R the radius of Saturn, $e = \frac{\rho R^3}{3a^3}$, $\therefore$ the greatest value of which ρ is susceptible is $0,1629078. \frac{a^3}{R^3}$; but this limit is not well defined, in consequence of the difficulty of obtaining the exact ratio of a to R, owing to the effects of irradiation, and the smallness of the apparent magnitudes; if $\frac{a}{R} = 2$ for the innermost ring, this limit = very nearly $\frac{13}{10}$. It is probable that the irradiation increases considerably the apparent magnitude of the ring, and it is likely that, in consequence of it, several rings are blended into one. As the centrifugal force c, arising from the motion of rotation $= \frac{S}{a^3}$, the motion of rotation is evidently equal to that of a satellite whose distance from the centre of Saturn is a. Robison shows, from a consideration of the distance and period of the second satellite, that the period of the ring is not the same as that of a satellite at the same distance. See Mechanical Philosophy, page 514.

Relatively to what is stated in page 135, it is shown in page 165 of the Third Book of the Celestial Mechanics, that if the ring was circular, the attraction of Saturn on an element of the ring is always negative, whatever may be the distance of the centre of Saturn from that of the ring; hence then it follows, that the centre of Saturn always repels that of the ring, consequently the curve which the second centre describes about the first is always convex towards Saturn, therefore eventually the second centre is elongated more and more from that of the planet, until its circumference touches the surface; and as a ring perfectly symmetrical in all its parts would be composed of an infinity of circumferences similar to that which we have just considered, its centre would be repelled by that of Saturn, provided that these two centres ceased to coincide, and then the ring would eventually be attached to the surface of Saturn.

Laplace's theory of the ring has been severely criticised by Professor Robison, who is so far from admitting Laplace's conclusions, that he thinks the inequalities in the form of the ring are incompatible with the equilibrium of forces among incoherent bodies, such as, according to our author, the parts composing the ring are: besides, as by supposition, there is no cohesion in it, any inequalities in the constitution of its different parts cannot influence the general motion of the whole in the manner he assumes, but merely by an inequality of gravitation, the effect of which would be to destroy the permanency of its construction, without securing, as Laplace imagines, the steadiness of its position; likewise, as he thinks, that the equilibrium of the fluid ring is one of instability, any, the slightest disturbance, would derange it. Robison supposes that the ring consists of coherent matter, the cohesive force being considerable, in order to counteract the centrifugal force, which is greater than the weight; its substance, according to him, is viscid, like to melted glass, and if the ring is not uniform, which is indicated from a consideration of its spots, but more massive on one side of the centre than the other, then the planet and the ring may revolve about a common centre, very nearly, but not accurately coinciding with the centre of the ring.

NOTES TO CHAPTER X.

(a) As the density ρ of the atmosphere is a function of the pressure p, according as we ascend in the atmosphere p and therefore ρ diminishes. See note (k) page 365, Volume I.

(b) It should follow from this, that at the surface of the atmosphere, the force of gravity would be equal to the centrifugal force arising from the motion of rotation. See as above.

(c) See notes page 492, Volume I. As $r^2 dv$ expresses the elementary area described by a molecule projected on the plane of the equator; (see page 467;) if r diminishes, dv and therefore the angular velocity of rotation must increase.

(d) See notes page 454 of Volume I. The mutual attraction of the molecules of the atmosphere is not taken into account here; however, it is easy to perceive, that in consequence of the rarity of the atmosphere, this attraction is inconsiderable. If r be the distance of a molecule dM of the atmosphere from the centre of gravity of the earth, (which we shall suppose spherical,) θ the angle which r makes with the axis of rotation, n the angular velocity of rotation, the centrifugal force of $dM = n^2 r.\ \sin.\ \theta$, and the element of its direction $= d.(r.\ \sin.\ \theta)$, therefore the integral of this force into the element of its direction, is $\frac{1}{2} n^2 r^2.\ \sin.\ ^2\theta$, $\therefore$ as $\frac{dp}{\rho} = P\delta x + Q\delta y + R\delta z$; page 454, Volume I., we have $\int \frac{dp}{\rho} = C + V + \frac{1}{2} n^2 r^2 . \sin.^2\theta$, V representing the sum of the molecules of the earth divided by their respective distances, *i. e.* as the earth is supposed to be spherical, $V = \frac{m}{r}$, being the integral of $-\int \left(\frac{dV}{dr}\right) . dr$, now at the exterior surface $p = 0$, ∴ we shall have $C = \frac{m}{r} + \frac{n^2}{2} . r^2 \sin.^2\theta, = \therefore \frac{2C}{m} = \frac{2}{r} + \frac{n^2}{m} . r^2 \sin.^2\theta$, or $c = \frac{2C}{m} = \frac{2}{r} + a r^2 \sin.^2\theta$, a denoting $\frac{n^2}{m}$, *i. e.* the ratio of the centrifugal force at the earth's equator to the force of gravity, the radius of the equator being supposed $= 1$, if R denotes the radius of the pole of the atmosphere, we have $c = \frac{2}{R}$, for θ then vanishes, $\therefore \frac{2}{R} = \frac{2}{r} + ar.^2 \sin.^2\theta$, and if R′ denote the radius of the equator of the earth's atmosphere, we have, as $\theta = 90$, $\frac{2}{R} = \frac{2}{R'} + aR'^2$, $\therefore aR'^3 = \frac{2.(R'-R)}{R}$; the greatest value of which R′ is susceptible is evidently that which belongs to the point in which the centrifugal force is equal to gravity, in which case we have

$$\frac{m}{R'^2} = n^2 . R', \; i.\,e. \; \frac{1}{R'^2} = aR',$$

and therefore $1 = aR'^3$, consequently $\frac{R'}{R} = \frac{2}{3}$; this ratio of R′ to R is the greatest possible, for supposing $aR'^3 = 1 - z$, z being necessarily positive or cypher, we shall have $\frac{R'}{R} = \frac{3-z}{2}$; it is evident that r increases with θ, and is a maximum at the equator, for differentiating $c = \frac{2}{r} + ar^2 \sin.^2\theta$ the equation of the surface, we obtain

$dr = \frac{ar.^3\, d\theta.\ \sin.\ \theta.\ \cos.\ \theta}{1-ar^3.\ \sin.\ ^2\theta}$; now the denominator of this fraction is always positive, for as $am.r.$ sin. θ is the centrifugal force of a molecule whose radius $= r$, (am being $= n^2$), this force resolved in the direction of r, $= amr.$ sin. $^2\theta$, and as it must be less than the gravity $\frac{m}{r^2}$, we have $ar^3.$ sin. $^2\theta < 1$, therefore r increases with θ, and consequently is a maximum at the equator.

The atmosphere has only one possible figure of equilibrium, for making the equation of the surface of the atmosphere to assume the form

$$r^3 - \frac{2r}{a\mathrm{R}.\ \sin.\ ^2\theta} + \frac{2}{a\ \sin.\ ^2\theta} = 0,$$

the values of r, from what precedes, which satisfy the problem, must be positive, and such that $1-ar^3.$ sin. $^2\theta$, is greater than cypher, but there is but one root of this kind, for if $r'\ r''\ r'''$ be the three values of r given by the preceding equation, and if two of them are positive, which is the greatest number that can be so, inasmuch as the absolute quantity in the preceding equation is positive, then as $1-ar^3$ sin. $^2\theta$ is >0, both r' and r'' must be positive and $<$ than $\frac{1}{\sqrt[3]{a.\ \sin.\ ^2\theta}}$; and as the second term is wanting in the given equation $r''' = -r'-r''$, $\therefore r'''$ is negative, and as it is $= -\ r'-r''$ it must be less than $\frac{-2}{\sqrt[3]{a\ \sin.\ ^2\theta}}$, $\therefore -r.'r.''r'''$ must be less than $\frac{2}{a\ \sin.\ ^2\theta}$; but as the absolute quantity is always equal to the product of the roots with the sign changed, $-r.'r.''r'''$ should be $= \frac{2}{a.\ \sin.\ ^2\theta}$, hence it appears that the supposition of their being two positive values of r is impossible, and therefore there is only one possible figure of equilibrium.

(*e*) The solar atmosphere only extends to the orbit of a planet which would circulate about the sun in a time equal to that of the rotation of this star, namely, in twenty-five days and one half; and according as the rotatory motion increases, the limit of atmosphere must be continually contracted. See note (VI.) of this Volume.

(*f*) Knowing the mass of the moon, and also the time in which it revolves on its axis, and likewise the mass of the earth, it is easy to obtain this distance; for if a be the distance of the earth from the moon, x the required distance, m the mass of the earth, and n the angular velocity of the moon, we have $\frac{m}{75.x^2} = \frac{m}{(a-x)^2} + n^2x.$

NOTES TO CHAPTER XI.

(*a*) In notes, page 414, the ratio of the disturbing force of the sun to the force of gravity was determined. Let f, f' represent the disturbing force of the sun on the moon, and on a particle of the terrestrial spheroid, of which the radius is r', r being the distance of the moon from the earth, c being the force which retains the moon in her orbit, and p, P the periods of the earth and moon, we have

$$f : c :: p^2 : P^2, \text{ and } c : f' :: \frac{r}{p^2} : \frac{r'}{P^2}, \therefore f : f' :: r : r';$$

and as we have the ratio of f to g, the force of gravity, (see notes page 448,) we can obtain the ratio of f' to g. According to Newton's estimation, this ratio expressed in numbers is that of 1 to 386046000, this gives the value of the additious force in places 90° distant from the sun, the ablatitious force in places to which the sun is vertical, and in their antipodes is twice greater; therefore the sum of the forces is to the force of gravity as 1 : 1286200; and this sum is the whole force which the sun exerts to raise the waters of the sea; for the effect is precisely the same whether the additious force depresses the water at places 90° from the sun, or elevates the water in the places beneath the sun, and in their antipodes. Now, as we have the ratio of c' the centrifugal force to g the force of gravity, and as we have the ratio of $g : 3f'$, we have the ratio of c' to $3f$, namely, that of 1 to 44527; hence, as according to Newton's theory, the centrifugal force makes the height of the water at the equator exceed the height at the pole by 85472 feet, by a proportion we find the height of the water under the sun, and in the opposite regions, 1 foot 11 $\frac{1}{30}$th, of an inch. He determines this elevation somewhat differently in the Systema Mundi, and makes the height = 9.2 inches.

