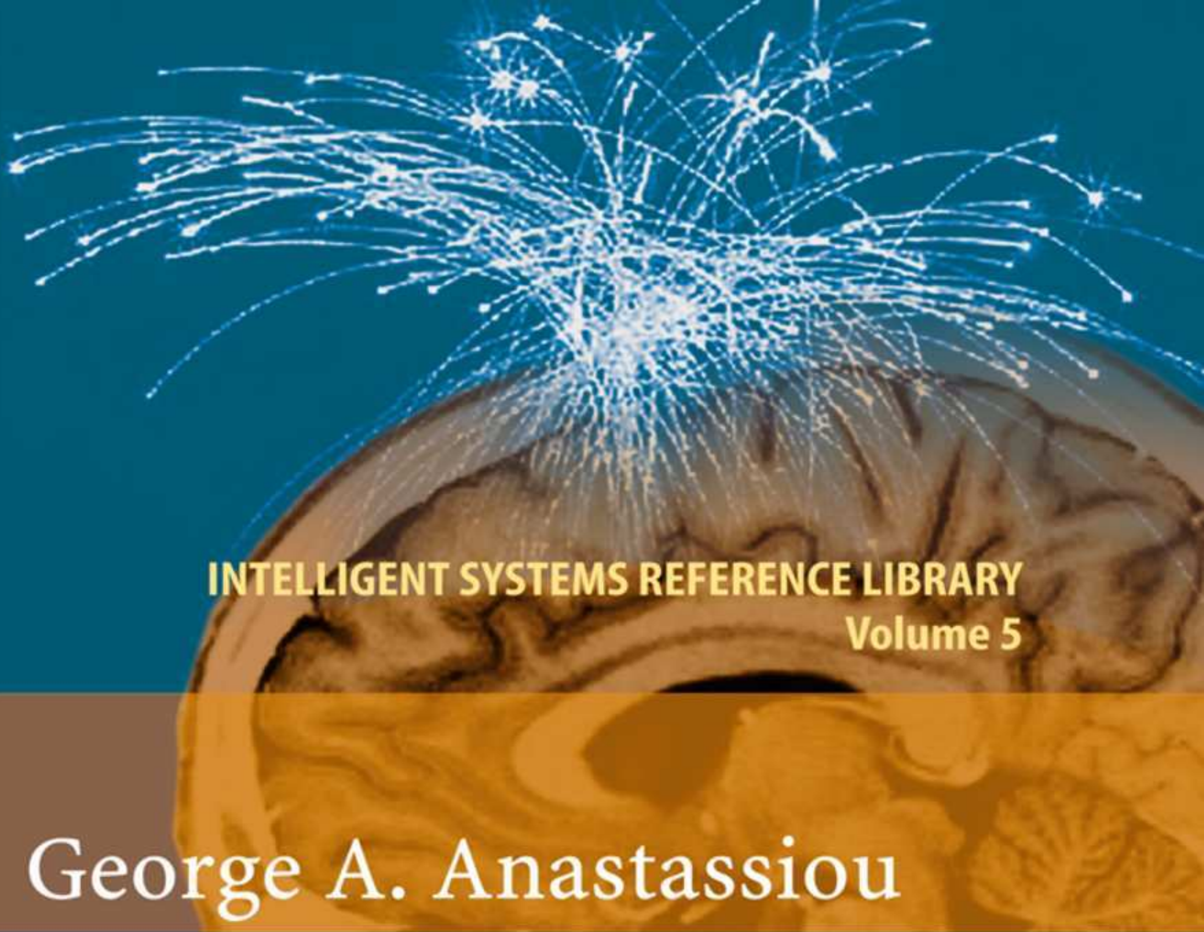

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Intelligent Mathematics: Computational Analysis

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Intelligent Mathematics: Computational Analysis

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Intelligent Mathematics: Computational Analysis

Springer

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ISBN 978-3-642-17097-3 e-ISBN 978-3-642-17098-0

DOI 10.1007/978-3-642-17098-0

Intelligent Systems Reference Library ISSN 1868-4394

Typeset & *Cover Design:* Scientific Publishing Services Pvt. Ltd., Chennai, India.

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

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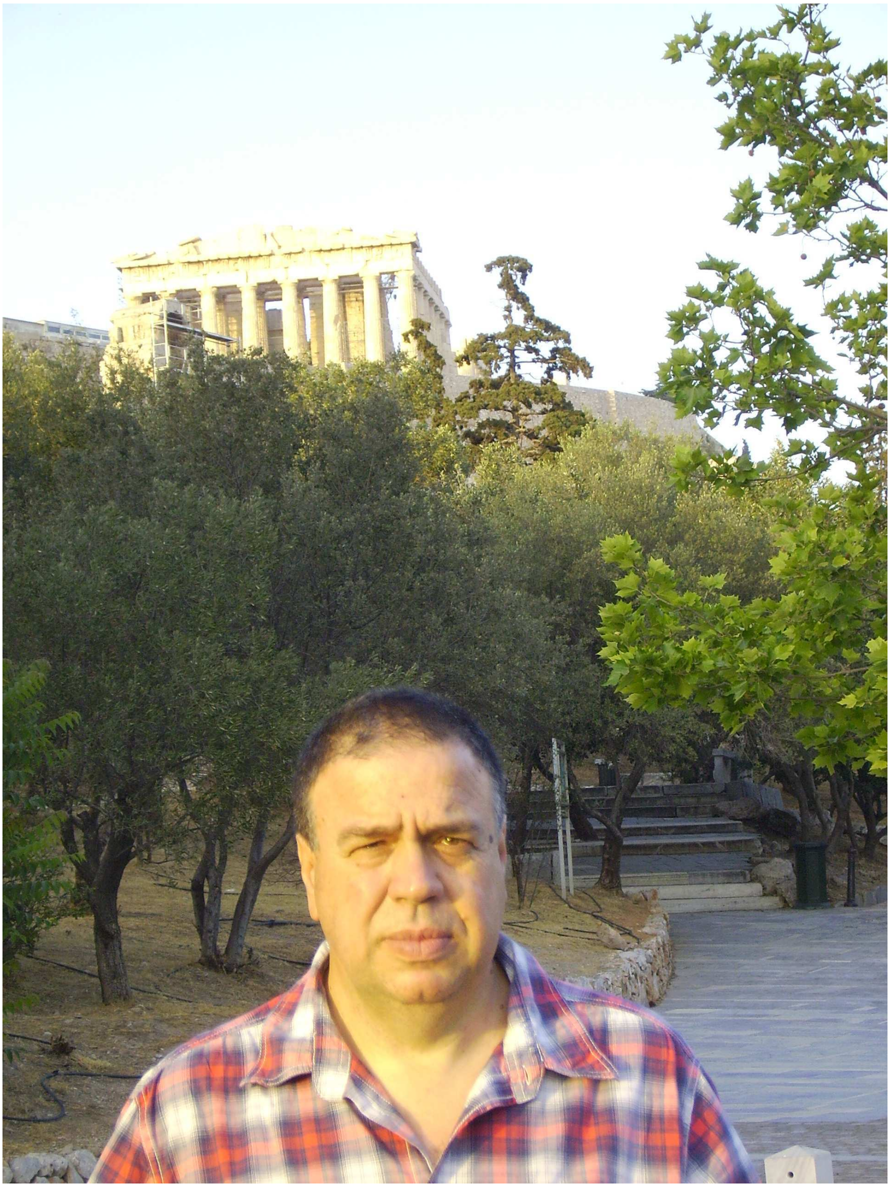
TO MY WIFE KOULA AND MY DAUGHTERS
ANGELA AND PEGGY



George A. Anastassiou, May 2010, Parthenon, Greece

Preface

Among others knowledge can be well described and expressed in an abstract way and can be computed using computational mathematical methods, then lead to real world conclusions. The strongly related to that Computational Analysis is a very large area which contains many different subareas and topics with results that are computational, constructive, and concrete with specific applications. Many of the exposed results here are quantitative, with precise rates of convergence given by optimal or nearly optimal inequalities. This monograph includes a great variety of topics of Computational Analysis deriving from author's research of the last 25 years. The chapters are a natural outgrowth of author's publications [8] and [11]-[92]. More precisely we present:

In Chapters 2-5 we present probabilistic wavelet like approximations.

In Chapter 6 we discuss constrained abstract approximation theory.

In Chapter 7 we talk about shape preserving weighted approximation.

Chapter 8 deals with non positive approximations to definite integrals.

Chapter 9 describes discrete best approximation in gauges sense.

Chapters 10-13 deal with the approximation theory of general Picard singular operators, including their global preservation property, as well us treating the corresponding fractional singular operators. In Chapter 14 we deal with the non-isotropic general Picard singular multivariate operators. In Chapter 15 we discuss the q-Gauss-Weierstrass singular q-integral operators.

Chapters 16-17 talk about quantitative approximations by shift-invariant univariate and multivariate integral operators. Chapter 18 gives nonlinear neural networks approximation.

Chapter 19 presents convergence with rates of positive linear operators.

Chapter 20 describes quantitative approximation by bounded linear operators.

Chapters 21-22 talk about univariate and multivariate quantitative approximation by stochastic positive linear operators acting on univariate and multivariate stochastic processes, respectively.

Chapter 23 deals with the right fractional calculus. In Chapters 24, 25 we give the quantitative fractional Korovkin theory of positive linear operators and its trigonometric aspect.

In Chapter 26 we give analytical inequalities. In Chapter 27 we give fractional Opial inequalities.

Chapter 28 presents fractional identities and inequalities regarding fractional integrals.

Chapter 29 deals with semigroup operator approximation, while Chapter 30 talks about simultaneous Feller probabilistic approximation. In Chapter 31 we deal with Fuzzy singular operator approximations.

In Chapter 32 we give transfers from real to fuzzy approximation. Chapter 33 talks about fuzzy wavelet and fuzzy neural networks approximations. Chapter 34 deals with fuzzy fractional calculus and fuzzy Ostrowski inequality. In Chapter 35 we talk about discrete fractional calculus and related inequalities.

In Chapter 36 we give the nabla discrete fractional calculus and related inequalities.

In Chapter 37 we study the q-inequalities, and in Chapter 38 we study q-fractional inequalities.

Chapters 39-41 deal with time scales: the delta approach, the nabla approach, their duality principle and related inequalities, respectively.

Chapters 42, 43 talk about the delta and nabla time scales fractional calculus and related inequalities, respectively. In Chapters 44, 45 we study the convergence with rates of approximate solutions to exact solution of multivariate Dirichlet problem and multivariate heat equation, respectively.

Finally Chapter 46 deals with uniqueness of solution of general evolution partial differential equation in multivariate time.

The chapters are self-contained and can be read independently one from the other and all necessary background is provided. An extensive list of references is given at the end. Several advanced graduate courses and seminars can be taught out of this book. The presented results are expected to find potential applications to fields like: applied and computational mathematics, stochastics, engineering, artificial intelligence, vision, complexity and machine learning. This monograph is the first written in mathematical computational analysis and is suitable for graduate students, researchers of the above mentioned disciplines, and for all science and engineering libraries.

The final preparation of this book took place during 2009-2010 in Memphis, Tennessee, USA.

I would like to thank my family for their dedication and love to me, which was the strongest support during the writing of this monograph. Also many thanks go to my typist and student Razvan Mezei for an excellent and on time technical job.

August 15, 2010

George A. Anastassiou
Memphis, TN, USA

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1
Introduction

Mathematics provides a complete language for describing systems and methods with rigor. So we are able to represent and manipulate knowledge in an abstract way and make computations that lead to useful conclusions for the real world.

Computational Analysis is a very large area-roof under which are housed many different subareas and topics of mathematical analysis and applications, as long as the results are computational, constructive, and concrete with specific and precise examples and applications. Many of our results are quantitative, with precise rates of convergence given via usually sharp and attained inequalities, in general via tight inequalities. This monograph includes very diverse topics of Computational Analysis emanating out of author's research of the last 25 years. So it is a natural outgrowth of author's publications [8] and [11]-[92].

The list of discussed topics includes:

Probabilistic wavelet Approximation and shape preservation, Neural networks Approximation, Classical Polynomial and Operator Constrained Approximation theory, Discrete Best Approximation, Approximation by singular integrals, Fractional Calculus, Approximation by fractional singular integrals, Convergence with rates of bounded and/ or positive linear operators, Quantitative Stochastic Operator Approximation, Quantitative Fractional approximation by positive linear operators including the trigonometric aspect, Quantitative Approximation by Convolution operators, Analytical

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 1–11.
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Inequalities and Fractional inequalities, Approximation by Semigroups of operators,

Fuzzy mathematics and Fuzzy fractional calculus,

Fuzzy wavelets and Fuzzy neural networks, Fuzzy Approximation, Fuzzy inequalities,

Discrete fractional calculus, q-calculus, q-fractional calculus, Time scales and inequalities,

Fractional time scales and related Inequalities, Approximation and uniqueness of solutions of well known partial differential equations.

This book presents multi-face mathematics research under one spirit: the computational and constructive.

A detailed description of the monograph follows.

In Chapter 2:

Continuous functions are approximated by wavelet like operators. These preserve convexity and r-convexity and transform continuous probability distribution functions into probability distribution functions at the same time preserving certain convexity conditions. The degree of this approximation is estimated by establishing some Jackson type inequalities.

In Chapter 3:

Shape-preserving properties of some naturally arising bivariate wavelet type operators are studied. Also the pointwise convergence of these operators with rates to the unit operator is examined, given via a Jackson type inequality. The simultaneous shape preservation of special wavelet type operators is discussed.

In Chapter 4:

Multivariate probabilistic distribution functions are approximated by some naturally arising wavelet type operators involving a scale function of compact support. These transform multivariate distribution functions to multivariate distribution functions. The degree of this approximation is given by establishing some sharp Jackson type inequalities.

In Chapter 5:

Naturally arising multivariate wavelet type operators, map continuous multivariate probabilistic distribution functions to multivariate probabilistic distribution functions and approximate them quantitatively with rates via Jackson type inequalities. The engaged scale functions here are not necessarily of compact support.

In Chapter 6:

Here we discuss the L-positive approximation. Namely we study the best approximation in an abstract constrained sense. When the involved bounded linear operator is a differential operator acting on a Sobolev space of functions we obtain Jackson type inequalities for simultaneous approximation with rates by multivariate polynomials and entire functions of exponential type.

In Chapter 7:

Results concerning shape preserving weighted uniform approximation on the real line are presented.

In Chapter 8:

The integral of a function over a finite interval, is approximated by Jackson-type approximators that are non-positive linear functionals. Several important cases are considered, in which approximations are given with rates by using higher order moduli of smoothness. Real applications of these results might be, e.g., in Communications and Medical Imaging.

In Chapter 9:

A discrete version is developed of the theory of best approximation in the "gauges" sense.

In Chapter 10:

Here we introduce and study the smooth Picard singular integral operators on the line of very general kind. We establish their convergence to the unit operator with rates. The estimates are mostly sharp and they are pointwise or uniform. The established inequalities involve the higher order modulus of smoothness. To prove optimality we use mainly the geometric moment theory method.

In Chapter 11:

We continue with the study of smooth Picard singular integral operators over the real line regarding their simultaneous global smoothness preservation property with respect to the L_p norm, $1 \leq p \leq \infty$, by involving higher order moduli of smoothness. Also we study their simultaneous approximation to the unit operator with rates involving the modulus of continuity with respect to the uniform norm. The produced Jackson type inequalities are almost sharp containing elegant constants, and they reflect the high order of differentiability of the engaged function.

In Chapter 12:

We continue further with the study of smooth Picard singular integral operators on the line regarding their convergence to the unit operator with rates in the L_p norm, $p \geq 1$. The related established inequalities involve the higher order L_p modulus of smoothness of the engaged function or its higher order derivative.

In Chapter 13:

Moreover we study the very general fractional smooth Picard singular integral operators on the real line, regarding their convergence to the unit operator with fractional rates in the uniform norm. The related established inequalities involve the higher order moduli of smoothness of the associated right and left Caputo fractional derivatives of the engaged function. Furthermore we produce a fractional Voronovskaya type of result giving the fractional asymptotic expansion of the basic error of our approximation. We finish with applications. Our operators are not in general positive.

In Chapter 14:

Here we study another type of Picard singular integral operators on $\mathbb{R}^n$ constructed by means of the concept of the nonisotropic β-distance and the q-exponential functions. The central role here is played by the concept of nonisotropic β-distance, which allows to improve and generalize the results given for classical Picard and q-Picard singular integral operators. In order to obtain the rate of convergence we introduce another type of modulus of continuity depending on the nonisotropic β-distance with respect to the uniform norm. Then we give the definition of β-Lebesgue points depending on nonisotropic β-distance and a pointwise approximation result shown at these points. Furthermore, we study the global smoothness preservation property of these Picard singular integral operators and prove a sharp inequality.

In Chapter 15:

We further introduce a generalization of Gauss- Weierstrass singular integral operators based on q-integers using the q-integral and we call them q-Gauss- Weierstrass integral operators. For these operators, we obtain a convergence property in a weighted function space using Korovkin theory. Then we estimate the rate of convergence of these operators in terms of a weighted modulus of continuity. We also prove optimal global smoothness preservation property of these operators.

In Chapter 16:

High order differentiable functions of one real variable are approximated by univariate shift-invariant integral operators wavelet-like, and their generalizations. The high order of this approximation is estimated by establishing some Jackson type inequalities, involving the modulus of continuity of the Nth order derivative of the function under approximation. At the end applications to Probability are given.

In Chapter 17:

High order differential functions of several variables are approximated by multivariate shift-invariant convolution type operators and their generalizations. The high order of this approximation is determined by giving some multivariate Jackson-type inequalities, engaging the first multivariate usual modulus of continuity of the Nth order partial derivatives of the multivariate function under approximation.

In Chapter 18:

Here by using the so-called max-product method, to associate to the Cardaliaguet-Euvrard linear operator a nonlinear neural network operator for which a Jackson-type approximation order is obtained. In some classes of functions, the order of approximation is essentially better than the order of approximation by the linear operator.

In Chapter 19:

We present here a generalized Shisha-Mond type inequality which implies a generalized Korovkin theorem. These are regarding the convergence with rates of a sequence of positive linear operators to the unit.

In Chapter 20:

This is a quantitative study for the rate of pointwise convergence of a sequence of bounded linear operators to an arbitrary operator in a very general setting involving the modulus of continuity. This is achieved through the Riesz representation theorem and the weak convergence of the corresponding signed measures to zero, studied quantitatively in various important cases.

In Chapter 21:

We introduce and study very general stochastic positive linear operators induced by general positive linear operators that are acting on continuous functions. These are acting on the space of real differentiable stochastic processes. Under some very mild, general and natural assumptions on the stochastic processes we produce related stochastic Shisha–Mond type inequalities of L^q-type $1 \leq q < \infty$ and corresponding stochastic Korovkin type theorems. These are regarding the stochastic q-mean convergence of a sequence of stochastic positive linear operators to the stochastic unit operator for various cases. All convergences are produced with rates and are given via the stochastic inequalities involving the stochastic modulus of continuity of the $n-th$ derivative of the engaged stochastic process, $n \geq 0$. The impressive fact is that the basic real Korovkin test functions assumptions are enough for the conclusions of our stochastic Korovkin theory. We give an application.

In Chapter 22:

We further introduce and study very general multivariate stochastic positive linear operators induced by general multivariate positive linear operators that are acting on multivariate continuous functions. These are acting on the space of real differentiable multivariate time stochastic processes.

Under some very mild, general and natural assumptions on the stochastic processes we produce related multidimensional stochastic Shisha–Mond type inequalities of L^q-type $1 \leq q < \infty$ and corresponding multidimensional stochastic Korovkin type theorems. These are regarding the stochastic q-mean convergence of a sequence of multivariate stochastic positive linear operators to the stochastic unit operator for various cases. All convergences are produced with rates and are given via the stochastic inequalities involving the maximum of the multivariate stochastic moduli of continuity of the nth order partial derivatives of the engaged stochastic process, $n \geq 0$. The astonishing fact here is that basic real Korovkin test functions assumptions are enough for the conclusions of our multidimensional stochastic Korovkin theory. We give an application.

In Chapter 23:

Here are presented fractional Taylor type formulae with fractional integral remainder and fractional differential formulae, regarding the right Caputo fractional derivative, the right generalized fractional derivative of Canavati type and their corresponding right fractional integrals. Then are given representation formulae of functions as fractional integrals of their above fractional derivatives, as well as of their right and left Weyl fractional derivatives. At the end, we mention some far reaching implications of our theory to mathematical analysis computational methods. Also we compare the right Caputo fractional derivative to the right Riemann-Liouville fractional derivative.

In Chapter 24:

We study quantitatively with rates the weak convergence of a sequence of finite positive measures to the unit measure. Equivalently we study quantitatively the pointwise convergence of sequence of positive linear operators to the unit operator, all acting on continuous functions. From there we derive with rates the corresponding uniform convergence of the last. Our inequalities for all of the above in their right hand sides contain the moduli of continuity of the right and left Caputo fractional derivatives of the involved function. From our uniform Shisha-Mond type inequality we derive the fractional Korovkin type theorem regarding the uniform convergence of positive linear operators to the unit. We give applications, especially to Bernstein polynomials for which we establish fractional quantitative results.

In the background we establish several fractional calculus results useful to approximation theory and not only.

In Chapter 25:

We study further quantitatively with rates the trigonometric weak convergence of a sequence of finite positive measures to the unit measure. Equivalently we study quantitatively the trigonometric pointwise convergence of sequence of positive linear operators to the unit operator, all acting on continuous functions on $[-\pi, \pi]$. From there we derive with rates the corresponding trigonometric uniform convergence of the last. Our inequalities for all of the above in their right hand sides contain the moduli of continuity of the right and left Caputo fractional derivatives of the involved function. From our uniform trigonometric Shisha-Mond type inequality we derive the trigonometric fractional Korovkin type theorem regarding the trigonometric uniform convergence of positive linear operators to the unit. We give applications, especially to Bernstein polynomials over $[-\pi, \pi]$ for which we establish fractional trigonometric quantitative results.

In Chapter 26:

Here are presented very general Taylor formulae, and then a representation formula. Based on the last we give new general inequalities of Opial type, Ostrowski type, Comparison of integral means, Information Theory Csiszar f- divergence type, and Grüss type.

In Chapter 27:

Here we present L_p, $p > 1$, fractional Opial type inequalities subject to high order boundary conditions. They involve the right and left Caputo, Riemann-Liouville fractional derivatives. These derivatives are blended together into the balanced Caputo, Riemann-Liouville, respectively, fractional derivatives. We give applications to a special case.

In Chapter 28:

We develop some integral identities and inequalities for the fractional integral. In particular we obtain Montgomery type identities for fractional integrals and a generalization to double fractional integrals. We further produce Ostrowski and Grüss type inequalities for fractional integrals.

In Chapter 29:

Some general representation formulae for (C_0) m-parameter operator semigroups with rates of convergence are obtained by the probabilistic approach and multiplier enlargement method. These cover all known representation formulae for (C_0) one-and m-parameter operator semigroups as special cases. When we consider special semigroups well-known convergence theorems for multivariate approximation operators are reobtained.

In Chapter 30:

A quantitative estimate for the simultaneous approximation of a function and its derivatives by the general Feller operator is established via the probabilistic approach. This covers the cases of some classical approximation operators such as the Bernstein, Szász, Baskakov and Gamma operator.

In Chapter 31:

We study the fuzzy global smoothness and fuzzy uniform convergence of fuzzy Picard, Gauss-Weierstrass and Poisson-Cauchy singular fuzzy integral operators to the fuzzy unit operator. These are given with rates involving the fuzzy modulus of continuity of a fuzzy derivative of the involved function. The produced fuzzy Jackson type inequalities are tight, containing elegant constants, and they reflect the order of the fuzzy differentiability of the engaged fuzzy function.

In Chapter 32:

Here we transfer basic real approximations to corresponding vectorial and fuzzy setting of: Bernstein polynomials, Bernstein-Durrmeyer operators, genuine Bernstein-Durrmeyer operators, Stancu type operators and special Stancu operators. These are convergences to the unit operator with rates. We also present the convergence with rates to zero of the difference of genuine Bernstein-Durrmeyer and special Stancu operators. All approximations involve Jackson type inequalities and moduli of smoothness of various orders. In order to transfer we develop basic and important general results at the vectorial and fuzzy level. Our technique goes from real to vectorial and then to fuzzy setting.

In Chapter 33:

Here are studied in terms of multivariate fuzzy high approximation to the multivariate unit several basic sequences of multivariate fuzzy wavelet type operators and multivariate fuzzy neural network operators. These operators are multivariate fuzzy analogs of earlier studied multivariate real ones. The produced results generalize earlier real ones into the fuzzy setting. Here the high order multivariate fuzzy pointwise convergence with rates to the multivariate fuzzy unit operator is established through multivariate fuzzy inequalities involving the multivariate fuzzy moduli of continuity of the Nth order ($N \geq 1$) H-fuzzy partial derivatives, of the engaged multivariate fuzzy number valued function. The purpose of embedding fuzziness into multivariate classical analysis is to better understand, explain and describe the imprecise, uncertain and chaotic phenomena of the real world and then derive useful conclusions.

In Chapter 34:

Here we study the right and left fuzzy fractional Riemann-Liouville integrals and the right and left fuzzy fractional Caputo derivatives. Then we present the right and left fuzzy fractional Taylor formulae. Based on these we establish a fuzzy fractional Ostrowski type inequality with applications. The last inequality provides an estimate for the deviation of a fuzzy real number valued function from its fuzzy average, and the related upper bounds are given in terms of the right and left fuzzy fractional derivatives of the involved function. The purpose of embedding fuzziness into fractional calculus and have them act together, is to better understand, explain and describe the imprecise, uncertain and chaotic phenomena of the real world and then derive important conclusions.

In Chapter 35:

Here we define a Caputo like discrete fractional difference and we compare it to the earlier defined Riemann-Liouville fractional discrete analog. Then we produce discrete fractional Taylor formulae and we estimate their remainders. Finally, we derive related discrete fractional Ostrowski, Poincare and Sobolev type inequalities.

In Chapter 36:

Here we define a Caputo like discrete nabla fractional difference and we produce discrete nabla fractional Taylor formulae. We estimate their remainders. Then we derive related discrete nabla fractional Opial, Ostrowski, Poincaré and Sobolev type inequalities.

In Chapter 37:

We give forward and reverse $q-$Hölder inequalities, $q-$Poincaré inequality, $q-$Sobolev inequality, $q-$reverse Poincaré inequality, $q-$reverse Sobolev inequality, $q-$Ostrowski inequality, $q-$Opial inequality and $q-$Hilbert-Pachpatte inequality. Some interesting background is mentioned and built at the beginning.

In Chapter 38:

Here we give q-fractional Poincare' type, Sobolev type and Hilbert-Pachpatte type integral inequalities, involving q-fractional derivatives of functions. We give also their q-generalized versions.

In Chapter 39:

Here we collect and develop necessary background on time scales that is required. Then we present time scales integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. We give also the generalized analogs of all these inequalities involving high order delta derivatives of functions on time scales. We finish with lots of applications: all these inequalities on the specific time scales $\mathbb{R}$, $\mathbb{Z}$ and $q^{\overline{\mathbb{Z}}}$, $q > 1$.

In Chapter 40:

Here we collect and develop necessary background on nabla time scales that is required. Then we present nabla time scales integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. We give also the generalized analogs of all these nabla inequalities involving high order nabla derivatives of functions on time scales. We finish with lots of applications: all these nabla inequalities on the specific time scales $\mathbb{R}$, $\mathbb{Z}$ and $q^{\overline{\mathbb{Z}}}$, $q > 1$. In most of these nabla inequalities the nabla differentiability order is any $n \in \mathbb{N}$, as opposed to delta time scales approach where n is always odd.

In Chapter 41:

Here we adopt([127]), develop further and use the principle of duality in time scales. Using this principle and based on a variety of important delta inequalities we produce the corresponding nabla ones. We give several applications.

In Chapter 42:

Here we develop the Delta Fractional Calculus on Time Scales. Then we produce related integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. Finally we give inequalities applications on the time scale $\mathbb{R}$.

In Chapter 43:

We also develop the Nabla Fractional Calculus on Time Scales. Then we produce related integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. Finally we give nabla fractional inequalities applications on the time scales $\mathbb{R}$, $\mathbb{Z}$.

In Chapter 44:

For the multidimensional Dirichlet problem of the Poisson equation on an arbitrary compact domain, this chapter examines convergence properties with rates of approximate solutions, obtained by a standard difference scheme over inscribed uniform grids. Sharp quantitative estimates are given by the use of second moduli of continuity of the second single partial derivatives of the exact solution. This is achieved by employing the probabilistic method of simple random walk.

In Chapter 45:

For the multidimensional Dirichlet problem of the heat equation on a cylinder, this chapter examines convergence properties with rates of approximate solutions, obtained by a naturally arising difference scheme over inscribed uniform grids. Sharp quantitative estimates are given by the use of first and second moduli of continuity of some first and second order partial derivatives of the exact solution. This is accomplished by using the probabilistic method of an appropriate random walk.

In Chapter 46:

The classical time dependent partial differential equations of mathematical physics involve evolution in one dimensional time. Space can be multidimensional, but time stays one dimensional. There are various mathematical cases (such as multiparameter Brownian motion) which suggest that there should be a mathematical theory of evolution in multidimensional time. We formulate a rather general class of equations that involve two "time dimensions" and we prove a uniqueness theorem in this context. We connect the latter to Opial type inequalities.

Chapters 2-5 rely on [85], [89], [90] and [87], respectively, which are joint works of author with X.M. Yu.

Chapter 6 relies on [75], which is joint work of the author with M. Ganzburg.

Chapter 7 relies on [74], which is joint work of the author with S. Gal and M. Ganzburg.

Chapter 8 relies on [70], which is joint work of the author with S. Gal.

Chapter 9 relies on [8], which is joint work of the author with S. Ali and O. Shisha.

Chapter 14 relies on [61], which is joint work of the author with A. Aral.

Chapter 15 relies on [62], which is joint work of the author with A. Aral.

Chapter 18 relies on [65], which is joint work of the author with L.Coroianu and S. Gal.

Chapter 28 relies on author's joint work [80], several coauthors.

Chapters 29, 30, rely on [92], [91], respectively, which are joint works of the author with Mi Zhou.

Chapters 44, 45, rely on [63], [64], respectively, which are joint works of the author with A.Bendikov.

And Chapter 46 relies on [76], which is joint work of the author with G. Ruiz Goldstein and J. Goldstein.

The rest of the chapters are based on individual works of the author.

The writing of this monograph was made to help the reader the most. The chapters are self-contained so that anyone of these can be read without using others and several graduate courses and seminars can be taught out of this book. All background needed to understand each chapter is usually found there. Also are given, per chapter, strong motivations and inspirations to write it.

We finish with a rich list of 288 related references. The exposed results are expected to find applications in most of the applied fields such as: applied and computational mathematics, stochastics, engineering, informatics, and especially in theoretical computer science such as artificial intelligence, vision, complexity and machine learning.

To the best of our knowledge this monograph is the first of the kind within computational analysis from the mathematical point of view and we hope is well received.

2

Convex Probabilistic Wavelet Like Approximation

Continuous functions are approximated by wavelet like operators. These preserve convexity and r-convexity and transform continuous probability distribution functions into probability distribution functions at the same time preserving certain convexity conditions. The degree of this approximation is estimated by presented Jackson type inequalities.

This chapter relies on [85].

2.1 Introduction

Let $\varphi(x)$ be a bounded continuous function on $\mathbb{R}$ with $\text{supp}\varphi(x) \subseteq [-a, a]$, $0 < a < +\infty$. Write

$$\varphi_{kj}(x) := 2^{\frac{k}{2}} \varphi\left(2^k x - j\right), \quad k, j \in \mathbb{Z}.$$

For $f \in C(\mathbb{R})$, we define the wavelet type operators

$$A_k(f)(x) := \sum_{j=-\infty}^{\infty} \langle f, \varphi_{kj} \rangle \varphi_{kj}(x), \quad k \in \mathbb{Z}, \tag{2.1}$$

where

$$\langle f, \varphi_{kj} \rangle := \int_{-\infty}^{\infty} f(t) \varphi_{kj}(t) \, dt,$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 13–27.
springerlink.com

and

$$B_k(f)(x) := \sum_{j=-\infty}^{\infty} 2^{-\frac{k}{2}} f\left(2^{-k}j\right)\varphi_{kj}(x), \quad k \in \mathbb{Z}. \tag{2.2}$$

In [86], was proved that if f is non-decreasing on $\mathbb{R}$, then, under certain conditions on φ, the linear operator functions $A_k(f)$ and $B_k(f)$ are non-decreasing and such that $|f(x) - A_k(f)|$ and $|f(x) - B_k(f)|$ can be estimated by $\omega_1(f, 2^{-k+1}a)$ or $\omega_2\left(f, 2^{-k+1}a\right)$, where $\omega_r(f,t)$, $r = 1, 2$, is the r-th modulus of smoothness of f on $\mathbb{R}$. Here

$$\omega_1(f,h) := \sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\le h}} |f(x) - f(y)|,$$

and

$$\omega_2(f,h) := \sup_{x,t:|t|\le h} |f(x) - 2f(x+t) + f(x+2t)|.$$

In this chapter, we are going to discuss the convex wavelet like approximation. We show that if f is convex on $\mathbb{R}$, then, under certain conditions on φ, the functions $A_k(f)$ and $B_k(f)$ are convex on $\mathbb{R}$ and also have the desired estimates for the degree of approximation. Moreover we consider the r-th convexity which is preserved by A_k and B_k for any positive integer r and we present similar results. We also discuss the case of coconvex probabilistic wavelet like approximation.

2.2 Convex Wavelet Like Approximation

Let

$$A(x) := \sum_{j=-\infty}^{\infty} C_j \varphi(x-j), \tag{2.3}$$

where $\{C_j\}$ is a sequence of real numbers. We first study the convexity of $A(x)$.

Lemma 2.1. Suppose that $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$, $\mathrm{supp}\varphi(x) \subseteq [-a,a]$, $0 < a < +\infty$, and satisfies

(i) $\sum_{j=-\infty}^{\infty} \varphi(x-j)$ and $\sum_{j=-\infty}^{\infty} j\varphi(x-j)$ are linear functions on $\mathbb{R}$.

(ii) there exist real numbers b_1 and b_2, $b_1 \le b_2$ such that $\varphi(x)$ is convex on $(-\infty, b_1]$ and $[b_2, +\infty)$ respectively, and $\varphi(x)$ is concave on $[b_1, b_2]$.

Then, if $\{C_j\}$ is a convex sequence, i.e., $\{C_j - C_{j-1}\}$ is non-decreasing, the function $A(x)$ defined by (2.3) is convex on $\mathbb{R}$.

Remark 2.2. We have some examples of $\varphi(x)$ which satisfy all the conditions of Lemma 2.1:

$$\varphi_1(x) = \begin{cases} 1+x, & -1 \le x < 0, \\ 1-x, & 0 \le x \le 1, \\ 0 & \text{otherwise}; \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 0, & x \le -\frac{3}{2}, \\ \frac{1}{2}\left(\frac{3}{2}+x\right)^2, & -\frac{3}{2} < x \le -\frac{1}{2}, \\ 1 + x - \left(x+\frac{1}{2}\right)^2, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2}\left(\frac{3}{2}-x\right)^2, & \frac{1}{2} \le x \le \frac{3}{2}, \\ 0, & x > \frac{3}{2}. \end{cases}$$

Indeed, the functions $\varphi_1(x)$ and $\varphi_2(x)$ satisfy

(i)' $\sum_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$, on $\mathbb{R}$ and

$$\sum_{j=-\infty}^{\infty} j\varphi(x-j) = x, \quad \text{on } \mathbb{R}.$$

We can also choose $b_1 = -\frac{1}{2}$ and $b_2 = \frac{1}{2}$ to have $\varphi_1(x)$ and $\varphi_2(x)$ satisfying (ii).

Proof. For any fixed x and $0 < \Delta x < \frac{1}{2}$, let j_1 be the integer such that $x - j_1 \le b_1 < x - j_1 + 1$ and j_2 be the integer such that $x - j_2 \le b_2 < x - j_2 + 1$. Because $b_1 \le b_2$, we have $j_2 \le j_1$. Since $0 < \Delta x < \frac{1}{2}$ and $\varphi(x)$ satisfies (ii), we see that

$$\varphi(x + 2\Delta x - j) - 2\varphi(x + \Delta x - j) + \varphi(x-j) \ge 0, \tag{2.4}$$

if $-\infty < j \le j_2 - 1$;

$$\varphi(x + 2\Delta x - j) - 2\varphi(x + \Delta x - j) + \varphi(x-j) \le 0, \tag{2.5}$$

if $j_2 + 1 \le j_1 - 1$ and $j_2 + 1 \le j \le j_1 - 1$;

$$\varphi(x + 2\Delta x - j) - 2\varphi(x + \Delta x - j) + \varphi(x-j) \ge 0, \tag{2.6}$$

if $j_1 + 1 \le j < +\infty$.

Suppose that $j_2 + 1 \le j_1 - 1$. From the property (i) of φ we have

$$\sum_{j=-\infty}^{\infty} [\varphi(x + 2\Delta x - j) - 2\varphi(x + \Delta x - j) + \varphi(x-j)] = 0, \tag{2.7}$$

and

$$\sum_{j=-\infty}^{\infty} j\,[\varphi(x + 2\Delta x - j) - 2\varphi(x + \Delta x - j) + \varphi(x-j)] = 0. \tag{2.8}$$

On the other hand, since $\{C_j\}$ is a convex sequence, $\{C_j - C_{j-1}\}$ is a non-decreasing sequence and then for $j \ge j_1 + 1$ we derive

$$\frac{C_j - C_{j_2}}{j - j_2} =$$

$$\frac{(C_j - C_{j-1}) + (C_{j-1} - C_{j-2}) + ... + (C_{j_1} - C_{j_1-1}) + ... + (C_{j_2+1} - C_{j_2})}{j - j_2} \geq$$

$$\frac{(C_{j_1} - C_{j_1-1}) + ... + (C_{j_2+1} - C_{j_2})}{j_1 - j_2} = \frac{C_{j_1} - C_{j_2}}{j_1 - j_2}.$$

That is

$$C_j - C_{j_2} - \frac{j - j_2}{j_1 - j_2}\left(C_{j_1} - C_{j_2}\right) \geq 0, \quad j_1 + 1 \leq j < +\infty. \tag{2.9}$$

Similarly we get

$$C_j - C_{j_2} - \frac{j - j_2}{j_1 - j_2}\left(C_{j_1} - C_{j_2}\right) \geq 0, \quad -\infty < j \leq j_2 - 1; \tag{2.10}$$

and

$$C_j - C_{j_2} - \frac{j - j_2}{j_1 - j_2}\left(C_{j_1} - C_{j_2}\right) \leq 0, \quad j_2 + 1 \leq j \leq j_1 - 1. \tag{2.11}$$

Obviously we see that

$$C_j - C_{j_2} - \frac{j - j_2}{j_1 - j_2}\left(C_{j_1} - C_{j_2}\right) = 0, \quad j = j_2 \text{ or } j = j_1. \tag{2.12}$$

It follows from (2.7) and (2.8) that

$$A\left(x + 2\Delta x\right) - 2A\left(x + \Delta x\right) + A\left(x\right) =$$

$$\sum_{j=-\infty}^{\infty} C_j \left[\varphi\left(x + 2\Delta x - j\right) - 2\varphi\left(x + \Delta x - j\right) + \varphi\left(x - j\right)\right] =$$

$$\sum_{j=-\infty}^{\infty} \left[C_j - C_{j_2} - \frac{j - j_2}{j_1 - j_2}\left(C_{j_1} - C_{j_2}\right)\right] \cdot$$

$$\left[\varphi\left(x + 2\Delta x - j\right) - 2\varphi\left(x + \Delta x - j\right) + \varphi\left(x - j\right)\right] =$$

$$\sum_{j=-\infty}^{j_2-1} + \sum_{j=j_2} + \sum_{j=j_2+1}^{j_1-1} + \sum_{j=j_1} + \sum_{j=j_1+1}^{\infty} := I_1 + I_2 + I_3 + I_4 + I_5.$$

From (2.4)-(2.6) and (2.9)-(2.12) we have

$$I_1 \geq 0, \;\; I_2 = 0, \;\; I_3 \geq 0, \;\; I_4 = 0 \;\; \text{and } I_5 \geq 0$$

which imply

$$A\left(x + 2\Delta x\right) - 2A\left(x + \Delta x\right) + A\left(x\right) \geq 0. \tag{2.13}$$

If $j_2 = j_1 - 1$, we have

$$A\left(x + 2\Delta x\right) - 2A\left(x + \Delta x\right) + A\left(x\right) = I_1 + I_2 + I_4 + I_5 = I_1 + I_5 \geq 0. \tag{2.14}$$

If $j_2 = j_1$, by (2.7) and (2.8), we observe that

$$A(x+2\Delta x) - 2A(x+\Delta x) + A(x) =$$

$$\sum_{j=-\infty}^{\infty} \left[C_j - C_{j_2} - (j - j_2)(C_{j_2+1} - C_{j_2})\right] \cdot$$

$$[\varphi(x+2\Delta x - j) - 2\varphi(x+\Delta x - j) + \varphi(x-j)] =$$

$$\sum_{j=-\infty}^{j_2-1} + \sum_{j=j_2} + \sum_{j=j_2+1}^{\infty} := I_1' + I_2' + I_3'.$$

Using a similar argument as before, we have

$$I_1' \geq 0, \;\; I_2' = 0 \;\; \text{and} \;\; I_3' \geq 0$$

which gives (2.13) again. Thus $A(x)$ is a convex function on $\mathbb{R}$. ■

Theorem 2.3. Suppose that $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$, supp $\varphi(x) \subseteq [-a, a]$, $0 < a < +\infty$ and satisfies

(i) $\sum_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$ on $\mathbb{R}$;

(ii) $\sum_{j=-\infty}^{\infty} j\varphi(x-j)$ is a linear function on $\mathbb{R}$;

(iii) there exist real numbers b_1 and b_2, $b_1 \leq b_2$ such that $\varphi(x)$ is convex on $(-\infty, b_1]$ and $[b_2, +\infty)$ respectively, and $\varphi(x)$ is concave on $[b_1, b_2]$.

Then, for $f \in C(\mathbb{R})$, if f is a convex function on $\mathbb{R}$, the linear wavelet operator function $A_k(f)$ defined by (2.1) are also convex on $\mathbb{R}$ and satisfy

$$|A_k(f)(x) - f(x)| \leq \omega_1\left(f, 2^{-k+1}a\right), \quad x \in \mathbb{R}, \;\; k \in \mathbb{Z}, \tag{2.15}$$

where $\omega_1(f, h)$ is the first modulus of continuity of f.

Remark 2.4. (1) Because $\varphi(x) = 0$ for $x \in (-\infty, -a) \cup (a, +\infty)$, if $\varphi(x)$ has property (iii), then

$$\varphi(x) \geq 0, \quad x \in \mathbb{R}. \tag{2.16}$$

Hence the linear wavelet operators A_k are positive.

(2) The condition (i) of Theorem 2.3 implies that [86]

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1.$$

Proof. First let us consider the convexity of $A_k(f)(x)$ on $\mathbb{R}$.

It follows from (2.16) that

$$\langle f, \varphi_{0j} \rangle = \int_{-\infty}^{\infty} f(t)\varphi(t-j)\, dt = \int_{-\infty}^{\infty} f(u+j)\varphi(u)\, du$$

is a convex sequence if f is convex on $\mathbb{R}$. Then, by Lemma 2.1, $A_0(f)(x)$ is a convex function on $\mathbb{R}$. Since for any $k \in \mathbb{Z}$ we have

$$A_k(f)(x) = A_0\left(f\left(2^{-k}\right)\right)\left(2^k x\right),$$

hence $A_k(f)(x)$ are convex functions on $\mathbb{R}$ as well.

From [86], we know that if $\varphi(x)$ is bounded and continuous with supp$\varphi(x) \subseteq [-a, a]$ and satisfies (i), then hold the estimates (2.15). ■

Theorem 2.5. Suppose that $\varphi(x)$ satisfies all the conditions of Theorem 2.3. Then, for $f \in C(\mathbb{R})$, if f is a convex function on $\mathbb{R}$, the linear wavelet operator functions $B_k(f)$ defined by (2.2) are also convex on $\mathbb{R}$ and satisfy

$$|B_k(f)(x) - f(x)| \leq \omega_1\left(f, 2^{-k}a\right), \quad x \in \mathbb{R}, \ k \in \mathbb{Z}. \tag{2.17}$$

Moreover, the inequalities (2.17) are sharp.

Proof. Since $\{f(j)\}$ is a convex sequence, from Lemma 2.1, we know that $B_0(f)(x)$ is a convex function on $\mathbb{R}$. Then, by

$$B_k(f)(x) = B_0\left(f\left(2^{-k}\right)\right)\left(2^k x\right),$$

it follows that $B_k(f)(x)$ are convex functions on $\mathbb{R}$ as well.

From [86] we know that inequalities (2.17) are valid. Now we prove the sharpness of (2.17).

Assume that there is a positive number $C < 1$ such that for any convex $f \in C(\mathbb{R})$ hold

$$|B_k(f)(x) - f(x)| \leq C\omega\left(f, 2^{-k}a\right), \quad x \in \mathbb{R}, \ k \in \mathbb{Z}. \tag{2.18}$$

Define

$$\varphi_3(x) = \begin{cases} x + \left[\frac{1}{1-c}\right] + 2, & -\left[\frac{1}{1-c}\right] - 2 \leq x \leq -\left[\frac{1}{1-c}\right] - 1, \\ -\left[\frac{1}{1-c}\right] - x, & -\left[\frac{1}{1-c}\right] - 1 < x \leq -\left[\frac{1}{1-c}\right], \\ 0 \quad \text{otherwise}, \end{cases}$$

where $[\cdot]$ is the integral part of the number. We have

$$\varphi_3(x) = \varphi_1\left(x + \left[\frac{1}{1-c}\right] + 1\right),$$

where $\varphi_1(x)$ is given in the remark of Theorem 2.3. Hence, from (ii)' we get that

$$\sum_{j=-\infty}^{\infty} \varphi_3(x-j) = \sum_{j=-\infty}^{\infty} \varphi_1\left(x + \left[\frac{1}{1-c}\right] + 1 - j\right) \equiv 1 \quad \text{on } \mathbb{R},$$

and

$$\sum_{j=-\infty}^{\infty} j\varphi_3(x-j) = \sum_{j=-\infty}^{\infty} j\varphi_1\left(x+\left[\frac{1}{1-c}\right]+1-j\right) = x+\left[\frac{1}{1-c}\right]+1 \quad \text{on } \mathbb{R},$$

i.e. $\varphi_3(x)$ satisfies the conditions (i) and (ii) of Theorem 2.3. Obviously, $\varphi_3(x)$ satisfies the condition (iii) of Theorem 2.3 and is a bounded continuous function on $\mathbb{R}$ with $\operatorname{supp}\varphi_3(x) \subseteq \left[-\left[\frac{1}{1-c}\right]-2, \left[\frac{1}{1-c}\right]+2\right]$. We consider $B_k(f)$ for such a φ_3.

Let

$$g(x) = (x-1)_+ \quad \text{on } \mathbb{R}.$$

Then $g(x) \in C(\mathbb{R})$ is a convex function, and

$$B_0(g)(x) = \sum_{j=-\infty}^{\infty} g(j)\varphi_3(x-j) = \sum_{j=2}^{\infty} g(j)\varphi_3(x-j).$$

From the definitions of g and φ_3, we have

$$B_0(g)(1) = \sum_{j=2}^{\infty} g(j)\varphi_3(1-j) = g\left(\left[\frac{1}{1-c}\right]+2\right) = \left[\frac{1}{1-c}\right]+1,$$

and then

$$B_0(g)(1) - g(1) = \left[\frac{1}{1-c}\right]+1. \tag{2.19}$$

On the other hand, we have

$$\omega(g,h) = h$$

and then from (2.18) we obtain

$$|B_0(g)(x) - g(x)| \le C \cdot \left(\left[\frac{1}{1-c}\right]+2\right), \qquad x \in \mathbb{R} \tag{2.20}$$

because of $a = \left[\frac{1}{1-c}\right]+2$ for φ_3. For $0 < C < 1$ it is easy to verify that $a-1 > Ca$, i.e.

$$\left[\frac{1}{1-c}\right]+1 > C\left(\left[\frac{1}{1-c}\right]+2\right).$$

Hence the equation (2.19) contradicts inequality (2.20). Thus inequality (2.17) is sharp for $k=0$. For any other $k \in \mathbb{Z}$ we can prove the sharpness of (2.17) by using a similar argument. ■

Theorem 2.6. Suppose that $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$, $\operatorname{supp}\varphi(x) \subseteq [-a,a]$, $0 < a < +\infty$ and satisfies

(i) $\sum_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$ on $\mathbb{R}$;
(ii) $\sum_{j=-\infty}^{\infty} j\varphi(x-j) = x$ on $\mathbb{R}$;
(iii) there exist real numbers b_1 and b_2, $b_1 \le b_2$ such that $\varphi(x)$ is convex on $(-\infty, b_1]$ and $[b_2, +\infty)$ respectively, and $\varphi(x)$ is concave on $[b_1, b_2]$.

Then, for $f \in C(\mathbb{R})$, if f is a convex function on $\mathbb{R}$, the linear wavelet operator functions $A_k(f)$ defined by (2.1) are also convex on $\mathbb{R}$ and satisfy

$$|A_k(f)(x) - f(x)| \le C\omega_2\left(f, 2^{-k+1}a\right), \quad x \in \mathbb{R},\, k \in \mathbb{Z},$$

where $\omega_2(f, h)$ is the second modulus of smoothness of f and C is an absolute constant.

Theorem 2.7. Suppose that $\varphi(x)$ satisfies all the conditions of Theorem 2.6. Then, for $f \in C(\mathbb{R})$, if f is a convex function on $\mathbb{R}$, the linear wavelet operator functions $B_k(f)$ defined by (2.2) are also convex on $\mathbb{R}$ and satisfy

$$|B_k(f)(x) - f(x)| \le C\omega_2\left(f, 2^{-k+1}a\right), \quad x \in \mathbb{R},\, k \in \mathbb{Z},$$

where $\omega_2(f, h)$ is the second modulus of smoothness of f and C is an absolute constant.

Theorem 2.6 and Theorem 2.7 come from Theorem 2.3, Theorem 2.5 and the results in [86].

2.3 r-th Convex Wavelet Approximation

Let $f \in C(\mathbb{R})$. If for any $x \in \mathbb{R}$ and $h > 0$ hold

$$\Delta_h^r f(x) := \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x+ih) \ge 0,$$

we say that $f(x)$ is r-th convex on $\mathbb{R}$. If for any $x \in \mathbb{R}$ and $h > 0$, $\Delta_h^r f(x) \le 0$, then $f(x)$ is r-th concave on $\mathbb{R}$.

For a real number sequence $\{C_j\}$, if for any $j \in \mathbb{Z}$ holds

$$\Delta^r C_j := \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} C_{j-r+i} \ge 0,$$

then we say that $\{C_j\}$ is an r-th convex sequence.

We shall discuss the r-th convex wavelet approximation. Here we only discuss the case of $r = 3$. For $r > 3$, we can use a similar method to deal with.

We need the following lemmas.

Lemma 2.8. Suppose that $\{C_j\}$ is a 3-th convex sequence of real numbers. Then for any fixed $j_1, j_2, j_3 \in \mathbb{Z}$ with $j_3 < j_2 < j_1$ we have

$$\left(\frac{C_j - C_{j_3}}{j - j_3} - \frac{C_{j_2} - C_{j_3}}{j_2 - j_3}\right) / (j - j_2) \geq$$

$$\left(\frac{C_{j_1} - C_{j_3}}{j_1 - j_3} - \frac{C_{j_2} - C_{j_3}}{j_2 - j_3}\right) / (j_1 - j_2), \quad j > j_1 \tag{2.21}$$

and

$$\left(\frac{C_j - C_{j_3}}{j - j_3} - \frac{C_{j_2} - C_{j_3}}{j_2 - j_3}\right) / (j - j_2) \leq$$

$$\left(\frac{C_{j_1} - C_{j_3}}{j_1 - j_3} - \frac{C_{j_2} - C_{j_3}}{j_2 - j_3}\right) / (j_1 - j_2), \quad j < j_1,\ j \neq j_3 \text{ and } j \neq j_2. \tag{2.22}$$

Proof. We first establish that

$$\left\{\frac{C_j - C_{j_3}}{j - j_3} - \frac{C_{j-1} - C_{j_3}}{j - j_3 - 1}\right\} \text{ is non-decreasing for } j \geq j_2 + 1. \tag{2.23}$$

Claim (2.23) is equivalent to that

$$I := \frac{C_j - C_{j_3}}{j - j_3} - 2\frac{C_{j-1} - C_{j_3}}{j - j_3 - 1} + \frac{C_{j-2} - C_{j_3}}{j - j_3 + 2} \geq 0 \quad \text{for } j \geq j_2 + 2. \tag{2.24}$$

Because

$$I = \frac{(j - j_3 - 1)(j - j_3 - 2)(C_j - C_{j_3}) - 2(j - j_3)(j - j_3 - 2)(C_{j-1} - C_{j_3})}{(j - j_3)(j - j_3 - 1)(j - j_3 - 2)} +$$

$$\frac{(j - j_3)(j - j_3 - 1)(C_{j-2} - C_{j_3})}{(j - j_3)(j - j_3 - 1)(j - j_3 - 2)}$$

$$= \frac{(j - j_3 - 1)(j - j_3 - 2)(C_j - 2C_{j-1} + C_{j-2})}{(j - j_3)(j - j_3 - 1)(j - j_3 - 2)} -$$

$$\frac{2(j - j_3 - 2)(C_{j-1} - C_{j-2}) + 2(C_{j-2} - C_{j_3})}{(j - j_3)(j - j_3 - 1)(j - j_3 - 2)},$$

for $j \geq j_2 + 2$ the claim (2.24) is equivalent to

$$J := (j - j_3 - 1)(j - j_3 - 2)(C_j - 2C_{j-1} + C_{j-2}) -$$

$$2(j - j_3 - 2)(C_{j-1} - C_{j-2}) + 2(C_{j-2} - C_{j_3}) \geq 0. \tag{2.25}$$

Since $\{C_j\}$ is a 3-th convex sequence, the sequence $\{C_j - 2C_{j-1} + C_{j-2}\}$ is non-decreasing, and then we obtain

$$(j - j_3 - 2)(C_{j-1} - C_{j-2}) - (C_{j-2} - C_{j_3}) =$$

$$\underbrace{(C_{j-1}-C_{j-2})+...+(C_{j-1}-C_{j-2})}_{j-j_3-2}-(C_{j-2}-C_{j-3})-$$
$$(C_{j-3}-C_{j-4})-...-(C_{j_3+1}-C_{j_3})=$$
$$[(C_{j-1}-C_{j-2})-(C_{j-2}-C_{j-3})]+[(C_{j-1}-C_{j-2})-(C_{j-3}-C_{j-4})]+...+$$
$$[(C_{j-1}-C_{j-2})-(C_{j_3+1}-C_{j_3})]=$$
$$[C_{j-1}-2C_{j-2}+C_{j-3}]+[(C_{j-1}-2C_{j-2}+C_{j-3})+(C_{j-2}-2C_{j-3}+C_{j-4})]+...$$
$$+[(C_{j-1}-2C_{j-2}+C_{j-3})+...+(C_{j_3+2}-2C_{j_3+1}+C_{j_3})]\leq$$
$$(C_j-2C_{j-1}+C_{j-2})(1+2+...+(j-j_3-2))=$$
$$\frac{(j-j_3-1)(j-j_3-2)}{2}(C_j-2C_{j-1}+C_{j-2}),$$

which gives (2.25). Hence we have (2.23).

From (2.23), it follows that

$$(j-j_2)^{-1}\left(\frac{C_j-C_{j_3}}{j-j_3}-\frac{C_{j_2}-C_{j_3}}{j_2-j_3}\right)=$$
$$(j-j_2)^{-1}\left[\left(\frac{C_j-C_{j_3}}{j-j_3}-\frac{C_{j-1}-C_{j_3}}{j-j_3-1}\right)+\left(\frac{C_{j-1}-C_{j_3}}{j-j_3-1}-\frac{C_{j-2}-C_{j_3}}{j-j_3-2}\right)+...\right.$$
$$\left.+\left(\frac{C_{j_1}-C_{j_3}}{j_1-j_3}-\frac{C_{j_1-1}-C_{j_3}}{j_1-j_3-1}\right)+...\left(\frac{C_{j_2+1}-C_{j_3}}{j_2-j_3+1}-\frac{C_{j_2}-C_{j_3}}{j_2-j_3}\right)\right]\geq$$
$$(j_1-j_2)^{-1}\left[\left(\frac{C_{j_1}-C_{j_3}}{j_1-j_3}-\frac{C_{j_1-1}-C_{j_3}}{j_1-j_3-1}\right)+...+\left(\frac{C_{j_2+1}-C_{j_3}}{j_2-j_3+1}-\frac{C_{j_2}-C_{j_3}}{j_2-j_3}\right)\right]$$
$$=(j_1-j_2)^{-1}\left(\frac{C_{j_1}-C_{j_3}}{j_1-j_3}-\frac{C_{j_2}-C_{j_3}}{j_2-j_3}\right),\quad j>j_1.$$

Thus (2.21) are valid. Similarly we can prove (2.22). ■

Lemma 2.9. Suppose that $\{C_j\}$ is a 3-th convex sequence of real numbers. Then for any fixed $j_1,j_2,j_3\in\mathbb{Z}$ with $j_3<j_2<j_1$ we have

$$Q_j:=C_j-\left\{C_{j_3}+(j-j_3)\frac{C_{j_2}-C_{j_3}}{j_2-j_3}+\right.$$
$$\left.(j-j_3)(j-j_2)\left[\frac{C_{j_1}-C_{j_3}}{j_1-j_3}-\frac{C_{j_2}-C_{j_3}}{j_2-j_3}\right]/(j_1-j_2)\right\}$$

$$\begin{array}{ll}\geq & 0,\quad j>j_1;=0,\ j=j_1;\\ \leq & 0,\quad j_2<j<j_1;=0,\ j=j_3;\\ \geq & 0,\quad j_3<j<j_2;=0,\ j=j_3;\\ \leq & 0,\quad j<j_3.\end{array}\tag{2.26}$$

Proof. It is easy to observe that

$$Q_j = 0, \ \text{ if } j = j_1, \ j_2 \text{ or } j_3;$$

and

$$Q_j = (j - j_3)(j - j_2)\left\{\left[\frac{C_j - C_{j_3}}{j - j_3} - \frac{C_{j_2} - C_{j_3}}{j_2 - j_3}\right] / (j - j_2) - \left[\frac{C_{j_1} - C_{j_3}}{j_1 - j_3} - \frac{C_{j_2} - C_{j_3}}{j_2 - j_3}\right] / (j_1 - j_2)\right\} \quad \text{if } j \neq j_3 \text{ or } j_2. \tag{2.27}$$

Then, if $j > j_1$, we have $j - j_3 > 0$, $j - j_2 > 0$ and (2.21), and therefore from (2.27) we have

$$Q_j \geq 0, \quad j \geq j_1.$$

If $j_2 < j < j_1$, we have $j - j_3 > 0$, $j - j_2 > 0$ and (2.22), and then by (2.27) we have

$$Q_j \leq 0, \quad j_2 \leq j \leq j_1.$$

If $j_3 < j < j_2$, because $j - j_3 > 0$, $j - j_2 < 0$ and (2.22), from (2.27) we have

$$Q_j \geq 0, \quad j_3 \leq j \leq j_2.$$

If $j < j_3$, because of $j - j_3 < 0$, $j - j_2 < 0$ and (2.22), from (2.27) we derive

$$Q_j \leq 0, \quad j \leq j_3.$$

■

Lemma 2.10. Suppose that $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$, $\text{supp}\varphi(x) \subseteq [-a, a]$, $0 < a < +\infty$, and satisfies

(i) $\sum_{j=-\infty}^{\infty} \varphi(x - j)$, $\sum_{j=-\infty}^{\infty} j\varphi(x - j)$ and $\sum_{j=-\infty}^{\infty} j^2\varphi(x - j)$ are quadratic functions on $\mathbb{R}$,

(ii) there exist real numbers b_1, b_2 and b_3, $b_1 \leq b_2 \leq b_3$ such that $\varphi(x)$ is 3-th convex on $(-\infty, b_1]$ and $[b_2, b_3]$ respectively, and $\varphi(x)$ is 3-th concave on $[b_1, b_2]$ and $[b_3, +\infty)$ respectively.

Then, if $\{C_j\}$ is a 3-th convex sequence, the function $A(x)$ defined by (2.3) is 3-th convex on $\mathbb{R}$.

Remark 2.11. The function $\varphi_2(x)$ in the remark of Lemma 2.1 satisfies all the conditions of Lemma 2.10 with $b_1 = -1$, $b_2 = 0$ and $b_3 = 1$.

Proof. For any fixed x and $0 < \Delta x < \frac{1}{3}$, let j_1 be the integer such that $x - j_1 \leq b_1 < x - j_1 + 1$, j_2 be the integer such that $x - j_2 \leq b_2 < x - j_2 + 1$ and j_3 be the integer such that $x - j_3 \leq b_3 < x - j_3 + 1$. Because $b_1 \leq b_2 \leq b_3$, we have $j_3 \leq j_2 \leq j_1$. Since $0 < \Delta x < \frac{1}{3}$ and $\varphi(x)$ satisfies (ii), we obtain

$$\Delta^3\varphi := \varphi(x + 3\Delta x - j) - 3\varphi(x + 2\Delta x - j) + 3\varphi(x + \Delta x - j) - \varphi(x - j) \leq 0, \tag{2.28}$$

if $-\infty < j \le j_3 - 1$;

$$\Delta^3 \varphi \ge 0, \tag{2.29}$$

if $j_3 + 1 \le j_2 - 1$ and $j_3 + 1 \le j \le j_2 - 1$;

$$\Delta^3 \varphi \le 0, \tag{2.30}$$

if $j_2 + 1 \le j_1 - 1$ and $j_2 + 1 \le j \le j_1 - 1$;

$$\Delta^3 \varphi \ge 0, \tag{2.31}$$

if $j_1 + 1 \le j < +\infty$.

Assume that $j_3 + 1 \le j_2 - 1$ and $j_2 + 1 \le j_1 - 1$. From the property (i) of φ we have

$$\sum_{j=-\infty}^{\infty} \left[\varphi\left(x + 3\Delta x - j\right) - 3\varphi\left(x + 2\Delta x - j\right) + 3\varphi\left(x + \Delta x - j\right) - \varphi\left(x - j\right)\right] = 0, \tag{2.32}$$

$$\sum_{j=-\infty}^{\infty} j\left[\varphi\left(x + 3\Delta x - j\right) - 3\varphi\left(x + 2\Delta x - j\right) + 3\varphi\left(x + \Delta x - j\right) - \varphi\left(x - j\right)\right] = 0, \tag{2.33}$$

and

$$\sum_{j=-\infty}^{\infty} j^2\left[\varphi\left(x + 3\Delta x - j\right) - 3\varphi\left(x + 2\Delta x - j\right) + 3\varphi\left(x + \Delta x - j\right) - \varphi\left(x - j\right)\right] = 0. \tag{2.34}$$

Then we obtain

$$\begin{gathered}
\Delta := A\left(x + 3\Delta x\right) - 3A\left(x + 2\Delta x\right) + 3A\left(x + \Delta x\right) - A\left(x\right) = \\
\sum_{j=-\infty}^{\infty} \left[C_j - C_{j_3} - \left(j - j_3\right)\frac{C_{j_2} - C_{j_3}}{j_2 - j_3} - \right. \\
\left. \left(j - j_3\right)\left(j - j_2\right)\left(\frac{C_{j_1} - C_{j_3}}{j_1 - j_3} - \frac{C_{j_2} - C_{j_3}}{j_2 - j_3}\right) / \left(j_1 - j_2\right)\right] \\
\cdot \left[\varphi\left(x + 3\Delta x - j\right) - 3\varphi\left(x + 2\Delta x - j\right) + 3\varphi\left(x + \Delta x - j\right) - \varphi\left(x - j\right)\right] = \\
\sum_{j=-\infty}^{j_3-1} + \sum_{j=j_3} + \sum_{j=j_3+1}^{j_2-1} + \sum_{j=j_2} + \sum_{j=j_2+1}^{j_1-1} + \sum_{j=j_1} + \sum_{j=j_1+1}^{\infty} := \\
\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6 + \Delta_7.
\end{gathered} \tag{2.35}$$

From Lemma 2.9 and (2.28)-(2.31), we have

$$\Delta_2 = \Delta_4 = \Delta_6 = 0,$$

and

$$\Delta_1 \geq 0, \quad \Delta_3 \geq 0, \quad \Delta_5 \geq 0, \quad \Delta_7 \geq 0$$

which implies

$$A(x+3\Delta x) - 3A(x+2\Delta x) + 3A(x+\Delta x) - A(x) \geq 0. \tag{2.36}$$

If $j_3 = j_2 - 1$ and $j_2 + 1 \leq j_1 - 1$, we have

$$\Delta = \Delta_1 + \Delta_2 + \Delta_4 + \Delta_5 + \Delta_6 + \Delta_7 \geq 0. \tag{2.37}$$

If $j_3 + 1 \leq j_2 - 1$ and $j_2 = j_1 - 1$, we obtain

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_6 + \Delta_7 \geq 0. \tag{2.38}$$

If $j_3 = j_2 - 1$ and $j_2 = j_1 - 1$, then we see that

$$\Delta = \Delta_1 + \Delta_2 + \Delta_4 + \Delta_6 + \Delta_7 \geq 0. \tag{2.39}$$

Now we consider the case that $j_3 = j_2$. We redefine $j_3 := j_2 - 1$, and still have (2.37) or (2.39). If $j_2 = j_1$, we redefine $j_1 := j_2 + 1$ and still have (2.38) or (2.39). In summary, $A(x)$ is a 3-th convex function on $\mathbb{R}$. ∎

Now we are ready to establish the following theorems for the 3-th convex wavelet like approximation.

Theorem 2.12. Suppose that $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$, $\text{supp}\varphi(x) \subseteq [-a, a]$, $0 < a < +\infty$, and satisfies

(i) $\sum_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$ on $\mathbb{R}$; $\sum_{j=-\infty}^{\infty} j\varphi(x-j)$ and $\sum_{j=-\infty}^{\infty} j^2\varphi(x-j)$ are quadratic functions on $\mathbb{R}$;

(ii) there exist real numbers b_1, b_2 and b_3, $b_1 \leq b_2 \leq b_3$ such that $\varphi(x)$ is 3-th convex on $(-\infty, b_1]$ and $[b_2, b_3]$ respectively, and $\varphi(x)$ is 3-th concave on $[b_1, b_2]$ and $[b_3, +\infty)$ respectively.

Then, for $f \in C(\mathbb{R})$, if f is a 3-th convex function on $\mathbb{R}$, the linear wavelet operators $A_k(f)$ defined by (2.1) and $B_k(f)$ defined by (2.2) are also 3-th convex functions on $\mathbb{R}$ and satisfy

$$|A_k(f)(x) - f(x)| \leq C\omega_1\left(f, 2^{-k+1}a\right), \tag{2.40}$$

$$|B_k(f)(x) - f(x)| \leq C\omega_1\left(f, 2^{-k+1}a\right), \quad x \in \mathbb{R}, k \in \mathbb{Z}, \tag{2.41}$$

where C is a constant only depending on φ.

Proof. It is easy to see that if f is 3-th convex, then $\{\langle f, \varphi_{0j}\rangle\}$ and $\{f(j)\}$ are 3-th convex sequences. Hence, by Lemma 2.10, $A_0(f)$ and $B_0(f)$ are 3-th convex, and so do $A_k(f)$ and $B_k(f)$.

For proving (2.40) and (2.41), we notice that under condition (ii) the function $\varphi(x)$ may not be always positive. But since $\varphi(x)$ is bounded and compactly supported, we can use a similar method as in [86] to obtain (2.40) and (2.41) with the constant C depending on φ. ∎

Theorem 2.13. Suppose that $\varphi(x)$ satisfies all the conditions in Theorem 2.12 except that (i) is replaced by
(i)' $\sum_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$ on $\mathbb{R}$;

$$\sum_{j=-\infty}^{\infty} j\varphi(x-j) = x \quad \text{on } \mathbb{R};$$

and

$$\sum_{j=-\infty}^{\infty} j^2\varphi(x-j) \quad \text{is a quadratic function on } \mathbb{R}.$$

Then, for $f \in C(\mathbb{R})$, if f is a 3-th convex function on $\mathbb{R}$, the linear wavelet operators $A_k(f)$ and $B_k(f)$ are also 3-th convex functions on $\mathbb{R}$ and satisfy

$$|A_k(f)(x) - f(x)| \le C\omega_2\left(f, 2^{-k+1}a\right),$$

$$|B_k(f)(x) - f(x)| \le C\omega_2\left(f, 2^{-k+1}a\right), \quad x \in \mathbb{R},\ k \in \mathbb{Z},$$

where C is a constant only depending on φ.

2.4 Coconvex Probabilistic Wavelet Like Approximation

In this section we are going to discuss the wavelet like approximation to some kind of special continuous distribution functions which are concave on $(x_0, +\infty)$.

Lemma 2.14. Suppose that $\varphi(x)$ satisfies all the conditions of Lemma 2.1. Let $f(x) \in C(\mathbb{R})$ be concave on $(x_0, +\infty)$. Then $B_0(f)(x)$ defined by (2.2) is concave on $(x_0 + a, +\infty)$.
Proof. Since $\text{supp}\varphi(x) \subseteq [-a, a]$, for $x \in (x_0 + a, +\infty)$ we have

$$B_0(f)(x) = \sum_{j=-\infty}^{\infty} f(j)\varphi(x-j) = \sum_{x-a\le j\le x+a} f(j)\varphi(x-j)$$
$$= \sum_{x_0<j} f(j)\varphi(x-j). \tag{2.42}$$

Let j_0 be the smallest integer such that $j_0 > x_0$, and

$$C_j := f(j), \quad j = j_0, j_0+1, \dots .$$

We define C_j for $j < j_0$ by the formula:

$$C_j = 2C_{j+1} - C_{j+2}, \quad j = j_0 - 1,\ j_0 - 2, \dots .$$

Because $f(x)$ is concave on $(x_0,+\infty)$, the sequence $\{C_j\}_{j=-\infty}^{\infty}$ is a concave sequence. Moreover, from $\mathrm{supp}\varphi(x) \subseteq [-a,a]$ and (2.42), we get that

$$B_0(f)(x) = \sum_{j=-\infty}^{\infty} C_j \varphi(x-j) \quad \text{for } x \in (x_0+a,+\infty).$$

But, by Lemma 2.1, the right-hand side of the above formula is a concave function on $\mathbb{R}$. Hence $B_0(f)(x)$ is concave on $(x_0+a,+\infty)$.

From

$$B_k(f)(x) = B_0\left(f\left(2^{-k}\right)\right)\left(2^k x\right),$$

it follows. ■

Lemma 2.15. Suppose that $\varphi(x)$ satisfies all the conditions of Lemma 2.1. Let $f(x) \in C(\mathbb{R})$ be concave on $(x_0,+\infty)$. Then, for any $k \in \mathbb{Z}$, $B_k(f)(x)$ defined by (2.2) is concave on $\left(x_0+2^{-k}a,+\infty\right)$.

From Theorem 4 in [86] and Lemma 2.15, we obtain

Theorem 2.16. Suppose that $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$, $\mathrm{supp}\varphi(x) \subseteq [-a,a]$, $0<a<+\infty$, and satisfies

(i) $\sum_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$ on $\mathbb{R}$; $\sum_{j=-\infty}^{\infty} j\varphi(x-j)$ is a linear function on $\mathbb{R}$;

(ii) there is a number b_0 such that $\varphi(x)$ is non-decreasing if $x \le b_0$ and is non-increasing if $x \ge b_0$;

(iii) there are real numbers b_1 and b_2, $b_1 \le b_2$ such that $\varphi(x)$ is convex on $(-\infty,b_1]$ and $[b_2,+\infty)$ respectively, and $\varphi(x)$ is concave on $[b_1,b_2]$.

Let $F(x)$ be a continuous distribution function on $\mathbb{R}$ that is concave on $(x_0,+\infty)$. Then the linear wavelet operator $B_k(F)$ defined by (2.2) are distribution functions which are concave on $\left(x_0+2^{-k}a,+\infty\right)$ and satisfy

$$|B_k(F)(x)-F(x)| \le \omega_1\left(F,2^{-k}a\right), \quad x \in \mathbb{R},\ k \in \mathbb{Z}. \tag{2.43}$$

The examples $\varphi_1(x)$ and $\varphi_2(x)$ showed in the Remark 2.2 of Lemma 2.1 satisfy all the conditions of Theorem 2.16.

If the condition (i) in Theorem 2.16 is replaced by

(i)' $\sum_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$ on $\mathbb{R}$,

$$\sum_{j=-\infty}^{\infty} j\varphi(x-j) = x \ \text{ on } \mathbb{R},$$

then inequality (2.43) can be replaced by

$$|B_k(F)(x)-F(x)| \le C\omega_2\left(F,2^{-k+1}a\right).$$

We can also obtain similar results for $A_k(F)(x)$.

Besides, we can use the same methods to discuss the coconvex probabilistic wavelet like approximation to the continuous distribution functions which have r-th derivatives concave or convex (depending on whether r is even or odd) on $(x_0,+\infty)$.

3

Bidimensional Constrained Wavelet Like Approximation

Shape-preserving properties of some naturally arising bivariate wavelet operators B_n are presented. Namely, let $f \in C^k(\mathbb{R}^2)$, $k > 0$, $r, s \geq 0$ all integers such that $r + s = k$. If

$$\frac{\partial^{r+s} f}{\partial x^r \partial y^r}(x, y) \geq 0,$$

then it is established, under mild conditions on B_n, that

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n f(x, y) \geq 0,$$

also pointwise convergence of $B_n(f)$ to f is given with rates through a Jackson type inequality. Related simultaneous shape-preserving results are also given for special type of wavelet operators B_n. This chapter relies on [89].

3.1 Introduction

Let $\varphi(x, y)$ be a bounded compactly supported function on $\mathbb{R}^2$ with supp $\varphi(x, y) \subseteq [-a, a] \times [-b, b]$, $0 < a, b < +\infty$, and $f(x, y) \in C(\mathbb{R}^2)$. For $n \in \mathbb{Z}$ we define

$$B_n(f)(x, y) =: \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(2^{-n} i, 2^{-n} j) \varphi(2^n x - i, 2^n y - j). \tag{3.1}$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 29–40.
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Since $\varphi(x,y)$ is compactly supported, there are only finite non-zero terms involved in the summations of (3.1). So $B_n(f)(x,y)$ are well-defined on $\mathbb{R}^2$.

We are concerned with the problems of shape-preserving approximation of $f(x,y)$ by $B_n(f)(x,y)$ on $\mathbb{R}^2$, and want to know when is valid

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x,y) \geq 0, \ if \ \frac{\partial^{r+s}}{\partial x^r \partial y^r} f(x,y) \geq 0, \ (x,y) \in \mathbb{R}^2.$$

We also consider the simultaneous shape-preserving approximation. That is, for each $r^* = 0, 1, \ldots, r, \ s^* = 0, 1, \ldots, s$, if

$$\varepsilon_{r^*,s^*} \cdot \frac{\partial^{r^*+s^*}}{\partial x^{r^*} \partial y^{r^*}} f(x,y) \geq 0,$$

where $\varepsilon_{r^*,s^*} = \pm 1, \ (x,y) \in \mathbb{R}^2$, then when hold for any $n \in \mathbb{Z}$ the same inequalities for $B_n(f)(x,y)$. Theorem 3.5 and Theorem 3.6 established later in this chapter can give some answers to these problems.

3.2 Results

We first prove some lemmas.

Lemma 3.1. Let $f(x) \in C(\mathbb{R})$ and k be a positive integer. Assume that $f^{(k)}(x) \in C(\mathbb{R}), \ f^{(k)}(x) \geq 0$ for $x \in \mathbb{R}$, and there are $-\infty =: x_0 < x_1 < x_2 \ldots < x_k < x_{k+1} := +\infty$ such that $f(x_i) = 0 \ (i = 1, 2, \ldots, k)$. Then

$$(-1)^{k+i} f(x) \geq 0, \ x_i < x < x_{i+1}, \ i = 0, 1, 2, \ldots, k \tag{3.2}$$

Proof. From the assumption, $f(x)$ has at least k zeros. Hence, by Rolle's theorem, $f^{(k-1)}(x)$ has at least one zero $x^{(0)}$.

Assume that $f^{(k-1)}(x)$ has only one zero $x^{(0)}$. Since $f^{(k)}(x) \geq 0$, $f^{(k-1)}(x)$ is non-decreasing and then $f^{(k-1)}(x) < 0$ for $x < x^{(0)}$ and $f^{(k-1)}(x) > 0$ for $x > x^{(0)}$.

Thus $f^{(k-2)}(x)$ is strictly decreasing if $x < x^{(0)}$ and is strictly increasing if $x > x^{(0)}$. From this it follows that $f^{(k-2)}(x)$ has at most two zeros. On the other hand, by the assumptions and Rolle's theorem, $f^{(k-2)}(x)$ has at least two zeros. Therefore, $f^{(k-2)}(x)$ has exactly two zeros and is negative if x is between these two zeros and is positive otherwise. Repeating this argument for $f^{(k-3)}(x), \ldots$, and $f(x)$, we know that $f(x)$ has exactly k zeros which are $x_1, x_2, \ldots, x_k$ and satisfies (3.2).

Now assume that there is another zero $x^{(1)} \neq x^{(0)}$ such that $f^{(k-1)}(x^{(1)})=$ 0. Because $f^{(k)}(x) \geq 0$, $f^{(k-1)}(x)$ is a non-decreasing which implies $f^{(k-1)}$ $(x)= 0$ whenever x is between $x^{(0)}$ and $x^{(1)}$.

Therefore we will have an interval $[x', x"]$ which contains $x^{(0)}$ and $x^{(1)}$ such that $f^{(k-1)}(x) = 0$ if $x \in [x', x"]$ and $f^{(k-1)}(x) \neq 0$ if $x < x'$ or

$x > x”$. Since $f^{(k-1)}(x)$ is non-decreasing, we have $f^{(k-1)}(x) < 0$ for $x < x'$ and $f^{(k-1)}(x) > 0$ for $x > x”$. From this it follows that $f^{(k-2)}(x)$ is strictly decreasing on $(-\infty, x')$ and is strictly increasing on $(x”, +\infty)$ and is constant on $[x', x”]$. Since $f^{(k-2)}(x)$ has at least two zeros, there are only two possibilities: (i) $f^{(k-2)}(x) = 0,\ x \in [x', x”]$ and $f^{(k-2)}(x) > 0$ for $x \notin [x', x”]$; (ii) $f^{(k-2)}(x)$ has exactly two zeros: one zero is on $(-\infty, x')$ while another one is on $(x”, +\infty)$.

If (i) holds, since $f^{(k-3)}(x)$ has as least three zeros by the assumptions and Rolle's theorem, we will have $f^{(k-3)}(x) = 0, x \in [x', x”]$ and $f^{(k-3)}(x) < 0$ for $x \in (-\infty, x')$ and $f^{(k-3)}(x) > 0$ for $x \in (x”, +\infty)$. Repeating these arguments, we obtain $f(x) = 0, x \in [x', x”]$ and $f(x) > 0$ for $x \in (-\infty, x') \cup (x”, +\infty)$ if k is even; and $f(x) < 0$ for $x \in (-\infty, x')$ and $f(x) > 0$ and $x \in (x”, +\infty)$ if k is odd. Thus we have $x_1, x_2, \ldots, x_k \in [x', x”]$ and (3.2) is satisfied.

If (ii) holds, then $f^{(k-2)}(x)$ has exactly two zeros, and $f^{(k-2)}(x)$ is negative whenever x is between these two zeros and is positive otherwise. Repeating the arguments we had at the beginning, we get that $f(x)$ has exactly k zeros which are $x_1, x_2, \ldots, x_k$ and satisfies (3.2). ■

Lemma 3.2. Suppose that $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$ with supp $\varphi(x) \subseteq [-a, a],\ 0 < a < +\infty$. If for some positive integer $k,\ \varphi^{(k-1)}(x) \in C(\mathbb{R})$ and there are k real numbers $-\infty =: x_0 < x_1 \leq x_2 \leq \ldots \leq x_k < x_{k+1} := +\infty$ such that $\varphi^{(k)}(x) \in C(x_i, x_{i+1})\ (i = 1, 2, \ldots, k)$ and

$$(-1)^i \varphi^{(k)}(x) \geq 0,\ x \in (x_i, x_{i+1}),\ i = 0, 1, 2, \ldots, k, \qquad (3.3)$$

then for any positive integer $s,\ s < k$, there are s real numbers $-\infty =: x'_0 < x'_1 \leq x'_2 \leq \ldots \leq x'_s < x'_{s+1} := +\infty$ such that

$$(-1)^i \varphi^{(s)}(x) \geq 0,\ x \in (x'_i, x'_{i+1}),\ i = 0, 1, 2, \ldots, s. \qquad (3.4)$$

Proof. Notice that we only need to prove the conclusion for $s = k - 1$.

Since $\varphi(x) = 0,\ x \notin [-a, a]$, we have $\varphi^{(k-1)}(x) = 0$ for $-\infty < x < -a$. From the assumption that $\varphi^{(k)}(x) \geq 0$ for $-\infty < x < x_1$, it follows that $\varphi^{(k-1)}(x)$ is non-decreasing on $(-\infty, x_1)$. Hence $\varphi^{(k-1)}(x) \geq 0$ for $x \in (-\infty, x_1)$. Since $\varphi^{(k)}(x) \leq 0$ for $x \in (x_1, x_2)$, $\varphi^{(k-1)}(x)$ is non-decreasing on $x \in (x_1, x_2)$. There are two possibilities: (i) $\varphi^{(k-1)}(x_2) < 0$ and (ii) $\varphi^{(k-1)}(x_2) \geq 0$. In the first case (i), since $\varphi^{(k-1)}(x_1) \geq 0$, $\varphi^{(k-1)}(x_2) < 0$ and $\varphi^{(k-1)}$ is non-increasing on (x_1, x_2), there is a $\xi \in [x_1, x_2)$ such that $\varphi^{(k-1)}(\xi) = 0,\ \varphi^{(k-1)}(x) \geq 0$ if $x < \xi$ and $\varphi^{(k-1)}(x) \leq 0$ if $\xi < x < x_2$. We take this ξ as x'_1 and then work on (x_2, x_3) to choose x'_2. If (ii) holds, we choose $x'_1 = x'_2 = x_2$, and we have $\varphi^{(k-1)}(x) \geq 0$ for $x \in (-\infty, x'_1)$ because $\varphi^{(k-1)}(x)$ is non-increasing on (x_1, x_2) and $\varphi^{(k-1)}(x_2) \geq 0$. Because $\varphi^{(k-1)}(x)$ is non-decreasing on (x_2, x_3) and $\varphi^{(k-1)}(x_2) \geq 0$, we have $\varphi^{(k-1)}(x) \geq 0$ on (x_2, x_3). Then we work on (x_3, x_4) to choose x'_3.

Repeating this process, suppose we have chosen $x'_1 \le x'_2 \le \ldots \le x'_{k-2} < x_{k-1}$ such that

$$(-1)^i \varphi^{(k-1)}(x) \ge 0,\ x \in (x'_i, x'_{i+1}),\ i = 0, 1, \ldots, k-3. \qquad (3.5)$$

Now we are going to choose x'_{k-1} such that

$$(-1)^i \varphi^{(k-1)}(x) \ge 0,\ x \in (x'_i, x'_{i+1}),\ i = k-2, k-1. \qquad (3.6)$$

Let k be even. From (3.3), we have

$$\varphi^{(k)}(x) \ge 0,\ x \in (x_{k-2}, x_{k-1}),$$

$$\varphi^{(k)}(x) \le 0,\ x \in (x_{k-1}, x_k)$$

and

$$\varphi^{(k)}(x) \ge 0,\ x \in (x_k, +\infty).$$

Therefore $\varphi^{(k-1)}(x)$ is non-decreasing on (x_{k-2}, x_{k-1}) and (x_k, ∞), and is non-increasing on (x_{k-1}, x_k). From (3.5), we have $\varphi^{(k-1)}(x'_{k-2}) \le 0$. If $\varphi^{(k-1)}(x_{k-1}) \le 0$, we may take $x'_{k-1} = x'_{k-2}$. Indeed, because $\varphi^{(k-1)}(x)$ is non-decreasing on (x_{k-2}, x_{k-1}), from $\varphi^{(k-1)}(x_{k-1}) \le 0$, we have $\varphi^{(k-1)}(x) \le 0$ on (x'_{k-1}, x_{k-1}). Because $\varphi^{(k-1)}(x)$ is non-increasing on (x_{k-1}, x_k), we still have $\varphi^{(k-1)}(x) \le 0$ on (x_{k-1}, x_k). Because $\varphi^{(k-1)}(x) = 0,\ x \in (a, +\infty)$ and $\varphi^{(k-1)}(x)$ is non-decreasing on $(x_k, +\infty)$, we obtain

$$\varphi^{(k-1)}(x) \le 0,\ x \in (x_k, +\infty). \qquad (3.7)$$

So it holds

$$\varphi^{(k-1)}(x) \le 0,\ x \in (x'_{k-1}, +\infty)$$

which gives (3.6).

If $\varphi^{(k-1)}(x_{k-1}) > 0$, noticing that $\varphi^{(k-1)}(x)$ is non-increasing on (x_{k-1}, x_k) and (3.7), we can find a $\xi \in (x_{k-1}, x_k]$ such that $\varphi^{(k-1)}(\xi) = 0,\ \varphi^{(k-1)}(x) \ge 0$ if $x_{k-1} < x < \xi$ and $\varphi^{(k-1)}(x) \le 0$ if $\xi < x \le x_k$. Take $x'_{k-1} := \xi$. Then we get

$$\varphi^{(k-1)}(x) \ge 0,\ x \in (x_{k-1}, x'_{k-1}) \qquad (3.8)$$

and

$$\varphi^{(k-1)}(x) \le 0,\ x \in (x'_{k-1}, +\infty). \qquad (3.9)$$

by (3.6). On the other hand, if $\varphi^{(k-1)}(x_{k-1}) > 0$, from the selection of x'_{k-2}, we know $x'_{k-2} < x_{k-1}$ and $\varphi^{(k-1)}(x'_{k-2}) = 0$. Then, since $\varphi^{(k-1)}(x)$ is non-decreasing on (x_{k-2}, x_{k-1}), we get

$$\varphi^{(k-1)}(x) \geq 0,\ x \in (x'_{k-2}, x_{k-1}).$$

From this and (3.8) we obtain

$$\varphi^{(k-1)}(x) \geq 0,\ x \in (x'_{k-2}, x'_{k-1}),$$

which with (3.9) gives (3.6).

We can use a similar method to choose x'_{k-1} if k is odd. This completes the proof of Lemma 3.2. ■

Lemma 3.3. Let r, s be non-negative integers, k be a positive integer and $r + s = k$. Assume that $\varphi(x, y)$ is a bounded compactly supported function on $\mathbb{R}^2$ with supp $\varphi(x, y) \subseteq [-a, a] \times [-b, b],\ 0 < a, b < +\infty$,

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r}\varphi(x, y) \in C(\mathbb{R}^2)$$

and satisfies the following conditions:

(i) for any fixed j and y,

$$\sum_{i=-\infty}^{\infty} p(i, j)\varphi(x - i, y - j)$$

is a polynomial of degree $< r$ with respect to x whenever $p(x, y)$ is a polynomial of degree $< r$ with respect to x.

(ii) for any fixed i and x,

$$\sum_{j=-\infty}^{\infty} p(i, j)\varphi(x - i, y - j)$$

is a polynomial of degree $< s$ with respect to y whenever $p(x, y)$ is a polynomial of degree $< s$ with respect to y.

(iii) There are k real numbers $-\infty =: x_0 < x_1 < x_2 \ldots < x_r < x_{r+1} := +\infty$ and $-\infty =: y_0 < y_1 < y_2 \ldots < y_s < y_{s+1} := +\infty$ such that

$$(-1)^{m+l}\frac{\partial^{r+s}\varphi}{\partial x^r \partial y^r}(x, y) \geq 0, \quad \begin{array}{l} x_m \leq x \leq x_{m+1}, \\ y_l \leq y \leq y_{l+1}, \\ m = 0, 1, \ldots, r \\ l = 0, 1, \ldots, s. \end{array} \tag{3.10}$$

Then if $f(x,y) \in C(\mathbb{R}^2)$,

$$\frac{\partial^{r+s} f}{\partial x^r \partial y^r}(x,y) \in C(\mathbb{R}^2)$$

and

$$\frac{\partial^{r+s} f}{\partial x^r \partial y^r}(x,y) \geq 0,\ (x,y) \in \mathbb{R}^2, \tag{3.11}$$

for the linear operators $B_n(f)(x,y)$ (defined by (3.1)) we also have

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x,y) \in C(\mathbb{R}^2)$$

and

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x,y) \geq 0,\ (x,y) \in \mathbb{R}^2. \tag{3.12}$$

Proof. Since $\varphi(x,y)$ is compactly supported, for any fixed $(x,y) \in \mathbb{R}^2$, the summations in (3.1) only involve finite non-zero terms. Therefore, if

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} \varphi(x,y) \in C(\mathbb{R}^2),$$

we have

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x,y) \in C(\mathbb{R}^2).$$

For simplicity we only prove (3.12) for $n = 0$. For the other cases, the arguments are the same.

Let $(\overline{x}, \overline{y})$ be a fixed point on $\mathbb{R}^2$, and i_m, j_l $(m = 1,2,\ldots,r;\ l = 1,2,\ldots,s)$ are the integers such that

$$\overline{x} - i_m \leq x_{r-m+1} < \overline{x} - i_m + 1,\ m = 1,2,\ldots,r \tag{3.13}$$

$$\overline{y} - j_l \leq y_{s-l+1} < \overline{y} - j_l + 1,\ l = 1,2,\ldots,s. \tag{3.14}$$

Since $x_m \leq x_{m+1}$ $(m = 1,\ldots,r-1)$ and $y_l \leq y_{l+1}$ $(l = 1,\ldots,s-1)$, we have $i_m \leq i_{m+1}$ $(m = 1,\ldots,r-1)$ and $j_l \leq j_{l+1}$ $(l = 1,\ldots,s-1)$. If for some m we have $i_m = i_{m+1}$, then we redefine $i_{m+1} := i_m + 1$. Hence after refinement we have $i_m < i_{m+1}$. (The refinement is going on in the order of m- to be explained later (*).) Similarly we may redefine j_l such that $j_l < j_{l+1}$.

For $f(x,y)$ and each fixed y, we can construct a polynomial $P_1(x,y)$ of degree $r-1$ with respect to x such that $f(i_m,y) = P_1(i_m,y)$, $m = 1,\ldots,r$. Then, for function $f(x,y) - P_1(x,y)$ and each fixed x, we can construct a polynomial $P_2(x, y)$ of degree $s - 1$ with respect to y such that $f(x, j_l) - P_1(x, j_l) = P_2(x, j_l),\ l = 1, \ldots, s$. Since $f(i_m, y) - P_1(i_m, y) \equiv 0$, we have $P_2(i_m, y) \equiv 0$.

Hence

$$F(x, y) := f(x, y) - P_1(x, y) - P_2(x, y) = 0,\ if\ x = i_m\ (m = 1, \ldots, r)$$

$$or\ y = j_l,\ (l = 1, 2, \ldots, s). \tag{3.15}$$

We also get

$$\frac{\partial^{r+s} F}{\partial x^r \partial y^r} = \frac{\partial^{r+s} f}{\partial x^r \partial y^r} \geq 0\ on\ \mathbb{R}^2, \tag{3.16}$$

and

$$\frac{\partial^s F}{\partial y^s}(x, y) = 0\ for\ x = i_m\ (m = 1, \ldots, r) \tag{3.17}$$

because of (3.15).

For fixed y, we apply Lemma 3.1 to $\frac{\partial^s F}{\partial y^s}(x, y)$. Because

$$\frac{\partial^r}{\partial x^r}\left(\frac{\partial^s F}{\partial y^s}\right)(x, y) \geq 0$$

by (3.16), and $\frac{\partial^s F}{\partial y^s}(i_m, y) = 0\ (m = 1, \ldots, r)$ by (3.17), we have

$$(-1)^{r+m}\frac{\partial^s F}{\partial y^s}(x, y) \geq 0, \quad \begin{array}{l} i_m < x < i_{m+1}, \\ m = 0, 1, \ldots, r. \end{array} \tag{3.18}$$

Here $i_0 := -\infty$ and $i_{r+1} := +\infty$.

Now let m be fixed and x be fixed with $i_m < x < i_{m+1}$. We apply Lemma 3.1 to $F(x, y)$ with respect to y. From (3.18) and (3.15), and Lemma 3.1, we have

$$(-1)^{r+s+m+l}F(x, y) \geq 0, \quad \begin{array}{l} i_m < x < i_{m+1}, \\ j_l < y < j_{l+1}, \\ m = 0, 1, \ldots, r;\ l = 0, 1, \ldots, s. \end{array} \tag{3.19}$$

Here $j_0 := -\infty$ and $j_{s+1} := +\infty$.

For the fixed $(\overline{x}, \overline{y})$, from the conditions (i) and (ii), we derive

$$\frac{\partial^{r+s} B_0(f)}{\partial x^r \partial y^s}(\overline{x}, \overline{y}) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(i, j)\varphi(x - i, y - j)\Big]_{x=\overline{x},\ y=\overline{y}}$$

$$= \frac{\partial^{r+s}}{\partial x^r \partial y^s} \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} F(i,j)\varphi(x-i, y-j)\Big]_{x=\overline{x},\ y=\overline{y}}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} F(i,j) \frac{\partial^{r+s}\varphi}{\partial x^r \partial y^s}(\overline{x}-i, \overline{y}-j)$$

$$= \sum_{l=0}^{s} \sum_{m=0}^{r} \sum_{j_l < j < j_{l+1}} \sum_{i_m < i < i_{m+1}} F(i,j) \frac{\partial^{r+s}\varphi}{\partial x^r \partial y^s}(\overline{x}-i, \overline{y}-j). \tag{3.20}$$

Here for the last equation we have used (3.15).

If there is some integer i such that $i_m < i < i_{m+1}$, then this i_{m+1} is not the refinement of the original i_{m+1} which satisfies (3.13). Hence in this case the i_{m+1} satisfies (3.13). Meanwhile even if i_m is a refinement, i_m still satisfies the inequality on the left-hand side of (3.13), because the refinement is greater than the original one. Thus, for $i_m < i < i_{m+1}$, by (3.13), we have

$$\overline{x} - i < \overline{x} - i_m < x_{r-m+1},$$

and noticing $i \le i_{m+1} - 1$, we have

$$\overline{x} - i \ge \overline{x} - i_{m+1} + 1 > x_{r-m}.$$

Hence

$$x_{r-m} < \overline{x} - i < x_{r-m+1},\ i_m < i < i_{m+1},\ m = 0, 1, \ldots, r. \tag{3.21}$$

Similarly we obtain

$$y_{s-l} < \overline{y} - j < y_{s-l+1},\ j_l < j < j_{l+1},\ l = 0, 1, \ldots, s. \tag{3.22}$$

From (3.21), (3.22) and (3.10), we have

$$(-1)^{r+s-m-l} \frac{\partial^{r+s}\varphi}{\partial x^r \partial y^s}(\overline{x}-i, \overline{y}-j) \ge 0, \quad \begin{array}{l} i_m < i < i_{m+1}, \\ j_l < j < j_{l+1}, \\ m = 0, 1, \ldots, r; \\ l = 0, 1, \ldots, s. \end{array}$$

From this, (3.19) and (3.20), we derive

$$\frac{\partial^{r+s}}{\partial x^r \partial y^s} B_0(f)(\overline{x}, \overline{y}) \ge 0.$$

■

(*)**A note on the last proof**. If m_0 is the smallest positive integer such that $i_{m_0} = i_{m_0+1}$, we redefine $i_{m_0+1} := i_{m_0} + 1$. If $i_{m_0+2} \leq$ the refinement of i_{m_0+1}, we redefine $i_{m_0+2} := i_{m_0} + 2$, and so on until we have some positive integer q such that $i_{m_0+q} \geq i_{m_0} + q$. Then we check for the next m_1 such that $i_{m_1} = i_{m_1+1}$. Do the same refinement. In this way, we can modify i_m such that $i_m < i_{m+1},\ m = 0, 1, \ldots, r$.

Lemma 3.4. Let r, s be non-negative integers, k be a positive integer and $r+s = k$. Assume that $\varphi(x)$ is a bounded compactly supported function on $\mathbb{R}$ with supp $\varphi(x) \subseteq [-a, a],\ 0 < a < +\infty$, $\varphi^{(k)}(x) \in C(\mathbb{R})$ and satisfies the following conditions:

(3.4.1) for each $k^* = 0, 1, \ldots, k-1$,

$$\sum_{i=-\infty}^{\infty} p(i)\varphi(x-i)$$

is a polynomial of degree k^* whenever $p(x)$ is a polynomial of degree k^*.

(3.4.2) there are k real numbers $-\infty =: x_0 < x_1 < x_2 \ldots < x_r < x_{r+1} := +\infty$ such that

$$(-1)^m \varphi^{(k)}(x) \geq 0,\ x_m \leq x \leq x_{m+1},\ m = 0, 1, \ldots, k.$$

Then, if $f(x, y) \in C(\mathbb{R}^2)$,

$$\frac{\partial^{r+s} f}{\partial x^r \partial y^r}(x, y) \in C(\mathbb{R}^2)$$

and

$$\varepsilon_{r^*,s^*} \cdot \frac{\partial^{r^*+s^*} f}{\partial x^{r^*} \partial y^{r^*}}(x, y) \geq 0,\ (x, y) \in \mathbb{R}^2 \tag{3.23}$$

where $r^* = 0, 1, \ldots, r,\ s^* = 0, 1, \ldots, s$ and $\varepsilon_{r^*,s^*} = \pm 1$, for the linear operators $B_n(f)(x, y)$ defined by (3.1) with $\varphi(x, y) := \varphi(x)\varphi(y)$, we also have

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x, y) \in C(\mathbb{R}^2)$$

and

$$\varepsilon_{r^*,s^*} \cdot \frac{\partial^{r^*+s^*}}{\partial x^{r^*} \partial y^{r^*}} B_n(f)(x, y) \geq 0,\ (x, y) \in \mathbb{R}^2,\quad \begin{array}{l} r^* = 0, 1, \ldots, r, \\ s^* = 0, 1, \ldots, s. \end{array} \tag{3.24}$$

Proof. Since $\varphi(x, y) := \varphi(x)\varphi(y)$ satisfies the condition (3.4.1), we have $\varphi(x, y)$ satisfy (i)' for any fixed j and y, for each $r^* = 0, 1, \ldots, r-1$,

$$\sum_{i=-\infty}^{\infty} p(i,j)\varphi(x-i, y-j)$$

is a polynomial of degree r^* with respect to x whenever $p(x,y)$ is a polynomial of degree r^* with respect to x.

(ii)' for any fixed i and x, for each $s^* = 0, 1, \ldots, s-1$,

$$\sum_{j=-\infty}^{\infty} p(i,j)\varphi(x-i, y-j)$$

is a polynomial of degree s^* with respect to y whenever $p(x,y)$ is a polynomial of degree s^* with respect to y.

By Lemma 3.2, from the condition (3.4.2), we have $\varphi(x,y)$ fulfill

(iii)' for each $r^* = 0, 1, \ldots, r$ and each $s^* = 0, 1, \ldots, s$, there are $k^* := r^* + s^*$ real numbers $-\infty =: x'_0 < x'_1 \le x'_2 \ldots \le x'_{r^*} < x'_{r^*+1} := +\infty$ and $-\infty =: y'_0 < y'_1 \le y'_2 \ldots \le y'_{s^*} < y'_{r^*+1} := +\infty$ such that

$$(-1)^{m+l} \frac{\partial^{r^*+s^*}\varphi}{\partial x^{r^*} \partial y^{s^*}}(x,y) = (-1)^m \varphi^{(r^*)}(x) \cdot (-1)^l \varphi^{(s^*)}(y) \ge 0, \quad \begin{array}{l} x'_m \le x \le x'_{m+1}, \\ y'_l \le y \le y'_{l+1}, \\ m = 0, 1, \ldots, r^* \\ l = 0, 1, \ldots, s^*. \end{array}$$

Then, from (i)', (ii)', and (iii)', by Lemma 3.3, we have (3.24) for $B_n(f)$ (x,y) if $f(x,y)$ satisfies (3.23). ■

For $f \in C(\mathbb{R}^2)$, $h > 0$ and $(x,y) \in \mathbb{R}^2$ we define the local modulus of continuity of f by

$$w_1(f, h; x, y) := \sup|f(x', y') - f(x,y)|, \ |x'-x| \le h, \ |y'-y| \le h.$$

Theorem 3.5. Let r, s be non-negative integers and k be a positive integer such that $r+s = k$. Assume that $\varphi(x,y)$ satisfies all the assumptions in Lemma 3.3 and the following additional condition:

(iv)

$$\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \varphi(x-i, y-j) \equiv 1 \ on \ \mathbb{R}^2.$$

Then, if $f(x,y) \in C(\mathbb{R}^2)$,

$$\frac{\partial^{r+s} f}{\partial x^r \partial y^r}(x,y) \in C(\mathbb{R}^2)$$

and

$$\frac{\partial^{r+s} f}{\partial x^r \partial y^r}(x,y) \geq 0,\ (x,y) \in \mathbb{R}^2,$$

for the linear operators $B_n(f)(x,y)$ defined by (3.1) we have

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x,y) \in C(\mathbb{R}^2)$$

and

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x,y) \geq 0,\ (x,y) \in \mathbb{R}^2,$$

and

$$|f(x,y) - B_n(f)(x,y)| \leq w_1(f, 2^{-n} \cdot d; x, y)\ (x,y) \in \mathbb{R}^2, \tag{3.25}$$

where $d = \max(a,b)$.

Proof. It is based on Lemma 3.3. Inequality (3.25) appears also in [84] where it is proved. ■

Theorem 3.6. Let r, s be non-negative integers and k be a positive integer such that $r + s = k$. Suppose that $\varphi(x)$ satisfies all the assumptions in Lemma 3.4 and the following additional condition:

$$\sum_{i=-\infty}^{\infty} \varphi(x-i) \equiv 1 \text{ on } \mathbb{R}.$$

Then, if $f(x,y) \in C(\mathbb{R}^2)$,

$$\frac{\partial^{r+s} f}{\partial x^r \partial y^r}(x,y) \in C(\mathbb{R}^2)$$

and

$$\varepsilon_{r^*,s^*} \cdot \frac{\partial^{r^*+s^*} f}{\partial x^{r^*} \partial y^{r^*}}(x,y) \geq 0,\ (x,y) \in \mathbb{R}^2$$

where $r^* = 0, 1, \ldots, r,\ s^* = 0, 1, \ldots, s$ and $\varepsilon_{r^*,s^*} = \pm 1$, for the linear operators $B_n(f)(x,y)$ defined by (3.1) with $\varphi(x,y) := \varphi(x)\varphi(y)$, we have

$$\frac{\partial^{r+s}}{\partial x^r \partial y^r} B_n(f)(x,y) \in C(\mathbb{R}^2)$$

and

$$\varepsilon_{r^*,s^*} \cdot \frac{\partial^{r^*+s^*}}{\partial x^{r^*} \partial y^{r^*}} B_n(f)(x,y) \geq 0, \ (x,y) \in \mathbb{R}^2, \quad \begin{matrix} r^* = 0,1,\ldots,r, \\ s^* = 0,1,\ldots,s. \end{matrix}$$

and

$$|f(x,y) - B_n(f)(x,y)| \leq w_1(f, 2^{-n} \cdot d; x,y) \ (x,y) \in \mathbb{R}^2,$$

where $d = \max(a,b)$.

Proof. It is based on Lemma 3.4. ■

Example 3.7. Take $\varphi(x,y) := \varphi(x)\varphi(y)$ where $\varphi(x)$ be the B-spline of order $k+2$:

$$\varphi(y) = B_{k+2}(x),$$

where $B_n(x)$ is defined inductively as follows

$$B_1(x) = \frac{1}{2}\left(\chi_{[-\frac{1}{2},\frac{1}{2}]}(x) + \chi_{(-\frac{1}{2},\frac{1}{2})}(x)\right),$$

$$B_n(x) = B_{n-1} * B_1(x), \ n = 2,3,\ldots.$$

Such a $\varphi(x,y)$ fulfills all the assumptions of Theorems 3.5 and 3.6

4

Multidimensional Probabilistic Scale Approximation

Multivariate probabilistic distribution functions are approximated by some naturally arising wavelet type operators involving a scale function. These transform multivariate distribution functions to multivariate distribution functions. The degree of this approximation is given by establishing some sharp Jackson type inequalities. This chapter relies on [90].

4.1 Introduction

We are interested in the problem of approximation to multivariable probabilistic distribution functions. It is known that ([255], pp. 107-108), a function $F(x_1, x_2, \ldots, x_r)$ is a probabilistic distribution function on $\mathbb{R}^r (r > 1)$ if and only if F is nondecreasing with respect to each variable $x_i (i = 1, 2, \ldots, r)$ and right continuous for all variables, and satisfies the following conditions:

(i)

$$F(-\infty, x_2, \ldots, x_r) = F(x_1, -\infty, x_3, \ldots, x_r) = \ldots = F(x_1, \ldots, x_{r-1}, -\infty) = 0,$$

$$F(+\infty, +\infty, \ldots, +\infty) = 1.$$

(ii) for every $(x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$ and all $\delta_i > 0 (i = 1, 2, \ldots, r)$ the inequality

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 41–56.
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$$F(x_1+\delta_1, x_2+\delta_2, \ldots, x_r+\delta_r) - \sum_{i=1}^{r} F(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1} \ldots, x_r+\delta_r) +$$

$$\sum_{i,j=1;\ i<j}^{r} F(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_i+\delta_i, \ldots, x_{j-1}+\delta_{j-1}, x_j, x_{j+1}+\delta_{j+1} \ldots, x_r+\delta_r)$$

$$+ \ldots + (-1)^r F(x_1, x_2, \ldots, x_r) \geq 0 \tag{4.1}$$

holds.

Let $\varphi(x_1, x_2, \ldots, x_r)$ be a bounded compactly supported function on $\mathbb{R}^r$ with

$$\text{supp}\ \varphi(x_1, x_2, \ldots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \ldots \times [-a_r, a_r],\ 0 < a_i < +\infty (i = 1, 2, \ldots, r)$$

and be right continuous with respect to all variables.

We want to approximate F on $\mathbb{R}^r$ by the linear combinations of translated dilates of $\varphi(x_1, x_2, \ldots, x_r)$.

Define

$$B_k(F)(x_1, x_2, \ldots, x_r) := \sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} F(2^{-k} j_1, 2^{-k} j_2, \ldots, 2^{-k} j_r)$$

$$\cdot \varphi(2^k x_1 - j_1, 2^k x_2 - j_2, \ldots, 2^k x_r - j_r) \tag{4.2}$$

on $\mathbb{R}^r$ for $k \in \mathbb{Z}$. Since φ is compactly supported, for any $(x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$ the summations in (4.2) only involve finite terms, so $B_k(F)$ is well-defined on $\mathbb{R}^r$.

In this chapter, we are going to use the linear operators $B_k(F)$ to approximate F and discuss under what conditions on φ the $B_k(F)$ give us probabilistic distribution functions if F is so. This is a generalization of [86] and [84] where it discussed the univariate case and bivariate case, respectively.

4.2 Main Result

Let $f(x_1, x_2, \ldots, x_r)$ be a bounded function on $\mathbb{R}^r$. For each $(x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$ and $h > 0$, we define the *first modulus of continuity*

$$w_1(f, h) := \sup_{|x_i' - x_i| \leq h,\ i=1,2,\ldots,r} |f(x_1', \ldots, x_r') - f(x_1, x_2, \ldots, x_r)|,$$

where the sup is taken over all $(x_1, x_2, \ldots, x_r), (x'_1, \ldots, x'_r)$ which satisfy $|x'_i - x_i| \le h$ for $i = 1, 2, \ldots, r$.

Theorem 4.1. Suppose $\varphi(x_1, x_2, \ldots, x_r)$ is a bounded compactly supported function on $\mathbb{R}^r$ with

$$\text{supp } \varphi(x_1, x_2, \ldots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \ldots \times [-a_r, a_r],\ 0 < a_i < +\infty (i = 1, 2, \ldots, r)$$

and is right continuous with respect to all variables, and satisfies the following conditions:

(i) For each i, $1 \le i \le r$ and any $(x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$,

$$\sum_{j=-\infty}^{\infty} \varphi(x_1, x_2, \ldots, x_{i-1}, x_i - j, x_{i+1}, \ldots, x_r) = C_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r),$$

where $C_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r)$ are independent of x_i $(i = 1, 2, \ldots, r)$.

(ii)
$$\sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_2=-\infty}^{\infty} \sum_{j_1=-\infty}^{\infty} \varphi(x_1 - j_1, x_2 - j_2, \ldots, x_r - j_r) \equiv 1$$

on $\mathbb{R}^r$.

(iii) With respect to each variable $\varphi(x_1, x_2, \ldots, x_r)$ is a two-pieces monotone function, which is nondecreasing first and then nonincreasing.

(iv) There is a point $(b_1, b_2, \ldots, b_r) \in \mathbb{R}^r$ such that for all $\delta_i > 0$ $(i = 1, 2, \ldots, r)$, holds the inequality

$$\varepsilon \cdot \Bigg[\varphi(x_1+\delta_1, x_2+\delta_2, \ldots, x_r+\delta_r) - \sum_{i=1}^{r} \varphi(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1}, \ldots,$$
$$x_r+\delta_r) + \sum_{i,j=1,\ i<j}^{r} \varphi(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1}, \ldots, x_{j-1}+\delta_{j-1}, x_j,$$
$$x_{j+1}+\delta_{j+1}, \ldots, x_r+\delta_r) + \ldots + (-1)^r \varphi(x_1, x_2, \ldots, x_r)\Bigg] \ge 0 \quad (4.3)$$

whenever

$$(x_1+\delta_1, x_2+\delta_2, \ldots, x_r+\delta_r) \textit{ and } (x_1, x_2, \ldots, x_r) \in \overline{J}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$$

where

$$J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) := \{(x_1, x_2, \ldots, x_r) \in \mathbb{R}^r;\ \text{sign}(x_i - b_i) = \varepsilon_i,\ \varepsilon_i = \pm 1\}$$

and

$$\varepsilon = \prod_{i=1}^{r}(-\varepsilon_i),\ \overline{J}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) \textit{ is the closure of } J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r).$$

Then, if $F(x_1, x_2, \ldots, x_r)$ is a probabilistic distribution function on $\mathbb{R}^r$, the linear operators $B_k(F)(x_1, x_2, \ldots, x_r)$ defined by (4.2) are also probabilistic distribution functions on $\mathbb{R}^r$. Besides, we have

$$\|B_k F - F\|_\infty \le w_1(F, 2^{-k} \cdot d), \tag{4.4}$$

$k \in \mathbb{Z}$, where $d := \max(a_1, a_2, \ldots, a_r)$. Moreover, the inequalities (4.4) are sharp for probabilistic distribution functions.

Examples 4.2. Here we want to present some examples of $\varphi(x_1, x_2, \ldots, x_r)$ which satisfy all the conditions in Theorem 4.1.

Let

$$\varphi_0(x) := \begin{cases} 1,\ -\frac{1}{2} \le x < \frac{1}{2}, \\ 0,\ \textit{otherwise.} \end{cases} \tag{4.5}$$

and

$$\varphi_1(x) := \begin{cases} x + 1,\ -1 \le x < 0, \\ 1 - x,\ 0 < x \le 1 \\ 0,\ \textit{otherwise.} \end{cases} \tag{4.6}$$

Define

$$\varphi_0(x_1, x_2, \ldots, x_r) = \varphi_0(x_1)\varphi_0(x_2)\ldots\varphi_0(x_r)$$

and

$$\varphi_1(x_1, x_2, \ldots, x_r) = \varphi_1(x_1)\varphi_1(x_2)\ldots\varphi_1(x_r).$$

Then $\varphi_0(x_1, x_2, \ldots, x_r)$ and $\varphi_1(x_1, x_2, \ldots, x_r)$ are the functions satisfying all the conditions in Theorem 4.1. Indeed, for $s = 0$ and 1, we have

$$\sum_{j=-\infty}^{\infty} \varphi_s(x_1, x_2, \ldots, x_{i-1}, x_i - j, x_{i+1}, \ldots, x_r) = \varphi_s(x_1)\ldots\varphi_s(x_{i-1})\varphi_s(x_{i+1})\ldots\varphi_s(x_r)$$

$$\cdot \sum_{j=-\infty}^{\infty} \varphi_s(x_i - j) = \varphi_s(x_1)\dots\varphi_s(x_{i-1})\varphi_s(x_{i+1})\dots\varphi_s(x_r),$$

and

$$\sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_2=-\infty}^{\infty} \sum_{j_1=-\infty}^{\infty} \varphi_s(x_1-j_1, x_2-j_2, \dots, x_r-j_r) = \prod_{i=1}^{r} \left(\sum_{j_i=-\infty}^{\infty} \varphi_s(x_i - j_i) \right) = 1.$$

Since $\varphi_s(x)$ $(s = 0, 1)$ are two-pieces monotone functions and $\varphi_s(x) \geq 0$, so $\varphi_s(x_1, x_2, \dots, x_r)$ satisfy the condition (iii). For the condition (iv), we may take $(b_1, b_2, \dots, b_r) = (0, 0, \dots, 0)$. For $\delta_i > 0$ $(i = 1, 2, \dots, r)$, let

$$\Delta_{\delta_1} f(x_1, x_2, \dots, x_r) := f(x_1 + \delta_1, x_2, \dots, x_r) - f(x_1, x_2, \dots, x_r),$$

$$\Delta_{\delta_2}\Delta_{\delta_1} f(x_1, x_2, \dots, x_r) := \Delta_{\delta_1} f(x_1, x_2+\delta_2, x_3 \dots, x_r) - \Delta_{\delta_1} f(x_1, x_2, x_3 \dots, x_r),$$

$$\dots,$$

$$\Delta_{\delta_r}\Delta_{\delta_{r-1}} \dots \Delta_{\delta_1} f(x_1, x_2, \dots, x_r) := \Delta_{\delta_{r-1}} \dots \Delta_{\delta_1} f(x_1, x_2, \dots, x_{r-1}, x_r + \delta r) -$$

$$\Delta_{\delta_{r-1}} \dots \Delta_{\delta_1} f(x_1, x_2, \dots, x_{r-1}, x_r).$$

Notice the following (in order to prove (iv) and other lemmas).

Lemma 4.3. For every $(x_1, x_2, x_3 \dots, x_r) \in \mathbb{R}^r$ and $\delta_i \in \mathbb{R}$ $(i = 1, 2, \dots, r)$, we have

$$f(x_1+\delta_1, x_2+\delta_2, \dots, x_r+\delta_r) - \sum_{i=1}^{r} f(x_1+\delta_1, \dots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1} \dots, x_r+\delta_r) +$$

$$\sum_{i,j=1;\ i<j}^{r} f(x_1+\delta_1, \dots, x_{i-1}+\delta_{i-1}, x_i, x_i+\delta_i, \dots, x_{j-1}+\delta_{j-1}, x_j, x_{j+1}+\delta_{j+1} \dots, x_r+\delta_r)$$

$$+ \dots + (-1)^r f(x_1, x_2, \dots, x_r)$$

$$= \Delta_{\delta_r}\Delta_{\delta_{r-1}} \dots \Delta_{\delta_1} f(x_1, x_2, \dots, x_r). \tag{4.7}$$

Proof. The proof is by induction on r. If $r = 1$, (4.7) is trivial. Suppose that equation (4.7) is valid for $r - 1$. Then we have

$$f(x_1+\delta_1, x_2+\delta_2, \ldots, x_r+\delta_r) - \sum_{i=1}^{r} f(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1} \ldots, x_r+\delta_r) +$$

$$\sum_{i,j=1;\ i<j}^{r} f(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_i+\delta_i, \ldots, x_{j-1}+\delta_{j-1}, x_j, x_{j+1}+\delta_{j+1} \ldots, x_r+\delta_r)$$

$$+ \ldots + (-1)^r f(x_1, x_2, \ldots, x_r) = \Big[f(x_1+\delta_1, x_2+\delta_2, \ldots, x_r+\delta_r) -$$

$$\sum_{i=1}^{r-1} f(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1} \ldots, x_r+\delta_r) +$$

$$\sum_{i,j=1;\ i<j}^{r-1} f(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_i+\delta_i, \ldots, x_{j-1}+\delta_{j-1}, x_j, x_{j+1}+\delta_{j+1} \ldots, x_r+\delta_r)$$

$$+ \ldots + (-1)^{r-1} f(x_1, x_2, \ldots, x_{r-1}, x_r+\delta_r) \Big] - \Big[f(x_1+\delta_1, \ldots, x_{r-1}+\delta_{r-1}, x_r) -$$

$$\sum_{i=1}^{r-1} f(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_i+\delta_i \ldots, x_{r-1}+\delta_{r-1}, x_r)$$

$$+ \ldots + (-1)^{r-2} \sum_{i=1}^{r-1} f(x_1, \ldots, x_{i-1}, x_i+\delta_i, x_{i+1}, \ldots, x_r)$$

$$+(-1)^{r-1} f(x_1, x_2, \ldots, x_r) \Big] = \Delta_{\delta_{r-1}} \ldots \Delta_{\delta_1} f(x_1, x_2, \ldots, x_{r-1}, x_r+\delta_r)$$

$$-\Delta_{\delta_{r-1}} \ldots \Delta_{\delta_1} f(x_1, x_2, \ldots, x_{r-1}, x_r) = \Delta_{\delta_r} \Delta_{\delta_{r-1}} \ldots \Delta_{\delta_1} f(x_1, x_2, \ldots, x_r).$$

■

In order to prove (iv) we need also

Lemma 4.4. Suppose that functions $f_i(x)$ $(i = 1, 2, \ldots, r)$ are defined on $\mathbb{R}$ and $f(x_1, x_2, \ldots, x_r) = \prod_{i=1}^r f_i(x_i)$ on $\mathbb{R}^r$. Then the left-hand side of (4.7) equals to $\prod_{i=1}^r [f_i(x_i+\delta_i) - f_i(x_i)]$.

Proof. By Lemma 4.3 we only need to verify

$$\Delta_{\delta_r} \Delta_{\delta_{r-1}} \ldots \Delta_{\delta_1} f(x_1, x_2, \ldots, x_r) = \prod_{i=1}^{r} [f_i(x_i+\delta_i) - f_i(x_i)]. \tag{4.8}$$

But this is done by induction on r. In fact, (4.8) is trivial for $r = 1$. Assume that (4.8) is valid for $r - 1$. Then

$$\Delta_{\delta_r} \Delta_{\delta_{r-1}} \ldots \Delta_{\delta_1} f(x_1, x_2, \ldots, x_r) = \Delta_{\delta_{r-1}} \ldots \Delta_{\delta_1} f(x_1, x_2, \ldots, x_{r-1}, x_r+\delta_r)$$

$$-\Delta_{\delta_{r-1}} \dots \Delta_{\delta_1} f(x_1, x_2, \dots, x_{r-1}, x_r) = \prod_{i=1}^{r-1} [f_i(x_i+\delta_i) - f_i(x_i)] \cdot f_r(x_r+\delta_r)$$

$$-\prod_{i=1}^{r-1} [f_i(x_i+\delta_i) - f_i(x_i)] \cdot f_r(x_r) = \prod_{i=1}^{r} [f_i(x_i+\delta_i) - f_i(x_i)].$$

■

1) **Back to the Example (4.2)**. For $s = 0, 1$, if $(x_1+\delta_1, x_2+\delta_2, \dots, x_r+\delta_r)$ and $(x_1, x_2, \dots, x_r) \in J(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ we have

$$\text{sign}[\varphi_s(x_i+\delta_i) - \varphi_s(x_i)] = -\varepsilon_i.$$

Hence

$$\text{sign} \prod_{i=1}^{r} [\varphi_s(x_i+\delta_i) - \varphi_s(x_i)] = \prod_{i=1}^{r} (-\varepsilon_i).$$

Thus, by Lemma 4.4, the functions $\varphi_s(x_1, x_2, \dots, x_r)$ satisfy the condition (iv) in Theorem 4.1.

2) **Another example**. In general, if we have $\varphi_s(x_1, x_2, \dots, x_r) = \prod_{i=1}^{r} f_i(x_i)$, where $f_i(x_i)$ $(i = 1, 2, \dots, r)$ are bounded compactly supported functions on $\mathbb{R}$ and right continuous and satisfy

(i)

$$\prod_{j=-\infty}^{\infty} f_i(x-j) \equiv 1 \textit{ on } \mathbb{R},$$

(ii) For each i, $i = 1, 2, \dots, r$, there is a number b_i such that $f_i(x)$ is nondecreasing if $x \leq b_i$ and $f_i(x)$ is nondecreasing if $x \geq b_i$.

Then $\varphi(x_1, x_2, \dots, x_r)$ satisfies all the conditions in Theorem 4.1.

3) **Auxiliary Results (in order to prove Theorem 4.1)**

Lemma 4.5. Suppose that $\varphi(x_1, x_2, \dots, x_r)$ is a bounded compactly supported function on $\mathbb{R}^r$ with

$$\text{supp}\ \varphi(x_1, x_2, \dots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \dots \times [-a_r, a_r], \quad 0 < a_i < +\infty (i = 1, 2, \dots, r)$$

and is right continuous with respect to all variables x_i. Then, for any sequence of $\{C_{j_1 j_2 \dots j_r}\}_{j_i=-\infty;\ i=1,2,\dots,r}^{\infty}$ real numbers, the function

$$A(x_1, x_2, \ldots, x_r) := \sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_2=-\infty}^{\infty} \sum_{j_1=-\infty}^{\infty} C_{j_1 j_2 \ldots j_r} \varphi(x_1 - j_1, x_2 - j_2, \ldots, x_r - j_r) \tag{4.9}$$

is also right continuous with respect to all variables x_i.

The proof of Lemma 4.5 is similar to the proof of Lemma 1 in [84].

Lemma 4.6 ([86]). Suppose that $\varphi(x)$ is a bounded function on $\mathbb{R}$ with supp $\varphi(x) \subseteq [-a, a]$, $0 < a < +\infty$ and satisfies the following conditions:

(i)

$$\sum_{j=-\infty}^{\infty} \varphi(x - j) \equiv C \text{ on } \mathbb{R}, \text{ where } C \text{ is a constant.}$$

(ii) There is a number a_0 such that $\varphi(x)$ is nondecreasing if $x \leq a_0$ and is nondecreasing if $x \geq a_0$. Then if $\{C_j\}_{j=-\infty}^{\infty}$ is a nondecreasing sequence, the function $A(x)$ defined by (4.9) is a nondecreasing function on $\mathbb{R}$.

By Lemma 4.5 and Lemma 4.6 we can prove

Lemma 4.7. Suppose that $\varphi(x_1, x_2, \ldots, x_r)$ is a bounded compactly supported function on $\mathbb{R}^r$ with supp $\varphi(x_1, x_2, \ldots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \ldots \times [-a_r, a_r]$, $0 < a_i < +\infty (i = 1, 2, \ldots, r)$ and $\varphi(x_1, x_2, \ldots, x_r)$ is right continuous with respect to all variables x_i $(i = 1, 2, \ldots, r)$. Then, the linear operators $B_k(f)(x_1, x_2, \ldots, x_r)$ defined by (4.2) are right continuous with respect to all variables x_i $(i = 1, 2, \ldots, r)$.

Lemma 4.8. Suppose that $\varphi(x_1, x_2, \ldots, x_r)$ is a bounded compactly supported function on $\mathbb{R}^r$ with supp $\varphi(x_1, x_2, \ldots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \ldots \times [-a_r, a_r]$, $0 < a_i < +\infty (i = 1, 2, \ldots, r)$ and satisfies the condition (i) and (iii) in Theorem 4.1. Then, if $(x_1, x_2, \ldots, x_r)$ is nondecreasing with respect to each variable x_i $(i = 1, 2, \ldots, r)$, so are the linear operators $B_k(f)(x_1, x_2, \ldots, x_r)$.

Similar to Lemma 5 in [86], we have

Lemma 4.9. Suppose that $\varphi(x_1, x_2, \ldots, x_r)$ is a bounded compactly supported function on $\mathbb{R}^r$ with supp $\varphi(x_1, x_2, \ldots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \ldots \times [-a_r, a_r]$, $0 < a_i < +\infty (i = 1, 2, \ldots, r)$ and satisfies the condition (iii) in Theorem 4.1. If $f(x_1, x_2, \ldots, x_r)$ satisfies

$$f(-\infty, x_2, \ldots, x_r) = f(x_1, -\infty, x_3, \ldots, x_r) = \ldots = f(x_1, \ldots, x_{r-1}, -\infty) = 0$$

and

$$f(+\infty, \dots, +\infty) = 1,$$

then for each fixed $k \in \mathbb{Z}$, we have

$$B_k(f)(-\infty, x_2, \dots, x_r) = B_k(f)(x_1, -\infty, x_3, \dots, x_r) = \dots$$

$$= B_k(f)(x_1, \dots, x_{r-1}, -\infty) = 0$$

and

$$B_k(f)(+\infty, \dots, +\infty) = 1.$$

Lemma 4.10. If for any $(x_1, x_2 \dots, x_r) \in \mathbb{R}^r$ and $0 < \delta_i < 1$ $(i = 1, 2, \dots, r)$, $f(x_1, x_2 \dots, x_r)$ satisfies the inequality (4.1), then for any $(x_1, x_2 \dots, x_r) \in \mathbb{R}^r$ and all $\delta_i > 0$ $(i = 1, 2, \dots, r)$, $f(x_1, x_2 \dots, x_r)$ satisfies (4.1).

Proof. We observe that

$$\Delta_{\delta_i}\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_r) =$$

$$\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f(x_1, \dots, x_{i-1}, x_i + \delta_i, x_{i+1}, \dots, x_r) -$$

$$\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_r)$$

$$= \Bigg[\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f(x_1, \dots, x_{i-1}, x_i + \delta_i, x_{i+1}, \dots, x_r) -$$

$$\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f\left(x_1, \dots, x_{i-1}, x_i + \frac{\delta_i}{2}, x_{i+1}, \dots, x_r\right)\Bigg]$$

$$+\Bigg[\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f\left(x_1, \dots, x_{i-1}, x_i + \frac{\delta_i}{2}, x_{i+1}, \dots, x_r\right)$$

$$-\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_r)\Bigg]$$

$$= \Delta_{\frac{\delta_i}{2}}\Delta_{\delta_{i-1}} \dots \Delta_{\delta_1} f\left(x_1, \dots, x_{i-1}, x_i + \frac{\delta_i}{2}, x_{i+1}, \dots, x_r\right)$$

$$+\Delta_{\frac{\delta_i}{2}}\Delta_{\delta_{i-1}}\dots\Delta_{\delta_1}f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_r),$$

true for any $1 \le i \le r$. Then we have

$$\Delta_{\delta_r}\dots\Delta_{\delta_{i+1}}\Delta_{\delta_i}\Delta_{\delta_{i-1}}\dots\Delta_{\delta_1}f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_r)$$

$$= \Delta_{\delta_r}\dots\Delta_{\delta_{i+1}}\Delta_{\frac{\delta_i}{2}}\Delta_{\delta_{i-1}}\dots\Delta_{\delta_1}f\left(x_1,\dots,x_{i-1},x_i+\frac{\delta_i}{2},x_{i+1},\dots,x_r\right)$$

$$+\Delta_{\delta_r}\dots\Delta_{\delta_{i+1}}\Delta_{\frac{\delta_i}{2}}\Delta_{\delta_{i-1}}\dots\Delta_{\delta_1}f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_r).$$

From this and Lemma 4.3, if $f(x_1,x_2\dots,x_r)$ satisfies (4.1) for any $(x_1,x_2\dots,x_r)\in\mathbb{R}^r$ and $0<\delta_i<1$ $(i=1,2,\dots,r)$, then $f(x_1,x_2\dots,x_r)$ satisfies (4.1) for any $(x_1,x_2\dots,x_r)\in\mathbb{R}^r$ and $0<\delta_i<2$ $(i=1,2,\dots,r)$. Repeating the argument gives that $f(x_1,x_2\dots,x_r)$ satisfies (4.1) for any $(x_1,x_2\dots,x_r)\in\mathbb{R}^r$ and all $\delta_i>0$ $(i=1,2,\dots,r)$. ∎

Lemma 4.11. Suppose that $f(x_1,x_2\dots,x_r)$ satisfies the inequality (4.1) for any $(x_1,x_2\dots,x_r)\in\mathbb{R}^r$ and all $\delta_i>0$ $(i=1,2,\dots,r)$. Let $j_i,\ j_i^{(0)}\in\mathbb{Z}$ $(i=1,2,\dots,r)$. If $j_i=j_i^{(0)}$ for some i $(1\le i\le r)$ then

$$J(f) := f(j_1,j_2,\dots,j_r)-\sum_{i=1}^{r}f(j_1,\dots,j_{i-1},j_i^{(0)},j_{i+1},\dots,j_r)$$

$$+\sum_{i,k=1,\ i<k}^{r}f(j_1,\dots,j_{i-1},j_i^{(0)},j_{i+1},\dots,j_{k-1},j_k^{(0)},j_{k+1},\dots,j_r)$$

$$+(-1)^r f(j_1^{(0)},j_2^{(0)},\dots,j_r^{(0)})=0. \tag{4.10}$$

If $j_i\ne j_i^{(0)}$ for any i $(1\le i\le r)$ and $\operatorname{sign}(j_i-j_i^{(0)})=\varepsilon_i,\ \varepsilon_i=\pm1$, then

$$\operatorname{sign}(J(f))=\prod_{i=1}^{r}\varepsilon_i. \tag{4.11}$$

Proof. Take $\delta_i := j_i - j_i^{(0)}$ $(i = 1, 2, \ldots, r)$. By Lemma 4.3, we have

$$J(f) = \Delta_{\delta_r} \ldots \Delta_{\delta_1} f(j_1^{(0)}, j_2^{(0)}, \ldots, j_r^{(0)}). \tag{4.12}$$

If $j_i = j_i^{(0)}$ for some i $(1 \leq i \leq r)$, then $\delta_i = 0$ from (4.12) we have $J(f) = 0$.

If $j_i \neq j_i^{(0)}$ for any i, $1 \leq i \leq r$, then from the following equation (which comes from the definition of Δ's)

$$\Delta_{\delta_r} \ldots \Delta_{\delta_{i+1}} \Delta_{\delta_i} \Delta_{\delta_{i-1}} \ldots \Delta_{\delta_1} f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r)$$

$$= -\Delta_{\delta_r} \ldots \Delta_{\delta_{i+1}} \Delta_{-\delta_i} \Delta_{\delta_{i-1}} \ldots \Delta_{\delta_1} f(x_1, \ldots, x_{i-1}, x_i + \delta_i, x_{i+1}, \ldots, x_r)$$

and the inequality (because f satisfies (4.1))

$$\Delta_{|\delta_r|} \ldots \Delta_{|\delta_1|} f(x_1, x_2, \ldots, x_r) \geq 0,$$

we have

$$\operatorname{sign}(J(f)) = \prod_{i=1}^{r} \varepsilon_i,$$

where $\varepsilon_i = \operatorname{sign}(j_i - j_i^{(0)}) = \operatorname{sign} \delta_i$. ■

Lemma 4.12. Suppose that $\varphi(x_1, x_2, \ldots, x_r)$ is a bounded compactly supported function on $\mathbb{R}^r$ with supp $\varphi(x_1, x_2, \ldots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \ldots \times [-a_r, a_r]$, $0 < a_i < +\infty (i = 1, 2, \ldots, r)$ and $\varphi(x_1, x_2, \ldots, x_r)$ satisfies the condition (i) and (iv) in Theorem 4.1. Then, if $f(x_1, x_2, \ldots, x_r)$ satisfy the inequality (4.1) for every $(x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$ and all $\delta_i > 0$ $(i = 1, 2, \ldots, r)$, so do the linear operators $B_k(f)(x_1, x_2, \ldots, x_r)$.

Proof. We prove this lemma for $B_0(f)$. It can be proved similarly for all other $B_k(f)$, since $B_k(f)(x_1, x_2, \ldots, x_r) = B_0(f(2^{-k} \cdot))(2^k x_1, 2^k x_2, \ldots, 2^k x_r)$.

By Lemma 4.3 and Lemma 4.10, we only need to show that for any fixed $(x_1^{(0)}, x_2^{(0)}, \ldots, x_r^{(0)}) \in \mathbb{R}^r$ and any fixed δ_i, $0 < \delta_i < 1$ $(i = 1, 2, \ldots, r)$,

$$\Delta_{\delta_r} \ldots \Delta_{\delta_1} B_0(f)(x_1^{(0)}, x_2^{(0)}, \ldots, x_r^{(0)}) \geq 0. \tag{4.13}$$

Let $j_i^{(0)}$ $(i = 1, 2, \ldots, r)$ be the integers such that

$$x_i^{(0)} + \delta_i - j_i^{(0)} - 1 \leq b_i < x_i^{(0)} + \delta_i - j_i^{(0)}. \tag{4.14}$$

From the condition (i) of Theorem 4.1, for any $(x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$ we have

$$\sum_{j_i=-\infty}^{\infty} \Delta_{\delta_r} \ldots \Delta_{\delta_{i+1}} \Delta_{\delta_i} \Delta_{\delta_{i-1}} \ldots \Delta_{\delta_1} \varphi(x_1, \ldots, x_{i-1}, x_i^{(0)} - j_i, x_{i+1}, \ldots, x_r)$$

$$= \Delta_{\delta_r} \ldots \Delta_{\delta_{i+1}} \Delta_{\delta_i} \Delta_{\delta_{i-1}} \ldots \Delta_{\delta_1} \sum_{j_i=-\infty}^{\infty} \varphi(x_1, \ldots, x_{i-1}, x_i^{(0)} - j_i, x_{i+1}, \ldots, x_r)$$

$$= \Delta_{\delta_r} \ldots \Delta_{\delta_{i+1}} \Delta_{\delta_i} \Delta_{\delta_{i-1}} \ldots \Delta_{\delta_1} C_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r) = 0.$$

Hence

$$\sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} f(j_1, \ldots, j_{i-1}, j_i^{(0)}, j_{i+1}, \ldots, j_r)$$

$$\Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_{i-1}^{(0)} - j_{i-1}, x_i^{(0)} - j_i, x_{i+1}^{(0)} - j_{i+1}, \ldots x_r^{(0)} - j_r)$$

$$= \sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_{i+1}=-\infty}^{\infty} \sum_{j_{i-1}=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} f(j_1, \ldots, j_{i-1}, j_i^{(0)}, j_{i+1}, \ldots, j_r) \cdot$$

$$\sum_{j_i=-\infty}^{\infty} \Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_{i-1}^{(0)} - j_{i-1}, x_i^{(0)} - j_i, x_{i+1}^{(0)} - j_{i+1}, \ldots x_r^{(0)} - j_r) = 0,$$

$$\sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} f(j_1, \ldots, j_{i-1}, j_i^{(0)}, j_{i+1}, \ldots, j_{k-1}, j_k^{(0)}, j_{k+1}, \ldots, j_r)$$

$$\cdot \Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_r^{(0)} - j_r) = 0, \ (i < k)$$

$$\ldots \ \ldots$$

$$\sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} f(j_1^{(0)}, j_2^{(0)}, \ldots, j_r^{(0)}) \Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_r^{(0)} - j_r) = 0.$$

Therefore, from the above equations and Lemma 4.11, we have

$$\Delta_{\delta_r} \ldots \Delta_{\delta_1} B_0(f) \varphi(x_1^{(0)}, \ldots, x_r^{(0)}) = \sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} f(j_1, \ldots, j_r)$$

$$\cdot \Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_r^{(0)} - j_r)$$

$$= \sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} J(f) \Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_r^{(0)} - j_r)$$

$$= \sum_{j_r=-\infty;\ j_r \neq j_r^{(0)}}^{\infty} \cdots \sum_{j_1=-\infty;\ j_1 \neq j_1^{(0)}}^{\infty} J(f) \Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_r^{(0)} - j_r). \tag{4.15}$$

If $j_i \neq j_i^{(0)}$ $(1 \leq i \leq r)$ and $\mathrm{sign}(j_i - j_i^{(0)}) = \varepsilon_i$, then from Lemma 4.11 we have

$$\mathrm{sign}(J(f)) = \prod_{i=1}^{r} \varepsilon_i. \tag{4.16}$$

On the other hand, from (4.14), we have

$$0 < x_i^{(0)} + \delta_i - j_i^{(0)} - b_i \leq 1. \tag{4.17}$$

If $j_i < j_i^{(0)}$ then $\varepsilon_i = \mathrm{sign}(j_i - j_i^{(0)}) = -1$ and, by (4.17),

$$0 < x_i^{(0)} + \delta_i - j_i^{(0)} - b_i < x_i^{(0)} + 1 - j_i^{(0)} - b_i \leq x_i^{(0)} - j_i - b_i < x_i^{(0)} + \delta_i - j_i - b_i$$

which give

$$\mathrm{sign}(x_i^{(0)} - j_i - b_i) = \mathrm{sign}(x_i^{(0)} + \delta_i - j_i - b_i) = \pm 1 = -\varepsilon_i.$$

If $j_i > j_i^{(0)}$ then $\varepsilon_i = \mathrm{sign}(j_i - j_i^{(0)}) = 1$ and, by (4.17),

$$x_i^{(0)} - j_i - b_i < x_i^{(0)} + \delta_i - j_i - b_i \leq x_i^{(0)} + \delta_i - j_i^{(0)} - 1 - b_i \leq 0$$

which give

$$\text{sign}(x_i^{(0)} - j_i - b_i) = -1 = -\varepsilon_i,$$

$$\text{sign}(x_i^{(0)} + \delta_i - j_i - b_i) = -1 = -\varepsilon_i$$

or

$$x_i^{(0)} + \delta_i - j_i = b_i.$$

Hence, combining all the above information, we have

$$(x_i^{(0)} + \delta_1 - j_1, \ldots, x_r^{(0)} + \delta_r - j_r)$$

and

$$(x_i^{(0)} - j_1, \ldots, x_r^{(0)} - j_r) \in \overline{J}(-\varepsilon_1, \ldots, -\varepsilon_r),$$

provided $\text{sign}(j_i - j_i^{(0)}) = \varepsilon_i$ $(i = 1, 2, \ldots, r)$. Thus, from the condition (iv), we have

$$\text{sign}\left(\Delta_{\delta_r} \ldots \Delta_{\delta_1} \varphi(x_1^{(0)} - j_1, \ldots, x_r^{(0)} - j_r)\right) = \prod_{i=1}^{r} \varepsilon_i.$$

From this, (4.15) and (4.16), we obtain

$$\Delta_{\delta_r} \ldots \Delta_{\delta_1} B_0(f)(x_1^{(0)}, \ldots, x_r^{(0)}) \geq 0.$$

■

4) **Proof of Theorem 4.1**. From Lemmas 4.7, 4.8, 4.9, and 4.12, we know that $B_k(F)(x_1, \ldots, x_r)$ are probabilistic distribution functions on $\mathbb{R}^r$ if F is so.

Proof of Inequality (4.4). Noticing $\text{supp}\, \varphi(x_1, x_2, \ldots, x_r) \subseteq [-a_1, a_1] \times [-a_2, a_2] \times \ldots \times [-a_r, a_r]$, by the condition (ii), we have

$$B_k(F)(x_1, \ldots, x_r) - F(x_1, \ldots, x_r) = \sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} \left[F(2^{-k}j_1, \ldots, 2^{-k}j_r) - F(x_1, \ldots, x_r)\right]$$

$$\cdot\varphi(2^k x_1 - j_1, .., 2^k x_r - j_r) = \sum_{2^k x_r - a_r \le j_r \le 2^k x_r + a_r} \cdots \sum_{2^k x_1 - a_1 \le j_1 \le 2^k x_1 + a_1} \Big[F(2^{-k} j_1, \ldots, 2^{-k} j_r)$$

$$-F(x_1, \ldots, x_r)\Big] \cdot \varphi(2^k x_1 - j_1, \ldots, 2^k x_r - j_r). \tag{4.18}$$

Since for $2^k x_i - a_i \le j_i \le 2^k x_i + a_i$ $(i = 1, \ldots, r)$, we have

$$|2^{-k} j_i - x_i| \le 2^{-k} a_i \quad (i = 1, \ldots, r),$$

then from the definition of $w_1(f, h)$ we have

$$\Big|F(2^{-k} j_1, \ldots, 2^{-k} j_r) - F(x_1, \ldots, x_r)\Big| \le w_1(F, 2^{-k} d), \tag{4.19}$$

where $d := \max(a_1, \ldots, a_r)$ and $2^k x_i - a_i \le j_i \le 2^k x_i + a_i$ $(i = 1, \ldots, r)$. On the other hand, since $\varphi(x_1, \ldots, x_r)$ is compactly supported and satisfies the condition (iii), we have $\varphi(x_1, \ldots, x_r) \ge 0$. It follows from this, (4.18), (4.19) and the condition (ii) that

$$|B_k(F)(x_1, \ldots, x_r) - F(x_1, \ldots, x_r)| \le$$

$$\sum_{2^k x_r - a_r \le j_r \le 2^k x_r + a_r} \cdots \sum_{2^k x_1 - a_1 \le j_1 \le 2^k x_1 + a_1} |F(2^{-k} j_1, \ldots, 2^{-k} j_r) - F(x_1, \ldots, x_r)|$$

$$\cdot\varphi(2^k x_1 - j_1, \ldots, 2^k x_r - j_r) \le w_1(F, 2^{-k} d).$$

Proof of Sharpness of Inequality (4.4). Take

$$\varphi(x_1, \ldots, x_r) = \varphi_0(x_1, \ldots, x_r) := \varphi_0(x_1)\varphi_0(x_2) \ldots \varphi_0(x_r)$$

with $\varphi_0(x)$ defined by (4.5). Define

$$f(x_1, \ldots, x_r) := f_0(x_1) f_0(x_2) \ldots f_0(x_r),$$

where

$$f_0(x) := \begin{cases} 1, \ x \ge 0 \\ 0, \ x \le -2^{-k-1} \\ 2^{k+1} x + 1, \ -2^{-k-1} < x < 0. \end{cases}$$

It is easy to verify that $f(x_1, \ldots, x_r)$ is continuous on $\mathbb{R}^r$ and is nondecreasing with respect to each variable with

$$f(-\infty, x_2, \ldots, x_r) = f(x_1, -\infty, x_3, \ldots, x_r) = \ldots = f(x_1, \ldots, x_{r-1}, -\infty) = 0,$$

$$f(+\infty, +\infty, \ldots, +\infty) = 1.$$

Besides, from Lemma 4.4, the function $f(x_1, \ldots, x_r)$ satisfies the inequality (4.1). Hence $f(x_1, \ldots, x_r)$ is a probabilistic distribution function on $\mathbb{R}^r$.

Consider $x_1 = x_2 = \ldots = x_r = -2^{-k-1}$. We have

$$B_k(f)(-2^{-k-1}, -2^{-k-1}, \ldots, -2^{-k-1}) - f(-2^{-k-1}, -2^{-k-1}, \ldots, -2^{-k-1})$$

$$= B_k(f)(-2^{-k-1}, -2^{-k-1}, \ldots, -2^{-k-1})$$

$$= \sum_{j_r=-\infty}^{\infty} \cdots \sum_{j_2=-\infty}^{\infty} \sum_{j_1=-\infty}^{\infty} f(2^{-k}j_1, 2^{-k}j_2, \ldots, 2^{-k}j_r)$$

$$\cdot \varphi\left(-\frac{1}{2} - j_1, -\frac{1}{2} - j_2, \ldots, -\frac{1}{2} - j_r\right) = f(0, \ldots, 0) = 1,$$

since $\varphi\left(-\frac{1}{2} - j_1, -\frac{1}{2} - j_2, \ldots, -\frac{1}{2} - j_r\right) = 0$ for any $(j_1, j_2, \ldots, j_r) \neq (0, \ldots, 0)$.

On the other hand, we have $d = a_1 = a_2 = ... = a_r = \frac{1}{2}$ and

$$w_1(f, 2^{-k}d) = w_1(f, 2^{-k-1}) = f(0, \ldots, 0) - f(-2^{-k-1}, -2^{-k-1}, \ldots, -2^{-k-1}).$$

Hence the inequalities (4.4) are sharp.

5

Multidimensional Probabilistic Approximation in Wavelet Like Structure

Let

$$\varphi_0(x,y) := \begin{cases} 1, & x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and $F(x,y)$ be a continuous probability distribution function on $\mathbb{R}^2$.

Then there exist linear wavelet type operators $L_n(F,x,y)$ which are also distribution functions and where the defining them wavelet function is $\varphi_0(x,y)$. These approximate $F(x,y)$ in the supnorm. The degree of this approximation is estimated by establishing a Jackson type inequality. Furthermore we give generalizations for the case of a wavelet function $\neq \varphi_0$, which is just any distribution function on $\mathbb{R}^2$, also we extend these results in $\mathbb{R}^r$, $r > 2$. This chapter relies on [87].

5.1 Introduction

There has been a great interest in the wavelet type approximations [138], [221]. There are very important and useful kind of approximations which only involve dilated translates of a basic function.

The aim here is to use wavelet like approximation to multivariate probabilistic distribution functions. This chapter is motivated by the following very important theorem of Analysis [134], p. 221. Let E be a locally

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compact Hausdorff space. It is proved that the discrete measures of the form

$$\sum_{i=1}^{n} \alpha_i \cdot d_{x_i},$$

where $\alpha_i \geq 0$, $x_i \in E$, d_{x_i} the unit (Dirac) measure, are dense in the weak*-topology in $M^+(E)$, the set of all positive Radon measures on E.

In this chapter we consider the following form of probabilistic discrete wavelets:

$$\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \alpha_{ij} \varphi\left(2^n x - i, 2^n y - j\right) \tag{*}$$

where

$$\varphi_0(x, y) = \begin{cases} 1, & x, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

is a basic probability distribution function. We prove, Theorem 5.2, that for any $F(x, y) \in C\left(\mathbb{R}^2\right)$ distribution function there exist linear operators $L_n(F, x, y)$ which are distribution functions of the above discrete form (*) and converge to $F(x, y)$ in the supnorm with the approximation errors bounded by $\omega_1\left(F, 2^{-n}\right)$, $n \in \mathbb{Z}$, $(x, y) \in \mathbb{R}^2$, where $\omega_1(F, t)$ is the first modulus of continuity of F. Then we extend this theory for linear operators of the similar form $\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \alpha_{ij} \varphi(2^n x - i,\ 2^n y - j)$, where φ now is any fixed distribution function on $\mathbb{R}^2$, see Theorem 5.3. We present also generalizations of these results in $\mathbb{R}^r$, $r > 2$, see Theorems 5.6, 5.7. For the univariate case see [88]. It is important to notice that the wavelet functions defining the operators L_n are not of compact support.

5.2 Results

Let $F(x, y)$ be a probability distribution function on $\mathbb{R}^2$. As we know [255], $F(x, y)$ satisfies the following conditions:

(i) $F(x, y)$ is non-decreasing with respect to each variable x and y, and is right-continuous with respect to both variables x and y;

(ii)

$$\lim_{\substack{x \to +\infty \\ y \to +\infty}} F(x, y) = 1 \quad \text{and} \quad \lim_{\substack{x \to -\infty \\ \text{or} \;\; y \to -\infty}} F(x, y) = 0; \tag{5.1}$$

(iii) for $h, k > 0$, holds

$$F(x + h, y + k) - F(x, y + k) - F(x + h, y) + F(x, y) \geq 0, \quad (x, y) \in \mathbb{R}^2. \tag{5.2}$$

Conversely, if a function $F(x, y)$ satisfies all the above conditions, then $F(x, y)$ is a distribution function on $\mathbb{R}^2$.

Let

$$\varphi_0(x,y) := \begin{cases} 1, & x, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\varphi_0(x,y)$ is a distribution function on $\mathbb{R}^2$. Call

$$L_n(F,x,y) := \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \left(C_{i,j}^{(n)} - C_{i,j-1}^{(n)} - C_{i-1,j}^{(n)} + C_{i-1,j-1}^{(n)} \right) \cdot \qquad (5.3)$$

$$\varphi_0(2^n x - i, 2^n y - j),$$

where $(x,y) \in \mathbb{R}^2$, $n \in \mathbb{Z}$, and

$$C_{i,j}^{(n)} := 2^{2n} \cdot \int_{2^{-n}j}^{2^{-n}(j+1)} \int_{2^{-n}i}^{2^{-n}(i+1)} F(u,v)\, dudv. \qquad (5.4)$$

In general, for any given distribution function $\varphi(x,y)$ on $\mathbb{R}^2$, we define

$$L_n(F,x,y;\varphi) :=$$

$$\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \left(C_{i,j}^{(n)} - C_{i,j-1}^{(n)} - C_{i-1,j}^{(n)} + C_{i-1,j-1}^{(n)} \right) \varphi(2^n x - i, 2^n y - j), \qquad (5.5)$$

where $(x,y) \in \mathbb{R}^2$, $n \in \mathbb{Z}$, and $C_{i,j}^{(n)}$ are defined by (5.4) too.

We first establish the following

Lemma 5.1. Let $F(x,y) \in C(\mathbb{R}^2)$ and $\varphi(x,y)$ be distribution functions on $\mathbb{R}^2$. Then the linear operators $L_n(F,x,y;\varphi)$ are well-defined by (5.4) and (5.5), and are distribution functions on $\mathbb{R}^2$.

Proof. Since $F(x,y)$ is a distribution function $F(x,y)$ is non-decreasing with respect to each variable x and y, and $F(x,y)$ satisfies (5.2). It follows from this and the definition of $C_{i,j}^{(n)}$ that

$$C_{i,j}^{(n)} - C_{i,j-1}^{(n)} \geq 0, \;\; C_{i,j}^{(n)} - C_{i-1,j}^{(n)} \geq 0$$

and

$$C_{i,j}^{(n)} - C_{i,j-1}^{(n)} - C_{i-1,j}^{(n)} + C_{i-1,j-1}^{(n)} \geq 0, \quad i,j,n \in \mathbb{Z}. \qquad (5.6)$$

On the other hand, since $F(x,y)$ satisfies (5.1), we get

$$\lim_{\substack{i\to+\infty \\ j\to+\infty}} C_{i,j}^{(n)} = \lim_{\substack{i\to+\infty \\ j\to+\infty}} 2^{2n} \int_{2^{-n}j}^{2^{-n}(j+1)} \int_{2^{-n}i}^{2^{-n}(i+1)} F(u,v)\, dudv$$

$$= \lim_{\substack{i\to+\infty \\ j\to+\infty}} \int_0^1 \int_0^1 F\left(2^{-n}(t+i), 2^{-n}(s+j)\right) dtds$$

$$= \int_0^1 \int_0^1 \lim_{\substack{i \to +\infty \\ j \to +\infty}} F\left(2^{-n}(t+i), 2^{-n}(s+j)\right) dtds = 1, \tag{5.7}$$

and

$$\lim_{\substack{x \to -\infty \\ \text{or } y \to -\infty}} = \int_0^1 \int_0^1 \lim_{\substack{i \to -\infty \\ \text{or } j \to -\infty}} F\left(2^{-n}(t+i), 2^{-n}(s+j)\right) dtds = 0. \tag{5.8}$$

Here we have used the fact that $F(x,y) \geq 0$ is non-decreasing with respect to each variable for taking the limit under the integrations.

It follows from (5.6), (5.7) and (5.8) that

$$\sum_{j=-m_1}^{m_2} \sum_{i=-k_1}^{k_2} \left(C_{i,j}^{(n)} - C_{i,j-1}^{(n)} - C_{i-1,j}^{(n)} + C_{i-1,j-1}^{(n)}\right)$$

$$= \sum_{j=-m_1}^{m_2} \sum_{i=-k_1}^{k_2} \left[(C_{i,j}^{(n)} - C_{i,j-1}^{(n)}) - \left(C_{i-1,j}^{(n)} - C_{i-1,j-1}^{(n)}\right) \right]$$

$$= \sum_{j=-m_1}^{m_2} \left(C_{k_2,j}^{(n)} - C_{k_2,j-1}^{(n)}\right) - \left(C_{-k_1-1,j}^{(n)} - C_{-k_1-1,j-1}^{(n)}\right)$$

$$= \sum_{j=-m_1}^{m_2} \left(C_{k_2,j}^{(n)} - C_{k_2,j-1}^{(n)}\right) - \sum_{j=-m_1}^{m_2} \left(C_{-k_1-1,j}^{(n)} - C_{-k_1-1,j-1}^{(n)}\right)$$

$$= C_{k_2,m_2}^{(n)} - C_{k_2,-m_1-1}^{(n)} - C_{-k_1-1,m_2}^{(n)} + C_{-k_1-1,-m_1-1}^{(n)}, \tag{5.9}$$

which tends to 1 as $m_1, m_2, k_1, k_2 \to +\infty$, and then the non-negative series

$$\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \left(C_{i,j}^{(n)} - C_{i,j-1}^{(n)} - C_{i-1,j}^{(n)} + C_{i-1,j-1}^{(n)}\right) = 1. \tag{5.10}$$

Therefore, because the distribution function $\varphi(x,y)$ has $0 \leq \varphi(x,y) \leq 1$, the summations in (5.5) are convergent absolutely and uniformly on $\mathbb{R}^2$, and so $L_n(F,x,y;\varphi)$ are well-defined on $\mathbb{R}^2$.

Since $\varphi(x,y)$ is non-decreasing with respect to each variable, by (5.6), $L_n(F,x,\ y;\varphi)$ are also non-decreasing with respect to each variable. The fact that $\varphi(x,y)$ is right-continuous with respect to both variables x and y and the summations in (5.5) are convergent absolutely and uniformly gives the right-continuity of $L_n(F,x,y;\varphi)$ with respect to both variables x and y. Besides, we have

$$\lim_{\substack{x \to +\infty \\ y \to +\infty}} L_n(F,x,y;\varphi) = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \left(C_{i,j}^{(n)} - C_{i,j-1}^{(n)} - C_{i-1,j}^{(n)} + C_{i-1,j-1}^{(n)}\right) \cdot$$

$$\lim_{\substack{x\to+\infty\\y\to+\infty}}\varphi\left(2^n x-i,2^n y-j\right)$$

$$=\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\left(C_{i,j}^{(n)}-C_{i,j-1}^{(n)}-C_{i-1,j}^{(n)}+C_{i-1,j-1}^{(n)}\right)=1$$

and

$$\lim_{\substack{x\to-\infty\\\text{or }\ y\to-\infty}}L_n\left(F,x,y;\varphi\right)=\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\left(C_{i,j}^{(n)}-C_{i,j-1}^{(n)}-C_{i-1,j}^{(n)}+C_{i-1,j-1}^{(n)}\right)\cdot$$
$$\lim_{\substack{x\to-\infty\\\text{or }\ y\to-\infty}}\varphi\left(2^n x-i,2^n y-j\right)=0.$$

Moreover, from (5.6) and that $\varphi(x,y)$ satisfies (5.2), we derive that $L_n(F,x,y;\varphi)$ satisfies (5.2). Thus, $L_n(F,x,y;\varphi)$ are distribution function on $\mathbb{R}^2$. ■

For $h>0$ and $F(x,y)\in C\left(\mathbb{R}^2\right)$, we define

$$\omega_1(F,h):=\sup_{\substack{|t|,|s|\le h\\(x,y)\in\mathbb{R}^2}}\left|F(x+t,y+s)-F(x,y)\right|.$$

Theorem 5.2. Assume that $F(x,y)\in C\left(\mathbb{R}^2\right)$ is a distribution function on $\mathbb{R}^2$. Then the linear operators $L_n(F,x,y)$ defined by (5.3) and (5.4) are distribution functions such that

$$\left|L_n(F,x,y)-F(x,y)\right|\le\omega_1\left(F,2^{-n}\right),\quad n\in\mathbb{Z},\ (x,y)\in\mathbb{R}^2. \tag{5.11}$$

Proof. By Lemma 5.1, we only need to prove (5.11).

Let us consider $n=0$ first. For any fixed x and y, suppose that i_0, j_0 are the integers such that

$$i_0\le x<i_0+1 \tag{5.12}$$

and

$$j_0\le y<j_0+1. \tag{5.13}$$

From the definition of $\varphi_0(x,y)$, we have

$$\varphi_0(x-i,y-j)=0,\quad i\ge i_0+1\text{ or }j\ge j_0+1$$

and

$$\varphi_0(x-i,y-j)=1,\quad i\le i_0\text{ and }j\le j_0.$$

It follows from these equations, (5.8) and (5.9) that

$$L_0(F,x,y)=\left(\sum_{j=-\infty}^{j_0}\sum_{i=-\infty}^{i_0}+\sum_{j=-\infty}^{j_0}\sum_{i=i_0+1}^{\infty}+\sum_{j=j_0+1}^{\infty}\sum_{i=-\infty}^{\infty}\right)$$

$$\left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)}\right) \varphi_0 \left(x-i, y-j\right)$$

$$= \sum_{j=-\infty}^{j_0} \sum_{i=-\infty}^{i_0} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)}\right) = C_{i_0,j_0}^{(0)}. \tag{5.14}$$

Hence we obtain

$$\left|L_0 \left(F, x, y\right) - F\left(x, y\right)\right| = \left|C_{i_0,j_0}^{(0)} - F\left(x, y\right)\right|$$

$$= \left|\int_{j_0}^{j_0+1} \int_{i_0}^{i_0+1} F\left(u, v\right) dudv - F\left(x, y\right)\right|$$

$$\leq \int_{j_0}^{j_0+1} \int_{i_0}^{i_0+1} \left|F\left(u, v\right) - F\left(x, y\right)\right| dudv \leq \omega_1 \left(F, 1\right). \tag{5.15}$$

The last inequality comes from (5.12) and (5.13). ■

Notice that

$$L_n \left(F, x, y\right) = L_0 \left(F\left(2^{-n}, 2^{-n}\right), 2^n x, 2^n y\right)$$

and

$$\omega_1 \left(F\left(2^{-n}, 2^{-n}\right), 1\right) = \sup_{\substack{|t|,|s| \leq h \\ (x,y)\in\mathbb{R}^2}} \left|F\left(2^{-n}\left(x+t\right), 2^{-n}\left(y+s\right)\right) - F\left(2^{-n}x, 2^{-n}y\right)\right|$$

$$= \sup_{\substack{|t|,|s| \leq h \\ (x,y)\in\mathbb{R}^2}} \left|F\left(x+2^{-n}t, y+2^{-n}s\right) - F\left(x, y\right)\right| = \omega_1 \left(F, 2^{-n}\right).$$

Then, by (5.15), we have

$$\left|L_n \left(F, x, y\right) - F\left(x, y\right)\right| = \left|L_0 \left(F\left(2^{-n}, 2^{-n}\right), 2^n x, 2^n y\right) - F\left(2^{-n}2^n x, 2^{-n}2^n y\right)\right|$$

$$\leq \omega_1 \left(F\left(2^{-n}, 2^{-n}\right), 1\right) = \omega_1 \left(F, 2^{-n}\right).$$

For the general case we have

Theorem 5.3. Assume that $F(x,y) \in C\left(\mathbb{R}^2\right)$ and $\varphi(x,y)$ are distribution functions on $\mathbb{R}^2$. Then the linear operators $L_n(F,x,y;\varphi)$ defined by (5.4) and (5.5) are distribution functions such that for any $a > 0$ holds

$$\left|L_n \left(F, x, y; \varphi\right) - F\left(x, y\right)\right| \leq$$

$$9\left\{\omega_1\left(F, 2^{-n}\left(a+1\right)\right) + \sup_{x,y\geq a}\left(1-\varphi\left(x,y\right)\right) + \sup_{\substack{x<-a \\ \text{or } y<-a}} \varphi\left(x,y\right)\right\}, \tag{5.16}$$

where $n \in \mathbb{Z}$, $(x,y) \in \mathbb{R}^2$.

Proof. For any fixed $a > 0$ and $(x, y) \in \mathbb{R}^2$, suppose that i_0, i_1, j_0 and j_1 are the integers such that

$$x - i_0 - 1 < a \leq x - i_0, \quad y - j_0 - 1 < a \leq y - j_0 \tag{5.17}$$

and

$$x - i_1 < -a \leq x - i_1 + 1, \quad y - j_1 < -a \leq y - j_1 + 1. \tag{5.18}$$

We have $i_0 \leq i_1$ and $j_0 \leq j_1$, and

$$\begin{aligned} L_0(F, x, y; \varphi) = &\left(\sum_{j=-\infty}^{j_0} \sum_{i=-\infty}^{i_0} + \sum_{j=-\infty}^{j_0} \sum_{i=i_0+1}^{i_1} + \sum_{j=-\infty}^{j_0} \sum_{i=i_1+1}^{\infty} + \sum_{j=j_0+1}^{j_1} \sum_{i=-\infty}^{i_0} + \right. \\ &\left. \sum_{j=j_0+1}^{j_1} \sum_{i=i_0+1}^{i_1} + \sum_{j=j_0+1}^{j_1} \sum_{i=i_1+1}^{\infty} + \sum_{j=j_1+1}^{\infty} \sum_{i=-\infty}^{\infty} \right) \\ &\left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right) \varphi(x - i, y - j) \\ &:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned} \tag{5.19}$$

For the terms in I_1, we have $i \leq i_0$ and $j \leq j_0$, and then from (5.17), we have $x - i \geq a$ and $y - j \geq a$. Hence, from (5.6), (5.10) and (5.14), we have

$$\begin{aligned} I_1 &= \sum_{j=-\infty}^{j_0} \sum_{i=-\infty}^{i_0} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right) \varphi(x - i, y - j) \\ &= \sum_{j=-\infty}^{j_0} \sum_{i=-\infty}^{i_0} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right) \\ &+ \sum_{j=-\infty}^{j_0} \sum_{i=-\infty}^{i_0} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right) \left(\varphi(x - i, y - j) - 1 \right) \\ &= C_{i_0,j_0}^{(0)} + I_1' \end{aligned} \tag{5.20}$$

where

$$I_1' := \sum_{j=-\infty}^{j_0} \sum_{i=-\infty}^{i_0} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right) \left(\varphi(x - i, y - j) - 1 \right)$$

and

$$|I_1'| \leq \sup_{x,y \geq a} \left(1 - \varphi(x, y) \right). \tag{5.21}$$

Noticing (5.6) and that $0 \le \varphi \le 1$, we get

$$|I_2| \le \sum_{j=-\infty}^{j_0} \sum_{i=i_0+1}^{i_1} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right).$$

From (5.8) and (5.9), the right-hand side of the above inequality equals

$$C_{i_1,j_0}^{(0)} - C_{i_0,j_0}^{(0)} = \int_{j_0}^{j_0+1} \int_{i_1}^{i_1+1} F(u,v)\, dudv - \int_{j_0}^{j_0+1} \int_{i_0}^{i_0+1} F(u,v)\, dudv$$

$$= \int_{j_0}^{j_0+1} \int_{i_0}^{i_0+1} \left[F(u + (i_1 - i_0), v) - F(u,v) \right] dudv \le \omega_1 (F, i_1 - i_0).$$

But from (5.17) and (5.18) we have $i_1 - i_0 \le 2a + 2$. Therefore

$$|I_2| \le 2\omega_1 (F, a+1). \tag{5.22}$$

Similarly we have

$$|I_4| \le \sum_{j=j_0+1}^{j_1} \sum_{i=-\infty}^{i_0} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right)$$

$$= C_{i_0,j_1}^{(0)} - C_{i_0,j_0}^{(0)} \le 2\omega_1 (F, a+1) \tag{5.23}$$

and

$$|I_5| \le \sum_{j=j_0+1}^{j_1} \sum_{i=i_0+1}^{i_1} \left(C_{i,j}^{(0)} - C_{i,j-1}^{(0)} - C_{i-1,j}^{(0)} + C_{i-1,j-1}^{(0)} \right)$$

$$= C_{i_1,j_1}^{(0)} - C_{i_1,j_0}^{(0)} - C_{i_0,j_1}^{(0)} + C_{i_0,j_0}^{(0)} \le 4\omega_1 (F, a+1). \tag{5.24}$$

For the terms in I_3 we have $i \ge i_1 + 1$ and then from (5.18) we have $x - i < -a$. Hence, from (5.6) and (5.10), we obtain

$$|I_3| \le \sup_{x<-a} \varphi(x,y). \tag{5.25}$$

Similarly, we derive

$$|I_6| \le \sup_{x<-a} \varphi(x,y) \tag{5.26}$$

and

$$|I_7| \le \sup_{y<-a} \varphi(x,y). \tag{5.27}$$

It follows from (5.19)-(5.27) that

$$|L_n(F,x,y;\varphi) - F(x,y)| \le \left| C_{i_0,j_0}^{(0)} - F(x,y) \right| + 8\omega_1 (F, a+1)$$

$$+\sup_{x,y\geq a}\left(1-\varphi\left(x,y\right)\right)+3\sup_{\substack{x<-a\\ \text{or } y<-a}}\varphi\left(x,y\right).$$

But, from (5.17), we also obtain

$$\left|C_{i_0,j_0}^{(0)}-F\left(x,y\right)\right|\leq\int_{j_0}^{j_0+1}\int_{i_0}^{i_0+1}\left|F\left(u,v\right)-F\left(x,y\right)\right|dudv\leq\omega_1\left(F,a+1\right).$$

Thus

$$\left|L_0\left(F,x,y;\varphi\right)-F\left(x,y\right)\right|\leq 9\omega_1\left(F,a+1\right)+\sup_{x,y\geq a}\left(1-\varphi\left(x,y\right)\right)+3\sup_{\substack{x<-a\\ \text{or } y<-a}}\varphi\left(x,y\right).$$

Noticing that

$$L_n\left(F,x,y;\varphi\right)=L_0\left(F\left(2^{-n},2^{-n}\right),2^nx,2^ny;\varphi\right)$$

and

$$\omega_1\left(F\left(2^{-n},2^{-n}\right),a+1\right)=\omega_1\left(F,2^{-n}\left(a+1\right)\right),$$

we obtain (5.16). ■

Let $a>0$ and $\varphi_a\left(x,y\right)$ be a distribution function such that $\varphi_a\left(x,y\right)=1$ if $x,y\geq a$ and $\varphi_a\left(x,y\right)=0$ if $x<-a$ or $y<-a$. Then, by Theorem 5.3, we get

Corollary 5.4. For distribution function $F\in C\left(\mathbb{R}^2\right)$, the linear operators $L_n(F,x,\ y;\varphi_a)$ are distribution functions such that

$$\left|L_n\left(F,x,y;\varphi_a\right)-F\left(x,y\right)\right|\leq 9\omega_1\left(F,2^{-n}\left(a+1\right)\right),$$

where $n\in\mathbb{Z}$, $(x,y)\in\mathbb{R}^2$.

Let $F\left(x_1,x_2,...,x_r\right)$ be a distribution function on $\mathbb{R}^r$, and $r>2$ be an integer. The necessary and sufficient conditions for F being a distribution function on $\mathbb{R}^r$, see [255], are

(i) F is non-decreasing with respect to each variable $x_1,x_2,...,x_r$ and is right-continuous with respect to all variables $x_1,x_2,...,x_r$;

(ii) $F\left(-\infty,x_2,...,x_r\right)=F\left(x_1,-\infty,x_3,...,x_r\right)=...=F\left(x_1,...,x_{r-1},-\infty\right)=0$ and $F\left(+\infty,+\infty,...,+\infty\right)=1$;

(iii) for every $\left(x_1,x_2,...,x_r\right)\in\mathbb{R}^r$ and all $\delta_i>0$ $(i=1,2,...,r)$, holds the inequality

$$F\left(x_1+\delta_1,x_2+\delta_2,...,x_r+\delta_r\right)-$$

$$\sum_{i=1}^{r}F\left(x_1+\delta_1,...,x_{i-1}+\delta_{i-1},x_i,x_{i+1}+\delta_{i+1},...,x_r+\delta_r\right)$$

$$+\sum_{\substack{i,j+1\\ i<j}}^{r}F(x_1+\delta_1,...,x_{i-1}+\delta_{i-1},x_i,x_{i+1}+\delta_{i+1},...,x_{j-1}+\delta_{j-1},x_j,$$

$$x_{j+1}+\delta_{j+1},...,x_r+\delta_r)+...+(-1)^r F(x_1,x_2,...,x_r)\geq 0. \tag{5.28}$$

Let

$$\varphi_0(x_1,x_2,...,x_r)=\begin{cases}1, & x_1,x_2,...,x_r\geq 0\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to verify that φ_0 satisfies all the above conditions and so φ_0 is a distribution function on $\mathbb{R}^r$. Define the following operators for $F\in C(\mathbb{R}^r)$:

$$L_n(F,x_1,...,x_r):=\sum_{i_1=-\infty}^{\infty}...\sum_{i_r=-\infty}^{\infty} d^{(n)}_{i_1,...,i_r}\varphi_0(2^n x_1-i_1,...,2^n x_r-i_r), \tag{5.29}$$

where

$$d^{(n)}_{i_1,...,i_r}:=C^{(n)}_{i_1,...,i_r}-\sum_{j=1}^{r}C^{(n)}_{i_1,...,i_{j-1},i_j-1,i_{j+1},...,i_r}+$$

$$\sum_{\substack{j,s=1\\ j<s}}^{r}C^{(n)}_{i_1,...,i_{j-1},i_j-1,i_{j+1},...,i_{s-1},i_s-1,i_{s+1},...,i_r}+...+(-1)^r C^{(n)}_{i_1-1,...,i_r-1} \tag{5.30}$$

with

$$C^{(n)}_{i_1,...,i_r}:=2^{rn}\int_{2^{-n}i_1}^{2^{-n}(i_1+1)}...\int_{2^{-n}i_r}^{2^{-n}(i_r+1)}F(u_1,...,u_r)\,du_1...du_r \tag{5.31}$$

for $i_1,...,i_r\in\mathbb{Z}$ and $n\in\mathbb{Z}$.

For any distribution function $\varphi(x_1,...,x_r)$ on $\mathbb{R}^r$ we define

$$L_n(F,x_1,...,x_r;\varphi):=\sum_{i_1=-\infty}^{\infty}...\sum_{i_r=-\infty}^{\infty} d^{(n)}_{i_1,...,i_r}\varphi(2^n x_1-i_1,...,2^n x_r-i_r), \tag{5.32}$$

where $d^{(n)}_{i_1,...,i_r}$ are defined by (5.30) and (5.31).

Similar to Lemma 5.1, Theorem 5.2, 5.3 and Corollary 5.4 we give

Lemma 5.5. Let $F(x_1,...,x_r)\in C(\mathbb{R}^r)$ and $\varphi(x_1,...,x_r)$ be distribution functions on $\mathbb{R}^r$. Then the linear operators $L_n(F,x_1,...,x_r;\varphi)$ are well-defined by (5.30), (5.31) and (5.32), and are distribution functions on $\mathbb{R}^r$.

Theorem 5.6. Assume that $F(x_1,...,x_r)\in C(\mathbb{R}^r)$ is a distribution function on $\mathbb{R}^r$. Then the linear operators $L_n(F,x_1,...,x_r)$ defined by (5.29), (5.30) and (5.31) are distribution functions such that

$$|L_n(F,x_1,...,x_r)-F(x_1,...,x_r)|\leq\omega_1\left(F,2^{-n}\right),$$

where $n\in\mathbb{Z}$, $(x_1,...,x_r)\in\mathbb{R}^r$, and

$$\omega_1(F,h):=\sup_{\substack{|t_i|\leq h\\ i=1,...,r\\ (x_1,...,x_r)\in\mathbb{R}^r}}|F(x_1+t_1,...,x_r+t_r)-F(x_1,...,x_r)|.$$

Theorem 5.7. Assume that $F(x_1,...,x_r) \in C(\mathbb{R}^r)$ and $\varphi(x_1,...,x_r)$ are distribution functions on $\mathbb{R}^r$. Then the linear operators $L_n(F,x_1,...,x_r;\varphi)$ defined by (5.30), (5.31) and (5.32) are distribution functions such that for any $a>0$ holds

$$|L_n(F,x_1,...,x_r;\varphi) - F(x_1,...,x_r)| \leq$$

$$C\left\{\omega_1\left(F,2^{-n}(a+1)\right) + \sup_{x_1,\ldots,x_r \geq a}(1-\varphi(x_1,...,x_r)) + \sup_{\substack{x_i<-a \\ i\in(1,\ldots,r)}}\varphi(x_1,...,x_r)\right\},$$

where $n \in \mathbb{Z}$, $(x_1,...,x_r) \in \mathbb{R}^r$, and C is an absolute constant.

Corollary 5.8. Assume that $F(x_1,...,x_r) \in C(\mathbb{R}^r)$ and $\varphi_a(x_1,...,x_r)$ are distribution functions on $\mathbb{R}^r$, and $\varphi_a(x_1,...,x_r) = 1$ if $x_1,...,x_r \geq a \geq 0$, $\varphi_a(x_1,...,x_r) = 0$ if $x_i < -a$, $i \in \{1,...,r\}$. Then the linear operators $L_n(F,x_1,...,x_r;\varphi_a)$ are distribution functions such that

$$|L_n(F,x_1,...,x_r;\varphi_a) - F(x_1,...,x_r)| \leq C\omega_1\left(F,2^{-n}(a+1)\right),$$

where $n \in \mathbb{Z}$, $(x_1,...,x_r) \in \mathbb{R}^r$, and C is an absolute constant.

6

About L-Positive Approximations

Let F be a normed space and let B be a subspace of F. Assume that $L\colon F \to L_\infty(\Omega)$, $\Omega \subset \mathrm{R}^m$, is a linear bounded operator and $M(L) = \{f \in F\colon Lf \geq 0 \text{ a.e. on } \Omega\}$. We establish some inequalities for best approximation of $f \in M(L)$ by elements from $B \cap M(L)$. In the case when L is a differential operator and F is the Sobolev space $W_p^\ell(\Omega)$ we obtain Jackson type estimates for simultaneous approximation of $f \in M(L)$ by multivariate polynomials and entire functions of exponential type from $M(L)$. This chapter relies on [75].

6.1 Introduction

Let F be a normed space with the norm $\|\cdot\|_F$ and let B be a subspace of F. Assume that M is a subset of F such that $M \cap B \neq \emptyset$. We define best approximation of f by elements from B (or from $B \cap M$) in the metric of F as

$$E(f, B, F) = \inf_{g \in B} \|f - g\|_F, \quad f \in F;$$

$$E_M(f, B, F) = \inf_{g \in B \cap M} \|f - g\|_F, \quad f \in M.$$

We are interested in efficient estimates of $E_M(f, B, F)$ for all $f \in M$. It is clear $E(f, B, F) \leq E_M(f, B, F)$, while for some sets M

$$E(f, B, F) = E_M(f, B, F), \quad f \in M. \tag{6.1}$$

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In particular, (6.1) holds if F is a Banach space and M is the set of elements from F which are invariant relative to a compact group of operators (see [169], [224], p. 26). Equality (6.1) is not valid for arbitrary $M \subset F$. So the problem of obtaining efficient estimates of $E_M(f, B, F)$ has attracted the attention of many authors. Mainly they have dealt with the classes of monotone and convex functions and with their generalizations.

Let M^k be the set of all functions defined on $[0, 1]$ such that $\Delta_h^k f(x) \geq 0$ for all $h \in [0, 1/k]$ and all $x \in [0, 1 - kh]$. In particular, M^1 and M^2 are the sets of monotone and convex functions respectively. Let $\mathcal{P}_n$ be the class of algebraic polynomials of degree n.

In the 1920s, S.N. Bernstein defined the class $M = \bigcap_{k=0}^{\infty} M^k$ of absolutely monotone functions and proved [112] that polynomials from M are dense in M.

The first Jackson type estimates of $E_{M^k}(f, \mathcal{P}_n, C[0, 1])$, $f \in M^k$, were given by O. Shisha [262] with much refinement given later by J. Roulier [256]. Further generalizations and improvements for the classes M^1 and M^2 were obtained by G.G. Lorentz and K. Zeller [217], R.A. DeVore [141], A.S. Shvedov [266, 267], K.A. Kopotun [212], Y. Hu, D. Leviatan, and X.M. Yu [192] and by many others. Analogous problems for spline approximation were considered by R.A. DeVore [142] and Y. Hu [191].

D. Leviatan [214] gave estimates of the degree of simultaneous approximation by monotone polynomials. Deep results in comonotone approximation were obtained by E. Passow, L. Raymon, and J.A. Roulier [240] and D.J. Newman [235].

The author and O. Shisha [83] generalized the problem of monotone approximation for functions from $M^k \cap C^k$, replacing the k-th derivative with a linear differential operator of order k. Some multidimensional results were obtained in [14, 268, 269]. In particular, the author [14] considered the class $M(L) = \{f \in C^k([0, 1]^2)\colon (Lf)(x) \geq 0\}$, where L is a differential operator, and derived an estimate of simultaneous approximation of $f \in M(L)$ by polynomials from $M(L)$ involving the bivariate first modulus of continuity.

There have been some negative results obtained in [218, 267]. In particular, G.G. Lorentz and K. Zeller [218] proved that there exists a k-monotone function $f_0 \in C^k[0, 1] \cap M^k$ satisfying the limit equality

$$\lim_{n\to\infty} E_{M^k}(f_0, \mathcal{P}_n, C[0, 1])/E(f_0, \mathcal{P}_n, C[0, 1]) = \infty. \tag{6.2}$$

In this chapter we generalize the set $M^k \cap C^k$ considering the class $M(L) = \{f \in F\colon (Lf)(x) \geq 0 \text{ a.e. on } \Omega\}$ where F is a normed space, $\Omega \subset \mathrm{R}^m$, and $L\colon F \to L_\infty(\Omega)$ is a linear bounded operator. We shall prove that under some conditions on L and $B \subset F$

$$E_{M(L)}(f, B, F) \leq CE(f, B, F), \quad f \in M(L), \tag{6.3}$$

where C is a constant independent on f and B. (6.2) shows that (6.3) is not valid for some unbounded operators.

