Geodesic Dome Angles

Introduction

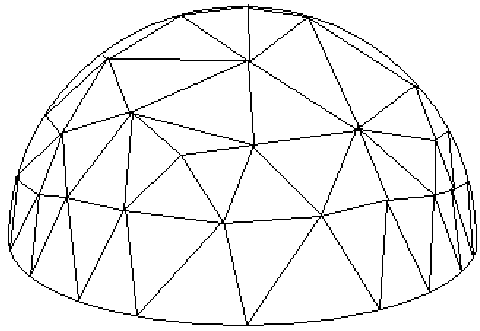


Illustration 1: Randomly framed dome

There are any number of ways of framing a dome. All you need to do is think of a random bunch of dots on the surface of a hemisphere and start joining them up into a network of triangles. Thing is you end up with all different shapes and sizes of triangles and if you wanted to build the dome and not have it collapse you would have to make some of the struts and hubs very substantial to take the irregularly distributed stresses. Geodesic domes do well because there are only a few different shapes of triangle of much the same size, and their arrangement distributes stress rather well so the dome frame is about as slight as it can be.

The Geodesic Dome

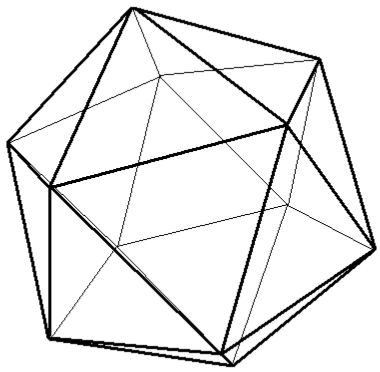


Illustration 2: 20-Faced Icosahedron

To work out how to frame a geodesic dome it helps to understand what a geodesic dome is. It starts off as a platonic solid, of which there are five: the familiar cube, the dodecahedron with twelve pentagonal faces, and the tetrahedron, octahedron and icosahedron with, respectively four, eight, and twenty triangular faces. Domes built on the icosahedron distribute their stresses well and have the smallest variety of frame shapes and sizes.

Subdividing the Faces

Working out the angles and strut lengths is basically the same for all the solids. A Geodesic dome approximates a sphere, so think of a sphere around the icosahedron with all twelve of its vertices touching. An icosahedron is not particularly spherical because each of its twenty faces is flat, but subdivide each face and project the new vertices out to the enclosing sphere, and the more you subdivide the primitive face the more spherical it becomes.

If the basic icosahedron is a 1-frequency, a dome built with a single subdivision of the basic face into four faces is called a 2-frequency dome, nine subdivided faces is a 3-frequency, and so on, and the nomenclature used is 2*v*, 3*v*, etc, where the *v* is really a Greek nu which is the symbol sometimes used in physics for frequency.

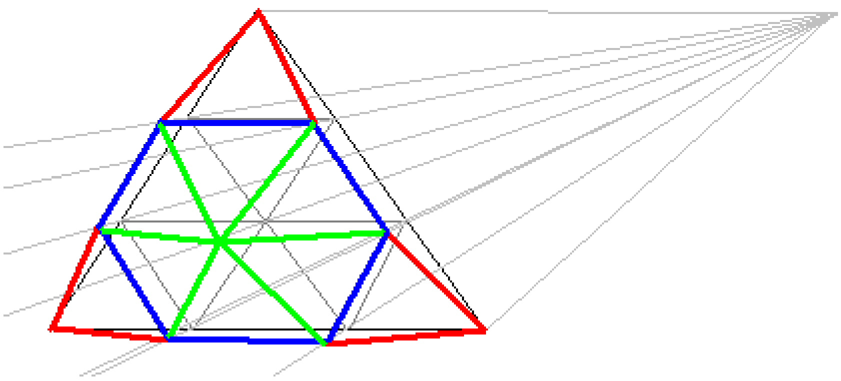


Illustration 3: Projecting the Subdivided Faces

Incidentally, it's called a geodesic dome because a geodesic, the shortest distance between two points on the surface of a sphere, is an arc of a great circle, that is, a circle with the sphere's diameter, and the straight lines of subdivided vertices on the face of the icosahedron lie on a great circle when they are projected out to the sphere, a result of them being projected from the icosahedron's center.