Newton deduces the force of the moon to move the sea from its proportion to that of the sun, which proportion he infers from the proportion of the motions of the sea, which arise from these forces. If L represent the force of the moon in the equator, and at its mean distance from the earth, S that of the sun in the same circumstances; as at the conjunction and opposition, the height of the tide is the sum of S and L, and in the quadratures it is produced by their difference, we have $L+S : L-S :: 45 : 25$, these numbers expressing the mean of the observed heights in syzygies and quadratures. If the sea covered the entire earth, supposed spherical, the figure which it would assume in consequence of the action of each luminary separately, would be that of an *oblong* spheroid, in which the elevation above the equicapacious sphere is double of the depression below this sphere; for in this case the capacity of the spheroid $\frac{4}{3}\pi.ab^2 = \frac{4}{3}\pi.r^3$, r being the radius of the equicapacious sphere, then $a-s=r=b+\delta$, we have $\frac{4}{3}\pi.(r+s)(r-\delta)^2 = \frac{4}{3}\pi.r^3$, $\therefore$ neglecting the squares and higher powers of δ and s, we have $r^3=r^3+r^2s-2r^2\delta$, $\therefore s=2\delta$, if the spheroid was oblate we would have $\delta=2s$, $\therefore$ in the first case, $r=\frac{a+2b}{3}$, in the other, $r=\frac{2a+b}{3}$.

And it is evident, from the expression $r=\frac{b^2}{1-e.^2\cos.^2\lambda}=$

$q.p.\ b.^2(1+e.^2 \cos.^2\lambda)$, that the difference between the elevation at any point λ and the greatest elevation, varies as $\sin.^2\lambda$; and it is easy to show, that the elevation of any point about the greatest depression, varies as $\cos.^2\lambda$, therefore it is easy to show, that the elevation of any point above the equicapacious sphere $= \delta.\left(\cos.^2\lambda - \frac{1}{3}\right)$, and the depression of any point beneath this sphere $= \delta.\left(\sin.^2\lambda - \frac{2}{3}\right)$. See page 382.

(*b*) Let S represent, as before, the mass of the sun, a its distance from the earth, r the radius of the terrestrial spheroid, and ϕ the distance of any place from the point where the sun is vertical, it is evident, from notes, page 410, that a particle of matter at that place is drawn towards the moon by a force $= \frac{3\,Sr}{a^3}.\cos.\phi$, and besides, its gravity towards the earth is increased by another force $= \frac{Sr}{a^3}$; and since, in the hemisphere opposite to the sun, ϕ is $> \frac{\pi}{2}$, π being the semicircumference, and therefore $\cos.\phi$ is negative; the effect of the force is to draw the particle from the earth in a direction opposite to that which it has in the other hemisphere. As $\frac{Sr}{a^3}$, is nearly always the same, it does not disturb the equilibrium of the waters. In order to obtain the whole force by which the action of the sun diminishes the gravity of a molecule, we should resolve $\frac{3\,Sr.\cos.\phi}{a^3}$ into a force in the direction of the radius vector, and another at right angles to the radius, then the whole force $=$ $\frac{3\,Sr.\cos.^2\phi - Sr}{a^3}$, and the other force draws the particles tangentially or horizontally, and it is $= \frac{3Sr}{a^3} \cdot \sin.\phi \cdot \cos.\phi$; if a' ϕ', represent corresponding quantities for the moon, its force in the direction of the radius $= \frac{3Lr}{a'^3} \cdot (\cos.^2\phi' - Lr)$. Newton does not take into account the effect of $\frac{3}{2}Lr^2 . \sin.^2\phi$; its effect is to increase the previous quantities.

(*c*) It follows from this construction, that near to high and low water the difference of the depths from those of high and low water, are as the squares of the times since high or low water.

(*d*) Besides, the value of $\cos.^2\phi$ relatively to different parts of the same sea, must be considerably different, in order that an oscillation may be produced; for the disturbance of the equilibrium of the waters of the sea is only produced by the inequality of action on different parts of the mass of waters; and this, combined with what is stated in page 143, explains why the tides of the Caspian and other inland seas are so inconsiderable.

(*e*) If L, S represent the actions of the moon and sun, or the difference between the respective semiaxes of the ellipsoids mentioned in the text, it is evident that in syzygies the total rise of the water arises from L+S, and in the quadratures this height is produced by L—S, for the height is regulated by the situation of the moon. From this it follows, that L is more than twice as great as S; indeed, it was observed by Newton, as stated in page 486, that L : S :: 7 : 2; or more accurate observations since his time, make

$$L+S : L-S :: 2 : 1, \text{ and } \therefore L : S :: 3 : 1.$$

Hence we can obtain the mass of the moon, for $\frac{S}{a^3}$ $\frac{L}{a'^3}$, are the forces of the sun and moon to move the waters;

therefore $\frac{S}{a^3} : \frac{L}{a'^3} :: 1 : 3$, and consequently $L = \frac{3Sa'^3}{a^3}$,

L in this way is found to be $\frac{1}{75.5}$ of the mass of the earth. In a given distance A of the sun from the moon, it is easy to determine the point where the elevation produced by the combined action of these luminaries is a maximum; for in that case we have

$$\delta.\left(\cos.^2\lambda - \frac{1}{3}\right) + \delta'.\left(\cos.^2\lambda' - \frac{1}{3}\right),$$

a maximum, (see note page 486;) and consequently $\delta.\, d\lambda.\sin.2\lambda + \delta'.d\lambda'.\sin.2\lambda' = 0$, but as $\lambda + \lambda' = A$, $d\lambda = -d\lambda'$, and $\therefore \delta.\sin.2\lambda = \delta'\sin.2\lambda'$; hence, if twice A be divided into two parts, such that the ratio of the sines may be given, half of each part will give the distances of S and L from the high water.

(*f*) If the harbour be not in the equator, it follows, from the expression for the elevation above the equicapacious sphere in page 487, that the difference of the semiaxes must be multiplied by $\cos.^2\lambda$.

It is evident, that as the place of high water coincides with the moon in the syzygies, and in the following quadrature, and is always between her place and that of the sun, that it must for some time be gradually left behind, and afterwards overtake the moon. To determine when the separation of the moon from the place of high water is a maximum, call x the distance between L and S, and y, the distance of the moon from the place of high water, then $\sin.2y : \sin.(2x-2y) :: \delta' : \delta$, therefore we have $\tan. 2y = \frac{\delta'(\sin.2x)}{\delta + \delta'\cos.2x}$, and $\therefore$ is a maximum when $2x = 90$, *i.e.* when $x = 45$, therefore in this case $2y$ is a maximum, and $dy = 0$, *i.e.* the motion of high water, or its separation from the sun to the eastward, is equal to the moon's easterly motion, *i.e.* in the octants the tide day is equal to a lunar day; and as the height of the lunar tide is proportional to Sr. cos. $^2\phi$, its momentary diminution is proportional to sin. ϕ cos. ϕ, or to sin. 2ϕ.

(g) According to Newton's theory, which we are at present assuming to be correct, the water at every instant assumes the figure of an oblong spheroid, of which the greater axis is directed to the luminary, when a luminary has north declination, the duration and magnitude of the superior tide will be greater than the duration and magnitude of the inferior tide; if the declination of the luminary was equal to the colatitude of the place, there would be only one tide in the day. For places in the equator, whatever be the place of the luminaries, the superior and inferior tides of the same day are the same, though from one day to another they differ, their value is L. cos. ^2d, d being the declination, at the pole there is no *daily* tide, but there is a gradual subsidence and rising by the moon's declining from the equator.

(h) The manner in which Laplace estimates the velocity of the propagation of gravity, is as follows: he supposes a force which, like light, though acting in a contrary direction, rushes towards the sun with an immense rapidity; the resistance which the planet experiences from this current in the direction of the tangent, he conceived to produce a perturbation in the elliptic motion, like to the aberration of light. Calling v the velocity of the gravific fluid which acts in the direction of r a radius vector drawn towards the sun, if δs represent the arc described by a planet in an inconceivably short interval of time, then the planet will be actuated by two forces in the direction of r and δs, which are respectively as v and $\frac{\delta s}{\delta t}$, the force in the direction of r being $\frac{M}{r^2}$, the resistance in the direction of the tangent $= \frac{M\delta s}{r.^2 v.\delta t}$, and if this force be resolved into two, one in the direction of r, and the other perpendicular to r, they will be respectively $\frac{M\,\delta r}{r^2 v.\delta t}$ and $\frac{\delta\phi}{r.v.\delta t}$, for the arc perpendicular to $r = r\delta\phi$, ϕ being the angle which r makes with the axis of x; therefore the entire force directed towards the centre $= \frac{M}{r^2}.\left(1+\frac{\delta r}{v.\delta t}\right)$; if this force be resolved into two, parallel to x and y, they will be $= \frac{M}{r^2}.\sin.\ \phi\left(1+\frac{\delta r}{v.\delta t}\right)$; $\frac{M}{r^2}.\cos.\ \phi\left(1+\frac{\delta r}{v.\delta t}\right)$, and the force $\frac{M\delta\phi}{r.v.\delta t}$, resolved parallel to x and y, gives

$\frac{M\delta\phi.\cos.\phi}{rv.\delta t}$; $\frac{M\delta\phi.\sin.\phi}{r.v.\delta t}$; therefore the entire force in the directions of x and y, are

$$\frac{M}{r^2}\left(\cos.\phi + \frac{\delta r.\cos.\phi - r\delta\phi.\sin.\phi}{v\delta t}\right)$$

$$\frac{M}{r^2}\left(\sin.\phi + \frac{\delta r.\sin.\phi + r\delta\phi.\cos.\phi}{v\delta t}\right);$$

therefore, by means of the equations in 273, we obtain

$$2\delta r.\delta\phi + r\delta^2\phi = -\frac{2gM}{rv}\delta t\delta\phi;\ \delta r^2 - r\delta\phi^2 =$$