This result and its generalizations are proved in Section 6.2. In Sections 6.3 and 6.4 we consider applications of (6.3) to simultaneous approximation, to convex and subharmonic approximation and to δ-monotone approximation. In particular, in Section 6.4 we obtain the main results concerning Jackson type estimates of simultaneous approximation and L-positive simultaneous approximation of $f \in M(L)$ by multivariate polynomials and entire functions of exponential type.

Throughout the chapter we shall use the following notation: R^m – the m-dimensional Euclidean space; $C(\Omega)$ – the space of continuous on $\Omega \subset \mathrm{R}^m$ functions f with the finite norm $\|f\|_{C(\Omega)} = \sup\limits_{\Omega} |f|$; $L_\infty(\mathrm{R}^m)$ – the space of measurable functions f defined on the measurable set $\Omega \subset \mathrm{R}^m$ with the finite norm $\|f\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}\limits_{\Omega}|f|$; $L_p(\Omega)$, $1 \le p < \infty$ – the space of measurable functions f defined on the measurable set $\Omega \subset \mathrm{R}^m$ with the finite norm $\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p dx\right)^{1/p}$.

6.2 L-Positive Approximation in a Normed Space

In this section we consider a general result on L-one-side approximation in a normed space. As corollaries we derive some estimates of L-positive approximation which are essentially the basis for all other results of the chapter.

6.2.1. A General Theorem

Let F be a normed space and let B be a subspace of F. We consider a family of linear bounded operators $L_\gamma\colon F \to L_\infty(\Omega_\gamma)$, $\gamma \in \Gamma$, where Γ is a set and $\{\Omega_\gamma\}_{\gamma\in\Gamma}$ is a family of subsets in R^m.

We define the following conditions:

(A1) $\sup\limits_{\gamma\in\Gamma} \|L\gamma\| < \infty$,

(A2) there exists an element $\rho \in B$ such that for every $\gamma \in \Gamma$

$$(L_\gamma \rho)(x) \ge 1 \quad \text{a.e. on } \Omega_\gamma. \tag{6.4}$$

Theorem 6.1. *If the family* $\{L_\gamma\}_{\gamma\in\Gamma}$ *satisfies conditions* (A1), (A2) *then for every* $f \in F$ *and any* $P \in B$ *there exist elements* $Q_i \in B$, $i = 1, 2$, *such that* $(-1)^{i+1}(L_\gamma(Q_i - f))(x) \ge 0$, $x \in \Omega_\gamma$, $\gamma \in \Gamma$, *and*

$$\|f - Q_i\|_F \le (1 + \|\rho\|_F \sup_{\gamma\in\Gamma} \|L_\gamma\|)\|f - P\|_F, \quad i = 1, 2. \tag{6.5}$$

Proof. Setting
$$Q_i = P + (-1)^{i+1}\lambda\rho, \quad i = 1, 2, \tag{6.6}$$
where $\lambda = \sup\limits_{\gamma\in\Gamma} \|L_\gamma\| \, \|f - P\|_F$, and taking into account (6.4), we have for every $\gamma \in \Gamma$

$$\begin{aligned}(-1)^{i+1}(L_\gamma(Q_i - f))(x) &= (-1)^{i+1}(L_\gamma(P - f))(x) + \lambda(L\rho)(x) \\ &\geq \lambda - \sup_{\gamma\in\Gamma} \|L_\gamma\| \, \|f - P\|_F = 0, \quad i = 1, 2.\end{aligned} \tag{6.7}$$

Furthermore,

$$\|f-Q_i\|_F \leq \|f-P\|+\lambda\|\rho\|_F = (1+\|\rho\|_F \sup_{\gamma\in\Gamma} \|L_\gamma\|)\|f-P\|, \; i = 1, 2. \tag{6.8}$$

(6.8) together with (6.7) completes the proof of Theorem 6.1. ■

6.2.2. Estimates of L-Positive Approximation.

Let us consider the sets of L_γ-positive and L_γ-negative elements from F

$$M^+(L_\gamma) = \{f \in F\colon (L_\gamma f)(x) \geq 0 \text{ a.e. on } \Omega_\gamma\}, \quad \gamma \in \Gamma,$$

$$M^-(L_\gamma) = \{f \in F\colon (L_\gamma f)(x) \leq 0 \text{ a.e. on } \Omega_\gamma\}, \quad \gamma \in \Gamma,$$

$$M^\pm = \bigcap_{\gamma\in\Gamma} M^\pm(L_\gamma).$$

Corollary 6.2. *If the family of operators* $\{L_\gamma\}_{\gamma\in\Gamma}$ *satisfies conditions* (A1), (A2) *then for every* $f \in M^\pm$

$$E_{M^\pm}(f, B, F) \leq (1 + \|\rho\|_F \sup_{\gamma\in\Gamma} \|L_\gamma\|)E(f, B, E). \tag{6.9}$$

Proof. We shall prove the case $f \in M^+$, the case $f \in M^-$ is similar. If $f \in M^+$ then according to Theorem 6.1 there exists $Q_1 \in B$ such that $(L_\gamma Q_1)(x) \geq (L_\gamma f)(x) \geq 0$ for a.a. $x \in \Omega_\gamma$ and all $\gamma \in \Gamma$. Thus $Q_1 \in M^+$ and (6.9) follows from (6.5). ■

For simplicity all the further results of the chapter will be formulated for L-positive approximation. The corresponding results for L-negative approximation can be easily reformulated.

Let $L\colon F \to L_\infty(\Omega)$, $\Omega \subset \mathrm{R}^m$, be a linear bounded operator satisfying the condition: there exists $\rho \in B$ such that

$$(L\rho)(x) \geq 1 \text{ a.e. on } \Omega. \tag{6.10}$$

We put
$$M(L) = \{f \in F\colon (Lf)(x) \geq 0 \text{ a.e. on } \Omega\}.$$

The next result evidently follows from Corollary 6.2 for a single operator.

Corollary 6.3. *For every* $f \in M(L)$

$$E_{M(L)}(f, B, F) \leq (1 + \|L\| \, \|\rho\|_F) E(f, B, F). \tag{6.11}$$

Remark 6.4. The constant in the right-hand side of (6.11) can be improved by replacing $\|\rho\|_F$ with $z = \inf\{\|\rho\|_F \colon \rho \in B,\ (L\rho)(x) \geq 1 \text{ a.e. on } \Omega\}$.

6.3 L-Positive Approximation in Functional Spaces

This section contains some applications of Corollaries 6.2 and 6.3 to L-positive approximation in functional spaces.

6.3.1. Classes of Functions

In the capacity of the normed space F we shall consider the space $L_p(\Omega)$ and the Sobolev space $W_p^\ell(\Omega)$, $1 \leq p \leq \infty$, with the norm [274, p. 122]

$$\|f\|_{W_p^\ell(\Omega)} = \sum_{|\alpha| \leq \ell} \|D^\alpha f\|_{L_p(\Omega)}.$$

Here Ω is a convex closed set in R^m; α is a sequence $(\alpha_1, \dots, \alpha_m)$, $\alpha_i \geq 0$, $1 \leq i \leq m$; $|\alpha| = \sum_{i=1}^m \alpha_i$; $\alpha! = \alpha_1! \cdots \alpha_m!$; $D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$, $|\alpha| \leq \ell$, are Sobolev derivatives for $1 \leq p < \infty$ [274, p. 121] and $D^\alpha f \in C(\Omega)$ for $p = \infty$, $|\alpha| \leq \ell$. Thus $W_\infty^\ell(\Omega)$ coincides with $C^\ell(\Omega)$.

In the capacity of the subspace B we shall consider the classes of algebraic polynomials, splines and entire functions of exponential type.

Let $\mathcal{P}_{n,m}$ be the class of algebraic polynomials in m variables and of degree at most n. Let $S_n^k[a, b]$ be the class of nonperiodical spline functions of degree n, that is, $S_n^k[a, b]$ is the class of all piecewise polynomial functions from $W_\infty^{k-1}[a, b]$ with n free knots of degree k.

Let $C^m = \mathrm{R}^m + i\mathrm{R}^m$ be the m-dimensional complex space. Assume that V is a centrally symmetric (with respect to the origin) convex body in R^m and V^* is the polar of V.

We say that an entire function $f(z)$ is of exponential type σV if for every $\varepsilon > 0$ there exists $A\varepsilon$ such that for every $z = (z_1, \dots, z_m) \in C^m$ we have

$$|f(z)| \leq A\varepsilon \exp\left(\sigma(1 + \varepsilon) \sup_{x \in V} \left| \sum_{i=1}^m x_i z_i \right| \right).$$

We denote by $B_{\sigma V}$ the class of all entire functions of exponential type σV. For example, if V is the cube $Q = \{x \in \mathrm{R}^m \colon |x_i| \leq 1,\ 1 \leq i \leq m\}$ then

$B_{\sigma Q}$ coincides with the class $B_{\sigma,m}$ of entire functions of exponential type σ.

6.3.2. L-Positive Simultaneous Approximation

Let Ω be a convex closed set in R^m and let

$$(L'f)(x) = \sum_{h\le|\alpha|\le\nu} a_\alpha(x) D^\alpha f(x) \tag{6.12}$$

be a differential operator where $a_\alpha(x)$, $h \le |\alpha| \le \nu$, are functions defined on Ω and satisfying the following conditions

(B1) there exists $\alpha^0 = (\alpha_1^0, \ldots, \alpha_m^0)$, $|\alpha^0| = h$, such that $a_{\alpha^0}(x) \ge 0$ or $a_{\alpha^0}(x) \le 0$, $x \in \Omega$;

(B2) $\sup\limits_{x\in\Omega} |a\alpha(x)/a_{\alpha^0}(x)| = C\alpha < \infty$, $h \le |\alpha| \le \nu$.

We put

$$M(L') = M(L', \Omega) = \{f \in W_p^\ell(\Omega)\colon (L'f)(x) \ge 0 \text{ a.e. on } \Omega\}.$$

Corollary 6.5. *Let L' satisfy conditions* (B1), (B2) *and let a subspace $B \subset W_p^\ell(\Omega)$ contain $x^{\alpha^0} = x_1^{\alpha_1^0} \cdots x_m^{\alpha_m^0}$. Here $1 \le p \le \infty$, $\ell \ge \mu$, where*

$$\mu = \begin{cases} \nu, & \text{if } p = \infty \\ [\nu + m/p] + 1, & \text{if } 1 \le p < \infty. \end{cases} \tag{6.13}$$

If $f \in M(L')$ then for every $P \in B$ there exists $Q \in B \cap M(L')$ such that for all α, $|\alpha| \le h - 1$, and for $\alpha = \alpha^0$

$$\|D^\alpha(f - Q)\|_{L_p(\Omega)} \le C \sum_{|\beta|\le\mu} \|D^\beta(f - P)\|_{L_p(\Omega)}. \tag{6.14}$$

Here

$$C \le (1 + C_0(\alpha^0!)^{-1} \max_{h\le|\alpha|\le\nu} C_\alpha \|x^{\alpha^0}\|_{W_p^\mu(\Omega)},$$

and C_0 depends only on m, p, ν, Ω; in particular, $C_0 = 1$ for $p = \infty$.

Moreover, for all $\alpha \ne \alpha^0$, $h \le |\alpha| \le \ell$,

$$\|D^\alpha(f - Q)\|_{L_p(\Omega)} = \|D^\alpha(f - P)\|_{L_p(\Omega)}. \tag{6.15}$$

Proof. We consider the operator

$$(Lf)(x) = |a_{\alpha^0}(x)|^{-1}(L'f)(x) = \sum_{h\le|\alpha|\le\nu} (a\alpha(x)/|a_{\alpha^0}(x)|) D^\alpha f(x).$$

It is clear $M(L) = M(L')$. Setting $\rho(x) = x^{\alpha^0}\mathrm{sgn}(a_{\alpha^0}(x))/\alpha^0!$ and using condition (B1), we have $(L\rho)(x) = 1$ on Ω, that is L satisfies (6.10). Furthermore using condition (B2), (6.13) and the embedding theorem for the Sobolev spaces [274, p. 124] we get for every $f \in W_p^\ell(\Omega)$, $\ell \geq \mu$,

$$\|Lf\|_{L_\infty(\Omega)} \leq \sum_{h \leq |\alpha| \leq \nu} C_\alpha \|D^\alpha f\|_{L_\infty(\Omega)}$$

$$\leq C_0 \left(\max_{h \leq |\alpha| \leq \nu} C_\alpha \right) \|f\|_{W_p^\mu(\Omega)}, \tag{6.16}$$

where C_0 is the embedding constant.

If $P \in B$ then using Theorem 6.1 for the single operator L and taking into account (6.6), (6.16) we can find $Q \in B \cap M(L')$ of the form $Q(x) = P(x) + Cx^{\alpha^0}$, where C is a constant such that

$$\sum_{|\beta| \leq \mu} \|D^\beta (f - Q)\|_{L_p(\Omega)}$$

$$\leq \left(1 + C_0 (\alpha_0!)^{-1} \max_{h \leq |\alpha| \leq \nu} C_\alpha \|x^{\alpha^0}\|_{W_p^\mu(\Omega)} \right) \sum_{|\beta| \leq \mu} \|D^\beta (f - P)\|_{L_p(\Omega)}.$$

the last yields (6.14) and (6.15) follows from the relations $D^\alpha Q = D^\alpha P$, $\alpha \neq \alpha^0$, $|h| \leq |\alpha| \leq \ell$. ■

Remark 6.6. We note that approximation in the metric of $W_p^\ell(\Omega)$ is equivalent in a certain sense to simultaneous approximation. In particular, the following inequality is a simple consequence of Corollary 6.5.

$$E_{M(L')}(f, B, W_p^\ell(\Omega))$$

$$\leq \left(1 + C_0 (\alpha^0!)^{-1} \max_{h \leq |\alpha| \leq \nu} C_\alpha \|x^{\alpha^0}\|_{W_p^\ell(\Omega)} \right) E(f, B, W_p^\ell(\Omega)). \tag{6.17}$$

Remark 6.7. Corollary 6.5 and inequality (6.17) hold in the following cases:

(a) Ω is a convex body in R^m, $B = \mathcal{P}_{n,m}$, $n \geq h$;

(b) $\Omega = \mathrm{R}^m$, $p = \infty$, $h = 0$, $B = B_{\sigma V}$;

(c) $\Omega = [a, b]$, $m = 1$, $B = S_n^k$, $k \geq \ell$.

In many cases we cannot use Corollary 6.5 or inequality (6.17). For example, if $m = 1$, $\Omega = [-1, 1]$, and

$$(L''f)(x) = (1 - x^2) d^2 f(x)/dx^2 + x df(x)/dx, \tag{6.18}$$

then L'' does not satisfy both conditions (B1), (B2). But in this case we can use the following result.

Corollary 6.8. *Let the differential operator*

$$(L''f)(x) = \sum_{|\alpha|\le\nu} a_\alpha(x) D^\alpha f(x)$$

satisfy conditions

(C1) $\sup_{0\le|\alpha|\le\nu} |a\alpha(x)| = d\alpha < \infty$,

(C2) *there exists an element* $\rho \in B$ *such that* $(L''\rho)(x) = 1$ *a.e. on* Ω.

Then for every $f \in W_p^\ell(\Omega)$, $1 \le p \le \infty$, $\ell \ge \mu$, (*where* μ *defined by* (6.13)),

$$E_{M(L'')}(f, B, W_p^\ell(\Omega))$$
$$\le \left(1 + C_0 \max_{0\le|\alpha|\le\nu} d_\alpha \|\rho\|_{W_p^\ell(\Omega)}\right) E(f, B, W_p^\ell(\Omega)).$$

The proof is similar to that of Corollary 6.5. For example, using Corollary 6.8 to operator (6.18) and $\rho = x^2/2$, we obtain the inequality ($f \in M(L'')$, $n \ge 2$),

$$E_{M(L'')}(f, \mathcal{P}_{n,1}, W_\infty^2[-1,1]) \le (7/2)E(f, \mathcal{P}_{n,1}, W_\infty^2[-1,1]).$$

6.3.3. Convex Simultaneous Approximation

Let Ω be a convex body in R^m and let M be the class of convex twice differentiable on Ω functions.

Corollary 6.9. *For any* $f \in M$ *and* $n \ge 2$ *it holds*

$$E_M(f, \mathcal{P}_{n,m}, W_\infty^2(\Omega)) \le (1 + \|\rho\|_{W_\infty^2(\Omega)})E(f, \mathcal{P}_{n,m}, W_\infty^2(\Omega)), \qquad (6.19)$$

where $\rho(x) = (1/2)\sum_{i=1}^m x_i^2$.

Proof. A function f belongs to M if and only if for every $\gamma \in \mathrm{R}^m$, $|\gamma| = 1$,

$$(L\gamma f)(x) = \sum_{1\le j,i\le m} \frac{\partial^2 f(x)}{\partial x_i \partial x_j}\gamma_i\gamma_j \ge 0, \quad x \in \Omega.$$

It is clear $\sup_{|\gamma|=1} \|L_\gamma\| \le 1$. Then setting $\rho(x) = (1/2)\sum_{i=1}^m x_i^2$ we have $(L_\gamma\rho)(x) = 1$ for every γ, $|\gamma| = 1$, and any $x \in \Omega$. Thus the family of operators $\{L_\gamma\}_{|\gamma|=1}$ satisfies conditions (A1), (A2), and (6.9) yields (6.19). ■

Some Jackson type estimates of $E_M(f, \mathcal{P}_{n,m}, C(\Omega))$ were obtained in [268].

6.3.4. Subharmonic Simultaneous Approximation

Let M be the class of subharmonic twice differentiable on Ω functions, where Ω is a bounded domain in R^m. It is known [184, p. 41] that $f \in M$ if and only if

$$(L''f)(x) = \sum_{i=1}^{m} \frac{\partial^2 f(x)}{\partial x_i^2} \geq 0, \quad x \in \Omega.$$

The operator L'' satisfies conditions (C1) and (C2) for $\rho(x) = (1/2)\sum_{i=1}^{m} x_i^2$ and $d_\alpha \leq 1$, $|\alpha| \leq 2$. Hence the following result is a consequence of Corollary 6.8.

Corollary 6.10. *For any $f \in M$ and $n \geq 2$ it holds*

$$E_M(f, \mathcal{P}_{n,m}, W_\infty^2(\Omega)) \leq (1 + \|\rho\|_{W_\infty^2(\Omega)}) E(f, \mathcal{P}_{n,m}, W_\infty^2(\Omega)),$$

where $\rho(x) = (1/2)\sum_{i=1}^{m} x_i^2$.

A.S. Shvedov [269] proved that every continuous function in a simply connected domain $\Omega \subset \mathrm{R}^m$ can be approximated uniformly on compact sets in Ω by subharmonic polynomials.

6.3.5. L Is a Convolution

Let $\Omega = \mathrm{R}^m$ or Ω be the m-dimensional torus T^m.

We consider the convolution

$$(Lf)(x) = \int_\Omega f(x-y)K(y)dy,$$

where $K \in L_1(\mathrm{R}^m)$, $f \in L_\infty(\mathrm{R}^m)$, if $\Omega = \mathrm{R}^m$, and $K \in L_q(T^m)$, $f \in L_p(T^m)$, $1/p + 1/q = 1$, $1 \leq p \leq \infty$, if $\Omega = T^m$.

The following result is a simple consequence of Corollary 6.3.

Corollary 6.11. *If B is a subspace of $L_p(\Omega)$ containing all the constants and $\mathcal{T}_k = \int_\Omega K(x)dx \neq 0$, then*

$$E_{M(L)}(f, B, L_p(\Omega)) \leq (1 + C\|K\|_{L_q(\Omega)}/|\mathcal{T}_k|) E(f, B, L_p(\Omega)).$$

Here $p = \infty$, $C = 1$, if $\Omega = R^m$, and $1 \leq p \leq \infty$, $C = (2\pi)^{m/p}$, if $\Omega = T^m$.

6.3.6. δ-Monotone Approximation

Let δ, $0 \le \delta \le b-a$, be a fixed number. We say that a function $f \in C[a,b]$ is δ-increasing if

$$(L_\gamma f)(x) = \frac{f(x)-f(\gamma)}{x-\gamma} \ge 0 \quad \text{for all } x, \gamma \in [a,b], |x-\gamma| \ge \delta, x \ne \gamma.$$

Let M_δ be the set of all δ-increasing functions. In particular, M_0 is the class of increasing functions on $[a,b]$. It is clear $M_\delta \supset M_0$. Equality (6.2) shows that the inequality

$$E_M(f, \mathcal{P}_{n,1}, C[a,b]) \le CE(f, \mathcal{P}_{n,1}, C[a,b]), \tag{6.20}$$

where C is a constant independent on n, is not valid for all $f \in M = M_0$. The following result shows that (6.20) holds for δ-increasing functions, $\delta>0$.

Corollary 6.12. *For every $f \in M_\delta$, $\delta > 0$, and $n \ge 1$*

$$E_{M_\delta}(f, \mathcal{P}_{n,1}, C[a,b]) \le (1 + (b-a)/\delta) E(f, \mathcal{P}_{n,1}, C[a,b]). \tag{6.21}$$

Proof. Setting for $\gamma \in [a,b]$

$$\Omega_\gamma = \{y \in [a,b] \colon |\gamma - y| \ge \delta\}$$

we have that $L_\gamma \colon C[a,b] \to C(\Omega_\gamma)$ is the family of bounded linear operators with $\sup_{\gamma \in [a,b]} \|L_\gamma\| \le 2/\delta$. Furthermore the function $\rho(x) = x - (b-a)/2$ satisfies the equality $(L_\gamma \rho)(x) = 1$, $\gamma \in [a,b]$, $x \in [a,b]$. Thus $\{L_\gamma\}_{\gamma \in [a,b]}$ satisfies conditions (A1), (A2) and (6.21) follows from (6.9). ■

Remark 6.13. Corollary 6.12 can be easily generalized to (δ, k)-monotone approximation.

6.4 Multidimensional Jackson Type Theorems for Simultaneous Approximation

This section contains multidimensional estimates of simultaneous approximation by polynomials and entire functions of exponential type involving the moduli of smoothness of arbitrary order. As consequences the corresponding estimates of L-positive approximation, where L is a differential operator, will be given.

Throughout the section we shall use the definitions of V, V^*, $\mathcal{P}_{n,m}$, $B_{\sigma V}$, $W_p^\ell(\Omega)$, D^α, $M(L', \Omega)$ and conditions (B1), (B2) given in 6.3.1, 6.3.2. We denote by C various constants not depending on essential parameters (like f, n, σ etc.)

6.4.1. Moduli of Smoothness

For a measurable function f defined on $\Omega \subset \mathrm{R}^m$ we put

$$\omega_{k,p}(f,\tau) = \omega_{k,p}(f,\tau)_\Omega = \sup_{|t|\le\tau} \|\Delta_t^k f\|_{L_p(\Omega_t)}$$
$$= \sup_{|t|\le\tau} \left\| \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f(x+st) \right\|_{L_p(\Omega_t)},$$

where $\Omega_t = \{x \in \mathrm{R}^m : x - jt \in \Omega,\ 0 \le j \le k\}$ is the domain of definition of the k-th difference Δ_t^k. In particular, when $f \in L_p(\mathrm{R}^m)$ we have

$$\omega_{k,p}(f,\tau)_{\mathrm{R}^m} = \sup_{|t|\le\tau} \|\Delta_t^k f\|_{L_p(\mathrm{R}^m)}.$$

We shall need the following properties of $\omega_{k,p}(f,\tau)$.

(a) for fixed $f \in L_p(\Omega)$ the quantity $\omega_{k,p}(f,\tau)$ is a nondecreasing function of τ defined on $[0,H]$ where $H = d/k$ and d is the diameter of Ω;

(b) for fixed τ the triangle inequality holds

$$\omega_{k,p}(f_1+f_2,\tau) \le \omega_{k,p}(f_1,\tau) + \omega_{k,p}(f_2,\tau),\ \ f_1, f_2 \in L_p(\Omega); \tag{6.22}$$

(c) for any $\lambda > 0$

$$\omega_{k,p}(f,\lambda\tau) \le (1+\lambda)^k \omega_{k,p}(f,\tau); \tag{6.23}$$

(d) for $f \in W_p^\ell(\Omega)$, $1 \le \ell \le k$, and $\tau > 0$ there holds

$$\omega_{k,p}(f,\tau) \le m^{\ell/2}\tau^\ell \max_{|\beta|=\ell} \omega_{k-\ell}(D^\beta f,\tau), \tag{6.24}$$

where $\omega_{0,p}(f,\tau) = \|f\|_{L_p(\Omega)}$;

(e) for $f \in L_p(\Omega)$

$$\omega_{k,p}(f,\tau) \le 2^j \omega_{k-j,p}(f,\tau), \quad 0 \le j \le k. \tag{6.25}$$

Concerning the proof of properties (a)–(e) we refer to [198, 277, p. 103].

6.4.2. Main Results

The following Theorems 6.14 and 6.15 are Jackson type estimates for simultaneous approximation by polynomials and entire functions of exponential type.

Theorem 6.14. *For any $k \ge 1$, $\ell \ge 0$, $n > k + \ell$, and $f \in W_p^\ell(V)$, $1 \le p \le \infty$, there exists a polynomial $p_n \in \mathcal{P}_{n,m}$ such that for every α, $0 \le |\alpha| \le \ell$,*

$$\|D^\alpha(f - P_n)\|_{L_p(V)} \le Cn^{|\alpha|-\ell} \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f, Hn^{-1}). \tag{6.26}$$

Theorem 6.15. *For any $k \geq 1$, $\ell \geq 0$, $\sigma > 0$, and $f \in W_p^\ell(R^m)$, $1 \leq p \leq \infty$, there exists a function $g_\sigma \in B_{\sigma V}$ such that for every α, $0 \leq |\alpha| \leq \ell$,*

$$\|D^\alpha(f - g_\sigma)\|_{L_p(\mathrm{R}^m)} \leq C\sigma^{|\alpha|-\ell} \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f, \sigma^{-1}). \tag{6.27}$$

Theorem 6.14 for $\ell = 0$, $p = \infty$, and Theorem 6.15 for $\ell = 0$ were proved in [168].

The following estimates of L-positive simultaneous approximation are the simple consequences of Theorems 6.14 and 6.15, Corollary 6.5 and Remark 6.7.

Let L' be differential operator (6.12) where a_α, $h \leq |\alpha| \leq \nu$, are functions defined on $V \subset \mathrm{R}^m$ and satisfying conditions (B1), (B2). Let μ be given by (6.13).

Corollary 6.16. *For every $f \in W_p^\ell(V) \cap M(L', V)$, $\ell \geq \mu$, and for any $n > k + \ell$, $k \geq 1$, there exists a polynomial $Q_n \in \mathcal{P}_{n,m} \cap M(L', V)$ such that for all α, $|\alpha| \leq \ell$,*

$$\|D^\alpha(f - Q_n)\|_{L_p(V)} \leq Cn^{s(\alpha)-\ell} \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f, Hn^{-1})$$

where $s(\alpha) = \mu$ for $|\alpha| \leq h - 1$ or $\alpha = \alpha_0$, and $s(\alpha) = |\alpha|$ for $\alpha \neq \alpha_0$, $h \leq |\alpha| \leq \ell$.

The close result for $m = 2$, $p = \infty$, $k = 1$, and $V = [0, 1]^2$ was obtained in [14].

Let L' be differential operator (6.12) with $h = 0$, where a_α, $0 \leq |\alpha| \leq \nu$, are functions defined on R^m and satisfying conditions (B1), (B2).

Corollary 6.17. *For every $f \in W_\infty^\ell(R^m) \cap M(L', R^m)$, $\ell \geq \nu$, there exists a function $g_\sigma \in B_{\sigma V} \cap M(L', R^m)$, $\sigma > 0$, $K \geq 1$, such that for every α, $|\alpha| \leq \ell$,*

$$\|D^\alpha(f - g_\sigma)\|_{L_\infty(\mathrm{R}^m)} \leq C\sigma^{d(\alpha)-\ell} \max_{|\beta|=\ell} \omega_{k,\infty}(D^\beta f, \sigma^{-1}).$$

As consequences of Theorems 6.14, 6.15 we prove the existence of Steklov type functions which are polynomials or entire functions of exponential type.

Corollary 6.18. *For any $f \in L_p(V)$, $1 \leq p \leq \infty$, and $k \geq 1$ there exist polynomials $P_n \in \mathcal{P}_{n,m}$, $n > k$, such that*

$$\|f - P_n\|_{L_p(V)} \leq C\omega_{k,p}(f, Hn^{-1}),$$

$$\max_{|\alpha|=k} \|D^\alpha P_n\|_{L_p(V)} \leq Cn^k \omega_{k,p}(f, Hn^{-1}).$$

Corollary 6.19. *For any $f \in L_p(R^m)$, $1 \leq p \leq \infty$, and $k \geq 1$ there exist functions $g_\sigma \in B_{\sigma V}$, $\sigma > 0$, such that*

$$\|f - g_\sigma\|_{L_p(\mathrm{R}^m)} \leq C\omega_{k,p}(f, \sigma^{-1}),$$

$$\max_{|\alpha|=k} \|D^\alpha g_\sigma\|_{L_p(\mathrm{R}^m)} \le C\sigma^k \omega_{k,p}(f,\sigma^{-1}).$$

First we prove Theorem 6.15. In proving Theorem 6.14 we use the idea [168] of reducing inequalities like (6.26) to inequalities like (6.27). With that end in view we construct two special "bridges" between Theorems 6.14 and 6.15. First of them is an estimate of best polynomial approximation of functions from $B_{\sigma V}$. The second one is an extension theorem preserving moduli of smoothness uniformly in τ. Finally, using these results and Theorem 6.15, we prove Theorem 6.14 and Corollaries 6.18, 6.19.

6.4.3. Proof of Theorem 6.15

Let d_1 be a function from $C^\infty(\mathrm{R}^m)$ with a support supp $d_1 \subset (1/2)V$, $\|d_1\|_{L_2(\mathrm{R}^m)} = 1$. Let us set

$$d_\sigma(x) = \sigma^{-m/2} d_1(x/\sigma); \quad \gamma_\sigma(t) = \sigma^m \hat{d}_1^2(\sigma t) = \hat{d}_\sigma^2(t), \tag{6.28}$$

where

$$\hat{\varphi}(t) = (2\pi)^{-m/2} \int_{\mathrm{R}^m} \varphi(y) \exp(-i\langle t, y\rangle) dy$$

denotes the Fourier transform of $\varphi \in L_2(\mathrm{R}^m)$.

We get from (6.28)

$$\gamma_\sigma(t) \ge 0; \quad \|\gamma_\sigma\|_{L_1(\mathrm{R}^m)} = 1; \tag{6.29}$$

$$\int_{\mathrm{R}^m} |t|^\lambda \gamma_\sigma(t) dt = \sigma^{-\lambda} \int_{\mathrm{R}^m} |t|^\lambda \hat{d}_1^2(t) dt = C\sigma^{-\lambda}, \quad \lambda > 0. \tag{6.30}$$

Let us now consider the multidimensional analogue of Korovkin means [168, 277, p. 258]

$$g_r(f,x) = \int_{\mathrm{R}^m} (I+(-1)^{r+1}\Delta_t^r) f(x) \gamma_\sigma(t) dt = \int_{\mathrm{R}^m} D_r(x-t) f(t) dt, \tag{6.31}$$

where I is the identity operator and

$$D_r(y) = \sum_{s=1}^{r} (-1)^{s-1} s^{-m} \binom{r}{s} \gamma_\sigma(-y/s), \quad r \ge 1. \tag{6.32}$$

We need the following properties of $g_r(f,x)$.

(1) For every $f \in L_p(\mathrm{R}^m)$, $1 \le p \le \infty$, the function $g_r(x) = g_r(f,x) \in B_{\sigma V} \cap L_p(\mathrm{R}^m)$. This fact was proved in [168].

(2) $g_r(1,x) = 1$. This follows from (6.29) and (6.31).

(3) For every $f \in W_p^\ell(\mathrm{R}^m)$ and any α, $|\alpha| \le \ell$,

$$g_r(D^\alpha f, x) = D^\alpha g_r(f,x). \tag{6.33}$$

To prove (6.33) we consider a function $\varphi \in C^\infty(\mathrm{R}^m)$ with a compact support. Using (6.31), (6.32) and the definition of Sobolev derivatives, we have

$$\int_{\mathrm{R}^m} D^\alpha g_r(f,x)\varphi(x)dx = (-1)^{|\alpha|}\int_{\mathrm{R}^m}\int_{\mathrm{R}^m} D_r(x)f(x-t)D^\alpha\varphi(x)dtdx$$

$$= \int_{\mathrm{R}^m} D_r(t)dt\int_{\mathrm{R}^m} D^\alpha f(x-t)\varphi(x)dx = \int_{\mathrm{R}^m} g_r(D^\alpha f,x)\varphi(x)dx.$$

Thus (6.33) is proved.

Using (6.29), properties (1)–(3) of $g_r(f,x)$, properties (6.23)–(6.25) of moduli of smoothness and the generalized Minkowski inequality [274, p. 271] we derive for every $f \in W_p^\ell(\mathrm{R}^m)$ and arbitrary $\tau > 0$

$$\|D^\alpha(f-g_{k+\ell})\|_{L_p(\mathrm{R}^m)} = \|D^\alpha f - g_{k+\ell}(D^\alpha f,\cdot)\|_{L_p(\mathrm{R}^m)}$$

$$\leq \int_{\mathrm{R}^m}\omega_{k+\ell,p}(D^\alpha f,t)\gamma_\sigma(t)dt \leq \omega_{k+\ell}(D^\alpha f,\tau)\int_{\mathrm{R}^m}(1+|t|/\tau)^{k+\ell}\gamma_\sigma(t)dt$$

$$\leq C\tau^{\ell-|\alpha|}\max_{|\beta|=\ell}\omega_{k+|\alpha|,p}(D^\beta f,\tau)\left(1+\tau^{-(k+\ell)}\int_{\mathrm{R}^m}|t|^{k+\ell}\gamma_\sigma(t)dt\right)$$

$$\leq C\tau^{\ell-|\alpha|}\max_{|\beta|=\ell}\omega_{k,p}(D^\beta f,\tau)\left(1+\tau^{-(k+\ell)}\int_{\mathrm{R}^m}|t|^{k+\ell}\gamma_\sigma(t)dt\right). \quad (6.34)$$

Setting $\tau = \left(\int_{\mathrm{R}^m}|t|^{k+\ell}\gamma_\sigma(t)dt\right)^{1/(k+\ell)}$ we have $\tau \leq C/\sigma$ by (6.30). Thus (6.27) follows from (6.34) and (6.23) for $g_\sigma(x) = g_{k+\ell}(f,x)$. ■

6.4.4. Polynomial Approximation of Entire Functions of Exponential Type

Lemma 6.20. *Let $g \in B_{V^*} \cap L_p(R^m)$, $1 \leq p \leq \infty$. Then for arbitrary $q \in (0,1)$, $k \geq 1$, $\ell \geq 1$, $n > k+\ell$, there exists a polynomial $F_n \in \mathcal{P}_{n,m}$ such that for every α,*

$$\|D^\alpha(g-F_n)\|_{L_p(qnV)} \leq C\exp(-bn)\max_{|\beta|=\ell}\omega_{k,p}(D^\beta g,1), \quad (6.35)$$

where $b > 0$ is independent on n and g.

Proof. In [168] showed that there exist polynomials $F_s \in \mathcal{P}_{s,m}$, $s > k+\ell$, such that

$$\|g-F_s\|_{C(qsV)} \leq C\exp(\langle -b,s\rangle)\omega_{k+\ell,\infty}(g,1), \quad (6.36)$$

where $b_1 > 0$ is independent on s and $g \in B_{V^*} \cap C(\mathrm{R}^m)$. Using the multidimensional Markov type inequality [281] we obtain from (6.36) for $n > k+\ell$

$$\|D^\alpha(g-F_n)\|_{L_p(qnV)} \leq Cn^{m/p}\|D^\alpha(g-F_n)\|_{C(qnV)}$$

$$\leq Cn^{m/p} \sum_{s=n+1}^{\infty} \|D^\alpha(F_s - F_{s+1})\|_{C(qnV)}$$

$$\leq Cn^{m/p-|\alpha|} \sum_{s=n+1}^{\infty} (s+1)^{2|\alpha|} \|g - F_s\|_{C(qsV)}$$

$$\leq C \exp(-bn) \omega_{k+\ell,\infty}(g, 1). \tag{6.37}$$

Thus we have from (6.37) and (6.24)

$$\|D^\alpha(g - F_n)\|_{L_p(qnV)} \leq C \exp(-bn) \max_{|\beta|=\ell} \omega_{k,\infty}(D^\beta g, 1). \tag{6.38}$$

Furthermore, $\Delta_t^k D^\beta g(x)$ is an entire function of exponential type d for each fixed $t \in \mathrm{R}^m$ and $|\beta| = \ell$. Here d is a diameter of V^*. Therefore, using the Nikolskii inequality for functions from $B_{d,m}$ [277, p. 235], we obtain

$$\max_{|\beta|=\ell} \omega_{k,\infty}(D^\beta g, 1) \leq C \max_{|\beta|=\ell} \omega_{k,p}(D^\beta g, 1). \tag{6.39}$$

Inequalities (6.38), (6.39) yield (6.35). ■

6.4.5. An Extension Theorem

Lemma 6.21. *For any $\ell \geq 0$, $k \geq 1$ and $p \in [1, \infty]$ there exists a bounded operator $T\colon W_p^\ell(V) \to W_p^\ell(\boldsymbol{R}^m)$ with the properties:*

(a) $Tf - f \in \mathcal{P}_{k+\ell-1,m}$ *on* V, $f \in W_p^\ell(V)$;

(b) *for any* $\tau \in [0, \operatorname{diam} V]$ *and* $0 \leq s \leq \ell$,

$$\max_{|\beta|=s} \omega_{k,p}(D^\beta Tf, \tau)_{\mathrm{R}^m} \leq C \max_{|\beta|=s} \omega_{k,p}(D^\beta f, \tau)_V. \tag{6.40}$$

The proof of the lemma is based on several lemmas. First of them is the extension theorem of Stein [274, p. 181]. We denote

$$|f|_{p,s,\Omega} = \max_{|\beta|=s} \|D^\alpha f\|_{L_p(\Omega)}.$$

Lemma 6.22. *There exists a linear operator E mapping functions on V to functions on $\boldsymbol{R}^m$ with the properties:*

(a) $(Ef)(x) = f(x)$ *for all* $x \in V$;

(b) *E maps $W_p^\ell(V)$ continuously into $W_p^\ell(\boldsymbol{R}^m)$ for all p, $1 \leq p \leq \infty$, and all $\ell = 0, 1, \ldots$, that is,*

$$\|Ef\|_{L_p(\mathrm{R}^m)} + |Ef|_{p,\ell,\mathrm{R}^m} \leq C(\|f\|_{L_p(V)} + |f|_{p,\ell,V}). \tag{6.41}$$

The next lemma considers relations between functionals and moduli of smoothness. We set ($\ell \geq 0$, $k \geq 1$)

$$K_{\ell,k}(f,\tau)_\Omega = \inf_{g \in W_p^{k+\ell}(\Omega)} (|f-g|_{p,\ell,\Omega} + \tau |g|_{p,k+\ell,\Omega}).$$

Lemma 6.23. *If Ω is an open convex set in $\boldsymbol{R}^m$, then for every $f \in W_p^\ell(\Omega)$, $1 \leq p \leq \infty$, and $s = 0, 1, \ldots, \ell$,*

$$C_1 K_{s,k}(f,\tau^k)_\Omega \leq \max_{|\alpha|=s} \omega_{k,p}(D^\alpha f, \tau^k)_\Omega \leq C_2 K_{2,k}(f,\tau^k)_\Omega. \tag{6.42}$$

For $s = 0$ inequalities (6.42) were proved by H. Johnen and K. Scherer [198]. The proof of Lemma 6.23 is similar to the case $s = 0$.

The last lemma considers estimates of polynomial approximation in the Sobolev spaces.

Lemma 6.24. *For every $h \geq 1$ there exists a polynomial operator $P_h \colon L_p(V) \to \mathcal{P}_{h-1,m}$, $1 \leq p \leq \infty$, such that for any $f \in W_p^s(V)$, $0 \leq s \leq h$,*

$$\|f - P_h(f)\|_{L_p(V)} + |f - P_h(f)|_{p,s,V} \leq C|f|_{p,s,V}. \tag{6.43}$$

Proof. Let $f \in L_p(V)$ and $P_h(f) \in \mathcal{P}_{h-1,m}$ be a polynomial satisfying the Whitney type inequality [120, 198]

$$\|f - P_h(f)\|_{L_p(V)} \leq C\omega_{h,p}(f,1). \tag{6.44}$$

If $s = h$ and $f \in W_p^h(V)$ then (6.43) follows from (6.44) and (6.24). Let now $0 \leq s \leq h-1$. Using (6.44) for $h = s$ and $f = P_h$ we get

$$\|P_h - P_s(P_h)\|_{L_p(V)} \leq C\omega_{s,p}(P_h,1). \tag{6.45}$$

Applying now estimates (6.44), (6.45), (6.22), (6.24), (6.25) and the Markov type inequality [281] we derive

$$|P_h(f)|_{p,s,V} = |P_h - P_s(P_h)|_{p,s,V} \leq C\omega_{s,p}(P_h,1)$$
$$\leq C(\omega_{s,p}(f,1) + \|f - P_h\|_{L_p(V)}) \leq C|f|_{p,s,V}. \tag{6.46}$$

And (6.23) and (6.46) yield (6.43). ■

Proof of Lemma 6.21. Let E be the bounded extension operator from Lemma 6.22 and let $P_{k+\ell}(f)$ be the polynomial operator from Lemma 6.24. We claim that the operator

$$Tf = E(f - P_{k+\ell}(f))$$

is the desired extension operator. It is obviously $Tf - f \in \mathcal{P}_{k+\ell-1,m}$ on $V, f \in L_p(V)$.

Furthermore, from (6.41) and (6.43) we obtain that T is a bounded operator from $W_p^s(V)$ into $W_p^s(\mathrm{R}^m)$, $s = 0, \dots, k+\ell$; moreover, for any $f \in W_p^s(V)$

$$|Tf|_{p,s,\mathrm{R}^m} \le C|f|_{p,s,V}. \tag{6.47}$$

It only remains to prove (6.40). Using Lemma 6.21 and (6.47) we get

$$\begin{aligned}\max_{|\alpha|=s} \omega_{k,p}(D^\alpha Tf, \tau)_{\mathrm{R}^m} &\le CK_{s,k}(Tf, \tau^k)_{\mathrm{R}^m} \\ \le C \inf_{\varphi \in W_p^{k+s}(V)} &(|Tf - T\varphi|_{p,s,\mathrm{R}^m} + \tau^k |T\varphi|_{p,k+s,\mathrm{R}^m}) \\ &\le C \inf_{\varphi \in W_p^{k+s}(V)} (|f - \varphi|_{p,s,V} + \tau^k |\varphi|_{p,k+s,V}) \\ &\le CK_{s,k}(f, \tau^k)_V \le C \max_{|\alpha|=s} \omega_{k,p}(D^\alpha f, \tau)_V. \end{aligned}$$

■

6.4.6. Proof of Theorem 6.14

Let $f \in W_p^\ell(V)$, $1 \le p \le \infty$, and let $f_1 = Tf \in W_p^\ell(\mathrm{R}^m)$ be the function satisfying properties (a), (b) of Lemma 6.21. According to Theorem 6.15 we can find $g \in B_{(n/2)V^*} \cap L_p(\mathrm{R}^m)$ satisfying the inequality ($|\alpha| \le \ell$)

$$\|D^\alpha(f_1 - g)\|_{L_p(\mathrm{R}^m)} \le Cn^{|\alpha|-\ell} \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f_1, n^{-1}). \tag{6.48}$$

We obtain from (6.48)

$$\max_{|\beta|=\ell} \omega_{k,p}(D^\beta g, n^{-1}) \le C \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f_1, n^{-1}). \tag{6.49}$$

Furthermore, the function $g_n(x) = g((2/n)x)$ belongs to B_{V^*}. Therefore using Lemma 6.20 for $q = 1/2$ we obtain that there exist polynomials $F_n \in \mathcal{P}_{n,m}$, $n > k+\ell$, such that

$$\|D^\alpha(g_n - F_n)\|_{L_p((n/2)V)} \le C \exp(-bn) \max_{|\beta|=\ell} \omega_{k,p}(D^\beta g_n, 1). \tag{6.50}$$

Setting $G_n(x) = F_n((n/2)x)$ and using (6.49), (6.50) we have

$$\begin{aligned}\|D^\alpha(g - G_n)\|_{L_p(V)} &= (n/2)^{|\alpha|-m/p} \|D^\alpha(g_n - F_n)\|_{L_p((n/2)V)} \\ &\le C \exp(-bn) \max_{|\beta|=\ell} \omega_{k,p}(D^\beta g, n^{-1}) \\ &\le C \exp(-bn) \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f_1, n^{-1}). \end{aligned} \tag{6.51}$$

Finally, setting $P_n = G_n + f - f_1$ we get from estimates (6.40), (6.48), (6.51) and (6.23) that

$$\|D^\alpha(f - P_n)\|_{L_p(V)} \le \|D^\alpha(f_1 - g)\|_{L_p(\mathrm{R}^m)} + \|D^\alpha(g - G_n)\|_{L_p(V)}$$

$$\leq Cn^{|\alpha|-\ell}(1 + n^{\ell-|\alpha|} \exp(-bn)) \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f_1, n^{-1})$$

$$\leq Cn^{|\alpha|-\ell} \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f_1, Hn^{-1})$$

$$\leq Cn^{|\alpha|-\ell} \max_{|\beta|=\ell} \omega_{k,p}(D^\beta f, Hn^{-1}). \qquad \blacksquare$$

6.4.7. Proofs of Corollaries 6.22, 6.23

Proof of Corollary 6.19. Denoting by d the diameter of V we have $B_{\sigma V} \subset B_{d\sigma,m}$. In the further estimates we need the Nikolskii inequality [277, p. 217]

$$\|D^\alpha g\|_{L_p(\mathrm{R}^m)} \leq (\sigma d/2)^{|\alpha|} \|\Delta_A^\alpha g\|_{L_p(\mathrm{R}^m)}, g \in B_{d\sigma,m} \cap L_p(\mathrm{R}^m), \tag{6.52}$$

where Δ_A^α denotes a mixed difference of order α, $A = (a_1, \ldots, a)$, $a = \pi/(2\sigma)$; and the Brudnyi inequality [120]

$$\|\Delta_A^\alpha g\|_{L_p(\mathrm{R}^m)} \leq C\omega_{|\alpha|,p}(g, |A|), \quad g \in L_p(\mathrm{R}^m). \tag{6.53}$$

We obtain from Theorem 6.15 that

$$\|f - g_\sigma\|_{L_p(\mathrm{R}^m)} \leq C\omega_{k,p}(f, \sigma^{-1}). \tag{6.54}$$

Using (6.52), (6.53), (6.54) and (6.25) we derive

$$\max_{|\alpha|=k} \|D^\alpha g_\sigma\|_{L_p(\mathrm{R}^m)} \leq C\sigma^k \max_{|\alpha|=k} \|\Delta_A^\alpha g_\sigma\|_{L_p(\mathrm{R}^m)}$$

$$\leq C\sigma^k \omega_{k,p}(g_\sigma, \sigma^{-1}) \leq C\sigma^k(\omega_{k,p}(f, \sigma^{-1}) + \|f - g_\sigma\|_{L_p(\mathrm{R}^m)})$$

$$\leq C\sigma^k \omega_{k,p}(f, \sigma^{-1}). \qquad \blacksquare$$

Proof of Corollary 6.18. Let $f \in L_p(V)$, $1 \leq p \leq \infty$, and let $f_1 \in L_p(\mathrm{R}^m)$ be the function satisfying properties (a), (b) of Lemma 6.21 for $\ell = 0$. Using Corollary 6.19 we can find $g \in B_{(n/2)V^*} \cap L_p(\mathrm{R}^m)$ such that

$$\|f_1 - g\|_{L_p(\mathrm{R}^m)} \leq C\omega_{k,p}(f_1, n^{-1}), \tag{6.55}$$

$$\max_{|\alpha|=k} \|D^\alpha g\|_{L_p(\mathrm{R}^m)} \leq Cn^k \omega_{k,p}(f_1, n^{-1}). \tag{6.56}$$

Using Lemma 6.20 we obtain that there exist $F_n \in \mathcal{P}_{n,m}$, $n > k$, such that (6.50) holds. Putting $G_n(x) = F_n((n/2)x)$ and using (6.55), (6.56) we get

$$\max_{|\alpha|=k} \|D^\alpha(g - G_n)\|_{L_p(V)} \leq C \exp(-bn)\omega_{k,p}(g, n^{-1}) \leq C\omega_{k,p}(f_1, n^{-1}). \tag{6.57}$$

Setting $P_n = G_n + f - f_1$ we derive from (6.55), (6.57) that

$$\|f - P_n\|_{L_p(V)} \leq C\omega_{k,p}(f, n^{-1}).$$

Furthermore, taking into account (6.56), (6.57) and the relations $D^\alpha P_n = D^\alpha G_n$, $|\alpha| = k$, we obtain

$$\max_{|\alpha|=k} \|D^\alpha P_n\|_{L_p(V)} \leq \max_{|\alpha|=k} (\|D^\alpha g\|_{L_p(\mathrm{R}^m)} + \|D^\alpha (g - P_n)\|_{L_p(V)})$$

$$\leq Cn^k \omega_{k,p}(f_1, n^{-1}) \leq Cn^k \omega_{k,p}(f, Hn^{-1}). \qquad \blacksquare$$

7
About Shape Preserving Weighted Uniform Approximation

Results concerning shape preserving weighted uniform approximation on the real line are presented. This chapter is based on [74].

7.1 Introduction

Shape preserving approximation by real polynomials of real variables on the compact interval $[a, b]$ in the classical non-weighted $L^p[a, b]$-norms with $0 < p \leq \infty$, is a well developed topic in mathematics (for a comprehensive treatment of the subject see for example the book [167]).

But studies concerning shape preserving weighted approximation on the real line seem to be almost nonexistent. An interesting rare article on the topic is [220].

The aim of this chapter is to show that the so-called L-positive approximation method developed in [75], see also Chapter 6, is powerful enough to produce new results in shape preserving weighted approximation.

7.2 Shape Preserving Weighted Uniform Approximation

For a continuous weight function $w : \mathbb{R} \to (0, 1]$, define the weighted space

$$C_w(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}; f - \text{ continuous on } \mathbb{R} \text{ and } \lim_{x \to \pm\infty} f(x)w(x) = 0\}.$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 89–91.
springerlink.com

It is a linear space endowed with the norm $\|f\|_{C_w(\mathbb{R})} = \sup\{w(x)|f(x)|; x \in \mathbb{R}\}$.

Also, for any $r \in \mathbb{N} \bigcup \{0\}$ define the space

$$C_w^r(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}; f^{(\gamma)} \in C_w(\mathbb{R}), \text{ for all } \gamma = 0, 1, ..., r\},$$

endowed with the norm $\|f\|_{C_w^r} = \max\{\|f^{(\gamma)}\|_{C_w(\mathbb{R})}; \gamma = 0, 1, ..., r\}$. Clearly we have $C_w^0(\mathbb{R}) = C_w(\mathbb{R})$.

In all what follows we will consider the exponential (Freud) weight

$$w_\alpha(x) = e^{-|x|^\alpha}, \text{ with } \alpha \geq 1.$$

The general results in [75], see also Chapter 6, will allow us to obtain in an easy way shape preserving results in weighted approximation. Thus, first we obtain the following results in simultaneous shape preserving weighted approximation.

Theorem 7.1. *Let $r \geq 0$ be an even number. For any $f \in C_{w_\alpha}^r(\mathbb{R})$ satisfying $f^{(j)}(x) \geq 0$, for all $x \in \mathbb{R}$ and $j = 0, 2, 4, ..., r$, there exists a sequence of polynomials $(P_n)_n$ with degree $(P_n) \leq n$, such that $P_n^{(j)}(x) \geq 0$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $j = 0, 2, 4, ..., r$ and*

$$\|f - P_n\|_{C_{w_\alpha}^r} \leq C E_n(f; C_{w_\alpha}^r(\mathbb{R})), \text{ for all } n \in \mathbb{N},$$

where $C > 0$ is independent of n and f and

$$E_n(f; C_{w_\alpha}^r(\mathbb{R})) := \inf\{\|f - P\|_{C_{w_\alpha}^r}; P \in \mathcal{P}_n\}.$$

Proof. If we fix r an even number and in Corollary 6.2 here, we take $L_\gamma(f) = f^{(\gamma)}$, $\gamma = 0, 2, 4, ..., r$, $F = C_{w_\alpha}^r(\mathbb{R})$ and define $\rho(x) = \sum_{j=0}^r x^{2j} \in C_{w_\alpha}^r(\mathbb{R})$, then we immediately obtain the conclusion in the theorem. ■

As an immediate consequence we obtain the following result.

Corollary 7.2. *Let $r \geq 0$ be an even number and $f \in C_{w_\alpha}^r(\mathbb{R})$ satisfying $f^{(j)}(x) \geq 0$, for all $x \in \mathbb{R}$ and $j = 0, 2, 4, ..., r$. There exists a sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ with degree $(P_n) \leq n$, such that for every $j = 0, 2, 4, ..., r$ we have*

$$\lim_{n\to\infty} \|P_n^{(j)} - f^{(j)}\|_{C_{w_\alpha}(\mathbb{R})} = 0 \text{ and } P_n^{(j)}(x) \geq 0, \forall x \in \mathbb{R}.$$

Proof. Taking into account Theorem 7.1, clearly that it is sufficient to prove that for any fixed even number r, we have

$$\lim_{n\to\infty} E_n(f; C_{w_\alpha}^r(\mathbb{R})) = 0.$$

For this purpose, let us denote by Q_n a polynomial of degree $\leq n$ attached to f such that

$$\|f - Q_n\|_{C_{w_\alpha}(\mathbb{R})} \leq c \inf_{Q \in \mathcal{P}_n} \|f - Q\|_{C_{w_\alpha}(\mathbb{R})},$$

with a constant $c \geq 1$. We clearly have $\lim_{n\to\infty} \|f - Q_n\|_{C_{w_\alpha}(\mathbb{R})} = 0$.

But according to a classical result of Freud ([161, Theorem 4.1]) (see also for example [225, p. 90, Theorem 4.1.7]), this immediately will imply that

$$\lim_{n\to\infty} \|f^{(j)} - Q_n^{(j)}\|_{C_{w_\alpha}(\mathbb{R})} = 0, \text{ for all } 1 \leq j \leq r.$$

Since

$$E_n(f; C^r_{w_\alpha}(\mathbb{R})) \leq \max_{0\leq j\leq r}\{\|f^{(j)} - Q_n^{(j)}\|_{C_{w_\alpha}(\mathbb{R})}\},$$

passing to limit with $n \to \infty$ we get the desired conclusion. ■

Remark 7.3. Given $r \in \mathbb{N}$ and f with $f^{(r)} \geq 0$ on $\mathbb{R}$ and denoting

$$E^r_n(f, C_{w_\alpha}(\mathbb{R})) := \inf\{\|f - P\|_{C_{w_\alpha}(\mathbb{R})}; P \in \mathcal{P}_n, \ P^{(r)}(x) \geq 0\},$$

the main result in [220, Theorem 1] is that we have

$$\lim_{n\to\infty} E^r_n(f, C_{w_\alpha}(\mathbb{R})) = 0,$$

or equivalently, that there exists a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ with degree $(P_n) \leq n$, such that we have

$$\lim_{n\to\infty} \|P_n - f\|_{C_{w_\alpha}(\mathbb{R})} = 0 \text{ and } P_n^{(r)}(x) \geq 0, \forall x \in \mathbb{R}.$$

It is clear that for even $r \in \mathbb{N}$, Corollary 7.2 is a simultaneous approximation-type result corresponding to Theorem 1 in [220].

Now, if for fixed $\delta \geq 0$ we define as in [75, p. 483] the set $M_\delta(\mathbb{R})$ of all δ-increasing functions, by the set of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the property

$$\frac{f(x) - f(\gamma)}{x - \gamma} \geq 0, \text{ for all } x, \gamma \in \mathbb{R}, |x - \gamma| \geq \delta, x \neq \gamma,$$

applying Corollary 6.3 here, we immediately obtain the following.

Theorem 7.4. *For any $\delta > 0$, $f \in C_{w_\alpha}(\mathbb{R}) \bigcap M_\delta(\mathbb{R})$, there exists a sequence of polynomials $(P_n)_n$ with degree $(P_n) \leq n$ such that $P_n \in M_\delta(\mathbb{R})$ for all $n \in \mathbb{N}$ and*

$$\|f - P_n\|_{w_\alpha} \leq C E_n(f; C_{w_\alpha}(\mathbb{R})), \text{ for all } n \in \mathbb{N},$$

where $C > 0$ is independent of f and n.

Remark 7.5. 1). Theorem 7.4 is the weighted correspondent of the non-weighted approximation result, see Corollary 6.12 here.

2) In fact, all the applicative results in the Sections 6.3 and 6.4 can be re-written in the weighted approximation setting, at least for Freud-type weights of one or several variables.

8

Jackson-Type Nonpositive Approximations for Definite Integrals

In this chapter the integral of a function over a finite interval, is approximated by Jackson-type approximations that are non-positive linear functionals. Several important cases are treated, in which approximations are given with rates by using higher order moduli of smoothness. Real applications of these results might be, e.g., in Communications and Medical Imaging. This chapter relies on [70].

8.1 Introduction

In this chapter we construct nonpositive linear functionals that approximate the integral $\int_0^1 f(y)\mu(dy)$, where μ is a probability measure on $[0,1]$, with the order $O\big(\omega_{p+1}\big(f;\frac{1}{n}\big)\big)$, $p \in \mathbb{N} \cup \{0\}$. These linear functionals are sums of suitable linear combinations of the integrals of dilated translates of f over successive subintervals of $[0,1]$. They are Jackson-type generalizations of approximants arising in Statistics and introduced in [20].

The chapter has been motivated by the interpretation of $\int_0^1 f(x)dx$ as work or energy from physics, and especially by applications in Signal Theory (time-limited and band-limited signals). In Communications and Medical Imaging, for example, we often measure work or energy of involved signals approximately, that is, by measuring work or energy of dilated translates of such signals over successive subintervals of the main finite interval, for example $[0,1]$.

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8.2 Main Results

Let $p \in \mathbb{N} \cup \{0\}$ be fixed and $f\colon [-p, p+1] \to \mathbb{R}$ be integrable on $[0,1]$ with respect to a probability measure denoted by μ.

By using the classical idea in approximation by trigonometric polynomials which produces better estimates, we introduce the integrated sums

$$R_{p+1,n}(f) = -\sum_{i=1}^{n}\sum_{k=1}^{p+1} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} d_{i,k}(f)(y)\chi_{[\frac{i-1}{n},\frac{i}{n}]}(y)\mu(dy)$$

where

$$d_{i,k}(f)(y) = (-1)^k \binom{p+1}{k} \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(y + k(u-y))\mu(du)$$

and

$$Q_{p+1,n}(f) = -\sum_{i=1}^{n}\sum_{k=1}^{p+1}(-1)^k \binom{p+1}{k} \int_{\frac{i-1}{n}}^{\frac{i}{n}} f\left(y + k\left(\frac{i}{n} - y\right)\right)\mu(dy).$$

These are, for $p \in \mathbb{N}$, nonpositive linear functions.

Remark 8.1. 1) For $p = 0$ we obtain the so-called sums-linear functionals introduced in [20].

2) For $p = 0$, $Q_{p+1,n}(f)$ becomes the Riemann sum $\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$, which means that $Q_{p+1,n}(f)$ represents in fact the Jackson-type generalization of the Riemann sum above.

Next we mention the usual $(p+1)$th uniform modulus of smoothness defined on the interval $I_p = [-p, p+1]$ by

$$\omega_{p+1}(f;\delta)_{I_p} := \sup\{|\Delta_h^{p+1} f(x)|; x, x+(p+1)h \in [-p, p+1], 0 \le h \le \delta\},$$

where

$$\Delta_h^{p+1} f(x) = \sum_{k=0}^{p+1} \binom{p+1}{k}(-1)^k \cdot f(x+kh).$$

Theorem 8.2. *It holds*

(i) $\left|R_{p+1,n}(f) - \int_0^1 f(y)\mu(dy)\right| \le \omega_{p+1}\left(f;\frac{1}{n}\right)_{I_p}, \ \forall n \in \mathbb{N}$,

(ii) $\left|Q_{p+1,n}(f) - \int_0^1 f(y)\mu(dy)\right| \le \omega_{p+1}\left(f;\frac{1}{n}\right)_{I_p}, \ \forall n \in \mathbb{N}$.

Proof. (i) We observe that

$$\left|R_{p+1,n}(f) - \int_0^1 f(y)\mu(dy)\right| = \left|-\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\sum_{k=1}^{p+1} n d_{i,k}(f)(y)\mu(dy) - \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(y)\mu(dy)\right|$$

$$= \left| \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left[\sum_{k=1}^{p+1} n d_{i,k}(f)(y) - f(y) \right] \mu(dy) \right| \leq \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left| \sum_{k=1}^{p+1} n d_{i,k}(f)(y) - f(y) \right| \mu(dy)$$

$$= \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left| n \int_{\frac{i-1}{n}}^{\frac{i}{n}} (-1)^k \binom{p+1}{k} f(y+k(u-y))\mu(dy) - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(y)\mu(du) \right| \mu(dy)$$

$$= \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left\{ n \left| \int_{\frac{i-1}{n}}^{\frac{i}{n}} [\Delta_{u-y}^{p+1} f(y)] \mu(du) \right| \right\} \mu(dy)$$

$$\leq \sum_{i=1}^{n} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |\Delta_{u-y}^{p+1} f(y)| \mu(du)\mu(dy)$$

$$(y+k(u-y) \in [-p, p+1], k = \overline{0, p+1}, u, v \in [0,1])$$

$$\leq \sum_{i=1}^{n} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \omega_{p+1}(f; |u-y|)_{I_p} \mu(du)\mu(dy) =: (*).$$

But $u, y \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$ implies $|u-y| \leq \frac{1}{n}$, so we derive

$$(*) \leq \sum_{i=1}^{n} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \frac{1}{n} \omega_{p+1}\left(f; \frac{1}{h}\right)_{I_p} \mu(du) = \omega_{p+1}\left(f; \frac{1}{h}\right)_{I_p},$$

which proves (i).

(ii) We obtain

$$\left| Q_{p+1,n}(f) - \int_0^1 f(y)\mu(dy) \right|$$

$$= \left| - \sum_{i=1}^{n} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \sum_{k=1}^{p+1} (-1)^k \binom{p+1}{k} f\left(y + k\left(\frac{i}{n} - y\right)\right) - \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(y)\mu(dy) \right) \right|$$

$$= \left| \sum_{i=1}^{n} \left\{ \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{\frac{i}{n}-y}^{p+1} f(y)\mu(dy) \right\} \right| \leq \sum_{i=1}^{n} \left\{ \int_{\frac{i-1}{n}}^{\frac{i}{n}} \omega_{p+1}\left(f; \left|\frac{i}{n} - y\right|\right)_{I_p} \mu(dy) \right\}$$

$$\leq \omega_{p+1}\left(f; \frac{1}{n}\right)_{I_p},$$

which establishes the theorem. ■

Remark 8.3. If $f \in C^{p+1}[-p, p+1]$, then the order of approximation of $\int_0^1 f(y)\ \mu(dy)$ by $R_{p+1,n}(f)$ and $Q_{p+1,n}(f)$ is $\mathcal{O}\left(\frac{1}{n^{p+1}}\right)$, which for $p \in \mathbb{N}$ big enough, cannot be obtained by the Riemann sums or by the classical quadrature formulas.

We now consider related L^1-results for $R_{p+1,n}(f)$ and $Q_{p+1,n}(f)$.

Theorem 8.4. *Let* $f \in L^1_\mu(\mathbb{R})$*. Then:*

(i) $\left|R_{p+1,n}(f) - \int_0^1 f(y)\mu(dy)\right| \leq 2\omega_{p+1}\left(f; \frac{1}{n}\right)_{L^1\mu(\mathbb{R})}$ $\forall n \in \mathbb{N}$, *where*

$$\omega_{p+1}\left(f; \frac{1}{n}\right)_{L^1\mu(\mathbb{R})} := \sup\left\{\|\Delta_h^{p+1} f(x)\|_{L^1\mu(\mathbb{R})}; 0 \leq h \leq \frac{1}{n}\right\}$$

$$= \sup\left\{\int_{-\infty}^{+\infty} |\Delta_h^{p+1} f(x)|\mu(dx); 0 \leq h \leq \frac{1}{n}\right\}.$$

(ii) *If* $\Delta_h^{p+1} f(y) \geq 0$, $\forall h \geq 0$, $y \in [-1, 2]$, *then*

$$\left|Q_{p+1,n}(f) - \int_0^1 f(y)\mu(dy)\right| \leq \omega_{p+1}\left(f; \frac{1}{n}\right)_{L^1\mu(\mathbb{R})}, \quad \forall n \in \mathbb{N}.$$

Proof. (i) We get

$$\left|R_{p+1,n}(f) - \int_0^1 f(y)\mu(dy)\right| \quad \text{(see the proof of Theorem 8.2)}$$

$$\leq \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|\sum_{k=1}^{p+1} n d_{i,k}(f)(y) - f(y)\right| \mu(dy) = \sum_{i=1}^{n} \left\{\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{u-y}^{p+1} f(y)\mu(du)\right| \mu(dy)\right\}$$

$$= \sum_{i=1}^{n} \left\{\int_0^1 \left[n \left|\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{u-y}^{p+1} f(y)\mu(du)\right| \cdot \chi_{[\frac{i-1}{n}, \frac{i}{n}]}(y)\right] \mu(dy)\right\}$$

$$\cdot n \sum_{i=1}^{n} \left\|\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{u-y}^{p+1} f(y)\mu(du)\right\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]} =: (*).$$

For any $y \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$, it holds

$$\left\|\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{u-y}^{p+1} f(y)\mu(du)\right\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]} \leq \left\|\int_{\frac{i-1}{n}}^{\frac{i}{n}} |\Delta_{u-y}^{p+1} f(y)|\mu(du)\right\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]}$$

$$\leq \left\|\int_{y-\frac{1}{n}}^{y} |\Delta_{u-y}^{p+1} f(y)|\mu(du)\right\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]} + \left\|\int_{y}^{y+\frac{1}{n}} |\Delta_{u-y}^{p+1} f(y)|\mu(du)\right\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]}$$

$$= \left\|\int_0^{\frac{1}{n}} |\Delta_{-v}^{p+1} f(y)|\mu(dv)\right\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]} + \left\|\int_0^{\frac{1}{n}} |\Delta_{v}^{p+1} f(y)|\mu(dv)\right\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]}$$

$$= \int_0^{\frac{1}{n}} \|\Delta_{-v}^{p+1} f(y)\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]}\mu(dv) + \int_0^{\frac{1}{n}} \|\Delta_{v}^{p+1} f(y)\|_{L^1_\mu[\frac{i-1}{n}, \frac{i}{n}]}\mu(dv).$$

Therefore, it follows

$$(*) \le n\left\{\int_0^{\frac{1}{n}} \|\Delta_{-v}^{p+1} f(y)\|_{L^1_\mu[0,1]}\mu(dv) + \int_0^{\frac{1}{n}} \|\Delta_{v}^{p+1} f(y)\|_{L^1_\mu[0,1]}\mu(dv)\right\}$$
$$\le n\left\{\int_0^{\frac{1}{n}} \|\Delta_{-v}^{p+1} f(y)\|_{L^1_\mu(\mathbb{R})}\mu(dv) + \int_0^{\frac{1}{n}} \|\Delta_{v}^{p+1} f(y)\|_{L^1_\mu(\mathbb{R})}\mu(dv)\right\}$$
$$\le 2n\int_0^{\frac{1}{n}} \|\Delta_{v}^{p+1} f(y)\|_{L^1_\mu(\mathbb{R})}\mu(dv) \le 2n\cdot\frac{1}{n}\omega_{p+1}\left(f;\frac{1}{n}\right)_{L^1_\mu(\mathbb{R})},$$

which establishes (i).

(ii) Similarly, we obtain

$$\begin{aligned}\left|Q_{p+1,n}(f) - \int_0^1 f(y)\mu(dy)\right| &= \left|\sum_{i=1}^n \left\{\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{\frac{i}{n}-y}^{p+1} f(y)\mu(dy)\right\}\right| \\ &= \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{\frac{i}{n}-y}^{p+1} f(y)\mu(dy) := (*).\end{aligned}$$

On the other hand, we have $\frac{i}{n} \le y + \frac{1}{n}$, which implies $\Delta_{\frac{i}{n}-y}^{p+1} f(y) \le \Delta_{\frac{1}{n}}^{p+1} f(y)$. Indeed, first let us assume $g \in C^{p+1}[0,p+2]$, as in (ii). Denote $F(h) = \Delta_h^{p+1} g(y)$, ($y \in [0,1]$ fixed), $h \in [0,1]$. By $F(h) \ge 0$, $\forall h \ge 0$. Also, by $g^{(k+1)}(y) = \lim\limits_{h\to 0} \frac{\Delta_h^{p+1} g(y)}{(p+1)!h^{p+1}}$, we obtain $g^{(p+1)}(y) \ge 0$, $\forall y \in [-1,2]$. Furthermore,

$$F'(h) = (p+1)\Delta_h^p g'(y+h) = (p+1)h^p g^{(p+1)}(\xi) \ge 0$$

(see, e.g., [67, p. 59–60]), so $\Delta_{\frac{i}{n}-y}^{p+1} g(y) = F\left(\frac{i}{n} - y\right) \le F\left(\frac{1}{n}\right) = \Delta_{\frac{1}{n}}^{p+1} g(y)$.

Also, the condition $\Delta_h^{p+1} f(y) \ge 0$, $\forall h \ge 0$, $\forall y \in [-1,2]$, implies for $p \ge 1$ that f is necessarily continuous on $[0,p+2]$. (If $p = 0$, it follows that f is nondecreasing and the theorem was proved in [20].) Then, denoting by $B_m(f)(y)$, the sequence of Bernstein polynomials on $[0,p+2]$, it is well known that $B_m^{(p+1)}(f)(y) \ge 0$, $\forall y \in [0,p+2]$, $\forall m \in \mathbb{N}$, so reasoning as above (because $B_m(f)(y) \in C^{p+1}[0,p+2]$), we get

$$\Delta_{\frac{i}{n}-y}^{p+1} B_m(f)(y) \le \Delta_{\frac{1}{n}}^{p+1} B_m(f)(y), \quad \forall m \in \mathbb{N},$$

by taking $g := B_m(f)$. Passing to the limit with $m \to +\infty$, we easily derive

$$\Delta_{\frac{i}{n}-y}^{p+1} f(y) \le \Delta_{\frac{1}{n}}^{p+1} f(y), \quad \forall y \in [0,1].$$

Hence

$$\begin{aligned}(*) &\le \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Delta_{\frac{1}{n}}^{p+1} f(y)\mu(dy) = \int_0^1 \Delta_{\frac{1}{n}}^{p+1} f(y)\mu(dy) \\ &= \|\Delta_{\frac{1}{n}}^{p+1} f(y)\|_{L^1_\mu[0,1]} \le \|\Delta_{\frac{1}{n}}^{p+1} f(y)\|_{L^1_\mu(\mathbb{R})}\end{aligned}$$

$$\leq \omega_{p+1}\left(f; \frac{1}{n}\right)_{L^1_\mu(\mathbb{R})},$$

which establishes the theorem. ■

Remark 8.5. 1) If in the formulas of $R_{p+1,n}(f)$ and $Q_{p+1,n}(f)$ we substitute $\int_0^1$ by $\int_a^b$ and $\frac{i}{n}$ by $a + \frac{b-a}{n} i$, then we easily obtain approximants to the integral $\int_a^b f(u)\mu(du)$.

2) Let us assume that $f\colon [A, B] \times [-p, p+1] \to \mathbb{R}$ satisfies the Lipschitz type condition

$$|f(t,u) - f(s,u)| \leq M|t-s|, \quad \forall s,t \in [A,B], \quad \forall u \in [-p, p+1],$$

where M is independent of s, t, u. Then, let us define $\tilde{R}_{p+1,n}(f)\colon [A,B] \to \mathbb{R}$ by

$$\tilde{R}_{p+1,n}(f)(x) = -\int_0^1 \sum_{i=1}^{n} \sum_{k=1}^{p+1} n\tilde{d}_{i,k}(f)(x,y)\chi_{[\frac{i-1}{n},\frac{i}{n}]}(y)\mu(dy)$$

where

$$\tilde{d}_{i,k}(f)(x,y) = (-1)^k \binom{p+1}{k} \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x, y + k(u-y))\mu(du).$$

Then we easily obtain

$$|\tilde{R}_{p+1,n}(f)(t) - \tilde{R}_{p+1,n}(f)(s)|$$

$$\leq \int_0^1 \sum_{i=1}^{n} \sum_{k=1}^{p+1} n\binom{p+1}{k} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |f(t, y+k(u-y)) - f(s, y+k(u-y))|\mu(du)\chi_{[\frac{i-1}{n},\frac{i}{n}]}(y)\mu(dy)$$

$$\leq M 2^{p+1}|t-s|, \quad \text{for all } s,t \in [A,B],$$

i.e., $\tilde{R}_{p+1,n}(f)(x)$ satisfies a kind of global smoothness preservation property. The same property is valid for the modified expression

$$\tilde{Q}_{p+1,n}(f)(x) = -\int_0^1 \sum_{i=1}^{n} \sum_{k=1}^{p+1} (-1)^k \binom{p+1}{k} f\left(x, y + k\left(\frac{i}{n} - y\right)\right) \chi_{[\frac{i-1}{n},\frac{i}{n}]}(y)\mu(dy).$$

9

Discrete Best L_1 Approximation Using the Gauges Way

A discrete theory is presented for the best approximation in the "gauges" sense. This chapter relies on [8].

9.1 Introduction

In [249], A. Pinkus and O. Shisha introduced novel measures of size ("gauges") of real functions of a real variable, continuous on $[0,1]$. In their simplest form, these measures can be described roughly as follows. If $f = 0$ throughout $[0,1]$, then these gauges of f, $|||f|||$ and $|||f|||_*$ are 0. Otherwise, $|||f|||$ is the largest of the areas of the (positive and negative) humps made up by the graph of f over $[0,1]$, while $|||f|||_*$ is the largest of the sum of areas of consecutive humps of the same sign (see Definition 9.1 below). Best approximation by polynomials (or other Chebyshev systems) can then be studied with respect to $|||\cdot|||$ and $|||\cdot|||_*$. The main point is that doing so, we can imitate successfully the classical Chebyshev theory of best approximation, much better that by using L_p norms, while, at the same time, $|||\cdot|||$ and $|||\cdot|||_*$ are basically integral measures of functions, a feature often desirable.

Such a continuous theory of best approximation with respect to $|||\cdot|||$ and $|||\cdot|||_*$ has been carried out in [249].

In this chapter we present the analogous discrete theory for real functions on finite subsets of $[0,1]$. In particular we prove (Corollary 9.12), that given f, continuous in $[0,1]$, an integer $n \geq 0$ and a sequence $(F_k)_{k=1}^{\infty}$ of finite subsets of $[0,1]$, each containing $0, 1$ and of cardinality $\geq n+2$ such that

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the maximal distance between consecutive points of $F_k \to 0$ as $k \to \infty$, the following relation holds (under a simple condition):

$$\lim_{k\to\infty} \min |||f - p|||_{F_k} = \min |||f - p||| ,$$

where $|||\cdot|||_{F_k}$ is the discrete version of $|||\cdot|||$, and where the minimum on both sides is taken over all polynomials p of degree $\leq n$.

The continuous theory of [249] has been further developed in [219], which contains also an outline of a discrete theory similar to this chapter, but with some of the proofs left out and with the underlying discrete gauges different from ours.

9.2 Background

We recall from [249].

Definition 9.1. Let f be a real function of a real variable, continuous in $[0,1]$. We put

$$|||f||| \text{ ("gauge of } f\text{") } =$$

$$\max\left\{\left|\int_a^b f(x)\,dx\right| : 0 \leq a \leq b \leq 1,\ f(x) > 0 \text{ on } (a,b) \text{ or } f(x) < 0 \text{ on } (a,b)\right\}$$

(see Note 9.13),

$$|||f|||_* \text{ ("star gauge of } f\text{") } =$$

$$\max\left\{\left|\int_a^b f(x)\,dx\right| : 0 \leq a \leq b \leq 1,\ f(x) \geq 0 \text{ on } (a,b) \text{ or } f(x) \leq 0 \text{ on } (a,b)\right\}.$$

As mentioned in [249], $|||\cdot|||$, $|||\cdot|||_*$ are not norms over $C([0,1])$.

Observe that the definitions of $|||f|||$ and $|||f|||_*$ make sense also if, for some $0 = x_0 < x_1 < ... < x_m = 1$, f is a real function, constant on $[x_j, x_{j+1})$, $j = 0, 1, ..., m-1$.

Definition 9.2. Given a finite set

$$F : 0 = x_0 < x_1 < ... < x_m = 1 \quad (m \geq 1)$$

and a real function f defined on F, we denote by f_F the real function, with domain $[0,1)$, which equals $f(x_j)$ on $[x_j, x_{j+1})$, $j = 0, 1, ..., m-1$, and set

$$|F| = m + 1, \quad |||f|||_F = |||f_F|||, \quad |||f|||_{*F} = |||f_F|||_* . \tag{9.1}$$

Observe that $|||f|||_F$ and $|||f|||_{*F}$ are independent of $f(1)$ but it is still natural to associate these "gauges" with a finite set including 1, as $x_m = 1$ determines the interval of constancy $[x_{m-1}, x_m)$ exactly as other x_j do (if $1 \leq j < m$). Also, one can consider a definition of $|||\cdot|||$ and $|||\cdot|||_*$ "symmetric" to Definition 9.2 where, in (9.1), $f_F(x) = f(x_{j+1})$ on each $(x_j, x_{j+1}]$, $j = 0, ..., m-1$.

(9.1) clearly implies

$$|||f|||_F \leq |||f|||_{*F} \leq \sum_{j=0}^{m-1} |f(x_j)|(x_{j+1} - x_j) \leq \max\{|f(x_j)| : 0 \leq j \leq m-1\}, \tag{9.2}$$

$$\begin{aligned} |||f|||_F &= 0 \text{ iff } f(x_j) = 0, \;\; j = 0, 1, ..., m-1, \\ |||f|||_{*F} &= 0 \text{ iff } f(x_j) = 0, \;\; j = 0, 1, ..., m-1, \end{aligned} \tag{9.3}$$

$$|||cf|||_F = |c| \cdot |||f|||_F, \quad |||cf|||_{*F} = |c| \cdot |||f|||_{*F} \quad \text{for every real } c.$$

For a fixed F, $|||\cdot|||_F$ and $|||\cdot|||_{*F}$ do not always satisfy the triangle inequality. Indeed, let $F = \left(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$, let $f = 1$ on F; $g(0) = g\left(\frac{1}{2}\right) = g(1) = 1$, $g\left(\frac{1}{4}\right) = g\left(\frac{3}{4}\right) = -\frac{1}{4}$. Then

$$|||f|||_F = |||f|||_{*F} = 1, \quad |||g|||_F = |||g|||_{*F} = \frac{1}{4},$$

$$|||f+g|||_F = |||f+g|||_{*F} = \frac{11}{8} > |||f|||_F + |||g|||_F = |||f|||_{*F} + |||g|||_{*F}.$$

9.3 More Background

If, for functions $f, f_1, f_2, f_3, ...$, we have $f_n(x) \to f(x)$ on some F, then it easily follows that

$$|||f_n - f|||_F \to 0, \quad |||f_n - f|||_{*F} \to 0;$$

but neither of the statements

$$|||f_n|||_F \to |||f|||_F, \quad |||f_n|||_{*F} \to |||f|||_{*F}$$

is valid. Indeed, let $F = \left(0, \frac{1}{3}, \frac{2}{3}, 1\right)$ and

$$\begin{aligned} f(0) &= f_n(0) = f\left(\frac{2}{3}\right) = f_n\left(\frac{2}{3}\right) = f(1) = f_n(1) = 1, \\ f\left(\frac{1}{3}\right) &= 0, \;\; f_n\left(\frac{1}{3}\right) = \frac{1}{n}; \quad n = 1, 2, ... \end{aligned}$$

Then

$$f_n(x) \to f(x) \quad \text{on } F, \tag{9.4}$$

but

$$|||f_n|||_F \to \frac{2}{3} \neq |||f|||_F = \frac{1}{3}.$$

Also, with the same F, f, let f_n be modified to

$$f_n\left(\frac{1}{3}\right) = -\frac{1}{n}, \quad n = 1, 2, \dots .$$

Then (9.4) but

$$|||f_n|||_{*F} \to \frac{1}{3} \neq |||f|||_{*F} = \frac{2}{3}.$$

9.4 Basic Result

Theorem 9.3. Let $f_n \to f$ on some F. Then

$$|||f|||_F \leq \varliminf_{n\to\infty} |||f_n|||_F \leq \varliminf_{n\to\infty} |||f_n|||_{*F} \,. \tag{9.5}$$

Proof. The last inequality follows from the first inequality in (9.2) (applied to f_n). The first example of Section 9.3 shows that the first inequality in (9.5) can be strict.

To establish that inequality, let $\varepsilon > 0$. We show that for some n_0,

$$|||f_n|||_F \geq |||f|||_F - \varepsilon \quad \text{for all } n \geq n_0.$$

We may suppose $f(x) \neq 0$ for some $x \in F - \{1\}$. Using the notation of Definition 9.2, let

$$|||f|||_F = \left|\sum_{j=r}^{s} f(x_j)(x_{j+1} - x_j)\right|, \quad 0 \leq r \leq s < m,$$

where

$$f(x_r) \neq 0, \quad f(x_r) f(x_t) > 0 \quad \text{whenever } r \leq t \leq s.$$

Choose n_0 so that

$$f_n(x_j) f(x_j) > 0, \quad |f_n(x_j) - f(x_j)| < \varepsilon; \quad j = r, r+1, \dots, s; \; n = n_0, n_0+1, \dots .$$

Then, for these n,

$$|||f_n|||_F \geq \sum_{j=r}^{s} |f_n(x_j)|(x_{j+1} - x_j) > \sum_{j=r}^{s} [|f(x_j)| - \varepsilon](x_{j+1} - x_j) \geq |||f|||_F - \varepsilon.$$

■

9.5 Main Result

Given an integer $n \geq 0$, we denote by π_n the set of all polynomials $\sum_{k=0}^{n} a_k x^k$, a_k real, considered as functions with domain $(-\infty, \infty)$.

Theorem 9.4. Let $n \geq 0$ be an integer and let F and f be as in Definition 9.2. Then:

(I) There exists a $p^* \in \pi_n$ for which $|||f - p^*|||_F \leq |||f - p|||_F$ for every $p \in \pi_n$.

(II) For some F, f, (I) becomes false if $|||\cdot|||_F$ is replaced by $|||\cdot|||_{*F}$.

Proof. To prove (I), we may assume $|F| \geq n + 3$. For otherwise, we can take as p^*, Lagrange's interpolation polynomial to f on $F - \{1\}$.

Call

$$C = \inf \{|||f - p|||_F : p \in \pi_n\} \tag{9.6}$$

and for $j = 1, 2, ...,$ let

$$p_j(x) \equiv \sum_{k=0}^{n} a_k^{(j)} x^k \in \pi_n \tag{9.7}$$

be such that

$$|||f - p_j|||_F \to C.$$

Then clearly the sequence $\max \{|p_j(x)| : x \in F - \{1\}\}$, $j = 1, 2, ...,$ is bounded, and hence by representing $p_j(x)$, $j = 1, 2, ...,$ as its own Lagrange's interpolation polynomial on $\{x_0, x_1, ..., x_n\}$, we see that for every a, b, $-\infty < a < b < \infty$, the sequence $\max \{|p_j(x)| : a \leq x \leq b\}$, $j = 1, 2, ...$ is bounded. Hence [234, p.56, Corollary 2] each of the sequence $\left(a_k^{(j)}\right)_{j=1}^{\infty}$ is bounded. Therefore there are integers $1 \leq h_1 < h_2 < ...$ such that, for $k = 0, 1, ..., n$, $a_k^{(h_j)}$ converges, say, to a_k. Put

$$p^*(x) \equiv \sum_{k=0}^{n} a_k x^k.$$

By (9.7), for every x,

$$p_{h_j}^*(x) \to p^*(x)$$

and hence, by Theorem 9.3,

$$|||f - p^*|||_F \leq C$$

which, by (9.6), yields (I).

To prove conclusion (II), let $F = \left(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$, let $f(0) = f\left(\frac{1}{2}\right) = 2$, $f\left(\frac{1}{4}\right) = f(1) = 0$ and $f\left(\frac{3}{4}\right) = -2$. If $c \leq 0$, then $|||f - c|||_{*F} \geq 1$. If $c > 0$, then $|||f - c|||_{*F} > \frac{1}{2}$.

In particular, for $n = 1, 2, ...,$ $\left|\left|\left|f - n^{-1}\right|\right|\right|_{*F} = 2^{-1} + (4n)^{-1}$. Hence among the numbers $|||f - c|||_{*F}$, $-\infty < c < \infty$, there is no minimal. ■

9.6 Preparation Results

We use the following result, essentially well-known. For the convenience of the reader we include proof.

Lemma 9.5. Let $-\infty < a < b < \infty$ and let f be a real function of a real variable, continuous in $[a,b]$, for which the set of $x \in [a,b]$ with $f(x) = 0$ is finite. Given $a \leq \alpha \leq x \leq \beta \leq b$ with $f(x) = 0$, set $\sigma(\alpha,\beta,x) = 2$ if $\alpha < x < \beta$ and f "does not change sign" at x, namely, there is $\delta > 0$ such that $\alpha \leq x - \delta < x + \delta \leq \beta$ and either $f(x) > 0$ throughout $I = (x-\delta, x+\delta) - \{x\}$ or $f(x) < 0$ throughout I; otherwise, set $\sigma(\alpha,\beta,x) = 1$.

Let

$$a \leq c_1 < c_2 < ..., c_n \leq b \quad (n \geq 2), \tag{9.8}$$

and let

$$(-1)^k f(c_k) \geq 0, \quad k = 1, 2, ..., n. \tag{9.9}$$

Then there are

$$c_1 \leq x_1 < x_2 < ... < x_m \leq c_n \quad (m \geq 1)$$

such that

$$f(x_k) = 0, \quad k = 1, 2, ..., m,$$

and

$$\sum_{k=1}^{m} \sigma(c_1, c_n, x_k) \geq n - 1. \tag{9.10}$$

Proof. Observe that if $a \leq \alpha_1 \leq \alpha_2 \leq x \leq \beta_2 \leq \beta_1 \leq b$, $f(x) = 0$, then

$$\sigma(\alpha_1, \beta_1, x) \geq \sigma(\alpha_2, \beta_2, x).$$

We prove Lemma 9.5 by induction. It is trivial for $n = 2$.

Let $N \geq 2$, and suppose Lemma 9.5 is true whenever $2 \leq n \leq N$. We prove it for $N + 1$.

Let

$$a \leq c_1 < c_2 < ... < c_{N+1} \leq b,$$

$$(-1)^k f(c_k) \geq 0, \quad k = 1, 2, ..., N + 1.$$

We prove (*): the existence of

$$c_1 \leq x_1 < x_2 < ... < x_m \leq c_{N+1}$$

such that

$$f(x_k) = 0, \quad j = 1, 2, ..., m,$$

and

$$\sum_{k=1}^{m} \sigma(c_1, c_{N+1}, x_k) \geq N.$$

By the induction hypothesis this is easily seen to hold in case $f(c_{N+1}) = 0$, or $f(c_{N+1}) f(c_N) \leq 0$. As $f(c_{N+1}) f(c_N) \leq 0$, we merely need to prove (*) when $f(c_{N+1}) \neq 0$, $f(c_N) = 0$. We can also assume not all of $f(c_1), ..., f(c_N)$ vanish. So let

$$f(c_r) \neq 0,\ f(c_{r+1}) = f(c_{r+2}) = ... = f(c_N) = 0, \quad 2 \leq r+1 \leq N.$$

If (c_r, c_{N+1}) contains a zero of f other than $c_{r+1}, c_{r+2}, ..., c_N$, then (*) is trivial if $r = 1$ and, otherwise, follows from the induction hypothesis, using it for $n = r$. So we may suppose f never vanishes in $\cup_{k=r}^{N} (c_k, c_{k+1})$. Also, sgn$f$ throughout (c_r, c_{r+1}) is $(-1)^r$. We may suppose that it is $(-1)^k$ throughout (c_k, c_{k+1}) for $k = r+1, ..., N$, for otherwise $\sigma(c_1, c_{N+1}, c_k) = 2$ for some $r < k \leq N$ and again (*) would follow trivially if $r = 1$, and, otherwise, would follow from the induction hypothesis, with $n = r$. However, sgn$f$ cannot be $(-1)^N$ throughout (c_N, c_{N+1}), because sgn$f(c_{N+1}) = (-1)^{N+1}$. ■

Corollary 9.6. Let $-\infty < a < b < \infty$ and let $f \neq 0$ belong to some π_k. Suppose (9.8) and (9.9). Then the number of zeros of f in $[c_1, c_n]$, multiplicities counted, is $\geq n-1$.

This follows from (9.10), nothing that, for $k = 1, 2, ..., m$, the multiplicity of x_k as a zero of f is $\geq \sigma(c_1, c_n, x_k)$.

9.7 Another Main Result

Theorem 9.7. Let $n \geq 0$ be an integer and let F and f be as in Definition 9.2, with $|F| \geq n+3$. There is a unique $p^* \in \pi_n$ minimizing $|||f-p|||_F$ among all $p \in \pi_n$. This p^* is characterized by the following property:

(**) There are integers

$$0 \leq u_1 \leq v_1 < u_2 \leq v_2 < ... < u_{n+2} \leq v_{n+2} < m, \ \ \sigma = \pm 1 \tag{9.11}$$

such that, for $k = 1, 2, ..., n+2$, $p^* \in \pi_n$ satisfies:

$$(-1)^k \sigma [f(x_j) - p^*(x_j)] \geq 0, \quad j = u_k, u_k+1, ..., v_k; \tag{9.12}$$

$$(-1)^k \sigma \sum_{j=u_k}^{v_k} [f(x_j) - p^*(x_j)](x_{j+1} - x_j) \geq |||f-p^*|||_F. \tag{9.13}$$

Remark 9.8. The second sentence of Theorem 9.7 is true even if $|F| = n+2$. For, in that case, let p^* be Lagrange's interpolation polynomial to f on $F - \{1\}$, and let $p \in \pi_n$ be a different polynomial. Then by (9.3),

$$|||f-p^*|||_F = 0 < |||f-p|||_F.$$

Proof. of Theorem 9.7.

(A) Assume the property (**) of the theorem. Let $p \in \pi_n$, $p \neq p^*$. We prove $|||f - p^*|||_F < |||f - p|||_F$. Thus p^* is the unique element of π_n minimizing $|||f - p|||_F$ among all $p \in \pi_n$.

Assume, on the contrary, $|||f - p^*|||_F \geq |||f - p|||_F$. We claim there exist w_k satisfying

$$u_k \leq w_k \leq v_k, \quad (-1)^k \sigma \left[p\left(x_{w_k}\right) - p^*\left(x_{w_k}\right)\right] \geq 0; \quad k = 1, 2, ..., n+2,$$

which implies by Corollary 9.6 that the number of zeros of $p - p^* \neq 0$, multiplicities counted, is $\geq n+1$, which is false.

Suppose our claim is false, and let k, $1 \leq k \leq n+2$, fulfill

$$(-1)^k \sigma \left[p\left(x_j\right) - p^*\left(x_j\right)\right] < 0, \quad j = u_k, u_{k+1}, ..., v_k. \tag{9.14}$$

By (9.14) and (9.12),

$$(-1)^k \sigma \left[f\left(x_j\right) - p\left(x_j\right)\right] > (-1)^k \sigma \left[f\left(x_j\right) - p^*\left(x_j\right)\right] \geq 0, \quad j = u_k, u_{k+1}, ..., v_k,$$

and hence

$$(-1)^k \sigma \sum_{j=u_k}^{v_k} \left[f\left(x_j\right) - p^*\left(x_j\right)\right]\left(x_{j+1} - x_j\right) <$$

$$(-1)^k \sigma \sum_{j=u_k}^{v_k} \left[f\left(x_j\right) - p\left(x_j\right)\right]\left(x_{j+1} - x_j\right) \leq |||f - p|||_F \leq |||f - p^*|||_F,$$

contradicting (9.13).

(B) Let $p^* \in \pi_n$ minimize $|||f - p|||_F$ among all $p \in \pi_n$ (existence of such p^* is guaranteed by Theorem 9.4, (I)). We shall prove (**) of the theorem, which, as shown above, establishes the second sentence of the theorem. We may assume $f - p^*$ does not vanish identically on $F - \{1\}$.

A maximal-definite sequence (mds) is a sequence of integers $I = (a, a+1, ..., b)$ where $0 \leq a \leq b < m$, satisfying:

(i) $\left[f\left(x_a\right) - p^*\left(x_a\right)\right]\left[f\left(x_b\right) - p^*\left(x_b\right)\right] \neq 0$;

(ii) $\sigma_a \left[f\left(x_j\right) - p^*\left(x_j\right)\right] \geq 0$ for every $j \in I$, where $\sigma_a = sgn\left[f\left(x_a\right) - p^*\left(x_a\right)\right]$;

(iii) $\sigma_a \sum_{j=a}^{b} \left[f\left(x_j\right) - p^*\left(x_j\right)\right]\left(x_{j+1} - x_j\right) \geq |||f - p^*|||_F$;

(iv) If s, t are integers, $0 \leq s \leq a \leq b \leq t < m$, and if $\sigma_a \left[f\left(x_j\right) - p^*\left(x_j\right)\right] \geq 0$ for every $s \leq j \leq t$, then $f\left(x_j\right) - p^*\left(x_j\right) = 0$ for every j satisfying $s \leq j \leq t$ but not $a \leq j \leq b$.

mds's are easily seen to be mutually disjoint. There are clearly integers a, b; $0 \leq a \leq b < m$, such that

$$\sigma_a \sum_{j=a}^{b} \left[f\left(x_j\right) - p^*\left(x_j\right)\right]\left(x_{j+1} - x_j\right) = |||f - p^*|||_{*F}$$

and such that (i) and (ii). Then $I = (a, a+1, ..., b)$ is an example of an mds.

Let all mds's be $(a_1, a_1+1, ..., b_1), (a_2, a_2+1, ..., b_2), ..., (a_r, a_r+1, ..., b_r)$ where $0 \le a_1 \le b_1 < a_2 \le b_2 < ... < a_r \le b_r < m$.

Let $\sigma = -\sigma_{a_1}$ and let r_1 be the largest j for which $\sigma_{a_1} = \sigma_{a_2} = ... = \sigma_{a_j}$. If $r_1 < r$, let

$$\begin{aligned}
\sigma_{a_{r_1+1}} &= \sigma_{a_{r_1+2}} = ... = \sigma_{a_{r_2}} = \sigma = (-1)^2 \sigma, \\
\sigma_{a_{r_2+1}} &= \sigma_{a_{r_2+2}} = ... = \sigma_{a_{r_3}} = -\sigma = (-1)^3 \sigma, \\
&\vdots \\
\sigma_{a_{r_{s-1}+1}} &= \sigma_{a_{r_{s-1}+2}} = ... = \sigma_{a_{r_s}} = (-1)^s \sigma.
\end{aligned}$$

If $r_1 = r$, set $s = 1$. Let

$$u_k = a_{r_k}, \quad v_k = b_{r_k}, \quad k = 1, 2, ..., s.$$

Then

$$0 \le u_1 \le v_1 < u_2 \le v_2 < ... < u_s \le v_s < m$$

and (9.12), (9.13) for $k = 1, 2, ..., s$. Hence (**) will follow once we show $s \ge n+2$. Suppose $s < n+2$. Set $y_0 = 0$, $y_s = 1$. If $s > 1$, then for every k, $1 \le k \le s-1$, we define y_k as follows. If there is j, $b_{r_k} < j < a_{r_k+1}$, for which $f(x_j) - p^*(x_j) = 0$, take the smallest such j, and denote by y_k the corresponding x_j. If there is no such j, put

$$y_k = \frac{1}{2}\left(x_{b_{r_k}} + x_{b_{r_k+1}}\right). \tag{9.15}$$

Thus, always,

$$0 = y_0 < y_1 < ... < y_s = 1.$$

Call

$$p(x) \equiv -\sigma \prod_{k=1}^{s-1} (y_k - x) \quad (\equiv -\sigma \text{ if } s = 1).$$

If $1 \le j \le s$, $y_{j-1} < x < y_j$, then $\operatorname{sgn} p(x) = (-1)^j \sigma$. This equality holds also if $y_{j-1} = x$, $j = 1$.

We show: for $\varepsilon > 0$ sufficiently small (to become clear from what follows),

$$|||f - p^* - \varepsilon p|||_F < |||f - p^*|||_F. \tag{9.16}$$

As $p \in \pi_n$, (9.16) contradicts the definition of p^*.

Given $\varepsilon > 0$ sufficiently small, let

$$|||f - p^* - \varepsilon p|||_F = \left|\sum_{j=u}^{v} [f(x_j) - p^*(x_j) - \varepsilon p(x_j)](x_{j+1} - x_j)\right|, \tag{9.17}$$

$0 \le u \le v < m$, where

$$f(x_j) - p^*(x_j) - \varepsilon p(x_j), \quad j = u, u+1, ..., v, \quad \text{are all } > 0 \text{ or all } < 0. \tag{9.18}$$

It is impossible for $[x_u, x_v]$ to contain a y_k, $k \ge 1$. For either such a y_k would be an x_j, $u \le j \le v$, $f(x_j) - p^*(x_j) = 0$, $p(x_j) = 0$, contradicting (9.18), or, by (9.15), we would have

$$u \le b_{r_k} < b_{r_k+1} \le v, \quad sgn\left[f\left(x_{b_{r_k}}\right) - p^*\left(x_{b_{r_k}}\right)\right] = (-1)^k \sigma,$$

$$sgn\left[f\left(x_{b_{r_k+1}}\right) - p^*\left(x_{b_{r_k+1}}\right)\right] = (-1)^{k+1} \sigma$$

which implies

$$sgn\left[f\left(x_{b_{r_k}}\right) - p^*\left(x_{b_{r_k}}\right) - \varepsilon p\left(x_{b_{r_k}}\right)\right] \ne$$

$$sgn\left[f\left(x_{b_{r_k+1}}\right) - p^*\left(x_{b_{r_k+1}}\right) - \varepsilon p\left(x_{b_{r_k+1}}\right)\right],$$

again contradicting (9.18).

So let

$$y_{k-1} \le x_u < x_{u+1} < ... < x_v < y_k$$

where $1 \le k \le s$ and where $y_{k-1} < x_u$ if $k > 1$. Observe that if $k < s$, then $v < a_{r_k+1}$ while if $k > 1$, then $u > b_{r_k-1}$. It follows that if $(u, u+1, ..., v)$ is a subsequence of an mds $(a, a+1, ..., b)$, then $\sigma_a = (-1)^k \sigma$.

(α) Assume

$$sgn\left[f(x_j) - p^*(x_j) - \varepsilon p(x_j)\right] = (-1)^k \sigma, \quad j = u, u+1, ..., v.$$

Then, by the above, for these j,

$$sgn\left[f(x_j) - p^*(x_j)\right] = (-1)^k \sigma, \quad |f(x_j) - p^*(x_j) - \varepsilon p(x_j)| < |f(x_j) - p^*(x_j)|$$

and hence, by (9.17), we have (9.16).

(β) Assume

$$sgn\left[f(x_j) - p^*(x_j) - \varepsilon p(x_j)\right] = (-1)^{k+1} \sigma, \quad j = u, u+1, ..., v.$$

Then, if $\varepsilon > 0$ is sufficiently small, for $j = u, u+1, ..., v$, $sgn\left[f(x_j) - p^*(x_j)\right]$ is $(-1)^{k+1} \sigma$ or 0. Also

$$\left|\sum_{j=u}^{v} \left[f(x_j) - p^*(x_j)\right](x_{j+1} - x_j)\right| < |||f - p^*|||_F, \tag{9.19}$$

for otherwise, as is easily seen, $(u, u+1, ..., v)$ would be a subsequence of an mds $(a, a+1, ..., b)$ with $\sigma_a = (-1)^{k+1} \sigma$, contradicting our statement preceding (α). But (9.19) and (9.17) imply (9.16) for $\varepsilon > 0$ sufficiently small. ∎

9.8 Conclusions

Theorem 9.9. Let $n \geq 0$ be an integer and let f be a real function of a real variable, continuous in $[0,1]$. For $k = 1, 2, ...,$ let

$$F_k : 0 = x_0^{(k)} < x_1^{(k)} < ... < x_{m(k)}^{(k)} = 1, \;\; m(k) \geq n+1,$$

be a finite subset of $[0,1]$ with

$$\delta_k \equiv \max\left\{\left(x_{j+1}^{(k)} - x_j^{(k)}\right) : 0 \leq j \leq m(k) - 1\right\} \to 0. \tag{9.20}$$

After Theorem 9.7 and Remark 9.8, given $k \geq 1$, consider the unique $p_k^* \in \pi_n$ minimizing $|||f - p|||_{F_k}$ among all $p \in \pi_n$. After Theorem 3.1 of [249], consider the unique $p^* \in \pi_n$ minimizing $|||f - p|||$ among all $p \in \pi_n$. Then

$$|||f - p^*||| \leq \varliminf_{k\to\infty} |||f - p_k^*|||_{F_k} \leq \varlimsup_{k\to\infty} |||f - p_k^*|||_{F_k} \leq |||f - p^*|||_* \,. \tag{9.21}$$

9.9 Proofs

In proving Theorem 9.9 we shall use the following two lemmas (see Note 9.14).

Lemma 9.10. Assume the first two sentences of Theorem 9.9. Then

$$|||f||| \leq \varliminf_{k\to\infty} |||f|||_{F_k} \leq \varlimsup_{k\to\infty} |||f|||_{F_k} \leq \varlimsup_{k\to\infty} |||f|||_{*F_k} \leq |||f|||_* \,.$$

Lemma 9.11. Repeat the first three sentences of Theorem 9.9. Then the sequence

$$\mu_k = \max\{|p_k^*(x)| : 0 \leq x \leq 1\}, \quad k = 1, 2, ...,$$

is bounded.

Proof. of Theorem 9.9. For $k = 1, 2, ...,$

$$|||f - p_k^*|||_{F_k} \leq |||f - p^*|||_{F_k} \leq |||f - p^*|||_{*F_k} \,;$$

hence, by Lemma 9.10,

$$\varlimsup_{k\to\infty} |||f - p_k^*|||_{F_k} \leq \varlimsup_{k\to\infty} |||f - p^*|||_{F_k} \leq |||f - p^*|||_*$$

which yields the last inequality in (9.21). ∎

For a real function g of a real variable, continuous in $[0,1]$, denote by $\omega_1(g,\cdot)$ the first modulus of continuity of $g(x)$, $0 \le x \le 1$.

Let $k \ge 1$. We prove that

$$|||f - p_k^*||| \le 2\delta_k M_k + \omega_1(f,\delta_k) + \omega_1(p_k^*,\delta_k) + |||f - p_k^*|||_{F_k}, \qquad (9.22)$$

where

$$M_k = \max\{|f(x) - p_k^*(x)| : 0 \le x \le 1\}. \qquad (9.23)$$

Let

$$|||f - p_k^*||| = \left| \int_a^b (f - p_k^*) \right|, \quad 0 \le a \le b \le 1, \qquad (9.24)$$

where $sgn(f - p_k^*)$ is constant (± 1) throughout (a,b).

If no point of F_k lies in (a,b), then (9.24) implies

$$|||f - p_k^*||| \le \delta_k M_k$$

and a fortiori (9.22). Thus we may suppose

$$x_u^{(k)} \le a < x_{u+1}^{(k)} < ... < x_v^{(k)} < b \le x_{v+1}^{(k)}, \quad 0 \le u < v < m(k).$$

Then (9.24) yields

$$|||f - p_k^*||| = \int_a^{x_{u+1}^{(k)}} |f - p_k^*| + \int_{x_{u+1}^{(k)}}^{x_v^{(k)}} |f - p_k^*| + \int_{x_v^{(k)}}^{b} |f - p_k^*| \le 2\delta_k M_k +$$

$$\sum_{j=u+1}^{v-1} \int_{x_j^{(k)}}^{x_{j+1}^{(k)}} \left[\left| f(x) - f\left(x_j^{(k)}\right) \right| + \left| f\left(x_j^{(k)}\right) - p_k^*\left(x_j^{(k)}\right) \right| + \left| p_k^*\left(x_j^k\right) - p_k^*(x) \right| \right] dx$$

(an "empty" sum means 0) which implies (9.22).

By the mean value theorem and A. A. Markoff's inequality [131, p. 94, problem 4],

$$\omega_1(p_k^*,\delta_k) \le 2n^2 \delta_k \sup\{\mu_j, j = 1,2,...\}, \quad k = 1,2,... .$$

By (9.20) and Lemma 9.11, the first three summands on the right side of (9.22) $\to 0$ as $k \to \infty$. Hence, by (9.22),

$$|||f - p^*||| \le \varliminf_{k\to\infty} |||f - p_k^*||| \le \varliminf_{k\to\infty} |||f - p_k^*|||_{F_k}. \qquad (9.25)$$

9.10 More Proofs

Proof. of Lemma 9.10. For $k = 1,2,...$, let f_k be the function whose graph is the polygon $P_0^{(k)} P_1^{(k)} ... P_{m(k)}^{(k)}$, where $P_j^{(k)}$ is the point $\left(x_j^{(k)}, f\left(x_j^{(k)}\right)\right)$ in the x, y plane, $j = 0, 1, ..., m(k)$. By (9.20), f_k converges uniformly to f on $[0, 1]$. Also, as one easily sees, for $k = 1, 2, ...$,

$$|||f_k||| \leq |||f|||_{F_k} + \varepsilon_k'; \quad |||f|||_{*F_k} \leq |||f_k|||_* + \varepsilon_k''$$

where $\varepsilon_k' \to 0$, $\varepsilon_k'' \to 0$. By [249], Lemma 9.10 follows. ■

Proof. of Lemma 9.11. The conclusion is obvious for $n = 0$. For let

$$p_k^*(x) \equiv a_k, \quad k = 1, 2, ... \,.$$

If $(\mu_k)_{k=1}^{\infty}$ is unbounded, then for some $k \geq 1$, $|f(x) - a_k| > |f(x)|$ and $f(x) - a_k$ has a fixed sign throughout $[0, 1]$, which clearly leads to a contradiction with the definition of p_k^*. Suppose $n > 0$. Let $k_0 \geq 1$ be such that if $k \geq k_0$, then δ_k of (9.20) is $\leq (40n^2)^{-1}$. We prove that, for all $k \geq k_0$,

$$\mu_k \leq 2\left(1 + 5n^2\right) M \tag{9.26}$$

where

$$M = \max\{|f(x)| : 0 \leq x \leq 1\}.$$

Indeed, let $k \geq k_0$ and suppose (9.26) is false. Put

$$S_k = \left\{x : 0 < x < 1, \frac{\mu_k}{2} < |p_k^*(x)|\right\}.$$

Then S_k is open and hence is the union of a set of open, disjoint intervals. One easily sees that there is an I belonging to this set and a real ξ such that $|p_k^*(\xi)| = \mu_k$ and such that ξ belongs to the closure of I. The length d of I must be $\geq (4n^2)^{-1}$. To prove this we may assume $I \neq (0, 1)$. Then I has an endpoint η with $|p_k^*(\eta)| = \frac{\mu_k}{2}$. Observe that, throughout I, $|p_k^*|$ is differentiable, being nowhere there 0. By the mean value theorem and A. A. Markoff's inequality referred to above, for some $\varsigma \in I$,

$$\mu_k [2(\xi - n)]^{-1} = \frac{(|p_k^*(\xi)| - |p_k^*(\eta)|)}{(\xi - \eta)} = \pm p_k^{*\prime}(\varsigma), \quad |p_k^{*\prime}(\varsigma)| \leq 2n^2 \mu_k;$$

hence

$$d \geq |\xi - \eta| \geq \left(4n^2\right)^{-1}.$$

Since $k \geq k_0$, I must intersect F_k. Let

$$x_u^{(k)} < x_{u+1}^{(k)} < ... < x_v^{(k)}, \quad 0 < u \leq v < m(k)$$

be all points of $F_k \cap I$. Then $x_v^{(k)} - x_u^{(k)} \geq (5n^2)^{-1}$, for otherwise either $x_u^{(k)} - x_{u-1}^{(k)}$ or $x_{v+1}^{(k)} - x_v^{(k)}$ would be $> (40n^2)^{-1}$.

For $j = u, u+1, ..., v$, we get

$$\left|f\left(x_j^{(k)}\right) - p_k^*\left(x_j^{(k)}\right)\right| \geq \left|p_k^*\left(x_j^{(k)}\right)\right| - \left|f\left(x_j^{(k)}\right)\right| >$$

$$\left(1+5n^2\right)M-\left|f\left(x_j^{(k)}\right)\right|\geq 5n^2M$$

and, as

$$\left|f\left(x_j^{(k)}\right)\right|<\left|p_k^*\left(x_j^{(k)}\right)\right|,$$

$$sgn\ p_k^*\left(x_j^{(k)}\right)=sgn\ p_k^*\left(x_u^{(k)}\right),$$

one has

$$sgn\left[f\left(x_j^{(k)}\right)-p_k^*\left(x_j^{(k)}\right)\right]=-sgn\ p_k^*\left(x_u^{(k)}\right).$$

Hence

$$|||f-p_k^*|||_{F_k}\geq\sum_{j=u}^{v}\left|f\left(x_j^{(k)}\right)-p_k^*\left(x_j^{(k)}\right)\right|\left(x_{j+1}^{(k)}-x_j^{(k)}\right)>M$$

$$\geq\sum_{j=0}^{m_k-1}\left|f\left(x_j^{(k)}\right)\right|\left(x_{j+1}^{(k)}-x_j^{(k)}\right)\geq|||f|||_{F_k},$$

contradicting the definition of p_k^*. ■

9.11 Final Conclusions

Theorem 9.9 implies

Corollary 9.12. Using the hypothesis and notation of Theorem 9.9, if

$$|||f-p^*|||=|||f-p^*|||_*, \tag{9.27}$$

then

$$\lim_{k\to\infty}|||f-p_k^*|||_{F_k}=|||f-p^*|||.$$

On the other hand, if (9.27) fails, then $|||f-p_k^*|||_{F_k}$ may diverge as $k\to\infty$, as the following example, with $n=0$, shows.

Consider the figure consisting of a plane coordinate system, the lines $y=8$ and $y=-8$, and the graph of a function $y=f(x)$, made up of the non-horizontal sided of four isosceles triangles. For $k=1,2,...$, let

$$F_k=\left(0,\frac{1}{k},\frac{2}{k},...,\frac{k-1}{k},1\right)$$

so that, as is easily seen,

$$p_k^*=0.$$

Also

$$\lim_{k\to\infty}|||f-p_{4k}^*|||_{F_k}=\lim_{k\to\infty}|||f|||_{F_{4k}}=1$$

while

$$\lim_{k\to\infty} \left|\left|\left|f - p^*_{4k+1}\right|\right|\right|_{F_{4k+1}} = \lim_{k\to\infty} |||f|||_{F_{4k+1}} = 2$$

so that $|||f - p^*_k|||_{F_k}$ diverges.

Note 9.13. Observe that 0 always belongs to the set whose maximum is taken (consider $0 \le a = b \le 1$). If $f(x) = 0$ throughout $[0, 1]$, then 0 is the unique element of this set and, so, $|||f||| = 0$.

Note 9.14. The first inequality of Lemma 9.10 is not used.

10

Quantitative Uniform Convergence of Smooth Picard Singular Integral Operators

In this chapter we study the smooth Picard singular integral operators on the line of very general kind. We establish their convergence to the unit operator with rates. The estimates are mostly sharp and they are pointwise and uniform. The presented inequalities involve the higher order modulus of smoothness. To prove optimality we apply mainly the geometric moment theory method. This chapter relies on [34].

10.1 Introduction

The rate of convergence of singular integrals has been studied earlier in [163], [164], [231], [16], [69], [68] and these motivate this chapter. Here we consider some very general operators, the *smooth Picard singular integral operators over* $\mathbb{R}$ and we study the degree of approximation to the unit operator with rates over smooth functions. We prove related inequalities involving the higher modulus of smoothness with respect to $\|\cdot\|_\infty$. The estimates are pointwise and uniform. Most of the times these are optimal in the sense that the inequalities are attained by basic functions. We apply the geometric moment theory method to give best upper bounds in the main theorems and also we give handy estimates there. The discussed operators are not in general positive.

Other motivation comes from [12], [13].

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 115–136.
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10.2 Results

In the next we study the following *smooth Picard singular integral operators* $P_{r,\xi}(f;x)$ defined as follows.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we put

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{10.1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$P_{r,\xi}(f;x) := \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) e^{-|t|/\xi} dt. \tag{10.2}$$

We suppose that $P_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$P_{r,\xi}(f;x) = \frac{1}{2\xi} \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} f(x+jt) e^{-|t|/\xi} dt \right). \tag{10.3}$$

We notice by $\frac{1}{2\xi}\int_{-\infty}^{\infty} e^{-|t|/\xi} dt = 1$ that $P_{r,\xi}(c,x) = c$, c constant and

$$P_{r,\xi}(f;x) - f(x) = \frac{1}{2\xi} \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} (f(x+jt) - f(x)) e^{-|t|/\xi} dt \right). \tag{10.4}$$

Since

$$\int_{-\infty}^{\infty} x^k e^{-|x|} dx = \begin{cases} 0, & k \text{ odd}, \\ 2k!, & k \text{ even}, \end{cases} \tag{10.5}$$

we get the useful here formula

$$\int_{-\infty}^{\infty} t^k e^{-|t|/\xi} dt = \begin{cases} 0, & k \text{ odd}, \\ 2k!\xi^{k+1}, & k \text{ even}. \end{cases} \tag{10.6}$$

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$ with the rth modulus of smoothness finite, i.e.

$$\omega_r(f^{(n)}, h) := \sup_{|t| \le h} \|\Delta_t^r f^{(n)}(x)\|_{\infty,x} < \infty, \quad h > 0, \tag{10.7}$$

where

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x+jt), \tag{10.8}$$

see [143], p. 44.

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}, \tag{10.9}$$

and the even function

$$G_n(t) := \int_0^{|t|} \frac{(|t| - w)^{n-1}}{(n-1)!} \omega_r(f^{(n)}, w) dw, \quad n \in \mathbb{N} \tag{10.10}$$

with

$$G_0(t) := \omega_r(f, |t|), \quad t \in \mathbb{R}. \tag{10.11}$$

Denote by $\lfloor \cdot \rfloor$ the integral part.

We present the first result

Theorem 10.1. *It holds that*

$$\left| P_{r,\xi}(f; x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x) \delta_{2m} \xi^{2m} \right| \leq \frac{1}{\xi} \int_0^{\infty} G_n(t) e^{-t/\xi} dt, \quad n \in \mathbb{N}. \tag{10.12}$$

In L.H.S.(10.12) *the sum collapses when $n = 1$.*

Proof. By Taylor's formula we get

$$f(x + jt) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (jt)^k + \int_0^{jt} \frac{(jt - z)^{n-1}}{(n-1)!} f^{(n)}(x + z) dz$$
$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (jt)^k + j^n \int_0^t \frac{(t - w)^{n-1}}{(n-1)!} f^{(n)}(x + jw) dw. \tag{10.13}$$

Multiplying both sides of (10.13) by α_j and summing up we obtain

$$\sum_{j=0}^{r} \alpha_j (f(x + jt) - f(x)) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k t^k + \mathcal{R}_n(0, t), \tag{10.14}$$

where

$$\mathcal{R}_n(0, t) := \int_0^t \frac{(t - w)^{n-1}}{(n-1)!} \tau(w) dw, \tag{10.15}$$

with

$$\tau(w) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x + jw) - \delta_n f^{(n)}(x).$$

Notice also that

$$-\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j} = (-1)^r\binom{r}{0}. \tag{10.16}$$

According to [16], p. 306, [12], we obtain

$$\tau(w) = \Delta_w^r f^{(n)}(x). \tag{10.17}$$

Therefore

$$|\tau(w)| \le \omega_r(f^{(n)}, |w|), \tag{10.18}$$

all $w \in \mathbb{R}$ independently of x. We do have after integration, see also (10.4), that

$$\begin{aligned} P_{r,\xi}(f;x) - f(x) &= \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j(f(x+jt)-f(x))\right)e^{-|t|/\xi}dt \\ &= \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k t^k + \mathcal{R}_n(0,t)\right)e^{-|t|/\xi}dt \\ &= \sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}t^k e^{-|t|/\xi}dt\right) + \mathcal{R}_n^*, \end{aligned} \tag{10.19}$$

where

$$\mathcal{R}_n^* := \frac{1}{2\xi}\int_{-\infty}^{\infty}\mathcal{R}_n(0,t)e^{-|t|/\xi}dt. \tag{10.20}$$

Here by (10.10) and (10.15) we get

$$|\mathcal{R}_n(0,t)| \le \int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w)|dw \le G_n(t). \tag{10.21}$$

Hence by (10.20) we find

$$\begin{aligned} |\mathcal{R}_n^*| &\le \frac{1}{2\xi}\int_{-\infty}^{\infty}G_n(t)e^{-|t|/\xi}dt \\ &= \frac{1}{\xi}\int_0^{\infty}G_n(t)e^{-t/\xi}dt. \end{aligned} \tag{10.22}$$

Using (10.6) we obtain

$$P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x)\delta_{2m}\xi^{2m} = \mathcal{R}_n^*. \tag{10.23}$$

Inequality (10.12) is now clear via (10.23) and (10.22).

Finally we would like to prove (10.21) with the use of (10.18). We have that for $t > 0$ it is obvious. Let $t < 0$, then

$$
\begin{aligned}
|\mathcal{R}_n(0,t)| &= \left| \int_t^0 \frac{(t-w)^{n-1}}{(n-1)!} \tau(w) dw \right| \\
&\leq \int_t^0 \frac{(w-t)^{n-1}}{(n-1)!} |\tau(w)| dw \leq \int_t^0 \frac{(-t-(-w))^{n-1}}{(n-1)!} \omega_r(f^{(n)}, |w|) dw \\
&= -\left(\int_t^0 \frac{(-t-(-w))^{n-1}}{(n-1)!} \omega_r(f^{(n)}, |-w|) d(-w) \right) \\
&= -\left(\int_{-t}^0 \frac{(-t-\theta)^{n-1}}{(n-1)!} \omega_r(f^{(n)}, |\theta|) d\theta \right) \\
&= \int_0^{-t} \frac{(-t-\theta)^{n-1}}{(n-1)!} \omega_r(f^{(n)}, |\theta|) d\theta \\
&= \int_0^{|t|} \frac{(|t|-\theta)^{n-1}}{(n-1)!} \omega_r(f^{(n)}, \theta) d\theta = G_n(t).
\end{aligned}
$$

The last completes the proof of Theorem 10.1. ■

Corollary 10.2. *Assume $\omega_r(f, \xi) < \infty$, $\xi > 0$. Then it holds for $n = 0$ that*

$$
|P_{r,\xi}(f;x) - f(x)| \leq \frac{1}{\xi} \int_0^\infty \omega_r(f,t) e^{-t/\xi} dt. \tag{10.24}
$$

Proof. We observe that

$$
\begin{aligned}
P_{r,\xi}(f;x) - f(x) &= \frac{1}{2\xi} \left(\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} \alpha_j (f(x+jt) - f(x)) \right) e^{-|t|/\xi} dt \right) \\
&= \frac{1}{2\xi} \left(\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} (f(x+jt) - f(x)) \right) e^{-|t|/\xi} dt \right. \\
&= \frac{1}{2\xi} \left(\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f(x+jt) \right. \right. \\
&\qquad \left. \left. - \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \right) f(x) \right) e^{-|t|/\xi} dt \right)
\end{aligned}
$$

$$\overset{(10.16)}{=} \frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f(x+jt)\right.\right.$$
$$\left.\left.+(-1)^r\binom{r}{0}f(x)\right)e^{-|t|/\xi}dt\right)$$
$$= \frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f(x+jt)\right)e^{-|t|/\xi}dt\right)$$
$$\overset{(10.8)}{=} \frac{1}{2\xi}\left(\int_{-\infty}^{\infty}((\Delta_t^r f)(x))e^{-|t|/\xi}dt\right).$$

I.e. we have proved

$$P_{r,\xi}(f;x)-f(x)=\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}(\Delta_t^r f(x))e^{-|t|/\xi}dt\right). \tag{10.25}$$

Hence by (10.25) we derive

$$\begin{aligned}|P_{r,\xi}(f;x)-f(x)| &\le \frac{1}{2\xi}\int_{-\infty}^{\infty}|\Delta_t^r f(x)|e^{-|t|/\xi}dt\\ &\le \frac{1}{2\xi}\int_{-\infty}^{\infty}\omega_r(f,|t|)e^{-|t|/\xi}dt\\ &= \frac{1}{\xi}\int_{0}^{\infty}\omega_r(f,t)e^{-|t|/\xi}dt.\end{aligned}$$

That is proving (10.24). ■

Inequality (10.12) is sharp.

Theorem 10.3. *Inequality (10.12) at $x=0$ is attained by $f(x)=x^{r+n}$, $r,n\in\mathbb{N}$ with $r+n$ even.*

Proof. As in [16], p. 307, [12], [265], p. 54 and (10.7), (10.8) we obtain

$$\omega_r(f^{(n)},t)=(r+n)(r+n-1)\cdots(r+1)r!t^r,$$

$t>0$. And

$$G_n(t)=r!|t|^{r+n},\quad t\in\mathbb{R}.$$

Also we have $f^{(k)}(0)=0$, $k=0,1,\ldots,n$. Thus the right hand side of (10.12) equals

$$\frac{r!}{\xi}\int_0^{\infty}t^{r+n}e^{-t/\xi}dt=r!(r+n)!\xi^{r+n}. \tag{10.26}$$

The left hand side of (10.12) equals

$$\begin{aligned}
|P_{r,\xi}(f;0)| &= \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(jt)\right)e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j (jt)^{r+n}\right)e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^{-n}(jt)^{r+n}\right)e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|\left(\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}j^{r}\right)\left(\int_{-\infty}^{\infty}t^{r+n}e^{-|t|/\xi}dt\right)\right| \\
&= \frac{1}{2\xi}\left|(\Delta_1^r x^r)(0)\int_{-\infty}^{\infty}t^{r+n}e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|r!\int_{-\infty}^{\infty}t^{r+n}e^{-|t|/\xi}dt\right| \\
&\overset{(10.6)}{=} \frac{1}{2\xi}\left|r!2(r+n)!\xi^{r+n+1}\right| = r!(r+n)!\xi^{r+n}.
\end{aligned}$$

I.e. we have established

$$|P_{r,\xi}(f;0)| = r!(r+n)!\xi^{r+n}. \tag{10.27}$$

Thus by (10.26) and (10.27) we have established the claim of the theorem.
Inequality (10.24) is sharp. ■

Corollary 10.4. *Inequality (10.24) is attained at $x = 0$ by $f(x) = x^r$, r even.*

Proof. Notice that $\Delta_t^r x^r = r!t^r$ and $\omega_r(f^{(n)}, t) = r!t^r$, $t > 0$. Thus

$$\text{R.H.S.}(10.24) = \frac{r!}{\xi}\int_0^{\infty} t^r e^{-t/\xi}dt = (r!)^2\xi^r.$$

Also $f(0)=0$. Therefore

$$\begin{aligned}
\text{L.H.S.}(10.24) &= |P_{r,\xi}(f;0)| = \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j j^r t^r\right)e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}j^r\right)t^r e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|(\Delta_1^r x^r)(0)\int_{-\infty}^{\infty}t^r e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|r!\int_{-\infty}^{\infty}t^r e^{-|t|/\xi}dt\right| \\
&\overset{(10.6)}{=} \frac{1}{2\xi}|r!2r!\xi^{r+1}| = (r!)^2\xi^r.
\end{aligned}$$

That is (10.24) is attained. ∎

Remark 10.5. On inequalities (10.12) and (10.24). We have the uniform estimates

$$\left\|P_{r,\xi}(f;x)-f(x)-\sum_{m=1}^{\lfloor\frac{n}{2}\rfloor}f^{(2m)}(x)\delta_{2m}\xi^{2m}\right\|_{\infty,x}\leq\frac{1}{\xi}\int_0^{\infty}G_n(t)e^{-t/\xi}dt, n\in\mathbb{N}, \tag{10.28}$$

and

$$\|P_{r,\xi}(f)-f\|_\infty\leq\frac{1}{\xi}\int_0^\infty\omega_r(f,t)e^{-t/\xi}dt,\quad n=0. \tag{10.29}$$

Remark 10.6. The following regards the convergence of operators $P_{r,\xi}$. From (10.10) we have

$$G_n(t)\leq\omega_r(f^{(n)},|t|)\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}dw,$$

i.e.

$$G_n(t)\leq\frac{|t|^n}{n!}\omega_r(f^{(n)},|t|). \tag{10.30}$$

Furthermore from (10.28) and (10.30) we obtain

$$\frac{1}{\xi}\int_0^\infty G_n(t)e^{-t/\xi}dt\leq\frac{1}{\xi n!}\int_0^\infty t^n\omega_r(f^{(n)},t)e^{-t/\xi}dt. \tag{10.31}$$

That is from (10.28) we get

$$\begin{aligned}
K_1 :=& \left\|P_{r,\xi}(f;x)-f(x)-\sum_{m=1}^{\lfloor\frac{n}{2}\rfloor}f^{(2m)}(x)\delta_{2m}\xi^{2m}\right\|_{\infty,x} \\
\leq& \frac{1}{\xi n!}\int_0^\infty t^n\omega_r(f^{(n)},t)e^{-t/\xi}dt,\quad n\in\mathbb{N}.
\end{aligned} \tag{10.32}$$

Using $\omega_r(f^{(n)},t) \le t^r \|f^{(r+n)}\|_\infty$, $t>0$ we find

$$\frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)},t)e^{-t/\xi}dt \le \frac{\|f^{(r+n)}\|_\infty}{\xi n!}\int_0^\infty t^{n+r}e^{-t/\xi}dt$$
$$= \frac{\|f^{(r+n)}\|_\infty}{n!}\xi^{n+r}(n+r)! = \left(\prod_{i=1}^r (n+i)\right)\|f^{(r+n)}\|_\infty \xi^{n+r}.$$

I.e.

$$\frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)},t)e^{-t/\xi}dt \le \left(\prod_{i=1}^r (n+i)\right)\|f^{(r+n)}\|_\infty \xi^{n+r}. \quad (10.33)$$

That is for $f \in C^{n+r}(\mathbb{R})$ we have

$$K_1 \le \prod_{i=1}^r (n+i)\|f^{(r+n)}\|_\infty \xi^{n+r}, \quad n \in \mathbb{N}. \quad (10.34)$$

Here is assumed that $\|f^{(r+n)}\|_\infty$ is finite.

One may use also that

$$\omega_r(f^{(n)},t) \le 2^r \|f^{(n)}\|_\infty.$$

Then

$$\frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)},t)e^{-t/\xi}dt \le \frac{2^r\|f^{(n)}\|_\infty}{\xi n!}\int_0^\infty t^n e^{-t/\xi}dt$$
$$= 2^r \|f^{(n)}\|_\infty \xi^n. \quad (10.35)$$

That is

$$K_1 \le 2^r \|f^{(n)}\|_\infty \xi^n, \quad n \in \mathbb{N}. \quad (10.36)$$

Here is assumed that $\|f^{(n)}\|_\infty < \infty$. Clearly from (10.34) or (10.36), given that $\left\|f^{(2m)}\right\|_\infty < \infty$, for $m = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor$, as $\xi \to 0$ we obtain that $P_{r,\xi} \to$ unit operator I pointwise as $\xi \to 0$ with rates, $n \in \mathbb{N}$.

Next using $\omega_r(f,\lambda t) \le (\lambda+1)^r \omega_r(f,t)$, $\lambda, t > 0$, we get from (10.29) that

$$\frac{1}{\xi}\int_0^\infty \omega_r(f,t)e^{-t/\xi}dt = \frac{1}{\xi}\int_0^\infty \omega_r\left(f,\xi\left(\frac{t}{\xi}\right)\right)e^{-t/\xi}dt$$
$$\le \omega_r(f,\xi)\int_0^\infty \left(1+\frac{t}{\xi}\right)^r e^{-t/\xi}dt/\xi$$
$$= \omega_r(f,\xi)\int_0^\infty (1+u)^r e^{-u}du$$
$$= \omega_r(f,\xi)\left(\sum_{k=0}^r \binom{r}{k}k!\right).$$

That is, we find for the case $n = 0$, see (10.29), that

$$\|P_{r,\xi}(f) - f\|_\infty \le \left(\sum_{k=0}^{r} \binom{r}{k} k!\right) \omega_r(f, \xi). \tag{10.37}$$

Here is assumed that $\omega_r(f, \xi) < \infty$. Now as $\xi \to 0$ we obtain

$$P_{r,\xi} \xrightarrow{u} I \text{ with rates}, n = 0.$$

Note 10.7. The operators $P_{r,\xi}$ *are not in general positive* and they are *of convolution type.*

Let $r = 2$, $n = 3$. Then $\alpha_0 = \frac{23}{8}$, $\alpha_1 = -2$, $\alpha_2 = \frac{1}{8}$. Consider $f(t) = t^2 \ge 0$ and $x = 0$. Then

$$P_{r,\xi}(t^2; 0) = -3\xi^2 < 0.$$

Next using Geometric Moment theory methods [200], [16] we find best upper bounds for the right hand side of (10.12) and (10.24).

Theorem 10.8. *Let ψ be a continuous and strictly increasing function on $\mathbb{R}_+$ such that $\psi(0) = 0$, and let*

$$\psi^{-1}\left(\frac{1}{\xi}\int_{\mathbb{R}_+} \psi(t) e^{-t/\xi} dt\right) =: d_\xi > 0, \quad \xi > 0. \tag{10.38}$$

Suppose $H_n := G_n \circ \psi^{-1}$ is concave on $\mathbb{R}_+$, $n \in \mathbb{Z}^+$. Then we obtain the best upper bound

$$\frac{1}{\xi}\int_{\mathbb{R}_+} G_n(t) e^{-t/\xi} dt \le G_n(d_\xi). \tag{10.39}$$

Corollary 10.9. *Consider the upper concave envelope $H_n^*(u)$ of $H_n(u)$. We derive the best upper bound*

$$\frac{1}{\xi}\int_{\mathbb{R}_+} G_n(t) e^{-t/\xi} dt \le H_n^*(\psi(d_\xi)), \quad n \in \mathbb{Z}_+. \tag{10.40}$$

Note 10.10. When H_n, $n \in \mathbb{Z}_+$ is concave, then $H_n^*(\psi(d_\xi)) = G_n(d_\xi)$.

Proof of Theorem 10.8. Here H_n is concave by assumption. It follows from the moment method of optimal distance [200], [16] that

$$\sup_{\mu \in \{\text{probability measures as in (10.38)}\}} \int_{\mathbb{R}_+} G_n(t) \mu(dt) = G_n(d_\xi).$$

Here is supposed that the last integrals are finite. Since by concavity of H_n the set

$$\Gamma_1 := \{(u, H_n(u)) \colon 0 \le u < \infty\}$$

describes the upper boundary of the convex hull conv Γ_0 of the curve

$$\Gamma_0 := \{(\psi(t), G_n(t)) \colon 0 \le t < \infty\}.$$

Notice here that $\frac{1}{\xi}e^{-t/\xi}dt$ is a probability measure on $\mathbb{R}_+$. ■

The fact that H_n can be a concave function is not strange at all, see [16], p. 310, Lemma 9.2.1(i) which we adjust here. Let g be a general modulus of smoothness function and consider

$$\tilde{G}_n(y) := \int_0^{|y|} \frac{(|y|-t)^{n-1}}{(n-1)!} g(t)dt, \tag{10.41}$$

all $y \in \mathbb{R}$, $n \in \mathbb{N}$.

Then we have

Lemma 10.11. *Let $\psi \in C^n((0,\infty))$ such that $\psi^{(k)}(0) \le 0$, for $k = 1, \ldots, n-1$ and $g(y)/\psi^{(n)}(y)$ is non-increasing, whenever $\psi^{(n)}(y) > 0$. Then $\tilde{H}_n := \tilde{G}_n \circ \psi^{-1}$ is a concave function, $n \in \mathbb{N}$.*

For the right hand side of inequality (10.12) we find the following simple upper bound without any special assumptions.

Theorem 10.12. *Call*

$$\tau_\xi := \xi((n+1)!)^{1/n+1}, \quad n \in \mathbb{N}, \ \xi > 0, \tag{10.42}$$

which the same as

$$\left(\frac{1}{\xi}\int_{\mathbb{R}_+} y^{n+1} e^{-y/\xi} dy\right)^{1/n+1} = \tau_\xi. \tag{10.43}$$

Let

$$G_n^*(y) := \int_0^{|y|} \frac{(|y|-t)^{n-1}}{(n-1)!} \omega_1(f^{(n)}, t)dt, \tag{10.44}$$

all $y \in \mathbb{R}$, where $\omega_1(f^{(n)}, t)$ is the first modulus of continuity of $f^{(n)}$ and is finite, $f \in C^n(\mathbb{R})$. Suppose also that

$$\int_{\mathbb{R}_+} G_n^*(y) e^{-y/\xi} dy < \infty.$$

Then

$$\frac{1}{\xi}\int_{\mathbb{R}_+} G_n(y) e^{-y/\xi} dy \le 2^r G_n^*(\tau_\xi), \quad r \in \mathbb{N}. \tag{10.45}$$

Proof. We have $\omega_r(f^{(n)}, |y|) \le 2^{r-1}\omega_1(f^{(n)}, |y|)$, for all $y \in \mathbb{R}$, see [143], p. 45. Furthermore by [143], p. 43 we find

$$\omega_1(f^{(n)}, |y|) \le \overline{\omega}_1(|y|) \le 2\omega_1(f^{(n)}, |y|),$$

for all $y \in \mathbb{R}$, where $\overline{\omega}_1$ is the least concave majorant of ω_1.

Thus
$$\omega_r(f^{(n)}, |y|) \le 2^{r-1}\overline{\omega}_1(|y|) \le 2^r \omega_1(f^{(n)}, |y|),$$
for all $y \in \mathbb{R}$. Put
$$\overline{G}_n(y) := \int_0^{|y|} \frac{(|y|-t)^{n-1}}{(n-1)!}\overline{\omega}_1(t)dt,$$
for all $y \in \mathbb{R}$. Therefore
$$\begin{aligned} G_n(y) &= \int_0^{|y|} \frac{(|y|-t)^{n-1}}{(n-1)!}\omega_r(f^{(n)}, t)dt \le 2^{r-1}G_n^*(y) \\ &\le 2^{r-1}\overline{G}_n(y) \le 2^r G_n^*(y), \quad \text{for all } y \in \mathbb{R}. \end{aligned}$$

The function $\psi(y) = y^{n+1}$ on $\mathbb{R}_+$ is continuous, strictly increasing and $\psi(0) = 0$. And $\psi^{(n)}(y) = (n+1)!y > 0$, for all $y \in \mathbb{R}_+ - \{0\}$, along with $\psi^{(k)}(0) = 0$, $k = 1, \ldots, n-1$. Since $\overline{\omega}_1(y)$ is concave on $\mathbb{R}_+$, this implies $\overline{\omega}_1(y)/y$ is decreasing in $y > 0$, so that $\overline{\omega}_1(y)/\psi^{(n)}(y)$ is decreasing on $(0, \infty)$.

Thus by Lemma 10.11 we get that $\overline{H}_n := \overline{G}_n \circ \psi^{-1}$ is a concave function on $\mathbb{R}_+$; and by Theorem 10.8 we derive
$$\frac{1}{\xi}\int_0^{\infty} \overline{G}_n(y)e^{-y/\xi}dy \le \overline{G}_n(\tau_\xi)$$
giving us
$$\begin{aligned} \frac{1}{\xi}\int_{\mathbb{R}_+} G_n(y)e^{-y/\xi}dy &\le 2^{r-1}\frac{1}{\xi}\int_{\mathbb{R}_+} \overline{G}_n(y)e^{-y/\xi}dy \\ &\le 2^{r-1}\overline{G}_n(\tau_\xi) \le 2^r G_n^*(\tau_\xi). \end{aligned}$$

The proof of the claim is now finished. ■

A related convergence theorem follows.

Theorem 10.13. *Let $f \in C(\mathbb{R})$ with $\omega_1(f, y)$ finite, $y > 0$. Then*
$$\|P_{r,\xi}(f) - f\|_\infty \le 2^r \omega_1(f, \xi). \tag{10.46}$$
I.e. as $\xi \to 0$ we get again $P_{r,\xi} \xrightarrow{u} I$, $n = 0$.

Proof. Notice
$$\frac{1}{\xi}\int_{\mathbb{R}_+} ye^{-y/\xi}dy = \xi. \tag{10.47}$$

We have again
$$\omega_r(f, |y|) \le 2^{r-1}\omega_1(f, |y|), \quad \forall y \in \mathbb{R},$$

see [143], p. 45. Furthermore

$$\omega_1(f,|y|) \le \overline{\omega}_1(|y|) \le 2\omega_1(f,|y|) \quad \forall y \in \mathbb{R},$$

where $\overline{\omega}_1$ is the least concave majorant of ω_1, see [143], p. 43. Thus

$$\omega_r(f,|y|) \le 2^{r-1}\overline{\omega}_1(|y|) \le 2^r \omega_1(f,|y|), \quad \forall y \in \mathbb{R}.$$

Notice that for $n = 0$ we obtain

$$\begin{aligned}|P_{r,\xi}(f;x) - f(x)| &= \frac{1}{2\xi}\left|\int_{\mathbb{R}}\left(\sum_{j=0}^{r}\alpha_j(f(x+jt)-f(x))\right)e^{-|t|/\xi}dt\right| \\ &\overset{(10.24)}{\le} \frac{1}{\xi}\int_0^{\infty}\omega_r(f,y)e^{-y/\xi}dy \\ &\le \frac{2^{r-1}}{\xi}\int_0^{\infty}\overline{\omega}_1(y)e^{-y/\xi}dy.\end{aligned}$$

The probability measure $\frac{1}{\xi}e^{-y/\xi}dy$ fulfills (10.47). By moment theory [200], [16] we get

$$\sup_{\mu \in \{\text{probability measures as in (10.47)}\}} \int_{\mathbb{R}_+} \overline{\omega}_1(y)\mu(dy) = \overline{\omega}_1(\xi) \le 2\omega_1(f,\xi).$$

Hence

$$|P_{r,\xi}(f;x) - f(y)| \le 2^{r-1} \cdot 2\omega_1(f,\xi) = 2^r \omega_1(f,\xi).$$

■

In the next we consider $f \in C^n(\mathbb{R})$, $n \ge 2$ even and the simple *smooth singular operator of symmetric convolution type*

$$P_\xi(f,x_0) := \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x_0+y)e^{-|y|/\xi}dy, \quad \text{for all } x_0 \in \mathbb{R},\ \xi > 0. \qquad (10.48)$$

That is

$$P_\xi(f;x_0) = \frac{1}{2\xi}\int_0^{\infty}\big(f(x_0+y)+f(x_0-y)\big)e^{-y/\xi}dy, \quad \text{for all } x_0 \in \mathbb{R},\ \xi > 0. \qquad (10.48)^*$$

We assume that f is such that

$$P_\xi(f;x_0) \in \mathbb{R}, \quad \forall x_0 \in \mathbb{R}, \forall \xi > 0 \ \text{ and } \ \omega_2(f^{(n)},h) < \infty,\ h > 0.$$

Note that $P_{1,\xi} = P_\xi$ and if $P_\xi(f;x_0) \in \mathbb{R}$ then $P_{r,\xi}(f;x_0) \in \mathbb{R}$. Let the central second order difference

$$(\tilde{\Delta}_y^2 f)(x_0) := f(x_0+y) + f(x_0-y) - 2f(x_0). \qquad (10.49)$$

Observe that

$$(\tilde{\Delta}^2_{-y} f)(x_0) = (\tilde{\Delta}^2_y f)(x_0).$$

Using Taylor's formula with Cauchy remainder we eventually obtain

$$(\tilde{\Delta}^2_y f)(x_0) = 2\sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} y^{2\rho} + \mathcal{R}_1, \tag{10.50}$$

where

$$\mathcal{R}_1 := \int_0^y (\tilde{\Delta}^2_t f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt. \tag{10.51}$$

Notice that

$$P_\xi(f; x_0) - f(x_0) = \frac{1}{2\xi} \int_0^\infty (\tilde{\Delta}^2_y f(x_0)) e^{-y/\xi} dy. \tag{10.52}$$

So immediately we derive

Proposition 10.14. *Assume $\omega_2(f,h) < \infty$, $h > 0$. Then*

$$|P_\xi(f; x_0) - f(x_0)| \le \frac{1}{2\xi} \int_0^\infty w_2(f,y) e^{-y/\xi} dy. \tag{10.53}$$

Hence

$$\|P_\xi(f) - f\|_\infty \le \frac{1}{2\xi} \int_0^\infty w_2(f,y) e^{-y/\xi} dy. \tag{10.54}$$

Furthermore we observe by (10.50) and (10.52) that

$$\begin{aligned} P_\xi(f; x_0) - f(x_0) &= \frac{1}{2\xi} \int_0^\infty \Bigg(2\sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} y^{2\rho} \\ &\quad + \int_0^y (\tilde{\Delta}^2_t f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt \Bigg) e^{-y/\xi} dy \\ &= \sum_{\rho=1}^{n/2} f^{(2\rho)}(x_0) \xi^{2\rho} \\ &\quad + \frac{1}{2\xi} \int_0^\infty \left(\int_0^y (\tilde{\Delta}^2_t f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy. \end{aligned}$$

Clearly we have the representation

$$\begin{aligned} K_2(x_0) \quad : &= P_\xi(f; x_0) - f(x_0) - \sum_{\rho=1}^{n/2} f^{(2\rho)}(x_0) \xi^{2\rho} \qquad (10.55) \\ &= \frac{1}{2\xi} \int_0^\infty \left(\int_0^y (\tilde{\Delta}^2_t f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy. \end{aligned}$$

Therefore

$$\begin{aligned}|K_2(x_0)| &\le \frac{1}{2\xi}\int_0^\infty\left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x_0)|\frac{(y-t)^{n-1}}{(n-1)!}dt\right)e^{-y/\xi}dy\\ &\le \frac{1}{2\xi}\int_0^\infty\left(\int_0^y \omega_2(f^{(n)},t)\frac{(y-t)^{n-1}}{(n-1)!}dt\right)e^{-y/\xi}dy.\end{aligned}$$

We have established that

Theorem 10.15. *Let $f \in C^n(\mathbb{R})$, n even, $P_\xi(f)$ real valued. Then*

$$\begin{aligned}|K_2(x_0)| &\le \frac{1}{2\xi}\int_0^\infty\left(\int_0^y w_2(f^{(n)},t)\frac{(y-t)^{n-1}}{(n-1)!}dt\right)e^{-y/\xi}dy\\ &\le \frac{1}{2\xi n!}\int_0^\infty \omega_2(f^{(n)},y)y^n e^{-y/\xi}dy. \qquad (10.56)\end{aligned}$$

Remark 10.16. The operators P_ξ are positive operators. From (10.54) we obtain

$$\begin{aligned}\frac{1}{2\xi}\int_0^\infty \omega_2(f,y)e^{-y/\xi}dy &= \frac{1}{2\xi}\int_0^\infty \omega_2\left(f,\xi\left(\frac{y}{\xi}\right)\right)e^{-y/\xi}dy\\ &\le \frac{1}{2\xi}\omega_2(f,\xi)\int_0^\infty\left(1+\frac{y}{\xi}\right)^2 e^{-y/\xi}dy = \frac{5}{2}\omega_2(f,\xi).\end{aligned}$$

I.e.

$$\|P_\xi(f)-f\|_\infty \le \frac{5}{2}\omega_2(f,\xi), \quad \xi>0. \qquad (10.57)$$

Acting similarly on the last part of inequality (10.56) it leads us to get

$$\|K_2\|_\infty \le \left(\frac{n^2+5n+5}{2}\right)\omega_2(f^{(n)},\xi)\xi^n, \quad \xi>0. \qquad (10.58)$$

Then from the inequality (10.57) as $\xi \to 0$ we obtain $P_\xi \xrightarrow{u} I$ with rates. And we get the uniform and pointwise convergence of $P_\xi \to I$ with rates from inequality (10.58), given that $\left\|f^{(2\rho)}\right\|_\infty < \infty, \rho = 1, ..., n/2$. Call here for $n \ge 2$ even

$$T_n(y) := \int_0^y \omega_2(f^{(n)},t)\frac{(y-t)^{n-1}}{(n-1)!}dt, \quad y \in \mathbb{R}_+. \qquad (10.59)$$

Then by (10.56) and (10.59) we have

$$|K_2(x_0)| \le \frac{1}{2\xi}\int_0^\infty T_n(y)e^{-y/\xi}dy, \qquad (10.60)$$

and

$$\|K_2\|_\infty \le \frac{1}{2\xi}\int_0^\infty T_n(y)e^{-y/\xi}dy. \qquad (10.61)$$

We put also

$$T_0(y) := \omega_2(y), \quad y > 0.$$

Optimality of Theorem 10.15 follows.

Proposition 10.17. *The first inequality of (10.56) is sharp, namely attained at $x_0 = 0$ by*

$$f_*(y) := \frac{|y|^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)}, \quad 0 < \alpha \leq 2, \ y \in \mathbb{R}, \ n \text{ even}. \tag{10.62}$$

Proof. See that $f_*^{(n)}(y) = |y|^\alpha$ and by Proposition 9.1.1, p. 298 of [16], [13] we get $\omega_2(f_*^{(n)}, |y|) = 2|y|^\alpha$. Also $f_*^{(k)}(0) = 0$, $k = 0, \ldots, n$. Then

$$\begin{aligned}
K_2(0) &= P_\xi(f_*; 0) = \frac{1}{\xi}\int_0^\infty \frac{y^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)} e^{-y/\xi} dy \\
&= \frac{\xi^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)} \int_0^\infty x^{\alpha+n} e^{-x} dx = \frac{\xi^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)} \Gamma(\alpha+n+1) \\
&= \frac{\xi^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)} \left(\prod_{i=1}^{n}(\alpha+i)\right) \Gamma(\alpha+1) = \Gamma(\alpha+1)\xi^{\alpha+n}.
\end{aligned}$$

That is

$$K_2(0) = \Gamma(\alpha+1)\xi^{\alpha+n} > 0.$$

On the other hand we observe that

$$\begin{aligned}
&\frac{1}{2\xi}\int_0^\infty \left(\int_0^y \omega_2(f_*^{(n)}, t) \frac{(y-t)^{n-1}}{(n-1)!} dt\right) e^{-y/\xi} dy \\
&= \frac{1}{2\xi(n-1)!}\int_0^\infty \left(\int_0^y (y-t)^{n-1} 2t^\alpha dt\right) e^{-y/\xi} dy \\
&= \frac{1}{\xi(n-1)!}\int_0^\infty \left(\int_0^y (y-t)^{n-1} (t-0)^{(\alpha+1)-1} dt\right) e^{-y/\xi} dy \\
&= \frac{\xi^{n+\alpha}}{(n-1)!}\int_0^\infty \left(\frac{\Gamma(n)\Gamma(\alpha+1)}{(\Gamma(n+\alpha+1)} \left(\frac{y}{\xi}\right)^{n+\alpha}\right) e^{-y/\xi} \frac{dy}{\xi} \\
&= \frac{\xi^{n+\alpha}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \int_0^\infty x^{n+\alpha} e^{-x} dx = \xi^{n+\alpha}\Gamma(\alpha+1).
\end{aligned}$$

That is proving equality in the first part of inequality (10.56). ■

It follows the optimality of inequality (10.53).

Proposition 10.18. *Inequality (10.53) is attained by* $f^*(y) = |y|^\alpha$, $y \in \mathbb{R}$, $0 < \alpha \le 2$ *at* $x_0 = 0$.

Proof. We notice that

$$P_\xi(f^*; 0) = \frac{1}{\xi}\int_0^\infty y^\alpha e^{-y/\xi} dy = \xi^\alpha \Gamma(\alpha+1) > 0.$$

Also we see again by Proposition 9.1.1, p. 298, [16], [13] that

$$\frac{1}{2\xi}\int_0^\infty \omega_2(f^*, y) e^{-y/\xi} dy = \frac{1}{\xi}\int_0^\infty y^\alpha e^{-y/\xi} dy.$$

That is proving equality to (10.53). ■

Next we present a Lipschitz type of related optimal result.

Theorem 10.19. *Let* $n \ge 2$ *even and* $f \in C^n(\mathbb{R})$ *such that*

$$\omega_2(f^{(n)}, |y|) \le 2A|y|^\alpha, \quad 0 < \alpha \le 2, \quad A > 0.$$

Then for $x_0 \in \mathbb{R}$ *we have*

$$\left| P_\xi(f; x_0) - f(x_0) - \sum_{\rho=1}^{n/2} f^{(2\rho)}(x_0)\xi^{2\rho} \right| \le \Gamma(\alpha+1) A \xi^{n+\alpha}. \tag{10.63}$$

Inequality (10.63) is sharp, namely it is attained at $x_0 = 0$ *by*

$$f_*(y) = \frac{A|y|^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)}.$$

Proof. For $y > 0$ we observe that

$$\begin{aligned} T_n(y) &= \int_0^y \omega_2(f^{(n)}, t)\frac{(y-t)^{n-1}}{(n-1)!} dt \\ &\le \int_0^y 2At^\alpha \frac{(y-t)^{n-1}}{(n-1)!} dt = \frac{2Ay^{n+\alpha}}{\prod_{i=1}^{n}(\alpha+i)}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2\xi}\int_0^\infty T_n(y) e^{-y/\xi} dy &\le \frac{A}{\xi \prod_{i=1}^{n}(\alpha+i)} \int_0^\infty y^{n+\alpha} e^{-y/\xi} dy \\ &= \frac{A\xi^{n+\alpha}}{\prod_{i=1}^{n}(\alpha+i)} \Gamma(n+\alpha+1) = \Gamma(\alpha+1) A \xi^{n+\alpha}. \end{aligned}$$

Using (10.60) we have proved (10.63).

Notice that $f_*^{(n)}(y) = A|y|^\alpha$, and by Proposition 9.1.1, p. 298, [16], [13] we find

$$\omega_2(f_*^{(n)}, |y|) = 2A|y|^\alpha.$$

Also $f_*^{(k)}(0) = 0$, $k = 0, \dots, n$. Then $K_2(0) = \Gamma(\alpha+1)A\xi^{\alpha+n} > 0$. That is proving equality to (10.63). ∎

Let $f \in C^n(\mathbb{R}), n \geq 2$ even, be such that $\omega_2(f^{(n)}, |t|) \leq g(t)$, where g is given arbitrary, bounded, even, positive function and Borel measurable. We consider the even function

$$\hat{T}_n(y) := \int_0^y g(t)\frac{(y-t)^{n-1}}{(n-1)!}dt, \quad y \in \mathbb{R}. \tag{10.64}$$

Theorem 10.20. *Let ψ be a function on $\mathbb{R}_+$ such that $\psi(0) = 0$, which is continuous and strictly increasing. Suppose that*

$$\psi^{-1}\left(\frac{1}{\xi}\int_0^\infty \psi(y)e^{-y/\xi}dy\right) = d_\xi > 0. \tag{10.65}$$

Suppose ($n \geq 2$ even) that $M_n(u) := \hat{T}_n(\psi^{-1}(u))$ is concave on $\mathbb{R}_+$. Then for any $x_0 \in \mathbb{R}$ we get

$$|K_2(x_0)| \leq \frac{1}{2}\hat{T}_n(d_\xi). \tag{10.66}$$

Proof. Here we are applying geometric moment theory, see [200], [16]. Notice that

$$\sup_{\mu \in (\mu \text{ be probability measures as in (10.65))}} \int_0^\infty \hat{T}_n(y)\mu(dy) = \hat{T}_n(d_\xi).$$

Since by the concavity of M_n, the set

$$\Gamma_1 := \{(u, M_n(u)) \colon 0 \leq u < \infty\}$$

is the upper boundary of the convex hull of the curve

$$\Gamma_0 := \{(\psi(y), \hat{T}_n(y)) \colon 0 \leq y < \infty\}.$$

Now theorem follows from (10.59) and (10.60). ∎

A more general result follows.

Theorem 10.21. *All here as in Theorem 10.20, but we consider now M_n^*, the upper concave envelope of the not necessarily concave M_n. Then*

$$|K_2(x_0)| \leq \frac{1}{2}M_n^*(\psi(d_\xi)), \quad \forall x_0 \in \mathbb{R}. \tag{10.67}$$

If M_n is concave then

$$\text{R.H.S.}(10.67) = \frac{1}{2}\hat{T}_n(d_\xi).$$

Let g be an arbitrary, continuous, even, positive function on $\mathbb{R}$ such that $g(0) = 0$. Let ψ be continuous, strictly increasing function on $\mathbb{R}_+$ with $\psi(0) = 0$ and $\hat{T}_n$ be as above, see (10.64).

Next we give sufficient conditions for $M_n = \hat{T}_n \circ \psi^{-1}$ to be concave on $\mathbb{R}_+$, $n \geq 2$ even. The result is similar to Theorem 9.1.3(ii), p. 302, [16], [13].

Theorem 10.22. *Suppose $\psi \in C^n((0,\infty))$, $n \geq 2$ even, that satisfies*

$$\psi^{(k)}(0) \leq 0, \quad \text{for } k = 0, \ldots, n-1.$$

Assume, further that $g(y)/\psi^{(n)}(y)$ is non-increasing on each interval where $\psi^{(n)}$ is positive. Then $M_n = \hat{T}_n \circ \psi^{-1}$ is concave. In particular $\hat{T}_n(y)/\psi(y)$ is non-increasing.

Finally we give to both operators $P_{r,\xi}$, P_ξ some alternative kind of estimates.

Theorem 10.23. *Assuming $f \in C^n(\mathbb{R})$ and $\omega_r(f^{(n)}, \xi) < \infty$, $\xi > 0$, $n \in \mathbb{N}$ and G_n as in (10.10). Then*

$$\frac{1}{\xi}\int_0^\infty G_n(t) e^{-t/\xi} dt \leq \delta(\xi), \tag{10.68}$$

where

$$\delta(\xi) := \omega_r(f^{(n)}, \xi)\xi^n \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)} \left[\lfloor e(n+r)! \rfloor - \lfloor e(n-k-1)! \rfloor\right] \right\}. \tag{10.69}$$

I.e. from (10.32) we have

$$K_1 \leq \delta(\xi). \tag{10.70}$$

That is as $\xi \to 0$ we get again $P_{r,\xi} \to I$, pointwise with rates, given that $\left\|f^{(2m)}\right\|_\infty < \infty, m = 1, ..., \lfloor n/2 \rfloor$.

Proof. We see that for $\xi > 0$

$$\omega_r(f^{(n)}, |w|) = \omega_r\left(f^{(n)}, \xi\left(\frac{|w|}{\xi}\right)\right) \leq \left(1 + \frac{|w|}{\xi}\right)^r \omega_r(f^{(n)}, \xi), \tag{10.71}$$

see [143], p. 45. Hence by (10.10) and (10.71) we observe

$$\begin{aligned} G_n(t) &\le \frac{\omega_r(f^{(n)},\xi)}{(n-1)!}\int_0^{|t|}(|t|-w)^{n-1}\left(1+\frac{w}{\xi}\right)^r dw \\ &= \frac{\omega_r(f^{(n)},\xi)}{\xi^r(n-1)!}\int_0^{|t|}(|t|-w)^{n-1}(w+\xi)^r dw \\ &= \frac{\omega_r(f^{(n)},\xi)}{\xi^r(n-1)!}\int_{\xi}^{(\xi+|t|)}\big((\xi+|t|)-z\big)^{n-1}z^r dz \\ &= \frac{\omega_r(f^{(n)},\xi)}{\xi^r(n-1)!}\left\{\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(\xi+|t|)^{n-k-1}\int_{\xi}^{\xi+|t|} z^{k+r}dz\right\} \\ &= \frac{\omega_r(f^{(n)},\xi)}{\xi^r}\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(k+r+1)}\big[(\xi+|t|)^{n+r}\right. \\ &\quad \left. -\,\xi^{r+k+1}(\xi+|t|)^{n-k-1}\big]\right\}. \end{aligned} \tag{10.72}$$

That is we find

$$G_n(t) \le \frac{\omega_r(f^{(n)},\xi)}{\xi^r}\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(k+r+1)}\right. \left.\big[(\xi+|t|)^{n+r}-\xi^{r+k+1}(\xi+|t|)^{n-k-1}\big]\right\}. \tag{10.73}$$

Therefore

$$\begin{aligned} &\frac{1}{\xi}\int_0^{\infty}G_n(t)e^{-t/\xi}dt \le \frac{\omega_r(f^{(n)},\xi)}{\xi^r} \\ &\quad\cdot\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\int_0^{\infty}\big((\xi+t)^{n+r}\right. \\ &\quad\left.-\,\xi^{r+k+1}(\xi+t)^{n-k-1}\big)e^{-t/\xi}d(t/\xi)\right\} \\ &= \frac{\omega_r(f^{(n)},\xi)}{\xi^r}\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\right. \\ &\quad\left.\cdot\left[\xi^{n+r}\int_0^{\infty}(1+x)^{n+r}e^{-x}dx-\xi^{r+n}\int_0^{\infty}(1+x)^{n-k-1}e^{-x}dx\right]\right\} \end{aligned}$$

$$= \omega_r(f^{(n)},\xi)\xi^n \Bigg\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)}$$

$$\cdot \left[\int_0^\infty (1+x)^{n+r} e^{-x} dx - \int_0^\infty (1+x)^{n-k-1} e^{-x} dx \right] \Bigg\}$$

$$= \omega_r(f^{(n)},\xi)\xi^n \Bigg\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)} \left[\sum_{j=0}^{n+r} \binom{n+r}{j} \int_0^\infty x^j e^{-x} dx \right.$$

$$\left. - \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} \int_0^\infty x^j e^{-x} dx \right]$$

$$= \omega_r(f^{(n)},\xi)\xi^n \Bigg\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)}$$

$$\cdot \left[\sum_{j=0}^{n+r} \binom{n+r}{j} j! - \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} j! \right]$$

$$= \omega_r(f^{(n)},\xi)\xi^n \Bigg\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)}$$

$$\cdot \left[\sum_{j=0}^{n+r} \frac{(n+r)!}{(n+r-j)!} - \sum_{j=0}^{n-k-1} \frac{(n-k-1)!}{(n-k-1-j)!} \right]$$

$$= \omega_r(f^{(n)},\xi)\xi^n \Bigg\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)}$$

$$\cdot \left[(n+r)! \sum_{j=0}^{n+r} \frac{1}{j!} - (n-k-1)! \sum_{j=0}^{n-k-1} \frac{1}{j!} \right] \Bigg\} = \delta(\xi). \quad (10.74)$$

Use now

$$m! \sum_{j=0}^{m} \frac{1}{j!} = \lfloor em! \rfloor, \quad m \in \mathbb{N}. \tag{10.75}$$

That is proving (10.68). ■

The counterpart of the last theorem follows.

Theorem 10.24. *Assuming $f \in C^n(\mathbb{R})$, n even and $\omega_2(f^{(n)},\xi) < \infty$, $\xi > 0$, and T_n as in (10.59). Then*

$$\frac{1}{2\xi} \int_0^\infty T_n(y) e^{-y/\xi} dy \le \tau(\xi), \tag{10.76}$$

where

$$\tau(\xi) := \frac{1}{2}\omega_2(f^{(n)},\xi)\xi^n\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(k+3)}\right.$$
$$\left.\left[\lfloor e(n+2)!\rfloor - \lfloor e(n-k-1)!\rfloor\right]\right\}. \quad (10.77)$$

I.e. from (10.61) we find

$$\|K_2\|_\infty \le \tau(\xi). \quad (10.78)$$

That is as $\xi \to 0$ we obtain again $P_\xi \to I$, pointwise with rates, given that $\left\|f^{(2\rho)}\right\|_\infty < \infty, \rho = 1, ..., \frac{n}{2}$.

Proof. We observe for $\xi > 0$ that

$$\omega_2(f^{(n)},t) \le \left(1+\frac{t}{\xi}\right)^2 \omega_2(f^{(n)},\xi), \quad t > 0, \quad (10.79)$$

see [143], p. 45. And by (10.59) and (10.79), we have, $y > 0$, that

$$T_n(y) \le \frac{\omega_2(f^{(n)},\xi)}{\xi^2(n-1)!}\int_0^y (y-t)^{n-1}(t+\xi)^2 dt. \quad (10.80)$$

That is for $y > 0$ we derive

$$T_n(y) \le \frac{\omega_2(f^{(n)},\xi)}{\xi^2}\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(k+3)}\left[(\xi+y)^{n+2} - \xi^{k+3}(\xi+y)^{n-k-1}\right]\right\}. \quad (10.81)$$

Therefore

$$\frac{1}{2\xi}\int_0^\infty T_n(y)e^{-y/\xi}dy \le \frac{1}{2}\omega_2(f^{(n)},\xi)\xi^n\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(k+3)}\right.$$
$$\left.\cdot\left[(n+2)!\sum_{j=0}^{n+2}\frac{1}{j!} - (n-k-1)!\sum_{j=0}^{n-k-1}\frac{1}{j!}\right]\right\} = \tau(\xi). \quad (10.82)$$

We used in the last (10.75). That is proving (10.76). ■

11

Global Smoothness and Simultaneous Approximation by Smooth Picard Singular Operators

In this chapter we study the smooth Picard singular integral operators over the real line regarding their simultaneous global smoothness preservation property with respect to the L_p norm, $1 \leq p \leq \infty$, by involving higher order moduli of smoothness. Also we study their simultaneous approximation to the unit operator with rates involving the first modulus of continuity with respect to the uniform norm. The established Jackson type inequalities are almost sharp containing elegant constants, and they reflect the high order of differentiability of the involved function. This chapter is based on [33].

11.1 Introduction

The global smoothness preservation property of singular integrals has been studied initially in [17] and later in [67]. The rate of convergence of singular integrals has been studied initially in [231], [163], [164], later in [23] and [69], [68], and also was studied in detail in [34], [36] over the real line, just for the Picard general type integral operators case. All the above-mentioned articles along with the earlier ones [12], [13] by the author motivate this chapter.

More precisely here we study the smooth Picard singular integral operators over $\mathbb{R}$ acting on highly smooth functions. We study first their simultaneous global smoothness preservation property with respect to $\| \cdot \|_p$,

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 137–150.
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$1 \le p \le \infty$, by using higher order moduli of smoothness. Then we study their simultaneous pointwise and uniform approximation to the unit operator with rates by using the first modulus of continuity. The established estimates are almost optimal and contain nice constants. The modulus of continuity in the estimates is with respect to the higher order derivative of the involved function. The studied operators are not in general positive.

11.2 Global Smoothness Preservation Results

Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable function and consider the Lebesgue integral

$$P_\xi(f;x) := \frac{1}{2\xi} \int_{-\infty}^{\infty} f(x+t) e^{-|t|/\xi} dt, \quad \xi > 0,\ x \in \mathbb{R}. \tag{11.1}$$

We would like to mention from [67], pp. 407–412 the following result regarding global smoothness preservation properties of P_ξ, see there (16.23), (16.36), (16.48).

Theorem 11.1. *Let $h > 0$.*

i) *Suppose that $\omega_m(f,h) < \infty$ and $P_\xi(f;x) \in \mathbb{R}$, then*

$$\omega_m(P_\xi f, h) \le \omega_m(f,h). \tag{11.2}$$

Inequality (11.2) is sharp, namely it is attained by $f(x) = x^m$.

ii) *Let $f \in L_1(\mathbb{R})$ then*

$$\omega_m(P_\xi f, h)_1 \le \omega_m(f,h)_1. \tag{11.3}$$

And

iii) *let $f \in L_p(\mathbb{R})$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\omega_m(P_\xi f, h)_p \le \frac{2}{p^{1/p} q^{1/q}} \omega_m(f,h)_p. \tag{11.4}$$

Above we use for $m \in \mathbb{N}$ the mth modulus of smoothness for $1 \le p \le \infty$,

$$\omega_m(f,h)_p := \sup_{0 \le t \le h} \|\Delta_t^m f(x)\|_{p,x}, \tag{11.5}$$

where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+jt), \tag{11.6}$$

see also [143], p. 44. Denote $\omega_m(f,h)_\infty = \omega_m(f,h)$. In [34], [36] we studied extensively the convergence properties to the unit of the following smooth Picard singular integral operator $P_{r,\xi}(f;x)$ defined next.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we call

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{11.7}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$.

We consider the Lebesgue integral

$$P_{r,\xi}(f;x) := \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right) e^{-|t|/\xi}dt. \tag{11.8}$$

Operators $P_{r,\xi}$ are not positive, see [34]. We notice that $\frac{1}{2\xi}\int_{-\infty}^{\infty} e^{-|t|/\xi}dt = 1$. We observe for $j = 1, \ldots, r$ that

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+jt)e^{-|t|/\xi}dt = P_{\xi j}(f;x). \tag{11.9}$$

And furthermore it holds

$$P_{r,\xi}(f;x) = \alpha_0 f(x) + \sum_{j=1}^{r}\alpha_j P_{\xi j}(f;x). \tag{11.10}$$

Notice that $P_{1,\xi} = P_\xi$. Assuming $P_{\xi j}(f;x) \in \mathbb{R}$, $j = 1, \ldots, r$, clearly one sees that $P_{r,\xi}(f;x) \in \mathbb{R}$.

The following global smoothness result holds.

Theorem 11.2. *Let $h > 0$, $f \colon \mathbb{R} \to \mathbb{R}$.*

i) *Suppose $P_{\xi j}(f;x) \in \mathbb{R}$, all $j = 1, \ldots, r$, $\xi > 0$, $x \in \mathbb{R}$ and $\omega_m(f,h) < \infty$. Then*

$$\omega_m(P_{r,\xi}f, h) \le \left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h). \tag{11.11}$$

ii) *Suppose $f \in L_1(\mathbb{R})$, then*

$$\omega_m(P_{r,\xi}f, h)_1 \le \left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h)_1. \tag{11.12}$$

iii) *Suppose $f \in L_p(\mathbb{R})$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\omega_m(P_{r,\xi}f, h)_p \le \frac{2\left(\sum_{j=0}^{r}|\alpha_j|\right)}{p^{1/p}q^{1/q}}\omega_m(f,h)_p. \tag{11.13}$$

Proof. i) We observe that

$$\begin{aligned}\omega_m(P_{r,\xi}f,h) &\overset{(11.10)}{=} \omega_m\left(a_0 f(x)+\sum_{j=1}^{r}\alpha_j P_{\xi j}(f;x),h\right)\\ &\leq |\alpha_0|\omega_m(f,h)+\sum_{j=1}^{r}|\alpha_j|\omega_m(P_{\xi j}f,h)\end{aligned}$$

$$\overset{(11.2)}{\leq} |\alpha_0|\omega_m(f,h)+\left(\sum_{j=1}^{r}|\alpha_j|\right)\omega_m(f,h)=\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h).$$

That is proving (11.11).

ii) Next we see

$$\begin{aligned}\omega_m(P_{r,\xi}f,h)_1 &\overset{(11.10)}{=} \omega_m\left(\alpha_0 f(x)+\sum_{j=1}^{r}\alpha_j P_{\xi j}(f;x),h\right)_1\\ &\leq |\alpha_0|\omega_m(f,h)_1+\sum_{j=1}^{r}|\alpha_j|\omega_m(P_{\xi j}f,h)_1\end{aligned}$$

$$\overset{(11.3)}{\leq} |\alpha_0|\omega_m(f,h)_1+\left(\sum_{j=1}^{r}|\alpha_j|\right)\omega_m(f,h)_1=\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h)_1.$$

That is proving (11.12).

iii) Finally we get

$$\begin{aligned}\omega_m(P_{r,\xi}f,h)_p &\overset{(11.10)}{=} \omega_m\left(\alpha_0 f(x)+\sum_{j=1}^{r}\alpha_j P_{\xi j}(f;x),h\right)_p\\ &\leq |\alpha_0|\omega_m(f,h)_p+\sum_{j=1}^{r}|\alpha_j|\omega_m(P_{\xi j}f,h)_p\\ &\overset{(11.4)}{\leq} |\alpha_0|\omega_m(f,h)_p+\left(\sum_{j=1}^{r}|\alpha_j|\right)\frac{2}{p^{1/p}q^{1/q}}\omega_m(f,h)_p =: (*).\end{aligned}$$

But it holds that

$$1\leq\frac{2}{p^{1/p}q^{1/q}}$$

by Corollary 13.3, p. 190, [185]. Hence we find

$$
\begin{aligned}
(*) \quad &\leq \quad |\alpha_0| \frac{2}{p^{1/p} q^{1/q}} \omega_m(f,h)_p + \left(\sum_{j=1}^{r} |\alpha_j| \right) \frac{2}{p^{1/p} q^{1/q}} \omega_m(f,h)_p \\
&= \quad \frac{2\left(\sum\limits_{j=0}^{r} |\alpha_j|\right)}{p^{1/p} q^{1/q}} \omega_m(f,h)_p .
\end{aligned}
$$

That is establishing (11.13). ■

Next we discuss about the derivatives of $P\xi(f;x)$ and $P_{r,\xi}(f;x)$ and their impact to simultaneous global smoothness preservation and convergence of these operators.

For the following differentiation result we use Theorem 24.5, pp. 193–194 of [9] and then the proof is easy.

Theorem 11.3. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n \in \mathbb{N}$. Furthermore suppose that $f^{(j)}(t)e^{-|t|} \in L_1(\mathbb{R})$ for all $j = 0, 1, \ldots, n-1$. Assume that there exist $g_j \geq 0$, $j = 1, 2, \ldots, n$, $g_j \in L_1(\mathbb{R})$ such that for each $x \in \mathbb{R}$ we have*

$$|f^{(j)}(x+t)|e^{-|t|} \leq g_j(t), \tag{11.14}$$

for almost all $t \in \mathbb{R}$, all $j = 1, 2, \ldots, n$. Then $f^{(j)}(x+t)e^{-|t|}$ defines a Lebesgue integrable function with respect to t for each $x \in \mathbb{R}$, all $j = 1, \ldots, n$, and

$$\left(\int_{-\infty}^{\infty} f(x+t) e^{-|t|} dt \right)^{(j)} = \int_{-\infty}^{\infty} f^{(j)}(x+t) e^{-|t|} dt, \tag{11.15}$$

for all $x \in \mathbb{R}$, all $j = 1, \ldots, n$.

We apply the last theorem to our case. First comes the related differentiation result about operator $P\xi$.

Theorem 11.4. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n \in \mathbb{N}$. Furthermore suppose that $f^{(j)}(t)e^{-|t|/\xi} \in L_1(\mathbb{R})$ for all $j = 0, 1, 2, \ldots, n-1$, $\xi > 0$. Suppose that there exist $g_{j,\xi} \geq 0$, $j = 1, 2, \ldots, n$, $g_{j,\xi} \in L_1(\mathbb{R})$ such that for each $x \in \mathbb{R}$ we have*

$$|f^{(j)}(x+t)|e^{-|t|/\xi} \leq g_{j,\xi}(t), \tag{11.16}$$

for almost all $t \in \mathbb{R}$, all $j = 1, 2, \ldots, n$. Then $f^{(j)}(x+t)e^{-|t|/\xi}$ defines a Lebesgue integrable function with respect to t for each $x \in \mathbb{R}$, all $j = 1, \ldots, n$, and

$$\left(P_\xi(f;x) \right)^{(j)} = P_\xi(f^{(j)};x), \tag{11.17}$$

for all $x \in \mathbb{R}$, all $j = 1, \ldots, n$.

Proof. As in Theorem 11.3. ■.

It follows the related differentiation result about $P_{r,\xi}$ operator.

Theorem 11.5. *Let $f \in C^{n-1}(\mathbb{R})$ such that $f^{(n)}$ exists, $n \in \mathbb{N}$, $r \in \mathbb{N}$. Furthermore suppose that $f^{(i)}(t)e^{-|t|/r\xi} \in L_1(\mathbb{R})$ for all $i = 0, 1, 2, \ldots, n-1$, $\xi > 0$. Suppose that there exist $g_{i,r\xi} \geq 0$, $i = 1, 2, \ldots, n$, $g_{i,r\xi} \in L_1(\mathbb{R})$ such that for each $x \in \mathbb{R}$ we have*

$$|f^{(i)}(x+t)|e^{-|t|/r\xi} \leq g_{i,r\xi}(t), \tag{11.18}$$

for almost all $t \in \mathbb{R}$, all $i = 1, 2, \ldots, n$. Then $f^{(i)}(x+t)e^{-|t|/j\xi}$ defines a Lebesgue integrable function with respect to t for each $x \in \mathbb{R}$, all $i = 1, \ldots, n$; $j = 1, \ldots, r$, and

$$\big(P_{r,\xi}(f;x)\big)^{(i)} = P_{r,\xi}(f^{(i)}, x), \tag{11.19}$$

for all $x \in \mathbb{R}$, all $i = 1, \ldots, n$.

Proof. By Theorem 11.4 and (11.10). ■

Using Theorems 11.1 and 11.4 we obtain the following simultaneous global smoothness result.

Theorem 11.6. *Let $h > 0$ and assumptions of Theorem 11.4 valid.*

i) *Suppose that $\omega_m(f^{(i)}, h) < \infty$, all $i = 0, 1, \ldots, n$, then*

$$\omega_m\big((P_\xi f)^{(i)}, h\big) \leq \omega_m(f^{(i)}, h), \tag{11.20}$$

for all $i = 0, 1, \ldots, n$.

ii) *Let $f^{(i)} \in L_1(\mathbb{R})$, $i = 0, 1, \ldots, n$ then*

$$\omega_m\big((P_\xi f)^{(i)}, h\big)_1 \leq \omega_m(f^{(i)}, h)_1, \tag{11.21}$$

for all $i = 0, 1, \ldots, n$.

And

iii) *Let $f^{(i)} \in L_p(\mathbb{R})$, $i = 0, 1, \ldots, n$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\omega_m\big((P_\xi f)^{(i)}, h\big)_p \leq \frac{2}{p^{1/p}q^{1/q}}\omega_m(f^{(i)}, h)_p, \tag{11.22}$$

for all $i = 0, 1, \ldots, n$.

Using Theorems 11.2 and 11.5 we get the more general simultaneous global smoothness result.

Theorem 11.7. *Let $h > 0$ and assumptions of Theorem 11.5 valid.*

i) *Assume that* $\omega_m(f^{(i)}, h) < \infty$, *all* $i = 0, 1, \ldots, n$, *then*

$$\omega_m\big((P_{r,\xi} f)^{(i)}, h\big) \le \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m(f^{(i)}, h), \tag{11.23}$$

for all $i = 0, 1, \ldots, n$.

ii) *Let* $f^{(i)} \in L_1(\mathbb{R})$, $i = 0, 1, \ldots, n$ *then*

$$\omega_m\big((P_{r,\xi} f)^{(i)}, h\big)_1 \le \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m(f^{(i)}, h)_1, \tag{11.24}$$

for all $i = 0, 1, \ldots, n$.

And

iii) *Let* $f^{(i)} \in L_p(\mathbb{R})$, $i = 0, 1, \ldots, n$, $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\omega_m\big((P_{r,\xi} f)^{(i)}, h\big)_p \le \frac{2\big(\sum_{j=0}^{r} |\alpha_j|\big)}{p^{1/p} q^{1/q}} \omega_m(f^{(i)}, h)_p, \tag{11.25}$$

for all $i = 0, 1, \ldots, n$.

11.3 Convergence Results

Here let $f \in C^n(\mathbb{R})$ with $\omega_1(f^{(n)}, h) < \infty$, $h > 0$, $n \in \mathbb{N}$. Suppose $P_{\xi j}(f; x) \in \mathbb{R}$ for $j = 1, \ldots, r \in \mathbb{N}$, $\xi > 0$, all $x \in \mathbb{R}$. From (11.10) we obtain

$$P_{r,\xi}(f; x) - f(x) = \sum_{j=1}^{r} \alpha_j \big(P_{\xi j}(f; x) - f(x)\big), \tag{11.26}$$

and

$$|P_{r,\xi}(f; x) - f(x)| \le \sum_{j=1}^{r} |\alpha_j| \, |P_{\xi j}(f; x) - f(x)|. \tag{11.27}$$

Here we have

$$P_{\xi j}(f; x) = \frac{1}{2\xi j} \int_{-\infty}^{\infty} f(x + t) e^{-|t|/\xi j} dt. \tag{11.28}$$

We set

$$\Delta_{\xi j}(f; x) := P_{\xi j}(f; x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x) (\xi j)^{2m}, \tag{11.29}$$

$j = 1, \ldots, r$, where $\lfloor \cdot \rfloor$ is the integral part of the number, $x \in \mathbb{R}$.