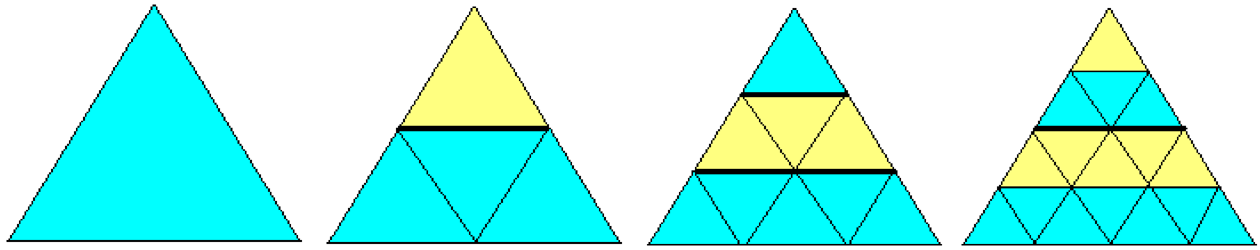


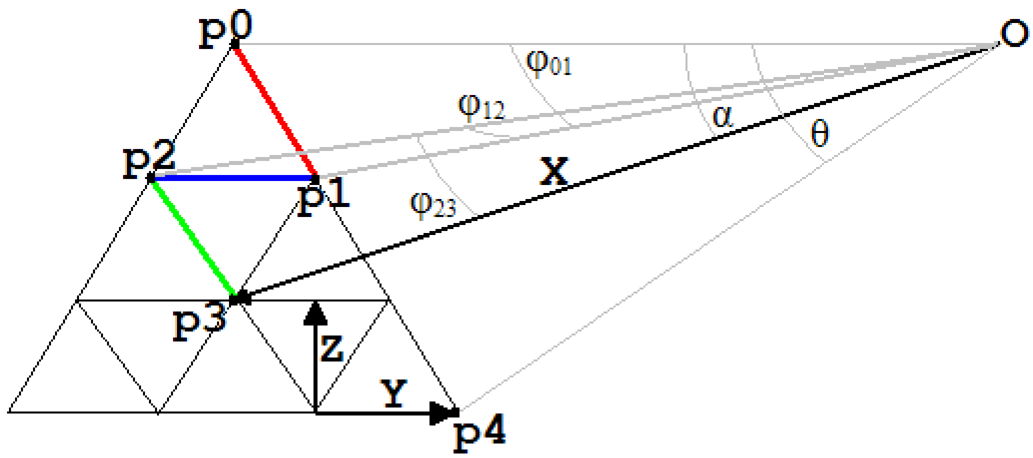
Illustration 4: 1v, 2v, 3v, and 4v subdivided primitive icosahedron faces

Notice that for odd-frequency subdivisions the primitive triangular face is divided into an odd number of rows of triangles, so for example the 3*v* dome has its basic face divided into three rows of one, three and five triangles. Because of the vertices on the mid-line, an even-frequency dome will split exactly into hemispheres, and because the vertices on the mid-line that form the base of the dome are all projected onto the same equatorial great circle the base of the dome is flat. This isn't so with the odd-frequency domes which split either above or below the central band of subdivided triangles. The result is that the base of an odd-frequency dome ripples as the base vertices ride up and down their great circles, all at a slight angle to the horizontal, most noticeably for the 3*v* dome because the central band of triangles is relatively wide, and less so the higher the dome frequency.

To describe whether the odd-frequency dome is split above or below the central band it's described as a fraction of the total rows of triangles. An icosahedron can be thought of as unfolding into top triangles, middle triangles and bottom triangles, and for an *v*-frequency dome those three sets of triangles all subdivided into *v* rows, making 3*v* rows in total. A dome split above the central band then has $\frac{(3\nu-1)/2}{3\nu}$ of the total rows, one split below the central band having $\frac{(3\nu+1)/2}{3\nu}$ of the total. A 3*v* dome is therefore either a 4/9th or a 5/9th, a 5*v* dome is a 7/15th or 8/15th, etc.