$-\frac{2gM}{r^2}\delta t^2\left(1+\frac{\delta r}{v\delta t}\right)$; if v be supposed to be infinite, these equations would give those of elliptic motion; multiplying by r and integrating, we obtain $r^2\delta\phi = A\delta t - 2gM\delta t\int\frac{\delta\phi}{r}$, if v is not infinite it is probably a function of r, however, relatively to the small variations of distance for each planet, we may assume it as constant, particularly as the integral $\int\frac{\delta\phi}{r}$ is extremely small, $\therefore \int\frac{\delta\phi}{v} = \frac{\phi}{v}$, and consequently $\delta\phi = \frac{A\delta t}{r^2} - \frac{2gM\phi\delta t}{r^2 v}$, $\therefore$ squaring and neglecting $\frac{1}{v^2}$, we obtain $\frac{r\delta\phi^2}{\delta t^2} = \frac{A^2}{r^3} - \frac{4gAM\phi}{r^3 v}$, which being substituted in the second of the foregoing equations, will give $0 = \frac{\delta^2 r}{\delta t^2} - \frac{A^2}{r^3} + \frac{4gAM\phi}{r^3 v} + \frac{2gM}{r^2}$, $\frac{\delta r}{v\delta t}$ being neglected as inconsiderable; making $\phi = nt + a\zeta, r = a(1+au)$, nt will be the mean motion, a the mean distance, and $a\zeta$, au very small numbers, $nT = 2\pi$, T being the time of a revolution, and as $\pi = \frac{AT}{2a^2}$, $n = \frac{A}{a^2}$; thus the preceding equations will become *q.p*

$$0 = n + \frac{a\delta\zeta}{\delta t} - \frac{A}{a^2}(1-2au) + \frac{2gMnt}{a^2 v},$$

$$0 = \frac{\delta^2 u}{\delta t^2} - \frac{A^2}{a a^4}(1-3au) + \frac{4g.A.Mnt}{a.a^4 v} + \frac{2gM}{a.a^3}(1-2au),$$ or

by substituting $A^2 = 2gMa$,

$$0 = \frac{\delta\zeta}{\delta t} + 2nu + \frac{n^3 at}{av}; \quad 0 = \frac{\delta^2 u}{\delta t^2} + n^2 u + \frac{2n^4 at}{av};$$

the form of the integral of the second of these equations is $u = D. \cos. (\beta t + b) + Et$, $\therefore$ as $r = a\ (1 + au)$, aD is the eccentricity, and $\beta t + b$ is the anomaly of the ellipse; if the epoch from which we reckon is the time of passing through perihelion, b must be $= 0$; hence if γ' denote the eccentricity, $u = \frac{\gamma'}{a} \cos.\ \beta t + Et$, and $\therefore \frac{\delta^2 u}{\delta t^2} = -\frac{\gamma}{a}$. $\beta^2. \cos.\ \beta t$, and $0 = \frac{n^2 - \beta^2}{a} \gamma \cos.\ \beta t + n^2 t\left(E + \frac{2n^2 a}{av}\right)$, $\therefore$ $\beta = n$; $E = -\frac{2n^2 a}{av}$, and $u = \frac{\gamma}{a}. \cos.\ nt - \frac{2n^2}{v}.t$, and by substituting we obtain

$$\delta\zeta = \frac{3n^2}{v} t\delta t - \frac{2n}{a}\gamma\,\delta t. \cos.\ nt, \text{ and } \zeta = \frac{3n^3}{2v}.t^2 - \frac{2\gamma}{a}. \sin.\ nt,$$

and consequently $r = a\left(1 + \gamma.\ \cos.\ \beta t - \frac{2n^2 a}{v} t\right)$, and $\phi =$ $nt - 2\gamma\ \sin.\ nt + \frac{3n^2 a}{2v} t^2$; nt being the mean motion, $2\gamma.$ $\sin.\ nt$ the first term of the equation of the centre, and $\frac{3n^2 a}{2v} t^2$ the secular equation proportional to the square of the time, which would appear to explain the secular equation of the moon. See page 63. If the secular equation of the moon was known, on the hypothesis *that it arose from this cause*, we could determine v, for if i be the number of months in the time t, then $nt = 2\pi i$, and the secular equation $= \frac{6na\pi^2 i^2}{v}$; in 2000 years $i = 2000 . \frac{525969}{39343}$, $\therefore \zeta$ the secular equation for 2000 years $=$ $\frac{6na\pi^2 (2000) .^2 (525969)^2}{(39343)^2 v}$; but if this ζ is $= 1°$, then $v =$ $\frac{6n}{1°} . \pi^2 .\ 4000000. \left(\frac{525969}{39343}\right)^2$, now $n = 32'', 94$ if the time is expressed in seconds, and $a = 0{,}0025138\ a$, a being the distance of the sun from the earth; therefore

$$\frac{6n}{1°} = \frac{32{,}94}{600} = 0{,}0549, \text{ and for one minute,}$$

$$v = \pi^2 549 . 1{,}00552 . \left(\frac{525969}{39343}\right)^2 a = 973753\, a,$$ *i.e.* in a minute of time the gravific fluid passes over one million of the earth's semidiameters; and its velocity is 8000000 greater than that of light, as the secular equations of the different planets, $\zeta = \frac{6na\pi^2 i^2}{v}$ vary as nai^2 or as $n^3 a$, we can find the secular equation for all planets, knowing that of any one.

(*i*) From the characteristic property of fluids, namely, the perfect mobility of its particles, it follows, that when a fluid mass is in equilibrio, each of its particles must likewise be in equilibrio, in consequence of the forces which solicit it. This is the general principle which Laplace applies to determine the relation which must exist between the forces which solicit the system when this condition is satisfied; and in determining the figure of the earth, he applies it to determine the equilibrium of a homogeneous fluid mass spread over a solid nucleus of any figure whatever. In the theory of the tides he introduces into the differential equations of the motion of fluids, the forces which disturb the equilibrium, namely, the attraction of the sun and moon; and secondly the attraction of the aqueous stratum, of which the interior radius is that of the spheroid of equilibrium, and the exterior that of the disturbed spheroid. The integrations of these differential equations present almost insuperable difficulties, even in the case in which the depth of the sea is assumed to be a function of the latitude; for even then the determination of the radius of the troubled spheroid would lead to a linear differential equation which cannot be integrated; however, the integration of this equation is not necessary, it is sufficient if we are able to satisfy it, for that part of the oscillations which depend on the primitive state of the sea must disappear very soon, from the action of exterior obstacles, so that, as without the action of the sun and moon, the sea would long since have attained a permanent state of equilibrium, it is only the action of these two stars which causes them to deviate from this state, and therefore it is solely necessary to consider the oscillations which depend on this action, now if the terms which produce these be developed, the part of the action of the star which disturbs the fluid molecule, is (neglecting the fourth powers of $\frac{1}{r}$, See page 166.)

$$\frac{3L}{2r^3}\left((\cos.\ \theta.\ \sin.\ v + \sin.\ \theta.\ \cos.\ v.)\ \cos.\ (nt+\tilde{\omega}-\psi)^2 - \frac{1}{3}\right),$$

where r is the distance of the attracting body from the centre of the earth, v its declination, ψ it right ascension, nt the rotatory motion of the earth, and ϖ the angle which a plane passing through x and r makes with the plane x, y, the preceding expression is equivalent to the three following:

$$\frac{L}{4r^3}\left(\sin.^2 v - \frac{1}{2}\cos.^2 v\right).(1+3\cos. 2\,\theta)$$

$$+\frac{3L}{r^3}.\sin.\theta.\cos.\theta.\sin.v.\cos.v.\cos.(nt+\varpi-\psi)$$

$$+\frac{3}{4}.\frac{L}{r^3}\sin.^2\theta.\cos.^2 v.\cos.2.(nt+\varpi-\psi).$$

Now, (as has been remarked,) the only oscillations which it is necessary to consider, are those which depend on the action of the sun and moon, for those which depend on the primitive state of the sea, must long since have disappeared, from the resistance which the waters of the sea have experienced in their motion. As r v and ψ vary with extreme slowness relatively to nt, the three preceding terms give rise to three different species of oscillations. The periods of the oscillation of the first species are very long; they are independent of the rotatory motion of the earth, and depend solely on the motion of L in its orbit. The periods of the oscillation of the second species depend principally on nt, the motion of rotation of the earth, their duration is very nearly a day. Finally, the periods of the oscillations of the third species, depend principally on $2nt$, their duration is about half a day. As the resulting equation which determines the oscillation of the sea is linear, it follows, from what is stated in page 286, Volume I., that these oscillations mix, without interfering with each other, therefore we may consider each separately.

With respect to the oscillations of the first species, they can be obtained in an approximate manner, if the spheroid covered by the sea is an ellipsoid of revolution, in which case the depth of the sea must be a function of the latitude. The part of these oscillations which depends on the motion of the nodes of the lunar orbit, may be very considerable; however, in consequence of the resistance which the waters of the ocean experience, the oscillations of this species are very much diminished, and their extent becomes very inconsiderable, so that, in virtue of these resistances, the oscillations are very nearly the same as if the sea should be in equilibrio under the attracting star.

With respect to the oscillations of the second species, they can be determined when the depth of the sea is supposed to be very nearly constant. The difference of the two tides of the same day depends on these oscillations; now it appears from observations that this difference is very small, and as it would seem to follow from the expression for the difference, that the height of the superior tides is $>r$ than that of the inferior, the depth of the sea is greater near to the poles than at the equator; but this depends on an hypothesis which we know not to be true, namely, that the sea is spread over the entire earth.

With respect to the oscillations of the third species, these also are easily determined, if the depth of the sea be supposed every where the same; according as the depth is increased, these oscillations approach to what they would be if the sea was in equilibrio under the attracting body.

In reference to what is stated in page 154, as to the effect of local circumstances, he shows, in the Thirteenth Book, that in consequence of the rotation of the earth, and of these local circumstances, the daily tide is reduced very nearly to a third, while the semidiurnal tide becomes at least sixteen times greater; however, when it is considered that the rotation of the earth destroys, in a sea of a *uniform* depth, the daily tide altogether, and likewise that if the depth of the sea was $\frac{1}{720}$th of the earth's radius, the height of the semidiurnal sea in the syzygies would be 11 metres, we should not be surprised at these results. See note (x) and Chapter XII.

It is easy to show, that in the hypothesis of a great depth, the two tides of the same day would be very different at Brest, if the declinations of the sun and moon were considerable; in fact, one tide would be eight times greater than the other; but according to observation they are very nearly equal, therefore the hypothesis of a great depth of sea is inadmissible.

Laplace proves, that the value of the quantity by which the sea is elevated, in consequence of the action of extraneous attracting bodies, ceases to be periodic, (which is a condition necessary in order to insure an equilibrium,) when the density of the sea surpasses that of the nucleus over which it is spread; when the contrary is the case, the equilibrium is stable, whatever may be the original agitation; but if otherwise, the stability of the equilibrium depends on the original disturbance.