In (11.29) the sum collapses when $n = 1$. Clearly we have

$$\Delta_\xi(f;x) = P_\xi(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x)\xi^{2m}, \quad x \in \mathbb{R}. \tag{11.30}$$

We call also

$$\delta_{2m} := \sum_{j=1}^{r} \alpha_j j^{2m}, \tag{11.31}$$

and

$$E_{r,\xi}(f;x) := P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x)\delta_{2m}\xi^{2m}, \quad x \in \mathbb{R}. \tag{11.32}$$

We observe that

$$E_{r,\xi}(f;x) = \sum_{j=1}^{r} \alpha_j \Delta_{\xi j}(f;x) \tag{11.33}$$

and

$$|E_{r,\xi}(f;x)| \le \sum_{j=1}^{r} |\alpha_j|\, |\Delta_{\xi j}(f;x)|, \quad x \in \mathbb{R}. \tag{11.34}$$

We study here the convergence of operators $P_{r,\xi}$ to the unit operator I with rates, $r \in \mathbb{N}$. We give first

Theorem 11.8. *It holds*

$$|\Delta_{\xi j}(f;x)| \le (\xi j)^n \left[j + \frac{1}{2} + \frac{1}{8j} \right] \omega_1(f^{(n)}, \xi), \quad j = 1, \ldots, r, \ \xi > 0 \tag{11.35}$$

and

$$|\Delta_\xi(f;x)| \le \frac{13}{8} \xi^n \omega_1(f^{(n)}, \xi). \tag{11.36}$$

That is we have

$$\|\Delta_{\xi j}(f)\|_\infty \le (\xi j)^n \left[j + \frac{1}{2} + \frac{1}{8j} \right] \omega_1(f^{(n)}, \xi), \tag{11.37}$$

and

$$\|\Delta_\xi(f)\|_\infty \le \frac{13}{8} \xi^n \omega_1(f^{(n)}, \xi). \tag{11.38}$$

Proof. Here let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$. By Taylor's formula, see Lemma 2, (2), p. 2 of [24] we have

$$f(x+t) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} t^k + \mathcal{R}_n(f; x, x+t), \tag{11.39}$$

where

$$\mathcal{R}_n(f;x,x+t) := \frac{1}{(n-1)!}\int_x^{x+t}(f^{(n)}(s)-f^{(n)}(x))(x+t-s)^{n-1}ds, \quad (11.40)$$

for all $x, t \in \mathbb{R}$.

Applying Theorem 6, (14), p. 4 of [24] we get

$$|\mathcal{R}_n(f;x,x+t)| \le \omega_1(f^{(n)},\xi)\left[\frac{|t|^{n+1}}{(n+1)!\xi}+\frac{|t|^n}{2n!}+\frac{\xi|t|^{n-1}}{8(n-1)!}\right], \quad (11.41)$$

all $t \in \mathbb{R}$, $\xi > 0$, $j = 1, \ldots, r$. From (11.39) we find

$$f(x+t) - \sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}t^k = \mathcal{R}_n(f;x,x+t) \quad (11.42)$$

and

$$\begin{aligned}\frac{1}{2\xi j}\int_{-\infty}^{\infty} f(x+t)e^{-|t|/\xi j}dt - \sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}\frac{1}{2\xi j}\int_{-\infty}^{\infty} t^k e^{-|t|/\xi j}dt \\ = \frac{1}{2\xi j}\int_{-\infty}^{\infty}\mathcal{R}_n(f;x,x+t)e^{-|t|/\xi j}dt. \quad (11.43)\end{aligned}$$

That is

$$\begin{aligned}P_{\xi j}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2\rfloor} f^{(2m)}(x)(\xi j)^{2m} \\ = \frac{1}{2\xi j}\int_{-\infty}^{\infty}\mathcal{R}_n(f;x,x+t)e^{-|t|/\xi j}dt. \quad (11.44)\end{aligned}$$

I.e. by (11.29) we obtain

$$\Delta_{\xi j}(f;x) = \frac{1}{2\xi j}\int_{-\infty}^{\infty}\mathcal{R}_n(f;x,x+t)e^{-|t|/\xi j}dt, \quad (11.45)$$

all $x \in \mathbb{R}$.

Furthermore we have

$$\begin{aligned}|\Delta_{\xi j}(f;x)| &\le \frac{1}{2\xi j}\int_{-\infty}^{\infty}|\mathcal{R}_n(f;x,x+t)|e^{-|t|/\xi j}dt \\ &\overset{(11.41)}{\le} \frac{\omega_1(f^{(n)},\xi)}{2\xi j}\int_{-\infty}^{\infty}\left[\frac{|t|^{n+1}}{(n+1)!\xi}+\frac{|t|^n}{2n!}+\frac{\xi|t|^{n-1}}{8(n-1)!}\right]e^{-|t|/\xi j}dt \\ &= \omega_1(f^{(n)},\xi)(\xi j)^n\left[j+\frac{1}{2}+\frac{1}{8j}\right]. \quad (11.46)\end{aligned}$$

Thus we have obtained (11.35). ■

The more general result follows.

Theorem 11.9. *It holds*

$$|E_{r,\xi}(f;x)| \le \left(\sum_{j=1}^{r} \binom{r}{j} \left[j + \frac{1}{2} + \frac{1}{8j}\right]\right) \xi^n \omega_1(f^{(n)}, \xi), \tag{11.47}$$

all $x \in \mathbb{R}$, $\xi > 0$, *and furthermore*

$$\|E_{r,\xi} f\|_\infty \le \left(\sum_{j=1}^{r} \binom{r}{j} \left[j + \frac{1}{2} + \frac{1}{8j}\right]\right) \xi^n \omega_1(f^{(n)}, \xi), \quad \xi > 0, \ n \in \mathbb{N}. \tag{11.48}$$

Proof. From (11.7), (11.34) and (11.35) we get

$$\begin{aligned} |E_{r,\xi}(f;x)| &\le \sum_{j=1}^{r} \binom{r}{j} \xi^n \left[j + \frac{1}{2} + \frac{1}{8j}\right] \omega_1(f^{(n)}, \xi) \\ &= \left(\sum_{j=1}^{r} \binom{r}{j} \left[j + \frac{1}{2} + \frac{1}{8j}\right]\right) \xi^n \omega_1(f^{(n)}, \xi). \end{aligned} \tag{11.49}$$

That is proving (11.47). ∎

Some alternative basic results follow.

Proposition 11.10. *All assumptions as above. Then*

$$|P_{\xi j}(f;x) - f(x)| \le \sum_{m=1}^{\lfloor n/2 \rfloor} |f^{(2m)}(x)| (\xi j)^{2m} + (\xi j)^n \left[j + \frac{1}{2} + \frac{1}{8j}\right] \omega_1(f^{(n)}, \xi), \tag{11.50}$$

and

$$|P_\xi(f;x) - f(x)| \le \sum_{m=1}^{\lfloor n/2 \rfloor} |f^{(2m)}(x)| \xi^{2m} + \frac{13}{8} \xi^n \omega_1(f^{(n)}, \xi), \quad x \in \mathbb{R}, \ \xi > 0, \ n \in \mathbb{N}. \tag{11.51}$$

Assuming that $\|f^{(2m)}\|_\infty < \infty$, $m = 1, \ldots, \lfloor n/2 \rfloor$ *we derive*

$$\|P_{\xi j} f - f\|_\infty \le \sum_{m=1}^{\lfloor n/2 \rfloor} \|f^{(2m)}\|_\infty (\xi j)^{2m} + (\xi j)^n \left[j + \frac{1}{2} + \frac{1}{8j}\right] \omega_1(f^{(n)}, \xi), \tag{11.52}$$

and

$$\|P_\xi f - f\|_\infty \le \sum_{m=1}^{\lfloor n/2 \rfloor} \|f^{(2m)}\|_\infty \xi^{2m} + \frac{13}{8} \xi^n \omega_1(f^{(n)}, \xi), \quad \xi > 0, \ n \in \mathbb{N}. \tag{11.53}$$

Proof. By (11.29) and (11.35), etc. ∎

We give

Corollary 11.11 ($n = 2$ case). *It holds*

$$|P_{\xi j}(f;x) - f(x)| \leq (\xi j)^2 \left[|f''(x)| + \left[j + \frac{1}{2} + \frac{1}{8j}\right]\omega_1(f'',\xi)\right], \quad (11.54)$$

and

$$|P_\xi(f;x) - f(x)| \leq \xi^2 \left(|f''(x)| + \frac{13}{8}\omega_1(f'',\xi)\right), \quad x \in \mathbb{R},\ \xi > 0. \quad (11.55)$$

Furthermore when $\|f''\|_\infty < \infty$ *we get*

$$\|P_{\xi j} f - f\|_\infty \leq (\xi j)^2 \left[\|f''\|_\infty + \left[j + \frac{1}{2} + \frac{1}{8j}\right]\omega_1(f'',\xi)\right], \quad (11.56)$$

and

$$\|P_\xi f - f\|_\infty \leq \xi^2 \left(\|f''\|_\infty + \frac{13}{8}\omega_1(f'',\xi)\right), \quad \xi > 0. \quad (11.57)$$

Proof. By Proposition 11.10. ∎

It follows

Corollary 11.12 ($n = 1$ case). *It holds*

$$|P_{\xi j}(f;x) - f(x)| \leq \xi \left[j^2 + \frac{j}{2} + \frac{1}{8}\right] \omega_1(f',\xi), \quad (11.58)$$

and

$$|P_\xi(f;x) - f(x)| \leq \frac{13}{8}\xi\omega_1(f',\xi), \quad x \in \mathbb{R},\ \xi > 0. \quad (11.59)$$

Furthermore we have

$$\|P_{\xi j} f - f\|_\infty \leq \xi \left[j^2 + \frac{j}{2} + \frac{1}{8}\right] \omega_1(f',\xi), \quad (11.60)$$

and

$$\|P_\xi f - f\|_\infty \leq \frac{13}{8}\xi\omega_1(f',\xi), \quad \xi > 0. \quad (11.61)$$

Proof. By proof of Theorem 11.8 for $n = 1$, see also (11.29). ∎

More generally we have

Proposition 11.13. *All assumptions as above. Then*

$$|P_{r,\xi}(f;x)-f(x)| \le \sum_{j=1}^{r}\sum_{m=1}^{\lfloor n/2\rfloor} |f^{(2m)}(x)|\binom{r}{j}\frac{\xi^{2m}}{j^{n-2m}} + \left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1(f^{(n)},\xi), \tag{11.62}$$

all $x\in\mathbb{R}$, $\xi>0$, $n\in\mathbb{N}$. Furthermore by assuming that $\|f^{(2m)}\|_\infty<\infty$, for $m=1,\ldots,\lfloor n/2\rfloor$ we derive

$$\|P_{r,\xi}f-f\|_\infty \le \sum_{j=1}^{r}\sum_{m=1}^{\lfloor n/2\rfloor} \|f^{(2m)}\|_\infty\binom{r}{j}\frac{\xi^{2m}}{j^{n-2m}} + \left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1(f^{(n)},\xi),\quad \xi>0,\ n\in\mathbb{N}. \tag{11.63}$$

Proof. From (11.27) and (11.50) we obtain

$$|P_{r,\xi}(f;x)-f(x)|$$

$$\le \sum_{j=1}^{r}\binom{r}{j}\left[\sum_{m=1}^{\lfloor n/2\rfloor}|f^{(2m)}(x)|\xi^{2m}j^{2m-n}+\xi^n\left[j+\frac{1}{2}+\frac{1}{8j}\right]\omega_1(f^{(n)},\xi)\right]$$

$$= \sum_{j=1}^{r}\sum_{m=1}^{\lfloor n/2\rfloor}|f^{(2m)}(x)|\binom{r}{j}\frac{\xi^{2m}}{j^{n-2m}} + \left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1(f^{(n)},\xi).$$ ∎

We have

Corollary 11.14 ($n=2$ case). *It holds*

$$|P_{r,\xi}(f;x)-f(x)|\le\xi^2\left\{(2^r-1)|f''(x)|+\left(\sum_{j=1}^{r}\binom{r}{j}\left(j+\frac{1}{2}+\frac{1}{8j}\right)\right)\omega_1(f'',\xi)\right\}, \tag{11.64}$$

all $x\in\mathbb{R}$, $\xi>0$. Furthermore by assuming that $\|f''\|_\infty<\infty$ we derive

$$\|P_{r,\xi}f-f\|_\infty \le \xi^2\left\{(2^r-1)\|f''\|_\infty + \left(\sum_{j=1}^{r}\binom{r}{j}\left(j+\frac{1}{2}+\frac{1}{8j}\right)\right)\omega_1(f'',\xi)\right\},\quad \xi>0. \tag{11.65}$$

Proof. By Proposition 11.13. ■

We also give

Corollary 11.15 ($n = 1$ case). *It holds*

$$|P_{r,\xi}(f;x) - f(x)| \le \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \xi \omega_1(f', \xi), \quad (11.66)$$

all $x \in \mathbb{R}$, $\xi > 0$. *And furthermore we get*

$$\|P_{r,\xi} f - f\|_\infty \le \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \xi \omega_1(f', \xi), \quad \xi > 0. \quad (11.67)$$

Proof. By use of (11.7), (11.27) and (11.58). ■

Next we present simultaneous approximation results of $P\xi$ to I with rates.

Theorem 11.16. *Let* $f \in C^{n+k}(\mathbb{R})$, $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$ *and* $\omega_1(f^{(n+i)}, h) < \infty$, $h > 0$, *for* $i = 0, 1, \ldots, k$. *We consider the assumptions of Theorem 11.4 as valid for* $n = k$ *there. Then*

1)

$$|(\Delta_\xi(f;x))^{(i)}| \le \frac{13}{8} \xi^n \omega_1(f^{(n+i)}, \xi), \quad (11.68)$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$, $n \in \mathbb{N}$,

2)

$$|(P_\xi(f;x))^{(i)} - f^{(i)}(x)| \le \sum_{m=1}^{\lfloor n/2 \rfloor} |f^{(2m+i)}(x)| \xi^{2m} + \frac{13}{8} \xi^n \omega_1(f^{(n+i)}, \xi), \quad (11.69)$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$, $n \in \mathbb{N}$,

3) $n = 2$ *case*,

$$|(P_\xi(f;x))^{(i)} - f^{(i)}(x)| \le \xi^2 \left(|f^{(2+i)}(x)| + \frac{13}{8} \omega_1(f^{(2+i)}, \xi) \right), \quad (11.70)$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$, *and*

4) $n = 1$ *case*,

$$|(P_\xi(f;x))^{(i)} - f^{(i)}(x)| \le \frac{13}{8} \xi \omega_1(f^{(1+i)}, \xi), \quad (11.71)$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$.

Proof. By using Theorems 11.4, 11.8 (11.36), Proposition 11.10 (11.51), Corollary 11.11 (11.55) and Corollary 11.12 (11.59). ■

We finish with operator $P_{r,\xi}$ simultaneous approximation results to I with rates.

Theorem 11.17. *Let $f \in C^{n+k}(\mathbb{R})$, $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and $\omega_1(f^{(n+i)}, h) < \infty$, $h > 0$, for $i = 0, 1, \ldots, k$. We consider the assumptions of Theorem 11.5 as valid for $n = k$ there. Then*

1)

$$|(E_{r,\xi}(f;x))^{(i)}| \leq \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \xi^n \omega_1(f^{(n+i)}, \xi), \tag{11.72}$$

for all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$, $n \in \mathbb{N}$,

2)

$$\begin{aligned} |(P_{r,\xi}(f;x))^{(i)} - f^{(i)}(x)| \quad \leq \quad & \sum_{j=1}^{r} \sum_{m=1}^{\lfloor n/2 \rfloor} |f^{(2m+i)}(x)| \binom{r}{j} \frac{\xi^{2m}}{j^{n-2m}} \\ & + \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \xi^n \omega_1(f^{(n+i)}, \xi), \end{aligned} \tag{11.73}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$, $n \in \mathbb{N}$,

3) *$n = 2$ case,*

$$\begin{aligned} |(P_{r,\xi}(f;x))^{(i)} - f^{(i)}(x)| \quad \leq \quad & \xi^2 \Bigg\{ (2^r - 1)|f^{(2+i)}(x)| \\ & + \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \omega_1(f^{(2+i)}, \xi) \Bigg\}, \end{aligned} \tag{11.74}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$, and

4) *$n = 1$ case,*

$$|(P_{r,\xi}(f;x))^{(i)} - f^{(i)}(x)| \leq \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \xi \omega_1(f^{(1+i)}, \xi), \tag{11.75}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \ldots, k$.

Proof. By using Theorems 11.5, 11.9 (11.47), Proposition 11.13 (11.62), Corollary 11.14 (11.64), and Corollary 11.15 (11.66). ■

12

Quantitative L_p Approximation by Smooth Picard Singular Operators

In this chapter we continue with the study of smooth Picard singular integral operators on the line regarding their convergence to the unit operator with rates in the L_p norm, $p \geq 1$. The related established inequalities involve the higher order L_p modulus of smoothness of the engaged function or its higher order derivative. This chapter relies on [36].

12.1 Introduction

The rate of convergence of singular integrals has been studied in [163], [164], [231], [69], [68], [16], [23], [34] and these articles motivate this chapter. Here we study the L_p, $p \geq 1$, convergence of smooth Picard singular integral operators over $\mathbb{R}$ to the unit operator with rates over smooth functions with higher order derivative in $L_p(\mathbb{R})$. These operators were introduced and studied in [34] with respect to $\|\cdot\|_\infty$. We establish related Jackson type inequalities involving the higher L_p modulus of smoothness of the engaged function or its higher order derivative. The discussed operators are not in general positive. Other motivation comes from [12], [13].

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 151–167.
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12.2 Results

In the next we deal with the *smooth Picard singular integral operators* $P_{r,\xi}(f;x)$ defined as follows.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \displaystyle\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{12.1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$.

Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in L_p(\mathbb{R})$, $1 \le p < \infty$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$P_{r,\xi}(f;x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-|t|/\xi} dt. \tag{12.2}$$

$P_{r,\xi}$ operators are not positive operators, see [34].

We notice by $\frac{1}{2\xi}\int_{-\infty}^{\infty} e^{-|t|/\xi} dt = 1$, that $P_{r,\xi}(c, x) = c$, c constant, and

$$P_{r,\xi}(f;x) - f(x) = \frac{1}{2\xi} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-|t|/\xi} dt \right). \tag{12.3}$$

We use also that

$$\int_{-\infty}^{\infty} t^k e^{-|t|/\xi} dt = \begin{cases} 0, & k \text{ odd}, \\ 2k!\xi^{k+1}, & k \text{ even}. \end{cases} \tag{12.4}$$

We need the rth L_p-modulus of smoothness

$$\omega_r(f^{(n)}, h)_p := \sup_{|t| \le h} \|\Delta_t^r f^{(n)}(x)\|_{p,x}, \quad h > 0, \tag{12.5}$$

where

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x + jt), \tag{12.6}$$

see [143], p. 44. Here we have that $\omega_r(f^{(n)}, h)_p < \infty$, $h > 0$.

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}, \tag{12.7}$$

and denote by $\lfloor \cdot \rfloor$ the integral part. Call

$$\tau(w,x) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x+jw) - \delta_n f^{(n)}(x). \tag{12.8}$$

Notice also that

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{12.9}$$

According to [16], p. 306, [12], we get

$$\tau(w,x) = \Delta_w^r f^{(n)}(x). \tag{12.10}$$

Thus

$$\|\tau(w,x)\|_{p,x} \le \omega_r(f^{(n)}, |w|)_p, \quad w \in \mathbb{R}. \tag{12.11}$$

Using Taylor's formula one has

$$\sum_{j=0}^{r} \alpha_j [f(x+jt) - f(x)] = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k t^k + \mathcal{R}_n(0,t,x), \tag{12.12}$$

where

$$\mathcal{R}_n(0,t,x) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!} \tau(w,x) dw, \quad n \in \mathbb{N}. \tag{12.13}$$

Using the above terminology we derive

$$\Delta(x) := P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x) \delta_{2m} \xi^{2m} = \mathcal{R}_n^*(x), \tag{12.14}$$

where

$$\mathcal{R}_n^*(x) := \frac{1}{2\xi} \int_{-\infty}^{\infty} \mathcal{R}_n(0,t,x) e^{-|t|/\xi} dt, \quad n \in \mathbb{N}. \tag{12.15}$$

In $\Delta(x)$, see (12.14), the sum collapses when $n = 1$.

We present the first result.

Theorem 12.1. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ and the rest as above. Then*

$$\|\Delta(x)\|_p \le \frac{2^{1/q} \tau^{1/p} \xi^n}{(rp+1)^{1/p} (q^2(n-1)+q)^{1/q} (n-1)!} w_r(f^{(n)}, \xi)_p, \tag{12.16}$$

where

$$\tau := \left[\int_0^{\infty} (1+u)^{rp+1} u^{np-1} e^{-(p/2)u} du - \left(\frac{2}{p}\right)^{np} \Gamma(np) \right] < \infty. \tag{12.17}$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_p \to 0.$

If additionally, $f^{(2m)} \in L_p(\mathbb{R})$, $m = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor$, then $\|P_{r,\xi}(f) - f\|_p \to 0$, as $\xi \to 0$.

Proof. We observe that

$$\begin{aligned} |\Delta(x)|^p &= \frac{1}{(2\xi)^p}\left|\int_{-\infty}^{\infty} \mathcal{R}_n(0,t,x)e^{-|t|/\xi}dt\right|^p \\ &\leq \frac{1}{(2\xi)^p}\left(\int_{-\infty}^{\infty} |\mathcal{R}_n(0,t,x)|e^{-|t|/\xi}dt\right)^p \qquad (12.18) \\ &\leq \frac{1}{(2\xi)^p}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w,x)|dw\right)e^{-|t|/\xi}dt\right)^p. \end{aligned}$$

Hence we have

$$I := \int_{-\infty}^{\infty} |\Delta(x)|^p dx \leq \frac{1}{(2\xi)^p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \gamma(t,x)e^{-|t|/\xi}dt\right)^p dx\right), \qquad (12.19)$$

where

$$\gamma(t,x) := \int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w,x)|dw \geq 0. \qquad (12.20)$$

Therefore

$$\begin{aligned} \text{R.H.S.}(12.19) &= \frac{1}{(2\xi)^p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \gamma(t,x)e^{-|t|/2\xi}e^{-|t|/2\xi}dt\right)^p dx\right) \\ &\leq \frac{1}{(2\xi)^p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \gamma^p(t,x)e^{-|pt|/2\xi}dt\right)\left(\int_{-\infty}^{\infty} e^{-|qt|/2\xi}dt\right)^{p/q} dx\right) \\ &= \frac{1}{(2\xi)^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \gamma^p(t,x)e^{-|pt|/2\xi}dt\right)dx\right) \\ &= \frac{2^{p-2}\xi^{-1}}{q^{p-1}}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \gamma^p(t,x)e^{-|pt|/2\xi}dt\right)dx\right). \end{aligned}$$

I.e.

$$\text{R.H.S.}(12.19) \leq \frac{2^{p-2}\xi^{-1}}{q^{p-1}}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \gamma^p(t,x)e^{-|pt|/2\xi}dt\right)dx\right). \qquad (12.21)$$

But we need to treat

$$\begin{aligned} \gamma(t,x) &\leq \left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw\right)^{1/p}\left(\int_0^{|t|}\left(\frac{(|t|-w)^{n-1}}{(n-1)!}\right)^q dw\right)^{1/q} \\ &= \frac{\left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw\right)^{1/p}}{(n-1)!}\frac{|t|^{(n-1+1/q)}}{(q(n-1)+1)^{1/q}}. \end{aligned}$$

I.e.

$$\gamma^p(t,x) \le \frac{\left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw\right)}{((n-1)!)^p} \frac{|t|^{np-1}}{(q(n-1)+1)^{p/q}}. \tag{12.22}$$

Consequently we have

$$\begin{aligned} \text{R.H.S.(12.21)} \;\le\; & \frac{2^{p-2}\xi^{-1}}{q^{p-1}}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{\left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw\right)|t|^{np-1}}{((n-1)!)^p(q(n-1)+1)^{p/q}} e^{-|pt|/2\xi} dt\right) dx\right) \\ =: \;& (*), \end{aligned}$$

(calling

$$c_1 := \frac{2^{p-2}}{\xi q^{p-1}((n-1)!)^p(q(n-1)+1)^{p/q}}) \tag{12.23}$$

and

$$\begin{aligned} (*) &= c_1\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} |\tau(sign(t)w,x)|^p dw\right)|t|^{np-1}e^{-|pt|/2\xi}\right)dx\right)dt\right) \\ &= c_1\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} |\Delta^r_{sign(t)w} f^{(n)}(x)|^p dw\right)|t|^{np-1}e^{-|pt|/2\xi}\right)dx\right)dt\right) \\ &= c_1\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} |\Delta^r_{sign(t)w} f^{(n)}(x)|^p dw\right)dx\right)|t|^{np-1}e^{-|pt|/2\xi}\right)dt\right) \\ &= c_1\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty} |\Delta^r_{sign(t)w} f^{(n)}(x)|^p dx\right)dw\right)|t|^{np-1}e^{-|pt|/2\xi}\right)dt\right) \\ &\le c_1\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} \omega_r(f^{(n)},w)_p^p dw\right)|t|^{np-1}e^{-|pt|/2\xi}\right)dt\right). \end{aligned} \tag{12.24}$$

So far we have proved

$$I \le c_1\left(\int_{-\infty}^{\infty}\left(\left(\int_0^{|t|} \omega_r(f^{(n)},w)_p^p dw\right)|t|^{np-1}e^{-|pt|/2\xi}\right)dt\right). \tag{12.25}$$

By [143], p. 45 we have

$$\begin{aligned} (\text{R.H.S.(12.25)}) \quad \le \quad & c_1(\omega_r(f^{(n)},\xi)_p)^p\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{w}{\xi}\right)^{rp} dw\right)\right. \\ & \left.\cdot\, |t|^{np-1}e^{-|pt|/2\xi}\right)dt\right) =: (**). \end{aligned} \tag{12.26}$$

But we see that

$$(**) = \left(\frac{\xi c_1}{rp+1}\right)\left(\omega_r(f^{(n)},\xi)_p\right)^p \mathcal{J}, \tag{12.27}$$

where

$$\mathcal{J} := \int_{-\infty}^{\infty} \left(\left(\left(1+\frac{|t|}{\xi}\right)^{rp+1} - 1 \right) |t|^{np-1} e^{-|pt|/2\xi} \right) dt$$
$$= 2\int_{0}^{\infty} \left(\left(1+\frac{t}{\xi}\right)^{rp+1} - 1 \right) t^{np-1} e^{-pt/2\xi} dt. \quad (12.28)$$

Here we find

$$\begin{aligned} \mathcal{J} &= 2\xi^{np} \int_0^{\infty} ((1+u)^{rp+1} - 1) u^{np-1} e^{-(p/2)u} du \\ &= 2\xi^{np} \left[\int_0^{\infty} (1+u)^{rp+1} u^{np-1} e^{-(p/2)u} du - \int_0^{\infty} u^{np-1} e^{-(p/2)u} du \right] \\ &= 2\xi^{np} \left[\int_0^{\infty} (1+u)^{rp+1} u^{np-1} e^{-(p/2)u} du - \left(\frac{2}{p}\right)^{np} \Gamma(np) \right]. \quad (12.29) \end{aligned}$$

Thus by (12.17) and (12.29) we obtain

$$\mathcal{J} = 2\xi^{np}\tau. \quad (12.30)$$

Using (12.27) and (12.30) we find

$$\begin{aligned} (**) &= \left(\frac{\xi c_1}{rp+1}\right) \left(\omega_r(f^{(n)}, \xi)_p\right)^p 2\xi^{np}\tau \quad (12.31) \\ &= \frac{2^{p/q}\tau\xi^{np}}{(rp+1)(q^2(n-1)+q)^{p/q}((n-1)!)^p} \left(\omega_r(f^{(n)}, \xi)_p\right)^p. \end{aligned}$$

I.e. we have established that

$$I \le \frac{2^{p/q}\tau\xi^{np}\omega_r(f^{(n)}, \xi)_p^p}{(rp+1)(q^2(n-1)+q)^{p/q}((n-1)!)^p}. \quad (12.32)$$

That is finishing the proof of the theorem. .

The counterpart of Theorem 12.1 follows, case of $p = 1$.

Theorem 12.2. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$. Then*

$$\|\Delta(x)\|_1 \le r! \left(\sum_{k=1}^{r+1} \left(\frac{\left(\prod_{j=1}^{k}(n-1+j)\right)}{k!(r+1-k)!} \right) \right) \xi^n \omega_r(f^{(n)}, \xi)_1. \quad (12.33)$$

Hence as $\xi \to 0$ we obtain $\|\Delta(x)\|_1 \to 0$.

If additionally, $f^{(2m)} \in L_1(\mathbb{R})$, $m = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor$, then $\|P_{r,\xi}(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. It follows

$$
\begin{aligned}
|\Delta(x)| &= \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\mathcal{R}_n(0,t,x)e^{-|t|/\xi}dt\right| \\
&\leq \frac{1}{2\xi}\int_{-\infty}^{\infty}|\mathcal{R}_n(0,t,x)|e^{-|t|/\xi}dt \qquad (12.34)\\
&\leq \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w,x)|dw\right)e^{-|t|/\xi}dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\Delta(x)\|_1 &= \int_{-\infty}^{\infty}|\Delta(x)|dx \leq \frac{1}{2\xi}\int_{-\infty}^{\infty}\Bigg(\int_{-\infty}^{\infty}\Bigg(\int_0^{|t|} \qquad (12.35)\\
&\qquad \frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w,x)|dw\Bigg)e^{-|t|/\xi}dt\Bigg)dx =: (*).
\end{aligned}
$$

But we see that

$$
\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(sign(t)w,x)|dw \leq \frac{|t|^{n-1}}{(n-1)!}\int_0^{|t|}|\tau(sign(t)w,x)|dw. \quad (12.36)
$$

Therefore it holds

$$
\begin{aligned}
(*) &\leq \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\frac{|t|^{n-1}}{(n-1)!}\int_0^{|t|}|\tau(sign(t)w,x)|dw\right)e^{-|t|/\xi}dt\right)dx \\
&= \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\frac{|t|^{n-1}}{(n-1)!}\int_0^{|t|}|\tau(sign(t)w,x)|dw\right)e^{-|t|/\xi}dx\right)dt \\
&= \frac{1}{2\xi(n-1)!}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}|\tau(sign(t)w,x)|dw\right)dx\right)|t|^{n-1}e^{-|t|/\xi}\Bigg)dt \\
&= \frac{1}{2\xi(n-1)!}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty}|\tau(sign(t)w,x)|dx\right)dw\right)|t|^{n-1}e^{-|t|/\xi}\Bigg)dt\right) \\
&\leq \frac{1}{2\xi(n-1)!}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_r(f^{(n)},w)_1dw\right)|t|^{n-1}e^{-|t|/\xi}\Bigg)dt\right). \qquad (12.37)
\end{aligned}
$$

That is, we get

$$
\|\Delta(x)\|_1 \leq \frac{1}{2\xi(n-1)!}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_r(f^{(n)},w)_1dw\right)|t|^{n-1}e^{-|t|/\xi}\Bigg)dt\right). \quad (12.38)
$$

Consequently we have

$$\begin{aligned}\|\Delta(x)\|_1 &\leq \frac{1}{2\xi(n-1)!}\omega_r(f^{(n)},\xi)_1\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{w}{\xi}\right)^r dw\right)|t|^{n-1}e^{-|t|/\xi}\right)dt\right) \\ &= \frac{\omega_r(f^{(n)},\xi)_1}{2(n-1)!(r+1)}\left(\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}e^{-|t|/\xi}dt\right) \\ &= \frac{\omega_r(f^{(n)},\xi)_1}{(n-1)!(r+1)}\left(\int_0^{\infty}\left(\left(1+\frac{t}{\xi}\right)^{r+1}-1\right)t^{n-1}e^{-t/\xi}dt\right) \\ &= \frac{\omega_r(f^{(n)},\xi)_1\xi^n}{(n-1)!(r+1)}\left(\int_0^{\infty}\left((1+t)^{r+1}-1\right)t^{n-1}e^{-t}dt\right). \qquad (12.39)\end{aligned}$$

We have gotten so far

$$\|\Delta(x)\|_1 \leq \frac{\omega_r(f^{(n)},\xi)_1\xi^n\cdot\lambda}{(n-1)!(r+1)}, \qquad (12.40)$$

where

$$\lambda := \int_0^{\infty}\left((1+t)^{r+1}-1\right)t^{n-1}e^{-t}dt. \qquad (12.41)$$

One easily finds that

$$\lambda = \sum_{k=0}^{r+1}\binom{r+1}{k}(n+k-1)!-(n-1)!. \qquad (12.42)$$

But then one sees that

$$\frac{\lambda}{(n-1)!} = \sum_{k=1}^{r+1}\binom{r+1}{k}\frac{(n+k-1)!}{(n-1)!}. \qquad (12.43)$$

We have proved (12.33). .

The case $n=0$ is met next.

Proposition 12.3. *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and the rest as above. Then*

$$\|P_{r,\xi}(f)-f\|_p \leq \left(\frac{2}{q}\right)^{1/q}\theta^{1/p}\omega_r(f,\xi)_p, \qquad (12.44)$$

where

$$\theta := \int_0^{\infty}(1+x)^{rp}e^{-(p/2)x}dx < \infty. \qquad (12.45)$$

Hence as $\xi\to 0$ we obtain $P_{r\xi}\to$ unit operator I in the L_p norm, $p>1$.

Proof. With some work we notice that, see also [34],

$$P_{r,\xi}(f;x)-f(x) = \frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\left((\Delta_t^r f)(x)\right)e^{-|t|/\xi}dt\right). \qquad (12.46)$$

And then

$$|P_{r,\xi}(f;x)-f(x)| \le \frac{1}{2\xi}\int_{-\infty}^{\infty}|\Delta_t^r f(x)|e^{-|t|/\xi}dt. \tag{12.47}$$

We next estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty}|P_{r,\xi}(f;x)-f(x)|^p dx\\
&\le \frac{1}{2^p\xi^p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\Delta_t^r f(x)|e^{-|t|/\xi}dt\right)^p dx\right)\\
&= \frac{1}{2^p\xi^p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\Delta_t^r f(x)|e^{-|t|/2\xi}e^{-|t|/2\xi}dt\right)^p dx\right)\\
&\le \frac{1}{2^p\xi^p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\Delta_t^r f(x)|^p e^{-|pt|/2\xi}dt\right)\left(\int_{-\infty}^{\infty}e^{-|qt|/2\xi}dt\right)^{p/q}dx\right)\\
&= \frac{1}{2^p\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\Delta_t^r f(x)|^p e^{-|pt|/2\xi}dt\right)dx\right)\\
&= \frac{1}{2^p\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\Delta_t^r f(x)|^p e^{-|pt|/2\xi}dx\right)dt\right)\\
&= \frac{1}{2^p\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\Delta_t^r f(x)|^p dx\right)e^{-|pt|/2\xi}dt\right)\\
&\le \frac{1}{2^p\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_{-\infty}^{\infty}\omega_r(f,|t|)_p^p e^{-|pt|/2\xi}dt\right)\\
&= \frac{1}{2^{p-1}\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_{0}^{\infty}\omega_r(f,t)_p^p e^{-pt/2\xi}dt\right)\\
&\le \frac{1}{2^{p-1}\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\omega_r(f,\xi)_p^p\left(\int_{0}^{\infty}\left(1+\frac{t}{\xi}\right)^{rp}e^{-pt/2\xi}dt\right)\\
&= \left(\frac{2}{q}\right)^{p/q}\omega_r(f,\xi)_p^p\left(\int_0^{\infty}(1+x)^{rp}e^{-(p/2)x}dx\right).
\end{aligned} \tag{12.48}$$

Clearly we have established (12.44). .

We also give

Proposition 12.4. *It holds*

$$\|P_{r,\xi}f-f\|_1 \le \lfloor er!\rfloor \omega_r(f,\xi)_1. \tag{12.49}$$

Hence as $\xi \to 0$ we get $P_{r,\xi} \to I$ in the L_1 norm.

Proof. We do have again

$$|P_{r,\xi}(f;x)-f(x)| \le \frac{1}{2\xi}\int_{-\infty}^{\infty}|\Delta_t^r f(x)|e^{-|t|/\xi}dt.$$

We estimate

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |P_{r,\xi}(f;x) - f(x)|dx \\
&\le \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |\Delta_t^r f(x)| e^{-|t|/\xi} dt\right) dx \\
&= \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |\Delta_t^r f(x)| dx\right) e^{-|t|/\xi} dt \\
&\le \frac{1}{2\xi}\int_{-\infty}^{\infty} \omega_r(f,|t|)_1 e^{-|t|/\xi} dt \\
&= \frac{1}{\xi}\int_{0}^{\infty} \omega_r(f,t)_1 e^{-t/\xi} dt \\
&\le \frac{\omega_r(f,\xi)_1}{\xi}\int_{0}^{\infty} \left(1+\frac{t}{\xi}\right)^r e^{-t/\xi} dt \\
&= \omega_r(f,\xi)_1 \int_0^{\infty} (1+x)^r e^{-x} dx = \omega_r(f,\xi)_1 \left(\sum_{k=0}^{r} \binom{r}{k} k!\right) \\
&= \omega_r(f,\xi)_1 \left(r!\sum_{k=0}^{r} \frac{1}{k!}\right) = \omega_r(f,\xi)_1 \lfloor er! \rfloor. \qquad (12.50)
\end{aligned}
$$

We have proved (12.49). .

In the next we consider $f \in C^n(\mathbb{R})$ with $f^{(n)} \in L_p(\mathbb{R})$, $n = 0$ or $n \ge 2$ even, $1 \le p < \infty$ and the similar *smooth singular operator of symmetric convolution type*

$$P_\xi(f;x) = \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+y) e^{-|y|/\xi} dy, \quad \text{for all } x \in \mathbb{R},\ \xi > 0. \qquad (12.51)$$

That is

$$P_\xi(f;x) = \frac{1}{2\xi}\int_{0}^{\infty} \big(f(x+y) + f(x-y)\big) e^{-y/\xi} dy, \qquad (12.51)^*$$

for all $x \in \mathbb{R}$, $\xi > 0$. Notice that $P_{1,\xi} = P_\xi$. Let the central second order difference

$$(\tilde{\Delta}_y^2 f)(x) := f(x+y) + f(x-y) - 2f(x). \qquad (12.52)$$

Notice that

$$(\tilde{\Delta}_{-y}^2 f)(x) = (\tilde{\Delta}_y^2 f)(x).$$

When $n \ge 2$ even using Taylor's formula with Cauchy remainder we eventually find

$$(\tilde{\Delta}_y^2 f)(x) = 2\sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x)}{(2\rho)!} y^{2\rho} + \mathcal{R}_1(x), \qquad (12.53)$$

where

$$\mathcal{R}_1(x) := \int_0^y (\tilde{\Delta}_t^2 f^{(n)})(x) \frac{(y-t)^{n-1}}{(n-1)!} dt. \tag{12.54}$$

Notice that

$$P_\xi(f;x) - f(x) = \frac{1}{2\xi} \int_0^\infty (\tilde{\Delta}_y^2 f(x)) e^{-y/\xi} dy. \tag{12.55}$$

Furthermore by (12.4), (12.53) and (12.55) we easily see that

$$\begin{aligned} K(x) := \quad & P_\xi(f;x) - f(x) - \sum_{\rho=1}^{n/2} f^{(2\rho)}(x) \xi^{2\rho} \qquad (12.56) \\ = \quad & \frac{1}{2\xi} \int_0^\infty \left(\int_0^y (\tilde{\Delta}_t^2 f^{(n)})(x) \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy. \end{aligned}$$

Therefore we have

$$|K(x)| \leq \frac{1}{2\xi} \int_0^\infty \left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)| \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy. \tag{12.57}$$

Here we estimate in L_p norm, $p \geq 1$, the error function $K(x)$. Notice that we have $\omega_2(f^{(n)}, h)_p < \infty$, $h > 0$, $n = 0$ or $n \geq 2$ even. Operators P_ξ are positive operators.

The related main L_p result here comes next.

Theorem 12.5. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \geq 2$ even and the rest as above. Then*

$$\|K(x)\|_p \leq \left(\frac{\tilde{\tau}^{1/p}}{(4p+2)^{1/p} (q^2(n-1)+q)^{1/q} (n-1)!} \right) \xi^n \omega_2(f^{(n)}, \xi)_p, \tag{12.58}$$

where

$$\tilde{\tau} := \left(\int_0^\infty (1+x)^{2p+1} x^{np-1} e^{-(p/2)x} dx - \left(\frac{2}{p} \right)^{np} \Gamma(np) \right) < \infty. \tag{12.59}$$

Hence as $\xi \to 0$ we get $\|K(x)\|_p \to 0$.

If additionally, $f^{(2\rho)} \in L_p(\mathbb{R})$, $\rho = 1, ..., \frac{n}{2}$, then $\|P_\xi(f) - f\|_p \to 0$, as $\xi \to 0$.

Proof. We observe that

$$|K(x)|^p \leq \frac{1}{2^p \xi^p} \left(\int_0^\infty \left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)| \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy \right)^p. \tag{12.60}$$

Call

$$\tilde{\gamma}(y,x) := \int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)| \frac{(y-t)^{n-1}}{(n-1)!} dt \geq 0, \tag{12.61}$$

then we have

$$|K(x)|^p \le \frac{1}{2^p \xi^p} \left(\int_0^\infty \tilde{\gamma}(y,x) e^{-y/\xi} dy \right)^p. \tag{12.62}$$

And hence

$$\begin{aligned}
\Lambda := \quad & \int_{-\infty}^{\infty} |K(x)|^p dx \le \frac{1}{2^p \xi^p} \int_{-\infty}^{\infty} \left(\int_0^\infty \tilde{\gamma}(y,x) e^{-y/\xi} dy \right)^p dx \\
= \quad & \frac{1}{2^p \xi^p} \left(\int_{-\infty}^{\infty} \left(\int_0^\infty \tilde{\gamma}(y,x) e^{-y/2\xi} e^{-y/2\xi} dy \right)^p dx \right) \\
& \text{(by Hölder's inequality)} \\
\le \quad & \frac{1}{2^p \xi^p} \left(\int_{-\infty}^{\infty} \left(\int_0^\infty (\tilde{\gamma}(y,x))^p e^{-py/2\xi} dy \right) \left(\int_0^\infty e^{-qy/2\xi} dy \right)^{p/q} dx \right) \\
= \quad & \frac{1}{2\xi q^{p/q}} \left(\int_{-\infty}^{\infty} \left(\int_0^\infty (\tilde{\gamma}(y,x))^p e^{-py/2\xi} dy \right) dx \right) =: (*).
\end{aligned} \tag{12.63}$$

By applying again Hölder's inequality we see that

$$\tilde{\gamma}(y,x) \le \frac{\left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)|^p dt \right)^{1/p}}{(n-1)!} \frac{y^{(n-1+\frac{1}{q})}}{(q(n-1)+1)^{1/q}}. \tag{12.64}$$

Therefore it holds

$$\begin{aligned}
(*) \quad \le \quad & \frac{1}{(q(n-1)+1)^{p/q} ((n-1)!)^p 2\xi q^{p/q}} \left(\int_0^\infty \left(\int_{-\infty}^{\infty} \left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)|^p dt \right) \right.\right. \\
& \left.\left. \cdot\, y^{pn-1} e^{-py/2\xi} \right) dx \right) dy \Bigg) =: (**).
\end{aligned} \tag{12.65}$$

We call

$$c_2 := \frac{1}{2\xi q^{p/q} ((n-1)!)^p (q(n-1)+1)^{p/q}}. \tag{12.66}$$

And thus

$$\begin{aligned}
(**) &= c_2\left(\int_0^\infty\left(\int_{-\infty}^\infty\left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)|^p dt\right)dx\right)y^{pn-1}e^{-py/2\xi}\right)dy\Bigg) \\
&= c_2\left(\int_0^\infty\left(\int_0^y\left(\int_{-\infty}^\infty |\tilde{\Delta}_t^2 f^{(n)}(x)|^p dx\right)dt\right)y^{pn-1}e^{-py/2\xi}\right)dy\Bigg) \\
&= c_2\left(\int_0^\infty\left(\int_0^y\left(\int_{-\infty}^\infty |\Delta_t^2 f^{(n)}(x-t)|^p dx\right)dt\right)y^{pn-1}e^{-py/2\xi}\right)dy\Bigg) \\
&= c_2\left(\int_0^\infty\left(\int_0^y\left(\int_{-\infty}^\infty |\Delta_t^2 f^{(n)}(x)|^p dx\right)dt\right)y^{pn-1}e^{-py/2\xi}\right)dy\Bigg) \\
&\le c_2\left(\int_0^\infty\left(\int_0^y \omega_2(f^{(n)},t)_p^p dt\right)y^{pn-1}e^{-py/2\xi}\right)dy\Bigg) \qquad (12.67) \\
&\le c_2\omega_2(f^{(n)},\xi)_p^p\left(\int_0^\infty\left(\int_0^y\left(1+\frac{t}{\xi}\right)^{2p}dt\right)y^{pn-1}e^{-py/2\xi}\right)dy\Bigg).
\end{aligned}$$

That is, so far we proved that

$$\Lambda \le c_2\omega_2(f^{(n)},\xi)_p^p\left(\int_0^\infty\left(\int_0^y\left(1+\frac{t}{\xi}\right)^{2p}dt\right)y^{pn-1}e^{-py/2\xi}\right)dy\Bigg). \qquad (12.68)$$

However

$$\text{R.H.S.}(12.68) = \frac{c_2\xi}{(2p+1)}\omega_2(f^{(n)},\xi)_p^p\left(\int_0^\infty\left(\left(1+\frac{y}{\xi}\right)^{2p+1}-1\right)y^{pn-1}e^{-py/2\xi}dy\right). \qquad (12.69)$$

Call

$$M := \int_0^\infty\left(\left(1+\frac{y}{\xi}\right)^{2p+1}-1\right)y^{pn-1}e^{-py/2\xi}dy. \qquad (12.70)$$

Thus

$$\begin{aligned}
M &= \xi^{pn}\int_0^\infty\big((1+x)^{2p+1}-1\big)x^{pn-1}e^{-(p/2)x}dx \qquad (12.71) \\
&= \xi^{pn}\left(\int_0^\infty(1+x)^{2p+1}x^{pn-1}e^{-(p/2)x}dx-\left(\frac{2}{p}\right)^{np}\Gamma(np)\right).
\end{aligned}$$

I.e. we get

$$M = \xi^{pn}\tilde{\tau}. \qquad (12.72)$$

Therefore it holds

$$\Lambda \le \frac{\tilde{\tau}\xi^{pn}\omega_2(f^{(n)},\xi)_p^p}{2(2p+1)((n-1)!)^p(q^2(n-1)+q)^{p/q}}. \qquad (12.73)$$

We have established (12.58). ■

The counterpart of Theorem 12.5 follows, $p = 1$ case.

Theorem 12.6. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in L_1(\mathbb{R})$, $n \geq 2$ even. Then*

$$\|K(x)\|_1 \leq n \left(\frac{(n+1)(n+2)}{6} + \frac{(n+1)}{2} + \frac{1}{2} \right) \xi^n \omega_2(f^{(n)}, \xi)_1. \tag{12.74}$$

Hence as $\xi \to 0$ we obtain $\|K(x)\|_1 \to 0$.

If additionally $f^{(2\rho)} \in L_1(\mathbb{R})$, $\rho = 1, ..., \frac{n}{2}$, then $\|P_\xi(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. Notice that

$$\tilde{\Delta}_t^2 f^{(n)}(x) = \Delta_t^2 f^{(n)}(x - t), \tag{12.75}$$

all $x, t \in \mathbb{R}$. Also it holds

$$\begin{aligned} \int_{-\infty}^{\infty} |\Delta_t^2 f^{(n)}(x-t)| dx &= \int_{-\infty}^{\infty} |\Delta_t^2 f^{(n)}(w)| dw \\ &\leq \omega_2(f^{(n)}, t)_1, \quad \text{all } t \in \mathbb{R}_+. \end{aligned} \tag{12.76}$$

Here we obtain

$$\begin{aligned} \|K(x)\|_1 &= \int_{-\infty}^{\infty} |K(x)| dx \\ &\overset{(12.57)}{\leq} \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)| \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy \right) dx \\ &\leq \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left(\frac{y^{n-1}}{(n-1)!} \left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)| dt \right) e^{-y/\xi} \right) dy \right) dx \\ &= \frac{1}{2\xi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} \left(\frac{y^{n-1}}{(n-1)!} \left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)| dt \right) e^{-y/\xi} \right) dx \right) dy \\ &= \frac{1}{2\xi} \left(\int_0^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x)| dt \right) dx \right) \frac{y^{n-1}}{(n-1)!} e^{-y/\xi} \right) dy \\ &= \frac{1}{2\xi} \left(\int_0^{\infty} \left(\int_0^y \left(\int_{-\infty}^{\infty} |\tilde{\Delta}_t^2 f^{(n)}(x)| dx \right) dt \right) \frac{y^{n-1}}{(n-1)!} e^{-y/\xi} \right) dy \end{aligned}$$

$$\begin{aligned}
&\overset{(12.75)}{=} \frac{1}{2\xi}\left(\int_0^\infty \left(\int_0^y \left(\int_{-\infty}^\infty |\Delta_t^2 f^{(n)}(x-t)|dx\right)dt\right)\frac{y^{n-1}}{(n-1)!}e^{-y/\xi}\right)dy\Bigg) \\
&\overset{(12.76)}{\leq} \frac{1}{2\xi}\left(\int_0^\infty \left(\int_0^y \omega_2(f^{(n)},t)_1 dt\right)\frac{y^{n-1}}{(n-1)!}e^{-y/\xi}\right)dy\Bigg) \\
&\leq \frac{\omega_2(f^{(n)},\xi)_1}{2\xi}\left(\int_0^\infty \left(\int_0^y \left(1+\frac{t}{\xi}\right)^2 dt\right)\frac{y^{n-1}}{(n-1)!}e^{-y/\xi}\right)dy\Bigg) \\
&= \frac{\omega_2(f^{(n)},\xi)_1}{6(n-1)!}\left(\int_0^\infty \left(\left(1+\frac{y}{\xi}\right)^3-1\right)y^{n-1}e^{-y/\xi}dy\right) \\
&= \frac{\xi^n\omega_2(f^{(n)},\xi)_1}{6(n-1)!}\left(\int_0^\infty ((1+x)^3-1)x^{n-1}e^{-x}dx\right) \\
&= \frac{\xi^n\omega_2(f^{(n)},\xi)_1}{6(n-1)!}\left(\int_0^\infty (x^{n+2}+3x^{n+1}+3x^n)e^{-x}dx\right) \\
&= \frac{\xi^n\omega_2(f^{(n)},\xi)_1}{6(n-1)!}((n+2)!+3(n+1)!+3n!) \\
&= n\left(\frac{(n+1)(n+2)}{6}+\frac{(n+1)}{2}+\frac{1}{2}\right)\xi^n\omega_2(f^{(n)},\xi)_1. \qquad (12.77)
\end{aligned}$$

We have proved (12.74). .

The related case here of $n = 0$ comes next.

Proposition 12.7. *Let $p, q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and the rest as above. Then*

$$\|P_\xi(f)-f\|_p \leq \frac{\rho^{1/p}}{2^{1/p}q^{1/q}}\omega_2(f,\xi)_p, \qquad (12.78)$$

where

$$\rho := \int_0^\infty (1+x)^{2p}e^{-(p/2)x}dx < \infty. \qquad (12.79)$$

Hence as $\xi \to 0$ we obtain $P_\xi \to I$ in the L_p norm, $p > 1$.

Proof. From (12.55) we find

$$|P_\xi(f;x)-f(x)|^p \leq \frac{1}{2^p\xi^p}\left(\int_0^\infty |\tilde{\Delta}_y^2 f(x)|e^{-y/\xi}dy\right)^p. \qquad (12.80)$$

We then estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty} |P_\xi(f;x) - f(x)|^p dx \\
&\quad \le \frac{1}{2^p \xi^p} \int_{-\infty}^{\infty} \left(\int_0^{\infty} |\tilde{\Delta}_y^2 f(x)| e^{-y/\xi} dy \right)^p dx \\
&\quad = \frac{1}{2^p \xi^p} \int_{-\infty}^{\infty} \left(\int_0^{\infty} |\tilde{\Delta}_y^2 f(x)| e^{-y/2\xi} e^{-y/2\xi} dy \right)^p dx \\
&\quad \le \frac{1}{2^p \xi^p} \left(\int_{-\infty}^{\infty} \left(\int_0^{\infty} |\tilde{\Delta}_y^2 f(x)|^p e^{-py/2\xi} dy \right) \left(\int_0^{\infty} e^{-qy/2\xi} dy \right)^{p/q} dx \right) \\
&\quad = \frac{1}{2^p \xi^p} \left(\frac{2\xi}{q} \right)^{p/q} \left(\int_{-\infty}^{\infty} \left(\int_0^{\infty} |\tilde{\Delta}_y^2 f(x)|^p e^{-py/2\xi} dy \right) dx \right) \\
&\quad = \frac{1}{2\xi q^{p/q}} \left(\int_0^{\infty} \left(\int_{-\infty}^{\infty} |\tilde{\Delta}_y^2 f(x)|^p dx \right) e^{-py/2\xi} dy \right) \\
&\quad = \frac{1}{2\xi q^{p/q}} \left(\int_0^{\infty} \left(\int_{-\infty}^{\infty} |\Delta_y^2 f(x-y)|^p dx \right) e^{-py/2\xi} dy \right) \\
&\quad = \frac{1}{2\xi q^{p/q}} \left(\int_0^{\infty} \left(\int_{-\infty}^{\infty} |\Delta_y^2 f(x)|^p dx \right) e^{-py/2\xi} dy \right) \\
&\quad \le \frac{1}{2\xi q^{p/q}} \left(\int_0^{\infty} \omega_2(f,y)_p^p e^{-py/2\xi} dy \right) \\
&\quad \le \frac{\omega_2(f,\xi)_p^p}{2\xi q^{p/q}} \left(\int_0^{\infty} \left(1 + \frac{y}{\xi}\right)^{2p} e^{-py/2\xi} dy \right) \\
&\quad = \frac{\omega_2(f,\xi)_p^p}{2q^{p/q}} \left(\int_0^{\infty} (1+x)^{2p} e^{-(p/2)x} dx \right). \qquad (12.81)
\end{aligned}$$

The proof of (12.78) is now evident. ∎

Also we give

Proposition 12.8. *It holds*

$$\|P_\xi f - f\|_1 \le \frac{5}{2} \omega_2(f,\xi)_1. \qquad (12.82)$$

Hence as $\xi \to 0$ we get $P_\xi \to I$ in the L_1 norm.

Proof. From (12.55) we have

$$|P_\xi(f;x) - f(x)| \le \frac{1}{2\xi} \int_0^{\infty} |\tilde{\Delta}_y^2 f(x)| e^{-y/\xi} dy. \qquad (12.83)$$

Hence we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |P_\xi(f;x) - f(x)| dx \\
&\quad \le \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\int_0^{\infty} |\tilde{\Delta}_y^2 f(x)| e^{-y/\xi} dy \right) dx \\
&\quad = \frac{1}{2\xi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} |\tilde{\Delta}_y^2 f(x)| dx \right) e^{-y/\xi} dy \\
&\quad = \frac{1}{2\xi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} |\Delta_y^2 f(x-y)| dx \right) e^{-y/\xi} dy \\
&\quad = \frac{1}{2\xi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} |\Delta_y^2 f(x)| dx \right) e^{-y/\xi} dy \\
&\quad \le \frac{1}{2\xi} \int_0^{\infty} \omega_2(f,y)_1 e^{-y/\xi} dy \\
&\quad \le \frac{\omega_2(f,\xi)_1}{2\xi} \int_0^{\infty} \left(1 + \frac{y}{\xi}\right)^2 e^{-y/\xi} dy \\
&\quad = \frac{\omega_2(f,\xi)_1}{2} \int_0^{\infty} (1+x)^2 e^{-x} dx = \frac{5}{2} \omega_2(f,\xi)_1. \qquad (12.84)
\end{aligned}
$$

We have established (12.82). ■

13

Approximation with Rates by Fractional Smooth Picard Singular Operators

In this chapter we study the very general fractional smooth Picard singular integral operators on the real line, regarding their convergence to the unit operator with fractional rates in the uniform norm. The related established inequalities involve the higher order moduli of smoothness of the associated right and left Caputo fractional derivatives of the involved function. Furthermore we present a fractional Voronovskaya type of result giving the fractional asymptotic expansion of the basic error of our approximation.

We finish with applications. The operators are not in general positive. This chapter relies on [60].

13.1 Background

We mention

Definition 13.1. Let $\nu \geq 0, n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number, $\lfloor \cdot \rfloor$ is the integral part), $f \in C^n(\mathbb{R})$. We call left Caputo fractional derivative ([145], [160], [179]) the function

$$D_{*x_0}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_{x_0}^{x} (x-t)^{n-\nu-1} f^{(n)}(t) dt, \tag{13.1}$$

$\forall x \geq x_0 \in \mathbb{R}$ fixed, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt, \nu > 0$.

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 169–190.
springerlink.com

We set $D^0_{*x_0} f(x) = f(x), \forall x \geq x_0$.
We suppose $D^\nu_{*x_0} f(x) = 0$, for $x < x_0$.

We need
Lemma 13.2. Let $\nu > 0, \nu \notin \mathbb{N}, n = \lceil \nu \rceil$, $f \in C^n(\mathbb{R})$, $\left\| f^{(n)} \right\|_\infty < \infty$, $x_0 \in \mathbb{R}$ fixed. Then $D^\nu_{*x_0} f(x_0) = 0$.

Proof. By Definition 13.1 we obtain

$$\begin{aligned} \left|D^\nu_{*x_0} f(x)\right| &\leq \frac{1}{\Gamma(n-\nu)} \int_{x_0}^x (x-t)^{n-\nu-1} \left| f^{(n)}(t) \right| dt \\ &\leq \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu)} \int_{x_0}^x (x-t)^{n-\nu-1} dt \\ &= \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu)} \frac{(x-x_0)^{n-\nu}}{(n-\nu)} \\ &= \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (x-x_0)^{n-\nu}, \forall x \geq x_0. \end{aligned}$$

The claim is now clear. ∎

We need the following left Caputo fractional Taylor formula.
Theorem 13.3. ([44,145]) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-\zeta)^{\alpha-1} D^\alpha_{*x_0} f(\zeta) d\zeta, \tag{13.2}$$

$\forall x \in \mathbb{R} : x \geq x_0$.

We also mention
Definition 13.4. ([160], [179]) Let $f \in C^m(\mathbb{R})$, $\alpha > 0$, $m = \lceil \alpha \rceil$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$D^\alpha_{x_0-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta, \tag{13.3}$$

$\forall x \leq x_0 \in \mathbb{R}$ fixed.
We suppose $D^\alpha_{x_0-} f(x) = 0, \forall x > x_0$.

We need
Lemma 13.5. Let $\alpha > 0, \alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $x_0 \in \mathbb{R}$ fixed. Then $D^\alpha_{x_0-} f(x_0) = 0$.

Proof.

$$\begin{aligned} \left|D^\alpha_{x_0-} f(x)\right| &\leq \frac{1}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta-x)^{m-\alpha-1} \left| f^{(m)}(\zeta) \right| d\zeta \\ &\leq \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta-x)^{m-\alpha-1} d\zeta \end{aligned}$$

$$= \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha)} \left. \frac{(\zeta - x)^{m-\alpha}}{(m-\alpha)} \right|_x^{x_0}$$

$$= \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (x_0 - x)^{m-\alpha}, \forall x \leq x_0,$$

proving the claim. ■

We need the following right Caputo fractional Taylor formula.

Theorem 13.6. ([44],[155]) Let $f \in C^m(\mathbb{R}), m = \lceil \alpha \rceil, \alpha > 0$. Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_x^{x_0} (\zeta - x)^{\alpha - 1} D_{x_0-}^{\alpha} f(\zeta) d\zeta, \qquad (13.4)$$

$\forall x \leq x_0$.

We further need

Theorem 13.7. Let $g \in C_b(\mathbb{R})$ (continuous and bounded), $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define

$$L(x, x_0) = \int_{x_0}^{x} (x - t)^{c-1} g(t) dt, \text{ for } x \geq x_0,$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

Proof. We notice that $L(x_0, x_0) = 0$. Assume $x \geq x_0$. Let $x_N \to x, x_{0N} \to x_0, N \in \mathbb{N}$ and assume without loss of generality that $x_N \geq x_{0N}$. We have $|x_N - x_{0N}| \leq |x_N| + |x_{0N}| \leq b_1 + b_2 =: d$, where b_1, b_2 the bounds of the convergent sequences x_N, x_{0N}. Clearly also $x - x_0 \leq |x| + |x_0| \leq d$. Then we have

$$L(x, x_0) = \int_0^{x - x_0} z^{c-1} g(x - z) dz = \int_0^d \chi_{[0, x - x_0]}(z) z^{c-1} g(x - z) dz,$$

where χ is the characteristic function.

So we have again

$$\begin{aligned} L(x_N, x_{0N}) &= \int_0^{x_N - x_{0N}} z^{c-1} g(x_N - z) dz \\ &= \int_0^d \chi_{[0, x_N - x_{0N}]}(z) z^{c-1} g(x_N - z) dz. \end{aligned}$$

We observe that

$$\chi_{[0, x_N - x_{0N}]}(z) \to \chi_{[0, x - x_0]}(z), \text{ a.e.},$$

and

$$\chi_{[0,x_N-x_{0N}]}(z)z^{c-1}g(x_N-z)\to\chi_{[0,x-x_0]}(z)z^{c-1}g(x-z)\,,\text{ a.e.}$$

Notice that

$$\chi_{[0,x_N-x_{0N}]}(z)z^{c-1}\left|g(x_N-z)\right|\le z^{c-1}\left\|g\right\|_\infty,$$

which is an integrable function.

Thus by Dominated Convergence theorem we derive

$$L(x_N,x_{0N})\to L(x,x_0)\,,\text{ as }N\to\infty.$$

Clearly now $L(x,x_0)$ is jointly continuous on $\mathbb{R}^2$. ■

We also mention

Theorem 13.8. Let $g\in C_b(\mathbb{R})$, $0<c<1$, $x,x_0\in\mathbb{R}$. Define

$$K(x,x_0)=\int_x^{x_0}(\zeta-x)^{c-1}g(\zeta)\,d\zeta,\text{ for }x\le x_0,$$

and $K(x,x_0)=0$, for $x>x_0$.

Then $K(x,x_0)$ is jointly continuous from $\mathbb{R}^2$ into $\mathbb{R}$.

Proof. Let $x_N\to x, x_{0N}\to x_0, N\in\mathbb{N}$ and without loss of generality we may assume that $x_N\le x_{0N}$. Here as in the proof of Theorem 13.7: $x_{0N}-x_N\le b_1+b_2=:d$, and $x_0-x\le d$. We have

$$\begin{aligned}K(x,x_0)&=\int_0^{x_0-x}z^{c-1}g(z+x)\,dz\\&=\int_0^d\chi_{[0,x_0-x]}(z)z^{c-1}g(z+x)\,dz,\end{aligned}$$

and

$$\begin{aligned}K(x_N,x_{0N})&=\int_0^{x_{0N}-x_N}z^{c-1}g(z+x_N)\,dz\\&=\int_0^d\chi_{[0,x_{0N}-x_N]}(z)z^{c-1}g(z+x_N)\,dz.\end{aligned}$$

We have

$$\chi_{[0,x_{0N}-x_N]}(z)\to\chi_{[0,x_0-x]}(z),\text{ a.e.,}$$

and

$$\chi_{[0,x_{0N}-x_N]}(z)z^{c-1}g(z+x_N)\to\chi_{[0,x_0-x]}(z)z^{c-1}g(z+x)\,,\text{ a.e.}$$

Notice that

$$\chi_{[0,x_{0N}-x_N]}(z)z^{c-1}\left|g\left(z+x_N\right)\right| \leq z^{c-1}\left\|g\right\|_{\infty},$$

which is integrable.

Thus by Dominated Convergence theorem we obtain

$$K(x_N, x_{0N}) \to K(x, x_0), \text{ as } N \to \infty.$$

Clearly now $K(x, x_0)$ is jointly continuous on $\mathbb{R}^2$. ■

Based on Theorems 13.7,13.8 we get

Proposition 13.9. Let $f \in C^m\left(\mathbb{R}\right)$, with $\left\|f^{(m)}\right\|_{\infty} < \infty$, $m = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $x, x_0 \in \mathbb{R}$. Then $D^{\alpha}_{*x_0}f(x), D^{\alpha}_{x_0-}f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.

We need

Definition 13.10. Let $f \in C^m\left(\mathbb{R}\right)$, $\left\|f^{(m)}\right\|_{\infty} < \infty$, $m = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $r \in \mathbb{N}$, $x, x_0 \in \mathbb{R}$. We define the difference

$$\left(\Delta^r_w\left(D^{\alpha}_{*x_0}f\right)\right)(x) := \sum_{j=0}^{r}\left(-1\right)^{r-j}\binom{r}{j}\left(D^{\alpha}_{*x_0}f\right)\left(x+jw\right), \tag{13.5}$$

$\forall w \in \mathbb{R}$,

and the rth modulus of smoothness,

$$\omega_r\left(D^{\alpha}_{*x_0}f, h\right) := \sup_{|t|\leq h}\left\|\left(\Delta^r_t\left(D^{\alpha}_{*x_0}f\right)\right)(x)\right\|_{\infty,x,\mathbb{R}}. \tag{13.6}$$

Notice that

$$\begin{aligned}\left|\left(\Delta^r_w\left(D^{\alpha}_{*x_0}f\right)\right)(x_0)\right| &\leq \left\|\left(\Delta^r_w\left(D^{\alpha}_{*x_0}f\right)\right)(x)\right\|_{\infty,x,\mathbb{R}} \\ &\leq \omega_r\left(D^{\alpha}_{*x_0}f, |w|\right).\end{aligned} \tag{13.7}$$

Similarly, we define the difference

$$\left(\Delta^r_w\left(D^{\alpha}_{x_0-}f\right)\right)(x) := \sum_{j=0}^{r}\left(-1\right)^{r-j}\binom{r}{j}\left(D^{\alpha}_{x_0-}f\right)\left(x+jw\right), \tag{13.8}$$

$\forall w \in \mathbb{R}$, and the rth modulus of smoothness,

$$\omega_r\left(D^{\alpha}_{x_0-}f, h\right) := \sup_{|t|\leq h}\left\|\left(\Delta^r_t\left(D^{\alpha}_{x_0-}f\right)\right)(x)\right\|_{\infty,x,\mathbb{R}}. \tag{13.9}$$

See again that

$$\begin{aligned}\left|\left(\Delta^r_w\left(D^{\alpha}_{x_0-}f\right)\right)(x_0)\right| &\leq \left\|\left(\Delta^r_w\left(D^{\alpha}_{x_0-}f\right)\right)(x)\right\|_{\infty,x,\mathbb{R}} \\ &\leq \omega_r\left(D^{\alpha}_{x_0-}f, |w|\right).\end{aligned} \tag{13.10}$$

As a related result we mention

Proposition 13.11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous.

Consider

$$G(x) = \omega_r (f(\cdot, x), \delta)_{[x,+\infty)}, \ \ \delta > 0, x \in \mathbb{R}.$$

(Here ω_r is defined over $[x, +\infty)$ instead of $\mathbb{R}$.)

Then G is continuous on $\mathbb{R}$.

Proof. Let $x_n \to x, x_n \leq x, \delta > 0$.

(The case $x_n \to x$, $x_n \geq x$ is similar.)

Then we can write

$$G(x_n) = \omega_r (f(\cdot, x_n), \delta)_{[x_n,+\infty)},$$

which is

$$G(x_n) = \max \{A_0, A_1, \ldots, A_{r+1}\}.$$

The $A_0, A_1, \ldots, A_{r+1}$ are described as follows.

(Here $\Delta_t^r f(u, x_n) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} f(u + jt, x_n)$.)

$$\begin{aligned}
A_0 &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in [x, +\infty) \text{ for all } j = 0, 1, \ldots r; |t| \leq \delta\}, \\
A_1 &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in [x, +\infty) \text{ for all } j = 0, 1, \ldots r - 1, \text{ and} \\
&\quad u + rt \in [x_n, x); |t| \leq \delta\}, \\
A_2 &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in [x, +\infty) \text{ for all } j = 0, 1, \ldots r - 2, \text{ and} \\
&\quad u + jt \in [x_n, x) \text{ for } j = r - 1, r; |t| \leq \delta\}, \\
&\ \vdots \\
A_{r-1} &= \sup \{|\Delta_t^r f(u, x_n)| : u, u + t \in [x, +\infty), \text{ and} \\
&\quad u + jt \in [x_n, x) \text{ for } j = 2, \ldots, r; |t| \leq \delta\}, \\
A_r &= \sup \{|\Delta_t^r f(u, x_n)| : u \in [x, +\infty), \text{ and} \\
&\quad u + jt \in [x_n, x) \text{ for } j = 1, \ldots, r; |t| \leq \delta\}, \\
A_{r+1} &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in [x_n, x) \text{ for } j = 0, \ldots, r; |t| \leq \delta\}.
\end{aligned}$$

Now, when $x_n \to x$, then $A_0 \to G(x)$; $A_l \to K_l(x) \leq G(x), l = 1, \ldots, r$; and $A_{r+1} \to 0$ (since $x_n \to x$).

In conclusion, $G(x_n) \to \max \{G(x), K_l(x), 0\}_{(l=1,\ldots,r)} = G(x)$. ∎

Proposition 13.12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous.

Consider

$$H(x) = \omega_r (f(\cdot, x), \delta)_{(-\infty,x]}, \ \ \delta > 0, x \in \mathbb{R}.$$

(Here ω_r is defined over $(-\infty, x]$ instead of $\mathbb{R}$.)

Then H is continuous on $\mathbb{R}$.

Proof. Let $x_n \geq x, x_n \to x$, (similarly is done $x_n \leq x$), $\delta > 0$.
Then we can write

$$H(x_n) = \omega_r (f(\cdot, x_n), \delta)_{(-\infty, x_n]},$$

which is

$$H(x_n) = \max \{B_0, B_1, \ldots, B_{r+1}\}.$$

The $B_0, B_1, \ldots, B_{r+1}$ are described as follows:

$$\begin{aligned}
B_0 &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in (-\infty, x] ; \text{ for all } j = 0, 1, \ldots r; |t| \leq \delta\}, \\
B_1 &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in (-\infty, x] \text{ for } j = 0, 1, \ldots r - 1, \text{ and} \\
&\quad u + rt \in (x, x_n] ; |t| \leq \delta\}, \\
B_2 &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in (-\infty, x] \text{ for } j = 0, 1, \ldots r - 2, \text{and} \\
&\quad u + jt \in (x, x_n], \text{ for } j = r - 1, r; |t| \leq \delta\}, \\
&\vdots \\
B_{r-1} &= \sup \{|\Delta_t^r f(u, x_n)| : u, u + t \in (-\infty, x], \text{and} \\
&\quad u + jt \in (x, x_n] \text{ for } j = 2, \ldots, r; |t| \leq \delta\}, \\
B_r &= \sup \{|\Delta_t^r f(u, x_n)| : u \in (-\infty, x], \text{ and} \\
&\quad u + jt \in (x, x_n] \text{ for } j = 1, \ldots, r; |t| \leq \delta\}, \quad \text{and} \\
B_{r+1} &= \sup \{|\Delta_t^r f(u, x_n)| : u + jt \in (x, x_n] \text{ for } j = 0, \ldots, r; |t| \leq \delta\}.
\end{aligned}$$

Now, when $x_n \to x$, then $B_0 \to H(x)$; $B_l \to T_l(x) \leq H(x), l = 1, \ldots, r$; and $B_{r+1} \to 0$ (since $x_n \to x$).
Hence $H(x_n) \to \max \{H(x), T_l(x), 0\}_{(l=1,\ldots,r)} = H(x)$. ■

From Propositions 13.9, 13.11, 13.12 we obtain

Proposition 13.13. Let $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $r \in \mathbb{N}$, $x \in \mathbb{R}$. Then $\omega_r (D_{*x}^\alpha f, h)_{[x,+\infty)}, \omega_r (D_{x-}^\alpha f, h)_{(-\infty,x]}$ are continuous functions of $x \in \mathbb{R}$, $h > 0$ fixed.

We make
Remark 13.14. Let g continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then we know that

$$\omega_r (g, t) \leq 2^r \|g\|_\infty < \infty.$$

Assuming that $(D_{*x}^\alpha f)(t), (D_{x-}^\alpha f)(t)$, are both continuous and bounded in $(x, t) \in \mathbb{R}^2$, i.e.

$$\begin{aligned}
\|D_{*x}^\alpha f\|_\infty &\leq K_1, \forall x \in \mathbb{R}; \\
\|D_{x-}^\alpha f\|_\infty &\leq K_2, \forall x \in \mathbb{R},
\end{aligned}$$

where $K_1, K_2 > 0$, we obtain

$$\begin{aligned}\omega_r\left(D_{*x}^{\alpha} f, \xi\right) &\leq 2^r K_1; \\ \omega_r\left(D_{x-}^{\alpha} f, \xi\right) &\leq 2^r K_2, \forall \xi \geq 0,\end{aligned}$$

for each $x \in \mathbb{R}$.

Therefore, for any $\xi \geq 0$,

$$\sup_{x \in \mathbb{R}}\left[\max\left(\omega_r\left(D_{*x}^{\alpha} f, \xi\right), \omega_r\left(D_{x-}^{\alpha} f, \xi\right)\right)\right] \leq 2^r \max\left(K_1, K_2\right) < \infty. \tag{13.11}$$

So in our setting for $f \in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_{\infty} < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, by Proposition 13.9, both $\left(D_{*x}^{\alpha} f\right)(t)$, $\left(D_{x-}^{\alpha} f\right)(t)$ are jointly continuous in (t, x) on $\mathbb{R}^2$. Assuming further that they are both bounded on $\mathbb{R}^2$ we get (13.11) valid. In particular, each of $\omega_r\left(D_{*x}^{\alpha} f, \xi\right), \omega_r\left(D_{x-}^{\alpha} f, \xi\right)$ is finite for any $\xi \geq 0$.

We need

Remark 13.15. Again let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$; $f^{(m)}(x) = 1, \forall x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. Notice $0 < m - \alpha < 1$. Then

$$D_{*x_0}^{\alpha} f(x) = \frac{\left(x - x_0\right)^{m-\alpha}}{\Gamma(m - \alpha + 1)}, \forall x \geq x_0.$$

Let us consider $x, y \geq x_0$, then

$$\begin{aligned}\left|D_{*x_0}^{\alpha} f(x) - D_{*x_0}^{\alpha} f(y)\right| &= \frac{1}{\Gamma(m - \alpha + 1)}\left|\left(x - x_0\right)^{m-\alpha} - \left(y - x_0\right)^{m-\alpha}\right| \\ &\leq \frac{|x - y|^{m-\alpha}}{\Gamma(m - \alpha + 1)}.\end{aligned}$$

So it is not strange to suppose that

$$\left|D_{*x_0}^{\alpha} f\left(x_1\right) - D_{*x_0}^{\alpha} f\left(x_2\right)\right| \leq K\left|x_1 - x_2\right|^{\beta}, \tag{13.12}$$

$K > 0$, $0 < \beta \leq 1$, $\forall x_1, x_2 \in \mathbb{R}$, any $x_0 \in \mathbb{R}$, here more generally $\left\|f^{(m)}\right\|_{\infty} < \infty$.

In general, one may assume

$$\begin{aligned}\omega_r\left(D_{x-}^{\alpha} f, \xi\right) &\leq M_1 \xi^{r-1+\beta_1}, \text{ and} \\ \omega_r\left(D_{*x}^{\alpha} f, \xi\right) &\leq M_2 \xi^{r-1+\beta_2},\end{aligned} \tag{13.13}$$

where $0 < \beta_1, \beta_2 \leq 1$, $\forall \xi > 0$, $r \in \mathbb{N}$; $M_1, M_2 > 0$; any $x \in \mathbb{R}$.

Setting $\beta = \min\left(\beta_1, \beta_2\right)$ and $M = \max\left(M_1, M_2\right)$, in that case we get

$$\sup_{x \in \mathbb{R}}\left\{\max\left(\omega_r\left(D_{x-}^{\alpha} f, \xi\right), \omega_r\left(D_{*x}^{\alpha} f, \xi\right)\right)\right\} \leq M \xi^{r-1+\beta} \rightarrow 0, \text{ as } \xi \rightarrow 0+. \tag{13.14}$$

13.2 Main Results

We need

Definition 13.16. Let $r \in \mathbb{N}$, $\alpha > 0$. We introduce the numbers

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-\alpha}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-\alpha}, & j = 0, \end{cases} \tag{13.15}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$.

Also denote

$$\delta_k = \sum_{j=1}^{r} \alpha_j j^k, k = 1, \ldots, m-1, \tag{13.16}$$

where $m = \lceil \alpha \rceil$.

We give

Theorem 13.17. Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\left\| f^{(m)} \right\|_\infty < \infty$, $x_0 \in \mathbb{R}$ fixed, $\xi > 0$. Then

i) if $t \geq 0$ we have

$$\begin{aligned} A \quad &: = A(t, x_0) := \sum_{j=0}^{r} \alpha_j \left[f(x_0 + jt) - f(x_0) \right] - \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left(\Delta_w^r \left(D_{*x_0}^\alpha f \right) \right)(x_0)\, dw, \end{aligned} \tag{13.17}$$

and

$$|A| \leq \omega_r \left(D_{*x_0}^\alpha f, \xi \right) \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{t^{k+\alpha}}{\xi^k \Gamma(\alpha + k + 1)} \right) \tag{13.18}$$

ii) if $t < 0$ we obtain

$$\begin{aligned} B \quad &: = B(t, x_0) := \sum_{j=0}^{r} \alpha_j \left[f(x_0 + jt) - f(x_0) \right] - \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k \\ &= \frac{1}{\Gamma(\alpha)} \int_t^0 (w-t)^{\alpha-1} \left(\Delta_w^r \left(D_{x_0-}^\alpha f \right) \right)(x_0)\, dw, \end{aligned} \tag{13.19}$$

and

$$|B| \leq \omega_r \left(D_{x_0-}^\alpha f, \xi \right) \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{|t|^{\alpha+k}}{\xi^k \Gamma(\alpha + k + 1)} \right). \tag{13.20}$$

Proof.
i) Let $t \geq 0$, we obtain

$$\sum_{j=0}^{r} \alpha_j \left[f(x_0 + jt) - f(x_0) \right] = \sum_{j=1}^{r} \alpha_j \left[f(x_0 + jt) - f(x_0) \right]$$

$$= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x_0+jt} ((x_0 + jt) - \zeta)^{\alpha-1} D_{*x_0}^{\alpha} f(\zeta)\, d\zeta \right]$$

$$= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{1}{\Gamma(\alpha)} \int_0^{jt} (jt - u)^{\alpha-1} D_{*x_0}^{\alpha} f(x_0 + u)\, du \right]$$

$$= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{1}{\Gamma(\alpha)} \int_0^{t} (jt - jw)^{\alpha-1} D_{*x_0}^{\alpha} f(x_0 + jw)\, j dw \right]$$

$$= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{j^{\alpha}}{\Gamma(\alpha)} \int_0^{t} (t - w)^{\alpha-1} D_{*x_0}^{\alpha} f(x_0 + jw)\, dw \right]$$

$$= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k + \frac{\sum_{j=1}^{r} \alpha_j j^{\alpha}}{\Gamma(\alpha)} \int_0^{t} (t - w)^{\alpha-1} D_{*x_0}^{\alpha} f(x_0 + jw)\, dw$$

$$= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k + \frac{\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j}}{\Gamma(\alpha)} \int_0^{t} (t - w)^{\alpha-1} D_{*x_0}^{\alpha} f(x_0 + jw)\, dw$$

$$= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - w)^{\alpha-1} \left\{ \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{*x_0}^{\alpha} f(x_0 + jw) \right) \right\} dw$$

$$= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k + \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - w)^{\alpha-1}$$
$$\cdot \left\{ \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{*x_0}^{\alpha} f(x_0 + jw) \right) + (-1)^r \left(D_{*x_0}^{\alpha} f(x_0) \right) \right\} dw$$

$$\begin{aligned}
&= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!}\delta_k t^k \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left\{ \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} D^{\alpha}_{*x_0} \left(f\left(x_0 + jw \right) \right) \right\} dw \\
&= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!}\delta_k t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left(\Delta_w^r \left(D^{\alpha}_{*x_0} f \right) \right)(x_0)\, dw.
\end{aligned}$$

We have proved that

$$\begin{aligned}
\sum_{j=0}^{r} \alpha_j \left[f\left(x_0 + jt\right) - f\left(x_0\right) \right] &= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!}\delta_k t^k \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left(\Delta_w^r \left(D^{\alpha}_{*x_0} f \right) \right)(x_0)\, dw,
\end{aligned}$$

that is (13.17) is true.

Next we observe that

$$\begin{aligned}
|A| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left| \Delta_w^r \left(D^{\alpha}_{*x_0} f \right)(x_0) \right| dw \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \omega_r \left(D^{\alpha}_{*x_0} f, \xi \frac{w}{\xi} \right) dw \\
&\leq \frac{\omega_r \left(D^{\alpha}_{*x_0} f, \xi \right)}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left(1 + \frac{w}{\xi} \right)^r dw
\end{aligned}$$

$$\begin{aligned}
&= \frac{\omega_r \left(D^{\alpha}_{*x_0} f, \xi \right)}{\Gamma(\alpha)} \left[\sum_{k=0}^{r} \binom{r}{k} \frac{1}{\xi^k} \int_0^t (t-w)^{\alpha-1} w^{(k+1)-1} dw \right] \\
&= \omega_r \left(D^{\alpha}_{*x_0} f, \xi \right) \left[\sum_{k=0}^{r} \binom{r}{k} \frac{k!}{\xi^k} \frac{t^{\alpha+k}}{\Gamma(\alpha+k+1)} \right],
\end{aligned}$$

proving inequality (13.18).

ii) Let $t < 0$. Then we get

$$\sum_{j=0}^{r} \alpha_j \left[f\left(x_0 + jt\right) - f\left(x_0\right) \right] = \sum_{j=1}^{r} \alpha_j \left[f\left(x_0 + jt\right) - f\left(x_0\right) \right]$$

$$
\begin{aligned}
&= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{1}{\Gamma(\alpha)} \int_{x_0+jt}^{x_0} (\zeta - x_0 - jt)^{\alpha-1} D_{x_0-}^{\alpha} f(\zeta)\, d\zeta \right] \\
&= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{1}{\Gamma(\alpha)} \int_{jt}^{0} (u - jt)^{\alpha-1} D_{x_0-}^{\alpha} f(x_0+u)\, du \right] \\
&= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{1}{\Gamma(\alpha)} \int_{t}^{0} (jw - jt)^{\alpha-1} D_{x_0-}^{\alpha} f(x_0+jw)\, jdw \right] \\
&= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k + \frac{j^{\alpha}}{\Gamma(\alpha)} \int_{t}^{0} (w - t)^{\alpha-1} D_{x_0-}^{\alpha} f(x_0+jw)\, dw \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k + \frac{\sum_{j=1}^{r} \alpha_j j^{\alpha}}{\Gamma(\alpha)} \int_{t}^{0} (w - t)^{\alpha-1} D_{x_0-}^{\alpha} f(x_0+jw)\, dw \\
&= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k + \frac{\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j}}{\Gamma(\alpha)} \int_{t}^{0} (w - t)^{\alpha-1} D_{x_0-}^{\alpha} f(x_0+jw)\, dw \\
&= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k + \frac{1}{\Gamma(\alpha)} \int_{t}^{0} (w - t)^{\alpha-1} \cdot \\
&\quad \left[\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{x_0-}^{\alpha} f(x_0+jw)\right) + (-1)^r \left(D_{x_0-}^{\alpha} f(x_0)\right) \right] dw
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t}^{0} (w - t)^{\alpha-1} \left[\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} D_{x_0-}^{\alpha} f(x_0+jw) \right] dw \\
&= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k + \frac{1}{\Gamma(\alpha)} \int_{t}^{0} (w - t)^{\alpha-1} \left(\Delta_w^r \left(D_{x_0-}^{\alpha} f\right)\right)(x_0)\, dw.
\end{aligned}
$$

We have proved $(t < 0)$

$$
\begin{aligned}
\sum_{j=0}^{r} \alpha_j \left[f(x_0+jt) - f(x_0)\right] &= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t}^{0} (w - t)^{\alpha-1} \left(\Delta_w^r \left(D_{x_0-}^{\alpha} f\right)\right)(x_0)\, dw.
\end{aligned}
$$

The last proves (13.19).

Next we notice that

$$\begin{aligned}
|B| &\leq \frac{1}{\Gamma(\alpha)} \int_t^0 (w-t)^{\alpha-1} \left|\Delta_w^r \left(D_{x_0-}^{\alpha} f\right)(x_0)\right| dw \\
&\leq \frac{1}{\Gamma(\alpha)} \int_t^0 (w-t)^{\alpha-1} \omega_r \left(D_{x_0-}^{\alpha} f, \xi \frac{|w|}{\xi}\right) dw \\
&\leq \frac{\omega_r \left(D_{x_0-}^{\alpha} f, \xi\right)}{\Gamma(\alpha)} \int_t^0 (w-t)^{\alpha-1} \left(1+\frac{|w|}{\xi}\right)^r dw \\
&= \frac{\omega_r \left(D_{x_0-}^{\alpha} f, \xi\right)}{\Gamma(\alpha)} \int_t^0 (w-t)^{\alpha-1} \left(1-\frac{w}{\xi}\right)^r dw \\
&= :(*).
\end{aligned}$$

We observe that

$$\int_t^0 (w-t)^{\alpha-1} \left(1-\frac{w}{\xi}\right)^r dw = \int_t^0 (w-t)^{\alpha-1} \left(\sum_{k=0}^r \binom{r}{k} (-1)^k \frac{w^k}{\xi^k}\right) dw$$

$$= \sum_{k=0}^r \binom{r}{k} \frac{1}{\xi^k} \int_t^0 (0-w)^{(k+1)-1} (w-t)^{\alpha-1} dw = \sum_{k=0}^r \frac{r!}{(r-k)!\xi^k} \frac{\Gamma(\alpha)}{\Gamma(\alpha+k+1)} |t|^{\alpha+k}.$$

Consequently we obtain

$$(*) = \omega_r \left(D_{x_0-}^{\alpha} f, \xi\right) \left(\sum_{k=0}^r \frac{r!}{(r-k)!\xi^k} \frac{|t|^{\alpha+k}}{\Gamma(\alpha+k+1)}\right),$$

proving (13.20). ■

In the next, let $\xi > 0$, $x, x_0 \in \mathbb{R}$, $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, with $\left\|f^{(m)}\right\|_\infty < \infty$.

Consider the Lebesgue integral

$$P_{r,\xi}(f,x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^r \alpha_j f(x+jt)\right) e^{-|t|/\xi} dt. \tag{13.21}$$

We assume $P_{r,\xi}(f,x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$.

Notice that

$$\frac{1}{2\xi} \int_{-\infty}^{\infty} e^{-|t|/\xi} dt = 1, \tag{13.22}$$

$$P_{r,\xi}(c,x) = c, c \text{ constant}, \tag{13.23}$$

and

$$\begin{aligned} P_{r,\xi}(f,x_0)-f(x_0) &= \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)\right)e^{-|t|/\xi}dt \\ &= \frac{1}{2\xi}\int_{-\infty}^{0}\left(\sum_{j=0}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)\right)e^{-|t|/\xi}dt \\ &\quad +\frac{1}{2\xi}\int_{0}^{\infty}\left(\sum_{j=0}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)\right)e^{-t/\xi}dt \\ &= :\Lambda. \end{aligned} \tag{13.24}$$

We have

$$\int_{-\infty}^{\infty}t^k e^{-|t|/\xi}dt=\begin{cases}0, & k \text{ odd},\\ 2k!\xi^{k+1}, & k \text{ even}.\end{cases} \tag{13.25}$$

We present

Theorem 13.18. Let $f\in C^m(\mathbb{R})$, $m=\lceil\alpha\rceil$, $\alpha>0$, with $\left\|f^{(m)}\right\|_{\infty}<\infty$, $\xi>0$, $x_0\in\mathbb{R}$. Then

1)

$$\left|P_{r,\xi}(f,x_0)-f(x_0)-\sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor}f^{(2\rho)}(x_0)\,\delta_{2\rho}\xi^{2\rho}\right| \tag{13.26}$$
$$\leq\lfloor er!\rfloor\cdot\xi^{\alpha}\cdot\max\left\{\omega_r\left(D_{x_0-}^{\alpha}f,\xi\right),\omega_r\left(D_{*x_0}^{\alpha}f,\xi\right)\right\}.$$

(Above if $m=1,2$ the sum disappears).

2)

$$\left\|P_{r,\xi}(f,\cdot)-f(\cdot)-\sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor}f^{(2\rho)}(\cdot)\,\delta_{2\rho}\xi^{2\rho}\right\|_{\infty} \tag{13.27}$$
$$\leq\lfloor er!\rfloor\cdot\xi^{\alpha}\cdot\sup_{x\in\mathbb{R}}\left\{\max\left(\omega_r\left(D_{x-}^{\alpha}f,\xi\right),\omega_r\left(D_{*x}^{\alpha}f,\xi\right)\right)\right\}.$$

We further give

Theorem 13.19. All as in Theorem 13.18. Additionally suppose that $\left\|f^{(2\rho)}\right\|_{\infty}<\infty$, $\rho=1,\ldots,\left\lfloor\frac{m-1}{2}\right\rfloor$. Then

$$\begin{aligned}\left\|P_{r,\xi}(f,\cdot)-f(\cdot)\right\|_{\infty} &\leq \sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor}\left\|f^{(2\rho)}\right\|_{\infty}\left|\delta_{2\rho}\right|\xi^{2\rho} \\ &\quad+\lfloor er!\rfloor\cdot\xi^{\alpha}\cdot\sup_{x\in\mathbb{R}}\left\{\max\left(\omega_r\left(D_{x-}^{\alpha}f,\xi\right),\omega_r\left(D_{*x}^{\alpha}f,\xi\right)\right)\right\}.\end{aligned} \tag{13.28}$$

Assuming further that both $(D^{\alpha}_{*x}f)(t)$, $(D^{\alpha}_{x-}f)(t)$ are bounded in $(t,x) \in \mathbb{R}^2$, we get, as $\xi \to 0+$, that $P_{r,\xi} \overset{u}{\to} I$ (uniformly), see (13.11).

Or, by assuming (13.13) we get (13.14), that is from (13.28) we obtain again $P_{r,\xi} \overset{u}{\to} I$ (unit operator), as $\xi \to 0+$.

Proof of Theorem 13.18. We use here heavily Theorem 13.17. We see that (see (13.24))

$$\begin{aligned}
\Lambda &= \frac{1}{2\xi}\int_{-\infty}^{0}\left\{\left[\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\delta_k t^k\right]e^{-|t|/\xi}\right. \\
&\quad \left.+\left[\frac{e^{-|t|/\xi}}{\Gamma(\alpha)}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D^{\alpha}_{x_0-}f\right)\right)(x_0)\,dw\right]\right\}dt \\
&\quad +\frac{1}{2\xi}\int_0^{\infty}\left\{\left[\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\delta_k t^k\right]e^{-t/\xi}\right. \\
&\quad \left.+\left[\frac{e^{-t/\xi}}{\Gamma(\alpha)}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D^{\alpha}_{*x_0}f\right)\right)(x_0)\,dw\right]\right\}dt \\
&= \frac{1}{2\xi}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\delta_k t^k\right]e^{-|t|/\xi}dt \\
&\quad +\frac{1}{2\xi\Gamma(\alpha)}\int_{-\infty}^{0}\left[e^{-|t|/\xi}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D^{\alpha}_{x_0-}f\right)\right)(x_0)\,dw\right]dt \\
&\quad +\frac{1}{2\xi\Gamma(\alpha)}\int_0^{+\infty}\left[e^{-t/\xi}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D^{\alpha}_{*x_0}f\right)\right)(x_0)\,dw\right]dt \\
&= \sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor} f^{(2\rho)}(x_0)\,\delta_{2\rho}\xi^{2\rho} + >> .
\end{aligned}$$

Hence

$$\theta(x_0) := P_{r,\xi}(f,x_0) - f(x_0) - \sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor} f^{(2\rho)}(x_0)\,\delta_{2\rho}\xi^{2\rho}$$

$$\begin{aligned}
&= \frac{1}{2\xi}\left[\int_{-\infty}^{0}\left[e^{-|t|/\xi}\frac{1}{\Gamma(\alpha)}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D^{\alpha}_{x_0-}f\right)\right)(x_0)\,dw\right]dt\right. \\
&\quad \left.+\int_0^{+\infty}\left[e^{-t/\xi}\frac{1}{\Gamma(\alpha)}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D^{\alpha}_{*x_0}f\right)\right)(x_0)\,dw\right]dt\right].
\end{aligned}$$

So that

$$\theta(x_0) = \frac{1}{2\xi}\left[\int_{-\infty}^{0} e^{-|t|/\xi}B(t,x_0)\,dt + \int_0^{+\infty} e^{-t/\xi}A(t,x_0)\,dt\right].$$

Consequently we derive

$$|\theta(x_0)| \leq \frac{1}{2\xi}\left[\int_{-\infty}^{0} e^{-|t|/\xi}\left|B(t,x_0)\right|dt+\int_{0}^{+\infty} e^{-t/\xi}\left|A(t,x_0)\right|dt\right]$$

$$\leq \frac{1}{2\xi}\left[\left(\int_{-\infty}^{0} e^{-|t|/\xi}\left(\sum_{k=0}^{r}\frac{r!}{(r-k)!}\frac{|t|^{\alpha+k}}{\xi^k\Gamma(\alpha+k+1)}\right)dt\right)\omega_r\left(D_{x_0-}^{\alpha}f,\xi\right)\right.$$
$$\left.+\left(\int_{0}^{+\infty} e^{-t/\xi}\left(\sum_{k=0}^{r}\frac{r!}{(r-k)!}\frac{t^{\alpha+k}}{\xi^k\Gamma(\alpha+k+1)}\right)dt\right)\omega_r\left(D_{*x_0}^{\alpha}f,\xi\right)\right]$$

(Call $\mathcal{M}(x_0) := \max\left\{\omega_r\left(D_{x_0-}^{\alpha}f,\xi\right),\omega_r\left(D_{*x_0}^{\alpha}f,\xi\right)\right\}$.)

$$\leq \frac{\mathcal{M}(x_0)}{2\xi}\left[\int_{-\infty}^{\infty} e^{-|t|/\xi}\left(\sum_{k=0}^{r}\frac{r!}{(r-k)!}\frac{|t|^{\alpha+k}}{\xi^k\Gamma(\alpha+k+1)}\right)dt\right]$$
$$= \frac{\mathcal{M}(x_0)}{2\xi}\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\alpha+k+1)\,\xi^k}\int_{-\infty}^{\infty} e^{-|t|/\xi}|t|^{\alpha+k}dt\right]$$
$$= \mathcal{M}(x_0)\xi^{\alpha}\cdot\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\alpha+k+1)}\int_{0}^{\infty} e^{-t/\xi}\left(\frac{t}{\xi}\right)^{\alpha+k}d\frac{t}{\xi}\right]$$
$$= \xi^{\alpha}\mathcal{M}(x_0)\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\alpha+k+1)}\int_{0}^{\infty} e^{-u}u^{(\alpha+k+1)-1}du\right]$$
$$= \xi^{\alpha}\mathcal{M}(x_0)\left(\sum_{k=0}^{r}\frac{r!}{(r-k)!}\right).$$

We found that

$$|\theta(x_0)| \leq \left(r!\sum_{k=0}^{r}\frac{1}{(r-k)!}\right)\xi^{\alpha}\mathcal{M}(x_0)$$
$$= \left(r!\sum_{k=0}^{r}\frac{1}{k!}\right)\xi^{\alpha}\mathcal{M}(x_0)$$
$$= \lfloor er!\rfloor\,\xi^{\alpha}\mathcal{M}(x_0),$$

that is proving (13.26). ■

Next we give a fractional Voronovskaya type result regarding singular integral operators.

Theorem 13.20. Here $f\in C^m(\mathbb{R})$, $m\in\mathbb{N}$, $m=\lceil\alpha\rceil$, $\alpha>0$, $\left\|f^{(m)}\right\|_{\infty}<\infty$, and $\left\|D_{x-}^{\alpha}f(y)\right\|_{\infty}\leq M_1$, $\left\|D_{*x}^{\alpha}f(y)\right\|_{\infty}\leq M_2$, where $M_1,M_2>0$, for any $x,y\in\mathbb{R}$.

Then

$$P_{r,\xi}\left(f,x\right)-f\left(x\right)-\sum_{\rho=1}^{\left\lfloor\frac{(m-1)}{2}\right\rfloor}f^{(2\rho)}\left(x\right)\delta_{2\rho}\xi^{2\rho}=o\left(\xi^{\alpha-\beta}\right), \tag{13.29}$$

$0<\beta<\alpha$, as $\xi\to 0+$.

I.e.

$$P_{r,\xi}\left(f,x\right)-f\left(x\right)=\sum_{\rho=1}^{\left\lfloor\frac{(m-1)}{2}\right\rfloor}f^{(2\rho)}\left(x\right)\xi^{2\rho}\left(\sum_{j=1}^{r}\alpha_j j^{2\rho}\right)+o\left(\xi^{\alpha-\beta}\right), \tag{13.30}$$

where $0<\beta<\alpha$.

(Above if $m=1,2$ the sum disappears.)

Proof. Since $f\in C^m\left(\mathbb{R}\right)$, $m=\lceil\alpha\rceil$, $\alpha>0$, by (13.2) and (13.4) we obtain

$$f\left(x\right)=\sum_{k=0}^{m-1}\frac{f^{(k)}\left(x_0\right)}{k!}\left(x-x_0\right)^k+\frac{D_{*x_0}^{\alpha}f\left(\zeta\right)}{\Gamma\left(\alpha+1\right)}\left(x-x_0\right)^{\alpha},$$

$\forall x\geq x_0$, here $x_0<\zeta<x$ and

$$f\left(x\right)=\sum_{k=0}^{m-1}\frac{f^{(k)}\left(x_0\right)}{k!}\left(x-x_0\right)^k+\frac{D_{x_0-}^{\alpha}f\left(\zeta\right)}{\Gamma\left(\alpha+1\right)}\left(x_0-x\right)^{\alpha},$$

$\forall x<x_0$, here $x<\zeta<x_0$.

So we find $(j=1,\ldots,r)$

$$f\left(x+jt\right)-f\left(x\right)=\sum_{k=1}^{m-1}\frac{f^{(k)}\left(x\right)}{k!}\left(jt\right)^k+\frac{D_{*x}^{\alpha}f\left(\zeta\right)}{\Gamma\left(\alpha+1\right)}\left(jt\right)^{\alpha},$$

for $x<\zeta<x+jt$, here $t\geq 0$.

Also it holds

$$f\left(x+jt\right)-f\left(x\right)=\sum_{k=1}^{m-1}\frac{f^{(k)}\left(x\right)}{k!}\left(jt\right)^k+\frac{D_{x-}^{\alpha}f\left(\zeta\right)}{\Gamma\left(\alpha+1\right)}\left(jt\right)^{\alpha},$$

for $x+jt<\zeta<x$, here $t<0$.

Notice that

$$\begin{aligned} P_{r,\xi}(f,x) - f(x) &= \frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j \int_{-\infty}^{\infty}(f(x+jt)-f(x))\,e^{-|t|/\xi}dt\right) \\ &= \frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j \left[\int_{-\infty}^{0}(f(x+jt)-f(x))\,e^{-|t|/\xi}dt\right.\right. \\ &\left.\left. + \int_{0}^{\infty}(f(x+jt)-f(x))\,e^{-t/\xi}dt\right]\right) \end{aligned}$$

$$\begin{aligned} = \frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j \left[\int_{-\infty}^{0}\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}(jt)^k + \frac{D_{x-}^{\alpha}f(\zeta)}{\Gamma(\alpha+1)}(jt)^{\alpha}\right)e^{-|t|/\xi}dt\right.\right. \\ \left.\left. + \int_{0}^{\infty}\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}(jt)^k + \frac{D_{*x}^{\alpha}f(\zeta)}{\Gamma(\alpha+1)}(jt)^{\alpha}\right)e^{-t/\xi}dt\right]\right) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j \left[\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}j^k\int_{-\infty}^{\infty}t^k e^{-|t|/\xi}dt\right)\right.\right. \\ &\left.\left. + \frac{j^{\alpha}}{\Gamma(\alpha+1)}\left(\int_{-\infty}^{0}t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-|t|/\xi}dt + \int_{0}^{\infty}t^{\alpha}\left(D_{*x}^{\alpha}f(\zeta)\right)e^{-t/\xi}dt\right)\right]\right) \\ &= \frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j \left[\left(2\sum_{\rho=1}^{\left\lfloor\frac{(m-1)}{2}\right\rfloor}f^{(2\rho)}(x)\,j^{2\rho}\xi^{2\rho+1}\right)\right.\right. \\ &\left.\left. + \frac{j^{\alpha}}{\Gamma(\alpha+1)}\left(\int_{-\infty}^{0}t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-|t|/\xi}dt + \int_{0}^{\infty}t^{\alpha}\left(D_{*x}^{\alpha}f(\zeta)\right)e^{-t/\xi}dt\right)\right]\right) \\ &= \sum_{\rho=1}^{\left\lfloor\frac{(m-1)}{2}\right\rfloor}f^{(2\rho)}(x)\left(\sum_{j=1}^{r}\alpha_j j^{2\rho}\right)\xi^{2\rho} \\ &+ \frac{\sum_{j=0}^{r}\alpha_j j^{\alpha}}{2\xi\Gamma(\alpha+1)}\left(\int_{-\infty}^{0}t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-|t|/\xi}dt + \int_{0}^{\infty}t^{\alpha}\left(D_{*x}^{\alpha}f(\zeta)\right)e^{-t/\xi}dt\right). \end{aligned}$$

We derive that

$$\begin{aligned} T \quad &: = P_{r,\xi}(f,x) - f(x) - \sum_{\rho=1}^{\left\lfloor\frac{(m-1)}{2}\right\rfloor}f^{(2\rho)}(x)\,\delta_{2\rho}\xi^{2\rho} \\ &= \frac{\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}}{2\xi\Gamma(\alpha+1)}\left[\int_{-\infty}^{0}t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-|t|/\xi}dt + \int_{0}^{\infty}t^{\alpha}\left(D_{*x}^{\alpha}f(\zeta)\right)e^{-t/\xi}dt\right]. \end{aligned}$$

We consider

$$\Delta_\xi := \frac{1}{\xi^\alpha} T.$$

Then we have

$$\Delta_\xi = \frac{\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j}}{2\xi^{\alpha+1}\Gamma(\alpha+1)} \left[\int_{-\infty}^{0} t^\alpha \left(D_{x-}^\alpha f(\zeta) \right) e^{-|t|/\xi} dt + \int_0^\infty t^\alpha \left(D_{*x}^\alpha f(\zeta) \right) e^{-t/\xi} dt \right]$$

$$= \frac{1}{2\xi^{\alpha+1}\Gamma(\alpha+1)} \left[\int_{-\infty}^{0} t^\alpha \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{x-}^\alpha f(\zeta) \right) \right) e^{-|t|/\xi} dt \right.$$
$$\left. + \int_0^\infty t^\alpha \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{*x}^\alpha f(\zeta) \right) \right) e^{-t/\xi} dt \right].$$

Set

$$\phi_\alpha(x,t) = \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{x-}^\alpha f(\zeta) \right),$$

and

$$\psi_\alpha(x,t) = \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{*x}^\alpha f(\zeta) \right).$$

Therefore

$$\Delta_\xi = \frac{1}{2\Gamma(\alpha+1)\xi^{\alpha+1}} \left[\int_{-\infty}^{0} t^\alpha \phi_\alpha(x,t) e^{-|t|/\xi} dt + \int_0^\infty t^\alpha \psi_\alpha(x,t) e^{-t/\xi} dt \right].$$

By theorem's assumptions we derive

$$|\phi_\alpha(x,t)| \leq \left(\sum_{j=1}^{r} \binom{r}{j} \right) M_1 = (2^r - 1) M_1,$$
$$|\psi_\alpha(x,t)| \leq (2^r - 1) M_2,$$

$\forall x, t \in \mathbb{R}$.

Call $M_3 = \max(M_1, M_2)$.

Thus

$$|\phi_\alpha(x,t)|, |\psi_\alpha(x,t)| \leq (2^r - 1) M_3,$$

$\forall x, t \in \mathbb{R}$.

Consequently we obtain

$$\begin{aligned}
|\Delta_\xi| &\leq \frac{(2^r-1)\,M_3}{2\Gamma(\alpha+1)\,\xi^{\alpha+1}}\left[\int_{-\infty}^{\infty}|t|^\alpha\, e^{-|t|/\xi}dt\right]\\
&= \frac{(2^r-1)\,M_3}{\Gamma(\alpha+1)\,\xi^{\alpha+1}}\left[\int_0^{\infty}u^\alpha e^{-u/\xi}du\right]\\
&= \frac{(2^r-1)\,M_3}{\Gamma(\alpha+1)}\left[\int_0^{\infty}\left(\frac{u}{\xi}\right)^\alpha e^{-u/\xi}d\frac{u}{\xi}\right]\\
&= \frac{(2^r-1)\,M_3}{\Gamma(\alpha+1)}\left[\int_0^{\infty}e^{-w}w^{(\alpha+1)-1}dw\right]\\
&= (2^r-1)\,M_3.
\end{aligned}$$

That is

$$|\Delta_\xi| \leq (2^r-1)\,M_3,$$

and

$$|T| \leq (2^r-1)\,M_3\xi^\alpha,$$

resulting into $T = O(\xi^\alpha)$.

However, let $0 < \beta < \alpha$, then easily we get

$$\frac{|T|}{\xi^{\alpha-\beta}} \leq (2^r-1)\,M_3\xi^\beta \to 0, \text{ as } \xi \to 0+.$$

I.e. $|T| = o\left(\xi^{\alpha-\beta}\right)$, proving the claim. ∎

13.3 Applications

Let $\alpha = \frac{1}{2}$, $\left\lceil\frac{1}{2}\right\rceil = 1$, $f \in C^1(\mathbb{R})$, $\|f'\|_\infty < \infty$, $\xi > 0$, $x_0 \in \mathbb{R}$.

Then by Theorem 13.18, (13.26), we derive

$$|P_{r,\xi}(f,x_0) - f(x_0)| \leq \lfloor er! \rfloor \cdot \sqrt{\xi} \cdot \max\left\{\omega_r\left(D_{x_0-}^{\frac{1}{2}}f,\xi\right), \omega_r\left(D_{*x_0}^{\frac{1}{2}}f,\xi\right)\right\}. \tag{13.31}$$

Consequently it holds

$$\|P_{r,\xi}(f) - f\|_\infty \leq \lfloor er! \rfloor \cdot \sqrt{\xi} \cdot \sup_{x\in\mathbb{R}}\left[\max\left\{\omega_r\left(D_{x-}^{\frac{1}{2}}f,\xi\right), \omega_r\left(D_{*x}^{\frac{1}{2}}f,\xi\right)\right\}\right]. \tag{13.32}$$

Above we suppose $\left(D_{x-}^{\frac{1}{2}}f\right)(y)$, $\left(D_{*x}^{\frac{1}{2}}f\right)(y)$ are bounded in $(x,y) \in \mathbb{R}^2$, for the convergence of $P_{r,\xi} \to I$, as $\xi \to 0+$.

By fractional Voronovskaya type Theorem 13.20, (13.29), under the above assumptions we get

$$P_{r,\xi}(f,x) - f(x) = o\left(\xi^{\frac{1}{2}-\beta}\right), \tag{13.33}$$

where $0 < \beta < \frac{1}{2}$.

Note 13.21. The integrals $P_{r,\xi}$ are not in general positive operators. Take $f(t) = t^2 \geq 0$, $r = 2$, $\alpha = 2.5$, $x = 0$. Then $\alpha_1 = -2$, $\alpha_2 = 2^{-2.5}$. We find

$$P_{2,\xi}\left(t^2, 0\right) = 2\xi^2 \left(-2 + 4 \cdot 2^{-2.5}\right) < 0,$$

proving the claim. ■

14

Multivariate Generalized Picard Singular Integral Operators

In this chapter, we study the type of Picard singular integral operators on $\mathbb{R}^n$ constructed by means of the nonisotropic β-distance and the q-exponential functions. The central role here is played by the concept of nonisotropic β-distance, which allows us to improve and generalize the results given for classical Picard and q-Picard singular integral operators. In order to obtain the rate of convergence we introduce a modulus of continuity depending on the nonisotropic β-distance with respect to the uniform norm. Then we give the definition of β-Lebesgue points depending on nonisotropic β-distance and a pointwise approximation result shown at these points. Futhermore, we present the global smoothness preservation property of these type of Picard singular integral operators and prove a sharp inequality. This chapter relies on [61].

14.1 Background

The q-analysis is extensively used in approximation theory, especially in the study of various sequences of linear positive operators such as Bernstein [248], Szász Mirakyan [98], Meyer, König and Zeller operators [276], Bleimann, Butzer and Hahn operators [100] and singular integral operators such as the Picard and Gauss-Weierstrass operators (see [99], [97] and [101]). In [97] we introduced a generalization of the well known Picard singular integral operators (see [67]) by using the q-analogue of the Euler

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 191–205.
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Gamma integral, and called the operators as $q-$the Picard singular integral operators. We have shown that these generalized operators have a more flexible rate of convergence than the classical Picard singular integral operators. Also these operators retain some approximation properties regarding, direct and pointwise approximation results in $L_p(\mathbb{R})$ and weighted $-L_p(\mathbb{R})$ spaces, global smoothness preservation properties and a Voronovskaya type theorem (see [99], [100], [97], [96], [95], [101]).

In this chapter, we introduce the multivariate variant of the q-Picard singular integral defined by (14.1) depending on the nonisotropic β-distance. Then we show that from the rate of convergence point of view these operators with this construction are more flexible than both of the classical Picard and q-Picard singular integral operators. That is, depending on our selection of the parameter q and the parameter β (which is defined below) the rate of convergence can be refined. Also we define a modulus of continuity which is harmonious with these operators. Finally for these operators a pointwise approximation result is shown and the global smoothness preservation property is given.

Recall that, the generalization of the Picard singular integral in the multivariate case given [67] and some approximation properties of them have been studied initially (see [17], [67], [69], [164] and [163]). Also the generalization of the classical Picard and Gauss-Weierstrass operators depending on β-distance and some pointwise approximation results have been presented in [96] and [95].

Now we give the concept of the nonisotropic β-distance. Let $n \in \mathbb{N}$ and $\beta_1, \beta_2, \cdots, \beta_n$ be positive numbers with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ and

$$||\mathbf{x}||_\beta = \left(|x_1|^{\frac{1}{\beta_1}} + \cdots + |x_n|^{\frac{1}{\beta_n}}\right)^{\frac{|\beta|}{n}}, \quad \mathbf{x} \in \mathbb{R}^n.$$

The expression $||\mathbf{x}||_\beta$ is called the nonisotropic β-distance between $\mathbf{x}$ and $\mathbf{0}$. Note that this distance has the following properties of homogeneity for positive t :

$$\left(\left|t^{\beta_1}x_1\right|^{\frac{1}{\beta_1}} + \cdots + \left|t^{\beta_n}x_n\right|^{\frac{1}{\beta_n}}\right)^{\frac{|\beta|}{n}} = t^{\frac{|\beta|}{n}} ||\mathbf{x}||_\beta .$$

Also, nonisotropic β-distance has following properties.

1. $||\mathbf{x}||_\beta = 0 \Leftrightarrow \mathbf{x} = 0$,
2. $\left|\left|t^\beta \mathbf{x}\right|\right|_\beta = t^{\frac{|\beta|}{n}} ||\mathbf{x}||_\beta$,
3. $||\mathbf{x}+\mathbf{y}||_\beta \leq M_\beta \left(||\mathbf{x}||_\beta + ||\mathbf{y}||_\beta\right)$,
   where $\beta_{\min} = \min\{\beta_1, \beta_2, \ldots \beta_n\}$ and $M_\beta = 2^{\left(1+\frac{1}{\beta_{\min}}\right)\frac{|\beta|}{n}}$, (see [193]).

It can be seen that nonisotropic β-distance becomes the ordinary Euclidean distance $|\mathbf{x}|$ for $\beta_i = \frac{1}{2}$, $i = 1, 2, \ldots, n$. Also, this distance does not satisfy the triangle inequality.

Now we recall that the $q-$generalizations of Picard singular integrals given in [97]. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $\lambda > 0$ and $0 < q < 1$, the $q-$generalizations of Picard singular integrals of f are

$$P_\lambda (f; q, x) \equiv P_\lambda (f; x) := \frac{(1-q)}{2 [\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} \frac{f(x+t)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt, \tag{14.1}$$

where the $q-$extension of exponential function e^x is

$$E_q (x) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;\, q)_n} x^n = (-x;\, q)_\infty\,, \tag{14.2}$$

with $(a;\, q)_n = \prod_{k=0}^{n} \left(1 - aq^k\right)$ and $(-x;\, q)_\infty = \prod_{k=0}^{\infty} \left(1 + xq^k\right)$.

For $q > 0$, $q-$number is

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & q \neq 1 \\ \lambda, & q = 1 \end{cases}$$

for all nonnegative λ. If λ is an integer, i.e. $\lambda = n$ for some n, we write $[n]_q$ and call it $q-$integer. Also, we define a $q-$factorial as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [1]_q\,, & n = 1, 2, ... \\ 1 & n = 0. \end{cases}.$$

For integers $0 \le k \le n$, the $q-$binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}.$$

For details see [170].

Another needed formula is $q-$extension of Euler integral representation for the gamma function given in [102] and [10] for $0 < q < 1$

$$c_q (x) \Gamma_q (x) = \frac{1-q}{\ln q^{-1}} q^{\frac{x(x-1)}{2}} \int_0^{\infty} \frac{t^{x-1}}{E_q((1-q)t)} dt\,, \quad \operatorname{Re} x > 0 \tag{14.3}$$

where $\Gamma_q (x)$ is the $q-$gamma function defined by

$$\Gamma_q (x) = \frac{(q;\, q)_\infty}{(q^x;\, q)_\infty} (1-q)^{1-x}\,, \;\; 0 < q < 1$$

and $c_q (x)$ satisfies the following conditions:

1. $c_q(x+1) = c_q(x)$
2. $c_q(n) = 1,\ n = 0,\ 1,\ 2, ...$
3. $\lim_{q \to 1^-} c_q(x) = 1.$

When $x = n+1$ with n a nonnegative integer, we obtain

$$\Gamma_q(n+1) = [n]_q!. \tag{14.4}$$

14.2 Construction of a Family of Singular Integral Operators

In order to introduce the new singular integral operators, we start with the following elementary lemma.

Lemma 14.1. For all $\lambda > 0$, $n \in \mathbb{N}$ and $\beta_i \in (0,\ \infty)$ $(i = 1, 2, \ldots n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ we have

$$\frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \mathcal{P}_\lambda(\beta, \mathbf{t})\, d\mathbf{t} = 1,$$

where

$$\mathcal{P}_\lambda(\beta, \mathbf{t}) = 1/E_q\left(\frac{(1-q)\,||\mathbf{t}||_\beta}{[\lambda]_q^{\frac{|\beta|}{n}}}\right), \tag{14.5}$$

and

$$c(n,\ \beta,\ q)^{-1} = \frac{n}{2\,|\beta|}\omega_{\beta,\,n-1}\Gamma_q(n)\,\frac{\ln q^{-1}}{(1-q)\,q^{\frac{n(n-1)}{2}}}. \tag{14.6}$$

Proof. The $\mathbf{t} = [\lambda]_q^\beta\,\mathbf{x}$ change of variable gives that

$$\frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \mathcal{P}_\lambda(\beta, \mathbf{t})\, d\mathbf{t} = c(n,\ \beta,\ q) \int_{\mathbb{R}^n} \frac{d\mathbf{x}}{E_q\left((1-q)\,||\mathbf{x}||_\beta\right)}.$$

We use generalized β-spherical coordinates ([193]) and consider the transformation

$$\begin{aligned}
x_1 &= (u\cos\theta_1)^{2\beta_1} \\
x_2 &= (u\sin\theta_1\cos\theta_2)^{2\beta_2} \\
&\ \ \vdots \\
x_{n-1} &= (u\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1})^{2\beta_{n-1}} \\
x_n &= (u\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1})^{2\beta_n},
\end{aligned}$$

where $0 \leq \theta_1, \theta_2, \cdots, \theta_{n-2} \leq \pi$, $0 \leq \theta_{n-1} \leq 2\pi$, $u \geq 0$. Denoting the Jacobian of this transformation by $J_\beta(u, \theta_1, \ldots, \theta_{n-1})$ we get

$$J_\beta(u, \theta_1, \ldots, \theta_{n-1}) = u^{2|\beta|-1}\Omega_\beta(\theta),$$

where $\Omega_\beta(\theta) = 2^n \beta_1 \ldots \beta_n \prod_{j=1}^{n-1} (\cos\theta_j)^{2\beta_j - 1} (\sin\theta_j)^{\sum_{k=j}^{j+1} 2\beta_k - 1}$. We can easily see that the integral

$$\omega_{\beta,\, n-1} = \int_{S^{n-1}} \Omega_\beta(\theta)\, d\theta \tag{14.7}$$

is finite, where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Thus we have

$$c(n,\ \beta,\ q) \int_{\mathbb{R}^n} \frac{d\mathbf{x}}{E_q\left((1-q)\,||\mathbf{x}||_\beta\right)} = c(n,\ \beta,\ q) \int_0^\infty \int_{S^{n-1}} \frac{u^{2|\beta|-1}\Omega_\beta(\theta)\, d\theta du}{E_q\left((1-q)\, u^{\frac{2|\beta|}{n}}\right)}.$$

Using (14.7), we derive

$$\int_{\mathbb{R}^n} \frac{d\mathbf{x}}{E_q\left((1-q)\,||\mathbf{x}||_\beta\right)} = c(n,\ \beta,\ q) \frac{n}{2|\beta|} \omega_{\beta,\, n-1} \int_0^\infty \frac{u^{n-1} du}{E_q((1-q)\, u)}$$

If we use (14.3) and choose

$$c(n,\ \beta,\ q)^{-1} = \frac{n}{2|\beta|} \omega_{\beta,\, n-1} \Gamma_q(n) \frac{\ln q^{-1}}{(1-q)\, q^{\frac{n(n-1)}{2}}}$$

then we have the desired result. ■

Definition 14.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. For $0 < q < 1, \lambda > 0$, $n \in \mathbb{N}$ and $\beta_i \in (0,\ \infty)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, the q-Picard integral depending on β-distance of f is

$$\begin{aligned} P_{\lambda,\beta}(f; q,\, \mathbf{x}) &\equiv P_{\lambda,\beta}(f;\, \mathbf{x}) \\ : &= \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} f(\mathbf{x} + \mathbf{t})\, \mathcal{P}_\lambda(\beta, \mathbf{t})\, d\mathbf{t}, \end{aligned} \tag{14.8}$$

where $\mathcal{P}_\lambda(\beta, \mathbf{t})$ and $c(n,\ \beta,\ q)$ defined as in (14.5) and (14.6), respectively.

Note that, if we take $\beta_i = \frac{1}{2}$, $i = 1, 2, \ldots, n$, it appears $P_{\lambda,\frac{1}{2}}(f; q,\, \mathbf{x})$ operators introduced in [97]. If we take $q \to 1$,then $P_{\lambda,\frac{1}{2}}(f; 1,\, \mathbf{x})$ operators are classical Picard singular integral (see [67]).

14.3 Approximation Properties of the Operator $P_{\lambda,\beta}(f;\cdot)$

In this section, we first introduce a nonisotropic modulus of continuity reflecting the nonisotropic β-distance and the operator $P_{\lambda,\beta}(f;\cdot)$. Then we estimate the rate of convergence. Secondly, we introduce β-Lebesgue points of f and give a pointwise approximation theorem on these points.

Definition 14.3. Let $f \in C(\mathbb{R}^n)$, $n \in \mathbb{N}$ and $\beta_i \in (0, \infty)$ $(i = 1, 2, \ldots n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. For every $\delta > 0$, nonisotropic moduli of continuity of f is

$$\omega_\beta(f;\ \delta) = \sup_{\substack{\mathbf{x}\in\mathbb{R}^n \\ ||\mathbf{h}||_\beta \le \delta}} |f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|.$$

Lemma 14.4. Let $f \in C(\mathbb{R}^n)$ and $\beta_i \in (0, 1]$ $(i = 1, 2, ..., n)$ with $|\beta| = \beta_1 + \cdots + \beta_n$. For $\delta > 0$ and $C > 0$, then

$$\omega_\beta\left(f;\ C^{\frac{|\beta|}{n}}\delta\right) \le (1+C)\,\omega_\beta(f;\ \delta).$$

Proof. For positive integer k, we can write

$$\begin{aligned}
\omega_\beta\left(f;\ k^{\frac{|\beta|}{n}}\delta\right) &= \sup_{\substack{\mathbf{x}\in\mathbb{R}^n \\ ||\mathbf{h}||_\beta \le \delta}} \left|f\left(\mathbf{x}+k^\beta\mathbf{h}\right) - f(\mathbf{x})\right| \\
&= \sup_{\substack{\mathbf{x}\in\mathbb{R}^n \\ ||\mathbf{h}||_\beta \le \delta}} \left|\sum_{s=1}^{k} f\left(\mathbf{x}+s^\beta\mathbf{h}\right) - f\left(\mathbf{x}+(s-1)^\beta\,\mathbf{h}\right)\right| \\
&\le \sup_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \left|\left|s^\beta\mathbf{h}-(s-1)^\beta\mathbf{h}\right|\right|_\beta \le \delta}} \sum_{s=1}^{k} \left|f\left(\mathbf{x}+s^\beta\mathbf{h}\right) - f\left(\mathbf{x}+(s-1)^\beta\,\mathbf{h}\right)\right| \\
&\le k\omega_\beta(f;\ \delta),
\end{aligned}$$

where $\left|\left|s^\beta\mathbf{h} - (s-1)^\beta\,\mathbf{h}\right|\right|_\beta \le ||\mathbf{h}||_\beta$, by $s^{\beta_i} - (s-1)^{\beta_i} \le 1$ for $i = 1, 2, ..., n$. Since $\omega_\beta(f;\ \delta)$ is a nondecreasing function of δ, we have

$$\omega_\beta\left(f;\ C^{\frac{|\beta|}{n}}\delta\right) \le (1+C)\,\omega_\beta(f;\ \delta).$$

■

Theorem 14.5 Let $0 < q < 1$, $\lambda > 0$, $n \in \mathbb{N}$ and $\beta_i \in (0, 1]$ $(i = 1, 2, \ldots n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. If $f \in C(\mathbb{R}^n)$, $\omega_\beta(f;\ \delta) < \infty$ for $\delta > 0$, then we have for every $\mathbf{x} \in \mathbb{R}^n$

$$|P_{\lambda,\beta}(f; q, \mathbf{x}) - f(\mathbf{x})| \le K(q,\beta)\,\omega_\beta\left(f;\ \ [\lambda]_q^{\frac{|\beta|}{n}}\right),$$

where

$$K(q,\beta)=1+\frac{q^{\frac{n(n-1)}{2}}\Gamma_q\left(n+\frac{n}{|\beta|}\right)c_q\left(n+\frac{n}{|\beta|}\right)}{\Gamma_q(n)q^{\frac{\left(n+\frac{n}{|\beta|}\right)\left(n+\frac{n}{|\beta|}-1\right)}{2}}}.$$

Proof. From Lemma 14.1 and definition of nonisotropic modulus of continuity, we can write

$$\begin{aligned}
& P_{\lambda,\beta}(f;q,\mathbf{x})-f(\mathbf{x}) \\
= & \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}}\int\limits_{\mathbb{R}^n}(f(\mathbf{x}+\mathbf{t})-f(\mathbf{x}))\,\mathcal{P}_\lambda(\beta,\mathbf{t})\,d\mathbf{t} \\
= & \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}}\int\limits_{\mathbb{R}^n}\omega_\beta\left(f;\ \ ||\mathbf{t}||_\beta\right)\mathcal{P}_\lambda(\beta,\mathbf{t})\,d\mathbf{t}.
\end{aligned}$$

Since

$$\omega_\beta\left(f;\ \ ||\mathbf{t}||_\beta\right)=\omega_\beta\left(f;\ \ \left(\frac{||\mathbf{t}||_\beta^{\frac{n}{|\beta|}}}{[\lambda]_q}\right)^{\frac{|\beta|}{n}}[\lambda]_q^{\frac{|\beta|}{n}}\right),$$

using Lemma 14.4 with $C=\frac{||\mathbf{t}||_\beta^{\frac{n}{|\beta|}}}{[\lambda]_q}$ for $\mathbf{t}\in\mathbb{R}^n$, we have

$$|P_{\lambda,\beta}(f;q,\mathbf{x})-f(\mathbf{x})|\leq\omega_\beta\left(f;\ \ [\lambda]_q^{\frac{|\beta|}{n}}\right)\left(1+\frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}[\lambda]_q}\int\limits_{\mathbb{R}^n}||\mathbf{t}||_\beta^{\frac{n}{|\beta|}}\mathcal{P}_\lambda(\beta,\mathbf{t})\,d\mathbf{t}\right).$$

We apply change of variable with

$$\begin{aligned}
\mathbf{t} &= [\lambda]_q^\beta\,\mathbf{y} \\
d\mathbf{t} &= [\lambda]_q^{|\beta|}\,d\mathbf{y},
\end{aligned}$$

where $\mathbf{y}\in\mathbb{R}^n$ such that $[\lambda]_q^\beta\,\mathbf{y}=\left([\lambda]_q^{\beta_1}y_1,\ldots,[\lambda]_q^{\beta_n}y_n\right)$ and then by using the generalized β-spherical coordinates as in Lemma 14.1, for $\mathbf{x}\in\mathbb{R}^n$ given we have

$$\begin{aligned}
|P_{\lambda,\beta}(f;q,\mathbf{x})-f(\mathbf{x})| &\leq \omega_\beta\left(f;\ \ [\lambda]_q^{\frac{|\beta|}{n}}\right)\left(1+c(n,\ \beta,\ q)\int\limits_{\mathbb{R}^n}\frac{||\mathbf{y}||_\beta^{\frac{n}{|\beta|}}}{E_q\left((1-q)\,||\mathbf{y}||_\beta\right)}d\mathbf{y}\right) \\
&= \omega_\beta\left(f;\ \ [\lambda]_q^{\frac{|\beta|}{n}}\right)\left(1+c(n,\ \beta,\ q)\int\limits_0^\infty\int\limits_{S^{n-1}}\frac{u^{2|\beta|-1}u^2\Omega_\beta(\theta)\,d\theta du}{E_q\left((1-q)\,u^{\frac{2|\beta|}{n}}\right)}\right).
\end{aligned}$$

By (14.7) we get

$$\left|P_{\lambda,\beta}\left(f;q,\mathbf{x}\right)-f\left(\mathbf{x}\right)\right|\leq\omega_{\beta}\left(f;\ \left[\lambda\right]_{q}^{\frac{|\beta|}{n}}\right)\left(1+c(n,\beta,q)\frac{n}{2\left|\beta\right|}\omega_{\beta,n-1}\int_{0}^{\infty}\frac{u^{n+\frac{n}{|\beta|}-1}du}{E_{q}\left(\left(1-q\right)u\right)}\right). \tag{14.9}$$

Also, using (14.3), we derive

$$\int_{0}^{\infty}\frac{u^{n+\frac{n}{|\beta|}-1}du}{E_{q}\left(\left(1-q\right)u\right)}=\frac{\Gamma_{q}\left(n+\frac{n}{|\beta|}\right)c_{q}\left(n+\frac{n}{|\beta|}\right)\ln q^{-1}}{\left(1-q\right)q^{\frac{\left(n+\frac{n}{|\beta|}\right)\left(n+\frac{n}{|\beta|}-1\right)}{2}}}.$$

Substituting this equality into (14.9) and using (14.6), we have desired result. ■

Remark 14.6. Let $X:=C_U\left(\mathbb{R}^n\right),\ n\geq 1$, be the space of uniformly continuous functions from $\mathbb{R}^n$ into $\mathbb{R}$. For $f\in X$, we consider the first order modulus of continuity of f by

$$\omega\left(f;\ \delta\right):=\sup_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{R}^n\\ ||\mathbf{t}-\mathbf{x}||\leq\delta}}\left|f\left(\mathbf{x}\right)-f\left(\mathbf{y}\right)\right|,\quad \delta>0.$$

Here $||\cdot||$ is an arbitrary norm in $\mathbb{R}^n$. We know that $\omega\left(f;\ \delta\right)$ is finite for all $\delta>0$ (see [67, pp. 297-298]) and trivially we see that

$$\lim_{\delta\downarrow 0}\omega\left(f;\ \delta\right)=0,\ \text{ iff } f\in X. \tag{14.10}$$

Also the above properties true for the Euclidean norm and its equivalent, the maximum norm.

If $f\in X$, where $\mathbb{R}^n$ is equipped with maximum norm, we observe the following: Let $\delta>0$ small enough, $\mathbf{x}=(x_1,...,x_n)\in\mathbb{R}^n$, and $||\mathbf{x}||_{\max}$ the maximum norm. Let $A=\left\{\mathbf{x},\mathbf{y}\in\mathbb{R}^n:||\mathbf{x}-\mathbf{y}||_{\beta}\leq\delta\right\}$. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ and $i=1,...,n$, we have

$$\left|x_i-y_i\right|\leq||\mathbf{x}-\mathbf{y}||_{\beta}^{\frac{n}{|\beta|}\beta_j},$$

and for $\mathbf{x},\mathbf{y}\in A$ we find

$$\left|x_i-y_i\right|\leq\delta^{\frac{n}{|\beta|}\beta_j}\leq\delta^{\frac{n}{|\beta|}\beta^*},\quad i=1,...,n,$$

where $\beta^*=\min\left\{\beta_1,...,\beta_n\right\}$. Thus we get

$$||\mathbf{x}-\mathbf{y}||_{\max}\leq\delta^{\frac{n}{|\beta|}\beta^*}.$$

That is $A \subset B$, where $B := \left\{ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : ||\mathbf{x}-\mathbf{y}||_{\max} \leq \delta^{\frac{n}{|\beta|}\beta^*} \right\}$. Hence

$$\begin{aligned} \omega_\beta(f;\,\delta) &= \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{R}^n \\ ||\mathbf{t}-\mathbf{x}||_\beta \leq \delta}} |f(\mathbf{x}) - f(\mathbf{y})| \\ &\leq \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{R}^n \\ ||\mathbf{t}-\mathbf{x}||_{\max} \leq \delta^{\frac{n}{|\beta|}\beta^*}}} |f(\mathbf{x}) - f(\mathbf{y})| =: \omega^{\max}\left(f;\,\delta^{\frac{n}{|\beta|}\beta^*}\right). \end{aligned}$$

Using above inequality and (14.10), for $f \in X$,

$$\lim_{\delta\downarrow 0} \omega_\beta(f;\;\delta) = 0. \tag{14.11}$$

Using Theorem 14.5 and (14.11), we can give following result.

Corollary 14.7. Let $P_{\lambda,\beta}(f,\cdot)$ be a positive linear operators, defined by (14.8). If $f \in X$, $n \in \mathbb{N}$, $\beta_i \in (0,\;1]$ $(i = 1,\,2,\ldots n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, $\lambda > 0$ and $0 < q < 1$, then

$$\lim_{\lambda\to 0} \|P_{\lambda,\beta}(f;q,\,\mathbf{x}) - f(\mathbf{x})\|_\infty = 0.$$

Now we introduce an analogy of the classical Lipschitz space $Lip_M(\alpha)$.

Definition 14.8. For a given $M > 0$, $n \in \mathbb{N}$, $\beta_i \in (0,\;\infty)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ and $0 \leq \alpha < 1$, we denote by $Lip_{M,\beta}(\alpha)$ the subset of all functions $f \in C(\mathbb{R}^n)$ such that

$$|f(\mathbf{t}) - f(\mathbf{x})| \leq M\,||\mathbf{t}-\mathbf{x}||_\beta^\alpha, \quad \text{for every } \mathbf{x}, \mathbf{t} \in \mathbb{R}^n.$$

Remark 14.9. Call $|\mathbf{t}_* - \mathbf{x}_*| = \max\{|t_1 - x_1|, \ldots, |t_n - x_n|\}$. We have

$$\begin{aligned} ||\mathbf{t}-\mathbf{x}||_\beta^{\frac{n}{|\beta|}} &= |t_1 - x_1|^{\frac{1}{\beta_1}} + \ldots + |t_n - x_n|^{\frac{1}{\beta_n}} \\ &\leq |t_* - x_*|^{\frac{1}{\beta_1}} + \ldots + |t_* - x_*|^{\frac{1}{\beta_n}} \\ &\leq n\,|t_* - x_*|^{\frac{1}{\beta_*}}, \end{aligned}$$

where $\frac{1}{\beta_*} = \min\left\{\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n}\right\}$ same as $\beta_* = \max\{\beta_1, ..., \beta_n\}$ if $|t_* - x_*| \leq 1$, and $\frac{1}{\beta_*} = \max\left\{\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n}\right\}$ same as $\beta_* = \min\{\beta_1, ..., \beta_n\}$ if $|t_* - x_*| > 1$. Therefore, we have

$$||\mathbf{t}-\mathbf{x}||_\beta^{\frac{n\beta_*}{|\beta|}} \leq n^{\beta_*}\,|t_* - x_*| \leq n^{\beta_*}\,|\mathbf{t}-\mathbf{x}|$$

and

$$||\mathbf{t}-\mathbf{x}||_\beta^\alpha \leq n^{\frac{|\beta|}{n}\alpha}\,|\mathbf{t}-\mathbf{x}|^{\frac{|\beta|}{n\beta_*}\alpha}.$$

If $f \in Lip_{M,\beta}(\alpha)$ then we have

$$|f(\mathbf{t}) - f(\mathbf{x})| \leq M n^{\frac{|\beta|}{n}\alpha} |\mathbf{t} - \mathbf{x}|^{\frac{|\beta|}{n\beta_*}\alpha}.$$

For small $\delta > 0$ the last implies

$$\omega^{Euclidean}(f;\, \delta) \leq M n^{\frac{|\beta|}{n}\alpha} \delta^{\frac{|\beta|}{n\beta_*}\alpha},$$

where $\beta_* = \max\{\beta_1, ..., \beta_n\}$, that is f is uniformly continuous.

Using Definition 14.3 and Definition 14.8, we have

$$\omega_\beta(f;\, \delta) \leq M\delta^\alpha \tag{14.12}$$

for any function $f \in Lip_{M,\beta}(\alpha)$.

Using Theorem 14.5 and (14.12), we can give following result.

Corollary 14.10. Let $P_{\lambda,\beta}(f, \cdot)$ be a positive linear operators, defined by (14.8). If $f \in Lip_{M,\beta}(\alpha)$ for some $0 \leq \alpha < 1$, $n \in \mathbb{N}$, $\beta_i \in (0,\ 1]$ $(i = 1, 2, \ldots n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, $\lambda > 0$ and $0 < q < 1$, then we have for every $\mathbf{x} \in \mathbb{R}^n$

$$|P_{\lambda,\beta}(f; q, \mathbf{x}) - f(\mathbf{x})| \leq MK(q, \beta)[\lambda]_q^{\frac{|\beta|}{n}\alpha},$$

where M is a positive constant independent of λ and $K(q, \beta)$ is defined as in Theorem 14.5.

Remark 14.11. As a consequence of Corollary 14.10 we can say that the convergence rate of the operators (14.8) to f is $\mathcal{O}\left([\lambda]_q^{\frac{|\beta|}{n}\alpha}\right)$, which can be made better depending on not only the chosen q but also the choice of β. Also, for suitable q and β this rate coincides with the rates of convergence of the q-Picard and classical Picard singular integral operators, respectively, to the identity.

Now we present a result which is a pointwise version of the theorem of approximation to the identity (see [275]). For this purpose we first give the following definition.

Definition 14.12. Let $f \in L_p(\mathbb{R}^n)$, $p > 1$ and $\beta_i \in (0,\ \infty)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. We say that $\mathbf{x}$ is β-Lebesgue point of f, if the condition

$$\lim_{h \to 0} \left(\frac{1}{h^{2|\beta|}} \int_{||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} \leq h} |f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})|^p \, d\mathbf{y} \right)^{\frac{1}{p}} = 0$$

holds.

Theorem 14.13. Let $n \in \mathbb{N}$ and $\beta_i \in (0, \ \infty)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, $\lambda > 0$ and $0 < q < 1$. If $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$, then

$$\lim_{\lambda \to 0} P_{\lambda,\beta}(f; q, \mathbf{x}) = f(\mathbf{x})$$

whenever $\mathbf{x}$ is a Lebesgue point of f.

Proof. Let $\mathbf{x}$ be a Lebesgue point of f. This means that for any $\varepsilon > 0$ one can find $\eta > 0$ such that $\eta > h$ implies that

$$\left(\frac{1}{h^{2|\beta|}} \int_{||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} \le h} |f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})|^p \, d\mathbf{y} \right)^{\frac{1}{p}} < \varepsilon.$$

Changing to generalized β-polar coordinates we can reinterpret the former condition as: if $\eta > h$ then

$$G_\beta(h) = \int_0^h s^{2|\beta|-1} g(s) \, ds < h^{2|\beta|} \varepsilon^p$$

where

$$g(s) = \int\limits_{S^{n-1}} \left| f\left(\mathbf{x} + (s\theta)^\beta \right) - f(\mathbf{x}) \right|^p \Omega_\beta(\theta) \, d\theta.$$

On the other hand, for all $\eta > 0$ we obtain

$$\begin{aligned} |P_{\lambda,\beta}(f; q, \mathbf{x}) - f(\mathbf{x})| &\le \frac{c(n, \ \beta, \ q)}{[\lambda]_q^{|\beta|}} \int\limits_{||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} < \eta} |f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})| \, \mathcal{P}_\lambda(\beta, \mathbf{y}) \, d\mathbf{y} \\ &\quad + \frac{c(n, \ \beta, \ q)}{[\lambda]_q^{|\beta|}} \int\limits_{||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} \ge \eta} |f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})| \, \mathcal{P}_\lambda(\beta, \mathbf{y}) \, d\mathbf{y} \\ &: = I_1 + I_2. \end{aligned}$$

To estimate I_1 first we use Hölder's inequality and later the generalized β-spherical coordinates, so we get

$$\begin{aligned} I_1 &\le \left(\frac{c(n, \ \beta, \ q)}{[\lambda]_q^{|\beta|}} \int\limits_{||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} \le \eta} |f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})|^p \, \mathcal{P}_\lambda(\beta, \mathbf{y}) \, d\mathbf{y} \right)^{\frac{1}{p}} \\ &= \left(\int_0^\eta \left\{ \int\limits_{S^{n-1}} \left| f\left(\mathbf{x} + (s\theta)^\beta \right) - f(\mathbf{x}) \right|^p \Omega_\beta(\theta) \, d\theta \right\} s^{2|\beta|-1} \mathcal{P}_\lambda^0(\beta, s) \, ds \right)^{\frac{1}{p}} \\ &= \left(\int_0^\eta g(s) \, s^{2|\beta|-1} \mathcal{P}_\lambda^0(\beta, s) \, ds \right)^{\frac{1}{p}}, \end{aligned}$$

where

$$\mathcal{P}_\lambda^0(\beta, s) = \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}\, E_q\left(\frac{(1-q)s^{\frac{2|\beta|}{n}}}{[\lambda]_q^{\frac{|\beta|}{n}}}\right)}.$$

Using integration by parts twice and the above observations we have

$$\begin{aligned} I_1 &\le \left(G_\beta(s)\,\mathcal{P}_\lambda^0(\beta, s)\Big|_0^\eta - \int_0^\eta G_\beta(s)\, d\left(\mathcal{P}_\lambda^0(\beta, s)\right)\right)^{\frac{1}{p}} \\ &\le \varepsilon\left(s^{2|\beta|}\mathcal{P}_\lambda^0(\beta, s)\Big|_0^\eta - \int_0^\eta s^{2|\beta|} d\left(\mathcal{P}_\lambda^0(\beta, s)\right)\right)^{\frac{1}{p}} \\ &\le \varepsilon\left(\eta^{2|\beta|}\mathcal{P}_\lambda^0(\beta, \eta) - \int_0^\infty s^{2|\beta|} d\left(\mathcal{P}_\lambda^0(\beta, s)\right)\right)^{\frac{1}{p}} \\ &\le \varepsilon\left(\eta^{2|\beta|}\mathcal{P}_\lambda^0(\beta, \eta) + 2|\beta| \int_0^\infty s^{2|\beta|-1}\mathcal{P}_\lambda^0(\beta, s)\, ds\right)^{\frac{1}{p}}. \end{aligned}$$

Because

$$2|\beta| \int_0^\infty s^{2|\beta|-1}\mathcal{P}_\lambda^0(\beta, s)\, ds = \frac{2|\beta|\, c(n,\ \beta,\ q)}{\omega_{\beta,\, n-1}\, [\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \mathcal{P}_\lambda(\beta, \mathbf{y})\, d\mathbf{y},$$

there exist a constant A such that $I_1 \le \varepsilon A$.

To estimate I_2, using Hölder's inequality for $\frac{1}{p} + \frac{1}{p'} = 1$ we have

$$I_2 \le \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \|f\|_p \|\chi_\eta \mathcal{P}_\lambda(\beta, \cdot)\|_{p'} + \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} |f(\mathbf{x})|\, \|\chi_\eta \mathcal{P}_\lambda(\beta, \cdot)\|_1,$$

where χ_η is the characteristic function of the set of $\mathbf{y}$ such that $||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} \ge \eta$. We observe that

$$\begin{aligned} \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \|\chi_\eta \mathcal{P}_\lambda(\beta, \cdot)\|_1 &= \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \int\limits_{||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} \ge \eta} \mathcal{P}_\lambda(\beta, \mathbf{y})\, d\mathbf{y} \\ &= c(n,\ \beta,\ q) \int\limits_{||\mathbf{y}||_\beta^{\frac{n}{2|\beta|}} \ge \frac{\eta}{\sqrt{[\lambda]_q}}} \frac{1}{E_q\left((1-q)\, ||\mathbf{y}||_\beta\right)} d\mathbf{y}. \end{aligned}$$

We notice that second summand tends to zero as $\lambda \to 0$. For the first summand we have

$$\begin{aligned}
\frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \left\|\chi_\eta \mathcal{P}_\lambda(\beta,\cdot)\right\|_{p'} &= \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \left(\int_{\|\mathbf{y}\|_\beta^{\frac{n}{2|\beta|}} \geq \eta} \mathcal{P}_\lambda(\beta,\mathbf{y}) \left[\mathcal{P}_\lambda(\beta,\mathbf{y})\right]^{\frac{p'}{p}} d\mathbf{y} \right)^{\frac{1}{p'}} \\
&\leq \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \left(\left\|\chi_\eta \mathcal{P}_\lambda(\beta,\cdot)\right\|_\infty^{\frac{p'}{p}} \int_{\|\mathbf{y}\|_\beta^{\frac{n}{2|\beta|}} \geq \eta} \mathcal{P}_\lambda(\beta,\mathbf{y})\, d\mathbf{y} \right)^{\frac{1}{p'}} \\
&= \left(\frac{c(n,\beta,\ q)}{[\lambda]_q^{|\beta|}} \left\|\chi_\eta \mathcal{P}_\lambda(\beta,\cdot)\right\|_\infty \right)^{\frac{1}{p}} \left(\frac{c(n,\beta,\ q)}{[\lambda]_q^{|\beta|}} \left\|\chi_\eta \mathcal{P}_\lambda(\beta,\cdot)\right\|_1 \right)^{\frac{1}{p'}}.
\end{aligned}$$

But by (14.2) we derive

$$\begin{aligned}
\frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \left\|\chi_\eta \mathcal{P}_\lambda(\beta,\cdot)\right\|_\infty &= \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \sup_{\|t\|_\beta^{\frac{n}{2|\beta|}} \geq \eta} 1/\ E_q \left(\frac{(1-q)\,\|\mathbf{t}\|_\beta}{[\lambda]_q^{\frac{|\beta|}{n}}} \right) \\
&\leq \frac{c(n,\ \beta,\ q)}{[\lambda]_q^{|\beta|}} \frac{[\lambda]_q^{\frac{|\beta|}{n}(n+1)}}{\prod_{k=0}^{n} \left([\lambda]_q^{\frac{|\beta|}{n}} + (1-q)\, q^k \eta^{\frac{2|\beta|}{n}} \right)} \\
&\leq c(n,\ \beta,\ q) \frac{[\lambda]_q^{\frac{|\beta|}{n}}}{\prod_{k=0}^{n} \left([\lambda]_q^{\frac{|\beta|}{n}} + (1-q)\, q^k \eta^{\frac{2|\beta|}{n}} \right)} \\
&\leq c(n,\ \beta,\ q)\, [\lambda]_q^{\frac{|\beta|}{n}} \to 0 \quad \text{as} \quad \lambda \to 0.
\end{aligned}$$

Thus the proof is completed. ■

14.4 Global Smoothness Preservation Property

In this section, we show that the q-Picard integral operators depending on the β-distance given by (14.8) satisfy the global smoothness preservation property. The global smoothness inequalities involve a different modulus of continuity given in [17] and [67].

Theorem 14.14. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ with $\omega_\beta(f;\ \delta) < \infty$, for any $\delta > 0$ and $\beta_i \in (0,\ \infty)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 +\ \beta_2 + \cdots + \beta_n$, such that $P_{\lambda,\beta}(f; q,\ \mathbf{x}) \in \mathbb{R}$ for $0 < q < 1$. Then we have

$$\omega_\beta\left(P_{\lambda,\beta}(f; q, \cdot)\,;\ \delta\right) \leq \omega_\beta(f;\ \delta)\,. \tag{14.13}$$

Proof. Notice that

$$P_{\lambda,\beta}(f;q,\mathbf{x}) - P_{\lambda,\beta}(f;q,\mathbf{y}) = \frac{c(n,\beta,q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} (f(\mathbf{x}+\mathbf{t}) - f(\mathbf{y}+\mathbf{t})) \mathcal{P}_\lambda(\beta,\mathbf{t})\, d\mathbf{t}.$$

By Lemma 14.1, we get

$$\begin{aligned} \left|P_{\lambda,\beta}(f;q,\mathbf{x}) - P_{\lambda,\beta}(f;q,\mathbf{y})\right| &= \frac{c(n,\beta,q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} |f(\mathbf{x}+\mathbf{t}) - f(\mathbf{y}+\mathbf{t})| \mathcal{P}_\lambda(\beta,\mathbf{t})\, d\mathbf{t} \\ &\le \omega_\beta(f;\delta). \end{aligned}$$

■

We finish with

Theorem 14.15. Inequality (14.13) is sharp, namely it is attained by the projection $f_*(\mathbf{x}) = x_j$, where $\mathbf{x} = (x_1,\ldots,x_j,\ldots,x_n) \in \mathbb{R}^n$ and $j \in \{1,\ldots n\}$ is fixed.

Proof. We see that

$$\begin{aligned} & P_{\lambda,\beta}(f_*;q,\mathbf{x}) - P_{\lambda,\beta}(f_*;q,\mathbf{y}) \\ =\; & \frac{c(n,\beta,q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} [(x_j+t_j) - (y_j+t_j)] \mathcal{P}_\lambda(\beta,\mathbf{t})\, d\mathbf{t} \\ =\; & x_j - y_j \\ =\; & f_*(\mathbf{x}) - f_*(\mathbf{y}). \end{aligned}$$

Hence for any $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ with $\|\mathbf{x}-\mathbf{y}\|_\beta \le \delta$, $\delta > 0$ we get

$$|P_{\lambda,\beta}(f_*;q,\mathbf{x}) - P_{\lambda,\beta}(f_*;q,\mathbf{y})| = |f_*(\mathbf{x}) - f_*(\mathbf{y})|$$

and

$$\omega_\beta(P_{\lambda,\beta}(f_*;q,\cdot);\delta) = \omega_\beta(f_*;\delta), \text{ for any } \delta > 0.$$

Further notice that

$$\begin{aligned} |x_i - y_i| &= \left(|x_i - y_i|^{\frac{1}{\beta_j}}\right)^{\beta_j} \le \|\mathbf{x}-\mathbf{y}\|_\beta^{\frac{n\beta_j}{|\beta|}} \\ &\le \delta^{\frac{n\beta_j}{|\beta|}} < \infty, \end{aligned}$$

and

$$\omega_\beta(f_*;\delta) < \infty.$$

At the and we observe that

$$\begin{aligned}
P_{\lambda,\beta}\left(f_{*};q,\,\mathbf{x}\right) &= \frac{c(n,\,\beta,\,\,q)}{[\lambda]_q^{|\beta|}}\int\limits_{\mathbb{R}^n}\left(x_j+t_j\right)\mathcal{P}_\lambda\left(\beta,\mathbf{t}\right)d\mathbf{t} \\
&= x_j+\frac{c(n,\,\beta,\,\,q)}{[\lambda]_q^{|\beta|}}\int\limits_{\mathbb{R}^n}t_j\mathcal{P}_\lambda\left(\beta,\mathbf{t}\right)d\mathbf{t} \\
&= x_j+\frac{c(n,\,\beta,\,\,q)}{[\lambda]_q^{|\beta|}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{t_j}{E_q\left(\frac{(1-q)||\mathbf{t}||_\beta}{[\lambda]_q^{\frac{|\beta|}{n}}}\right)}dt_1...dt_j...dt_n \\
&= x_j.
\end{aligned}$$

That is $P_{\lambda,\beta}\left(f_{*};q,\,\mathbf{x}\right)=x_j\in\mathbb{R}$. So f_* fulfills all the assumptions of Theorem 14.14. ■

15

Approximation by q-Gauss-Weierstrass Singular Integral Operators

In this chapter, we present a generalization of Gauss-Weierstrass operators based on q-integers using the q-integral and we call them q-Gauss- Weierstrass integral operators. For these operators, we obtain a convergence property in a weighted function space using Korovkin theory. Then we estimate the rate of convergence of these operators in terms of a weighted modulus of continuity. We also give optimal global smoothness preservation property of these operators. This chapter is based on [62].

15.1 Introduction

Recently, in [97] a q-generalization of Gauss-Weierstrass and Picard singular integral operators was introduced by using the q-analogue of the Euler Gamma integral. In [61], was given a different generalization of q-Picard singular integral operators by using the nonisotropic β-distance.

In this chapter, we introduce a q-generalization of Gauss-Weierstrass singular integral operators by using the q-integral. In 1910, Jackson [195] defined and studied the q-integral. He also was the first to develop q-calculus in a systematic way. Nowadays there is a significant increase of activity in the area of the q-calculus due to its applications in mathematics and physics.

The aim of this chapter is to derive the weighted approximation error of the q-type Gauss-Weierstrass singular integral operators for functions of polynomial growth. This estimate will be in terms of a weighted modulus

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 207–214.
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of continuity that we give below. Also we give a direct approximation result for these functions. We finally prove the optimal global smoothness of these operators by using the usual modulus of continuity.

Next we provide a summary of the mathematical notations and definitions used in this chapter. All of the results can be found in [170] and [201]. Throughout this chapter, we fix $q \in (0,1)$.

For $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$,

$$[n]_q = \frac{1-q^n}{1-q}, \qquad (\alpha;q)_n = \prod_{k=0}^{n-1}\left(1-\alpha q^k\right), \quad n=1,2,..., \ (-x;q)_\infty = \prod_{k=0}^{\infty}\left(1+xq^k\right),$$

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n = 1, 2, ... \\ 1 & n = 0. \end{cases}.$$

The q-derivative $D_q f$ of a real valued function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0, \tag{15.1}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The q-Jackson integrals and the q-improper integrals of a real valued function are defined as (see [195] and [211])

$$\int_0^a f(x)\, d_q x = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n, \ a \in \mathbb{R},$$

and

$$\int_0^{\frac{\infty}{A}} f(x)\, d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0, \tag{15.2}$$

provided the sums converge absolutely.

One can define the Jackson integral in a generic interval $[a,b]$ as

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x.$$

There are two important q-analogues of the exponential function:

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (-(1-q)\,x;q)_\infty \tag{15.3}$$

and

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)\,x;q)_\infty}. \tag{15.4}$$

Note that for $q \in (0,1)$ the series expansion of $e_q(x)$ has radius of convergence $\frac{1}{1-q}$. To the opposite, the series expansion of $E_q(x)$ converges for every real x.

The q-gamma integral is defined by [211]

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx)\, d_q x, \qquad t > 0 \tag{15.5}$$

which satisfies the following functional equation:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t),$$

where $[t]_q = \frac{1-q^t}{1-q}$ and $\Gamma_q(1) = 1$.

The change of variable formula (see [201]) for $u(x) = \gamma x^\beta$ is

$$\int_{u(a)}^{u(b)} f(u)\, d_q u = \int_a^b f(u(x))\, D_{q^{1/\beta}} u(x)\, d_{q^{1/\beta}} x. \tag{15.6}$$

15.2 Description of the Operators

Definition 15.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $n \in \mathbb{N}$, $q \in (0,\ 1)$ and $x \in \mathbb{R}$, the q-Gauss-Weierstrass integral of f is

$$\mathcal{W}_n(f; q, x) := \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} f(x+t)\, E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t. \tag{15.7}$$

Lemma 15.2. The operator $\mathcal{W}_n$ satisfies, for every $k \in \mathbb{N}$,

$$\mathcal{W}_n\left(t^k; q, x\right) = \sum_{j=0}^{k} \binom{k}{j} \frac{2^j \Gamma_{q^2}\left(\frac{j+1}{2}\right)}{[n]_q^{\frac{j}{2}}\, \Gamma_{q^2}\left(\frac{1}{2}\right)} x^{k-j}.$$

Proof. From (15.7) we obtain

$$\begin{aligned} \mathcal{W}_n\left(t^k; q, x\right) &= \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} (t+x)^k E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t \quad (15.8) \\ &= \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} t^j E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t. \end{aligned}$$

Using (15.1) we can write the q-derivative of the equality $t = 2\frac{\sqrt{u}}{\sqrt{[n]_q}}$ as

$$\begin{aligned} D_{q^2}(t) &= \frac{2}{\sqrt{[n]_q}} \frac{\sqrt{u} - \sqrt{q^2 u}}{(1-q^2)\, u} \\ &= \frac{2}{(q+1)\sqrt{[n]_q}\sqrt{u}}. \end{aligned} \tag{15.9}$$

Also, using the change of variable formula (15.6) for q-integral with $\beta = \frac{1}{2}$, then from (15.9) and (15.5) we derive

$$\begin{aligned}
\int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} t^j E_{q^2}\left(-q^2[n]_q \frac{t^2}{4}\right) d_q t &= \frac{2^{j+1}}{(q+1)[n]_q^{\frac{j+1}{2}}} \int_0^{\frac{1}{1-q^2}} u^{\frac{j-1}{2}} E_{q^2}\left(-q^2 u\right) d_{q^2} u \\
&= \frac{2^{j+1}\Gamma_{q^2}\left(\frac{j+1}{2}\right)}{(q+1)[n]_q^{\frac{j+1}{2}}},
\end{aligned}$$

for $j = 0,...,k$. From (15.8) we have desired result. ∎

Remark 15.3. Note that q-Gauss-Weierstrass operators $\mathcal{W}_n$ given by (15.7) can be rewritten via an improper integral by using definition (15.2). From (15.3) we can easily see that $E_q\left(-\frac{q^n}{1-q}\right) = 0$ for $n \leq 0$. Thus we can write

$$\mathcal{W}_n(f; q, x) = \frac{\sqrt{[n]_q}(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\infty}{\sqrt{[n]_q}\sqrt{1-q^2}}} f(x+t) E_{q^2}\left(-q^2[n]_q \frac{t^2}{4}\right) d_q t.$$

15.3 Approximation Properties in a Weighted Space

In this section, by using a Bohman-Korovkin type theorem proved in [162], we present the direct approximation property of the operator $\mathcal{W}_n$ given by (15.7).

Let us denote by $B_2(\mathbb{R})$ the weighted space of real-valued functions f defined on $\mathbb{R}$ with the property $|f(x)| \leq M_f\left(1+x^2\right)$ for all $x \in \mathbb{R}$, where M_f is a constant depending on the function f. We consider the weighted subspace $C_2(\mathbb{R})$ of $B_2(\mathbb{R})$ given by

$$C_2(\mathbb{R}) = \{f \in B_2(\mathbb{R}) : f \text{ continuous on } \mathbb{R}\}.$$

We also consider the space of functions $C_2^k(\mathbb{R}) = \left\{f \in C_2(\mathbb{R}) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} = k_f \in \mathbb{R}\right\}$ equipped with the norm $\|f\|_2 = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{1+x^2}$.

Theorem 15.4. Let T_n be a sequence of linear positive operators mapping $C_2(\mathbb{R})$ into $B_2(\mathbb{R})$ and satisfying the conditions

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{T}_n(t^\nu; x) - x^\nu|}{1+x^2} = 0, \quad \text{for } \nu = 0, 1, 2.$$

Then, for any $f \in C_2^k(\mathbb{R})$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{T}_n(f; x) - f(x)|}{1+x^2} = 0,$$

and there exists a function $f^* \in C_2(\mathbb{R}) \backslash C_2^k(\mathbb{R})$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}_+} \frac{|\mathcal{T}_n(f^*;x) - f^*(x)|}{1+x^2} \geq 1.$$

For $f \in C_2^k(\mathbb{R})$, we consider the weighted modulus of continuity defined in [286] given by

$$\Omega_2(f,\ \delta) = \sup_{x\in\mathbb{R},\ |h|\leq\delta} \frac{|f(x+h) - f(x)|}{1+(h+x)^2}.$$

This function has the following properties:

1. $\Omega_2(f,\ \delta) \leq 2\|f\|_2$,
2. $\Omega_2(f,\ m\delta) \leq m\Omega_2(f,\ \delta)$, $m \in \mathbb{N}$,
3. $\lim_{\delta\to 0} \Omega_2(f,\ \delta) = 0$.

Note that, we can not find a rate of convergence in terms of usual first modulus of continuity $\omega_1(f,\ \delta)$ of the function f because the modulus of continuity $\omega_1(f;\ \delta)$ on the infinite interval does not tend to zero as $\delta \to 0$. For this reason we consider the weighted modulus of continuity $\Omega_2(f,\ \delta)$.

Remark 15.5. Since any linear and positive operator is monotone, Lemma 15.2 guarantee that $\mathcal{W}_n(f) \in C_2(\mathbb{R})$ for each $f \in C_2(\mathbb{R})$.
Notice that, if we choose $q = 1$ then the operators $\mathcal{W}_n$ turn out to be the classical Gauss-Weierstrass singular integral operators.
Since for a fixed value of q with $0 < q < 1$,

$$\lim_{n\to\infty} [n]_q = \frac{1}{1-q},$$

to ensure the convergence properties of $\mathcal{W}_n$ we will assume $q = q_n$ as a sequence such that $q_n \to 1$ as $n \to \infty$ for $0 < q_n < 1$ and so that $[n]_{q_n} \to \infty$ as $n \to \infty$. An example of such a sequence is $q_n = 1 - 1/na^n$, where $a > 3$ (see [247]).

Theorem 15.6. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$.* For each $f \in C_2^k(\mathbb{R})$ we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{W}_n(f;q_n,\ x) - f(x)|}{1+x^2} = 0.$$

Proof. Clearly, $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{W}_n(1;q_n,\ x)-1|}{1+x^2} = 0$. From Lemma 15.2 we obtain

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{|\mathcal{W}_n(t;q_n,\ x) - x|}{1+x^2} = 0.$$

Also, using Lemma 15.2 again, we can write

$$\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(t^2;q_n,\,x\right)-x^2\right|}{1+x^2}\leq\sup_{x\in\mathbb{R}}\frac{|x|}{1+x^2}\frac{4}{\sqrt{[n]_{q_n}}\Gamma_{q_n^2}\left(\frac{1}{2}\right)}+\frac{4\Gamma_{q_n^2}\left(\frac{3}{2}\right)}{[n]_{q_n}\Gamma_{q_n^2}\left(\frac{1}{2}\right)},$$

which implies that

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(t^2;q_n,\,x\right)-x^2\right|}{1+x^2}=0.$$

Since the conditions of Theorem 15.4 are satisfied, we get for any $f\in C_2^k\left(\mathbb{R}\right)$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(f;q_n,\,x\right)-f\left(x\right)\right|}{1+x^2}=0.$$

■

Theorem 15.7. For $f\in C_2^k\left(\mathbb{R}\right),\,n\in\mathbb{N}$ we have

$$\sup_{x\in\mathbb{R}}\frac{\left|\mathcal{W}_n\left(f;q,\,x\right)-f\left(x\right)\right|}{1+x^2}\leq\left(1+\frac{12}{\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{8\Gamma_{q^2}\left(\frac{3}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\right)\Omega_2\left(f,\,\frac{1}{\sqrt{[n]_q}}\right).$$

Proof. From the properties of Ω_2 it is obvious that for any $\lambda>0$,

$$\Omega_2\left(f,\,\lambda\delta\right)\leq\left(\lambda+1\right)\Omega_2\left(f,\,\delta\right).$$

For $\delta>0$, if we use the definition of Ω_2 and the last inequality with $\lambda=\frac{t}{\delta}$ we have

$$\begin{aligned}\left|f\left(x+t\right)-f\left(x\right)\right| &\leq \left(1+\left(t+x\right)^2\right)\Omega_2\left(f,\,t\right)\\ &\leq \left(1+\left(t+x\right)^2\right)\left(1+\frac{t}{\delta}\right)\Omega_2\left(f,\,\delta\right).\end{aligned}$$

By the linearity and monotonicity of $\mathcal{W}_n$ applied to last inequality we get

$$\left|\mathcal{W}_n\left(f;q,\,x\right)-f\left(x\right)\right|$$

$$\leq\frac{\sqrt{[n]_q}(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}}\left(1+\left(t+x\right)^2\right)\left(1+\frac{t}{\delta}\right)E_{q^2}\left(-q^2\left[n\right]_q\frac{t^2}{4}\right)d_qt\;\Omega_2\left(f,\,\delta\right).$$

We can use the identity $\left(1+\left(x+t\right)^2\right)\left(1+\frac{t}{\delta}\right)=\left(1-\frac{x}{\delta}\right)\left(1+\left(t+x\right)^2\right)+\frac{1}{\delta}\left(t+x\right)+\frac{1}{\delta}\left(t+x\right)^3$, to rewrite the RHS above as follows

$$\left(\left(1-\frac{x}{\delta}\right)\left(1+\mathcal{W}_n\left(t^2;q,\,x\right)\right)+\frac{1}{\delta}\mathcal{W}_n\left(t;q,\,x\right)+\frac{1}{\delta}\mathcal{W}_n\left(t^3;q,\,x\right)\right)\Omega_2\left(f,\,\delta\right).$$

Using Lemma 15.2 and simple algebraic manipulations, the above expression becomes

$$\left\{ \left(1+x^2\right)\left(1+\frac{2}{\delta\sqrt{[n]_q}\Gamma_{q^2}\left(\frac{1}{2}\right)}\right)+4x\left(\frac{1}{\sqrt{[n]_q}_q\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{2\Gamma_{q^2}\left(\frac{3}{2}\right)}{\delta\,[n]_q\,\Gamma_{q^2}\left(\frac{1}{2}\right)}\right)\right.$$
$$\left.+\frac{4\Gamma_{q^2}\left(\frac{3}{2}\right)}{[n]_q\,\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{8}{\delta\,[n]_q^{3/2}\,\Gamma_{q^2}\left(\frac{1}{2}\right)}\right\}\Omega_2\left(f,\ \delta\right).$$

Putting together the above inequalities, we obtain, after dividing by $\left(1+x^2\right)$ and choosing $\delta=\frac{1}{\sqrt{[n]_q}}$,

$$\begin{aligned}
&\frac{\left|\mathcal{W}_n\left(f;q,\,x\right)-f\left(x\right)\right|}{1+x^2}\\
\leq\ &\left\{1+\frac{2}{\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{2}{\sqrt{[n]_q}\Gamma_{q^2}\left(\frac{1}{2}\right)}\left(1+2\Gamma_{q^2}\left(\frac{3}{2}\right)\right)\right.\\
&\left.+\frac{4}{[n]_q\,\Gamma_{q^2}\left(\frac{1}{2}\right)}\left(\Gamma_{q^2}\left(\frac{3}{2}\right)+2\right)\right\}\Omega_2\left(f,\ \frac{1}{\sqrt{[n]_q}}\right)\\
\leq\ &\left\{1+\frac{12}{\Gamma_{q^2}\left(\frac{1}{2}\right)}+\frac{8\Gamma_{q^2}\left(\frac{3}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\right\}\Omega_2\left(f,\ \frac{1}{\sqrt{[n]_q}}\right).
\end{aligned}$$

This completes the proof. ■

Remark 15.8. If $f\in C_2^k\left(\mathbb{R}\right),\ n\in\mathbb{N}$, then the weighted convergence rate of the operators of (15.7) to f is $\frac{1}{\sqrt{[n]_{q_n}}}$ for $0<q_n<1$ *and* $q_n\to 1$ as $n\to\infty$. Also this convergence rate can be made better depending on the choice of q_n and is at least as fast as $\frac{1}{\sqrt{n}}$.

Remark 15.9. We define the usual modulus of continuity

$$\omega_1\left(f;\ \delta\right)=\sup_{x\in\mathbb{R},|h|\leq\delta}\left|f\left(x+h\right)-f\left(x\right)\right|,$$

where $f:\mathbb{R}\to\mathbb{R}$.
Then for $f\left(x\right)=x$ we get trivially that

$$\omega_1\left(x;\ \delta\right)=\delta.$$

Here we consider $f : \mathbb{R} \to \mathbb{R}$ with $\omega_1(f;\ \delta) < \infty$, for any $\delta > 0$, and such that $\mathcal{W}_n(f; q, x)$ exists for any $x \in \mathbb{R}$. We notice that

$$\begin{aligned} & \mathcal{W}_n(f; q, x+h) - \mathcal{W}_n(f; q, x) \\ = & \frac{\sqrt{[n]_q}(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} (f(x+h+t) - f(x+t)) E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t. \end{aligned}$$

Thus it holds

$$\begin{aligned} & |\mathcal{W}_n(f; q, x+h) - \mathcal{W}_n(f; q, x)| \\ \leq & \frac{\sqrt{[n]_q}(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} |f(x+h+t) - f(x+t)| E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t \\ \leq & \omega_1(f;\ \delta) \frac{\sqrt{[n]_q}(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} E_{q^2}\left(-q^2 [n]_q \frac{t^2}{4}\right) d_q t \\ = & \omega_1(f;\ \delta). \end{aligned}$$

Therefore

$$\omega_1(\mathcal{W}_n(f; q, \cdot);\ \delta) \leq \omega_1(f;\ \delta), \quad \text{for any } \delta > 0, \tag{15.10}$$

proving the global smoothness preservation property of $\mathcal{W}_n$.
We know by Lemma 15.2 for $k = 1$ that

$$\mathcal{W}_n(t; q, x) = x + \frac{2}{\sqrt{[n]_q}\Gamma_{q^2}\left(\frac{1}{2}\right)},$$

hence

$$\omega_1(\mathcal{W}_n(t; q, x);\ \delta) = \omega_1(x;\ \delta) = \delta,$$

proving that (15.10) holds with equality. Hence (15.10) is a sharp inequality.

16

Quantitative Approximation by Univariate Shift-Invariant Integral Operators

High order differentiable functions of one real variable are approximated by univariate shift-invariant integral operators wavelet-like, and their generalizations. The high order of this approximation is estimated by establishing some Jackson type inequalities, involving the modulus of continuity of the Nth order derivative of the function under approximation. At the end we give applications to Probability. This chapter is based on [28].

16.1 Background

Here we follow [67], p. 281, see also [79]. Let $X := C_U(\mathbb{R})$ be the space of uniformly continuous real valued functions on $\mathbb{R}$ and $C(\mathbb{R})$ the space of continuous functions from $\mathbb{R}$ into itself. $C^N(\mathbb{R})$, $N \geq 1$, denotes the space of N times continuously differentiable functions on $\mathbb{R}$. Let $\{\ell_k\}_{k\in\mathbb{Z}}$ be a sequence of positive linear operators that map X into $C(\mathbb{R})$ such that

$$(\ell_k f)(x) := \big(\ell_0(f(2^{-k}\cdot))\big)(x), \quad x \in \mathbb{R}, \quad f \in X.$$

Let φ be a real valued function of compact support $\subseteq [-a, a]$, $a > 0$, $\varphi \geq 0$, φ is Lebesgue measurable and such that

$$\int_{-\infty}^{\infty} \varphi(x - u)\, du = 1, \quad \text{any } x \in \mathbb{R},$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 215–237.
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which is the same as

$$\int_{-\infty}^{\infty} \varphi(u)\,du = 1.$$

Example 16.1. i)

$$\varphi(x) := \chi_{[-\frac{1}{2},\frac{1}{2})}(x) = \begin{cases} 1, & x \in \left[-\frac{1}{2}, \frac{1}{2}\right), \\ 0, & \text{elsewhere,} \end{cases}$$

the characteristic function;

ii)

$$\varphi(x) := \begin{cases} 1-x, & 0 \le x \le 1, \\ 1+x, & -1 \le x \le 0, \\ 0, & \text{elsewhere,} \end{cases}$$

the hat function.

Let $\{\mathcal{L}_k\}_{k\in\mathbb{Z}}$ be the sequence of positive linear operators acting on X and defined by

$$(\mathcal{L}_k f)(x) := \int_{-\infty}^{\infty} (\ell_k f)(u)\varphi(2^k x - u)\,du. \qquad (*)$$

Notice that

$$(\mathcal{L}_k f)(x) = \big(\mathcal{L}_0(f(2^{-k}\cdot))\big)(2^k x),$$

for any $k \in \mathbb{Z}$, and $x \in \mathbb{R}$. Clearly operators $\mathcal{L}_k$ can also act on $C^N(\mathbb{R})$.

Notice that φ is a scaling-like function and the operators $\mathcal{L}_k$ are wavelet-like integral operators. Operators $(*)$ under mild assumptions that are very natural are shift invariant, possess the global smoothness preservation property, converge to the unit operator, and preserve continuous probability distribution functions. For all these see again [67], Chapter 10, and [79]. Applications of operators $(*)$ were given in the above mentioned references. Namely there the specialized general operators were denoted by $\{A_k\}_{k\in\mathbb{Z}}$, $\{B_k\}_{k\in\mathbb{Z}}$, $\{L_k\}_{k\in\mathbb{Z}}$, $\{\Gamma_k\}_{k\in\mathbb{Z}}$. These were mentioned and studied in [79], and fulfill all the above nice properties of operators $(*)$. For their precise definitions, see here Theorems 16.3, 16.5, 16.7, 16.9, next.

In [67], Chapter 10, p. 293, and initially in [79], it was proved the following motivating result.

Theorem 16.2. *For any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$ it holds*

$$|(A_k f)(x) - f(x)| \le \omega_1\left(f, \frac{a}{2^{k-1}}\right),$$

$$|(B_k f)(x) - f(x)| \le \omega_1\left(f, \frac{a}{2^{k}}\right), \qquad (**)$$

$$|(L_k f)(x) - f(x)| \le \omega_1\left(f, \frac{a+1}{2^{k}}\right),$$

and

$$|(\Gamma_k f)(x) - f(x)| \leq \omega_1 \left(f, \frac{a+1}{2^k} \right),$$

where ω_1 is the first usual modulus of continuity with respect to the supremum norm, i.e.,

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in R \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0.$$

In this chapter, see Theorems 16.3, 16.5, 16.7, 16.9, we present inequalities similar to (∗∗), but more complicated, involving $\omega_1(f^{(N)}, \cdot)$ for $f \in C^N(\mathbb{R})$, $N \geq 1$. That is studying the high order approximation to the unit of the particular operators A_k, B_k, L_k, Γ_k. Then in several propositions we continue the same study for the more general operators $A_{k,j}$, $B_{k,j}$, $L_{k,j}$, $\Gamma_{k,j}$ and $I^A_{k,q}$, $I^B_{k,q}$, $I^L_{k,q}$, $I^\Gamma_{k,q}$. These operators are naturally built on the operators A_k, B_k, L_k, Γ_k and studied in Chapter 14, pp. 373–389 of [67], see also [66] where first appeared.

At the end, see Corollaries 16.27, 16.28, 16.29, 16.30, we give applications to the above mentioned results to $F \in C^1(\mathbb{R})$ probability distribution functions. The resulting inequalities involve $\omega_1(f, \cdot)$, where now f is the probability density function of F.

16.2 Main Results

We give the first result.

Theorem 16.3. *Let $f \in C^N(\mathbb{R})$, $N \geq 1$. Let φ be a real valued function of compact support $\subseteq [-a, a]$, $a > 0$, $\varphi \geq 0$, φ is continuous and even. Furthermore it is supposed that*

$$\int_{-\infty}^{\infty} \varphi(x-u)\, du = 1, \quad \text{for any } x \in \mathbb{R},$$

same as

$$\int_{-\infty}^{\infty} \varphi(u)\, du = 1.$$

Define

$$r_k^f(u) := 2^k \int_{-\infty}^{\infty} f(t)\varphi(2^k t - u)\, dt, \quad u \in \mathbb{R}, \tag{16.1}$$

and

$$(A_k f)(x) = \int_{-\infty}^{\infty} r_k^f(u)\varphi(2^k x - u)\, du, \tag{16.2}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$.

Then

$$|(A_kf)(x)-f(x)|\le\sum_{i=1}^{N}\frac{|f^{(i)}(x)|}{i!}\,\frac{a^i}{2^{i(k-1)}}+\frac{a^N}{N!2^{N(k-1)}}\omega_1\left(f^{(N)},\frac{a}{2^{k-1}}\right),\quad(16.3)$$

for any $k\in\mathbb{Z}$*, and any* $x\in\mathbb{R}$*. Inequality* (16.3) *is attained when* f *is a constant function.*

Remark 16.4. If $f^{(N)}$ is uniformly continuous or bounded and continuous then $\omega_1\big(f^{(N)},\frac{a}{2^{k-1}}\big)$ is finite, and as $k\to+\infty$ we obtain that

$$(A_kf)(x)\to f(x)\quad(16.4)$$

pointwise with rates. If f is bounded then A_k is bounded too.

Proof. of Theorem 16.3. Since $f\in C^N(\mathbb{R})$, $N\ge1$ we have by Taylor's formula that

$$f(t)=f(x)+\sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}(t-x)^i+\int_x^t\big(f^{(N)}(s)-f^{(N)}(x)\big)\frac{(t-s)^{N-1}}{(N-1)!}\,ds,$$

for all $t,x\in\mathbb{R}$.

Thus

$$\begin{aligned}f(t)\varphi(2^kt-u)\;&=\;f(x)\varphi(2^kt-u)+\sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}\varphi(2^kt-u)(t-x)^i\\&\quad+\varphi(2^kt-u)\int_x^t\big(f^{(N)}(s)-f^{(N)}(x)\big)\frac{(t-s)^{N-1}}{(N-1)!}\,ds.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}f(t)\varphi(2^kt-u)\,dt\\&=\;f(x)\int_{-\infty}^{\infty}\varphi(2^kt-u)dt+\sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}\int_{-\infty}^{\infty}\varphi(2^kt-u)(t-x)^i\,dt\\&\quad+\int_{-\infty}^{\infty}\varphi(2^kt-u)\left(\int_x^t\big(f^{(N)}(s)-f^{(N)}(x)\big)\frac{(t-s)^{N-1}}{(N-1)!}\,ds\right)dt.\end{aligned}$$

Consequently we get

$$\begin{aligned}r_k^f(u)\;&=\;2^k\int_{-\infty}^{\infty}f(t)\varphi(2^kt-u)\,dt\\&=\;f(x)\int_{-\infty}^{\infty}\varphi(2^kt-u)2^k\,dt+\sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}\int_{-\infty}^{\infty}\varphi(2^kt-u)(t-x)^i2^k\,dt\\&\quad+\int_{-\infty}^{\infty}\varphi(2^kt-u)\left(\int_x^t\big(f^{(N)}(s)-f^{(N)}(x)\big)\frac{(t-s)^{N-1}}{(N-1)!}\,ds\right)2^k\,dt.\end{aligned}$$

Hence

$$r_k^f(u) - f(x) = \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} \int_{-\infty}^{\infty} \varphi(2^k t - u)(t-x)^i 2^k \, dt$$
$$+ \int_{-\infty}^{\infty} \varphi(2^k t - u) \left(\int_x^t \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{(t-s)^{N-1}}{(N-1)!} ds \right) 2^k dt.$$

That is,

$$(A_k f)(x) - f(x) = \int_{-\infty}^{\infty} r_k^f(u)\varphi(2^k x - u) \, du - f(x) \int_{-\infty}^{\infty} \varphi(2^k x - u) \, du$$
$$= \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} \int_{-\infty}^{\infty} \varphi(2^k x - u) \left(\int_{-\infty}^{\infty} \varphi(2^k t - u)(t-x)^i 2^k dt \right) du + \mathcal{R},$$

where

$$\mathcal{R} := \int_{-\infty}^{\infty} \varphi(2^k x - u) \left(\int_{-\infty}^{\infty} \varphi(2^k t - u) \right.$$
$$\left. \left(\int_x^t \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{(t-s)^{N-1}}{(N-1)!} ds \right) 2^k dt \right) du.$$

Here, φ is of compact support, so that $\varphi \neq 0$ if $-a \leq 2^k x - u \leq a$, that is if $-a + 2^k x \leq u \leq a + 2^k x$. Thus $\varphi \neq 0$ if $\left|x - \frac{u}{2^k}\right| \leq \frac{a}{2^k}$, and similarly $\varphi \neq 0$ if $\left|t - \frac{u}{2^k}\right| \leq \frac{a}{2^k}$, $k \in \mathbb{Z}$. But then

$$|t - x| \leq \left|t - \frac{u}{2^k}\right| + \left|x - \frac{u}{2^k}\right| \leq \frac{a}{2^{k-1}},$$

and

$$|t - x|^i \leq \frac{a^i}{2^{i(k-1)}}, \quad i = 1, \ldots, N.$$

Therefore

$$\begin{aligned} |(A_k f)(x) - f(x)| &\leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \int_{-\infty}^{\infty} \varphi(2^k x - u) \\ &\quad \left(\int_{-\infty}^{\infty} \varphi(2^k t - u) \frac{a^i}{2^{i(k-1)}} 2^k dt \right) du + |\mathcal{R}| \qquad (16.5) \\ &= \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{a^i}{2^{i(k-1)}} + |\mathcal{R}|, \quad \text{for any } x \in \mathbb{R}. \end{aligned}$$

Next we estimate $\mathcal{R}$. We derive

$$|\mathcal{R}| \leq \int_{-\infty}^{\infty} \varphi(2^k x - u) \left(\int_{-\infty}^{\infty} \varphi(2^k t - u) \lambda(t,x) 2^k \, dt \right) du, \qquad (16.6)$$

where

$$\lambda(t,x) := \left|\int_x^t \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{(t-s)^{N-1}}{(N-1)!}\, ds\right|.$$

But we need to estimate first $\lambda(t,x)$.