Central Angles

Working out the dome angles starts by realizing that the angles that the struts subtend at the center of the dome are the same before the vertices are projected onto the sphere as afterwards. Working out the angles for the vertices is not too difficult on the plane face, and with those angles, working out all the other dome lengths and angles is not too difficult either. Consider the face of an icosahedron of unit radius:



For an icosahedron $\tan(\theta/2)=1/\tau$ where $\tau=\frac{1+\sqrt{5}}{2}$ is the golden ratio of classical architecture.

This makes the icosahedron vertex angle θ approximately 63.4º. Vectors to any of the vertices can be expressed as a sum of the three mutually perpendicular vectors **X**, **Y** and **Z**. Vector **X** locates the center of the equilateral triangular face. With a little trigonometry it can be seen that the side length of the triangular face is $2\sin(\theta/2)$ and the height of the face is $\sqrt{3}\sin(\theta/2)$ so

$$X^2 = \cos^2\alpha = 1 - \sin^2\alpha = 1 - \frac{4}{3}\sin^2\theta/2$$

where $\sin\alpha = \frac{2}{\sqrt{3}}\sin(\theta/2)$ - it's going to be the square of the magnitudes that's useful. **Y** and **Z** are vectors one vertex along and up respectively such that

$$Y^2 = \frac{4}{\nu^2}\sin^2(\theta/2) = \frac{3}{\nu^2}(1 - X^2)$$

and

$$Z^2 = \frac{1}{3}\sin^2(\theta/2) = \frac{3}{4}Y = \frac{9}{4\nu^2}(1 - X^2) \quad .$$

From the vector identity $\boldsymbol{r_0}\cdot\boldsymbol{r_1} = r_0 r_1 \cos(\phi_{01})$ the vertex angles follow. The vertices are easy to express as a sum of **X**, **Y** and **Z** and because these vectors are mutually perpendicular their magnitudes and their scalar products are all simple to calculate.

$$\boldsymbol{p_0} = \boldsymbol{X} + 2\boldsymbol{Z} \qquad \boldsymbol{p_1} = \boldsymbol{X} + \boldsymbol{Y}/2 + \boldsymbol{Z} \qquad \boldsymbol{p_2} = \boldsymbol{X} - \boldsymbol{Y}/2 + \boldsymbol{Z} \qquad \boldsymbol{p_3} = \boldsymbol{X}$$

For φ_{01}, $\boldsymbol{p_0}\cdot\boldsymbol{p_1} = p_0 p_1 \cos\phi_{01} = X^2 + 2Z^2 = \sqrt{X^2 + 4z^2}\sqrt{X^2 + Y^2/4 + Z^2}\cos\phi_{01}$

which substituting for Y^2 and Z^2 and rearranging gives the red angle as

$$\cos\phi_{01} = \frac{(X^2+1)/2}{\sqrt{(2X^2+1)/3}} \;\Rightarrow\; \phi_{01} = 20.08^o$$

For φ_{12}, $\boldsymbol{p_1}\cdot\boldsymbol{p_2} = p_1 p_2 \cos\phi_{12} = X^2 - Y^2/4 + Z^2 = (X^2 + Y^2/4 + Z^2)\cos\phi_{12}$

again, substituting and rearranging gives the blue angle as

$$\cos\phi_{12} = \frac{(5\mathrm{X}^2+1)/2}{2\mathrm{X}^2+1} \;\Rightarrow\; \phi_{12} = 23.28^o$$