He likewise shows, from the relations which exist between the depth of the sea and the oscillations of the second species, that these oscillations must disappear for the entire earth when the depth of the sea is constant; but no admissible law of the depth of the sea can render the oscillations of the third species equal to nothing for the whole earth.

In some harbours the oscillations of the second species may be insensible, while in others the oscillations of the third species can hardly be recognized.

The reason why the principle stated in page 155 is applicable to the tides, is, that the forces become the same after the interval of half a day.

This principle being combined with that of the coexistence of very small oscillations already adverted to, enables us to obtain an expression for the height of the tides, of which the arbitrary quantities comprise the effect of the local circumstances of the port and each harbour; for this purpose Laplace reduced into a series of sines and cosines of angles, increasing proportionally to the time, the expression of the solar and lunar forces. He considers each term of the series as representing the action of a particular star, which moves uniformly at a constant distance in the plane of the equator; hence arise several species of partial tides, of which the periods are nearly half a day, an entire day, half a year, an entire year, eighteen years and a half.

When the sun and moon do not move in the plane of the equator, then the effect produced may be conceived to be made up of the action of *several* stars respectively moving in the plane of the equator at different distances and at different periods; and the total tide due to the action of the sun is the combination of the partial tides due to the action of each of those stars.

(*l*) Each observation has for its analytical expression a function of the elements which we want to determine, and if these elements are very nearly known, this function becomes a linear function of their corrections. By putting it equal to an observation, we form what is called an equation of condition; and if there be a considerable number of like observations, they are combined so as to form as many final equations as there are elements; and then, by resolving these equations, we determine the corrections of the elements. The artifice consists in combining the equations of condition in the most advantageous manner; for this purpose, it is to be observed, that the formation of a final equation by means of equations of condition, is effected by multiplying each of them by an indeterminate factor, and then combining these products; but it is necessary to select the system of factors which gives the smallest error; now it is evident, that if we multiply each error of which an element determined by a system is still susceptible by the probability of this error, *the most advantageous system* is that in which the sum of these products, taken positively, is a *minimum;* for a positive or negative error may be considered as a loss. Therefore, by forming this sum of products, the condition of the *minimum* will determine the most advantageous system of factors, and the *minimum* of error to be apprehended on each element. In the analytic theory of probabilities, Laplace shows that this system is that of the coefficients of the elements in each equation of condition, so that a first final equation is formed by multiplying respectively each equation of condition by the coefficient of its first element, and then combining all these equations thus multiplied. A second final equation is formed by employing the coefficients of the second element, and so on. In the same work he gives the expression of the minimum of error, whatever may be the *number* of the elements. This *minimum* gives the probability of the errors of which the corrections of these elements are still susceptible, and which is proportional to the number of which the hyperbolic logarithm is unity, raised to a power of which the exponant is the square of the errors taken negatively, and divided by the square of the minimum of the error, multiplied by 2π. The coefficient of the negative square of the error may therefore be considered as the modulus of the probability of errors, since the error remaining the same, the probability decreases with rapidity, when it increases, so that the result obtained *inclines* towards truth so much the more as the modulus is greater. Laplace, for this reason, terms this modulus *the weight of the result;* and by a remarkable analogy of those weights with those of bodies, referred to their common centre of gravity, it happens that if the *same* element is furnished by different compound systems, each consisting of a great number of observations, the most advantageous mean result of them all taken together is the sum of the products of each partial result by its weight, this sum being divided by the sum of all the weights; moreover, the total weight of different systems is the sum of their partial weights, so that the probability of the errors of the mean result of their aggregate sum is proportional to the number of which the hyperbolic logarithm is unity, raised to a power of which the exponent is the square of the error taken negatively, and multiplied by the sum of the weights. Indeed, each weight depends on the law of probability of the errors in each system, and almost always this law is unknown; but Laplace fortunately succeeded in eliminating the factor which contains it, by means of the sum of the squares of the deviations of the observations of the system from their mean result. It were therefore desirable, in order to perfect our information on the results obtained from a collection of a great number of observations, that at the side of each result the weight which corresponds to it should be written. In order to facilitate the computation, Laplace developed the analytical expression when there were only four elements to determine. But as the number of elements increases, this expression becomes more and more complicated. He gives a very simple means of determining the weight of a result, whatever be the number of elements, and then a regular process of arriving at our object is preferable to the employment of analytical formula. When by this means the exponential which represents the law of the probability of the errors of the result is obtained, the integral of the product of this exponential by the differential of the error, being taken within definite limits, will give the probability that the error of the result is comprised within those limits, by multiplying it by the square root of the weight of the result divided by 2π. See Celestial Mechanics, page 82, and Volume I., page 473.

(*m*) Observation agrees with theory in making the diminution of the total tide, reckoning from the maximum, to be proportional to the square of the times. Likewise the solstitial tides are less than those in equinoxes in the proportion of the square of the cosine of declination to radius, which is exactly the proportion between them which can be inferred from theory. In like manner, agreeable to the formula in page 494, the variations of distance must have some influence on the height and retardation of the tides, in which there is also a perfect conformability between theory and observation.

The ratio of the action of the moon to that of the sun can be determined either from the syzygial heights compared with the heights in quadratures, or from the variation of retardation in syzygies and quadrature, or from the actual diminution of the heights in these positions of the sun and moon.

If *e* be the proportion of the mass of the moon, ÷ded by the cube of its mean distance from the earth to the mass of the sun, divided by the cube of its mean distance from the earth, it is $q.p=3$.

The proportion of the solar to the lunar action is $\frac{1}{3}$d in the harbour of Brest; but it would be nothing at a harbour constructed at the extremity of two canals, whose embouchures being near each other, are so situated that the solar tide employs a day and a half to arrive by one canal at the harbour, and only a quarter of a day to arrive by the other. The low water of the second corresponds to the high water of the first canal; therefore, if, at the common termination of the two canals, the tides are of equal height, the sea, as far as the action of the sun is concerned, will then be stationary; but as the lunar day surpasses the solar, the low lunar tide of one canal does not correspond with the high lunar tide of the other, so that at their common extremity they will not destroy each other.

The number of observations from which Laplace deduced the ratio of the heights in solstitial syzygies to those in the syzygies of quadratures, in the Fourth Book of the Celestial Mechanics, was twenty-four, made respectively in the quadratures and syzygies of these luminaries, whereas the number from which he deduced the corresponding proportions in 1820, were 128 in each; therefore a greater degree of accuracy was to be expected from the last; however, an inspection of the results from ancient and modern observation, shows that there is a perfect conformity between them.

(*o*) Suppose a canal communicating by means of its two extremities with the ocean, the tide in any harbour situated on the banks of this canal will be the result of undulations transmitted by its two *embouchures*, but its situation may be such, and the undulations of the tides may arrive at it at such different times, that the *maximum* of the one may coincide with the minimum of the other; and if they are equal, it is evident that, in consequence of these undulations, there is no tide in this harbour, but there will be a tide produced by the oscillations of second species, of which, as the period is twice as long, will not so correspond that the maximum of those which arrive by one embouchure may correspond with the minimum of those which come by the other. In this case there will be no tide on the day when the sun and moon are in the plane of the equator, but when the moon has declination, there will be only one tide in the lunar day, so that, if the high water is at the rising, the low water will happen at the setting of the sun, and *vice versa*. See Princip. Math., Vol. III. Prop. 24.

(*p*) See notes, page 489.

If, as is stated in page 166, there is any tide depending on the fourth power of the distance of the moon from the earth, it would be evinced in the difference between the action of the moon, in the new compared with its action in full moon, and between its action in the northern and southern quadratures; and it is certain, from the theory of probabilities, that the increased number of observations can supply their want of accuracy, so that, by means of them, we can appreciate inequalities much less than the errors of which they are susceptible. The differences above-mentioned ought to be sensibly indicated in the numerous observations of the height of the tides discussed by Bouvard. The terms divided by the cube of the distance, which are the only ones hitherto considered, do not indicate any difference between the lunar tides of full and new moon: but a comparison of a great number of observations proves, that the terms divided by the fourth power of the distance indicate an excess of the full moon tides over those of the new moon, both in the equinoxes and also in the solstices; and, conformably to theory, the excess is greater in the equinox than in the solstices.

Bouvard having separated, in the computation of the solstitial syzygies, the tides in which the declination of the moon was southern, from the tides in which the declination of the moon was northern, found, from taking the sum of a great number of each, that the action of the southern moon on the sea exceeded the action of the northern moon.

Newton thus accounts for this phenomenon: there are two inlets to this port; and if, through one of those inlets, a tide arrives at Batsha at the third hour after the moon passes the meridian, and through the other, six hours after, if these tides are equal, as one is flowing while the other is ebbing, the water must stagnate, this is the case when the moon is on the equator, but when the moon declines to the north of the equator; the morning tide exceeds the evening, as appears by what is already stated in notes, page 490, so that two greater and two lesser tides arrive at Batsha by turns. The difference of these will produce an ebbing and flowing, which will attain its maximum at the middle, between the two greatest tides, and be lowest at the middle, between the two lowest tides; therefore, at the setting of the moon it is high, and at the rising it is low water; when the moon is at the other side of the water, or the evening exceeds the morning tide, the case is reversed, and it is high water as the moon rises, and low water when she sets.

NOTES TO CHAPTER XII.

(*a*) Besides the oscillations of the atmosphere due to the attractions of the sun and moon, there are also movements excited in it by the variations of the solar heat; but it is impossible to subject these last to analysis. The first mentioned oscillations are given by an analysis similar to that which determines the oscillations of the sea when the depth is uniform.

The oscillations in the atmosphere ought to produce corresponding oscillations in the heights of the barometer; and indeed it is only by means of the variations of the barometer that the existence of the very inconsiderable wind, which is produced by the action of the sun and moon in an atmosphere already considerably agitated by other causes, can be indicated. These barometric observations ought to be made within the tropics, where, as is stated in page 173, the changes arising from irregular causes are fewer; indeed, the gravity of the mercury in the barometer must be affected, however, not so much as the more distant air.

The principle referred to here, is that stated in page 155.

(*b*) See notes to the preceding Chapter, page 501.

(*e*) Since, on the day of the syzygy, the lunar action combines with the greatest diurnal variation, and on the day of quadrature, it is greatest when the diurnal variation is least, the difference of these heights must be evidently equal to twice the lunar action, and therefore equal twice the height of the atmospheric lunar tide.