Case of $x \le t$. Then $t - s \ge 0$. And

$$\begin{aligned}
\lambda(t,x) &\le \int_x^t |f^{(N)}(s) - f^{(N)}(x)| \frac{(t-s)^{N-1}}{(N-1)!} ds \\
&\le \int_x^t \omega_1\left(f^{(N)}, |s-x|\right) \frac{(t-s)^{N-1}}{(N-1)!}\, ds \\
&\le \int_x^t \omega_1\left(f^{(N)}, (t-x)\right) \frac{(t-s)^{N-1}}{(N-1)!}\, ds = \omega_1\left(f^{(N)}, (t-x)\right) \frac{(t-x)^N}{N!}.
\end{aligned}$$

So when $x \le t$ we get

$$\lambda(t,x) \le \omega_1\left(f^{(N)}, (t-x)\right) \frac{(t-x)^N}{N!}.$$

Case of $x \ge t$. Then $t - s \le 0$. And

$$\begin{aligned}
\lambda(t,x) &\le \int_t^x |f^{(N)}(s) - f^{(N)}(x)| \frac{(s-t)^{N-1}}{(N-1)!}\, ds \\
&\le \int_t^x \omega_1\left(f^{(N)}, (x-t)\right) \frac{(s-t)^{N-1}}{(N-1)!}\, ds \\
&= \omega_1\left(f^{(N)}, (x-t)\right) \frac{(x-t)^N}{N!}.
\end{aligned}$$

That is, when $x \ge t$ we get

$$\lambda(t,x) \le \omega_1\left(f^{(N)}, (x-t)\right) \frac{(x-t)^N}{N!}.$$

Thus in general we have

$$\lambda(t,x) \le \omega_1\left(f^{(N)}, |t-x|\right) \frac{|t-x|^N}{N!}, \tag{16.7}$$

for any $t, x \in \mathbb{R}$. Hence by (16.6) and (16.7) we observe

$$\begin{aligned}
|\mathcal{R}| &\le \int_{-\infty}^{\infty} \varphi(2^k x - u)\left(\int_{-\infty}^{\infty} \varphi(2^k t - u)\omega_1\left(f^{(N)}, |t-x|\right) \frac{|t-x|^N}{N!} 2^k\, dt\right) du \\
&\le \frac{a^N}{N! 2^{N(k-1)}} \omega_1\left(f^{(N)}, \frac{a}{2^{k-1}}\right) \int_{-\infty}^{\infty} \varphi(2^k x - u)\left(\int_{-\infty}^{\infty} \varphi(2^k t - u) 2^k\, dt\right) du \\
&= \frac{a^N}{N! 2^{N(k-1)}} \omega_1\left(f^{(N)}, \frac{a}{2^{k-1}}\right).
\end{aligned}$$

So that

$$|\mathcal{R}| \leq \frac{a^N}{N!2^{N(k-1)}}\omega_1\left(f^{(N)}, \frac{a}{2^{k-1}}\right), \tag{16.8}$$

for any $k \in \mathbb{Z}$.

Finally putting together (16.5) and (16.8) we obtain (16.3). ■

Next we present

Theorem 16.5. *Let $f \in C^N(\mathbb{R})$, $N \geq 1$. Let φ be a real valued Lebesgue measurable function of compact support $\subseteq [-a, a]$, $a > 0$, $\varphi \geq 0$, such that*

$$\int_{-\infty}^{\infty} \varphi(x-u)\, du = 1, \quad \text{for any } x \in \mathbb{R},$$

same as

$$\int_{-\infty}^{\infty} \varphi(u)\, du = 1.$$

Define

$$(B_k f)(x) = \int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right)\varphi(2^k x - u)\, du, \tag{16.9}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$. Then

$$|(B_k f)(x) - f(x)| \leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!}\frac{a^i}{2^{ki}} + \frac{a^N}{N!2^{kN}}\omega_1\left(f^{(N)}, \frac{a}{2^k}\right), \tag{16.10}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$. Inequality (16.10) *is attained when f is a constant function.*

Remark 16.6. If $f^{(N)}$ is uniformly continuous or bounded and continuous, then as $k \to +\infty$ we obtain that

$$(B_k f)(x) \to f(x) \tag{16.11}$$

pointwise with rates. If f is bounded then $B_k f$ is bounded too.

Proof. of Theorem 16.5. Since $f \in C^N(\mathbb{R})$, $N \geq 1$, by Taylor's formula we have

$$\begin{aligned} f\left(\frac{u}{2^k}\right) &= f(x) + \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!}\left(\frac{u}{2^k} - x\right)^i \\ &\quad + \int_x^{u/2^k} \left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!}\, dt. \end{aligned}$$

Then

$$\begin{aligned} f\left(\frac{u}{2^k}\right)\varphi(2^k x - u) &= f(x)\varphi(2^k x - u) + \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!}\left(\frac{u}{2^k} - x\right)^i \varphi(2^k x - u) \\ &\quad + \varphi(2^k x - u)\int_x^{u/2^k} \left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!}\, dt. \end{aligned}$$

Consequently we obtain

$$\begin{aligned}(B_k f)(x) := \quad & \int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right)\varphi(2^k x - u)\,du = f(x)\int_{-\infty}^{\infty}\varphi(2^k x - u)\,du \\ & + \sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}\int_{-\infty}^{\infty}\left(\frac{u}{2^k} - x\right)^i \varphi(2^k x - u)\,du + \mathcal{R},\end{aligned}$$

where

$$\mathcal{R} := \int_{-\infty}^{\infty}\varphi(2^k x - u)\left(\int_x^{u/2^k}\left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!}\,dt\right)du. \quad (16.12)$$

That is

$$(B_k f)(x) - f(x) = \sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}\int_{-\infty}^{\infty}\left(\frac{u}{2^k} - x\right)^i \varphi(2^k x - u)du + \mathcal{R}, \quad (16.13)$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$. Here again, $\varphi \neq 0$ if $\left|x - \frac{u}{2^k}\right| \leq \frac{a}{2^k}$. Thus

$$|(B_k f)(x) - f(x)| \leq \sum_{i=1}^{N}\frac{|f^{(i)}(x)|}{i!}\,\frac{a^i}{2^{ki}} + |\mathcal{R}|. \quad (16.14)$$

We put

$$\Gamma_u(x) := \left|\int_x^{u/2^k}\left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!}\,dt\right|.$$

I.e.,

$$|\mathcal{R}| \leq \int_{-\infty}^{\infty}\varphi(2^k x - u)\Gamma_u(x)\,du.$$

Next we need to estimate $\Gamma_u(x)$.
i) if $x \leq \frac{u}{2^k}$, then

$$\begin{aligned}\Gamma_u(x) \quad & \leq \quad \int_x^{u/2^k}|f^{(N)}(t) - f^{(N)}(x)|\frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!}\,dt \\ & \leq \quad \int_x^{u/2^k}\omega_1\left(f^{(N)}, |t - x|\right)\frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!}\,dt \\ & \leq \quad \omega_1\left(f^{(N)}, \left|\frac{u}{2^k} - x\right|\right)\int_x^{u/2^k}\frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!}\,dt \\ & \leq \quad \omega_1\left(f^{(N)}, \frac{a}{2^k}\right)\frac{\left(\frac{u}{2^k} - x\right)^N}{N!} \leq \omega_1\left(f^{(N)}, \frac{a}{2^k}\right)\frac{a^N}{2^{kN}N!}\,.\end{aligned}$$

I.e., when $x \le \frac{u}{2^k}$, we have

$$\Gamma_u(x) \le \frac{a^N}{N!2^{kN}} \omega_1 \left(f^{(N)}, \frac{a}{2^k} \right).$$

ii) If $x \ge \frac{u}{2^k}$, then

$$\begin{aligned} \Gamma_u(x) &\le \int_{u/2^k}^{x} |f^{(N)}(t) - f^{(N)}(x)| \frac{\left(t - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} \, dt \\ &\le \int_{u/2^k}^{x} \omega_1\big(f^{(N)}, |t-x|\big) \frac{\left(t - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} \, dt \\ &\le \omega_1 \left(f^{(N)}, \left| x - \frac{u}{2^k} \right| \right) \int_{u/2^k}^{x} \frac{\left(t - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} \, dt \\ &= \omega_1 \left(f^{(N)}, \left| x - \frac{u}{2^k} \right| \right) \frac{\left(x - \frac{u}{2^k}\right)^{N}}{N!} \le \frac{a^N}{N!2^{kN}} \omega_1 \left(f^{(N)}, \frac{a}{2^k} \right). \end{aligned}$$

I.e.,

$$\Gamma_u(x) \le \frac{a^N}{N!2^{kN}} \omega_1 \left(f^{(N)}, \frac{a}{2^k} \right),$$

when $x \ge \frac{u}{2^k}$. That is

$$\Gamma_u(x) \le \frac{a^N}{N!2^{kN}} \omega_1 \left(f^{(N)}, \frac{a}{2^k} \right) =: \lambda \ge 0,$$

always true for any $x \in \mathbb{R}$. Consequently we have

$$|\mathcal{R}| \le \int_{-\infty}^{\infty} \varphi(2^k x - u) \lambda \, du = \lambda,$$

i.e.,

$$|\mathcal{R}| \le \lambda. \tag{16.15}$$

Clearly now (16.14) and (16.15) imply (16.10). ∎

It follows the related

Theorem 16.7. *Let $f \in C^N(\mathbb{R})$, $N \ge 1$. Let φ be a real valued Lebesgue measurable function of compact support $\subseteq [-a, a]$, $a > 0$, $\varphi \ge 0$, such that*

$$\int_{-\infty}^{\infty} \varphi(x-u) \, du = 1, \quad \textit{for any } x \in \mathbb{R}.$$

Define

$$c_k^f(u) := 2^k \int_0^{2^{-k}} f\left(t + \frac{u}{2^k} \right) dt, \quad u \in \mathbb{R}, \tag{16.16}$$

and

$$(L_k f)(x) := \int_{-\infty}^{\infty} c_k^f(u)\varphi(2^k x - u)\, du, \tag{16.17}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$.

Then

$$|(L_k f)(x) - f(x)| \leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{(a+1)^i}{2^{ki}} + \frac{(a+1)^N}{N!2^{kN}} \omega_1 \left(f^{(N)}, \frac{a+1}{2^k} \right), \tag{16.18}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$. Inequality (16.18) *is attained when f is a constant function.*

Remark 16.8. If $f^{(N)}$ is uniformly continuous or bounded and continuous, then as $k \to +\infty$ we obtain that

$$(L_k f)(x) \to f(x) \tag{16.19}$$

pointwise with rates. If f is bounded then $L_k f$ is bounded too.

Proof. of Theorem 16.7. Since $f \in C^N(\mathbb{R})$, $N \geq 1$, by Taylor's formula we have

$$\begin{aligned} f\left(t + \frac{u}{2^k}\right) &= f(x) + \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} \left(t + \frac{u}{2^k} - x\right)^i \\ &+ \int_x^{t+\frac{u}{2^k}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t + \frac{u}{2^k} - s\right)^{N-1}}{(N-1)!}\, ds. \end{aligned}$$

Then it follows

$$\begin{aligned} c_k^f(u) &= 2^k \int_0^{2^{-k}} f\left(t + \frac{u}{2^k}\right) dt \\ &= f(x) + \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} 2^k \int_0^{2^{-k}} \left(t + \frac{u}{2^k} - x\right)^i dt \\ &+ 2^k \int_0^{2^{-k}} \left(\int_x^{t+\frac{u}{2^k}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t + \frac{u}{2^k} - s\right)^{N-1}}{(N-1)!}\, ds \right) dt. \end{aligned}$$

So that

$$\begin{aligned}
&\int_{-\infty}^{\infty} c_k^f(u)\varphi(2^k x-u)\,du \\
&= f(x)\int_{-\infty}^{\infty}\varphi(2^k x-u)du \\
&\quad+\sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}2^k\int_{-\infty}^{\infty}\varphi(2^k x-u)\left(\int_0^{2^{-k}}\left(t+\frac{u}{2^k}-x\right)^i dt\right)du \\
&\quad+2^k\int_{-\infty}^{\infty}\varphi(2^k x-u)\left(\int_0^{2^{-k}}\left(\int_x^{t+\frac{u}{2^k}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\right.\right. \\
&\quad\left.\left.\times\ \frac{\left(t+\frac{u}{2^k}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)du.
\end{aligned}$$

That is we derive

$$\begin{aligned}
(L_k f)(x)-f(x) \quad = \quad & \sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}2^k\int_{-\infty}^{\infty}\varphi(2^k x-u) \\
& \left(\int_0^{2^{-k}}\left(t+\frac{u}{2^k}-x\right)^i dt\right)du+\mathcal{R},
\end{aligned} \tag{16.20}$$

where

$$\begin{aligned}
\mathcal{R} := \quad & 2^k\int_{-\infty}^{\infty}\varphi(2^k x-u)\left(\int_0^{2^{-k}}\left(\int_x^{t+\frac{u}{2^k}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\right.\right. \\
& \left.\left.\times\ \frac{\left(t+\frac{u}{2^k}-s\right)^{N-1}}{(N-1)!}ds\right)dt\right)du.
\end{aligned}$$

Again $\varphi \neq 0$, if $\left|\frac{u}{2^k}-x\right| \leq \frac{a}{2^k}$. Therefore

$$\begin{aligned}
\left|\int_0^{2^{-k}}\left(t+\frac{u}{2^k}-x\right)^i dt\right| \quad &\leq \quad \int_0^{2^{-k}}\left(|t|+\left|\frac{u}{2^k}-x\right|\right)^i dt \\
&\leq \quad \int_0^{2^{-k}}\left(\frac{1}{2^k}+\frac{a}{2^k}\right)^i dt=\frac{(a+1)^i}{2^k 2^{ki}}.
\end{aligned}$$

Consequently from (16.20) we obtain

$$|(L_k f)(x)-f(x)| \leq \sum_{i=1}^{N}\frac{|f^{(i)}(x)|}{i!}\,\frac{(a+1)^i}{2^{ki}}+|\mathcal{R}|. \tag{16.21}$$

We put

$$\delta(x,t,u) := \int_x^{t+\frac{u}{2^k}}\left(f^{(N)}(s)-f^{(N)}(x)\right)\frac{\left(t+\frac{u}{2^k}-s\right)^{N-1}}{(N-1)!}\,ds. \tag{16.22}$$

It follows that

$$|\mathcal{R}| \leq 2^k \int_{-\infty}^{\infty} \varphi(2^k x - u) \left(\int_0^{2^{-k}} |\delta(x,t,u)| dt \right) du. \tag{16.23}$$

Next we estimate $\delta(x,t,u)$.
i) Let $t + \frac{u}{2^k} \geq x$. Then

$$\begin{aligned} |\delta(x,t,u)| &\leq \int_x^{t+\frac{u}{2^k}} |f^{(N)}(s) - f^{(N)}(x)| \frac{\left(t + \frac{u}{2^k} - s\right)^{N-1}}{(N-1)!} ds \\ &\leq \int_x^{t+\frac{u}{2^k}} \omega_1\left(f^{(N)}, (s-x)\right) \frac{\left(t + \frac{u}{2^k} - s\right)^{N-1}}{(N-1)!} ds \\ &\leq \int_x^{t+\frac{u}{2^k}} \omega_1\left(f^{(N)}, \left(t + \frac{u}{2^k} - x\right)\right) \frac{\left(t + \frac{u}{2^k} - s\right)^{N-1}}{(N-1)!} ds \\ &\leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \int_x^{t+\frac{u}{2^k}} \frac{\left(t + \frac{u}{2^k} - s\right)^{N-1}}{(N-1)!} ds \\ &= \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{\left(t + \frac{u}{2^k} - x\right)^{N}}{N!}. \end{aligned}$$

That is when $t + \frac{u}{2^k} \geq x$ we derive

$$|\delta(x,t,u)| \leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{\left(t + \frac{u}{2^k} - x\right)^{N}}{N!}.$$

ii) Let $t + \frac{u}{2^k} \leq x$. Then

$$\begin{aligned} |\delta(x,t,u)| &\leq \int_{t+\frac{u}{2^k}}^{x} |f^{(N)}(s) - f^{(N)}(x)| \frac{\left(s - \left(t + \frac{u}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \\ &\leq \int_{t+\frac{u}{2^k}}^{x} \omega_1\left(f^{(N)}, (x-s)\right) \frac{\left(s - \left(t + \frac{u}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \\ &\leq \int_{t+\frac{u}{2^k}}^{x} \omega_1\left(f^{(N)}, \left(x - \frac{u}{2^k} - t\right)\right) \frac{\left(s - \left(t + \frac{u}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \\ &\leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \int_{t+\frac{u}{2^k}}^{x} \frac{\left(s - \left(t + \frac{u}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \\ &= \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{\left(x - \left(t + \frac{u}{2^k}\right)\right)^{N}}{N!}. \end{aligned}$$

That is, when $t + \frac{u}{2^k} \leq x$ we obtain

$$|\delta(x,t,u)| \leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{\left(x - \left(t + \frac{u}{2^k}\right)\right)^{N}}{N!}.$$

That is it is always true that

$$\begin{aligned} |\delta(x,t,u)| &\leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{\left|t+\frac{u}{2^k}-x\right|^N}{N!} \\ &\leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{(a+1)^N}{N!2^{kN}} =: \lambda \geq 0. \end{aligned} \tag{16.24}$$

I.e.,

$$|\delta(x,t,u)| \leq \lambda. \tag{16.25}$$

By (16.23) and (16.25) we find that

$$|\mathcal{R}| \leq 2^k \int_{-\infty}^{\infty} \varphi(2^k x - u) \left(\int_0^{2^{-k}} \lambda \, dt \right) du = \lambda.$$

That is

$$|\mathcal{R}| \leq \frac{(a+1)^N}{N!2^{kN}} \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right). \tag{16.26}$$

Finally from (16.21) and (16.26) we get (16.18). ■

The last main result follows.

Theorem 16.9. *Let $f \in C^N(\mathbb{R})$, $N \geq 1$. Let φ be a real valued Lebesgue measurable function of compact support $\subseteq [-a,a]$, $a > 0$, $\varphi \geq 0$, such that*

$$\int_{-\infty}^{\infty} \varphi(x-u)\, du = 1, \quad \text{for any } x \in \mathbb{R}.$$

Define

$$\begin{aligned} \gamma_k^f(u) &:= \sum_{j=0}^{n} w_j f\left(\frac{u}{2^k} + \frac{j}{2^k n}\right), \quad n \in \mathbb{N},\ w_j \geq 0, \\ \sum_{j=0}^{n} w_j &= 1, \quad u \in \mathbb{R} \quad \text{and} \\ (\Gamma_k f)(x) &:= \int_{-\infty}^{\infty} \gamma_k^f(u) \varphi(2^k x - u)\, du, \end{aligned} \tag{16.27}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$.

Then

$$|(\Gamma_k f)(x) - f(x)| \leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{(a+1)^i}{2^{ki}} + \frac{(a+1)^N}{N!2^{kN}} \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right), \tag{16.28}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$. Inequality (16.28) is attained when f is a constant function.

Remark 16.10. If $f^{(N)}$ is uniformly continuous or bounded and continuous, then as $k \to +\infty$ we get that

$$(\Gamma_k f)(x) \to f(x) \tag{16.29}$$

pointwise with rates. If f is bounded, then $\Gamma_k f$ is bounded too.

Proof. of Theorem 16.9. Since $f \in C^N(\mathbb{R})$, $N \geq 1$, by Taylor's formula we obtain

$$\begin{aligned}
&\sum_{j=0}^{n} w_j f\left(\frac{u}{2^k} + \frac{j}{2^k n}\right) \\
&\quad = \sum_{j=0}^{n} w_j f(x) + \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{n} w_j \left(\frac{u}{2^k} + \frac{j}{2^k n} - x\right)^i \\
&\quad\quad + \sum_{j=0}^{n} w_j \int_x^{\frac{u}{2^k}+\frac{j}{2^k n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{2^k} + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!}\, dt.
\end{aligned}$$

So that

$$\begin{aligned}
\gamma_k^f(u) - f(x) \quad &= \quad \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{n} w_j \left(\frac{u}{2^k} + \frac{j}{2^k n} - x\right)^i \\
&\quad + \sum_{j=0}^{n} w_j \int_x^{\frac{u}{2^k}+\frac{j}{2^k n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{2^k} + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!}\, dt.
\end{aligned}$$

Therefore we obtain

$$(\Gamma_k f)(x) - f(x) = \sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{n} w_j \int_{-\infty}^{\infty} \varphi(2^k x - u) \left(\frac{u}{2^k} + \frac{j}{2^k n} - x\right)^i du + \mathcal{R}, \tag{16.30}$$

where

$$\begin{aligned}
\mathcal{R} := \quad & \sum_{j=0}^{n} w_j \int_{-\infty}^{\infty} \varphi(2^k x - u) \\
& \left(\int_x^{\frac{u}{2^k}+\frac{j}{2^k n}} (f^{(N)}(t) - f^{(N)}(x)) \frac{\left(\frac{u}{2^k} + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!} dt\right) du.
\end{aligned}$$

Consequently we observe

$$\begin{aligned}
|(\Gamma_k f)(x) - f(x)| \quad &\leq \quad \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \sum_{j=0}^{n} w_j \int_{-\infty}^{\infty} \varphi(2^k x - u) \frac{(a+1)^i}{2^{ki}}\, du + |\mathcal{R}| \\
&= \quad \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{(a+1)^i}{2^{ki}} + |\mathcal{R}|.
\end{aligned} \tag{16.31}$$

We set

$$\varepsilon(x,u,j) := \int_x^{\frac{u}{2^k}+\frac{j}{2^k n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{u}{2^k} + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!}\, dt. \tag{16.32}$$

It follows that

$$|\mathcal{R}| \le \sum_{j=0}^{n} w_j \int_{-\infty}^{\infty} \varphi(2^k x - u) |\varepsilon(x,u,j)|\, du. \tag{16.33}$$

Next we estimate $\varepsilon(x,u,j)$.

i) *Case of* $\frac{u}{2^k} + \frac{j}{2^k n} \ge x$. Then

$$\begin{aligned} |\varepsilon(x,u,j)| &\le \int_x^{\frac{u}{2^k}+\frac{j}{2^k n}} |f^{(N)}(t) - f^{(N)}(x)| \frac{\left(\frac{u}{2^k} + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!}\, dt \\ &\le \omega_1\left(f^{(N)}, \left(\frac{u}{2^k} + \frac{j}{2^k n} - x\right)\right) \cdot \frac{\left(\frac{u}{2^k} + \frac{j}{2^k n} - x\right)^N}{N!} \\ &\le \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \cdot \frac{(a+1)^N}{N! 2^{kN}}. \end{aligned}$$

I.e., when $\frac{u}{2^k} + \frac{j}{2^k n} \ge x$ we derive

$$|\varepsilon(x,u,j)| \le \frac{(a+1)^N}{N! 2^{kN}} \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right).$$

ii) *Case of* $\frac{u}{2^k} + \frac{j}{2^k n} \le x$. Then

$$\begin{aligned} |\varepsilon(x,u,j)| &\le \int_{\frac{u}{2^k}+\frac{j}{2^k n}}^{x} |f^{(N)}(t) - f^{(N)}(x)| \frac{\left(t - \left(\frac{u}{2^k} + \frac{j}{2^k n}\right)\right)^{N-1}}{(N-1)!}\, dt \\ &\le \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{\left(x - \left(\frac{u}{2^k} + \frac{j}{2^k n}\right)\right)^N}{N!} \\ &\le \frac{(a+1)^N}{N! 2^{kN}} \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right). \end{aligned}$$

I.e., when $\frac{u}{2^k} + \frac{j}{2^k n} \le x$, we get again

$$|\varepsilon(x,u,j)| \le \frac{(a+1)^N}{N! 2^{kN}} \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right).$$

So always it holds

$$|\varepsilon(x,u,j)| \le \frac{(a+1)^N}{N! 2^{kN}} \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right). \tag{16.34}$$

Therefore we obtain

$$\begin{aligned} |\mathcal{R}| &\leq \sum_{j=0}^{n} w_j \int_{-\infty}^{\infty} \varphi(2^k x - u) \frac{(a+1)^N}{N! 2^{kN}} \omega_1 \left(f^{(N)}, \frac{a+1}{2^k} \right) du \\ &= \frac{(a+1)^N}{N! 2^{kN}} \omega_1 \left(f^{(N)}, \frac{a+1}{2^k} \right) := \lambda. \end{aligned} \tag{16.35}$$

I.e.,

$$|\mathcal{R}| \leq \lambda. \tag{16.36}$$

Finally from (16.31) and (16.36) we establish (16.28). ■

Remark 16.11. Here we define the following operators (see [67], p. 375 and [66])

$$\mathcal{L}_{k,j}(f;x) := \int_{-\infty}^{\infty} \ell_k(f, 2^k x - ju)\varphi(u)\, du, \quad j \in \mathbb{N},\ x \in \mathbb{R},\ k \in \mathbb{Z}.$$

Notice that $\mathcal{L}_{k,1} = \mathcal{L}_k$, any $k \in \mathbb{Z}$. As in [67], p. 381 and [66] we see that

$$\mathcal{L}_{k,j}(f;x) = \int_{-\infty}^{\infty} \ell_k(f,u) \frac{1}{j} \varphi\left(\frac{1}{j}(2^k x - u) \right) du,$$

and

$$\int_{-\infty}^{\infty} \frac{1}{j} \varphi\left(\frac{1}{j}(x-u) \right) du = 1, \quad \text{all } j \in \mathbb{N},\ \text{any } x \in \mathbb{R}.$$

By calling

$$\varphi_j^*(\cdot) := \frac{1}{j} \varphi\left(\frac{1}{j} \cdot \right), \quad j \in \mathbb{N}$$

we observe that *supp* $\varphi_j^* \subseteq [-ja, ja]$ and $\mathcal{L}_{k,j}(\varphi) \equiv \mathcal{L}_k(\varphi_j^*)$, furthermore φ_j^* inherits all the properties of φ. Denote here $\beta_k(f,u) := f\left(\frac{u}{2^k}\right)$, $k \in \mathbb{Z}$.

Based on the above comments and as in [67], p. 383 and [66], we define

$$(A_{k,j} f)(x) := \int_{-\infty}^{\infty} r_k^f (2^k x - ju)\varphi(u)\, du, \tag{16.37}$$

$$(B_{k,j} f)(x) := \int_{-\infty}^{\infty} \beta_k(f, 2^k x - ju)\varphi(u)\, du, \tag{16.38}$$

$$(L_{k,j} f)(x) := \int_{-\infty}^{\infty} c_k^f (2^k x - ju)\varphi(u)\, du, \tag{16.39}$$

and

$$(\Gamma_{k,j} f)(x) = \int_{-\infty}^{\infty} \gamma_k^f (2^k x - ju)\varphi(u)\, du. \tag{16.40}$$

Clearly $A_k = A_{k,1}$, $B_k = B_{k,1}$, $L_k = L_{k,1}$ and $\Gamma_k = \Gamma_{k,1}$.

Furthermore one can rewrite the above operators as follows ($j \in \mathbb{N}$)

$$(A_{k,j}f)(x) = \int_{-\infty}^{\infty} r_k^f(u)\varphi_j^*(2^k x - u)\,du, \tag{16.41}$$

$$(B_{k,j}f)(x) = \int_{-\infty}^{\infty} \beta_k(f,u)\varphi_j^*(2^k x - u)\,du, \tag{16.42}$$

$$(L_{k,j}f)(x) = \int_{-\infty}^{\infty} c_k^f(u)\varphi_j^*(2^k x - u)\,du, \tag{16.43}$$

and

$$(\Gamma_{k,j}f)(x) = \int_{-\infty}^{\infty} \gamma_k^f(u)\varphi_j^*(2^k x - u)\,du. \tag{16.44}$$

We present

Proposition 16.12. *Same assumptions as in Theorem* 16.3. *Then*

$$|(A_{k,j}(f)(x) - f(x)| \leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!}\,\frac{j^i a^i}{2^{i(k-1)}} + \frac{j^N a^N}{N!2^{N(k-1)}}\omega_1\left(f^{(N)}, \frac{ja}{2^{k-1}}\right), \tag{16.45}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$.

Proposition 16.13. *Same assumptions as in Theorem* 16.5. *Then*

$$|(B_{k,j}f)(x) - f(x)| \leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!}\,\frac{j^i a^i}{2^{ki}} + \frac{j^N a^N}{N!2^{kN}}\omega_1\left(f^{(N)}, \frac{ja}{2^{k}}\right), \tag{16.46}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$.

Proposition 16.14. *Same assumptions as in Theorem* 16.7. *Then*

$$|(L_{k,j}f)(x) - f(x)| \leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!}\,\frac{(ja+1)^i}{2^{ki}} + \frac{(ja+1)^N}{N!2^{kN}}\omega_1\left(f^{(N)}, \frac{ja+1}{2^{k}}\right), \tag{16.47}$$

for any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$.

Proposition 16.15. *Same assumptions as in Theorem* 16.9. *Then*

$$|(\Gamma_{k,j}f)(x) - f(x)| \leq \sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!}\,\frac{(ja+1)^i}{2^{ki}} \tag{16.48}$$

$$+ \frac{(ja+1)^N}{N!2^{kN}}\omega_1\left(f^{(N)}, \frac{ja+1}{2^{k}}\right), \ k \in \mathbb{Z}, \ x \in \mathbb{R}.$$

Note 16.16. Inequalities (16.45)–(16.48) are attained when f is a constant function.

Remark 16.17. We mention the generalized Jackson's like operators motivated from classical Approximation Theory, see [67], p. 377, and [66],

$$(I_{k,q}f)(x) := -\sum_{j=1}^{q}(-1)^j\binom{q}{j}(\mathcal{L}_{k,j}f)(x), \quad q \in \mathbb{N}. \tag{16.49}$$

We apply

$$-\sum_{j=1}^{q}(-1)^j\binom{q}{j} = 1.$$

Applications of the above general operator are (see [67], p. 384 and [66]),

$$(I_{k,q}^{A}f)(x) := -\sum_{j=1}^{q}(-1)^j\binom{q}{j}(A_{k,j}f)(x), \tag{16.50}$$

$$(I_{k,q}^{B}f)(x) := -\sum_{j=1}^{q}(-1)^j\binom{q}{j}(B_{k,j}f)(x), \tag{16.51}$$

$$(I_{k,q}^{L}f)(x) := -\sum_{j=1}^{q}(-1)^j\binom{q}{j}(L_{k,j}f)(x), \tag{16.52}$$

and

$$(I_{k,q}^{\Gamma}f)(x) := -\sum_{j=1}^{q}(-1)^j\binom{q}{j}(\Gamma_{k,j}f)(x), \tag{16.53}$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$.

From [67], p. 378, and [66], we get that

$$|(I_{k,q}f)(x) - f(x)| \leq \sum_{j=1}^{q}\binom{q}{j}|(\mathcal{L}_{k,j}f)(x) - f(x)|. \tag{16.54}$$

Inequality (16.54) is attained when f is a constant. Notice also that $\sum_{j=1}^{q}\binom{q}{j} = 2^q - 1$.

Applying the above we obtain:

Proposition 16.18. *Same assumptions as in Theorem 16.3. Then*

$$\begin{aligned} |(I_{k,q}^{A}f)(x) - f(x)| \quad &\leq \quad (2^q - 1)\left[\sum_{i=1}^{N}\frac{|f^{(i)}(x)|a^i q^i}{i!2^{i(k-1)}}\right. \\ &\quad \left. + \frac{a^N q^N}{N!2^{N(k-1)}}\omega_1\left(f^{(N)}, \frac{qa}{2^{k-1}}\right)\right], \end{aligned} \tag{16.55}$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$, $q \in \mathbb{N}$ fixed.

Proof. We have that

$$
\begin{aligned}
|(I^A_{k,q}f)(x) - f(x)| &\le \sum_{j=1}^{q} \binom{q}{j} |(A_{k,j}f)(x) - f(x)| \\
&\overset{(16.45)}{\le} \sum_{j=1}^{q} \binom{q}{j} \left[\sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{j^i a^i}{2^{i(k-1)}} + \frac{j^N a^N}{N! 2^{N(k-1)}} \omega_1 \left(f^{(N)}, \frac{ja}{2^{k-1}} \right) \right] \\
&\le \left(\sum_{j=1}^{q} \binom{q}{j} \right) \left[\sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{q^i a^i}{2^{i(K-1)}} + \frac{q^N a^N}{N! 2^{N(k-1)}} \omega_1 \left(f^{(N)}, \frac{qa}{2^{k-1}} \right) \right] \\
&= (2^q - 1) \left[\sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{q^i a^i}{2^{i(k-1)}} + \frac{q^N a^N}{N! 2^{N(k-1)}} \omega_1 \left(f^{(N)}, \frac{qa}{2^{k-1}} \right) \right].
\end{aligned}
$$

■

Proposition 16.19. *Same assumptions as in Theorem 16.5. Then*

$$
\begin{aligned}
|(I^B_{k,q}f)(x) - f(x)| \quad &\le \quad (2^q - 1) \left[\sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \right. \\
&\left. \cdot \frac{a^i q^i}{2^{ki}} + \frac{a^N q^N}{N! 2^{kN}} \omega_1 \left(f^{(N)}, \frac{qa}{2^k} \right) \right], \qquad (16.56)
\end{aligned}
$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$, $q \in \mathbb{N}$ fixed.

Proof. We observe that

$$
\begin{aligned}
|(I^B_{k,q}f)(x) - f(x)| &\le \sum_{j=1}^{q} \binom{q}{j} |(B_{k,j}f)(x) - f(x)| \\
&\le \sum_{j=1}^{q} \binom{q}{j} \left[\sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{j^i a^i}{2^{ki}} + \frac{j^N a^N}{N! 2^{kN}} \omega_1 \left(f^{(N)}, \frac{ja}{2^k} \right) \right] \\
&\le (2^q - 1) \left[\sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \frac{q^i a^i}{2^{ki}} + \frac{q^N a^N}{N! 2^{kN}} \omega_1 \left(f^{(N)}, \frac{qa}{2^k} \right) \right].
\end{aligned}
$$

■

Proposition 16.20. Same assumptions as in Theorem 16.7. Then

$$
\begin{aligned}
|(I^L_{k,q}f)(x) - f(x)| \quad &\le \quad (2^q - 1) \left[\sum_{i=1}^{N} \frac{|f^{(i)}(x)|}{i!} \right. \qquad (16.57) \\
&\left. \cdot \frac{(qa+1)^i}{2^{ki}} + \frac{(qa+1)^N}{N! 2^{kN}} \omega_1 \left(f^{(N)}, \frac{qa+1}{2^k} \right) \right],
\end{aligned}
$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$, $q \in \mathbb{N}$ fixed.

Proof. We see that

$$\begin{aligned}|(I_{k,q}^{L}f)(x)-f(x)| &\leq \sum_{j=1}^{q}\binom{q}{j}|(L_{k,j}f)(x)-f(x)| \\ &\leq \left(\sum_{j=1}^{q}\binom{q}{j}\right)\left[\sum_{i=1}^{N}\frac{|f^{(i)}(x)|}{i!}\frac{(qa+1)^{i}}{2^{ki}}+\frac{(qa+1)^{N}}{N!2^{kN}}\omega_1\left(f^{(N)},\frac{qa+1}{2^{k}}\right)\right].\end{aligned}$$

■

Proposition 16.21. *Same assumptions as in Theorem* 16.9. *Then*

$$\begin{aligned}|(I_{k,q}^{\Gamma}f)(x)-f(x)| \quad \leq \quad & (2^{q}-1)\left[\sum_{i=1}^{N}\frac{|f^{(i)}(x)|}{i!}\right. \\ & \left.\cdot\frac{(qa+1)^{i}}{2^{ki}}+\frac{(qa+1)^{N}}{N!2^{kN}}\omega_1\left(f^{(N)},\frac{qa+1}{2^{k}}\right)\right],\end{aligned} \tag{16.58}$$

any $k \in \mathbb{Z}$, any $x \in \mathbb{R}$, and $q \in \mathbb{N}$ fixed.

Proof. Similar to Proposition 16.20, using Proposition 16.15. ■

Note 16.22. Inequalities (16.55)–(16.58) are attained when f is a constant function.

Inequalities (16.55)–(16.58) improve a lot in the case of $N = 1$. We use that $\sum_{j=1}^{q}\binom{q}{j}j = q2^{q-1}$.

We give

Proposition 16.23. *Here $f \in C^1(\mathbb{R})$. Same assumptions as in Theorem* 16.3. *Then*

$$|(I_{k,q}^{A}f)(x)-f(x)| \leq \frac{aq}{2^{k-q}}\left(|f'(x)|+\omega_1\left(f',\frac{qa}{2^{k-1}}\right)\right), \tag{16.59}$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$, $q \in \mathbb{N}$ fixed.

Proof. We have again

$$\begin{aligned}|(I_{k,q}^{A}f)(x)-f(x)| &\overset{(16.54)}{\leq} \sum_{j=1}^{q}\binom{q}{j}|(A_{k,j}f)(x)-f(x)| \\ &\overset{(16.45)}{\leq} \sum_{j=1}^{q}\binom{q}{j}\frac{ja}{2^{k-1}}\left(|f'(x)|+\omega_1\left(f',\frac{qa}{2^{k-1}}\right)\right) \\ &= \frac{a}{2^{k-1}}\left(|f'(x)|+\omega_1\left(f',\frac{qa}{2^{k-1}}\right)\right)\left(\sum_{j=1}^{q}\binom{q}{j}j\right) \\ &= \frac{a}{2^{k-1}}\left(|f'(x)|+\omega_1\left(f',\frac{qa}{2^{k-1}}\right)\right)q2^{q-1}.\end{aligned}$$

■

Proposition 16.24. *Let $f \in C^1(\mathbb{R})$. Same assumptions as in Theorem* 16.5. *Then*

$$|(I^B_{k,q}f)(x) - f(x)| \leq \frac{aq}{2^{k-q+1}}\left(|f'(x)| + \omega_1\left(f', \frac{qa}{2^k}\right)\right), \tag{16.60}$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$, $q \in \mathbb{N}$ fixed.

Proof. We have again

$$\begin{aligned}
|(I^B_{k,q}f)(x) - f(x)| &\overset{(16.54)}{\leq} \sum_{j=1}^{q}\binom{q}{j}|(B_{k,j}f)(x) - f(x)| \\
&\overset{(16.46)}{\leq} \sum_{j=1}^{q}\binom{q}{j}\frac{ja}{2^k}\left(|f'(x)| + \omega_1\left(f', \frac{qa}{2^k}\right)\right) \\
&= \frac{a}{2^k}\left(|f'(x)| + \omega_1\left(f', \frac{qa}{2^k}\right)\right)\left(\sum_{j=1}^{q}\binom{q}{j}j\right) \\
&= \frac{aq2^{q-1}}{2^k}\left(|f'(x)| + \omega_1\left(f', \frac{qa}{2^k}\right)\right).
\end{aligned}$$

■

Proposition 16.25. *Let $f \in C^1(\mathbb{R})$. Same assumptions as in Theorem* 16.7. *Then*

$$\begin{aligned}
|(I^L_{k,q}f)(x) - f(x)| \leq\; & \frac{(aq2^{q-1} + 2^q - 1)}{2^k} \\
& \cdot\left(|f'(x)| + \omega_1\left(f', \frac{qa+1}{2^k}\right)\right),
\end{aligned} \tag{16.61}$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$, $q \in \mathbb{N}$ fixed.

Proof. We observe again

$$\begin{aligned}
|(I^L_{k,q}f)(x) - f(x)| &\overset{(16.54)}{\leq} \sum_{j=1}^{q}\binom{q}{j}|(L_{k,j}f)(x) - f(x)| \\
&\overset{(16.47)}{\leq} \sum_{j=1}^{q}\binom{q}{j}\left(\frac{ja+1}{2^k}\right)\left(|f'(x)| + \omega_1\left(f', \frac{qa+1}{2^k}\right)\right) \\
&= \frac{\left(|f'(x)| + \omega_1\left(f', \frac{qa+1}{2^k}\right)\right)}{2^k}\left(a\sum_{j=1}^{q}\binom{q}{j}j + \sum_{j=1}^{q}\binom{q}{j}\right) \\
&= \frac{\left(|f'(x)| + \omega_1\left(f', \frac{qa+1}{2^k}\right)\right)}{2^k}(aq2^{q-1} + 2^q - 1).
\end{aligned}$$

■

Finally we give

Proposition 16.26. *Let $f \in C^1(\mathbb{R})$. Same assumptions as in Theorem* 16.9. *Then*

$$|(I^{\Gamma}_{k,q} f)(x) - f(x)| \leq \frac{(aq2^{q-1} + 2^q - 1)}{2^k} \left(|f'(x)| + \omega_1 \left(f', \frac{qa+1}{2^k} \right) \right), \quad (16.62)$$

any $k \in \mathbb{Z}$, and any $x \in \mathbb{R}$, $q \in \mathbb{N}$ fixed.

Proof. Similar to Proposition 16.25. ■

16.3 Applications

Next we present applications to Probability. Let $F \in C^1(\mathbb{R})$ be a probability distribution function and $f = F'$ be the corresponding probability density function. Here we assume additionally that φ is a continuous function on $[-a, a]$.

By Remark 14.2.3(III), p. 389 of [67] and [66], we have that $A_{k,j}$, $B_{k,j}$, $L_{k,j}$, $\Gamma_{k,j}$ operators map continuous probabilistic distribution functions to continuous probabilistic distribution functions, for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$.

Corollary 16.27. (to Theorem 16.3 and Proposition 16.12). *It holds that*

$$|(A_{k,j} F)(x) - F(x)| \leq \frac{ja}{2^{k-1}} \left(f(x) + \omega_1 \left(f, \frac{ja}{2^{k-1}} \right) \right), \quad (16.63)$$

any $k \in \mathbb{Z}$, $x \in \mathbb{R}$, $j \in \mathbb{N}$.

Corollary 16.28. (to Theorem 16.5 and Proposition 16.13). *It holds that*

$$|(B_{k,j} F)(x) - F(x)| \leq \frac{ja}{2^k} \left(f(x) + \omega_1 \left(f, \frac{ja}{2^k} \right) \right), \quad (16.64)$$

any $k \in \mathbb{Z}$, $x \in \mathbb{R}$, $j \in \mathbb{N}$.

Corollary 16.29. (to Theorem 16.7 and Proposition 16.14). *It holds that*

$$|(L_{k,j} F)(x) - F(x)| \leq \frac{ja+1}{2^k} \left(f(x) + \omega_1 \left(f, \frac{ja+1}{2^k} \right) \right), \quad (16.65)$$

any $k \in \mathbb{Z}$, $x \in \mathbb{R}$, $j \in \mathbb{N}$.

Finally,

Corollary 16.30. (to Theorem 16.9 and Proposition 16.15). *It holds that*

$$|(\Gamma_{k,j}F)(x) - F(x)| \leq \frac{ja+1}{2^k}\left(f(x) + \omega_1\left(f, \frac{ja+1}{2^k}\right)\right), \tag{16.66}$$

any $k \in \mathbb{Z}$, $x \in \mathbb{R}$, $j \in \mathbb{N}$.

17

Quantitative Approximation by Multivariate Shift-Invariant Convolution Operators

High order differentiated functions of several variables are approximated by multivariate shift-invariant convolution type operators and their generalizations. The high order of this approximation is determined by giving some multivariate Jackson-type inequalities, involving the first multivariate usual modulus of continuity of the Nth order partial derivatives of the multivariate function to be approximated. This chapter follows [30].

17.1 Background

Here we use [67, p. 297], see also [78]. Let $X := C_U(\mathbb{R}^r)$, $r \geq 1$, be the space of uniformly continuous real valued functions on $\mathbb{R}^r$, and $C(\mathbb{R}^r)$ the space of continuous functions from $\mathbb{R}^r$ into $\mathbb{R}$. $C^N(\mathbb{R}^r)$, $N \geq 1$, denotes the space of N times continuously differentiable functions from $\mathbb{R}^r$ into $\mathbb{R}$. Let $\{\ell_k\}_{k\in\mathbb{Z}}$ be a sequence of positive linear operators that map X into $C(\mathbb{R}^r)$ with the property

$$(\ell_k f)(\vec{x}) := \big(\ell_0(f(2^{-k}\cdot))\big)(\vec{x}), \quad \vec{x} \in \mathbb{R}^r,\ f \in X.$$

Let φ be a real valued function of compact support $\subseteq \times_{i=1}^{r}[-a_i, a_i]$, $a_i > 0$, $\varphi \geq 0$, φ is Lebesgue measurable and such that

$$\int_{\mathbb{R}^r} \varphi(\vec{x} - \vec{u})d\vec{u} = 1, \quad \text{any } \vec{x} \in \mathbb{R}^r,$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 239–260.
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which is the same as

$$\int_{\mathbb{R}^r} \varphi(\vec{u})d\vec{u} = 1.$$

Examples. i) For $i = 1, \ldots, r$ consider the characteristic function

$$\varphi_i(x) := \chi_{[-\frac{1}{2},\frac{1}{2})}(x) = \begin{cases} 1, & x \in \left[-\frac{1}{2}, \frac{1}{2}\right) \\ 0, & \text{else.} \end{cases}$$

Define

$$\varphi^*(\vec{x}) := \prod_{i=1}^{r} \varphi_i(x_i), \quad \text{all } \vec{x} := (x_1, \ldots, x_r) \in \mathbb{R}^r.$$

Then φ^* fulfills above requirements for φ.

ii) For $i = 1, \ldots, r$ consider the hat function

$$\varphi_i(x_i) := \begin{cases} 1 + x_i, & -1 \le x_i \le 0, \\ 1 - x_i, & 0 \le x_i \le 1. \end{cases}$$

Define

$$\tilde{\varphi}(x_1, x_2, \ldots, x_r) := \prod_{i=1}^{r} \varphi_i(x_i) \ge 0, \quad \text{for all } (x_1, \ldots, x_r) \in \mathbb{R}^r.$$

Then $\tilde{\varphi}$ fulfills above requirements for φ.

Let $\{\mathcal{L}_k\}_{k\in\mathbb{Z}}$ be the sequence of positive linear operators acting on X and defined by

$$(\mathcal{L}_k f)(\vec{x}) := \int_{\mathbb{R}^r} (\ell_k f)(\vec{u})\varphi(2^k\vec{x} - \vec{u})d\vec{u}. \tag{$*$}$$

Notice that

$$(\mathcal{L}_k f)(\vec{x}) = \big(\mathcal{L}_0(f(2^{-k}\cdot))\big)(2^k\vec{x}), \quad \text{for any } k \in \mathbb{Z}, \text{ and } \vec{x} \in \mathbb{R}^r.$$

Clearly operators $\mathcal{L}_k$ can also act on $C^N(\mathbb{R}^r)$. See that φ is a multivariate scaling-like function and the operators $\mathcal{L}_k$ are convolution type or wavelet-like multivariate integral operators.

Operators $(*)$ under mild natural assumptions are shift invariant, possess the global smoothness preservation property, converge to the unit operator, and preserve continuous probability distribution functions. For these see again [67], Chapter 11 and [78]. Applications of operators $(*)$ were presented in the above mentioned references. In fact there the general specialized operators were denoted by $\{A_k\}_{k\in\mathbb{Z}}$, $\{B_k\}_{k\in\mathbb{Z}}$, $\{L_k\}_{k\in\mathbb{Z}}$, $\{\Gamma_k\}_{k\in\mathbb{Z}}$. These were first mentioned and studied in [78], and fulfill all the above nice properties of operators $(*)$. For their precise definition, see Theorems 17.3, 17.5, 17.7, 17.9, next.

In [67, Chapter 11, p. 318], and initially in [78], it was established the following motivating result.

Theorem 17.1. *For any* $k \in \mathbb{Z}$, $a := \max(a_1, \ldots, a_r)$, $\vec{x} \in \mathbb{R}^r$, *it holds*

$$\begin{aligned}
|(A_k f)(\vec{x}) - f(\vec{x})| &\le \omega_1\left(f, \frac{a}{2^{k-1}}\right), \\
|(B_k f)(\vec{x}) - f(\vec{x})| &\le \omega_1\left(f, \frac{a}{2^{k}}\right), \\
|(L_k f)(\vec{x}) - f(\vec{x})| &\le \omega_1\left(f, \frac{1+a}{2^{k}}\right), \\
|(\Gamma_k f)(\vec{x}) - f(\vec{x})| &\le \omega_1\left(f, \frac{1+a}{2^{k}}\right), \quad f \in X, \qquad (**)
\end{aligned}$$

where ω_1 *is the first usual multivariate modulus of continuity defined as follows.*

Definition 17.2. Let $f \in C(\mathbb{R}^r)$ which is bounded or uniformly continuous, we define $(h > 0)$

$$\omega_1(f, h) := \sup_{\text{all } x_i, x_i' \in \mathbb{R} | x_i - x_i'| \le h, \text{ for } i=1,\ldots,r} |f(x_1, \ldots, x_r) - f(x_1', \ldots, x_r')|. \quad (***)$$

From $(**)$ we get pointwise and uniform convergence to unit operator of operators A_k, B_k, L_k, Γ_k.

In this chapter, see Theorems 17.3, 17.5, 17.7, 17.9, we present inequalities similar to $(**)$, but much more complicated, involving $\omega_1(f_{\tilde{a}}, \cdot)$, $\tilde{a}$: $|\tilde{a}| = N$. Here $f_{\tilde{a}}$ denotes an Nth order partial derivative of $f \in C^N(\mathbb{R}^r)$, $N \ge 1$. That is studying the high order approximation to the unit of the particular general multivariate operators A_k, B_k, L_k, Γ_k. Then in several propositions we continue the same study for the more general multivariate operators $A_{k,j}$, $B_{k,j}$, $L_{k,j}$, $\Gamma_{k,j}$ and $I_{k,q}^A$, $I_{k,q}^B$, $I_{k,q}^L$, $I_{k,q}^\Gamma$. These operators are naturally built on the multivariate operators A_k, B_k, L_k, Γ_k and were studied in Chapter 15, pp. 399–400 of [67], see also [72] where first appeared.

17.2 Main Results

We give the first result:

Theorem 17.3. *Let* $f \in C^N(\mathbb{R}^r)$, N *and* $r \ge 1$. *Let* φ *be a real valued function of compact support* $\subseteq \times_{i=1}^r [-a_i, a_i]$, $a_i > 0$, $\varphi \ge 0$, φ *is continuous and even,* $\varphi(-\vec{x}) = \varphi(\vec{x})$, $\forall \vec{x} \in \mathbb{R}^r$. *Furthermore it is supposed that*

$$\underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{r-\text{fold}} \varphi(x_1 - u_1, \ldots, x_r - u_r) du_1 \cdots du_r = 1,$$

for any $\vec{x} := (x_1, \ldots, x_r) \in \mathbb{R}^r$*, which is the same as*

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \varphi(u_1, \ldots, u_r) du_1 \cdots du_r = 1,$$

in short

$$\int_{\mathbb{R}^r} \varphi(\vec{u}) d\vec{u} = 1.$$

Define

$$r_k^f(\vec{u}) := 2^{kr} \int_{\mathbb{R}^r} f(\vec{t}) \varphi(2^k \vec{t} - \vec{u}) d\vec{t} \tag{17.1}$$

for any $\vec{u} \in \mathbb{R}^r$*, and*

$$(A_k f)(\vec{x}) := \int_{\mathbb{R}^r} r_k^f(\vec{u}) \varphi(2^k \vec{x} - \vec{u}) d\vec{u}, \tag{17.2}$$

for any $k \in \mathbb{Z}$*, and any* $\vec{x} \in \mathbb{R}^r$*.*

Here we further assume that all of the partial derivatives of f *of order* N*, denoted by*

$$f_{\tilde{a}} := \frac{\partial^{\tilde{a}} f}{\partial x^{\tilde{a}}} \left(\tilde{a} := (a_1, \ldots, a_r), a_i \in \mathbb{Z}^+, i = 1, \ldots, r \colon |\tilde{a}| := \sum_{i=1}^r a_i = N \right),$$

are uniformly continuous or bounded and continuous on $\mathbb{R}^r$*. Denote* $a := \max(a_1, \ldots, a_r)$*. Then*

$$\begin{aligned} |(A_k f)(\vec{x}) - f(\vec{x})| \quad &\leq \quad \sum_{j=1}^N \frac{a^j}{j! 2^{(k-1)j}} \left(\left(\sum_{i=1}^r \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right) \\ &\quad + \frac{a^N r^N}{N! 2^{(k-1)N}} \max_{\tilde{a} \colon |\tilde{a}| = N} \omega_1 \left(f_{\tilde{a}}, \frac{a}{2^{k-1}} \right), \end{aligned} \tag{17.3}$$

for any $k \in \mathbb{Z}$*, and any* $\vec{x} \in \mathbb{R}^r$*. Inequality (17.3) is attained when* f *is a constant function.*

Remark 17.4. (i) If the Nth order partials $f_{\tilde{a}}$ are uniformly continuous or bounded and continuous then $\omega_1\big(f_{\tilde{a}}, \frac{a}{2^{k-1}}\big)$ are finite, and as $k \to +\infty$ we get that

$$(A_k f)(\vec{x}) \to f(\vec{x}),$$

pointwise with rates. If f is bounded then $(A_k f)$ is bounded too.

(ii) When $N = 1$, inequality (17.3) becomes

$$|(A_k f)(\vec{x}) - f(\vec{x})| \leq \frac{a}{2^{k-1}} \left\{ \left(\sum_{i=1}^r \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1, \ldots, r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{a}{2^{k-1}} \right) \right\}, \tag{17.4}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$.

Proof of Theorem 17.3. We observe that

$$(A_k f)(\vec{x}) - f(\vec{x}) = \int_{\mathbb{R}^r} (r_k^f(\vec{u}) - f(\vec{x}))\varphi(2^k \vec{x} - \vec{u}) d\vec{u},$$

where

$$r_k^f(\vec{u}) - f(\vec{x}) = \int_{\mathbb{R}^r} \left(f\left(\frac{\vec{y}}{2^k}\right) - f(\vec{x}) \right) \varphi(\vec{y} - \vec{x}) d\vec{y}.$$

Put

$$g_{\frac{\vec{y}}{2^k}}(t) := f\left(\vec{x} + t\left(\frac{\vec{y}}{2^k} - \vec{x}\right)\right), \quad 0 \le t \le 1.$$

Thus for $j = 1, \ldots, N$ we have

$$g_{\frac{\vec{y}}{2^k}}^{(j)}(t) = \left\{ \left(\sum_{i=1}^{r} \left(\frac{y_i}{2^k} - x_i \right) \frac{\partial}{\partial x_i} \right)^j f \right\} \left(x_1 + t\left(\frac{y_1}{2^k} - x_1\right), \ldots, x_r + t\left(\frac{y_r}{2^k} - x_r\right) \right)$$

and

$$g_{\frac{\vec{y}}{2^k}}(0) = f(\vec{x}).$$

Through Taylor's formula we get

$$f\left(\frac{\vec{y}}{2^k}\right) = g_{\frac{\vec{y}}{2^k}}(1) = \sum_{j=0}^{N} \frac{g_{\frac{\vec{y}}{2^k}}^{(j)}(0)}{j!} + \mathcal{R}_N\left(\frac{\vec{y}}{2^k}, 0\right),$$

where

$$\mathcal{R}_N\left(\frac{\vec{y}}{2^k}, 0\right) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{N-1}} \left(g_{\frac{\vec{y}}{2^k}}^{(N)}(t_N) - g_{\frac{\vec{y}}{2^k}}^{(N)}(0) \right) dt_N \right) \cdots \right) dt_1.$$

Consequently

$$\left(f\left(\frac{\vec{y}}{2^k}\right) - f(\vec{x}) \right) \varphi(\vec{y} - \vec{u}) = \sum_{j=1}^{N} \frac{g_{\frac{\vec{y}}{2^k}}^{(j)}(0)}{j!} \varphi(\vec{y} - \vec{u}) + \mathcal{R},$$

where

$$\mathcal{R} := \mathcal{R}_N\left(\frac{\vec{y}}{2^k}, 0\right) \varphi(\vec{y} - \vec{u}).$$

Since φ has a compact support it holds

$$\left| \frac{y_i}{2^k} - x_i \right| \le \frac{a_i}{2^{k-1}}, \quad i = 1, \ldots, r.$$

Therefore we derive

$$\left| g_{\frac{\vec{y}}{2^k}}^{(j)}(0) \right| \le \left(\frac{a}{2^{k-1}} \right)^j \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right).$$

Hence

$$|r_k^f(\vec{u}) - f(\vec{x})| \le \sum_{j=1}^{N} \frac{1}{j!} \int_{\mathbb{R}^r} \left| g_{\frac{\vec{y}}{2^k}}^{(j)}(0) \right| \varphi(\vec{y}-\vec{u}) d\vec{y} + \mathcal{R}^*,$$

where

$$\mathcal{R}^* := \int_{\mathbb{R}^r} \left| \mathcal{R}_N \left(\frac{\vec{y}}{2^k}, 0 \right) \right| \varphi(\vec{y}-\vec{u}) d\vec{y}.$$

That is,

$$|r_k^f(\vec{u}) - f(\vec{x})| \le \sum_{j=1}^{N} \frac{a^j}{j! 2^{j(k-1)}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right) + \mathcal{R}^*. \tag{17.5}$$

Let $0 \le t_N \le 1$, then

$$\begin{aligned} \left| g_{\frac{\vec{y}}{2^k}}^{(N)}(t_N) - g_{\frac{\vec{y}}{2^k}}^{(N)}(0) \right| &= \Bigg| \left\{ \left(\sum_{i=1}^{r} \left(\frac{y_i}{2^k} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f \right\} \\ &\quad \left(x_1 + t_N \left(\frac{y_1}{2^k} - x_1 \right), \ldots, x_r + t_N \left(\frac{y_r}{2^k} - x_r \right) \right) \\ &\quad - \left\{ \left(\sum_{i=1}^{r} \left(\frac{y_i}{2^k} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f \right\} (x_1, \ldots, x_r) \Bigg| \\ &\le \frac{a^N r^N}{2^{(k-1)N}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a}{2^{k-1}} \right). \end{aligned}$$

Consequently we find

$$\begin{aligned} \left| \mathcal{R}_N \left(\frac{\vec{y}}{2^k}, 0 \right) \right| &\le \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{N-1}} \left| g_{\frac{\vec{y}}{2^k}}^{(N)}(t_N) - g_{\frac{\vec{y}}{2^k}}^{(N)}(0) \right| dt_N \right) \cdots \right) dt_1 \\ &\le \frac{a^N r^N}{N! 2^{(k-1)N}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a}{2^{k-1}} \right) =: \lambda. \end{aligned} \tag{17.6}$$

That is

$$\left| \mathcal{R}_N \left(\frac{\vec{y}}{2^k}, 0 \right) \right| \le \lambda.$$

Therefore

$$\mathcal{R}^* \le \lambda \int_{\mathbb{R}^r} \varphi(\vec{y}-\vec{u}) d\vec{y} = \lambda.$$

That is

$$\mathcal{R}^* \le \lambda. \tag{17.7}$$

We have established that

$$|r_k^f(\vec{u}) - f(\vec{x})| \overset{(17.5),(17.6),(17.7)}{\leq} \sum_{j=1}^{N} \frac{a^j}{j!2^{j(k-1)}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right)$$
$$+ \frac{a^N r^N}{N!2^{(k-1)N}} \max_{\tilde{a}:\ |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a}{2^{k-1}} \right) =: \gamma. \tag{17.8}$$

It follows

$$|(A_k f)(\vec{x}) - f(\vec{x})| \overset{(17.8)}{\leq} \int_{\mathbb{R}^r} |r_k^f(\vec{u}) - f(\vec{x})| \varphi(2^k \vec{x} - \vec{u}) d\vec{u}$$
$$\leq \gamma \int_{\mathbb{R}^r} \varphi(2^k x - \vec{u}) d\vec{u} = \gamma,$$

proving (17.3). ■

Next we give

Theorem 17.5. *Let $f \in C^N(\mathbb{R}^r)$, N and $r \geq 1$. Let φ be a real valued function of compact support $\subseteq \times_{i=1}^r [-a_i, a_i]$, $a_i > 0$, $\varphi \geq 0$, φ is Lebesgue measurable and*

$$\int_{\mathbb{R}^r} \varphi(\vec{x} - \vec{u}) d\vec{u} = 1, \quad \text{for any } \vec{x} \in \mathbb{R}^r.$$

The last is the same as

$$\int_{\mathbb{R}^r} \varphi(\vec{u}) d\vec{u} = 1.$$

Define

$$\beta_k(f, \vec{u}) = f\left(\frac{\vec{u}}{2^k}\right), \quad \text{any } \vec{u} \in \mathbb{R}^r, \tag{17.9}$$

and

$$(B_k f)(\vec{x}) := \int_{\mathbb{R}^r} \beta_k(f, \vec{u}) \varphi(2^k \vec{x} - \vec{u}) d\vec{u}, \tag{17.10}$$

for any $k \in \mathbb{Z}$, and any $\vec{x} \in \mathbb{R}^r$.

Here we further assume that all partials $f_{\tilde{a}}$, $|\tilde{a}| = N$, are uniformly continuous or bounded and continuous on $\mathbb{R}^r$. Denote $a := \max(a_1, \ldots, a_r)$. Then

$$|(B_k f)(\vec{x}) - f(\vec{x})| \leq \sum_{j=1}^{N} \frac{a^j}{j!2^{kj}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right)$$
$$+ \frac{a^N r^N}{N!2^{kN}} \max_{\tilde{a}:\ |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a}{2^k} \right), \tag{17.11}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, which is attained by constant functions.

Remark 17.6. (i) Since the Nth order partials $f_{\tilde{\alpha}}$ are uniformly continuous or bounded and continuous and $k \to +\infty$ we find that

$$(B_k f)(\vec{x}) - f(\vec{x}),$$

pointwise with rates. If f is bounded then $(B_k f)$ is bounded too.

(ii) When $N = 1$, inequality (17.11) becomes

$$|(B_k f)(\vec{x}) - f(\vec{x})| \le \frac{a}{2^k} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\dots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{a}{2^k} \right) \right\}, \quad (17.12)$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$.

Proof of Theorem 17.5. Put

$$g_{\frac{\vec{u}}{2^k}}(t) := f\left(\vec{x} + t \left(\frac{\vec{u}}{2^k} - \vec{x} \right) \right), \quad \text{all } 0 \le t \le 1.$$

Then for $j = 1, 2, \dots, N$ we get that

$$g^{(j)}_{\frac{\vec{u}}{2^k}}(t) = \left\{ \left(\sum_{i=1}^{r} \left(\frac{u_i}{2^k} - x_i \right) \frac{\partial}{\partial x_i} \right)^j f \right\} \left(\vec{x} + t \left(\frac{\vec{u}}{2^k} - \vec{x} \right) \right),$$

$$g_{\frac{\vec{u}}{2^k}}(0) = f(\vec{x}).$$

By Taylor's formula we derive

$$f\left(\frac{\vec{u}}{2^k} \right) = g_{\frac{\vec{u}}{2^k}}(1) = \sum_{j=0}^{N} \frac{g^{(j)}_{\frac{\vec{u}}{2^k}}(0)}{j!} + \mathcal{R}_N\left(\frac{\vec{u}}{2^k}, 0 \right),$$

where

$$\mathcal{R}_N\left(\frac{\vec{u}}{2^k}, 0 \right) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{N-1}} \left(g^{(N)}_{\frac{\vec{u}}{2^k}}(t_N) - g^{(N)}_{\frac{\vec{u}}{2^k}}(0) \right) dt_N \right) \cdots \right) dt_1.$$

Thus

$$f\left(\frac{\vec{u}}{2^k} \right) \varphi(2^k x - \vec{u}) = \sum_{j=0}^{N} \frac{g^{(j)}_{\frac{\vec{u}}{2^k}}(0)}{j!} \varphi(2^k \vec{x} - \vec{u}) + \varphi(2^k \vec{x} - \vec{u}) \mathcal{R}_N\left(\frac{\vec{u}}{2^k}, 0 \right).$$

Consequently we observe that

$$(B_k f)(\vec{x}) - f(\vec{x}) = \sum_{j=1}^{N} \int_{\mathbb{R}^r} \frac{g^{(j)}_{\frac{\vec{u}}{2^k}}(0)}{j!} \varphi(2^k \vec{x} - \vec{u}) d\vec{u} + \mathcal{R},$$

where

$$\mathcal{R} := \int_{\mathbb{R}^r} \varphi(2^k\vec{x} - \vec{u})\mathcal{R}_N\left(\frac{\vec{u}}{2^k}, 0\right) d\vec{u}.$$

Since φ is of compact support we have

$$\left|x_i - \frac{u_i}{2^k}\right| \le \frac{a_i}{2^k}, \quad i = 1, \dots, r.$$

Furthermore we get

$$\left|g^{(j)}_{\frac{\vec{u}}{2^k}}(0)\right| \le \left(\frac{a}{2^k}\right)^j \left(\left(\sum_{i=1}^r \left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right),$$

and

$$\begin{aligned}\left|\sum_{j=1}^N \int_{\mathbb{R}^r} \frac{g^{(j)}_{\frac{\vec{u}}{2^k}}(0)}{j!}\varphi(2^k\vec{x} - \vec{u})d\vec{u}\right| &\le \sum_{j=1}^N \int_{\mathbb{R}^r} \frac{|g^{(j)}_{\frac{\vec{u}}{2^k}}(0)|}{j!}\varphi(2^k\vec{x} - \vec{u})d\vec{u} \\ &\le \sum_{j=1}^N \frac{a^j}{2^{kj}j!}\left(\left(\sum_{i=1}^r \left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right)\int_{\mathbb{R}^r}\varphi(2^k\vec{x} - \vec{u})d\vec{u} \\ &= \sum_{j=1}^N \frac{a^j}{2^{kj}j!}\left(\left(\sum_{i=1}^r \left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right).\end{aligned}$$

That is,

$$|(B_k f)(\vec{x}) - f(\vec{x})| \le \sum_{j=1}^N \frac{a^j}{j!2^{kj}}\left(\left(\sum_{i=1}^r \left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right) + |\mathcal{R}|. \tag{17.13}$$

Next we estimate $|\mathcal{R}|$, $0 \le t_N \le 1$. We observe that

$$\begin{aligned}\left|g^{(N)}_{\frac{\vec{u}}{2^k}}(t_N) - g^{(N)}_{\frac{\vec{u}}{2^k}}(0)\right| &= \left|\left\{\left(\sum_{i=1}^r \left(\frac{u_i}{2^k} - x_i\right)\frac{\partial}{\partial x_i}\right)^N f\right\}\left(\vec{x} + t_N\left(\frac{\vec{u}}{2^k} - \vec{x}\right)\right)\right. \\ &\quad \left. - \left\{\left(\sum_{i=1}^r \left(\frac{u_i}{2^k} - x_i\right)\frac{\partial}{\partial x_i}\right)^N f\right\}(\vec{x})\right| \\ &\le \frac{a^N r^N}{2^{kN}} \max_{\tilde{a}:\,|\tilde{a}|=N} \omega_1\left(f_{\tilde{a}}, \frac{a}{2^k}\right).\end{aligned}$$

Thus

$$\left|\mathcal{R}_N\left(\frac{\vec{u}}{2^k},0\right)\right| \leq \int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{N-1}}\left|g^{(N)}_{\frac{\vec{u}}{2^k}}(t_N)-g^{(N)}_{\frac{\vec{u}}{2^k}}(0)\right|dt_N\right)\cdots\right)dt_1$$
$$\leq \int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{N-1}}\frac{a^Nr^N}{2^{kN}}\max_{\tilde{a}:\,|\tilde{a}|=N}\omega_1\left(f_{\tilde{a}},\frac{a}{2^k}\right)dt_N\right)\cdots\right)dt_1$$
$$= \frac{a^Nr^N}{N!2^{kN}}\max_{\tilde{a}:\,|\tilde{a}|=N}\omega_1\left(f_{\tilde{a}},\frac{a}{2^k}\right)=:\rho. \tag{17.14}$$

Consequently we have

$$|\mathcal{R}|\leq\rho\int_{\mathbb{R}^r}\varphi(2^k\vec{x}-\vec{u})d\vec{u}=\rho,$$

that is,

$$|\mathcal{R}|\leq\rho. \tag{17.15}$$

Finally combining (17.13), (17.14), (17.15) we produce (17.11). ∎

It follows the related

Theorem 17.7. *Let f, φ, a as in Theorem 17.5. Define*

$$c_k^f(\vec{u}):=2^{kr}\int_{2^{-k}\vec{u}}^{2^{-k}(\vec{u}+\vec{1})}f(\vec{t})d\vec{t}=2^{kr}\int_0^{2^{-\vec{k}}}f\left(\vec{t}+\frac{\vec{u}}{2^k}\right)d\vec{t},\ \text{any } \vec{u}\in\mathbb{R}^r, \tag{17.16}$$

and

$$(L_kf)(\vec{x}):=\int_{\mathbb{R}^r}c_k^f(\vec{u})\varphi(2^k\vec{x}-\vec{u})d\vec{u}, \tag{17.17}$$

for any $k\in\mathbb{Z}$, and any $\vec{x}\in\mathbb{R}^r$. Then

$$|(L_kf)(\vec{x})-f(\vec{x})|\leq\sum_{j=1}^N\frac{(a+1)^j}{j!2^{kj}}\left(\left(\sum_{i=1}^r\left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right)$$
$$+\frac{(a+1)^Nr^N}{N!2^{kN}}\max_{\tilde{a}:\,|\tilde{a}|=N}\omega_1\left(f_{\tilde{a}},\frac{a+1}{2^k}\right), \tag{17.18}$$

which is attained by constant functions.

Remark 17.8. (i) Since the Nth order partials $f_{\tilde{a}}$ are uniformly continuous or bounded and continuous and $k\to+\infty$ we obtain that

$$(L_kf)(\vec{x})\to f(\vec{x}),$$

pointwise with rates. If f is bounded then (L_kf) is bounded too.

(ii) When $N=1$, inequality (17.18) becomes

$$|(L_kf)(\vec{x})-f(\vec{x})|\leq\left(\frac{a+1}{2^k}\right)\left\{\left(\sum_{i=1}^r\left|\frac{\partial f(\vec{x})}{\partial x_i}\right|\right)+r\max_{i\in\{1,\dots,r\}}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{a+1}{2^k}\right)\right\}, \tag{17.19}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$.

Proof of Theorem 17.7. We see that

$$(L_k f)(\vec{x}) - f(\vec{x}) = \int_{\mathbb{R}^r} (c_k^f(\vec{u}) - f(\vec{x}))\varphi(2^k \vec{x} - \vec{u}) d\vec{u}.$$

We set

$$g_{\vec{t}+\frac{\vec{u}}{2^k}}(\tau) := f\left(\vec{x} + \tau\left(\vec{t} + \frac{\vec{u}}{2^k} - \vec{x}\right)\right), \quad 0 \le \tau \le 1.$$

Thus

$$g^{(j)}_{\vec{t}+\frac{\vec{u}}{2^k}}(\tau) = \left\{\left(\sum_{i=1}^r \left(t_i + \frac{u_i}{2^k} - x_i\right)\frac{\partial}{\partial x_i}\right)^j f\right\}\left(\vec{x} + \tau\left(\vec{t} + \frac{\vec{u}}{2^k} - \vec{x}\right)\right),$$

and

$$g_{\vec{t}+\frac{\vec{u}}{2^k}}(0) = f(\vec{x}).$$

By Taylor's formula we obtain

$$f\left(\vec{t} + \frac{\vec{u}}{2^k}\right) = g_{\vec{t}+\frac{\vec{u}}{2^k}}(1) = \sum_{j=0}^N \frac{g^{(j)}_{\vec{t}+\frac{\vec{u}}{2^k}}}{j!}(0) + \mathcal{R}_N\left(\vec{t} + \frac{\vec{u}}{2^k}, 0\right),$$

where

$$\mathcal{R}_N\left(\vec{t} + \frac{\vec{u}}{2^k}, 0\right) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{N-1}} \left(g^{(N)}_{\vec{t}+\frac{\vec{u}}{2^k}}(t_N) - g^{(N)}_{\vec{t}+\frac{\vec{u}}{2^k}}(0)\right) dt_N\right)\cdots\right) dt_1.$$

Then

$$c_k^f(\vec{u}) - f(\vec{x}) = \sum_{j=1}^N 2^{kr} \frac{\int_{\vec{0}}^{2^{-\vec{k}}} g^{(j)}_{\vec{t}+\frac{\vec{u}}{2^k}}(0) d\vec{t}}{j!} + 2^{kr}\int_{\vec{0}}^{2^{-\vec{k}}} \mathcal{R}_N\left(\vec{t} + \frac{\vec{u}}{2^k}, 0\right) d\vec{t}.$$

Here $0 \le t_i \le 2^{-k}$, and

$$\left|x_i - \frac{u_i}{2^k}\right| \le \frac{a_i}{2^k}, \quad i = 1, \ldots, r.$$

Furthermore we have $(j = 1, \ldots, N)$

$$\left|g^{(j)}_{\vec{t}+\frac{\vec{u}}{2^k}}(0)\right| \le \left(\frac{a+1}{2^k}\right)^j \left(\left(\sum_{i=1}^r \left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right).$$

Thus

$$\sum_{j=1}^N \frac{2^{kr}}{j!}\int_{\vec{0}}^{2^{-\vec{k}}} \left|g^{(j)}_{\vec{t}+\frac{\vec{u}}{2^k}}(0)\right| d\vec{t} \le \sum_{j=1}^N \frac{(a+1)^j}{j! 2^{kj}} \left(\left(\sum_{i=1}^r \left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right). \qquad (17.20)$$

Let $0 \le \tau_N \le 1$, then

$$\begin{aligned}\left| g^{(N)}_{\vec{t}+\frac{\vec{u}}{2^k}}(\tau_N) - g^{(N)}_{\vec{t}+\frac{\vec{u}}{2^k}}(0) \right| &= \left| \left\{ \left(\sum_{i=1}^{r} \left(t_i + \frac{u_i}{2^k} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f \right\} \left(\vec{x} + \tau_N \left(\vec{t} + \frac{\vec{u}}{2^k} - \vec{x} \right) \right) \right. \\ & \quad \left. - \left\{ \left(\sum_{i=1}^{r} \left(t_i + \frac{u_i}{2^k} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f \right\} (\vec{x}) \right| \\ &\le \frac{(a+1)^N r^N}{2^{kN}} \max_{\tilde{a}:\,|\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a+1}{2^k} \right).\end{aligned}$$

Moreover it holds

$$\begin{aligned}\left| \mathcal{R}_N \left(\vec{t} + \frac{\vec{u}}{2^k}, 0 \right) \right| &\le \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{N-1}} \left| g^{(N)}_{\vec{t}+\frac{\vec{u}}{2^k}}(t_N) - g^{(N)}_{\vec{t}+\frac{\vec{u}}{2^k}}(0) \right| dt_N \right) \cdots \right) dt_1 \\ &\le \frac{(a+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\,|\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a+1}{2^k} \right). \qquad (17.21)\end{aligned}$$

From (17.20) and (17.21) we find

$$\begin{aligned}|c_k^f(\vec{u}) - f(\vec{x})| &\le \sum_{j=1}^{N} \frac{(a+1)^j}{j! 2^{kj}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right) \\ &+ \frac{(a+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\,|\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a+1}{2^k} \right) =: \rho. \qquad (17.22)\end{aligned}$$

Finally we have

$$|(L_k f)(\vec{x}) - f(\vec{x})| \overset{(17.22)}{\le} \rho \int_{\mathbb{R}^r} \varphi(2^k \vec{x} - \vec{u}) d\vec{u} = \rho. \qquad \blacksquare$$

The last main result follows.

Theorem 17.9. *Let f, φ, a as in Theorem 17.5. Define*

$$(\Gamma_k f)(\vec{x}) := \int_{\mathbb{R}^r} \gamma_k^f(\vec{u}) \varphi(2^k \vec{x} - \vec{u}) d\vec{u}, \quad \vec{x} \in \mathbb{R}^r, \qquad (17.23)$$

where

$$\gamma_k^f(\vec{u}) := \sum_{j_1=0}^{n_1} \cdots \sum_{j_r=0}^{n_r} w_{j_1,\ldots,j_r} \cdot f \left(\frac{u_1}{2^k} + \frac{j_1}{2^k n_1}, \ldots, \frac{u_r}{2^k} + \frac{j_r}{2^k n_r} \right) (n_1, \ldots, n_r) \in \mathbb{N}^r,$$

$$w_{j_1,\ldots,j_r} \ge 0, \ \sum_{j_1=0}^{n_1} \cdots \sum_{j_r=0}^{n_r} w_{j_1,\ldots,j_r} = 1, \ \vec{u} \in \mathbb{R}^r. \qquad (17.24)$$

Then

$$|(\Gamma_k f)(\vec{x}) - f(\vec{x})| \leq \sum_{j=1}^{N} \frac{(a+1)^j}{j!2^{kj}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right)$$
$$+ \frac{(a+1)^N r^N}{N!2^{kN}} \max_{\tilde{a}\,:\,|\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a+1}{2^k} \right), \tag{17.25}$$

which is attained by constant functions.

Remark 17.10. (i) Since the Nth order partials $f_{\tilde{a}}$ are uniformly continuous or bounded and continuous and $k \to +\infty$ we get that

$$(\Gamma_k f)(\vec{x}) \to f(\vec{x}),$$

pointwise with rates. If f is bounded then $(\Gamma_k f)$ is bounded too.

(ii) When $N = 1$, inequality (17.25) becomes

$$|(\Gamma_k f)(\vec{x}) - f(\vec{x})| \leq \left(\frac{a+1}{2^k} \right) \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{a+1}{2^k} \right) \right\}, \tag{17.26}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$.

Proof of Theorem 17.9. We see that

$$(\Gamma_k f)(\vec{x}) - f(\vec{x}) = \int_{\mathbb{R}^r} (\gamma_k^f(\vec{u}) - f(\vec{x}))\varphi(2^k\vec{x} - \vec{u})d\vec{u},$$

where

$$\gamma_k^f(\vec{u}) - f(\vec{x}) = \sum_{\vec{j}=\vec{0}}^{\vec{n}} w_{\vec{j}} \left(f\left(\frac{\vec{u}}{2^k} + \frac{\vec{j}}{2^k\vec{n}} \right) - f(\vec{x}) \right).$$

Put

$$g_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(\tau) := f\left(\vec{x} + \tau \left(\frac{\vec{u}}{2^k} + \frac{\vec{j}}{2^k\vec{n}} - \vec{x} \right) \right), \quad 0 \leq \tau \leq 1.$$

Thus $(j = 1, \ldots, N)$

$$g^{(j)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(\tau) = \left\{ \left(\sum_{i=1}^{r} \left(\frac{u_i}{2^k} + \frac{j_i}{2^k n_i} - x_i \right) \frac{\partial}{\partial x_i} \right)^j f \right\} \left(\vec{x} + \tau \left(\frac{\vec{u}}{2^k} + \frac{\vec{j}}{2^k\vec{n}} - \vec{x} \right) \right),$$

and

$$g_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(0) = f(\vec{x}).$$

By Taylor's formula we obtain

$$f\left(\frac{\vec{u}}{2^k} + \frac{\vec{j}}{2^k\vec{n}} \right) - f(\vec{x}) = g_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(1) - f(\vec{x}) = \sum_{j=1}^{N} \frac{g^{(j)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(0)}{j!} + \mathcal{R}_N\left(\frac{\vec{u}}{2^k} + \frac{\vec{j}}{2^k\vec{n}}, 0 \right),$$

where

$$\mathcal{R}_N\left(\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}},0\right):=\int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{N-1}}\left(g^{(N)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(t_N)-g^{(N)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(0)\right)dt_N\right)\cdots\right)dt_1.$$

Therefore

$$\gamma_k^f(\vec{u})-f(\vec{x})=\sum_{j=1}^{N}\sum_{\vec{j}=\vec{0}}^{\vec{n}}w_{\vec{j}}\frac{g^{(j)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(0)}{j!}+\sum_{\vec{j}=\vec{0}}^{\vec{n}}w_{\vec{j}}\mathcal{R}_N\left(\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}},0\right).$$

Again here it holds that

$$\left|x_i-\frac{u_i}{2^k}\right|\le\frac{a_i}{2^k},\ \ \frac{j_i}{n_i}\le 1;\quad i=1,\ldots,r,$$

and $a:=\max(a_1,\ldots,a_r)$. Furthermore, we have $(j=1,\ldots,N)$

$$\left|g^{(j)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(0)\right|\le\frac{(a+1)^j}{2^{kj}}\left(\left(\sum_{i=1}^{r}\left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right).$$

Thus

$$\sum_{j=1}^{N}\sum_{\vec{j}=\vec{0}}^{\vec{n}}w_{\vec{j}}\frac{|g^{(j)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(0)|}{j!}\le\sum_{j=1}^{N}\frac{(a+1)^j}{j!2^{kj}}\left(\left(\sum_{i=1}^{r}\left|\frac{\partial}{\partial x_i}\right|\right)^j f(\vec{x})\right).\tag{17.27}$$

Let $0\le\tau_N\le 1$, then

$$\begin{aligned}&\left|g^{(N)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(\tau_N)-g^{(N)}_{\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}}(0)\right|\\&\quad=\left|\left\{\left(\sum_{i=1}^{r}\left(\frac{u_i}{2^k}+\frac{j_i}{2^kn_i}-x_i\right)\frac{\partial}{\partial x_i}\right)^N f\right\}\left(\vec{x}+\tau_N\left(\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}}-\vec{x}\right)\right)\right.\\&\left.-\left\{\left(\sum_{i=1}^{r}\left(\frac{u_i}{2^k}+\frac{j_i}{2^kn_i}-x_i\right)\frac{\partial}{\partial x_i}\right)^N f\right\}(\vec{x})\right|\le\frac{(a+1)^N r^N}{2^{kN}}\max_{\tilde{a}:\,|\tilde{a}|=N}\omega_1\left(f_{\tilde{a}},\frac{a+1}{2^k}\right).\end{aligned}$$

Clearly we obtain

$$\left|\mathcal{R}_N\left(\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}},0\right)\right|\le\frac{(a+1)^N r^N}{N!2^{kN}}\max_{\tilde{a}:\,|\tilde{a}|=N}\omega_1\left(f_{\tilde{a}},\frac{a+1}{2^k}\right)=:\theta.\tag{17.28}$$

That is,

$$\sum_{\vec{j}=\vec{0}}^{\vec{n}}w_{\vec{j}}\left|\mathcal{R}_N\left(\frac{\vec{u}}{2^k}+\frac{\vec{j}}{2^k\vec{n}},0\right)\right|\le\theta.\tag{17.29}$$

From (17.27), (17.28) and (17.29) we find

$$|\gamma_k^f(\vec{u}) - f(\vec{x})| \leq \sum_{j=1}^{N} \frac{(a+1)^j}{j!2^{kj}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\vec{x}) \right)$$
$$+ \frac{(a+1)^N r^N}{N!2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{a+1}{2^k} \right) =: \rho. \tag{17.30}$$

Finally we notice that

$$|(\Gamma_k f)(\vec{x}) - f(\vec{x})| \overset{(17.30)}{\leq} \rho \int_{\mathbb{R}^r} \varphi(2^k \vec{x} - \vec{u}) d\vec{u} = \rho. \qquad \blacksquare$$

Remark 17.11. Here we define the following multivariate operators (see [67, p. 394], and [72])

$$\mathcal{L}_{k,j}(f;\vec{x}) := \int_{\mathbb{R}^r} \ell_k\big(f; 2^k\vec{x} - j\vec{u}\big)\varphi(\vec{u})d\vec{u}, \quad k \in \mathbb{Z},\ j \in \mathbb{N},\ \vec{x} \in \mathbb{R}^r. \tag{17.31}$$

Notice that $\mathcal{L}_{k,1} = \mathcal{L}_k$, any $k \in \mathbb{Z}$. As in [67, p. 394], and [72] we notice that

$$\mathcal{L}_{k,j}(f;\vec{x}) = \int_{\mathbb{R}^r} (\ell_k f)(\vec{u}) \frac{1}{j^r} \varphi\left(\frac{1}{j}(2^k\vec{x} - \vec{u}) \right) d\vec{u}, \quad k \in \mathbb{Z},\ \vec{x} \in \mathbb{R}^r.$$

We see that

$$\int_{\mathbb{R}^r} \frac{1}{j^r} \varphi\left(\frac{1}{j}(\vec{x} - \vec{u}) \right) d\vec{u} = 1, \quad \text{all } j \in \mathbb{N},\ \vec{x} \in \mathbb{R}^r,\ r \geq 1.$$

Put

$$\varphi_j^*(\cdot) := \frac{1}{j^r} \varphi\left(\frac{1}{j} \cdot \right), \quad j \in \mathbb{N},$$

then supp $\varphi_j^* \subseteq \times_{i=1}^r [-ja_i, ja_i]$, $a_i > 0$. Moreover φ_j^* inherits all other properties of φ.

Clearly now we have that

$$\mathcal{L}_{k,j}(\varphi) = \mathcal{L}_k(\varphi_j^*).$$

According to the above comments and as in [67, p. 399], and [72], we define

$$(A_{k,j} f)(\vec{x}) := \int_{\mathbb{R}^r} r_k^f(\vec{u}) \varphi_j^*(2^k\vec{x} - \vec{u}) d\vec{u}, \tag{17.32}$$

$$(B_{k,j} f)(\vec{x}) := \int_{\mathbb{R}^r} f\left(\frac{\vec{u}}{2^k} \right) \varphi_j^*(2^k\vec{x} - \vec{u}) d\vec{u}, \tag{17.33}$$

$$(L_{k,j} f)(\vec{x}) := \int_{\mathbb{R}^r} c_k^f(\vec{u}) \varphi_j^*(2^k\vec{x} - \vec{u}) d\vec{u}, \tag{17.34}$$

and

$$(\Gamma_{k,j} f)(\vec{x}) := \int_{\mathbb{R}^r} \gamma_k^f(\vec{u}) \varphi_j^*(2^k \vec{x} - \vec{u}) d\vec{u}, \tag{17.35}$$

for any $\vec{x} \in \mathbb{R}^r$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Clearly

$$A_k = A_{k,1}, \quad B_k = B_{k,1}, \quad L_k = L_{k,1} \quad \text{and} \quad \Gamma_k = \Gamma_{k,1}.$$

Here f, φ are as in Theorems 17.3, 17.5, 17.7, 17.9, respectively.

We present

Proposition 17.12. *Same assumptions as in Theorem 17.3. Then*

$$\begin{aligned} |(A_{k,j} f)(\vec{x}) - f(\vec{x})| \leq \sum_{\rho=1}^{N} \frac{j^\rho a^\rho}{\rho! 2^{(k-1)\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) \\ + \frac{j^N a^N r^N}{N! 2^{(k-1)N}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{ja}{2^{k-1}} \right), \end{aligned} \tag{17.36}$$

any $k \in \mathbb{Z}$, and any $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$. Inequality (17.36) is attained when f is a constant function.

Corollary 17.13. *Same assumptions as in Theorem 17.3, $N = 1$. Then*

$$|(A_{k,j} f)(\vec{x}) - f(\vec{x})| \leq \frac{ja}{2^{k-1}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{ja}{2^{k-1}} \right) \right\}, \tag{17.37}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$.

Proposition 17.14. *Same assumptions as in Theorem 17.5. Then*

$$\begin{aligned} |(B_{k,j} f)(\vec{x}) - f(\vec{x})| \quad \leq \quad & \sum_{\rho=1}^{N} \frac{j^\rho a^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) \\ & + \frac{j^N a^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{ja}{2^k} \right), \end{aligned} \tag{17.38}$$

any $k \in \mathbb{Z}$, and any $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$. Inequality (17.38) is attained when f is a constant function.

Corollary 17.15. *Same assumptions as in Theorem 17.5, $N = 1$. Then*

$$|(B_{k,j} f)(\vec{x}) - f(\vec{x})| \leq \frac{ja}{2^k} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{ja}{2^k} \right) \right\}, \tag{17.39}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$.

Proposition 17.16. *Same assumptions as in Theorem 17.7. Then*

$$|(L_{k,j}f)(\vec{x}) - f(\vec{x})| \leq \sum_{\rho=1}^{N} \frac{(ja+1)^{\rho}}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^{\rho} f(\vec{x}) \right) + \frac{(ja+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{ja+1}{2^k} \right), \qquad (17.40)$$

any $k \in \mathbb{Z}$, and any $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$. Inequality (17.40) is attained when f is a constant function.

Corollary 17.17. *Same assumptions as in Theorem 17.7, $N = 1$. Then*

$$|(L_{k,j}f)(\vec{x}) - f(\vec{x})| \leq \left(\frac{ja+1}{2^k} \right) \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{ja+1}{2^k} \right) \right\}, \qquad (17.41)$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$.

Proposition 17.18. *Same assumptions as in Theorem 17.9. Then*

$$|(\Gamma_{k,j}f)(\vec{x}) - f(\vec{x})| \leq \sum_{\rho=1}^{N} \frac{(ja+1)^{\rho}}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^{\rho} f(\vec{x}) \right) + \frac{(ja+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{ja+1}{2^k} \right), \qquad (17.42)$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$. Inequality (17.42) is attained by constant functions.

Corollary 17.19. *Same assumptions as in Theorem 17.9, $N = 1$. Then*

$$|(\Gamma_{k,j}f)(\vec{x}) - f(\vec{x})| \leq \left(\frac{ja+1}{2^k} \right) \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{ja+1}{2^k} \right) \right\}, \qquad (17.43)$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $j \in \mathbb{N}$.

Remark 17.20. We mention the generalized multivariate Jackson's like operators motivated from classical Approximation Theory, see [67, p. 395], and [72],

$$I_{k,q}(f;\vec{x}) := -\sum_{j=1}^{q} (-1)^j \binom{q}{j} \mathcal{L}_{k,j}(f;\vec{x}), \quad \text{for all } \vec{x} \in \mathbb{R}^r,\ q \in \mathbb{N}. \qquad (17.44)$$

We apply

$$-\sum_{j=1}^{q} (-1)^j \binom{q}{j} = 1.$$

Applications of the last general operator are (see [67, p. 400] and [72]),

$$I_{k,q}^{A}(f;\vec{x}) := -\sum_{j=1}^{q}(-1)^j \binom{q}{j} A_{k,j}(f;\vec{x}), \tag{17.45}$$

$$I_{k,q}^{B}(f;\vec{x}) := -\sum_{j=1}^{q}(-1)^j \binom{q}{j} B_{k,j}(f;\vec{x}), \tag{17.46}$$

$$I_{k,q}^{L}(f;\vec{x}) := -\sum_{j=1}^{q}(-1)^j \binom{q}{j} L_{k,j}(f;\vec{x}), \tag{17.47}$$

and

$$I_{k,q}^{\Gamma}(f;\vec{x}) := -\sum_{j=1}^{q}(-1)^j \binom{q}{j} \Gamma_{k,j}(f;\vec{x}), \tag{17.48}$$

any $\vec{x} \in \mathbb{R}^r$.

From [67, p. 396] and [72], we have that

$$|I_{k,q}(f;\vec{x}) - f(\vec{x})| \le \sum_{j=1}^{q} \binom{q}{j} |(\mathcal{L}_{k,j}f)(\vec{x}) - f(\vec{x})|. \tag{17.49}$$

Inequality (17.49) is attained when f is a constant. We use also that

$$\sum_{j=1}^{q} \binom{q}{j} = 2^q - 1.$$

Applying the last we obtain

Proposition 17.21. *Same assumptions as in Theorem 17.3. Then*

$$\begin{aligned} |(I_{k,q}^{A}f)(\vec{x}) - f(\vec{x})| \quad \le \quad & (2^q - 1)\left[\sum_{\rho=1}^{N} \frac{q^\rho a^\rho}{\rho! 2^{(k-1)\rho}} \left(\left(\sum_{i=1}^{r}\left|\frac{\partial}{\partial x_i}\right|\right)^\rho f(\vec{x})\right)\right. \\ & \left. + \frac{q^N a^N r^N}{N! 2^{(k-1)N}} \max_{\tilde{a}\,:\,|\tilde{a}|=N} \omega_1\left(f_{\tilde{a}}, \frac{qa}{2^{k-1}}\right)\right], \end{aligned} \tag{17.50}$$

any $k \in \mathbb{Z}$, and any $\vec{x} \in \mathbb{R}^r$. Inequality (17.50) is attained when f is a constant function.

Proof. We notice that

$$|(I_{k,q}^A f)(\vec{x}) - f(\vec{x})| \overset{(17.49)}{\leq} \sum_{j=1}^{q} \binom{q}{j} |(A_{k,j} f)(\vec{x}) - f(\vec{x})|$$

$$\overset{(17.36)}{\leq} \sum_{j=1}^{q} \binom{q}{j} \left[\sum_{\rho=1}^{N} \frac{j^\rho a^\rho}{\rho! 2^{(k-1)\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{j^N a^N r^N}{N! 2^{(k-1)N}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{ja}{2^{k-1}} \right) \right]$$

$$\leq \left(\sum_{j=1}^{q} \binom{q}{j} \right) \left[\sum_{\rho=1}^{N} \frac{q^\rho a^\rho}{\rho! 2^{(k-1)\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{q^N a^N r^N}{N! 2^{(k-1)N}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa}{2^{k-1}} \right) \right]$$

$$= (2^q - 1) \left[\sum_{\rho=1}^{N} \frac{q^\rho a^\rho}{\rho! 2^{(k-1)\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{q^N a^N r^N}{N! 2^{(k-1)N}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa}{2^{k-1}} \right) \right]. \quad \blacksquare$$

Proposition 17.22. *Same assumptions as in Theorem 17.5. Then*

$$|(I_{k,q}^B f)(\vec{x}) - f(\vec{x})| \leq (2^q - 1) \left[\sum_{\rho=1}^{N} \frac{q^\rho a^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{q^N a^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa}{2^k} \right) \right], \tag{17.51}$$

any $k \in \mathbb{Z}$, and any $\vec{x} \in \mathbb{R}^r$. Inequality (17.51) is attained when f is a constant function.

Proof. We observe that

$$|(I_{k,q}^B f)(\vec{x}) - f(\vec{x})| \overset{(17.49)}{\leq} \sum_{j=1}^{q} \binom{q}{j} |(B_{k,j} f)(\vec{x}) - f(\vec{x})|$$

$$\overset{(17.38)}{\leq} \sum_{j=1}^{q} \binom{q}{j} \left[\sum_{\rho=1}^{N} \frac{j^\rho a^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{j^N a^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{ja}{2^k} \right) \right]$$

$$\leq \left(\sum_{j=1}^{q} \binom{q}{j} \right) \left[\sum_{\rho=1}^{N} \frac{q^\rho a^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{q^N a^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa}{2^k} \right) \right]$$

$$= (2^q - 1) \left[\sum_{\rho=1}^{N} \frac{q^\rho a^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{q^N a^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa}{2^k} \right) \right]. \quad \blacksquare$$

Proposition 17.23. *Same assumptions as in Theorem 17.7. Then*

$$|(I_{k,q}^L f)(\vec{x}) - f(\vec{x})| \leq (2^q - 1) \left[\sum_{\rho=1}^{N} \frac{(qa+1)^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{(qa+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa+1}{2^k} \right) \right], \tag{17.52}$$

any $k \in \mathbb{Z}$, and any $\vec{x} \in \mathbb{R}^r$. Inequality (17.52) is attained by constant functions.

Proof. We see that

$$|(I_{k,q}^L f)(\vec{x}) - f(\vec{x})| \overset{(17.49)}{\leq} \sum_{j=1}^{q} \binom{q}{j} |(L_{k,j} f)(\vec{x}) - f(\vec{x})|$$

$$\overset{(17.40)}{\leq} \sum_{j=1}^{q} \binom{q}{j} \left[\sum_{\rho=1}^{N} \frac{(ja+1)^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{(ja+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{ja+1}{2^k} \right) \right]$$

$$\leq \left(\sum_{j=1}^{q} \binom{q}{j} \right) \left[\sum_{\rho=1}^{N} \frac{(qa+1)^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{(qa+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa+1}{2^k} \right) \right]$$

$$= (2^q - 1) \left[\sum_{\rho=1}^{N} \frac{(qa+1)^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{(qa+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa+1}{2^k} \right) \right]. \quad \blacksquare$$

Proposition 17.24. *Same assumptions as in Theorem 17.9. Then*

$$|(I_{k,q}^\Gamma f)(\vec{x}) - f(\vec{x})| \leq (2^q - 1) \left[\sum_{\rho=1}^{N} \frac{(qa+1)^\rho}{\rho! 2^{k\rho}} \left(\left(\sum_{i=1}^{r} \left| \frac{\partial}{\partial x_i} \right| \right)^\rho f(\vec{x}) \right) + \frac{(qa+1)^N r^N}{N! 2^{kN}} \max_{\tilde{a}:\, |\tilde{a}|=N} \omega_1 \left(f_{\tilde{a}}, \frac{qa+1}{2^k} \right) \right], \quad (17.53)$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$. Inequality (17.53) is attained by constant functions.

Proof. Similar to Proposition 17.23, by the use of Proposition 17.18. ■

Inequalities (17.50)–(17.53) improve greatly in the case of $N = 1$. We use that

$$\sum_{j=1}^{q} \binom{q}{j} j = q 2^{q-1}.$$

We present

Proposition 17.25. *Same assumptions as in Theorem 17.3, $N = 1$. Then*

$$|(I_{k,q}^A f)(\vec{x}) - f(\vec{x})| \leq \frac{aq}{2^{k-q}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\dots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^{k-1}} \right) \right\}, \quad (17.54)$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $q \in \mathbb{N}$.

Proof. We have again

$$|(I_{k,q}^A f)(\vec{x}) - f(\vec{x})| \overset{(17.49)}{\leq} \sum_{j=1}^{q} \binom{q}{j} |(A_{k,j}f)(\vec{x}) - f(\vec{x})|$$

$$\overset{(17.37)}{\leq} \sum_{j=1}^{q} \binom{q}{j} \left[\frac{ja}{2^{k-1}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{ja}{2^{k-1}} \right) \right\} \right]$$

$$\leq \sum_{j=1}^{q} \binom{q}{j} \left[\frac{ja}{2^{k-1}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^{k-1}} \right) \right\} \right]$$

$$= \left(\sum_{j=1}^{q} \binom{q}{j} j \right) \left[\frac{a}{2^{k-1}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^{k-1}} \right) \right\} \right]$$

$$= \frac{q 2^{q-1} a}{2^{k-1}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^{k-1}} \right) \right\}. \quad \blacksquare$$

Proposition 17.26. *Same assumptions as in Theorem 17.5, $N = 1$. Then*

$$|(I_{k,q}^B f)(\vec{x}) - f(\vec{x})| \leq \frac{qa}{2^{k-q+1}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^k} \right) \right\}, \tag{17.55}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $q \in \mathbb{N}$.

Proof. We observe again

$$|(I_{k,q}^B f)(\vec{x}) - f(\vec{x})| \overset{(17.49)}{\leq} \sum_{j=1}^{q} \binom{q}{j} |(B_{k,j}f)(\vec{x}) - f(\vec{x})|$$

$$\overset{(17.39)}{\leq} \sum_{j=1}^{q} \binom{q}{j} \left[\frac{ja}{2^{k}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^{k}} \right) \right\} \right]$$

$$= \left(\sum_{j=1}^{q} \binom{q}{j} j \right) \left[\frac{a}{2^{k}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^{k}} \right) \right\} \right]$$

$$= \frac{q 2^{q-1} a}{2^{k}} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa}{2^{k}} \right) \right\}. \quad \blacksquare$$

Proposition 17.27. *Same assumptions as in Theorem 17.7, $N = 1$. Then*

$$|(I_{k,q}^L f)(\vec{x}) - f(\vec{x})| \leq \frac{(aq2^{q-1} + 2^q - 1)}{2^k} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa+1}{2^k} \right) \right\}, \tag{17.56}$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $q \in \mathbb{N}$.

Proof. We notice again

$$
\begin{aligned}
|(I_{k,q}^L f)(\vec{x}) - f(\vec{x})| &\overset{(17.49)}{\leq} \sum_{j=1}^{q} \binom{q}{j} |(L_{k,j}f)(\vec{x}) - f(\vec{x})| \\
&\overset{(17.41)}{\leq} \sum_{j=1}^{q} \binom{q}{j} \left[\frac{(ja+1)}{2^k} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa+1}{2^k} \right) \right\} \right] \\
&= \frac{\left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa+1}{2^k} \right) \right\}}{2^k} \cdot \left[a \left(\sum_{j=1}^{q} \binom{q}{j} j \right) + \sum_{j=1}^{q} \binom{q}{j} \right] \\
&= \frac{\left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa+1}{2^k} \right) \right\}}{2^k} \cdot [aq2^{q-1} + 2^q - 1]. \qquad \blacksquare
\end{aligned}
$$

Proposition 17.28. *Same assumptions as in Theorem 17.9, $N = 1$. Then*

$$
\begin{aligned}
|(I_{k,q}^{\Gamma} f)(\vec{x}) - f(\vec{x})| \leq \; & \frac{(aq2^{q-1} + 2^q - 1)}{2^k} \left\{ \left(\sum_{i=1}^{r} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \right) \right. \\
& \left. + r \max_{i \in \{1,\ldots,r\}} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{qa+1}{2^k} \right) \right\}, \qquad (17.57)
\end{aligned}
$$

any $k \in \mathbb{Z}$, $\vec{x} \in \mathbb{R}^r$, $q \in \mathbb{N}$.

Proof. Similar to Proposition 17.27, with the use of (17.43). ■

18

Approximation by a Nonlinear Cardaliaguet-Euvrard Neural Network Operator of Max-Product Kind

The aim of this chapter is that by using the so-called max-product method, to associate to Cardaliaguet-Euvrard linear operator, a nonlinear neural network operator, for which a Jackson-type approximation order is obtained. In some classes of functions, the order of approximation is essentially better than the order of approximation of the corresponding linear operator. This chapter relies on [65].

18.1 Introduction

Based on the Open Problem 5.5.4, pp. 324-326 in Gal [167], we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated case), Baskakov operators (truncated and nontruncated case), Meyer-König and Zeller operators and Bleimann-Butzer-Hahn operators.

This idea applied, for example, to the linear Bernstein operators $B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n)$, where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, works as follows. Writing in the equivalent form $B_n(f)(x) = \frac{\sum_{k=0}^{n} p_{n,k}(x) f(k/n)}{\sum_{k=0}^{n} p_{n,k}(x)}$ and then replacing the sum operator Σ by the maximum operator $\bigvee$, one obtains the

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nonlinear Bernstein operator of max-product kind

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)},$$

where the notation $\bigvee_{k=0}^{n} p_{n,k}(x)$ means $max\{p_{n,k}(x); k \in \{0, ..., n\}\}$ and similarly for the numerator.

For this max-product operator nice approximation and shape preserving properties can be found in e.g. Bede, Coroianu & Gal [108].

For example, it is proved that for some classes of functions (like those of concave functions), the order of approximation given by the max-product Bernstein operators, are essentially better than the approximation order of their linear counterparts.

The aim of this chapter is to use the same idea to the neural network operators of Cardaliaguet-Euvrard-type introduced and studied in e.g. Cardaliaguet & Euvrard [128], Anastassiou [18], [19], [22], Zhang, Cao, & Xu [288] (see also the references cited there). We will obtain that in the class of Lipschitz functions with positive values, the new obtained nonlinear neural network operator has essentially better approximation property than its linear counterpart.

Thus, by following Cardaliaguet & Euvrard [128], for $b : \mathbb{R} \to \mathbb{R}_+$ a centered bell-shaped function (that is, nondecreasing on $(-\infty, 0]$, nonincreasing on $[0, +\infty)$), with compact support $[-T, T]$, $T > 0$ (that is $b(x) > 0$ for all $x \in (-T, T)$) and therefore such that $I = \int_{-T}^{T} b(x)dx > 0$, the Cardaliaguet-Euvrard neural network is defined by

$$C_{n,\alpha}(f)(x) = \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I \cdot n^{1-\alpha}} \cdot b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right),$$

where $0 < \alpha < 1$, $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded or uniformly continuous on $\mathbb{R}$

Denoting by $CB(\mathbb{R})$ the space of all real-valued continuous and bounded functions on $\mathbb{R}$ and $CB_+(\mathbb{R}) = \{f : \mathbb{R} \to [0, \infty); f \in CB(\mathbb{R})\}$, applying the max-product method as in the above case of Bernstein polynomials, the corresponding max-product Cardaliaguet-Euvrard network operator will be formally given by

$$C_{n,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] f\left(\frac{k}{n}\right)}{\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right]}, x \in \mathbb{R}, f \in CB_+(\mathbb{R}).$$

Remark 18.1. For any $x \in \mathbb{R}$, denoting $J_{T,n}(x) = \{k \in \mathbb{Z}; -n^2 \leq k \leq n^2, n^{1-\alpha}(x - k/n) \in (-T, T)\}$, then we can write as a well defined operator

$$C_{n,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] f\left(\frac{k}{n}\right)}{\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right]}, x \in \mathbb{R}, n > \max\{T + |x|, T^{-1/\alpha}\}, \tag{18.1}$$

where $J_{T,n}(x) \neq \emptyset$, for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$. Indeed, we have

$$\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] > 0, \text{ for all } x \in \mathbb{R} \text{ and } n > \max\{T + |x|, T^{-1/\alpha}\},$$

because by e.g. Anastassiou [18], relationships (2)-(4), pp. 238-239, if $n \geq T + |x|$ then $-n^2 \leq nx - Tn^\alpha \leq nx + Tn^\alpha \leq n^2$, while $n^{1-\alpha}|x - k/n| < T$ is equivalent to $nx - Tn^\alpha < k < nx + Tn^\alpha$. This implies that if $(nx + Tn^\alpha) - (nx - Tn^\alpha) = 2Tn^\alpha > 2$ and $n \geq T + |x|$, then $J_{T,n}(x) \neq \emptyset$, which proves our assertion.

The plan of this chapter goes as follows: in Section 18.2 we present some auxiliary results, in Section 18.3 we obtain the main approximation result, while in Section 18.4 we compare the approximation result in Section 18.3 with that for the corresponding linear neural Cardaliaguet-Euvrard network operator.

18.2 Auxiliary Results

Remark 18.2. From the consideration in the last Remark of Section 18.1, it is clear that $C_{n,\alpha}^{(M)}(f)(x)$ is a well-defined function for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$ and it is continuous on $\mathbb{R}$ if b is continuous on $\mathbb{R}$.

In addition, $C_{n,\alpha}^{(M)}(e_0)(x) = 1$, where $e_0(x) = 1$, for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$.

In what follows we will see that for $f \in CB_+(\mathbb{R})$, the $C_{n,\alpha}^{(M)}$ operator fulfils similar properties with those of the $B_n^{(M)}(f)$ operator in Bede & Gal [110].

Lemma 18.3. *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$, $0 < \alpha < 1$ and $C_{n,\alpha}^{(M)}$ be defined as in Section 18.1.*

(i) If $|f(x)| \leq c$ for all $x \in \mathbb{R}$ then $|C_{n,\alpha}^{(M)}(f)(x)| \leq c$, for all $x \in \mathbb{R}$ and $n > \{T + |x|, T^{-1/\alpha}\}$ and $C_{n,\alpha}^{(M)}(f)(x)$ is continuous at any point $x \in \mathbb{R}$, for all $n > \max\{T + |x|, T^{-1/\alpha}\}$;

(ii) If $f, g \in CB_+(\mathbb{R})$ satisfy $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $C_{n,\alpha}^{(M)}(f)(x) \leq C_{n,\alpha}^{(M)}(g)(x)$ for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$;

(iii) $C_{n,\alpha}^{(M)}(f+g)(x) \leq C_{n,\alpha}^{(M)}(f)(x) + C_{n,\alpha}^{(M)}(g)(x)$ for all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$;

(iv) For all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$, we have

$$|C_{n,\alpha}^{(M)}(f)(x) - C_{n,\alpha}^{(M)}(g)(x)| \leq C_{n,\alpha}^{(M)}(|f - g|)(x);$$

(v) $C_{n,\alpha}^{(M)}$ is positive homogenous, that is $C_{n,\alpha}^{(M)}(\lambda f)(x) = \lambda C_{n,\alpha}^{(M)}(f)(x)$ for all $\lambda \geq 0$, $x \in \mathbb{R}$, $n > \max\{T + |x|, T^{-1/\alpha}\}$ and $f \in CB_+(\mathbb{R})$.

Proof. (i) Immediate by the formula of definition for $C_{n,\alpha}^{(M)}$ in (18.1).

(ii) Let $f, g \in CB_+(\mathbb{R})$ be with $f \leq g$ and fix $x \in \mathbb{R}$, $n > \max\{T+|x|, T^{-1/\alpha}\}$. Since $J_{T,n}(x)$ is independent of f and g, by (18.1) we immediately get the conclusion.

(iii) By (18.1) and by the sublinearity of $\bigvee$, it is immediate.

(iv) Let $f, g \in CB_+(\mathbb{R})$. We have $f = f - g + g \leq |f - g| + g$, which by $(i) - (iii)$ successively implies $C_{n,\alpha}^{(M)}(f)(x) \leq C_{n,\alpha}^{(M)}(|f - g|)(x) + C_{n,\alpha}^{(M)}(g)(x)$, that is $C_{n,\alpha}^{(M)}(f)(x) - C_{n,\alpha}^{(M)}(g)(x) \leq C_{n,\alpha}^{(M)}(|f - g|)(x)$, for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$.

Writing now $g = g - f + f \leq |f - g| + f$ and applying the above reasonings, it follows $C_{n,\alpha}^{(M)}(g)(x) - C_{n,\alpha}^{(M)}(f)(x) \leq C_{n,\alpha}^{(M)}(|f - g|)(x)$, which combined with the above inequality gives $|C_{n,\alpha}^{(M)}(f)(x) - C_{n,\alpha}^{(M)}(g)(x)| \leq C_{n,\alpha}^{(M)}(|f - g|)(x)$, for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$.

(v) By (18.1) it is immediate. ■

Remark 18.4. By (18.1) it is easy to see that instead of (ii), $C_{n,\alpha}^{(M)}$ satisfies the stronger condition

$$C_{n,\alpha}(f \vee g)(x) = C_{n,\alpha}(f)(x) \vee C_{n,\alpha}(g)(x),$$

for all $f, g \in CB_+(\mathbb{R}), x \in \mathbb{R}, n > \max\{T + |x|, T^{-1/\alpha}\}$.

Corollary 18.5. *For all $f \in CB_+(\mathbb{R})$, $0 < \alpha < 1$, $b(x)$ as in the statement of Lemma 18.3, $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$, we have*

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq \left[\frac{1}{\delta} C_{n,\alpha}^{(M)}(\Phi_x)(x) + 1\right] \omega_1(f; \delta)_{\mathbb{R}},$$

where $\delta > 0$, $\Phi_x(u) = |x - u|$ for all $x, u \in \mathbb{R}$, and $\omega_1(f; \delta)_{\mathbb{R}} = \max\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x - y| \leq \delta\}$.

Proof. Indeed, denoting $e_0(x) = 1$, from the identity valid for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$,

$$C_{n,\alpha}^{(M)}(f)(x) - f(x) = [C_{n,\alpha}^{(M)}(f)(x) - f(x) \cdot C_{n,\alpha}^{(M)}(e_0)(x)] + f(x)[C_{n,\alpha}^{(M)}(e_0)(x) - 1],$$

by Lemma 18.3 it easily follows

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq$$

$$|C_{n,\alpha}^{(M)}(f(x))(x) - C_{n,\alpha}^{(M)}(f(u))(x)| + |f(x)| \cdot |C_{n,\alpha}^{(M)}(e_0)(x) - 1| \leq$$
$$C_{n,\alpha}^{(M)}(|f(u) - f(x)|)(x) + |f(x)| \cdot |C_{n,\alpha}^{(M)}(e_0)(x) - 1|.$$

Now, since for all $u, x \in \mathbb{R}$ we have

$$|f(u) - f(x)| \leq \omega_1(f; |u - x|)_{\mathbb{R}} \leq \left[\frac{1}{\delta}|u - x| + 1\right] \omega_1(f; \delta)_{\mathbb{R}},$$

replacing above and taking into account that $C_{n,\alpha}^{(M)}(e_0) = 1$, we immediately obtain the estimate in the statement. ■

Remark 18.6. Therefore, to get an approximation property for $C_{n,\alpha}^{(M)}$, it is enough to obtain a good estimate for

$$E_{n,\alpha}(x) = C_{n,\alpha}^{(M)}(\Phi_x)(x) = \frac{\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] |x - k/n|}{\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right]},$$

for all $x \in \mathbb{R}$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$.

18.3 Approximation Results

In this section we obtain an approximation result for the operator $C_{n,\alpha}^{(M)}(f)$. For this purpose, first of all we need to calculate the denominators of $C_{n,\alpha}^{(M)}(f)(x)$ and of $E_{n,\alpha}(x)$, that is we will exactly calculate the expression

$$\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] = \bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right].$$

In this sense, we present the following.

Lemma 18.7. *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$ and $0 < \alpha < 1$.*

Then for any $j \in \mathbb{Z}$ with $-n^2 \leq j \leq n^2$, all $x \in [j/n, (j+1)/n]$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$, we have

$$\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] =$$

$$\max\left\{b\left[n^{1-\alpha}\left(x - \frac{j}{n}\right)\right], b\left[n^{1-\alpha}\left(x - \frac{j+1}{n}\right)\right]\right\} > 0.$$

Proof. Let $j \in \mathbb{Z}$ with $-n^2 \leq j \leq n^2$, $x \in [j/n, (j+1)/n]$ and $n > \max\{T + |x|, T^{-1/\alpha}\}$. We can write

$$\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] =$$

$$\max\left\{\bigvee_{k=-n^2}^{j} b\left[n^{1-\alpha}\left(x-\frac{k}{n}\right)\right], \bigvee_{k=j+1}^{n^2} b\left[n^{1-\alpha}\left(x-\frac{k}{n}\right)\right]\right\}.$$

We observe that for $k \in \{-n^2, ..., j\}$ we have $n^{1-\alpha}(x-k/n) \geq n^{1-\alpha}(x-j/n) \geq 0$ and since b is nonincreasing on $[0, +\infty)$, it easily follows that

$$\bigvee_{k=-n^2}^{j} b\left[n^{1-\alpha}\left(x-\frac{k}{n}\right)\right] = b\left[n^{1-\alpha}\left(x-\frac{j}{n}\right)\right].$$

Similarly, observing that for $k \in \{j+1, ..., n^2\}$ we have $n^{1-\alpha}(x-k/n) \leq n^{1-\alpha}(x-(j+1)/n) \leq 0$, since $b(x)$ is nondecreasing on $(-\infty, 0]$, it easily follows that

$$\bigvee_{k=j+1}^{n^2} b\left[n^{1-\alpha}\left(x-\frac{k}{n}\right)\right] = b\left[n^{1-\alpha}\left(x-\frac{j+1}{n}\right)\right].$$

It remains to prove that for $x \in [j/n, (j+1)/n]$ and $n > \max\{T+|x|, T^{-1/\alpha}\}$ we have $j, j+1 \in J_{T,n}(x)$. Indeed, since $x \in [j/n, (j+1)/n]$ is equivalent to $j \leq nx \leq j+1$, we evidently get $j < nx + Tn^{\alpha} \leq n^2$, for all $n \geq T + |x|$ and $j+1 \leq nx+1 < nx+Tn^{\alpha} \leq n^2$, for all $n > \max\{T+|x|, T^{-1/\alpha}\}$. Also, because $-n^2 \leq nx - Tn^{\alpha} \leq j+1-Tn^{\alpha} < j < j+1$, for all $n > \max\{T+|x|, T^{-1/\alpha}\}$, we get that $j, j+1 \in J_{T,n}(x)$ for all $n > \max\{T+|x|, T^{-1/\alpha}\}$, which proves the lemma. ■

Remark 18.8. The formula in the statement of Lemma 18.7 is valid for all $x \in [-n, +n]$ only. Indeed, since in Lemma 18.7 we suppose that $n > |x| + T$, it follows that we cannot have the complementary possibilities for x, $x \in (n, +\infty)$ or $x \in (-\infty, -n)$, because in both cases this would imply the contradiction $|x| > n > |x| + T$.

Theorem 18.9. *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$ and $0 < \alpha < 1$. In addition, suppose that the following requirements are fulfilled:*

(i) There exist $0 < m_1 \leq M_1 < \infty$ such that $m_1(T-x) \leq b(x) \leq M_1(T-x)$ for all $x \in [0, T]$;

(ii) There exist $0 < m_2 \leq M_2 < \infty$ such that $m_2(x+T) \leq b(x) \leq M_2(x+T)$ for all $x \in [-T, 0]$.

Then for all $f \in CB_{+}(R)$, $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$ satisfying $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$, we have the estimate

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq c\omega_1\left(f; n^{\alpha-1}\right)_{\mathbb{R}},$$

where

$$c = 2\left(\max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} + 1\right).$$

Proof. Let $x \in \mathbb{R}$ and let $j \in \mathbb{Z}$ with $-n^2 \leq j \leq n^2 - 1$ such that $x \in [j/n, (j+1)/n]$. Also, let $k_x \in J_{T,n}(x)$ be such that

$$\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] |x - \frac{k}{n}| = b\left[n^{1-\alpha}\left(x - \frac{k_x}{n}\right)\right] \left|x - \frac{k_x}{n}\right|.$$

It follows that

$$E_{n,\alpha}(x) = \frac{b\left[n^{1-\alpha}\left(x - \frac{k_x}{n}\right)\right] |x - \frac{k_x}{n}|}{\bigvee_{k \in J_{T,n}(x)} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right]}.$$

Taking into account Lemma 18.7 we immediately obtain

$$E_{n,\alpha}(x) = \min\left\{\frac{b\left[n^{1-\alpha}\left(x - \frac{k_x}{n}\right)\right] |x - \frac{k_x}{n}|}{b\left[n^{1-\alpha}\left(x - \frac{j}{n}\right)\right]}, \frac{b\left[n^{1-\alpha}\left(x - \frac{k_x}{n}\right)\right] |x - \frac{k_x}{n}|}{b\left[n^{1-\alpha}\left(x - \frac{j+1}{n}\right)\right]}\right\},$$

for all $n > \max\{T + |x|, T^{-1/\alpha}\}$.

In order to prove the estimate in the theorem we distinguish the following two cases: 1) $k_x > j$ and 2) $k_x \leq j$.

Case 1) Taking into account condition (ii), since $k_x \in J_{T,n}(x)$ and $j + 1 \in J_{T,n}(x)$, by $x - k_x/n \leq 0$ and $x - (j+1)/n \leq 0$, we immediately get

$$\begin{aligned} E_{n,\alpha}(x) &\leq \frac{b\left[n^{1-\alpha}\left(x - \frac{k_x}{n}\right)\right] \left(\frac{k_x}{n} - x\right)}{b\left[n^{1-\alpha}\left(x - \frac{j+1}{n}\right)\right]} \leq \frac{M_2}{m_2} \cdot \frac{[T + n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](\frac{k_x}{n} - x)}{T + n^{1-\alpha}\left(x - \frac{j+1}{n}\right)} \\ &\leq \frac{M_2}{m_2} \cdot \frac{[T + n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](\frac{k_x}{n} - x)}{T + n^{1-\alpha}\left(\frac{-1}{n}\right)} \\ &= \frac{M_2}{m_2} \cdot \frac{n^{\alpha}[T + n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](\frac{k_x}{n} - x)}{Tn^{\alpha} - 1}. \end{aligned}$$

Since $[T + n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](\frac{k_x}{n} - x) = -n^{1-\alpha}\left(\frac{k_x}{n} - x - \frac{T}{2n^{1-\alpha}}\right)^2 + \frac{T^2}{4n^{1-\alpha}} \leq \frac{T^2}{4n^{1-\alpha}}$, it easily follows that

$$E_{n,\alpha}(x) \leq \frac{M_2}{4m_2} \cdot \frac{T^2 n^{2\alpha-1}}{Tn^{\alpha} - 1} = \frac{M_2}{4m_2} \cdot \frac{T^2 n^{\alpha}}{Tn^{\alpha} - 1} \cdot n^{\alpha-1}.$$

Supposing, in addition, that $n > (2/T)^{1/\alpha}$ (where clearly $(2/T)^{1/\alpha} > T^{-1/\alpha}$), it follows that

$$\frac{n^{\alpha}}{Tn^{\alpha} - 1} = \frac{1}{T}\left(1 + \frac{1/T}{n^{\alpha} - 1/T}\right) \leq \frac{1}{T}\left(1 + \frac{1/T}{2/T - 1/T}\right) = \frac{2}{T},$$

which implies

$$E_{n,\alpha}(x) \leq \frac{TM_2}{2m_2} \cdot n^{\alpha-1},$$

for all $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$.

Case 2) Taking into account condition (i), since $k_x \in J_{T,n}(x)$ and $j \in J_{T,n}(x)$, by $x - k_x/n \geq 0$ and $x - j/n \geq 0$, we immediately get

$$E_{n,\alpha}(x) \leq \frac{b\left[n^{1-\alpha}\left(x - \frac{k_x}{n}\right)\right]\left(x - \frac{k_x}{n}\right)}{b\left[n^{1-\alpha}\left(x - \frac{j}{n}\right)\right]} \leq \frac{M_1}{m_1} \cdot \frac{[T - n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](x - \frac{k_x}{n})}{T - n^{1-\alpha}\left(x - \frac{j}{n}\right)}$$

$$\leq \frac{M_1}{m_1} \cdot \frac{[T - n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](x - \frac{k_x}{n})}{T - n^{1-\alpha}\left(\frac{1}{n}\right)}$$

$$= \frac{M_1}{m_1} \cdot \frac{n^{\alpha}[T - n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](x - \frac{k_x}{n})}{Tn^{\alpha} - 1}.$$

Since $[T - n^{1-\alpha}\left(x - \frac{k_x}{n}\right)](x - \frac{k_x}{n}) = -n^{1-\alpha}\left(x - \frac{k_x}{n} - \frac{T}{2n^{1-\alpha}}\right)^2 + \frac{T^2}{4n^{1-\alpha}} \leq \frac{T^2}{4n^{1-\alpha}}$, reasoning exactly as in the Case 1), we obtain

$$E_{n,\alpha}(x) \leq \frac{TM_1}{2m_1} \cdot n^{\alpha-1},$$

for all $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$.

Now, applying Corollary 18.5 for $\delta = \max\{\frac{TM_2}{2m_2} \cdot n^{1-\alpha}, \frac{TM_1}{2m_1} \cdot n^{1-\alpha}\}$ and from the property $\omega_1(f, \lambda\delta)_{\mathbb{R}} \leq (\lambda + 1)\omega_1(f, \delta)_{\mathbb{R}}$, we obtain the desired conclusion. ■

Corollary 18.10. *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$ and $0 < \alpha < 1$. If $0 < \lim_{x \nearrow T} \frac{b(x)}{T-x} < \infty$ and $0 < \lim_{x \searrow -T} \frac{b(x)}{T+x} < \infty$ then for all $f \in CB_+(R)$, $x \in \mathbb{R}$ and for all all $n \in \mathbb{N}$ satisfying $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$ there exists $c \in R_+$ independent of n and f such that*

$$\left|f(x) - C_{n,\alpha}^{(M)}(f)(x)\right| \leq c\omega_1\left(f; n^{\alpha-1}\right)_{\mathbb{R}}.$$

Proof. Let us consider the function $g : [0, T] \to \mathbb{R}$, $g(x) = \frac{b(x)}{T-x}$ if $x \in [0, T)$ and $g(T) = \lim_{x \nearrow T} \frac{b(x)}{T-x}$. From our assumptions we get that g is continuous and strictly positive. By the Weierstrass' theorem it follows that g attains its minimum and maximum. Hence there exist $0 < m_1 \leq M_1 < \infty$ such that $m_1 \leq g(x) \leq M_1$ for all $x \in [0, T]$. It follows that $m_1(T - x) \leq b(x) \leq M_1(T - x)$ *for all* $x \in [0, T)$. Since $b(T) = 0$ we easily get that $m_1(T - x) \leq b(x) \leq M_1(T - x)$ *for all* $x \in [0, T]$.

Now, let us consider the function $h : [-T, 0]$, $h(x) = \frac{b(x)}{T+x}$ if $x \in (-T, 0]$ and $h(-T) = \lim_{x \searrow -T} \frac{b(x)}{T+x}$ Again, it is easy to prove that there exist $0 < m_2 \leq M_2 < \infty$ such that $m_2(x + T) \leq b(x) \leq M_2(x + T)$ *for all* $x \in [-T, 0]$.

From the above considerations, applying Theorem 18.9 we easily obtain the desired conclusion. ■

In what follows, we will give some examples of bell-shaped functions for which we can apply Theorem 18.9.

Example 18.11. Let us consider $b : \mathbb{R} \to [0, \infty)$, $b(x) = 1 + x$ if $x \in [-1, 0]$, $b(x) = 1 - x$ if $x \in [0, 1]$, $b(x) = 0$ elsewhere. Using the same notations as in Theorem 18.9 we have $T = 1$ and $m_1 = M_1 = m_2 = M_2 = 1$. By Theorem 18.9, it follows that for all $f \in CB_+(R)$, $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$ satisfying $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$, we have the estimate

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq 3\omega_1 \left(f; n^{\alpha-1}\right)_{\mathbb{R}}.$$

Example 18.12. Let us consider $b : \mathbb{R} \to [0, \infty)$, $b(x) = 1 - x^2$ if $x \in [-1, 1]$, $b(x) = 0$ elsewhere. We have $T = 1$, $m_1 = m_2 = 1$, $M_1 = M_2 = 2$. By Theorem 18.9, it follows that for all $f \in CB_+(R)$, $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$ satisfying $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$, we have the estimate

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq 4\omega_1 \left(f; n^{\alpha-1}\right)_{\mathbb{R}}.$$

Example 18.13. Let us consider $b : \mathbb{R} \to [0, \infty)$, $b(x) = \cos x$ if $x \in [-\pi/2, \pi/2]$, $b(x) = 0$ elsewhere. Since for $t \in [0, \pi/2]$ we have $2t/\pi \leq \sin t \leq t$ it follows that $(2/\pi)(\pi/2 - x) \leq \sin(\pi/2 - x) = \cos x \leq \pi/2 - x$ for all $x \in [0, \pi/2]$ and $(2/\pi)(\pi/2 + x) \leq \sin(\pi/2 + x) = \cos x \leq \pi/2 + x$ for all $x \in [-\pi/2, 0]$. From the above inequalities it follows that $T = \pi/2$, $m_1 = m_2 = 2/\pi$ and $M_1 = M_2 = 1$. Applying Theorem 18.9, we obtain

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq 7\omega_1 \left(f; n^{\alpha-1}\right)_{\mathbb{R}}.$$

Remark 18.14. In what follows we will prove that in general, if the bell-shaped function b satisfies the hypothesis of Theorem 18.9, then the order of approximation of the expression $E_{n,\alpha}(x)$ in Theorem 18.9. cannot be improved. Firstly, let us notice that from the conclusion of Theorem 18.9 it suffices to prove that we cannot improve the order of approximation of the expression $E_{n,\alpha}(x)$ for the case when $b(x) = T + x$ if $x \in [-T, 0]$, $b(x) = T - x$ if $x \in [0, T]$, $b(x) = 0$ elsewhere. Without any loss of generality we may assume that $T = 1$. For $n \in \mathbb{N}$, $n > (2/T)^{1/\alpha}$, take $x_n = 1/2n$. It is easy to check that for all $n \geq 2$, we have $n > \max\{T + |x_n|, (2/T)^{1/\alpha}\}$. Since $x_n \in (0, 1/n)$,by Lemma 18.7, it follows that $\bigvee_{k=-n^2}^{k=n^2} b\left[n^{1-\alpha}\left(x_n - \frac{k}{n}\right)\right] = \max\left\{b(n^{1-\alpha}x_n), b\left[n^{1-\alpha}\left(x_n - \frac{1}{n}\right)\right]\right\}$. Through simple calculus we get

$$\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x_n - \frac{k}{n}\right)\right] = \frac{2n^\alpha - 1}{2n^\alpha}.$$

This, immediately implies

$$E_{n,\alpha}(x_n) = \frac{\bigvee_{k=-n^2}^{k=n^2} b\left[n^{1-\alpha}\left(x_n - \frac{k}{n}\right)\right] \left|x_n - \frac{k}{n}\right|}{(2n^\alpha - 1)/2n^\alpha}.$$

From the above equality it follows that for all $k \in \mathbb{Z}$, $-n^2 \le k \le n^2$, we have

$$E_{n,\alpha}(x_n) \ge \frac{b\left[n^{1-\alpha}\left(x_n - \frac{k}{n}\right)\right]|x_n - \frac{k}{n}|}{(2n^\alpha - 1)/2n^\alpha}. \tag{18.2}$$

Let us take $k_n = [\frac{5n^\alpha+3}{6}] - 1$. It is easy to check that $-n^2 \le k_n \le n^2$. Also, for n sufficiently large we have $x_n \le k_n/n$. Then,

$$\begin{aligned}
& b\left[n^{1-\alpha}\left(x_n - \frac{k_n}{n}\right)\right]|x_n - \frac{k_n}{n}| \\
&= [1 + n^{1-\alpha}\left(\frac{1}{2n} - \frac{k_n}{n}\right)](\frac{k_n}{n} - x_n) = -n^{1-\alpha}\left(\frac{k_n}{n} - \frac{1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}} \\
&= -n^{1-\alpha}\left(\frac{2k_n - 1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}}.
\end{aligned}$$

Since

$$\begin{aligned}
& \frac{2k_n - 1}{2n} - \frac{1}{2n^{1-\alpha}} \\
&= \frac{2\left([\frac{5n^\alpha+3}{6}] - 1\right) - 1}{2n} - \frac{1}{2n^{1-\alpha}} \ge \frac{2\left(\frac{5n^\alpha+3}{6} - 2\right) - 1}{2n} - \frac{1}{2n^{1-\alpha}} = \frac{1}{3n^{1-\alpha}} - \frac{2}{n},
\end{aligned}$$

it follows that for $n \ge 6^{1/\alpha}$ we have $\frac{2k_n-1}{2n} - \frac{1}{2n^{1-\alpha}} \ge 0$. Therefore, for $n \ge 6^{1/\alpha}$ we have

$$\begin{aligned}
& -n^{1-\alpha}\left(\frac{2k_n - 1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}} \\
&= -n^{1-\alpha}\left(\frac{2\left([\frac{5n^\alpha+3}{6}] - 1\right) - 1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}} \\
&\ge -n^{1-\alpha}\left(\frac{2 \cdot \frac{5n^\alpha+3}{6} - 1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}} = \frac{5}{36} \cdot n^{\alpha-1}.
\end{aligned}$$

Taking into account relation (18.2) and the above inequality, we get

$$\begin{aligned}
E_{n,\alpha}(x_n) &\ge \frac{b\left[n^{1-\alpha}\left(x_n - \frac{k_n}{n}\right)\right]|x_n - \frac{k_n}{n}|}{(2n^\alpha - 1)/2n^\alpha} = \frac{-n^{1-\alpha}\left(\frac{2k_n-1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}}}{(2n^\alpha - 1)/2n^\alpha} \\
&\ge \frac{\frac{5}{36} \cdot n^{\alpha-1}}{(2n^\alpha - 1)/2n^\alpha} = \frac{5n^\alpha}{18(2n^\alpha - 1)} \cdot n^{\alpha-1}.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \frac{5n^\alpha}{18(2n^\alpha-1)} = \frac{5}{36}$, it follows that for n sufficiently large we get

$$E_{n,\alpha}(x_n) \ge \frac{1}{8} \cdot n^{\alpha-1},$$

which implies the desired conclusion.

18.4 Conclusion

The linear Cardaliaguet-Euvrard operators $C_{n,\alpha}(f)(x)$ were introduced in Cardaliaguet & Euvrard [128], were it is proved the convergence on compacta to the approximated function. The results were of qualitative type. The first quantitative type estimates in the approximation by $C_{n,\alpha}(f)(x)$ was obtained in Anastassiou [18], [19], [22] and then improved in Zhang, Cao, & Xu [288], where at the page 1164 the following type of quantitative estimate is obtained :

$$|C_{n,\alpha}(f)(x) - f(x)| \leq \frac{C_1}{n^\alpha} + C_2\omega_1\left(f; n^{\alpha-1}\right)_{\mathbb{R}},$$

for all $n > \max\{T + |x|, T^{-1/\alpha}\}$, where $C_1, C_2 > 0$ are constants independent on n but depending on b and f.

If we suppose now that f is a Lipschitz function on $\mathbb{R}$, that is there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$, for all $x, y \in \mathbb{R}$, from the above estimate we get the following order of approximation by the linear Cardaliaguet-Euvrard operator :

$$|C_{n,\alpha}(f)(x) - f(x)| = \mathcal{O}\left(\frac{1}{n^\alpha}\right) + \mathcal{O}\left(\frac{1}{n^{1-\alpha}}\right), \text{ for all } n > \max\{T + |x|, T^{-1/\alpha}\}.$$

On the other hand, for $f \in CB_+(\mathbb{R})$ a Lipschitz function, in the case of max-product Cardaliaguet-Euvrard operator, by Theorem 18.9 we get the order of approximation

$$|C_{n,\alpha}^{(M)}(f)(x) - f(x)| = \mathcal{O}\left(\frac{1}{n^{1-\alpha}}\right), \text{ for all } n > \max\{T + |x|, (2/T)^{1/\alpha}\}.$$

It is clear that for $\frac{1}{2} \leq \alpha < 1$, we get the same order of approximation $\mathcal{O}\left(\frac{1}{n^{1-\alpha}}\right)$ for both operators $C_{n,\alpha}(f)(x)$ and $C_{n,\alpha}^{(M)}(f)(x)$, while for $0 < \alpha < \frac{1}{2}$, the approximation order obtained by the max-product operator $C_{n,\alpha}^{(M)}(f)(x)$ is essentially better than that obtained by the linear operator $C_{n,\alpha}(f)(x)$.

This shows the advantage we can have by using the max-product Cardaliaguet-Euvrard operator.

19

A Generalized Shisha - Mond Type Inequality

We present here a generalized Shisha-Mond type inequality which implies a generalized Korovkin theorem. These are regarding the convergence with rates of a sequence of positive linear operators to the unit. This chapter is based on [39].

19.1 Results

We give the following definition

Definition 19.1. Let Q be a connected compact Hausdorff space and $C(Q,\mathbb{R})$ the collection of all continuous $f : Q \to \mathbb{R}$. Let $g \in C(Q,\mathbb{R})$ be fixed and define *the g-pseudomodulus of continuity of* $f \in C(Q,\mathbb{R})$ as

$$w_g(f,h) := \sup_{x,y}\{|f(x)-f(y)| : |g(x)-g(y)| \leq h\}, \tag{19.1}$$

here $h \geq 0$.

Thus $w_g(g,h) \leq h$. The quantity $w_g(f,h)$ enjoys most of the basic properties of the usual modulus of continuity $\omega_1(f,h)$ (positively homogeneous as a function of f, non-decreasing nonnegative and subadditive in h). However, $w_g(f,\cdot)$ is an upper-semicontinuous function and in general not a continuous one.

Example 19.2. Let

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$$g(x) = \begin{cases} 0, \ 0 \le x \le 1; \\ x-1, \ 1 \le x \le 2; \\ 1, \ 2 \le x \le 3 \end{cases} \tag{19.2}$$

and $f(x) = x$ then

$$w_g(f,h) = \begin{cases} 1+h, \ 0 \le h < 1; \\ 3, \ h \ge 1. \end{cases} \tag{19.3}$$

Obviously, $w_g(f,\cdot)$ is discontinuous.

Consider a sequence of positive linear operators $L_n : C(Q,\mathbb{R}) \to C(Q,\mathbb{R})$, such that the sequence of functions $\{L_n(1)\}_{n\in\mathbb{N}}$ is uniformly bounded. In particular, $|f| \le \tilde{g}$ implies $|L_n(f)| \le L_n(\tilde{g})$. The following result is a useful generalization, similar proof, of a result due to Shisha and Mond (1968), [264], who took $Q = [a,b] \subset \mathbb{R}$ and $g(x) = x$.

We have

Theorem 19.3. *It holds*

$$\|L_n(f) - f\| \le \|f\| \, \|L_n(1) - 1\| + w_g(f,\rho_n)(1 + \|L_n(1)\|), \tag{19.4}$$

where

$$\rho_n := (\|L_n((g - g(y))^2)(y)\|)^{1/2}.$$

Here $\|\cdot\|$ stands for the supremum norm. If $L_n(1) = 1$, then (19.4) simplifies to

$$\|L_n(f) - f\| \le 2w_g(f,\rho_n). \tag{19.5}$$

As an application one has the following theorem,similar to the well-known theorem due to Korovkin (1953), see [213], however it is more general.

Corollary 19.4. *Let $Q = [a,b] \subset \mathbb{R}$ and let $\{L_n : C([a,b]) \to C([a,b])\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators. Suppose that $g \in C([a,b])$ is 1-1 function and further that $L_n(1) \xrightarrow{u} 1$, $L_n(g) \xrightarrow{u} g$, and $L_n(g^2) \xrightarrow{u} g^2$. Then $L_n(f) \xrightarrow{u} f$, for all $f \in C([a,b])$, u here stands for uniform convergence.*

Proof. Notice that

$$\rho_n^2 \le \|L_n(g^2) - g^2\| + 2\|g\| \, \|L_n(g) - g\| + \|g\|^2 \|L_n(1) - 1\|. \tag{19.6}$$

Now apply Theorem 19.3. ■

20

Quantitative Approximation by Bounded Linear Operators

This is a quantitative study for the rate of pointwise convergence of a sequence of bounded linear operators to an arbitrary operator in a very general setting involving the modulus of continuity. This is accomplished via the Riesz representation theorem and the weak convergence of the corresponding signed measures to zero, studied quantitatively in various important cases. This chapter relies on [25].

20.1 Introduction

This chapter has been greatly motivated by the following result, see [189] and [199], that solves a problem of P. Lévy.

Theorem 20.1. Let $\{\mu_a\}$ be a bounded net (or sequence) of signed Borel measures on $[0,1]$; i.e., there is a number $M > 0$ such that $|\int f d\mu_a| \leq M\|f\|_\infty$ for $f \in C[0,1]$. Define $K_a(x) = \mu_a[0,x]$ for $0 \leq x \leq 1$. Then the following are equivalent:

i) $\lim\limits_a \int f d\mu_a = 0$ for each $f \in C[0,1]$ (i.e., $\{\mu_a\}$ converges weakly to zero).

ii) $\lim\limits_a \big(\int |K_a| dx + |K_a(1)|\big) = 0$,

where λ stands for the Lebesgue measure on $[0,1]$.

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20.2 Results

We give the first result:

Theorem 20.2. Let $f \in C[-1,1]$ and $\{\mu_m\}_{m\in\mathbb{N}}$ be a sequence of nontrivial finite Borel signed measures on $[-1,1]$. Put $M_m := \mu_m[-1,1]$ and write $|\mu_m| = \mu_m^+ + \mu_m^-$, where μ_m^+, μ_m^- are the positive and negative parts, respectively, in the Jordan–Hahn decomposition of $\mu_m = \mu^+ - \mu^-$. Then

$$\left|\int_{-1}^{1} f d\mu_m\right| \leq |f(0)|\,|M_m| + \{1 + |\mu_m|[-1,1]\}\omega_1\left(f, \int_{-1}^{1} |t| d|\mu_m|\right), \qquad (20.1)$$

where ω_1 stands for the first modulus of continuity. Moreover,

$$0 < \int_{-1}^{1} |t| d|\mu_m| < +\infty.$$

If $M_m \to 0$ and $\int_{-1}^{1} |t| d|\mu_m| \to 0$, as $m \to +\infty$, then $\{\mu_m\}_{m\in\mathbb{N}}$ converges weakly to zero.

Proof. Let $f \in C[-1,1]$. Then

$$\int_{-1}^{1} f(t) d\mu_m(t) = \int_{-1}^{1} f(t) d(\mu_m^+ - \mu_m^-)(t) = \int_{-1}^{1} f(t) d\mu_m^+(t) - \int_{-1}^{1} f(t) d\mu_m^-(t),$$

and

$$\left|\int_{-1}^{1} f(t) d\mu_m(t)\right| \leq \int_{-1}^{1} |f(t)| d|\mu_m|(t).$$

We see that

$$\int_{-1}^{1} f d\mu_m = \int_{-1}^{1} (f - f(0)) d\mu_m + f(0)\mu_m[-1,1].$$

Therefore

$$\left|\int_{-1}^{1} f d\mu_m\right| \leq \left|\int_{-1}^{1} (f - f(0)) d\mu_m\right| + |f(0)|\,|M_m|$$

$$\leq \int_{-1}^{1} |f - f(0)| d|\mu_m|(t) + |f(0)|\,|M_m|$$

(by Corollary 7.1.1, p. 209, [16])

$$\leq \omega_1(f, h_m) \int_{-1}^{1} \left\lceil \frac{|t|}{h_m} \right\rceil d|\mu_m|(t) + |f(0)|\,|M_m|$$

($\lceil \cdot \rceil$ is the ceiling of the number)

$$\leq \omega_1(f, h_m)\left\{|\mu_m|[-1,1] + \frac{1}{h_m}\int_{-1}^{1}|t|d|\mu_m|\right\} + |f(0)|\,|M_m|$$

$$\text{(by choosing } h_m = \int_{-1}^{1}|t|d|\mu_m|\text{)}$$

$$= \omega_1\left(f, \int_{-1}^{1}|t|d|\mu_m|\right)\{1 + |\mu_m|[-1,1]\} + |f(0)|\,|M_m|.$$

We have established (20.1). If $M_m \to 0$ and $\int_{-1}^{1}|t|d|\mu_m| \to 0$, as $m \to \infty$, we obtain $\int_{-1}^{1} f d\mu_m \to 0$, as $m \to \infty$. ■

Theorem 20.3. Let $f \in C^1[-1,1]$ and let $\{\mu_m\}_{m\in\mathbb{N}}$ be a sequence of non-trivial Borel signed measures on $[-1,1]$. We suppose that each μ_m is bounded, and put $M_m := \mu_m[-1,1]$. Define $K_m(x) := \mu_m[-1,x]$, $-1 \leq x \leq 1$, K_m is of bounded variation. Then

$$\left|\int_{-1}^{1} f d\mu_m\right| \leq |f(1)|\,|M_m| + |f'(0)|\left|\int_{-1}^{1} K_m(x)dx\right| +$$

$$\left(1 + \int_{-1}^{1}|K_m(x)|dx\right)\cdot\omega_1\left(f', \int_{-1}^{1}|K_m(x)|\,|x|dx\right),\ \forall m \in \mathbb{N}. \tag{20.2}$$

Here ω_1 is the first modulus of continuity. If $M_m \to 0$, $\int_{-1}^{1}|K_m(x)|dx \to 0$, as $m \to \infty$, then $\{\mu_m\}_{m\in\mathbb{N}}$ converges weakly to zero.
Proof. By integration by parts,

$$\int_{-1}^{1} f d\mu_m = \int_{-1}^{1} f dK_m + f(-1)K_m(-1)$$

$$= -\int_{-1}^{1} K_m df + f(1)K_m(1) - f(-1)K_m(-1) + f(-1)K_m(-1)$$

$$= -\int_{-1}^{1} K_m(x)f'(x)dx + f(1)M_m$$

$$= -\left[\int_{-1}^{1} K_m(x)(f'(x) - f'(0))dx + f'(0)\int_{-1}^{1} K_m(x)dx\right] + f(1)M_m.$$

Hence

$$\left|\int_{-1}^{1} f d\mu_m\right| \leq |f(1)|\,|M_m| + |f'(0)|\left|\int_{-1}^{1} K_m(x)dx\right| + \int_{-1}^{1}|K_m(x)|\,|f'(x) - f'(0)|dx$$

$$\leq |f(1)|\,|M_m| + |f'(0)|\left|\int_{-1}^{1} K_m(x)dx\right| + \omega_1(f', h_m)\cdot\int_{-1}^{1}|K_m(x)|\left\lceil\frac{|x|}{h_m}\right\rceil dx.$$

Here $\lceil\cdot\rceil$ is the ceiling of the number, and the last inequality comes from Corollary 7.1.1, p. 209, [16]. Because

$$\left\lceil \frac{|x|}{h_m} \right\rceil \leq 1 + \frac{|x|}{h_m},$$

we obtain that

$$\int_{-1}^{1} |K_m(x)| \left\lceil \frac{|x|}{h_m} \right\rceil dx \leq \int_{-1}^{1} |K_m(x)| \left(1 + \frac{|x|}{h_m}\right) dx$$

$$= \int_{-1}^{1} |K_m(x)| dx + \frac{1}{h_m} \int_{-1}^{1} |K_m(x)|\, |x| dx.$$

Clearly $K_m \not\equiv 0$, by $K_m(x) \neq 0$ and may be zero only at some points. Here we choose

$$h_m = \int_{-1}^{1} |K_m(x)|\, |x| dx.$$

It is obvious that $h_m > 0$, and h_m is finite by $\|K_m\|_\infty < \infty$. Combining these together, we have established the validity of (20.2). Under the special assumptions $M_m \to 0$ and $\int_{-1}^{1} |K_m(x)| dx \to 0$, as $m \to \infty$, we obtain that $\int_{-1}^{1} f d\mu_m \to 0$, as $m \to \infty$. ■

Finally we have the following theorem which for $n = 1$ implies Theorem 20.3.

Theorem 20.4. Let $f \in C^n[-1,1]$, $n \geq 1$, and $\{\mu_m\}_{m\in\mathbb{N}}$ be a sequence of nontrivial Borel signed measures on $[-1,1]$. Suppose that each μ_m is bounded, and set $M_m := \mu_m[-1,1]$. Define $K_m(x) := \mu_m[-1,x]$, $-1 \leq x \leq 1$. Then

$$\left| \int_{-1}^{1} f d\mu_m \right| \leq |f(1)|\, |M_m| + \sum_{k=0}^{n-1} \frac{|f^{(k+1)}(0)|}{k!} \left| \int_{-1}^{1} K_m(x) x^k dx \right|$$

$$+ \left(\frac{1}{(n-1)!} \int_{-1}^{1} |K_m(x)|\, |x|^{n-1} dx + \frac{1}{n!} \right) \cdot \omega_1 \left(f^{(n)}, \int_{-1}^{1} |K_m(x)|\, |x|^n dx \right). \tag{20.3}$$

If $M_m \to 0$, $\int_{-1}^{1} |K_m(x)| dx \to 0$ with $m \to \infty$, then $\{\mu_m\}_{m\in\mathbb{N}}$ converges weakly to zero.

Proof. Let $f \in C^n[-1,1]$, $n \geq 1$. We have that

$$f'(x) = \sum_{k=0}^{n-1} \frac{f^{(k+1)}(0)}{k!} x^k + \int_0^x (f^{(n)}(t) - f^{(n)}(0)) \frac{(x-t)^{n-2}}{(n-2)!} dt, \tag{20.4}$$

for any $-1 \leq x \leq 1$. We also have again

$$\int_{-1}^{1} f d\mu_m = -\int_{-1}^{1} K_m(x) f'(x) dx + f(1) M_m. \tag{20.5}$$

So combining (20.4) and (20.5), we obtain that

$$\int_{-1}^{1} f d\mu_m = -\sum_{k=0}^{n-1} \frac{f^{(k+1)}(0)}{k!} \int_{-1}^{1} K_m(x) x^k dx$$

$$-\int_{-1}^{1} K_m(x) \left(\int_0^x (f^{(n)}(t) - f^{(n)}(0)) \frac{(x-t)^{n-2}}{(n-2)!} dt \right) dx + f(1) M_m.$$

In particular, we observe that

$$\left| \int_0^x (f^{(n)}(t) - f^{(n)}(0)) \frac{(x-t)^{n-2}}{(n-2)!} dt \right|$$

$$\leq \omega_1(f^{(n)}, h_m) \left| \int_0^x \left\lceil \frac{|t|}{h_m} \right\rceil \frac{|x-t|^{n-2}}{(n-2)!} dt \right| = \omega_1(f^{(n)}, h_m) \phi_{n-1}(|x|).$$

Here

$$\phi_{n-1}(|x|) := \int_0^{|x|} \left\lceil \frac{t}{h_m} \right\rceil \frac{(|x|-t)^{n-2}}{(n-2)!} dt, \quad x \in \mathbb{R}$$

(see (7.1.13) in Remark 7.1.3, p. 210, [16]). Therefore,

$$\left| \int_{-1}^{1} f d\mu_m \right| = |f(1)| \, |M_m| + \sum_{k=0}^{n-1} \frac{|f^{(k+1)}(0)|}{k!} \left| \int_{-1}^{1} K_m(x) x^k dx \right|$$

$$+ \omega_1(f^{(n)}, h_m) \cdot \int_{-1}^{1} |K_m(x)| \phi_{n-1}(|x|) \, dx. \tag{20.6}$$

By (7.2.9), p. 217, [16], we find that

$$\phi_{n-1}(|x|) \leq \frac{|x|^{n-1}}{(n-1)!} \left(1 + \frac{|x|}{n h_m} \right) = \frac{|x|^{n-1}}{(n-1)!} + \frac{|x|^n}{n! h_m}, \quad x \in \mathbb{R}.$$

Hence

$$|K_m(x)| \phi_{n-1}(|x|) \leq |K_m(x)| \frac{|x|^{n-1}}{(n-1)!} + \frac{|K_m(x)| \, |x|^n}{n! h_m}.$$

Consequently, we obtain

$$\int_{-1}^{1} |K_m(x)| \phi_{n-1}(|x|) dx \leq \frac{1}{(n-1)!} \int_{-1}^{1} |K_m(x)| \, |x|^{n-1} dx$$

$$+ \frac{1}{n! h_m} \int_{-1}^{1} |K_m(x)| \, |x|^n dx$$

$$\left(\text{by picking } h_m = \int_{-1}^{1} |K_m(x)| \, |x|^n dx \right)$$

$$= \frac{1}{(n-1)!} \int_{-1}^{1} |K_m(x)| \, |x|^{n-1} dx + \frac{1}{n!},$$

that is,

$$\int_{-1}^{1} |K_m(x)| \phi_{n-1}(|x|) dx \leq \frac{1}{(n-1)!} \int_{-1}^{1} |K_m(x)| \, |x|^{n-1} dx + \frac{1}{n!}. \qquad (20.7)$$

So by combining (20.6) and (20.7), we obtain (20.3). If $\int_{-1}^{1} |K_m(x)| dx \to 0$, as $m \to \infty$, then

$$\int_{-1}^{1} |K_m(x)| \, |x|^N dx \to 0, \quad \text{as } m \to +\infty, \qquad \text{for any } N \in \mathbb{N}.$$

Assuming also $M_m \to 0$, as $m \to \infty$ we get $\int_{-1}^{1} f d\mu_m \to 0$. ■

Remark 20.5. Let $f \in C[-1,1]$ and assume L_m, T are bounded linear operators from $C[-1,1]$ into itself such that $L_m(f) \to T(f)$ uniformly as $m \to \infty$, i.e., $K_m(f) = (L_m - T)(f) \to 0$, uniformly, as $m \to \infty$. By the Riesz representation theorem, we have that

$$(K_m(f))(x_0) = \int_{-1}^{1} f(t) \mu_{mx_0}(dt), \quad x_0 \in [-1,1], \quad \forall f \in C[-1,1], \qquad (20.8)$$

where μ_{mx_0} is a unique finite Baire signed measure, see [257], p. 310, Theorem 8. Here $\mu_{mx_0}[-1,1] =: M_{mx_0} \in \mathbb{R}$. So the pointwise convergence $(K_m(f))(x_0) \to 0$ as $m \to \infty$ is equivalent to the weak convergence of μ_{mx_0} to zero. The last implies $M_{mx_0} \to 0$ as $m \to \infty$. Clearly, Theorems 20.2, 20.3, and 20.4 provide estimates and rates of pointwise convergence to zero for the sequence K_m. Equivalently, this chapter presents a quantitative study of the pointwise convergence of operators L_m to T as $m \to +\infty$.

21

Quantitative Stochastic Korovkin Theory

Here we study very general stochastic positive linear operators induced by general positive linear operators that are acting on continuous functions. These are acting on the space of real differentiable stochastic processes. Under some very mild, general and natural assumptions on the stochastic processes we produce related stochastic Shisha–Mond type inequalities of L^q-type $1 \leq q < \infty$ and corresponding stochastic Korovkin type theorems. These are regarding the stochastic q-mean convergence of a sequence of stochastic positive linear operators to the stochastic unit operator for various cases. All convergences are produced with rates and are given via the stochastic inequalities involving the stochastic modulus of continuity of the $n-th$ derivative of the engaged stochastic process, $n \geq 0$. The impressive fact is that the basic real Korovkin test functions assumptions are enough for the conclusions of our stochastic Korovkin theory. We give an application. This chapter is based on [38].

21.1 Introduction

Motivation for this chapter are [15], [16], [279], [280]. We introduce the stochastic positive linear operator M, see (21.1), based on a general positive linear operator $\tilde{L}$ from $C([a,b])$ into itself. The operator M is acting on a wide space of differentiable real valued stochastic processes X.

We give the definition of q-mean first modulus of continuity, $1 \leq q < \infty$, see (21.16), and we prove important properties of it, such as in

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 281–298.
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Proposition 21.9. Here we suppose that $X^{(n)}(x,\omega)$ is continuous in $x \in [a,b]$, uniformly with respect to $\omega \in \Omega$—the probability space, $n \geq 0$. We assume also the integrability conditions (21.32) or the one in Assumption 21.23. We first give the pointwise stochastic Shisha–Mond type inequalities, see (21.33), (21.47), (21.57) and (21.68). Then we derive the corresponding uniform stochastic Shisha–Mond type inequalities (21.34), (21.48), (21.58) and (21.69). From these we establish the stochastic Korovkin type Theorems 21.20, 21.27, 21.33 and 21.39. These are regarding the q-mean convergence of a sequence of stochastic positive linear operators $\{M_N\}_{N\in\mathbb{N}}$ as in (21.1) to the stochastic unit operator I.

The impressive fact here is that the basic Korovkin real assumptions are enough to enforce our conclusions at the stochastic setting. So our stochastic inequalities that involve the q-mean first modulus of continuity of $X^{(n)}$ describe quantitatively and with rates the above convergence. At the end we give an application regarding the stochastic Bernstein operators where we apply the stochastic inequality (21.69).

21.2 Main Results

Concepts 21.1. Let $\tilde{L}$ be a positive linear operator from $C([a,b])$ into itself. Let $X(t,\omega)$ be a stochastic process from $[a,b]\times(\Omega,\mathcal{B},P)$ into $\mathbb{R}$, where $(\Omega,\mathcal{B},P)$ is a probability space. Here we suppose that $X(\cdot,\omega)\in C^n([a,b])$, for each $\omega\in\Omega$ and $X^{(k)}(t,\cdot)$ is measurable for all $k=0,1,\ldots,n$, for each $t\in[a,b]$, $n\geq 0$.

Define

$$M(X)(t,\omega) := \tilde{L}(X(\cdot,\omega))(t), \quad \forall\omega\in\Omega, \quad \forall t\in[a,b], \tag{21.1}$$

and assume that it is a random variable in ω. Clearly M is a positive linear operator on stochastic processes.

We make

Remark 21.2. By the Riesz representation theorem we have that there exists μ_t unique, completed Borel measure on $[a,b]$ with

$$m_t := \mu_t([a,b]) = \tilde{L}(1)(t) \geq 0, \tag{21.2}$$

such that

$$\tilde{L}(f)(t) = \int_{[a,b]} f(x)d\mu_t(x), \tag{21.3}$$

for each $t\in[a,b]$ and all $f\in C([a,b])$. Consequently we have that

$$M(X)(t,\omega) = \int_{[a,b]} X(x,\omega)d\mu_t(x), \quad \forall(t,\omega)\in[a,b]\times\Omega, \tag{21.4}$$

and X as above.

We make

Remark 21.3. Let $n \geq 1$. Using the Taylor formula with $t \in [a,b]$ fixed momentarily, we have

$$X(s,\omega) = \sum_{k=0}^{n} \frac{X^{(k)}(t,\omega)}{k!}(s-t)^k + \int_t^s \left(X^{(n)}(x,\omega) - X^{(n)}(t,\omega)\right)\frac{(s-x)^{n-1}}{(n-1)!}dx, \quad \forall s \in [a,b]. \tag{21.5}$$

Therefore we obtain

$$\begin{aligned} M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t) &= \int_{[a,b]} X(s,\omega)\mu_t(ds) - X(t,\omega)\tilde{L}(1)(t) \\ &= \sum_{k=1}^{n} \frac{X^{(k)}(t,\omega)}{k!}\left(\tilde{L}((\cdot - t)^k)(t)\right) \\ &+ \int_{[a,b]} \left(\int_t^s \left(X^{(n)}(x,\omega) - X^{(n)}(t,\omega)\right)\frac{(s-x)^{n-1}}{(n-1)!}dx\right)\mu_t(ds), \end{aligned} \tag{21.6}$$

for each $t \in [a,b]$.

Furthermore we get

$$\begin{aligned} &|M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)| \\ &\leq \sum_{k=1}^{n} \frac{|X^{(k)}(t,\omega)|}{k!}|\tilde{L}((\cdot - t)^k)(t)| \\ &+ \frac{1}{(n-1)!}\int_{[a,b]} \left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|\mu_t(ds), \end{aligned} \tag{21.7}$$

for each $t \in [a,b]$.

We also make

Remark 21.4. Here we are working on the remainder of (21.7). Let $p,q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, i.e. $p = \frac{q}{q-1}$. We notice by Hölder's inequality that

$$\begin{aligned} &\left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right| \\ &\leq \left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|^q dx\right|^{1/q} \frac{|t-s|^{\frac{qn-1}{q}}(q-1)^{\frac{q-1}{q}}}{(qn-1)^{\frac{q-1}{q}}}. \end{aligned} \tag{21.8}$$

Thus we have

$$\begin{aligned} &\left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|^q \\ &\leq \left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|^q dx\right| \frac{|t-s|^{qn-1}(q-1)^{q-1}}{(qn-1)^{q-1}}. \end{aligned} \tag{21.9}$$

Applying again Hölder's inequality we derive

$$\begin{aligned}
\Gamma := \quad & \frac{1}{(n-1)!}\left(\int_\Omega\left\{\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|\right.\right.\right. \\
& \left.\left.\left. \times |s-x|^{n-1}dx\right|\mu_t(ds)\right\}^q P(d\omega)\right)^{1/q} \\
\leq \quad & \frac{(\tilde{L}(1)(t))^{\frac{q-1}{q}}}{(n-1)!}\left(\int_\Omega\left\{\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|\right.\right.\right. \\
& \left.\left.\left. \times |s-x|^{n-1}dx\right|^q \mu_t(ds)\right\}P(d\omega)\right)^{1/q} \\
\overset{\text{by (21.9)}}{\leq} \quad & c_0(t,q,n)\left(\int_\Omega\left(\int_{[a,b]}\left(\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q dx\right|\right.\right.\right. \\
& \left.\left.\left. \times |t-s|^{qn-1}\right)\mu_t(ds)\right)P(d\omega)\right)^{1/q} =: (*), \qquad (21.10)
\end{aligned}$$

where

$$c_0(t,q,n) = \frac{1}{(n-1)!}\cdot\left(\frac{\tilde{L}(1)(t)(q-1)}{qn-1}\right)^{1-\frac{1}{q}}. \qquad (21.11)$$

Here $\varphi(x,\omega) := |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q \geq 0$, is a real valued random variable

for each $x \in [a,b]$, as well continuous in x, and thus by Proposition 3.3(i), [32], it is jointly measurable in (x,ω). And from the proof of Proposition 3.3, [32], the integral $\int_t^s \varphi(x,\omega)dx$ is a real valued random variable.

Thus

$$\lambda(s,\omega) := \left|\int_t^s \varphi(x,\omega)dx\right| |t-s|^{qn-1} \qquad (21.12)$$

is a real valued random variable, which is continuous in $s \in [a,b]$, i.e. it is Borel measurable on $[a,b]$. Again by Proposition 3.3(i), [32], $\lambda(s,\omega)$ is jointly measurable in (s,ω).

Therefore by applying Tonelli–Fubini's theorem, see [150], p. 104, we get that

$$\begin{aligned}(*) &= c_0(t,q,n)\left(\int_{[a,b]}\left(\int_\Omega\left(\left|\int_t^s |X^{(n)}(x,\omega)\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. - X^{(n)}(t,\omega)|^q dx\right| |t-s|^{qn-1}\right)P(d\omega)\right)\mu_t(ds)\right)^{1/q} \quad (21.13) \\ &= c_0(t,q,n)\left(\int_{[a,b]}\left(\int_\Omega\left(\left|\int_t^s |X^{(n)}(x,\omega)\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. - X^{(n)}(t,\omega)|^q dx\right| P(d\omega)\right)|t-s|^{qn-1}\mu_t(ds)\right)^{1/q}\end{aligned}$$

(again by applying Tonelli–Fubini's theorem)

$$\begin{aligned}&= c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\int_\Omega |X^{(n)}(x,\omega)\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. - X^{(n)}(t,\omega)|^q P(d\omega)\right)dx\right|\right)|t-s|^{qn-1}\right)\mu_t(ds)\right)^{1/q}. \quad (21.14)\end{aligned}$$

Thus so far we have shown that

Lemma 21.5. *It holds*

$$\begin{aligned}\Gamma := &\frac{1}{(n-1)!}\left(\int_\Omega\left(\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)\right.\right.\right. \\ &\left.\left.\left. - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|\mu_t(ds)\right)^q P(d\omega\right)^{1/q} \\ \le\ & c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\int_\Omega |X^{(n)}(x,\omega)\right.\right.\right.\right. \quad (21.15) \\ &\left.\left.\left.\left. - X^{(n)}(t,\omega)|^q P(d\omega)\right)dx\right|\right)|t-s|^{qn-1}\right)\mu_t(ds)\right)^{1/q}, \quad q>1,\ n\ge 1.\end{aligned}$$

We give

Definition 21.6. We define the q-mean first modulus of continuity of X by

$$\begin{aligned}\Omega_1(X,\delta)_{L^q} := \ &\sup\left\{\left(\int_\Omega |X(x,\omega)-X(y,\omega)|^q P(d\omega)\right)^{1/q} : \right. \quad (21.16) \\ &\left. x,y\in[a,b],\ |x-y|\le\delta\right\}, \quad \delta>0, \quad 1\le q<\infty.\end{aligned}$$

Definition 21.7. Let $1 \leq q < \infty$. Let $X(x,\omega)$ be a real stochastic process. We call X a q-mean uniformly continuous stochastic process (or random function) over $[a,b]$, iff $\forall \varepsilon > 0$ $\exists \delta > 0$: whenever $|x-y| \leq \delta$; $x, y \in [a,b]$ implies that

$$\int_{\Omega} |X(x,s) - X(y,s)|^q P(ds) \leq \varepsilon. \tag{21.17}$$

We denote it as $X \in C_{\mathbb{R}}^{Uq}([a,b])$.

It holds

Proposition 21.8. *Let $X \in C_{\mathbb{R}}^{Uq}([a,b])$, then $\Omega_1(X,\delta)_{L^q} < \infty$, any $\delta > 0$.*

Proof. Similar to the proof of Proposition 3.1, [32]. ∎

Also it holds

Proposition 21.9. *Let $X(t,\omega)$ be a stochastic process from $[a,b] \times (\Omega, \mathcal{B}, P)$ into $\mathbb{R}$. The following are true*:

(i) $\Omega_1(X,\delta)_{L^q}$ *is nonnegative and nondecreasing in* $\delta > 0$.

(ii) $\lim_{\delta \downarrow 0} \Omega_1(X,\delta)_{L^q} = \Omega_1(X,0)_{L^q} = 0$, *iff* $X \in C_{\mathbb{R}}^{Uq}([a,b])$.

(iii) $\Omega_1(X,\delta_1+\delta_2)_{L^q} \leq \Omega_1(X,\delta_1)_{L^q} + \Omega_1(X,\delta_2)_{L^q}$, $\delta_1, \delta_2 > 0$.

(iv) $\Omega_1(X,n\delta)_{L^q} \leq n\Omega_1(X,\delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(v)
$$\Omega_1(X,\lambda\delta)_{L^q} \leq \lceil \lambda \rceil \Omega_1(X,\delta)_{L^q} \leq (\lambda+1)\Omega_1(X,\delta)_{L^q},$$
$\lambda > 0$, $\delta > 0$, *where* $\lceil \cdot \rceil$ *is the ceiling of the number.*

(vi) $\Omega_1(X+Y,\delta)_{L^q} \leq \Omega_1(X,\delta)_{L^q} + \Omega_1(Y,\delta)_{L^q}$, $\delta > 0$.

(vii) $\Omega_1(X,\cdot)_{L^q}$ *is continuous on* $\mathbb{R}_+$ *for* $X \in C_{\mathbb{R}}^{Uq}([a,b])$.

Proof. Obvious. ∎

We give

Remark 21.10. By Proposition 21.9(v) we find

$$\Omega_1(X,|x-y|)_{L^q} \leq \left\lceil \frac{|x-y|}{\delta} \right\rceil \Omega_1(X,\delta)_{L^q}, \quad \forall x,y \in [a,b], \text{ any } \delta > 0. \tag{21.18}$$

Assumption 21.11. Let $n \geq 0$.

Here we suppose that $X^{(n)}(x,\omega)$ is *continuous in $x \in [a,b]$, uniformly with respect to $\omega \in \Omega$*. I.e. $\forall \varepsilon > 0$ $\exists \delta > 0$: whenever $|x-y| \leq \delta$; $x,y \in [a,b]$, then

$$|X^{(n)}(x,\omega) - X^{(n)}(y,\omega)| \leq \varepsilon, \quad \forall \omega \in \Omega.$$

We denote this by $X^{(n)} \in C_{\mathbb{R}}^{U}([a,b])$, *the space of continuous in* x*, uniformly with respect to* ω*, stochastic processes.*

Hence here $X^{(n)}(\cdot,\omega) \in C([a,b])$, $\forall \omega \in \Omega$ and $X^{(n)}$ is q-mean uniformly continuous in $t \in [a,b]$, that is $X^{(n)} \in C_{\mathbb{R}}^{Uq}([a,b])$, for any $1 \le q < \infty$.

We make

Remark 21.12. We continue work on the remainder of (21.7). We observe the following ($q > 1$),

$$
\begin{aligned}
& c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\int_\Omega |X^{(n)}(x,\omega)\right.\right.\right.\right. \\
& \quad \left.\left.\left.\left. - X^{(n)}(t,\omega)|^q P(d\omega)\right)dx\right|\right)|t-s|^{qn-1}\right)\mu_t(ds)\Bigg)^{1/q} \\
& \le c_0(t,q,n)\left(\int_{[a,b]}\left(\left(\left|\int_t^s \Omega_1^q\left(X^{(n)},|x-t|_{L^q}\right)dx\right|\right)|t-s|^{qn-1}\right)\mu_t(ds)\right)^{1/q} \\
& \qquad \text{(let } h>0\text{)} \\
& \overset{\text{(by (21.18))}}{\le} \quad c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\left\lceil\frac{|x-t|}{h}\right\rceil^q\right.\right.\right.\right. \\
& \quad \left.\left.\left.\left. \times\, \Omega_1^q(X^{(n)},h)_{L^q}\right)dx\right|\right)|t-s|^{qn-1}\right)\mu_t(ds)\Bigg)^{1/q} \\
& \le\ \Omega_1(X^{(n)},h)_{L^q}\, c_0(t,q,n)\left(\int_{[a,b]}\right. \\
& \quad \left.\left(\left|\int_t^s\left(1+\frac{|x-t|}{h}\right)^q dx\right|\right)|t-s|^{qn-1}\right)\mu_t(ds)\Bigg)^{1/q} \\
& =:\ (**).
\end{aligned} \tag{21.19}
$$

Put

$$
\tau := 2^{1-\frac{1}{q}} c_0(t,q,n)\Omega_1(X^{(n)},h)_{L^q}. \tag{21.20}
$$

Hence we have

$$(**) \leq \tau \left(\int_{[a,b]} \left(\left(\left| \int_t^s \left(1 + \frac{|x-t|^q}{h^q} \right) dx \right| \right) |t-s|^{qn-1} \right) \mu_t(ds) \right)^{1/q}$$

$$\leq \tau \left(\int_{[a,b]} \left(\left(|s-t| + \left(\frac{1}{h^q} \int_t^s |x-t|^q dx \right) \right) |t-s|^{qn-1} \right) \mu_t(ds) \right)^{1/q} \quad (21.21)$$

$$= \tau \left(\int_{[a,b]} \left(\left(|s-t| + \frac{1}{h^q} \frac{|t-s|^{q+1}}{(q+1)} \right) |t-s|^{qn-1} \right) \mu_t(ds) \right)^{1/q} \quad (21.22)$$

$$= \tau \left(\left(\int_{[a,b]} |s-t|^{qn} \mu_t(ds) \right) + \frac{1}{h^q(q+1)} \left(\int_{[a,b]} |t-s|^{q(n+1)} \mu_t(ds) \right) \right)^{1/q}$$

$$\leq \tau \left[m_t^{1/n+1} \left(\int_{[a,b]} |s-t|^{q(n+1)} \mu_t(ds) \right)^{n/n+1} + \frac{1}{h^q(q+1)} \left(\int_{[a,b]} |t-s|^{q(n+1)} \mu_t(ds) \right) \right]^{1/q} =: (***). \quad (21.23)$$

We set and suppose that

$$h := \left(\frac{1}{(q+1)} \int_{[a,b]} |t-s|^{q(n+1)} \mu_t(ds) \right)^{1/q(n+1)} = \left(\frac{1}{(q+1)} \tilde{L}(|t-\cdot|^{q(n+1)})(t) \right)^{1/q(n+1)} > 0. \quad (21.24)$$

That is

$$h^{q(n+1)} = \frac{1}{(q+1)} \left(\int_{[a,b]} |t-s|^{q(n+1)} \mu_t(ds) \right) > 0. \quad (21.25)$$

Therefore

$$(***) = \tau \left[m_t^{1/n+1} h^{qn} (q+1)^{n/n+1} + h^{qn} \right]^{1/q} = \tau h^n \left[m_t^{1/n+1} (q+1)^{n/n+1} + 1 \right]^{1/q}. \quad (21.26)$$

We have shown that

$$\Gamma = \frac{1}{(n-1)!} \left(\int_\Omega \left(\int_{[a,b]} \left| \int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)| |s-x|^{n-1} dx \right| \mu_t(ds) \right)^q P(d\omega) \right)^{1/q} \leq \tau h^n \left[m_t^{1/n+1} (q+1)^{n/n+1} + 1 \right]^{1/q}. \quad (21.27)$$

We have established

Lemma 21.13. *It holds*

$$\Gamma \leq \left[((\tilde{L}(1))(t))^{1/(n+1)}(q+1)^{n/(n+1)}+1\right]^{1/q}$$
$$\cdot \frac{1}{(n-1)!(q+1)^{\frac{n}{q(n+1)}}}\left((\tilde{L}(|\cdot -t|^{q(n+1)})(t)\right)^{\frac{n}{q(n+1)}} \cdot \left(\frac{2(q-1)\tilde{L}(1)(t)}{qn-1}\right)^{1-\frac{1}{q}}$$
$$\cdot\, \Omega_1\left(X^{(n)}, \frac{1}{(q+1)^{\frac{1}{q(n+1)}}}\cdot\left(\tilde{L}(|\cdot -t|^{q(n+1)})(t)\right)^{\frac{1}{q(n+1)}}\right)_{L^q}, \qquad (21.28)$$

$q>1$, $n \geq 1$.

We make

Remark 21.14. Here we observe that

$$|M(X)(t,\omega)-X(t,\omega)| \leq |M(X)(t,\omega)-X(t,\omega)\tilde{L}(1)(t)| + |X(t,\omega)|\,|\tilde{L}(1)(t)-1|. \qquad (21.29)$$

Combining (21.29) with (21.7) we have

$$|M(X)(t,\omega)-X(t,\omega)|$$
$$\leq |X(t,\omega)|\,|\tilde{L}(1)(t)-1| + \sum_{k=1}^{n}\frac{|X^{(k)}(t,\omega)|}{k!}|\tilde{L}((\cdot - t)^k)(t)| \qquad (21.30)$$
$$+\frac{1}{(n-1)!}\left(\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|\mu_t(ds)\right),$$

$\forall t \in [a,b]$.

We need

Definition 21.15. Denote by

$$(EX)(t) := \int_\Omega X(t,\omega)P(d\omega), \quad \forall t\in[a,b], \qquad (21.31)$$

the expectation operator.

We make

Assumption 21.16. We suppose that

$$(E|X^{(k)}|^q)(t) < \infty, \quad \forall t \in [a,b] \qquad (21.32)$$

and for all $k=0,1,\ldots,n$; $n \geq 0$.

Based on all the above it holds

Theorem 21.17. *Suppose Concepts* 21.1, $1 < q < \infty$, *Assumptions 21.11 and 21.16,* $n \geq 1$. *Then*

$$\begin{aligned}
&(E(|M(X) - X|^q)(t))^{1/q} \\
&\leq ((E|X|^q)(t))^{1/q} |\tilde{L}(1)(t) - 1| + \sum_{k=1}^{n} \frac{((E|X^{(k)}|^q)(t))^{1/q}}{k!} |\tilde{L}((\cdot - t)^k)(t)| \\
&\quad + \left(\frac{2(q-1)\tilde{L}(1)(t)}{qn-1} \right)^{1-\frac{1}{q}} \cdot \frac{1}{(n-1)!(q+1)^{\frac{n}{q(n+1)}}} \qquad (21.33) \\
&\quad \cdot \left[(\tilde{L}(1)(t))^{1/(n+1)} (q+1)^{n/(n+1)} + 1 \right]^{1/q} \cdot \left(\tilde{L}(|\cdot - t|^{q(n+1)})(t) \right)^{\frac{n}{q(n+1)}} \\
&\quad \cdot \Omega_1 \left(X^{(n)}, \frac{1}{(q+1)^{\frac{1}{q(n+1)}}} \left(\tilde{L}(|\cdot - t|^{q(n+1)})(t) \right)^{\frac{1}{q(n+1)}} \right)_{L^q}, \ \forall t \in [a, b].
\end{aligned}$$

Note 21.18. If $\tilde{L}(|\cdot - t|^{q(n+1)})(t) = 0$, then (21.33) holds trivially as equality.

We further present

Corollary 21.19. *Suppose Concepts 21.1,* $1 < q < \infty$, *Assumptions 21.11 and 21.16,* $n \geq 1$. *Then*

$$\begin{aligned}
&\|E(|M(X) - X|^q)\|_\infty^{1/q} \\
&\leq \|E(|X^q|)\|_\infty^{1/q} \|\tilde{L}1 - 1\|_\infty + \sum_{k=1}^{n} \frac{\|E(|X^{(k)}|^q)\|_\infty^{1/q}}{k!} \|\tilde{L}((\cdot - t)^k)(t)\|_\infty \\
&\quad + \left(\frac{2(q-1)\|\tilde{L}(1)\|_\infty}{qn-1} \right)^{1-\frac{1}{q}} \frac{1}{(n-1)!(q+1)^{n/q(n+1)}} \\
&\quad \cdot \left(\|(\tilde{L}(1))^{\frac{1}{n+1}} (q+1)^{\frac{n}{n+1}} + 1\|_\infty \right)^{1/q} \|\tilde{L}(|\cdot - t|^{q(n+1)})(t)\|_\infty^{n/q(n+1)} \\
&\quad \cdot \Omega_1 \left(X^{(n)}, \frac{1}{(q+1)^{\frac{1}{q(n+1)}}} \|\tilde{L}(|\cdot - t|^{q(n+1)})(t)\|_\infty^{\frac{1}{q(n+1)}} \right)_{L^q}. \qquad (21.34)
\end{aligned}$$

We present a Korovkin ([213]) type theorem for stochastic processes in our general setting.

Theorem 21.20. *Let* $\{\tilde{L}_N\}_{N \in \mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N \in \mathbb{N}}$, *on stochastic processes all as in Concepts 21.1,* $1 < q < \infty$, *Assumptions 21.11 and 21.16,* $n \geq 1$. *Additionally suppose that* $\{\tilde{L}_N(1)\}_{N \in \mathbb{N}}$ *is bounded and* $\|\tilde{L}_N(|\cdot - t|^{q(n+1)})(t)\|_\infty \to 0$, *along with* $\tilde{L}_N 1 \xrightarrow{u} 1$, *as* $N \to \infty$. *Then*

$$\|E(|M_N(X) - X|^q)\|_\infty \to 0,$$

as $N \to \infty$, *for all* X *as in Concepts 21.1 and Assumptions 21.11, 21.16,* $n \geq 1$. *I.e.*

$$M_N \underset{N \to \infty}{\overset{\text{“}q\text{-mean”}}{\longrightarrow}} I,$$

the unit operator, with rates and in our setting.

Proof. By Corollary 21.19 and the fact

$$\|\tilde{L}_N((\cdot - t)^k)(t)\|_\infty \leq \|\tilde{L}_N(1)\|_\infty^{\frac{q(n+1)-k}{q(n+1)}} \|\tilde{L}_N(|\cdot - t|^{q(n+1)})(t)\|_\infty^{\frac{k}{q(n+1)}}, \quad (21.35)$$

for $k = 1, \ldots, n$. .

We need

Lemma 21.21. *Let* $\varphi(s,x) \not\equiv 0$ *jointly continuous in* $(s,x) \in [a,b]^2$. *Consider*

$$\gamma(s) := \int_t^s \varphi(s,x)dx, \quad (21.36)$$

where t *is fixed in* $[a,b]$. *Then* $\gamma(s)$ *is continuous in* $s \in [a,b]$.