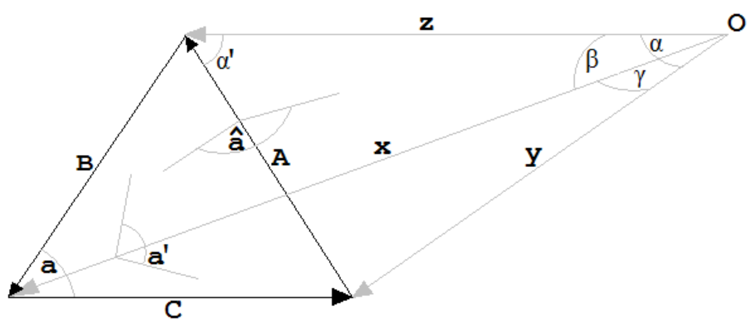
And for φ_{23}, $\boldsymbol{p_2}\cdot\boldsymbol{p_3} = p_2 p_3 \cos\phi_{23} = X^2 = \sqrt{X^2 + Y^2/4 + Z^2}\sqrt{X^2}\cos\phi_{23}$

which gives the green angle as

$$\cos\phi_{23} = \sqrt{\frac{3\mathrm{X}^2}{2\mathrm{X}^2+1}} \;\Rightarrow\; \phi_{23} = 23.80^o$$

In this way it's quite straightforward to find the central angles for any frequency dome built on any platonic solid, all you really need to know is the vertex angle for the solid.

Frame Angles



Once you know the angles that the struts subtend at the center of the dome all of the other angles follow. In the diagram the panel's three edges are represented by vectors **A**, **B** and **C** which subtend angles α, β and γ respectively at the center of the dome. The vectors **x**, **y** and **z** locate the panel vertices.

Strut Length

The strut length is the only dimension that depends on the size of the dome, all of the angles, for a given design of dome, are independent of the size. By trigonometry, the strut lengths for a dome of unit radius are

$$A=2\sin(\alpha/2) \quad B=2\sin(\beta/2) \quad C=2\sin(\gamma/2)$$

These just scale linearly with the radius of the actual dome.

Mitre Angle

The mitre angles where the struts meet the hubs are the simplest as they are just the compliment of the central angles

$$\alpha'=(180-\alpha)/2 \quad \beta'=(180-\beta)/2 \quad \gamma'=(180-\gamma)/2$$

Corner Angle

The angles at the corner of the frame in the plane of the frame follow from the vector product of the edge vectors. Writing the edge vectors as

$$\boldsymbol{A}=\boldsymbol{z}-\boldsymbol{y} \quad \boldsymbol{B}=\boldsymbol{x}-\boldsymbol{z} \quad \boldsymbol{C}=\boldsymbol{y}-\boldsymbol{x}$$

and expanding the vector product, using the vector identity for the unit-length vertex vectors

$$\boldsymbol{x}\cdot\boldsymbol{y}=\cos\gamma \quad \boldsymbol{y}\cdot\boldsymbol{z}=\cos\alpha \quad \boldsymbol{z}\cdot\boldsymbol{x}=\cos\beta$$

gives

$$\boldsymbol{B}\cdot\boldsymbol{C}=-B\,C\cos a=(\boldsymbol{x}-\boldsymbol{z})\cdot(\boldsymbol{y}-\boldsymbol{x})=\cos\gamma-1-\cos\alpha+\cos\beta=-4\sin(\beta/2)\sin(\gamma/2)\cos a$$

and rearranging gives

$$\cos a=\frac{1+\cos\alpha-\cos\beta-\cos\gamma}{4\sin(\beta/2)\sin(\gamma/2)} \quad \cos b=\frac{1+\cos\beta-\cos\gamma-\cos\alpha}{4\sin(\gamma/2)\sin(\alpha/2)} \quad \cos c=\frac{1+\cos\gamma-\cos\alpha-\cos\beta}{4\sin(\alpha/2)\sin(\beta/2)}$$