The diurnal variation which has been observed being regulated by the *solar* day, indicates evidently that this variation is due to the action of the sun; however, when we consider the smallness of the effects due to the *combined* attractions of the sun and moon, the attractive force of the sun *alone* must be considered as almost insensible, therefore it must be by the *action of heat*, that the sun produces the daily variation of the barometer. It is, however, as has been already remarked, impossible to submit to analysis the effects of this action on the height of the barometer; it is principally apparent at the equator; however, notwithstanding the inconstancy of our climates, it is also indicated, though less sensibly, to observations without the tropics; besides the maximum and minimum mentioned in the text, there is a second maximum at eleven o'clock, P. M., and a second minimum at four o'clock, A. M. See *Essai Philosophique sur les Probabilities*, page 123, *5me* edition.

(*d*) By comparing the heights at nine A. M., with those of the *same days*, at three P. M., he found that its mean value for each month remained constantly positive for each of seventy-two months, reckoning from the 1st of January 1817 to the 1st of January 1823, its mean value in these seventy-two months is very nearly $\frac{8}{10}$ of a millimeter, which is much less than at the equator; it is remarkable that the mean result of the diurnal variations of the barometer from nine A. M., to three P. M., is only 0,5428 for the three months of November, December, and January, and that it increases to 1^m,0563 for the three following months; nothing similar to this occurs in the following six months.

(*e*) As there is a *calorific* quality accompanying the *colorific* action of light in the spectrum, so in every modification of the rays of light a calorific quality is a concomitant. Its existence is clearly established by means of the photometer, an instrument which is contrived to point out the power of illumination by the slight elevation of temperature which it occasions. It consists of a differential thermometer, having one of its balls *diaphonous*, and the other blown of a deep black enamel, and when the light incident on the two balls is of the same intensity, the temperature of the black ball will rise more than that of the other, owing to its absorbing a greater number of calorific rays; and *vice versa*, if the two balls were precisely the same, it is evident that the one which was most *illuminated* would be that whose temperature would be most *increased*.

(*f*) Before he applied the calculus of probabilities to this phenomenon, he determined the law of the probability of the anomalies of the diurnal variation, which may arise from chance, and then, by applying it to the observations of this phenomenon, he found that there was more than 300,000 to 1 that it was produced by a regular cause. The following is the outline of the method for determining the probability of the mean error of a great number of values of the diurnal variation: let n denote a great number of values of the diurnal variation of the barometer, the sum of them all divided by n gives the mean value; if e denotes the sum of the squares of the differences of this mean value from each of their values, and u the mean error of a great number s of values of the diurnal variation, the probability of u will be proportional to $c^{-s.\frac{n}{2e}.u^2}$, as in this case, $n=1584$, and $\therefore e = 5473{,}98$, and $\therefore \frac{n}{2e} = 0{,}144685$, and if s expresses the number of diurnal variations near the syzygies, we have $s=792$, and the probability of the mean error u will be proportional to $e^{-114{,}59u^2}$, and the probability of a mean error u' near the quadratures, is proportional to $e^{-114{,}59u'^2}$; $\therefore$ if z denotes the excess of u' over u, by the method of the work already cited, the probability of z will be proportional to $-114{,}59.\frac{z^2}{2}$.

It is to be observed here, that the observations employed by Laplace, are taken without any reference to the time of year, therefore the partial lunar tides which would depend on the declinations of the moon and on its parallax, disappear in the collection of these observations. The analytic expression for the lunar tide, like to that for the sea, is expressed by the formula

$$R. \cos. \{2nt+2\varpi-2mt-2(m't-mt)-2\lambda'\}$$

R depends on the action of the moon on the atmosphere, whether direct or transmitted by the sea, mt $m't$ represent the mean motions of the sun and moon, nt the rotation of the earth, ϖ the longitude of the place, $nt+\varpi-mt$ is the horary angle of the sun, and λ' is an indeterminate constant quantity.

The combined action of the sun and moon must cause a tendency in the air as well as the ocean to move westward; however, as the rate is, according to Laplace, only *four miles* during each revolution of the earth on its axis, it is evidently too small to be subjected to observation.

(*h*) At the parallel of 25°, the mean temperature is 4° of the centigrade thermometer lower than at the equator. This difference of heat may be supposed to graduate through the atmosphere to the height of 10,000 feet; therefore the expansion of air at the equator, which draws to it a meridional wind, will amount to a column of 100 feet. The velocity of the current thence produced, must be $8.\sqrt{100}$, or 80 feet in a second, *i.e.* 54 miles in an hour; but as the velocity of a point in a parallel of 24° is seven miles an hour faster than on the parallel of 25°, when the wind arrives at the parallel of 24, it will seem to a spectator to have acquired a tendency of seven miles an hour to the west; at its arrival at the parallels of 23°, 22°, 21°, &c. it will gain continual though decreasing additions to its apparently westerly course, which, at the equator, will be increased to 104 miles in an hour. The same obtains also for the southern hemisphere; however, as the mean temperature for a given latitude is greater on the northern than on the southern side of the equator, inasmuch as a larger land surface is presented to the action of the solar rays in the northern hemisphere, the mean path of the easterly current of the air is 3° to the north of the equator. It is also to be observed, that the sun is not always vertical to the same place; therefore, though the hottest region for the entire year is 3° north of the equator, still its position must in some measure be dependent on the seasons. In the summer months it shifts towards the tropic of Cancer; during winter the hottest parallel passes to the other side of the equator; hence, in the progress of summer the trade wind oscillates about a point towards the north, and it declines towards the south with the advance of winter. But the trade winds experience a much more considerable modification, arising from the circumstance of the sun acting more powerfully upon the land within the torrid zones than upon the water; hence, when he moves towards the northern hemisphere great heat is communicated to the desarts of Africa, the consequence of this greater heat acquired in the sands of these desarts than in the seas which lie to the east and north east of them, is a rarefaction in the columns of air incumbent on them, and therefore a tendency in the adjacent columns which are more moderately heated to flow in and displace the heated air; this changes the direction of the wind. These periodical winds are called monsoons; and on the north side of the equator, in the Arabian and Indian seas, it is north west during the summer months, from April to October, and in the opposite direction, or south east, during the winter months; on the south side of the equator it is the direct contrary, being north east in summer, and south west in winter. In order that the equilibrium between the parts of the atmosphere may be preserved, it is necessary that in the upper regions of the atmosphere there should be a perpetual current towards the poles. As these streams, after they pass the tropics, descend towards the surface, with the celerity due to the equatorial regions, they will appear to blow to the west with the excess of their previous velocity over that of the parallel which they reach. This is the reason why, in places above the latitude of 30° the prevailing wind is westerly, and this is also the reason why westerly winds are generally warm, as coming from a warmer region; and on the same principle the north and east winds are cold, as they originate in regions nearer to the arctic circle.

Jupiter's atmosphere must be much more agitated than ours is by the moon, from the joint attractions of the *four* satellites; however, the effect of the sun's action cannot be so considerable, in consequence of its much greater distance.

NOTES TO CHAPTER XIII.

(*a*) THERE are two cases in which there would be no precession of the equinoxes, namely, first if the earth was a perfect sphere; in which case the solar force, on any particle in the hemisphere turned towards the sun, which is proportional to the distance of the particle from the plane of the circle of light and darkness, is equal and contrary to the force by which similarly situated particles in the opposite hemisphere are drawn; therefore the solar forces in the opposite hemispheres balance each other; see page 407, from which it is evident, that the mean quantity of the solar force is $\frac{3S}{a^3}. r. \cos. \delta$, where S denotes the mass of the sun, a the mean distance of the sun from the earth, r the radius of the equator, and δ the declination of the sun. The part of this force, which is perpendicular to the plane of the ring $= \frac{3S}{a^3}. r. \sin. \delta. \cos. \delta.$; or secondly, if the axis of the earth was always perpendicular to the ecliptic, in which case the action of an external body would be absolutely equal on the two parts of the spheroid above and below the ecliptic, therefore it would not produce any alteration in any of its motions. Since to each of the moons mentioned in the text we can apply what has been stated respecting the lunar orbit, which, in consequence of the solar action, intersects at each of its revolutions the plane of the ecliptic in a point anterior to that in which it met the ecliptic at a previous revolution, which causes the lunar nodes to retrograde; in the very same manner, each point of the ring intersects the ecliptic in a point anterior to that at which it had intersected it twenty-four hours before; and from the action of all these moons on the globe of the earth, there will result every day a small retrogradation or angular motion of the intersection of the equator and ecliptic, which, on account of the rapidity of the earth's revolution, and the greatness of its mass relatively to that of the ring, must be very small; however, as this retrogradation is repeated 365 times in the course of the year, there results at the end of the year a retrograde motion of several seconds, produced by the sole action of the sun. The part of the solar action which is perpendicular to the ring $= \frac{3\,S}{a^3} \cdot r.$ sin. δ. cos. δ. Hence, if $F = \frac{S}{a^2}$ the force with which the sun acts on a particle at the centre of the earth, and if t T represent the times of the diurnal and annual revolutions of the earth, and e the centrifugal force we have

$$F : e :: \frac{a}{T^2} : \frac{r}{t^2} \text{ and } \therefore F = \frac{e.t.^2 a}{T.^2 r}$$

and the part of the solar action perpendicular to the plane of the ring

$$= \frac{3t^2}{T^2} \cdot \frac{e\,a}{r} \cdot \text{sin. } \delta. \text{ cos. } \delta.$$