Proof. Easy. ■

We make

Remark 21.22. Let $n \geq 1$. By (21.7) we derive

$$\begin{aligned}
&\int_\Omega |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)|P(d\omega) \\
&\quad \leq \sum_{k=1}^n \frac{E|X^{(k)}|(t)}{k!}|\tilde{L}((\cdot - t)^k)(t)| + \frac{1}{(n-1)!}\left(\int_\Omega \left(\int_{[a,b]} \left| \int_t^s |X^{(n)}(x,\omega) \right.\right.\right. \\
&\quad\quad \left.\left.\left. - X^{(n)}(t,\omega)| \, |s-x|^{n-1}dx \right| \mu_t(ds)\right) P(d\omega)\right). \quad (21.37)
\end{aligned}$$

(The integrand function is jointly continuous in (x,s) and measurable in ω, therefore is jointly measurable in (s,ω) and also nonnegative. Use also Lemma 21.21. Therefore we can apply twice Tonelli–Fubini's theorem to get)

$$\begin{aligned}
= \quad &\sum_{k=1}^n \frac{E|X^{(k)}|(t)}{k!}|\tilde{L}((\cdot - t)^k)(t)| + \frac{1}{(n-1)!}\left(\int_{[a,b]} \left| \int_t^s \left(\int_\Omega |X^{(n)}(x,\omega) \right.\right.\right. \\
&\quad \left.\left.\left. - X^{(n)}(t,\omega)|P(d\omega)\right)|s-x|^{n-1})dx \right| \mu_t(ds)\right) \quad (21.38)
\end{aligned}$$

$\Big($set

$$\mathcal{J} := \sum_{k=1}^{n} \frac{(E|X^{(k)}|)(t)}{k!} |\tilde{L}((\cdot - t)^k)(t)|\Big) \tag{21.39}$$

$$\leq \mathcal{J} + \frac{1}{(n-1)!}\left(\int_{[a,b]}\left|\int_t^s \Omega_1(X^{(n)}, |x-t|)_{L^1}|s-x|^{n-1}dx\right|\mu_t(ds)\right)$$

$$\underset{(h>0)}{\leq} \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{(n-1)!}\left(\int_{[a,b]}\left|\int_t^s\left(1+\frac{|x-t|}{h}\right)\right.\right.$$

$$\left.\left.\cdot\, |s-x|^{n-1}dx\right|\mu_t(ds)\right) \tag{21.40}$$

$$\leq \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{(n-1)!}\left(\int_{[a,b]}\left\{\left|\int_t^s |s-x|^{n-1}dx\right|\right.\right.$$

$$\left.\left. + \frac{1}{h}\left|\int_t^s |x-t|\,|s-x|^{n-1}dx\right|\right\}\mu_t(ds)\right) \tag{21.41}$$

$$= \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{(n-1)!}\left(\int_{[a,b]}\left(\frac{|t-s|^n}{n} + \frac{1}{h}\frac{|t-s|^{n+1}}{n(n+1)}\right)\mu_t(ds)\right)$$

$$= \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{(n-1)!}\left[\frac{\tilde{L}(|\cdot - t|^n)(t)}{n} + \frac{\tilde{L}(|\cdot - t|^{n+1})(t)}{hn(n+1)}\right]$$

$$\leq \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{(n-1)!}\left[\frac{1}{n}(\tilde{L}(1)(t))^{1/(n+1)}((\tilde{L}(|\cdot - t|^{n+1}))(t))^{n/(n+1)}\right.$$

$$\left. + \frac{1}{hn(n+1)}(\tilde{L}(|\cdot - t|^{n+1})(t))\right]. \tag{21.42}$$

(Now take

$$h := (\tilde{L}(|\cdot - t|^{n+1})(t))^{1/(n+1)} > 0, \tag{21.43}$$

i.e.

$$h^{n+1} = \tilde{L}(|\cdot - t|^{n+1})(t)) \tag{21.44}$$

$$= \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1} h^n}{n!}\left[(\tilde{L}(1)(t))^{1/(n+1)} + \frac{1}{(n+1)}\right].$$

We have proved that

$$\int_\Omega |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)|P(d\omega) \tag{21.45}$$

$$\leq \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{n!}h^n\left(((\tilde{L}(1))(t))^{1/(n+1)} + \frac{1}{n+1}\right).$$

Also by (21.29) we derive

$$\int_\Omega |M(X)(t,\omega) - X(t,\omega)|P(d\omega)$$
$$\leq (E|X|)(t)|\tilde{L}(1)(t) - 1|$$
$$+ \int_\Omega |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)|P(d\omega). \qquad (21.46)$$

Assumption 21.23. *Here we assume* $(E|X^{(k)}|)(t) < \infty$, $\forall t \in [a,b]$, *all* $k = 0, 1, \ldots, n$, $n \geq 0$.

From the above is derived

Theorem 21.24. *Suppose Concepts 21.1 and Assumptions 21.11, 21.23,* $n \geq 1$. *Then*

$$E(|M(X) - X|)(t)$$
$$\leq (E|X|)(t)|\tilde{L}(1)(t) - 1| + \sum_{k=1}^{n} \frac{(E|X^{(k)}|)(t)}{k!} |\tilde{L}((\cdot - t)^k)(t)|$$
$$+ \frac{1}{n!} \left(((\tilde{L}(1))(t))^{1/(n+1)} + \frac{1}{n+1} \right) (\tilde{L}(|\cdot - t|^{n+1})(t))^{n/(n+1)}$$
$$\cdot \Omega_1 \left(X^{(n)}, (\tilde{L}(|\cdot - t|^{n+1})(t))^{1/(n+1)} \right)_{L^1}, \quad \forall t \in [a,b]. \qquad (21.47)$$

Note 21.25. If $\tilde{L}(|\cdot - t|^{n+1})(t) = 0$, then (21.47) holds trivially as equality.

We further present

Corollary 21.26. *Suppose Concepts 21.1 and Assumptions 21.11, 21.23,* $n \geq 1$. *Then*

$$\|E(|M(X) - X)|)\|_\infty \leq \|E|X|\,\|_\infty \|\tilde{L}1 - 1\|_\infty \qquad (21.48)$$
$$+ \sum_{k=1}^{n} \frac{\|E(|X^{(k)}|)\|_\infty}{k!} \|\tilde{L}((\cdot - t)^k)(t)\|_\infty + \frac{1}{n!} \left\| (\tilde{L}(1))^{1/(n+1)} + \frac{1}{n+1} \right\|_\infty$$
$$\cdot \|\tilde{L}(|\cdot - t|^{n+1})(t)\|_\infty^{n/(n+1)} \Omega_1 \left(X^{(n)}, \|\tilde{L}(|\cdot - t|^{n+1})(t)\|_\infty^{1/(n+1)} \right)_{L^1}.$$

The following Korovkin type theorem for stochastic processes in our general setting is valid.

Theorem 21.27. *Let* $\{\tilde{L}_N\}_{N \in \mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N \in \mathbb{N}}$ *on stochastic processes, all as in Concepts 21.1, Assumptions 21.11 and 21.23,* $n \geq 1$. *Additionally suppose that* $\{\tilde{L}_N(1)\}_{N \in \mathbb{N}}$ *is bounded and* $\|\tilde{L}_N(|\cdot - t|^{n+1})(t)\|_\infty \to 0$, *along with* $\tilde{L}_n 1 \xrightarrow{u} 1$, *as* $N \to \infty$. *Then*

$$\|E(|M_N(X) - X|)\|_\infty \to 0, \quad \text{as } N \to \infty,$$

for all X *as in Concepts 21.1 and Assumptions 21.11, 21.23,* $n \geq 1$. *I.e.*
$$M_N \underset{N\to+\infty}{\overset{\text{"1-mean"}}{\longrightarrow}} I \text{ with rates.}$$

Proof. By Corollary 21.26 and the fact

$$\|\tilde{L}_N((\cdot - t)^k)(t)\|_\infty \leq \|\tilde{L}_N(1)\|_\infty^{1-\frac{k}{n+1}} \|\tilde{L}_N(|\cdot - t|^{n+1})(t)\|_\infty^{\frac{k}{n+1}}, \tag{21.49}$$

for $k = 1, \ldots, n$. ■

Note 21.28. We observe that $M_N \overset{\text{"q-mean"}}{\longrightarrow} I$ implies $M_N \overset{\text{"1-mean"}}{\longrightarrow} I$, according to Theorems 21.20 and 21.27, $n \geq 1$.

Next we specialize in the $n = 0$ case. We do first the subcase $q > 1$. For that we make

Remark 21.29. We have that

$$\begin{aligned}\Delta(t,\omega) := \quad & M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t) \\ = \quad & \int_{[a,b]} (X(s,\omega) - X(t,\omega))\mu_t(ds).\end{aligned} \tag{21.50}$$

Let $q > 1$, then by Hölder's inequality we have

$$\begin{aligned}|\Delta(t,\omega)|^q \quad & \leq \quad \left(\int_{[a,b]} |X(s,\omega) - X(t,\omega)|\mu_t(ds)\right)^q \\ & \leq \quad m_t^{q-1} \int_{[a,b]} |X(s,\omega) - X(t,\omega)|^q \mu_t(ds).\end{aligned} \tag{21.51}$$

Therefore we derive

$$\begin{aligned}&\left(\int_\Omega |\Delta(t,\omega)|^q P(d\omega)\right)^{1/q} \leq m_t^{1-\frac{1}{q}} \\ &\cdot \left(\int_\Omega \left(\int_{[a,b]} |X(s,\omega) - X(t,\omega)|^q \mu_t(ds)\right) P(d\omega)\right)^{1/q}\end{aligned} \tag{21.52}$$

(the integrand function is nonnegative, continuous in s, measurable in ω, therefore jointly measurable in (s,ω) and by Tonelli–Fubini's theorem we get)

$$= m_t^{1-\frac{1}{q}} \left(\int_{[a,b]} \left(\int_\Omega |X(s,\omega) - X(t,\omega)|^q P(d\omega) \right) \mu_t(ds) \right)^{1/q} \quad (21.53)$$

$$\leq m_t^{1-\frac{1}{q}} \left(\int_{[a,b]} \Omega_1^q (X, |s-t|)_{L^q} \mu_t(ds) \right)^{1/q}$$

$$\text{(take } h > 0) \quad (21.54)$$

$$\leq m_t^{1-\frac{1}{q}} \Omega_1(X,h)_{L^q} \left(\int_{[a,b]} \left(1 + \frac{|s-t|}{h} \right)^q \mu_t(ds) \right)^{1/q}$$

$$\leq 2^{1-\frac{1}{q}} m_t^{1-\frac{1}{q}} \Omega_1(X,h)_{L^q} \left(m_t + \frac{1}{h^q} \int_{[a,b]} |s-t|^q \mu_t(ds) \right)^{1/q}$$

$$\left(\text{choose } h := \left(\int_{[a,b]} |s-t|^q d\mu_t(s) \right)^{1/q} > 0 \right) \quad (21.55)$$

$$= 2^{1-\frac{1}{q}} m_t^{1-\frac{1}{q}} \Omega_1 \left(X, \left(\int_{[a,b]} |s-t|^q d\mu_t(s) \right)^{1/q} \right)_{L^q} (m_t + 1)^{1/q}. \quad (21.56)$$

We have established

Theorem 21.30. *Suppose Concepts 21.1 and Assumptions 21.11, 21.16 for* $n = 0$, $1 < q < \infty$. *Then*

$$\begin{aligned} &\left(E(|M(X) - X|^q)(t)\right)^{1/q} \\ &\leq \left(E(|X|^q)(t)\right)^{1/q} |\tilde{L}(1)(t) - 1| \quad (21.57) \\ &\quad + (2\tilde{L}(1)(t))^{1-\frac{1}{q}} ((\tilde{L}(1))(t) + 1)^{1/q} \Omega_1 \big(X, (\tilde{L}(|\cdot - t|^q)(t))^{1/q}\big)_{L^q}, \end{aligned}$$

$\forall t \in [a,b]$.

Note 21.31. Inequality (21.57) is trivially true and holds as equality when (see (21.55)) $h = 0$.

We give

Corollary 21.32. *Suppose Concepts 21.1 and Assumptions 21.11, 21.16 for* $n = 0$, $1 < q < \infty$. *Then*

$$\begin{aligned} &\|E(|M(X) - X|^q)\|_\infty^{1/q} \leq \|E(|X|^q)\|_\infty^{1/q} \|\tilde{L}1 - 1\|_\infty \quad (21.58) \\ &\quad + (2\|\tilde{L}(1)\|_\infty)^{1-\frac{1}{q}} \|\tilde{L}(1) + 1\|_\infty^{1/q} \Omega_1 \big(X, \|\tilde{L}(|\cdot - t|^q)(t)\|_\infty^{1/q}\big)_{L^q}. \end{aligned}$$

We present the next Korovkin type result.

Theorem 21.33. *Let* $\{\tilde{L}_N\}_{N \in \mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N \in \mathbb{N}}$ *on stochastic processes, all as in Concepts 21.1,* $1 < q < \infty$, *Assumptions 21.11,* *21.16 for $n = 0$. Additionally suppose that $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ is bounded and $\|\tilde{L}_N(|\cdot -t|^q)(t)\|_\infty \to 0$, along with $\tilde{L}_N 1 \overset{u}{\to} 1$, as $N \to \infty$. Then*

$$\|E(|M_N(X) - X|^q)\|_\infty \to 0, \quad as\ N \to \infty,$$

for all X as in Concepts 21.1 and Assumptions 21.11, 21.16, $n = 0$. I.e. $M_N \underset{N\to\infty}{\overset{\text{"q-mean"}}{\longrightarrow}} I$ with rates in our setting.

Note 21.34. The rate of convergence in Theorem 21.20 is much higher than of Theorem 21.33 because of the assumed differentiability of X, see and compare inequalities (21.34), (21.35) and (21.58).

We make

Remark 21.35. Let $\Delta(t,\omega)$ as in (21.50). Then

$$\begin{aligned}
&\int_\Omega |\Delta(t,\omega)|P(d\omega) \\
\le\ & \int_\Omega \left(\int_{[a,b]} |X(s,\omega) - X(t,\omega)|\mu_t(ds)\right) P(d\omega) && (21.59)\\
& \text{(by Tonelli–Fubini's theorem)} \\
=\ & \int_{[a,b]} \left(\int_\Omega |X(s,\omega) - X(t,\omega)|P(d\omega)\right) \mu_t(ds) && (21.60)\\
\le\ & \int_{[a,b]} \Omega_1(X, |s-t|)_{L^1} \mu_t(ds) && (21.61)\\
\le\ & \Omega_1(X,h)_{L^1} \int_{[a,b]} \left(1 + \frac{|s-t|}{h}\right) \mu_t(ds) && (21.62)\\
=\ & \Omega_1(X,h)_{L^1} \left(m_t + \frac{1}{h}\int_{[a,b]} |s-t|\mu_t(ds)\right) && (21.63)\\
\le\ & \Omega_1(X,h)_{L^1} \left(m_t + \frac{1}{h} m_t^{1/2} \left(\int_{[a,b]} (s-t)^2 \mu_t(ds)\right)^{1/2}\right) && (21.64)
\end{aligned}$$

$\Big($pick

$$\begin{aligned}
h :=\ & \left(\int_{[a,b]} (s-t)^2 \mu_t(ds)\right)^{1/2} > 0\Big) && (21.65)\\
=\ & \Omega_1(X,h)_{L^1}(m_t + \sqrt{m_t}). && (21.66)
\end{aligned}$$

That is we get

$$\int_\Omega |\Delta(t,\omega)|P(d\omega) \le \left(\tilde{L}(1)(t) + \sqrt{\tilde{L}(1)(t)}\right)\Omega_1\left(X, ((\tilde{L}(\cdot - t)^2)(t))^{1/2}\right)_{L^1}. \quad (21.67)$$

We have proved

Theorem 21.36. *Suppose Concepts 21.1 and Assumptions 21.11, 21.23 for* $n = 0$. *Then*

$$(E(|M(X) - X|))(t) \leq (E|X|)(t)|\tilde{L}(1)(t) - 1| \quad (21.68)$$
$$+ \left(\tilde{L}(1)(t) + \sqrt{\tilde{L}(1)(t)}\right)\Omega_1\left(X, ((\tilde{L}(\cdot - t)^2)(t))^{1/2}\right)_{L^1}, \ \forall t \in [a, b].$$

Note 21.37. Inequality (21.68) is trivially true and holds as equality when (see (21.65)) $h = 0$.

We give (see also [264])

Corollary 21.38. *Suppose Concepts 21.1 and Assumptions 21.11, 21.23 for* $n = 0$. *Then*

$$\|E(|M(X) - X|)\|_\infty \leq \|E(X)\|_\infty \|\tilde{L}1 - 1\|_\infty$$
$$+ \|\tilde{L}1 + \sqrt{\tilde{L}1}\|_\infty \Omega_1\left(X, \|(\tilde{L}((\cdot - t)^2))(t)\|_\infty^{1/2}\right)_{L^1}. \quad (21.69)$$

We present a final Korovkin (see [213]) type result.

Theorem 21.39. *Let* $\{\tilde{L}_N\}_{N \in \mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N \in \mathbb{N}}$ *on stochastic processes, all as in Concepts 21.1 and Assumptions 21.11, 21.23 for* $n = 0$. *Additionally assume that* $\{\tilde{L}_N(1)\}_{N \in \mathbb{N}}$ *is bounded and*

$$\tilde{L}_N 1 \xrightarrow{u} 1, \quad \tilde{L}_N id \xrightarrow{u} id, \quad \tilde{L}_N id^2 \xrightarrow{u} id^2, \quad \text{as } N \to \infty. \quad (21.70)$$

Then

$$\|E(|M_N(X) - X|)\|_\infty \to 0, \quad \text{as } N \to \infty, \quad (21.71)$$

for all X *as in Concepts 21.1 and Assumptions 21.11, 21.23 for* $n = 0$. *I.e.*

$$M_N \underset{N \to \infty}{\overset{\text{"1-mean"}}{\longrightarrow}} I$$ *with rates in our setting.*

Proof. We use Corollary 21.38. By [264] we have that

$$\|(\tilde{L}_N((\cdot - t)^2))(t)\|_\infty \leq \|\tilde{L}_N(x^2)(t) - t^2\|_\infty + 2c\|L_N(x)(t) - t\|_\infty$$
$$+ c^2\|L_N(1)(t) - 1\|_\infty, \quad (21.72)$$

where $c := \max(|a|, |b|)$, $\forall N \in \mathbb{N}$. Thus by assuming the basic Korovkin conditions (21.70) we get by (21.72) that $\|(\tilde{L}_N((\cdot - t)^2))(t)\|_\infty \to 0$, as $N \to \infty$, etc. .

We make also

Remark 21.40. 1) If $X^{(n)}$ fulfills a Lipschitz type condition then our results become more specific and simplify.

2) In the special important case of $\tilde{L}(1)(t) = 1$, $\forall t \in [a,b]$, all of our results here simplify a lot and take an elegant form. Furthermore in this case, supposing Assumption 21.16 we need to impose (21.32) only for $k = 1, \ldots, n$ and supposing Assumption 21.23 we need to impose it only for $k = 1, \ldots, n$, $n \geq 1$.

We finish by giving

Application 21.41. Let $f \in C([0,1])$ and the Bernstein polynomial

$$B_N(f)(t) := \sum_{k=0}^{n} f\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k}, \quad \forall t \in [0,1], \ \forall N \in \mathbb{N}. \tag{21.73}$$

We have that

$$B_N((\cdot - t)^2)(t) = \frac{t(1-t)}{N}, \quad \forall t \in [0,1], \tag{21.74}$$

and

$$\|B_N((\cdot - t)^2)(t)\|_\infty^{1/2} \leq \frac{1}{2\sqrt{N}}, \quad \forall N \in \mathbb{N}. \tag{21.75}$$

Clearly B_N is an example of an $\tilde{L}_N$ as in Concepts 21.1. Define the corresponding application of M_N by

$$\begin{aligned} \tilde{B}_N(X)(t,\omega) := \quad & B_N(X(\cdot,\omega))(t) \qquad (21.76) \\ = \quad & \sum_{k=0}^{N} X\left(\frac{k}{N},\omega\right)\binom{N}{k} t^k (1-t)^{N-k}, \quad \forall t \in [0,1], \end{aligned}$$

for all $\omega \in \Omega$, $N \geq 1$, where X is as in Concepts 21.1 and Assumptions 21.11, 21.13 for $n = 0$. Since $B_N(1)(t) = 1$ by (21.69) we derive that

$$\|E(|\tilde{B}_N(X) - X|)\|_\infty \leq 2\Omega_1\left(X, \frac{1}{2\sqrt{N}}\right)_{L^1}, \quad N \geq 1, \tag{21.77}$$

for all X as above. Thus as $N \to \infty$ we obtain

$$\|E(|\tilde{B}_N(X) - X|)\|_\infty \to 0, \tag{21.78}$$

i.e. $\tilde{B}_N \overset{\text{“1-mean”}}{\longrightarrow} I$ with rates, which is the expected conclusion given by Theorem 21.39. If X is of Lipschitz type of order 1 i.e. if $\Omega_1(X,\delta)_{L^1} \leq K\delta$, where $K > 0$, $\forall \delta > 0$, then

$$\|E(|\tilde{B}_N(X) - X|)\|_\infty \leq \frac{K}{\sqrt{N}}, \quad \forall N \geq 1. \tag{21.79}$$

One can give many similar other applications of the above theory.

22

Quantitative Multidimensional Stochastic Korovkin Theory

Here we study very general multivariate stochastic positive linear operators induced by general multivariate positive linear operators that are acting on multivariate continuous functions. These are acting on the space of real differentiable multivariate time stochastic processes. Under some very mild, general and natural assumptions on the stochastic processes we present related multidimensional stochastic Shisha–Mond type inequalities of L^q-type $1 \leq q < \infty$ and corresponding multidimensional stochastic Korovkin type theorems. These are regarding the stochastic q-mean convergence of a sequence of multivariate stochastic positive linear operators to the stochastic unit operator for various cases. All convergences are given with rates and are shown via the stochastic inequalities involving the maximum of the multivariate stochastic moduli of continuity of the nth order partial derivatives of the engaged stochastic process, $n \geq 0$. The astonishing fact here is that basic real Korovkin test functions assumptions are enough for the conclusions of the multidimensional stochastic Korovkin theory. We give an application. This chapter relies on [40].

22.1 Introduction

Motivation for this chapter are [15], [16], [279], [280]. We introduce the multivariate stochastic positive linear operator M, see (22.4), based on a general multivariate positive linear operator $\tilde{L}$ from $C(Q)$ into itself, Q

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convex compact $\subseteq \mathbb{R}^k$, $k > 1$. The operator M is acting on a wide space of differentiable real valued multidimensional time stochastic processes X.

We give the definition of multidimensional q-mean first modulus of continuity, $1 \leq q < \infty$, see (22.1), and we prove important properties of it, such as in Propositions 22.3 and 22.4. Here we suppose that $X_\alpha(x, \omega)$, $|\alpha| = n$, is continuous in $x \in Q$, uniformly with respect to $\omega \in \Omega$-the probability space, $n \geq 0$. We assume also the integrability Assumptions 22.11 and 22.23.

We first give the pointwise multidimensional stochastic Shisha–Mond type inequalities, see (22.31), (22.50), (22.54), (22.112), (22.126) and (22.136). Then we derive the corresponding uniform multidimensional stochastic Shisha–Mond type inequalities (22.51), (22.55), (22.113), (22.127) and (22.137). From these we prove the multivariate stochastic Korovkin type Theorems 22.22, 22.31, 22.36 and 22.41. These are regarding the q-mean convergence of a sequence of multivariate stochastic positive linear operators $\{M_N\}_{N \in \mathbb{N}}$ as in (22.4) to the stochastic unit operator I.

The impressive thing here is that basic Korovkin multidimensional real assumptions are enough to enforce the conclusions at the stochastic setting. So the multidimensional stochastic inequalities that involve the multidimensional q-mean first modulus of continuity of X_α, $|\alpha| = n$, describe quantitatively and with rates the above convergence. At the end we give an application regarding the multivariate stochastic Bernstein operators where we apply the multidimensional stochastic inequality (22.127).

22.2 Background

We give

Definition 22.1. Let Q be a compact convex subset of $\mathbb{R}^k$, $k > 1$. Let $X(t, \omega)$ be a stochastic process from $Q \times (\Omega, \mathcal{B}, P)$ into $\mathbb{R}$, where $(\Omega, \mathcal{B}, P)$ is a probability space. We define the *q-mean multivariate first moduli of continuity of X* by

$$\Omega_1(X, \delta)_{L^q} := \sup \left\{ \left(\int_\Omega |X(x, \omega) - X(y, \omega)|^q P(d\omega) \right)^{1/q} : x, y \in Q, \right.$$
$$\left. \|x - y\|_{\ell^1} \leq \delta \right\}, \quad \delta > 0, \ 1 \leq q < \infty. \tag{22.1}$$

We mention

Definition 22.2. Let $1 \leq q < \infty$. Let $X(x, \omega)$, $x \in Q$, $\omega \in \Omega$ be a multivariate real stochastic process. We call X a *q-mean uniform continuous multivariate stochastic process over Q*, iff $\forall \varepsilon > 0$ $\exists \delta > 0$: whenever

$\|x-y\|_{\ell^1} \le \delta$; $x, y \in Q$ implies that

$$\int_\Omega |X(x,s) - X(y,s)|^q P(ds) \le \varepsilon. \tag{22.2}$$

We denote it as $X \in C_{\mathbb{R}}^{Uq}(Q)$.

It holds

Proposition 22.3. *Let $X \in C_{\mathbb{R}}^{Uq}(Q)$, then $\Omega_1(X,\delta)_{L^q} < \infty$, $\forall \delta > 0$.*

Proof. Let $\varepsilon_0 > 0$ be arbitrary but fixed. Then there exists $\delta_0 > 0 : \|x-y\|_{\ell^1} \le \delta_0, x, y \in Q$, implies

$$\int_\Omega |X(x,s) - X(y,s)|^q \, P(ds) \le \varepsilon_0 < \infty.$$

That is $\Omega_1(X,\delta_0) \le \varepsilon_0^{1/q} < \infty$. Let now $\delta > 0$ arbitrary, $x, y \in Q : \|x-y\|_{\ell^1} \le \delta$. Choose $n_0 \in \mathbb{N} : n_0\delta_0 \ge \delta$ and set $x_i := x + \frac{i}{n_0}(y-x), 0 \le i \le n_0$. Then

$$\begin{aligned}
&\left(\int_\Omega |X(x,\omega) - X(y,\omega)|^q \, P(d\omega)\right)^{1/q} \\
\le\ &\left(\int_\Omega |X(x,\omega) - X(x_1,\omega)|^q \, P(d\omega)\right)^{1/q} \\
&+ \left(\int_\Omega |X(x_1,\omega) - X(x_2,\omega)|^q \, P(d\omega)\right)^{1/q} \\
&+ \ldots + \left(\int_\Omega |X(x_{n_0-1},\omega) - X(y,\omega)|^q \, P(d\omega)\right)^{1/q} \\
\le\ & n_0 \Omega_1(X,\delta_0) \le n_0 \varepsilon_0^{1/q} < \infty,
\end{aligned}$$

since $\|x_i - x_{i+1}\| = \frac{1}{n_0}\|x-y\|_{\ell^1} \le \frac{1}{n_0}\delta \le \delta_0$, $0 \le i \le n_0$. Therefore $\Omega_1(X,\delta) \le n_0 \varepsilon_0^{1/q} < \infty$. ∎

Also it holds

Proposition 22.4. *Let $X(t,\omega)$ be a multivariate stochastic process from $Q \times (\Omega, \mathcal{B}, P)$ into $\mathbb{R}$.*

The following are true.

(i) *$\Omega_1(X,\delta)_{L^q}$ is nonnegative and nondecreasing in $\delta > 0$.*

(ii) *$\lim_{\delta \downarrow 0} \Omega_1(X,\delta)_{L^q} = \Omega_1(X,0)_{L^q} = 0$, iff $X \in C_{\mathbb{R}}^{Uq}(Q)$.*

(iii) *$\Omega_1(X,\delta_1+\delta_2)_{L^q} \le \Omega_1(X,\delta_1)_{L^q} + \Omega_1(X,\delta_2)_{L^q}$, $\delta_1, \delta_2 > 0$.*

(iv) $\Omega_1(X, n\delta)_{L^q} \le n\Omega_1(X,\delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(v) $\Omega_1(X, \lambda\delta)_{L^q} \le \lceil\lambda\rceil\Omega_1(X,\delta)_{L^q} \le (\lambda+1)\Omega_1(X,\delta)_{L^q}$, $\lambda > 0$, $\delta > 0$, *where* $\lceil\cdot\rceil$ *is the ceiling of the number.*

(vi) $\Omega_1(X+Y, \delta)_{L^q} \le \Omega_1(X,\delta)_{L^q} + \Omega_1(Y,\delta)_{L^q}$, $\delta > 0$.

(vii) $\Omega_1(X, \cdot)_{L^q}$ *is continuous on* $\mathbb{R}_+$ *for* $X \in C_{\mathbb{R}}^{Uq}(Q)$.

Proof. (i) is obvious.

(ii) Clearly $\Omega_1(X,0)_{L^q} = 0$.

($\Rightarrow$) Let $\lim_{\delta\downarrow 0}\Omega_1(X,\delta)_{L^q} = 0$. Then $\forall\varepsilon > 0$, $\varepsilon^{1/q} > 0$ and $\exists\delta > 0$, $\Omega_1(X,\delta)_{L^q} \le \varepsilon^{1/q}$. I.e. for any $x, y \in Q$: $\|x-y\|_{\ell^1} \le \delta$ we get

$$\int_\Omega \|X(x,s) - X(y,s)|^q P(ds) \le \varepsilon.$$

That is $X \in C_{\mathbb{R}}^{Uq}(Q)$.

($\Leftarrow$) Let $x \in C_{\mathbb{R}}^{Uq}(Q)$. Then $\forall\varepsilon > 0$ $\exists\delta > 0$: whenever $\|x-y\|_{\ell^1} \le \delta$, $x, y \in Q$, it implies

$$\int_\Omega |X(x,s) - X(y,s)|^q P(ds) \le \varepsilon.$$

I.e. $\forall\varepsilon > 0$ $\exists\delta > 0$: $\Omega_1(X,\delta)_{L^q} \le \varepsilon^{1/q}$. That is $\Omega_1(X,\delta)_{L^q} \to 0$ as $\delta\downarrow 0$.

(iii) Let $x_1, x_2 \in Q$: $\|x_1 - x\|_{\ell^1} \le \delta_1 + \delta_2$. Set

$$x = \frac{\delta_2}{\delta_1+\delta_2}x_1 + \frac{\delta_1}{\delta_1+\delta_2}x_2,$$

clearly by convexity of Q we have that $x \in \overline{x_1x_2}$. Then easily we find that $\|x - x_1\|_{\ell^1} \le \delta_1$ and $\|x_2 - x\|_{\ell^1} \le \delta_2$. We have

$$\begin{aligned}
&\left(\int_\Omega |X(x_1,\omega) - X(x_2,\omega)|^q P(d\omega)\right)^{1/q} \\
&\le \left(\int_\Omega |X(x_1,\omega) - X(x,\omega)|^q P(d\omega)\right)^{1/q} + \left(\int_\Omega |X(x,\omega) - X(x_2,\omega)|^q P(d\omega)\right)^{1/q} \\
&\le \Omega_1\big(X, \|x_1 - x\|_{\ell^1}\big)_{L^q} + \Omega_1\big(X, \|x_2 - x\|_{\ell^1}\big)_{L^q} \le \Omega_1(X,\delta_1)_{L^q} + \Omega_1(X,\delta_2)_{L^q}.
\end{aligned}$$

Therefore (iii) is true.

(iv) and (v) are obvious.

(vi) Notice that

$$\begin{aligned}
&\left(\int_\Omega |(X(x,\omega) + Y(x,\omega)) - (X(y,\omega) + Y(y,\omega))|^q P(d\omega)\right)^{1/q} \\
&\le \left(\int_\Omega |X(x,\omega) - X(y,\omega)|^q P(d\omega)\right)^{1/q} + \left(\int_\Omega |Y(x,\omega) - Y(y,\omega)|^q P(d\omega)\right)^{1/q}.
\end{aligned}$$

That is (vi) is now clear.

(vii) By (iii) we obtain

$$\left|\Omega_1(X,\delta_1+\delta_2)_{L^q}-\Omega_1(X,\delta_1)_{L^q}\right|\le\Omega_1(X,\delta_2)_{L^q}.$$

Let now $X\in C_{\mathbb{R}}^{Uq}(Q)$, then by (ii) $\lim_{\delta_2\downarrow 0}\Omega_1(f,\delta_2)_{L^q}=0$. That is proving the continuity of $\Omega_1(X,\cdot)_{L^q}$ on $\mathbb{R}_+$. .

We make

Remark 22.5. By Proposition 22.4(v) we derive

$$\Omega_1\big(X,\|x-y\|_{\ell^1}\big)_{L^q}\le\left\lceil\frac{\|x-y\|_{\ell^1}}{\delta}\right\rceil\Omega_1(X,\delta)_{L^q}, \tag{22.3}$$

$\forall x,y\in Q$, any $\delta>0$.

22.3 Main Results

We introduce

Concepts 22.6. Let Q be a compact convex subset of $\mathbb{R}^k$, $k>1$ and let $\tilde{L}$ be a positive linear operator from $C(Q)$ into itself. Let $X(t,\omega)$ be a multivariate stochastic process from $Q\times(\Omega,\mathcal{B},P)$ into $\mathbb{R}$, where $(\Omega,\mathcal{B},P)$ is a probability space.

Here we suppose that $X(\cdot,\omega)\in C^n(Q)$, for each $\omega\in\Omega$, and that $X_\alpha(t,\cdot)=\frac{\partial^\alpha X}{\partial x^\alpha}(t,\cdot)$ is measurable for each $t\in Q$, for all $\alpha=(\alpha_1,\dots,\alpha_k)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,\dots,k$, $|\alpha|=\sum_{i=1}^k\alpha_i=\rho$, $0\le\rho\le n$, $n\ge 0$.

Define

$$M(X)(t,\omega):=\tilde{L}\big(X(\cdot,\omega)\big)(t),\quad\forall t\in Q,\ \ \forall\omega\in\Omega, \tag{22.4}$$

and assume that it is a random variable in ω. Clearly M is a positive linear operator on stochastic processes.

We make

Remark 22.7. By the Riesz representation theorem we have that there exists μ_t unique, completed Borel measure on Q with

$$m_t:=\mu_t(Q)=\tilde{L}(1)(t)\ge 0, \tag{22.5}$$

such that

$$\tilde{L}(f)(t)=\int_Q f(x)d\mu_t(x), \tag{22.6}$$

$\forall t\in Q$ and $\forall f\in C(Q)$. Consequently we have that

$$M(X)(t,\omega)=\int_Q X(x,\omega)d\mu_t(x), \tag{22.7}$$

$\forall(t,\omega) \in Q \times \Omega$, and X as above.

We make

Remark 22.8. Denote by

$$\Delta(t,\omega) := M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t) = \int_Q \big(X(s,\omega) - X(t,\omega)\big)\mu_t(ds), \tag{22.8}$$

and

$$|\Delta(t,\omega)| \le \int_Q |X(s,\omega) - X(t,\omega)|\mu_t(ds), \quad \forall(t,\omega) \in Q \times \Omega. \tag{22.9}$$

Therefore we have

$$\int_\Omega |\Delta(t,\omega)|P(d\omega) \le \int_\Omega \left(\int_Q |X(s,\omega) - X(t,\omega)|\mu_t(ds)\right)P(d\omega).$$

(By [9], p. 156 the function under integration is jointly measurable in (s,ω) (Q is a separable metric space). It is also nonnegative. Thus by Tonelli–Fubini theorem, [150], p. 104 we have

$$= \int_Q \left(\int_\Omega |X(s,\omega) - X(t,\omega)|P(d\omega)\right)\mu_t(ds). \tag{22.10}$$

Let $0 \le r \le 1$ and

$$G(r,s,\omega) := X(t + r(s-t),\omega), \quad n \in \mathbb{N}. \tag{22.11}$$

Then by Taylor's formula we obtain

$$X(s_1,\dots,s_k,\omega) = G(1,s,\omega) = \sum_{j=0}^{n} \frac{G^{(j)}(0,s,\omega)}{j!} + \mathcal{R}_n(0,s,\omega), \tag{22.12}$$

$$\mathcal{R}_n(0,s,\omega) := \int_0^1 \left(\int_0^{r_1} \cdots \left(\int_0^{r_{n-1}} \big(G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)\big)dr_n\right)\cdots\right)dr_1, \tag{22.13}$$

where

$$G^{(j)}(r,s,\omega) = \left[\left(\sum_{i=1}^{k}(s_i - t_i)\frac{\partial}{\partial x_i}\right)^j X\right]\big(t_1 + r(s_1 - t_1),\dots,t_k + r(s_k - t_k),\omega\big). \tag{22.14}$$

Thus

$$\left|X(s_1,\ldots,s_k,\omega)-X(t_1,\ldots,t_k,\omega)\right|\leq\sum_{j=1}^{n}\frac{|G^{(j)}(0,s,\omega)|}{j!}$$
$$+\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}\left|G^{(n)}(r_n,s,\omega)-G^{(n)}(0,s,\omega)\right|dr_n\right)\cdots dr_1\right). \tag{22.15}$$

Call $\varphi(r_n,\omega):=|G^{(n)}(r_n,s,\omega)-G^{(n)}(0,s,\omega)|\geq 0$. We notice that φ is continuous in $r_n\in[0,1]$ and measurable in $\omega\in\Omega$, therefore by [9], p. 156 is jointly measurable in (r_n,ω). Next $\int_0^{r_{n-1}}\varphi(r_n,\omega)dr_n$ is continuous in r_{n-1} and measurable in ω, thus jointly measurable in (r_{n-1},ω), etc., the same is true for the rest of the repeated integrals in the remainder of (22.15), also all are nonnegative. Hence we can apply Tonelli–Fubini theorem, [150], p. 104 to obtain:

$$\int_\Omega|X(s,\omega)-X(t,\omega)|P(d\omega)\leq\sum_{j=1}^{n}\frac{\int_\Omega|G^{(j)}(0,s,\omega)|P(d\omega)}{j!}+\Lambda, \tag{22.16}$$

where

$$\Lambda:=\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}\left(\int_\Omega|G^{(n)}(r_n,s,\omega)-G^{(n)}(0,s,\omega)|P(d\omega)\right)dr_n\cdots\right)dr_1. \tag{22.17}$$

We further derive

$$\int_\Omega|G^{(n)}(r_n,s,\omega)-G^{(n)}(0,s,\omega)|dP(\omega)$$
$$\leq\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{j=1}^{k}|s_j-t_j|^{\alpha_j}\right)\cdot\int_\Omega\left|X_\alpha(t+r_n(s-t),\omega)-X_\alpha(t,\omega)\right|dP(\omega) \tag{22.18}$$
$$\leq\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{j=1}^{k}|s_j-t_j|^{\alpha_j}\right)\cdot\Omega_1\left(X_\alpha,r_n\|s-t\|_{\ell^1}\right)_{L^1} \tag{22.19}$$

$$\overset{(22.3)}{\underset{(\text{let } h>0)}{\leq}} \sum_{|\alpha|=n} \frac{n!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{j=1}^{k}|s_j-t_j|^{\alpha_j}\right)$$

$$\cdot\,\Omega_1(X_\alpha,h)_{L^1}\left\lceil\frac{r_n\|s-t\|_{\ell^1}}{h}\right\rceil \tag{22.20}$$

$$\left(\text{Call } w:=\max_{\alpha:\ |\alpha|=n}\Omega_1(X_\alpha,h)_{L^1}.\right) \tag{22.21}$$

$$\leq \sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{j=1}^{k}|s_j-t_j|^{\alpha_j}\right) w\left\lceil\frac{r_n\|s-t\|_{\ell^1}}{h}\right\rceil. \tag{22.22}$$

That is we get

$$\int_\Omega\left|G^{(n)}(r_n,s,\omega)-G^{(n)}(0,s,\omega)\right|dP(\omega)$$

$$\leq\ w\left(\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{j=1}^{k}|s_j-t_j|^{\alpha_j}\right)\left\lceil\frac{r_n\|s-t\|_{\ell^1}}{h}\right\rceil\right), \tag{22.23}$$

where $0\leq r_n\leq 1$,

$$\|s-t\|_{\ell^1}=\sum_{i=1}^{k}|s_i-t_i|.$$

Therefore we find

$$\begin{aligned}\Lambda\quad &\leq\quad w\left\{\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k1}\left(\prod_{j=1}^{k}|s_j-t_j|^{\alpha_j}\right)\right.\\ &\qquad\left.\cdot\int_0^1\left(\int_0^{r_1}\cdots\left(\left\lceil\frac{r_n\|s-t\|_{\ell^1}}{h}\right\rceil dr_n\right)\cdots\right)dr_1\right\}\\ &\qquad\text{(by change of variable)}\\ &=\quad w\phi_n(\|s-t\|_{\ell^1}),\quad\text{for } s\neq t.\end{aligned} \tag{22.24}$$

Here we use

$$\phi_n(x):=\int_0^{|x|}\int_0^{x_1}\cdots\left(\int_0^{x_{n-1}}\left\lceil\frac{x_n}{h}\right\rceil dx_n\right)\cdots dx_1, \tag{22.25}$$

see [16], p. 210. Clearly it holds

$$\Lambda\leq w\phi_n(\|s-t\|_{\ell^1}),\quad\forall s,t\in Q. \tag{22.26}$$

We need

Assumption 22.9. Let $n\geq 0$.

Here we suppose that $X_\alpha(x,\omega)$, $|\alpha| = n$, is *continuous in* $x \in Q$, *uniformly with respect to* $\omega \in \Omega$. I.e. $\forall \varepsilon > 0\ \exists \delta > 0$: whenever $\|x - y\|_{\ell^1} \le \delta$; $x, y \in Q$, then

$$\left|X_\alpha(x,\omega) - X_\alpha(y,\omega)\right| \le \varepsilon, \quad \forall \omega \in \Omega.$$

We denote this by $X_\alpha \in C^U_{\mathbb{R}}(Q)$, *the space of continuous in* x, *uniformly with respect to* ω, *multivariate stochastic processes.*

Hence here $X\alpha(\cdot,\omega) \in C(Q)$, $\forall \omega \in \Omega$, and $X\alpha$ is q-mean uniformly continuous multivariate stochastic process over Q, that is $X\alpha \in C^{Uq}_{\mathbb{R}}(Q)$, for any $1 \le q < \infty$, α: $|\alpha| = n$. The last implies that $w < \infty$.

We also use

Definition 22.10. Denote by

$$(EX)(t) := \int_\Omega X(t,\omega)P(d\omega), \quad \forall t \in Q, \tag{22.27}$$

the expectation operator.

We need

Assumption 22.11. Here we assume $(E|X_\alpha|)(t) < \infty$, $\forall t \in Q$, all $\alpha{=}(\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, k$, $|\alpha| = \sum_{i=1}^k \alpha_i = \rho$, $0 \le \rho \le n$, $n \ge 0$.

Assumption 22.11 clearly implies that

$$\int_\Omega |G^{(j)}(0,s,\omega)|P(d\omega) < \infty, \quad j = 1, \dots, n, \quad \forall s, t \in Q.$$

We put together things into

Remark 22.12. Here all elements are as in Concepts 22.6, Assumptions 22.9 and 22.11, $n \ge 1$. We proved that

$$\begin{aligned}\int_\Omega |X(s,\omega) - X(t,\omega)|P(d\omega) \quad &\le \quad \sum_{j=1}^n \frac{\int_\Omega |G^{(j)}(0,s,\omega)|P(d\omega)}{j!} \\ &\quad + w\phi_n\left(\|s-t\|_{\ell^1}\right), \quad \forall s,t \in Q.\end{aligned} \tag{22.28}$$

Integrating (22.28) against μ_t and using Tonelli–Fubini's theorem we obtain

$$\begin{aligned}&\int_\Omega \left|M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)\right|P(d\omega) \\ &\quad \le \sum_{j=1}^n \frac{\int_\Omega \left(\int_Q |G^{(j)}(0,s,\omega)|\mu_t(ds)\right)P(d\omega)}{j!} \\ &\qquad + w\int_Q \phi_n\left(\|s-t\|_{\ell^1}\right)\mu_t(ds), \quad \forall t \in Q.\end{aligned} \tag{22.29}$$

We notice also that

$$|M(X)(t,\omega) - X(t,\omega)| \leq |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)| + |X(t,\omega)|\,|\tilde{L}(1)(t) - 1|. \tag{22.30}$$

We have established the following $\mathcal{L}_1$ result.

Theorem 22.13. *Here all elements are as in Concepts 22.6, Assumptions 22.9 and 22.11, $n \geq 1$. Then*

$$E\big(|M(X) - X|\big)(t) \leq (E(X))(t)|\tilde{L}(1)(t) - 1| + \sum_{j=1}^{n} \frac{\int_\Omega \big(\int_Q |G^{(j)}(0,s,\omega)|\mu_t(ds)\big) P(d\omega)}{j!} + w \int_Q \phi_n\big(\|s-t\|_{\ell^1}\big)\mu_t(ds), \quad \forall t \in Q. \tag{22.31}$$

Note 22.14. By Assumption 22.11 clearly

$$\int_\Omega \left(\int_Q |G^{(j)}(0,s,\omega)|\mu_t(ds) \right) P(d\omega) < \infty, \quad \forall t \in Q.$$

We make

Remark 22.15. From [16], p. 210, (7.1.18) we find

$$\phi_n(x) \leq \left(\frac{|x|^{n+1}}{(n+1)!h} + \frac{|x|^n}{2n!} + \frac{h|x|^{n-1}}{8(n-1)!} \right), \tag{22.32}$$

and from [16], p. 217, (7.2.9) we have

$$\phi_n(x) \leq \frac{|x|^n}{n!} \left(1 + \frac{|x|}{(n+1)h} \right), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{22.33}$$

Therefore we get

$$\int_Q \phi_n\big(\|s-t\|_{\ell^1}\big)\mu_t(ds) \leq \left(\frac{\int_Q \|s-t\|_{\ell^1}^{n+1}\mu_t(ds)}{(n+1)!h} + \frac{\int_Q \|s-t\|_{\ell^1}^{n}\mu_t(ds)}{2n!} + \frac{h\int_Q \|s-t\|_{\ell^1}^{n-1}\mu_t(ds)}{8(n-1)!} \right) \tag{22.34}$$

and

$$\int_Q \phi_n\big(\|s-t\|_{\ell^1}\big)\mu_t(ds) \leq \frac{1}{n!} \left(\int_Q \|s-t\|_{\ell^1}^{n}\mu_t(ds) + \frac{\int_Q \|s-t\|_{\ell^1}^{n+1}\mu_t(ds)}{(n+1)h} \right). \tag{22.35}$$

Using Hölder's inequality we have

$$\int_Q \|s-t\|_{\ell^1}^n d\mu_t(ds) \leq \left(\tilde{L}(1)(t)\right)^{\frac{1}{n+1}} \left(\int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds)\right)^{\frac{n}{n+1}}, \tag{22.36}$$

and

$$\int_Q \|s-t\|_{\ell^1}^{n-1} \mu_t(ds) \leq \left(\tilde{L}(1)(t)\right)^{\frac{2}{n+1}} \left(\int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds)\right)^{\frac{n-1}{n+1}}, \quad \forall n \in \mathbb{N}. \tag{22.37}$$

Consequently we obtain

$$\begin{aligned} \text{R.H.S.}(22.34) \quad \leq \quad & \left(\frac{\int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds)}{(n+1)!h} + \frac{\left(\tilde{L}(1)(t)\right)^{\frac{1}{n+1}} \left(\int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds)\right)^{\frac{n}{n+1}}}{2n!} \right. \\ & \left. + \frac{h\left(\tilde{L}(1)(t)\right)^{\frac{2}{n+1}} \left(\int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds)\right)^{\frac{n-1}{n+1}}}{8(n-1)!} \right). \end{aligned} \tag{22.38}$$

Choose and suppose

$$h := r \left(\frac{1}{\tilde{L}(1)(t)} \int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds) \right)^{\frac{1}{n+1}} > 0, \tag{22.39}$$

where $r > 0$ and $\tilde{L}(1)(t) > 0$. Hence

$$\text{R.H.S.}(22.38) = \frac{\tilde{L}(1)(t)}{r^{n+1} n!} h^n \left(\frac{nr^2}{8} + \frac{r}{2} + \frac{1}{n+1} \right). \tag{22.40}$$

So we get that

$$\int_Q \phi_n\left(\|s-t\|_{\ell^1}\right) \mu_t(ds) \leq \frac{\tilde{L}(1)(t)}{r^{n+1} n!} h^n \left(\frac{nr^2}{8} + \frac{r}{2} + \frac{1}{n+1} \right), \quad \forall n \in \mathbb{N}. \tag{22.41}$$

Furthermore we have for the choice of h as in (22.39) that

$$\begin{aligned} \text{R.H.S.}(22.35) \quad \leq \quad & \frac{1}{n!} \left(\left(\tilde{L}(1)(t)\right)^{\frac{1}{n+1}} \left(\int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds) \right)^{\frac{n}{n+1}} \right. \\ & \left. + \frac{\int_Q \|s-t\|_{\ell^1}^{n+1} \mu_t(ds)}{(n+1)h} \right) \\ = \quad & \frac{\tilde{L}(1)(t) h^n}{r^{n+1} n!} \left(r + \frac{1}{(n+1)} \right). \end{aligned} \tag{22.42}$$

That is we obtain

$$\int_Q \phi_n\left(\|s-t\|_{\ell^1}\right) \mu_t(ds) \leq \frac{\tilde{L}(1)(t) h^n}{r^{n+1} n!} \left(r + \frac{1}{n+1} \right), \quad \forall n \in \mathbb{N}. \tag{22.43}$$

We conclude that

$$\int_Q \phi_n\big(\|s-t\|_{\ell^1}\big)\mu_t(ds) \le \frac{\tilde{L}(1)(t)h^n}{r^{n+1}n!}\min\left(\left(\frac{nr^2}{8}+\frac{r}{2}+\frac{1}{n+1}\right),\left(r+\frac{1}{n+1}\right)\right), \quad \forall n \in \mathbb{N}. \tag{22.44}$$

Notice that

$$\min\left(\left(\frac{nr^2}{8}+\frac{r}{2}+\frac{1}{n+1}\right),\left(r+\frac{1}{n+1}\right)\right) = \begin{cases} \frac{nr^2}{8}+\frac{r}{2}+\frac{1}{n+1}, & \text{if } 0<r\le \frac{4}{n} \\ r+\frac{1}{n+1}, & \text{if } r>\frac{4}{n}. \end{cases} \tag{22.45}$$

When e.g. $r=1$ we get

$$\min\left(\left(\frac{n}{8}+\frac{1}{2}+\frac{1}{n+1}\right),\frac{n+2}{n+1}\right) = \begin{cases} \frac{n}{8}+\frac{1}{2}+\frac{1}{n+1}, & \text{if } n\le 4 \\ \frac{n+2}{n+1}, & \text{if } n>4. \end{cases} \tag{22.46}$$

We need

Remark 22.16. Here for $j=1,\ldots,n$ we have

$$|G^{(j)}(0,s,\omega)| = \left|\left[\left(\sum_{i=1}^k (s_i-t_i)\frac{\partial}{\partial x_i}\right)^j X\right](t,\omega)\right| \tag{22.47}$$

$$\le \sum_{\alpha:=(\alpha_1,\ldots,\alpha_k),\alpha_i\in\mathbb{Z}^+,i=1,\ldots,k,|\alpha|:=\sum_{i=1}^k \alpha_i=j} \frac{j!}{\prod_{i=1}^k \alpha_i!}\left(\prod_{i=1}^k |s_i-t_i|^{\alpha_i}\right)|X_\alpha(t,\omega)|.$$

Therefore we find

$$\int_Q |G^{(j)}(0,s,\omega)|\mu_t(ds) \le \sum_{\alpha=(\alpha_1,\ldots,\alpha_k),\alpha_i\in\mathbb{Z}^+,i=1,\ldots,k,|\alpha|=\sum_{i=1}^k \alpha_i=j} \frac{j!}{\prod_{i=1}^k \alpha_i!} \cdot\left(\int_Q\left(\prod_{i=1}^k |s_i-t_i|^{\alpha_i}\right)\mu_t(ds)\right)|X_\alpha(t,\omega)|. \tag{22.48}$$

Consequently we derive

$$\int_\Omega \left(\int_Q |G^{(j)}(0,s,\omega)| \mu_t(ds) \right) P(d\omega) \leq \sum_{\alpha=(\alpha_1,\ldots,\alpha_k),\alpha_i \in \mathbb{Z}^+, i=1,\ldots,k, |\alpha|=\sum_{i=1}^k \alpha_i = j}$$

$$\cdot \frac{j!}{\prod_{i=1}^k \alpha_i!} \left(\int_Q \left(\prod_{i=1}^k |s_i - t_i|^{\alpha_i} \right) \mu_t(ds) \right) (E|X_\alpha|)(t). \tag{22.49}$$

From Theorem 22.13 and Remarks 22.15 and 22.16 we conclude the general result.

Theorem 22.17. *Here all elements are as in Concepts 22.6, Assumptions 22.9 and 22.11, $n \geq 1$, $r > 0$, $L(1)(t) > 0$, $t \in Q$. Then*

$$E\big(|M(X) - X|\big)(t) \leq (E|X|)(t) |\tilde{L}(1)(t) - 1|$$

$$+ \sum_{j=1}^n \left\{ \sum_{\alpha=(\alpha_1,\ldots,\alpha_k),\alpha_i \in \mathbb{Z}^+, i=1,\ldots,k, |\alpha|=\sum_{i=1}^k \alpha_i = j} \frac{(E|X_\alpha|(t))}{\prod_{i=1}^k \alpha_i!} \tilde{L}\left(\prod_{i=1}^k |\cdot - t_i|^{\alpha_i} \right)(t) \right\}$$

$$+ \frac{(\tilde{L}(1)(t))^{\frac{1}{n+1}}}{rn!} \left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{n+1})(t) \right)^{\frac{n}{n+1}} \min\left(\frac{nr^2}{8} + \frac{r}{2} + \frac{1}{n+1}, r + \frac{1}{n+1} \right)$$

$$\cdot \left\{ \max_{\alpha:\, |\alpha|=n} \Omega_1 \left(X_\alpha, \frac{r}{(\tilde{L}(1)(t))^{\frac{1}{n+1}}} \left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{n+1})(t) \right)^{\frac{1}{n+1}} \right)_{L^1} \right\}. \tag{22.50}$$

Note 22.18. If

$$\tilde{L}\big(\|\cdot - t\|_{\ell^1}^{n+1}\big)(t) = \int_Q \|s - t\|_{\ell^1}^{n+1} \mu_t(ds) = 0$$

then μ_t takes all of its mass $\tilde{L}(1)(t)$ at $\{t\}$, elsewhere is zero. In that case $M(X)(t,\omega) = X(t,\omega)\tilde{L}(1)(t)$ and

$$|M(X)(t,\omega) - X(t,\omega)| = |X(t,\omega)|\,|\tilde{L}(1)(t) - 1|,$$

and

$$E\big(|M(X) - X|\big)(t) = (E|X|)(t)|\tilde{L}(1)(t) - 1|.$$

That is proving (22.50) trivially true.

A further general global conclusion follows.

Theorem 22.19. *Here all elements are as in Concepts 22.6, Assumptions 22.9 and 22.11, $n \geq 1$, $r > 0$, $\tilde{L}(1)(t) > 0$, $\forall t \in Q$. Then*

$$\|E(|M(X) - X|)\|_\infty \leq \|E|X|\|_\infty \|\tilde{L}(1) - 1\|_\infty$$

$$+ \sum_{j=1}^{n} \left\{ \sum_{\alpha=(\alpha_1,\ldots,\alpha_k),\alpha_i \in \mathbb{Z}^+, i=1,\ldots,k, |\alpha|=\sum_{i=1}^k \alpha_i = j} \frac{\|E|X_\alpha|\|_\infty}{\prod_{i=1}^k \alpha_i!} \left\| \tilde{L}\left(\prod_{i=1}^k |\cdot - t_i|^{\alpha_i} \right)(t) \right\|_\infty \right\}$$

$$+ \frac{\|\tilde{L}(1)\|_\infty^{\frac{1}{n+1}}}{rn!} \left\| \tilde{L}(\|\cdot - t\|_{\ell^1}^{n+1})(t) \right\|_\infty^{\frac{n}{n+1}}$$
$$\times \min\left(\frac{nr^2}{8} + \frac{r}{2} + \frac{1}{n+1}, r + \frac{1}{n+1} \right) \left\{ \max_{\alpha:\, |\alpha|=n} \Omega_1 \left(X_\alpha, \right.\right.$$
$$\left.\left. \frac{r}{\left(\inf_{t \in Q} \tilde{L}(1)\right)^{\frac{1}{n+1}}} \left\| \tilde{L}(\|\cdot - t\|_{\ell^1}^{n+1})(t) \right\|_\infty^{\frac{1}{n+1}} \right)_{L^1} \right\}. \tag{22.51}$$

For our related convergence result we give

Remark 22.20. Here we choose and suppose momentarily that

$$h := \left(\int_Q \|s - t\|_{\ell^1}^{n+1} \mu_t(ds) \right)^{\frac{1}{n+1}} > 0. \tag{22.52}$$

Then from (22.35) and inequality (22.42) we get

$$\int_Q \phi_n\left(\|s - t\|_{\ell^1}\right) \mu_t(ds) \leq \frac{h^n}{n!} \left((\tilde{L}(1)(t))^{\frac{1}{n+1}} + \frac{1}{n+1} \right). \tag{22.53}$$

Consequently reasoning as before we give the general multivariate Shisha–Mond type inequality, see [264].

Theorem 22.21. *Here all elements are as in Concepts 22.6, Assumptions 22.9 and 22.11, $n \geq 1$. Then*

i)

$$E\left(|M(X) - X|\right)(t) \leq (E|X|)(t) |\tilde{L}(1)(t) - 1|$$
$$+ \sum_{j=1}^{n} \left\{ \sum_{\alpha=(\alpha_1,\ldots,\alpha_k),\alpha_i \in \mathbb{Z}^+, i=1,\ldots,k, |\alpha|=\sum_{i=1}^k \alpha_i = j} \frac{(E|X_\alpha|(t))}{\prod_{i=1}^k \alpha_i!} \tilde{L}\left(\prod_{i=1}^k |\cdot - t_i|^{\alpha_i} \right)(t) \right\}$$
$$+ \frac{\left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{n+1})(t)\right)^{\frac{n}{n+1}}}{n!} \left((\tilde{L}(1)(t))^{\frac{1}{n+1}} + \frac{1}{n+1} \right) \left\{ \max_{\alpha:\, |\alpha|=n} \Omega_1\left(X_\alpha, \right.\right.$$
$$\left.\left. \left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{n+1})(t)\right)^{\frac{1}{n+1}}\right)_{L^1} \right\}, \quad \forall t \in Q. \tag{22.54}$$

Also it holds

ii)

$$\begin{aligned}&\big\|E\big(|M(X)-X|\big)\big\|_\infty \le \|E|X|\|_\infty \|\tilde{L}(1)(t)-1\|_\infty \\ &+\sum_{j=1}^{n}\left\{\sum_{\alpha=(\alpha_1,\ldots,\alpha_k),\alpha_i\in\mathbb{Z}^+,i=1,\ldots,k,|\alpha|=\sum_{i=1}^k\alpha_i=j}\frac{\|E|X_\alpha|\|_\infty}{\prod_{i=1}^k\alpha_i!}\left\|\tilde{L}\Big(\prod_{i=1}^k|\cdot-t_i|^{\alpha_i}\Big)(t)\right\|_\infty\right\} \\ &+\frac{\big\|\tilde{L}(\|\cdot-t\|_{\ell^1}^{n+1})(t)\big\|_\infty^{\frac{n}{n+1}}}{n!}\left\|(\tilde{L}(1))^{\frac{1}{n+1}}+\frac{1}{n+1}\right\|_\infty \\ &\times\left\{\max_{\alpha:\,|\alpha|=n}\Omega_1\big(X_\alpha,\big\|\tilde{L}(\|\cdot-t\|_{\ell^1}^{n+1})(t)\big\|_\infty^{\frac{1}{n+1}}\big)_{L^1}\right\}. \qquad (22.55)\end{aligned}$$

The following Korovkin type theorem (see [213]) for multivariate stochastic processes in our general setting is valid.

Theorem 22.22. *Let $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ be a sequence of positive linear operators and the induced sequence of positive linear operators $\{M_N\}_{N\in\mathbb{N}}$ on multivariate stochastic processes, all as in Concepts 22.6, Assumptions 22.9 and 22.11, $n\ge 1$. Additionally assume that $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ is bounded and $\|\tilde{L}_N(\|\cdot-t\|_{\ell^1}^{n+1})(t)\|_\infty\to 0$, along with $\tilde{L}_N 1\xrightarrow{u}1$, as $N\to\infty$. Then*

$$\big\|E(|M_N(X)-X|)\big\|_\infty\to 0,\quad \text{as } N\to\infty,$$

for all X as in Concepts 22.6 and Assumptions 22.9, 22.11, $n\ge 1$. I.e. $M_N \underset{N\to\infty}{\overset{\text{“1-mean”}}{\longrightarrow}} I$ unit operator with rates.

Proof. By Theorem 22.21(ii), inequality (22.55), and the fact

$$\begin{aligned}&\big\|\tilde{L}_N(\|\cdot-t\|_{\ell^1}^{j})(t)\big\|_\infty \\ &\quad\le \big\|\tilde{L}_N(1)\big\|_\infty^{1-\frac{j}{n+1}}\big\|\tilde{L}_N(\|\cdot-t\|_{\ell^1}^{n+1})(t)\big\|_\infty^{\frac{j}{(n+1)}}, \qquad (22.56)\end{aligned}$$

for $j=1,\ldots,n$. Also we use

$$\left\|\tilde{L}_N\left(\prod_{i=1}^k|\cdot-t_i|^{\alpha_i}\right)(t)\right\|_\infty\le\frac{\prod_{i=1}^k\alpha_i!}{j!}\big\|\tilde{L}_N(\|\cdot-t\|_{\ell^1}^{j})(t)\big\|_\infty, \qquad (22.57)$$

$\forall\alpha=(\alpha_1,\ldots,\alpha_k)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,\ldots,k$; $|\alpha|=\sum_{i=1}^k\alpha_i=j$, for $j=1,\ldots,n$. ■

We need for Theorem 22.29 later, etc.

Assumption 22.23. We assume that

$$\big(E|X_\alpha|^q\big)(t) < \infty, \quad \forall t \in Q,$$

all $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$, $|\alpha| = \sum_{i=1}^{k} \alpha_i = \rho$, $0 \le \rho \le n$, $n \ge 0$, $1 < q < \infty$.

Next we treat case of $1 < q < \infty$, $n > 1$.

We make

Remark 22.24. By (22.8) we have

$$|\Delta(t,\omega)|^q \le \left(\int_Q |X(s,\omega) - X(t,\omega)| \mu_t(ds) \right)^q, \tag{22.58}$$

and

$$\left(\int_\Omega |\Delta(t,\omega)|^q P(d\omega) \right)^{\frac{1}{q}} \le \left(\int_\Omega \left(\int_Q |X(s,\omega) - X(t,\omega)| \mu_t(ds) \right)^q P(d\omega) \right)^{\frac{1}{q}}. \tag{22.59}$$

By (22.15) we obtain

$$\int_Q |X(s,\omega) - X(t,\omega)| \mu_t(ds) \le \sum_{j=1}^{n} \frac{\int_Q |G^{(j)}(0,s,\omega)| \mu_t(ds)}{j!} + \int_Q \left(\int_0^1 \left(\int_0^{r_1} \cdots \left(\int_0^{r_{n-1}} \left| G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega) \right| dr_n \right) \cdots \right) dr_1 \right) \mu_t(ds). \tag{22.60}$$

Thus

$$\left(\int_\Omega \left(\int_Q |X(s,\omega) - X(t,\omega)| \mu_t(ds) \right)^q P(d\omega) \right)^{\frac{1}{q}} \le \sum_{j=1}^{n} \frac{1}{j!} \left(\int_\Omega \left(\int_Q |G^{(j)}(0,s,\omega)| \mu_t(ds) \right)^q P(d\omega) \right)^{\frac{1}{q}} + K, \tag{22.61}$$

where

$$K := \left(\int_\Omega \left\{ \int_Q \left(\int_0^1 \left(\int_0^{r_1} \cdots \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)| dr_n \right) \cdots \right) dr_1 \right) \mu_t(ds) \right\}^q P(d\omega) \right)^{\frac{1}{q}}. \tag{22.62}$$

We notice the following by Hölder's inequality

$$\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)| dr_n$$
$$\leq r_{n-1}^{1-\frac{1}{q}} \left(\int_0^{r_{n-1}} G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q dr_n \right)^{\frac{1}{q}}, \quad (22.63)$$

and

$$\int_0^{r_{n-2}} \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)| dr_n \right) dr_{n-1}$$
$$\leq \int_0^{r_{n-2}} r_{n-1}^{1-\frac{1}{q}} \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q dr_n \right)^{\frac{1}{q}} dr_{n-1}$$
$$\leq \left(\frac{r_{n-2}^2}{2} \right)^{1-\frac{1}{q}} \left(\int_0^{r_{n-2}} \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) \right.\right.$$
$$\left.\left. - G^{(n)}(0,s,\omega)|^q dr_n \right) dr_{n-1} \right)^{\frac{1}{q}}. \quad (22.64)$$

Similarly by Hölder's inequality we derive

$$\int_0^{r_{n-3}} \left(\int_0^{r_{n-2}} \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)| dr_n \right) dr_{n-1} \right) dr_{n-2}$$
$$\leq \left(\frac{r_{n-3}^3}{3!} \right)^{1-\frac{1}{q}} \left(\int_0^{r_{n-3}} \left(\int_0^{r_{n-2}} \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) \right.\right.\right.$$
$$\left.\left.\left. - G^{(n)}(0,s,\omega)|^q dr_n \right) dr_{n-1} \right) dr_{n-2} \right)^{\frac{1}{q}}, \quad (22.65)$$

and finally

$$\int_0^{r_1} \left(\int_0^{r_2} \left(\int_0^{r_3} \cdots \left(\int_0^{r_{n-3}} \left(\int_0^{r_{n-2}} \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) \right.\right.\right.\right.\right.$$
$$\left.\left.\left. - G^{(n)}(0,s,\omega)| dr_n \right) dr_{n-1} \right) dr_{n-2} \right) \cdots dr_2$$
$$\leq \left(\frac{r_1^{n-1}}{(n-1)!} \right)^{1-\frac{1}{q}} \left(\int_0^{r_1} \cdots \left(\int_0^{r_{n-1}} |G^{(n)}(r_n,s,\omega) \right.\right.$$
$$\left.\left. - G^{(n)}(0,s,\omega)|^q dr_n \right) \cdots dr_2 \right)^{\frac{1}{q}}. \quad (22.66)$$

Consequently we get

$$\int_Q\left(\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}|G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|dr_n\right)dr_{n-1}\right)\cdots\right)dr_1\right)\mu_t(ds) \leq \frac{1}{(n!)^{1-\frac{1}{q}}}\left(\int_Q\left(\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}|G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q dr_n\right)\cdots\right)dr_1\right)^{\frac{1}{q}}\mu_t(ds)\right), \tag{22.67}$$

and

$$\left[\int_Q\left(\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}|G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|dr_n\right)dr_{n-1}\right)\cdots\right)dr_1\right)\mu_t(ds)\right]^q \leq \left(\frac{\tilde{L}(1)(t)}{n!}\right)^{q-1}\left[\int_Q\left(\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}|G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q dr_n\right)\cdots\right)dr_1\right)\mu_t(ds)\right]. \tag{22.68}$$

Therefore we find

$$K \leq Z \tag{22.69}$$

where

$$Z := c\left[\int_\Omega\left(\int_Q\left(\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}|G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q dr_n\right)\cdots\right)dr_2\right)dr_1\right)\mu_t(ds)\right)P(d\omega)\right]^{\frac{1}{q}}, \tag{22.70}$$

with

$$c := \left(\frac{\tilde{L}(1)(t)}{n!}\right)^{1-\frac{1}{q}}. \tag{22.71}$$

Clearly here, see [9], p. 156, etc.,

$$F(r_1,s,\omega) := \int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}|G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q dr_n\right)\cdots\right)dr_2 \tag{22.72}$$

is jointly measurable in (r_1, ω) and nonnegative. The same is true for all other similar functions building F.

Next we treat

$$I := \int_\Omega \left(\int_Q \mathcal{X}(s,\omega)\mu_t(ds) \right) P(d\omega), \tag{22.73}$$

where

$$\mathcal{X}(s,\omega) := \int_0^1 F(r_1, s, \omega) dr_1 \tag{22.74}$$

is measurable in $\omega \in \Omega$. We will prove that $\mathcal{X}(s,\omega)$ is continuous in $s \in Q$.

We notice the following.

Here $|G^{(n)}(r_n, s, \omega) - G^{(n)}(0, s, \omega)|^q$ is seen easily to be jointly continuous in $(r_n, s) \in [0,1] \times Q$. Also, by Lemma 22.25 next, the function

$$\Gamma(r_{n-1}, s, \omega) := \int_0^{r_{n-1}} |G^{(n)}(r_n, s, \omega) - G^{(n)}(0, s, \omega)|^q dr_n \tag{22.75}$$

is continuous in $s \in Q$, $\forall \omega \in \Omega$. Of course Γ is continuous in $r_{n-1} \in [0,1]$ and measurable in ω.

By Lemma 22.26 next, Γ is jointly continuous in (r_{n-1}, s).

We need

Lemma 22.25. *Let $\varphi(r,s)$ jointly continuous in $(r,s) \in [0,1] \times Q$. Then*

$$\gamma(s) := \int_0^\tau \varphi(r,s) dr, \quad \tau \in [0,1] \tag{22.76}$$

is continuous in $s \in Q$.

Proof. We have valid that $\forall \varepsilon > 0$ $\exists \delta > 0$: whenever $\|(r_1, s_1) - (r_2, s_2)\|_{\ell^1} \leq \delta$, for (r_1, s_1), $(r_2, s_2) \in [0,1] \times Q$, then $|\varphi(r_1, s_1) - \varphi(r_2, s_2)| \leq \varepsilon$. Hence for the same ε, δ we observe that

$$\begin{aligned} |\gamma(s_1,\omega) - \gamma(s_2,\omega)| &\leq \int_0^\tau |\varphi(r, s_1) - \varphi(r, s_2)| dr \qquad (22.77) \\ &\leq \varepsilon\tau, \quad \text{whenever } \|(r, s_1) - (r, s_2)\|_{\ell^1} \leq \delta, \quad \text{any } r \in [0,1]. \end{aligned}$$

That is proving $\gamma(s,\omega)$ is continuous in $s \in Q$. ∎

Also we need

Lemma 22.26. *Let φ be jointly continuous in $[0,1] \times Q$. Then*

$$\eta(r,s) := \int_0^r \varphi(\theta, s) d\theta, \tag{22.78}$$

is jointly continuous in $(r,s) \in [0,1] \times Q$.

Proof. Here η is continuous in $r \in [0,1]$, and by Lemma 22.25 is continuous in $s \in Q$. Also $\exists M > 0$ such that $\|\varphi\|_\infty \le M$. Let $r_n \to r$, $s_n \to s$, then we see that

$$\begin{aligned}
|\eta(r_n, s_n) - \eta(r,s)| &= \left| \int_0^{r_n} \varphi(\theta, s_n) d\theta - \int_0^r \varphi(\theta, s) d\theta \right| \\
&= \left| \int_0^{r_n} \varphi(\theta, s_n) d\theta - \int_0^r \varphi(\theta, s_n) d\theta \right. \\
&\quad \left. + \int_0^r \varphi(\theta, s_n) d\theta - \int_0^r \varphi(\theta, s) d\theta \right| \\
&\le A_n + B_n,
\end{aligned}$$

where

$$A_n := \left| \int_0^{r_n} \varphi(\theta, s_n) d\theta - \int_0^r \varphi(\theta, s_n) d\theta \right|,$$

and

$$B_n := \int_0^r |\varphi(\theta, s_n) - \varphi(\theta, s)| d\theta.$$

We have always that

$$\begin{aligned}
\left| \int_0^{r_n} \varphi(\theta, s_n) d\theta - \int_0^r \varphi(\theta, s_n) d\theta \right| &\le \int_r^{r_n} |\varphi(\theta, s_n)| d\theta \\
&\le M|r_n - r| \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

I.e. $A_n \to 0$, as $n \to \infty$.

Next we observe that

$$|\varphi(\theta, s_n) - \varphi(\theta, s)| \le 2M < \infty.$$

Also, by continuity

$$|\varphi(\theta, s_n) - \varphi(\theta, s)| \to 0, \quad \text{as } n \to \infty, \quad \text{for any } \theta \in [0, r].$$

Thus, by Dominated convergence theorem, we get $B_n \to 0$, as $n \to \infty$. The claim has been established. ∎

We make

Remark 22.27. We are continuing from Remark 22.24. So by using Lemma 22.26 repeatedly, we conclude that $F(r_1, s, \omega)$ is jointly continuous in $(r_1, s) \in [0,1] \times Q$ and measurable in $\omega \in \Omega$. Finally, we have by Lemma 22.25, that the function $\mathcal{X}(s, \omega) \ge 0$ is continuous in $s \in Q$ and measurable in ω, therefore by [9], p. 156, is jointly measurable in (s, ω).

Hence by Tonelli–Fubini's theorem, [150], p. 104 we have

$$I = \int_Q \left(\int_\Omega \mathcal{X}(s, \omega) P(d\omega) \right) \mu_t(ds). \tag{22.79}$$

Again, by applying n times Tonelli–Fubini theorem, we get

$$Z = c\Bigg(\int_Q\Bigg(\int_0^1\Bigg(\int_0^{r_1}\cdots\Bigg(\int_0^{r_{n-1}}\Bigg(\int_\Omega |G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q P(d\omega)\Bigg)dr_n\Bigg)\cdots\Bigg)dr_1\Bigg)\mu_t(ds)\Bigg)^{1/q}, \ \infty > q > 1. \tag{22.80}$$

Notice that

$$\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!} = k^n \tag{22.81}$$

and use next that x^q, $x \geq 0$ is convex.

We see that

$$\int_\Omega |G^{(n)}(r_n,s,\omega) - G^{(n)}(0,s,\omega)|^q P(d\omega)$$
$$\leq \int_\Omega\Bigg\{\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\Bigg(\prod_{j=1}^k |s_j-t_j|^{\alpha_j}\Bigg) \times |X_\alpha(t+r_n(s-t),\omega) - X_\alpha(t,\omega)|\Bigg\}^q P(d\omega) \tag{22.82}$$
$$\leq k^{n(q-1)}\Bigg\{\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\Bigg(\prod_{j=1}^k |s_j-t_j|^{q\alpha_j}\Bigg) \times\Bigg(\int_\Omega |X_\alpha(t+r_n(s-t),\omega) - X_\alpha(t,\omega)|^q P(d\omega)\Bigg)\Bigg\} \tag{22.83}$$
$$\leq k^{n(q-1)}\Bigg\{\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\Bigg(\prod_{j=1}^k |s_j-t_j|^{q\alpha_j}\Bigg) \times \Omega_1^q\big(X_\alpha, r_n\|s-t\|_{\ell^1}\big)_{L^q}\Bigg\} \tag{22.84}$$

(let $h > 0$)

$$\leq k^{n(q-1)}\Bigg\{\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\Bigg(\prod_{j=1}^k |s_j-t_j|^{q\alpha_j}\Bigg) \times\left\lceil\frac{r_n\|s-t\|_{\ell^1}}{h}\right\rceil^q \Omega_1^q(X_\alpha,h)_{L^q}\Bigg\}. \tag{22.85}$$

(Put

$$w := \max_{\alpha:\ |\alpha|=n} \Omega_1(X_\alpha, h)_{L^q}), \tag{22.86}$$

$$\leq w^q k^{n(q-1)} \left\{ \sum_{|\alpha|=n} \frac{n!}{\alpha! \cdots \alpha_k!} \left(\prod_{j=1}^{k} |s_j - t_j|^{q\alpha_j} \right) \right\} \times \left\lceil \frac{r_n \|s-t\|_{\ell^1}}{h} \right\rceil^q. \tag{22.87}$$

So we get that

$$\int_\Omega |G^{(n)}(r_n, s, \omega) - G^{(n)}(0, s, \omega)|^q P(d\omega)$$
$$\leq w^q k^{n(q-1)} \left\{ \sum_{|\alpha|=n} \frac{n!}{\alpha_1! \cdots \alpha_k!} \left(\prod_{j=1}^{k} |s_j - t_j|^{q\alpha_j} \right) \right\} \times \left\lceil \frac{r_n \|s-t\|_{\ell^1}}{h} \right\rceil^q, \quad 1 < q < \infty,\ h > 0,\ 0 \leq r_n \leq 1. \tag{22.88}$$

Thus we conclude that

$$\left(\int_0^1 \left(\int_0^{r_1} \cdots \left(\int_0^{r_{n-1}} \left(\int_\Omega |G^{(n)}(r_n, s, \omega) - G^{(n)}(0, s, \omega)|^q P(d\omega) \right) dr_n \right) \cdots \right) dr_1 \right)$$
$$\leq w^q k^{n(q-1)} \left\{ \sum_{|\alpha|=n} \frac{n!}{\alpha_1! \cdots \alpha_k!} \left(\prod_{j=1}^{k} (|s_j - t_j|^q)^{\alpha_j} \right) \right\} \tag{22.89}$$
$$\times \left(\int_0^1 \left(\int_0^{r_1} \cdots \left(\int_0^{r_{n-1}} \left\lceil \frac{r_n \|s-t\|_{\ell^1}}{h} \right\rceil^q dr_n \right) \cdots \right) dr_1 \right).$$

Then

$$\text{R.H.S.}(22.89) \leq w^q k^{n(q-1)} \left\{ \sum_{|\alpha|=n} \frac{n!}{\alpha_1! \cdots \alpha_k!} \times \left(\prod_{j=1}^{k} (|s_j - t_j|^q)^{\alpha_j} \right) \right\} \left(\int_0^1 \left(\int_0^{r_1} \times \cdots \left(\int_0^{r_{n-1}} \left(1 + \frac{r_n \|s-t\|_{\ell^1}}{h} \right)^q dr_n \right) \cdots \right) dr_1 \right) \tag{22.90}$$

$$\overset{(s\neq t\text{ case})}{=} \frac{w^q k^{n(q-1)}}{\|s-t\|_{\ell^1}^n}\left\{\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{j=1}^{k}(|s_j-t_j|^q)^{\alpha_j}\right)\right\} \tag{22.91}$$
$$\times\left(\int_0^{\|s-t\|_{\ell^1}}\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}\left(1+\frac{r_n}{h}\right)^q dr_n\right)\cdots\right)dr_1\right)$$

$$\leq \frac{w^q k^{n(q-1)}2^{q-1}}{\|s-t\|_{\ell^1}^n}\left[\left(\sum_{j=1}^{k}(|s_j-t_j|^q)\right)^n\right] \tag{22.92}$$
$$\times\left(\int_0^{\|s-t\|_{\ell^1}}\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}\left(1+\frac{r_n^q}{h^q}\right)dr_n\right)\cdots\right)dr_1\right)$$
$$\leq w^q k^{n(q-1)}2^{q-1}\|s-t\|_{\ell^1}^{n(q-1)}\left\{\frac{\|s-t\|_{\ell^1}^n}{n!}\right.$$
$$\left.+\frac{1}{h^q}\frac{\|s-t\|_{\ell^1}^{q+n}}{(q+1)\cdots(q+n)}\right\}. \tag{22.93}$$

That is we get that $(1<q<\infty)$

$$\left(\int_0^1\left(\int_0^{r_1}\cdots\left(\int_0^{r_{n-1}}\left(\int_\Omega |G^{(n)}(r_n,s,\omega)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-G^{(n)}(0,s,\omega)|^q P(d\omega)\right)dr_n\right)\cdots\right)dr_1\right)$$
$$\leq w^q k^{n(q-1)}2^{q-1}\|s-t\|_{\ell^1}^{n(q-1)}\left\{\frac{\|s-t\|_{\ell^1}^n}{n!}\right.$$
$$\left.+\frac{1}{h^q}\frac{\|s-t\|_{\ell^1}^{q+n}}{(q+1)\cdots(q+n)}\right\},\quad \forall s,t\in Q \tag{22.94}$$

(trivially true when $s=t$).

Therefore using (22.80) we obtain

$$Z \leq cwk^{n\left(1-\frac{1}{q}\right)}2^{1-\frac{1}{q}}\left[\int_Q\left(\frac{\|s-t\|_{\ell^1}^{nq}}{n!}\right.\right.$$
$$\left.\left.+\frac{1}{h^q}\frac{\|s-t\|_{\ell^1}^{(n+1)q}}{(q+1)\cdots(q+n)}\right)\mu_t(ds)\right]^{1/q}. \tag{22.95}$$

Consequently

$$\begin{aligned} K &\leq \left(\frac{\tilde{L}(1)(t)}{n!}\right)^{1-\frac{1}{q}} w k^{n\left(1-\frac{1}{q}\right)} 2^{1-\frac{1}{q}} \left[\frac{1}{n!}\int_Q \|s-t\|_{\ell^1}^{nq}\mu_t(ds)\right. \\ &\quad \left. + \frac{1}{h^q(q+1)\cdots(q+n)}\int_Q \|s-t\|_{\ell^1}^{(n+1)q}\mu_t(ds)\right]^{1/q} \\ &\leq \left(\frac{2k^n\tilde{L}(1)(t)}{n!}\right)^{1-\frac{1}{q}} w \left[\frac{(\tilde{L}(q)(t))^{\frac{1}{n+1}}}{n!}\left(\int_Q \|s-t\|_{\ell^1}^{(n+1)q}\mu_t(ds)\right)^{\frac{n}{n+1}}\right. \\ &\quad \left. + \frac{\int_Q \|s-t\|_{\ell^1}^{(n+1)q}\mu_t(ds)}{h^q(q+1)\cdots(q+n)}\right]^{1/q}. \qquad (22.96) \end{aligned}$$

(Call and assume momentarily

$$\begin{aligned} h :=& \quad \left(\int_Q \|s-t\|_{\ell^1}^{(n+1)q}\mu_t(ds)\right)^{\frac{1}{(n+1)q}} > 0.) \qquad (22.97) \\ =& \quad \left(\frac{2k^n\tilde{L}(1)(t)}{n!}\right)^{1-\frac{1}{q}} w \left[\frac{(\tilde{L}(q)(t))^{\frac{1}{n+1}}}{n!}h^{nq}\right. \\ &\quad \left. + \frac{h^{(n+1)q}}{h^q(q+1)\cdots(q+n)}\right]^{1/q} \qquad (22.98) \end{aligned}$$

$$= \left(\frac{2k^n\tilde{L}(1)(t)}{n!}\right)^{1-\frac{1}{q}} w h^n \left[\frac{(\tilde{L}(1)(t))^{\frac{1}{n+1}}}{n!} + \frac{1}{(q+1)\cdots(q+n)}\right]^{1/q}. \qquad (22.99)$$

I.e. we have that

$$\begin{aligned} K &\leq \left(\frac{2k^n\tilde{L}(1)(t)}{n!}\right)^{1-\frac{1}{q}} w h^n \left[\frac{(\tilde{L}(1)(t))^{\frac{1}{n+1}}}{n!}\right. \\ &\quad \left. + \frac{1}{(q+1)\cdots(q+n)}\right]^{1/q}, \quad n \in \mathbb{N}. \qquad (22.100) \end{aligned}$$

We continue with

Remark 22.28. We have by Hölder's inequality that $(1 < q < \infty)$

$$\begin{aligned} &\left(\int_Q |G^{(j)}(0,s,\omega)|\mu_t(ds)\right)^q \\ &\qquad \leq \left(\tilde{L}(1)(t)\right)^{q-1}\left(\int_Q |G^{(j)}(0,s,\omega)|^q\mu_t(ds)\right), \qquad (22.101) \end{aligned}$$

and

$$\left(\int_\Omega\left(\int_Q |G^{(j)}(0,s,\omega)|\mu_t(ds)\right)^q P(d\omega)\right)^{\frac{1}{q}}$$

$$\leq \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}}\left(\int_\Omega\left(\int_Q |G^{(j)}(0,s,\omega)|^q\mu_t(ds)\right)P(d\omega)\right)^{\frac{1}{q}} \tag{22.102}$$

(by Tonelli–Fubini's theorem)

$$= \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}}\left(\int_Q\left(\int_\Omega |G^{(j)}(0,s,\omega)|^q P(d\omega)\right)\mu_t(ds)\right)^{\frac{1}{q}}. \tag{22.103}$$

We do have again

$$|G^{(j)}(0,s,\omega)| \leq \sum_{|\alpha|=j}\frac{j!}{\alpha_1!\cdots\alpha_k!}\prod_{i=1}^k |s_i-t_i|^{\alpha_i}|X_\alpha(t,\omega)|. \tag{22.104}$$

Furthermore,

$$|G^{(j)}(0,s,\omega)|^q \leq \left\{\sum_{|\alpha|=j}\frac{j!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{i=1}^k |s_i-t_i|^{\alpha_i}\right)|X_\alpha(t,\omega)|\right\}^q \tag{22.105}$$

$$\leq (k^j)^{q-1}\left\{\sum_{|\alpha|=j}\frac{j!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{i=1}^k |s_i-t_i|^{q\alpha_i}\right)|X_\alpha(t,\omega)|^q\right\}. \tag{22.106}$$

Consequently we obtain

$$\int_\Omega |G^{(j)}(0,s,\omega)|^q P(d\omega) \tag{22.107}$$

$$\leq k^{j(q-1)}\left\{\sum_{|\alpha|=j}\frac{j!}{\alpha_1!\cdots\alpha_k!}\left(\prod_{i=1}^k |s_i-t_i|^{q\alpha_i}\right)\left(E|X_\alpha|^q\right)(t)\right\}.$$

Hence

$$\left(\int_Q\left(\int_\Omega |G^{(j)}(0,s,\omega)|^q P(d\omega)\right)\mu_t(ds)\right)^{\frac{1}{q}}$$

$$\leq k^{j\left(1-\frac{1}{q}\right)}\left\{\sum_{|\alpha|=j}\frac{j!}{\alpha_1!\cdots\alpha_k!}\right.$$

$$\left.\times\left(\int_Q\left(\prod_{i=1}^k |s_i-t_i|^{q\alpha_i}\right)\mu_t(ds)\right)\left(E|X_\alpha|^q\right)(t)\right\}^{\frac{1}{q}} \tag{22.108}$$

$$= k^{j\left(1-\frac{1}{q}\right)} \left\{ \sum_{|\alpha|=j} \frac{j!(E|X_\alpha|^q)(t)}{\alpha_1! \cdots \alpha_k!} \left(\tilde{L}\left(\prod_{i=1}^{k} |\cdot - t_i|^{q\alpha_i} \right) \right)(t) \right\}^{\frac{1}{q}}. \tag{22.109}$$

I.e. we got that

$$\begin{aligned} &\left(\int_\Omega \left(\int_Q |G^{(j)}(0,s,\omega)|\mu_t(ds) \right)^q P(d\omega) \right)^{\frac{1}{q}} \\ &\leq \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}} k^{j\left(1-\frac{1}{q}\right)} \left\{ \sum_{|\alpha|=j} \frac{j!(E|X_\alpha|^q)(t)}{\alpha_1! \cdots \alpha_k!} \right. \\ &\qquad \left. \times \left(\tilde{L}\left(\prod_{i=1}^{k} |\cdot - t_i|^{q\alpha_i} \right) \right)(t) \right\}^{\frac{1}{q}}. \end{aligned} \tag{22.110}$$

Finally we easily obtain $(1 < q < \infty)$

$$\begin{aligned} &\left(E(|M(X) - X|^q)(t)\right)^{\frac{1}{q}} \leq \left((E|X|^q)(t)\right)^{\frac{1}{q}} |\tilde{L}(1)(t) - 1| \\ &+ \left(\int_\Omega \left(\int_Q |X(s,\omega) - X(t,\omega)|\mu_t(ds) \right)^q P(d\omega) \right)^{\frac{1}{q}}, \quad \forall t \in Q. \end{aligned} \tag{22.111}$$

Putting things together we have the following L_q Shisha–Mond [264] type result regarding multivariate stochastic processes.

Theorem 22.29. *Let $\tilde{L}\colon C(Q) \hookrightarrow C(Q)$ positive linear operator, where $Q \subseteq \mathbb{R}^k$ compact and convex $k > 1$. Let*

$$M(X)(t,\omega) := \tilde{L}(X(\cdot,\omega))(t), \quad \forall t \in Q, \ \forall \omega \in \Omega$$

a probability space. Here $X(s,\omega) \in C^n(Q)$, $n \geq 1$, in s and measurable in ω. Also the partials X_α, $1 \leq |\alpha| \leq n$ are measurable in ω. We assume $E(|X|^q)(t) < \infty$, $(E|X_\alpha|^q)(t) < \infty$, $\forall t \in Q$, $1 < q < \infty$ and all α such that $1 \leq |\alpha| \leq n$. We further assume that $X_\alpha \in C^U_{\mathbb{R}}(Q)$, all $\alpha\colon |\alpha| = n$. Also $M(X)(t,\omega)$ is supposed to be measurable in $\omega \in \Omega$. Then

1)

$$\left(E(|M(X)-X|^q)(t)\right)^{\frac{1}{q}} \le |(E|X|^q)(t)|^{\frac{1}{q}}|\tilde{L}(1)(t)-1|$$
$$+\left(\sum_{j=1}^{n} \frac{k^{j\left(1-\frac{1}{q}\right)}}{(j!)^{1-\frac{1}{q}}} \left\{ \sum_{|\alpha|=j} \frac{(E|X_\alpha|^q)(t)}{\alpha_1!\cdots\alpha_k!} \right.\right.$$
$$\left.\left. \times \left(\tilde{L}\left(\prod_{i=1}^{k} |\cdot - t_i|^{q\alpha_i}\right)\right)(t) \right\}^{\frac{1}{q}}\right) \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}}$$
$$+\left(\frac{2k^n \tilde{L}(1)(t)}{n!}\right)^{1-\frac{1}{q}} \left[\frac{(\tilde{L}(1)(t))^{\frac{1}{n+1}}}{n!} + \frac{1}{(q+1)\cdots(q+n)}\right]^{\frac{1}{q}}$$
$$\times \left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{(n+1)q})(t)\right)^{\frac{n}{(n+1)q}} \left\{ \max_{\alpha:\, |\alpha|=n} \right.$$
$$\left. \Omega_1\left(X_\alpha, \left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{(n+1)q})(t)\right)^{\frac{1}{(n+1)q}}\right)_{L^q} \right\}, \quad \forall t \in Q. \tag{22.112}$$

Also we have

2)

$$\|E(|M(X)-X|^q)\|_\infty^{\frac{1}{q}} \le \|E(|X^q|)\|_\infty^{\frac{1}{q}} \|\tilde{L}1 - 1\|_\infty$$
$$+\left(\sum_{j=1}^{n} \frac{k^{j(1-\frac{1}{q})}}{(j!)^{1-\frac{1}{q}}} \left\{ \sum_{|\alpha|=j} \frac{\|E(|X_\alpha|^q)\|_\infty}{\alpha_1!\cdots\alpha_k!} \right.\right.$$
$$\left.\left. \times \left\| \left(\tilde{L}\left(\prod_{i=1}^{k} |\cdot - t_i|^{q\alpha_i}\right)\right)(t) \right\|_\infty \right\}^{\frac{1}{q}}\right) \|\tilde{L}(1)\|_\infty^{1-\frac{1}{q}}$$
$$+\left(\frac{2k^n \|\tilde{L}(1)\|_\infty}{n!}\right)^{1-\frac{1}{q}} \left\| \frac{(\tilde{L}(1))^{\frac{1}{n+1}}}{n!} + \frac{1}{(q+1)\cdots(q+n)} \right\|_\infty^{\frac{1}{q}}$$
$$\times \left\|\left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{(n+1)q})(t)\right\|_\infty^{\frac{n}{(n+1)q}} \left\{ \max_{\alpha:\, |\alpha|=n} \right.$$
$$\left. \times\, \Omega_1\left(X_\alpha, \|\left(\tilde{L}(\|\cdot - t\|_{\ell^1}^{(n+1)q})\right)(t)\|_\infty^{\frac{1}{(n+1)q}}\right)_{L^q} \right\}. \tag{22.113}$$

Proof. Comes by Concepts 22.6, Assumptions 22.9, 22.23, and Remarks 22.24, 22.27, 22.28. For the case of $h = 0$, see (22.97), inequality (22.112) holds trivially as equality. ■

Note 22.30. When $\tilde{L}(1)(t) = 1$, $\forall t \in Q$, then the assumption

$$(E|X|^q)(t) < \infty, \quad \forall t \in Q, \quad 1 \le q < \infty$$

in Theorems 22.13, 22.17, 22.19, 22.21, 22.29 is redundant.

The following general Korovkin type theorem (see [213]) is valid for L_q convergence of multivariate stochastic processes.

Theorem 22.31. *Let* $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N\in\mathbb{N}}$ *on multivariate stochastic processes, all as in the assumption of Theorem 22.29. Additionally assume that* $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ *is bounded and*

$$\left\| \left(\tilde{L}_N(\|\cdot - t\|_{\ell^1}^{q(n+1)})\right)(t) \right\|_\infty \to 0,$$

along with $\tilde{L}_N 1 \overset{n}{\to} 1$ *as* $N \to \infty$. *Then* $\|E(|M_N(X) - X|^q)\|_\infty \to 0$, *as* $N \to \infty$, *for all* X *as in the assumptions of Theorem 22.29. I.e.* $M_N \overset{\text{“}q\text{-mean”}}{\longrightarrow} I$, *the unit operator, with rates and in our setting.*

Proof. By inequality (22.113). Observe here that $(1 < q < \infty)$

$$\begin{aligned} &\|\tilde{L}_N(\|\cdot - t\|_{\ell^1}^{qj})(t)\|_\infty \\ &\quad \le \|\tilde{L}_N(1)\|_\infty^{1-\frac{j}{n+1}} \|\tilde{L}_N(\|\cdot - t\|_{\ell^1}^{q(n+1)})(t)\|_\infty^{\frac{j}{(n+1)}}, \end{aligned} \tag{22.114}$$

for $j = 1, \ldots, n$. Notice that

$$\left(\sum_{i=1}^k |\cdot - t_i|^q\right)^j \le \left(\sum_{i=1}^k |\cdot - t_i|\right)^{qj} = \|\cdot - t\|_{\ell^1}^{qj}, \tag{22.115}$$

and it is clearly true that

$$\left\| \tilde{L}_N\left(\left(\prod_{i=1}^k |\cdot - t_i|^{q\alpha_i}\right)\right)(t) \right\|_\infty \le \frac{\alpha_1! \cdots \alpha_k!}{j!} \|\tilde{L}_N(\|\cdot - t\|_{\ell^1}^{qj}))(t)\|_\infty, \tag{22.116}$$

$\forall \alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$; $|\alpha| = j$, $j = 1, \ldots, n$. ■

Note 22.32. We observe that $M_N \overset{\text{“}q\text{-mean”}}{\longrightarrow} I$ implies $M_N \overset{\text{“}1\text{-mean”}}{\longrightarrow} I$, according to Theorems 22.22 and 22.31.

Next we specialize in the $n = 0$ case. We do first the subcase of $q = 1$. For that we make

Remark 22.33. We have that

$$\Delta(t,\omega) := M(x)(t,\omega) - X(t,\omega)\tilde{L}(1)(t) = \int_Q \big(X(s,\omega) - X(t,\omega)\big)\mu_t(ds). \tag{22.117}$$

Then

$$\begin{aligned}
&\int_\Omega |\Delta(t,\omega)|P(d\omega) \\
&\quad\leq \int_\Omega \left(\int_Q |X(s,\omega)-X(t,\omega)|\mu_t(ds)\right) P(d\omega) \qquad (22.118)\\
&\qquad\qquad \text{(by Tonelli–Fubini's theorem)}\\
&\quad= \int_Q \left(\int_\Omega |X(s,\omega)-X(t,\omega)|P(d\omega)\right)\mu_t(ds) \qquad (22.119)\\
&\quad\leq \int_Q \Omega_1\big(X,\|s-t\|_{\ell^1}\big)\mu_t(ds) \qquad (22.120)\\
&\qquad\qquad (h>0)\\
&\quad\leq \Omega_1(X,h)_{L^1}\int_Q\left(1+\frac{\|s-t\|_{\ell^1}}{h}\right)\mu_t(ds) \qquad (22.121)\\
&\quad= \Omega_1(X,h)_{L^1}\left(\tilde{L}(1)(t)+\frac{1}{h}\int_Q \|s-t\|_{\ell^1}\mu_t(ds)\right). \qquad (22.122)
\end{aligned}$$

(Choose and suppose momentarily

$$\begin{aligned}
h :=& \int_Q \|s-t\|_{\ell^1}\mu_t(ds) > 0) \qquad (22.123)\\
=& \ \Omega_1\big(X,\tilde{L}(\|\cdot-t\|_{\ell^1})(t)\big)_{L^1}(\tilde{L}(1)(t)+1). \qquad (22.124)
\end{aligned}$$

I.e. we got

$$\int_\Omega |\Delta(t,\omega)|P(d\omega) \leq (\tilde{L}(1)(t)+1)\Omega_1\big(X,\tilde{L}(\|\cdot-t\|_{\ell^1})(t)\big)_{L^1}. \qquad (22.125)$$

We have proved

Theorem 22.34. *Here all elements are as in Concepts 22.6, Assumptions 22.9, 22.11 when $n=0$. Then*

1)

$$\begin{aligned}
\big(E(|M(X)-X|)\big)(t) \leq& (E|X|)(t)|\tilde{L}(1)(t)-1| \qquad (22.126)\\
&+ (\tilde{L}(1)(t)+1)\Omega_1\big(X,\tilde{L}(\|\cdot-t\|_{\ell^1})(t)\big)_{L^1}, \quad \forall t\in Q.
\end{aligned}$$

Also it holds

2)

$$\begin{aligned}
\|E(|M(X)-X|)\|_\infty \quad\leq\quad & \|E|X|\|_\infty\|\tilde{L}1-1\|_\infty \qquad (22.127)\\
&+ \|\tilde{L}(1)+1\|_\infty\Omega_1\big(X,\|\tilde{L}(\|\cdot-t\|_{\ell^1}(t)\|_\infty\big)_{L^1}.
\end{aligned}$$

Note 22.35. Inequality (22.126) holds trivially as equality when $h=0$, see (22.123).

The implied by (22.127) Korovkin type result follows.

Theorem 22.36. *Let $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ be a sequence of positive linear operators and the induced sequence of positive linear operators $\{M_N\}_{N\in\mathbb{N}}$ on multivariate stochastic processes, all as in Concepts 22.6, Assumptions 22.9, 22.11 when $n=0$. Additionally assume that $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ is bounded and $\tilde{L}_N 1 \overset{u}{\to} 1$, $\|\tilde{L}_N(\|\cdot - t\|_{\ell^1})(t)\|_\infty \to 0$, as $N\to\infty$. Then*

$$\|E(|M_N(X)-X|)\|_\infty \to 0, \quad \text{as } N\to\infty,$$

for all X as in Concepts 22.6, Assumptions 22.9, 22.11 when $n=0$. I.e.

$$M_N \underset{N\to\infty}{\overset{\text{“1-mean”}}{\longrightarrow}} I \text{ with rates in our setting.}$$

Finally we treat the subcase $q>1$ when $n=0$.

Remark 22.37. Let $\Delta(t,\omega)$ as in (22.117), then by Hölder's inequality ($1<q<\infty$) we have

$$\begin{aligned} |\Delta(t,\omega)|^q &\le \left(\int_Q |X(s,\omega)-X(t,\omega)|\mu_t(ds)\right)^q \\ &\le \left(\tilde{L}(1)(t)\right)^{q-1}\int_Q |X(s,\omega)-X(t,\omega)|^q\mu_t(ds). \end{aligned} \quad (22.128)$$

Therefore we get

$$\left(\int_\Omega |\Delta(t,\omega)|^q P(d\omega)\right)^{\frac{1}{q}} \le \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_\Omega\left(\int_Q |X(s,\omega)-X(t,\omega)|^q\mu_t(ds)\right)P(d\omega)\right)^{\frac{1}{q}} \quad (22.129)$$

(by Tonelli–Fubini's theorem we get)

$$= \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}}\left(\int_Q\left(\int_\Omega |X(s,\omega)-X(t,\omega)|^q P(d\omega)\right)\mu_t(ds)\right)^{\frac{1}{q}} \quad (22.130)$$

$$\leq \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}} \left(\int_Q \Omega_1^q\left(X, \|s-t\|_{\ell^1}\right)_{L^q} \mu_t(ds)\right)^{\frac{1}{q}} \tag{22.131}$$

(take $h > 0$)

$$\leq \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}} \Omega_1(X, h)_{L^q} \times \left(\int_Q \left(1 + \frac{\|s-t\|_{\ell^1}}{h}\right)^q \mu_t(ds)\right)^{\frac{1}{q}} \tag{22.132}$$

$$\leq 2^{1-\frac{1}{q}} \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}} \Omega_1(X, h)_{L^q} \Bigg[(\tilde{L}(1))(t) + \frac{1}{h^q} \int_Q \|s-t\|_{\ell^1}^q \mu_t(ds)\Bigg]^{\frac{1}{q}} \tag{22.133}$$

(choose and suppose momentarily

$$h := \left(\int_Q \|s-t\|_{\ell^1}^q \mu_t(ds)\right)^{\frac{1}{q}} > 0 \tag{22.134}$$

$$= \left(2\tilde{L}(1)(t)\right)^{1-\frac{1}{q}} \Omega_1\left(X, (\tilde{L}(\|\cdot - t\|_{\ell^1}^q)(t))^{\frac{1}{q}}\right)_{L^q} (\tilde{L}(1)(t) + 1)^{\frac{1}{q}}. \tag{22.135}$$

We have established

Theorem 22.38. *Suppose Concepts 22.6, Assumptions 22.9, 22.23 when $n = 0$, $1 < q < \infty$. Then*

1)

$$\left(E(|M(X) - X|^q)(t)\right)^{\frac{1}{q}} \leq \left((E|X|^q)(t)\right)^{\frac{1}{q}} |\tilde{L}(1)(t) - 1| + \left(2\tilde{L}(1)(t)\right)^{1-\frac{1}{q}} (\tilde{L}(1)(t) + 1)^{\frac{1}{q}} \Omega_1\left(X, (\tilde{L}(\|\cdot - t\|_{\ell^1}^q)(t))^{\frac{1}{q}}\right)_{L^q}, \ \forall t \in Q. \tag{22.136}$$

2)

$$\|E(|M(X) - X|^q)\|_\infty^{\frac{1}{q}} \leq \|E(|X|^q)\|_\infty^{\frac{1}{q}} \|\tilde{L}(1) - 1\|_\infty + \left(2\|\tilde{L}(1)\|_\infty\right)^{1-\frac{1}{q}} \|\tilde{L}1 + 1\|_\infty^{\frac{1}{q}} \Omega_1\left(X, \|\tilde{L}(\|\cdot - t\|_{\ell^1}^q)(t)\|_\infty^{\frac{1}{q}}\right)_{L^q}. \tag{22.137}$$

Note 22.39. When $h = 0$, see (22.134), then inequality (22.136) is trivially valid as equality.

Note 22.40. When $\tilde{L}(1) = 1$ then the assumption $(E|X|^q)(t) < \infty$, $\forall t \in Q$, $1 \leq q < \infty$, in Theorems 22.34, 22.38 is redundant.

We give the final Korovkin type related result based on (22.137).

Theorem 22.41. *Let $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ be a sequence of positive linear operators and the induced sequence of positive linear operators $\{M_N\}_{N\in\mathbb{N}}$ in multivariate stochastic processes, all as in Concepts 22.6, Assumptions 22.9, 22.23 for $n = 0$, $1 < q < \infty$. Additionally assume that $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ is* *bounded and* $\|\tilde{L}_N(\|\cdot - t\|_{\ell^1}^q)(t)\|_\infty \to 0$*, along with* $\tilde{L}_N(1) \xrightarrow{u} 1$*, as* $N \to \infty$*. Then*

$$\|E(|M_N(X) - X|^q)\|_\infty \to 0, \quad \text{as } N \to \infty,$$

for all X *as in Concepts 22.6, Assumptions 22.9, 22.23,* $n = 0$*. I.e.* $M_N \underset{N\to\infty}{\overset{\text{“}q\text{-mean”}}{\longrightarrow}} I$ *with rates in our setting.*

Note 22.42. We observe again that $M_N \overset{\text{“}q\text{-mean”}}{\longrightarrow} I$ implies $M_N \overset{\text{“1-mean”}}{\longrightarrow} I$, according to Theorems 22.36 and 22.41.

Note 22.43. The rate of convergence in Theorems 22.22, 22.31 is much higher than in the corresponding Theorems 22.36, 22.41 because of the assumed differentiability of X, see and compare inequalities (22.55) versus (22.127), and (22.113) versus (22.137).

Note 22.44. If X_α, $|\alpha| = n \in \mathbb{Z}_+$, fulfills a Lipschitz type condition then our results become more specific and simplify.

We finish this chapter with

Application 22.45. Here we will apply inequality (22.127) of Theorem 22.34. Let $f \in C([0,1]^2)$, the two-dimensional Bernstein polynomials of f are defined by

$$\begin{aligned} B_{m,\overline{n}}(f;t_1,t_2) := & \sum_{k=0}^{m}\sum_{\ell=0}^{\overline{n}} f\left(\frac{k}{m},\frac{\ell}{\overline{n}}\right)\binom{m}{k}\binom{\overline{n}}{\ell} t_1^k(1-t_1)^{m-k} \\ & \times t_2^\ell(1-t_2)^{\overline{n}-\ell}, \end{aligned} \tag{22.138}$$

for all $t := (t_1,t_2) \in [0,1]^2$, it is known that $B_{m,\overline{n}}(f) \to f$ uniformly on $[0,1]^2$. Clearly

$$B_{m,\overline{n}}(1;t_1,t_2) = 1, \quad \forall (t_1,t_2) \in [0,1]^2, \ \ \forall (m,\overline{n}) \in \mathbb{N}^2. \tag{22.139}$$

By using Schwarz's inequality repeatedly and maximizing we obtain

$$\begin{aligned} B_{m,\overline{n}}(\|\cdot - t\|_{\ell^1})(t) &\le \left(B_{m,\overline{n}}(\|\cdot - t\|_{\ell^1}^2)(t)\right)^{\frac{1}{2}} \qquad (22.140) \\ &\le \frac{1}{2}\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{\overline{n}}}\right), \quad \forall (m,\overline{n}) \in \mathbb{N}^2, \ \forall t \in [0,1]^2. \end{aligned}$$

That is

$$\|B_{m,\overline{n}}(\|\cdot - t\|_{\ell^1})(t)\|_\infty \le \frac{1}{2}\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{\overline{n}}}\right). \tag{22.141}$$

Here $B_{m,\overline{n}}$ is an example of an $\tilde{L}$ operator as in Concepts 22.6. Define the corresponding application of M by

$$\tilde{B}_{m,\overline{n}}(X)(t,\omega) := B_{m,\overline{n}}(X(\cdot,\omega))(t)$$
$$= \sum_{k=0}^{m}\sum_{\ell=0}^{\overline{n}} X\left(\frac{k}{m},\frac{\ell}{\overline{n}},\omega\right)\binom{m}{k}\binom{\overline{n}}{\ell} t_1^k(1-t_1)^{m-k} t_2^{\ell}(1-t_2)^{\overline{n}-\ell},$$
$$\forall t=(t_1,t_2)\in[0,1]^2, \quad \forall \omega\in\Omega, \quad \forall (m,\overline{n})\in\mathbb{N}^2, \tag{22.142}$$

where X is as in Concepts 22.6 and Assumptions 22.9, 22.11 for $n=0$.

By (22.127) we obtain

$$\|E(|\tilde{B}_{m,\overline{n}}(X)-X|)\|_\infty \le 2\Omega_1\left(X,\frac{1}{2}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{\overline{n}}}\right)\right)_{L^1}, \ \forall (m,\overline{n})\in\mathbb{N}^2, \tag{22.143}$$

for all X as above. Thus, as $m,\overline{n}\to\infty$, we get

$$\|E(|\tilde{B}_{m,\overline{n}}(X)-X|)\|_\infty \to 0, \tag{22.144}$$

i.e. $\tilde{B}_{m,\overline{n}} \overset{\text{“1-mean”}}{\longrightarrow} I$ with rates. If X is of Lipschitz type of order 1 i.e. if $\Omega_1(X,\delta)_{L^1} \le K\delta$, where $K>0$, $\forall \delta>0$, then

$$\|E(|\tilde{B}_{m,\overline{n}}(X)-X|)\|_\infty \le K\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{\overline{n}}}\right), \quad \forall (m,\overline{n})\in\mathbb{N}^2. \tag{22.145}$$

One can give many similar other applications of the above theory.

23

About the Right Fractional Calculus

Here we present fractional Taylor type formulae with fractional integral remainder and fractional differential formulae, regarding the right Caputo fractional derivative, the right generalized fractional derivative of Canavati type ([126]) and their corresponding right fractional integrals.

Then we give representation formulae of functions as fractional integrals of their above fractional derivatives, as well as of their right and left Weyl fractional derivatives.

At the end, we mention some far reaching implications of this theory to mathematical analysis computational methods.

Also we compare the right Caputo fractional derivative to right Riemann-Liouville fractional derivative. This chapter relies on [44].

23.1 About the Right Caputo Fractional Derivative

We start with (for this section see also [160], [179], [259])

Definition 23.1. Let $f \in L_1([a,b])$, $\alpha > 0$. We define the right Riemann-Liouville fractional operator of order α by

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta - x)^{\alpha-1} f(\zeta) d\zeta, \tag{23.1}$$

$\forall x \in [a,b]$, where Γ is the gamma function. We set $I_{b-}^0 := I$ (the identity operator).

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 333–354.
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We mention

Theorem 23.2. Let $f \in L_1([a,b]), \alpha > 0$. Then $I_{b-}^{\alpha} f(x)$ exists almost everywhere on $[a,b]$ and $I_{b-}^{\alpha} f \in L_1([a,b])$.

Proof. Define $k : \Omega := [a,b] \times [a,b] \to \mathbb{R}$ by $k(\zeta, x) = (\zeta - x)_+^{\alpha-1}$, that is,

$$k(\zeta,x) = \begin{cases} (\zeta - x)^{\alpha-1}, & \text{if } a \le x \le \zeta \le b, \\ 0, & \text{if } a \le \zeta \le x \le b. \end{cases}$$

Then k is measurable on Ω, and we have

$$\begin{aligned} \int_a^b k(\zeta,x)dx &= \int_a^{\zeta} k(\zeta,x)dx + \int_{\zeta}^b k(\zeta,x)dx \\ &= \int_a^{\zeta} (\zeta - x)^{\alpha-1} dx \\ &= \frac{(\zeta - a)^{\alpha}}{\alpha}. \end{aligned}$$

Because the repeated integral

$$\begin{aligned} \int_a^b \left(\int_a^b k(\zeta,x) \left| f(\zeta) \right| dx \right) d\zeta &= \int_a^b \left| f(\zeta) \right| \left(\int_a^b k(\zeta,x)dx \right) d\zeta \\ &= \int_a^b \left| f(\zeta) \right| \frac{(\zeta - a)^{\alpha}}{\alpha} d\zeta \\ &= \alpha^{-1} \int_a^b (\zeta - a)^{\alpha} \left| f(\zeta) \right| d\zeta \\ &\le \frac{(b-a)^{\alpha}}{\alpha} \int_a^b \left| f(\zeta) \right| d\zeta \\ &= \frac{(b-a)^{\alpha}}{\alpha} \left\| f(\zeta) \right\|_{L_1(a,b)} < \infty \end{aligned}$$

Therefore the function $H : \Omega \to \mathbb{R}$ such that $H(\zeta, x) := k(\zeta, x) f(\zeta)$ is integrable over Ω by Tonelli's theorem. Hence, by Fubini's theorem we obtain that $\int_a^b k(\zeta,x) f(\zeta) d\zeta$ is an integrable function on $[a,b]$, as a function of $x \in [a,b]$. That is $I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta - x)^{\alpha-1} f(\zeta) d\zeta$ is integrable on $[a,b]$.

Thus $I_{b-}^{\alpha} f$ exists a.e. on $[a,b]$. ■

We further need

Lemma 23.3. Let $\alpha \ge 1$ and $f \in L_1([a,b])$. Then $I_{b-}^{\alpha} f \in C([a,b])$

Proof. For $\alpha = 1$ is trivial, thus we assume $\alpha > 1$.
Let $x, y \in [a, b] : x \geq y$ and $x \to y$.
We notice that

$$\left|I_{b-}^{\alpha} f(x) - I_{b-}^{\alpha} f(y)\right| = \frac{1}{\Gamma(\alpha)} \left| \int_x^b (\zeta - x)^{\alpha-1} f(\zeta) d\zeta - \int_y^b (\zeta - y)^{\alpha-1} f(\zeta) d\zeta \right|$$

$$= \frac{1}{\Gamma(\alpha)} \left| \int_x^b (\zeta - x)^{\alpha-1} f(\zeta) d\zeta - \int_y^x (\zeta - y)^{\alpha-1} f(\zeta) d\zeta - \int_x^b (\zeta - y)^{\alpha-1} f(\zeta) d\zeta \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_x^b \left| (\zeta - x)^{\alpha-1} - (\zeta - y)^{\alpha-1} \right| |f(\zeta)| \, d\zeta + \int_y^x (\zeta - y)^{\alpha-1} |f(\zeta)| \, d\zeta \right]$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_x^b \left| (\zeta - x)^{\alpha-1} - (\zeta - y)^{\alpha-1} \right| |f(\zeta)| \, d\zeta + (x - y)^{\alpha-1} \|f(\zeta)\|_{L_1([a,b])} \right].$$

As $x \to y$ we get $(\zeta - x)^{\alpha-1} \to (\zeta - y)^{\alpha-1}$, thus

$$\left| (\zeta - x)^{\alpha-1} - (\zeta - y)^{\alpha-1} \right| \to 0,$$

and also

$$\left| (\zeta - x)^{\alpha-1} - (\zeta - y)^{\alpha-1} \right| \leq 2 (b - a)^{\alpha-1}.$$

Hence

$$\left| (\zeta - x)^{\alpha-1} - (\zeta - y)^{\alpha-1} \right| |f(\zeta)| \leq 2 (b - a)^{\alpha-1} |f(\zeta)| \in L_1([a, b]),$$

and also $\left| (\zeta - x)^{\alpha-1} - (\zeta - y)^{\alpha-1} \right| |f(\zeta)| \to 0$ as $x \to y$, for almost all $\zeta \in [a, b]$.

Therefore by Dominated Convergence Theorem we conclude, as $x \to y$, that $\int_x^b \left| (\zeta - x)^{\alpha-1} - (\zeta - y)^{\alpha-1} \right| |f(\zeta)| \, d\zeta \to 0$.

Consequently, $\left| I_{b-}^{\alpha} f(x) - I_{b-}^{\alpha} f(y) \right| \to 0$ as $x \to y$.

Therefore $I_{b-}^{\alpha} f \in C([a, b])$. ∎

We also have

Theorem 23.4. Let $\alpha, \beta \geq 0$, $f \in L_1([a, b])$. Then

$$I_{b-}^{\alpha} I_{b-}^{\beta} f = I_{b-}^{\alpha+\beta} f = I_{b-}^{\beta} I_{b-}^{\alpha} f, \tag{23.2}$$

valid almost everywhere on $[a, b]$. If additionally $f \in C([a, b])$ or $\alpha + \beta \geq 1$, then we have identity true on all of $[a, b]$.

Proof. Since $I_{b-}^{0} := I$ (the identity operator), if $\alpha = 0$ or $\beta = 0$ or both are zero, then the statement of the theorem is trivially true. So we suppose $\alpha, \beta > 0$.

We observe that

$$I_{b-}^{\alpha} I_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{x}^{b} (t-x)^{\alpha-1} \left(\int_{t}^{b} (\tau - t)^{\beta-1} f(\tau) d\tau \right) dt.$$

The above integrals exist a.e. on $[a, b]$. So if $I_{b-}^{\alpha} I_{b-}^{\beta} f(x)$ exists we apply Fubini's theorem to interchange the order of integration and get that

$$\begin{aligned} I_{b-}^{\alpha} I_{b-}^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{x}^{b} \left(\int_{x}^{\tau} (t-x)^{\alpha-1} (\tau - t)^{\beta-1} f(\tau) dt \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{x}^{b} f(\tau) \left(\int_{x}^{\tau} (\tau - t)^{\beta-1} (t-x)^{\alpha-1} dt \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{x}^{b} f(\tau) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\tau - x)^{\alpha+\beta-1} d\tau \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_{x}^{b} f(\tau) (\tau - x)^{(\alpha+\beta)-1} d\tau \\ &= I_{b-}^{\alpha+\beta} f(x). \end{aligned}$$

That is

$$I_{b-}^{\alpha} I_{b-}^{\beta} f(x) = I_{b-}^{\alpha+\beta} f(x) \tag{23.3}$$

true, whenever any of the two sides exists, which is true a. e. on $[a, b]$.

Clearly, if $f \in C([a, b])$ then $I_{b-}^{\beta} f \in C([a, b])$, therefore $I_{b-}^{\alpha} I_{b-}^{\beta} f \in C([a, b])$ and $I_{b-}^{\alpha+\beta} f \in C([a, b])$.

Since in (23.3) two continuous functions coincide a.e., they must be equal everywhere.

At last, if $f \in L_1([a, b])$ and $\alpha + \beta \geq 1$, we get $I_{b-}^{\alpha+\beta} f \in C([a, b])$ by Lemma 23.3. Hence, since $I_{b-}^{\alpha+\beta} f(x)$ is defined and existing for any $x \in [a, b]$, by Fubini's theorem as before, equals to $I_{b-}^{\alpha} I_{b-}^{\beta} f(x)$, for all $x \in [a, b]$, proving the claim. ■

We need

Definition 23.5. Let $f \in AC^{m}([a, b])$ (space of functions from $[a, b]$ into $\mathbb{R}$ with $m - 1$ derivative absolutely continuous function on $[a, b]$), $m \in \mathbb{N}$, where $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number).

We define the **right Caputo fractional derivative of order** $\alpha > 0$, by

$$D_{b-}^{\alpha} f(x) := (-1)^{m} I_{b-}^{m-\alpha} f^{(m)}(x), \tag{23.4}$$

that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta. \tag{23.5}$$

Note 23.6. Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, with $\alpha > 0$, then $f^{(m-1)} \in AC([a,b])$, which implies that $f^{(m)}$ exists a.e. on $[a,b]$ and that $f^{(m)} \in L_1([a,b])$.

Consequently if $f \in AC^m([a,b])$, then $D_{b-}^{\alpha} f(x)$ exists a.e. on $[a,b]$ and $D_{b-}^{\alpha} f \in L_1([a,b])$.

Observe that when $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x),$$

$\forall x \in [a,b]$.

We need

Definition 23.7. Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. We define the **right Riemann-Liouville fractional derivative** by

$$\mathcal{D}_{b-}^{\alpha} f(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_x^b (t-x)^{m-\alpha-1} f(t) dt, \tag{23.6}$$

$$\mathcal{D}_{b-}^{0} f(x) := I \text{ (the identity operator)}.$$

We present

Theorem 23.8. Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. Then

$$\mathcal{D}_{b-}^{\alpha} \left(f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right) = \left(D_{b-}^{\alpha} f \right)(x), \tag{23.7}$$

a.e. on $[a,b]$.

If L.H.S.(23.7) exists at $x \in [a,b]$, then L.H.S.(23.7)=R.H.S.(23.7).

If R.H.S.(23.7) exists at $x \in [a,b]$, then L.H.S.(23.7)=R.H.S.(23.7).

Proof. We have that

$$\begin{aligned}
&\mathcal{D}_{b-}^{\alpha} \left(f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right) \\
= \;& (-1)^m D^m I_{b-}^{m-\alpha} \left(f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right) \\
= \;& (-1)^m \frac{d^m}{dx^m} \int_x^b \frac{(\zeta - x)^{m-\alpha-1}}{\Gamma(m-\alpha)} \left(f(\zeta) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (\zeta - b)^k \right) d\zeta.
\end{aligned}$$

Next we use integration by parts repeatedly to obtain

$$\begin{aligned}
&\int_x^b \frac{(\zeta-x)^{m-\alpha-1}}{\Gamma(m-\alpha)}\left(f(\zeta)-\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(\zeta-b)^k\right)d\zeta \\
=\ &\int_x^b \frac{\left(f(\zeta)-\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(\zeta-b)^k\right)}{\Gamma(m-\alpha+1)}d\,(\zeta-x)^{m-\alpha} \\
=\ &\left(f(\zeta)-\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(\zeta-b)^k\right)(\zeta-x)^{m-\alpha}\Bigg|_x^b \\
&-\int_x^b \frac{(\zeta-x)^{m-\alpha}}{\Gamma(m-\alpha+1)}\left(f'(\zeta)-\sum_{k=1}^{m-1}\frac{f^{(k)}(b)}{(k-1)!}(\zeta-b)^{k-1}\right)d\zeta \\
=\ &-\int_x^b \frac{(\zeta-x)^{m-\alpha}}{\Gamma(m-\alpha+1)}\left(f'(\zeta)-\sum_{k=1}^{m-1}\frac{f^{(k)}(b)}{(k-1)!}(\zeta-b)^{k-1}\right)d\zeta.
\end{aligned}$$

Thus we have

$$\begin{aligned}
L \quad : \quad &= I_{b-}^{m-\alpha}\left(f(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right) \\
&= (-1)I_{b-}^{m-\alpha+1}\left(f'(x)-\sum_{k=1}^{m-1}\frac{f^{(k)}(b)}{(k-1)!}(x-b)^{k-1}\right).
\end{aligned}$$

Under our assumptions we can perform the above m times, to derive

$$L=(-1)^m I_{b-}^{2m-\alpha}\left(f^{(m)}(x)\right).$$

That is

$$(-1)^m I_{b-}^{m-\alpha}\left(f(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right)=I_{b-}^{m}I_{b-}^{m-\alpha}f^{(m)}(x), \text{ a.e.}$$

Consequently we have

$$\begin{aligned}
&\mathcal{D}_{b-}^{\alpha}\left(f(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right) \\
=\ &(-1)^m D^m I_{b-}^{m-\alpha}\left(f(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right) \\
&\overset{a.e.}{=} D^m I_{b-}^{m} I_{b-}^{m-\alpha} f^{(m)}(x) \\
=\ &(-1)^m I\ I_{b-}^{m-\alpha} f^{(m)}(x) \\
=\ &(-1)^m I_{b-}^{m-\alpha} f^{(m)}(x) \\
=\ &D_{b-}^{\alpha} f(x).
\end{aligned}$$

Above we used that $D^m I_{b-}^m = (-1)^m I$ on $L_1([a,b])$. ■

Next we give the comparison result.

Theorem 23.9. Suppose $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha \geq 0$. Assume any of $\mathcal{D}_{b-}^{\alpha} f(x)$, $\left(D_{b-}^{\alpha} f\right)(x)$ exists for some $x \in [a,b]$.
Then

$$\mathcal{D}_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(b)(-1)^k (b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{23.8}$$

So, if $f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$, then

$$\mathcal{D}_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} f(x). \tag{23.9}$$

Proof.
We apply Theorem 23.8. So, by (23.7) we obtain

$$D_{b-}^{\alpha} f(x) = \mathcal{D}_{b-}^{\alpha} f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} \mathcal{D}_{b-}^{\alpha}\left((x-b)^k\right). \tag{23.10}$$

We find

$$\begin{aligned}
\mathcal{D}_{b-}^{\alpha}(x-b)^k &= \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_x^b (t-x)^{m-\alpha-1}(t-b)^k dt \\
&= \frac{(-1)^{m+k}}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_x^b (b-t)^{(k+1)-1}(t-x)^{(m-\alpha)-1} dt \\
&= \frac{(-1)^{m+k}}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \frac{k!\Gamma(m-\alpha)}{\Gamma(k+1+m-\alpha)} (b-x)^{k+m-\alpha} \\
&= \frac{(-1)^{m+k} k!}{\Gamma(k+1+m-\alpha)} \left(\frac{d}{dx}\right)^m (b-x)^{k+m-\alpha}. \qquad (23.11)
\end{aligned}$$

But it holds

$$\begin{aligned}
\left((b-x)^{(k+m-\alpha)}\right)^{(m)} &= (-1)^m (k-\alpha+m)\ldots(k-\alpha+1)(b-x)^{k-\alpha} \\
&= (-1)^m \frac{\Gamma(k-\alpha+m+1)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha} \qquad (23.12)
\end{aligned}$$

Therefore

$$\frac{\mathcal{D}_{b-}^{\alpha}(x-b)^k}{k!} = \frac{(-1)^k (b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)} \tag{23.13}$$

Finally we have

$$D_{b-}^{\alpha} f(x) = \mathcal{D}_{b-}^{\alpha} f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)(-1)^k (b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{23.14}$$

We further need

Theorem 23.10. Let $f \in AC^n([a,b]), n \in \mathbb{N}$. Then

$$I_{b-}^{n} f^{(n)}(x) = (-1)^n \left\{ f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right\}, \tag{23.15}$$

$\forall x \in [a,b]$, where

$$I_{b-}^{n} f^{(n)}(x) := \frac{1}{(n-1)!} \int_x^b (t-x)^{n-1} f^{(n)}(t) dt. \tag{23.16}$$

Furthermore

$$D^n I_{b-}^n = (-1)^n I \tag{23.17}$$

on $L_1([a,b])$.

Proof. Since $f \in AC^n([a,b])$, then by Taylor's theorem we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t) dt, \tag{23.18}$$

and

$$\begin{aligned} K \quad : \quad &= f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \\ &= \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t) dt, \end{aligned} \tag{23.19}$$

That is

$$\begin{aligned} K &= \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1} f^{(n)}(t) dt, \\ &= (-1)^n I_b^n f^{(n)}(x) \end{aligned} \tag{23.20}$$

■

We continue with the right Caputo fractional Taylor formula with integral remainder.

Theorem 23.11. Let $f \in AC^m([a,b]), x \in [a,b], \alpha > 0, m = \lceil \alpha \rceil$. Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta - x)^{\alpha-1} D_{b-}^{\alpha} f(\zeta) d\zeta. \tag{23.21}$$

Proof. We see that

$$\begin{aligned} I_{b-}^{\alpha} D_{b-}^{\alpha} f(x) &= I_{b-}^{\alpha} (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x) \\ &= (-1)^m I_{b-}^{\alpha} I_{b-}^{m-\alpha} f^{(m)}(x) \\ &= (-1)^m I_{b-}^{\alpha+m-\alpha} f^{(m)}(x) \\ &= (-1)^m I_{b-}^{m} f^{(m)}(x) \\ &= (-1)^{2m} \left[f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k \right] \\ &= f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k. \end{aligned}$$

Therefore

$$\begin{aligned} f(x) &= \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + I_{b-}^{\alpha} D_{b-}^{\alpha} f(x) \\ &= \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta - x)^{\alpha-1} D_{b-}^{\alpha} f(\zeta) d\zeta. \end{aligned}$$

■

Next we mention

Theorem 23.12. Let $f \in AC^m([a,b]), \alpha > 0, m = \lceil \alpha \rceil < \beta$. Then

$$I_{b-}^{\beta-\alpha} f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)(-1)^k}{\Gamma(k+1+\beta-\alpha)}(b-x)^{k+\beta-\alpha} + I_{b-}^{\alpha} D_{b-}^{\alpha} f(x), \tag{23.22}$$

$\forall x \in [a,b]$.
That is

$$\begin{aligned} \frac{1}{\Gamma(\beta-\alpha)} \int_x^b (\zeta - x)^{\beta-\alpha-1} f(\zeta) d\zeta = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)(-1)^k (b-x)^{k+\beta-\alpha}}{\Gamma(k+1+\beta-\alpha)} \\ + \frac{1}{\Gamma(\beta)} \int_x^b (\zeta - x)^{\beta-1} D_{b-}^{\alpha} f(\zeta) d\zeta, \end{aligned} \tag{23.23}$$

$\forall x \in [a,b]$.

Proof. It holds

$$\begin{aligned} I_{b_-}^{\beta-\alpha}(b-x)^k &= \frac{1}{\Gamma(\beta-\alpha)}\int_x^b (b-\zeta)^{(k+1)-1}(\zeta-x)^{(\beta-\alpha)-1}d\zeta \\ &= \frac{1}{\Gamma(\beta-\alpha)}\frac{\Gamma(k+1)\Gamma(\beta-\alpha)}{\Gamma(k+1+\beta-\alpha)}(b-x)^{k+\beta-\alpha} \\ &= \frac{k!}{\Gamma(k+1+\beta-\alpha)}(b-x)^{k+\beta-\alpha}. \qquad (23.24) \end{aligned}$$

We see that

$$\begin{aligned} I_{b_-}^{\beta} D_{b_-}^{\alpha} f(x) &= (-1)^m I_{b_-}^{\beta} I_{b_-}^{m-\alpha} f^{(m)}(x) \\ &= (-1)^m I_{b_-}^{\beta-\alpha+m} f^{(m)}(x) \\ &= (-1)^m I_{b_-}^{\beta-\alpha}\left(I_{b_-}^{m} f^{(m)}(x)\right) \\ &= (-1)^{2m} I_{b_-}^{\beta-\alpha}\left\{f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right\} \\ &= I_{b_-}^{\beta-\alpha} f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!} I_{b_-}^{\beta-\alpha}(x-b)^k \end{aligned}$$

$$\begin{aligned} &= I_{b_-}^{\beta-\alpha} f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(-1)^k I_{b_-}^{\beta-\alpha}(b-x)^k \\ &\overset{(23.24)}{=} I_{b_-}^{\beta-\alpha} f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(b)(-1)^k}{\Gamma(k+1+\beta-\alpha)}(b-x)^{k+\beta-\alpha}. \qquad (23.25) \end{aligned}$$

■

We also give

Theorem 23.13. Let $f \in AC^m([a,b]), m := \lceil \alpha \rceil, 0 < \alpha < \beta \leq m$. Then

$$I_{b_-}^{\alpha} D_{b_-}^{\beta} f(x) = -I_{b_-}^{(1+\alpha-\beta)} f'(x) + \sum_{k=0}^{m-2}\frac{f^{(k+1)}(b)(-1)^k}{\Gamma(k+2+\alpha-\beta)}(b-x)^{(k+1+\alpha-\beta)}, \quad (23.26)$$

almost everywhere in $[a,b]$.

That is, we have

$$I_{b_-}^{(1+\alpha-\beta)} f'(x) = \sum_{k=0}^{m-2}\frac{f^{(k+1)}(b)(-1)^k}{\Gamma(k+2+\alpha-\beta)}(b-x)^{(k+1+\alpha-\beta)} - I_{b_-}^{\alpha} D_{b_-}^{\beta} f(x), \quad (23.27)$$

almost everywhere in $[a, b]$.

Hence

$$\frac{1}{\Gamma(1+\alpha-\beta)}\int_x^b (\zeta-x)^{\alpha-\beta} f'(\zeta)d\zeta = \sum_{k=0}^{m-2}\frac{f^{(k+1)}(b)(-1)^k}{\Gamma(k+2+\alpha-\beta)}(b-x)^{(k+1+\alpha-\beta)}$$
$$-\frac{1}{\Gamma(\alpha)}\int_x^b (\zeta-x)^{\alpha-1} D_{b-}^{\beta} f(\zeta)d\zeta, \tag{23.28}$$

almost everywhere in $[a, b]$. All the above are complete identities if $m+\alpha-\beta \geq 1$.

Proof. Notice $\lceil\alpha\rceil = \lceil\beta\rceil = m$. We observe that

$$\begin{aligned}
I_{b-}^{\alpha} D_{b-}^{\beta} f(x) &= (-1)^m I_{b-}^{\alpha} I_{b-}^{m-\beta} f^{(m)}(x) \\
&\overset{(a.e.)}{=} (-1)^m I_{b-}^{\alpha+m-\beta} f^{(m)}(x) \\
&= (-1)^m I_{b-}^{m+\alpha-\beta} \left(f'(x)\right)^{(m-1)} \\
&= (-1)^m I_{b-}^{(m-1)+(\alpha-\beta+1)} \left(f'(x)\right)^{(m-1)} \\
&= (-1)^m I_{b-}^{(m-1)+(1-(\beta-\alpha))} \left(f'(x)\right)^{(m-1)} \\
&\overset{(a.e.)}{=} (-1)^m I_{b-}^{(1-(\beta-\alpha))} I_{b-}^{(m-1)} \left(f'(x)\right)^{(m-1)} \\
&= (-1)^m I_{b-}^{(1-(\beta-\alpha))} (-1)^{m-1} \left\{ f'(x) - \sum_{k=0}^{m-2} \frac{f^{(k+1)}(b)}{k!}(x-b)^k \right\} \\
&= I_{b-}^{(1+\alpha-\beta)} \left\{ -f'(x) + \sum_{k=0}^{m-2} \frac{f^{(k+1)}(b)}{k!}(-1)^k (b-x)^k \right\} \\
&= -I_{b-}^{(1+\alpha-\beta)} f'(x) + \sum_{k=0}^{m-2} \frac{f^{(k+1)}(b)}{k!}(-1)^k I_{b-}^{(1+\alpha-\beta)} (b-x)^k \\
&= -I_{b-}^{(1+\alpha-\beta)} f'(x) + \sum_{k=0}^{m-2} \frac{f^{(k+1)}(b)}{k!}(-1)^k \\
&\quad \frac{1}{\Gamma(1+\alpha-\beta)}\int_x^b (b-\zeta)^{(k+1)-1}(\zeta-x)^{(1+\alpha-\beta)-1} d\zeta \\
&= -I_{b-}^{(1+\alpha-\beta)} f'(x) + \sum_{k=0}^{m-2} \frac{f^{(k+1)}(b)}{k!}(-1)^k \frac{1}{\Gamma(1+\alpha-\beta)} \\
&\quad \frac{k!\Gamma(1+\alpha-\beta)}{\Gamma(k+1+1+\alpha-\beta)}(b-x)^{(k+1+\alpha-\beta)}
\end{aligned}$$

$$= -I_{b-}^{(1+\alpha-\beta)} f'(x) + \sum_{k=0}^{m-2} \frac{f^{(k+1)}(b)(-1)^k}{\Gamma(k+2+\alpha-\beta)} (b-x)^{(k+1+\alpha-\beta)}. \quad (23.29)$$

■

We further present

Proposition 23.14. Let $f \in C([a,b]), \alpha > 0, m = \lceil \alpha \rceil < \beta$. Then

$$D_{b-}^{\alpha} I_{b-}^{\beta} f(x) = I_{b-}^{\beta-\alpha} f(x), \quad (23.30)$$

$\forall x \in [a,b]$.

Proof. Call $\beta = m + v, v > 0$. We notice that

$$\begin{aligned} D_{b-}^{\alpha} I_{b-}^{\beta} f(x) &= (-1)^m I_{b-}^{m-\alpha} \left(I_{b-}^{\beta} f(x) \right)^{(m)} \\ &= (-1)^m I_{b-}^{m-\alpha} D^m I_{b-}^{m+v} f(x) \\ &= (-1)^m I_{b-}^{m-\alpha} D^m I_{b-}^{m} I_{b-}^{v} f(x) \\ &= (-1)^{2m} I_{b-}^{m-\alpha} I I_{b-}^{v} f(x) \\ &= I_{b-}^{m-\alpha} I_{b-}^{v} f(x) \\ &= I_{b-}^{m+v-\alpha} f(x) \\ &= I_{b-}^{\beta-\alpha} f(x). \end{aligned}$$

■

Also we have

Proposition 23.15. Let $n \in \mathbb{N}$ such that $n \leq m-1 < \alpha \leq m, m = \lceil \alpha \rceil, f \in AC^{m-n}([a,b])$. Then

$$D_{b-}^{\alpha} I_{b-}^{n} f(x) = D_{b-}^{(\alpha-n)} f(x), \quad (23.31)$$

$\forall x \in [a,b]$.

Proof. Set $m = k + n$. We observe that

$$\begin{aligned}
D_{b-}^{\alpha} I_{b-}^{n} f(x) &= (-1)^m I_{b-}^{m-\alpha} D^m I_{b-}^{n} f(x) \\
&= (-1)^m I_{b-}^{m-\alpha} D^k D^n I_{b-}^{n} f(x) \\
&= (-1)^{m+n} I_{b-}^{m-\alpha} D^k f(x) \\
&= (-1)^{m+n} I_{b-}^{m-\alpha} D^{m-n} f(x) \\
&= (-1)^{m+n} I_{b-}^{(m-n)-(\alpha-n)} f^{(m-n)}(x) \\
\text{(notice } \lceil \alpha - n \rceil &= m - n) \\
&= (-1)^{m+n} (-1)^{m-n} D_{b-}^{(\alpha-n)} f(x) \\
&= D_{b-}^{(\alpha-n)} f(x).
\end{aligned}$$

■

23.2 About the Right Generalized Fractional Derivative

Here see also [126], [23], p.539-545.

Let $v > 0$, $n := [v], \alpha = v - n, 0 < \alpha < 1$, here $[\cdot]$ is the integer part, $f \in C([a,b])$, call the **right Riemann-Liouville fractional integral operator** by

$$\left(J_{b-}^{v} f\right)(x) := \frac{1}{\Gamma(v)} \int_x^b (\zeta - x)^{v-1} f(\zeta) d\zeta, \tag{23.32}$$

$x \in [a,b]$. Define the subspace of functions

$$C_{b-}^{v}([a,b]) := \left\{ f \in C^n([a,b]) : J_{b-}^{1-\alpha} f^{(n)} \in C^1([a,b]) \right\} \tag{23.33}$$

Define the **right generalized** $v-$**fractional derivative** of f over $[a,b]$ as

$$D_{b-}^{v} f := (-1)^{n-1} D J_{b-}^{1-\alpha} f^{(n)}. \tag{23.34}$$

Notice that

$$J_{b-}^{1-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (\zeta - x)^{-\alpha} f^{(n)}(\zeta) d\zeta \tag{23.35}$$

exists for $f \in C_{b-}^{v}([a,b])$, and

$$D_{b-}^{v} f(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (\zeta - x)^{-\alpha} f^{(n)}(\zeta) d\zeta. \tag{23.36}$$

That is,

$$\left(D_{b-}^{v} f\right)(x) = \frac{(-1)^{n-1}}{\Gamma(n-v+1)} \frac{d}{dx} \int_x^b (\zeta - x)^{n-v} f^{(n)}(\zeta) d\zeta. \tag{23.37}$$

If $v \in \mathbb{N}$, then $\alpha = 0, n = v$, and

$$D_{b-}^{v} f(x) = (-1)^n f^{(n)}(x). \tag{23.38}$$

Lemma 23.16. Let $f \in C([a,b]), v \geq 1, n = [v], \alpha = v - n$. Then

$$\left(\left(J_{b-}^{v} f\right)(x)\right)^{(k)} = (-1)^k J_{b-}^{v-k} f(x), \tag{23.39}$$

$k = 0, 1, \ldots, n-1$.

Also

$$\left(\left(J_{b-}^{v} f\right)(x)\right)^{(n)} = (-1)^n J_{b-}^{\alpha} f(x), \tag{23.40}$$

if $\alpha > 0$,

and

$$\left(J_{b-}^{v} f\right)^{(n)} = (-1)^n f, \text{ if } \alpha = 0. \tag{23.41}$$

Proof. Clear by Proposition 23.14. ■

Theorem 23.17. $J_{b-}^{v} : C([a,b]) \hookrightarrow C([a,b]), v > 0$ is (1-1).

Proof. Let $f \in C([a,b])$ such that $J_{b-}^{v} f = 0$.
If $0 < v < 1$, then $J_{b-}^{1} f = J_{b-}^{1-v} J_{b-}^{v} f = 0$, Hence $J_{b-}^{1} f = 0$.
That is $(-1) f = (J_{b-}^{1} f)' = 0$, and $f = 0$.
If now $v \geq 1$, then $v = n + \alpha$, (where $n := [v], \alpha := v - n, n \geq 1$, and $0 \leq \alpha < 1$).
If $\alpha = 0$, then $J_{b-}^{n} f = 0$, hence $(-1)^n f = (J_{b-}^{n} f)^{(n)} = 0$, so that $f = 0$.
If $\alpha > 0$, then $J_{b-}^{\alpha} \left(J_{b-}^{n} f\right) = J_{b-}^{n+\alpha} f = J_{b-}^{v} f = 0$.
Hence by first case of this proof we get

$$J_{b-}^{n} f = 0.$$

And as in the second case of this proof we get $f = 0$.
So theorem's proof now is complete. ■

Remark 23.18. Let $f \in C_{b-}^{v}([a,b])$. We notice that

$$\begin{aligned} J_{b-}^{1}\left(D_{b-}^{v} f\right)(x) &= \int_{x}^{b}\left(D_{b-}^{v} f\right)(\zeta) d\zeta \\ &= (-1)^{n-1} \int_{x}^{b} \frac{d}{d\zeta}\left(J_{b-}^{1-\alpha} f^{(n)}\right)(\zeta) d\zeta \\ &= (-1)^{n-1}\left[\left(J_{b-}^{1-\alpha} f^{(n)}\right)(b)-\left(J_{b-}^{(1-\alpha)} f^{(n)}\right)(x)\right] \\ &= (-1)^{n} J_{b-}^{(1-\alpha)} f^{(n)}(x). \end{aligned} \tag{23.42}$$

That is

$$\begin{aligned} J_{b-}^{(1-\alpha)} f^{(n)}(x) &= (-1)^{n} J_{b-}^{1}\left(D_{b-}^{v} f\right)(x) \\ &= (-1)^{n} J_{b-}^{(1-\alpha)}\left(J_{b-}^{\alpha}\left(D_{b-}^{v} f\right)\right)(x) \end{aligned} \tag{23.43}$$

Hence by $J_{b-}^{1-\alpha}$ being (1-1) we obtain

$$f^{(n)}(x) = (-1)^{n} J_{b-}^{\alpha}\left(D_{b-}^{v} f\right)(x). \tag{23.44}$$

Therefore

$$\begin{aligned} J_{b-}^{n} f^{(n)}(x) &= (-1)^{n} J_{b-}^{n} J_{b-}^{\alpha}\left(D_{b-}^{v} f\right)(x) \\ &= (-1)^{n} J_{b-}^{n+\alpha}\left(D_{b-}^{v} f\right)(x). \end{aligned}$$

Thus

$$J_{b-}^{n} f^{(n)}(x) = (-1)^{n}\left(J_{b-}^{v} D_{b-}^{v} f\right)(x). \tag{23.45}$$

Let now $v \geq 1$, then

$$J_{b-}^{n} f^{(n)}(x) = (-1)^{n}\left\{f(x)-\sum_{k=0}^{n-1} \frac{f^{(k)}(b_{-})}{k!}(x-b)^{k}\right\}. \tag{23.46}$$

Therefore

$$f(x)-\sum_{k=0}^{n-1} \frac{f^{(k)}(b_{-})}{k!}(x-b)^{k} = \left(J_{b-}^{v} D_{b-}^{v} f\right)(x). \tag{23.47}$$

That is

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b_{-})}{k!}(x-b)^{k} + \left(J_{b-}^{v} D_{b-}^{v} f\right)(x). \tag{23.48}$$

If $0 < v < 1$, then $n = 0$.

Then clearly we get

$$f(x) = \left(J_{b-}^{v} D_{b-}^{v} f\right)(x). \tag{23.49}$$

We have proved the following Taylor fractional formulae

Theorem 23.19. Let $f \in C_{b-}^{v}([a,b]), v > 0, n := [v]$. Then

1. If $v \geq 1$, we get

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b_-)}{k!}(x-b)^k + \left(J_{b-}^{v} D_{b-}^{v} f\right)(x), \tag{23.50}$$

   $\forall x \in [a,b]$.

2. If $0 < v < 1$, we obtain

$$f(x) = J_{b-}^{v} D_{b-}^{v} f(x), \tag{23.51}$$

   $\forall x \in [a,b]$.

23.3 About the Right and Left Weyl Fractional Derivatives

Here we use concepts and some results from [179], [226].

Definition 23.20. Consider the class of **good** functions

$$E = \left\{ f \in C^{\infty}(\mathbb{R}) : \lim_{x \to +\infty} x^N f^{(k)}(x) = 0, \forall k \in \mathbb{Z}_+, \forall N \in \mathbb{N} \right\} \tag{23.52}$$

The **right Weyl fractional integral** for $f \in E$ is given by

$$W^{-v} f(x) := \frac{1}{\Gamma(v)} \int_x^{\infty} (\xi - x)^{v-1} f(\xi) d\xi, \tag{23.53}$$

$v > 0, \forall x \in \mathbb{R}$, and it exists. We set $W^0 := I$.

Let $v \geq 0$, and let $v = n - \lambda$, where $n \in \mathbb{N}$ and $0 < \lambda \leq 1$.

We define the **right Weyl fractional derivative** as

$$W^v f = (-1)^n D^n W^{-\lambda} f \tag{23.54}$$

for $f \in E$.

That is we have

$$W^v f(x) = \frac{(-1)^n}{\Gamma(\lambda)} \frac{d^n}{dx^n} \int_x^{\infty} (\xi - x)^{\lambda - 1} f(\xi) d\xi, \tag{23.55}$$

$\forall x \in \mathbb{R}, \forall f \in E.$

Remark 23.21. In [226] it is proved that

$$W^{\alpha} W^{\beta} = W^{\alpha+\beta}, \forall \alpha, \beta \in \mathbb{R}, \tag{23.56}$$

So that $W^0 = -DW^{-1}$.
Let $v > 0$, then $W^{-v} W^{v} = W^0 = I$.
Thus $W^{-v} W^{v} f = f, \forall f \in E$.
I.e.

$$f(x) = W^{-v}(W^{v} f)(x), \forall x \in \mathbb{R}. \tag{23.57}$$

More precisely we get from the above

Theorem 23.22. It holds

$$f(x) = \frac{1}{\Gamma(v)} \int_x^{\infty} (\xi - x)^{v-1} \left(W^{v} f\right)(\xi) d\xi, \tag{23.58}$$

$\forall x \in \mathbb{R}, \forall f \in E.$

One can rewrite the last one as

$$f(x) = \frac{1}{\Gamma(v)} \int_0^{\infty} z^{v-1} \left(W^{v} f\right)(x+z) dz, \tag{23.59}$$

$\forall x \in \mathbb{R}, \forall f \in E.$

We need further

Definition 23.23. Next we consider also the **alternative class of good functions**

$$E^* = \left\{ f \in C^{\infty}(\mathbb{R}) : \lim_{x \to -\infty} x^N f^{(k)}(x) = 0, \forall N \in \mathbb{N}, \forall k \in \mathbb{Z}_+ \right\} \tag{23.60}$$

(Notice that the Schwartz class of test functions in distribution theory equals $E \cap E^*$.)

We define the **left Weyl fractional integral,** $v > 0$,

$$W_*^{-v} f(x) := \frac{1}{\Gamma(v)} \int_{-\infty}^{x} (x - \xi)^{v-1} f(\xi) d\xi, \tag{23.61}$$

$\forall f \in E^*$. We set $W_*^0 := I$.

Fact. It is known $(\mu, v \geq 0)$, see [179], that

$$W_*^{-v} W_*^{-\mu} = W_*^{-(v+\mu)}. \tag{23.62}$$

Definition 23.24. For $v \geq 0, v = n - \lambda, n \in \mathbb{N}, 0 < \lambda \leq 1$, we define the **left Weyl fractional derivative** as

$$W_*^v f(x) \quad : \quad = D^n W_*^{-\lambda} f(x) \quad = \frac{1}{\Gamma(\lambda)} \frac{d^n}{dx^n} \int_{-\infty}^{x} (x-\xi)^{\lambda-1} f(\xi) d\xi, \tag{23.63}$$

$\forall x \in \mathbb{R}, \forall f \in E^*$.

For $f \in E^*$, we notice that $g(x) := f(-x) \in E$.

Remark 23.25. We see that

$$\begin{aligned} W_*^{-v} W_*^v &= W_*^{-v} D^n W_*^{-\lambda} \overset{(23.65)}{=} W_*^{-v} W_*^{-\lambda} D^n \\ &= W_*^{-(v+\lambda)} D^n = W_*^{-n} D^n \overset{(23.68)}{=} I. \end{aligned} \tag{23.64}$$

We want to prove

$$D^n W_*^{-\lambda} = W_*^{-\lambda} D^n \tag{23.65}$$

Indeed we notice that

$$W_*^{-\lambda} f(x) = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} z^{\lambda-1} f(x-z) dz \tag{23.66}$$

Thus

$$\begin{aligned} \left(W_*^{-\lambda} f(x) \right)^{(n)} &= \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} z^{\lambda-1} f^{(n)}(x-z) dz \\ &= W_*^{-\lambda} f^{(n)}(x), \end{aligned} \tag{23.67}$$

proving (23.65).

Also we want to prove

$$W_*^{-n} D^n = I, \forall f \in E^*. \tag{23.68}$$

Indeed we have

$$\begin{aligned} W_*^{-n} f^{(n)}(x) &= \frac{1}{(n-1)!} \int_{-\infty}^{x} (x-\xi)^{n-1} f^{(n)}(\xi) d\xi \\ &= f(x), \text{ by } f \in E^*, \forall x \in \mathbb{R}, \end{aligned} \tag{23.69}$$

see also next Remark 23.27, proving (23.68).

So from (23.64) we derived that

$$W_*^{-v} W_*^v f(x) = f(x), \forall f \in E^*, \forall x \in \mathbb{R}. \tag{23.70}$$

The last gives

Theorem 23.26. It holds

$$f(x) = \frac{1}{\Gamma(v)} \int_{-\infty}^{x} (x-\xi)^{v-1} (W_*^v f)(\xi) d\xi, \tag{23.71}$$

$\forall f \in E^*, \forall x \in \mathbb{R}$.

One can rewrite (23.71) as

$$f(x) = \frac{1}{\Gamma(v)} \int_{0}^{+\infty} z^{v-1} (W_*^v f)(x-z) dz, \tag{23.72}$$

$\forall f \in E^*, \forall x \in \mathbb{R}$.

As related material we make
Remark 23.27. Let $f \in C^n(\mathbb{R}), n \in \mathbb{N}$.

I) The following are equivalent

$$f(x) = \frac{1}{(n-1)!} \int_{-\infty}^{x} (x-t)^{n-1} f^{(n)}(t) dt, \forall x \in \mathbb{R}, \tag{23.73}$$

$$\Longleftrightarrow$$

$$\lim_{a \to -\infty} f^{(k)}(a)(x-a)^k = 0, \forall x \in \mathbb{R}, \text{ all } k = 0, 1, \ldots, n-1, \tag{23.74}$$

$$\Longleftrightarrow$$

$$\lim_{a \to -\infty} f^{(k)}(a)(x-a)^k = 0, \text{ for some } x \in \mathbb{R}, \text{ all } k = 0, 1, \ldots, n-1, \tag{23.75}$$

$$\Longleftrightarrow$$

$$\lim_{a \to -\infty} a^k f^{(k)}(a) = 0, \text{ all } k = 0, 1, \ldots, n-1. \tag{23.76}$$

And

$$\lim_{a \to -\infty} a^{n-1} f^{(k)}(a) = 0, \text{ all } k = 0, 1, \ldots, n-1, \tag{23.77}$$

implies (23.73).

This equivalence is established mainly by the use of Taylor's formula with integral remainder, etc.

The subclass of functions $f \in C^n(\mathbb{R})$ with (23.73) valid is rich.

II) Similarly, the following are equivalent

$$(-1)^n f(x) = \frac{1}{(n-1)!} \int_{x}^{+\infty} (t-x)^{n-1} f^{(n)}(t) dt, \forall x \in \mathbb{R}, \tag{23.78}$$

$$\Longleftrightarrow$$

$$\lim_{b \to +\infty} f^{(k)}(b)(x-b)^k = 0, \forall x \in \mathbb{R}, \text{ all } k = 0, 1, \ldots, n-1, \tag{23.79}$$

$$\Longleftrightarrow$$

$$\lim_{b \to +\infty} f^{(k)}(b)(x-b)^k = 0, \text{ some } x \in \mathbb{R}, \text{ all } k = 0, 1, \ldots, n-1, \tag{23.80}$$

$$\Longleftrightarrow$$

$$\lim_{b \to +\infty} b^k f^{(k)}(b) = 0, \text{ all } k = 0, 1, \ldots, n-1. \tag{23.81}$$

And

$$\lim_{b \to +\infty} b^{n-1} f^{(k)}(b) = 0, \text{ all } k = 0, 1, \ldots, n-1, \tag{23.82}$$

implies (23.78).

The subclass of $f \in C^n(\mathbb{R})$ as in (23.78) is also rich.

23.4 Consequences

1. By Theorem 23.11, for $f \in AC^m([a,b]), x \in [a,b], \alpha > 0, m = \lceil \alpha \rceil$, and $f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$, we obtain

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta - x)^{\alpha - 1} D_{b-}^{\alpha} f(\zeta) d\zeta. \tag{23.83}$$

And when $f^{(k)}(a) = 0, k = 0, 1, \ldots, m-1$, by Corollary 3.6, p.40, [145], we get

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \zeta)^{\alpha - 1} D_{*a}^{\alpha} f(\zeta) d\zeta, \tag{23.84}$$

where $D_{*a}^{\alpha} f$ is the left Caputo fractional derivative of f of order α and anchored at a.

If both $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$, then by the above we find that

$$f(x) = \frac{1}{2\Gamma(\alpha)} \left[\int_a^x (x - \zeta)^{\alpha - 1} D_{*a}^{\alpha} f(\zeta) d\zeta + \int_x^b (\zeta - x)^{\alpha - 1} D_{b-}^{\alpha} f(\zeta) d\zeta \right], \tag{23.85}$$

$\forall x \in [a, b]$.

2. Let $f \in C_{b-}^{v}([a,b]), v \geq 1, n := [v], f^{(k)}(b) = 0,\ k = 0, 1, \ldots, n-1$. Then, by Theorem 23.19, part (1), we derive

$$\begin{aligned} f(x) &= (J_{b-}^{v} D_{b-}^{v} f)(x) \\ &= \frac{1}{\Gamma(v)} \int_x^b (\zeta - x)^{v-1} \left(D_{b-}^{v} f \right)(\zeta) d\zeta, \end{aligned} \tag{23.86}$$

$\forall x \in [a,b]$.

Also let $f \in C_a^v([a,b]) := \{ f \in C^n([a,b]) \mid$ $\gamma(x) := \left(\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} f^{(n)}(t) dt \right) \in C^1([a,b]) \right\}$, $v \geq 1, n := [v]$, with $f^{(k)}(a) = 0,\ k = 0, 1, \ldots, n-1$. Then by Theorem 25.1, part (1), p. 540 of [23], we get

$$f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} D_a^v f(t) dt, \tag{23.87}$$

$\forall x \in [a,b]$.

Here $C_a^v([a,b])$ and $D_a^v f := \gamma'$ are as in $p.540$ of [23].
If $f \in C_a^v([a,b]) \cap C_{b-}^v([a,b]), v \geq 1, n := [v]$, with $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, \ldots, n-1$, then

$$f(x) = \frac{1}{2\Gamma(v)} \left[\int_a^x (x-t)^{v-1} D_a^v f(t) dt + \int_x^b (\zeta - x)^{v-1} \left(D_{b-}^v f \right)(\zeta) d\zeta \right]. \tag{23.88}$$

3. Let I be an interval $\subset \mathbb{R}$ of finite or infinite length, $x_0 \in I$, and μ a positive finite measure on the Borel $\sigma-$algebra of I. Let $f \in C^m(I), m := \lceil \alpha \rceil, \alpha > 0$. Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x - \zeta)^{\alpha-1} D_{*x_0}^{\alpha} f(\zeta) d\zeta, \tag{23.89}$$

$\forall x \in I : x \geq x_0$.
Also it holds

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_x^{x_0} (\zeta - x)^{\alpha-1} D_{x_0-}^{\alpha} f(\zeta) d\zeta, \tag{23.90}$$

$\forall x \in I : x \leq x_0$.

Consequently we obtain

$$\begin{aligned}
\int_I f(x)d\mu(x) &= \int_{\{x\in I: x\le x_0\}} f(x)d\mu(x) + \int_{\{x\in I: x\ge x_0\}} f(x)d\mu(x) \\
&= \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{\{x\in I: x\le x_0\}} (x-x_0)^k d\mu(x) + \\
&\quad \frac{1}{\Gamma(\alpha)} \int_{\{x\in I: x\le x_0\}} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} D^{\alpha}_{x_0-} f(\zeta) d\zeta \right) d\mu(x) + \\
&\quad + \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{\{x\in I: x\ge x_0\}} (x-x_0)^k d\mu(x) + \qquad (23.91) \\
&\quad \frac{1}{\Gamma(\alpha)} \int_{\{x\in I: x\ge x_0\}} \left(\int_{x_0}^{x} (x-\zeta)^{\alpha-1} D^{\alpha}_{*x_0} f(\zeta) d\zeta \right) d\mu(x).
\end{aligned}$$

So we derive

$$\begin{aligned}
\int_I f(x)d\mu(x) &= \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_I (x-x_0)^k d\mu(x) + \\
&\quad \frac{1}{\Gamma(\alpha)} \Bigg\{ \int_{\{x\in I: x\le x_0\}} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} D^{\alpha}_{x_0-} f(\zeta) d\zeta \right) d\mu(x) + \\
&\quad \int_{\{x\in I: x> x_0\}} \left(\int_{x_0}^{x} (x-\zeta)^{\alpha-1} D^{\alpha}_{*x_0} f(\zeta) d\zeta \right) d\mu(x) \Bigg\}, \qquad (23.92)
\end{aligned}$$

etc.

In (23.92) we assume that all integrals exist.

We can do similar things with the generalized right and left $v-$ fractional derivatives; see Section 23.2 and [23], p. 540, and Section 23.4, Part 2.

One can exploit in analogous ways Theorem 23.22 and Theorem 23.26, regarding the right and left Weyl fractional derivatives.

24

Fractional Convergence Theory of Positive Linear Operators

In this chapter we study quantitatively with rates the weak convergence of a sequence of finite positive measures to the unit measure. Equivalently we study quantitatively the pointwise convergence of sequence of positive linear operators to the unit operator, all acting on continuous functions. From there we obtain with rates the corresponding uniform convergence of the latter. The inequalities for all of the above in their right hand sides contain the moduli of continuity of the right and left Caputo fractional derivatives of the involved function. From the uniform Shisha-Mond type inequality we derive the fractional Korovkin type theorem regarding the uniform convergence of positive linear operators to the unit. We give applications, especially to Bernstein polynomials for which we establish fractional quantitative results.

In the background we prove several fractional calculus results useful to approximation theory and not only. This chapter relies on [43].

24.1 Introduction

In this chapter among others we are motivated by the following results

Theorem 24.1. (P. P. Korovkin [213], (1960), p. 14) Let $[a,b]$ be a closed interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators mapping $C([a,b])$ into itself. Suppose that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$. Then $(L_n f)$ converges uniformly to f on $[a,b]$ for all functions $f \in C([a,b])$.

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 355–376.
springerlink.com

Let $f \in C([a,b])$ and $0 \leq h \leq b-a$. The first modulus of continuity of f at h is given by

$$\omega_1(f,h) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq h}} |f(x) - f(y)| .$$

If $h > b-a$, then we define $\omega_1(f,h) = \omega_1(f, b-a)$.

Another motivation is the following

Theorem 24.2. (Shisha and Mond [264], (1968)) Let $[a,b] \subset \mathbb{R}$ a closed interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$ into itself. For $n = 1, \ldots,$ suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, \ldots,$ we have

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \|L_n 1 - 1\|_\infty + \|L_n 1 + 1\|_\infty \omega_1(f, \mu_n) \tag{24.1}$$

where

$$\mu_n = \left\|L_n\left((t-x)^2\right)(x)\right\|_\infty^{\frac{1}{2}}$$

and $\|\cdot\|_\infty$ stands for the sup-norm over $[a,b]$.

One can easily see, for $n = 1, 2, \ldots$

$$\mu_n^2 \leq \left\|L_n\left(t^2; x\right) - x^2\right\|_\infty + 2c\,\|L_n(t;x) - x\|_\infty + c^2\,\|L_n(1;x) - 1\|_\infty ,$$

where $c = \max(|a|, |b|)$.

Thus, given the Korovkin assumptions (see Theorem 24.1) as $n \to \infty$ we get $\mu_n \to 0$, and by (24.1) that $\|L_n f - f\|_\infty \to 0$ for any $f \in C([a,b])$. That is one derives the Korovkin conclusion in a quantitative way and with rates of convergence.

One more motivation follows

Theorem 24.3. (See Corollary 7.2.2, p. 219, [16]) Consider the positive linear operator

$$L : C^n([a,b]) \to C([a,b]), n \in \mathbb{N}.$$

Let

$$\begin{aligned} c_k(x) &= L((t-x)^k, x), k = 0, 1, \ldots, n; \\ d_n(x) &= \left[L(|t-x|^n, x)\right]^{\frac{1}{n}};\ c(x) = \max(x-a, b-x) \quad \left(c(x) \geq \frac{b-a}{2}\right). \end{aligned}$$

Let $f \in C^n([a,b])$ such that $\omega_1\left(f^{(n)}, h\right) \leq w$, where w, h are fixed positive numbers, $0 < h < b-a$. Then

$$|L(f,x) - f(x)| \leq |f(x)|\,|c_0(x) - 1| + \sum_{k=1}^{n} \frac{\left|f^{(k)}(x)\right|}{k!} |c_k(x)| + R_n. \tag{24.2}$$

Here

$$R_n = w\phi_n(c(x)) \left(\frac{d_n(x)}{c(x)}\right)^n = \frac{w}{n!}\theta_n\left(\frac{h}{c(x)}\right) d_n^n(x), \text{ where } \theta_n\left(\frac{h}{u}\right) = n!\phi_n(u)/u^n,$$

with

$$\phi_n(x) = \int_0^{|x|} \left\lceil \frac{t}{h} \right\rceil \frac{(|x| - t)^{n-1}}{(n-1)!} dt, \ (x \in \mathbb{R}),$$

$\lceil \cdot \rceil$ is the ceiling of the number.

Inequality (24.2) is sharp. It is approximately attained by $w\phi_n((t-x)_+)$ and a measure μ_x supported by $\{x, b\}$ when $x - a \leq b - x$, also approximately attained by $w\phi_n((x-t)_+)$ and a measure μ_x supported by $\{x, a\}$ when $x - a \geq b - x$: in each case with masses $c_0(x) - \left(\frac{d_n(x)}{c(x)}\right)^n$ and $\left(\frac{d_n(x)}{c(x)}\right)^n$, respectively.

Using the last method and its refinements one gets nice and simple results for specific operators.

For example from Corollary 7.3.4, p. 230, [16], we obtain:

let $f \in C^1([0,1])$ and consider the Bernstein polynomials

$$(B_n f)(t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}, \quad t \in [0,1],$$

then $\|B_n f - f\|_\infty \leq \frac{0.78125}{\sqrt{n}} \omega_1 \left(f', \frac{1}{4\sqrt{n}}\right)$ .So $B_n f \xrightarrow{\mu} f$ as $n \to \infty$ with rates.

In this chapter we study quantitatively the rate of weak convergence of a sequence of finite positive measures to the unit measure given the existence and presence of the left and right Caputo fractional derivatives of the involved function. That is in the right hand sides of the derived inequalities appear the first moduli of continuity of the above mentioned fractional derivatives, see Theorem 24.25 and Corollary 24.26.

Then via the Riesz representation theorem we transfer Theorem 24.25 into the language of quantitative pointwise convergence of a sequence of positive linear operators to the unit operator, all operators acting from $C([a,b])$ into itself, see Theorem 24.27, Corollary 24.28 and Theorem 24.30.

From there we obtain quantitative results with respect to the sup-norm $\|\cdot\|_\infty$, regarding the uniform convergence of positive linear operators to the unit. Again in the right hand side of our inequalities we have moduli of continuity with respect to right and left Caputo derivatives of the engaged function. For the latter see Theorem 24.32, a Shisha-Mond type result. From there we derive the latter Korovkin type convergence theorem at the fractional level, see Theorem 24.33.

We give many of applications of the fractional Shisha-Mond and Korovkin theory, see Corollaries 24.35-24.38.

In the background section we present many interesting fractional results which by themselves have their own merit.

In approximation theory the involvement of fractional derivatives is very rare, almost nothing exists. The only fractional articles that exist are of V. Dzyadyk [153] of 1959, F. Nasibov [233] of 1962, J. Demjanovic [140] of 1975, and of M. Jaskolski [196] of 1989, all regarding estimates to best approximation of functions by algebraic and trigonometric polynomials.

24.2 Background

We mention

Definition 24.4. Let $v \geq 0$, $n = \lceil v \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [145], p. 38, [160], [259]) the function

$$D_{*a}^{v} f(x) = \frac{1}{\Gamma(n-v)} \int_a^x (x-t)^{n-v-1} f^{(n)}(t) dt, \tag{24.3}$$

$\forall x \in [a,b]$, where Γ is the gamma function $\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt$, $v > 0$.

We set $D_{*a}^{0} f(x) = f(x), \forall x \in [a,b]$.

Example 24.5. Take $v = \frac{1}{2}$, then $n = 1$ and $f(t) = t^\beta \in C([0,1])$, $0 < \beta \leq \frac{1}{2}$, $t \in [0,1]$. See that $f'(t) = \beta t^{\beta-1} \in L_1([0,1])$.

We see that

$$D_{*0}^{\frac{1}{2}} f(x) = \frac{\beta}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-t)^{-\frac{1}{2}} t^{\beta-1} dt. \tag{24.4}$$

By setting $t = xs$, $dt = xds$,

$$D_{*0}^{\frac{1}{2}} f(x) = \frac{\beta}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 (x-xs)^{-\frac{1}{2}} x^{\beta-1} s^{\beta-1} x ds = \frac{\Gamma(\beta+1)}{\Gamma\left(\beta+\frac{1}{2}\right)} x^{\beta-\frac{1}{2}}.$$

Let $0 < \beta < \frac{1}{2}$, then $D_{*0}^{\frac{1}{2}} f(0) = +\infty$. Let $\beta = \frac{1}{2}$, then

$$D_{*0}^{\frac{1}{2}} f(0) = \frac{\sqrt{\pi}}{2} > 0, \tag{24.5}$$

a positive real number!

Conclusion: In general for $D_{*a}^{v} f(a)$ we do not know what it is, it could be infinite, or finite non-zero, or zero! (see next).

Lemma 24.6. Let $v > 0$, $v \notin \mathbb{N}$, $n = \lceil v \rceil$, $f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_\infty([a,b])$. Then $D_{*a}^{v} f(a) = 0$.

Proof. By (24.3) we derive

$$\left| D_{*a}^{v} f(x) \right| \leq \frac{1}{\Gamma(n-v)} \int_a^x (x-t)^{n-v-1} \left| f^{(n)}(t) \right| dt \leq \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-v+1)} (x-a)^{n-v}.$$

That is

$$\left| D_{*a}^{v} f(x) \right| \leq \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-v+1)} (x-a)^{n-v}, \ \forall x \in [a,b]. \tag{24.6}$$

That is $D_{*a}^{v} f(a) = 0$. ■

We need

Definition 24.7. (see also [160], [155], [44]) Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta, \tag{24.7}$$

$\forall x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$.

Lemma 24.8. Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$.

Proof. As in Lemma 24.6. ■

Lemma 24.9. Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$; μ is a positive finite measure on the Borel σ-algebra of $[a,b]$, $x_0 \in [a,b]$. Then

$$\begin{aligned} E_{x_0} \quad : \quad &= \int_{[a,b]} f(x) d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,b]} (x - x_0)^k d\mu(x) \qquad (24.8) \\ &= \frac{1}{\Gamma(\alpha)} \left\{ \int_{[a,x_0]} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} \left(D_{x_0-}^{\alpha} f(\zeta) - D_{x_0-}^{\alpha} f(x_0) \right) d\zeta \right) d\mu(x) + \right. \\ &\left. \int_{(x_0,b]} \left(\int_{x_0}^{x} (x - \zeta)^{\alpha-1} \left(D_{*x_0}^{\alpha} f(\zeta) - D_{*x_0}^{\alpha} f(x_0) \right) d\zeta \right) d\mu(x) \right\}. \end{aligned}$$

Proof. From [145], p. 40, we get by left Caputo fractional Taylor formula that

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - \zeta)^{\alpha-1} D_{*x_0}^{\alpha} f(\zeta) d\zeta, \tag{24.9}$$

for all $x_0 < x \leq b$.

Also from [44], using the right Caputo fractional Taylor formula we have

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (\zeta - x)^{\alpha-1} D_{x_0-}^{\alpha} f(\zeta) d\zeta, \tag{24.10}$$

for all $a \leq x \leq x_0$.

Consequently we find

$$\int_{[a,b]} f(x) d\mu(x) = \int_{[a,x_0]} f(x) d\mu(x) + \int_{(x_0,b]} f(x) d\mu(x)$$

$$= \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,b]} (x-x_0)^k d\mu(x) + \tag{24.11}$$
$$\frac{1}{\Gamma(\alpha)} \left\{ \int_{[a,x_0]} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} D_{x_0-}^{\alpha} f(\zeta) d\zeta \right) d\mu(x) + \right.$$
$$\left. \int_{(x_0,b]} \left(\int_{x_0}^{x} (x-\zeta)^{\alpha-1} D_{*x_0}^{\alpha} f(\zeta) d\zeta \right) d\mu(x) \right\}.$$

Notice also that $D_{x_0-}^{\alpha} f(x_0) = D_{*x_0}^{\alpha} f(x_0) = 0$.
The proof of (24.8) is now complete. ■

Convention 24.10. We suppose that

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{24.12}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{24.13}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 24.11. Let $f \in C^n([a,b])$, $n = \lceil v \rceil, v > 0$. Then $D_{*a}^{v} f(x)$ is continuous in $x \in [a,b]$.

Proof. We notice that

$$D_{*a}^{v} f(x) = \frac{1}{\Gamma(n-v)} \int_0^{x-a} z^{n-v-1} f^{(n)}(x-z) dz, \tag{24.14}$$

and

$$D_{*a}^{v} f(y) = \frac{1}{\Gamma(n-v)} \int_0^{y-a} z^{n-v-1} f^{(n)}(y-z) dz.$$

Here $a \le x \le y \le b$, and $0 \le x-a \le y-a$.

Hence it holds

$$D_{*a}^{v} f(y) - D_{*a}^{v} f(x) = \frac{1}{\Gamma(n-v)} \left[\int_0^{x-a} z^{n-v-1} \left(f^{(n)}(y-z) - f^{(n)}(x-z) \right) dz \right.$$
$$\left. + \int_{x-a}^{y-a} z^{n-v-1} f^{(n)}(y-z) dz \right]. \tag{24.15}$$

We have that

$$|D_{*a}^{v} f(y) - D_{*a}^{v} f(x)| \le \frac{1}{\Gamma(n-v)} \left[\frac{(x-a)^{n-v}}{(n-v)} \omega_1(f^{(n)}, |y-x|) \right.$$
$$\left. + \frac{\left\| f^{(n)} \right\|_{\infty}}{(n-v)} \left((y-a)^{n-v} - (x-a)^{n-v} \right) \right]$$
$$\le \frac{1}{\Gamma(n-v)} \left[\frac{(b-a)^{n-v}}{(n-v)} \omega_1(f^{(n)}, |y-x|) + \frac{\left\| f^{(n)} \right\|_{\infty}}{(n-v)} \left((y-a)^{n-v} - (x-a)^{n-v} \right) \right].$$

So as $y \to x$ the last expression goes to zero. As a result,

$$D_{*a}^{v} f(y) \to D_{*a}^{v} f(x), \tag{24.16}$$

proving the claim. ∎

Proposition 24.12. Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.

Proof. As in Proposition 24.11. ∎

We also mention

Proposition 24.13. Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{24.17}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proof. Fix $x : x \geq y_0 \geq x_0$; $x, x_0, y_0 \in [a,b]$.Then

$$\left| D_{*x_0}^{\alpha} f(x) - D_{*y_0}^{\alpha} f(x) \right| = \frac{1}{\Gamma(m-\alpha)} \left| \int_{x_0}^{y_0} (x-t)^{m-\alpha-1} f^{(m)}(t)dt \right| \tag{24.18}$$

$$\leq \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha)} \left(\int_{x_0}^{y_0} (x-t)^{m-\alpha-1} dt \right) = \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} \left((x-y_0)^{m-\alpha} - (x-x_0)^{m-\alpha} \right)$$

$\to 0$, as $y_0 \to x_0$, proving continuity of $D_{*x_0}^{\alpha} f$ in $x_0 \in [a,b]$. ∎

Proposition 24.14. Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)d\zeta, \tag{24.19}$$

for all $x, x_0 \in [a,b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

Proof. As in Proposition 24.13. ∎

We need

Proposition 24.15. Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)dt, \text{ for } x \geq x_0, \tag{24.20}$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in (x, x_0) on $[a,b]^2$.

Proof. We notice that $L(x_0, x_0) = 0$.

Suppose $x \geq x_0$, then

$$L(x, x_0) = \int_0^{x-x_0} z^{c-1} g(x-z)dz = \int_0^{b-a} \chi_{[0,x-x_0]}(z) z^{c-1} g(x-z)dz, \tag{24.21}$$

where χ is the characteristic function.

Let $x_N \to x, x_{0N} \to x_0, N \in \mathbb{N}$ and assume without loss of generality that $x_N \geq x_{0N}$.

So we have again

$$L(x_N, x_{0N}) = \int_0^{x_N - x_{0N}} z^{c-1} g(x_N - z) dz = \int_0^{b-a} \chi_{[0, x_N - x_{0N}]}(z) z^{c-1} g(x_N - z) dz. \tag{24.22}$$

We have that

$$\chi_{[0, x_N - x_{0N}]}(z) \to \chi_{[0, x - x_0]}(z), \quad a.e., \tag{24.23}$$

and

$$\chi_{[0, x_N - x_{0N}]}(z) z^{c-1} g(x_N - z) \to \chi_{[0, x - x_0]}(z) z^{c-1} g(x - z), \quad a.e. \tag{24.24}$$

Notice that

$$\chi_{[0, x_N - x_{0N}]}(z) z^{c-1} |g(x_N - z)| \leq z^{c-1} \|g\|_\infty, \tag{24.25}$$

which is an integrable function.

Thus by Dominated Convergence theorem we obtain

$$L(x_N, x_{0N}) \to L(x, x_0), \text{ as } N \to \infty. \tag{24.26}$$

Clearly now $L(x, x_0)$ is jointly continuous on $[a, b]^2$. ■

We mention

Proposition 24.16. Let $g \in C([a, b])$, $0 < c < 1$, $x, x_0 \in [a, b]$. Define

$$K(x, x_0) = \int_x^{x_0} (\zeta - x)^{c-1} g(\zeta) d\zeta, \text{ for } x \leq x_0, \tag{24.27}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a, b]^2$ into $\mathbb{R}$.

Proof. As in Proposition 24.15. ■

Based on Propositions 24.15, 24.16 we get

Corollary 24.17. Let $f \in C^m([a, b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a, b]$.Then $D^\alpha_{*x_0} f(x)$, $D^\alpha_{x_0 -} f(x)$ are jointly continuous functions in (x, x_0) from $[a, b]^2$ into $\mathbb{R}$.

We need

Theorem 24.18. Let $f : [a, b]^2 \to \mathbb{R}$ be jointly continuous. Consider

$$G(x) = \omega_1(f(\cdot, x), \delta, [x, b]),$$

$\delta > 0$, $x \in [a, b]$.

Then G is continuous on $[a, b]$.

Proof. **(i)** Let $x_n \to x$, $a \leq x_n \leq x$, and $0 < \delta \leq b - x$ first (The case when $x_n \to x$ with $x_n \geq x$ is similar). Then we can write

$$G(x_n) = \max(A, B, C),$$

where

$$\begin{aligned} A &= \sup\{|f(u, x_n) - f(v, x_n)|; u, v \in [x, b], |u - v| \leq \delta\}, 0 < \delta \leq b - x, \\ B &= \sup\{|f(u, x_n) - f(v, x_n)|; u \in [x_n, x], v \in [x, b], |u - v| \leq \delta\}, \\ C &= \sup\{|f(u, x_n) - f(v, x_n)|; u, v \in [x_n, x], |u - v| \leq \delta\}. \end{aligned}$$

Now, when $x_n \to x$, then $A \to G(x), B \to K(x) \leq G(x), C \to 0$ (since also u converges to v).

In conclusion, $G(x_n) \to \max\{G(x), K(x), 0\} = G(x)$.

(ii) If $\delta > b - x$, then $\omega_1(f(\cdot, x), \delta, [x, b]) = \omega_1(f(\cdot, x), b - x, [x, b])$, a case covered by (i).

That is proving the claim. ∎

Theorem 24.19. Let $f : [a, b]^2 \to \mathbb{R}$ be jointly continuous. Then

$$H(x) = \omega_1(f(\cdot, x), \delta, [a, x]),$$

$x \in [a, b]$, is continuous in $x \in [a, b], \delta > 0$.