Hub Radial Angle

Looking down on the hub towards the center of the dome the angle that the struts radiate from the hub is not actually the same as the frame corner angle because the struts radiate at an angle, α', to the hub axis. This is the dihedral angle between the planes $\boldsymbol{x}\times\boldsymbol{y}$ and $\boldsymbol{x}\times\boldsymbol{z}$, so using the vector and scalar triple product identities

$$\boldsymbol{x}\times\boldsymbol{y}\cdot\boldsymbol{x}\times\boldsymbol{z}=\sin\gamma\sin\beta\cos a'=\boldsymbol{x}\cdot\boldsymbol{z}\times(\boldsymbol{x}\times\boldsymbol{y})=\boldsymbol{x}\cdot(\cos\alpha\,\boldsymbol{x}-\cos\beta\,\boldsymbol{y})=\cos\alpha-\cos\beta\cos\gamma$$

thus

$$\cos a'=\frac{\cos\alpha-\cos\beta\cos\gamma}{\sin\beta\sin\gamma} \quad \cos b'=\frac{\cos\beta-\cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} \quad \cos c'=\frac{\cos\gamma-\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

Strut Bevel Angle

The bevel angle, $\hat{a}$ on the top of the strut is the dihedral angle between the plane $\boldsymbol{y}\times\boldsymbol{z}$ and the plane $\boldsymbol{B}\times\boldsymbol{C}$ such that

$$\boldsymbol{y}\times\boldsymbol{z}\cdot\boldsymbol{B}\times\boldsymbol{C}=4\ \sin\alpha\ \sin\beta/2\ \sin\gamma/2\ \sin a\ \cos\hat{a}$$

The left hand side can be expanded directly and expressed in terms of $\cos\beta'$ and $\cos\gamma'$ but it's a little neater to use the vector triple product identities to rearrange it so

$$\boldsymbol{y}\times\boldsymbol{z}\cdot\boldsymbol{B}\times\boldsymbol{C}=\boldsymbol{B}\cdot\boldsymbol{C}\times(\boldsymbol{y}\times\boldsymbol{z})=\boldsymbol{B}\cdot((\boldsymbol{z}\cdot\boldsymbol{C})\ \boldsymbol{y}-(\boldsymbol{C}\cdot\boldsymbol{y})\ \boldsymbol{z})=(\boldsymbol{x}-\boldsymbol{z})\cdot((\boldsymbol{z}\cdot(\boldsymbol{y}-\boldsymbol{x}))\ \boldsymbol{y}-((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{y})\ \boldsymbol{z})$$

which after substituting for the dot products of the vertex vectors and a little rearrangement gives

$$\cos\hat{a}=\frac{(1-\cos\gamma)(1-\cos\beta)-(\cos\alpha-\cos\gamma)(\cos\alpha-\cos\beta)}{4\sin\alpha\ \sin\beta/2\ \sin\gamma/2\ \sin a}$$

$$\cos\hat{b}=\frac{(1-\cos\alpha)(1-\cos\gamma)-(\cos\beta-\cos\alpha)(\cos\beta-\cos\gamma)}{4\sin\beta\ \sin\gamma/2\ \sin\alpha/2\ \sin b}$$

$$\cos\hat{c}=\frac{(1-\cos\beta)(1-\cos\alpha)-(\cos\gamma-\cos\beta)(\cos\gamma-\cos\alpha)}{4\sin\gamma\ \sin\alpha/2\ \sin\beta/2\ \sin c}$$

Examples

2v Icosahedral Dome

A	B	α	β	α'	β'	a	b	â	$\hat{b}$	a'	b'
.6180	.6180	36.00°	36.00°	72°	72°	60°	60°	79°	79°	63°	63°
.6180	.5466	36.00°	31.72°	72°	74°	69°	56°	83°	79°	72°	58°

3v Icosahedral Dome

A	B	α	β	α'	β'	a	b	â	$\hat{b}$	a'	b'
.4036	.4124	23.28°	23.80°	78°	78°	59°	61°	83°	83°	60°	62°
.4036	.3486	23.28°	20.08°	78°	80°	71°	55°	86°	83°	72°	56°

Note that for both the 2v and 3v dome there are only two different triangular faces, and the triangles are isosceles so the C dimensions are all the same as the B dimensions.