(b) Besides the motion round the line of the nodes which the force $\frac{S}{3\,a^3} \cdot r$ sin. δ. cos. δ has a tendency to produce, the ring in twenty-four hours revolves on an axis perpendicular to its plane; therefore, since these two forces act on it simultaneously, the consequence will be, that the ring will neither revolve on this axis, nor on the line of the nodes, but on an axis which lies in the same plane with each, dividing the angular distance between them in such a manner that the sine of the angular distance between them is inversely as the angular velocity about that axis; for the composition of angular motions follows the general law of the composition of forces; but as the axes are perpendicular to each other, the sine of the angular distance of the new axis from the line of the nodes is equal to the cosine of the angular distance of the first axis of rotation from the second. Hence, if $\pi\ \tilde{\omega}'$ represent the angles which, in consequence of the earth's rotation and of the solar force, the equator and axis of the earth describe in an indefinitely small portion of time, the axis of the earth will be changed by the simultaneous action of the two forces, by an angle of which the tangent $= \frac{\tilde{\omega}}{\pi}$; but these forces are not of the same kind, for that which produces $\tilde{\omega}'$ acts incessantly, while the other acts only once; hence it follows, that as the quantity $\tilde{\omega}$ is continually renewed, the position of the earth's axis is continually changing. However, though this axis is continually shifting its position, neither the angular velocity of the axis or its inclination would undergo any change, if $\frac{3\,S}{a^3}\,.\,r.\cos.\,\delta.$ was constant. For if $\tilde{\omega}$ as before, represents the angular velocity of a body, and if $\frac{3\,S}{a^3}\,r.\ \cos.\ \delta$ would generate in 1″ an angular velocity $= a$, then if 1″ be divided into n parts, the velocity produced in each of these parts $= \frac{a}{n}$, hence, from what has been just stated, by compounding the angular velocities $\tilde{\omega}$ and $\frac{a}{n}$, of which the axes are at *right* angles to each other, the resulting angular velocity $= \sqrt{\tilde{\omega}^2 + \frac{a^2}{n^2}}$, compounding this with the angular velocity generated in the second, third, fourth, &c. intervals, the compound angular velocity becomes

$$\sqrt{\bar{\omega}^2+\frac{2a^2}{n^2}}\ \sqrt{\bar{\omega}^2+\frac{3a^2}{n^2}}, \text{ \&c. } \sqrt{\bar{\omega}^2+\frac{na^2}{n^2}} =$$

$$\sqrt{\bar{\omega}^2+\frac{a^2}{n}} = \text{(when } n \text{ is indefinitely increased) } \bar{\omega},$$

hence it follows, that when the axis of rotation is *at right angles* to the axis about which $\frac{3S}{a^3} r.$ sin. δ cos. δ has a tendency to produce a motion of rotation, the angular velocity is uniform, when $\frac{3 Sr \cos. \delta}{a^3}$ is *a uniform force.* Neither is the inclination to the line of the nodes altered. For suppose this force to generate an angular velocity a in 1″, if this time be divided into n parts, then $\frac{a}{n}$ will be the velocity generated in each of them; consequently, from what has been just established, it follows, that the tangent of the angle contained between two successive positions of the axes of rotation $= \frac{a}{n\bar{\omega}}$, which, when n is increased indefinitely, is the expression for the arc between them, and since, by what preceds, $\bar{\omega}$ remains constant, at every successive interval, angles $= \frac{a}{n\bar{\omega}}$ will be added to this angle; therefore, at the end of 1″ the two axes will be inclined at an angle $= \frac{na}{n\bar{\omega}} = \frac{a}{\bar{\omega}}$, and as this obtains for each successive interval $=$ 1″, the axis of rotation will shift its position with an angular velocity $= \frac{a}{\bar{\omega}}$, and as the angle $\frac{a}{\bar{\omega}}$ is very small, the axis of rotation at the end of 1″ will deviate from the solstitial colure by an angle which is indefinitely small with respect to $\frac{a}{\bar{\omega}}$, therefore this axis will describe a circle round the pole of the ecliptic, moving *in antecedentia* with an angular velocity equal to that of the line of the equinoxes. This would be the case if $\frac{3S}{a^3} r. \sin. \delta \cos. \delta$ was constant, which, however, is not the case, for it is 0 at the equinoxes, besides, as the arc described by the pole is $\perp$ to the plane passing through the sun and the earth's axis, it is not always in the direction of a tangent to the circle whose centre exists in a perpendicular to the plane of the ecliptic; hence, strictly speaking, neither the angular motion of the pole of the equator, nor its inclination to the ecliptic, is invariable, however, the changes are confined within very narrow limits. This is the cause of the solar inequality, of precession, &c.

The decomposition of motion adverted to in page 184, is, in fact, an application of the principle of D'Alembert, explained in page 287, Vol. I.

(*c*) Differentiating the expression $\frac{3t^2}{T^2} . \frac{ca}{r} . \sin. \delta \cos \delta$. with respect to t and δ, and then integrating, the precession for the entire year, comes out $= 360. \frac{3t}{2T} . \frac{ca}{r}$. cos. of obliquity, it appears from this expression, that in order to obtain the exact quantity of the precession, we should know the compression of the earth.

(*d*) It appears, from what has been already stated, that (every thing else being the same) the retrogradation is proportional to the cosine of the inclination of the plane of the ring to that in which the external body moves; and as, in the case of the moon, this inclination is continually varying, the precession and inclination of the axis is subject to continual change from the lunar action. It also follows, from this, that the greatest inclination of the ecliptic to the equator is in the new moon of spring, and the full moon of autumn, the moon being at the same time in its ascending node; or in the full moon of spring, and new moon of autumn, the moon being then in the descending node. The least obliquity has place in the first and last quarter, at the beginning of summer or winter, the moon being at this time 90° from her node.

(*e*) Strictly speaking, the inequalities produced by the action of the moon are of two kinds, the period of the first being equal to that of the moon in her orbit, and that of the second equal to the time of a revolution of the moon's nodes. Hence it follows, that there are limits within which the variations of the precessional motion and obliquity of the ecliptic are contained; the inclination to the ecliptic returning to its former value in the time of a revolution of the moon's nodes.

(*f*) Subtracting the expression for the lunar precession from the entire annual precession produced by the combined action of the sun and moon, we obtain the ratio of the solar annual precession to that of the lunar; which, as it involves the ratio of the sun's to the moon's mass, enables us to determine the relative proportions of these qualities.

In reference to what is stated in page 189, it is to be remarked, that the cause of D'Alembert's error arose from his supposing that as the molecules of the sea, with which the earth is in a great measure covered, yield to the action of the stars, they could not contribute to the motions of the earth's axis, so that, in computing those motions, he employed the ellipticity of the spheroid, which was covered by the ocean, which ellipticity he supposed to be less than that of the surface of the sea. But Laplace, by subjecting to analysis the oscillations of the fluid spread over the terrestrial spheroid, and also the pressure which it exerts on the surface of the spheroid, proved that this fluid transmits to the terrestrial axis the same motions as if it constituted a solid mass with the earth. He also, by means of the principle of the conservation of areas, showed that the action of the stars on the sea, in whatever manner it was spread over the spheroid, produced on the nutation and precession the same effects as if the sea consolidated itself about the spheroid.

(*g*) The theories of Newton relatively to the figure of the earth and the seas, are those which suppose the earth homogeneous, the sea having the same density as the earth which it covers, and that the waters of the ocean assume every moment the figure in which they would be in equilibrio under the action of the sun.

(*h*) The effect of the action of each of the planets is to induce a motion of the common section of the planes of the two orbits of the earth and planet, while their mutual inclination is not altered; see page 25, Vol. II. In the case of precession and nutation, the variation is in the equator and earth's axis; but in this case the variation is in the ecliptic, to which the axis is referred. Laplace proved, by a careful analysis, that if the earth was perfectly spherical, the variation of the obliquity of the true ecliptic to the equator, which is caused by the attractions of the planets, would be much more considerable than they are, and from the same cause the variation of the length of the tropical year, which would be caused by the sole motion of the ecliptic, is reduced to a fourth of what it would be if the earth was a sphere. However, the *sidereal* year remains invariable.

(*i*) If the right ascension of a star reckoned from the true equinox be converted into time, it will be expressed by two terms, one of which gives the mean rotation, and the other is the correction, which is variable; this would seem to imply that the rotation of the earth was variable. However, as has been remarked, this is only an illusion, for the term which is added to the mean rotation, is independent altogether of the motion of the earth on its axis.

It follows, ∴ from the rotation being given by one sole term, compared with what is stated in Vol. I., page 465, that if the earth revolves on a principal axis the rotation is perfectly uniform. Even if the axis of rotation was not a principal one, still the actions of sun and moon would not affect its motion, as appears from what is just stated; but in this case, in consequence of the centrifugal forces, the rotation cannot be uniform. However, as from the observation of a long series of years, no irregularity has been discovered in the rotation of the earth, we must conclude that it revolves about a principal axis, which is confirmed from a consideration of the variations of the obliquity and of the precession which result from it; for if the axis of rotation deviated $1''$ from the principal axis, the obliquity and precession, or what is the same thing, the latitudes and longitudes of the stars would experience, in the course of six months, variations of $2''$ and $5''$, which would be indicated by observations. The uniformity of rotation is likewise proved from the following consideration, namely, that if it was deranged by the actions of the sun and moon, the centrifugal force which depends on the rotation, would experience, in the course of a month and year, variations depending on the different positions of the sun and moon; therefore the gravity which is diminished by the centrifugal force, and consequently the length of a pendulum which vibrates seconds, would be liable to analogous variations; but no change has been observed in the length of a pendulum vibrating seconds under a given latitude.

(*k*) When a body descends from a considerable height, or moves from the equator towards the poles, it brings into its new situation more velocity than it can retain, consequently it must impart some of it to the general mass of the earth; the contrary obtains when a body recedes from the axis of the earth. In general, the momentum of rotation of the entire mass of the earth is to the change of the momentum of rotation of the displaced body, as the velocity of diurnal rotation to the variation in that velocity, arising from the motion of the body. In this way, the continual degradation of mountains and alluvial deposits produced by rain, &c. which is incessantly going on, should cause an increase in the length of the day.

In addition to what is stated in page 194, it may be remarked, that there is a general compensation of the effects produced by the current of air from the poles to the equator, (which is the cause of the trade winds,) which tends to diminish the motion of the earth, by a contrary current in the upper regions of the air, which sets in from the equator to the poles.

Some geologists maintain, that the level of the sea was once 15000 feet higher than at present, from which it follows, that a mass equal to the 440th of the whole earth must have been degraded from being above the level of the present sea, to being underneath it; and if the density of water was equal to the mean density of the earth, it would be easy to show that, in consequence of this degradation, the duration of a revolution on the earth's axis must have been deminished by 5′,682; see preceding page. As, however, the mean density is to that of water as 4.71 : 1, this acceleration is reduced to 1.′12″ This change on the surface, or even in the interior of the earth, would also produce great changes in the position of the axis of rotation; it may, if an explosive force existed in the interior of the earth, as was suggested in notes, page 470, have changed by the action of such a force continually its position, and with it that of the earth's equator; and that such a force was formerly in very active operation, appears to be indicated by many facts in the natural history of the earth, and of the mineral kingdom.