Proof. As in Theorem 24.18. ∎

We make

Remark 24.20. Let μ be a finite positive measure on Borel $\sigma-$algebra of $[a, b]$. Let $\alpha > 0$, then by Hölder's inequality we find

$$\int_{[a,x_0]} (x_0 - x)^\alpha d\mu(x) \leq \left(\int_{[a,x_0]} (x_0 - x)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu([a, x_0])^{\frac{1}{(\alpha+1)}}, \tag{24.28}$$

and

$$\int_{(x_0,b]} (x - x_0)^\alpha d\mu(x) \leq \left(\int_{(x_0,b]} (x - x_0)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu((x_0, b])^{\frac{1}{(\alpha+1)}}. \tag{24.29}$$

Let now $m = \lceil \alpha \rceil, \alpha \notin \mathbb{N}$, $\alpha > 0$, $k = 1, \ldots, m - 1$. Then by applying again Hölder's inequality we get

$$\int_{[a,b]} |x - x_0|^k \, d\mu(x) \leq \left(\int_{[a,b]} |x - x_0|^{\alpha+1} \, d\mu(x) \right)^{\frac{k}{(\alpha+1)}} \mu([a, b])^{\frac{\alpha+1-k}{(\alpha+1)}}. \tag{24.30}$$

Terminology 24.21. Let $L_N : C([a, b]) \to C([a, b])$, $N \in \mathbb{N}$, be a sequence of positive linear operators. By Riesz representation theorem (see [257], p. 304) we have

$$L_N(f, x_0) = \int_{[a,b]} f(t) d\mu_{Nx_0}(t), \tag{24.31}$$

$\forall x_0 \in [a,b]$, where μ_{Nx_0} is a unique positive finite measure on $\sigma-$Borel algebra of $[a,b]$. Set

$$L_N(1,x_0) = \mu_{Nx_0}([a,b]) = M_{Nx_0}. \tag{24.32}$$

We make

Remark 24.22. Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil v \rceil, v > 0, v \notin \mathbb{N}$. Then as in the proof of Lemma 24.6, we have

$$|D_{*a}^v f(x)| \le \frac{\left\|f^{(n)}\right\|_\infty}{\Gamma(n-v+1)}(x-a)^{n-v}, \forall x \in [a,b]. \tag{24.33}$$

Thus we observe

$$\omega_1(D_{*a}^v f, \delta) = \sup_{\substack{x,y\in[a,b]\\|x-y|\le\delta}} |D_{*a}^v f(x) - D_{*a}^v f(y)|$$

$$\le \sup_{\substack{x,y\in[a,b]\\|x-y|\le\delta}} \left(\frac{\left\|f^{(n)}\right\|_\infty}{\Gamma(n-v+1)}(x-a)^{n-v} + \frac{\left\|f^{(n)}\right\|_\infty}{\Gamma(n-v+1)}(y-a)^{n-v} \right) \tag{24.34}$$

$$\le \frac{2\left\|f^{(n)}\right\|_\infty}{\Gamma(n-v+1)}(b-a)^{n-v}. \tag{24.35}$$

Consequently

$$\omega_1(D_{*a}^v f, \delta) \le \frac{2\left\|f^{(n)}\right\|_\infty}{\Gamma(n-v+1)}(b-a)^{n-v}. \tag{24.36}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil, \alpha > 0, \alpha \notin \mathbb{N}$, then

$$\omega_1(D_{b-}^\alpha f, \delta) \le \frac{2\left\|f^{(m)}\right\|_\infty}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{24.37}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil, \alpha > 0, \alpha \notin \mathbb{N}$, we obtain

$$\sup_{x_0\in[a,b]} \omega_1(D_{*x_0}^\alpha f, \delta)_{[x_0,b]} \le \frac{2\left\|f^{(m)}\right\|_\infty}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}, \tag{24.38}$$

and

$$\sup_{x_0\in[a,b]} \omega_1(D_{x_0-}^\alpha f, \delta)_{[a,x_0]} \le \frac{2\left\|f^{(m)}\right\|_\infty}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{24.39}$$

We also make

Remark 24.23. Let $L_N : C([a,b]) \to C([a,b])$, $N \in \mathbb{N}$, be a sequence of positive linear operators. Using (24.31) and Hölder's inequality we obtain ($x \in [a,b], k = 1, \ldots, m-1, m = \lceil \alpha \rceil, \alpha \notin \mathbb{N}, \alpha > 0$) for $k = 1, \ldots, m-1$ that

$$\left\| L_N(|\cdot - x|^k, x) \right\|_\infty \leq \left\| L_N(|\cdot - x|^{\alpha+1}, x) \right\|_\infty^{\frac{k}{(\alpha+1)}} \| L_N 1 \|_\infty^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}. \tag{24.40}$$

Also we see that

$$C([a,b]) \ni |\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot) \leq |\cdot - x|^{\alpha+1}, \forall x \in [a,b], \tag{24.41}$$

and

$$C([a,b]) \ni |\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot) \leq |\cdot - x|^{\alpha+1}, \forall x \in [a,b]. \tag{24.42}$$

By positivity of L_N we obtain

$$\left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x) \right\|_\infty \leq \left\| L_N(|\cdot - x|^{\alpha+1}, x) \right\|_\infty, \tag{24.43}$$

and

$$\left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x) \right\|_\infty \leq \left\| L_N(|\cdot - x|^{\alpha+1}, x) \right\|_\infty. \tag{24.44}$$

So if the right hand side of each of (24.43), (24.44) tends to zero, so do the left hand sides of these.

We also make

Remark 24.24. Let $\alpha > 0, \alpha \notin \mathbb{N}$. Take $a \leq x \leq x_0$, then

$$(x_0 - x)^{\alpha+1} \leq (x_0 - x)^{\alpha+1} 1 + 0.$$

Similarly, for $x_0 \leq x \leq b$, we get

$$(x - x_0)^{\alpha+1} \leq 0 + (x - x_0)^{\alpha+1} \cdot 1.$$

So we have

$$|\cdot - x|^{\alpha+1} \leq |\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot) + |\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), \forall x \in [a,b]. \tag{24.45}$$

Thus, by positivity of L_N,we get

$$\begin{aligned} \left\| L_N\left(|\cdot - x|^{\alpha+1}, x\right) \right\|_\infty &\leq \left\| L_N\left(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x\right) \right\|_\infty \\ &+ \left\| L_N\left(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x\right) \right\|_\infty. \end{aligned} \tag{24.46}$$

So if both $\left\| L_N\left(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x\right) \right\|_\infty, \left\| L_N\left(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x\right) \right\|_\infty \to 0$, as $N \to \infty$, then $\left\| L_N\left(|\cdot - x|^{\alpha+1}, x\right) \right\|_\infty \to 0$.

24.3 Main Results

We present the first main result

Theorem 24.25. Let $f \in AC^m([a,b])$, $f^{(m)} \in L_\infty([a,b]), m = \lceil \alpha \rceil, \alpha \notin \mathbb{N}, \alpha > 0; r_1, r_2 > 0, \mu$ is a positive finite measure on the Borel $\sigma-$algebra of $[a,b], x_0 \in [a,b]$. Then

$$\left| \int_{[a,b]} f(x) d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,b]} (x-x_0)^k d\mu(x) \right|$$
$$\leq \frac{1}{\Gamma(\alpha+1)} \left\{ \left[\mu([a,x_0])^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1)r_1} \right] \right.$$
$$\omega_1 \left(D_{x_0-}^\alpha f, r_1 \left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} \tag{24.47}$$
$$\left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} + \left[(\mu((x_0,b]))^{\frac{1}{(\alpha+1)}} \right.$$

$$+ \left. \frac{1}{(\alpha+1)r_2} \right] \omega_1 \left(D_{*x_0}^\alpha f, r_2 \left(\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]}$$
$$\left. \left(\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right\}.$$

Proof. By (24.8) we derive

$$|E_{x_0}| \leq \frac{1}{\Gamma(\alpha)} \left\{ \int_{[a,x_0]} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} \right. \right.$$
$$\left| D_{x_0-}^\alpha f(\zeta) - D_{x_0-}^\alpha f(x_0) \right| d\zeta \Big) d\mu(x) + \tag{24.48}$$
$$\left. \int_{(x_0,b]} \left(\int_{x_0}^x (x-\zeta)^{\alpha-1} \left| D_{*x_0}^\alpha f(\zeta) - D_{*x_0}^\alpha f(x_0) \right| d\zeta \right) d\mu(x) \right\} = (*).$$

Let $h_1, h_2 > 0$, then

$$(*) \leq \frac{1}{\Gamma(\alpha)} \left\{ \left[\int_{[a,x_0]} \left(\int_x^{x_0} (\zeta-x)^{\alpha-1} \left(1 + \frac{x_0-\zeta}{h_1} \right) d\zeta \right) \right. \right. \tag{24.49}$$
$$\left. d\mu(x) \right] \omega_1 \left(D_{x_0-}^\alpha f, h_1 \right)_{[a,x_0]} + \left[\int_{(x_0,b]} \right.$$
$$\left. \left. \left(\int_{x_0}^x (x-\zeta)^{\alpha-1} \left(1 + \frac{\zeta-x_0}{h_2} \right) d\zeta \right) d\mu(x) \right] \omega_1 \left(D_{*x_0}^\alpha f, h_2 \right)_{[x_0,b]} \right\}.$$

That is,

$$\begin{aligned}|E_{x_0}| \quad &\leq \quad \frac{1}{\Gamma(\alpha)} \left\{ \left[\int_{[a,x_0]} \left(\frac{(x_0-x)^{\alpha}}{\alpha} + \frac{1}{h_1} \int_x^{x_0} (x_0-\zeta)^{2-1} (\zeta-x)^{\alpha-1} d\zeta \right) \right.\right. \\ &d\mu(x) \Big] \, \omega_1 \left(D^{\alpha}_{x_0-} f, h_1 \right)_{[a,x_0]} + \left[\int_{(x_0,b]} \right. \\ &\left.\left. \left(\frac{(x-x_0)^{\alpha}}{\alpha} + \frac{1}{h_2} \int_{x_0}^{x} (x-\zeta)^{\alpha-1} (\zeta-x_0)^{2-1} d\zeta \right) d\mu(x) \right] \omega_1 \left(D^{\alpha}_{*x_0} f, h_2 \right)_{[x_0,b]} \right\}\end{aligned} \tag{24.50}$$

$$\begin{aligned}&= \frac{1}{\Gamma(\alpha)} \left\{ \left[\int_{[a,x_0]} \left(\frac{(x_0-x)^{\alpha}}{\alpha} + \frac{1}{h_1} \frac{(x_0-x)^{\alpha+1}}{\alpha(\alpha+1)} \right) d\mu(x) \right] \omega_1 \left(D^{\alpha}_{x_0-} f, h_1 \right)_{[a,x_0]} \right. \\ &+ \left[\int_{(x_0,b]} \left(\frac{(x-x_0)^{\alpha}}{\alpha} + \frac{1}{h_2} \frac{(x-x_0)^{\alpha+1}}{\alpha(\alpha+1)} \right) d\mu(x) \right] \\ &\left. \times \, \omega_1 \left(D^{\alpha}_{*x_0} f, h_2 \right)_{[x_0,b]} \right\}.\end{aligned} \tag{24.51}$$

Hence

$$|E_{x_0}| \leq \frac{1}{\Gamma(\alpha)} \left\{ \left[\frac{1}{\alpha} \int_{[a,x_0]} (x_0-x)^{\alpha} d\mu(x) + \right.\right.$$

$$\begin{aligned}&\frac{1}{h_1 \alpha(\alpha+1)} \int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \Big] \, \omega_1 \left(D^{\alpha}_{x_0-} f, h_1 \right)_{[a,x_0]} + \left[\frac{1}{\alpha} \int_{(x_0,b]} (x-x_0)^{\alpha} d\mu(x) + \right. \\ &\left.\left. \frac{1}{h_2 \alpha(\alpha+1)} \int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) \right] \omega_1 \left(D^{\alpha}_{*x_0} f, h_2 \right)_{[x_0,b]} \right\}.\end{aligned} \tag{24.52}$$

Momentarily we suppose positive choices of

$$h_1 = r_1 \left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} > 0, \tag{24.53}$$

and

$$h_2 = r_2 \left(\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} > 0. \tag{24.54}$$

Consequently we obtain

$$\begin{aligned}|E_{x_0}| \;\leq\; &\frac{1}{\Gamma(\alpha+1)} \left\{ \left[(\mu([a,x_0]))^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1) r_1} \right] \omega_1 \left(D^{\alpha}_{x_0-} f, h_1 \right)_{[a,x_0]} \left(\frac{h_1}{r_1} \right)^{\alpha} \right. \\ &\left. + \left[(\mu((x_0,b]))^{\frac{1}{(\alpha+1)}} + \frac{1}{r_2(\alpha+1)} \right] \omega_1 \left(D^{\alpha}_{*x_0} f, h_2 \right)_{[x_0,b]} \left(\frac{h_2}{r_2} \right)^{\alpha} \right\},\end{aligned} \tag{24.55}$$

proving (24.47).

Next we examine special cases. If $\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)=0$, then $(x-x_0)=0$, a.e. on $(x_0,b]$, that is $x=x_0$ a.e. on $(x_0,b]$,

more precisely $\mu\{x\in(x_0,b]:x\neq x_0\}=0$, hence $\mu(x_0,b]=0$.

Therefore μ concentrates on $[a,x_0]$.

In that case inequality (24.47) is written and holds as

$$\left|\int_{[a,x_0]}f(x)d\mu(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\int_{[a,x_0]}(x-x_0)^k d\mu(x)\right|$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left\{\left[(\mu([a,x_0]))^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r_1}\right]\right. \tag{24.56}$$

$$\left.\omega_1\left(D^{\alpha}_{x_0-}f,r_1\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}}\right\}.$$

Since $(b,b]=\emptyset$ and $\mu(\emptyset)=0$, in the case of $x_0=b$, we get again (24.56) written for $x_0=b$. So inequality (24.56) is a valid inequality when $\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\neq 0$.

If additionally we suppose that $\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)=0$, then $(x_0-x)=0$, a.e. on $[a,x_0]$, that is $x=x_0$ a.e. on $[a,x_0]$, which means $\mu\{x\in[a,x_0]:x\neq x_0\}=0$. Hence $\mu=\delta_{x_0}M$, where δ_{x_0} is the unit Dirac measure and $M=\mu([a,b])>0$.

In the last case we get that $L.H.S(24.56)=R.H.S.(24.56)=0$, that is (24.56) is valid trivially.

Finally let us go the other way around. Let us suppose that $\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)=0$, then reasoning similarly as before we get that μ over $[a,x_0]$ concentrates at x_0. That is $\mu=\delta_{x_0}\mu([a,x_0])$, on $[a,x_0]$.

In the last case (24.47) is written and it holds as

$$\left|\int_{(x_0,b]}f(x)d\mu(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\int_{(x_0,b]}(x-x_0)^k d\mu(x)\right|$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left\{\left[(\mu((x_0,b]))^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r_2}\right]\right. \tag{24.57}$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f,r_2\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}}\right\}.$$

If $x_0=a$ then (24.57) can be redone and rewritten, just replace $(x_0,b]$ by $[a,b]$ all over.

So inequality (24.57) is valid when

$$\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\neq 0.$$

If additionally we assume that $\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)=0$, then as before $\mu(x_0,b]=0$. Hence (24.57) is trivially true, in fact $L.H.S.(24.57)=R.H.S.(24.57)=0$.

The proof of (24.47) now has completed in all possible cases. ■

We continue in a special case.

In the assumptions of Theorem 24.25, when $r=r_1=r_2>0$, and by calling $M=\mu([a,b])\geq\mu([a,x_0]),\mu((x_0,b])$, we obtain

Corollary 24.26. It holds

$$\left|\int_{[a,b]}f(x)d\mu(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\int_{[a,b]}(x-x_0)^k d\mu(x)\right|$$

$$\leq \frac{1}{\Gamma(\alpha+1)}\left[M^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r}\right]\Bigg[\omega_1\left(D^{\alpha}_{x_0-}f,r\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}$$
$$\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}+\omega_1\left(D^{\alpha}_{*x_0}f,r\left(\int_{[x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}$$
$$\left(\int_{[x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\Bigg]. \tag{24.58}$$

Based on Theorem 24.25, Corollary 24.26 and (24.31), we get

Theorem 24.27. Let $f\in AC^m([a,b])$, $f^{(m)}\in L_\infty([a,b])$, $m=\lceil\alpha\rceil,\alpha\notin\mathbb{N},\alpha>0;r>0$,and $L_N:C([a,b])\to C([a,b]),N\in\mathbb{N}$, a sequence of positive linear operators, $x_0\in[a,b]$. Then

$$\left|L_N(f,x_0)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}L_N((x-x_0)^k,x_0)\right|\leq\frac{1}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r}\right]$$

$$\Bigg[\omega_1\left(D^{\alpha}_{x_0-}f,r\left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x),x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]} \tag{24.59}$$
$$\left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x),x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}+$$
$$\omega_1\left(D^{\alpha}_{*x_0}f,r\left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x),x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}$$
$$\left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x),x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\Bigg].$$

Corollary 24.28. (to Theorem 24.27) It holds

$$\left|L_N(f,x_0)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}L_N((x-x_0)^k,x_0)\right|\leq\frac{1}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r}\right]$$
$$\left[\omega_1\left(D^\alpha_{x_0-}f,r\left(L_N\left(|x-x_0|^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\right.$$
$$\left.+\omega_1\left(D^\alpha_{*x_0}f,r\left(L_N\left(|x-x_0|^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right]\left(L_N\left(|x-x_0|^{\alpha+1},x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}. \tag{24.60}$$

We make

Remark 24.29. Let $f\in AC([a,b])$, $f'\in L_\infty([a,b])$, $0<\alpha<1, x_0\in[a,b]; L_N: C([a,b])\to C([a,b]), N\in\mathbb{N}$, sequence of positive linear operators. Then by Theorem 24.27 and

$$|L_N(f,x_0)-f(x_0)|\leq|L_N(f,x_0)-f(x_0)L_N(1,x_0)|+|f(x_0)|\,|L_N(1,x_0)-1|, \tag{24.61}$$

we obtain

Theorem 24.30. Let $f\in AC([a,b])$, $f'\in L_\infty([a,b])$, $0<\alpha<1, x_0\in[a,b]; L_N: C([a,b])\to C([a,b]), N\in\mathbb{N}$, sequence of positive linear operators. Then

$$\begin{aligned}|L_N(f,x_0)-f(x_0)| \leq\; & |f(x_0)|\,|L_N(1,x_0)-1|+\\ & \frac{1}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r}\right]\\ & \left[\omega_1\left(D^\alpha_{x_0-}f,r\left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x),x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\right.\\ & \left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x),x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}+ \end{aligned} \tag{24.62}$$

$$\omega_1\left(D^\alpha_{*x_0}f,r\left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x),x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}$$
$$\left.\left(L_N\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x),x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right].$$

We make

Remark 24.31. We see that

$$R.H.S.(24.59) \leq \frac{1}{\Gamma(\alpha+1)}\left[\|L_N(1)\|_\infty^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r}\right]$$
$$\left[\sup_{x\in[a,b]}\omega_1\left(D_{x-}^\alpha f, r\left\|L_N\left(|\cdot-x|^{\alpha+1}\chi_{[a,x]}(\cdot),x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[a,x]}\right.$$
$$\left\|L_N\left(|\cdot-x|^{\alpha+1}\chi_{[a,x]}(\cdot),x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}+$$
$$\sup_{x\in[a,b]}\omega_1\left(D_{*x}^\alpha f, r\left\|L_N\left(|\cdot-x|^{\alpha+1}\chi_{[x,b]}(\cdot),x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,b]}$$
$$\left.\left\|\left(L_N\left(|\cdot-x|^{\alpha+1}\chi_{[x,b]}(\cdot),x\right)\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}\right]=\Theta. \tag{24.63}$$

So that

$$Z := \left\|L_N(f,x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x)}{k!}L_N((\cdot-x)^k,x)\right\|_\infty \leq \Theta. \tag{24.64}$$

We further observe that

$$|L_N(f,x)-f(x)| \leq Z+|f(x)|\,|L_N(1,x)-1|+\sum_{k=1}^{m-1}\frac{\left|f^{(k)}(x)\right|}{k!}\left|L_N((\cdot-x)^k,x)\right|$$
$$\leq |f(x)|\,|L_N(1,x)-1|+\sum_{k=1}^{m-1}\frac{\left|f^{(k)}(x)\right|}{k!}\left|L_N((\cdot-x)^k,x)\right|+\Theta. \tag{24.65}$$

We have established the main result, a Shisha-Mond type inequality at the fractional level.

Theorem 24.32. Let $f \in AC^m([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil\alpha\rceil, \alpha \notin \mathbb{N}, \alpha > 0, r > 0$,and $L_N : C([a,b]) \to C([a,b]), N \in \mathbb{N}$, a sequence of positive linear operators, $x \in [a,b]$. Then

$$\|L_N f-f\|_\infty \leq \|f\|_\infty\|L_N 1-1\|_\infty+\sum_{k=1}^{m-1}\frac{\left\|f^{(k)}\right\|_\infty}{k!}\left\|L_N((\cdot-x)^k,x)\right\|_\infty+\frac{1}{\Gamma(\alpha+1)}$$
$$\left(\|L_N(1)\|_\infty^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r}\right)\left[\sup_{x\in[a,b]}\right. \tag{24.66}$$
$$\omega_1\left(D_{x-}^\alpha f, r\left\|L_N(|\cdot-x|^{\alpha+1}\chi_{[a,x]}(\cdot),x)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[a,x]}\left\|L_N(|\cdot-x|^{\alpha+1}\chi_{[a,x]}(\cdot),x)\right\|_\infty^{\frac{\alpha}{(\alpha+1)}}+$$
$$\left.\sup_{x\in[a,b]}\omega_1\left(D_{*x}^\alpha f, r\left\|L_N(|\cdot-x|^{\alpha+1}\chi_{[x,b]}(\cdot),x)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,b]}\left\|L_N(|\cdot-x|^{\alpha+1}\chi_{[x,b]}(\cdot),x)\right\|_\infty^{\frac{\alpha}{(\alpha+1)}}\right].$$

Next we derive the following Korovkin type convergence result at fractional level.

Theorem 24.33. Let $\alpha \notin \mathbb{N}, \alpha > 0, m = \lceil \alpha \rceil$,and $L_N : C([a,b]) \to C([a,b]), N \in \mathbb{N}$, a sequence of positive linear operators. Assume $L_N 1 \xrightarrow{u} 1$ (uniformly), and $\left\| L_N (|\cdot - x|^{\alpha+1}, x) \right\|_\infty \to 0$, as $N \to \infty$. Then $L_N f \xrightarrow{u} f, \forall f \in AC^m([a,b])$, $f^{(m)} \in L_\infty([a,b])$. (The second condition means $\left(L_N (|\cdot - x|^{\alpha+1}) \right)(x) \xrightarrow{u} 0, x \in [a,b]$.)

Proof. Since $\|L_N 1 - 1\|_\infty \to 0$ we get $\|L_N 1 - 1\|_\infty \le K$, for some $K > 0$. We write $L_N 1 = L_N 1 - 1 + 1$, hence

$$\|L_N 1\|_\infty \le \|L_N 1 - 1\|_\infty + \|1\|_\infty \le K + 1, \forall N \in \mathbb{N}.$$

That is $\|L_N 1\|_\infty$ is bounded.

So we are using inequality (24.66).

By assumption $\left\| L_N (|\cdot - x|^{\alpha+1}, x) \right\|_\infty \to 0$ and (24.40) we get $\left\| L_N (|\cdot - x|^k, x) \right\|_\infty \to 0$, for all $k = 1, \ldots, m-1$.

Also by (24.43) and (24.44) we obtain that

$$\left\| L_N (|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x) \right\|_\infty \to 0, \text{ and } \left\| L_N (|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x) \right\|_\infty \to 0,$$

as $N \to \infty$.

Additionally by (24.38) and (24.39) we derive that

$$\sup_{x \in [a,b]} \omega_1 (D_{x-}^\alpha f, \cdot)_{[a,x]}, \sup_{x \in [a,b]} \omega_1 (D_{*x}^\alpha f, \cdot)_{[x,b]} \le \frac{2 \left\| f^{(m)} \right\|_\infty}{\Gamma(m - \alpha + 1)} (b-a)^{m-\alpha}, \tag{24.67}$$

so they are bounded.

Thus based on the above, from (24.66), we derive that $\|L_N f - f\|_\infty \to 0$, proving the claim. ■

We make

Remark 24.34. Based on Corollary 24.17 and Theorems 24.18, 24.19, given that $f \in C^m([a,b])$, we get that

$$\begin{aligned}
(i) \qquad & \sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, r \left\| L_N (|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x) \right\|_\infty^{\frac{1}{(\alpha+1)}} \right)_{[a,x]} \\
= \ & \omega_1 \left(D_{x_1 -}^\alpha f, r \left\| L_N (|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x) \right\|_\infty^{\frac{1}{(\alpha+1)}} \right)_{[a,x_1]} \qquad (24.68) \\
\to \ & 0, \text{ as } \left\| L_N (|\cdot - x|^{\alpha+1}, x) \right\|_\infty \to 0, \text{ as } N \to \infty,
\end{aligned}$$

for some $x_1 \in [a,b]$.

Similarly

$$\begin{aligned}(ii) \qquad & \sup_{x\in[a,b]} \omega_1 \left(D^{\alpha}_{*x} f, r \left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x) \right\|_{\infty}^{\frac{1}{(\alpha+1)}} \right)_{[x,b]} \\ = \quad & \omega_1 \left(D^{\alpha}_{*x_2} f, r \left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x) \right\|_{\infty}^{\frac{1}{(\alpha+1)}} \right)_{[x_2,b]} \qquad (24.69) \\ \to \quad & 0, \text{ as } \left\| L_N(|\cdot - x|^{\alpha+1}, x) \right\|_{\infty} \to 0, \text{ as } N \to \infty,\end{aligned}$$

for some $x_2 \in [a,b]$.

We give

Corollary 24.35. Here $L_N : C([a,b]) \to C([a,b]), N \in \mathbb{N}$,positive linear operators. Let $0 < \alpha < 1, r > 0, f \in AC([a,b]),\ f' \in L_\infty([a,b])$. Then

$$\|L_N f - f\|_\infty \le \|f\|_\infty \|L_N 1 - 1\|_\infty + \frac{1}{\Gamma(\alpha+1)} \left(\|L_N(1)\|_\infty^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1)r} \right) \Bigg[\Bigg\{ \sup_{x\in[a,b]}$$

$$\omega_1 \left(D^{\alpha}_{x-} f, r \left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x) \right\|_{\infty}^{\frac{1}{(\alpha+1)}} \right)_{[a,x]} \Bigg\} \left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x) \right\|_{\infty}^{\frac{\alpha}{(\alpha+1)}}$$

$$+ \left\{ \sup_{x\in[a,b]} \omega_1 \left(D^{\alpha}_{*x} f, r \left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x) \right\|_{\infty}^{\frac{1}{(\alpha+1)}} \right)_{[x,b]} \right\}$$

$$\left\| L_N(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x) \right\|_{\infty}^{\frac{\alpha}{(\alpha+1)}} \Bigg]. \qquad (24.70)$$

24.4 Application

Consider $f \in C([0,1])$ and the Bernstein polynomials $(B_N f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \binom{N}{k} t^k$ $(1-t)^{N-k}, \forall t \in [0,1], N \in \mathbb{N}$.

We have $B_N 1 = 1$, and B_N are positive linear operators.

Here let $0 < \alpha < 1, r > 0$ and take $f \in AC([0,1]),\ f' \in L_\infty([0,1])$.

Applying Corollary 24.35 we obtain

Corollary 24.36. It holds

$$\|B_N f - f\|_\infty \le \frac{1}{\Gamma(\alpha+1)} \left(1 + \frac{1}{(\alpha+1)r} \right) \qquad (24.71)$$

$$\Bigg[\sup_{x\in[0,1]} \omega_1 \left(D^{\alpha}_{x-} f, r \left\| B_N(|\cdot - x|^{\alpha+1} \chi_{[0,x]}(\cdot), x) \right\|_{\infty}^{\frac{1}{(\alpha+1)}} \right)_{[0,x]} \left\| B_N(|\cdot - x|^{\alpha+1} \chi_{[0,x]}(\cdot), x) \right\|_{\infty}^{\frac{\alpha}{(\alpha+1)}} +$$

$$\sup_{x\in[0,1]} \omega_1 \left(D^{\alpha}_{*x} f, r \left\| B_N(|\cdot - x|^{\alpha+1} \chi_{[x,1]}(\cdot), x) \right\|_{\infty}^{\frac{1}{(\alpha+1)}} \right)_{[x,1]} \left\| B_N(|\cdot - x|^{\alpha+1} \chi_{[x,1]}(\cdot), x) \right\|_{\infty}^{\frac{\alpha}{(\alpha+1)}} \Bigg],$$

$\forall N \in \mathbb{N}$.

Next let $\alpha = \frac{1}{2}$, and $r = \frac{1}{\alpha+1}$, that is $r = \frac{2}{3}$. Notice $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$.

Corollary 24.37. Let $f \in AC([0,1])$, $f' \in L_\infty([0,1])$, $N \in \mathbb{N}$. Then

$$\|B_N f - f\|_\infty \leq \frac{4}{\sqrt{\pi}} \left[\sup_{x\in[0,1]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{2}{3} \left\| B_N(|\cdot - x|^{\frac{3}{2}} \chi_{[0,x]}(\cdot), x) \right\|_\infty^{\frac{2}{3}} \right)_{[0,x]} \right.$$
$$\left\| B_N(|\cdot - x|^{\frac{3}{2}} \chi_{[0,x]}(\cdot), x) \right\|_\infty^{\frac{1}{3}} + \sup_{x\in[0,1]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{2}{3} \left\| B_N(|\cdot - x|^{\frac{3}{2}} \chi_{[x,1]}(\cdot), x) \right\|_\infty^{\frac{2}{3}} \right)_{[x,1]}$$
$$\left. \left\| B_N(|\cdot - x|^{\frac{3}{2}} \chi_{[x,1]}(\cdot), x) \right\|_\infty^{\frac{1}{3}} \right]. \tag{24.72}$$

Here we have

$$|t - x|^{3/2} \chi_{[0,x]}(t) = \begin{cases} (x-t)^{3/2}, & \text{for } 0 \leq t \leq x, \\ 0, & \text{for } x < t \leq 1, \end{cases} \tag{24.73}$$

and

$$|t - x|^{3/2} \chi_{[x,1]}(t) = \begin{cases} (t-x)^{3/2}, & \text{for } x \leq t \leq 1, \\ 0, & \text{for } 0 \leq t < x. \end{cases} \tag{24.74}$$

Consequently for $x \in [0,1]$, we find

$$B_N\left(|\cdot - x|^{\frac{3}{2}} \chi_{[0,x]}(\cdot)\right)(x) = \sum_{k=0}^{[xN]} \left(x - \frac{k}{N}\right)^{3/2} \binom{N}{k} x^k (1-x)^{N-k}, \tag{24.75}$$

and

$$B_N\left(|\cdot - x|^{\frac{3}{2}} \chi_{[x,1]}(\cdot)\right)(x) = \sum_{k=\lceil xN \rceil}^{N} \left(\frac{k}{N} - x\right)^{3/2} \binom{N}{k} x^k (1-x)^{N-k}. \tag{24.76}$$

One further has

$$B_N\left(|\cdot - x|^{\frac{3}{2}} \chi_{[0,x]}(\cdot)\right)(x), B_N\left(|\cdot - x|^{\frac{3}{2}} \chi_{[x,1]}(\cdot)\right)(x) \tag{24.77}$$

$$\leq B_N\left(|\cdot - x|^{\frac{3}{2}}\right)(x) = \sum_{k=0}^{N} \left|x - \frac{k}{N}\right|^{3/2} \binom{N}{k} x^k (1-x)^{N-k} \tag{24.78}$$

(by discrete Hölder's inequality)

$$\leq \left(\sum_{k=0}^{N} \left(x - \frac{k}{N}\right)^2 \binom{N}{k} x^k (1-x)^{N-k} \right)^{\frac{3}{4}} \tag{24.79}$$

$$= \left(\frac{1}{N} x(1-x)\right)^{\frac{3}{4}} \leq \frac{1}{(4N)^{3/4}}, \forall x \in [0,1]. \tag{24.80}$$

We have shown

Corollary 24.38. Let $f \in AC([0,1])$, $f' \in L_\infty([0,1])$, $N \in \mathbb{N}$. Then

$$\|B_N f - f\|_\infty \leq \frac{2^{\frac{3}{2}}}{\sqrt{\pi}\sqrt[4]{N}} \left[\sup_{x\in[0,1]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[0,x]} + \sup_{x\in[0,1]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[x,1]} \right]. \tag{24.81}$$

Notice $\frac{2^{\frac{3}{2}}}{\sqrt{\pi}} \approx 1.59$.

So as $N \to \infty$ we derive again that $B_N f \xrightarrow{u} f$ with rates.

Discussion 24.39. From (24.81), Corollary 24.17, and Theorems 24.18, 24.19 we obtain that

$$\|B_N f - f\|_\infty \leq \frac{2^{\frac{3}{2}}}{\sqrt{\pi}\sqrt[4]{N}} \left[\omega_1 \left(D_{x_1-}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[0,x_1]} + \omega_1 \left(D_{*x_2}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[x_2,1]} \right], \tag{24.82}$$

for some $x_1, x_2 \in [0,1]$, $f \in C^1([0,1])$.

That is

$$\|B_N f - f\|_\infty \leq \frac{2^{\frac{3}{2}}}{\sqrt{\pi}\sqrt[4]{N}} \left[\omega_1 \left(D_{x_1-}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[0,1]} + \omega_1 \left(D_{*x_2}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[0,1]} \right]. \tag{24.83}$$

Further we suppose that $D_{x_1-}^{\frac{1}{2}} f$ and $D_{*x_2}^{\frac{1}{2}} f$ are Lipschitz functions of order 1, that is

$$\left| D_{x_1-}^{\frac{1}{2}} f(x) - D_{x_1-}^{\frac{1}{2}} f(y) \right| \leq K_1 |x-y|, \tag{24.84}$$

and

$$\left| D_{*x_2}^{\frac{1}{2}} f(x) - D_{*x_2}^{\frac{1}{2}} f(y) \right| \leq K_2 |x-y|, \forall x,y \in [0,1], \text{ and } K_1, K_2 > 0. \tag{24.85}$$

Then from (24.83) we get

$$\|B_N f - f\|_\infty \leq \frac{2^{\frac{3}{2}} (K_1 + K_2)}{3\sqrt{\pi} N^{\frac{3}{4}}}. \tag{24.86}$$

Assume next that f' is a Lipschitz function of order 1, that is

$$|f'(x) - f'(y)| \leq K_3 |x-y|, \forall x,y \in [0,1], \text{ and } K_3 > 0. \tag{24.87}$$

Then from Section 24.1 the Introduction, we get

$$\|B_N f - f\|_\infty \leq \frac{0.1953125 K_3}{N}, N \in \mathbb{N}. \tag{24.88}$$

In [250], T. Popoviciu for $f \in C([0,1])$ proved that

$$\|B_N f - f\|_\infty \le \frac{5}{4}\omega_1\left(f, \frac{1}{\sqrt{N}}\right) = 1.25\omega_1\left(f, \frac{1}{\sqrt{N}}\right). \tag{24.89}$$

If f is a Lipschitz function of order 1, that is

$$|f(x) - f(y)| \le K_4 |x-y|, \tag{24.90}$$

$\forall x, y \in [0,1]$, and $K_4 > 0$, then we have

$$\|B_N f - f\|_\infty \le \frac{1.25K_4}{\sqrt{N}}. \tag{24.91}$$

We also notice that

$$\frac{1}{N} < \frac{1}{N^{\frac{3}{4}}} < \frac{1}{N^{\frac{1}{2}}}, \text{ for } N \in \mathbb{N} \setminus \{1\}. \tag{24.92}$$

So looking at (24.88), (24.86) and (24.91), we observe that as the used in the estimates differentiability of f increases so the resulting speed of convergence of $B_N f$ to f increases, in fact at the used $\frac{1}{2}-$derivative the speed is in between the corresponding speeds for f and f'. Of course in the last argument we supposed that f, $D_{x_1-}^{\frac{1}{2}} f, D_{*x_2}^{\frac{1}{2}} f$ and f' are all Lipschitz functions. If f' is a Lipschitz function or just $f \in C^1([0,1])$, not necessarily $D_{x_1-}^{\frac{1}{2}} f, D_{*x_2}^{\frac{1}{2}} f$ are Lipschitz ones.

25

Fractional Trigonometric Convergence Theory of Positive Linear Operators

In this chapter we study quantitatively with rates the trigonometric weak convergence of a sequence of finite positive measures to the unit measure. Equivalently we study quantitatively the trigonometric pointwise convergence of sequence of positive linear operators to the unit operator, all acting on continuous functions on $[-\pi, \pi]$. From there we obtain with rates the corresponding trigonometric uniform convergence of the latter. The inequalities for all of the above in their right hand sides contain the moduli of continuity of the right and left Caputo fractional derivatives of the involved function. From these uniform trigonometric Shisha-Mond type inequality we derive the trigonometric fractional Korovkin type theorem regarding the trigonometric uniform convergence of positive linear operators to the unit. We give applications, especially to Bernstein polynomials over $[-\pi, \pi]$ for which we establish fractional trigonometric quantitative results. This chapter relies on [46].

25.1 Introduction

In this chapter among other we are motivated by the following results.

Theorem 25.1 (P.P.Korovkin [213], (1960)). *Let* $L_n : C([-\pi, \pi]) \to C([-\pi, \pi])$, $n \in \mathbb{N}$, *be a sequence of positive linear operators. Suppose* $L_n(1) \overset{u}{\to} 1(uniformly)$, $L_n(\cos t) \overset{u}{\to} \cos t, L_n(\sin t) \overset{u}{\to} \sin t$, *as* $n \to \infty$. *Then* $L_n f \overset{u}{\to} f$, *for every* $f \in C([-\pi, \pi])$ *that is* $2\pi-$ *periodic.*

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 377–397.
springerlink.com

Let $f \in C([a,b])$ and $0 \leq h \leq b-a$. The first modulus of continuity of f at h is given by

$$\omega_1(f,h) = \sup\{|f(x)-f(y)|; x,y \in [a,b], |x-y| \leq h\}$$

If $h > b-a$, then we define

$$\omega_1(f,h) = \omega_1(f,b-a).$$

Another motivation is the following.

Theorem 25.2 (Shisha and Mond [263], (1968)). *Let $L_1, L_2, \ldots,$ be linear positive operators, whose common domain D consists of real functions with domain $(-\infty,\infty)$. Suppose $1, \cos x, \sin x, f$ belong to D, where f is an everywhere continuous, 2π-periodic function,with modulus of continuity ω_1. Let $-\infty < a < b < \infty$, and suppose that for $n = 1,2,\ldots, L_n(1)$ is bounded in $[a,b]$.*

Then for $n = 1,2,\ldots,$

$$\|L_n(f) - f\|_\infty \leq \|f\|_\infty \|L_n(1) - 1\|_\infty + \|L_n(1) + 1\|_\infty \,\omega_1(f,\mu_n), \tag{25.1}$$

where

$$\mu_n = \pi \left\| \left(L_n \left(sin^2 \left(\frac{t-x}{2} \right) \right) \right) (x) \right\|_\infty^{1/2},$$

and $\|\cdot\|_\infty$ stands for the sup norm over $[a,b]$.

In particular, if $L_n(1) = 1$, then (25.1) reduces to

$$\|L_n(f) - f\|_\infty \leq 2\omega_1(f,\mu_n).$$

One can easily see that, for $n = 1,2,\ldots,$

$$\mu_n^2 \leq \left(\frac{\pi^2}{2}\right) \left[\|L_n(1) - 1\|_\infty\right.$$

$$\left. + \|(L_n(\cos t))(x) - \cos x\|_\infty + \|(L_n(\sin t))(x) - \sin x\|_\infty\right],$$

so the last along with (25.1) prove Korovkin's Theorem 25.1 in a quantitative way and with rates of convergence.

One more motivation follows.

Theorem 25.3 (see [16], p. 217). *Let $f \in C^n([-\pi,\pi])$, $n \geq 1$, and μ a measure on $[-\pi,\pi]$ of mass $m > 0$. Set*

$$\beta := \left(\int \left(sin \frac{|t|}{2} \right)^{n+1} \cdot \mu(dt) \right)^{1/(n+1)} \tag{25.2}$$

and denote by $w := \omega_1(f^{(n)},\beta)$ the modulus of continuity of $f^{(n)}$ at β. Then

$$\left| \int f\,d\mu - f(0) \right| \leq |f(0)| \cdot |m-1| + \sum_{k=1}^{n} \frac{|f^{(k)}(0)|}{k!} \cdot \left| \int t^k \mu(dt) \right|$$

$$+w[m^{1/(n+1)} + \pi/(n+1)] \cdot \frac{\pi^n \beta^n}{n!}.$$

Final motivation is [43]. A great aid for fractional calculus is [259].

In this chapter we study quantitatively the rate of trigonometric weak convergence of a sequence of finite positive measures to the unit measure given the existence and presence of the left and right Caputo fractional derivatives of the involved function. That is in the right hand sides of the derived inequalities appear the first moduli of continuity of the above mentioned fractional derivatives, see Theorem 25.23 and Corollary 25.24.

Then via the Riesz representation theorem we transfer Theorem 25.23 into the language of quantitative trigonometric pointwise convergence of a sequence of positive linear operators to the unit operator, all operators acting from $C([-\pi, \pi])$ into itself, see Theorem 25.25, Corollary 25.26 and Theorem 25.28.

From there we derive quantitative results with respect to the sup-norm $\|\cdot\|_\infty$, regarding the trigonometric uniform convergence of positive linear operators to the unit. Again in the right hand side of our inequalities we have moduli of continuity with respect to right and left Caputo derivatives of the engaged function. For the last see Theorem 25.30, a trigonometric Sisha-Mond type result. From there we obtain the first trigonometric Korovkin type convergence theorem at the fractional level, see Theorem 25.31.

We give applications of the fractional trigonometric Sisha-Mond and trigonometric Korovkin theory, see Corollaries 25.34 - 25.36, etc.

In approximation theory the involvement of fractional derivatives is very rare, almost nothing exists, with the exception of the recent [43]. The few fractional articles that exist are of V. Dzyadyk [153] of 1959, F. Nasibov [233] of 1962, J. Demjanovic [140] of 1975, and of M. Jaskolski [196] of 1989, all regarding estimates to best approximation of functions by algebraic and trigonometric polynomials.

25.2 Background

We need

Definition 25.4. Let $\upsilon \geq 0, n = \lceil \upsilon \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [145], p. 38, [160], [259] the function

$$D_{*a}^{\upsilon} f(x) = \frac{1}{\Gamma(n-\upsilon)} \int_a^x (x-t)^{n-\upsilon-1} f^{(n)}(t) dt, \tag{25.3}$$

$\forall x \in [a,b]$, where Γ is the gamma function $\Gamma(\upsilon) = \int_0^\infty e^{-t} t^{\upsilon-1} dt, \upsilon > 0$.

We set $D_{*a}^0 f(x) = f(x), \forall x \in [a,b]$.

Lemma 25.5 ([43]). *Let* $v > 0, v \notin \mathbb{N}, n = \lceil v \rceil, f \in C^{n-1}([a,b])$ *and* $f^{(n)} \in L_\infty([a,b])$. *Then* $D_{*a}^{v} f(a) = 0$.

Definition 25.6 (see also [160], [155], [44]). Let $f \in AC^m([a,b]), m = \lceil \alpha \rceil, \alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta, \tag{25.4}$$

$\forall x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$.

Lemma 25.7 ([43]). *Let* $f \in C^{m-1}([a,b]), f^{(m)} \in L_\infty([a,b]), m = \lceil \alpha \rceil, \alpha > 0$. *Then* $D_{b-}^{\alpha} f(b) = 0$.

We also need

Lemma 25.8 ([43]). *Let* $f \in AC^m([a,b]), m = \lceil \alpha \rceil, \alpha > 0; \mu$ *is a positive finite measure on the Borel* σ*-algebra of* $[a,b], x_0 \in [a,b]$. *Then*

$$E_{x_0}([a,b]) := \int_{[a,b]} f(x) d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,b]} (x - x_0)^k d\mu(x)$$

$$= \frac{1}{\Gamma(\alpha)} \left\{ \int_{[a,x_0]} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} (D_{x_0-}^{\alpha} f(\zeta) - D_{x_0-}^{\alpha} f(x_0)) d\zeta \right) d\mu(x) + \right.$$

$$\left. \int_{(x_0,b]} \left(\int_{x_0}^{x} (x - \zeta)^{\alpha-1} (D_{*x_0}^{\alpha} f(\zeta) - D_{*x_0}^{\alpha} f(x_0)) d\zeta \right) d\mu(x) \right\}. \tag{25.5}$$

Convention 25.9. We suppose that

$$D_{*x_0}^{\alpha} f(x) = 0, \quad for \quad x < x_0, \tag{25.6}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \quad for \quad x > x_0, \tag{25.7}$$

for all $x, x_0 \in (a,b]$.

We mention

Proposition 25.10 ([43]). *Let* $f \in C^n([a,b]), n = \lceil v \rceil, v > 0$. *Then* $D_{*a}^{v} f(x)$ *is continuous in* $x \in [a,b]$.

Also we have

Proposition 25.11 ([43]). *Let* $f \in C^m([a,b]), m = \lceil v \rceil, v > 0$. *Then* $D_{b-}^{\alpha} f(x)$ *is continuous in* $x \in [a,b]$.

We further mention

Proposition 25.12 ([43]). *Let* $f \in C^{m-1}([a,b]), f^{(m)} \in L_\infty([a,b]), m = \lceil \alpha \rceil, \alpha > 0$ *and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x - t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{25.8}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proposition 25.13 ([43]). *Let* $f \in C^{m-1}([a,b]), f^{(m)} \in L_\infty([a,b]), m = \lceil \alpha \rceil, \alpha > 0$ *and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta, \tag{25.9}$$

for all $x, x_0 \in [a,b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

We need

Proposition 25.14 ([43]). *Let* $g \in C([a,b]), 0 < c < 1, x, x_0 \in [a,b]$. *Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t) dt, \text{ for } x \geq x_0, \tag{25.10}$$

and $L(x, x_0) = 0$, *for* $x < x_0$.

Then L is jointly continuous in (x, x_0) on $[a,b]^2$.

We mention

Proposition 25.15 ([43]). *Let* $g \in C([a,b]), 0 < c < 1, x, x_0 \in [a,b]$. *Define*

$$K(x, x_0) = \int_{x}^{x_0} (\zeta - x)^{c-1} g(\zeta) d\zeta, \text{ for } x \leq x_0, \tag{25.11}$$

and $K(x, x_0) = 0$, *for* $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a,b]^2$ into $\mathbb{R}$.

Based on Propositions 25.14, 25.15 we obtain

Corollary 25.16 ([43]). *Let* $f \in C^m([a,b]), m = \lceil \alpha \rceil, \alpha > 0, x, x_0 \in [a,b]$. *Then* $D_{*x_0}^{\alpha} f(x), D_{x_0-}^{\alpha} f(x)$ *are jointly continuous functions in* (x, x_0) *from* $[a,b]^2$ *into* $\mathbb{R}$.

We need

Theorem 25.17 ([43]). *Let* $f : [a,b]^2 \to \mathbb{R}$ *be jointly continuous. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x,b]),$$

$\delta > 0, x \in [a,b]$.

Then G is continuous on $[a,b]$.

Also it holds

Theorem 25.18 ([43]). *Let* $f : [a,b]^2 \to \mathbb{R}$ *be jointly continuous. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta, [a,x]),$$

$x \in [a,b]$, *is continuous in* $x \in [a,b], \delta > 0$.

We make

Remark 25.19. Let μ be a finite positive measure on Borel σ-algebra of $[-\pi, \pi]$. Let $\alpha > 0$, then by Hölder's inequality we obtain ($x_0 \in [-\pi, \pi]$),

$$\int_{[-\pi, x_0]} (x_0 - x)^{\alpha} d\mu(x) \leq 2^{\alpha} \left(\int_{[-\pi, x_0]} \left(\frac{(x_0 - x)}{2} \right)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}}$$

$$\mu([-\pi, x_0])^{\frac{1}{(\alpha+1)}} \leq (2\pi)^{\alpha} \left(\int_{[-\pi, x_0]} (\sin((x_0 - x)/4))^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu([-\pi, x_0])^{\frac{1}{(\alpha+1)}}, \tag{25.12}$$

by $|t| \leq \pi \sin(|t|/2), t \in [-\pi, \pi]$.

Similarly we get

$$\int_{(x_0, \pi]} (x - x_0)^{\alpha} d\mu(x) \leq 2^{\alpha} \left(\int_{(x_0, \pi]} \left(\frac{(x - x_0)}{2} \right)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}}$$

$$\mu((x_0, \pi])^{\frac{1}{(\alpha+1)}} \leq (2\pi)^{\alpha} \left(\int_{(x_0, \pi]} (\sin((x - x_0)/4))^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu((x_0, \pi])^{\frac{1}{(\alpha+1)}}. \tag{25.13}$$

Let now $m = \lceil \alpha \rceil, \alpha \in \mathbb{N}, \alpha > 0, k = 1, \ldots, m - 1$. Then again by Hölder's inequality we find

$$\int_{[-\pi, \pi]} |x - x_0|^k d\mu(x)$$

$$\leq 2^k \left(\int_{[-\pi, \pi]} \left(\frac{|x - x_0|}{2} \right)^{\alpha+1} d\mu(x) \right)^{\frac{k}{(\alpha+1)}} (\mu([-\pi, \pi]))^{\frac{\alpha+1-k}{(\alpha+1)}}$$

$$\leq (2\pi)^k \left(\int_{[-\pi, \pi]} (\sin(|x - x_0|/4))^{\alpha+1} d\mu(x) \right)^{\frac{k}{(\alpha+1)}} \mu([-\pi, \pi])^{\frac{\alpha+1-k}{(\alpha+1)}}, \tag{25.14}$$

Terminology 25.20. Here $C([-\pi, \pi])$ denotes all the real valued continuous functions on $[-\pi, \pi]$. Let $L_N : C([-\pi, \pi]) \to C([-\pi, \pi]), N \in \mathbb{N}$, be a sequence of positive linear operators. By Riesz representation theorem (see [257], p. 304) we have

$$L_N(f, x_0) = \int_{[-\pi, \pi]} f(t) d\mu_{N x_0}(t), \tag{25.15}$$

$\forall x_0 \in [-\pi, \pi]$, where $\mu_{N x_0}$ is a unique positive finite measure on a Borel algebra of $[-\pi, \pi]$. Put

$$L_N(1, x_0) = \mu_{N x_0}([-\pi, \pi]) = M_{N x_0}. \tag{25.16}$$

We make

Remark 25.21([43]). Let $f \in C^{n-1}([a, b]), f^{(n)} \in L_{\infty}([a, b]), n = \lceil v \rceil v > 0, v \notin \mathbb{N}$.

Then we have

$$|D_{*a}^{v} f(x)| \leq \frac{\|f^{(n)}\|_{\infty}}{\Gamma(n - v + 1)} (x - a)^{n-v}, \forall x \in [a, b]. \tag{25.17}$$

Thus we see that

$$\omega_1(D^{v}_{*a}f,\delta)=\sup_{x,y\in[a,b]\,|x-y|\le\delta}|D^{v}_{*a}f(x)-D^{v}_{*a}f(y)|$$

$$\le \sup_{x,y\in[a,b]\,|x-y|\le\delta}\left(\frac{\|f^{(n)}\|_\infty}{\Gamma(n-v+1)}(x-a)^{n-v}+\frac{\|f^{(n)}\|_\infty}{\Gamma(n-v+1)}(y-a)^{n-v}\right) \tag{25.18}$$

$$\le \frac{2\|f^{(n)}\|_\infty}{\Gamma(n-v+1)}(b-a)^{n-v}. \tag{25.19}$$

Consequently

$$\omega_1(D^{v}_{*a}f,\delta)\le \frac{2\|f^{(n)}\|_\infty}{\Gamma(n-v+1)}(b-a)^{n-v}. \tag{25.20}$$

Similarly, let $f\in C^{m-1}([a,b]), f^{(m)}\in L_\infty([a,b]), m=\lceil\alpha\rceil, \alpha>0, \alpha\notin\mathbb{N}$, then

$$\omega_1(D^{\alpha}_{b-}f,\delta)\le \frac{2\parallel f^{(m)}\parallel_\infty}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{25.21}$$

So for $f\in C^{m-1}([a,b]), f^{(m)}\in L_\infty([a,b]), m=\lceil\alpha\rceil, \alpha>0, \alpha\notin\mathbb{N}$, we find

$$\sup_{x_0\in[a,b]}\omega_1(D^{\alpha}_{*x_0}f,\delta)_{[x_0,b]}\le \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}, \tag{25.22}$$

and

$$\sup_{x_0\in[a,b]}\omega_1(D^{\alpha}_{x_0-}f,\delta)_{[a,x_0]}\le \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{25.23}$$

We also make

Remark 25.22. Let $L_N: C([-\pi,\pi])\to C([-\pi,\pi]), N\in\mathbb{N}$, be a sequence of positive linear operators. Using (25.15) and Hölder's inequality we obtain ($x\in[-\pi,\pi], k=1,\dots,m-1, m=\lceil\alpha\rceil, \alpha\notin\mathbb{N}, \alpha>0$) for $k=1,\dots,m-1$ that

$$\left\|L_N\left(|\cdot-x|^k,x\right)\right\|_\infty\le(2\pi)^k\left(\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\frac{k}{(\alpha+1)}}\right)$$

$$\|L_N1\|_\infty^{(\alpha+1-k)/(\alpha+1)}. \tag{25.24}$$

Notice that for any $x\in[-\pi,\pi]$ we get

$$C([-\pi,\pi])\ni|\cdot-x|\mathcal{X}_{[-\pi,\pi]}(\cdot)\le|\cdot-x|\in C([-\pi,\pi]),$$

therefore

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{25.25}$$

Consequently, by positivity of L_N we derive

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty \leq \left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty. \tag{25.26}$$

Similarly, for any $x\in[-\pi,\pi]$ we have

$$C([-\pi,\pi]) \ni |\cdot-x|\mathcal{X}_{[x,\pi]} \leq |\cdot-x| \in C([-\pi,\pi]),$$

thus

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{25.27}$$

Therefore

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty \leq \left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty. \tag{25.28}$$

So if the right hand side of (25.26),(25.28) goes to zero, so do their left hand sides.

In fact we notice that

$$\left(\sin\frac{|\cdot-x|}{4}\right)^{\alpha+1} = \left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1} + \left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, \tag{25.29}$$

for every $x\in[-\pi,\pi]$.

Therefore it holds

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty \leq \left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty + \left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty. \tag{25.30}$$

Consequently, if both

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty,$$

$$\left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty \to 0,$$

as $N \to +\infty$, then

$$\left\| L_N \left(\left(\sin \left(\frac{|\cdot - x|}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty \to 0.$$

25.3 Main Results

We present the first main result

Theorem 25.23. *Let $f \in AC^m([-\pi,\pi]), f^{(m)} \in L_\infty([-\pi,\pi]), m = \lceil \alpha \rceil, \alpha \notin \mathbb{N}, \alpha > 0; r_1, r_2 > 0, \mu$ is a positive finite measure on the Borel σ- algebra of $[-\pi,\pi], x_0 \in [-\pi,\pi]$. Then*

$$\left| \int_{[-\pi,\pi]} f(x) d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[-\pi,\pi]} (x - x_0)^k d\mu(x) \right|$$
$$\leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left\{ \left[(\mu([-\pi, x_0]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1) r_1} \right] \right.$$
$$\left(\int_{[-\pi,x_0]} \left(\sin \left(\frac{x_0 - x}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$\omega_1 \left(D^\alpha_{x_0-} f, r_1 \left(\int_{[-\pi,x_0]} \left(\sin \left(\frac{x_0 - x}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[-\pi,x_0]}$$
$$+ \left[(\mu((x_0, \pi]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1) r_2} \right] \left(\int_{(x_0,\pi]} \left(\sin \left(\frac{x - x_0}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$\left. \omega_1 \left(D^\alpha_{*x_0} f, r_2 \left(\int_{(x_0,\pi]} \left(\sin \left(\frac{x - x_0}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,\pi]} \right\}. \quad (25.31)$$

Proof. By (25.5) we get

$$E_{x_0}([-\pi,\pi])$$
$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_{[-\pi,x_0]} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} |D^\alpha_{x_0-} f(\zeta) - D^\alpha_{x_0-} f(x_0)| d\zeta \right) d\mu(x) \right.$$

$$+\int_{(x_0,\pi]}\left(\int_{x_0}^{x}(x-\zeta)^{\alpha-1}|D_{*x_0}^{\alpha}f(\zeta)-D_{*x_0}^{\alpha}f(x_0)|d\zeta\right)d\mu(x)\Bigg\}=(*). \quad (25.32)$$

Let $h_1, h_2 > 0$, then

$$(*)\le\frac{1}{\Gamma(\alpha)}\Bigg\{\Bigg[\int_{[-\pi,x_0]}\left(\int_{x}^{x_0}(\zeta-x)^{\alpha-1}\left(1+\frac{x_0-\zeta}{h_1}\right)d\zeta\right)$$
$$d\mu(x)\Bigg]\,\omega_1(D_{x_0-}^{\alpha}f,h_1)_{[-\pi,x_0]}$$
$$+\left[\int_{(x_0,\pi]}\left(\int_{x_0}^{x}(x-\zeta)^{\alpha-1}\left(1+\frac{\zeta-x_0}{h_2}\right)d\zeta\right)d\mu(x)\right]\omega_1(D_{*x_0}^{\alpha}f,h_2)_{[x_0,\pi]}\Bigg\}. \quad (25.33)$$

That is,

$$E_{x_0}([-\pi,\pi])$$
$$\le\frac{1}{\Gamma(\alpha)}\Bigg\{\Bigg[\int_{[-\pi,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\left(\int_{x}^{x_0}(x_0-\zeta)^{2-1}(\zeta-x)^{\alpha-1}d\zeta\right)\right.$$
$$d\mu(x)\Bigg]\omega_1(D_{x_0-}^{\alpha}f,h_1)_{[-\pi,x_0]}$$
$$+\left[\int_{(x_0,\pi]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\int_{x_0}^{x}(x-\zeta)^{\alpha-1}(\zeta-x_0)^{2-1}d\zeta\right)d\mu(x)\right]$$
$$\omega_1(D_{*x_0}^{\alpha}f,h_2)_{[x_0,\pi]}\Bigg\} \quad (25.34)$$
$$=\frac{1}{\Gamma(\alpha)}\Bigg\{\left[\int_{[-\pi,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\frac{(x_0-x)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu(x)\right]$$
$$\omega_1(D_{x_0-}^{\alpha}f,h_1)_{[-\pi,x_0]}$$
$$+\left[\int_{(x_0,\pi]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\frac{(x-x_0)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu(x)\right]\omega_1(D_{*x_0}^{\alpha}f,h_2)_{[x_0,\pi]}\Bigg\}. \quad (25.35)$$

Therefore

$$E_{x_0}([-\pi,\pi])\le\frac{1}{\Gamma(\alpha)}\Bigg\{\Bigg[\frac{1}{\alpha}\int_{[-\pi,x_0]}(x_0-x)^{\alpha}d\mu(x)$$
$$+\frac{1}{h_1\alpha(\alpha+1)}\int_{[-\pi,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\Bigg]\omega_1(D_{x_0-}^{\alpha}f,h_1)_{[-\pi,x_0]}$$
$$+\left[\frac{1}{\alpha}\int_{(x_0,\pi]}(x-x_0)^{\alpha}d\mu(x)+\frac{1}{h_2\alpha(\alpha+1)}\int_{(x_0,\pi]}(x-x_0)^{\alpha+1}d\mu(x)\right]$$
$$\omega_1(D_{*x_0}^{\alpha}f,h_2)_{[x_0,\pi]}\Bigg\}. \quad (25.36)$$

Momentarily we suppose positive choices of

$$h_1 = r_1 \left(\int_{[-\pi,x_0]} \left(\sin \left(\frac{x_0 - x}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} > 0, \tag{25.37}$$

$$h_2 = r_2 \left(\int_{(x_0,-\pi]} \left(\sin \left(\frac{x - x_0}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} > 0. \tag{25.38}$$

Consequently, by (25.12),(25.13) and (25.36), we derive

$$E_{x_0}([-\pi,\pi]) \leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left\{ \left[(\mu([-\pi,x_0]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_1} \right] \left(\frac{h_1}{r_1} \right)^\alpha \right.$$
$$\omega_1(D^\alpha_{x_0-}f, h_1)_{[-\pi,x_0]} + \left[(\mu((x_0,\pi]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_2} \right] \left(\frac{h_2}{r_2} \right)^\alpha$$
$$\left. \omega_1(D^\alpha_{*x_0}f, h_2)_{[x_0,\pi]} \right\}, \tag{25.39}$$

proving (25.31).

Next we examine the special cases. If

$$\int_{(x_0,\pi]} \left(\sin \left(\frac{x - x_0}{4} \right) \right)^{\alpha+1} d\mu(x) = 0,$$

then $\sin\left(\frac{x-x_0}{4}\right) = 0$, a.e. on $(x_0,\pi]$, that is $x = x_0$ a.e. on $(x_0,\pi]$, more precisely $\mu\{x \in (x_0,\pi] : x \neq x_0\} = 0$, hence $\mu(x_0,\pi] = 0$. Therefore μ concentrates on $[-\pi,x_0]$. In that case (25.31) is written and holds as

$$\left| \int_{[-\pi,x_0]} f(x)d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[-\pi,x_0]} (x-x_0)^k d\mu(x) \right|$$
$$\leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left\{ \left[(\mu([-\pi,x_0]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_1} \right] \right.$$
$$\left(\int_{[-\pi,x_0]} \left(\sin \left(\frac{x_0 - x}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$\omega_1 \left(D^\alpha_{x_0-}f, r_1 \left(\int_{[-\pi,x_0]} \left(\sin \left(\frac{x_0 - x}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[-\pi,x_0]}. \tag{25.40}$$

Since $(\pi,\pi] = \emptyset$ and $\mu(\emptyset) = 0$, in the case of $x_0 = \pi$, we get again (25.40) written for $x_0 = \pi$. So inequality (25.40) is a valid inequality when

$$\int_{[-\pi,x_0]} \left(\sin \left(\frac{(x_0 - x)}{4} \right) \right)^{\alpha+1} d\mu(x) \neq 0.$$

If additionally we suppose that

$$\int_{[-\pi,x_0]}\left(\sin\left(\frac{(x_0-x)}{4}\right)\right)^{\alpha+1}d\mu(x)=0,$$

then $\sin(\frac{x_0-x}{4})=0$, a.e. on $[-\pi,x_0]$, that is $x=x_0$ a.e. on $[-\pi,x_0]$, which means $\mu\{x\in[-\pi,x_0]:x\neq x_0\}=0$. Hence $\mu=\delta_{x_0}M$, where δ_{x_0} is the unit Dirac measure and $M=\mu([-\pi,\pi])>0$.

In the last case we obtain L.H.S (25.40)=R.H.S (25.40)=0, that is (25.40) is valid trivially.

At last we go the other way around. Let us suppose that

$$\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu(x)=0,$$

then reasoning similarly as before, we get that μ over $[-\pi,x_0]$ concentrates at x_0. That is $\mu=\delta_{x_0}\mu([-\pi,x_0])$, on $[-\pi,x_0]$.

In the last case (25.31) is written and holds as

$$\begin{aligned}&\left|\int_{(x_0,\pi]}f(x)d\mu(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\int_{(x_0,\pi]}(x-x_0)^k d\mu(x)\right|\\&\leq\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left\{\left[(\mu((x_0,\pi]))^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r_2}\right]\right.\\&\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\\&\left.\omega_1\left(D_{*x_0}^\alpha f,r_2\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right\}.\end{aligned}\tag{25.41}$$

If $x_0=-\pi$, then (25.41) can be redone and rewritten, just replace $(x_0,\pi]$ by $[-\pi,\pi]$ all over. So inequality (25.41) is valid when

$$\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\neq0.$$

If additionally we assume that

$$\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)=0,$$

then as before $\mu(x_0,\pi]=0$. Hence (25.41) is trivially true, in fact L.H.S (25.41)= R.H.S (25.41)=0. The prof of (25.31) now is completed in all possible cases. ∎

We continue in a special case.

In the assumptions of Theorem 25.23, when $r = r_1 = r_2 > 0$, and by calling $M = \mu([-\pi,\pi]) \geq \mu([-\pi,x_0]), \mu((x_0,\pi])$, we obtain

Corollary 25.24. *It holds*

$$\left| \int_{[-\pi,\pi]} f(x)d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[-\pi,\pi]} (x-x_0)^k d\mu(x) \right|$$

$$\leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left[M^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r} \right]$$

$$\left[\left(\int_{[-\pi,x_0]} \left(\sin\left(\frac{x_0-x}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right.$$

$$\omega_1 \left(D^\alpha_{x_0-} f, r \left(\int_{[-\pi,x_0]} \left(\sin\left(\frac{x_0-x}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[-\pi,x_0]}$$

$$+ \left(\int_{[x_0,\pi]} \left(\sin\left(\frac{x-x_0}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left. \omega_1 \left(D^\alpha_{*x_0} f, r \left(\int_{[x_0,\pi]} \left(\sin\left(\frac{x-x_0}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,\pi]} \right]. \tag{25.42}$$

Based on Theorem 25.23, Corollary 25.24 and (25.15), we get

Theorem 25.25. *Let* $f \in AC^m([-\pi,\pi]), f^{(m)} \in L_\infty([-\pi,\pi]), m = \lceil\alpha\rceil, \alpha \notin \mathbb{N}, \alpha > 0; r > 0$, *and* $L_N : C([-\pi,\pi]) \to C([-\pi,\pi]), n \in \mathbb{N}$, *a sequence of positive linear operators,* $x_0 \in [-\pi,\pi]$. *Then*

$$\left| L_N(f,x_0) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} L_N((x-x_0)^k, x_0) \right|$$

$$\leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r} \right]$$

$$\left[\left(L_N \left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4} \right) \right)^{\alpha+1}, x_0 \right) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right.$$

$$\omega_1 \left(D^\alpha_{x_0-} f, r \left(L_N \left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4} \right) \right)^{\alpha+1}, x_0 \right) \right)^{\frac{1}{(\alpha+1)}} \right)_{[-\pi,x_0]}$$

$$+\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f,r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right]. \tag{25.43}$$

Corollary 25.26 (to Theorem 25.25). *It holds*

$$\left|L_N(f,x_0)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}L_N((x-x_0)^k,x_0)\right|$$
$$\leq\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r}\right]$$
$$\left[\omega_1\left(D^{\alpha}_{x_0-}f,r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}\right.$$
$$\left.\omega_1\left(D^{\alpha}_{*x_0}f,r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right]$$
$$\left(L_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{\alpha}{(\alpha+1)}}. \tag{25.44}$$

We make

Remark 25.27. Let $f\in AC([-\pi,\pi]), f'\in L_\infty([-\pi,\pi]), 0<\alpha<1, x_0\in[-\pi,\pi]; L_N: C([-\pi,\pi])\to C([-\pi,\pi]), N\in\mathbb{N}$, sequence of positive linear operators. Then by Theorem 25.25 and

$$|L_N(f,x_0)-f(x_0)|\leq|L_N(f,x_0)-f(x_0)L_N(1,x_0)|+|f(x_0)||L_n(1,x_0)-1|, \tag{25.45}$$

we obtain

Theorem 25.28. *Let* $f\in AC([-\pi,\pi]), f'\in L_\infty([-\pi,\pi]), 0<\alpha<1, r>0, x_0\in[-\pi,\pi]; L_N: C([-\pi,\pi])\to C([-\pi,\pi]), N\in\mathbb{N}$, *sequence of positive linear operators. Then*

$$|L_N(f,x_0)-f(x_0)|\leq|f(x_0)|\,|L_N(1,x_0)-1|$$
$$+\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\omega_1\left(D^{\alpha}_{x_0-}f,r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}$$

$$+\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f,r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right]. \tag{25.46}$$

We make

Remark 25.29. We see that

$$R.H.S(25.43)\le\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\left[\|L_N(1)\|_{\infty}^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_{\infty}^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\sup_{x\in[-\pi,\pi]}\omega_1\left(D^{\alpha}_{x-}f,r\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{[-\pi,x]}$$

$$+\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_{\infty}^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\sup_{x\in[-\pi,\pi]}\omega_1\left(D^{\alpha}_{*x}f,r\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_{\infty}^{\frac{1}{\alpha+1}}\right)_{[x,\pi]}\right]=:\theta. \tag{25.47}$$

So that

$$Z:=\left\|L_N(f,x_0)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}L_N((\cdot-x)^k,x)\right\|_{\infty}\le\theta. \tag{25.48}$$

We further observe that

$$|L_N(f,x)-f(x)|\le Z+|f(x)|\ |L_N(1,x)-1|+\sum_{k=0}^{m-1}\frac{|f^{(k)}(x)|}{k!}\left|L_N((\cdot-x)^k,x)\right|$$

$$\leq |f(x)|\,|L_N(1,x)-1| + \sum_{k=1}^{m-1} \frac{|f^{(k)}(x)|}{k!} \left| L_N((\cdot - x)^k, x) \right| + \theta. \tag{25.49}$$

We have proved the main result, a Shisha-Mond type trigonometric inequality at the fractional level.

Theorem 25.30. *Let* $f \in AC^m([-\pi,\pi]), f^{(m)} \in L_\infty([-\pi,\pi]), m = \lceil \alpha \rceil, \alpha \notin \mathbb{N}, \alpha > 0, r > 0$ *and* $L_N : C([-\pi,\pi]) \to C([-\pi,\pi]), N \in \mathbb{N}$*, a sequence of positive linear operators,* $x \in [-\pi,\pi]$*. Then*

$$\|L_N f - f\|_\infty \leq \|f\|_\infty \|L_N 1 - 1\|_\infty + \sum_{k=1}^{m-1} \frac{\|f^{(k)}\|_\infty}{k!} \|L_N((\cdot - x)^k, x)\|_\infty$$

$$+ \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left[\|L_N(1)\|_\infty^{1/(\alpha+1)} + \frac{2\pi}{(\alpha+1)r} \right]$$

$$\left[\left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)} \right.$$

$$\sup_{x\in[-\pi,\pi]} \omega_1 \left(D_{x-}^\alpha f, r \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\frac{1}{\alpha+1}} \right)_{[-\pi,x]}$$

$$+ \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left. \sup_{x\in[-\pi,\pi]} \omega_1 \left(D_{*x}^\alpha f, r \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\frac{1}{\alpha+1}} \right)_{[x,\pi]} \right]. \tag{25.50}$$

Next we give the following trigonometric Korovkin type convergence result at fractional level.

Theorem 25.31. *Let* $\alpha \notin \mathbb{N}, \alpha > 0, m = \lceil \alpha \rceil$*, and* $L_N : C([-\pi,\pi]) \to C([-\pi,\pi]), N \in \mathbb{N}$*, a sequence of positive linear operators. Suppose* $L_N 1 \xrightarrow{n} 1$ *(uniformly), and*

$$\left\| L_N \left(\left(\sin\left(\frac{|\cdot - x|}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty \to 0,$$

as $N \to \infty$*. Then* $L_N f \xrightarrow{u} f, \forall f \in AC^m([-\pi,\pi]), f^{(m)} \in L_\infty([-\pi,\pi])$*. (The second condition means* $\left(L_N \left(\left(\sin\left(\frac{|\cdot - x|}{4} \right) \right)^{\alpha+1} \right) \right)(x) \xrightarrow{u} 0, x \in [-\pi,\pi]$*.)*

Proof. Since $\|L_N 1 - 1\|_\infty \to 0$ we get $\|L_N 1 - 1\|_\infty \leq K$, for some $K > 0$. We write $L_N 1 = L_N 1 - 1 + 1$, hence

$$\|L_N 1\|_\infty \leq \|L_N 1 - 1\|_\infty + \|1\|_\infty \leq K + 1, \forall N \in \mathbb{N}.$$

That is $\|L_N 1\|_\infty$ is bounded. So we are using inequality (25.50). By assumption $\|L_N((\sin(\frac{|\cdot-x|}{4}))^{\alpha+1}, x)\|_\infty \to 0$, as $N \to \infty$ and (25.24) we get $\|L_N(|\cdot -x|^k, x)\|_\infty \to 0$ for $k = 1, \ldots, m-1$. Also by (25.26) and (25.28) we get that

$$\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty,$$

and

$$\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0,$$

as $N \to \infty$.

Additionally by (25.22) and (25.23) we obtain that

$$\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{x-}f, \cdot\right)_{[-\pi,x]}, \sup_{x\in[-\pi,\pi]} \omega_1(D^\alpha_{*x}f; \cdot)_{[x,\pi]} \le \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\alpha+1)}(2\pi)^{m-\alpha},$$

so they are bounded.

Thus based on the above, from (25.50), we derive that $\|L_N f - f\|_\infty \to 0$, proving the claim. ■

We make

Remark 25.32. Based on Corollary 25.16 and Theorem 25.17, 25.18, given that $f \in C^m([-\pi,\pi])$, we get that,

$$(i) \quad \sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{x-}f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[-\pi,x]}$$

$$= \omega_1\left(D^\alpha_{x_1-}f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_1]} \to 0, \tag{25.51}$$

as

$$\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0,$$

when $N \to \infty$, for some $x_1 \in [-\pi,\pi]$.

Similarly

$$(ii) \quad \sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{*x}f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[x,\pi]}$$

$$= \omega_1\left(D^\alpha_{*x_2}f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x_2,\pi]} \to 0, \tag{25.52}$$

as

$$\left\| L_N \left(\left(\sin \left(\frac{|\cdot - x|}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty \to 0,$$

when $N \to \infty$, for some $x_2 \in [-\pi, \pi]$.

Corollary 25.33. *Here* $L_N : C([-\pi,\pi]) \to C([-\pi,\pi]), N \in \mathbb{N}$, *positive linear operators. Let* $0 < \alpha < 1, r > 0, f \in AC([-\pi,\pi]), f' \in L_\infty([-\pi,\pi])$. *Then*

$$\|L_N f - f\|_\infty \leq \|f\|_\infty \|L_N 1 - 1\|_\infty + \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left[\|L_N(1)\|_\infty^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r} \right]$$

$$\left[\left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)} \right.$$

$$\left\{ \sup_{x \in [-\pi,\pi]} \omega_1 \left(D_{x-}^\alpha f, r \left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\frac{1}{\alpha+1}} \right)_{[-\pi,x]} \right\}$$

$$+ \left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left. \left\{ \sup_{x \in [-\pi,\pi]} \omega_1 \left(D_{*x}^\alpha f, r \left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\frac{1}{\alpha+1}} \right)_{[x,\pi]} \right\} \right]. \tag{25.53}$$

25.4 Application

Consider the Bernstein polynomials on $[-\pi, \pi]$ for $f \in C([-\pi,\pi])$:

$$(B_N f)(x) = \sum_{k=0}^{N} \binom{N}{k} f\left(-\pi + \frac{2\pi k}{N} \right) \left(\frac{x+\pi}{2\pi} \right)^k \left(\frac{\pi - x}{2\pi} \right)^{N-k},$$

$N \in \mathbb{N}$, any $x \in [-\pi,\pi]$. There are positive linear operators from $C([-\pi,\pi])$ into itself. Here let $0 < \alpha < 1, r > 0$ and take $f \in AC([-\pi,\pi]), f' \in L_\infty([-\pi,\pi])$. Setting $g(t) = f(2\pi t - \pi), t \in [0,1]$, we have $g(0) = f(-\pi), g(1) = f(\pi)$, and

$$(B_N g)(t) = \sum_{k=0}^{N} \binom{N}{k} g\left(\frac{k}{N} \right) t^k (1-t)^{N-k} = (B_N f)(x), x \in [-\pi,\pi].$$

Here $x=\varphi(t)=2\pi t-\pi$ is an $1-1$ and onto map from [0,1] onto $[-\pi,\pi]$. Clearly here $g\in AC([0,1])$ and $g'\in L_\infty([0,1])$.

Observe also that

$$(B_N((\cdot-x)^2))(x)=[(B_N((\cdot-t)^2))(t)](2\pi)^2=\frac{(2\pi)^2}{N}t(1-t)$$

$$=\frac{(2\pi)^2}{N}\left(\frac{x+\pi}{2\pi}\right)\left(\frac{\pi-x}{2\pi}\right)=\frac{1}{N}(x+\pi)(\pi-x)\le\frac{\pi^2}{N},\forall x\in[-\pi,\pi].$$

I.e.

$$(B_N((\cdot-x)^2))(x)\le\frac{\pi^2}{N},\forall x\in[-\pi,\pi].$$

In particular $(B_N1)(x)=1,\forall x\in[-\pi,\pi]$.

Applying Corollary 25.33 we obtain

Corollary 25.34. *It holds*

$$\|B_Nf-f\|_\infty\le\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[1+\frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\left\{\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{x-}^\alpha f,r\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[-\pi,x]}\right\}$$

$$+\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\left\{\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{*x}^\alpha f,r\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[x,\pi]}\right\}\right], \tag{25.54}$$

$\forall N\in\mathbb{N}$.

Next let $\alpha=\frac{1}{2}$, and $r=\frac{1}{\alpha+1}$, that is $r=\frac{2}{3}$. Notice $\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$.

Corollary 25.35. *Let $f\in AC([-\pi,\pi]),f'\in L_\infty([-\pi,\pi]),n\in\mathbb{N}$. Then*

$$\|B_Nf-f\|_\infty\le2\sqrt{2}(2\pi+1)\left[\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\frac{3}{2}},x\right)\right\|_\infty^{\frac{1}{3}}\right.$$

$$\left\{\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{2}{3}\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\frac{3}{2}},x\right)\right\|_\infty^{\frac{2}{3}}\right)_{[-\pi,x]}\right\}$$

$$+\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\frac{3}{2}},x\right)\right\|_\infty^{\frac{1}{3}}$$

$$\left\{\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{2}{3}\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\frac{3}{2}},x\right)\right\|_\infty^{\frac{2}{3}}\right)_{[x,\pi]}\right\}\Bigg], \tag{25.55}$$

$\forall \quad N\in\mathbb{N}$.

By $|\sin x|<|x|, \forall x\in\mathbb{R}-\{0\}$, in particular $\sin x\le x$, for $x\ge 0$, we get

$$\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{3/2}\le\left(\frac{|\cdot-x|}{4}\right)^{3/2}=\frac{1}{8}|\cdot-x|^{3/2}.$$

Therefore

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\frac{3}{2}},x\right)\right\|_\infty\le\frac{1}{8}\left\|B_N\left(|\cdot-x|^{\frac{3}{2}},x\right)\right\|_\infty. \tag{25.56}$$

We see that

$$B_N(|\cdot-x|^{3/2},x)=\sum_{k=0}^N\left|x+\pi-\frac{2\pi k}{N}\right|^{3/2}\binom{N}{K}\left(\frac{x+\pi}{2\pi}\right)^k\left(\frac{\pi-x}{2\pi}\right)^{N-k}$$

(by discrete Hölder's inequality)

$$\le\left[\sum_{k=0}^N\left(x+\pi-\frac{2\pi k}{N}\right)^2\binom{N}{k}\left(\frac{x+\pi}{2\pi}\right)^k\left(\frac{\pi-x}{2\pi}\right)^{N-k}\right]^{3/4}$$

$$=\left(B_N\left((\cdot-x)^2,x\right)\right)^{3/4}\le\frac{\pi^{3/2}}{N^{3/4}},\ \forall \quad x\in[-\pi,\pi]. \tag{25.57}$$

Consequently it holds

$$\|B_N(|\cdot-x|^{3/2},x)\|_\infty\le\frac{\pi^{3/2}}{N^{3/4}}, \tag{25.58}$$

and

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{3/2},x\right)\right\|_\infty\le\frac{\pi^{3/2}}{8N^{3/4}},\ \forall \quad N\in\mathbb{N}. \tag{25.59}$$

Therefore we obtain

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{3/2},x\right)\right\|_\infty,$$

$$\left\| B_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{3/2}, x \right) \right\|_\infty$$

$$\leq \left\| B_N \left(\left(\sin \left(\frac{|\cdot - x|}{4} \right) \right)^{3/2}, x \right) \right\|_\infty \leq \frac{\pi^{3/2}}{8N^{3/4}}, \ \forall \quad N \in \mathbb{N}. \tag{25.60}$$

We have established

Corollary 25.36. *Let* $f \in AC([-\pi,\pi]), f' \in L_\infty([-\pi,\pi]), N \in \mathbb{N}$. *Then*

$$\|B_N f - f\|_\infty \leq \frac{(2\pi+1)\sqrt{2\pi}}{\sqrt[4]{N}} \left[\sup_{x\in[-\pi,\pi]} \omega_1 \left(D_{x-}^{1/2} f, \frac{\pi}{6\sqrt{N}} \right)_{[-\pi,x]} \right.$$

$$\left. + \sup_{x\in[-\pi,\pi]} \omega_1 \left(D_{*x}^{1/2} f, \frac{\pi}{6\sqrt{N}} \right)_{[x,\pi]} \right]. \tag{25.61}$$

So as $N \to \infty$ we derive that $B_N f \xrightarrow{u} f$ with rates.

Discussion 25.37. From (25.61), Corollary 25.16, and Theorems 25.17,25.18 we obtain that

$$\|B_N f - f\|_\infty \leq \frac{(2\pi+1)\sqrt{2\pi}}{\sqrt[4]{N}} \left[\omega_1 \left(D_{x_1-}^{1/2} f, \frac{\pi}{6\sqrt{N}} \right)_{[-\pi,x_1]} \right.$$

$$\left. + \omega_1 \left(D_{*x_2}^{1/2} f, \frac{\pi}{6\sqrt{N}} \right)_{[x_2,\pi]} \right], \tag{25.62}$$

for some $x_1, x_2 \in ([-\pi,\pi]), f \in C^1([-\pi,\pi])$.

Hence

$$\|B_N f - f\|_\infty \leq \left(\frac{2\pi+1}{\sqrt[4]{N}} \right) (\sqrt{2\pi}) \left[\omega_1 \left(D_{x_1-}^{1/2} f, \frac{\pi}{6\sqrt{N}} \right)_{[-\pi,x]} \right.$$

$$\left. + \omega_1 \left(D_{*x_2}^{1/2} f, \frac{\pi}{6\sqrt{N}} \right)_{[-\pi,\pi]} \right]. \tag{25.63}$$

Further we suppose that $D_{x_1-}^{1/2} f$ and $D_{*x_2}^{1/2}$ are Lipschitz functions of order 1, that is

$$\left| D_{x_1-}^{1/2} f(x) - D_{x_1-}^{1/2} f(y) \right| \leq K_1 |x-y|, \tag{25.64}$$

and

$$\left| D_{*x_2-}^{1/2} f(x) - D_{*x_2-}^{1/2} f(y) \right| \leq K_2 |x-y|, \tag{25.65}$$

$\forall x, y \in [-\pi,\pi]$, and $K_1, K_2 > 0$. Then from (25.63) we find

$$\|B_N f - f\|_\infty \leq \frac{\pi\sqrt{2\pi}\,(2\pi+1)}{6N^{3/4}} (K_1 + K_2). \tag{25.66}$$

26

Extended Integral Inequalities

Here we present very general Taylor formulae, and then a representation formula. Based on the latter we give general integral inequalities of Opial type, Ostrowski type, Comparison of integral means, Information Theory Csiszar f- divergence type, and Grüss type. This chapter is based on [45].

26.1 Introduction

We are motivated by the following inequalities.

First an Opial type inequality

Theorem 26.1 ([5]),p.8). *Let* $f(t)$ *be absolutely continuous in* $[0, a]$*, and* $f(0) = 0$*. Then*

$$\int_0^a |f(t)f'(t)|dt \leq \frac{a}{2}\int_0^a (f'(t))^2 dt. \tag{26.1}$$

Inequality (26.1) is attained iff $f(t) = ct,\ c > 0$.

Theorem 26.2 ([238.Ostrowski,1938]). *Let* $f : [a, b] \in \mathbb{R}$ *be continuous on [a,b] and differentiable on (a,b), whose derivative* $f'(a, b) \to \mathbb{R}$ *is bounded on (a,b),i.e.,* $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < +\infty$*. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty, \tag{26.2}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 399–422.
springerlink.com

Theorem 26.3 ([180, Grüss, 1935]). *Let* f, g *integrable functions from [a,b] into* $\mathbb{R}$, *such that* $m \leq f(x) \leq M$, $\rho \leq g(x) \leq \sigma$, *for all* $x \in [a, b]$, *where* $m, M, \rho, \sigma \in \mathbb{R}$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dt \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right| \\ \leq \frac{1}{4}(M-m)(\sigma-\rho). \tag{26.3}$$

Here we present very general Taylor formulae, see Theorem 26.4, 26.5. Based on Theorem 26.4 we produce a general representation formula, see Theorem 26.9. Then based on Theorem 26.9 we prove new very general inequalities of: Opial type, see Theorem 26.14; Ostrowski type, see Theorem 26.18, sharp inequality (26.45); comparison of integral means, see Theorem 26.22; Information Theory inequalities, see Theorems 26.26, 26.27; and Grüss type inequalities, see Theorem 26.30.

For all these formulas and inequalities we give applications when the power function g is $e^x, \sin x, \cos x, \tan x$.

26.2 Results

We present the first result, a general Taylor formula

Theorem 26.4. *Let* $f, f', \ldots, f^{(n)}; g, g'$ *be continuous from* $[a, b](or[b, a])$ *into* $\mathbb{R}, n \in \mathbb{N}$. *Assume* $(g^{-1})^{(k)}, k = 0, 1, \ldots, n$ *are continuous. Then*

$$f(b) = f(a) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(a))}{k!}(g(b) - g(a))^k + \mathcal{R}_n(a, b), \tag{26.4}$$

where

$$\mathcal{R}_n(a, b) = \frac{1}{(n-1)!} \int_a^b (g(b) - g(s))^{n-1} (f \circ g^{-1})^{(n)}(g(s))g'(s)ds \\ = \frac{1}{(n-1)!} \int_{g(a)}^{g(b)} (g(b) - t)^{n-1} (fog^{-1})^{(n)}(t)dt. \tag{26.5}$$

Proof. Call $l = fog^{-1}$. Then $l, l', \ldots, l^{(n)}$ are continuous from $g([a, b])$ into $f([a, b])$. Here $g([a, b]) = [c, d]$, from some $c, d \in \mathbb{R}$. Clearly $g(a), g(b) \in [c, d]$. So we can apply Taylor formula for l at $g(a)$ and $g(b)$. Thus we derive

$$l(g(b)) = l(g(a)) + \sum_{k=1}^{n-1} \frac{l^{(k)}(g(a))}{k!}(g(b) - g(a))^k$$

$$+\frac{1}{(n-1)!}\int_{g(a)}^{g(b)}(g(b)-t)^{n-1}l^{(n)}(t)dt. \tag{26.6}$$

That is proving the claim, since $l((g)) = f(b), l(g(a)) = f(a)$, also apply change of variable for the remainder. ■

The counterpart of previous theorem follows

Theorem 26.5. *Assume $g; f, f', \ldots, f^{(n-1)}$ are continuous on $[a,b], n \in \mathbb{N}$. Also assume $(g^{-1})^{(k)}, k = 0, 1, \ldots, n-1$ are continuous. Assume $f^{(n)}$ exists in (a,b) and $(g^{-1})^{(n)}$ exists in $(g([a,b]))^0$.*

Let $\alpha, \beta \in [a,b]$, then there exists $\gamma \in (\alpha, \beta)$ or $\gamma \in (\beta, \alpha)$ such that

$$f(\beta) = f(\alpha) + \sum_{k=1}^{n-1}\frac{(f\circ g^{-1})^{(k)}}{k!}(g(\alpha))\cdot(g(\beta)-g(\alpha))^k$$
$$+ \quad \frac{(f\circ g^{-1})^{(n)}}{n!}(g(\gamma))\cdot(g(\beta)-g(\alpha))^n. \tag{26.7}$$

Proof. Here $(f\circ g^{-1})^{(k)}, k = 0, 1, \ldots, n-1$ are continuous on $g([a,b])$ and $(f\circ g^{-1})^{(n)}$ exists in $(g([a,b]))^0$. Let $\alpha, \beta \in [a,b]$, we apply Taylor's formula for $g(\alpha), g(\beta) \in g([a,b]), (f = f\circ g^{-1}\circ g)$ to the function fog^{-1}. We obtain

$$f(\beta) = f(\alpha) + \sum_{k=1}^{n-1}\frac{(f\circ g^{-1})^{(k)}}{k!}(g(\alpha))\cdot(g(\beta)-g(\alpha))^k$$
$$+ \quad \frac{(f\circ g^{-1})^{(n)}}{n!}(\gamma_1)\,(g(\beta)-g(\alpha))^n, \tag{26.8}$$

where γ_1 between $g(\alpha)$ and $g(\beta)$.

By intermediate value theorem there exists γ between α, β such that $g(\gamma) = \gamma_1$, proving the claim. ■

Remark 26.6. Here we assume that $f^{(k)}(a) = 0, k = 0, 1, \ldots, n-1$.
By $f = f\circ g^{-1}\circ g$ we get $f(a) = (f\circ g^{-1})(g(a)) = 0$.

Also

$$(f\circ g^{-1})'(g(a)) = f'(a)\cdot((g^{-1})'(g(a)) = 0, \tag{26.9}$$

and

$$(f\circ g^{-1})''(g(a)) = f''(a)\cdot((g^{-1})'(g(a)))^2 + f'(a)\cdot(g^{-1})''(g(a)) = 0. \tag{26.10}$$

Furthermore we obtain

$$(f\circ g^{-1})'''(g(a)) = f'''(a)\cdot((g^{-1})'(g(a)))^3$$
$$+3f''(a)\cdot(g^{-1})'(g(a))\cdot(g^{-1})''(g(a)) \quad + f'(a)\cdot(g^{-1})'''(g(a)) = 0. \tag{26.11}$$

So we have in general that $(f\circ g^{-1})^{(k)}(g(a)) = 0$, all $k = 0, 1, \ldots, n-1$.

Therefore by (26.4) and (26.5) we obtain

$$f(b) = \frac{1}{(n-1)!} \int_a^b (g(b) - g(s))^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(s)) \cdot g'(s) ds. \tag{26.12}$$

Similarly from Taylor formula directly applied on f, we have

$$f(b) = \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} f^{(n)}(t) dt. \tag{26.13}$$

Consequently, if $f^{(k)}(a) = 0, k = 0, 1, \ldots, n-1$ and f, g as in Theorem 26.4, it holds

$$\int_a^b (b-t)^{n-1} f^{(n)}(t) dt = \int_a^b (g(b) - g(t))^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) dt. \tag{26.14}$$

Next we go reverse, we suppose $(f \circ g^{-1})^{(k)}(g(a)) = 0, k = 0, 1, \ldots, n-1$ and $(g^{-1})'(g(a)) \neq 0$, then $f^{(k)}(a) = 0, k = 0, 1, \ldots, n-1$.

Next we apply Theorems 26.4, 26.5 for $g(x) = e^x$. One can give similar applications for $g = \sin, \cos, \tan$, etc, over suitable intervals.

Proposition 26.6. *Let $f^{(n)}$ continuous, from [a,b] (or [a,b]) into $\mathbb{R}, n \in \mathbb{N}$. Then*

$$f(b) = f(a) + \sum_{k=1}^{n-1} \frac{[(f \circ ln)^{(k)}(e^a)]}{k!} \cdot (e^b - e^a)^k + \mathcal{R}_n(a, b), \tag{26.15}$$

where

$$\begin{aligned} \mathcal{R}_n(a, b) &= \frac{1}{(n-1)!} \int_{e^a}^{e^b} (e^b - t)^{n-1} (f \circ ln)^{(n)}(t) dt \\ &= \frac{1}{(n-1)!} \int_a^b (e^b - e^s)^{n-1} (f \circ ln)^{(n)}(e^s) \cdot e^s ds. \end{aligned} \tag{26.16}$$

We continue with

Proposition 26.7. *Let $f, f', \ldots, f^{(n-1)}$ are continuous on [a,b] and $f^{(n)}$ exists in $(a, b), n \in \mathbb{N}$. If $\alpha, \beta \in [a, b]$, then there exists γ between α, β such that*

$$f(\beta) = f(\alpha) + \sum_{k=1}^{n-1} \frac{(f \circ ln)^{(k)}}{k!} (e^\alpha) \cdot (e^\beta - e^\alpha)^k + \frac{(f \circ ln)^{(n)}}{n!} (e^\gamma) \cdot (e^\beta - e^\alpha)^n. \tag{26.17}$$

Next we present inequalities based on the above Taylor formula (26.4).

We make

Remark 26.8. Let f, g as in Theorem 26.4, and any $x, y \in [a, b]$. Then, by (26.4) and (26.5) we obtain

$$f(x) = f(y) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(y))}{k!} \cdot (g(x) - g(y))^k$$

$$+\frac{1}{n-1!} \int_y^x (g(x) - g(s))^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(s)) \cdot g'(s) ds. \quad (26.18)$$

Integrating (26.18) over [a,b] with respect to y, we get

$$f(x) = \frac{1}{b-a} \int_a^b f(y) dy$$

$$+\sum_{k=1}^{n-1} \frac{1}{k!(b-a)} \int_a^b (f \circ g^{-1})^{(k)}(g(y)) \cdot (g(x) - g(y))^k dy$$

$$+\frac{1}{(n-1)!(b-a)} \int_a^b \int_y^x (g(x) - g(t))^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) dt, \quad (26.19)$$

$\forall\, x \in [a, b]$.
Define the kernel

$$K(t, x) = \begin{cases} t - a, \ a \le t \le x \le b; \\ t - b, \ a \le x < t \le b. \end{cases} \quad (26.20)$$

By letting $* := (g(x) - g(t))^{(n-1)} \cdot (fog^{-1})^{(n)}(g(t)) \cdot g'(t)$ we find

$$\int_a^b \left(\int_y^x * \, dt \right) dy = \int_a^x \left(\int_y^x * \, dt \right) dy + \int_x^b \left(\int_y^x * \, dr \right) dy$$

$$= \int_a^x \left(\int_a^t * \, dy \right) dt - \int_x^b \left(\int_x^y * \, dt \right) dy$$

$$= \int_a^x * \left(\int_a^t dy \right) dt - \int_x^b * \left(\int_t^b dy \right) dt$$

$$= \int_a^x * \, (t-a) dt + \int_x^b *(t-b) dt = \int_a^b * \, K(t, x) dt \quad (26.21)$$

Above, we have that

$$\left\{ \begin{array}{l} a \le y \le x \\ y \le t \le x \\ \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} a \le t \le x \\ a \le y \le t \\ \end{array} \right\}$$

and

$$\left\{\begin{array}{l} x \le y \le b \\ x \le t \le y \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x \le t \le b \\ t \le y \le b \end{array}\right\} \tag{26.22}$$

Therefore we obtain

$$\int_a^b \int_y^x (g(x)-g(y))^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) dt$$

$$= \int_a^b (g(x)-g(t))^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) \cdot K(t,x) dt. \tag{26.23}$$

We have proved the following representation formula

Theorem 26.9. *Let $f, f', \ldots, f^{(n)}; g, g'$ be continuous from $[a,b]$ into $\mathbb{R}, n \in \mathbb{N}$. Suppose $(g^{-1})^{(k)}, k = 0, 1, \ldots, n$ are continuous. Then*

$$f(x) = \frac{1}{b-a} \int_a^b f(y) dy$$

$$+\frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_a^b (f \circ g^{-1})^{(k)}(g(y)) \cdot (g(x)-g(y))^k dy \right\}$$

$$+\frac{1}{(n-1)!(b-a)} \int_a^b (g(x)-g(t))^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t)$$

$$\cdot K(t,x) dt, \tag{26.24}$$

$\forall\, x \in [a,b]$.

Same applications of last theorem follow

Theorem 26.10. *Let $f \in C^n([a,b]), n \in \mathbb{N}$. Then*

$$f(x) = \frac{1}{b-a} \int_a^b f(y) dy + \frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_a^b (f \circ ln)^{(k)}(e^y) \cdot (e^x - e^y)^k dy \right\}$$

$$+\frac{1}{(n-1)!(b-a)} \int_a^b (e^x - e^t)^{n-1} \cdot (f \circ ln)^{(n)}(e^t) \cdot e^t \cdot K(t,x) dt, \tag{26.25}$$

$\forall\, x \in [a,b]$.

Theorem 26.11. *Let $f \in C^n([-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon]), n \in \mathbb{N}, \varepsilon > 0$ small. Then*

$$f(x) = \frac{1}{\pi - 2\varepsilon} \int_{\frac{-\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} f(y) dy$$

$$+\frac{1}{\pi - 2\varepsilon} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_{\frac{-\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} (f \circ \sin^{-1})^{(k)}(\sin\, y) \cdot (\sin\, x - \sin\, y)^k dy \right\}$$

$$+\frac{1}{(n-1)!(\pi-2\varepsilon)}\int_{\frac{-\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}(\sin\, x-\sin\, t)^{n-1}(f\circ\sin^{-1})^{(n)}(\sin\, t)\cos\, t$$
$$\cdot K(t,x)dt, \tag{26.26}$$

$\forall\, x\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right].$

Theorem 26.12. *Let* $f\in C^n([\varepsilon,\pi-\varepsilon]), n\in\mathbb{N},\varepsilon>0$ *small. Then*

$$f(x)=\frac{1}{\pi-2\varepsilon}\int_{\varepsilon}^{\pi-\varepsilon}f(y)dy$$
$$+\frac{1}{\pi-2\varepsilon}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{\varepsilon}^{\pi-\varepsilon}(f\circ\cos^{-1})^{(k)}(\cos\, y)\cdot(\cos\, x-\cos\, y)^k dy\right\}$$
$$-\frac{1}{(n-1)!(\pi-2\varepsilon)}\int_{\varepsilon}^{\pi-\varepsilon}(\cos\, x-\cos\, t)^{n-1}(f\circ\cos^{-1})^{(n)}(\cos\, t)\sin\, t\cdot K(t,x)dt, \tag{26.27}$$

$\forall\, x\in[\varepsilon,\pi-\varepsilon].$

Theorem 26.13. *Let* $f\in C^n([-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon]), n\in\mathbb{N},\varepsilon>0$ *small. Then*

$$f(x)=\frac{1}{\pi-2\varepsilon}\int_{\frac{-\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}f(y)dy$$
$$+\frac{1}{\pi-2\varepsilon}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{\frac{-\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}(f\circ\tan^{-1})^{(k)}(\tan\, y)\cdot(\tan\, x-\tan\, y)^k dy\right\}$$
$$+\frac{1}{(n-1)!(\pi-2\varepsilon)}\int_{\frac{-\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}(\tan\, x-\tan\, t)^{n-1}(f\circ\tan^{-1})^{(n)}(\tan\, t)$$
$$\cdot\sec^2 t\cdot K(t,x)dt, \tag{26.28}$$

$\forall\, x\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right].$

Next we present an Opial type inequality

Theorem 26.14. *Let* $f,f',\ldots,f^{(n)};g,g'$ *be continuous from* $[a,b]$ *into* $\mathbb{R}, n\in\mathbb{N}$. *Suppose* $(g^{-1})^{(k)}, k=0,1,\ldots,n$ *are continuous. Further assume that* $f^{(k)}(a)=0, k=0,1,\ldots,n-1$. *Here* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\int_a^x|f(w)|\left|(f\circ g^{-1})^{(n)}(g(w))\right|\left|g'(w)\right|dw$$
$$\leq\frac{1}{2^{1/q}(n-1)!}\left(\int_a^x\left(\int_a^w|g(w)-g(s)|^{(n-1)p}ds\right)dw\right)^{1/p}$$
$$\left(\int_a^x\left|(f\circ g^{-1})^{(n)}(g(w))\right|^q\left|g'(w)\right|^q dw\right)^{2/q}, \tag{26.29}$$

$\forall\, x\in[a,b].$

Proof. By assumptions we have

$$f(x) = \frac{1}{(n-1)!} \int_a^x (g(x) - g(s))^{n-1} (f \circ g^{-1})^{(n)}(g(s)) \cdot g'(s) ds, \tag{26.30}$$

$\forall\, x \in [a, b]$.
By Hölder's inequality we get

$$\begin{aligned} |f(x)| &\le \frac{1}{(n-1)!} \int_a^x |(g(x) - g(s))|^{n-1} \left|(f \circ g^{-1})^{(n)}(g(s))\right| \cdot |g'(s)| ds \\ &\le \frac{1}{(n-1)!} \left(\int_a^x |(g(x) - g(s))|^{p(n-1)}\, ds \right)^{1/p} \\ &\left(\int_a^x \left|(f \circ g^{-1})^{(n)}(g(s))\right|^q \left|g'(s)\right|^q ds \right)^{1/q} =: (*). \end{aligned} \tag{26.31}$$

We put

$$z(x) = \int_a^x \left|(f \circ g^{-1})^{(n)}(g(s))\right|^q \left|g'(s)\right|^q ds \ge 0, \tag{26.32}$$

$z(a) = 0$.
Hence

$$z'(x) = \left|(f \circ g^{-1})^{(n)}(g(x))\right|^q \left|g'(x)\right|^q \ge 0,$$

and

$$(z'(x))^{1/q} = \left|(f \circ g^{-1})^{(n)}(g(x))\right| \left|g'(x)\right| \ge 0, \tag{26.33}$$

$\forall\, x \in [a, b]$.
Consequently we obtain

$$\begin{aligned} &|f(w)|\ |(f \circ g^{-1})^{(n)}(g(w))|\, |g'(w)| \\ &\le \frac{1}{(n-1)!} \left(\int_a^w |g(w) - g(s)|^{p(n-1)} ds \right)^{1/p} (z(w) z'(w))^{1/q}, \end{aligned} \tag{26.34}$$

$\forall\, w \in [a, b]$.
Thus

$$\begin{aligned} &\int_a^x |f(w)|\ |(f \circ g^{-1})^{(n)}(g(w))|\, |g'(w)| dw \\ &\le \frac{1}{(n-1)!} \left\{ \int_a^x \left[\left(\int_a^w |g(w) - g(s)|^{p(n-1)}\, ds \right)^{1/p} (z(w) z'(w))^{1/q} \right] dw \right\} \end{aligned} \tag{26.35}$$

(we apply again Hölder's inequality)

$$\le \frac{1}{(n-1)!} \left\{ \left(\int_a^x \left(\int_a^w |g(w) - g(s)|^{p(n-1)}\, ds \right) dw \right)^{1/p} \right.$$

$$\left.\left(\int_a^x z(w)z'(w)dw\right)^{1/q}\right\} \tag{26.36}$$

$$= \frac{1}{(n-1)!}\left(\int_a^x\left(\int_a^w |g(w)-g(s)|^{p(n-1)}\,ds\right)dw\right)^{1/p}\left(\frac{z^2(x)}{2}\right)^{1/q}$$

$$= \frac{1}{2^{1/q}(n-1)!}\left(\int_a^x\left(\int_a^w |g(w)-g(s)|^{p(n-1)}\,ds\right)dw\right)^{1/p}$$

$$\left(\int_a^x \left|(f\circ g^{-1})^{(n)}(g(w))\right|^q \left|g'(w)\right|^q dw\right)^{2/q}, \tag{26.37}$$

$\forall\, x\in[a,b]$, proving the claim. ■

Next we apply Theorem 26.14 to obtain

Proposition 26.15. *Let* $f\in C^n([a,b]), n\in\mathbb{N}, p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. *Suppose* $f^{(k)}(a)=0, k=0,1,\dots,n-1$. *Then*

$$\int_a^x |f(w)|\;|(f\circ ln)^{(n)}(e^w)|\,e^w dw$$

$$\le \frac{1}{2^{1/q}(n-1)!}\left(\int_a^x\left(\int_a^w (e^w-e^s)^{p(n-1)}ds\right)dw\right)^{1/p}$$

$$\left(\int_a^x \left|(f\circ ln)^{(n)}(e^w)\right|^q e^{qw}dw\right)^{2/q}, \tag{26.38}$$

$\forall\, x\in[a,b]$.

Proposition 26.16. *Let* $f\in C^n([-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon]), n\in\mathbb{N}, \varepsilon>0$ *small;* $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. *Suppose* $f^{(k)}(-\frac{\pi}{2}+\varepsilon)=0, k=0,1,\dots,n-1$. *Then*

$$\int_{-\frac{\pi}{2}+\varepsilon}^x |f(w)|\;\left|(f\circ\sin^{-1})^{(n)}(\sin\,w)\right|\;\cos\,w\,dw$$

$$\le \frac{1}{2^{1/q}(n-1)!}\left(\int_{-\frac{\pi}{2}+\varepsilon}^x\left(\int_{-\frac{\pi}{2}+\varepsilon}^w (\sin\,w-\sin\,s)^{(n-1)p}ds\right)dw\right)^{1/p}$$

$$\left(\int_{-\frac{\pi}{2}+\varepsilon}^x \left|(f\circ\sin^{-1})^{(n)}(\sin\,w)\right|^q(\cos\,w)^q dw\right)^{2/q}, \tag{26.39}$$

$\forall\, x\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]$.

One can give in any other similar applications to Theorem 26.14.

We make

Remark 26.17. By Theorem 26.9 we find

$$\mathcal{E}_n(x) := f(x)-\frac{1}{b-a}\int_a^b f(y)dy$$

$$-\frac{1}{(b-a)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_a^b (f\circ g^{-1})^{(k)}(g(y))\cdot(g(x)-g(y))^k dy\right\}$$

$$=\frac{1}{(n-1)!(b-a)}\int_a^b (g(x)-g(t))^{n-1}\cdot(f\circ g^{-1})^{(n)}(g(t))\cdot g'(t)$$

$$\cdot K(t,x)dt =: I_n(f)(x), \qquad (26.40)$$

$\forall\, x\in[a,b]$.

Hence

$$|\mathcal{E}_n(x)| = |I_n(f)(x)| \le \frac{1}{(n-1)!(b-a)}\int_a^b |(g(x)-g(t))|^{n-1}$$

$$|(f\circ g^{-1})^{(n)}(g(t))|\,|g'(t)|\,|K(t,x)|dt =: \Theta(x). \qquad (26.41)$$

We distinguish the following cases.

(i)

$$\Theta(x)\le\frac{1}{(n-1)!(b-a)}\left\|(f\circ g^{-1})^{(n)}\circ g\right\|_\infty \|g'\|_\infty$$

$$\left[\int_a^x |g(x)-g(t)|^{n-1}(t-a)dt+\int_x^b |g(x)-g(t)|^{n-1}(b-t)dt\right]=\Theta_1(x), \quad (26.42)$$

(ii)

$$\Theta(x)\le\frac{1}{(n-1)!(b-a)}\,\|g(x)-g(\cdot)\|_\infty^{n-1}\left\|g'\right\|_\infty \max((x-a),$$

$$(b-x))\left\|(f\circ g^{-1})^{(n)}og\right\|_1=\Theta_2(x), \qquad (26.43)$$

and

(iii) By Hölder's inequality we find

$$\Theta(x)\le\frac{1}{(n-1)!(b-a)}\,\|(g(x))-g(\cdot))^{n-1}g'(\cdot)$$

$$K(\cdot,x)\,\|_q\|\,(f\circ g^{-1})^{(n)}og\,\|_p =: \Theta_3(x), \qquad (26.44)$$

where $p,q>1:\frac{1}{p}+\frac{1}{q}=1$.

We have proved the following general Ostrowski type inequality

Theorem 26.18. *All assumptions as in Theorem 26.9. Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$|\mathcal{E}_n(x)|\le\min\left(\Theta_1(x),\Theta_2(x),\Theta_3(x)\right), \qquad (26.45)$$

$\forall\, x\in[a,b]$.

Inequality (26.45) for a fixed $x \in [a,b]$ is attained by an $\hat{f} \in C^n([a,b])$ such that

$$|(\hat{f} \circ g^{-1})^{(n)}(g(t))| = A(|g(x) - g(t)|^{n-1}|g'(t)|\ |K(t,x)|)^{q/p}, \tag{26.46}$$

where $A > 0$, and

$$[(\hat{f} \circ g^{-1})^{(n)}(g(t)) \cdot (g(x) - g(t))^{n-1} \cdot g'(t) \cdot K(t,x)]$$

of fixed sign , $\forall\, t \in [a,b]$.

Next we apply Theorem 26.18.

We get

Theorem 26.19. *Let $f, \in C^n([a,b]), n \in \mathbb{N}, p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(y)dy - \frac{1}{(b-a)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_a^b (f \circ ln)^{(k)}(e^y)\cdot(e^x - e^y)^k dy\right| \right.$$

$$\leq \frac{1}{(n-1)!(b-a)} \min\left\{ e^b \left\|(f \circ ln)^{(n)}(e^{(\cdot)})\right\|_\infty \left[\int_a^x (e^x - e^t)^{n-1}(t-a)dt \right.\right.$$

$$\left. + \int_x^b (e^t - e^x)^{n-1}(b-t)dt\right],$$

$$e^b(\max(e^x - e^a, e^b - e^x))^{n-1} \cdot \max((x-a),(b-x)) \cdot \left\|(f \circ ln)^{(n)}(e^{(\cdot)})\right\|_1,$$

$$\left. \left\|(e^x - e^{(\cdot)})^{(n-1)} \cdot e^{(\cdot)} \cdot K(\cdot,x)\right\|_q \cdot \left\|(f \circ ln)^{(n)}(e^{(\cdot)})\right\|_p \right\}, \tag{26.47}$$

$\forall\, x \in [a,b]$.

Inequality (26.47) for a fixed $x \in [a,b]$ is attained by an $\hat{f} \in C^n([a,b])$ such that

$$|(\hat{f} \circ ln)^{(n)}(e^t)| = A(|g^x - e^t|^{n-1} \cdot e^t \cdot |K(t,x)|)^{q/p}, \tag{26.48}$$

where $A > 0$, and

$$[(\hat{f} \circ ln)^{(n)}(e^t) \cdot (e^x - e^t)^{n-1} \cdot e^t \cdot K(t,x)]$$

of fixed sign , $\forall\, t \in [a,b]$.

Theorem 26.20. *Let $f, \in C^n([\varepsilon, \pi - \varepsilon]), n \in \mathbb{N}, \varepsilon > 0$ small; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| f(x) - \frac{1}{\pi - 2\varepsilon}\int_\varepsilon^{\pi-\varepsilon} f(y)dy \right.$$

$$-\frac{1}{(\pi - 2\varepsilon)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_\varepsilon^{\pi-\varepsilon}(f o \cos^{-1})^{(k)}(\cos\, y)\cdot(\cos\, x - \cos\, y)^k dy\ \right|$$

$$\leq \frac{1}{(n-1)!(\pi-2\varepsilon)} \min\left\{ \left\|(f\circ \cos^{-1})^{(n)} o \cos\right\|_\infty \left[\int_\varepsilon^x (\cos t - \cos x)^{n-1}(t-\varepsilon)dt \right.\right.$$
$$\left. + \int_x^{\pi-\varepsilon} (\cos x - \cos t)^{n-1}(\pi-\varepsilon-t)dt \right],$$
$$\|\cos x - \cos\cdot\|_\infty^{n-1} \max((x-\varepsilon),(\pi-\varepsilon-x)) \left\|(f\circ\cos^{-1})^{(n)} \circ \cos\right\|_1,$$
$$\left. \left\|(\cos x - \cos(\cdot))^{(n-1)} \sin(\cdot) K(\cdot,x)\right\|_q \left\|(f\circ\cos^{-1})^{(n)}\circ\cos\right\|_p \right\}, \tag{26.49}$$

$\forall\, x \in [\varepsilon, \pi-\varepsilon]$.

Inequality (26.49) for a fixed $x \in [\varepsilon, \pi-\varepsilon]$ is attained by an $\hat{f} \in C^n([\varepsilon,\pi-\varepsilon])$ such that

$$|(\hat{f}\circ\cos^{-1})^{(n)}(\cos t)| = A(|\cos x - \cos t|^{n-1}|\sin t|\,|K(t,x)|)^{q/p}, \tag{26.50}$$

where $A > 0$, and

$$[(\hat{f}\circ\cos^{-1})^{(n)}(\cos t)\cdot(\cos x - \cos t)^{n-1}\cdot \sin t \cdot K(t,x)]$$

of fixed sign , $\forall\, t \in [\varepsilon, \pi-\varepsilon]$.

We need to make

Remark 26.21. Let f, g as in Theorem 26.9. Let μ be a finite positive measure of mass $m > 0$ on $([c,d], \mathcal{P}([c,d])), [c,d] \subseteq [a,b]$, where $\mathcal{P}$ stands for the power set. Integrating (26.24) against μ we find

$$\int_{[c,d]} f(x)d\mu(x) = \frac{1}{(b-a)}\left(\int_a^b f(y)dy\right) m$$
$$+\frac{1}{(b-a)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{[c,d]}\left(\int_a^b (f\circ g^{-1})^{(k)}(g(y))\cdot(g(x)-g(y))^k dy\right)d\mu(x)\right\}$$
$$+\frac{1}{(n-1)!(b-a)}\left(\int_{[c,d]}\left(\int_a^b (g(x)-g(t))^{n-1}\cdot(f\circ g^{-1})^{(n)}(g(t))\cdot g'(t)\right.\right.$$
$$\left.\left.\cdot K(t,x)dt\right)d\mu(x)\right). \tag{26.51}$$

Therefore we have

$$M_n(f) := \frac{1}{m}\int_{[c,d]} f(x)d\mu(x) - \frac{1}{(b-a)}\int_a^b f(y)dy$$
$$-\frac{1}{m(b-a)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{[c,d]}\left(\int_a^b (f\circ g^{-1})^{(k)}(g(y))\cdot(g(x)-g(y))^k dy\right)d\mu(x)\right\}$$
$$=\frac{1}{(n-1)!(b-a)m}\left(\int_{[c,d]}\left(\int_a^b (g(x)-g(t))^{n-1}\cdot(f\circ g^{-1})^{(n)}(g(t))\cdot g'(t)\right.\right.$$
$$\left.\left.\cdot K(t,x)dt\right)d\mu(x)\right) =: J(f). \tag{26.52}$$

One can estimate $J(f)$.

We derive the following comparison of integral means result

Theorem 26.22. *Let the assumptions of Theorem 26.9. Let μ be a finite positive measure of mass $m > 0$ on $([c,d], \mathcal{P}([c,d])), [c,d] \subseteq [a,b]$. Then*

$$|M_n(f)| \leq \frac{1}{(n-1)!(b-a)m} \min\left\{ \left(\int_{[c,d]} \left[\int_a^x |(g(x)-g(t))|^{n-1} \cdot (t-a)dt \right.\right.\right.$$

$$\left.\left. + \int_x^b |g(x)-g(t)|^{n-1} \cdot (b-t)dt \right] d\mu(x) \right) \left\| (f \circ g^{-1})^{(n)} \circ g \cdot g' \right\|_{\infty},$$

$$\left(\int_{[c,d]} (\|g(x)-g(\cdot)\|_{\infty}^{n-1} (\max(x-a), b-x)) d\mu(x) \right) \left\| (f \circ g^{-1})^{(n)} \circ g \cdot g' \right\|_1,$$

$$\left.\left(\int_{[c,d]} \left\| (g(x)-g(\cdot))^{n-1} \cdot K(\cdot,x) \right\|_q d\mu(x) \right) \left\| (f \circ g^{-1})^{(n)} \circ g \cdot g' \right\|_p, \right\} \quad (26.53)$$

where $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

We give some applications of Theorem 26.22.

Theorem 26.23. *Let $f \in C^n([a,b]), n \in \mathbb{N}$. Let μ be a finite positive measure of mass $m > 0$ on $([c,d], \mathcal{P}([c,d])), [c,d] \subseteq [a,b]$. Then*

$$\left| \frac{1}{m} \int_{[c,d]} f(x) d\mu(x) - \frac{1}{(b-a)} \int_a^b f(y) dy \right.$$

$$\left. - \frac{1}{m(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_{[c,d]} \left(\int_a^b (f o l n)^{(k)}(e^y)(e^x - e^y)^k dy \right) d\mu(x) \right\} \right|$$

$$\leq \frac{1}{(n-1)!(b-a)m} \min\left\{ \left(\int_{[c,d]} \left[\int_a^x [e^x - e^t)^{n-1}(t-a)dt \right.\right.\right.$$

$$\left.\left. + \int_x^b (e^t - e^x)^{n-1}(b-t)dt \right] d\mu(x) \right) \left\| (f \circ ln)^{(n)}(e^{(\cdot)}) \cdot e^{(\cdot)} \right\|_{\infty},$$

$$\left(\int_{[c,d]} \left(\max(e^x - e^a, e^b - e^x) \right)^{(n-1)} (\max(x-a, b-x)) d\mu(x) \right)$$

$$\left\| (f \circ ln)^{(n)}(e^{(\cdot)}) \cdot e^{(\cdot)} \right\|_1,$$

$$\left.\left(\int_{[c,d]} \left\| (e^x - e^{(\cdot)})^{n-1} \cdot K(\cdot,x) \right\|_q d\mu(x) \right) \left\| (f \circ ln)^{(n)}(e^{(\cdot)}) \cdot e^{(\cdot)} \right\|_p \right\}, \quad (26.54)$$

where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Theorem 26.24. *Let $f \in C^n([-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon]), n \in \mathbb{N}, \varepsilon > 0$ small. Let μ be a finite positive measure of mass $m > 0$ on $([c,d], \mathcal{P}([c,d])), [c,d] \subseteq [-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon]$. Then*

$$\Bigg| \frac{1}{m}\int_{[c,d]} f(x)d\mu(x) - \frac{1}{\pi - 2\varepsilon}\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} f(y)dy$$

$$-\frac{1}{m(\pi-2\varepsilon)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{[c,d]}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}(f\circ\tan^{-1})^{(k)}(\tan y)\right.\right.$$

$$\left.\left.\cdot(\tan x - \tan y)^k dy\right) d\mu(x)\right\}\Bigg|$$

$$\leq \frac{1}{(n-1)!(\pi-2\varepsilon)m}\min\left\{\left(\int_{[c,d]}\left[\int_{-\frac{\pi}{2}+\varepsilon}^{x}(\tan x - \tan t)^{n-1}(t+\frac{\pi}{2}-\varepsilon)dt\right.\right.\right.$$

$$\left.\left.+\int_{x}^{\frac{\pi}{2}-\varepsilon}(\tan t - \tan x)^{n-1}(\frac{\pi}{2}-\varepsilon-t)dt\right] d\mu(x)\right)$$

$$\|(f\circ\tan^{-1})^{(n)}(\tan(\cdot))\cdot\sec^2(\cdot)\|_\infty,$$

$$\left(\int_{[c,d]}\left(\max(\tan x - \tan(-\frac{\pi}{2}-\varepsilon), \tan\left(\frac{\pi}{2}-\varepsilon\right) - \tan x)\right)^{(n-1)}\right.$$

$$\left.\left(\max\left(x+\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}-\varepsilon-x\right)\right) d\mu(x)\right)\|\left(f\circ\tan^{-1}\right)^{(n)}(\tan(\cdot))\cdot\sec^2(\cdot)\|_1\ ,$$

$$\left(\int_{[c,d]}\left\|(\tan x - \tan(\cdot))^{n-1}\cdot K(\cdot,x)\right\|_q d\mu(x)\right)$$

$$\left\|((f\circ\tan^{-1})^{(n)}(\tan(\cdot))\cdot\sec^2(\cdot)\right\|_p\Bigg\}, \tag{26.55}$$

where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Background 26.25. Next we follow [137]. This is related to Information theory. Let f be a convex function from $(0, +\infty)$ into $\mathbb{R}$, which is strictly convex at 1 with $f(1) = 0$. Let $(X, \mathcal{A}, \lambda)$ be a measure space, where λ is a finite or a σ-finite measure on $(X, \mathcal{A})$. And let μ_1, μ_2 be two probability measures on $(X, \mathcal{A})$ such that $\mu_1 \ll \lambda, \mu_2 \ll \lambda$ (absolutely continuous), e.g.$\lambda = \mu_1 + \mu_2$. Denote by $p = \frac{d\mu_1}{d\lambda}, q = \frac{d\mu_2}{d\lambda}$ the Radon-Nikodym derivatives of μ_1, μ_2 with respect to λ (densities). Here, we assume that

$0 < a \leq \frac{p}{q} \leq b$, *a.e.on X and $a \leq 1 \leq b$.*

The quantity

$$\Gamma_f(\mu_1, \mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x) \tag{26.56}$$

was introduced by Csiszar in 1967 (see [137]), and is called the *f-divergence* of the probability measures μ_1 and μ_2. By Lemma 1.1 of [137], the integral (26.56) is well defined, and $\Gamma_f(\mu_1, \mu_2) \geq 0$,with equality only when $\mu_1 = \mu_2$. Furthermore $\Gamma_f(\mu_1, \mu_2)$ does not depend on the choice of λ. Here, by assuming $f(1) = 0$, we can consider $\Gamma_f(\mu_1, \mu_2)$ the f-divergence, as a measure of the difference between the probability measures μ_1, μ_2.

Here we give a representation on estimates for $\Gamma_f(\mu_1, \mu_2)$ via formula (26.24). We give

Theorem 26.26. *All as in Background 26.25 and Theorem 26.9. Then*

$$\Gamma_f(\mu_1, \mu_2) = \frac{1}{b-a} \int_a^b f(y)dy$$
$$+\frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\int_X g(x) \left(\int_a^b \left(f \circ g^{-1}\right)^{(k)} (f(y)) \right.\right.\right.$$
$$\left.\left.\left. \cdot \left(g\left(\frac{p(x)}{q(x)}\right) - g(y) \right)^k dy \right) d\lambda(x) \right) \right\} + G_n, \tag{26.57}$$

where

$$G_n := \frac{1}{(n-1)!(b-a)} \left(\int_X g(x) \left(\int_a^b \left(g\left(\frac{p(x)}{q(x)}\right) - g(t) \right) \right)^{n-1} \right.$$
$$\left. \cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) \cdot K\left(t, \frac{p(x)}{q(x)}\right) dt \right) d\lambda(x) \Bigg). \tag{26.58}$$

Proof. By (26.24) we obtain that

$$q(x) f\left(\frac{p(x)}{q(x)}\right) = \frac{q(x)}{b-a} \int_a^b f(y)dy$$
$$+\frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{q(x)}{k!} \int_a^b (f \circ g^{-1})^{(k)}(g(y)) \cdot \left(g\left(\frac{p(x)}{q(x)}\right) - g(y) \right)^k dy \right\}$$
$$+\frac{q(x)}{(n-1)!(b-a)} \int_a^b \left(g\left(\frac{p(x)}{q(x)}\right) - g(t) \right)^{n-1} \cdot (f \circ g^{-1})^{(n)}(g(t))$$
$$\cdot g'(t) \cdot K\left(t, \frac{p(x)}{q(x)}\right) dt, \tag{26.59}$$

a.e. on X.

Integrating (26.59) against λ we derive (26.58). ∎

Next we estimate G_n, that is we estimate $\Gamma_f(\mu_1, \mu_2)$.

Theorem 26.27. *All assumptions as in Theorem 26.26. Then*

$$|G_n| \leq \frac{1}{(n-1)!(b-a)} \min \Bigg\{ \left(\int_X g(x) \left(\int_a^b \left| g\left(\frac{p(x)}{q(x)}\right) - g(t) \right|^{n-1} \left| K\left(t, \frac{p(x)}{q(x)}\right) \right| dt \right) d\lambda(x) \right) \left\| ((f \circ g^{-1})^{(n)} \circ g) \cdot g' \right\|_\infty ,$$

$$\left(\int_X g(x) \left\| \left(g\left(\frac{p(x)}{q(x)}\right) - g(\cdot) \right)^{n-1} \cdot K\left(\cdot, \frac{p(x)}{q(x)}\right) \right\|_{p_2} d\lambda(x) \right) \left\| ((f \circ g^{-1})^{(n)} \circ g) \cdot g' \right\|_{p_1} ,$$

$$\left(\int_X g(x) \left\| g\left(\frac{p(x)}{q(x)}\right) - g(\cdot) \right\|_\infty^{n-1} \cdot \max\left(\frac{p(x)}{q(x)} - a, b - \frac{p(x)}{q(x)} \right) d\lambda(x) \right) \left\| ((f \circ g^{-1})^{(n)} \circ g) \cdot g' \right\|_1 \Bigg\}, \tag{26.60}$$

where $p_1, p_2 > 1$, such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Proof. By (26.57), (26.58). ■

In the following we apply Theorem 26.27.

Theorem 26.28. *All as in Background 26.25 with $f \in C^n([a,b]), n \in \mathbb{N}$. Then*

$$\Bigg| \Gamma_f(\mu_1, \mu_2) - \frac{1}{b-a} \int_a^b f(y) dy - \frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\int_X g(x) \left(\int_a^b (f \circ ln)^{(k)} (e^y) \cdot \left(e^{\frac{p(x)}{q(x)}} - e^y \right)^k dy \right) d\lambda(x) \right) \right\} \Bigg|$$

$$\leq \frac{1}{(n-1)!(b-a)} \min \Bigg\{ \left(\int_X g(x) \left(\int_a^b \left| e^{\frac{p(x)}{q(x)}} - e^t \right|^{n-1} \left| K\left(t, \frac{p(x)}{q(x)}\right) \right| dt \right) d\lambda(x) \right) \left\| \left((f \circ ln)^{(n)} \circ e^{(\cdot)} \right) \cdot e^{(\cdot)} \right\|_\infty ,$$

$$\left(\int_X g(x) \left\| \left(e^{\frac{p(x)}{q(x)}} - e^{(\cdot)} \right)^{n-1} \cdot K\left(\cdot, \frac{p(x)}{q(x)}\right) \right\|_{p_2} d\lambda(x) \right) \left\| \left((f \circ ln)^{(n)} \circ e^{(\cdot)} \right) \cdot e^{(\cdot)} \right\|_{p_1} ,$$

$$\left(\int_X g(x) \left\| e^{\frac{p(x)}{q(x)}} - e^{(\cdot)} \right\|_\infty^{n-1} \max\left(\frac{p(x)}{q(x)} - a, b - \frac{p(x)}{q(x)} \right) d\lambda(x) \right)$$

$$\left\| \left((f \circ ln)^{(n)} \circ e^{(\cdot)} \right) \cdot e^{(\cdot)} \right\|_1 \Bigg\}, \tag{26.61}$$

where $p_1, p_2 > 1$, such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Theorem 26.29. *All as in Background 26.25 with* $a = -\frac{\pi}{2} + \varepsilon, b = \frac{\pi}{2} - \varepsilon, \varepsilon > 0$ *small. Here* $f \in C^n([-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon]), n \in \mathbb{N}$.

Then

$$\Bigg| \Gamma_f(\mu_1, \mu_2) - \frac{1}{\pi - 2\varepsilon} \int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} f(y)dy$$

$$-\frac{1}{\pi - 2\varepsilon} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\int_X g(x) \left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} \left(f \circ \sin^{-1}\right)^{(k)} (\sin\, y) \right.\right.\right.$$

$$\left.\left.\left. \cdot \left(\sin\left(\frac{p(x)}{q(x)}\right) - \sin\, y \right)^k dy \right) d\lambda(x) \right) \right\}$$

$$\leq \frac{1}{(n-1)!(\pi - 2\varepsilon)} \min \left\{ \left(\int_X g(x) \left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} \left| \sin\left(\frac{p(x)}{q(x)}\right) - \sin\, t \right|^{n-1} \right.\right.\right.$$

$$\left.\left. \left| K\left(t, \frac{p(x)}{q(x)}\right) \right| dt \right) d\lambda(x) \right) \left\| \left(\left(f \circ \sin^{-1}\right)^{(n)} \circ \sin \right) \cdot \cos \right\|_\infty,$$

$$\left(\int_X g(x) \left\| \left(\sin\left(\frac{p(x)}{q(x)}\right) - \sin(\cdot) \right)^{n-1} \cdot K\left(\cdot, \frac{p(x)}{q(x)}\right) \right\|_{p_2} d\lambda(x) \right)$$

$$\left\| \left(\left(f \circ \sin^{-1}\right)^{(n)} \circ \sin \right) \cdot \cos \right\|_{p_1},$$

$$\left(\int_X g(x) \left\| \sin\left(\frac{p(x)}{q(x)}\right) - \sin(\cdot) \right\|_\infty^{n-1} \right.$$

$$\left. \cdot \max\left(\frac{p(x)}{q(x)} - a, b - \frac{p(x)}{q(x)} \right) d\lambda(x) \right) \left\| \left((f \circ \sin^{-1})^{(n)} \circ \sin \right) \cdot \cos \right\|_1 \Bigg\}, \tag{26.62}$$

where $p_1, p_2 > 1$, *such that* $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Next we give a very general Grüss type inequality

Theorem 26.30. *Let* $f, h, f'; h', \ldots, f^{(n)}, h^{(n)}; g, g'$ *be continuous from [a,b], into* $\mathbb{R}, n \in \mathbb{N}$. *Suppose* $(g^{-1})^{(k)}, k = 0, 1, \ldots, n$ *are continuous.*

Then

$$\left| \frac{1}{(b-a)} \int_a^b f(x)h(x)dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x)dx \right) \left(\int_a^b h(x)dx \right) \right.$$

$$-\frac{1}{2(b-a)^2} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\int_a^b \left(\int_a^b \left[h(x) \cdot (f \circ g^{-1})^{(k)}(g(y)) \right.\right.\right.\right.$$

$$+f(x)\cdot\left(h\circ g^{-1}\right)^{(k)}(g(y))\Big]\cdot(g(x)-g(y))^k\,dy\Big)\,dx\Big)\Big\}\Big|$$

$$\leq \min\Bigg\{\frac{1}{2(n-1)!(b-a)^2}\left(\int_a^b\left(\int_a^b\Big[|h(x)|\cdot\Big\|\Big((f\circ g^{-1})^{(n)}\circ g\Big)\cdot g'\Big\|_\infty\right.\right.$$

$$\left.\left.+|f(x)|\cdot\|((h\circ g^{-1})^{(n)}\circ g)\cdot g'\|_\infty\Big]\cdot|g(x)-g(t)|^{n-1}\ |K(t,x)|dt\right)dx\right),$$

$$\frac{\|g(x)-g(t)\|^{n-1}_{\infty,(x,t)\in[a,b]^2}}{2(n-1)!}\cdot\Big[\|h\|_\infty\cdot\|((f\circ g^{-1})^{(n)}\circ g)\cdot g'\|_1$$

$$+\|f\|_\infty\cdot((h\circ g^{-1})^{(n)}\circ g)\cdot g'\|_1\Big],$$

$$\frac{1}{2(n-1)!(b-a)^{(1+\frac{1}{r})}}\Big[\Big\{\|h\|_{p,[a,b]}\cdot\|((f\circ g^{-1})^{(n)}\circ g)\cdot g'\|_{q,[a,b]}$$

$$+\|f\|_{p,[a,b]}\cdot\|((h\circ g^{-1})^{(n)}\circ g)\cdot g'\|_{q,[a,b]}\Big\}\,\|(g(x)-g(t))^{n-1}$$

$$\cdot K(t,x)\|_{r,[a,b]^2}\Big]\Bigg\},\tag{26.63}$$

where $p,q,r>1$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$.

Proof. Here f,h,g are as in Theorem 26.9.

Therefore by (26.24) we have

$$f(x)=\frac{1}{b-a}\int_a^b f(y)dy$$

$$+\frac{1}{(b-a)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_a^b(f\circ g^{-1})^{(k)}(g(y))\cdot(g(x)-g(y))^k\,dy\right\}$$

$$+\frac{1}{(n-1)!(b-a)}\int_a^b(g(x)-g(t))^{n-1}\cdot(f\circ g^{-1})^{(n)}(g(t))\cdot g'(t)$$

$$\cdot K(t,x)dt,\tag{26.64}$$

$\forall\, x\in[a,b]$.

We also have

$$h(x)=\frac{1}{b-a}\int_a^b h(y)dy$$

$$+\frac{1}{(b-a)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_a^b(h\circ g^{-1})^{(k)}(g(y))\cdot(g(x)-g(y))^k\,dy\right\}$$

$$+\frac{1}{(n-1)!(b-a)}\int_a^b(g(x)-g(t))^{n-1}$$

$$\cdot(h\circ g^{-1})^{(n)}(g(t))\cdot g'(t)\cdot K(t,x)dt,\tag{26.65}$$

$\forall\, x\in[a,b]$.

We further have $\forall\, x \in [a,b])$ that

$$f(x)h(x) = \frac{h(x)}{b-a} \int_a^b f(y)dy$$

$$+\frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_a^b h(x)(f \circ g^{-1})^{(k)}(g(y)) \cdot (g(x)-g(y))^k \, dy \right\}$$

$$+\frac{1}{(n-1)!(b-a)} \int_a^b h(x) \left(g(x)-g(t)\right)^{n-1}$$

$$\cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) \cdot K(t,x)dt, \qquad (26.66)$$

and

$$f(x)h(x) = \frac{f(x)}{b-a} \int_a^b h(y)dy$$

$$+\frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_a^b f(x)(h \circ g^{-1})^{(k)}(g(y)) \cdot (g(x)-g(y))^k \, dy \right\}$$

$$+\frac{1}{(n-1)!(b-a)} \int_a^b f(x) \left(g(x)-g(t)\right)^{n-1}$$

$$\cdot (h \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) \cdot K(t,x)dt. \qquad (26.67)$$

Then we integrate to find

$$\int_a^b f(x)h(x)dx = \frac{1}{b-a} \left(\int_a^b f(x)dx \right) \left(\int_a^b h(x)dx \right)$$

$$+\frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\int_a^b \left(\int_a^b h(x)(f \circ g^{-1})^{(k)}(g(y)) \cdot (g(x)-g(y))^k \, dy \right) dx \right) \right\}$$

$$+\frac{1}{(n-1)!(b-a)} \left(\int_a^b \left(\int_a^b h(x) \left(g(x)-g(t)\right)^{n-1} \right.\right.$$

$$\left.\left. \cdot (f \circ g^{-1})^{(n)}(g(t)) \cdot g'(t) \cdot K(t,x)dt \right) dx \right), \qquad (26.68)$$

and

$$\int_a^b f(x)h(x)dx = \frac{1}{b-a} \left(\int_a^b f(x)dx \left(\int_a^b h(x)dx \right)\right.$$

$$+\frac{1}{(b-a)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\int_a^b \left(\int_a^b f(x)(h \circ g^{-1})^{(k)}(g(y)) \cdot (g(x)-g(y))^k \, dy \right) dx \right) \right\}$$

$$+\frac{1}{(n-1)!(b-a)} \left(\int_a^b \left(\int_a^b f(x) \left(g(x)-g(t)\right)^{n-1} \right.\right.$$

$$\left(h \circ g^{-1}\right)^{(n)} (g(t)) \cdot g'(t) \cdot K(t,x)dt\Big) \, dx\Big) . \tag{26.69}$$

By adding (26.68), (26.69) and dividing by $2(b-a)$, we obtain

$$\Delta_n(f,h) := \frac{1}{(b-a)} \int_a^b f(x)h(x)dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x)dx \right) \left(\int_a^b h(x)dx \right)$$

$$-\frac{1}{2(b-a)^2} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \left(\int_a^b \left(\int_a^b \left[h(x) \cdot (f \circ g^{-1})^{(k)}(g(y)) \right. \right. \right. \right.$$

$$\left. \left. \left. \left. + f(x) \cdot \left(h \circ g^{-1}\right)^{(k)} (g(y)) \right] \cdot (g(x)-g(y))^k \, dy \right) dx \right) \right\}$$

$$= \frac{1}{2(n-1)!(b-a)^2} \left(\int_a^b \left(\int_a^b \left[h(x) \cdot \left(f \circ g^{-1}\right)^{(n)} (g(t)) \right. \right. \right.$$

$$\left. \left. \left. + f(x) \cdot (h \circ g^{-1})^{(n)}(g(t)) \right] \cdot (g(x)-g(t))^{n-1} \cdot g'(t) \cdot K(t,x)dt \right) dx \right) . \tag{26.70}$$

Therefore we obtain the estimates

i)

$$|\Delta_n(f,h)| \leq \frac{1}{2(n-1)!(b-a)^2} \left(\int_a^b \left(\int_a^b \left[|h(x)| \cdot \| \left(\left(f \circ g^{-1}\right)^{(n)} \circ g \right) \cdot g' \|_\infty \right. \right. \right.$$

$$\left. \left. \left. + |f(x)| \cdot \left\| \left((h \circ g^{-1})^{(n)} \circ g \right) \cdot g' \right\|_\infty \right] \cdot |g(x)-g(t)|^{n-1} \, |K(t,x)|dt \right) dx \right) , \tag{26.71}$$

also we have

ii)

$$|\Delta_n(f,h)| \leq \frac{\|g(x)-g(t)\|^{n-1}_{\infty,(x,t)\in[a,b]^2}}{2(n-1)!} \left[\|h\|_\infty \cdot \left\| ((f \circ g^{-1})^{(n)} \circ g) \cdot g' \right\|_1 \right.$$

$$\left. + \, \|f\|_\infty \cdot \left\| ((h \circ g^{-1})^{(n)} \circ g) \cdot g' \right\|_1 \right] , \tag{26.72}$$

finally, by the generalized Hölder inequality, we obtain that

iii)

$$|\Delta_n(f,h)| \leq \frac{1}{2(n-1)!(b-a)^2} \left[\left\{ \left(\int_a^b \int_a^b |h(x)|^p dt \, dx \right)^{1/p} \right. \right.$$

$$\left(\int_a^b \int_a^b | \left(f \circ g^{-1}\right)^{(n)} (g(t)) \cdot g'(t)|^q dt \, dx \right)^{1/q} \tag{26.73}$$

$$+\left(\int_a^b\int_a^b|f(x)|^p dt\,dx\right)^{1/p}\left(\int_a^b\int_a^b|\left(h\circ g^{-1}\right)^{(n)}(g(t))\cdot g'(t)|^q dt\,dx\right)^{1/q}\Bigg\}$$

$$\left\|(g(x)-g(t))^{n-1}K(t,x)\right\|_{r,[a,b]^2}\Big]$$

$$=\frac{1}{2(n-1)!(b-a)^2}\Big[\Big\{(b-a)^{\frac{1}{p}}\cdot\|h\|_{p,[a,b]}\cdot(b-a)^{\frac{1}{q}}\|\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\cdot g'\|_{q,[a,b]}$$

$$+(b-a)^{\frac{1}{p}}\cdot\|f\|_{p,[a,b]}\cdot(b-a)^{\frac{1}{q}}\|((h\circ g^{-1})^{(n)}\circ g)\cdot g'\|_{q,[a,b]}\Big\} \tag{26.74}$$

$$\|(g(x)-g(t))^{n-1}\cdot K(t,x)\|_{r,[a,b]^2}\Big]$$

$$=\frac{(b-a)^{1-\frac{1}{r}}}{2(n-1)!(b-a)^2}\Big[\Big\{\|h\|_{p,[a,b]}\cdot\|\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\cdot g'\|_{q,[a,b]}+$$

$$\|f\|_{p,[a,b]}\cdot\|((h\circ g^{-1})^{(n)}\circ g)\cdot g'\|_{q,[a,b]}\Big\}\ \|(g(x)-g(t))^{n-1}\cdot K(t,x)\|_{r,[a,b]^2}\Big]. \tag{26.75}$$

That is we derive

$$|\Delta_n(f,h)|$$

$$\leq\frac{1}{2(n-1)!(b-a)^{(1+\frac{1}{r})}}\Big[\Big\{\|h\|_{p,[a,b]}\cdot\|\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\cdot g'\|_{q,[a,b]}$$

$$+\|f\|_{p,[a,b]}\cdot\|((h\circ g^{-1})^{(n)}\circ g)\cdot g'\|_{q,[a,b]}\Big\}\ \|(g(x)-g(t))^{n-1}\cdot K(t,x)\|_{r,[a,b]^2}\Big]. \tag{26.76}$$

The proof of the theorem now is completed. ■

Finally we apply last Theorem 26.30 to derive specific Grüss type inequalities.

Theorem 26.31. *Let $f,h\in C^n([a,b]), n\in\mathbb{N}$.*

Then

$$\Bigg|\frac{1}{(b-a)}\int_a^b f(x)h(x)dx-\frac{1}{(b-a)^2}\left(\int_a^b f(x)dx\right)\left(\int_a^b h(x)dx\right)$$

$$-\frac{1}{2(b-a)^2}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left(\int_a^b\left(\int_a^b\Big[h(x)(f\circ ln)^{(k)}(e^y)\right.\right.\right.$$

$$\left.\left.\left.+f(x)\left(h\circ ln\right)^{(k)}(e^y)\Big]\left(e^x-e^y\right)^k dy\right)dx\right)\right\}\Bigg|$$

$$\leq\min\Bigg\{\frac{1}{2(n-1)!(b-a)^2}\left(\int_a^b\left(\int_a^b\Big[|h(x)|\left\|\left((f\circ ln)^{(n)}\circ e^{(\cdot)}\right)\right\|_\infty\right.\right.$$

$$+|f(x)|\|((h\circ ln)^{(n)}\circ e^{(\cdot)})\cdot e^{(\cdot)}\|_\infty\Big]|e^x-e^t|^{n-1}\ |K(t,x)|dt)\,dx),$$

$$\frac{(e^b-e^a)^{n-1}}{2(n-1)!}\Big[\|h\|_\infty\|((f\circ ln)^{(n)}\circ e^{(\cdot)})\cdot e^{(\cdot)}\|_1$$

$$+\|f\|_\infty\|((h\circ ln)^{(n)}\circ e^{(\cdot)})\cdot e^{(\cdot)}\|_1\Big],$$

$$\frac{1}{2(n-1)!(b-a)^{(1+\frac{1}{r})}}\Big[\Big\{\|h\|_{p,[a,b]}\|((f\circ ln)^{(n)}\circ e^{(\cdot)})\cdot e^{(\cdot)}\|_{q,[a,b]}$$

$$+\|f\|_{p,[a,b]}\|((h\circ ln)^{(n)}\circ e^{(\cdot)})\cdot g^{(\cdot)}\|_{q,[a,b]}\Big\}\ \|(e^x-e^t)^{n-1}K(t,x)\|_{r,[a,b]^2}\Big]\Big\},\tag{26.77}$$

where $p,q,r>1$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$.

Theorem 26.32. *Let* $f,h\in C^n([-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon]), n\in\mathbb{N},\varepsilon>0$ *small. Then*

$$\Bigg|\frac{1}{(\pi-2\varepsilon)}\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}f(x)h(x)dx$$

$$-\frac{1}{(\pi-2\varepsilon)^2}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}f(x)dx\right)\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}h(x)dx\right)$$

$$-\frac{1}{2(\pi-2\varepsilon)^2}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\Big[h(x)\left(fo\sin^{-1}\right)^{(k)}(\sin\,y)\right.\right.\right.$$

$$\left.\left.\left.+f(x)\left(h\circ\sin^{-1}\right)^{(k)}(\sin\,y)\Big](\sin\,x-\sin\,y)^k\,dy\right)dx\right)\right\}\Bigg|$$

$$\leq\min\left\{\frac{1}{2(n-1)!(\pi-2\varepsilon)^2}\right.$$

$$\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\Big[|h(x)|\,\left\|\left(\left(f\circ\sin^{-1}\right)^{(n)}\circ\sin(\cdot)\right)\cos(\cdot)\right\|_\infty\right.\right.$$

$$\left.\left.+|f(x)|\left\|\left(\left(h\circ\sin^{-1}\right)^{(n)}\circ\sin(\cdot)\right)\cos(\cdot)\right\|_\infty\Big]\,|\sin\,x-\sin\,t|^{n-1}\,|K(t,x)|dt\right)dx\right),$$

$$\frac{\left(\sin\left(\frac{\pi}{2}-\varepsilon\right)-\sin\left(-\frac{\pi}{2}+\varepsilon\right)\right)^{n-1}}{2(n-1)!}\Big[\|h\|_\infty\left\|((f\circ\sin^{-1})^{(n)}\circ\sin)\cos(\cdot)\right\|_1$$

$$+\|f\|_\infty\left\|((h\circ\sin^{-1})^{(n)}\circ\sin)\cos(\cdot)\right\|_1\Big],$$

$$\frac{1}{2(n-1)!(\pi-2\varepsilon)^{(1+\frac{1}{r})}}\Big[\Big\{\|f\|_{p,[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon]}$$

$$\left\|\left(\left(f\circ\sin^{-1}\right)^{(n)}\circ\sin(\cdot)\right)\cos(\cdot)\right\|_{q,[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon]}$$

$$+\|f\|_{p,[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon]}\left\|\left(\left(h\circ\sin^{-1}\right)^{(n)}\circ\sin(\cdot)\right)\cos(\cdot)\right\|_{q,[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon]}\Big\}$$

$$\left\|(\sin\, x - \sin\, t)^{n-1} K(t,x)\right\|_{r,\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]^2}\Big]\Big\} \tag{26.78}$$

where $p, q, r > 1$, such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Theorem 26.33. *Let* $f, h \in C^n([\varepsilon, \pi - \varepsilon]), n \in \mathbb{N}, \varepsilon > 0$ *small. Then*

$$\Bigg|\frac{1}{\pi - 2\varepsilon}\int_{\varepsilon}^{\pi-\varepsilon} f(x)h(x)dx - \frac{1}{(\pi-2\varepsilon)^2}\left(\int_{\varepsilon}^{\pi-\varepsilon} f(x)dx\right)\left(\int_{\varepsilon}^{\pi-\varepsilon} h(x)dx\right)$$

$$-\frac{1}{2(\pi-2\varepsilon)^2}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left(\int_{\varepsilon}^{\pi-\varepsilon}\left(\int_{\varepsilon}^{\pi-\varepsilon}\Big[h(x)\left(f\circ\cos^{-1}\right)^{(k)}(\cos\, y)\right.\right.\right.$$

$$\left.\left.\left.+f(x)\left(h\circ\cos^{-1}\right)^{(k)}(\cos\, y)\Big]\,(\cos\, x - \cos\, y)^k\, dy\right)dx\right)\right\}\Bigg|$$

$$\leq \min\left\{\frac{1}{2(n-1)!(\pi-2\varepsilon)^2}\right.$$

$$\left(\int_{\varepsilon}^{\pi-\varepsilon}\left(\int_{\varepsilon}^{\pi-\varepsilon}\Big[|h(x)|\,\left\|\left(\left(f\circ\cos^{-1}\right)^{(n)}\circ\cos\right)\cdot\sin\right\|_{\infty}\right.\right.$$

$$\left.\left.+|f(x)|\left\|\left(\left(h\circ\cos^{-1}\right)^{(n)}\circ\cos\right)\sin\right\|_{\infty}\Big]\,|\cos\, x - \cos\, t|^{n-1}\,|K(t,x)|dt\right)dx\right),$$

$$\frac{(\cos\varepsilon - \cos(\pi-\varepsilon))^{n-1}}{2(n-1)!}\Big[\|h\|_{\infty}\left\|((f\circ\cos^{-1})^{(n)}\circ\cos)\sin\right\|_1$$

$$+\|f\|_{\infty}\left\|\left(\left(h\circ\cos^{-1}\right)^{(n)}\circ\cos\right)\sin\right\|_1\Big],$$

$$\frac{1}{2(n-1)!(\pi-2\varepsilon)^{(1+\frac{1}{r})}}\left[\left\{\|h\|_{p,[\varepsilon,\pi-\varepsilon]}\left\|\left(\left(f\circ\cos^{-1}\right)^{(n)}\circ\cos\right)\sin\right\|_{q,[\varepsilon,\pi-\varepsilon]}\right.\right.$$

$$\left.+\|f\|_{p[\varepsilon,\pi-\varepsilon]}\|((h\circ\cos^{-1})^{(n)}\circ\cos)\sin\|_{q,[\varepsilon,\pi-\varepsilon]}\right\}$$

$$\left.\left.\left\|(\cos\, x - \cos\, t)^{n-1}\, K(t,x)\right\|_{r,[\varepsilon,\pi-\varepsilon]^2}\right]\right\}, \tag{26.79}$$

where $p, q, r > 1$, *such that* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Theorem 26.34. *Let* $f, h \in C^n\left(\left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]\right), \varepsilon > 0$ *small,* $n \in \mathbb{N}$. *Then*

$$\Bigg|\frac{1}{(\pi-2\varepsilon)}\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} f(x)h(x)dx$$

$$-\frac{1}{(\pi-2\varepsilon)^2}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} f(x)dx\right)\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} h(x)dx\right)$$

$$-\frac{1}{2(\pi-2\varepsilon)^2}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\Big[h(x)\left(f\circ\tan^{-1}\right)^{(k)}(\tan\, y)\right.\right.\right.$$

$$+f(x)\left(h\circ\tan^{-1}\right)^{(k)}(\tan\, y)\Big]\,(\tan\, x-\tan\, y)^k\,dy\Big)\,dx\Big)\Big\}\Big|$$

$$\leq \min\left\{\frac{1}{2(n-1)!(\pi-2\varepsilon)^2}\right.$$

$$\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\left(\int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon}\Big[|h(x)|\left\|\left(\left(f\circ\tan^{-1}\right)^{(n)}\circ\tan\right)\sec^2\right\|_{\infty}\right.\right.$$

$$\left.\left.+|f(x)|\left\|\left(\left(h\circ\tan^{-1}\right)^{(n)}\circ\tan\right)\sec^2\right\|_{\infty}\Big]\,|\tan\, x-\tan\, t|^{n-1}\,|K(t,x)|dt\right)dx\right),$$

$$\frac{\left(\tan\left(\frac{\pi}{2}-\varepsilon\right)-\tan\left(-\frac{\pi}{2}+\varepsilon\right)\right)^{n-1}}{2(n-1)!}\left[\|h\|_{\infty}\left\|\left(\left(f\circ\tan^{-1}\right)^{(n)}\circ\tan\right)\sec^2\right\|_1\right.$$

$$\left.+\|f\|_{\infty}\left\|((h\circ\tan^{-1})^{(n)}\circ\tan)\sec^2\right\|_1\right],$$

$$\frac{1}{2(n-1)!(\pi-2\varepsilon)^{(1+\frac{1}{r})}}\left[\left\{\|f\|_{p,\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]}\right.\right.$$

$$\left\|\left(\left(f\circ\tan^{-1}\right)^{(n)}\circ\tan\right)\sec^2\right\|_{q,\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]}$$

$$\left.+\|f\|_{p,\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]}\left\|\left(\left(ho\tan^{-1}\right)^{(n)}o\tan\right)\sec^2\right\|_{q,\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]}\right\}$$

$$\left.\left.\left\|(\tan\, x-\tan\, t)^{n-1}K(t,x)\right\|_{r,\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]^2}\right]\right\}, \tag{26.80}$$

where $p,q,r>1$, such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$.

27

Balanced Fractional Opial Integral Inequalities

Here we study L_p, $p > 1$, fractional Opial integral inequalities subject to high order boundary conditions. They engage the right and left Caputo, Riemann-Liouville fractional derivatives. These derivatives are mixed together into the balanced Caputo, Riemann-Liouville, respectively, fractional derivative.

We give applications to a special case. This chapter relies on [41].

27.1 Background

This chapter is motivated by the well known theorem of Z. Opial [237], 1960, which follows

Theorem 27.1. *Let* $x(t) \in C^1([0,h])$ *be such that* $x(0) = x(h) = 0$, *and* $x(t) > 0$ *in* $(0,h)$. *Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{27.1}$$

In (27.1), *the constant* $\frac{h}{4}$ *is the best possible. Inequality* (27.1) *holds as equality for the optimal function*

$$x(t) = \begin{cases} ct, & 0 \leq t \leq h/2, \\ c(h-t) & \frac{h}{2} \leq t \leq h, \end{cases}$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 423–433.
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where $c > 0$ is an arbitrary constant.

To prove easier Theorem 27.1, Beesack [111] proved the following famous Opial type inequality which is used very commonly.

This is another motivation for this chapter.

Theorem 27.2. *Let $x(t)$ be absolutely continuous in $[0,a]$, and $x(0) = 0$. Then*

$$\int_0^a \left|x(t)\,x'(t)\right| dt \le \frac{a}{2} \int_0^a \left(x'(t)\right)^2 dt. \tag{27.2}$$

Inequality (27.2) is sharp, it is attained by $x(t) = ct$, $c > 0$ is an arbitrary constant.

Opial type inequalities are used a lot in proving uniqueness of solutions to differential equations, also to give upper bounds to their solutions.

By themselves have made a great subject of intensive research and there exists a great literature about them.

Typical and great sources on them are the monographs [5], [42].

We need (see also [44], [155], [160], [179], [259])

Definition 27.3. Let $f \in AC^m([a,b])$ (space of functions from $[a,b]$ into $\mathbb{R}$ with $m-1$ derivative absolutely continuous function on $[a,b]$), $m \in \mathbb{N}$, where $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number).

We define the right Caputo fractional derivative of order $\alpha > 0$, by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta. \tag{27.3}$$

We set $D_{b-}^0 f(x) = f(x)$, $\forall x \in [a,b]$.

Note 27.4. Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, with $\alpha > 0$, then $f^{(m-1)} \in AC([a,b])$, which implies that $f^{(m)}$ exists a.e. on $[a,b]$ and that $f^{(m)} \in L_1([a,b])$.

Consequently if $f \in AC^m([a,b])$, then $D_{b-}^{\alpha} f(x)$ exists a.e. on $[a,b]$ and $D_{b-}^{\alpha} f \in L_1([a,b])$, see [44].

Observe that when $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x), \ \forall x \in [a,b]. \tag{27.4}$$

We continue with the right Caputo fractional Taylor formula with integral remainder, see [44].

Theorem 27.5. *Let $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta - x)^{\alpha-1} D_{b-}^{\alpha} f(\zeta)\, d\zeta. \tag{27.5}$$

We need also (see [145], p.38)

Definition 27.6. Let $f \in AC^m([a,b]), m \in \mathbb{N}$, where $m = \lceil \alpha \rceil$, $\alpha > 0$. We define the left Caputo fractional derivative of order $\alpha > 0$, by

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{27.6}$$

$\forall x \in [a,b]$. We set $D_{*a}^0 f(x) = f(x)$, $\forall x \in [a,b]$.

Again here $D_{*a}^{\alpha} f$ exists a.e. on $[a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b])$, see [145], pp.13. When $\alpha = m \in \mathbb{N}$, then

$$D_{*a}^m f(x) = f^{(m)}(x), \ \forall x \in [a,b]. \tag{27.7}$$

We continue with the left Caputo fractional Taylor formula with integral remainder, see [145], p.40.

Theorem 27.7. *Let $f \in AC^m([a,b])$, $m \in \mathbb{N}$, where $m = \lceil \alpha \rceil$, $\alpha > 0$, $x \in [a,b]$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} D_{*a}^{\alpha} f(\tau)\, d\tau. \tag{27.8}$$

Above Γ is the gamma function,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt, \ \alpha > 0.$$

We introduce the following balanced Caputo fractional derivative

Definition 27.8. Let $f \in AC^m([a,b]), m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x \in [a,b]$. We define

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{*a}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{27.9}$$

In this chapter we establish L_p, $p > 1$, Opial type inequalities involving the balanced Caputo fractional derivative subject to high order boundary conditions, more precisely by assuming that

$$f^{(k)}(a) = f^{(k)}(b) = 0, \ k = 0, 1, \ldots, m-1. \tag{27.10}$$

We extend these results to Riemann-Liouville fractional derivatives.

27.2 Results

We present the main result

Theorem 27.9. *Let* $f \in AC^m([a,b])$, $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Suppose*

$$f^{(k)}(a) = f^{(k)}(b) = 0, \;\; k = 0, 1, \ldots, m-1;$$

$$p, \; q > 1 : \;\; \frac{1}{p} + \frac{1}{q} = 1, \text{ and } \alpha > \frac{1}{q}.$$

(i) Case of $1 < q \leq 2$. *Then*

$$\int_a^b |f(\omega)| \, |D^\alpha f(\omega)| \, d\omega \leq$$

$$\frac{2^{-\left(\alpha + \frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}} \cdot \tag{27.11}$$

$$\left(\int_a^b |D^\alpha f(\omega)|^q \, d\omega\right)^{2/q}.$$

(ii) Case of $q > 2$. *Then*

$$\int_a^b |f(\omega)| \, |D^\alpha f(\omega)| \, d\omega \leq$$

$$\frac{2^{-\left(\alpha + \frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}} \cdot \tag{27.12}$$

$$\left(\int_a^b |D^\alpha f(\omega)|^q \, d\omega\right)^{2/q}.$$

(iii) When $p = q = 2$, $\alpha > \frac{1}{2}$, *then*

$$\int_a^b |f(\omega)| \, |D^\alpha f(\omega)| \, d\omega \leq$$

$$\frac{2^{-\left(\alpha + \frac{1}{2}\right)} (b-a)^{\alpha}}{\Gamma(\alpha)\left[\sqrt{2\alpha(2\alpha-1)}\right]} \cdot \tag{27.13}$$

$$\left(\int_a^b (D^\alpha f(\omega))^2 \, d\omega\right).$$

Remark 27.10. Let us say that $\alpha = 1$, then by (27.13) we derive

$$\int_a^b |f(\omega)| \left|f'(\omega)\right| d\omega \leq$$
$$\frac{(b-a)}{4} \left(\int_a^b \left(f'(\omega)\right)^2 d\omega \right), \tag{27.14}$$

that is reproving and recovering Opial's inequality (27.1), see [237], see also Olech's result [236].

Proof of Theorem 27.9. Let $x \in [a,b]$. We have by assumption $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, m-1$ and Theorem 27.7 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} D_{*a}^{\alpha} f(\tau)\, d\tau, \tag{27.15}$$

and by assumption $f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$ and Theorem 27.5 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} D_{b-}^{\alpha} f(\tau)\, d\tau. \tag{27.16}$$

Using Hölder's inequality on (27.15) we obtain

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} \left|D_{*a}^{\alpha} f(\tau)\right| d\tau$$
$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x \left((x-\tau)^{\alpha-1}\right)^p d\tau \right)^{1/p} \left(\int_a^x \left|D_{*a}^{\alpha} f(\tau)\right|^q d\tau \right)^{1/q}$$
$$= \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left(\int_a^x \left|D_{*a}^{\alpha} f(\tau)\right|^q d\tau \right)^{1/q}. \tag{27.17}$$

Put

$$z(x) := \int_a^x \left|D_{*a}^{\alpha} f(\tau)\right|^q d\tau, \quad (z(a) = 0).$$

Then

$$z'(x) = \left|D_{*a}^{\alpha} f(x)\right|^q,$$

and

$$\left|D_{*a}^{\alpha} f(x)\right| = \left(z'(x)\right)^{1/q}, \text{ all } a \leq x \leq b.$$

Therefore by (27.17) we have

$$|f(\omega)| \left|D_{*a}^{\alpha} f(\omega)\right| \leq \frac{1}{\Gamma(\alpha)}$$
$$\frac{(\omega-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left(z(\omega) z'(\omega)\right)^{1/q}, \tag{27.18}$$

all $a \leq \omega \leq x$.

Next working similarly with (27.16) we derive

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} \left|D_{b-}^{\alpha} f(\tau)\right| d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b \left((\tau - x)^{\alpha-1} \right)^p d\tau \right)^{1/p} \left(\int_x^b \left|D_{b-}^{\alpha} f(\tau)\right|^q d\tau \right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left(\int_x^b \left|D_{b-}^{\alpha} f(\tau)\right|^q d\tau \right)^{1/q}. \quad (27.19)$$

Put

$$\lambda(x) := \int_x^b \left|D_{b-}^{\alpha} f(\tau)\right|^q d\tau = -\int_b^x \left|D_{b-}^{\alpha} f(\tau)\right|^q d\tau, \quad (\lambda(b) = 0).$$

Then

$$\lambda'(x) = -\left|D_{b-}^{\alpha} f(x)\right|^q$$

and

$$\left|D_{b-}^{\alpha} f(x)\right| = \left(-\lambda'(x)\right)^{1/q}, \text{ all } a \leq x \leq b.$$

Therefore by (27.19) we have

$$|f(\omega)| \left|D_{b-}^{\alpha} f(\omega)\right| \leq \frac{1}{\Gamma(\alpha)}$$

$$\frac{(b-\omega)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left(-\lambda(\omega)\lambda'(\omega)\right)^{1/q}, \quad (27.20)$$

all $x \leq \omega \leq b$.

Next we integrate (27.18) over $[a, x]$ to get

$$\int_a^x |f(\omega)| \left|D_{*a}^{\alpha} f(\omega)\right| d\omega \leq \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}}$$

$$\int_a^x (\omega - a)^{\frac{p(\alpha-1)+1}{p}} \left(z(\omega) z'(\omega)\right)^{1/q} d\omega \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}}$$

$$\left(\int_a^x (\omega - a)^{p(\alpha-1)+1} d\omega \right)^{1/p} \left(\int_a^x z(\omega) z'(\omega) d\omega \right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}} \frac{(x-a)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{1/p}} \frac{z(x)^{2/q}}{2^{1/q}}$$

$$= \frac{2^{-1/q} (x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2) \right]^{1/p}} \left(\int_a^x |D_{*a}^{\alpha} f(\omega)|^q \, d\omega \right)^{2/q}. \quad (27.21)$$

So we have established

$$\int_a^x |f(\omega)| \left| D_{*a}^{\alpha} f(\omega) \right| d\omega \leq$$

$$\frac{2^{-1/q} (x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2) \right]^{1/p}} \left(\int_a^x |D_{*a}^{\alpha} f(\omega)|^q \, d\omega \right)^{2/q}, \quad (27.22)$$

for all $a \leq x \leq b$.

By (27.22) we derive

$$\int_a^{\frac{a+b}{2}} |f(\omega)| \left| D_{*a}^{\alpha} f(\omega) \right| d\omega \leq$$

$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}} 2^{-\left[\frac{(p(\alpha-1)+2)}{p} + \frac{1}{q} \right]}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2) \right]^{1/p}}$$

$$\left(\int_a^{\frac{a+b}{2}} |D_{*a}^{\alpha} f(\omega)|^q \, d\omega \right)^{2/q}. \quad (27.23)$$

Similarly we integrate (27.20) over $[x, b]$ to get

$$\int_x^b |f(\omega)| \left| D_{b-}^{\alpha} f(\omega) \right| d\omega \leq \frac{1}{\Gamma(\alpha) (p(\alpha-1)+1)^{1/p}}$$

$$\int_x^b (b-\omega)^{\frac{p(\alpha-1)+1}{p}} \left(-\lambda(\omega) \lambda'(\omega) \right)^{1/q} d\omega \leq$$

$$\frac{1}{\Gamma(\alpha) (p(\alpha-1)+1)^{1/p}}$$

$$\left(\int_x^b (b-\omega)^{p(\alpha-1)+1} d\omega \right)^{1/p} \left(\int_x^b -\lambda(\omega) \lambda'(\omega) \, d\omega \right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha) (p(\alpha-1)+1)^{1/p}} \frac{(b-x)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{1/p}} \frac{(\lambda(x))^{2/q}}{2^{1/q}}. \quad (27.24)$$

We have proved that

$$\int_x^b |f(\omega)| \left| D_{b-}^{\alpha} f(\omega) \right| d\omega \leq$$

$$\frac{2^{-1/q}(b-x)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}}\left(\int_x^b\left|D_{b-}^{\alpha}f(\omega)\right|^q d\omega\right)^{2/q}, \tag{27.25}$$

for all $a \leq x \leq b$.

By (27.25) we obtain

$$\int_{\frac{a+b}{2}}^{b}|f(\omega)|\left|D_{b-}^{\alpha}f(\omega)\right|d\omega \leq$$

$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}}\,2^{-\left[\frac{(p(\alpha-1)+2)}{p}+\frac{1}{q}\right]}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}}$$

$$\left(\int_{\frac{a+b}{2}}^{b}\left|D_{b-}^{\alpha}f(\omega)\right|^q d\omega\right)^{2/q}. \tag{27.26}$$

Adding (27.23) and (27.26) we obtain

$$\int_a^b|f(\omega)|\left|D^{\alpha}f(\omega)\right|d\omega \leq$$

$$\frac{2^{-\left(\alpha+\frac{1}{p}\right)}(b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}}$$

$$\left[\left(\int_a^{\frac{a+b}{2}}\left|D_{*a}^{\alpha}f(\omega)\right|^q d\omega\right)^{2/q}+\left(\int_{\frac{a+b}{2}}^{b}\left|D_{b-}^{\alpha}f(\omega)\right|^q d\omega\right)^{2/q}\right] =: (*) \tag{27.27}$$

Suppose $1 < q \leq 2$, then $\frac{2}{q} \geq 1$.

Therefore we get

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)}(b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}}$$

$$\left[\int_a^{\frac{a+b}{2}}\left|D_{*a}^{\alpha}f(\omega)\right|^q d\omega+\int_{\frac{a+b}{2}}^{b}\left|D_{b-}^{\alpha}f(\omega)\right|^q d\omega\right]^{2/q} = \tag{27.28}$$

$$\frac{2^{-\left(\alpha+\frac{1}{p}\right)}(b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}} \tag{27.29}$$

$$\left(\int_a^b\left|D^{\alpha}f(\omega)\right|^q d\omega\right)^{2/q}.$$

So for $1 < q \leq 2$ we have proved (27.11).

Assume now $q > 2$, then $0 < \frac{2}{q} < 1$.
Therefore we derive

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)} 2^{1-\frac{2}{q}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}}$$

$$\left[\int_a^{\frac{a+b}{2}} |D_{*a}^{\alpha} f(\omega)|^q \, d\omega + \int_{\frac{a+b}{2}}^{b} |D_{b-}^{\alpha} f(\omega)|^q \, d\omega\right]^{2/q} =$$

$$\frac{2^{-\left(\alpha+\frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}} \tag{27.30}$$

$$\left(\int_a^b |D^{\alpha} f(\omega)|^q \, d\omega\right)^{2/q}.$$

So when $q > 2$ we have established (27.12).

(iii) The case of $p = q = 2$, *see* (27.13), *is obvious, it derives from* (27.11) *immediately.* ■

We need (see [44], [155], [160], [145], p.22)

Definition 27.11. Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. We define the right Riemann-Liouville fractional derivative by

$$\mathcal{D}_{b-}^{\alpha} f(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_x^b (t-x)^{m-\alpha-1} f(t) \, dt, \tag{27.31}$$

$$\mathcal{D}_{b-}^{0} f(x) := I(x) \ \text{ (the identity operator)}.$$

We also define the left Riemann-Liouville fractional derivative by

$$\mathcal{D}_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_a^x (x-t)^{m-\alpha-1} f(t) \, dt, \tag{27.32}$$

$$\mathcal{D}_{a+}^{0} f(x) := I(x).$$

We further define the new balanced Riemann-Liouville fractional derivative

$$\mathcal{D}^{\alpha} f(x) := \begin{cases} \mathcal{D}_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ \mathcal{D}_{a+}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{27.33}$$

Remark 27.12. Let now $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. In [43] we have proved that $D_{b-}^{\alpha} f(x)$, $D_{*a}^{\alpha} f(x)$ are continuous functions in $x \in [a,b]$. Of course $C^m([a,b]) \subset AC^m([a,b])$, so that $f \in AC^m([a,b])$.

Thus by Theorem 9 of [44], we obtain that also $\mathcal{D}_{b-}^{\alpha} f(x)$ exists and continuous for every $x \in [a,b]$. Furthermore if $f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$ we get

$$\mathcal{D}_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} f(x), \tag{27.34}$$

$\forall x \in [a,b]$.

Similarly, by [145], p.39, we obtain that $\mathcal{D}_{a+}^{\alpha} f(x)$ exists and continuous in $x \in [a,b]$. Furthermore if $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, m-1$ we have

$$\mathcal{D}_{a+}^{\alpha} f(x) = D_{*a}^{\alpha} f(x), \tag{27.35}$$

$\forall x \in [a,b]$.

So if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$ we get that

$$D^{\alpha} f(x) = \mathcal{D}^{\alpha} f(x), \tag{27.36}$$

$\forall x \in [a,b]$.

So by Theorem 27.9 we obtain the corresponding results for the balanced Riemann-Liouville fractional derivative

Theorem 27.13. *Let $f \in C^m([a,b])$, $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Suppose $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\alpha > \frac{1}{q}$.*

(i) Case of $1 < q \leq 2$. Then

$$\int_a^b |f(\omega)| |\mathcal{D}^{\alpha} f(\omega)| \, d\omega \leq$$

$$\frac{2^{-\left(\alpha + \frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}} \cdot$$

$$\left(\int_a^b |\mathcal{D}^{\alpha} f(\omega)|^q \, d\omega \right)^{2/q}. \tag{27.37}$$

(ii) Case of $q > 2$. Then

$$\int_a^b |f(\omega)| |\mathcal{D}^{\alpha} f(\omega)| \, d\omega \leq$$

$$\frac{2^{-\left(\alpha + \frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{1/p}} \cdot$$

$$\left(\int_a^b |\mathcal{D}^{\alpha} f(\omega)|^q \, d\omega \right)^{2/q}. \tag{27.38}$$

(iii) When $p = q = 2, \alpha > \frac{1}{2}$, then

$$\int_a^b |f(\omega)| |\mathcal{D}^{\alpha} f(\omega)| \, d\omega \leq$$

$$\frac{2^{-\left(\alpha+\frac{1}{2}\right)}(b-a)^{\alpha}}{\Gamma(\alpha)\left[\sqrt{2\alpha(2\alpha-1)}\right]}\left(\int_a^b(\mathcal{D}^{\alpha}f(\omega))^2\,d\omega\right). \tag{27.39}$$

Conclusion 27.14. According to the monographs [5], [42], the presented method of involving balanced fractional derivatives into Opial type inequalities, subject to boundary conditions, could be expanded to all possible directions, by producing interesting results and applications. Especially all these results proved here, and similar that can be proved, are expected to have wide applications to fractional differential equations.

28

Montgomery Identities for Fractional Integrals and Fractional Inequalities

In this chapter we develop some integral identities and inequalities for the fractional integral. We obtain Montgomery identities for fractional integrals and a generalization for double fractional integrals. We also give Ostrowski and Grüss inequalities for fractional integrals. This chapter is based on [80].

28.1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [230]:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P_1(x,t) f'(t)\, dt, \tag{28.1}$$

where $P_1(x, t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \le t \le x, \\ \dfrac{t-b}{b-a}, & x < t \le b, \end{cases} \tag{28.2}$$

Assume now that $w : [a, b] \to [0, \infty)$ is some probability density function, i.e. is a positive integrable function satisfying $\int_a^b w(t)\ dt = 1$, and $W(t) = \int_a^x w(x)\ dx$ for $t \in [a, b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. The

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following identity (given by Pečarić in [241]) is the weighted generalization of the Montgomery identity:

$$f(x) = \int_a^b w(t)f(t)\,dt + \int_a^b P_w(x,t)f'(t)\,dt, \tag{28.3}$$

where the weighted Peano kernel is

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases}$$

In [107], [148], the authors obtain two identities which generalized (28.1) for functions of two variables. In fact, for function $f : [a,b] \times [c,d] \to \mathbb{R}$ such that the partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ all exist and are continuous on $[a,b] \times [c,d]$, so for all $(x,y) \in [a,b] \times [c,d]$ we have:

$$\begin{aligned}(d-c)(b-a)f(x,y) &= \int_c^d \int_a^b f(s,t)\,ds\,dt + \int_c^d \int_a^b \frac{\partial f(s,t)}{\partial s} p(x,s)\,ds\,dt \\ &+ \int_a^b \int_c^d \frac{\partial f(s,t)}{\partial t} q(y,t)\,dt\,ds \\ &+ \int_c^d \int_a^b \frac{\partial^2 f(s,t)}{\partial s \partial t} p(x,s)q(y,t)\,ds\,dt, \end{aligned} \tag{28.4}$$

where

$$p(x,s) = \begin{cases} s-a, & a \le s \le x, \\ s-b, & x < s \le b, \end{cases} \quad \textit{and} \quad q(y,t) = \begin{cases} t-c, & c \le t \le y, \\ t-d, & y < t \le d. \end{cases} \tag{28.5}$$

28.2 Fractional Calculus

We give some necessary definitions and mathematical preliminaries of the fractional calculus theory which are used further in this chapter.

Definition 28.1. The Riemann-Liouville integral operator of order $\alpha > 0$ with $a \ge 0$ is defined as

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \\ J_a^0 f(x) &= f(x). \end{aligned} \tag{28.6}$$

Properties of the operator can be found in [228]. In case of $\alpha = 1$, the fractional integral reduces to the classical integral.

28.3 Montgomery Identities for Fractional Integrals

Montgomery identities can be generalized in the fractional integrals forms. The main results of generalization are given in the following lemmas.

Lemma 28.2. Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, then the following Montgomery identity for fractional integrals holds:

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^{\alpha}(P_2(x,b)f'(b)), \quad \alpha \geq 1, \tag{28.7}$$

where $P_2(x,t)$ is the fractional Peano kernel is defined:

$$P_2(x,t) = \begin{cases} \dfrac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t \leq x, \\ \dfrac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x < t \leq b. \end{cases} \tag{28.8}$$

Proof. In order to prove Montgomery identity for fractional integrals in relation (28.7), by using the properties of fractional integrations and relation (28.8), we have

$$\begin{aligned} \Gamma(\alpha) J_a^{\alpha}(P_1(x,b)f'(b)) &= \int_a^b (b-t)^{\alpha-1} P_1(x,t) f'(t)\, dt \\ &= \int_a^x \frac{t-a}{b-a}(b-t)^{\alpha-1} f'(t)\, dt \\ &\quad + \int_x^b \frac{t-b}{b-a}(b-t)^{\alpha-1} f'(t)\, dt \\ &= \int_a^x (b-t)^{\alpha-1} f'(t)\, dt \\ &\quad - \frac{1}{b-a}\int_a^b (b-t)^{\alpha} f'(t)\, dt. \end{aligned} \tag{28.9}$$

Next, by integration by parts and using (28.9), we get

$$\begin{aligned} &\Gamma(\alpha)\, J_a^{\alpha}\,(P_1(x,b)f'(b)) \\ &= (b-x)^{\alpha-1} f(x) - \frac{\alpha}{b-a}\Gamma(\alpha) J_a^{\alpha} f(b) \\ &\quad + (\alpha-1)\int_a^x (b-t)^{\alpha-2} f(t)\, dt \\ &= (b-x)^{\alpha-1} f(x) - \frac{1}{b-a}\Gamma(\alpha) J_a^{\alpha} f(b) \\ &\quad + \Gamma(\alpha) J_a^{\alpha-1}(P_1(x,b)f(b)), \end{aligned} \tag{28.10}$$

finally, from (28.10) for $\alpha \geq 1$, we obtain

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^{\alpha}(P_2(x,b)f'(b)),$$

and the proof is completed. ∎

Remark 28.3. Let $\alpha = 1$ then formula (28.7) reduces to the classic Montgomery identity (28.1).

Lemma 28.4. Let $w : [a,b] \to [0,\infty)$ be a probability density function, i.e. $\int_a^b w(t)\, dt = 1$, and set $W(t) = \int_a^t w(x)\, dx$ for $a \leq t \leq b$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$, $\alpha \geq 1$, then the generalization of the weighted Montgomery identity for fractional integrals is in the following form:

$$f(x) = (b-x)^{1-\alpha}\Gamma(\alpha) J_a^{\alpha}(w(b)f(b)) - J_a^{\alpha-1}(Q_w(x,b)f(b)) + J_a^{\alpha}(Q_w(x,b)f'(b)). \tag{28.11}$$

Where the weighted fractional Peano kernel is

$$Q_w(x,t) = \begin{cases} (b-x)^{1-\alpha}\Gamma(\alpha)W(t), & a \leq t \leq x, \\ (b-x)^{1-\alpha}\Gamma(\alpha)(W(t)-1), & x < t \leq b. \end{cases} \tag{28.12}$$

Proof. From the fractional calculus and relation (28.12), we have

$$\begin{aligned} J_a^{\alpha}(Q_w(x,b)f'(b)) &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} Q_w(x,t) f'(t)\, dt \\ &= (b-x)^{1-\alpha}\Big(\int_a^b (b-t)^{\alpha-1} W(t) f'(t)\, dt \\ &\quad - \int_x^b (b-t)^{\alpha-1} f'(t)\, dt\Big). \end{aligned} \tag{28.13}$$

Using integration by parts in (28.13) and $W(a) = 0, W(b) = 1$, we get

$$\begin{aligned} \int_a^b (b-t)^{\alpha-1} W(t) f'(t)\, dt = & -\Gamma(\alpha) J_a^{\alpha}(w(b)f(b)) \\ & +(\alpha-1)\int_a^b (b-t)^{\alpha-2} W(t) f(t)\, dt, \end{aligned} \tag{28.14}$$

and

$$\int_x^b (b-t)^{\alpha-1} f'(t)\, dt = -(b-x)^{\alpha-1} f(x) + (\alpha-1)\int_x^b (b-t)^{\alpha-2} f(t)\, dt. \tag{28.15}$$

We apply (28.14) and (28.15) in to (28.13), to obtain

$$
\begin{aligned}
J_a^{\alpha}(Q_w(x,b)f'(b)) &= (b-x)^{1-\alpha}\Big[-\Gamma(\alpha)J_a^{\alpha}(w(b)f(b)) \\
&\quad -(\alpha-1)\int_x^b (b-t)^{\alpha-2}f(t)\,dt + (b-x)^{\alpha-1}f(x) \\
&\quad +(\alpha-1)\int_a^b (b-t)^{\alpha-2}W(t)f(t)\,dt\Big] \\
&= f(x) - \Gamma(\alpha)(b-x)^{1-\alpha}J_a^{\alpha}(w(b)f(b)) + (b-x)^{1-\alpha}(\alpha-1) \\
&\quad \times\Big[\int_a^x (b-t)^{\alpha-2}W(t)f(t)\,dt \\
&\quad + \int_x^b (b-t)^{\alpha-2}(W(t)-1)f(t)\,dt\Big] \\
&= f(x) - \Gamma(\alpha)(b-x)^{1-\alpha}J_a^{\alpha}(w(b)f(b)) \\
&\quad +J_a^{\alpha-1}(Q_w(x,b)f(b)).
\end{aligned} \tag{28.16}
$$

Finally, we derive that

$$
f(x) = (b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha}(w(b)f(b)) - J_a^{\alpha-1}(Q_w(x,b)f(b)) + J_a^{\alpha}(Q_w(x,b)f'(b)), \tag{28.17}
$$

proving the claim. ■

Remark 28.5. Let $\alpha = 1$ then the weighted generalization of the Montgomery identity for fractional integrals in (28.11) reduces to the weighted generalization of the Montgomery identity for integrals in (28.3).

Lemma 28.6. Let function $f : [a,b]\times[c,d] \to \mathbb{R}$ have continuous partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s\partial t}$ on $[a,b]\times[c,d]$, for all $(x,y)\in[a,b]\times[c,d]$ and $\alpha,\beta \geq 2$, then the following two variables Montgomery identity for fractional integrals holds:

$$
\begin{aligned}
(d-c)\,(b-a)\,f(x,y) &= (b-x)^{1-\alpha}(d-y)^{1-\beta}\Gamma(\alpha)\Gamma(\beta)\Big[J_{a,c}^{\alpha,\beta}\Big(q(y,d)\,\frac{\partial}{\partial t}f(b,d)\Big) \\
&\quad +J_{c,a}^{\beta,\alpha}\left(f(b,d) + p(x,b)\,\frac{\partial f(b,d)}{\partial s} + p(x,b)\,q(y,d)\,\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right) \\
&\quad -J_{c,a}^{\beta,\alpha-1}\left(p(x,b)\,f(b,d) + p(x,b)\,q(y,d)\,\frac{\partial f(b,d)}{\partial t}\right) \\
&\quad -J_{c,a}^{\beta-1,\alpha}\left(q(y,d)\,f(b,d) + p(x,b)\,q(y,d)\,\frac{\partial f(b,d)}{\partial s}\right) \\
&\quad +J_{c,a}^{\beta-1,\alpha-1}\Big(p(x,b)\,q(y,d)\,f(b,d)\Big)\Big],
\end{aligned}
$$

where

$$
J_{c,a}^{\beta;\alpha}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_c^y\int_a^x (x-s)^{\alpha-1}(y-t)^{\beta-1}f(s,t)\,ds\,dt,
$$

also, $p(x,s)$ and $q(y,t)$ are defined by (28.5).

Proof. Plug into (28.4), instead of f the function $g(x,y) = f(x,y)(b-x)^{\alpha-1}(d-y)^{\beta-1}$. ■

28.4 An Ostrowski Type Fractional Inequality

In 1938, Ostrowski proved the following interesting integral inequality [238]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{1}{(b-a)^2}\left(x - \frac{a+b}{2}\right)^2 \right](b-a)M, \qquad (28.18)$$

where $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \le M$, for every $x \in [a,b]$. Now we extend it to fractional integrals.

Theorem 28.7. Let $f : [a,b] \to \mathbb{R}$ be a differentiable on $[a,b]$ and $|f'(x)| \le M$, for every $x \in [a,b]$ and $\alpha \ge 1$. Then the following Ostrowski fractional inequality holds:

$$\left| f(x) - \frac{\Gamma(\alpha)}{b-a} \quad (b-x)^{1-\alpha} J_a^\alpha f(b) + J_a^{\alpha-1} P_2(x,b) f(b) \right| \qquad (28.19)$$
$$\le \frac{M}{\alpha(\alpha+1)}\left[(b-x)\left(2\alpha\left(\frac{b-x}{b-a}\right) - \alpha - 1 \right) + (b-a)^\alpha (b-x)^{1-\alpha} \right].$$

Proof. From Lemma 28.2 we have

$$\left| f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^\alpha f(b) + J_a^{\alpha-1}(P_2(x,b) f(b)) \right| = \left| J_a^\alpha (P_2(x,b) f'(b)) \right|. \qquad (28.20)$$

Therefore, from (28.20) and (28.6) and $|f'(x)| \le M$, we obtain

$$\frac{1}{\Gamma(\alpha)} \left| \int_a^b (b-t)^{\alpha-1} P_2(x,t) f'(t)\,dt \right| \le \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \left|P_2(x,t)\right| \left|f'(t)\right| dt$$
$$\le \frac{M}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \left|P_2(x,t)\right| dt$$
$$\le M \frac{(b-x)^{1-\alpha}}{b-a} \left(\int_a^x (b-t)^{\alpha-1}(t-a)\,dt + \int_x^b (b-t)^\alpha\,dt \right)$$
$$= \frac{M}{\alpha(\alpha+1)}\left[(b-x)\left(2\alpha\left(\frac{b-x}{b-a}\right) - \alpha - 1 \right) + (b-a)^\alpha (b-x)^{1-\alpha} \right].$$

The last proves inequality (28.19). ■

28.5 A Grüss Type Fractional Inequality

In 1935, Grüss proved one of the most celebrated integral inequalities [180], which can be stated as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(M-m)(N-n), \tag{28.21}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the conditions

$$m \leq f(x) \leq M, \qquad n \leq g(x) \leq N,$$

for all $x \in [a,b]$, where m, M, n, N are given real constants.

We give

Proposition 28.8. Provided that $f(x)$ and $g(x)$ are two integrable functions for all $x \in [a,b]$, and satisfy the conditions

$$m \leq (b-x)^{\alpha-1} f(x) \leq M, \qquad n \leq (b-x)^{\alpha-1} g(x) \leq N,$$

where $\alpha > 1/2$, and m, M, n, N are real constants. Then the following Grüss fractional inequality holds

$$\left|\frac{\Gamma(2\alpha-1)}{(b-a)\Gamma^2(\alpha)} J_a^{2\alpha-1}(fg)(b) - \frac{1}{(b-a)^2} J_a^{\alpha} f(b) J_a^{\alpha} g(b)\right| \leq \frac{1}{4\,\Gamma^2(\alpha)}(M-m)(N-n). \tag{28.22}$$

Proof. If replace $h(x) = (b-x)^{\alpha-1} f(x)$ and $k(x) = (b-x)^{\alpha-1} g(x)$ in (28.21), we will get (28.22). ∎

In [42] are contained many related fractional inequalities.

29

Representations for (C_0) $m-$Parameter Operator Semigroups

In this chapter some general representation formulae for (C_0) m-parameter operator semigroups with rates of convergence are given by the probabilistic approach and multiplier enlargement method. These cover all known representation formulae for (C_0) one-and m-parameter operator semigroups as special cases. When we consider special semigroups well-known convergence theorems for multivariate approximation operators are regained. This chapter is based on [92].

29.1 History

Recently the study of representation formulae for (C_0) operator semigroups has attracted much attention (Shaw [260], [261], Butzer-Hahn [123], Pfeifer [243]-[245] and Chen-Zhou [130]). They gave some general formulae that include earlier (Post Widder, Hille-Phillips [188] and Chung [135]) concrete representation formulae. But most of the work done so far is confined to one-parameter case, while Shaw's method for multi-parameter case is not an easy one to get new formulae and the results are without rates of convergence. In this chapter we try to give some general representation formulae for (C_0) m-parameter operator semigroups. The main idea is the use of probabilistic setting in the representation of operator semigroups, initiated by Chung [135] and developed by Butzer-Hahn [123] and Pfeifer [243], and so-called multiplier enlargement method by Hsu-Wang [190], [278] and

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 443–467.
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Shaw [260], [261]. At the same time by introducing a modified second modulus of continuity of operator semigroup and a Steklov-type element we prove quantitative estimates of the obtained formulae.

All existent representation formulae of (C_0) one-and multi-parameter operator semigroups are special cases of our results. In particular Shaw's formulae [260], [261] for m-parameter operator semigroups are special cases of our results when specifying the random vectors considered. Also with our method it is easier to obtain new formulae.

We finish with examples to show the application of the results in multivariate operator approximation theory when we consider particular operator semigroups.

29.2 Background

Let $\mathcal{X}$ be a Banach space with elements $f,\ g, ...,$ having norm $\|f\|,\ \|g\|, ...,$ and $\mathcal{E}(\mathcal{X})$ be the Banach algebra of endomorphism of $\mathcal{X}$. If $T \in \mathcal{E}(\mathcal{X})$, $\|T\|$ also denotes the norm of T. Let $\mathcal{R}^m$ be the m-dimensional Euclidean space supplied with the usual definition of arithmetical operations and metric. We write $t = (t_1, ..., t_m) \in \mathcal{R}^m$, $\overline{t} = t_1 + ... + t_m$, $\left|\overline{t}\right| = |t_1| + ... + |t_m|$ and denote the unit vectors by $e_1, ...e_m$, where $e_k = (0, ...1, ..., 0)$ with 1 in the k-th place and 0 elsewhere. Further, let

$$\mathcal{R}_+^m = \{t \in \mathcal{R}^m;\ t_k \geq 0,\ k = 1, ..., m\},$$

the first closed 2^m-ant in $\mathcal{R}^m$. $\mathcal{Z}_+$ denotes the set of all non-negative integers and

$$\mathcal{Z}_+^m = \{n = (n_1, ..., n_m);\ n_k \in \mathcal{Z}_+,\ k = 1, ..., m\},$$

while $\mathcal{N}$ is the set of all positive integers.

A family of bounded linear operators $\{T(t); t \in \mathcal{R}_+^m\}$ on $\mathcal{X}$ is called a (C_0) m-parameter operator semigroup in $\mathcal{E}(\mathcal{X})$ when the following three conditions are satisfied:

$$i)\ T(t+s) = T(t)T(s), \quad t, s \in \mathcal{R}_+^m; \tag{29.1}$$

$$ii)\ T(0) = I \quad \text{(identity operator)}; \tag{29.2}$$

$$iii)\ s - \lim_{t \in \mathcal{R}_+^m, t \to 0} T(t)f = f, \quad f \in \mathcal{X}. \tag{29.3}$$

It is known that $\{T(t); t \in \mathcal{R}_+^m\}$ is the direct product of $m(C_0)$ one-parameter operator semigroups in $\mathcal{E}(\mathcal{X})$:

$$T(t) = \prod\nolimits_{k=1}^{m} T_k(t_k), \tag{29.4}$$

where $T_k(t_k) = T(t_k e_k)$. The operators $\{T_k(t_k); 0 \leq t_k < \infty\}$ $(k = 1, ..., m)$ commute with each other.

Let A_k be the infinitesimal generator of $\{T_k(t_k); 0 \leq t_k < \infty\}$ with domain $D(A_k)$, $k = 1, ..., m$. Then if $f \in D(A_k)$ so does $T(t)f$ for each $t \in \mathcal{R}_+^m$ and

$$A_k T(t) f = T(t) A_k f.$$

Further if $f \in D(A_j)$ and $f \in D(A_j A_k)$ then $f \in D(A_k A_j)$ and $A_k A_j f = A_j A_k f$, $(j, k = 1, ..., m)$. In the following we use the notation

$$D^2 := \bigcap_{k,j=1}^{m} D(A_k A_j).$$

D^2 is a linear subspace of $\mathcal{X}$.

To each $k = 1, ..., m$, there correspond two numbers $M_k \geq 1$ and $\omega_k \geq 0$ such that

$$\|T_k(t_k)\| \leq M_k e^{\omega_k t_k},\ 0 \leq t_k < \infty.$$

Thus we have the inequality

$$\|T(t)\| \leq M \exp(\omega(t_1 + ... + t_m)) = M e^{\omega \overline{t}},\ t \in \mathcal{R}_+^m, \tag{29.5}$$

where $M = M_1 ... M_m$ and $\omega = \max\{\omega_k, 1 \leq k \leq m\}$.

In the following we always mean $\{T(t); t \in \mathcal{R}_+^m\}$ satisfies (29.5), unless otherwise specified.

For the above definition and properties about operator semigroup we refer to Butzer-Berens [122], Hille-Phillips [188] or W, Köhnen [210].

Let (Ω, A, P) be a probability space. For every real-valued random variable X defined on (Ω, A, P), $E(X)$ denotes its expectation. If $\xi = E(X)$ exists then $\sigma^2 = \sigma^2(X) = E[(X - \xi)^2]$ is called the variance of X. Let further $\Psi_x(u) = E(u^X)$, $u \geq 0$ and $\Psi_X^*(u) = E(e^{uX})$, $u \in \mathcal{R}$ denote the probability-generating function and the moment-generating function of X respectively.

We need to consider m-dimensional random vectors, also denoted by $X, Y, ...$, on (Ω, A, P). For m-dimensional random vector $X = (X_{01}, ..., X_{0m})$, we also use $E(X)$ to denote its expectation:

$$E(X) := (E(X_{01}), ..., E(X_{0m}))$$

and denote

$$\sigma_i^2(X) := \sigma^2(X_{0i}).$$

It is not difficult to extend the integration theory about extended-Pettis integral developed in [243] to mutivariate case.

Let $\{T(t); t \in \mathcal{R}_+^m\}$ be as above and X is a $\mathcal{R}_+^m$-valued random vector such that

$$\Psi_{\bar{X}}^*(\omega) < \infty,\ \bar{X} = X_{01} + ... + X_{0m},$$

then for every $f \in \mathcal{X}$ define

$$E[T(X)f] := \int_\Omega T(X) f dP,$$

which exists in the Bochner sense in $\mathcal{X}$ by the strong continuity of $\{T(t); t \in \mathcal{R}_+^m\}$ and (29.5). Moreover, the map $E[T(X)] : f \to E(T(X)f]$ on $\mathcal{X}$ defines a bounded linear operator $E(T(X)] \in \mathcal{E}(\mathcal{X})$ with

$$\|E(T(X)]\| \leq M\Psi_{\bar{X}}^*(\omega).$$

$E[T(X)]$ is called the expectation of $T(X)$ and is understood as an extended Pettis integral following [243].

If X, Y are are independent $\mathcal{R}_+^m$-valued random vectors such that $\Psi_{\bar{X}}^*(\omega) < \infty$, $\Psi_{\bar{Y}}^*(\omega) < \infty$ then $E[T(X)]$, $E[T(Y)]$ and $E[T(X+Y)]$ exist in $\mathcal{E}(\mathcal{X})$ and there holds

$$E[T(X) \circ T(Y)] = E[T(X+Y)] = E[T(X)] \circ E[T(Y)],$$

where "$\circ$" denotes composition.

For the above please read [243], [244], [245] and the references cited there.

29.3 Basic Results

We need a Taylor's expansion integral formula for (C_0) m-parameter operator semigroups.

Lemma 29.1. Suppose $\{T(t); t \in \mathcal{R}_+^m\}$ is a (C_0) m-parameter operator semigroup satisfying (29.5). Then for every $g \in D^2$ and $s, t \in \mathcal{R}_+^m$, there holds

$$\begin{aligned} &T(t)g - T(s)g \\ &\quad = T(s)[(t_1 - s_1)A_1 g + ... + (t_m - s_m)A_m g] \qquad (29.6) \\ &\qquad + \int_0^1 (1-u)T(s + u(t-s))((t_1 - s_1)A_1 + ... + (t_m - s_m)A_m)^2 g du. \end{aligned}$$

Proof. Let $G(u) = T(s + u(t-s))g \in \mathcal{X}$, $u \in [0,1]$, then

$$\begin{aligned} G'(u) &:= \frac{dG(u)}{du} \\ &= T(s + u(t-s))[(t_1 - s_1)A_1 + ... + (t_m - s_m)A_m]g. \end{aligned}$$

and

$$G''(u) = T(s + u(t-s))[(t_1 - s_1)A_1 + ... + (t_m - s_m)A_m]^2 g.$$

Now (29.6) follows from the Taylor formula with integral remainder for Banach space valued functions (see, e.g., [146], Theorem 8.14.[130]).

For our purpose we need a second modulus of continuity $\omega_2(Tf, \delta)$ and the Steklov operator $J_h(f)(h > 0)$for (C_0) m-parameter operator semigroup $\{T(t); t \in \mathcal{R}_+^m\}$ and $f \in \mathcal{X}$. ■

Definition 29.2.

$$\omega_2\left(Tf,\delta\right)=\sup_{\substack{t=(t_1,\ldots,t_m)\\0\le t_i,t_j\le\delta}}\left\{\left\|\left(T(t)-I\right)^2 f\right\|,\left\|\left(T_i(t_i)-I\right)\left(T_j(t_j)-I\right)f\right\|\right\}.$$

When $\delta\to 0$, by the strong continuity of $\{T(t);\ t\in\mathcal{R}_+^m\},\ \omega_2(Tf,\delta)\to 0$.

Definition 29.3.

$$\begin{aligned}J_h(f)=&(\frac{2}{h})^{2m}\underbrace{\int_0^{h/2}\ldots\int_0^{h/2}}_{2m}[2T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m)\\&-T(2\xi_1+2\eta_1,\ldots,2\xi_m+2\eta_m)]f d\xi_1 d\eta_1\ldots d\xi_m d\eta_m.\end{aligned}$$

The integral may be considered as multi-$\mathcal{X}$-valued Riemann integral.
We have following

Lemma 29.4.

i) $J_h(f)\in D^2$, for all $f\in\mathcal{X}$;
ii) $\|f-J_h(f)\|\le\varepsilon_2(Tf,\ h)$;
iii) $\|A_iA_jJ_h(f)\|\le 9Me^{2(m-1)h\omega}\omega_2(Tf,\ h)/h^2,\ 1\le i,\ j\le m.$

Proof. *i)* Let

$$J_1=\underbrace{\int_0^{h/2}\ldots\int_0^{h/2}}_{2m}T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m)f d\xi_1 d\eta_1\ldots d\xi_m d\eta_m, \tag{29.7}$$

$$\begin{aligned}J_2=&\underbrace{\int_0^{h/2}\ldots\int_0^{h/2}}_{2m}T(2\xi_1+2\eta_1,\ldots,2\xi_m+2\eta_m)f d\xi_1 d\eta_1\ldots d\xi_m d\eta_m,\\=&(\frac{1}{2})^{2m}\underbrace{\int_0^{h}\ldots\int_0^{h}}_{2m}T(\xi_1+\eta_1,\ldots,\xi_m+\eta_m)f d\xi_1 d\eta_1\ldots d\xi_m d\eta_m.\end{aligned} \tag{29.8}$$

It is not difficult to show that $J_1 \in D^2,\ J_2 \in D^2$ (cf.[122, p.10]) and hence (i) holds.

$$
\begin{aligned}
& ii)\ \|f - J_h(f)\| \\
=& \Big\|(2/h)^{2m} \underbrace{\int_0^{h/2} \cdots \int_0^{h/2}}_{2m} [f - 2T(\xi_1+\eta_1, ..., \xi_m+\eta_m) \\
& + T(2\xi_1+2\eta_1, ..., 2\xi_m+2\eta_m)] f d\xi_1 d\eta_1 d\eta_1 ... d\xi_m d\eta_m \Big\| \\
& \Big\|(2/h)^{2m} \underbrace{\int_0^{h/2} \cdots \int_0^{h/2}}_{2m} [T(\xi_1+\eta_1, ..., \xi_m+\eta_m) - I]^2 f \\
& \times d\xi_1 d\eta_1 ... d\xi_m d\eta_m \Big\| \\
\leq & (2/h)^{2m} \underbrace{\int_0^{h/2} \cdots \int_0^{h/2}}_{2m} \Big\|[T(\xi_1+\eta_1, ..., \xi_m+\eta_m) - I]^2 f\Big\| \\
& \times d\xi_1 d\eta_1 ... d\xi_m d\eta_m \\
\leq & \omega_2(Tf,\ h).
\end{aligned}
$$

iii) When $i \neq j$, similar to one parameter operator semigroup case (ibid.), we can show

$$
\begin{aligned}
A_i A_j J_1 =& \underbrace{\int_0^{h/2} \cdots \int_0^{h/2}}_{2m-2} \prod_{k \neq i,j} T_k(\xi_k+\eta_k) T_i(\eta_i) T_j(\eta_j) \\
& \times (T_i(h/2) - I)(T_j(h/2) - I) f \prod_{k \neq i,j} d\xi_k d\eta_k d\eta_i d\eta_j
\end{aligned}
$$

and

$$
\begin{aligned}
A_i A_j J_2 =& (1/2)^{2m} \underbrace{\int_0^{h} \cdots \int_0^{h}}_{2m-2} \prod_{k \neq i,j} T_k(\xi_k+\eta_k) T_i(\eta_i) T_j(\eta_j) \\
& \times (T_i(h) - I)(T_j(h) - I) f \prod_{k \neq i,j} d\xi_k d\eta_k d\eta_i d\eta_j.
\end{aligned}
$$

Therefore

$$\|A_iA_jJ_h(f)\| = \left\|(2/h)^{2m}[2A_iA_jJ_1 - A_iA_jJ_2]\right\|$$
$$\leq (\frac{2}{h})^{2m}\{2\underbrace{\int_0^{h/2}\cdots\int_0^{h/2}}_{2m-2}\prod_{k\neq i,j} M_k e^{\omega_k(\xi_k+\eta_k)} M_i e^{\omega_i\eta_i} M_j e^{\omega_j\eta_j}$$
$$\times \left\|(T_i(\frac{h}{2}) - I)(T_j(\frac{h}{2}) - I)f\right\| \prod_{k\neq i,j} d\xi_k d\eta_k d\eta_i d\eta_j$$
$$+ (\frac{1}{2})^{2m}\underbrace{\int_0^{h}\cdots\int_0^{h}}_{2m-2}\prod_{k\neq i,j} M_k e^{\omega_k(\xi_k+\eta_k)} M_i e^{\omega_i\eta_i} M_j e^{\omega_j\eta_j}$$
$$\times \|(T_i(h) - I)(T_j(h) - I)f\| \prod_{k\neq i,j} d\xi_k d\eta_k d\eta_i d\eta_j\}$$
$$\leq (2/h)^{2m} M\{2e^{(2m-2)\omega h/2}(h/2)^{2m-2} + (1/2)^{2m} e^{(2m-2)\omega h} h^{2m-2}\}$$
$$\times \omega_2(Tf,\ h)$$
$$\leq Me^{(2m-2)\omega h}\{2(\frac{h}{2})^{-2} + \frac{1}{h^2}\}\omega_2(Tf,\ h)$$
$$= 9Me^{2(m-1)\omega h}\omega_2(Tf,\ h)/h^2.$$

When $i = j$, the same estimate holds. ■

Lemma 29.5. For any $\mathcal{R}_+^m$-valued random vector $Y = (Y_{01}, ..., Y_{0m})$ with $E(Y) = x = (x_1, ..., x_m)$ and $f \in \mathcal{X}$ there holds.

$$\|E[T(Y)]f - T(x)f\| = \|E[T(Y)f] - T(x)f\|$$
$$\leq M\omega_2(Tf,\ h)\{2E(e^{\omega\overline{Y}}) \tag{29.9}$$
$$+ \frac{9}{2}mMe^{2\omega\overline{x}}e^{2(m-1)h\omega}$$
$$\times [E(e^{p\omega\overline{Y}})]^{1/p}[\sum_{i=1}^{m}(E((Y_{0i} - x_i)^{2q}))^{1/q}]/h^2\},$$

where $p > 1,\ q > 1,\ 1/p + 1/q = 1,\ h > 0.$

If $\omega = 0$, we have

$$\|E[T(Y)]f - T(x)f\| \leq 2M\omega_2(Tf,\ h)[1 + \frac{9mM}{4h^2}\sum_{i=1}^{m}\sigma^2(Y_{0i})] \tag{29.10}$$

Proof. It holds

$$\|E[T(Y)]f - T(x)f\| = \|E[T(Y)f] - T(x)f\|$$
$$\leq \|E[T(Y)f] - E[T(Y)J_nf]\| + \|E[T(Y)J_nf] - T(x)J_nf\|$$
$$+ \|T(x)J_nf - T(x)f\| \tag{29.11}$$
$$:= I_1 + I_2 + I_3.$$

$$I_1 \le E[\|T(Y)(J_n f - f)\|] \le E[Me^{\omega\overline{Y}}\|J_n f - f\|] \le ME(e^{\omega\overline{Y}})\omega_2(Tf,\ h) \tag{29.12}$$

by Lemma 29.4.

$$I_3 \le Me^{\omega\overline{x}}\omega_2(Tf,\ h) \le ME(e^{\omega\overline{Y}})\omega_2(Tf,\ h) \tag{29.13}$$

by Jensen's inequality.

Note that $g := J_h(f) \in D^2$, by Lemma 29.4. Apply Lemma 29.1, we get

$$\begin{aligned}
I_2 = &\Big\| E\{T(x)[(Y_{01} - x_1)A_1 + ... + (Y_{0m} - x_m)A_m]g + \int_0^1 (1-u) \\
&\times T(x + u(Y - x))[(Y_{01} - x_1)A_1 + ... + (Y_{0m} - x_m)A_m]^2 g du\}\Big\| \\
&\Big\| E\{\int_0^1 (1-u)T(x + u(Y - x))[(Y_{01} - x_1)A_1 + ... \\
&+(Y_{0m} - x_m)A_m]^2 g du\}\Big\| \\
\le& E\{\int_0^1 (1-u)\,\|T(x + u(Y - x))\| \\
&\times \Big\|[(Y_{01} - x_1)A_1 + ... + (Y_{0m} - x_m)A_m]^2 g\Big\|\, du\} \\
\le& ME\{\int_0^1 (1-u)\exp(\omega\overline{(x + u(Y - x))} \\
&\times \Big\|[(Y_{01} - x_1)A_1 + ... + (Y_{0m} - x_m)A_m]^2 g\Big\|\, du\} \\
\le& \frac{1}{2}Me^{2\omega\overline{x}}E\{e^{\omega\overline{Y}}\sum_{i=1}^m\sum_{j=1}^m |Y_{0i} - x_i\,\|Y_{0j} - x_j\,|\,\|A_iA_jg\|\} \\
\le& \frac{1}{2}Me^{2\omega\overline{x}}E\{e^{\omega\overline{Y}}\sum_{i=1}^m\sum_{j=1}^m \frac{1}{2}[(Y_{0i} - x_i)^2 + (Y_{0j} - x_j)^2]\}9Me^{2(m-1)h\omega} \\
&\times \omega_2(Tf,\ h)/h^2 \quad \text{(by Lemma 29.4, } iii\text{)} \quad) \\
&= \frac{9}{2}M^2e^{2\omega\overline{x}}e^{2(m-1)h\omega}m\sum_{i=1}^m E\left[e^{\omega\overline{Y}}(Y_{0i} - x_i)^2\right]\omega_2(Tf,\ h)/h^2 \\
&\le \frac{9}{2}mM^2e^{2\omega\overline{x}}e^{2(m-1)h\omega}[E(e^{p\omega\overline{Y}})]^{1/p}[\sum_{i=1}^m (E((Y_{0i} - x_i)^{2q}))^{1/q}] \\
&\times \omega_2(Tf,\ h)/h^2,
\end{aligned} \tag{29.14}$$

by Hölder's inequality.

Therefore by (29.11)-(29.14) we get (29.9).

If $\omega = 0$ we have $I_1 \le M\omega_2(Tf,\ h)$, $I_3 \le M\omega_2(Tf,\ h)$ and $I_2 \le \frac{9}{2}M^2m \times \sum_{i=1}^m \sigma^2(Y_{0i})\omega_2(Tf,\ h)/h^2$ so (29.10) follows. ∎

29.4 Main Results

Here comes the first main result

Theorem 29.6. Let $X = (X_{01}, ..., X_{0m})$ be an $\mathcal{R}_+^m$-valued random vector with $E(X) = x = (x_1, ..., x_m)$ and there exists a $\delta > 0$ such that $\Psi_{\bar{X}}^*(\delta) < \infty$. Then for any (C_0) m-parameter operator semigroup satisfying (29.5), there holds for all $n > \max(p\omega/\delta,\ 1/\delta^2)$ that

$$\begin{aligned}
&\|\{E[T(X/n)]\}^n f - T(x)f\| \\
\leq & 2M\omega_2(Tf, 1/\sqrt{n})\{e^{\omega\bar{x}} \exp[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi_{\bar{X}(\delta)}^*] \qquad (29.15)\\
&+ 2^{1/q} 9M\frac{m^2q^2}{e^2}e^{3\omega\bar{x}}e^{2(m-1)\omega/\sqrt{n}}\exp[(\frac{2np\omega^2}{e^2(n\delta-p\omega)^2} \\
&+ \frac{2ne^{\delta\bar{x}}}{e^2q(\delta\sqrt{n}-1)^2}\Psi_{\bar{X}}^*(\delta)]\},
\end{aligned}$$

where $p,\ q > 1,\ 1/p + 1/q = 1$ is an arbitrary conjugate pair.

When $\omega = 0$

$$\|\{E[T(X/n)]\}^n f - T(x)f\| \leq 2M\omega_2(Tf, \frac{1}{\sqrt{n}})[1 + \frac{9}{4}mM\sum_{i=1}^m \sigma^2(X_{0i})]. \quad (29.16)$$

Note. All the right hand sides of (29.15) and (29.16) are finite.

Proof of Theorem 29.6: Let X_k be a sequence of independent random vectors identically distributed as X, and $Y = \frac{1}{n}\sum_{k=1}^n X_k$, then

$$E(Y) = \frac{1}{n}\sum_{k=1}^n E(X_k) = x,$$
$$E[T(Y)f] = \{E[T(X/n)]\}^n f.$$

For $u > 0$ we get

$$\begin{aligned}
\Psi_{\bar{Y}}^*(u) &= E(e^{\frac{u}{n}\sum_{k=1}^n \overline{X_k}}) = (E(e^{\frac{u}{n}\bar{X}}))^n \\
&\leq (1 + \frac{u}{n}E(\bar{X}) + E(\frac{u^2\bar{X}^2}{2n^2}e^{\frac{u}{n}\bar{X}}))^n \\
&\leq (1 + \frac{u}{n}\bar{x} + \frac{u^2}{2n^2}(\frac{2}{\delta - u/n})^2 e^{-2}E(e^{\delta\bar{X}}))^n \\
&\leq e^{u\bar{x}}\exp[\frac{2nu^2}{e^2(n\delta-u)^2}\Psi_{\bar{X}}^*(\delta)],
\end{aligned}$$

when $u/n < \delta$.

Above we made use of the inequalities (see also Pfeifer[244, p.275])

$$r^\alpha e^{\eta r} \leq (\frac{\alpha}{\delta-\eta})^\alpha e^{-\alpha}e^{\delta r} \text{ (when } \eta < \delta,\ r > 0,\ \alpha > 0) \quad (29.17)$$

and

$$(1+r)^n \leq e^{nr}. \tag{29.18}$$

So when $n > p\omega/\delta \geq \omega/\delta$,

$$E(e^{\omega\bar{Y}}) \leq e^{\omega\bar{x}} \exp\{\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi^*_{\bar{X}}(\delta)\}$$

and

$$\begin{aligned}[E(e^{p\omega\bar{Y}})]^{1/p} &\leq \{e^{p\omega\bar{x}} \exp[\frac{2np^2\omega^2}{e^2(n\delta-p\omega)^2}\Psi^*_{\bar{X}}(\delta)]\}^{1/p} \\ &= e^{\omega\bar{x}} \exp[\frac{2np\omega^2}{e^2(n\delta-p\omega)^2}\Psi^*_{\bar{X}}(\delta)].\end{aligned}$$

Observe that for $Y = (Y_{01}, ..., Y_{0m})$ we obtain

$$\begin{aligned}&E((Y_{0i}-x_i)^{2q}) = E((\frac{1}{n}\sum_{k=1}^{n} X_{ki} - x_i)^{2q}) \\ &\leq (\frac{2q}{\sqrt{n}})^{2q} e^{-2q} E(e^{\sqrt{n}|\frac{1}{n}\sum_{k=1}^{n} X_{ki}-x_i|}) \text{ by (29.17)}) \\ &\leq (\frac{2q}{\sqrt{n}})^{2q} e^{-2q} [E(e^{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_{ki}-x_i)}) + E(e^{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(x_i-X_{ki})})] \\ &\leq (\frac{2q}{\sqrt{n}})^{2q} e^{-2q} 2\exp[E(\frac{1}{2}(X_{0i}-x_i)^2 e^{\frac{1}{\sqrt{n}}|X_{0i}-x_i|})] \\ &\text{(by Taylor's expansion and (29.18))} \\ &\leq (\frac{2q}{\sqrt{n}})^{2q} e^{-2q} 2\exp[(\frac{2}{\delta-1/\sqrt{n}})^2 e^{-2} E(\frac{1}{2}e^{\delta|X_{0i}-x_i|})] \\ &\text{(by (29.17) when } 1/\sqrt{n} < \delta) \\ &\leq (\frac{2q}{\sqrt{n}})^{2q} e^{-2q} 2\exp[\frac{2n}{(\delta\sqrt{n}-1)^2} e^{-2} e^{\delta\bar{x}} \Psi^*_{\bar{X}}(\delta)],\ 1 \leq i \leq m.\end{aligned}$$

Hence we proved that

$$[E((Y_{0i}-x_i)^{2q})]^{1/q} \leq \frac{4q^2}{n} e^{-2} 2^{1/q} \exp[\frac{2n}{q(\delta\sqrt{n}-1)^2} e^{-2} e^{\delta\bar{x}} \Psi^*_{\bar{X}}(\delta)].$$

Now apply Lemma 29.5 and take $h = 1/\sqrt{n}$:

$$
\begin{aligned}
&\|\{E[T(X/n)]\}^n f - T(x)f\| = \|E[T(Y)]f - T(x)f\| \\
\leq & M\omega_2(Tf, 1/\sqrt{n})\{2e^{\omega\bar{x}} \exp[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi^*_{\bar{X}}(\delta)] \\
&+ \frac{9}{2}mMe^{2\omega\bar{x}}e^{2(m-1)\frac{\omega}{\sqrt{n}}}e^{\omega\bar{x}}\exp[\frac{2np\omega^2}{e^2(n\delta-p\omega)^2}\Psi^*_{\bar{X}}(\delta)] \\
&\times m\frac{4q^2}{n}e^{-2}2^{1/q}\exp[\frac{2n}{q(\delta\sqrt{n}-1)^2}e^{-2}e^{\delta\bar{x}}\Psi^*_{\bar{X}}(\delta)]n\} \\
=&2M\omega_2(Tf, 1/\sqrt{n})\{e^{\omega\bar{x}}\exp[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi^*_{\bar{X}}(\delta)] \\
&+ 2^{1/q}9M\frac{m^2q^2}{e^2}e^{3\omega\bar{x}}e^{2(m-1)\omega/\sqrt{n}}\exp[(\frac{2np\omega^2}{e^2(n\delta-p\omega)^2} \\
&+ \frac{2ne^{\delta\bar{x}}}{e^2q(\delta\sqrt{n}-1)^2}\Psi^*_{\bar{X}}(\delta)]\}.
\end{aligned}
$$

When $\omega = 0$, noting that $\sigma^2(Y_{0i}) = \sigma^2(X_{0i})/n$ by (29.10), we get (29.16).

A ramification of Theorem 29.6 follows

Theorem 29.7. Let N be a $\mathcal{Z}_+$-valued random variable with $E(N) = \eta,\ \eta > 0$, and let $Y = (Y_{01}, ..., Y_{0m})$ be an $\mathcal{R}^m_+$-valued random vector independent of N with $E(Y) = \gamma = (\gamma_1, ..., \gamma_m)$. Suppose that there exists $\delta > 0$ such that

$$\Psi_N(\Psi^*_{\bar{Y}}(\delta)) < \infty.$$

Then when $n > \max(p\omega/\delta,\ 1/\delta^2)$ there holds

$$
\begin{aligned}
&\|\{\Psi_N[E(T(Y/n))]\}^n f - T(\eta\gamma)f\| \\
\leq & 2M\omega_2(Tf, 1/\sqrt{n})\{e^{\omega\eta\bar{\gamma}}\exp[\frac{2n\omega^2}{e^2(n\delta-\omega)^2}\Psi_N(\Psi^*_{\bar{Y}}(\delta))] \\
&+ 2^{1/q}9M\frac{m^2q^2}{e^2}e^{3\omega\eta\bar{\gamma}}e^{2(m-1)\omega/\sqrt{n}}\exp[(\frac{2np\omega^2}{e^2(n\delta-p\omega)^2} \\
&+ \frac{2ne^{\delta\eta\bar{\gamma}}}{e^2q(\delta\sqrt{n}-1)^2}\Psi_N(\Psi^*_{\bar{X}}(\delta))]\},
\end{aligned}
\tag{29.19}
$$

where $p,\ q > 1,\ 1/p + 1/q = 1$ is an arbitrary conjugate pair.

If $\omega = 0$, there holds

$$
\begin{aligned}
&\|\Psi_N[E(T(Y/n))]\}^n f - T(\eta\gamma)f\| \\
&\leq 2M\omega_2(Tf, 1/\sqrt{n})\{1 + \frac{9}{4}mM\sum_{i=1}^m[\eta\sigma^2(Y_{0i}) + \sigma^2(N)\gamma_i^2]\}.
\end{aligned}
\tag{29.20}
$$

Proof. Consider $Y_k \overset{i.i.d.}{\sim} Y$, which are also independent of N. In Theorem 29.6, take $X = \sum_{k=1}^{N} Y_k$ (as usual, an empty sum equals 0), then

$$E[T(\frac{1}{n}X)] = E[T(\frac{1}{n}\sum_{k=1}^{N} Y_k)] = \sum_{l=0}^{\infty} P(N=l)E[T(\frac{1}{n}\sum_{k=1}^{l} Y_k)]$$
$$= \sum_{l=0}^{\infty} P(N=l)[E(T(\frac{1}{n}Y))]^l = \Psi_N(E(T(\frac{1}{n}Y))),$$
$$E(X) = \sum_{l=0}^{\infty} P(N=l)E[\sum_{k=l}^{l} Y_k] = E(N)E(Y) = \eta\gamma.$$

Also

$$\Psi^*_{\bar{X}}(\delta) = E(e^{\delta\bar{X}}) = E(e^{\delta\sum_{k=1}^{N} Y_k}) = \sum_{l=0}^{\infty} P(N=l)E(e^{\delta\sum_{k=1}^{l}\bar{Y}_k})$$
$$= \sum_{l=0}^{\infty} P(N=l)(E(e^{\delta\bar{Y}}))^l = \Psi_N(\Psi^*_{\bar{Y}}(\delta)).$$

By $X = (X_{01}, ..., X_{0m})$ we get

$$\sigma^2(X_{0i}) = \sigma^2(\sum_{k=1}^{N} Y_{ki}) = \sum_{l=0}^{\infty} P(N=l)E((\sum_{k=1}^{l} Y_{ki})^2) - \eta^2\gamma_i^2$$
$$= \sum_{l=0}^{\infty} P(N=l)(lE(Y_{0i}^2) + l(l-1)\gamma_i^2) - \eta^2\gamma_i^2$$
$$= \eta\sigma^2(Y_{0i}) + \sigma^2(N)\gamma_i^2.$$

Then (29.19), (29.20) follow by (29.15). (29.16). ■

An application of Lemma 29.5 comes next

Theorem 29.8. For each positive real number r, let N_r be a $\mathcal{Z}_+$-valued random variable with $E(N_r) = \tau\eta$, where $\eta \in \mathcal{R}_+$ is fixed. Let X be a $\mathcal{R}_+^m$-valued random vector with $E(X) = \gamma = (\gamma_1, ..., \gamma_m)$, independent of N_r. Assume that there exists a $\delta > 0$ such that $\Psi^*_{\bar{X}}(\delta) < \infty$ and further there are $p > 1,\ q > 1$ with $1/p + 1/q = 1$ such that

$$\lim_{\tau\to\infty} \sup \Psi_{N_\tau}(\Psi^*_{\bar{X}}(\frac{p\omega}{\tau})) = d_1 < \infty, \tag{29.21}$$

$$\lim_{\tau\to\infty} \sup \tau\{E[(\frac{1}{\tau}N_\tau - \eta)^{2q}]\}^{1/q} = d_2 < \infty \tag{29.22}$$

and

$$\lim_{\tau\to\infty} \sup \Psi^*_{N_\tau}(\frac{2}{e^2(\sqrt{\tau}\delta - 1)^2} e^{\delta\overline{\gamma}} \Psi^*_{\bar{X}}(\delta)) = d_3 < \infty. \tag{29.23}$$

Then for $\tau > 1/\delta^2$ there holds

$$\begin{aligned}
&\|\Psi_{N_\tau}(E(T(X/\tau)])f - T(\eta\gamma)f\| \\
\leq &M\omega_2(Tf,\ 1/\sqrt{\tau})\{2d_1 \qquad (29.24)\\
&+ 9mMe^{2\omega\eta\bar{\gamma}}e^{2(m-1)\omega/\sqrt{\tau}}d_1^{1/p}[m2^{1/q}(\frac{2q}{e})^2 d_3^{1/q} + d_2\sum_{i=1}^m \gamma_i^2]\}.
\end{aligned}$$

If $\omega = 0$, then

$$\begin{aligned}
&\|\Psi_{N_\tau}(E[T(X/\tau)])f - T(\eta\gamma)f\| \\
&\leq 2M\omega_2(Tf,\ 1/\sqrt{\tau})\{1 + \frac{9}{4}mM\sum_{i=1}^m[\eta\sigma^2(X_{0i}) + \gamma_i^2\frac{1}{\tau}\sigma^2(N_\tau)]\}. \qquad (29.25)
\end{aligned}$$

Proof. Let the random vectors $X_k \overset{i.i.d.}{\sim} X$, which are also independent of N_τ. Consider $Y_\tau = \frac{1}{\tau}\sum_{k=1}^{N_\tau} X_k$, where $Y_\tau = (Y_{01}, ..., Y_{0m})$, then apply Lemma 29.5 with $h = 1/\sqrt{\tau}$. We derive

$$\begin{aligned}
E[T(Y_\tau)]f &= \sum_{l=0}^\infty P(N_\tau = l)E[T(\frac{1}{\tau}\sum_{k=1}^l X_k)]f = \Psi_{N_\tau}(E[T(X/\tau)]f),\\
E(Y_\tau) &= \sum_{l=0}^\infty P(N_\tau = l)\frac{1}{\tau}E(\sum_{k=1}^l X_k) = \frac{1}{\tau}E(N_\tau)E(X) = \eta\gamma,\\
E(e^{\omega\bar{Y}_\tau}) &\leq E(e^{p\omega\bar{Y}_\tau}) = \Psi_{N_\tau}(\Psi^*_{\bar{X}}(\frac{p\omega}{\tau})) \leq d_1.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&E((Y_{0i} - \eta\gamma_i)^{2q}))^{1/q} \\
&= \{E([\frac{1}{\tau}\sum_{k=1}^{N_\tau} X_{ki} - \frac{1}{\tau}N_\tau\gamma_i + \frac{1}{\tau}N_\tau\gamma_i - \eta\gamma_i]^{2q})\}^{1/q} \\
&\leq 2[E((\frac{1}{\tau}\sum_{k=1}^{N_\tau}(X_{ki} - \gamma_i))^{2q})]^{1/q} + 2\gamma_i^2[E((\frac{1}{\tau}N_\tau - \eta)^{2q})]^{1/q} \\
&=: 2I_1 + 2I_2.
\end{aligned}$$

We notice that

$$\begin{aligned}
I_1^q &= E[(\frac{1}{\tau}\sum_{k=1}^{N_\tau}(X_{ki}-\gamma_i))^{2q}] \\
&\le (\frac{2q}{\sqrt{\tau}})^{2q}e^{-2q}E[e^{\frac{1}{\sqrt{\tau}}|\sum_{k=1}^{N_\tau}(X_{ki}-\gamma_i)|}] \text{ (by (29.17))} \\
&\le (\frac{2q}{e\sqrt{\tau}})^{2q}\{E[e^{\frac{1}{\sqrt{\tau}}\sum_{k=1}^{N_\tau}(X_{ki}-\gamma_i)}]+E[e^{\frac{1}{\sqrt{\tau}}\sum_{k=1}^{N_\tau}(\gamma_i-X_{ki})}]\} \\
&\le (\frac{2q}{e\sqrt{\tau}})^{2q}\{E[(E(e^{\frac{1}{\sqrt{\tau}}(X_{0i}-\gamma_i)}))^{N_\tau}]+E[(E(e^{\frac{1}{\sqrt{\tau}}(\gamma_i-X_{oi})}))^{N_\tau}]\} \\
&\le (\frac{2q}{e\sqrt{\tau}})^{2q}\{E[(E(1+\frac{1}{\sqrt{\tau}}(X_{0i}-\gamma_i)+\frac{1}{2\tau}(X_{0i}-\gamma_i)^2e^{\frac{1}{\sqrt{\tau}}|X_{0i}-\gamma_i|}))^{N_\tau}] \\
&+E[(E(1+\frac{1}{\sqrt{\tau}}(\gamma_i-X_{0i})+\frac{1}{2\tau}(X_{0i}-\gamma_i)^2e^{\frac{1}{\sqrt{\tau}}|X_{0i}-\gamma_i|}))^{N_\tau}]\} \\
&\le 2(\frac{2q}{e\sqrt{\tau}})^{2q}E\{(\exp[E(X_{0i}-\gamma_i)^2e^{\frac{1}{\sqrt{\tau}}|X_{0i}-\gamma_i|}])^{N_\tau/2\tau}\}(\frac{1}{\sqrt{\tau}}<\delta) \\
&\le 2(\frac{2q}{e\sqrt{\tau}})^{2q}E\{(\exp[(\frac{2}{\delta-1/\sqrt{\tau}})^2e^{-2}E(e^{\delta|X_{0i}-\gamma_i|})])^{N_\tau/2\tau}\} \\
&\le 2(\frac{2q}{e\sqrt{\tau}})^{2q}E\{(\exp[(\frac{2\sqrt{\tau}}{e(\sqrt{\tau}\delta-1)})^2e^{\delta\bar{\gamma}}\Psi_{\bar{X}}^*(\delta)])^{N_\tau/2\tau}\} \\
&\le 2(\frac{2q}{e\sqrt{\tau}})^{2q}d_3.
\end{aligned}$$

So that

$$I_1 \le 2^{1/q}\frac{4q^2}{e^2\tau}d_3^{1/q}$$

and

$$I_2 = \gamma_i^2(E((\frac{1}{\tau}N_\tau-\eta)^{2q}))^{1/q} \le \gamma_i^2 d_2/\tau.$$

Therefore by Lemma 29.5, inequality (29.9), for $Y = Y_\tau$ and $h = 1/\sqrt{\tau}$, we obtain

$$\begin{aligned}
&\|\Psi_{N_\tau}[E(T(X/\tau))]f - T(\eta\gamma)f\| \\
\le& M\omega_2(Tf,\ 1/\sqrt{\tau})\{2d_1+\frac{9}{2}mMe^{2\omega\eta\bar{\gamma}}e^{2(m-1)\omega/\sqrt{\tau}}d_1^{1/p} \\
&\times 2\sum_{i=1}^{m}[2^{1/q}(\frac{4q^2}{e^2\tau}d_3^{1/q})+d_2\frac{\gamma_i^2}{\tau}]\tau\} \\
=& M\omega_2(Tf,\ 1\sqrt{\tau})\{2d_1+9mMe^{2\omega\eta\bar{\gamma}}e^{2(m-1)\omega/\sqrt{\tau}}d_1^{1/p} \\
&\times[m2^{1/q}(\frac{2q}{e})^2d_3^{1/q}+d_2\sum_{i=1}^{m}\gamma_i^2]\}.
\end{aligned}$$

If $\omega = 0$, we apply Lemma 29.5, inequality (29.10). Observe that

$$\begin{aligned}\sigma^2(Y_{0i}) &= \sigma^2(\frac{1}{\tau}\sum_{k=1}^{N_\tau} X_{ki}) = \frac{1}{\tau^2}\sum_{l=0}^{\infty} P(N_\tau = l)E((\sum_{k=1}^{l} X_{ki})^2) - \eta^2\gamma_i^2 \\ &= \frac{1}{\tau^2}\sum_{l=0}^{\infty} P(N_\tau = l)(lE(X_{0i}^2) + l(l-1)\gamma_i^2) - \eta^2\gamma_i^2 \\ &= \frac{1}{\tau^2}E(N_\tau)E(X_{0i}^2) + \frac{1}{\tau^2}(E(N_\tau^2))\gamma_i^2 - \frac{1}{\tau^2}(E(N_\tau))\gamma_i^2 - \eta^2\gamma_i^2 \\ &= \frac{1}{\tau}[\eta\sigma^2(X_{0i}) + \gamma_i^2\frac{1}{\tau}\sigma^2(N_\tau)].\end{aligned}$$

By (29.10), when $h = 1\sqrt{\tau}$, we obtain (29.25).

Another generalization of Theorem 29.6 is presented next. ■

Theorem 29.9. Let $n = (N_1, ..., N_m)$ be a $\mathcal{Z}_+^m$-valued random vector with $E(N) = \eta = (\eta_1, ..., \eta_m)$. For each i $(1 \leq i \leq m)$, let $\{Y_{ki}\}_{k=1}^{\infty}$ be a sequence of *i.i.d.* real-valued random variables distributed as Y a fixed random variable with $E(Y) = \gamma$. N and Y_{ki} are assumed to be independent. Also suppose that there exists a $\delta > 0$ such that

$$\Psi_{\bar{N}}(\Psi_Y^*(\delta)) < \infty.$$

Then for $n > \max(p\omega/\delta, 1/\delta^2)$ there holds

$$\left\|\{E[T(\sum_{k_1=1}^{N_1}\frac{1}{n}Y_{k_1 1}, ..., \sum_{k_m=1}^{N_m}\frac{1}{n}Y_{k_m m})]\}^n f - T(\gamma\eta)f\right\| \tag{29.26}$$
$$\leq 2M\omega_2(Tf, 1/\sqrt{n})\{e^{\omega\gamma\bar{\eta}}\exp[\frac{2\eta\omega^2}{e^2(n\delta-\omega)^2}\Psi_{\bar{N}}(\Psi_Y^*(\delta))] + 2^{1/q}9M\frac{m^2q^2}{e^2}$$
$$\times e^{3\omega\gamma\bar{\eta}}e^{2(m-1)\omega/\sqrt{n}}\exp[(\frac{2np\omega^2}{e^2(n\delta-p\omega)^2} + \frac{2ne^{\delta\gamma\bar{\eta}}}{e^2q(\delta\sqrt{n}-1)^2}\Psi_{\bar{N}}(\Psi_Y^*(\delta))]\},$$

where p, $q > 1$, $1/p + 1/q = 1$ is an arbitrary conjugate pair.

If $\omega = 0$, there holds

$$\left\|\{E[T(\sum_{k_1=1}^{N_1}\frac{1}{n}Y_{k_1 1}, ..., \sum_{k_m=1}^{N_m}\frac{1}{n}Y_{k_m m})]\}^n f - T(\gamma\eta)f\right\|$$
$$\leq 2M\omega_2(Tf, 1/\sqrt{n})\{1 + \frac{9}{4}mM\sum_{i=1}^{m}[\eta_i\sigma^2(Y) + \sigma^2(N_i)\gamma^2]\}. \tag{29.27}$$

Proof. In Theorem 29.6, take $X := (\sum_{k_1=1}^{N_1} Y_{k_1}1, ..., \sum_{k_m=1}^{N_m} Y_{k_m}m)$ and let $X_k \overset{i.i.d.}{\sim} X$ then

$$E(X) = (E[\sum_{k_1=1}^{N_1} Y_{k_1 1}], ..., E[\sum_{k_m=1}^{N_m} Y_{k_m m}]) = (EN_1EY, ..., EN_mEY) = \gamma\eta.$$

We derive

$$\begin{aligned}
\Psi^*_{\bar{X}}(\delta) &= E(e^{\sigma\bar{X}}) = E(e^{\delta(\sum_{k_1=1}^{N_1} Y_{k_1 1}+\ldots+\sum_{k_m=1}^{N_m} Y_{k_m m})}) \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,...,l_m))E(e^{\delta\sum_{k_1=1}^{l_1} Y_{k_1 1}+\ldots+\delta\sum_{k_m=1}^{l_m} Y_{k_m m}}) \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,...,l_m))E(e^{\delta\sum_{k_1=1}^{l_1} Y_{k_1 1}})...E(e^{\delta\sum_{k_m=1}^{l_m} Y_{k_m m}}) \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,...,l_m))(E(e^{\delta Y}))^{l_1}...(E(e^{\delta Y}))^{l_m} \\
&= \sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty} P(N=(l_1,...,l_m))(E(e^{\delta Y}))^{l_1+\ldots+l_m} \\
&= E((E(e^{\delta Y}))^{N_1+\ldots+N_m}) \\
&= \Psi_{\bar{N}}(\Psi^*_Y(\delta)).
\end{aligned}$$

Then (29.26) is implied by (29.15).

If $\omega = 0$ we see that

$$\sigma^2(X_{0i}) = \sigma^2(\sum_{k_i=1}^{N_1} Y_{k,i}) = \eta_i\sigma^2(Y) + \sigma^2(N_i)\gamma^2.$$

similarly established as the fact at the end of the proof of Theorem 29.7. Then (29.27) is implied by (29.16). ■

29.5 Further Results: Multiplier Enlargement Formulae

In this section we modify the formulae obtained in the previous section by so-called multiplier enlargement method (see [154]) initiated by Hsu-Wang [190], [278] in 60's and also used by Shaw [260], [261] in the representation of operator semigroups. The modified representation formulae have a larger of applications and when we specify the random vectors (variables) considered, the representation formulae for m-parameter operator semigroups of Shaw [261] are reobtained. For simplicity we only consider equibounded operator semigroups i.e.

$$\|T(t)\| \leq M, \text{ all } t \in \mathcal{R}^m_+.$$

Here we only need to give two versions related to Theorems 29.6 and 29.9. Others can be similarly obtained.

Theorem 29.10. Assume $\|T(t)\| \leq M$, all $t \in \mathcal{R}_+^m$, and α_n is a sequence of positive real numbers with $\lim_{n\to\infty} \inf \alpha_n > 0$. For each $n \in \mathcal{N}$ let $X(n)$ be a $\mathcal{R}_+^m$-valued random vector with $E[X(n)] = x/\alpha_n$. Assume $\lim_{n\to\infty} \sup \alpha_n \sigma_i^2(X(n)) < \infty$, $i = 1, ..., m$. Then there holds

$$\left\| \{E[T(\frac{\alpha_n}{n} X(n))]\}^n f - T(x)f \right\|$$
$$\leq 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})(1 + \frac{9}{4} mM\alpha_n \sum_{i=1}^{m} \sigma_i^2(X(n))[. \tag{29.28}$$

Proof. For each fixed n, let $X_k \overset{i.i.d.}{\sim} X(n),\ k = 1, ..., n$, and consider

$$Y := \frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k.$$

Then

$$E(Y) = E(\frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k) = \alpha_n E(X(n)) = \alpha_n x/\alpha_n = x$$

and

$$E[T(Y)]f = E[T(\frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k)]f = \{E[T(\frac{\alpha_n}{n} X(n))]\}^n f.$$

Furthermore

$$\sigma_i^2(Y) = \sigma_i^2(\frac{1}{n} \sum_{k=1}^{n} \alpha_n X_k) = \frac{\alpha_n^2}{n^2} n\sigma_i^2(X(n)) = \frac{\alpha_n^2}{n} \sigma_i^2(X(n)).$$

Now take $h = (\alpha_n/n)^{1/2}$, then by (29.10), we get (29.28). ∎

Theorem 29.11. Let α_n be a sequence of positive real numbers, satisfying

$$\lim_{n\to\infty} \alpha_n/n = 0 \text{ and } \lim_{n\to\infty} \inf \alpha_n > 0.$$

For each $n \in \mathcal{N}$, let $N(n) := (N_1(n), ..., N_m(n))$ be a $\mathcal{Z}_+^m$-valued random vector with $E(N(n)) = (1/\alpha_n)\eta = (1/\alpha_n)(\eta_1, ..., \eta_m)$ and

$$\lim_{n\to\infty} \sup \alpha_n \sigma_i^2(N(n)) < \infty,\ \text{all } i = 1, ..., m.$$

For each i $(1 \leq i \leq m)$, $\{Y_{ki}(n)\}_{k=1}^{\infty}$ is a sequence of *i.i.d.* random variables, distributed as $Y(n)$, where $Y(n)$ is a fixed real-valued random variable for each $n \in \mathcal{N}$ and $E(Y(n)) = \alpha_n \gamma$. Assume that $Y_{0i}(n)$ $(i = 1, ..., m),\ N(n)$ are altogether independent. Suppose also that

$$\lim_{n\to\infty} \sup \sigma^2(Y(n))/\alpha_n^2 < \infty.$$

Consider the equibounded operator semigroup $\{T(t); t \in \mathcal{R}_+^m\}$ with

$$\|T(t)\| \leq M,\ \text{all } t \in \mathcal{R}_+^m.$$

Then there holds

$$\left\| \{E[T(\sum_{k_1=1}^{N_1(n)} \frac{1}{n} Y_{k_1 1}(n), ..., \sum_{k_m=1}^{N_m(n)} \frac{1}{n} Y_{k_m m}(n))]\}^n f - T(\gamma\eta) f \right\| \tag{29.29}$$
$$\leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2})\{1 + \frac{9}{4} mM \sum_{i=1}^{m} [\frac{\eta_i}{\sigma_n^2} \sigma^2(Y(n)) + \gamma^2 \alpha_n \sigma_i^2(N(n))]\}.$$

Proof. We want to apply Lemma 29.5. Let

$$X_k \overset{i.i.d.}{\sim} X := (\sum_{K_1=1}^{N_1(n)} Y_{k_1 1}(n), ..., \sum_{k_m=1}^{N_m(n)} Y_{k_m m}(n))$$

and

$$Y = \frac{1}{n} \sum_{k=1}^{n} X_k.$$

Then

$$E(Y) = E(X) = (E(\sum_{k_1=1}^{N_i(n)} Y_{k_1 1}(n)), ..., E(\sum_{k_m=1}^{N_m(n)} Y_{k_m m}(n)))$$
$$= (\frac{1}{\alpha_n} \eta_1 \alpha_n \gamma, ..., \frac{1}{\alpha_n} \eta_m \alpha_n \gamma) = (\eta_1 \gamma_1, ..., \eta_m \gamma) = \gamma\eta.$$

And

$$E[T(Y)]f = \{E[T(\sum_{k_1=1}^{N_1(n)} \frac{1}{n} Y_{k_1 1}(n), ..., \sum_{k_m=1}^{N_m(n)} \frac{1}{n} Y_{k_m m}(n))]\}^n f.$$

Furthermore

$$\alpha_i^2(Y) = \sigma_i^2(\frac{1}{n} \sum_{k=1}^{n} X_k) = \frac{1}{n} \sigma^2(\sum_{k_i=1}^{N_i(n)} Y_{k_i i}(n))$$
$$= \frac{1}{n}[E(N_i(n))\sigma^2(Y_{0i}(n)) + \sigma^2(N_i(n))(E(Y_{0i}(n)))^2]$$
$$= \frac{1}{n}[\frac{1}{\alpha_n} \eta_i \sigma^2(Y(n)) + \sigma^2(N_i(n)) \alpha_n^2 \gamma^2].$$

Choose

$$h := (\alpha_n/n)^{1/2}$$

then by (29.10) of Lemma 29.5 we obtain

$$\left\| \{E[T(\sum_{k_1=1}^{N_i(n)} \frac{1}{n} Y_{k_1 1}(n), ..., \sum_{k_m=1}^{N_m(n)} \frac{1}{n} Y_{k_m m}(n))]\}^n f - T(\gamma\eta) f \right\|$$
$$\leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2})\{1 + \frac{9mM}{4} \frac{n}{\alpha_n} \sum_{i=1}^{m} \frac{1}{n}[\frac{1}{\alpha_n}\eta_i \sigma^2(Y_{0i}(n))$$
$$+ \sigma^2(N_i(n))\alpha_n^2 \gamma^2]\}$$
$$\leq 2M\omega_2(Tf, (\alpha_n/n)^{1/2})\{1 + \frac{9}{4} mM \sum_{i=1}^{m} [\frac{\eta_i}{\alpha_n^2} \sigma^2(Y(n))$$
$$+ \gamma^2 \alpha_n \sigma_i^2(N(n))]\}.$$

■

29.6 Applications

In this section we specify the random vectors (variables) and α_n of Theorems 29.6-29.11 to derive some concrete representation formulae for (C_0) m-parameter operator semigroups. We also illustrate how to get the results on multivariate approximation operators from the corresponding ones on operator semigroups. Unless otherwise mentioned all (C_0) m-parameter operator semigroups considered satisfy (29.5).

Example 29.12. Take $X = (X_{01}, ..., X_{0m})$ that follows the multi-point distribution $EX = x = (x_1, ..., x_m)$:

$$P(X = e_i) = x_i \qquad (e_i = (0, ..., 1, ..., 0))$$

and

$$P(X = 0) = 1 - \bar{x}, \text{ where } 0 < \bar{x} < 1 \ (\bar{x} = x_1 + ... + x_m).$$

Then

$$\Psi_{\bar{X}}^*(\delta) = E(e^{\delta \bar{X}}) = P(\bar{X} = 0) + P(\bar{X} = 1)e^{\delta} = 1 - \bar{x} + \bar{x}e^{\delta} < \infty.$$

Furthermore we have

$$E[T(X/n)] = 1 + \sum_{i=1}^{m} x_i (T_i(1/n) - I).$$

Hence by Theorem 29.6 there is a constant $K = K(\omega, M, x, \delta, m)$ such that

$$\left\| (I + \sum_{i=1}^{m} x_i (T_i(1/n) - I))^n f - T(x) f \right\|$$
$$\leq K\omega_2(Tf, 1/\sqrt{n}) \to 0 \quad (n \to \infty). \tag{29.30}$$

From the above result on operator semigroups we are able recover the approximation theorem for multivariate Bernstein operators as follows. Choose

$$\begin{aligned}\mathcal{X} &:= BUC(\mathcal{R}^m)\\ &:= \{f;\ \ f \text{ is a bounded uniformly continuous function from } \mathcal{R}^m \text{ into } \mathcal{R}\}\end{aligned}$$

and define

$$T(t)f(x) := f(x+t) = f(x_1+t_1, ..., x_m+t_m)$$

for each $f \in \mathcal{X}$ and $x \in \mathcal{R}^m$, then $\{T(t); t \in \mathcal{R}_+^m\}$ is a (C_0) m-parameter operator semigroup in $\mathcal{E}(\mathcal{X})$.

Now let $x = 0$, $t = (t_1, ..., t_m)$, $0 < \bar{t} < 1,\ 0 < t_i < 1,\ i = 1, ..., m$. Then

$$\begin{aligned}&\{I + \sum_{i=1}^{m} t_i[T_i(1/n) - I]\}^n f(0)\\ &= \sum_{k \in \mathcal{Z}_+^m, \bar{k} \le n} f(k_1/n, ..., k_m/n) \frac{n!}{k_1!...k_m(n-k_1-...-k_m)!}\\ &\quad \times t_1^{k_1}...t_m^{k_m}(1-\bar{t})^{n-\bar{k}}\\ &= B_n^f(t_1, ..., t_m),\end{aligned}$$

where $B_n^f(t_1, ..., t_m)$ is m-variate Bernstein operator over a simplex (cf. [216]). So by (29.30), we derive that

$$\lim_{n\to\infty} B_n^f(t_1, ..., t_m) = T(t)f(0) = f(t_1, ..., t_m), \quad \text{uniformly.}$$

Remark 29.13. The fact that the approximation theorem for Bernstein operator can be derived from simple operator semigroup consideration has been observed by many authors, see, e.g., [122, p.28], [207] and [244]. When consider other representation formulae for m-parameter operator semigroups in the following examples we may derive other known convergence theorems for multivariate approximation operators, but we avoid to go into detail here.

Example 29.14. Let α_n be a sequence of positive real numbers with $\lim_{n\to\infty} \inf \alpha_n > 0$ and $\lim_{n\to\infty} \alpha_n/n = 0$. For each $n \in \mathcal{N}$ take $X(n) = (X_{01}(n), ..., X_{0m}(n))$ to be modified multi-point distribution:

$$\begin{aligned}&P(X(n) = e_i) = x_i/\alpha_n,\ 1 \le i \le m,\\ &P(X(n) = 0) = 1 - \overline{x}/\alpha_n,\ (0 < \bar{x}/\alpha_n < 1 \text{ and } x_i > 0).\end{aligned}$$

Then

$$E[X(n)] = \frac{x}{\alpha_n},\ (x = (x_1, ..., x_m))$$

and

$$\sigma_i^2(X(n)) = E(X_i^2(n)) - (E(X_i(n)))^2 = \frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2}.$$

For equibounded (C_0) m-parameter operator semigroup $\{T(t);\ t \in \mathcal{R}_+^m\}$ with $\|T(t)\| \le M$, all $t \in \mathcal{R}_+^m$, we have

$$\begin{aligned} E[T(\frac{\alpha_n}{n}X(n))] &= T(0)P(X(n)=0) + \sum_{i=1}^m T(\frac{\alpha_n}{n}e_i)P(X(n)=e_i) \\ &= I + \sum_{i=1}^m \frac{x_i}{\alpha_n}(T_i(\frac{\alpha_n}{n}) - I). \end{aligned}$$

Thus by Theorem 29.10, we derive

$$\begin{aligned} &\left\| \{I + \sum_{i=1}^m \frac{x_i}{\alpha_n}(T_i(\frac{\alpha_n}{n}) - I)\}^n f - T(x)f \right\| \\ &\le 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})[1 + \frac{9}{4}mM\alpha_n \sum_{i=1}^m (\frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2})] \qquad (29.31) \\ &= 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})[1 + \frac{9}{4}mM \sum_{i=1}^m (x_i - \frac{x_i^2}{\alpha_n})] \to 0 \quad (n \to \infty). \end{aligned}$$

Remark 29.15. (29.30) is the special case of (29.31) when $\alpha_n \equiv 1$, but (29.30) is true for arbitrary (C_0) m-parameter operator semigroup.

Inequalities (29.31) and the following (29.32)-(29.34) are Shaw's formulae [260], [261] supplied with rates of convergence.

Example 29.16. Assume α_n as in Example 29.14. For each n, $(X(n) := (X_{01}(n), ..., X_{0m}(n))$ follows the negative multi-point distribution:

$$P(X(n) = (k_1, ..., k_m)) = \binom{\bar{k}}{k}(1 + \frac{1}{\alpha_n}\bar{x})^{-1} \prod_{i=1}^m (\frac{x_i}{\alpha_n + \overline{x}})^{k_i},$$

for all $k = (k_1, ..., k_m) \in \mathcal{Z}_+^m = \{(n_1, ..., n_m),\ n_i \in \mathcal{Z}_+,\ 1 \le i \le m\}$, where $x = (x_1, ...x_m) \in \mathcal{R}_+^m$, fixed; and

$$\binom{n}{k} = \frac{n(n-1)...(n-\overline{k}+1)}{k_1!...k_m!}.$$

Then

$$P(X_{0i}(n) = k_i) = (1 + \frac{1}{\alpha_n}x_i)^{-1}(\frac{x_i}{\alpha_n + x_i})^{k_i},$$

(see [159, p.165 (8.4)])

$$E(X_{0i}(n)) = \sum_{k_i=0} k_i(1 + \frac{1}{\alpha_n}x_i)^{-1}(\frac{x_i}{\alpha_n + x_i})^{k_i} = x_i/\alpha_n$$

and

$$\sigma_i^2(X(n)) = \frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n}.$$

For equibounded (C_0) m-parameter operator semigroup $\{T(t);\ t \in \mathcal{R}_+^m\}$ with $\|T(t)\| \leq M$, all $t \in \mathcal{R}_+^m$, we have

$$\begin{aligned}
&E[T(\frac{\alpha_n}{n}X(n))] \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} T(\frac{\alpha_n}{n}(k_1, ..., k_m))\binom{\bar{k}}{k}(1+\frac{1}{\alpha_n}\bar{x})^{-1}\prod_{i=1}^{m}(\frac{x_i}{\alpha_n+\bar{x}})^{k_i} \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \binom{\bar{k}}{k}(1+\frac{1}{\alpha_n}\bar{x})^{-1}\prod_{i=1}^{m}(\frac{x_iT_i(\alpha_n/n)}{\alpha_n+\bar{x}})^{k_i} \\
&= (1+\frac{1}{\alpha_n}\bar{x})^{-1}[I - \frac{x_1T_1(\alpha_n/n)+...+x_mT_m(\alpha_n/n)}{\alpha_n+\bar{x}}]^{-1} \\
&= \{I + \frac{I}{\alpha_n}\bar{x} - \frac{x_1}{\alpha_n}T_1(\alpha_n/n) - \cdots - \frac{x_m}{\alpha_n}T_m(\alpha_n/n)\}^{-1} \\
&= \{I - \sum_{i=1}^{m}\frac{x_i}{\alpha_n}(T_i(\alpha_n/n) - I)\}^{-1}.
\end{aligned}$$

By Theorem 29.10 we get

$$\begin{aligned}
&\left\|\{I - \sum_{i=1}^{m}\frac{x_i}{\alpha_n}(T_i(\alpha_n/n) - I)\}^{-n}f - T(x)f\right\| \\
&\leq 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})[1 + \frac{9}{4}mM\alpha_n\sum_{i=1}^{m}(\frac{x_i}{\alpha_n} + \frac{x_i^2}{\alpha_n^2})] \qquad (29.32) \\
&= 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})[1 + \frac{9}{4}mM\sum_{i=1}^{m}(x_i + \frac{x_i^2}{\alpha_n})] \to 0 \quad (n \to \infty).
\end{aligned}$$

Example 29.17. In Theorem 29.11 take $N(n)$ that follows the multi-point distribution:

$$P(N(n) = e_i) = x_i/\alpha_{n:}\ 1 \leq i \leq m,$$
$$P(N(n) = 0) = 1 - \bar{x}/\alpha_n,\ \text{where } x = (x_1, ..., x_m) \in \mathcal{R}_+^m,\ \text{fixed.}$$

Here α_n is as in Example 29.14. Let $Y_{0i}(n)$, $1 \leq i \leq m$, be exponentially distributed with density $\frac{1}{\alpha_n}e^{-\nu/\alpha_n}$, $\nu \in \mathcal{R}_+$. Then

$$E(N(n)) = \frac{1}{\alpha_n}x = (\frac{1}{\alpha_n}x_1, ..., \frac{1}{\alpha_n}x_m),\ E(Y_{0i}(n)) = \alpha_n,$$
$$\sigma^2(N_i(n)) = \frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2} \text{ and } \sigma^2(Y_{0i}(n)) = \alpha_n^2.$$

Also

$$E[T_i(\frac{1}{n}Y_{0i}(n))] = \int_0^\infty T_i(v/n)\frac{1}{\alpha_n}e^{-v/\alpha_n}dv = (I - \frac{\alpha_n}{n}A_i)^{-1}$$

(cf.[188, p.360] and [261, p.226, -3 lines]).

Furthermore we derive

$$\begin{aligned}
&E[T(\sum_{k_1=1}^{N_1(n)}\frac{1}{n}Y_{k_1 l}(n), ..., \sum_{k_m=1}^{N_m(n)}\frac{1}{n}Y_{k_m m}(n))] \\
&= T(0)P(N(n) = 0) + \sum_{i=1}^{m} E[T_i(\frac{1}{n}Y_{k_i}(n))]P(N(n) = e_i) \\
&= I + \sum_{i=1}^{m}\frac{x_i}{\alpha_n}[(I - \frac{\alpha_n}{n}A_i)^{-1} - I].
\end{aligned}$$

By (29.29) of Theorem 29.11 for equibounded (C_0) m-parameter operator semigroup $\{T(t); t \in \mathcal{R}_+^m\}$ with $\|T(t)\| \le M$, all $t \in \mathcal{R}_+^m$, we obtain

$$\begin{aligned}
&\left\|\{I + \sum_{i=1}^{m}(x_i/\alpha_n)[(I - \frac{\alpha_n}{n}A_i)^{-1} - I]\}^n f - T(x)f\right\| \\
&\le 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})\{1 + \frac{9}{4}mM\sum_{i=1}^{m}\left[\frac{x_i}{\alpha_n^2}\alpha_n^2 + \alpha_n\left(\frac{x_i}{\alpha_n} - \frac{x_i^2}{\alpha_n^2}\right)\right]\} \quad (29.33) \\
&= 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})[1 + \frac{9}{4}mM\sum_{i=1}^{m}(2x_i - x_i^2/\alpha_n)] \to 0 \quad (n \to \infty).
\end{aligned}$$

Example 29.18. Take α_n, $Y_{0i}(n)$ as in Example 29.17. Let $N(n)$ be the negative multi-point distribution:

$$P(N(n) = (l_1, ..., l_m)) = \binom{\bar{l}}{l}(1 + \frac{1}{\alpha_n}\bar{x})^{-1}\prod_{i=1}^{m}(\frac{x_i}{\alpha_n + \bar{x}})^{l_i},$$

for all $l = (l_1, ..., l_m) \in \mathcal{Z}_+^m$, where $x = (x_1, ..., x_m) \in \mathcal{R}_+^m$, fixed. Then

$$\begin{aligned}
E(N(n)) &= \frac{1}{\alpha_n}x = (\frac{1}{\alpha_n}x_1, ..., \frac{1}{\alpha_n}x_m),\ E(Y_{0i}(n)) = \alpha_n, \\
\sigma^2(N_i(n)) &= \frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n} \text{ and } \sigma^2(Y_{0i}(n)) = \alpha_n^2.
\end{aligned}$$

Furthermore we notice that

$$
\begin{aligned}
&E[T(\sum_{k_1=1}^{N_1(n)} \frac{1}{n} Y_{k_1 l}(n), ..., \sum_{k_m=1}^{N_m(n)} \frac{1}{n} Y_{k_m m}(n))] \\
&= \sum_{l \in \mathcal{Z}_+^m}^{m} P(N(n) = l) E[T(\sum_{k_1=1}^{l_1} \frac{1}{n} Y_{k_1 1}(n)), ..., T(\sum_{k_m=1}^{l_m} \frac{1}{n} Y_{k_m m}(n))] \\
&= \sum_{l \in \mathcal{Z}_+^m} \binom{\bar{l}}{l} (1 + \frac{1}{\alpha_n} \bar{x})^{-1} \prod_{i=1}^{m} (\frac{x_i E[T_i(Y/n)]}{\alpha_n + \bar{x}})^{l_i} \\
&= \{I + \frac{1}{\alpha_n} \bar{x} I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} E[T_i(Y/n)]\}^{-1} \\
&= \{I + \frac{1}{\alpha_n} \bar{x} I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} (I - \frac{\alpha_n}{n} A_i)^{-1}\}^{-1} \\
&= \{I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} [(I - \frac{\alpha_n}{n} A_i)^{-1} - I]\}^{-1}.
\end{aligned}
$$

Thus by (29.29) of Theorem 29.11, for equibounded (C_0) m-parameter operator semigroups $\{T(t);\, t \in \mathcal{R}_+^m\}$ with $\|T(t)\| \leq M$, all $t \in \mathcal{R}_+^m$, there holds

$$
\begin{aligned}
&\left\| \{I - \sum_{i=1}^{m} \frac{x_i}{\alpha_n} [(I - \frac{\alpha_n}{n} A_i)^{-1} - I]\}^{-n} f - T(x) f \right\| \\
&\leq 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})\{1 + \frac{9}{4} mM \sum_{i=1}^{m} [\frac{x_i}{\alpha_n^2} \alpha_n^2 + \alpha_n (\frac{x_i^2}{\alpha_n^2} + \frac{x_i}{\alpha_n})]\} \qquad (29.34) \\
&= 2M\omega_2(Tf,\ (\alpha_n/n)^{1/2})[1 + \frac{9}{4} mM \sum_{i=1}^{m} (2x_i + \frac{x_i^2}{\alpha_n})] \to 0 \quad (n \to \infty).
\end{aligned}
$$

Example 29.19. Take N to be the non-negative integer-valued random variable that follows the geometric distribution over $\mathcal{Z}_+$:

$$
P(N = k) = \frac{1}{1+\eta} (\frac{\eta}{1+\eta})^k, \text{ for all } k \in \mathcal{Z}_+, \text{ where } \eta > 0 \text{ is a parameter.}
$$

Let also $Y \equiv (x_1, ..., x_m) \in \mathcal{R}_+^m$. Then

$$
E(N) = \eta > 0,\ EY \equiv (x_1, ..., x_m) = x.
$$

Furthermore,

$$
\Psi_N(\Psi_{\bar{X}}^*(\delta)) = E(e^{\delta(x_1 + ... + x_m)N}) = \frac{1}{1 + \eta - e^{\delta(x_1 + ... + x_m)} \eta} < \infty,
$$

for $\delta < \frac{1}{\bar{x}} \ln(1 + 1/\eta)$.

So by Theorem 29.7, there is a constant $K = K(M, \omega, \delta, \eta, x)$ such that for sufficiently large n

$$\begin{aligned}
&\|\{\Psi_N(E[T(Y/n)])\}^n f - T(\eta x)f\| \\
&\left\|\{I + \eta[I - T(x_1/n, ..., x_m/n)]\}^{-n} f - T(\eta x)f\right\| \\
&\leq K\omega_2(Tf,\ 1\sqrt{n}) \to 0\ (n \to \infty).
\end{aligned} \tag{29.35}$$

Example 29.20. In Theorem 29.8, take $p = q = 2$, and N_τ to be the Poisson process $(\tau \in \mathcal{R}_+)$:

$$P(N_\tau = k) = e^{-\eta\tau}\frac{(\eta\tau)^k}{k!},\ \text{for all}\ k \in \mathcal{Z}_+^m,\ \text{where}\ \eta > 0\ \text{is a parameter.}$$

Consider $X \equiv (x_1, ..., x_m) = x \in \mathcal{R}_+^m$. Then

$$E(N_\eta) = \eta\tau,\ \ \Psi_{N_\tau}(s) = e^{(s-1)\eta\tau},\ \text{and}\ \Psi_{\bar{X}}^*(\delta) = e^{\delta\bar{x}} < \infty.$$

Furthermore note that

$$\begin{aligned}
d_1 &= \lim_{\tau\to\infty} \sup \Psi_{N_\tau}(\Psi_{\bar{X}}^*(\frac{2\omega}{\tau})) = \lim_{\tau\to\infty} \sup \exp[(e^{\frac{2\omega}{\tau}\bar{x}} - 1)\eta\tau] \\
&\leq \lim_{\tau\to\infty} \sup \exp[\eta\tau\frac{2\omega}{\tau}\bar{x}e^{\frac{2\omega}{\tau}\bar{x}}] < \infty,
\end{aligned}$$

$$d_2 = \lim_{\tau\to\infty} \sup \tau\{E[(\frac{1}{\tau}N_\tau - \eta)^4]\}^{1/2} = \lim_{\tau\to\infty} \sup \tau(3\frac{\eta^2}{\tau^2} + \frac{\eta}{\tau^3})^{1/2} < \infty$$

and

$$\begin{aligned}
d_3 &= \lim_{\tau\to\infty} \sup \Psi_{N_\tau}^*(\frac{2}{e^2(\sqrt{\tau}\delta - 1)^2}e^{\delta\bar{x}}\Psi_{\bar{X}}^*(\delta)) \\
&= \lim_{\tau\to\infty} \sup \exp\{(\exp[\frac{2}{e^2(\sqrt{\tau}\delta - 1)^2}e^{2\delta\bar{x}}] - 1)\eta\tau\} \\
&\leq \lim_{\tau\to\infty} \sup \exp\{\frac{2}{e^2(\sqrt{\tau}\delta - 1)^2}e^{2\delta\bar{x}} \exp[\frac{2}{e^2(\sqrt{\tau}\delta - 1)^2}e^{2\delta\bar{x}}]\eta\tau\} < \infty.
\end{aligned}$$

So by (29.24) of Theorem 29.8 there exists a constant $K = K(\delta, M, \omega, d_1, d_2, d_3)$ such that for sufficiently large n we have

$$\begin{aligned}
&\|\Psi_{N_\tau}(E(T(X/\tau)))f - T(\eta x)f\| \\
&= \|\exp[\eta\tau(T(x_1/\tau, ..., x_m/\tau) - I)]f - T(\eta x)f\| \\
&\leq K\omega_2(Tf,\ 1\sqrt{\tau}) \to 0 \qquad (\tau \to \infty).
\end{aligned} \tag{29.36}$$

30

Simultaneous Approximation Using the Feller Probabilistic Operator

In this chapter a quantitative estimate for the simultaneous approximation of a function and its derivatives by the Feller probabilistic operator is given using probabilistic approach. This covers the cases of some classical approximation operators such as the Bernstein, Szász, Baskakov and Gamma operator. This chapter relies on [91].

30.1 Basics

For a sequence of i.i.d. non-negative r.v.'s X_1, $X_2, ...$, with $E[X_1] = x$, the Feller operator (cf. [158, p.218],) is defined by

$$F_n(f, x) := E\left[f(\frac{S_n}{n})\right] := \int_0^{\infty} f(t)dP(\frac{1}{n}S_n \leq t), \tag{30.1}$$

where $S_n = \sum_{i=1}^{n} X_i$, $P\left(\frac{1}{n}S_n \leq t\right)$ is the distribution function of $\frac{1}{n}S_n$ and f is a continuous function.

The Feller operator F_n contains some well-known classical operators such as Bernstein, Szász, Baskakov and Gamma operator as special cases, and has been studied by many authors about various approximation properties (see, e.g. [182], [204], [205], [272], [285], and their citations.)

The purpose here is to investigate the simultaneous approximation of a function and its derivatives by the Feller operator. A quantitative estimate is obtained by using probabilistic methods. The general setting allows us

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 469–485.
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to prove results similar to already known ones for the above mentioned specific classical operators, as well as to produce another different related result. This is demonstrated in the last Section 30.4.

30.2 The Main Result

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X(x)$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$ with $E[X(x)] = x \geq 0$. The variance and the moment generating function of $X(x)$ will be denoted by $\sigma^2(x) := E[(X(x) - x)^2]$ and $\Psi^*_{X(x)}(t) := E[\exp(tX(x))]$, respectively. For each fixed x, let $X_n \equiv X_n(x)$, $n = 1, 2, ...$, be a sequence of independent r.v.'s identically distributed as $X(x)$. Define the corresponding Feller operator $F_n(f, x)$ as in (30.1). Denote by $\mathbb{Z}_+$ the set of all non-negative integers, $D := \frac{\partial}{\partial_x}$ and $\omega_A(f, \delta)$ the first modulus of continuity of function f in the interval $[0, \; A]$:

$$\omega_A(f, \delta) := \sup\{|f(u) - f(v)|; \quad u, v \in [0, A], \; |u - v| \leq \delta\}.$$

The main result follows.

Theorem 30.1. Let $r \in \mathcal{Z}_+$ and $A > 0$ be fixed. Suppose for each fixed t that

$$P(X(x) > t) \in C^r((0, \; A)) \tag{30.2}$$

and there exist two positive constants M and α such that

$$\left| D^k P(X(x) > t) \right| \leq M e^{-\alpha t} \tag{30.3}$$

uniformly for all $0 \leq k \leq r$ and $0 < x < A$.

Suppose further that $f \in C^r([0, \infty))$ and

$$\left| f^{(k)}(t) \right| \leq K e^{\beta t}, \; 0 \leq k \leq r, \tag{30.4}$$

for some constants β and $K > 0$.

Then for each $x \in (0, \; A)$ there holds for large n that

$$\left| D^r F_n(f, \; x) - f^{(r)}(x) \right| \leq \frac{2M^r}{\alpha^r}(1 + \sigma(x))\omega_A(f^{(r)}, \frac{1}{\sqrt{n}})$$
$$+ \{r! K e^{\beta A} \sum_{k=1}^{r-1} k M^k (\frac{2}{\alpha})^{k+1} + \frac{r(r-1)}{2} K e^{\beta A} + 1\} \frac{1}{n} \tag{30.5}$$

Comment. By assumption (30.3) it is easy to see that $\Psi^*_{X(x)}(\alpha/2) < \infty$ and consequently $\sigma(x) < \infty$. So the l.h.s of (30.5) is finite and tends to 0 as $n \to \infty$.

30.3 Proof of Theorem 30.1

To prove the Theorem 30.1 we need the following two lemmas.

Lemma 30.2. Under the hypotheses of Theorem 30.1, there holds for large n that

$$D^r E[f(\frac{S_n(x)}{n})]$$
$$= \sum_{k=0}^{r} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n}) a_{rk} \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j), \tag{30.6}$$

where $(n)_k := n...(n-k+1)$ and $a_{rk} \equiv a_{rk}(x, t_1, ..., t_k,\ X_1, ..., X_k)$'s satisfy the recurrence relation

$$a_{rk} = Da_{r-1,k} + a_{r-1,k-1} DP(X_k(x) > t_k),\ (r > 0) \tag{30.7}$$

with initial condition $a_{00} = 1$ and the conventions $a_{k0} = 0$ for $k > 0$ and $a_{k,-1} = a_{k,k+1} = 0$ for $k \ge 0$.

Furthermore we have,

$$\underbrace{\int_0^\infty ... \int_0^\infty}_{k} a_{rk} \prod_{i=1}^{k} dt_i = \begin{cases} 1, & \text{if } k = r, \\ 0, & \text{if } 0 < k < r, \end{cases} \tag{30.8}$$

and

$$|a_{rk}| \le \begin{cases} M^r e^{-\alpha(t_1+...+t_r)}, & k = r, \\ r! M^k e^{-\alpha(t_1+...+t_k)}, & 0 < k < r. \end{cases} \tag{30.9}$$

Proof. Consider the general function $a(y) \equiv a(y, t_1, ..., t_k)$ satisfying $a(y) \in C^1((0,\ A))$ and

$$\left| D^i a(y) \right| \le M e^{-\alpha(t_1+...+t_k)} \tag{30.10}$$

for $i = 0, 1$ and for all $y \in (0,\ A)$. If $k < r$ put

$$I(y) := \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n}) a(y) \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(y) \le t_j)$$

then

$$DI(x) = \lim_{y\to x} \frac{1}{y-x}(I(y) - I(x))$$

$$= \lim_{y\to x} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n}) \frac{a(y) - a(x)}{y - x} \prod_{i=1}^k dt_i \prod_{j=k+1}^n dP(X_j(y) \le t_j)$$

$$+ \lim_{y\to x} \frac{1}{y-x} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n}) a(x) \prod_{i=1}^k dt_i$$
$$\times \left[\prod_{j=k+1}^n dP(X_j(y) \le t_j) - \prod_{j=k+1}^n dP(X_j(x) \le t_j)\right]$$
$$=: D_1 + D_2. \tag{30.11}$$

We see that when $n > \frac{2\beta}{\alpha}$

$$\left| f^{(k)}(\frac{t_1 + ... + t_n}{n}) Da(x) \right| \le K e^{\frac{\beta}{n}(t_1+...+t_n)} M e^{-\alpha(t_1+...+t_k)}$$
$$\le MK e^{-\frac{\alpha}{2}(t_1+...+t_k)} e^{\frac{\alpha}{2}(t_{k+1}+...+t_n)},$$

and

$$\lim_{y\to x} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} e^{-\frac{\alpha}{2}(t_1+...+t_k)} e^{\frac{\alpha}{2}(t_{k+1}+...+t_n)} \prod_{i=1}^k dt_i \prod_{j=k+1}^n dP(X_j(y) \le t_j)$$
$$= (\frac{2}{\alpha})^k (\Psi^*_{X(x)}(\frac{\alpha}{2}))^{n-k} < \infty,$$

by Lebesgue convergence theorem and using the condition (30.3).

New Proposition 11.18 of [258, p.270] implies that

$$D_1 = \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n}) Da(x) \prod_{i=1}^k dt_i \prod_{j=k+1}^n dP(X_j(x) \le t_j). \tag{30.12}$$

Note 30.3. We encounter several integral operations such as changing integral with limit or changing integration orders. The conditions (30.2)-(30.4) and the exponential bounds of the integral functions will guarantee the validity of those operations, which can be taken care of similar to the above we will not go into further details each time.

Back to (30.11) we get

$$\begin{aligned} D_2 = & \sum_{j=k+1}^{n} \lim_{y\to x} \frac{1}{y-x} \underbrace{\int_0^\infty \dots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1+\dots+t_n}{n}\right) a(x) \prod_{i=1}^{k} dt_i \\ & \times \Big[\prod_{\mu=k+1}^{j} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu) \\ & - \prod_{\mu=k+1}^{j-1} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j}^{n} dP(X_\nu(x) \le t_\nu)\Big] \\ = & \sum_{j=k+1}^{n} \lim_{y\to x} \frac{1}{y-x} \underbrace{\int_0^\infty \dots \int_0^\infty}_{n-1} a(x) \\ & \times \int_0^\infty f^{(k)}\left(\frac{t_1+\dots+t_n}{n}\right) d[P(X_j(y) \le t_j) - P(X_j(x) \le t_j)] \\ & \times \prod_{i=1}^{k} dt_i \prod_{\mu=k+1}^{j-1} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu). \end{aligned} \tag{30.13}$$

Notice that

$$\begin{aligned} & \int_0^\infty f^{(k)}\left(\frac{t_1+\dots+t_n}{n}\right) d[P(X_j(y) \le t_j) - P(X_j(x) \le t_j)] \\ = & \int_0^\infty f^{(k)}\left(\frac{t_1+\dots+t_n}{n}\right) d[-P(X_j(y) > t_j) + P(X_j(x) > t_j)] \\ = & f^{(k)}\left(\frac{t_1+\dots+t_n}{n}\right)[-P(X_j(y) > t_j) + P(X_j(x) > t_j)] \Big|_{t_j=0^-}^{t_j=\infty} \\ & + \int_0^\infty \frac{1}{n} f^{(k+1)}\left(\frac{t_1+\dots+t_n}{n}\right)[P(X_j(y) > t_j) - P(X_j(x) > t_j)] dt_j \\ = & \int_0^\infty \frac{1}{n} f^{(k+1)}\left(\frac{t_1+\dots+t_n}{n}\right)[P(X_j(y) > t_j) - P(X_j(x) > t_j)] dt_j. \end{aligned} \tag{30.14}$$

Using (30.14) in (30.13) we obtain

$$D_2 = \sum_{j=k+1}^{n} \lim_{y \to x} \underbrace{\int_0^\infty \ldots \int_0^\infty}_{n} \frac{1}{n} f^{(k+1)}\left(\frac{t_1 + \ldots + t_n}{n}\right) a(x)$$
$$\times \frac{[P(X_j(y) > t_j) - P(X_j(x) > t_j)]}{y - x} dt_j$$
$$\times \prod_{i=1}^{k} dt_i \prod_{\mu=k+1}^{j-1} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu)$$
$$= \sum_{j=k+1}^{n} \frac{1}{n} \underbrace{\int_0^\infty \ldots \int_0^\infty}_{n} f^{(k+1)}\left(\frac{t_1 + \ldots + t_n}{n}\right) a(x) DP(X_j(x) > t_j) dt_j \prod_{i=1}^{k} dt_i$$
$$\times \prod_{\mu=k+1}^{j-1} dP(X_\mu(x) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu)$$
$$= \frac{n-k}{n} \underbrace{\int_0^\infty \ldots \int_0^\infty}_{n} f^{(k+1)}\left(\frac{t_1 + \ldots + t_n}{n}\right) a(x) DP(X_{k+1}(x) > t_{k+1})$$
$$\times \prod_{i=1}^{k+1} dt_i \prod_{j=k+2}^{n} dP(X_j(x) \le t_j), \tag{30.15}$$

the last step being true due to the fact that X_j's are identically distributed.

Combining (30.11), (30.12) and (30.15) we derive

$$D \underbrace{\int_0^\infty \ldots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \ldots + t_n}{n}\right) a(x) \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j)$$
$$= \underbrace{\int_0^\infty \ldots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \ldots + t_n}{n}\right) Da(x) \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j)$$
$$+ \frac{n-k}{n} \underbrace{\int_0^\infty \ldots \int_0^\infty}_{n} f^{(k+1)}\left(\frac{t_1 + \ldots + t_n}{n}\right) a(x) DP(X_{k+1}(x) > t_{k+1})$$
$$\times \prod_{i=1}^{k+1} dt_i \prod_{j=k+2}^{n} dP(X_j(x) \le t_j). \tag{30.16}$$

Now we are ready to start the proof of (30.6). It is easy to prove that

$$E[f(\frac{S_n(x)}{n})] = \underbrace{\int_0^\infty \ldots \int_0^\infty}_{n} \ f\left(\frac{t_1 + \ldots + t_n}{n}\right) \prod_{j=1}^{n} dP(X_j(x) \le t_j).$$

n

In (30.16) take $k = 0$, $a(x) = a_{00} \equiv 1$ we obtain

$$DE[f(\frac{S_n(x)}{n})]$$
$$= \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f'(\frac{t_1 + ... + t_n}{n}) DP(X_1(x) > t_1) dt_1 \prod_{j=2}^n dP(X_j(x) \le t_j).$$

So (30.6) (with (30.7)) is true for $r = 0, 1$.

Assume (30.6) is true for $r - 1$, i.e.

$$D^{r-1} E[f(\frac{S_n(x)}{n})]$$
$$= \sum_{k=0}^{r-1} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n}) a_{r-1,k} \prod_{i=1}^k dt_i \prod_{j=k+1}^n dP(X_j(x) \le t_j).$$

Substitute $a(x)$ in (30.16) for each $a_{r-1,k}$ we obtain

$$D^r E[f(\frac{S_n(x)}{n})] \tag{30.17}$$
$$= \sum_{k=0}^{r-1} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n}) D a_{r-1,k} \prod_{i=1}^k dt_i \prod_{j=k+1}^n dP(X_j(x) \le t_j)$$
$$+ \sum_{k=0}^{r-1} \frac{(n)_k}{n^k} \frac{n-k}{n} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{k+1)}(\frac{t_1 + ... + t_n}{n}) a_{r-1,k} DP(X_{k+1}(x) > t_{k+1})$$
$$\times \prod_{i=1}^{k+1} dt_i \prod_{j=k+2}^n dP(X_j(x) \le t_j)$$
$$= \sum_{k=0}^{r} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + ... + t_n}{n})[D a_{r-1,k} + a_{r-1,k-1} DP(X_k(x) > t_k)]$$
$$\times \prod_{i=1}^k dt_i \prod_{j=k+1}^n dP(X_j(x) \le t_j)$$

and (30.6) follows for r.

To prove (30.8) we need the following facts. Because

$$x = \int_0^\infty t dP(X(x) \le t) = \int_0^\infty t d(-P(X(x) > t),$$

integration by parts yields

$$x = \int_0^\infty P(X(x) > t) dt.$$

Thus

$$1 = \int_0^\infty DP(X(x) > t)dt \tag{30.18}$$

and

$$0 = \int_0^\infty D^i P(X(x) > t)dt \text{ for } 2 \leq i \leq r. \tag{30.19}$$

By introduction on r it is not difficult to show that a_{rk} is the sum of the terms of the form $D^{i_1}P(X_1(x) > t_1)...D^{i_k}P(X_k(x) > t_k)$ with $i_1 + ... + i_k = r > 0$. If $k < r$ then at least one of i_j 's is greater than 1, so by (30.19)

$$\underbrace{\int_0^\infty ... \int_0^\infty}_{k} D^{i_1}P(X_1(x) > t_1)...D^{i_k}P(X_k(x) > t_k) \prod_{i=1}^{k} dt_i$$

$$= \prod_{j=1}^{k} \int_0^\infty D^{i_j}P(X_j(x) > t_j)dt_i = 0. \tag{30.20}$$

If $k = r$ we notice that

$$a_{rr} = Da_{r-1,r} + a_{r-1,r-1}DP(X_r(x) > t_r) = a_{r-1,r-1}DP(X_r(x) > t_r) = \cdots$$
$$= DP(X_1(x) > t_1) \cdots DP(X_r(x) > t_r) \tag{30.21}$$

and so

$$\underbrace{\int_0^\infty ... \int_0^\infty}_{r} a_{rr} \prod_{i=1}^{r} dt_i = \prod_{i=1}^{r} \int_0^\infty DP(X_i(x) > t_i)dt_i = 1 \tag{30.22}$$

by (30.19). Now (30.20) and (30.22) prove (30.8).

Finally we come to (30.9). Denote d_{rk} the number of terms of the form $D^{i_1}P(X_1(x) > t)...(D^{i_k}P(X_k(x) > t_k)$ in a_{rk} when a_{rk} is decomposed as the sum of such terms.

Let

$$d_r = \max\{d_{rk};\ 0 \leq k \leq r\},$$

then $Da_{r-1,k}$ counts at most $kd_{r-1} \leq (r-1)d_{r-1}$ such terms, and $a_{r-1,k-1}DP$ $(X_k(x) > t_k)$ gives no more than d_{r-1} terms. By (30.7) we find that

$$d_r \leq (r-1)d_{r-1} + d_{r-1} = rd_{r-1}.$$

Note that $d_1 = 1$ we get

$$d_r \leq r!. \tag{30.23}$$

Moreover by condition (30.2) of Theorem 30.1

$$\left| D^{i_1}P(X_1(x) > t_1)...D^{i_k}P(X_k(x) > t_k) \right| \leq M^k e^{-\alpha(t_1+...+t_k)}.$$

Together we have for $k < r$ that

$$|a_{rk}| \leq r!M^k e^{-\alpha(t_1+...+t_k)}$$

and by (30.21) $|a_{rr}| \le M^r e^{-\alpha(t_1+\ldots+t_r)}$, therefore (30.9) holds. Q.E.D. ■

Lemma 30.4. Under the hypotheses of Theorem 30.1, there holds for large n that

$$\left| E[f(\frac{S_n(x)}{n})] - f(x) \right| \le (1+\sigma(x))\omega_A(f, \frac{1}{\sqrt{n}}) + 2Ke^{\beta A}(\rho_A(x))^n, \qquad (30.24)$$

where $(\rho_A(x))^2 = \inf_{t>0} E[e^{t(X(x)-A)}] < 1$.

Proof. We get

$$\begin{aligned} &\left| E[f(\frac{S_n(x)}{n})] - f(x) \right| \\ &\le \int_0^A |f(t)-f(x)|\, dP(\frac{1}{n}S_n(x) \le t) + \int_A^\infty |f(t)-f(x)|\, dP(\frac{1}{n}S_n(x) \le t) \\ &:= R_1 + R_2, \qquad (30.25) \end{aligned}$$

$$\begin{aligned} R_1 &\le \int_0^A \omega_A(f, |t-x|) dP(\frac{1}{n}S_n(x) \le t) \\ &\le \omega_A(f, \frac{1}{\sqrt{n}}) \int_0^A (1+\sqrt{n}\,|t-x|) dP(\frac{1}{n}S_n(x) \le t) \\ &\le \omega_A(f, \frac{1}{\sqrt{n}})(1+\sqrt{n}(E[(\frac{1}{n}S_n(x)-x)^2])^{1/2}) \\ &= (1+\sigma(x))\omega_A(f, \frac{1}{\sqrt{n}}) \qquad (30.26) \end{aligned}$$

and

$$\begin{aligned} R_2 &\le 2K \int_A^\infty e^{\beta t} dP(\frac{1}{n}S_n(x) \le t) \\ &\le 2K(E[e^{\frac{2\beta}{n}S_n(x)}])^{1/2}(P(\frac{1}{n}S_n(x) \ge A))^{1/2}. \end{aligned}$$

Furthermore, Theorem 1 of [132] (see also [205, Lemma 3]) leads to

$$P(\frac{1}{n}S_n(x) \ge A) \le (\rho_A(x))^{2n},$$

where $\rho_A(x)$ is as in (30.24). At the same time Theorem 3.1 of [244] implies that

$$E[e^{\frac{2\beta}{n}S_n(x)}] \le e^{2\beta x} \exp\{\frac{2(2\beta)^2 \Psi^*_{X(x)}(\alpha/2)}{e^2 n((\alpha/2)-2\beta/n)^2}\}, \quad (n > 4\beta/\alpha). \qquad (30.27)$$

Note here $\Psi^*_{X(x)}(\alpha/2) < \infty$ due to (30.2), and thus when

$n \geq \max\{64\beta\Psi^*_{X(x)}(\alpha/2)/e^2\alpha^2(A-x),\ 8\beta/\alpha\}$ we have

$$E[e^{\frac{2\beta}{n}S_n(x)}] \leq e^{2\beta A}. \tag{30.28}$$

Hence there holds for large n that

$$R_2 \leq 2Ke^{\beta A}(\rho_A(x))^n. \tag{30.29}$$

Now (30.24) follows from (30.25), (30.26) and (30.29) ■

The proof of Theorem 30.1

In the case of $r = 0$ inequality (30.5) can be easily derived from Lemma 30.4. We thus suppose $r \geq 1$ in the following.

By (30.6) of Lemma 30.2 there hold

$$\left|D^r F_n(f,\ x) - f^{(r)}(x)\right| = \left|D^r E[f(\frac{S_n(x)}{n})] - f^{(r)}(x)\right|$$

$$\leq \frac{(n)_r}{n^r}\left|\underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(r)}(\frac{t_1+...+t_n}{n})a_{rr}\prod\nolimits_{i=1}^r dt_i \prod\nolimits_{j=r+1}^n dP(X_j(x) \leq t_j) - f^{(r)}(x)\right|$$

$$+\left|\sum_{k=0}^{r-1}\frac{(n)_k}{n^k}\underbrace{\int_0^\infty ... \int_0^\infty}_{n} f^{(k)}(\frac{t_1+...t_n}{n})a_{rk}\prod\nolimits_{i=1}^k dt_i \prod\nolimits_{j=k+1}^n dP(X_j(x) \leq t_j)\right|$$

$$+\left|1-\frac{(n)_r}{n^r}\right|\left|f^{(r)}(x)\right|$$

$$:= I_1 + I_2 + I_3. \tag{30.30}$$

First we treat

$$I_1 = \frac{(n)_r}{n^r}\left|\underbrace{\int_0^\infty ... \int_0^\infty}_{r+1} f^{(r)}(\frac{t_1+...+t_r}{n}+\frac{n-r}{n}t)a_{rr}\prod\nolimits_{i=1}^r dt_i\right. \tag{30.31}$$

$$\left.\times dP(\frac{1}{n-r}S_{n-r}(x) \leq t) - f^{(r)}(x)\right|$$

$$\overset{(30.8)}{=} \frac{(n)_r}{n^r}\left|\underbrace{\int_0^\infty ... \int_0^\infty}_{r+1} (f^{(r)}(\frac{t_1+...t_r}{n}+\frac{n-r}{n}t) - f^{(r)}(x))a_{rr}\prod\nolimits_{i=1}^r dt_i\right.$$

$$\left.\times dP(\frac{1}{n-r}S_{n-r}(x) \leq 1)\right|$$

$$\overset{(30.9)}{\leq} \underbrace{\int_0^\infty ... \int_0^\infty}_{r+1} \left| f^{(r)}(\frac{t_1 + ...t_r}{n} + \frac{n-r}{n}t) - f^{(r)}(x) \right| M^r e^{-\alpha(t_1+...+t_r)} \prod_{i=1}^r dt_i$$
$$\times dP(\frac{1}{n-r}S_{n-r}(x) \leq t)$$
$$= \int_0^\infty \int_0^\infty \left| f^{(r)}(\frac{s}{n} + \frac{n-r}{n}t) - f^{(r)}(x) \right| \frac{1}{(r-1)!} M^r s^{r-1} e^{-\alpha s} ds$$
$$\times dP(\frac{1}{n-r}S_{n-r}(x) \leq t), \text{ (let } t_1 + ... + t_r = s \text{ and } t_i = t_i \text{ for } i < r)$$
$$= \int_0^{\frac{A+x}{2}} \int_0^{\frac{(A-x)n}{2}} . + \int_0^{\frac{A+x}{2}} \int_{\frac{(A-x)n}{2}}^\infty . + \int_{\frac{A+x}{2}}^\infty \int_0^\infty . =: I_{11} + I_{12} + I_{13}.$$

We obtain

$$I_{11} \leq \omega_A(f^{(r)}, \lambda) \int_0^\infty \int_0^\infty \{1 + \frac{1}{\lambda}(\frac{s}{n} + \frac{rt}{n} + |t-x|)\} \frac{1}{(r-1)!} M^r s^{r-1} e^{-\alpha s}$$
$$\times dsdP(\frac{1}{n-r}S_{n-r}(x) \leq t), \ (\lambda > 0)$$
$$= \frac{M^r}{\alpha^r}\omega_A(f^{(r)}, \lambda)\{1 + \frac{1}{\lambda}E[\left|\frac{1}{n-r}S_{n-r}(x) - x\right|] + \frac{rx}{\lambda n} + \frac{r}{\lambda na}\}$$
$$\leq \frac{M^r}{\alpha^r}\omega_A(f^{(r)}, \frac{1}{\sqrt{n}})\{1 + \frac{\sqrt{n}}{\sqrt{n-r}}\sigma(x) + \frac{rx}{\sqrt{n}} + \frac{r}{\alpha\sqrt{n}}\}, \ (\lambda = \frac{1}{\sqrt{n}})$$
$$\leq \frac{2M^r}{\alpha^r}(1 + \sigma(x))\omega_A(f^{(r)}, \frac{1}{\sqrt{n}}), \tag{30.32}$$

for $n \geq (2rx)^2 + (2r/\alpha)^2 + 4r/3$. Furthermore

$$I_{12} \leq \int_0^{\frac{A+x}{2}} \int_{\frac{(A-x)n}{2}} K(e^{\frac{\beta s}{n} + \beta t} + e^{\beta x}) \frac{1}{(r-1)!} M^r s^{r-1} e^{-\alpha s} dsdP(\frac{1}{n-r}S_{n-r}(x) \leq t)$$
$$\leq \frac{2}{(r-1)!} M^r K e^{\beta A} \int_{\frac{(A-x)n}{2}}^\infty s^{r-1} e^{-\frac{\alpha}{2}s} ds, \ (n > 2\beta/\alpha). \tag{30.33}$$

Using inequality (see (3.6) of [244])

$$s^{r-1} \leq (\frac{4(r-1)}{e\alpha})^{r-1} e^{\frac{\alpha}{4}s}, \ (s > 0)$$

it is straight forward to show that

$$I_{12} \leq C\rho_1^n,$$

where $C = 8M^r K e^{\beta A}(4(r-1)/e\alpha)^{r-1}/\alpha(r-1)!$ and $\rho_1 = e^{-\alpha(A-x)/8} < 1$. Thus when n is large enough there holds

$$I_{12} \leq \frac{1}{2n}. \tag{30.34}$$

Also it holds

$$\begin{aligned}
I_{13} &\leq \int_{\frac{A+x}{2}}^{0} \int_{0}^{\infty} 2Ke^{\frac{\beta s}{n}+\beta t} \frac{1}{(r-1)!} M^r s^{r-1} e^{-\alpha s} ds dP(\frac{1}{n-r} S_{n-r}(x) \leq t) \\
&\leq \frac{2}{(r-1)!} KM^r \int_{0}^{\infty} s^{r-1} e^{-\frac{\alpha}{2}s} ds \int_{\frac{A+x}{2}}^{\infty} e^{\beta t} dP(\frac{1}{n-r} S_{n-r}(x) \leq t), \quad (n > \frac{2\beta}{\alpha}) \\
&\leq \frac{2^{r+1} KM^r}{\alpha^r} (E[e^{\frac{2\beta}{n-r} S_{n-r}(x)}])^{\frac{1}{2}} (P(\frac{1}{n-r} S_{n-r}(x) \geq \frac{A+x}{2}))^{\frac{1}{2}} \\
&\leq \frac{2^{r+1} KM^r}{\alpha^r} e^{\beta A} (\rho_{\frac{A+x}{2}}(x))^{n-r},
\end{aligned}$$

when n is sufficiently large, which can be proved similarly as (30.29). Thus when n is large enough there holds

$$I_{13} \leq \frac{1}{2n} \tag{30.35}$$

In summary we have for large n that

$$I_1 \leq \frac{2M^r}{\alpha^r}(1+\sigma(x))\omega_A(f^{(r)}, \tfrac{1}{\sqrt{n}}) + \frac{1}{n}. \tag{30.36}$$

Next we are going to estimate I_2. By (30.9) of Lemma 30.2 we have for $k < r$ that

$$\begin{aligned}
&\left| \underbrace{\int_0^\infty \dots \int_0^\infty}_{n} f^{(k)}(\frac{t_1 + \dots + t_n}{n}) a_{rk} \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \leq t_j) \right| \\
&= \left| \underbrace{\int_0^\infty \dots \int_0^\infty}_{k+1} (f^{(k)}(\frac{t_1 + \dots + t_k}{n} + \frac{n-k}{n} t) - f^{(k)}(\frac{n-k}{n} t)) a_{rk} \right. \\
&\quad \left. \times \prod_{i=1}^{k} dt_i dP(\frac{1}{n-k} S_{n-k}(x) \leq t) \right| \\
&\leq \underbrace{\int_0^\infty \dots \int_0^\infty}_{k+1} \left| f^{(k)}(\frac{t_1 + \dots + t_k}{n} + \frac{n-k}{n} t) - f^{(k)}(\frac{n-k}{n} t) \right| r! M^k e^{-\alpha(t_1 + \dots + t_k)} \\
&\quad \times \prod_{i=1}^{k} dt_i dP(\frac{1}{n-k} S_{n-k}(x) \leq t)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \left| f^{(k)}(\frac{s}{n} + \frac{n-k}{n}t) - f^{(k)}(\frac{n-k}{n}t) \right| \frac{r!M^k}{(k-1)!} s^{k-1} e^{-\alpha s} \\
&\quad \times dsdP(\frac{1}{n-k} S_{n-k}(x) \le t) \\
&\le \int_0^\infty \int_0^\infty K e^{\beta \frac{s}{n} + \beta t} \frac{s}{n} \frac{r!}{(k-1)!} M^k s^{k-1} e^{-\alpha s} dsdP(\frac{1}{n-k} S_{n-k}(x) \le t) \\
&\quad \text{(by mean value theorem and (30.4))} \\
&\le \frac{1}{n} K M^k k r! (\frac{2}{\alpha})^{k+1} E[e^{\frac{\beta}{n-k} S_{n-k}(x)}] \le r! K e^{\beta A} k M^k (\frac{2}{\alpha})^{k+1} \frac{1}{n},
\end{aligned}
$$

when $n \ge (32\beta\Psi^*_{X(x)}(\frac{\alpha}{2})/\alpha^2 e^2 (A-x)) + (4\beta/\alpha) + r$, similarly shown as (30.28). Therefore when n large enough there holds

$$
I_2 \le r! K e^{\beta A} \sum_{k=1}^{r-1} k M^k (\frac{2}{\alpha})^{k+1} \frac{1}{n}. \tag{30.37}
$$

Finally (cf. [208, p.27]),

$$
I_3 \le \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right| \le \frac{r(r-1)}{2n} K e^{\beta A}. \tag{30.38}
$$

Now (30.5) follows from (30.30) and (30.36)-(30.38). ■

30.4 Applications

When specifying the underlying r.v.'s, the Feller operator (30.1) collapses to various concrete operators. We discuss four such operators in this section to demonstrate the applications of the general results.

Example 30.5. (Bernstein operator) Let $X(x)$ have the Bernoulli distribution:

$$
P(X(x) = 1) = x, \ P(X(x) = 0) = 1 - x \ (0 < x < 1),
$$

then (30.1) becomes the Bernstein operator:

$$
B_n(f, \ x) = \sum_{k=1}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}.
$$

Furthermore for $0 < x < 1$

$$
P(X(x) > t) = \begin{cases} 1, & t < 0, \\ x, & 0 \le t < 1 \\ 0, & t \ge 1, \end{cases}
$$

$$
DP(X(x) > t) = \begin{cases} 0, & t < 0, \\ 1, & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases}
$$

and $D^j P(X(x) > t) = 0$ $(j \geq 2)$. By (30.6) of Lemma 30.2 we have

$$\left| D^r B_n(f,\ x) - f^{(r)}(x) \right|$$

$$\leq \frac{(n)_r}{n^r} \left| \underbrace{\int_0^1 ... \int_0^1}_{n} f^{(r)}\left(\frac{t_1 + ... + t_n}{n}\right) \right.$$

$$\times \prod_{i=1}^{r} dt_i \prod_{j=r+1}^{n} dP(X_j(x) \leq t_j) - f^{(r)}(x) \left| + \right| \frac{(n)_r}{n^r} - 1 \left| \left| f^{(r)}(x) \right| \right.$$

$$\leq \frac{(n)_r}{n^r} \underbrace{\int_0^1 ... \int_0^1}_{n} \left| f^{(r)}\left(\frac{t_1 + ... + t_n}{n}\right) - f^{(r)}(x) \right| \prod_{i=1}^{r} dt_i \prod_{j=r+1}^{n} dP(X_j(x) \leq t_j)$$

$$+ \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right|$$

$$= \frac{(n)_r}{n^r} \underbrace{\int_0^1 ... \int_0^1}_{r} \int_0^1 \left| f^{(r)}\left(\frac{t_1 + ... + t_r}{n} + \frac{n-r}{n} t\right) - f^{(r)}(x) \right|$$

$$\times \prod_{i=1}^{r} dt_i dP\left(\frac{1}{n-r} S_{n-r}(x) \leq t\right) + \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right|$$

$$\leq \frac{(n)_r}{n^r} \omega_1(f^{(r)}, \lambda) \underbrace{\int_0^1 ... \int_0^1}_{r} \int_0^1 \left(1 + \frac{1}{\lambda} \left| \frac{t_1 + ... + t_r}{n} + \frac{n-r}{n} t - x \right| \right)$$

$$\times \prod_{i=1}^{r} dt_i dP\left(\frac{1}{n-r} S_{n-r}(x) \leq t\right) + \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right|$$

$$\leq \frac{(n)_r}{n^r} \omega_1(f^{(r)}, \lambda) \{1 + \frac{1}{n\lambda} + \frac{rx}{n\lambda} + \frac{1}{\lambda} (E[(\frac{1}{n-r} S_{n-r}(x) - x)^2])^{1/2}\}$$

$$+ \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right|$$

$$\leq \frac{(n)_r}{n^r} \omega_1(f^{(r)}, \frac{1}{\sqrt{n}}) \{1 + \frac{r(1+x)}{\sqrt{n}} + \frac{\sqrt{x(1-x)}\sqrt{n}}{\sqrt{n-r}}\}$$

$$+ \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right|,\ (\lambda = \frac{1}{\sqrt{n}})$$

$$= (1 + \sqrt{x(1-x)} + a_n) \omega_1(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right|,$$

where

$$a_n := \frac{(n)_r}{n^r} - 1 + \left(\frac{(n)_r}{n^r} \frac{\sqrt{n}}{\sqrt{n-r}} - 1\right) \sqrt{x(1-x)} + \frac{(n)_r}{n^r} \frac{r(1+x)}{\sqrt{n}} \to 0,\ (n \to \infty).$$

I.e.

$$\left|D^r B_n(f,\ x) - f^{(r)}(x)\right| \leq (1+\sqrt{x(1-x)}+a_n)\omega_1(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|.$$

For the simultaneous approximation by the Bernstein operator B_n see [174], [208] and [222].

Example 30.6. (Szász operator) Let $X(x)$ follow the Poisson distribution:

$$P(X(x) = k) = e^{-x}\frac{x^k}{k!},\ (k = 0, 1, 2, ...)$$

then (30.1) becomes the Szász operator:

$$S_n(f,\ x) = e^{-nx}\sum_{k=0}^{\infty} f(\frac{k}{n})\frac{n^k}{k!}x^k.$$

Furthermore we derive

$$P(X(x) > t) = 1 - P(X(x) \leq t) = e^{-x}\sum_{k=[t]+1}^{\infty}\frac{x^k}{k!}$$

and $DP(X(x) > t) = \frac{1}{[t]!}x^{[t]}e^{-x}$. From now on '[ ]' will denote the integer part function. So by Leibniz's formula we get

$$D^k P(X(x) > t) = x^{[t]}e^{-x}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{k-1-j}}{([t]-j)!}x^{-j}.$$

Now by Stirling's formula we can find constants M and $\alpha > 0$ such that (30.3) holds.

Therefore by Theorem 30.1 we have

$$\left|D^r S_n(f,\ x) - f^{(r)}(x)\right| = O(\omega_A(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{1}{n}),\ (n \to \infty)$$

for x and f in Theorem 30.1.

The simultaneous approximation of the Szász operator has been studied by many authors. One can find related expositions in [6], [121], [209], and the papers cited there.

Example 30.7. (Baskakov operator) Let $X(x)$ have the geometric distribution:

$$P(X(x) = k) = \frac{1}{1+x}(\frac{x}{1+x})^k,\ (k = 0, 1, 2, ...)$$

then (30.1) becomes the (special) Baskakov operator (cf. [205] or [285]):

$$B_n^*(f,\ x) = (1+x)^{-n} \sum_{n=0}^{\infty} f(\frac{k}{n}) \binom{n+k-1}{k} (\frac{x}{1+x})^k.$$

We have for $0 < x < A$ that

$$P(X(x) > t) = \sum_{k=[t]+1}^{\infty} \frac{1}{1+x} (\frac{x}{1+x})^k = (\frac{x}{1+x})^{[t]+1}.$$

Also it holds

$$\left| D^k P(X(x) > t) \right| = \left| \sum_{j=0}^{k} \binom{k}{j} (D^j x^{[t]+1}) D^{k-j} (1+x)^{-([t]+1)} \right|$$

$$\leq \sum_{j=0}^{k} \binom{k}{j} ([t]+1)...([t]-j+2) x^{[t]+1-j} ([t]+1)...([t]+k-j)(1+x)^{-([t]+1-j)-k}$$

$$\leq ([t]+k)^k \sum_{j=0}^{k} \binom{k}{j} (\frac{x}{1+x})^{[t]+1-j} (\frac{1}{1+k})^k$$

$$\leq 2^r ([t]+r)^r (\frac{A}{1+A})^{[t]-r}, \text{ for all } k \leq r.$$

Now it is clear that there exist M and $\alpha > 0$ such that (30.3) holds. By Theorem 30.1, we have

$$\left| D^r B_n^*(f,\ x) - f^{(r)}(x) \right| = O(\omega_A(f^{(r)}, \tfrac{1}{\sqrt{n}}) + \frac{1}{n}), \ (n \to \infty)$$

for x and f as in Theorem 30.1.

Note that the simultaneous approximation of general Baskakov operators has been studied in [209].

Example 30.8. (Gamma operator) Let $X(x)$ follow the exponential distribution with density:

$$g(\upsilon, x) = x^{-1} e^{-\upsilon/x}, \ \upsilon > 0, \ 0 < a \leq x \leq b < \infty,$$

then (30.1) becomes the Gamma operator:

$$G_n(f,\ x) = \frac{x^{-n}}{(n-1)!} \int_0^{\infty} f(\frac{\upsilon}{n}) \upsilon^{n-1} e^{-\upsilon/x} d\upsilon.$$

Now

$$P(X(x) > t) = \int_t^{\infty} \frac{1}{x} e^{-\upsilon/x} d\upsilon = e^{-t/x}$$

and it is easy to show that for $k \leq r$ there exist b_{kj} 's such that

$$\left|D^k P(X(x) > t)\right| = \left|\frac{1}{x^{2k}}\left(\sum_{j=0}^{k} b_{kj} x^j t^{k-j}\right) e^{-t/x}\right|$$
$$\leq \frac{(r+1)\max(|b_{kj}|\,; k \leq r, j \leq k)(\max(b,1))^r}{(\min(a,1))^{2r}} t^r e^{-\frac{1}{b}t}.$$

Thus we can find M and $\alpha > 0$ satisfying (30.3) and by Theorem 30.1 there holds

$$\left|D^r G_n(f,\ x) - f^{(r)}(x)\right| = O(\omega_A(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{1}{n}),\ (n \to \infty)$$

for x and f as in Theorem 30.1.

31

Global Smoothness Preservation and Uniform Convergence of Singular Integral Operators in the Fuzzy Sense

In this chapter, we study the fuzzy global smoothness and fuzzy uniform convergence of fuzzy Picard, Gauss- Weierstrass and Poisson- Cauchy singular fuzzy integral operators to the fuzzy unit operator. These are given with rates involving the fuzzy modulus of continuity of a fuzzy derivative of the involved function. The established fuzzy Jackson type inequalities are tight, containing elegant constants, and they reflect the order of the fuzzy differentiability of the involved fuzzy function. This chapter is based on [55].

31.1 Fuzzy Real Analysis Background

We use the following background

Definition 31.1 (see [283]) Let $\mu : \mathrm{R} \rightarrow [0, 1]$ with the following properties

(i) is normal, i.e., $\exists x_0 \in \mathrm{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x)\ \mu(y)\}$, $\forall x, y \in \mathrm{R}$, $\forall \lambda \in [0, 1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on R, i.e. $\forall x_0 \in \mathrm{R}$ and $\forall \epsilon > 0$, $\exists$ neighborhood $V(x_0) :\ \mu(x) \leq \mu(x_0) + \epsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{supp(\mu)}$ is compact in R (where $supp(\mu) := \{x \in \mathrm{R} :\ \mu(x) > 0\}$).

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 487–501.
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We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \ \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \ \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0, 1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$ ([172]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \forall r \in [0, 1],$$

where

- $[u]^r + [v]^r$ means the usual addition of two integrals (as subsets of $\mathbb{R}$) and
- $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [283]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = [u_-^{(r)}, \ u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}, \ u_-^{(r)}, \ u_+^{(r)} \in \mathbb{R}, \ \forall r \in [0, 1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u, v) := \sup_{r \in [0,1]} \max \left\{ |u_-^{(r)} - v_-^{(r)}|, \ |u_+^{(r)} - v_+^{(r)}| \right\},$$

where

$$[v]^r = [v_-^{(r)}, \ v_+^{(r)}]; \ u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [283], [284].

Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g) := \sup_{x\in \mathbb{R}} D(f(x), g(x)).$$

Here Σ^* stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \forall u \in \mathbb{R}_{\mathcal{F}}.$$

We need

Remark 31.2 ([29]). Here $r \in [0,1]$ $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, \ldots,$ $m \in \mathbb{N}$. Assume that

$$\sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}\, for\ i = 1, \ldots,\ m.$$

Then one sees easily that

$$\sup_{r\in[0,1]} \max\left(\sum_{i=1}^m x_i^{(r)}, \sum_{i=1}^m y_i^{(r)}\right) \leq \sum_{i=1}^m \sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right).$$

Definition 31.3. Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, we define the fuzzy modulus of continuity of f by

$$w_1^{(\mathcal{F})}(f,\delta) = \sup_{x,y\in\mathbb{R},\ |x-y|\leq\delta} D(f(x),\ f(y)),\ \delta > 0.$$

Note 31.4. For $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, we use

$$[f]^r = [f_-^{(r)},\ f_+^{(r)}],$$

where $f_{\pm}^{(r)} : \mathbb{R} \to \mathbb{R}$, $\forall r \in [0,1]$.

Let $g : \mathbb{R} \to \mathbb{R}$, define

$$w_1(g,\delta) = \sup_{x,y\in\mathbb{R},\ |x-y|\leq\delta} |g(x) - g(y)|,\ \delta > 0.$$

We need

Proposition 31.5. Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Suppose that $w_1^{(\mathcal{F})}(f,\delta)$, $w_1(f_-^{(r)},\delta)$, $w_1(f_+^{(r)},\delta)$ are finite for any $\delta > 0$, $r \in [0,1]$.

Then

$$w_1^{(\mathcal{F})}(f,\delta) = \sup_{r\in[0,1]} \max\{w_1(f_-^{(r)},\delta),\ w_1(f_+^{(r)},\delta)\}.$$

Proof. By Proposition 1 of [37]. ■

We define by $C_{\mathcal{F}}^U(\mathbb{R})$, the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$.

We mention

Proposition 31.6([37]) Let $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$. Then $w_1^{(\mathcal{F})}(f,\delta) < \infty$, for any $\delta > 0$.

Proposition 31.7([37]) It holds

$$\lim_{\delta \to 0} w_1^{(\mathcal{F})}(f,\delta) = w_1^{(\mathcal{F})}(f,0) = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$.

Proposition 31.8([37]) Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.

Note 31.9 It is clear by Propositions 31.5, 31.7, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$).

We need

Definition 31.10. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : \ x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.

Definition 31.11([283]) Let $T := [x_0, \ x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is supposed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist in $\mathbb{R}_{\mathcal{F}}$ in an neighborhood of x.

Definition 31.12. We denote by $C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \in \mathbb{N}$, the space of all N-times fuzzy continuously differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$.

Here higher order fuzzy derivatives are defined via Definition 31.11 in the obvious way, as in the ordinary real case.

We mention

Theorem 31.13 ([202]) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable, $0 \le r \le 1$, $t \in [a,b]$. Clearly

$$[f(t)]^r = [f(t)_-^{(r)}, \ f(t)_+^{(r)}] \subseteq \mathbb{R}.$$

Then $(f)_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^r = [(f(t)_-^{(r)})', \ (f(t)_+^{(r)})'].$$

That is

$$(f')_{\pm}^{(r)} = (f_{\pm}^{(r)})', \ \forall r \in [0,1].$$

Remark 31.14 ([35]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \ge 1$. Then by Theorem 31.13 we obtain $f_{\pm}^{(r)} \in C^N(\mathbb{R})$ and

$$[f^{(i)}(t)]^r = [(f(t)_-^{(r)})^{(i)}, \ (f(t)_+^{(r)})^{(i)}],$$

for $i = 0, 1, 2 \ldots, N$, and in particular we have

$$(f^{(i)})_{\pm}^{(r)} = (f_{\pm}^{(r)})^{(i)},$$

for any $r \in [0,1]$.

For the definition of general fuzzy integral we follow [10] next.

Definition 31.15. Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to \mathrm{R}_{\mathcal{F}}$ *measurable* iff $\forall$ closed $B \subseteq \mathrm{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \ \textit{all} \ w \in \Omega$$

is measurable, see [206].

Theorem 31.16 ([206]) For $F : \Omega \to \mathrm{R}_{\mathcal{F}}$,

$$F(w) = \{(F^{(r)}(w)_-, \ F_+^{(r)}(w))|0 \le r \le 1\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall r \in [0,1], \ F_-^{(r)}, \ F_+^{(r)}$ are measurable.

Following [206], given that for each $r \in [0,1]$, $F_-^{(r)}, \ F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_-^{(r)} d\mu, \ \int_A F_+^{(r)} \right) \middle| 0 \le r \le 1 \right\}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 31.17 ([206]) A measurable function $F : \Omega \to \mathrm{R}_{\mathcal{F}}$,

$$F(w) = \{(F^{(r)}(w)_-, \ F_+^{(r)}(w))|0 \le r \le 1\}$$

is *integrable* if for each $r \in [0,1]$, $F_{\pm}^{(r)}$ are integrable, or equivalently, if $F_{\pm}^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \ \int_A F_+^{(r)} \right) \middle| 0 \le r \le 1 \right\}.$$

By [206] F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here

$$\|u\|_{\mathcal{F}} := D(u, \widetilde{0}), \ \forall u \in \mathrm{R}_{\mathcal{F}}.$$

We need also

Theorem 31.18 ([206]) Let $F, G : \Omega \to \mathrm{R}_{\mathcal{F}}$ be integrable. Then

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F, G)d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}} d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Remark 31.19. Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)}\right],$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu,$$

$\forall r \in [0, 1]$, respectively.

Notation 31.20. In this chapter we define the fuzzy singular integral operators: Picard P_ξ, Gauss- Weierstrass W_ξ, and the Poisson- Cauchy M_ξ, $\xi > 0$.

Their real analogs are defined and denoted exactly the same way in [33], [81], [82], and we are motivated from there.

Here their fuzzy or real versions for convenience are denoted with the same symbols $P\xi$, $W\xi$, $M\xi$, respectively. According to the context we understand if the operator on hand is fuzzy or real one.

Related work was done in [31].

31.2 Main Results

Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy measurable function and consider the fuzzy Lebesgue integrals,

$$P_\xi(f, x) := \frac{1}{2\xi} \odot \int_{-\infty}^{\infty} f(x + t) \odot e^{-\frac{|t|}{\xi}} dt, \tag{31.1}$$

$\xi > 0,\ x \in \mathrm{R}$, also consider

$$W_\xi(f,x) := \frac{1}{\sqrt{\pi\xi}} \odot \int_{-\infty}^{\infty} f(x+t) \odot \mathrm{e}^{-\frac{t^2}{\xi}} dt, \tag{31.2}$$

$$M_\xi(f,x) := \frac{\Gamma(\beta)\ \alpha\ \xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)} \odot \int_{-\infty}^{\infty} f(x+t) \odot \frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta} dt, \tag{31.3}$$

$\alpha \in \mathrm{N},\ \beta > \frac{1}{2\alpha}$.

Here the gamma function is

$$\Gamma(\beta) = \int_0^\infty \mathrm{e}^{-t} t^{\beta-1} dt,\ \beta > 0.$$

We present the global smoothness preservation property

Theorem 31.21. Let $h > 0$. Suppose $w_1^{(\mathcal{F})}(f,h) < \infty$; $P_\xi(f,x),\ W_\xi(f,x)$, $M_\xi(f,x) \in \mathrm{R}_{\mathcal{F}}$, then

(i)

$$w_1^{(\mathcal{F})}(P_\xi f, h) \le w_1^{(\mathcal{F})}(f,h) \tag{31.4}$$

(ii)

$$w_1^{(\mathcal{F})}(W_\xi f, h) \le w_1^{(\mathcal{F})}(f,h) \tag{31.5}$$

(iii)

$$w_1^{(\mathcal{F})}(M_\xi f, h) \le w_1^{(\mathcal{F})}(f,h) \tag{31.6}$$

Proof. (i) Notice that

$$D\left(P\xi(f,x), P\xi(f,y)\right) = \frac{1}{2\xi} D\left(\int_{-\infty}^{\infty} f(x+t)\odot \mathrm{e}^{-\frac{|t|}{\xi}} dt,\ \int_{-\infty}^{\infty} f(y+t)\odot \mathrm{e}^{-\frac{|t|}{\xi}} dt\right) \le$$

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} D\left(f(x+t), f(y+t)\right) \mathrm{e}^{-\frac{|t|}{\xi}} dt \le$$

$$\frac{1}{2\xi} w_1^{(\mathcal{F})}(f,|x-y|) \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{|t|}{\xi}} dt = w_1^{(\mathcal{F})}(f,|x-y|),$$

taking the supremum over all $x,y : |x-y| \le h$ we prove the claim.

Properties (ii), (iii) follow similarly. ∎

Remark 31.22. We observe that ($r \in [0,1]$)

$$[P\xi(f,x)]^r = \frac{1}{2\xi}\left[\int_{-\infty}^{\infty} f(x+t)\odot \mathrm{e}^{-\frac{|t|}{\xi}} dt\right]^r$$

$$= \frac{1}{2\xi}\left[\left(\int_{-\infty}^{\infty} f(x+t)\odot \mathrm{e}^{-\frac{|t|}{\xi}} dt\right)_-^{(r)},\ \left(\int_{-\infty}^{\infty} f(x+t)\odot \mathrm{e}^{-\frac{|t|}{\xi}} dt\right)_+^{(r)}\right] =$$

$$= \frac{1}{2\xi}\left[\int_{-\infty}^{\infty} f_-^{(r)}(x+t)\mathrm{e}^{-\frac{|t|}{\xi}}dt,\ \int_{-\infty}^{\infty} f_+^{(r)}(x+t)\mathrm{e}^{-\frac{|t|}{\xi}}dt\right]$$

$$= \left[P\xi(f_-^{(r)},x), P\xi(f_+^{(r)},x)\right].$$

I.e. we proved

$$(P_\xi(f,x))_\pm^{(r)} = P_\xi(f_\pm^{(r)},x), \tag{31.7}$$

$\forall r \in [0,1],\ \forall x \in \mathrm{R}$.

Similarly are valid

$$(W_\xi(f))_\pm^{(r)} = W_\xi(f_\pm^{(r)}), \tag{31.8}$$

$$(M_\xi(f))_\pm^{(r)} = M_\xi(f_\pm^{(r)}), \tag{31.9}$$

$\forall r \in [0,1]$.

Assumption 31.23. From now we suppose $f \in C_{\mathcal{F}}^n(\mathrm{R})$, with $w_1^{(\mathcal{F})}(f^{(n)},h) < \infty,\ h>0,\ n \in \mathrm{N}$.

Assume further that $P\xi(f,x),\ W\xi(f,x),\ M\xi(f,x) \in \mathrm{R}_{\mathcal{F}},\ \forall \xi > 0,\ \forall x \in \mathrm{R}$, with $\beta > \frac{n+2}{2\alpha}$.

Here $\lfloor \cdot \rfloor$ denotes the integral part of the number.

We give the following convergence results.

Theorem 31.24. It holds

(i)

$$D\left(P_\xi(f,x),f(x)\right) \le \sum_{m=1}^{\lfloor n/2\rfloor} D(f^{(2m)}(x),\widetilde{0})\xi^{2m} + \frac{13}{8}\xi^n w_1^{(\mathcal{F})}(f^{(n)},\xi),\ x\in\mathrm{R},\ \xi>0. \tag{31.10}$$

(ii) Assuming $D^*(f^{(2m)},\widetilde{0}) < \infty,\ m = 1,\ldots,\ \lfloor n/2\rfloor$, we obtain

$$D^*\left(P_\xi f, f\right) \le \sum_{m=1}^{\lfloor n/2\rfloor} D^*(f^{(2m)},\widetilde{0})\xi^{2m} + \frac{13}{8}\xi^n w_1^{(\mathcal{F})}(f^{(n)},\xi),\ \xi>0. \tag{31.11}$$

(iii)

$$D\left(W_\xi(f,x),f(x)\right) \le \sum_{m=1}^{\lfloor n/2\rfloor} \frac{D(f^{(2m)},\widetilde{0})}{m!}\left(\frac{\xi}{4}\right)^m + \frac{2\xi^{(n-1)/2}}{(n-1)!\sqrt{\pi}}$$

$$\left[\left(\frac{1}{2(n+1)}+\frac{\xi}{8}\right)\prod_{s=0}^{\lfloor\frac{n-1}{2}\rfloor-1}\frac{(n-2-2s)}{2}M_{n-1} + \frac{\sqrt{\xi}}{2n}\prod_{s=0}^{\lfloor\frac{n}{2}\rfloor-1}\frac{(n-1-2s)}{2}M_n\right]w_1^{(\mathcal{F})}(f^{(n)},\xi), \tag{31.12}$$

$\forall x \in \mathrm{R},\ \xi > 0.$

Here

$$M_n = \begin{cases} \frac{1}{2},\ n - odd \\ \frac{\sqrt{\pi}}{2},\ n - even, \end{cases} \tag{31.13}$$

for $n = 1$, we put

$$\prod_{l=0}^{\lfloor n/2 \rfloor - 1} = 1$$

(iv) Assuming that $D^*(f^{(2m)}, \widetilde{0}) < \infty,\ m = 1, \ldots,\ \lfloor n/2 \rfloor$, we obtain

$$D^*\left(W_\xi f, f\right) \leq \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{D(f^{(2m)}, \widetilde{0})}{m!} \left(\frac{\xi}{4}\right)^m + \frac{2\xi^{(n-1)/2}}{(n-1)!\sqrt{\pi}}$$

$$\left[\left(\frac{1}{2(n+1)} + \frac{\xi}{8}\right) \prod_{s=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \frac{(n-2-2s)}{2} M_{n-1} \right. \tag{31.14}$$

$$\left. + \frac{\sqrt{\xi}}{2n} \prod_{s=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{(n-1-2s)}{2} M_n \right] w_1^{(\mathcal{F})}(f^{(n)}, \xi),$$

$\forall \xi > 0.$

(v)

$$D\left(M_\xi(f,x), f(x)\right) \leq \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{D(f^{(2m)}(x), \widetilde{0})}{(2m)!} \cdot \frac{\Gamma\left(\frac{2m+1}{2\alpha}\right)\Gamma\left(\beta - \frac{2m+1}{2\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)} \xi^{2m} +$$

$$\frac{\xi^n}{(n-1)!\ \Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)} \left[\frac{1}{(n+1)n} \Gamma\left(\frac{n+2}{2\alpha}\right) \Gamma\left(\beta - \frac{n+2}{2\alpha}\right) + \right.$$

$$\left. \frac{1}{2n}\ \Gamma\left(\frac{n+1}{2\alpha}\right) \Gamma\left(\beta - \frac{n+1}{2\alpha}\right) + \frac{1}{8}\ \Gamma\left(\frac{n}{2\alpha}\right) \Gamma\left(\beta - \frac{n+2}{2\alpha}\right) \right] w_1^{(\mathcal{F})}(f^{(n)}, \xi), \tag{31.15}$$

$\forall x \in \mathrm{R},\ \xi > 0.$

(vi) Assuming that $D^*(f^{(2m)}, \widetilde{0}) < \infty,\ m = 1, \ldots,\ \lfloor n/2 \rfloor$, we get

$$D^*\left(M_\xi f, f\right) \leq \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{D(f^{(2m)}, \widetilde{0})}{2m!} \cdot \frac{\Gamma\left(\frac{2m+1}{2\alpha}\right)\Gamma\left(\beta - \frac{2m+1}{2\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)} \xi^{2m} +$$

$$\frac{\xi^n}{(n-1)!\,\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\left[\frac{1}{(n+1)n}\Gamma\left(\frac{n+2}{2\alpha}\right)\Gamma\left(\beta-\frac{n+2}{2\alpha}\right)+\right.$$
$$\left.\frac{1}{2n}\,\Gamma\left(\frac{n+1}{2\alpha}\right)\Gamma\left(\beta-\frac{n+1}{2\alpha}\right)+\frac{1}{8}\,\Gamma\left(\frac{n}{2\alpha}\right)\Gamma\left(\beta-\frac{n}{2\alpha}\right)\right]w_1^{(\mathcal{F})}(f^{(n)},\xi) \tag{31.16}$$

$\forall \xi > 0$.

Proof. (i) We notice that

$$D\left(P_\xi(f,x),f(x)\right)=\sup_{r\in[0,1]}\max\left\{\left|\left(P_\xi(f,x)\right)_-^{(r)}-f_-^{(r)}(x)\right|,\ \left|\left(P_\xi(f,x)\right)_+^{(r)}-f_+^{(r)}(x)\right|\right\}$$
$$=\sup_{r\in[0,1]}\max\left\{\left|P_\xi\left(f_-^{(r)},x\right)-f_-^{(r)}(x)\right|,\ \left|P_\xi\left(f_+^{(r)},x\right)-f_+^{(r)}(x)\right|\right\}$$

(by Proposition 1 of [33], see there (51))

$$\leq\sup_{r\in[0,1]}\max\left\{\sum_{m=1}^{\lfloor n/2\rfloor}\left|\left(f_-^{(r)}\right)^{(2m)}(x)\right|\xi^{2m}+\frac{13}{8}\,\xi^n w_1\left((f_-^{(r)})^{(n)},\xi\right),\right.$$
$$\left.\sum_{m=1}^{\lfloor n/2\rfloor}\left|\left(f_+^{(r)}\right)^{(2m)}(x)\right|\xi^{2m}+\frac{13}{8}\,\xi^n w_1\left((f_+^{(r)})^{(n)},\xi\right)\right\}=$$
$$\sup_{r\in[0,1]}\max\left\{\sum_{m=1}^{\lfloor n/2\rfloor}\left|\left(f^{(2m)}(x)\right)_-^{(r)}\right|\xi^{2m}+\frac{13}{8}\,\xi^n w_1\left((f^{(n)})_-^{(r)},\xi\right),\right.$$
$$\left.\sum_{m=1}^{\lfloor n/2\rfloor}\left|\left(f^{(2m)}(x)\right)_+^{(r)}\right|\xi^{2m}+\frac{13}{8}\,\xi^n w_1\left((f^{(n)})_+^{(r)},\xi\right)\right\}\leq$$
$$\sum_{m=1}^{\lfloor n/2\rfloor}\xi^{2m}\sup_{r\in[0,1]}\max\left\{\left|\left(f^{(2m)}(x)\right)_-^{(r)}\right|,\left|\left(f^{(2m)}(x)\right)_+^{(r)}\right|\right\}$$
$$+\frac{13}{8}\,\xi^n\sup_{r\in[0,1]}\max\left\{w_1\left((f^{(n)})_-^{(r)},\xi\right),w_1\left((f^{(n)})_+^{(r)},\xi\right)\right\}=$$
$$\sum_{m=1}^{\lfloor n/2\rfloor}D^*(f^{(2m)}(x),\widetilde{0})\xi^{2m}+\frac{13}{8}\,\xi^n w_1^{(\mathcal{F})}(f^{(n)},\xi),$$

proving the claim.

(iii) Proved as in (i) now using Proposition 1 of [81], see there (52).

(v) Proved as in (i) by using Proposition 1 of [82], see there (62). ∎

Remark 31.25. As $\xi \to 0$ from (i), (iii), (v) we derive that

$$D\left(P\xi(f,x), f(x)\right) \to 0, \ n \in \mathrm{N};$$
$$D\left(W\xi(f,x), f(x)\right) \to 0, \ n \in \mathrm{N} - \{1\};$$
$$D\left(M\xi(f,x), f(x)\right) \to 0, \ n \in \mathrm{N}.$$

Also by assuming $D^*(f^{(2m)}, \widetilde{0}) < \infty, \ m = 1, \ldots, \ \lfloor n/2 \rfloor$ we obtain

$$D^*\left(P\xi f, f\right) \to 0, \ n \in \mathrm{N};$$
$$D^*\left(W\xi f, f\right) \to 0, \ n \in \mathrm{N} - \{1\};$$
$$D^*\left(M\xi f, f\right) \to 0, \ n \in \mathrm{N}.$$

We give
Corollary 31.26. ($n = 2$ case)
It holds

(i)

$$D\left(P_\xi(f,x), f(x)\right) \le \xi^2 \left(D(f''(x), \widetilde{0}) + \frac{13}{8} w_1^{(\mathcal{F})}(f'', \xi) \right), \tag{31.17}$$

$x \in \mathrm{R}, \ \xi > 0.$

(ii) when $D^*(f'', \widetilde{0}) < \infty$, we get

$$D^*\left(P_\xi f, f\right) \le \xi^2 \left(D^*(f'', \widetilde{0}) + \frac{13}{8} w_1^{(\mathcal{F})}(f'', \xi) \right), \ \xi > 0. \tag{31.18}$$

(iii)

$$D\left(W_\xi(f,x), f(x)\right) \le D(f''(x), \widetilde{0}) \left(\frac{\xi}{4}\right) + w_1^{(\mathcal{F})}(f'', \xi) \left[\left(\frac{1}{6} + \frac{\xi}{8} \right) \sqrt{\frac{\xi}{\pi}} + \frac{\xi}{8} \right], \tag{31.19}$$

$x \in \mathrm{R}, \ \xi > 0.$

(iv) when $D^*(f'', \widetilde{0}) < \infty$, we find

$$D^*\left(W_\xi f, f\right) \le D^*(f'', \widetilde{0}) \left(\frac{\xi}{4}\right) + w_1^{(\mathcal{F})}(f'', \xi) \left[\left(\frac{1}{6} + \frac{\xi}{8} \right) \sqrt{\frac{\xi}{\pi}} + \frac{\xi}{8} \right], \ \xi > 0. \tag{31.20}$$

(v)

$$D\left(M_{\xi}(f,x),f(x)\right) \leq \frac{\xi^2}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\left\{\frac{D(f''(x),\widetilde{0})}{2}\;\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+w_1^{(\mathcal{F})}(f'',\xi)\right.$$

$$\left.\left[\frac{1}{6}\;\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)+\frac{1}{4}\;\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{1}{8}\;\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta-\frac{1}{\alpha}\right)\right]\right\}, \tag{31.21}$$

$\forall x \in \mathrm{R},\ \forall \xi > 0.$

(vi) when $D^*(f''(x),\widetilde{0}) < \infty$, we get

$$D^*\left(M_{\xi}f,f\right) \leq \frac{\xi^2}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\left\{\frac{D^*(f'',\widetilde{0})}{2}\;\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+w_1^{(\mathcal{F})}(f'',\xi)\right.$$

$$\left.\left[\frac{1}{6}\;\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)+\frac{1}{4}\;\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{1}{8}\;\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta-\frac{1}{\alpha}\right)\right]\right\}, \tag{31.22}$$

$\forall \xi > 0.$

Corollary 31.27. ($n = 1$ case)
It holds

(i)

$$D\left(P_{\xi}(f,x),f(x)\right) \leq \frac{13}{8}\;\xi\;w_1^{(\mathcal{F})}(f',\xi),\ \forall x \in \mathrm{R},\ \forall \xi > 0. \tag{31.23}$$

(ii)

$$D^*\left(P_{\xi}f,f\right) \leq \frac{13}{8}\;\xi\;w_1^{(\mathcal{F})}(f',\xi),\ \forall \xi > 0. \tag{31.24}$$

(iii)

$$D\left(W_{\xi}(f,x),f(x)\right) \leq w_1^{(\mathcal{F})}(f',\xi)\left[\left(\frac{1}{4}+\frac{\xi}{8}\right)+\frac{1}{2}\sqrt{\frac{\xi}{\pi}}\right],\ \forall x \in \mathrm{R},\ \forall \xi > 0. \tag{31.25}$$

(iv)

$$D^*\left(W_{\xi}f,f\right) \leq w_1^{(\mathcal{F})}(f',\xi)\left[\left(\frac{1}{4}+\frac{\xi}{8}\right)+\frac{1}{2}\sqrt{\frac{\xi}{\pi}}\right],\ \forall \xi > 0. \tag{31.26}$$

If $f' \in C^U_{\mathcal{F}}(\mathrm{R})$, then as $\xi \to 0$, we obtain

$$D\left(W\xi(f,x),f(x)\right) \to 0,\ D^*\left(W\xi f,f\right) \to 0.$$

(v)

$$D\left(M_{\xi}(f,x),f(x)\right) \leq \frac{\xi}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}$$
$$\left[\frac{1}{2}\ \Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{1}{2}\ \Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta-\frac{1}{\alpha}\right)+\frac{1}{8}\ \Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)\right] w_1^{(\mathcal{F})}(f',\xi), \tag{31.27}$$

$\forall x \in \mathrm{R},\ \forall \xi > 0.$

(vi)

$$D\left(M_{\xi}f,f\right) \leq \frac{\xi}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}$$
$$\left[\frac{1}{2}\ \Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{1}{2}\ \Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta-\frac{1}{\alpha}\right)+\frac{1}{8}\ \Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)\right] w_1^{(\mathcal{F})}(f',\xi), \tag{31.28}$$

$\forall \xi > 0.$

Next we cover case of $n = 0$.
We make
Remark 31.28.
We have

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{|t|}{\xi}}\,dt = 1,$$
$$\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{t^2}{\xi}}\,dt = 1,$$

and

$$\frac{\Gamma(\beta)\ \alpha\ \xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\frac{dt}{(t^{2\alpha}+\xi^{2\alpha})^{\beta}} = 1, \tag{31.29}$$

$\alpha \in \mathrm{N},\ \beta > \frac{1}{2\alpha};\ \xi > 0.$
Put

$$K_{1,\xi}(t) := \frac{1}{2\xi}\mathrm{e}^{-\frac{|t|}{\xi}},$$
$$K_{2,\xi}(t) := \frac{1}{\sqrt{\pi\xi}}\mathrm{e}^{-\frac{t^2}{\xi}},$$
$$K_{3,\xi}(t) := \frac{\Gamma(\beta)\ \alpha\ \xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\cdot\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^{\beta}},\ t \in \mathrm{R}. \tag{31.30}$$

Here let $f \in C_{\mathcal{F}}(\mathrm{R})$ which is fuzzy bounded.
Set

$$L_{1,\xi} = P_{\xi},\ L_{2,\xi} = W_{\xi},\ L_{3,\xi} = M_{\xi},\ \xi > 0. \tag{31.31}$$

So we have the fuzzy Lebesgue integrals

$$L_{j,\xi}(f,x) = \int_{-\infty}^{\infty} f(x+t) \odot K_{j,\xi}(t)dt, \tag{31.32}$$

with the real integrals

$$\int_{-\infty}^{\infty} K_{j,\xi}(t)dt = 1, \ all\ j = 1,2,3, \ \forall x \in \mathrm{R}.$$

Notice that ($r \in [0,1]$)

$$[f(x)]^r = [f(x)_-^{(r)}, \ f(x)_+^{(r)}]$$
$$= \left[\int_{-\infty}^{\infty} (f(x))_-^{(r)} K_{j,\xi}(t)dt, \ \int_{-\infty}^{\infty} (f(x))_+^{(r)} K_{j,\xi}(t)dt\right]$$
$$= \left[\int_{-\infty}^{\infty} (f(x) \odot K_{j,\xi}(t))_-^{(r)} dt, \ \int_{-\infty}^{\infty} (f(x) \odot K_{j,\xi}(t))_+^{(r)} dt\right]$$
$$= \left[\int_{-\infty}^{\infty} f(x) \odot K_{j,\xi}(t)dt\right]^r.$$

Therefore

$$f(x) = \int_{-\infty}^{\infty} f(x) \odot K_{j,\xi}(t)dt. \tag{31.33}$$

Hence we have

$$D\left(L_{j,\xi}(f,x), f(x)\right) = D\left(\int_{-\infty}^{\infty} f(x+t) \odot K_{j,\xi}(t)dt, \ \int_{-\infty}^{\infty} f(x) \odot K_{j,\xi}(t)dt\right) \leq$$

$$\int_{-\infty}^{\infty} D\left(f(x+t), f(x)\right) K_{j,\xi}(t)dt \leq \int_{-\infty}^{\infty} w_1^{(\mathcal{F})}(f,|t|)\ K_{j,\xi}(t)dt =$$

(for $j = 1,3$ we have next)

$$\int_{-\infty}^{\infty} w_1^{(\mathcal{F})}\left(f, \xi\frac{|t|}{\xi}\right)\ K_{j,\xi}(t)dt \leq$$

(by [37], Proposition 2-(3))

$$w_1^{(\mathcal{F})}(f,\xi)\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)K_{j,\xi}(t)dt = w_1^{(\mathcal{F})}(f,\xi)\left[1+\frac{1}{\xi}\int_{-\infty}^{\infty}|t|\ K_{j,\xi}(t)dt\right].$$

We have proved so far that

$$D\left(L_{j,\xi}(f,x), f(x)\right) \leq w_1^{(\mathcal{F})}(f,\xi)\left[1+\frac{2}{\xi}\int_{0}^{\infty} t\ K_{j,\xi}(t)dt\right], \tag{31.34}$$

for $j = 1, 3$.

Similarly one establishes that

$$D\left(L_{2,\xi}(f,x), f(x)\right) \leq w_1^{(\mathcal{F})}(f, \sqrt{\xi}) \left[1 + \frac{2}{\sqrt{\xi}} \int_0^{\infty} t\ K_{2,\xi}(t)dt\right]. \tag{31.35}$$

We see that

$$\frac{2}{\xi} \int_0^{\infty} t\ K_{1,\xi}(t)dt = 1, \tag{31.36}$$

$$\frac{2}{\sqrt{\xi}} \int_0^{\infty} t\ K_{2,\xi}(t)dt = \frac{1}{\sqrt{\pi}}, \tag{31.37}$$

$$\frac{2}{\xi} \int_0^{\infty} t\ K_{3,\xi}(t)dt = \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta - \frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)},\ \beta > \frac{1}{\alpha},\ \alpha \in \mathrm{N}. \tag{31.38}$$

We have proved

Theorem 31.29. Let $f \in C_{\mathcal{F}}(\mathrm{R})$ which is fuzzy bounded. Then $(\xi > 0)$

(i)

$$D^*\left(P_\xi f, f\right) \leq 2w_1^{(\mathcal{F})}(f, \xi), \tag{31.39}$$

(ii)

$$D^*\left(W_\xi f, f\right) \leq \left(1 + \frac{1}{\sqrt{\pi}}\right) w_1^{(\mathcal{F})}(f, \sqrt{\xi}), \tag{31.40}$$

(iii)

$$D^*\left(M_\xi f, f\right) \leq \left(1 + \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta - \frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)}\right) w_1^{(\mathcal{F})}(f, \xi),\ \beta > \frac{1}{\alpha},\ \alpha \in \mathrm{N}. \tag{31.41}$$

Given that also $f \in C_{\mathcal{F}}^U(\mathrm{R})$, as $\xi \to 0$, we get $D^*\left(P_\xi f, f\right) \to 0,\ D^*\left(W_\xi f, f\right) \to 0$ and $D^*\left(M_\xi f, f\right) \to 0$, with rates.

32

Real Approximations Transferred to Vectorial and Fuzzy Setting

Here we transfer basic real approximations to corresponding vectorial and fuzzy setting of: Bernstein polynomials, Bernstein-Durrmeyer operators, genuine Bernstein-Durrmeyer operators, Stancu type operators and special Stancu operators. These are convergences to the unit operator with rates. We also give the convergence with rates to zero of the difference of genuine Bernstein-Durrmeyer and special Stancu operators. All approximations involve Jackson type inequalities and moduli of smoothness of various orders. In order to transfer we develop basic and important general results at the vectorial and fuzzy level. Our technique goes from real to vectorial and then to fuzzy setting. This chapter is based on [58].

32.1 Results

Let $(X, \|\cdot\|)$ be a normed vector space over K, where $K = \mathbb{R}$ or $K = \mathbb{C}$. Similar to the real case we give the following

Definition 32.1. (see also [166]) For $f : [0,1] \longrightarrow X$ we define the first modulus of continuity

$$\omega_1 (f, \delta) = \sup \{ \| f(v) - f(u) \| ; \ u, v \in [0,1], \ |v - u| \le \delta \}, \tag{32.1}$$

and the second Ditzian-Totik modulus of smoothness

$$\omega_2^{\varphi} (f, \delta) = \sup \{ \sup \{ \| f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x)) \| ;$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 503–522.
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$$x \in I_{2,h}\}\,,\ \ h \in [0,\delta]\}\,, \tag{32.2}$$

where $I_{2,h} = \left[\frac{h^2}{1+h^2}, \frac{1}{1+h^2}\right]$, $\varphi(x) = \sqrt{x(1-x)}$, $0 < \delta \leq 1$.

We need

Theorem 32.2. ([232], [166]) *Let $(X, \|\cdot\|)$ be a normed space over K, where $K = \mathbb{R}$ or $\mathbb{C}$ and denote by $X^* = \{x^* : X \longrightarrow K;\ x^*$ is linear and continuous$\}$.*

Here $\||x^*\|| = \sup\{|x^*(x)| :\ \ \|x\| = 1\}$.

$$\|x\| = \sup\{|x^*(x)|\,;\ x^* \in X^*,\ \ \||x^*\|| \leq 1\}\,. \tag{32.3}$$

We need

Definition 32.3. Let continuous function $f : [0,1] \longrightarrow X$. The vectorial Bernstein-Durrmeyer operators are defined by

$$D_n^{\upsilon}(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 f(t)\,p_{n,k}(t)\,dt, \tag{32.4}$$

$0 \leq k \leq n$, $p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}$, $x \in [0,1]$.

Here the integral $\int_a^b g(t)\,dt$ $(g : [0,1] \longrightarrow X)$ is defined as the limit for $m \longrightarrow \infty$ in the norm $\|\cdot\|$ of all (usual) Riemann sums $\sum_{i=0}^{m} (x_{i+1} - x_i)\, f(\xi_i)$.

Put $\|f\|_\infty = \sup\{\|f(x)\|\,;\ x \in [0,1]\}$.

We present

Theorem 32.4. *Let $f : [0,1] \longrightarrow X$ continuous. Then there are universal constants c_1, $c_2 > 0$ such that*

$$c_1\left(\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right) + \omega_1\left(f, \frac{1}{n}\right)\right)$$
$$\leq \|D_n^{\upsilon} f - f\|_\infty \leq c_2\left(\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right) + \omega_1\left(f, \frac{1}{n}\right)\right). \tag{32.5}$$

Proof. By [177], [178], we have for $g \in C([0,1])$ that

$$c_1\left(\omega_2^{\varphi}\left(g, \frac{1}{\sqrt{n}}\right) + \omega_1\left(g, \frac{1}{n}\right)\right) \leq$$
$$\|D_n(g) - g\|_\infty \leq c_2\left(\omega_2^{\varphi}\left(g, \frac{1}{\sqrt{n}}\right) + \omega_1\left(g, \frac{1}{n}\right)\right),$$

here D_n is the real Bernstein-Durrmeyer operators (when $X = \mathbb{R}$).

Let $x^* \in X^*$ be fixed with $|||x^*||| \leq 1$. Then $g = x^* \circ f : [0,1] \longrightarrow \mathbb{R}$ is continuous.

Therefore

$$c_1 \left(\omega_2^{\varphi} \left(x^* \circ f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(x^* \circ f, \frac{1}{n} \right) \right) \leq$$

$$\| D_n (x^* \circ f) - x^* \circ f \|_{\infty} \leq c_2 \left(\omega_2^{\varphi} \left(x^* \circ f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(x^* \circ f, \frac{1}{n} \right) \right).$$

Because x^* is linear and continuous, it commutes with $\sum$ and integral $\int$, and therefore

$$D_n (x^* \circ f)(x) - x^* \circ f(x) = x^* \left(D_n^{\upsilon} f(x) - f(x) \right).$$

Also, since

$$\omega_1 \left(g, \frac{1}{n} \right) = \sup \left\{ |x^* (f(v) - f(u))| ;\ v, u \in [0,1], \right.$$

$$\left. |v - u| \leq \frac{1}{n} \right\} \leq \sup \left\{ |||x^*||| \cdot \| f(v) - f(u) \| ; \right.$$

$$\left. v, u \in [0,1],\ \ |v - u| \leq \frac{1}{n} \right\} \leq \omega_1 \left(f, \frac{1}{n} \right),$$

that is

$$\omega_1 \left(g, \frac{1}{n} \right) \leq \omega_1 \left(f, \frac{1}{n} \right).$$

Also we have

$$\omega_2^{\varphi} \left(g, \frac{1}{\sqrt{n}} \right) = \sup \left\{ \sup \left\{ |x^* (f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x)))| ; \right. \right.$$

$$\left. \left. x \in I_{2,h} \right\},\ h \in \left[0, \frac{1}{\sqrt{n}} \right] \right\} \leq \sup \left\{ \sup \left\{ |||x^*||| \cdot \| f(x + h\varphi(x)) - 2f(x) \right. \right.$$

$$\left. \left. + f(x - h\varphi(x)) \| ;\ x \in I_{2,h} \right\},\ h \in \left[0, \frac{1}{\sqrt{n}} \right] \right\} \leq \omega_2^{\varphi} \left(f, \frac{1}{\sqrt{n}} \right).$$

I.e.

$$\omega_2^{\varphi} \left(g, \frac{1}{\sqrt{n}} \right) \leq \omega_2^{\varphi} \left(f, \frac{1}{\sqrt{n}} \right).$$

Therefore

$$\left| x^* \left(D_n^{\upsilon} (f)(x) - f(x) \right) \right| \leq c_2 \left(\omega_2^{\varphi} \left(f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(f, \frac{1}{n} \right) \right),$$

hence

$$\sup_{\substack{x^* \in X^* \\ |||x^*||| \leq 1}} \left| x^* \left(D_n^{\upsilon} (f)(x) - f(x) \right) \right| \leq c_2 \left(\omega_2^{\varphi} \left(f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(f, \frac{1}{n} \right) \right),$$

and by Theorem 32.2 we obtain

$$\left\| D_n^{\upsilon} (f)(x) - f(x) \right\| \leq c_2 \left(\omega_2^{\varphi} \left(f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(f, \frac{1}{n} \right) \right),$$

$\forall x \in [0,1]$, that is

$$\left\| D_n^{\upsilon} (f) - f \right\|_{\infty} \leq c_2 \left(\omega_2^{\varphi} \left(f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(f, \frac{1}{n} \right) \right).$$

We also have

$$c_1 \left(\omega_2^{\varphi} \left(x^* \circ f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(x^* \circ f, \frac{1}{n} \right) \right) \leq$$

$$\sup \left\{ \left| x^* \left(D_n^{\upsilon} (f)(x) - f(x) \right) \right| ; \ x \in [0,1] \right\} \leq$$

$$\sup \left\{ |||x^*||| \cdot \left\| D_n^{\upsilon} (f)(x) - f(x) \right\| ; \ x \in [0,1] \right\}$$

$$\leq \left\| D_n^{\upsilon} (f) - f \right\|_{\infty}.$$

Next, for any $x \in I_{2,h}$, $h \in \left[0, \frac{1}{\sqrt{n}}\right]$, $\upsilon, u \in [0,1]$ with $|\upsilon - u| \leq \frac{1}{n}$, we have

$$c_1 \left| x^* \left(f(x + h\varphi(x)) - 2f(x) + \right. \right.$$

$$\left. \left. f(x - h\varphi(x)) \right) \right| + \left| x^* \left(f(\upsilon) - f(u) \right) \right|) \leq$$

$$c_1 \left(\omega_2^{\varphi} \left(x^* \circ f, \frac{1}{\sqrt{n}} \right) + \omega_1 \left(x^* \circ f, \frac{1}{n} \right) \right) \leq \left\| D_n^{\upsilon} (f) - f \right\|_{\infty}.$$

Therefore

$$c_1 \sup_{\substack{x^* \in X^* \\ |||x^*||| \leq 1}} \left\{ \left| x^* \left(f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x)) \right) \right| \right\}$$

$$\leq \left\| D_n^{\upsilon} (f) - f \right\|_{\infty},$$

also it holds

$$c_1 \sup_{\substack{x^* \in X^* \\ |||x^*||| \leq 1}} \left\{ \left| x^* \left(f(\upsilon) - f(u) \right) \right| \right\} \leq \left\| D_n^{\upsilon} (f) - f \right\|_{\infty}.$$

Consequently by Theorem 32.2 we derive

$$c_1 \|f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))\|$$
$$\leq \|D_n^{\upsilon} f - f\|_{\infty},$$

and

$$c_1 \|f(v) - f(u)\| \leq \|D_n^{\upsilon} f - f\|_{\infty}.$$

The last imply

$$\frac{c_1}{2}\left(\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right) + \omega_1\left(f, \frac{1}{n}\right)\right) \leq \|D_n^{\upsilon} f - f\|_{\infty},$$

finishing the proof of the theorem. ■

We make

Remark 32.5. Let us recall a few facts concerning fuzzy-number valued functions.

Given a set $X \neq \emptyset$, a fuzzy subset of X is a mapping $u : X \longrightarrow [0,1]$ and obviously any classical subset A of X can be considered as a fuzzy subset of X defined by $\mathcal{X}_A : X \longrightarrow [0,1]$, $\mathcal{X}_A(x) = 1$, if $x \in A$, $\mathcal{X}_A(x) = 0$ if $x \in X \backslash A$. (see e.g. [287]).

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of real axis $\mathbb{R}$ (i.e. $u : \mathbb{R} \longrightarrow [0,1]$), satisfying the following properties:

(i) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is normal i.e. $\exists x_u \in \mathbb{R}$ with $u(x_u) = 1$;

(ii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$);

(iii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is upper semi-continuous on $\mathbb{R}$;

(iv) $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy real numbers (see e.g. [149]).

Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$, because any real number $x_0 \in \mathbb{R}$, can be described as the fuzzy number whose value is 1 for $x = x_0$ and 0 otherwise.

For $0 < r \leq 1$ and $u \in \mathbb{R}_{\mathcal{F}}$ define $[u]^r = \{x \in \mathbb{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[u]^r$ is a bounded closed interval. For u, $v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we have the sum $u \oplus v$ and the product $\lambda \odot u$ defined by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \odot u]^r = \lambda [u]^r$, $\forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see e.g. [149], [283]).

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}_+ \cup \{0\}$ by

$$D(u,v) = \sup_{r \in [0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}, \tag{32.6}$$

where $[u]^r = [u_-^r, u_+^r]$, $[v]^r = [v_-^r, v_+^r]$.

The following properties are known ([149]):

$D(u \oplus w, v \oplus w) = D(u, v), \forall u, v, w \in \mathbb{R}_{\mathcal{F}}$
$D(k \odot u, k \odot v) = |k| D(u, v), \forall u, v \in \mathbb{R}_{\mathcal{F}}, \forall k \in \mathbb{R};$
$D(u \oplus v, w \oplus e) \le D(u, w) + D(v, e), \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$ and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Also, we need the following Riemann integral, as particular case of the Henstock integral introduced by [283].

A function $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$, $[a, b] \subset \mathbb{R}$ is called Riemann integrable on $[a, b]$, if there exists $I \in \mathbb{R}_{\mathcal{F}}$, with the property: $\forall \epsilon > 0$, $\exists \delta > 0$, such that for any division of $[a, b]$, $d : a = x_0 < \ldots < x_n = b$ of norm $\nu(d) < \delta$, and for any points $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$, we have

$$D\left(\sum_{i=0}^{n-1*} f(\xi_i) \odot (x_{i+1} - x_i), I\right) < \epsilon,$$

where $\sum^*$ means sum with respect to $\oplus$. Then we denote $I = (FR) \int_a^b f(x)\, dx$.

An important result for our reasonings will be the following known result.

Theorem 32.6. (see e.g. [283]) $\mathbb{R}_{\mathcal{F}}$, *can be embedded in* $\mathcal{B} = \overline{C}([0, 1]) \times \overline{C}([0, 1])$, *where* $\overline{C}([0, 1])$ *is the class of all real valued bounded functions* $f : [0, 1] \longrightarrow \mathbb{R}$ *such that* f *is left continuous for any* $x \in (0, 1]$, f *has right limit for any* $x \in [0, 1)$ *and* f *is right continuous at 0. With the norm* $\|\cdot\| = \sup_{x \in [0,1]} |f(x)|$, $\overline{C}([0, 1])$ *is a Banach space. Denote* $\|\cdot\|_{\mathcal{B}}$ *the usual product norm i.e.* $\|(f, g)\|_{\mathcal{B}} = \max\{\|f\|, \|g\|\}$. *Let us denote the embedding by* $j : \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{B}$, $j(u) = (u_-, u_+)$. *Then* $j(\mathbb{R}_{\mathcal{F}})$ *is a closed convex cone in* $\mathbb{B}$ *and* j *satisfies the following properties:*

(i) $j(s \odot u \oplus t \odot v) = s \cdot j(u) + t \cdot j(v)$ *for all* $u, v \in \mathbb{R}_{\mathcal{F}}$, *and* $s, t \ge 0$ *(here "$\cdot$" and "$+$" denote the scalar multiplication and addition in* $\mathbb{B}$*);*

(ii) $D(u, v) = \|j(u) - j(v)\|_{\mathbb{B}}$ *(i.e.* j *embeds* $\mathbb{R}_{\mathcal{F}}$ *in* $\mathbb{B}$ *isometrically and isomorphically).*

Let $f : [0, 1] \longrightarrow \mathbb{R}_{\mathcal{F}}$ *fuzzy continuous, we define the fuzzy Bernstein-Durrmeyer operators as*

$$D_n^{\mathcal{F}}(f, x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \odot (FR) \int_0^1 f(t) \odot p_{n,k}(t)\, dt, \qquad (32.7)$$

the integral $(FR) \int_a^b f(t)\, dt$ *is defined as the limit for* $m \longrightarrow \infty$ *in the distance* D, *of all the usual fuzzy Riemann sums* $\sum_{i=0}^{m} {}^* (x_{i+1} - x_i) \odot f(\xi_i)$ *(here the sums are with respect to operation* $\oplus$*).*

Also, let us define the following fuzzy moduli of continuity of f :

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup\{D(f(x + h), f(x));$$

$$x,\ x+h\in[0,1]\,,\ 0\le h\le\delta\}\,, \tag{32.8}$$

and

$$\omega_2^{(\mathcal{F})\varphi}(f,\delta)=\sup\{D\left(f\left(x+h\varphi\left(x\right)\right)\oplus f\left(x-h\varphi\left(x\right)\right),2\odot f\left(x\right)\right);$$

$$x,\ x\pm h\varphi(x)\in[0,1]\}\,,\ \varphi(x)=\sqrt{x(1-x)}. \tag{32.9}$$

We give

Theorem 32.7. *Let $f:[0,1]\longrightarrow\mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then there exist universal constants $c_1,\,c_2>0$ such that $\forall n\in\mathbb{N}$ we have*

$$c_1\left(\omega_2^{(\mathcal{F})\varphi}\left(f,\frac{1}{\sqrt{n}}\right)+\omega_1^{(\mathcal{F})}\left(f,\frac{1}{n}\right)\right)\le$$

$$\sup\left\{D\left(D_n^{\mathcal{F}}(f)(x),f(x)\right);\ x\in[0,1]\right\}\le$$

$$c_2\left(\omega_2^{(\mathcal{F})\varphi}\left(f,\frac{1}{\sqrt{n}}\right)+\omega_1^{(\mathcal{F})}\left(f,\frac{1}{n}\right)\right). \tag{32.10}$$

Proof. Define $g:[0,1]\longrightarrow X$ by $g(x)=j(f(x))\,,\ x\in[0,1]\,,$ where j is given by Theorem 32.6 and $X=\overline{C}([0,1])\times\overline{C}([0,1])$ endowed with the norm in Theorem 32.6, denoted by $\|\cdot\|_{\mathcal{B}}$. By Theorem 32.6, (ii), we see that

$$\|g(x+h)-g(x)\|=\|j(f(x+h))-j(f(x))\|$$

$$=D(f(x+h),f(x))\,,$$

also

$$\|g(x+h\varphi(x))+g(x-h\varphi(x))-2g(x)\|=$$

$$\|j(f(x+h\varphi(x))\oplus f(x-h\varphi(x)))-j(2\odot f(x))\|$$

$$=D(f(x+h\varphi(x))\oplus f(x-h\varphi(x)),2\odot f(x))\,,$$

which imply

$$\omega_1(g,\delta)=\omega_1^{(\mathcal{F})}(f,\delta)\,,$$

and

$$\omega_2^{\varphi}(g,\delta)=\omega_2^{(\mathcal{F})\varphi}(f,\delta)\,,\ \forall\delta>0.$$

Because j is linear over the positive scalars and j commutes with the fuzzy integral, we get

$$D_n^{\upsilon}(g)(x)=j\left(D_n^{\mathcal{F}}(f)(x)\right),$$

and furthermore

$$\|D_n^{\upsilon}(g)(x)-g(x)\|=\left\|j\left(D_n^{\mathcal{F}}(f)(x)\right)-j(f(x))\right\|_{\mathcal{B}}$$

$$= D\left(\left(D_n^{\mathcal{F}}(f)\right)(x), f(x)\right).$$

Since f and j are continuous we get that g is continuous.

Hence by Theorem 32.4 we have

$$c_1\left(\omega_2^{\varphi}\left(g, \frac{1}{\sqrt{n}}\right) + \omega_1\left(g, \frac{1}{n}\right)\right) \leq$$

$$\|D_n^{v} g - g\|_{\infty} \leq c_2\left(\omega_2^{\varphi}\left(g, \frac{1}{\sqrt{n}}\right) + \omega_1\left(g, \frac{1}{n}\right)\right),$$

the last proves the theorem. ■

We use

Definition 32.8. Let $f \in C([0,1])$, we define

$$\omega_2(f,h) = \sup\left\{\left|f(u) - 2f\left(\frac{u+v}{2}\right) + f(v)\right|,\right. \tag{32.11}$$

$$\left. u,\ v \in [0,1],\ |u-v| \leq 2h\right\},\ h > 0.$$

Let $f \in C([0,1],\ X)$, $(X,\ \|\cdot\|)$ normed vector space, we also define

$$\omega_2^{v}(f,h) = \sup\left\{\left\|f(u) - 2f\left(\frac{u+v}{2}\right) + f(v)\right\|,\right. \tag{32.12}$$

$$\left. u,\ v \in [0,1],\ |u-v| \leq 2h\right\},\ h > 0.$$

Let $f \in C([0,1],\ \mathbb{R}_{\mathcal{F}})$, we further define

$$\omega_2^{(\mathcal{F})}(f,h) = \sup\left\{D\left(f(u) \oplus f(v), 2 \odot f\left(\frac{u+v}{2}\right)\right),\right.$$

$$\left. u,\ v \in [0,1],\ |u-v| \leq 2h\right\},\ h > 0. \tag{32.13}$$

We make

Remark 32.9. Let $f \in C([0,1],\ X)$, $x^* \in X^*$, with $|||x^*||| \leq 1$, then $g = x^* \circ f : [0,1] \longrightarrow \mathbb{R}$ is continuous.

We observe

$$\omega_2(g,h) = \omega_2(x^* \circ f, h) =$$

$$\sup\left\{\left|x^*\left(f(u) - 2f\left(\frac{u+v}{2}\right) + f(v)\right)\right|,\right.$$

$$\left. u,\ v \in [0,1],\ |u-v| \leq 2h\right\} \leq$$

$$\sup\left\{|||x^*||| \cdot \left\|f(u) - 2f\left(\frac{u+v}{2}\right) + f(v)\right\|,\right.$$

$$u,\ v \in [0,1]\,,\ \ |u-v| \le 2h\} \le$$

$$\sup\left\{\left\|f(u) - 2f\left(\frac{u+v}{2}\right) + f(v)\right\|,\right.$$

$$u,\ v \in [0,1]\,,\ \ |u-v| \le 2h\} = \omega_2^v(f,h)\,.$$

That is, we got that

$$\omega_2(x^* \circ f, h) \le \omega_2^v(f,h)\,,\ h > 0. \tag{32.14}$$

Next, let $X = \overline{C}([0,1])^2$and j as in Theorem 32.6.

Let $f \in C([0,1]\,,\ \mathbb{R}_{\mathcal{F}})\,,$ and consider $g(x) = j(f(x))\,,\ x \in [0,1]\,,$ i.e. $g \in C([0,1]\,,\ X)\,.$

By Theorem 32.6, (ii), we observe that

$$\left\|g(u) - 2g\left(\frac{u+v}{2}\right) + g(v)\right\| =$$

$$\left\|j(f(u) \oplus f(v)) - j\left(2 \odot f\left(\frac{u+v}{2}\right)\right)\right\|_{\mathcal{B}} =$$

$$D\left(f(u) \oplus f(v)\,, 2 \odot f\left(\frac{u+v}{2}\right)\right),$$

which implies

$$\omega_2^v(j \circ f, h) = \omega_2^{(\mathcal{F})}(f,h)\,,\ h > 0. \tag{32.15}$$

We use

Definition 32.10. Let $f \in C([0,1])\,,$ we define the Bernstein polynomial operators,

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x)\, f\left(\frac{k}{n}\right),\ x \in [0,1]\,, \tag{32.16}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ n \in \mathbb{N}.$$

Let $f \in C([0,1]\,, X)\,,\ (X, \|\cdot\|)$ normed vector space. We define also the vectorial Bernstein operators,

$$B_n^v(f,x) = \sum_{k=0}^{n} p_{n,k}(x)\, f\left(\frac{k}{n}\right),\ x \in [0,1]\,,\ n \in \mathbb{N}. \tag{32.17}$$

Let $f \in C([0,1]\,,\ \mathbb{R}_{\mathcal{F}})\,,$ we define further the fuzzy Bernstein operators

$$B_n^{\mathcal{F}}(f,x) = \sum_{k=0}^{n}{}^{*} p_{n,k}(x) \odot f\left(\frac{k}{n}\right),\ x \in [0,1]\,,\ n \in \mathbb{N}. \tag{32.18}$$

We make

Remark 32.11. Let $x^* \in X^*$, $f \in C([0,1], X)$.

Then

$$B_n(x^* \circ f)(x) - x^* \circ f(x) = x^*(B_n^{\upsilon}(f)(x) - f(x)). \tag{32.19}$$

Let j as in Theorem 32.6 and $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$.

Then

$$\|(B_n^{\upsilon}(j \circ f))(x) - (j \circ f)(x)\| = \left\| j\left(B_n^{(\mathcal{F})}(f)(x)\right) - j(f(x)) \right\|_{\mathcal{B}} =$$

$$D\left(\left(B_n^{\mathcal{F}}(f)\right)(x), f(x)\right). \tag{32.20}$$

We mention the celebrated major result

Theorem 32.12. ([239], p.97) *For $f \in C([0,1])$, $n \in \mathbb{N}$, we have*

$$\|B_n(f) - f\|_{\infty} \leq \omega_2\left(f, \frac{1}{\sqrt{n}}\right), \tag{32.21}$$

a sharp inequality.

We present

Theorem 32.13. *For $f \in C([0,1], X)$, $(X, \|\cdot\|)$ a normed vector space, $n \in \mathbb{N}$, we have*

$$\sup_{x \in [0,1]} \|(B_n^{\upsilon} f)(x) - f(x)\| \leq \omega_2^{\upsilon}\left(f, \frac{1}{\sqrt{n}}\right). \tag{32.22}$$

Proof. Let $x^* \in X^*$ be fixed with $|||x^*||| \leq 1$. Then $x^* \circ f \in C([0,1])$ and by (32.21) and (32.14) we have

$$\|B_n(x^* \circ f) - x^* \circ f\|_{\infty} \leq \omega_2\left(x^* \circ f, \frac{1}{\sqrt{n}}\right)$$

$$\leq \omega_2^{\upsilon}\left(f, \frac{1}{\sqrt{n}}\right).$$

That is

$$|x^*((B_n^{\upsilon} f)(x) - f(x))| \leq \omega_2^{\upsilon}\left(f, \frac{1}{\sqrt{n}}\right), \ \forall x \in [0,1].$$

Thus

$$\sup_{\substack{x^* \in X^* \\ |||x^*||| \leq 1}} |x^*((B_n^{\upsilon} f)(x) - f(x))| \leq \omega_2^{\upsilon}\left(f, \frac{1}{\sqrt{n}}\right), \ \forall x \in [0,1].$$

By Theorem 32.2 we derive

$$\|(B_n^{\upsilon} f)(x) - f(x)\| \le \omega_2^{\upsilon}\left(f, \frac{1}{\sqrt{n}}\right),$$

$\forall x \in [0,1]$, proving the claim. ■

Next we give

Theorem 32.14. *For $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$, $n \in \mathbb{N}$, we have*

$$\sup_{x\in[0,1]} D\left(\left(B_n^{\mathcal{F}} f\right)(x), f(x)\right) \le \omega_2^{(\mathcal{F})}\left(f, \frac{1}{\sqrt{n}}\right). \tag{32.23}$$

Proof. Consider j, X as in Theorem 32.6. Then by Theorem 32.13 and (32.15) we obtain

$$\sup_{x\in[0,1]} \|(B_n^{\upsilon}(j \circ f))(x) - (j \circ f)(x)\| \le \omega_2^{\upsilon}\left(j \circ f, \frac{1}{\sqrt{n}}\right)$$

$$= \omega_2^{(\mathcal{F})}\left(f, \frac{1}{\sqrt{n}}\right). \tag{32.24}$$

Clearly from (32.20) and (32.24) we get (32.23). ■

We need

Definition 32.15. ([239], p.151) Let $f \in C([0,1])$, $n \in \mathbb{N}$. We define the Durrmeyer type operators (the genuine Bernstein-Durrmeyer operators)

$$M_n^{-1,-1}(f,x) = f(0)(1-x)^n + f(1)x^n +$$

$$(n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 f(t)\, p_{n-2,k-1}(t)\, dt. \tag{32.25}$$

Similarly we define

Definition 32.16. Let $f \in C([0,1], X)$, where $(X, \|\cdot\|)$ a normed vector space.

We define

$$^{\upsilon}M_n^{-1,-1}(f,x) = f(0)(1-x)^n + f(1)x^n +$$

$$(n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 f(t)\, p_{n-2,k-1}(t)\, dt. \tag{32.26}$$

Definition 32.17. Let $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$.