NOTES TO CHAPTER XIV.

(*a*) What is termed the mean axis in the text, is the second principal axis of rotation; and it is shown, in the Celestial Mechanics, Vol. II., page 370, that if the moon was homogeneous, the excess of the first above the second is to the excess of the second above the third, as 40 : 10, *i. e.* 4 : 1. But as the ratio of these axes deduced from substituting numerical values for the terms of the proportion, and of the principal moments referred to them, does not agree with observation, it follows that the moon is not homogeneous: see note (*e*).

(*b*) Laplace obtained the difference between the motion of rotation and revolution of the moon, by the integration of a differential equation of the second order; this quantity is composed entirely of *periodic* terms, and contains two constant arbitrary quantities; therefore, there results from this a libration, of which the extent is also arbitrary. Hence it follows, that the *mean* motion of rotation of the moon is equal to her mean motion of revolution. The difference between the motions of rotation and revolution should be comprised between the greatest and least of the values of which the periodic quantity was susceptible. It is necessary, to secure the stability of equilibrium, that the periodic terms which multiply the time should be real; for if they were imaginary, the arguments which depend on them would be changed into exponentials and arcs of circles susceptible of indefinite increase, or at least the slightest cause might produce it. The condition of this reality requires that the greatest of the principal axes be directed to the earth. According to the theory of probabilities, the chances are almost that of infinity to one that this alone, of all possible cases, namely, the equality of rotation and revolution, should accurately obtain at the commencement; and even if it did, it would not continue long, if the moon was not a perfectly homogeneous sphere; for the attraction of the earth would derange its rotation, and besides the mean motion of the moon, to which it is supposed to be equal, is not perfectly invariable. The perturbations of the motion of revolution depend on the action of the sun and of the planets on the centre of gravity of the earth, while the perturbations of rotation result from the action of the earth on the elongated figure of the moon, and it is impossible that these two actions, produced by different bodies, and having different arguments, should be perfectly equal.

In general, it may be remarked, that the librations of the moon are of two kinds, the first arising from inequalities in its motion, the second from the attraction of the earth; the first cause gives the apparent or optical libration, the second gives the real or mechanical one. The libration in latitude is altogether optical, that in longitude is partly optical and partly real; it is the most considerable libration.

(*c*) It is not at all probable that these nodes coincided accurately at the commencement, no more than that the motions of rotation and revolution were perfectly equal; but a consideration of the arbitrary inequalities introduced by the integration of the second differences, shows, that if the difference between their positions was originally inconsiderable, the terrestrial attraction would establish and maintain the coincidence of their mean nodes. From the expressions previously established, three conditions are given relative to the moments of inertia of the lunar spheroid; and by a comparison of them with those *furnished* by the theory of the figure of this spheroid, it appears that these conditions cannot be satisfied by supposing the moon homogeneous and fluid, nor on the hypothesis that it is originally fluid, and of a variable density; hence it follows that the moon has not the figure which it would have, if it was primitively fluid, consequently it must have been at its origin a hard body of irregular figure, which is confirmed by a consideration of its spots. Newton determines the ratio of the greater to the lesser lunar axis, (on the supposition that the moon is fluid,) from knowing the height to which the sea is elevated by the lunar action; for the force of the earth to raise the lunar fluid is to the corresponding force of the moon to raise the waters of our ocean, in a ratio compounded the accelerating gravity of the moon to the earth to the accelerating gravity of the earth to the moon, and of the diameter of the moon to that of the earth, which by substituting numerical values become the ratio of 1081 to 100; and as the tide by the lunar action alone is raised $8\frac{3}{5}$ feet, the lunar fluid ought to be raised 93 feet, $\therefore$ the major axis should exceed the minor by 186 feet, and as the lunar equator is inclined at a very inconsiderable angle to the plane of its orbit, the effect of the rotation on its axis ought to increase this excess.

(*d*) See notes page 435.

(*e*) The position of this meridian being determined, we are enabled to establish every circumstance connected with the moon's rotation, from a computation of a great number of longitudes and latitudes as seen from the centre of the moon, by means of observations of a spot made at different epochs, it is found that these longitudes and latitudes differ from each other, and vary with the time; hence it follows that the moon revolves on an axis inclined to the ecliptic. As a comparison of the latitudes indicates but inconsiderable changes, the axis of rotation does not differ much from that of the ecliptic, i. e. the lunar equator is inclined at an inconsiderable angle to the ecliptic.

NOTES TO CHAPTER XV.

(*a*) According to what is established in page 290, vol. X, it follows that the common centre of gravity is either in perfect repose, or has a uniform rectilinear motion in space. But there is *only one case* in which the centre would remain in perfect repose, while there is an infinite number in favour of a motion in some one direction or other with some determinate velocity; it is ∴ much more probable that our sun and the fixed stars, which are bodies of the same nature, have a proper motion in space, than that they are absolutely at rest. With respect to the sun, its motion of translation may be, with great probability, inferred from its rotatory motion; it is likewise probable for the stars, as will appear from the following note.

(*b*) Herchell found, that if we suppose the sun to be in motion towards that region of the heavens in which the constellation Hercules is situated, there should arise a separation between several stars situated on that side, while, on the other hand, there would arise a contraction between those which are situated on the opposite side; and he found, that out of forty-two stars which appear to have experienced particular motions, there were upwards of thirty, *part* of whose motions corresponds to what should result from the motion of our sun towards the one, and from the other. He specified that only *part* of their motions arose from this, for as the stars have proper motions of their own in different directions, it is evident that their apparent motion results from their true motion combined with that of the sun. It is evident from the principle of universal gravitation, adverted to in the text, that the stars, which we may consider as the centres of so many different systems, must revolve about some common centre, for otherwise, as they exert attractive forces on each other, they must tend to approach towards each other; and though in consequence of their immense distance, this tendency may be extremely feeble, still as it would be caused by a motion continually accelerated, after a great lapse of time, they would all meet *in the common centre* of gravity. See page 446, vol. 1.

When the proper motion of the star is in an opposite direction to that of the sun, it is in the most favourable circumstances to be be observed, for in that case the apparent motion is = to the sum of these two motions. Suppose, then, that a star = to our sun moved with an = and contrary motion, they will be at the same distance from the centre of our system, and the apparent motion from the sun, considered as immoveable, will be double of the true motion, hence A the arc described in any time = half f, the distance of the star from the sun multiplied into ϕ, the apparent motion of the star in that time, i. e. $A = \frac{f \cdot \phi}{2}$, in 50 years ϕ is observed to be $= 45''$, and $\therefore$ in one year $\phi = 0'',9$ and $\frac{\phi}{2} = 0'',45. = 0,00000218166$, which multiplied by f (= 300000 semidiameters of the earth's orbit) gives $A = 0,654$, in one year, but an arc of the earth's orbit = 0,654 subtends an angle at the sun = 37°, 30′, which is described by the earth in 38 days $\therefore$ the velocity of the star will be to that of the earth inversely as the times *i. e.* :: 38 : 365 $\therefore$ 1 : 9; now it appears from what is stated in Notes, page 371, that the velocities of bodies which describe circles about a common centre, are inversely as the square roots of their distances from the centre, ∴ if the star revolved about the sun in virtue of the central force of the sun, its velocity would be $\frac{1}{\sqrt{f}}$ $= \frac{1}{648}$ of that of the earth, *i. e.* 61 times less than the actual velocity, it is ∴ 61 times more than it would be, if it depended on the central force of the sun; hence we must conclude, either that it results from a central body, of which the mass is much $>r$ than that of the sun, or to which the star is much nearer than to the sun; as the last case is extremely improbable, it follows that the stars revolve about a central body of which the mass is much greater than that of the sun.

(*c*) It was proposed by means of these double stars to find the parallax of these stars, for as the spaces which intervene between them appear to enlarge according as the earth in its annual route approaches nearest to them, and therefore ought to be least six months afterwards, when the earth is at the greatest distance from them, Herchell made a great number of observations in order to deduce their relative parallax, supposing that the least star is the most remote, and as they have the same longitude, latitude, altitude, &c., they are all affected by refraction, aberration, &c. in the same manner, so that their relative position and apparent distance is not at all changed.

(*d*) Bessel announced in 1812, that a consideration of the observations of Bradley proved, that the double stars constitute a particular system by themselves. Several stars of this kind evince by their motion a mutual dependence on each other, particularly the two stars mentioned in the text. This system of two stars moves with considerable velocity, they seem to be connected with each other by the law of attraction, and in 60 years they appear to have described a considerable part of their orbit round the common centre of gravity.

(*e*) From a consideration of the observations of previous astronomers he inferred the position for the year 1800, also the annual motion, the time of revolution, which he thought to be = 400, the semiaxis major which he assumed = 25″, and the annual parallax = 0″,46: it is evident from the formulæ previously established in page 374, that if the axis major and period are known we can obtain the ratio of the sum of their masses to that of the earth.

Note to page 205.

(*a*) It thus appears that the laws of motion and general properties of matter are the same in every part of the universe, and that all are explained by the *one* principle of the mutual gravitation of bodies; it is likewise evident that the existence of this force was not hypothetically assumed, but was deduced as a necessary consequence of the laws of Kepler, combined with the laws of motion.

NOTES TO CHAPTER I. BOOK V.

(*b*) In fact, on the supposition that the Zodiac originated in Egypt, and that it was first invented in order to serve as a sort of kalendar to point out the different circumstances of the rural year, then there are two ways of reconciling what is indicated by the signs of the Zodiac with the climate of Egypt and its agriculture, either by making the Zodiac to have originated at a period long anterior to that at which it is at present supposed to commence; or by supposing that the constellations of the Zodiac are named, not from their rising with the sun, or the commencement of the day, but from their setting, or the beginning of night. The former hypothesis would make the world to be created at a time long anterior to that which we know from all history both sacred and profane, and also from contemporary records, it actually was; besides it would assign to the human race a duration longer than what Laplace himself admits it had, see page 50. Likewise, it may be remarked, that in these rude times, when the observations of the stars were made by the naked eye, it is much more likely that the stars were observed at night, when they are easily seen, and not in the day-time, when they are with difficulty discerned; the latter then is the true mode of reconciling the names of these signs with the different circumstances of the year: indeed on the first hypothesis if we consider the positions which the signs are observed to have in the Zodiac, their names do not indicate any thing connected with the climate of Egypt; for if Capricornus was originally at the lowest point, then the sign Virgo, representing a gleaner, could not indicate the harvest; for three thousand years ago, the principal star of this sign rose for Memphis, 45 days after the summer solstice, and set about 15 days before the autumnal equinox, during which time Egypt was inundated by the Nile. Besides, Capricornus is represented half goat and half fish, and Aquarius, in the most ancient Zodiac, by a simple urn. The sign Pisces, from their very denomination, can only designate the rainy season and an abundance of waters, and notwithstanding all this, the principal stars belonging to these constellations, rise and set heliacally at the very time Egypt is most dry. But according to the second of the preceding suppositions, there is a striking correspondence between these signs and the different circumstances of the Egyptian year, for then Aries is placed at the autumnal equinox, and Libra at the vernal; Capricornus at the summer solstitial point, then Aquarius, and after them Pisces; the Nile begins to rise in June, or a little before it; now this phenomenon, combined with the motion of the sun, through the highest point of his course, could not be better indicated than by an animal half a fish and half a quadruped, remarkable for seeking always the highest points of the mountains. The months of August and September, during which Egypt is overflowed, could not be better designated than by the Urn and Pisces. To these signs succeeds Aries, symbol of the reviving of nature, which excites animals to reproduction: the Bull, emblematic of labour, which in Egypt begins in November; after this sign comes that of Twins, which is a symbol of the regeneration of all natural productions. The return of the sun or its retrogradation, was thus represented by the Cancer, which is vulgarly supposed to march backwards. The Lion, which succeeds the Crab, might denote the reviving force of the Sun, and the sign Virgo or the gleaner, denotes the harvest, which in Egypt takes place in February and March. For the remaining three signs, Libra, the balance, denotes the return of equal day and night. The Scorpion, the maladies caused by the southern winds in the following months. Sagittarius, the season for hunting, &c.

Now that the first point of Aries should be at the autumnal equinox, we should go back 12960 years; for in that time 180° will be performed very nearly at the rate of a degree every 72 years; to this is to be added, the quantity by which the first point of Aries has retrograded since the time of Hipparchus, which would make up the era 15000 years anterior to the present time; but on the hypothesis that the signs were denominated from their setting with the sun; then we would have only to go back to the era usually assigned to the commencement of the Zodiac, and at the same time retain the correspondence between the signs and the seasons, and the circumstances of the year.

(*c*) If the revolution of the sun be supposed equal to $365^{d}\frac{1}{4}$ or $365\frac{235}{940}$, as the lunar year $= 354\frac{348}{940}$, their difference is $10^{d}\frac{887}{940}$, which multiplied by 19, gives $206^{d}\frac{673}{940}$ which = seven lunar months, each of which consists of $29^{d}\frac{499}{940}$.

It was by this period that the Chinese reckoned the years of the empire, and of the emperors who reigned over them; and this period of 60 years corresponds to our century or period of 100 years: each of the years of this period has a particular name composed of two terms, which are applied in the following manner; there are two series of terms, one consisting of ten and the other of twelve terms; the first of the one are combined with the first of the other, so that as one series has ten terms, and the other twelve, after the first series is exhausted, its first term is combined with the eleventh term of the second series, and the second term of the first series with the twelfth term of the second series, and this goes on until the first term of the first series concurs with the first term of the second series; but this, as is evident from the theory of combinations, does not take place until after sixty different combinations with respect to the days; the first day of each year bears the name of the year, after which we reckon them by the names composed of the sexagenary period which is recommenced whenever it is necessary.

The Luni Solar period of 600 years, to which we adverted in page 68, was invented by the Chaldean astronomers. This supposes a tolerably accurate knowledge of the solar year, and also of a lunation; for in 600 years, each consisting of $365^d\ 5^h\ 51', 36''$ there are exactly 7421 lunations, each of which consists of $29^d\ 12^h\ 44'\ 3''$, but if the motions of the sun and moon were the same then as at the present day, at the end of this period there would be a considerable aberration.

(*d*) Such an exact situation of the pyramids could not be the effect of chance; we infer from it that they had accurate means of finding the meridian line, which is extremely difficult to trace accurately, as is evident from the error which Tycho Brache committed, in tracing the meridian line at the observatory of Uraniburgh. According to some historians, the Pyramids were observatories from which the Egyptian priests surveyed the heavens.

(*e*) The rising and inundation of the Nile, an event which excited the attention of all Egypt, was at the commencement of this empire announced by the Heliacal rising of Sirius. It is probable that it was on this account that they made their years to commence then, which, according to their estimation of the length of the year, made its commencement continually to retrograde, so that if it commenced for any one year at the summer solstice, four years after it would commence a day sooner, on the hypothesis that the true length of the year exceeded 365^d by the fourth part of a day; in this way the commencement of the year would retrograde continually, and in 1461 years take place at every season of the year, at the end of which time it would recommence at the summer salstice, for $\frac{1461}{4} = 365 + \frac{1}{4}$.

(*d*) *to page* 219.] For $\frac{365}{7} = 52 + \frac{1}{7}$, hence it appears that the last day of the year is of the same denomination as the first, and ∴ if the first day of the week denotes the first year, the second day of the week will represent the second year, and so on.

(*e*) In fact, previously to the time of Thales, who derived all his information on these subjects from the Egyptian priests, their astronomy consisted only in having given denominations to some constellations, and in having noted the heliacal rising and setting of certain stars. This is all which is furnished by Hesiod and Homer, their most ancient writers. Thales, at his return from Egypt, made them acquainted with some of the important astronomical truths known to the ancients.

NOTES TO CHAPTER II.

This constancy of the inclination of the lunar orbit to the plane of the ecliptic, adverted to in page 238, was remarked by Kepler at the conclusion of his Epitome of the Copernican Astronomy; but the reason which he assigned for it was very remarkable. "It is agreed," says he, "that the moon, a secondary planet and satellite of the earth, is inclined at an invariable angle to the plane of the earth's orbit, whatever be the variations which this plane experiences in its position with respect to the fixed stars; and if ancient observations on the greatest latitudes of the moon, and on the obliquity of the ecliptic are irreconcileable with this hypothesis, it should be rejected sooner than call them in question." Here the reasons of suitableness and harmony have conducted Kepler to a just result; but how often have they bewildered him; when we give ourselves up to imagination and conjecture, it is only by a lucky chance that we can light on truth; but the almost total impossibility of arriving at it in the midst of the errors with which it is almost always encumbered, ought to induce us to ascribe all the merit of its discovery to him who establishes it solidly by observation and computation, the sole bases of human knowledge.

(a) Knowing the duration of a total and central eclipse of the moon, and also the periodic time of the moon, the angle which the semisection of the shadow subtends at the earth is known; hence, as the apparent diameter of the sun, and also the horizontal parallax of the moon are known, we obtain an expression for the horizontal parallax of the sun.—See Brinkley's Astronomy, page 255.

NOTE TO CHAPTER III.

(*a*) A celebrated peripatetic philosopher, John, surnamed the Grammarian, who was in high favour with the Saracen general who took the city, requested as a present the royal library. The general replied, that it was not in his power to grant such a request without the knowledge and consent of the Caliph; he accordingly wrote to Omar, who was then Caliph, and the answer has been given in the text. This account is, however, now doubted.

NOTES TO CHAPTER V.

(*a*) The very first applications of analysis to the motions of the moon furnish an example of this superiority. For they give with the greatest ease not only the inequality of the variation, which is obtained with the greatest difficulty by the synthetic method, but likewise the evection which Newton did not even suppose was caused at all by the law of gravity. It would certainly be impossible to obtain by means of synthesis, the numerous lunar inequalities, the values of which, determined by analysis, represent observations as exactly as our very best tables, which are formed by combining an immense number of observations with theory.

(*b*) The endeavours of geometers to demonstrate Euclid's twentieth axiom about parallel lines, have been hitherto unsuccessful. However no person questions the truth of this axiom, or of the theorems which Euclid has deduced from it. The perception of extension contains ∴ a peculiar property, which is self-evident, without which we could not rigorously establish the doctrine of parallels. The motion of a limited extension, for example of a circle, does not involve any thing which depends on its absolute magnitude; but if we conceive its radius to be diminished, we are forced to diminish also in the same proportion its circumference, and the sides of all the inscribed figures.

This proportionality was, according to Laplace, an axiom much more obvious than that of Euclid. It is curious to observe, that agreeably to what is stated in page 322, this axiom is pointed out in the results of universal gravitation.

NOTES TO CHAPTER VI.

(*a*) There are altogether forty-three motions in the same direction in our system, which have been recognized, and by the application of the analysis of probabilities which has been adverted to in note (*l*), page 498, it is found that there is more than four thousand million of millions to one, that this disposition is not the effect of chance.

(*b*) It is clear that if the inclinations of the orbits of a comet to the plane of the ecliptic, be supposed to increase insensibly from cypher, after it exceeds a right angle, its motion will be in a contrary direction from what it was previously to its attaining 90°.

(*c*) See chapter II. of this volume, and also notes, page 390.

(*d*) See notes, page 521.

(*e*) At present the orbit described by each of the satellites in space, is a species of epiclyoid resulting from the twofold motion with which the satellite is actuated, namely, its own motion about Jupiter, with which is to be combined the motion of Jupiter about the sun, in which also the satellite participates, and the result of the two, is evidently a species of epiclyoid; if, however, the action of Jupiter ceased suddenly, then each satellite would be under the immediate influence of the sun's action, and the species of conic section, which it would describe, would depend on the ratio which the velocity by which it was actuated bore to the velocity in a circle at the same distance.—See notes, page 376.

(*f*) See the sixth and last note. seventh

(*g*) It is worthy of observation, that this Scholium was not published with the first edition of the Principia; up to that time Newton was only engaged in mathematical investigations, and, according to Laplace, he would have consulted more his own glory had he always confined his attention to those sciences.

THE END.

ERRATA.

Page	37	line	14	*after the words* may be *read* also.
——	86	——	5	*from bottom, for* millioneth *read* thousandth.
——	157	——	9	*for* solstices *read* quadratures.
——	175	——	6	*for* one *read* nine.
——	201	——	5	*for* millioneth *read* thousanth.
——	236	——	17	*for* their *read* its.
——	300	——	18	*after* fall *read* through.
——	330	——	11	*for* each *read* some.
——	347	——	11	*for* Albatenus *read* Albatenius.
——	ib.	——	16	*for* Strato *read* Strabo.
——	466	——	9	*for* principal *read* principle.