We define

$$^{(\mathcal{F})}M_n^{-1,-1}(f,x) = f(0) \odot (1-x)^n \oplus f(1) \odot x^n \oplus$$

$$(n-1) \odot \sum_{k=1}^{n-1}{}^{*} p_{n,k}(x) \odot (FR)\int_0^1 f(t) \odot p_{n-2,k-1}(t)\, dt. \qquad (32.27)$$

We use

Theorem 32.18. ([239], p.155) *For $f \in C([0,1])$, $n \in \mathbb{N}$, we have*

$$\left\| M_n^{-1,-1}(f) - f \right\|_\infty \leq \frac{5}{4}\omega_2\left(f, \frac{1}{\sqrt{n+1}}\right). \qquad (32.28)$$

We make

Remark 32.19. Let $x^* \in X^*$, $f \in C([0,1],\ X)$.

Then

$$M_n^{-1,-1}(x^* \circ f)(x) - x^* \circ f(x) = x^*\left({}^{\upsilon}M_n^{-1,-1}(f)(x) - f(x)\right). \qquad (32.29)$$

Let j as in Theorem 32.6 and $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$.

Then

$$\left\| \left({}^{\upsilon}M_n^{-1,-1}(j \circ f)\right)(x) - (j \circ f)(x) \right\| =$$

$$\left\| j\left({}^{(\mathcal{F})}M_n^{-1,-1}(f)(x)\right) - j(f(x)) \right\|_{\mathcal{B}} =$$

$$D\left(\left({}^{(\mathcal{F})}M_n^{-1,-1}(f)\right)(x), f(x)\right). \qquad (32.30)$$

We present

Theorem 32.20. *For $f \in C([0,1], X)$, $(X, \|\cdot\|)$ a normed vector space, $n \in \mathbb{N}$, we have*

$$\sup_{x\in[0,1]} \left\| \left({}^{\upsilon}M_n^{-1,-1} f\right)(x) - f(x) \right\| \leq \frac{5}{4}\omega_2^{\upsilon}\left(f, \frac{1}{\sqrt{n+1}}\right). \qquad (32.31)$$

Proof. Let $x^* \in X^*$ be fixed with $|||x^*||| \leq 1$. Then $x^* \circ f \in C([0,1])$ and by (32.28) and (32.14) we have

$$\left\| M_n^{-1,-1}(x^* \circ f) - x^* \circ f \right\|_\infty \leq \frac{5}{4}\omega_2\left(x^* \circ f, \frac{1}{\sqrt{n+1}}\right)$$

$$\leq \frac{5}{4}\omega_2^{\upsilon}\left(f, \frac{1}{\sqrt{n+1}}\right).$$

That is

$$\left|x^*\left({}^{\upsilon}M_n^{-1,-1}(f)\right)(x)-f(x)\right|\leq\frac{5}{4}\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{n+1}}\right),$$

$\forall x\in[0,1]$.

Thus

$$\sup_{\substack{x^*\in X^*\\ \left|\left\|x^*\right\|\right|\leq 1}}\left|x^*\left({}^{\upsilon}M_n^{-1,-1}(f)\right)(x)-f(x)\right|\leq\frac{5}{4}\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{n+1}}\right),$$

$\forall x\in[0,1]$.

By Theorem 32.2 we derive

$$\left\|\left({}^{\upsilon}M_n^{-1,-1}f\right)(x)-f(x)\right\|\leq\frac{5}{4}\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{n+1}}\right),$$

$\forall x\in[0,1]$, proving the claim. ■

Next we present

Theorem 32.21. *For $f\in C([0,1],\mathbb{R}_{\mathcal{F}})$, $n\in\mathbb{N}$, we have*

$$\sup_{x\in[0,1]}D\left(\left({}^{(\mathcal{F})}M_n^{-1,-1}f\right)(x),f(x)\right)\leq\frac{5}{4}\omega_2^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{n+1}}\right).\qquad(32.32)$$

Proof. Consider j, X as in Theorem 32.6. Then by Theorem 32.20 and (32.15) we obtain

$$\sup_{x\in[0,1]}\left\|\left({}^{\upsilon}M_n^{-1,-1}(j\circ f)\right)(x)-(j\circ f)(x)\right\|\leq$$

$$\frac{5}{4}\omega_2^{\upsilon}\left(j\circ f,\frac{1}{\sqrt{n+1}}\right)=\frac{5}{4}\omega_2^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{n+1}}\right).\qquad(32.33)$$

Clearly from (32.30) and (32.33) we get (32.32). ■

We need

Definition 32.22. ([175]) For $f\in C([0,1])$, $m\in\mathbb{N}$, and $0\leq\beta\leq\gamma$, we define the Stancu-type positive linear operators

$$\left(L_{m0}^{\langle 0\beta\gamma\rangle}f\right)(x)=\sum_{k=0}^{m}f\left(\frac{k+\beta}{m+\gamma}\right)p_{m,k}(x),\qquad(32.34)$$

$x\in[0,1]$, $p_{m,k}(x)=\begin{pmatrix}m\\k\end{pmatrix}x^k(1-x)^{m-k}$.

We also give

Definition 32.23. For $f \in C([0,1], X)$, $(X, \|\cdot\|)$ a normed vector space, $m \in \mathbb{N}$, and $0 \leq \beta \leq \gamma$, we define

$$\left({}^{v}L_{m0}^{\langle 0\beta\gamma\rangle} f\right)(x) = \sum_{k=0}^{m} f\left(\frac{k+\beta}{m+\gamma}\right) p_{m,k}(x), \tag{32.35}$$

$x \in [0,1]$.

Definition 32.24. Let $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$, $m \in \mathbb{N}$, $0 \leq \beta \leq \gamma$, we define

$$\left({}^{\mathcal{F}}L_{m0}^{\langle 0\beta\gamma\rangle} f\right)(x) = \sum_{k=0}^{m}{}^{*} f\left(\frac{k+\beta}{m+\gamma}\right) \odot p_{m,k}(x), \tag{32.36}$$

$x \in [0,1]$.

We use

Theorem 32.25. (Gonska and Meier [175]) *For* $f \in C([0,1])$, $h > 0$, $0 \leq \beta \leq \gamma$, $m \in \mathbb{N}$, *and* $x \in [0,1]$ *it holds*

$$\left|\left(L_{m0}^{\langle 0\beta\gamma\rangle} f\right)(x) - f(x)\right| \leq$$

$$\left[3 + \max\left\{h^{-2}, 1\right\} \frac{\left(\left(\gamma^2 - m\right)x^2 + mx + \beta^2\right)}{(m+\gamma)^2}\right] \omega_2(f,h)$$

$$+\frac{2|\beta - \gamma x|}{(m+\gamma)} \max\left\{h^{-1}, 1\right\} \omega_1(f,h). \tag{32.37}$$

We obtain

Corollary 32.26. For $\mathbb{N} \ni m > \lceil \gamma^2 \rceil$ ($\lceil \cdot \rceil$ is the ceiling), $f \in C([0,1])$ we obtain

$$\left\|L_{m0}^{\langle 0\beta\gamma\rangle} f - f\right\|_{\infty} \leq \left[3 + \frac{\left(m^3 + 4m\beta^2\left(m - \gamma^2\right)\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right]$$

$$\omega_2\left(f, \frac{1}{\sqrt{m}}\right) + \frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)} \omega_1\left(f, \frac{1}{\sqrt{m}}\right). \tag{32.38}$$

Proof. Choose $h = \frac{1}{\sqrt{m}}$ into (32.37) and maximize the right hand side of (32.37). ■

We make

Remark 32.27. Let $f \in C([0,1], X)$, $x^* \in X^*$, with $\||x^*\|| \leq 1$, and $x^* \circ f \in C([0,1])$ we easily get

$$\omega_1(x^* \circ f, \delta) \leq \omega_1(f, \delta), \ \delta > 0. \tag{32.39}$$

Also see that

$$L_{m0}^{\langle 0\beta\gamma\rangle}\left(x^*\circ f\right)(x)-x^*\circ f(x)=$$
$$x^*\left({}^{\upsilon}L_{m0}^{\langle 0\beta\gamma\rangle}(f)(x)-f(x)\right). \tag{32.40}$$

Let j as in Theorem 32.6 and $f\in C\left([0,1],\mathbb{R}_{\mathcal{F}}\right)$.

Then

$$\left\|\left({}^{\upsilon}L_{m0}^{\langle 0\beta\gamma\rangle}(j\circ f)\right)(x)-(j\circ f)(x)\right\|=$$
$$\left\|j\left({}^{\mathcal{F}}L_{m0}^{\langle 0\beta\gamma\rangle}(f)(x)\right)-j(f(x))\right\|_{\mathcal{B}}=$$
$$D\left(\left({}^{\mathcal{F}}L_{m0}^{\langle 0\beta\gamma\rangle}(f)\right)(x),f(x)\right). \tag{32.41}$$

We give

Theorem 32.28. *For $f\in C([0,1],X)$, $(X,\|\cdot\|)$ a normed vector space, $m\in\mathbb{N}$, $m>\lceil\gamma^2\rceil$, we get :*

$$\sup_{x\in[0,1]}\left\|\left({}^{\upsilon}L_{m0}^{\langle 0\beta\gamma\rangle}f\right)(x)-f(x)\right\|\le$$
$$\left[3+\frac{\left(m^3+4m\beta^2\left(m-\gamma^2\right)\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right]\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{m}}\right)$$
$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1\left(f,\frac{1}{\sqrt{m}}\right). \tag{32.42}$$

Proof. Let $x^*\in X^*$ be fixed with $|||x^*|||\le 1$. Then $x^*\circ f\in C([0,1])$ and by (32.38) and (32.14), (32.39), we have

$$\left\|L_{m0}^{\langle 0\beta\gamma\rangle}\left(x^*\circ f\right)-x^*\circ f\right\|_{\infty}\le\left[3+\left(\frac{m^3+4m\beta^2\left(m-\gamma^2\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right)\right]$$
$$\omega_2\left(x^*\circ f,\frac{1}{\sqrt{m}}\right)+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1\left(x^*\circ f,\frac{1}{\sqrt{m}}\right)\le$$
$$\left[3+\left(\frac{m^3+4m\beta^2\left(m-\gamma^2\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right)\right]\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{m}}\right)$$
$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1\left(f,\frac{1}{\sqrt{m}}\right).$$

That is

$$\left|x^*\left({}^{\upsilon}L_{m0}^{\langle 0\beta\gamma\rangle}(f)(x)-f(x)\right)\right|\le$$
$$\left[3+\left(\frac{m^3+4m\beta^2\left(m-\gamma^2\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right)\right]\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{m}}\right)$$

$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1\left(f,\frac{1}{\sqrt{m}}\right),\ \forall x\in[0,1].$$

Thus

$$\sup_{\substack{x^*\in X^*\\ |||x^*|||\leq 1}}\left|x^*\left({}^{\upsilon}L_{m0}^{\langle 0\beta\gamma\rangle}(f)(x)-f(x)\right)\right|\leq$$

$$\left[3+\left(\frac{m^3+4m\beta^2\left(m-\gamma^2\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right)\right]\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{m}}\right)$$

$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1\left(f,\frac{1}{\sqrt{m}}\right),\ \forall x\in[0,1].$$

By Theorem 32.2 we derive

$$\left\|\left({}^{\upsilon}L_{m0}^{\langle 0\beta\gamma\rangle}(f)\right)(x)-f(x)\right\|\leq$$

$$\left[3+\frac{\left(m^3+4m\beta^2\left(m-\gamma^2\right)\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right]\omega_2^{\upsilon}\left(f,\frac{1}{\sqrt{m}}\right)$$

$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1\left(f,\frac{1}{\sqrt{m}}\right),\ \forall x\in[0,1].$$

proving the claim. ■

We present

Theorem 32.29. *For $f\in C([0,1],\mathbb{R}_{\mathcal{F}})$, $m\in\mathbb{N}: m>\lceil\gamma^2\rceil$, we obtain*

$$\sup_{x\in[0,1]}D\left(\left({}^{\mathcal{F}}L_{m0}^{\langle 0\beta\gamma\rangle}f\right)(x),f(x)\right)\leq$$

$$\left[3+\left(\frac{m^3+4m\beta^2\left(m-\gamma^2\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right)\right]\omega_2^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{m}}\right)$$

$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{m}}\right). \tag{32.43}$$

Proof. Consider j, X as in Theorem 32.6. Then by Theorem 32.28 and (32.15), and

$$\omega_1(j\circ f,\delta)=\omega_1^{(\mathcal{F})}(f,\delta),\ \delta>0,$$

we get

$$\sup_{x\in[0,1]}\left\|\left({}^{\upsilon}L_{m0}^{\langle 0\beta\gamma\rangle}(j\circ f)\right)(x)-(j\circ f)(x)\right\|\leq$$

$$\left[3+\left(\frac{m^3+4m\beta^2\left(m-\gamma^2\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right)\right]\omega_2^{\upsilon}\left(j\circ f,\frac{1}{\sqrt{m}}\right)$$

$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1\left(j\circ f,\frac{1}{\sqrt{m}}\right)=$$

$$\left[3+\left(\frac{m^3+4m\beta^2\left(m-\gamma^2\right)}{4\left(m-\gamma^2\right)(m+\gamma)^2}\right)\right]\omega_2^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{m}}\right) \tag{32.44}$$

$$+\frac{2(\beta+\gamma)\sqrt{m}}{(m+\gamma)}\omega_1^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{m}}\right),\ m\in\mathbb{N}:m>\left\lceil\gamma^2\right\rceil.$$

for clearly from (32.41) and (32.44) we get (32.43). ■

We use

Definition 32.30. Let $f\in C([0,1])$, and $x\in[0,1]$ such that $x+4h\in[0,1]$, where $h>0$. We define the modulus of smoothness of order 4 as,

$$\begin{aligned}\omega_4(f,\delta)=\sup\{&|(f(x)+f(x+4h)+6f(x+2h))\\&-4(f(x+h)+f(x+3h))|:\\&x,\ x+4h\in[0,1],\ 0<h\le\delta\},\ \delta>0.\end{aligned} \tag{32.45}$$

Clearly $\omega_4(f,\cdot)$ is a non-decreasing function, and $\omega_4(f,\delta)\longrightarrow 0$, as $\delta\longrightarrow 0$.

Definition 32.31. Let $f\in C([0,1],X)$, $(X,\|\cdot\|)$ a normed vector space. We also define

$$\begin{aligned}\omega_4^{\upsilon}(f,\delta)=\sup\{&\|(f(x)+f(x+4h)+6f(x+2h))\\&-4(f(x+h)+f(x+3h))\|:\\&x,\ x+4h\in[0,1],\ 0<h\le\delta\},\ \delta>0.\end{aligned} \tag{32.46}$$

Definition 32.32. Let $f\in C([0,1],\mathbb{R}_{\mathcal{F}})$, we further define

$$\begin{aligned}\omega_4^{(\mathcal{F})}(f,\delta)=\sup\{&D((f(x)\oplus f(x+4h)\oplus 6\odot f(x+2h)),\\&4\odot(f(x+h)\oplus f(x+3h))):\\&x,\ x+4h\in[0,1],\ 0<h\le\delta\},\ \delta>0.\end{aligned} \tag{32.47}$$

We make

Remark 32.33. Let $f\in C([0,1],\ X)$, $x^*\in X^*$, with $|||x^*|||\le 1$, then $x^*\circ f\in C([0,1])$.

We observe that

$$\omega_4(x^*\circ f,\delta)=\sup\{|x^*[(f(x)+f(x+4h)+6f(x+2h))$$

$$-4(f(x+h)+f(x+3h))]|:$$
$$x,\ x+4h\in[0,1],\ 0<h\le\delta\}\le$$
$$\sup\{\||x^*\||\cdot\|(f(x)+f(x+4h)+6f(x+2h))$$
$$-4(f(x+h)+f(x+3h))\|:$$
$$x,\ x+4h\in[0,1],\ 0<h\le\delta\}\le$$
$$\sup\{\|(f(x)+f(x+4h)+6f(x+2h))$$
$$-4(f(x+h)+f(x+3h))\|:$$
$$x,\ x+4h\in[0,1],\ 0<h\le\delta\}=\omega_4^{v}(f,\delta)$$

We have established that

$$\omega_4(x^*\circ f,\delta)\le\omega_4^{v}(f,\delta),\ \delta>0. \tag{32.48}$$

Next, let $X=\overline{C}([0,1])^2$ and j as in Theorem 32.6. Let $f\in C([0,1],\mathbb{R}_{\mathcal{F}})$, and consider $g(x)=j(f(x))$, $x\in[0,1]$, i.e. $g\in C([0,1],\ X)$.

By Theorem 32.6, (ii), we notice

$$\|(g(x)+g(x+4h)+6g(x+2h))$$
$$-4(g(x+h)+g(x+3h))\|=$$
$$\|j(f(x)\oplus f(x+4h)\oplus 6\odot f(x+2h))$$
$$-j(4\odot(f(x+h)\oplus f(x+3h)))\|_{\mathcal{B}}=$$
$$D((f(x)\oplus f(x+4h)\oplus 6\odot f(x+2h)),$$
$$4\odot(f(x+h)\oplus f(x+3h))),$$

which implies

$$\omega_4^{v}(j\circ f,\delta)=\omega_4^{(\mathcal{F})}(f,\delta),\ \delta>0. \tag{32.49}$$

We use

Definition 32.34. Let $f\in C([0,1])$, we define the special Stancu operator ([271])

$$S_n(f,x)=\frac{2(n!)}{(2n)!}\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}(nx)_k(n-nx)_{n-k}, \tag{32.50}$$

where $(a)_0=1$, $(a)_b=\prod_{k=0}^{b-1}(a-k)$, $a\in\mathbb{R}$, $b\in\mathbb{N}$, $n\in\mathbb{N}$, $x\in[0,1]$.

Definition 32.35. Let $f\in C([0,1],X)$, $(X,\|\cdot\|)$ a normed vector space.

We also define

$$S_n^{\upsilon}(f,x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(nx)_k(n-nx)_{n-k}, \tag{32.51}$$

$n \in \mathbb{N}$, $x \in [0,1]$.

Definition 32.36. Let $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$.
We further define

$$S_n^{(\mathcal{F})}(f,x) = \frac{2(n!)}{(2n)!} \odot \sum_{k=0}^{n}{}^{*} f\left(\frac{k}{n}\right) \odot \left(\binom{n}{k}(nx)_k(n-nx)_{n-k}\right), \tag{32.52}$$

$n \in \mathbb{N}$, $x \in [0,1]$.

We make
Remark 32.37. Let $x^* \in X^*$, $f \in C([0,1], X)$.
Then

$$\begin{aligned} &S_n(x^* \circ f)(x) - M_n^{-1,-1}(x^* \circ f)(x) = \\ &x^*\left(S_n^{\upsilon}(f)(x) - {}^{\upsilon}M_n^{-1,-1}(f)(x)\right). \end{aligned} \tag{32.53}$$

Let j as in Theorem 32.6 and $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$.
Then

$$\begin{aligned} &\left\|(S_n^{\upsilon}(j \circ f))(x) - \left({}^{\upsilon}M_n^{-1,-1}(j \circ f)\right)(x)\right\| = \\ &\left\|j\left(S_n^{(\mathcal{F})}(f)(x)\right) - j\left({}^{(\mathcal{F})}M_n^{-1,-1}(f)(x)\right)\right\|_{\mathcal{B}} = \\ &D\left(\left(S_n^{(\mathcal{F})}(f)\right)(x), \left({}^{(\mathcal{F})}M_n^{-1,-1}(f)\right)(x)\right). \end{aligned} \tag{32.54}$$

We mention
Theorem 32.38. ([176], p. 75) *Let* $f \in C([0,1])$, $n \in \mathbb{N}$. *Then*

$$\left|\left(S_n - M_n^{-1,-1}\right)(f;x)\right| \le c_1\omega_4\left(f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right), \tag{32.55}$$

$\forall x \in [0,1]$, *where* $c_1 > 0$ *is an absolute constant independent of* n, f *and* x.

We give
Theorem 32.39. *Let* $f \in C([0,1], X)$, $(X, \|\cdot\|)$ *a normed vector space,* $n \in \mathbb{N}$, $x \in [0,1]$.
Then

$$\left\|\left(S_n^{\upsilon} - {}^{\upsilon}M_n^{-1,-1}\right)(f;x)\right\| \le c_1\omega_4^{\upsilon}\left(f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right), \tag{32.56}$$

where $c_1 > 0$ *is a universal constant.*

Proof. Let $x^* \in X^*$ be fixed with $|||x^*||| \leq 1$. Then $x^* \circ f \in C([0,1])$ and by (32.55) and (32.48) we have

$$\left|\left(S_n - M_n^{-1,-1}\right)(x^* \circ f; x)\right| \leq c_1 \omega_4 \left(x^* \circ f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right)$$
$$\leq c_1 \omega_4^{\upsilon} \left(f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right).$$

That is by (32.53) we obtain

$$\left|x^* \left(S_n^{\upsilon}(f)(x) - {}^{\upsilon}M_n^{-1,-1}(f)(x)\right)\right| \leq$$
$$c_1 \omega_4^{\upsilon} \left(f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right).$$

Hence

$$\sup_{\substack{x^* \in X^* \\ |||x^*||| \leq 1}} \left|x^* \left(S_n^{\upsilon}(f)(x) - {}^{\upsilon}M_n^{-1,-1}(f)(x)\right)\right| \leq$$
$$c_1 \omega_4^{\upsilon} \left(f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right),$$

$\forall x \in [0,1]$.

By Theorem 32.2 we derive (32.56). ■

Finally we give

Theorem 32.40. *Let* $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$, $n \in \mathbb{N}$, $x \in [0,1]$. *Then*

$$D\left(\left(S_n^{(\mathcal{F})}(f)\right)(x), \left({}^{(\mathcal{F})}M_n^{-1,-1}(f)\right)(x)\right) \leq$$
$$c_1 \omega_4^{(\mathcal{F})} \left(f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right), \tag{32.57}$$

where $c_1 > 0$ *is a universal constant.*

Proof. Consider j, X as in Theorem 32.6. Then by Theorem 32.39 and (32.49) we get that

$$\left\|\left(S_n^{\upsilon} - {}^{\upsilon}M_n^{-1,-1}\right)(j \circ f; x)\right\| \leq$$
$$c_1 \omega_4^{\upsilon} \left(j \circ f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right) =$$
$$c_1 \omega_4^{(\mathcal{F})} \left(f, \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right). \tag{32.58}$$

Clearly from (32.54) and (32.58) we obtain (32.57). ■

33

High Order Multivariate Approximation by Multivariate Wavelet Type and Neural Network Operators in the Fuzzy Sense

Here we study in terms of multivariate fuzzy high approximation to the multivariate unit several basic sequences of multivariate fuzzy wavelet type operators and multivariate fuzzy neural network operators. These operators are multivariate fuzzy analogs of earlier studied multivariate real ones. The produced results generalize earlier real ones into the fuzzy setting. Here the high order multivariate fuzzy pointwise convergence with rates to the multivariate fuzzy unit operator is established through multivariate fuzzy inequalities involving the multivariate fuzzy moduli of continuity of the Nth order ($N \geq 1$) H-fuzzy partial derivatives, of the engaged multivariate fuzzy number valued function. The purpose of embedding fuzziness into multivariate classical analysis is to better understand, explain and describe the imprecise, uncertain and chaotic phenomena of the real world and then derive useful conclusions. This chapter relies on [49].

33.1 Fuzzy Real Analysis Background

We need the following background

Definition 33.1(see [283]) Let $\mu : \mathrm{R} \to [0,1]$ with the following properties

(i) is normal, i.e., $\exists x_0 \in \mathrm{R};\ \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x)\ \mu(y)\}$, $\forall x, y \in \mathrm{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

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(iii) μ is upper semicontinuous on R, i.e. $\forall x_0 \in$ R and $\forall \epsilon > 0$, $\exists$ neighborhood $V(x_0)$: $\mu(x) \leq \mu(x_0) + \epsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{supp(\mu)}$ is compact in R (where $supp(\mu) := \{x \in \mathrm{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathrm{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathrm{R}_{\mathcal{F}}$, for any $x_0 \in$ R, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathrm{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathrm{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathrm{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0, 1]$, $[\mu]^r$ is a closed and bounded interval of R ([172]).

For $u, v \in \mathrm{R}_{\mathcal{F}}$ and $\lambda \in$ R, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \forall r \in [0, 1],$$

where

- $[u]^r + [v]^r$ means the usual addition of two integrals (as subsets of R) and
- $\lambda [u]^r$ means the usual product between a scalar and a subset of R (see, e.g., [283]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = [u_-^{(r)}, \ u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in$ R, $\forall r \in [0, 1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathrm{R}_{\mathcal{F}} \times \mathrm{R}_{\mathcal{F}} \to \mathrm{R}_+$ by

$$D(u, v) := \sup_{r \in [0,1]} \max \left\{ |u_-^{(r)} - v_-^{(r)}|, \ |u_+^{(r)} - v_+^{(r)}| \right\},$$

where

$$[v]^r = [v_-^{(r)}, \ v_+^{(r)}]; \ u, v \in \mathrm{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [283], [284].

Let $f, g : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g) := \sup_{x \in \mathbb{R}^m} D(f(x), g(x)).$$

Here Σ^* stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \forall u \in \mathbb{R}_{\mathcal{F}}.$$

We need

Remark 33.2 ([29]). Here $r \in [0,1]$ $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}, i = 1, \ldots, \ m \in \mathbb{N}$. Assume that

$$\sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}\,, for \ i = 1, \ldots, \ m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max\left(\sum_{i=1}^m x_i^{(r)}, \sum_{i=1}^m y_i^{(r)}\right) \leq \sum_{i=1}^m \sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right).$$

Definition 33.3 Let $f \in C(\mathbb{R}^m), \ m \in \mathbb{N}$, which is bounded or uniformly continuous, we define $(h > 0)$

$$w_1(f,h) := \sup_{all \ x_i, \ x_i' \in \mathbb{R}, \ |x_i - x_i'| \leq h, \ for \ i=1,\ldots, \ m} |f(x_1, \ldots, \ x_m) - f(x_1', \ldots, \ x_m')|.$$

Definition 33.4 Let $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, we define the fuzzy modulus of continuity of f by

$$w_1^{(\mathcal{F})}(f,\delta) = \sup_{x,y \in \mathbb{R}^m, \ |x_i - y_i| \leq \delta, \ for \ i=1,\ldots, \ m} D(f(x), \ f(y)), \ \delta > 0,$$

where $x = (x_1, \ldots, \ x_m), \ y = (y_1, \ldots, \ y_m)$.

For $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, we use

$$[f]^r = [f_-^{(r)}, \ f_+^{(r)}],$$

where $f_\pm^{(r)} : \mathbb{R}^m \to \mathbb{R}, \ \forall \ r \in [0,1]$.

We need

Proposition 33.5 Let $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. Suppose that $w_1^{(\mathcal{F})}(f,\delta), \ w_1(f_-^{(r)}, \delta)$, $w_1(f_+^{(r)}, \delta)$ are finite for any $\delta > 0, \ r \in [0,1]$.

Then

$$w_1^{(\mathcal{F})}(f,\delta) = \sup_{r\in[0,1]} \max\{w_1(f_-^{(r)},\delta),\ w_1(f_+^{(r)},\delta)\}.$$

Proof. By Proposition 1 of [37]. ■

We define by $C_{\mathcal{F}}^U(\mathrm{R}^m)$, the space of fuzzy uniformly continuous functions from $\mathrm{R}^m \to \mathrm{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathrm{R}^m)$ is the space of fuzzy continuous functions on R^m, and $C_b(\mathrm{R}^m, \mathrm{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 33.6([37]) Let $f \in C_{\mathcal{F}}^U(\mathrm{R}^m)$. Then $w_1^{(\mathcal{F})}(f,\delta) < \infty$, for any $\delta > 0$.

Proposition 33.7([37]) It holds

$$\lim_{\delta\to 0} w_1^{(\mathcal{F})}(f,\delta) = w_1^{(\mathcal{F})}(f,0) = 0,$$

iff $f \in C_{\mathcal{F}}^U(\mathrm{R}^m)$.

Proposition 33.8([37]) Let $f \in C_{\mathcal{F}}(\mathrm{R}^m)$. Then $f_\pm^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over R^m, respectively in $\pm$.

Note 33.9 It is clear by Propositions 33.5, 33.7, that if $f \in C_{\mathcal{F}}^U(\mathrm{R}^m)$, then $f_\pm^{(r)} \in C_U(\mathrm{R}^m)$ (uniformly continuous on R^m).

We need

Definition 33.10 Let $x, y \in \mathrm{R}_{\mathcal{F}}$. If there exists $z \in \mathrm{R}_{\mathcal{F}} :\ x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.

Definition 33.11([283]) Let $T := [x_0,\ x_0 + \beta] \subset \mathrm{R}$, with $\beta > 0$. A function $f : T \to \mathrm{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathrm{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h\to 0+} \frac{f(x+h) - f(x)}{h},\quad \lim_{h\to 0+} \frac{f(x) - f(x-h)}{h}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x),\ f(x) - f(x-h)$ exist in $\mathrm{R}_{\mathcal{F}}$ in an neighborhood of x.

Definition 33.12 We denote by $C_{\mathcal{F}}^N(\mathrm{R}^m)$, $N \in \mathrm{N}$, the space of all N-times fuzzy continuously differentiable functions from R^m into $\mathrm{R}_{\mathcal{F}}$.

Here fuzzy partial derivatives are defined via Definition 33.11 in the obvious way as in the ordinary real case.

We mention

Theorem 33.13([202]) Let $f : [a,b] \subseteq \mathrm{R} \to \mathrm{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b],\ 0 \le r \le 1$. Clearly

$$[f(t)]^r = [f(t)_-^{(r)},\ f(t)_+^{(r)}] \subseteq \mathrm{R}.$$

Then $(f(t))_\pm^{(r)}$ are differentiable and

$$[f'(t)]^r = [(f(t)_-^{(r)})', \ (f(t)_+^{(r)})'].$$

That is

$$(f')_{\pm}^{(r)} = (f_{\pm}^{(r)})', \ \forall r \in [0,1].$$

Remark 33.14 (see also [35]) Let $f \in C^N(\mathrm{R}, \mathrm{R}_{\mathcal{F}}), \ N \geq 1$. Then by Theorem 33.13 we obtain $f_{\pm}^{(r)} \in C^N(\mathrm{R})$ and

$$[f^{(i)}(t)]^r = [(f(t)_-^{(r)})^{(i)}, \ (f(t)_+^{(r)})^{(i)}],$$

for $i = 0, 1, 2 \ldots, N$, and in particular we have

$$(f^{(i)})_{\pm}^{(r)} = (f_{\pm}^{(r)})^{(i)},$$

for any $r \in [0,1]$.

Let $f \in C_{\mathcal{F}}^N(\mathrm{R}^m)$, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\widetilde{\alpha_1}, \ldots, \ \widetilde{\alpha_m}), \ \widetilde{\alpha}_i \in \mathrm{Z}^+, \ i = 1, \ldots, \ m$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^{m} \widetilde{\alpha}_i \leq N, \ N > 1.$$

Then by Theorem 33.13 we get that

$$\left(f_{\pm}^{(r)}\right)_{\widetilde{\alpha}} = (f_{\widetilde{\alpha}})_{\pm}^{(r)}, \ \forall r \in [0,1],$$

and any $\widetilde{\alpha} : \ |\widetilde{\alpha}| \leq N$. Here $f_{\pm}^{(r)} \in C^N(\mathrm{R}^m)$.

For the definition of general fuzzy integral we follow [206] next.

Definition 33.15 Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to \mathrm{R}_{\mathcal{F}}$ *measurable* iff $\forall$ closed $B \subseteq \mathrm{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \ \textit{all} \ w \in \Omega$$

is measurable, see [206].

Theorem 33.16 ([206]) For $F : \Omega \to \mathrm{R}_{\mathcal{F}}$,

$$F(w) = \{(\ F_-^{(r)}(w), \ F_+^{(r)}(w))|0 \leq r \leq 1\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall r \in [0,1], \ F_-^{(r)}, \ F_+^{(r)}$ are measurable.

Following [206], given that for each $r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} \right) \middle| 0 \le r \le 1 \right\}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 33.17 ([206]) A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \{(F_-^{(r)}(w),\ F_+^{(r)}(w))|0 \le r \le 1\}$$

is *integrable* if for each $r \in [0,1]$, $F_\pm^{(r)}$ are integrable, or equivalently, if $F_\pm^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} \right) \middle| 0 \le r \le 1 \right\}.$$

By [206] F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here

$$\|u\|_{\mathcal{F}} := D(u, \widetilde{0}),\ \forall u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 33.18 ([206]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG) d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu \right) \le \int_A D(F, G) d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \le \int_A \|F\|_{\mathcal{F}} d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)}\right],$$

that is

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu,$$

$\forall r \in [0,1]$, respectively.

We use

Notation 33.19 We denote

$$\left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^2 f(\overrightarrow{x}) := D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0}\right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0}\right).$$

In general we denote $(j = 1, \ldots, N)$

$$\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x}) := \sum_{(j_1,\ldots,j_m)\in Z_+^m:\ \sum_{i=1}^m j_i = j} \frac{j!}{j_1!\, j_2! \ldots j_m!} D\left(\frac{\partial^j f(x_1,\ldots,x_m)}{\partial x_1^{j_1} \partial x_2^{j_2} \ldots \partial x_m^{j_m}}, \widetilde{0}\right).$$

Notation 33.20 In this chapter we define the multivariate fuzzy wavelet type operators $A_k, B_k, C_k, D_k,\ k \in Z$, in Theorems 33.23, 33.21, 33.25, 33.27, respectively. Their real analogs are defined exactly the same way in Chapter 9 of [23] and we keep here for these operators the same notations $A_k, B_k, C_k, D_k,\ k \in Z$ (as in [23]).

Also the multivariate fuzzy neural network operators F_n, G_n are defined here in Subsections 33.2.2 and 33.2.3. Their real analogs are defined exactly the same way in Chapter 3 of [23], using there and here also the same notations $F_n, G_n,\ n \in N$.

In this chapter for convenience we are using indiscriminately, whether it is real or fuzzy operator: $A_k, B_k, C_k, D_k, F_n, G_n$. What it really is, it is understood by the context.

We were also motivated by [21], [22], [31].

33.2 Main Results

33.2.1 Convergence with Rates of Multivariate Fuzzy Wavelet Type Operators

We present the first main result on multivariate fuzzy wavelet type operators.

Theorem 33.21 Let $f \in C_{\mathcal{F}}^N(\mathrm{R}^m)$, $m, N \in \mathrm{N}$; $\overrightarrow{x} \in \mathrm{R}^m$ and $k \in \mathrm{Z}$. Let $\varphi \geq 0$ be a bounded function on R^m of compact support

$$\subseteq \prod_{i=1}^{m} [-a_i, a_i], \ 0 < a_i < +\infty, \ a := \max(a_1, \ldots, \ a_m).$$

Assume that

$$\sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_m=-\infty}^{\infty} \varphi(x_1 - j_1, \ldots, \ x_m - j_m) = 1,$$

all $\overrightarrow{x} := (x_1, \ldots, \ x_m) \in \mathrm{R}^m$, in short

$$\sum_{\overrightarrow{j}=-\infty}^{\infty} \varphi(\overrightarrow{x} - \overrightarrow{j}) = 1,$$

all $\overrightarrow{x} \in \mathrm{R}^m$, where $\overrightarrow{j} := (j_1, \ldots, \ j_m)$.

Set

$$B_k(f)(x_1, \ldots, \ x_m) := \sum_{j_1=-\infty}^{\infty}{}^{*} \cdots \sum_{j_m=-\infty}^{\infty}{}^{*} f\left(\frac{j_1}{2^k}, \ldots, \ \frac{j_m}{2^k}\right) \odot \varphi(2^k x_1 - j_1, \ldots, \ 2^k x_m - j_m),$$

any $k \in \mathrm{Z}$, all $(x_1, \ldots, \ x_m) \in \mathrm{R}^m$; in short

$$B_k(f)(\overrightarrow{x}) = \sum_{\overrightarrow{j}=-\infty}^{\infty}{}^{*} f\left(\frac{\overrightarrow{j}}{2^k}\right) \odot \varphi(2^k \overrightarrow{x} - \overrightarrow{j}),$$

any $k \in \mathrm{Z}$, all $\overrightarrow{x} \in \mathrm{R}^m$.

Here we further suppose that all of the fuzzy partial derivatives of f of order N, denoted by

$$f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}} \ \left(\widetilde{\alpha} := (\widetilde{\alpha_1}, \ldots, \ \widetilde{\alpha_m}), \ \widetilde{\alpha}_i \in \mathrm{Z}^+, \ i = 1, \ldots, \ m : \ |\widetilde{\alpha}| = \sum_{i=1}^{m} \alpha_i = N\right),$$

are fuzzy continuous and fuzzy bounded, or fuzzy uniformly continuous on R^m.

Then

$$D\Bigg((B_k(f))(\overrightarrow{x}), \ f(\overrightarrow{x})\Bigg) \leq \sum_{j=1}^{N} \frac{a^j}{j! \ 2^{kj}} \left(\left(\sum_{i=1}^{m} D\Big(\frac{\partial}{\partial x_i}, \widetilde{0}\Big)\right)^j f(\overrightarrow{x})\right) +$$

$$\frac{a^N \ m^N}{N! \ 2^{kN}} \max_{\widetilde{\alpha}: \ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})} \left(f_{\widetilde{\alpha}}, \frac{a}{2^k}\right), \tag{33.1}$$

any $k \in \mathbb{Z}$, which is attained by constant fuzzy functions.

Remark 33.22 (i) Clearly here $B_k f \to f$ pointwise over $\mathbb{R}^m$, as $k \to \infty$, convergence with respect to metric D.

(ii) Given that $f \in C^N_{\mathcal{F}b}(\mathbb{R}^m)$ (i.e., all of f and its fuzzy partial derivatives up to order N are fuzzy continuous and fuzzy bounded) we obtain

$$D^*(B_k f, f) \leq \sum_{j=1}^{N} \frac{a^j}{j!\ 2^{kj}} \left(\sum_{i=1}^{m} D^*\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f + \frac{a^N\ m^N}{N!\ 2^{kN}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{a}{2^k}\right),$$

any $k \in \mathbb{Z}$.

That is $B_k f \to f$, fuzzy uniformly over $\mathbb{R}^m$, as $k \to \infty$.

(iii) When $N = 1$ from (33.1) we get that

$$D\left((B_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\right) \leq \frac{a}{2^{kj}} \left\{ \sum_{i=1}^{m} D\left(\frac{\partial f(\overrightarrow{x})}{\partial x_i}, \widetilde{0}\right) + m \cdot \max_{i \in \{1,\ldots,\ m\}} w_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \frac{a}{2^k}\right) \right\},$$

any $k \in \mathbb{Z}$.

Proof. (of Theorem 33.21)

Since φ is of compact support $(B_k f)$ is a finite sum. Thus for $r \in [0,1]$ we have

$$[B_k(f)(\overrightarrow{x})]^r = \sum_{\overrightarrow{j}=-\infty}^{\infty} \left[f\left(\frac{\overrightarrow{j}}{2^k}\right)\right]^r \varphi(2^k \overrightarrow{x} - \overrightarrow{j}) =$$

$$\sum_{\overrightarrow{j}=-\infty}^{\infty} \left[\left(f\left(\frac{\overrightarrow{j}}{2^k}\right)\right)_-^{(r)}, \left(f\left(\frac{\overrightarrow{j}}{2^k}\right)\right)_+^{(r)} \right] \varphi(2^k \overrightarrow{x} - \overrightarrow{j}) =$$

$$\left[\sum_{\overrightarrow{j}=-\infty}^{\infty} \left(f\left(\frac{\overrightarrow{j}}{2^k}\right)\right)_-^{(r)} \varphi(2^k \overrightarrow{x} - \overrightarrow{j}),\ \sum_{\overrightarrow{j}=-\infty}^{\infty} \left(f\left(\frac{\overrightarrow{j}}{2^k}\right)\right)_+^{(r)} \varphi(2^k \overrightarrow{x} - \overrightarrow{j}) \right] =$$

$$\left[B_k\left((f)_-^{(r)} \right)(\overrightarrow{x}),\ B_k\left((f)_+^{(r)} \right)(\overrightarrow{x}) \right].$$

That is,

$$(B_k f)_\pm^{(r)} = B_k\left(f_\pm^{(r)}\right),\ \forall r \in [0,1].$$

We see that

$$D\left((B_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\right) = \sup_{r \in [0,1]} \max \left\{ \left| (B_k f)_-^{(r)}(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x}) \right|,\ \left| (B_k f)_+^{(r)}(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x}) \right| \right\}$$

$$= \left\{ \left| B_k\left(f_-^{(r)}\right)(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x}) \right|, \ \left| B_k\left(f_+^{(r)}\right)(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x}) \right| \right\}.$$

Clearly here, $(f_{\widetilde{\alpha}})_{\pm}^{(r)}$ are continuous and bounded, or uniformly continuous on R^m, $|\widetilde{\alpha}| = N$, $\forall r \in [0,1]$. Also $f_{\pm}^{(r)} \in C^N(\mathrm{R}^m)$, $\forall r \in [0,1]$.

By Remark 33.14, we observe that

$$\left(f_{\pm}^{(r)}\right)_{\widetilde{\alpha}} = (f_{\widetilde{\alpha}})_{\pm}^{(r)},$$

for any $r \in [0,1]$, and any $\widetilde{\alpha}: \ |\widetilde{\alpha}| \le N$, where

$$f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}},$$

with $\widetilde{\alpha} := (\widetilde{\alpha_1}, \dots, \ \widetilde{\alpha_m})$, $\widetilde{\alpha}_i \in \mathrm{Z}^+$, $i = 1, \dots, \ m$, and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^m \alpha_i \le N.$$

Therefore we can apply Theorem 9.1 of [23], p.201 to get

$$D\left((B_k f)(\overrightarrow{x}), \ f(\overrightarrow{x})\right) \le \sup_{r\in[0,1]} \max \Bigg\{$$

$$\sum_{j=1}^N \frac{a^j}{j! \ 2^{kj}} \left(\left(\sum_{i=1}^m \left| \frac{\partial}{\partial x_i} \right| \right)^j f_-^{(r)}(\overrightarrow{x}) \right) + \frac{a^N \ m^N}{N! \ 2^{kN}} \max_{\widetilde{\alpha}: \ |\widetilde{\alpha}|=N} w_1\left((f_-^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^k} \right),$$

$$\sum_{j=1}^N \frac{a^j}{j! \ 2^{kj}} \left(\left(\sum_{i=1}^m \left| \frac{\partial}{\partial x_i} \right| \right)^j f_+^{(r)}(\overrightarrow{x}) \right) + \frac{a^N \ m^N}{N! \ 2^{kN}} \max_{\widetilde{\alpha}: \ |\widetilde{\alpha}|=N} w_1\left((f_+^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^k} \right) \Bigg\} \le$$

$$\sum_{j=1}^N \frac{a^j}{j! \ 2^{kj}} \sup_{r\in[0,1]} \max \left\{ \left(\left(\sum_{i=1}^m \left| \frac{\partial}{\partial x_i} \right| \right)^j f_-^{(r)}(\overrightarrow{x}) \right), \ \left(\left(\sum_{i=1}^m \left| \frac{\partial}{\partial x_i} \right| \right)^j f_+^{(r)}(\overrightarrow{x}) \right) \right\}$$

$$+ \frac{a^N \ m^N}{N! \ 2^{kN}} \max_{\widetilde{\alpha}: \ |\widetilde{\alpha}|=N} \sup_{r\in[0,1]} \max \left\{ w_1\left((f_-^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^k} \right), \ w_1\left((f_+^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^k} \right) \right\} =: (*).$$

The following example of $m = 2$ and $j = 2$ will help us derive a general conclusion.

We have

$$\sup_{r\in[0,1]} \max \left\{ \left(\left| \frac{\partial}{\partial x_1} \right| + \left| \frac{\partial}{\partial x_2} \right| \right)^2 f_-^{(r)}(x_1, x_2), \ \left(\left| \frac{\partial}{\partial x_1} \right| + \left| \frac{\partial}{\partial x_2} \right| \right)^2 f_+^{(r)}(x_1, x_2) \right\} =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\frac{\partial^2 f_-^{(r)}(x_1,x_2)}{\partial x_1^2}\right| + \left|\frac{\partial^2 f_-^{(r)}(x_1,x_2)}{\partial x_2^2}\right| + 2\left|\frac{\partial^2 f_-^{(r)}(x_1,x_2)}{\partial x_1 \partial x_2}\right|,\right.$$

$$\left.\left|\frac{\partial^2 f_+^{(r)}(x_1,x_2)}{\partial x_1^2}\right| + \left|\frac{\partial^2 f_+^{(r)}(x_1,x_2)}{\partial x_2^2}\right| + 2\left|\frac{\partial^2 f_+^{(r)}(x_1,x_2)}{\partial x_1 \partial x_2}\right|\right\} \leq$$

$$\sup_{r\in[0,1]} \max\left\{\left|\frac{\partial^2 f_-^{(r)}(x_1,x_2)}{\partial x_1^2}\right|, \left|\frac{\partial^2 f_+^{(r)}(x_1,x_2)}{\partial x_1^2}\right|\right\} +$$

$$\sup_{r\in[0,1]} \max\left\{\left|\frac{\partial^2 f_-^{(r)}(x_1,x_2)}{\partial x_2^2}\right|, \left|\frac{\partial^2 f_+^{(r)}(x_1,x_2)}{\partial x_2^2}\right|\right\} +$$

$$2\sup_{r\in[0,1]} \max\left\{\left|\frac{\partial^2 f_-^{(r)}(x_1,x_2)}{\partial x_1 \partial x_2}\right|, \left|\frac{\partial^2 f_+^{(r)}(x_1,x_2)}{\partial x_1 \partial x_2}\right|\right\} =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\right)_-^{(r)}\right|, \left|\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\right)_+^{(r)}\right|\right\} +$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}\right)_-^{(r)}\right|, \left|\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}\right)_+^{(r)}\right|\right\} +$$

$$2\sup_{r\in[0,1]} \max\left\{\left|\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right)_-^{(r)}\right|, \left|\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right)_+^{(r)}\right|\right\} =$$

$$D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0}\right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0}\right).$$

That is we have

$$\sup_{r\in[0,1]} \max\left\{ \left(\left|\frac{\partial}{\partial x_1}\right| + \left|\frac{\partial}{\partial x_2}\right|\right)^2 f_-^{(r)}(x_1,x_2),\ \left(\left|\frac{\partial}{\partial x_1}\right| + \left|\frac{\partial}{\partial x_2}\right|\right)^2 f_+^{(r)}(x_1,x_2)\right\} \leq$$

$$D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0}\right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0}\right) =: \left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^2 f(\overrightarrow{x}).$$

So in general we obtain

$$\sup_{r\in[0,1]} \max\left\{\left(\left(\sum_{i=1}^{m}\left|\frac{\partial}{\partial x_1}\right|\right)^j f_-^{(r)}(\overrightarrow{x})\right), \left(\left(\sum_{i=1}^{m}\left|\frac{\partial}{\partial x_1}\right|\right)^j f_+^{(r)}(\overrightarrow{x})\right)\right\}$$

$$\leq \left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x}).$$

Therefore we derive

$$(*) \leq \sum_{j=1}^{N} \frac{a^j}{j!\ 2^{kj}} \left(\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i},\ \widetilde{0} \right) \right)^j f(\overrightarrow{x}) \right) + \frac{a^N\ m^N}{N!\ 2^{kN}}$$

$$\max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} \sup_{r\in[0,1]} \max \left\{ w_1 \left((f_{\widetilde{\alpha}})_-^{(r)}, \frac{a}{2^k} \right),\ w_1 \left((f_{\widetilde{\alpha}})_+^{(r)}, \frac{a}{2^k} \right) \right\} =$$

$$\sum_{j=1}^{N} \frac{a^j}{j!\ 2^{kj}} \left(\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i},\ \widetilde{0} \right) \right)^j f(\overrightarrow{x}) \right) + \frac{a^N\ m^N}{N!\ 2^{kN}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})} \left(f_{\widetilde{\alpha}}, \frac{a}{2^k} \right),$$

proving the claim. ■

We continue with

Theorem 33.23 Let $f \in C_{\mathcal{F}}^N(\mathrm{R}^m) \cap C_b(\mathrm{R}^m, \mathrm{R}_{\mathcal{F}}),\ m, N \in \mathrm{N};\ \overrightarrow{x} \in \mathrm{R}^m$ and $k \in \mathrm{Z}$. Let $\varphi \geq 0$ a continuous function on R^m of compact support

$$\subseteq \prod_{i=1}^{m} [-a_i, a_i],\ 0 < a_i < +\infty,\ a := \max(a_1, \ldots,\ a_m).$$

Assume that

$$\sum_{\overrightarrow{j}=-\infty}^{\infty} \varphi(\overrightarrow{x} - \overrightarrow{j}) = 1,$$

(then $\int_{\mathrm{R}^m} \varphi(\overrightarrow{x}) d\overrightarrow{x} = 1$).

Define

$$A_k(f)(\overrightarrow{x}) := \sum_{\overrightarrow{j}=-\infty}^{\infty}{}^{*} \alpha_{k\overrightarrow{j}}(f) \odot \varphi(2^k \overrightarrow{x} - \overrightarrow{j}),$$

where

$$\alpha_{k\overrightarrow{j}}(f) := \int_{\mathrm{R}^m} f\left(\frac{\overrightarrow{u}}{2^k} \right) \odot \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u},\ k \in \mathrm{Z}.$$

Here we suppose that all of the fuzzy partial derivatives of f of order N, denoted by

$$f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}} \left(\widetilde{\alpha} := (\widetilde{\alpha_1}, \ldots,\ \widetilde{\alpha_m}),\ \widetilde{\alpha}_i \in \mathrm{Z}^+,\ i = 1, \ldots,\ m :\ |\widetilde{\alpha}| = \sum_{i=1}^{m} \alpha_i = N \right)$$

are fuzzy continuous and fuzzy bounded, or fuzzy uniformly continuous on R^m.

Then

$$D\left((A_kf)(\overrightarrow{x}),\ f(\overrightarrow{x})\right) \leq \sum_{j=1}^{N} \frac{a^j}{j!\ 2^{(k-1)j}} \left(\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x})\right) +$$

$$\frac{a^N\ m^N}{N!\ 2^{(k-1)N}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{a}{2^{k-1}}\right), \tag{33.32}$$

any $k \in \mathrm{Z}$, which is attained by constant fuzzy functions.

Remark 33.24 (i) Clearly here $A_kf \to f$ pointwise over R^m, as $k \to \infty$, convergence with respect to metric D.

(ii) Given that $f \in C^N_{\mathcal{F}b}(\mathrm{R}^m)$ we get

$$\begin{aligned} D^*(A_kf, f) \quad \leq \quad & \sum_{j=1}^{N} \frac{a^j}{j!\ 2^{(k-1)j}} \left(\left(\sum_{i=1}^{m} D^*\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f\right) \\ & + \frac{a^N\ m^N}{N!\ 2^{(k-1)N}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{a}{2^{k-1}}\right), \end{aligned}$$

any $k \in \mathrm{Z}$.

That is $A_kf \to f$, fuzzy uniformly over R^m, as $k \to \infty$.

(iii) When $N = 1$ from (33.2) we obtain that

$$D\left((A_kf)(\overrightarrow{x}),\ f(\overrightarrow{x})\right) \leq \frac{a}{2^{k-1}}\left\{\sum_{i=1}^{m} D\left(\frac{\partial f(\overrightarrow{x})}{\partial x_i}, \widetilde{0}\right) + m \cdot \max_{i \in \{1,\dots,\ m\}} w_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \frac{a}{2^{k-1}}\right)\right\},$$

any $k \in \mathrm{Z}$.

Proof. (of Theorem 33.23)

Since φ is of compact support (A_kf) is a finite sum. Furthermore $\varphi(\overrightarrow{u} - \overrightarrow{j})$ is non zero when

$$\overrightarrow{u} - \overrightarrow{j} \in \prod_{i=1}^{m}[-a_i, a_i],$$

that is when

$$\overrightarrow{u} \in \prod_{i=1}^{m}[j_i - a_i, j_i + a_i].$$

Consequently we have

$$\alpha_{k\overrightarrow{j}}(f) = \int_{j_1-a_1}^{j_1+a_1} \int_{j_2-a_2}^{j_2+a_2} \cdots \int_{j_m-a_m}^{j_m+a_m} f\left(\frac{\overrightarrow{u}}{2^k}\right) \odot \varphi(\overrightarrow{u} - \overrightarrow{j})d\overrightarrow{u}.$$

For $r \in [0, 1]$ we have

$$[\alpha_{k\overrightarrow{j}}(f)]^r = \left[\int_{\prod_{i=1}^m [j_i - a_i, j_i + a_i]} f\left(\frac{\overrightarrow{u}}{2^k}\right) \odot \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u}\right]^r =$$

$$\left[\int_{\prod_{i=1}^m [j_i - a_i, j_i + a_i]} \left(f\left(\frac{\overrightarrow{u}}{2^k}\right)\right)_-^{(r)} \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u}, \int_{\prod_{i=1}^m [j_i - a_i, j_i + a_i]} \left(f\left(\frac{\overrightarrow{u}}{2^k}\right)\right)_+^{(r)} \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u}\right].$$

We notice that

$$[A_k(f)(\overrightarrow{x})]^r = \sum_{\overrightarrow{j}=-\infty}^{\infty} \left[\alpha_{k\overrightarrow{j}}(f)\right]^r \varphi(2^k\overrightarrow{x} - \overrightarrow{j}) =$$

$$\sum_{\overrightarrow{j}=-\infty}^{\infty} \left[\int_{\prod_{i=1}^m [j_i - a_i, j_i + a_i]} \left(f\left(\frac{\overrightarrow{u}}{2^k}\right)\right)_-^{(r)} \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u},\right.$$

$$\left.\int_{\prod_{i=1}^m [j_i - a_i, j_i + a_i]} \left(f\left(\frac{\overrightarrow{u}}{2^k}\right)\right)_+^{(r)} \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u}\right] \varphi(2^k\overrightarrow{x} - \overrightarrow{j}) =$$

$$\left[\sum_{\overrightarrow{j}=-\infty}^{\infty} \left(\int_{\prod_{i=1}^m [j_i - a_i, j_i + a_i]} \left(f\left(\frac{\overrightarrow{u}}{2^k}\right)\right)_-^{(r)} \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u}\right) \varphi(2^k\overrightarrow{x} - \overrightarrow{j}),\right.$$

$$\left.\sum_{\overrightarrow{j}=-\infty}^{\infty} \left(\int_{\prod_{i=1}^m [j_i - a_i, j_i + a_i]} \left(f\left(\frac{\overrightarrow{u}}{2^k}\right)\right)_+^{(r)} \varphi(\overrightarrow{u} - \overrightarrow{j}) d\overrightarrow{u}\right) \varphi(2^k\overrightarrow{x} - \overrightarrow{j})\right]$$

$$= \left[\left(A_k\left(f_-^{(r)}\right)\right)(\overrightarrow{x}), \left(A_k\left(f_+^{(r)}\right)\right)(\overrightarrow{x})\right].$$

I.e. we proved that

$$(A_k f)_\pm^{(r)} = A_k\left(f_\pm^{(r)}\right), \ \forall r \in [0,1].$$

So we have

$$D\left((A_k(f))(\overrightarrow{x}), \ f(\overrightarrow{x})\right) = \sup_{r\in[0,1]} \max\left\{\left|(A_k f)_-^{(r)}(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x})\right|, \left|(A_k f)_+^{(r)}(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x})\right|\right\}$$

$$= \sup_{r\in[0,1]} \max\left\{\left|A_k\left(f_-^{(r)}\right)(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x})\right|, \left|A_k\left(f_+^{(r)}\right)(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x})\right|\right\}$$

(by Theorem 9.2, p.206, [23])

$$\leq \sup_{r\in[0,1]} \max\left\{ \sum_{j=1}^{N} \frac{a^j}{j!\ 2^{(k-1)j}} \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_-^{(r)}(\overrightarrow{x}) \right) + \frac{a^N\ m^N}{N!\ 2^{(k-1)N}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1 \left((f_-^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^{k-1}} \right), \right.$$

$$\left. \sum_{j=1}^{N} \frac{a^j}{j!\ 2^{(k-1)j}} \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_+^{(r)}(\overrightarrow{x}) \right) + \frac{a^N\ m^N}{N!\ 2^{(k-1)N}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1 \left((f_+^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^{k-1}} \right) \right\}$$

$$\leq \sum_{j=1}^{N} \frac{a^j}{j!\ 2^{(k-1)j}} \sup_{r\in[0,1]} \max\left\{ \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_-^{(r)}(\overrightarrow{x}) \right),\ \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_+^{(r)}(\overrightarrow{x}) \right) \right\}$$

$$+ \frac{a^N\ m^N}{N!\ 2^{(k-1)N}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} \sup_{r\in[0,1]} \max\left\{ w_1 \left((f_-^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^{k-1}} \right),\ w_1 \left((f_+^{(r)})_{\widetilde{\alpha}}, \frac{a}{2^{k-1}} \right) \right\} \leq$$

$$\sum_{j=1}^{N} \frac{a^j}{j!\ 2^{(k-1)j}} \left(\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i},\ \widetilde{0} \right) \right)^j f(\overrightarrow{x}) \right) + \frac{a^N\ m^N}{N!\ 2^{(k-1)N}}$$

$$\max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} \sup_{r\in[0,1]} \max\left\{ w_1 \left((f_{\widetilde{\alpha}})_-^{(r)}, \frac{a}{2^{k-1}} \right),\ w_1 \left((f_{\widetilde{\alpha}})_+^{(r)}, \frac{a}{2^{k-1}} \right) \right\} =$$

$$\sum_{j=1}^{N} \frac{a^j}{j!\ 2^{(k-1)j}} \left(\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i},\ \widetilde{0} \right) \right)^j f(\overrightarrow{x}) \right) + \frac{a^N\ m^N}{N!\ 2^{(k-1)N}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})} \left(f_{\widetilde{\alpha}}, \frac{a}{2^{k-1}} \right),$$

proving the claim. ∎

We continue with

Theorem 33.25 All assumptions here as in Theorem 33.21. Set

$$\gamma_{k\overrightarrow{j}}(f) := 2^{mk} \int_{2^{-k}\overrightarrow{j}}^{2^{-k}(\overrightarrow{j}+\overrightarrow{1})} f(\overrightarrow{t})d\overrightarrow{t} = 2^{mk} \int_{\overrightarrow{0}}^{2^{\overrightarrow{-k}}} f\left(\overrightarrow{t} + \frac{\overrightarrow{j}}{2^k} \right) d\overrightarrow{t},$$

and

$$C_k(f)(\overrightarrow{x}) := \sum_{\overrightarrow{j}=-\infty}^{\infty} \gamma_{k\overrightarrow{j}}(f) \odot \varphi(2^k\overrightarrow{x} - \overrightarrow{j}),$$

all $\overrightarrow{x} \in \mathrm{R}^m$ and $k \in \mathrm{Z}$. Then

$$D\left(C_k(f)(\overrightarrow{x}),\ f(\overrightarrow{x}) \right) \leq \sum_{j=1}^{N} \frac{(a+1)^j}{j!\ 2^{kj}} \left(\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i}, \widetilde{0} \right) \right)^j f(\overrightarrow{x}) \right) +$$

$$\frac{(a+1)^N\ m^N}{N!\ 2^{kN}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{a+1}{2^k}\right), \tag{33.3}$$

any $k \in \mathbb{Z}$, which is attained by constant fuzzy functions.

Remark 33.26 (i) Clearly here $C_k f \to f$ pointwise over $\mathbb{R}^m$, as $k \to \infty$, convergence with respect to metric D.

(ii) Given that $f \in C^N_{\mathcal{F}b}(\mathbb{R}^m)$, we derive

$$D^*(C_k f, f) \le \sum_{j=1}^{N} \frac{(a+1)^j}{j!\ 2^{kj}} \left(\left(\sum_{i=1}^{m} D^*\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f\right) + \frac{(a+1)^N\ m^N}{N!\ 2^{kN}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{a+1}{2^k}\right),$$

any $k \in \mathbb{Z}$.

That is $C_k f \to f$, fuzzy uniformly over $\mathbb{R}^m$, as $k \to \infty$.

(iii) When $N = 1$ from (33.3) we obtain that

$$D\left((C_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\right) \le \left(\frac{a+1}{2^k}\right) \left\{ \sum_{i=1}^{m} D\left(\frac{\partial f(\overrightarrow{x})}{\partial x_i}, \widetilde{0}\right) + m \cdot \max_{i \in \{1,\dots,\ m\}} w_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \frac{a+1}{2^k}\right)\right\},$$

any $k \in \mathbb{Z}$.

Proof. (of Theorem 33.25) We observe for $r \in [0,1]$ that

$$[\gamma_{k\overrightarrow{j}}(f)]^r := 2^{mk}\left[\int_{\overrightarrow{0}}^{2^{\overrightarrow{-k}}} f\left(\overrightarrow{t} + \frac{\overrightarrow{j}}{2^k}\right) d\overrightarrow{t}\right]^r =$$

$$2^{mk}\left[\int_{\overrightarrow{0}}^{2^{\overrightarrow{-k}}} f_-^{(r)}\left(\overrightarrow{t} + \frac{\overrightarrow{j}}{2^k}\right) d\overrightarrow{t}, \int_{\overrightarrow{0}}^{2^{\overrightarrow{-k}}} f_+^{(r)}\left(\overrightarrow{t} + \frac{\overrightarrow{j}}{2^k}\right) d\overrightarrow{t}\right]$$

$$= \left[\gamma_{k\overrightarrow{j}}\left(f_-^{(r)}\right), \gamma_{k\overrightarrow{j}}\left(f_+^{(r)}\right)\right].$$

That is we proved that

$$\left(\gamma_{k\overrightarrow{j}}(f)\right)_{\pm}^{(r)} = \gamma_{k\overrightarrow{j}}\left(f_{\pm}^{(r)}\right).$$

Hence we obtain

$$[C_k(f)(\overrightarrow{x})]^r = \sum_{\overrightarrow{j}=-\infty}^{\infty} \left[\gamma_{k\overrightarrow{j}}(f)\right]^r \varphi(2^k\overrightarrow{x} - \overrightarrow{j}) =$$

$$\sum_{\overrightarrow{j}=-\infty}^{\infty} \left[\gamma_{k\overrightarrow{j}}\left(f_-^{(r)}\right),\ \gamma_{k\overrightarrow{j}}\left(f_+^{(r)}\right)\right] \varphi(2^k\overrightarrow{x} - \overrightarrow{j}) =$$

$$\left[\sum_{\overrightarrow{j}=-\infty}^{\infty} \gamma_{k\overrightarrow{j}}\left(f_{-}^{(r)}\right)\varphi(2^{k}\overrightarrow{x}-\overrightarrow{j}),\ \sum_{\overrightarrow{j}=-\infty}^{\infty} \gamma_{k\overrightarrow{j}}\left(f_{+}^{(r)}\right)\varphi(2^{k}\overrightarrow{x}-\overrightarrow{j})\right]$$

$$=\left[\left(C_k\left(f_{-}^{(r)}\right)\right)(\overrightarrow{x}),\ \left(C_k\left(f_{+}^{(r)}\right)\right)(\overrightarrow{x})\right].$$

So we have established that

$$(C_k f(\overrightarrow{x}))_{\pm}^{(r)} = C_k\left(f_{\pm}^{(r)}\right)(\overrightarrow{x}),\ \forall r\in[0,1].$$

Then we use Theorem 9.3, p.211, [23] and we follow the same steps as in the proof of Theorem 33.21. ■

We also give

Theorem 33.27 All assumptions here as in Theorem 33.21. Set

$$(D_k f)(\overrightarrow{x}) := \sum_{\overrightarrow{j}=-\infty}^{\infty}{}^{*}\ \delta_{k\overrightarrow{j}}(f)\odot\varphi(2^{k}\overrightarrow{x}-\overrightarrow{j}),$$

where

$$\delta_{k\overrightarrow{j}}(f) := \sum_{\overrightarrow{l}=\overrightarrow{0}}^{\overrightarrow{n}}{}^{*}\ w_{\overrightarrow{l}}\odot f\left(\frac{\overrightarrow{j}}{2^k}+\frac{\overrightarrow{l}}{2^k\overrightarrow{n}}\right),$$

$\overrightarrow{l}\in Z_{+}^{m},\ \overrightarrow{n}\in N^{m},\ w_{\overrightarrow{l}}\geq 0,$

$$\sum_{\overrightarrow{l}=\overrightarrow{0}}^{\overrightarrow{n}} w_{\overrightarrow{l}} = 1,$$

$k\in Z,\ \overrightarrow{j}\in Z^{m},\ \overrightarrow{x}\in R^{m}.$

That is

$$\delta_{k,j_1,\ldots,\,j_r}(f)=\sum_{l_1=0}^{n_1}{}^{*}\sum_{l_2=0}^{n_2}{}^{*}\cdots\sum_{l_r=0}^{n_m}{}^{*} w_{l_1,\ldots,\,l_m}\odot f\left(\frac{j_1}{2^k}+\frac{l_1}{2^k\ n_1},\ldots,\ \frac{j_m}{2^k}+\frac{l_m}{2^k\ n_m}\right),$$

$$w_{l_1,\ldots,\,l_m}\geq 0,\quad \sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\cdots\sum_{l_r=0}^{n_m} w_{l_1,\ldots,\,l_m}=1.$$

Then

$$D\left(D_k(f)(\overrightarrow{x}),\ f(\overrightarrow{x})\right)\leq\sum_{j=1}^{N}\frac{(a+1)^j}{j!\ 2^k j}\left(\left(\sum_{i=1}^{m} D\Big(\frac{\partial}{\partial x_i},\widetilde{0}\Big)\right)^{j} f(\overrightarrow{x})\right)+$$

$$\frac{(a+1)^N\ m^N}{N!\ 2^{kN}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{a+1}{2^k}\right), \tag{33.4}$$

any $k \in \mathrm{Z}$, which is attained by constant functions.

Remark 33.28 (i) Clearly here $D_k f \to f$ pointwise over R^m, as $k \to \infty$, convergence with respect to metric D.

(ii) Given that $f \in C^N_{\mathcal{F}b}(\mathrm{R}^m)$, we get

$$D^*(D_k f, f) \leq \sum_{j=1}^{N} \frac{(a+1)^j}{j!\ 2^{kj}} \left(\left(\sum_{i=1}^{m} D^*\Big(\frac{\partial}{\partial x_i}, \widetilde{0}\Big)\right)^j f\right) + \frac{(a+1)^N\ m^N}{N!\ 2^{kN}} \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{a+1}{2^k}\right),$$

any $k \in \mathrm{Z}$.

That is $D_k f \to f$, fuzzy uniformly over R^m, as $k \to \infty$.

(iii) When $N = 1$ from (33.4) we derive

$$D\Big((D_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\Big) \leq \left(\frac{a+1}{2^k}\right)\left\{\sum_{i=1}^{m} D\Big(\frac{\partial f(\overrightarrow{x})}{\partial x_i}, \widetilde{0}\Big) + m \cdot \max_{i\in\{1,\ldots,\ m\}} w_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \frac{a+1}{2^k}\right)\right\},$$

any $k \in \mathrm{Z}$.

Proof. (of Theorem 33.27) We notice that

$$[\delta_{k\overrightarrow{j}}(f)]^r = \sum_{\overrightarrow{l}=\overrightarrow{0}}^{\overrightarrow{n}} w_{\overrightarrow{l}} \left[f\left(\frac{\overrightarrow{j}}{2^k} + \frac{\overrightarrow{l}}{2^k\overrightarrow{n}}\right)\right]^r =$$

$$\sum_{\overrightarrow{l}=\overrightarrow{0}}^{\overrightarrow{n}} w_{\overrightarrow{l}} \left[f_-^{(r)}\left(\frac{\overrightarrow{j}}{2^k} + \frac{\overrightarrow{l}}{2^k\overrightarrow{n}}\right), f_+^{(r)}\left(\frac{\overrightarrow{j}}{2^k} + \frac{\overrightarrow{l}}{2^k\overrightarrow{n}}\right)\right] =$$

$$\left[\sum_{\overrightarrow{l}=\overrightarrow{0}}^{\overrightarrow{n}} w_{\overrightarrow{l}} f_-^{(r)}\left(\frac{\overrightarrow{j}}{2^k} + \frac{\overrightarrow{l}}{2^k\overrightarrow{n}}\right), \sum_{\overrightarrow{l}=\overrightarrow{0}}^{\overrightarrow{n}} w_{\overrightarrow{l}} f_+^{(r)}\left(\frac{\overrightarrow{j}}{2^k} + \frac{\overrightarrow{l}}{2^k\overrightarrow{n}}\right)\right] =$$

$$= \left[\delta_{k\overrightarrow{j}}\left(f_-^{(r)}\right), \delta_{k\overrightarrow{j}}\left(f_+^{(r)}\right)\right].$$

That is

$$\left(\delta_{k\overrightarrow{j}}(f)\right)_{\pm}^{(r)} = \delta_{k\overrightarrow{j}}\left(f_{\pm}^{(r)}\right),\ \forall r \in [0,1].$$

Furthermore we observe that

$$[D_k(f)(\overrightarrow{x})]^r = \sum_{\overrightarrow{j}=-\infty}^{\infty} \left[\delta_{k\overrightarrow{j}}(f)\right]^r \varphi(2^k\overrightarrow{x} - \overrightarrow{j}) =$$

$$\sum_{\overrightarrow{j}=-\infty}^{\infty} \left[\delta_{k\overrightarrow{j}}\left(f_-^{(r)}\right),\ \delta_{k\overrightarrow{j}}\left(f_+^{(r)}\right)\right]\varphi(2^k\overrightarrow{x}-\overrightarrow{j}) =$$

$$\left[\sum_{\overrightarrow{j}=-\infty}^{\infty} \delta_{k\overrightarrow{j}}\left(f_-^{(r)}\right)\varphi(2^k\overrightarrow{x}-\overrightarrow{j}),\ \sum_{\overrightarrow{j}=-\infty}^{\infty} \delta_{k\overrightarrow{j}}\left(f_+^{(r)}\right)\varphi(2^k\overrightarrow{x}-\overrightarrow{j})\right]$$

$$= \left[\left(D_k\left(f_-^{(r)}\right)\right)(\overrightarrow{x}),\ \left(D_k\left(f_+^{(r)}\right)\right)(\overrightarrow{x})\right].$$

I.e. we proved that

$$(D_k f)_{\pm}^{(r)} = D_k\left(f_{\pm}^{(r)}\right),\ \forall r \in [0,1].$$

Then by using Theorem 9.4, p.214, [23] and following similar steps as in the proof of Theorem 33.21. we finish proof. ∎

We further give

Theorem 33.29 Let $f \in C_{\mathcal{F}}(\mathrm{R}^m),\ m \in \mathrm{N}$, which fuzzy uniformly continuous or fuzzy bounded.

(i) Under the notations and assumptions of Theorem 33.21, $N = 0$, we get

$$D\Big((B_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\Big) \le w_1^{(\mathcal{F})}\left(f, \frac{a}{2^k}\right),\ k \in \mathrm{Z}. \tag{33.5}$$

(ii) Under the notations and assumptions of Theorem 33.23, $N = 0$, we derive

$$D\Big((A_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\Big) \le w_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right),\ k \in \mathrm{Z}. \tag{33.6}$$

(iii) Under the notations and assumptions of Theorem 33.25, $N = 0$, we get

$$D\Big((C_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\Big) \le w_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right),\ k \in \mathrm{Z}. \tag{33.7}$$

(iv) Under the notations and assumptions of Theorem 33.27, $N = 0$, we obtain

$$D\Big((D_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\Big) \le w_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right),\ k \in \mathrm{Z}. \tag{33.8}$$

All inequalities (33.5)-(33.8) are attained by fuzzy constant functions.

Proof. We notice that

$$D\left((B_k f)(\overrightarrow{x}),\ f(\overrightarrow{x})\right) = \sup_{r\in[0,1]} \max\left\{ \left| (B_k f)_-^{(r)}(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x}) \right|,\ \left| (B_k f)_+^{(r)}(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x}) \right| \right\}$$

$$= \sup_{r\in[0,1]} \max\left\{ \left| \left(B_k\left(f_-^{(r)}\right)\right)(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x}) \right|,\ \left| \left(B_k\left(f_+^{(r)}\right)\right)(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x}) \right| \right\} \leq$$

(by [90], or (9.35) of [23], p.219)

$$\sup_{r\in[0,1]} \max\left\{ w_1\left(f_-^{(r)}, \frac{a}{2^k}\right),\ w_1\left(f_+^{(r)}, \frac{a}{2^k}\right) \right\} = w_1^{(\mathcal{F})}\left(f, \frac{a}{2^k}\right),\ k \in \mathrm{Z},$$

proving (i).

The rest (ii)-(iv) are proved similarly by the use of Proposition 9.1, p.219 of [23]. ■

33.2.2 Convergence with Rates of Multivariate Fuzzy Cardaliaguet- Euvrard Neural Network Operators

We use the following (see [128])

Definition 33.30 A function $b : \mathrm{R} \to \mathrm{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to R. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell- shaped function. A bell- shaped function is said to be centered if its center is zero.

Definition 33.31 (see [128]) A function $b : \mathrm{R}^d \to \mathrm{R} (d \geq 1)$ is said to be a d- dimensional bell-shaped function if it is integrable and its integral is not zero, and if for all $i = 1, \ldots,\ d$,

$$t \to b(x_1, \ldots, t, \ldots, x_d)$$

is centered bell- shaped function, where $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathrm{R}^d$ arbitrary.

Example 33.32 (From [128]) Let b a centered bell- shaped function over R, then $(x_1, \ldots, x_d) \to b(x_1) \ldots b(x_d)$ is a d- dimensional bell-shaped function.

Assumption 33.33 Here $b(\overrightarrow{x})$ is of compact support

$$\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i],\ T_i > 0$$

and it may have jump discontinuities there. Set $I := \int_{\mathcal{B}} b(\overrightarrow{x}) d\overrightarrow{x}$. Note that $I > 0$. Let $f : \mathrm{R}^d \to \mathrm{R}_{\mathcal{F}}$ be a fuzzy continuous and fuzzy bounded function or a fuzzy uniformly continuous function.

In this subsection we study the D-metric pointwise convergence with rates over R^d, to the fuzzy unit operator, of the multivariate fuzzy Cardaliaguet- Euvrard neural network operators,

$$(F_n(f))(\overrightarrow{x}) = \sum_{k_1=-n^2}^{n^2}{}^{*} \cdots \sum_{k_d=-n^2}^{n^2}{}^{*} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) \odot$$

$$\frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, \ n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{I \cdot n^{\alpha \cdot d}}, \tag{33.9}$$

where $0 < \alpha < 1$ and $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathrm{R}^d, \ n \in \mathrm{N}$.

For the real related operators see [128], [23], p.90.

The terms in the fuzzy multiple sum (33.9) can be nonzero iff simultaneously

$$\left| n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right) \right| \leq T_i,$$

all $i = 1, \ldots, d$ i.e., $\left| x_i - \frac{k_i}{n} \right| \leq \frac{T_i}{n^{1-\alpha}}$, all $i = 1, \ldots, \ d$ iff

$$nx_i - T_i \cdot n^{\alpha} \leq k_i \leq nx_i + T_i \cdot n^{\alpha}, \ all \ i = 1, \ldots, \ d. \tag{33.10}$$

To have the order

$$-n^2 \leq nx_i - T_i \cdot n^{\alpha} \leq k_i \leq nx_i + T_i \cdot n^{\alpha} \leq n^2, \tag{33.11}$$

we need $n \geq T_i + |x_i|$, all $i = 1, \ldots, \ d$. So (33.11) is true when we consider

$$n \geq \max_{\{i=1,\ldots, \ d\}} (T_i + |x_i|). \tag{33.12}$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (33.11) it is enough to suppose that $n \geq 2T^*$, where $T^* := \max\{T_1, \ldots, \ T_d\} > 0$. Take

$$\widetilde{I}_i := [nx_i - T_i n\alpha, nx_i + T_i n\alpha], \ i = 1, \ldots, \ d, \ n \in \mathrm{N}.$$

The length of $\widetilde{I}_i$ is $2T_i n^{\alpha}$. By Proposition 2.1, p.61 of [23] we obtain that the cardinality of $\{k_i \in \mathrm{Z} \ \textit{that belong to} \ \widetilde{I}_i\} := card(k_i) \geq \max(2T_i n^{\alpha} - 1, 0)$, any $i \in \{1, \ldots, \ d\}$. In order to have $card(k_i) \geq 1$ we need $2T_i n^{\alpha} - 1 \geq 1$ iff $n \geq T_i^{-1/\alpha}$, any $i \in \{1, \ldots, \ d\}$.

Therefore, a sufficient condition for causing the order (33.11) along with the interval $\widetilde{I}_i$ to contain at least one integer for all $i = 1, \ldots, \ d$ is that

$$n \geq \max_{\{i=1,\ldots,\ d\}} \left(T_i + |x_i|, T_i^{-1/\alpha} \right). \tag{33.13}$$

Clearly as $n \to +\infty$ we get that $card(k_i) \to +\infty$, all $i = 1, \ldots,\ d$. Also notice that $card(k_i)$ equals to the the cardinality of integers in $[\lceil nx_i - T_i \cdot n^\alpha \rceil, [nx_i + T_i \cdot n^\alpha]]$ for all $i = 1, \ldots,\ d$.

Here denotes $\lceil\ \rceil$ the ceiling of the number, while $[\ ]$ denotes the integral part.

We set $b^* := b(\overrightarrow{0})$ the maximum of $b(\overrightarrow{x})$. From now on in this chapter we will assume (33.13). Consequently

$$(F_n(f))(\overrightarrow{x}) = \sideset{}{^*}\sum_{k_1=\lceil nx_1 - T_1 \cdot n^\alpha \rceil}^{[nx_1+T_1\cdot n^\alpha]} \cdots \sideset{}{^*}\sum_{k_d=\lceil nx_d - T_d \cdot n^\alpha \rceil}^{[nx_d+T_d\cdot n^\alpha]} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) \odot$$
$$\frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots,\ n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{I \cdot n^{\alpha \cdot d}}, \tag{33.14}$$

all $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathrm{R}^d,\ n \in \mathrm{N}$, where

$$I = \int_{-T_1}^{T_1} \cdots \int_{-T_d}^{T_d} b(x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Denote by $\|\cdot\|_\infty$ the maximum norm on $\mathrm{R}^d,\ d \geq 1$. So if

$$\left| n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right) \right| \leq T_i,$$

all $i = 1, \ldots, d$, we find that

$$\left\| \left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right) \right\|_\infty \leq \frac{T^*}{n^{1-\alpha}},$$

where $\overrightarrow{k} := (k_1, \ldots, k_d)$.

We notice that ($r \in [0,1]$)

$$[(F_n(f))(\overrightarrow{x})]^r = \sum_{k_1=\lceil nx_1 - T_1 \cdot n^\alpha \rceil}^{[nx_1+T_1\cdot n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d \cdot n^\alpha \rceil}^{[nx_d+T_d\cdot n^\alpha]} \left[f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) \right]^r$$
$$\frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots,\ n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{I \cdot n^{\alpha \cdot d}} =$$

$$\sum_{k_1=\lceil nx_1-T_1\cdot n^{\alpha}\rceil}^{[nx_1+T_1\cdot n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_d\cdot n^{\alpha}\rceil}^{[nx_d+T_d\cdot n^{\alpha}]} \left[f_-^{(r)}\left(\frac{k_1}{n},\ldots,\frac{k_d}{n}\right), f_+^{(r)}\left(\frac{k_1}{n},\ldots,\frac{k_d}{n}\right)\right]$$

$$\frac{b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,\ n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}{I\cdot n^{\alpha\cdot d}} =$$

$$\left[\sum_{k_1=\lceil nx_1-T_1\cdot n^{\alpha}\rceil}^{[nx_1+T_1\cdot n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_d\cdot n^{\alpha}\rceil}^{[nx_d+T_d\cdot n^{\alpha}]} f_-^{(r)}\left(\frac{k_1}{n},\ldots,\frac{k_d}{n}\right) \frac{b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,\ n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}{I\cdot n^{\alpha\cdot d}},\right.$$

$$\left.\sum_{k_1=\lceil nx_1-T_1\cdot n^{\alpha}\rceil}^{[nx_i+T_i\cdot n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_d\cdot n^{\alpha}\rceil}^{[nx_d+T_d\cdot n^{\alpha}]} f_+^{(r)}\left(\frac{k_1}{n},\ldots,\frac{k_d}{n}\right) \frac{b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,\ n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}{I\cdot n^{\alpha\cdot d}}\right]$$

$$= \left[\left(F_n(f_-^{(r)})\right)(\overrightarrow{x}),\ \left(F_n(f_+^{(r)})\right)(\overrightarrow{x})\right].$$

We have established that

$$\left(F_n(f)\right)_{\pm}^{(r)} = F_n\left(f_{\pm}^{(r)}\right),\ \forall r\in[0,1]. \tag{33.15}$$

We need

Definition 33.34 Let $f:\mathrm{R}^d\to\mathrm{R}_{\mathcal{F}}$. We call

$$w_1^{(\mathcal{F})}(f,h)_{\infty} := \sup_{all\ \overrightarrow{x},\overrightarrow{y}\in\mathrm{R}^d,\ \|\overrightarrow{x}-\overrightarrow{y}\|_{\infty}\le h} D\left(f(\overrightarrow{x}),f(\overrightarrow{y})\right),$$

$h>0$, the first multidimensional fuzzy modulus of continuity of f with respect to $\|\cdot\|_{\infty}$.

We need

Proposition 33.35 Suppose that $w_1^{(\mathcal{F})}(f,h)_{\infty},\ w_1(f_-^{(r)},h)_{\infty},\ w_1(f_+^{(r)},h)_{\infty}$ are finite for any $h>0$, any $r\in[0,1]$. Here w_1 is the usual real modulus of continuity.

Then

$$w_1^{(\mathcal{F})}(f,h)_{\infty} = \sup_{r\in[0,1]} \max\left\{w_1(f_-^{(r)},h)_{\infty},\ w_1(f_+^{(r)},h)_{\infty}\right\}. \tag{33.16}$$

Proof. By [37]. ■

We present

Theorem 33.36 Let $\overrightarrow{x}\in\mathrm{R}^d$, then it holds that

$$D\left(\left(F_n(f)\right)(\overrightarrow{x}), f(\overrightarrow{x}) \right) \leq D\left(f(\overrightarrow{x}), \widetilde{0}\right) \cdot$$

$$\left| \sum_{k_1=\lceil nx_1 - T_1 \cdot n^\alpha \rceil}^{[nx_1+T_1\cdot n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d \cdot n^\alpha \rceil}^{[nx_d+T_d\cdot n^\alpha]} \frac{1}{I \cdot n^{\alpha d}} \cdot b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right) \right) - 1 \right| +$$

$$\frac{b^*}{I} \cdot \prod_{i=1}^{d} \left(2T_i + \frac{1}{n^\alpha}\right) \cdot w_1^{(\mathcal{F})}\left(f, \frac{T^*}{n^{1-\alpha}}\right)_\infty . \qquad (33.17)$$

Proof. We notice that

$$D\left(\left(F_n(f)\right)(\overrightarrow{x}), f(\overrightarrow{x}) \right) = \sup_{r\in[0,1]} \max\left\{ \left| \left(F_n(f)\right)_-^{(r)}(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x}) \right|, \left| \left(F_n(f)\right)_+^{(r)}(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x}) \right| \right\}$$

$$= \sup_{r\in[0,1]} \max\left\{ \left| F_n\left(f_-^{(r)}\right)(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x}) \right|, \left| F_n\left(f_+^{(r)}\right)(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x}) \right| \right\}$$

(by Theorem 3.1, p.92 of [23])

$$\leq \sup_{r\in[0,1]} \max\left\{ |f_-^{(r)}(\overrightarrow{x})| \cdot \left| \sum_{k_1=\lceil nx_1 - T_1 \cdot n^\alpha \rceil}^{[nx_1+T_1\cdot n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d \cdot n^\alpha \rceil}^{[nx_d+T_d\cdot n^\alpha]} \frac{1}{I \cdot n^{\alpha d}} \cdot \right.\right.$$

$$\left. b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right) \right) - 1 \right| + \frac{b^*}{I} \cdot \prod_{i=1}^{d} \left(2T_i + \frac{1}{n^\alpha}\right) \cdot w_1^{(\mathcal{F})}\left(f_-^{(r)}, \frac{T^*}{n^{1-\alpha}}\right)_\infty ,$$

$$|f_+^{(r)}(\overrightarrow{x})| \cdot \left| \sum_{k_1=\lceil nx_1 - T_1 \cdot n^\alpha \rceil}^{[nx_1+T_1\cdot n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d \cdot n^\alpha \rceil}^{[nx_d+T_d\cdot n^\alpha]} \frac{1}{I \cdot n^{\alpha d}} \cdot \right.$$

$$\left.\left. b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right) \right) - 1 \right| + \frac{b^*}{I} \cdot \prod_{i=1}^{d} \left(2T_i + \frac{1}{n^\alpha}\right) \cdot w_1^{(\mathcal{F})}\left(f_+^{(r)}, \frac{T^*}{n^{1-\alpha}}\right)_\infty \right\}$$

$$\leq \left(\sup_{r\in[0,1]} \max\left\{ |f_-^{(r)}(\overrightarrow{x})|, |f_+^{(r)}(\overrightarrow{x})| \right\} \right) \cdot \left| \sum_{k_1=\lceil nx_1 - T_1 \cdot n^\alpha \rceil}^{[nx_1+T_1\cdot n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d \cdot n^\alpha \rceil}^{[nx_d+T_d\cdot n^\alpha]} \frac{1}{I \cdot n^{\alpha d}} \cdot \right.$$

$$\left. b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right) \right) - 1 \right| + \frac{b^*}{I} \cdot \prod_{i=1}^{d} \left(2T_i + \frac{1}{n^\alpha}\right) \cdot$$

$$\sup_{r\in[0,1]} \max\left\{ w_1\left(f_-^{(r)}, \frac{T^*}{n^{1-\alpha}}\right)_\infty, w_1\left(f_+^{(r)}, \frac{T^*}{n^{1-\alpha}}\right)_\infty \right\} =$$

$$D\left(f(\overrightarrow{x}),\widetilde{0}\right)\cdot\left|\sum_{k_1=\lceil nx_1-T_1\cdot n^\alpha\rceil}^{[nx_1+T_1\cdot n^\alpha]}\cdots\sum_{k_d=\lceil nx_d-T_d\cdot n^\alpha\rceil}^{[nx_d+T_d\cdot n^\alpha]}\frac{1}{I\cdot n^{\alpha d}}\cdot\right.$$

$$\left.b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,\ n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)-1\right|+\frac{b^*}{I}\cdot\prod_{i=1}^{d}\left(2T_i+\frac{1}{n^\alpha}\right)w_1^{(\mathcal{F})}\left(f,\frac{T^*}{n^{1-\alpha}}\right)_\infty,$$

proving the claim. ■

We need

Lemma 33.37 ([23], p.95) It holds true that ($\overrightarrow{x}\in\mathrm{R}^d$)

$$S_n(\overrightarrow{x}):=\sum_{k_1=\lceil nx_1-T_1\cdot n^\alpha\rceil}^{[nx_1+T_1\cdot n^\alpha]\ \ *}\cdots\sum_{k_d=\lceil nx_d-T_d\cdot n^\alpha\rceil}^{[nx_d+T_d\cdot n^\alpha]\ \ *}\frac{1}{I\cdot n^{\alpha d}}\cdot$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,\ n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)\to 1,$$

pointwise, as $n\to+\infty$.

Remark 33.38 Given that $f\in C_{\mathcal{F}}^{U}(\mathrm{R}^d)$ (fuzzy uniformly continuous functions), as $n\to\infty$, we get $D\left((F_n(f))(\overrightarrow{x}),f(\overrightarrow{x})\right)\to 0$, $\forall\overrightarrow{x}\in\mathrm{R}^d$, pointwise with rates.

The next related result follows:

Theorem 33.39 Let $\overrightarrow{x}\in\mathrm{R}^d$, $f\in C_{\mathcal{F}}^{N}(\mathrm{R}^d)$, $N\in\mathrm{N}$, such that all of its fuzzy partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha}:|\widetilde{\alpha}|=N$, are fuzzy uniformly continuous or fuzzy continuous and fuzzy bounded. Then

$$D\left((F_n(f))(\overrightarrow{x}),f(\overrightarrow{x})\right)\le D\left(f(\overrightarrow{x}),\widetilde{0}\right)\cdot\left|\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^\alpha\rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^\alpha]}\frac{1}{I\cdot n^{\alpha d}}\cdot\right.$$

$$\left.b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)-1\right|+\frac{b(\overrightarrow{0})}{I}\cdot\left(\prod_{i=1}^{d}\left(2T_i+\frac{1}{n^\alpha}\right)\right)\cdot$$

$$\left\{\sum_{j=1}^{N}\frac{(T^*)^j}{j!\ n^{j(1-\alpha)}}\left[\left(\sum_{i=1}^{m}D\left(\frac{\partial}{\partial x_i},\ \widetilde{0}\right)\right)^j f(\overrightarrow{x})\right]\right\}+\frac{(T^*)^N\cdot d^N}{N!\ n^{N(1-\alpha)}}\cdot\frac{b(\overrightarrow{0})}{I}\cdot$$

$$\left(\prod_{i=1}^{d}\left(2T_i+\frac{1}{n^\alpha}\right)\right)\cdot\max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N}w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right)_\infty. \tag{33.18}$$

As $n\to\infty$, we get $D\left((F_n(f))(\overrightarrow{x}),f(\overrightarrow{x})\right)\to 0$ pointwise with rates.

Proof. As before we have

$$D\left(\left(F_n(f)\right)(\overrightarrow{x}), f(\overrightarrow{x}) \right) = \sup_{r\in[0,1]} \max\left\{ \left| \left(F_n\left(f_-^{(r)}\right)\right)(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x}) \right|, \right.$$

$$\left. \left| \left(F_n\left(f_+^{(r)}\right)\right)(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x}) \right| \right\} \leq^{([23],p.103)} \sup_{r\in[0,1]} \max\left\{ |f_-^{(r)}(\overrightarrow{x})| \cdot \left| \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}]} \frac{1}{I\cdot n^{\alpha d}} \cdot \right.\right.$$

$$\left. b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right) \right) - 1 \right| + \frac{b(\overrightarrow{0})}{I} \cdot \left(\prod_{i=1}^{d} \left(2T_i + \frac{1}{n^{\alpha}} \right) \right)$$

$$\cdot \left[\sum_{j=1}^{N} \frac{(T^*)^j}{j!\; n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_-^{(r)}(\overrightarrow{x}) \right) \right] + \frac{(T^*)^N \cdot d^N}{N!\; n^{N(1-\alpha)}} \cdot \frac{b(\overrightarrow{0})}{I}$$

$$\cdot \left(\prod_{i=1}^{d} \left(2T_i + \frac{1}{n^{\alpha}} \right) \right) \cdot \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1 \left((f_-^{(r)})_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right)_{\infty},$$

$$|f_+^{(r)}(\overrightarrow{x})| \cdot \left| \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}]} \frac{1}{I\cdot n^{\alpha d}} \cdot b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right) \right) - 1 \right| + \frac{b(\overrightarrow{0})}{I} \cdot \left(\prod_{i=1}^{d} \left(2T_i + \frac{1}{n^{\alpha}} \right) \right)$$

$$\cdot \left[\sum_{j=1}^{N} \frac{(T^*)^j}{j!\; n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_+^{(r)}(\overrightarrow{x}) \right) \right] + \frac{(T^*)^N \cdot d^N}{N!\; n^{N(1-\alpha)}} \cdot \frac{b(\overrightarrow{0})}{I}$$

$$\left. \cdot \left(\prod_{i=1}^{d} \left(2T_i + \frac{1}{n^{\alpha}} \right) \right) \cdot \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1 \left((f_+^{(r)})_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right)_{\infty} \right\} \leq$$

$$\sup_{r\in[0,1]} \max\left\{ |f_-^{(r)}(\overrightarrow{x})|,\ |f_+^{(r)}(\overrightarrow{x})| \right\} \cdot \left| \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}]} \frac{1}{I\cdot n^{\alpha d}} \cdot b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right) \right) - 1 \right| +$$

$$\frac{b(\overrightarrow{0})}{I} \cdot \left(\prod_{i=1}^{d} \left(2T_i + \frac{1}{n^{\alpha}} \right) \right) \cdot \left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j!\; n^{j(1-\alpha)}} \cdot \right.$$

$$\left. \sup_{r\in[0,1]} \max\left\{ \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_-^{(r)}(\overrightarrow{x}) \right), \left(\left(\sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_+^{(r)}(\overrightarrow{x}) \right) \right\} + \right.$$

$$\frac{(T^*)^N \cdot d^N}{N!\; n^{N(1-\alpha)}} \cdot \frac{b(\overrightarrow{0})}{I} \cdot \left(\prod_{i=1}^{d} \left(2T_i + \frac{1}{n^{\alpha}} \right) \right) \cdot$$

$$\max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} \sup_{r\in[0,1]} \max\left\{ w_1\left((f_{\widetilde{\alpha}})_-^{(r)}, \frac{T^*}{n^{1-\alpha}}\right)_\infty,\ w_1\left((f_{\widetilde{\alpha}})_+^{(r)}, \frac{T^*}{n^{1-\alpha}}\right)_\infty \right\} \le$$

$$D\left(f(\overrightarrow{x}), \widetilde{0}\right) \cdot \left| \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^\alpha \rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^\alpha]} \frac{1}{I\cdot n^{\alpha d}} \cdot b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right) - 1 \right| + \frac{b(\overrightarrow{0})}{I} \cdot \left(\prod_{i=1}^d \left(2T_i + \frac{1}{n^\alpha}\right)\right)$$

$$\cdot\left\{ \sum_{j=1}^N \frac{(T^*)^j}{j!\ n^{j(1-\alpha)}} \left[\left(\sum_{i=1}^m D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x})\right]\right\} + \frac{(T^*)^N \cdot d^N}{N!\ n^{N(1-\alpha)}} \cdot \frac{b(\overrightarrow{0})}{I} \cdot$$

$$\left(\prod_{i=1}^d \left(2T_i + \frac{1}{n^\alpha}\right)\right) \cdot \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right)_\infty,$$

proving the claim. ∎

33.2.3 The Multivariate Fuzzy "Squashing Operators" and Their Fuzzy Convergence to the Unit with Rates

We use

Definition 33.40 Let the nonnegative function $S : \mathrm{R}^d \to \mathrm{R},\ d \ge 1,\ S$ has compact support

$$\mathcal{B} := \prod_{i=1}^d [-T_i, T_i],\ T_i > 0$$

and is nondecreasing for each coordinate. S can be continuous only on either $\prod_{i=1}^d(-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [128]). Assume that

$$I^* := \int_{\mathcal{B}} S(\overrightarrow{t})d\overrightarrow{t} > 0. \tag{33.19}$$

Example 33.41 Let $\hat{S}$ as above when $d = 1$. Then

$$\hat{\hat{S}}(\overrightarrow{x}) := \hat{S}(x_1)\ldots\hat{S}(x_d),\quad \overrightarrow{x} = (x_1, \ldots,\ x_d) \in \mathrm{R}^d,$$

is a multivariate "squashing function".

Let $f : \mathrm{R}^d \to \mathrm{R}_{\mathcal{F}}$ be either a fuzzy uniformly continuous or a fuzzy continuous and fuzzy bounded function. Let $\overrightarrow{x},\ \overrightarrow{x'} \in \mathcal{B}$ such that $x_{ik} \le x'_{ik}$ for some $i_k \in \{1, \ldots, d\};\ k = 1, \ldots,\ r \le d$. Then

$$S(x_1, \ldots, x_{i_1}, \ldots, x_{i_2}, \ldots, x_{i_3}, \ldots, x_{i_k}, \ldots, x_d)$$
$$\le S(x_1, \ldots, x'_{i_1}, \ldots, x'_{i_2}, \ldots, x'_{i_3}, \ldots, x'_{i_k}, \ldots, x_d).$$

Clearly

$$\max_{\overrightarrow{x} \in \mathcal{B}} S(\overrightarrow{x}) = S(\overrightarrow{T}), \quad \overrightarrow{T} := (T_1, \ldots, T_d).$$

For $\overrightarrow{x} \in \mathrm{R}^d$ we define the multivariate fuzzy "squashing operator"

$$(G_n(f))(\overrightarrow{x}) = \sum_{k_1=-n^2}^{n^2}{}^{*} \cdots \sum_{k_d=-n^2}^{n^2}{}^{*} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) \odot$$

$$\frac{S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, \ n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{I^* \cdot n^{\alpha \cdot d}}, \tag{33.20}$$

where $0 < \alpha < 1$ and $n \in \mathrm{N}$:

$$n \geq \max_{i \in \{1, \ldots, \ d\}} \{T_i + |x_i|, \ T_i^{-1/\alpha}\}. \tag{33.21}$$

It is clear that

$$(G_n(f))(\overrightarrow{x}) = \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha} \rceil}^{[n\overrightarrow{x} - \overrightarrow{T} n^{\alpha}]}{}^{*} f\left(\frac{\overrightarrow{k}}{n}\right) \odot \frac{S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{I^* \cdot n^{\alpha \cdot d}}. \tag{33.22}$$

For the real analog of G_n see [128], [23], p.112.
We notice the following ($r \in [0, 1]$)

$$[G_n(f)]^r = \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n\alpha \rceil}^{[n\overrightarrow{x} - \overrightarrow{T} n\alpha]} \left[f\left(\frac{\overrightarrow{k}}{n}\right)\right]^r \cdot \frac{S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{I^* \cdot n^{\alpha \cdot d}} =$$

$$\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n\alpha \rceil}^{[n\overrightarrow{x} - \overrightarrow{T} n\alpha]} \left[f_-^{(r)}\left(\frac{\overrightarrow{k}}{n}\right), f_+^{(r)}\left(\frac{\overrightarrow{k}}{n}\right)\right] \cdot \frac{S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{I^* \cdot n^{\alpha \cdot d}} =$$

$$\left[\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n\alpha \rceil}^{[n\overrightarrow{x} - \overrightarrow{T} n\alpha]} f_-^{(r)}\left(\frac{\overrightarrow{k}}{n}\right) \cdot \frac{S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{I^* \cdot n^{\alpha \cdot d}},\right.$$

$$\left.\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n\alpha \rceil}^{[n\overrightarrow{x} - \overrightarrow{T} n\alpha]} f_+^{(r)}\left(\frac{\overrightarrow{k}}{n}\right) \cdot \frac{S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{I^* \cdot n^{\alpha \cdot d}}\right].$$

That is we proved

$$\left(G_n(f)\right)_{\pm}^{(r)} = G_n\left(f_{\pm}^{(r)}\right), \ \forall r \in [0,1]. \tag{33.23}$$

Here we study the fuzzy pointwise convergence with rates of $\left(G_n(f)\right)(\overrightarrow{x}) \to f(\overrightarrow{x})$, as $n \to \infty$, $\overrightarrow{x} \in \mathrm{R}^d$. This is given in the next result.

Theorem 33.42 Under the above terms and assumptions we obtain

$$D\left(\left(G_n(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \leq D\left(f(\overrightarrow{x}), \widetilde{0}\right) \cdot \left| \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha d}} \cdot \right.$$

$$\left. S\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right) - 1\right| + \frac{S(\overrightarrow{T})}{I^*} \cdot \left(\prod_{i=1}^{d}\left(2T_i + \frac{1}{n^{\alpha}}\right)\right) \cdot w_1^{(\mathcal{F})}\left(f, \frac{T^*}{n^{1-\alpha}}\right)_{\infty}. \tag{33.24}$$

Proof. Based on (33.23), Theorem 3.3, p.113 of [23]. It is similar to the proof of Theorem 33.36 here. ■

We need

Lemma 33.43 ([23], p.114) It holds

$$D_n(\overrightarrow{x}) := \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha d}} \cdot S\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right) \to 1, \tag{33.25}$$

pointwise, as $n \to \infty$, where $\overrightarrow{x} \in \mathrm{R}^d$.

Remark 33.44 Let $f \in C_{\mathcal{F}}^U(\mathrm{R}^d)$ then, as $n \to \infty$, we get $D\left(\left(G_n(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \to 0$, $\forall \overrightarrow{x} \in \mathrm{R}^d$, pointwise with rates.

We finish with

Theorem 33.45 Let $\overrightarrow{x} \in \mathrm{R}^d$, $f \in C_{\mathcal{F}}^N(\mathrm{R}^d)$, $N \in \mathrm{N}$, such that all of its fuzzy partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha}: |\widetilde{\alpha}| = N$, are fuzzy uniformly continuous or fuzzy continuous and fuzzy bounded. Then

$$D\left(\left(G_n(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \leq D\left(f(\overrightarrow{x}), \widetilde{0}\right) \cdot \left| \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha d}} \cdot \right.$$

$$\left. S\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right) - 1\right| + \frac{S(\overrightarrow{T})}{I^*} \cdot \left(\prod_{i=1}^{d}\left(2T_i + \frac{1}{n^{\alpha}}\right)\right) \cdot$$

$$\left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j!\, n^{j(1-\alpha)}} \left[\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i}, \widetilde{0} \right) \right)^j f(\overrightarrow{x}) \right] \right\} + \frac{(T^*)^N \cdot d^N}{N!\, n^{N(1-\alpha)}} \cdot \frac{S(\overrightarrow{T})}{I^*} \cdot$$

$$\left(\prod_{i=1}^{d} \left(2T_i + \frac{1}{n^\alpha} \right) \right) \cdot \max_{\widetilde{\alpha}:\ |\widetilde{\alpha}|=N} w_1^{(\mathcal{F})} \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right)_\infty . \tag{33.26}$$

As $n \to \infty$, we get $D\Big((G_n(f))(\overrightarrow{x}), f(\overrightarrow{x}) \Big) \to 0$, pointwise with rates.

Proof. Similar to the proof of Theorem 33.39 here, based on Theorem 3.4, p.117 of [23]. ■

34

Fuzzy Fractional Calculus and the Ostrowski Integral Inequality

Here we introduce and study the right and left fuzzy fractional Riemann- Liouville integrals and the right and left fuzzy fractional Caputo derivatives. Then we present the right and left fuzzy fractional Taylor formulae. Based on these we establish a fuzzy fractional Ostrowski type inequality with applications. The last inequality provides an estimate for the deviation of a fuzzy real number valued function from its fuzzy average, and the related upper bounds are given in terms of the right and left fuzzy fractional derivatives of the involved function. The purpose of embedding fuzziness into fractional calculus and have them act together, is to better understand, explain and describe the imprecise, uncertain and chaotic phenomena of the real world and then derive useful conclusions. This chapter is based on [54].

34.1 Fuzzy Mathematical Analysis Background

We need the following basic background

Definition 34.1. (see [283]) Let $\mu : \mathrm{R} \to [0,1]$ with the following properties

(i) is normal, i.e., $\exists x_0 \in \mathrm{R};\ \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x)\ \mu(y)\}$, $\forall x, y \in \mathrm{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 553–574.
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(iii) μ is upper semicontinuous on R, i.e. $\forall x_0 \in \mathbb{R}$ and $\forall \epsilon > 0$, $\exists$ neighborhood $V(x_0): \ \mu(x) \le \mu(x_0) + \epsilon, \ \forall x \in V(x_0)$.

(iv) The set $\overline{supp(\mu)}$ is compact in R (where $supp(\mu) := \{x \in \mathbb{R}: \ \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \le 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R}: \ \mu(x) \ge r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R}: \ \mu(x) \ge 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of R ([172]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \forall r \in [0,1],$$

where

- $[u]^r + [v]^r$ means the usual addition of two integrals (as subsets of R) and
- $\lambda [u]^r$ means the usual product between a scalar and a subset of R (see, e.g., [283]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \le r_1 \le r_2 \le 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, \ u_+^{(r)}]$, where $u_-^{(r)} \le u_+^{(r)}, \ u_-^{(r)}, \ u_+^{(r)} \in \mathbb{R}, \ \forall r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ |u_-^{(r)} - v_-^{(r)}|, \ |u_+^{(r)} - v_+^{(r)}| \right\},$$

where

$$[v]^r = [v_-^{(r)}, \ v_+^{(r)}]; \ u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [284], [283].

Here Σ^* stands for fuzzy summation and $\widetilde{0} : \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \forall u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^*(f,g) := \sup_{x \in [a,b]} D(f,g),$$

where $f, \ g : [a,b] \to \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 34.2. Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, we define the (first) fuzzy modulus of continuity of f by

$$w_1^{(\mathcal{F})}(f,\delta) = \sup_{x,y \in [a,b], \ |x-y| \le \delta} D(f(x), \ f(y)), \ \delta > 0.$$

We define $C_{\mathcal{F}}^U([a,b])$ the space of uniformly continuous functions from $[a,b] \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}([a,b])$ the space of fuzzy continuous functions on $[a,b]$.

It is clear that

$$C_{\mathcal{F}}^U([a,b]) = C_{\mathcal{F}}([a,b]).$$

We mention

Proposition 34.3. ([37]) Let $f \in C_{\mathcal{F}}^U([a,b])$. Then $w_1^{(\mathcal{F})}(f,\delta) < \infty$, any $\delta > 0$.

Proposition 34.4. ([37]) It holds

$$\lim_{\delta \to 0} w_1^{(\mathcal{F})}(f,\delta) = w_1^{(\mathcal{F})}(f,0) = 0,$$

iff $f \in C_{\mathcal{F}}^U([a,b])$.

Proposition 34.5. ([37]) Here $[f]^r = [f_-^{(r)}, \ f_+^{(r)}], \ r \in [0,1]$. If $f \in C_{\mathcal{F}}([a,b])$ then $f_\pm^{(r)} \in C([a,b)$, for $r \in [0,1]$, in fact these are equicontinuous families, respectively in $\pm$. Furthermore f is a fuzzy bounded function.

We need

Definition 34.6. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : \ x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.

Definition 34.7 ([283]) Let $T := [x_0, \ x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \ \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h)-f(x),\ f(x)-f(x-h)$ exist in $\mathbb{R}_{\mathcal{F}}$ in an neighborhood of x.

We denote by $C^N_{\mathcal{F}}([a,b]),\ N \geq 1$, the space all N-times continuously fuzzy differentiable functions from $[a,b]$ into $\mathbb{R}_{\mathcal{F}}$.

We mention

Theorem 34.8 ([202]) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b],\ 0 \leq r \leq 1$. Clearly

$$[f(t)]^r = [f(t)^{(r)}_-,\ f(t)^{(r)}_+] \subseteq \mathbb{R}.$$

Then $(f(t))^{(r)}_{\pm}$ are differentiable and

$$[f'(t)]^r = [(f(t)^{(r)}_-)',\ (f(t)^{(r)}_+)'].$$

I.e.

$$f'^{(r)}_{\pm} = (f^{(r)}_{\pm})',\ \forall r \in [0,1].$$

Remark 34.9 ([35]) Let $f \in C^N_{\mathcal{F}}([a,b]),\ N \geq 1$. Then by Theorem 34.8 we obtain

$$[f^{(i)}(t)]^r = [(f(t)^{(r)}_-)^{(i)},\ (f(t)^{(r)}_+)^{(i)}],$$

for $i = 0,1,2\ldots,N$, and in particular we have that

$$(f^{(i)})^{(r)}_{\pm} = (f^{(r)}_{\pm})^{(i)},$$

for any $r \in [0,1]$, all $i = 0,1,2\ldots,N$.

Note 34.10 ([35]) Let $f \in C^N_{\mathcal{F}}([a,b]),\ N \geq 1$. Then by Theorem 34.8 we have $f^{(r)}_{\pm} \in C^N_{\mathcal{F}}([a,b])$, for any $r \in [0,1]$.

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \delta/2$), see [283].

Definition 34.11 ([165], p. 644) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P^* (v-u) \odot f(\xi),\ I\right) < \epsilon.$$

We write

$$I := (FR)\int_a^b f(x)dx.$$

We mention

Theorem 34.12 ([172]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then

$$(FR)\int_a^b f(x)dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR)\int_a^b f(x)dx\right]^r = \left[\int_a^b (f)_-^{(r)}(x)dx,\ (f)_+^{(r)}(x)dx\right],$$

$\forall r \in [0,1]$.

Theorem 34.13 ([283]) Let $f \in C_{\mathcal{F}}([a,b])$ and $c \in [a,b]$. Then

$$(FR)\int_a^b f(x)dx = (FR)\int_a^c f(x)dx + (FR)\int_c^b f(x)dx.$$

Theorem 34.14 ([172]) Let $f, g : [a,b] \in C_{\mathcal{F}}([a,b])$ and $c_1, c_2 \in \mathbb{R}$. Then

$$(FR)\int_a^b (c_1 f(x) + c_2 g(x))dx = c_1(FR)\int_a^b f(x)dx + c_2(FR)\int_a^b g(x)dx.$$

Also we need

Lemma 34.15 ([26]) If $f, g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ are fuzzy continuous functions, then the function $F : [a,b] \to \mathbb{R}_+$ defined by $F(x) := D(f(x), g(x))$ is continuous on $[a,b]$, and

$$D\left((FR)\int_a^b f(x)dx, (FR)\int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx.$$

For the definition of general fuzzy integral we follow [206] next.

Definition 34.16. Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to \mathbb{R}_{\mathcal{F}}$ *measurable* iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x),\ all\ w \in \Omega$$

is measurable, see [206].

Theorem 34.17 ([206]) For $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \{(F_-^{(r)}(w),\ F_+^{(r)}(w))|0 \leq r \leq 1\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall r \in [0,1],\ F_-^{(r)},\ F_+^{(r)}$ are measurable.

Following [206], given that for each $r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_-^{(r)} d\mu,\ \int_A F_+^{(r)} \right) \middle| 0 \le r \le 1 \right\}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 34.18 ([206]) A measurable function $F : \Omega \to \mathrm{R}_{\mathcal{F}}$,

$$F(w) = \{(F_-^{(r)}(w),\ F_+^{(r)}(w)) | 0 \le r \le 1\}$$

is *integrable* if for each $r \in [0,1]$, $F_\pm^{(r)}$ are integrable, or equivalently, if $F_\pm^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu,\ \int_A F_+^{(r)} \right) \middle| 0 \le r \le 1 \right\}.$$

By [206], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D(u, \widetilde{0}),\ \forall u \in \mathrm{R}_{\mathcal{F}}.$$

We need also

Theorem 34.19 ([206]) Let $F, G : \Omega \to \mathrm{R}_{\mathcal{F}}$ be integrable. Then

(1) Let $a, b \in \mathrm{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG) d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu,\ \int_A G d\mu \right) \le \int_A D(F, G) d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \le \int_A \|F\|_{\mathcal{F}} d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu,\ \int_A F_+^{(r)}\right],$$

that is,

$$\left(\int_A F d\mu\right)_\pm^{(r)} = \int_A F_\pm^{(r)} d\mu,\ \forall r \in [0,1],$$

respectively.

Let $f \in C_{\mathcal{F}}([a,b]),\ \nu > 0$.

We define the Fuzzy Fractional left Riemann- Liouville operator as

$$J_a \nu f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t)dt,\ x \in [a,b],$$
$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann- Liouville operator as

$$\mathrm{I}_{b-}\nu f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)dt,\ x \in [a,b],$$
$$\mathrm{I}_{b-}^0 f := f.$$

Above, Γ is the gamma function

$$\Gamma(\nu) := \int_0^\infty \mathrm{e}^{-t} t^{\nu-1} dt.$$

We mention

Definition 34.20. Let $f : [a,b] \to \mathrm{R}_{\mathcal{F}}$ is called fuzzy absolutely continuous iff $\forall \epsilon > 0,\ \exists \delta > 0$: for every finite, pairwise disjoint, family

$$(c_k, d_k)_{k=1}^n \subseteq (a,b) \ with \ \sum_{k=1}^n (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^n D(f(d_k), f(c_k)) < \epsilon.$$

We denote the related space of functions by $AC_{\mathcal{F}}([a,b])$.

If $f \in AC_{\mathcal{F}}([a,b])$, then $f \in C_{\mathcal{F}}([a,b])$.

It holds

Proposition 34.21. $f \in AC_{\mathcal{F}}([a,b]) \iff f_\pm^{(r)} \in AEC([a,b]),\ \forall r \in [0,1]$ (absolutely equicontinuous).

Proof. Let $f \in AC_{\mathcal{F}}([a,b])$, then $\forall \epsilon > 0,\ \exists \delta > 0$: for every finite, pairwise disjoint, family

$$(c_k, d_k)_{k=1}^n \subseteq (a,b) \textit{ with } \sum_{k=1}^n (d_k - c_k) < \delta$$

we obtain

$$\sum_{k=1}^n D(f(d_k), f(c_k)) < \epsilon.$$

The last condition means

$$\sum_{k=1}^n \sup_{r\in[0,1]} \max\left\{|f_-^{(r)}(d_k) - f_-^{(r)}(c_k)|,\ |f_+^{(r)}(d_k) - f_+^{(r)}(c_k)|\right\} < \epsilon.$$

But we have

$$\sup_{r\in[0,1]} \max\left\{\sum_{k=1}^n |f_-^{(r)}(d_k) - f_-^{(r)}(c_k)|,\ \sum_{k=1}^n |f_+^{(r)}(d_k) - f_+^{(r)}(c_k)|\right\}$$
$$\leq \sum_{k=1}^n \sup_{r\in[0,1]} \max\left\{|f_-^{(r)}(d_k) - f_-^{(r)}(c_k)|,\ |f_+^{(r)}(d_k) - f_+^{(r)}(c_k)|\right\}.$$

From the above we derive

$$\sum_{k=1}^n |f_-^{(r)}(d_k) - f_-^{(r)}(c_k)| < \epsilon$$
$$\sum_{k=1}^n |f_+^{(r)}(d_k) - f_+^{(r)}(c_k)| < \epsilon,$$

$\forall r \in [0,1]$, proving the claim. ■

Remark 34.22. So, if $f \in AC_{\mathcal{F}}([a,b])$, then f is of bounded variation in the fuzzy sense.

Clearly here $f_{\pm}^{(r)}$ are differentiable a.e., for any $r \in [0,1]$.

Hence by Theorem 34.8 we get

$$(f_{\pm}^{(r)})' = (f')_{\pm}^{(r)},$$

a.e. on $[a,b]$, and

$$(f_{\pm}^{(r)})' \in L_1([a,b]),\ \forall r \in [0,1].$$

Let $f_* : [0,1] \to \mathrm{R}_{\mathcal{F}}$, given by

$$f_*(x) := u \odot \mathrm{e}^{-x},$$

where $u \in \Re_{\mathcal{F}}$ is fixed.

Clearly f_* is a Lipschitz function in the fuzzy sense: indeed we have (by Lemma 2.2, [71])

$$D(\mathrm{e}^{-x} \odot u, \mathrm{e}^{-y} \odot u) \leq |\mathrm{e}^{-x} - \mathrm{e}^{-y}| \cdot D(u, \widetilde{0}) \leq D(u, \widetilde{0}) \cdot |x - y|, \ \forall x, y \in [0,1].$$

That is

$$D(f_*(x), f_*(y)) \leq D(u, \widetilde{0}) \cdot |x - y|, \ \forall x, y \in [0,1].$$

Therefore $f_* \in AC_{\mathcal{F}}([0,1])$, but f_* is nowhere H-differentiable ([109]).

Consequently fuzzy absolutely continuity does not necessarily imply H-differentiability a.e.

34.2 Main Results

We mention

Definition 34.23. We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.

Let $f \in C^n_{\mathcal{F}}([a,b]), \ n = \lceil \nu \rceil, \ \nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling).

$$D^{\nu\mathcal{F}}_{*a} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t) dt$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} (f^{(n)})^{(r)}_-(t) dt, \right. \right.$$

$$\left. \left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} (f^{(n)})^{(r)}_+(t) dt, \right) \right| 0 \leq r \leq 1 \right\} =$$

$$\left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} (f^{(r)}_-)^{(n)}(t) dt, \right. \right.$$

$$\left. \left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} (f^{(r)}_+)^{(n)}(t) dt, \right) \right| 0 \leq r \leq 1 \right\}. \tag{34.1}$$

So, we obtain

$$\left[D^{\nu\mathcal{F}}_{*a} f(x) \right]^r = \left[\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} (f^{(r)}_-)^{(n)}(t) dt, \right.$$

$$\frac{1}{\Gamma(n-\nu)}\int_a^x (x-t)^{n-\nu-1}(f_+^{(r)})^{(n)}(t)dt,\Bigg)\Bigg],\ 0\leq r\leq 1. \tag{34.2}$$

That is

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)}\int_a^x (x-t)^{n-\nu-1}(f_{\pm}^{(r)})^{(n)}(t)dt$$
$$= \left(D_{*a}^{\nu}(f_{\pm}^{(r)})\right)(x),$$

see [145], [42].

I.e. we get that

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}(f_{\pm}^{(r)})\right)(x),$$

$\forall x\in[a,b]$, in short

$$\left(D_{*a}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}(f_{\pm}^{(r)}),\ \forall r\in[0,1]. \tag{34.3}$$

We use

Lemma 34.24. We prove that $D_{*a}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x\in[a,b]$.

Proof. Without loss of generality we may assume $a\leq x\leq y\leq b$, that is $0\leq x-a\leq y-a$.

So, we have

$$D\left(D_{*a}^{\nu\mathcal{F}} f(x),\ D_{*a}^{\nu\mathcal{F}} f(y)\right) = \frac{1}{\Gamma(n-\nu)}$$

$$D\left(\int_a^x (x-t)^{n-\nu-1}\odot f^{(n)}(t)dt,\ \int_a^y (y-t)^{n-\nu-1}\odot f^{(n)}(t)dt\right)$$

$$=\frac{1}{\Gamma(n-\nu)}D\left(\int_0^{x-a} z^{n-\nu-1}\odot f^{(n)}(x-z)dz,\ \int_0^{y-a} z^{n-\nu-1}\odot f^{(n)}(y-z)dz\right)=$$

$$\frac{1}{\Gamma(n-\nu)}D\Bigg(\int_0^{x-a} z^{n-\nu-1}\odot f^{(n)}(x-z)dz,$$

$$\int_0^{x-a} z^{n-\nu-1}\odot f^{(n)}(y-z)dz\oplus\int_{x-a}^{y-a} z^{n-\nu-1}\odot f^{(n)}(y-z)dz\Bigg)$$

$$\leq\frac{1}{\Gamma(n-\nu)}\Bigg\{D\left(\int_0^{x-a} z^{n-\nu-1}\odot f^{(n)}(x-z)dz,\ \int_0^{x-a} z^{n-\nu-1}\odot f^{(n)}(y-z)dz\right)+$$

$$D\left(\int_{x-a}^{y-a} z^{n-\nu-1}\odot f^{(n)}(y-z)dz,\ \widetilde{0}\right)\Bigg\}\leq$$

$$\frac{1}{\Gamma(n-\nu)}\Bigg\{\int_{0}^{x-a} z^{n-\nu-1}D\left(f^{(n)}(x-z),f^{(n)}(y-z)\right)dz$$

$$+\int_{x-a}^{y-a} z^{n-\nu-1}D\left(f^{(n)}(y-z),\widetilde{0}\right)dz\Bigg\}\leq$$

$$\frac{1}{\Gamma(n-\nu)}\Bigg\{\left(\int_{0}^{x-a} z^{n-\nu-1}dz\right)w_1^{(\mathcal{F})}(f^{(n)},y-x)+D^*(f^{(n)},\widetilde{0})\int_{x-a}^{y-a} z^{n-\nu-1}dz\Bigg\}=$$

$$\frac{1}{\Gamma(n-\nu)}\Bigg\{\frac{(x-a)^{n-\nu}}{n-\nu}w_1^{(\mathcal{F})}(f^{(n)},y-x)+D^*(f^{(n)},\tilde{0})\cdot\left(\frac{(y-a)^{n-\nu}-(x-a)^{n-\nu}}{n-\nu}\right)\Bigg\}$$

$$\leq\frac{1}{\Gamma(n-\nu)}\Bigg\{\frac{(b-a)^{n-\nu}}{n-\nu}w_1^{(\mathcal{F})}(f^{(n)},y-x)+\frac{D^*(f^{(n)},\tilde{0})}{(n-\nu)}\left((y-a)^{n-\nu}-(x-a)^{n-\nu}\right)\Bigg\}\to 0,$$

as $y\to x$, by noticing $f^{(n)}\in C_{\mathcal{F}}^{U}([a,b])$. ■

It follows the Fuzzy fractional left Caputo Taylor formula.

Theorem 34.25 Let $\nu>0,\ n=\lceil\nu\rceil,\ f\in C_{\mathcal{F}}^{n}([a,b]),\ a\leq x\leq b.$

Then

$$f(x)=\sum_{k=0}^{n-1\,*}\frac{(x-a)^k}{k!}\odot f^{(k)}(a)\oplus$$

$$\frac{1}{\Gamma(\nu)}\odot\int_a^x (x-t)^{\nu-1}\odot\left(D_{*a}^{\nu\mathcal{F}}f\right)(t)dt. \tag{34.4}$$

Proof. We obtain (see [145], p.40, [42], p.616)

$$f_{\pm}^{(r)}(x)=\sum_{k=0}^{n-1}\frac{(f_{\pm}^{(r)})^{(k)}(a)}{k!}(x-a)^k+\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}D_{*a}\nu f_{\pm}^{(r)}(t)dt$$

$$=\sum_{k=0}^{n-1}\frac{(f^{(k)})_{\pm}^{(r)}(a)}{k!}(x-a)^k+\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\left(D_{*a}^{\nu\mathcal{F}}f\right)_{\pm}^{(r)}(t)dt.$$

Here it holds $b-a \geq 0$, $x-a \geq 0$, for $x \in [a,b]$, and $(f^{(k)})_-^{(r)}(t) \leq (f^{(k)})_+^{(r)}(t)$, $\forall t \in [a,b]$, all $k = 0,1,\ldots, n$, $\forall r \in [0,1]$.

We observe that

$$[f(x)]^r = [f_-^{(r)}(x),\ f_+^{(r)}(x)] = \left[\sum_{k=0}^{n-1} \frac{(f^{(k)})_-^{(r)}(a)}{k!}(x-a)^k + \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\left(D_{*a}^{\nu\mathcal{F}} f\right)_-^{(r)}(t)dt,\right.$$

$$\left.\sum_{k=0}^{n-1} \frac{(f^{(k)})_+^{(r)}(a)}{k!}(x-a)^k + \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\left(D_{*a}^{\nu\mathcal{F}} f\right)_+^{(r)}(t)dt\right]$$

$$= \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}\left[(f^{(k)})_-^{(r)}(a),\ (f^{(k)})_+^{(r)}(a)\right] +$$

$$\frac{1}{\Gamma(\nu)}\left[\int_a^x (x-t)^{\nu-1}\left(D_{*a}^{\nu\mathcal{F}} f\right)_-^{(r)}(t)dt,\ \int_a^x (x-t)^{\nu-1}\left(D_{*a}^{\nu\mathcal{F}} f\right)_+^{(r)}(t)dt\right].$$

Lemma 34.24 implies that $\left(D_{*a}^{\nu\mathcal{F}} f\right)_\pm^{(r)}$, $r \in [0,1]$ are in $C([a,b])$.

Furthermore

$$(x-t)^{\nu-1}\left(D_{*a}^{\nu\mathcal{F}} f\right)_\pm^{(r)}(t),$$

are Lebesgue integrable, $r \in [0,1]$.

Thus we get

$$\int_a^x (x-t)^{\nu-1} \odot \left(D_{*a}^{\nu\mathcal{F}} f\right)(t)dt \in \mathrm{R}_{\mathcal{F}}.$$

So we obtain $\forall r \in [0,1]$ that

$$[f(x)]^r = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}[f^{(k)}(a)]^r + \frac{1}{\Gamma(\nu)}\left[\int_a^x (x-t)^{\nu-1} \odot \left(D_{*a}^{\nu\mathcal{F}} f\right)(t)dt\right]^r =$$

$$\left[\sum_{k=0}^{n-1}{}^{*} \frac{(x-a)^k}{k!} \odot f^{(k)}(a) \oplus \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot \left(D_{*a}^{\nu\mathcal{F}} f\right)(t)dt\right]^r,$$

proving the claim. ∎

We need

Definition 34.26 We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.

Let $f \in C_{\mathcal{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$.

$$D_{b-}^{\nu\mathcal{F}} f(x) \quad : \quad = \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)dt$$

$$= \quad \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} (f^{(n)})_-^{(r)}(t)dt, \right.\right.$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} (f^{(n)})_+^{(r)}(t)dt \right) \middle| 0 \le r \le 1 \right\}$$

$$= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} (f_-^{(r)})^{(n)}(t)dt, \right.\right. \tag{34.5}$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} (f_+^{(r)})^{(n)}(t)dt \right) \middle| 0 \le r \le 1 \right\}$$

We obtain

$$[D_{b-}^{\nu\mathcal{F}} f(x)]^r = \left[\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} (f_-^{(r)})^{(n)}(t)dt, \right.$$

$$\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} (f_+^{(r)})^{(n)}(t)dt \right],$$

$0 \le r \le 1$.

That is

$$\left(D_{b-}^{\nu\mathcal{F}} f(x) \right)_\pm^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} (f_\pm^{(r)})^{(n)}(t)dt = \left(D_{b-}^{\nu} (f_\pm^{(r)}) \right)(x),$$

see [44].

I.e. we get that

$$\left(D_{b-}^{\nu\mathcal{F}} f(x) \right)_\pm^{(r)} = \left(D_{b-}^{\nu} (f_\pm^{(r)}) \right)(x), \tag{34.6}$$

$\forall x \in [a,b]$, in short

$$\left(D_{b-}^{\nu\mathcal{F}} f \right)_\pm^{(r)} = D_{b-}^{\nu} (f_\pm^{(r)}), \ \forall r \in [0,1].$$

Clearly

$$D_{b-}^{\nu} (f_-^{(r)}) \le D_{b-}^{\nu} (f_+^{(r)}), \ \forall r \in [0,1].$$

It follows the fractional fuzzy right Caputo Taylor formula.

Theorem 34.27 Let $\nu > 0,\quad n = \lceil \nu \rceil\ ,\ f \in C^n_{\mathcal{F}}([a,b]),\ a \le x \le b.$
Then

$$f(x) \oplus \sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(b-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(b) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(b-x)^{2m}}{(2m)!} \odot f^{(2m)}(b) \oplus$$

$$\frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot \left(D^{\nu\mathcal{F}}_{b-} f \right)(t) dt =: B. \tag{34.7}$$

Setting

$$A := \sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(b-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(b),$$

we get $f(x) = B - A$, as H-difference.
Above $[\cdot]$ denotes the integral part.

Proof. We obtain (see [44])

$$f^{(r)}_{\pm}(x) = \sum_{k=0}^{n-1} \frac{(f^{(r)}_{\pm})^{(k)}(b)}{k!}(x-b)^k + \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} D_{b-}\nu f^{(r)}_{\pm}(t) dt =$$

$$= \sum_{k=0}^{n-1} \frac{(f^{(k)})^{(r)}_{\pm}(b)}{k!}(x-b)^k + \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} \left(D^{\nu\mathcal{F}}_{b-} f \right)^{(r)}_{\pm}(t) dt.$$

Equivalently we have

$$f^{(r)}_{\pm}(x) + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m+1}}{(2m+1)!} (f^{(r)}_{\pm})^{(2m+1)}(b) =$$

$$\sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m}}{(2m)!} (f^{(r)}_{\pm})^{(2m)}(b) + \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} \left(D^{\nu\mathcal{F}}_{b-} f \right)^{(r)}_{\pm}(t) dt.$$

Here $b - x \ge 0$ for any $x \in [a,b]$ and

$$(f^{(k)})^{(r)}_{-}(t) \le (f^{(k)})^{(r)}_{+}(t),\ \forall t \in [a,b],$$

all $k = 0, 1, \ldots,\ n,\ \forall r \in [0,1]$.
We observe that

$$\left[f(x) \oplus \sum_{m=0}^{\left[\frac{n-1}{2}\right]}{}^{*} \frac{(b-x)^{2m+1}}{(2m+1)!} f^{(2m+1)}(b) \right]^{r} = [f(x)]^{r} + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m+1}}{(2m+1)!} \left[f^{(2m+1)}(b) \right]^{r} =$$

$$\left[(f(x))_{-}^{(r)}, \ (f(x))_{+}^{(r)} \right] + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m+1}}{(2m+1)!} \cdot \left[\left(f^{(2m+1)}(b) \right)_{-}^{(r)}, \ \left(f^{(2m+1)}(b) \right)_{+}^{(r)} \right] =$$

$$\left[(f(x))_{-}^{(r)}, \ (f(x))_{+}^{(r)} \right] + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m+1}}{(2m+1)!} \cdot \left[\left(f_{-}^{(r)} \right)^{(2m+1)}(b), \left(f_{+}^{(r)} \right)^{(2m+1)}(b) \right] =$$

$$\left[(f(x))_{-}^{(r)} + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m+1}}{(2m+1)!} \cdot \left(f_{-}^{(r)} \right)^{(2m+1)}(b), \ (f(x))_{+}^{(r)} + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m+1}}{(2m+1)!} \cdot \left(f_{+}^{(r)} \right)^{(2m+1)}(b) \right] =$$

$$\left[\sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m}}{(2m)!} \cdot \left(f_{-}^{(r)} \right)^{(2m)}(b) + \frac{1}{\Gamma(\nu)} \int_{x}^{b} (t-x)^{\nu-1} \left(D_{b-}^{\nu\mathcal{F}} f \right)_{-}^{(r)}(t) dt, \right.$$

$$\left. \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m}}{(2m)!} \cdot \left(f_{+}^{(r)} \right)^{(2m)}(b) + \frac{1}{\Gamma(\nu)} \int_{x}^{b} (t-x)^{\nu-1} \left(D_{b-}^{\nu\mathcal{F}} f \right)_{+}^{(r)}(t) dt \right]$$

$$= \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m}}{(2m)!} \cdot \left[\left(f_{-}^{(r)} \right)^{(2m)}(b), \ \left(f_{+}^{(r)} \right)^{(2m)}(b) \right]$$

$$+ \frac{1}{\Gamma(\nu)} \left[\int_{x}^{b} (t-x)^{\nu-1} \left(D_{b-}^{\nu\mathcal{F}} f \right)_{-}^{(r)}(t) dt, \ \int_{x}^{b} (t-x)^{\nu-1} \left(D_{b-}^{\nu\mathcal{F}} f \right)_{+}^{(r)}(t) dt \right] =: (*)$$

By Lemma 34.28 next, we get that $\left(D_{b-}^{\nu\mathcal{F}} f \right)_{\pm}^{(r)}$, $r \in [0,1]$ are in $C([a,b])$. Furthermore

$$(t-x)^{\nu-1} \left(D_{b-}^{\nu\mathcal{F}} f \right)_{\pm}^{(r)}(t)$$

are Lebesgue integrable, $r \in [0,1]$.

Thus we get

$$\int_x^b (t-x)^{\nu-1} \odot \left(D_{b-}^{\nu\mathcal{F}} f\right)(t)dt \in \mathrm{R}_{\mathcal{F}}.$$

So we have

$$(*) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m}}{(2m)!} \cdot \left[\left(f^{(2m)}(b)\right)_-^{(r)}, \left(f^{(2m)}(b)\right)_+^{(r)}\right] +$$

$$+\frac{1}{\Gamma(\nu)}\left[\int_x^b (t-x)^{\nu-1}\left(D_{b-}^{\nu\mathcal{F}} f\right)_-(r)(t)dt, \int_x^b (t-x)^{\nu-1}\left(D_{b-}^{\nu\mathcal{F}} f\right)_+^{(r)}(t)dt\right] =$$

$$\sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(b-x)^{2m}}{(2m)!} \cdot \left[f^{(2m)}(b)\right]^r + \frac{1}{\Gamma(\nu)}\left[\int_x^b (t-x)^{\nu-1} \odot \left(D_{b-}^{\nu\mathcal{F}} f\right)(t)dt\right]^r =$$

$$\left[\sum_{m=0}^{\left[\frac{n-1}{2}\right]}{}^* \frac{(b-x)^{2m}}{(2m)!} \odot f^{(2m)}(b) \oplus \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot \left(D_{b-}^{\nu\mathcal{F}} f\right)(t)dt\right]^r,$$

for any $r \in [0,1]$.

We have proved that

$$\left[f(x) \oplus \sum_{m=0}^{\left[\frac{n-1}{2}\right]}{}^* \frac{(b-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(b)\right]^r =$$

$$\left[\sum_{m=0}^{\left[\frac{n-1}{2}\right]}{}^* \frac{(b-x)^{2m}}{(2m)!} \odot f^{(2m)}(b) \oplus \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot \left(D_{b-}^{\nu\mathcal{F}} f\right)(t)dt\right]^r,$$

$\forall r \in [0,1]$, establishing the claim. ∎

We need

Lemma 34.28 $D_{b-}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.

Proof. Without loss of generality we suppose $a \le y \le x \le b$, that is $0 \le b-x \le b-y$.

So, we have

$$D\left(D_{b-}^{\nu\mathcal{F}} f(x), D_{b-}^{\nu\mathcal{F}} f(y)\right) = \frac{1}{\Gamma(n-\nu)}$$

$$D\left(\int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)dt, \int_y^b (t-y)^{n-\nu-1} \odot f^{(n)}(t)dt\right)$$

$$= \frac{1}{\Gamma(n-\nu)} D\left(\int_0^{b-x} z^{n-\nu-1} \odot f^{(n)}(z+x)dz,\ \int_0^{b-y} z^{n-\nu-1} \odot f^{(n)}(z+y)dz\right)$$

$$= \frac{1}{\Gamma(n-\nu)} D\Bigg(\int_0^{b-x} z^{n-\nu-1} \odot f^{(n)}(z+x)dz,$$

$$\int_0^{b-x} z^{n-\nu-1} \odot f^{(n)}(z+y)dz \oplus \int_{b-x}^{b-y} z^{n-\nu-1} \odot f^{(n)}(z+y)dz\Bigg) \leq \frac{1}{\Gamma(n-\nu)}$$

$$\Bigg\{D\left(\int_0^{b-x} z^{n-\nu-1} \odot f^{(n)}(z+x)dz,\ \int_0^{b-x} z^{n-\nu-1} \odot f^{(n)}(z+y)dz\right) +$$

$$D\Bigg(\tilde{0},\ \int_{b-x}^{b-y} z^{n-\nu-1} \odot f^{(n)}(z+y)dz\Bigg)\Bigg\} \leq$$

$$\frac{1}{\Gamma(n-\nu)}\left\{\int_0^{b-x} z^{n-\nu-1} D\left(f^{(n)}(z+x),\ f^{(n)}(z+y)\right)dz + \int_{b-x}^{b-y} z^{n-\nu-1} D\left(f^{(n)}(z+y),\ \tilde{0}\right)dz\right\} \leq$$

$$\frac{1}{\Gamma(n-\nu)}\left\{w_1^{(\mathcal{F})}(f^{(n)}, x-y)\left(\int_0^{b-x} z^{n-\nu-1}dz\right) + D^*(f^{(n)}, \tilde{0})\left(\int_{b-x}^{b-y} z^{n-\nu-1}dz\right)\right\} =$$

$$\frac{1}{\Gamma(n-\nu)}\left\{w_1^{(\mathcal{F})}(f^{(n)}, x-y)\frac{(b-x)^{n-\nu}}{(n-\nu)} + \frac{D^*(f^{(n)}, \tilde{0})}{(n-\nu)}\left[(b-y)^{n-\nu} - (b-x)^{n-\nu}\right]\right\} \leq$$

$$\frac{1}{\Gamma(n-\nu)}\left\{w_1^{(\mathcal{F})}(f^{(n)}, x-y)\frac{(b-a)^{n-\nu}}{(n-\nu)} + \frac{D^*(f^{(n)}, \tilde{0})}{(n-\nu)}\left[(b-y)^{n-\nu} - (b-x)^{n-\nu}\right]\right\} \to 0$$

as $y \to x$, by noticing $f^{(n)} \in C_{\mathcal{F}}^U([a,b])$. ■

We next give a fuzzy- fractional Ostrowski inequality, motivated by [238], [27].

Theorem 34.29 Let $\nu > 0,\ n = \lceil \nu \rceil,\ f \in C_{\mathcal{F}}^n([a,b]),\ c \in [a,b]$.

Then

1)

$$D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(x)dx, f(c)\right) \leq \frac{1}{b-a}$$

$$\left\{\sum_{k=1}^{n-1} \frac{D^*(f^{(k)}(c), \tilde{0})}{(k+1)!}\left[(b-c)^{k+1} + (c-a)^{k+1}\right] + \right.$$

$$\left.\frac{1}{\Gamma(\nu+2)}\left[(b-c)^{\nu+1}\sup_{t\in[c,b]} D\left((D_{*c}^{\nu\mathcal{F}} f)(t), \tilde{0}\right) + (c-a)^{\nu+1}\sup_{t\in[a,c]} D\left((D_{c-}^{\nu\mathcal{F}} f)(t), \tilde{0}\right)\right]\right\} \tag{34.8}$$

2) If $f^{(k)}(c) = \tilde{0}$, $k = 1, \ldots, n$.
Then

$$D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(x)dx, f(c)\right) \leq \frac{1}{(b-a)\Gamma(\nu+2)}$$

$$\left[(b-c)^{\nu+1} \sup_{t\in[c,b]} D\left((D_{*c}^{\nu\mathcal{F}} f)(t), \tilde{0}\right) + (c-a)^{\nu+1} \sup_{t\in[a,c]} D\left((D_{c-}^{\nu\mathcal{F}} f)(t), \tilde{0}\right)\right]. \tag{34.9}$$

Proof. Let $c \in [a,b]$.
We observe that

$$D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(x)dx, f(c)\right) =$$

$$D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(x)dx, f(c) \odot \frac{1}{b-a}\int_a^b 1dx\right) =$$

$$D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(x)dx, \frac{1}{b-a} \odot (FR)\int_a^b f(c)dx\right) =$$

$$\frac{1}{b-a}D\left((FR)\int_a^b f(x)dx, (FR)\int_a^b f(c)dx\right) \leq \frac{1}{b-a}\int_a^b D\left((f(x), f(c)\right)dx$$

$$= \frac{1}{b-a}\left[\int_a^c D\left((f(x), f(c)\right)dx + \int_c^b D\left((f(x), f(c)\right)dx\right].$$

Notice that ($f \in C_{\mathcal{F}}^n([a,b])$, $\nu > 0$, $n = \lceil \nu \rceil$, then $f \in C_{\mathcal{F}}^n([c,b])$)

$$f(x) = \sum_{k=0}^{n-1}{}^{*} \frac{(x-c)^k}{k!} \odot f^{(k)}(c) \oplus \frac{1}{\Gamma(\nu)} \odot \int_c^x (x-t)^{\nu-1} \odot \left(D_{*c}^{\nu\mathcal{F}} f\right)(t)dt,$$

all $c \leq x \leq b$.
Also here $f \in C_{\mathcal{F}}^n([a,c])$, thus we obtain

$$f(x) \oplus \sum_{m=0}^{\left[\frac{n-1}{2}\right]}{}^{*} \frac{(c-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(c) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]}{}^{*} \frac{(c-x)^{2m}}{(2m)!} \odot f^{(2m)}(c) \oplus$$

$$\frac{1}{\Gamma(\nu)} \odot \int_x^c (t-x)^{\nu-1} \odot \left(D_{c-}^{\nu\mathcal{F}} f\right)(t)dt,$$

all $a \le x \le c$.

We observe that $(a \le x \le c)$,

$$D\left((f(x), f(c)\right) = D\left(f(x) \oplus \sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(c-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(c),\right.$$

$$\left. f(c) \oplus \sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(c-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(c) \right) = D\left(\sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(c-x)^{2m}}{(2m)!} \odot f^{(2m)}(c) \oplus \right.$$

$$\left. \frac{1}{\Gamma(\nu)} \odot \int_x^c (t-x)^{\nu-1} \odot \left(D_{c-}^{\nu\mathcal{F}} f\right)(t)dt,\ f(c) \oplus \sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(c-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(c) \right) =$$

$$D\left(\sum_{m=1}^{\left[\frac{n-1}{2}\right]_*} \frac{(c-x)^{2m}}{(2m)!} \odot f^{(2m)}(c) \oplus \frac{1}{\Gamma(\nu)} \odot \int_x^c (t-x)^{\nu-1} \odot \left(D_{c-}^{\nu\mathcal{F}} f\right)(t)dt,\right.$$

$$\left. \sum_{m=0}^{\left[\frac{n-1}{2}\right]_*} \frac{(c-x)^{2m+1}}{(2m+1)!} \odot f^{(2m+1)}(c) \right) \le$$

$$\sum_{k=1}^{n-1} D\left(\frac{(c-x)^k}{k!} \odot f^{(k)}(c), \tilde{0} \right) + \frac{1}{\Gamma(\nu)} D\left(\int_x^c (t-x)^{\nu-1} \odot \left(D_{c-}^{\nu\mathcal{F}} f\right)(t)dt, \tilde{0} \right) =$$

$$\sum_{k=1}^{n-1} \frac{(c-x)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) + \frac{1}{\Gamma(\nu)} D\left(\int_x^c (t-x)^{\nu-1} \odot \left(D_{c-}^{\nu\mathcal{F}} f\right)(t)dt, \int_x^c \tilde{0}dt \right) \le$$

$$\sum_{k=1}^{n-1} \frac{(c-x)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) + \frac{1}{\Gamma(\nu)} \int_x^c (t-x)^{\nu-1} D\left(\left(D_{c-}^{\nu\mathcal{F}} f\right)(t), \tilde{0} \right) dt \le$$

$$\sum_{k=1}^{n-1} \frac{(c-x)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) + \frac{1}{\Gamma(\nu)} \sup_{t\in[a,c]} D\left(\left(D_{c-}^{\nu\mathcal{F}} f\right)(t), \tilde{0} \right) \frac{(c-x)\nu}{\nu}.$$

That is, we have proved that

$$D\left((f(x), f(c)\right) \le \sum_{k=1}^{n-1} \frac{(c-x)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) +$$

$$\frac{(c-x)\nu}{\Gamma(\nu+1)} \sup_{t\in[a,c]} D\left(\left(D_{c-}^{\nu\mathcal{F}} f\right)(t), \tilde{0} \right),\ a \le x \le c.$$

So we get

$$\int_a^c D\left((f(x), f(c)\right) dx \leq \sum_{k=1}^{n-1} \frac{(c-x)^{k+1}}{(k+1)!} D\left(f^{(k)}(c), \tilde{0}\right) + \sup_{t \in [a,c]} D\left(\left(D_{c-}^{\nu \mathcal{F}} f\right)(t), \tilde{0}\right) \frac{(c-a)\nu}{\Gamma(\nu+2)}.$$

We also have $(c \leq x \leq b)$

$$D\left((f(x), f(c)\right) = D\left(\sum_{k=0}^{n-1 *} \frac{(x-c)^k}{k!} \odot f^{(k)}(c) \oplus \frac{1}{\Gamma(\nu)} \odot \int_c^x (x-t)^{\nu-1} \odot \left(D_{*c}^{\nu \mathcal{F}} f\right)(t)dt,\ f(c)\right) =$$

$$D\left(\sum_{k=1}^{n-1 *} \frac{(x-c)^k}{k!} \odot f^{(k)}(c) \oplus \frac{1}{\Gamma(\nu)} \odot \int_c^x (x-t)^{\nu-1} \odot \left(D_{*c}^{\nu \mathcal{F}} f\right)(t)dt,\ \tilde{0}\right) \leq$$

$$\sum_{k=1}^{n-1} \frac{(x-c)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) + \frac{1}{\Gamma(\nu)} D\left(\int_c^x (x-t)^{\nu-1} \odot \left(D_{*c}^{\nu \mathcal{F}} f\right)(t)dt, \int_c^x \tilde{0}dt\right) \leq$$

$$\sum_{k=1}^{n-1} \frac{(x-c)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) + \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} D\left(D_{*c}^{\nu \mathcal{F}} f(t), \tilde{0}\right) dt$$

$$\leq \sum_{k=1}^{n-1} \frac{(x-c)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) + \sup_{t \in [c,b]} D\left(D_{*c}^{\nu \mathcal{F}} f(t), \tilde{0}\right) \frac{(x-c)\nu}{\Gamma(\nu+1)}.$$

I.e. we derive that

$$D\left((f(x), f(c)\right) \leq \sum_{k=1}^{n-1} \frac{(x-c)^k}{k!} D\left(f^{(k)}(c), \tilde{0}\right) + \sup_{t \in [c,b]} D\left(D_{*c}^{\nu \mathcal{F}} f(t), \tilde{0}\right) \frac{(x-c)\nu}{\Gamma(\nu+1)},$$

all $c \leq x \leq b$.

Consequently we obtain that

$$\int_c^b D\left((f(x), f(c)\right) dx \leq \sum_{k=1}^{n-1} \frac{(b-c)^{k+1}}{(k+1)!} D\left(f^{(k)}(c), \tilde{0}\right) + \frac{(b-c)^{\nu+1}}{\Gamma(\nu+2)} \sup_{t \in [c,b]} D\left(D_{*c}^{\nu \mathcal{F}} f(t), \tilde{0}\right).$$

So we have proved $(c \in [a,b])$

$$D\left(\frac{1}{b-a} \odot (FR) \int_a^b f(x)dx, f(c)\right) \leq \frac{1}{b-a} \Bigg[$$

$$\sum_{k=1}^{n-1} \frac{(c-a)^{k+1}}{(k+1)!} D\left(f^{(k)}(c), \tilde{0}\right) + \sup_{t \in [a,c]} D\left(D_{c-}^{\nu \mathcal{F}} f(t), \tilde{0}\right) \frac{(c-a)^{\nu+1}}{\Gamma(\nu+2)} +$$

$$\sum_{k=1}^{n-1} \frac{(b-c)^{k+1}}{(k+1)!} D\left(f^{(k)}(c), \tilde{0}\right) + \frac{(b-c)^{\nu+1}}{\Gamma(\nu+2)} \sup_{t\in[c,b]} D\left(D_{*c}^{\nu\mathcal{F}} f(t), \tilde{0}\right)\Bigg] =$$

$$\frac{1}{b-a}\left\{ \sum_{k=1}^{n-1} \frac{D(f^{(k)}(c), \tilde{0})}{(k+1)!} \left[(b-c)^{k+1} + (c-a)^{k+1}\right] + \frac{1}{\Gamma(\nu+2)}\right.$$

$$\left.\left[(b-c)^{\nu+1} \sup_{t\in[c,b]} D\left((D_{*c}^{\nu\mathcal{F}} f)(t), \tilde{0}\right) + (c-a)^{\nu+1} \sup_{t\in[a,c]} D\left((D_{c-}^{\nu\mathcal{F}} f)(t), \tilde{0}\right)\right] \right\},$$

proving the claim. ■

Applications to Theorem 34.29 follow

Corollary 34.30 Let $\nu = \frac{1}{2},\ f \in C'_{\mathcal{F}}([a,b]),\ c \in [a,b]$.

Then

$$D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(x)dx, f(c)\right) \leq \frac{4}{3\sqrt{\pi}(b-a)}$$

$$\left[(b-c)^{1.5} \sup_{t\in[c,b]} D\left((D_{*c}^{\frac{1}{2}\mathcal{F}} f)(t), \tilde{0}\right) + (c-a)^{1.5} \sup_{t\in[a,c]} D\left((D_{c-}^{\frac{1}{2}\mathcal{F}} f)(t), \tilde{0}\right)\right]. \tag{34.10}$$

Proof. Notice

$$\Gamma(2.5) = \frac{3\sqrt{\pi}}{4},$$

etc. ■

Corollary 34.31 Let $\nu = \frac{3}{2},\ f \in C^2_{\mathcal{F}}([a,b]),\ c \in [a,b]$.

Then

$$D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(x)dx, f(c)\right) \leq \frac{1}{b-a}\left\{ \frac{D\left(f'(c), \tilde{0}\right)}{2} \left[(b-c)^2 + (c-a)^2\right] + \frac{8}{15\sqrt{\pi}}\right.$$

$$\left.\left[(b-c)^{2.5} \sup_{t\in[c,b]} D\left((D_{*c}^{\frac{3}{2}\mathcal{F}} f)(t), \tilde{0}\right) + (c-a)^{2.5} \sup_{t\in[a,c]} D\left((D_{c-}^{\frac{3}{2}\mathcal{F}} f)(t), \tilde{0}\right)\right] \right\}. \tag{34.11}$$

Proof. See that

$$\Gamma(3.5) = \frac{15\sqrt{\pi}}{8},$$

etc. ■

Corollary 34.32 Let $\nu = \frac{5}{2},\ f \in C^3_{\mathcal{F}}([a,b]),\ c \in [a,b]$.

Then

$$D\left(\frac{1}{b-a}\odot(FR)\int_a^b f(x)dx, f(c)\right) \leq \frac{1}{b-a}\Bigg\{$$

$$\frac{D\left(f'(c),\tilde{0}\right)}{2}\left[(b-c)^2+(c-a)^2\right]+\frac{D\left(f''(c),\tilde{0}\right)}{6}\left[(b-c)^3+(c-a)^3\right]+\frac{16}{105\sqrt{\pi}}$$

$$\left[(b-c)^{3.5}\sup_{t\in[c,b]} D\left((D_{*c}^{\frac{5}{2}\mathcal{F}}f)(t),\tilde{0}\right)+(c-a)^{3.5}\sup_{t\in[a,c]} D\left((D_{c-}^{\frac{5}{2}\mathcal{F}}f)(t),\tilde{0}\right)\right]\Bigg\}. \tag{34.12}$$

Proof. Notice

$$\Gamma(4.5)=\frac{105\sqrt{\pi}}{16},$$

etc. ■

35

About Discrete Fractional Calculus with Inequalities

Here we define a Caputo like discrete fractional difference and we compare it to the earlier defined Riemann-Liouville fractional discrete analog. Then we present discrete fractional Taylor formulae and we estimate their remainders. Finally we give related discrete fractional Ostrowski, Poincare and Sobolev type inequalities. This chapter is based on [48].

35.1 Background

We make

Definition 35.1. We use [104], [106], [227].
Let $\nu > 0$. The ν-th fractional sum of f is defined by

$$\Delta^{-\nu} f(t,a) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s).$$

Here f is defined for $s = a \bmod (1)$ and $\Delta^{-\nu} f$ is defined for $t = (a+\nu) \bmod (1)$; in particular $\Delta^{-\nu}$ maps functions defined on $\mathbb{N}_\alpha$ to functions defined on $\mathbb{N}_{\alpha+\nu}$, where $\mathbb{N}_t = \{t, t+1, t+2, ...\}$.

Here $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$.

From now in this context for convenience we set $\Delta^{-\nu} f(t,a) = \Delta^{-\nu} f(t)$.

We need

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 575–585.
springerlink.com

Theorem 35.2. ([104]) Let f be a real-valued function defined on $\mathbb{N}_a$ and let $\mu, \nu > 0$. Then

$$\Delta^{-\nu}\left(\Delta^{-\mu} f(t)\right) = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}\left(\Delta^{-\nu} f(t)\right), \qquad \forall\, t \in \mathbb{N}_{a+\mu+\nu}.$$

We make

Definition 35.3. Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$, $\lceil . \rceil$ ceiling of number. Set $\nu = m - \mu$.

The μ-th fractional Caputo like difference is defined as

$$\Delta_*^{\mu} f(t) = \Delta^{-\nu}\left(\Delta^m f(t)\right) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} \left(\Delta^m f\right)(s), \quad \forall\, t \in \mathbb{N}_{a+\nu}.$$

Here Δ^m is the m-th order forward difference operator

$$\left(\Delta^m f\right)(s) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(s+k).$$

We mention

Theorem 35.4. ([106]) For $\nu > 0$ and p a positive integer we have

$$\Delta^{-\nu}\Delta^{p} f(t) = \Delta^{p}\Delta^{-\nu} f(t) - \sum_{k=0}^{\nu-1} \frac{(t-a)^{(\nu-p+k)}}{\Gamma(\nu+k-p+1)} \Delta^k f(a),$$

where f is defined on $\mathbb{N}_a$.

Remark 35.5. Let $\mu > 0$ and $m-1 < \mu < m$, $m = \lceil \mu \rceil$, where m is a positive integer, $\nu = m - \mu > 0$. Then by Theorem 35.4 we obtain

$$\Delta^{-\nu}\Delta^{m} f(t) = \Delta^{m}\Delta^{-\nu} f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{(\nu-m+k)}}{\Gamma(\nu+k-m+1)} \Delta^k f(a),$$

where f is defined on $\mathbb{N}_a$.

So we have established

$$\Delta_*^{\mu} f(t) = \Delta^{m}\Delta^{-\nu} f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{(\nu-m+k)}}{\Gamma(\nu+k-m+1)} \Delta^k f(a),$$

that is

$$\Delta^{m}\Delta^{-\nu} f(t) = \Delta_*^{\mu} f(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{(\nu-m+k)}}{\Gamma(\nu+k-m+1)} \Delta^k f(a), \tag{35.1}$$

where f is defined on $\mathbb{N}_a$.

Definition 35.6. ([106]) The μ-th fractional Riemann-Liouville type difference is defined by

$$\Delta^{\mu} f(t) := \Delta^{m-\nu} f(t) := \Delta^{m} \left(\Delta^{-\nu} f(t)\right),$$

where $\mu > 0$, $m-1 < \mu < m$, $\nu = m - \mu > 0$.

Remark 35.7. Consequently from (35.1) we obtain

$$\Delta^{\mu} f(t) = \Delta_{*}^{\mu} f(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{(\nu-m+k)}}{\Gamma(\nu+k-m+1)} \Delta^{k} f(a), \tag{35.2}$$

where f is defined on $\mathbb{N}_a$.

35.2 Results

We give the following Caputo type fractional Taylor's difference formula.

Theorem 35.8. For $\mu > 0$, μ non-integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$, it holds:

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^{k} f(a) + \frac{1}{\Gamma(\mu)} \sum_{s=a+\nu}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_{*}^{\mu} f(s), \ \forall t \in \mathbb{N}_{a+m}, \tag{35.3}$$

where f is defined on $\mathbb{N}_a$ with $a \in \mathbb{Z}^{+}$, $\mathbb{Z}^{+} := \{0, 1, 2, ...\}$.

Proof. Notice that by Definition 35.3,
$\Delta_{*}^{\mu} f(t) = \Delta^{-\nu} (\Delta^{m} f(t)) = \Delta^{-(m-\mu)} (\Delta^{m} f(t))$, $\forall\ t \in \mathbb{N}_{a+\nu}$.

Consequently we get $\Delta^{-\mu} \Delta_{*}^{\mu} f(t) = \Delta^{-\mu} \Delta^{-(m-\mu)} (\Delta^{m} f(t))$ (by Theorem 35.2) $= \Delta^{-(\mu+(m-\mu))} (\Delta^{m} f(t)) = \Delta^{-m} (\Delta^{m} f(t))$, $\forall\ t \in \mathbb{N}_{a+\nu+\mu}$.

So that

$$\Delta^{-\mu} \Delta_{*}^{\mu} f(t) = \Delta^{-m} (\Delta^{m} f(t)), \qquad \forall\ t \in \mathbb{N}_{a+m}. \tag{35.4}$$

We see that

$$(t-s-1)^{(m-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s-m+1)} = (t-s-1)(t-s-2)\dots(t-s-m+1), \tag{35.5}$$

the falling factorial, here we have $t - s - m + 1 > 0$.

Therefore we obtain

$$\Delta^{-m} (\Delta^{m} f(t)) = \frac{1}{(m-1)!} \sum_{s=a}^{t-m} (t-s-1)^{(m-1)} \Delta^{m} f(s). \tag{35.6}$$

By ([1], p. 28, Theorem 1.8.5) the discrete Taylor's formula we derive

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^{k} f(a) + \frac{1}{(m-1)!} \sum_{s=a}^{t-m} (t-s-1)^{(m-1)} \Delta^{m} f(s), \tag{35.7}$$

where $t^{(k)} = t(t-1)\dots(t-k+1)$.

From the last we find

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) + \Delta^{-\mu} \Delta_*^{\mu} f(t), \tag{35.8}$$

where f is defined on $\mathbb{N}_a$, $\forall\, t \in \mathbb{N}_{a+m}$, proving the claim. ■

We make

Remark 35.9. Here $[a,b]$ denotes the discrete interval $[a,b] = [a, a+1, a+2, \dots, b]$, where $a < b$ and $a, b \in \{0, 1, \dots\}$.

Let $\mu > 0$ be non integer such that $m - 1 < \mu < m$, i.e. $m = \lceil \mu \rceil$. Consider a function f defined on $[a,b]$. Then clearly the fractional discrete Taylor's formula (35.3) is valid only for $t \in [a+m, b]$, $a + m < b$.

We use

Theorem 35.10. ([106]) Let p be a positive integer and let $\nu > p$. Then

$$\Delta^p \left(\Delta^{-\nu} f(t) \right) = \Delta^{-(\nu - p)} f(t). \tag{35.9}$$

We make

Remark 35.11. Let $\mu > p$, where $p \in \mathbb{N}$. Then

$$\Delta^p \left(\Delta^{-\mu} \Delta_*^{\mu} f(t) \right) \overset{(35.9)}{=} \Delta^{-(\mu - p)} \left(\Delta_*^{\mu} f(t) \right), \qquad \forall\, t \in \mathbb{N}_{a+m-p}. \tag{35.10}$$

Also notice that

$$\Delta^p \left(\frac{(t-a)^{(k)}}{k!} \right) = \frac{(t-a)^{(k-p)}}{(k-p)!}, \qquad \text{for } k \geq p. \tag{35.11}$$

By the last we obtain the following discrete Caputo type fractional extended Taylor's formula.

Theorem 35.12. Let $\mu > p$, $p \in \mathbb{N}$, μ not integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$. Then

$$\Delta^p f(t) = \sum_{k=p}^{m-1} \frac{(t-a)^{(k-p)}}{(k-p)!} \Delta^k f(a) + \frac{1}{\Gamma(\mu - p)} \sum_{s=a+\nu}^{t-\mu+p} (t-s-1)^{(\mu-p-1)} \Delta_*^{\mu} f(s), \tag{35.12}$$

$\forall\, t \in \mathbb{N}_{a+m-p}$, f is defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$.

Note 35.13. Assuming that f is defined on $[a,b]$, then (35.12) is valid only for $[a+m-p, b]$, with $a + m - p < b$.

Notice for $p = 0$ applied on (35.12) we get (35.3).

We give

Proposition 35.14. For $\mu > 0$, μ not an integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$, f is defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$; and $\Delta^k f(a) = 0$, for $k = 0, ..., m-1$, we get

$$f(t) = \frac{1}{\Gamma(\mu)} \sum_{s=a+\nu}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_*^\mu f(s), \qquad \forall\, t \in \mathbb{N}_{a+m}. \tag{35.13}$$

Proof. By (35.3). ■

Also we present

Proposition 35.15. Let $\mu > p$, $p \in \mathbb{N}$, μ non-integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$; f is defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$. Suppose that $\Delta^k f(a) = 0$, $k = p, ..., m-1$. Then

$$\Delta^p f(t) = \frac{1}{\Gamma(\mu - p)} \sum_{s=a+\nu}^{t-\mu+p} (t-s-1)^{(\mu-p-1)} \Delta_*^\mu f(s), \qquad \forall\, t \in \mathbb{N}_{a+m-p}. \tag{35.14}$$

Proof. By (35.12). ■

We make

Remark 35.16. We want to calculate

$$\sum_{s=a+\nu}^{t-\mu} (t-s-1)^{(\mu-1)} = \sum_{s=a+\nu}^{t-\mu} \frac{\Gamma(t-s)}{\Gamma(t-s-\mu+1)} = \sum_{s=a+\nu}^{t-\mu-1} \frac{\Gamma(t-s)}{\Gamma(t-s-\mu+1)} + \Gamma(\mu). \tag{35.15}$$

We notice that

$$\frac{\Gamma(x+1)}{\Gamma(k+1)\Gamma(x-k+1)} = \frac{\Gamma(x+2)}{\Gamma(k+2)\Gamma(x-k+1)} - \frac{\Gamma(x+1)}{\Gamma(k+2)\Gamma(x-k)} \tag{35.16}$$

with $x > k$, $x, k \in \mathbb{R}$; $k > -1$, $x > -1$.

That is

$$\frac{\Gamma(x+1)}{\Gamma(x-k+1)} = \frac{1}{(k+1)} \left(\frac{\Gamma(x+2)}{\Gamma(x-k+1)} - \frac{\Gamma(x+1)}{\Gamma(x-k)} \right). \tag{35.17}$$

We find $A := \sum_{s=a+\nu}^{t-\mu-1} \frac{\Gamma(t-s)}{\Gamma(t-s-\mu+1)} =$
(by (35.17) for $x := t-s-1 \geq \mu > 0$, $k := \mu - 1 > -1$, and $x > k$)
$\frac{1}{\mu} \sum_{s=a+\nu}^{t-\mu-1} \left[\frac{\Gamma(t-s+1)}{\Gamma(t-s+1-\mu)} - \frac{\Gamma(t-s)}{\Gamma(t-s-\mu)} \right] =$
$\frac{1}{\mu} \left[\left(\frac{\Gamma(t-a-\nu+1)}{\Gamma(t-a-\nu+1-\mu)} - \frac{\Gamma(t-a-\nu)}{\Gamma(t-a-\nu-\mu)} \right) + \left(\frac{\Gamma(t-a-\nu)}{\Gamma(t-a-\nu-\mu)} - \frac{\Gamma(t-a-\nu-1)}{\Gamma(t-a-\nu-1-\mu)} \right) + \right.$
$\left. \left(\frac{\Gamma(t-a-\nu-1)}{\Gamma(t-a-\nu-1-\mu)} - \frac{\Gamma(t-a-\nu-2)}{\Gamma(t-a-\nu-2-\mu)} \right) + ... \left(... - \frac{\Gamma(\mu+1)}{\Gamma(1)} \right) \right] =$
$\left[\frac{\Gamma(t-a-\nu+1)}{\mu\Gamma(t-a-\nu+1-\mu)} - \Gamma(\mu) \right].$

That is

$$A = \frac{\Gamma(t-a-\nu+1)}{\mu\Gamma(t-a-\nu+1-\mu)} - \Gamma(\mu). \tag{35.18}$$

Consequently we found

$$\sum_{s=a+\nu}^{t-\mu} (t-s-1)^{(\mu-1)} = \frac{\Gamma(t-a-\nu+1)}{\mu\Gamma(t-a+1-m)} = \frac{(t-a-\nu)^{(\mu)}}{\mu}. \tag{35.19}$$

Using (35.19) we give

Corollary 35.17 (to Theorem 35.8) Let $\mu > 0$, μ non-integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$, $t \in \mathbb{N}_{a+m}$, f defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$. Then

$$\left| f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) \right| \leq \frac{(t-a-\nu)^{(\mu)}}{\Gamma(\mu+1)} \cdot \max_{s \in \{a+\nu, a+\nu+1, \ldots, t-\mu\}} |\Delta_*^\mu f(s)|. \tag{35.20}$$

Similarly we obtain

Corollary 35.18 (to Theorem 35.12) Let $\mu > p$, $p \in \mathbb{N}$, μ non-integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$, $t \in \mathbb{N}_{a+m-p}$, f defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$. Then

$$\left| \Delta^p f(t) - \sum_{k=p}^{m-1} \frac{(t-a)^{(k-p)}}{(k-p)!} \Delta^k f(a) \right| \leq \frac{(t-a-\nu)^{(\mu-p)}}{\Gamma(\mu-p+1)} \cdot \max_{s \in \{a+\nu, \ldots, t-\mu+p\}} \left|\Delta_*^\mu f(s)\right|. \tag{35.21}$$

We use

Lemma 35.19. Let $a > \nu$, $a, \nu > -1$, $a, \nu \in \mathbb{R}$, $a \leq b$. Then

$$\sum_{r=a}^{b} r^{(\nu)} = \frac{1}{(\nu+1)} \left(\frac{\Gamma(b+2)}{\Gamma(b-\nu+1)} - \frac{\Gamma(a+1)}{\Gamma(a-\nu)} \right) = \left(\frac{(b+1)^{(\nu+1)} - a^{(\nu+1)}}{\nu+1} \right). \tag{35.22}$$

Proof. We have

$\sum_{r=a}^{b} r^{(\nu)} = \sum_{r=a}^{b} \frac{\Gamma(r+1)}{\Gamma(r-\nu+1)} \overset{\text{(by (35.17))}}{=} \frac{1}{(\nu+1)} \sum_{r=a}^{b} \left(\frac{\Gamma(r+2)}{\Gamma(r-\nu+1)} - \frac{\Gamma(r+1)}{\Gamma(r-\nu)} \right) =$
$\frac{1}{(\nu+1)} \left(\sum_{r=a}^{b} \left\{ \left(\frac{\Gamma(a+2)}{\Gamma(a-\nu+1)} - \frac{\Gamma(a+1)}{\Gamma(a-\nu)} \right) + \left(\frac{\Gamma(a+3)}{\Gamma(a+2-\nu)} - \frac{\Gamma(a+2)}{\Gamma(a+1-\nu)} \right) + \right.\right.$
$\left.\left. \left(\frac{\Gamma(a+4)}{\Gamma(a+3-\nu)} - \frac{\Gamma(a+3)}{\Gamma(a+2-\nu)} \right) + \ldots + \left(\frac{\Gamma(b+1)}{\Gamma(b-\nu)} - \frac{\Gamma(b)}{\Gamma(b-1-\nu)} \right) + \left(\frac{\Gamma(b+2)}{\Gamma(b-\nu+1)} - \frac{\Gamma(b+1)}{\Gamma(b-\nu)} \right) \right\} \right) =$
$\frac{1}{(\nu+1)} \left(\frac{\Gamma(b+2)}{\Gamma(b-\nu+1)} - \frac{\Gamma(a+1)}{\Gamma(a-\nu)} \right),$

proving the claim. ∎

Next we present a discrete fractional Ostrowski type inequality.

Theorem 35.20. Let $\mu > p$, $p \in \mathbb{Z}^+$, μ not an integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$. Here f is defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$ and $j \in [a+m-p+1, b]$, with $a+m-p < b \in \mathbb{N}$. Assume that $\Delta^k f(a) = 0$, for $k \in [p+1, \ldots, m-1]$.

Then

$$\left| \left(\frac{1}{(b-a-m+p)} \sum_{j=a+m-p+1}^{b} \Delta^p f(j) \right) - \Delta^p f(a) \right| \leq$$

$$\frac{1}{(b-a-m+p)\,\Gamma(\mu-p+2)}\left[(b-a-\nu+1)^{(\mu-p+1)}-\Gamma(\mu-p+2)\right]$$

$$\cdot\left(\max_{t\in\{a+\nu,\ldots,b-\mu+p\}}\left|\Delta_*^{\mu}f(t)\right|\right). \tag{35.23}$$

Proof. By (35.12) we have

$$\Delta^{p}f(j)-\Delta^{p}f(a)=\frac{1}{\Gamma(\mu-p)}\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{(\mu-p-1)}\,\Delta_*^{\mu}f(s), \tag{35.24}$$

for all $j\in[a+m-p+1,b]$.

We derive that

$$\frac{1}{b-(a+m-p)}\sum_{j=a+m-p+1}^{b}\Delta^{p}f(j)-\Delta^{p}f(a)\quad=$$

$$\frac{1}{b-(a+m-p)}\sum_{j=a+m-p+1}^{b}\left(\Delta^{p}f(j)-\Delta^{p}f(a)\right)\quad=$$

$$\frac{1}{(b-(a+m-p))\,\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left(\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{(\mu-p-1)}\,\Delta_*^{\mu}f(s)\right). \tag{35.25}$$

Therefore we get

$\left|\frac{1}{b-(a+m-p)}\sum_{j=a+m-p+1}^{b}\Delta^{p}f(j)-\Delta^{p}f(a)\right|=$
$\frac{1}{(b-a-m+p)}\left|\sum_{j=a+m-p+1}^{b}\left(\Delta^{p}f(j)-\Delta^{p}f(a)\right)\right|\leq$
$\frac{1}{(b-a-m+p)}\sum_{j=a+m-p+1}^{b}\left|\Delta^{p}f(j)-\Delta^{p}f(a)\right|\leq$
$\frac{1}{(b-a-m+p)\Gamma(\mu-p)}\sum_{j=a+m-p+1}^{b}\left(\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{(\mu-p-1)}\cdot\left|\Delta_*^{\mu}f(s)\right|\right)\overset{(\text{by }(35.19))}{\leq}$
$\frac{1}{(b-a-m+p)\Gamma(\mu-p+1)}\sum_{j=a+m-p+1}^{b}(j-a-\nu)^{(\mu-p)}\cdot\max_{s\in\{a+\nu,\ldots j-\mu+p\}}\left|\Delta_*^{\mu}f(s)\right|\leq$

$$\frac{1}{(b\text{-}a\text{-}m+p)\,\Gamma(\mu\text{-}p+1)}\left(\sum_{j=a+m-p+1}^{b}(j\text{-}a\text{-}\nu)^{(\mu\text{-}p)}\right)\cdot\max_{s\in\{a+\nu,\ldots,b\text{-}\mu+p\}}\left|\Delta_*^{\mu}f(s)\right|=:(*) \tag{35.26}$$

Next we use (35.22). We notice that

$$\sum_{j=a+m-p+1}^{b}(j-a-\nu)^{(\mu-p)}=\sum_{r=\mu-p+1}^{b-a-\nu}r^{(\mu-p)}=$$

$$\frac{1}{(\mu-p+1)}\left(\frac{\Gamma(b-a-\nu+2)}{\Gamma(b-a-m+p+1)}-\Gamma(\mu-p+2)\right). \tag{35.27}$$

Therefore

$$(*) = \frac{1}{(b-a-m+p)\,\Gamma(\mu-p+2)}\left(\frac{\Gamma(b-a-\nu+2)}{\Gamma(b-a-m+p+1)} - \Gamma(\mu-p+2)\right)$$
$$\cdot\left(\max_{t\in\{a+\nu,\dots,b-\mu+p\}}|\Delta_*^{\mu}f(t)|\right). \qquad (35.28)$$

The last completes the proof. ∎

Next we present a discrete fractional Poincaré inequality.

Theorem 35.21. Let $\mu > p$, $p \in \mathbb{Z}^+$, μ non-integer, $m = \lceil\mu\rceil$, $\nu = m-\mu$. Suppose that $\Delta^k f(a) = 0$, $k = p,...,m-1$, f defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$. Let $\gamma,\delta > 1 : \frac{1}{\gamma}+\frac{1}{\delta} = 1$. Then

$$\sum_{j=a+m-p}^{b}|\Delta^p f(j)|^{\delta} \le \frac{1}{(\Gamma(\mu-p))^{\delta}}\left[\sum_{j=a+m-p}^{b}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{\delta}{\gamma}}\right]$$
$$\cdot\left(\sum_{s=a+\nu}^{b-\mu+p}|\Delta_*^{\mu}f(s)|^{\delta}\right). \qquad (35.29)$$

Proof. We have

$$\Delta^p f(j) = \frac{1}{\Gamma(\mu-p)}\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{(\mu-p-1)}\,\Delta_*^{\mu}f(s), \quad \forall\, j \in [a+m-p,b]. \qquad (35.30)$$

Let $\gamma,\delta > 1$ such that $\frac{1}{\gamma}+\frac{1}{\delta} = 1$.

We observe that

$$|\Delta^p f(j)| \le \frac{1}{\Gamma(\mu-p)}\sum_{s=a+\nu}^{j-\mu+p}(j-s-1)^{(\mu-p-1)}\,|\Delta_*^{\mu}f(s)|$$

(by discrete Hölder's inequality)

$$\le \frac{1}{\Gamma(\mu-p)}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{s=a+\nu}^{j-\mu+p}|\Delta_*^{\mu}f(s)|^{\delta}\right)^{\frac{1}{\delta}}. \qquad (35.31)$$

That is, it holds

$$|\Delta^p f(j)|^{\delta} \le \frac{1}{(\Gamma(\mu-p))^{\delta}}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{\delta}{\gamma}}\cdot\left(\sum_{s=a+\nu}^{j-\mu+p}|\Delta_*^{\mu}f(s)|^{\delta}\right)$$
$$\le \frac{1}{(\Gamma(\mu-p))^{\delta}}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{\delta}{\gamma}}$$

$$\cdot\left(\sum_{s=a+\nu}^{b-\mu+p}\left|\Delta_*^{\mu}f(s)\right|^{\delta}\right),\quad \forall\ j\in[a+m-p,b]. \tag{35.32}$$

Applying $\sum_{j=a+m-p}^{b}$ on (35.32) we establish (35.29). ■

It follows a discrete Sobolev type fractional inequality.

Theorem 35.22. Let $\mu>p$, $p\in\mathbb{Z}^+$, μ non-integer, $m=\lceil\mu\rceil$, $\nu=m-\mu$. Suppose that $\Delta^k f(a)=0$, $k=p,...,m-1$; f defined on $\mathbb{N}_a$, $a\in\mathbb{Z}^+$. Let $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, and $r\geq 1$. Then

$$\left(\sum_{j=a+m-p}^{b}\left|\Delta^p f(j)\right|^r\right)^{\frac{1}{r}}\leq$$

$$\frac{1}{\Gamma(\mu-p)}\left[\sum_{j=a+m-p}^{b}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{r}{\gamma}}\right]^{\frac{1}{r}}\cdot\left(\sum_{s=a+\nu}^{b-\mu+p}\left|\Delta_*^{\mu}f(s)\right|^{\delta}\right)^{\frac{1}{\delta}}. \tag{35.33}$$

Proof. By (35.31) and discrete Hölder's inequality, we have

$$\left|\Delta^p f(j)\right|\leq\frac{1}{\Gamma(\mu-p)}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{1}{\gamma}}$$

$$\cdot\left(\sum_{s=a+\nu}^{b-\mu+p}\left|\Delta_*^{\mu}f(s)\right|^{\delta}\right)^{\frac{1}{\delta}},\qquad \forall\ j\in[a+m-p,b], \tag{35.34}$$

where $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$.

Hence, by $r\geq 1$ we derive

$$\left|\Delta^p f(j)\right|^r\leq\frac{1}{(\Gamma(\mu-p))^r}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{r}{\gamma}}$$

$$\cdot\left(\sum_{s=a+\nu}^{b-\mu+p}\left|\Delta_*^{\mu}f(s)\right|^{\delta}\right)^{\frac{r}{\delta}},\qquad \forall\ j\in[a+m-p,b]. \tag{35.35}$$

Consequently we obtain

$$\sum_{j=a+m-p}^{b}\left|\Delta^p f(j)\right|^r\leq\frac{1}{(\Gamma(\mu-p))^r}\left[\sum_{j=a+m-p}^{b}\left(\sum_{s=a+\nu}^{j-\mu+p}\left((j-s-1)^{(\mu-p-1)}\right)^{\gamma}\right)^{\frac{r}{\gamma}}\right]$$

$$\cdot\left(\sum_{s=a+\nu}^{b-\mu+p}\left|\Delta_*^{\mu}f(s)\right|^{\delta}\right)^{\frac{r}{\delta}}. \tag{35.36}$$

The last proves the claim. ■

We finish with the following discrete fractional average Sobolev type inequality.

Theorem 35.23. Let $0 < \mu_1 < \mu_2 < ... < \mu_k$; $m_l = \lceil \mu_l \rceil$, $\nu_l = m_l - \mu_l$, $l = 1, ..., k$, $k \in \mathbb{N}$. Assume that $\Delta^{\tau} f(a) = 0$, for $\tau = 0, 1, ..., m_k - 1$: f is defined on $\mathbb{N}_a$, $a \in \mathbb{Z}^+$. Let $r \geq 1$; $C_l(s) > 0$ defined on $[a + \nu_l, b - \mu_l]$, $l = 1, ..., k$. Put $B_l := \sum_{s=a+\nu_l}^{b-\mu_l} C_l(s) \left(\Delta_*^{\mu_l} f(s)\right)^2$,

$\delta^* := \max_{1 \leq l \leq k} \left\{ \frac{1}{(\Gamma(\mu_l))^2} \left[\sum_{j=a+m_l}^{b} \left(\sum_{s=a+\nu_l}^{j-\mu_l} \left((j-s-1)^{(\mu_l - 1)} \right)^2 \right)^{\frac{r}{2}} \right]^{\frac{2}{r}} \right\}$,

$\varrho^* := \max_{1 \leq l \leq k} \left\| \left(\frac{1}{C_l(s)} \right) \right\|_{\infty, [a+\nu_l, b-\mu_l]}$.

Then

$$\|f\|_{r,[a+m_k,b]} \leq \sqrt{\delta^* \varrho^*} \left(\frac{\sum_{l=1}^{k} B_l}{k} \right)^{\frac{1}{2}}. \tag{35.37}$$

Proof. We see that also $\Delta^{\tau} f(a) = 0$, $\tau = 0, 1, ..., m_l - 1$, $l = 1, ..., k - 1$. So the assumptions of Theorem 35.22 are fulfilled for f and fractional orders μ_l, $l = 1, ..., k$. Thus by choosing $p = 0$ and $\gamma = \delta = 2$ we apply (35.33), for $l = 1, ..., k$, to obtain

$$\left(\sum_{j=a+m_l}^{b} |f(j)|^r \right)^{\frac{1}{r}} \leq \frac{1}{\Gamma(\mu_l)} \left[\sum_{j=a+m_l}^{b} \left(\sum_{s=a+\nu_l}^{j-\mu_l} \left((j-s-1)^{(\mu_l-1)} \right)^2 \right)^{\frac{r}{2}} \right]^{\frac{1}{r}}$$

$$\cdot \left(\sum_{s=a+\nu_l}^{b-\mu_l} \left(\Delta_*^{\mu_l} f(s)\right)^2 \right)^{\frac{1}{2}}. \tag{35.38}$$

Therefore

$$\left(\sum_{j=a+m_l}^{b} |f(j)|^r \right)^{\frac{2}{r}} \leq$$

$$\frac{1}{(\Gamma(\mu_l))^2} \left[\sum_{j=a+m_l}^{b} \left(\sum_{s=a+\nu_l}^{j-\mu_l} \left((j-s-1)^{(\mu_l-1)} \right)^2 \right)^{\frac{r}{2}} \right]^{\frac{2}{r}} \cdot \left(\sum_{s=a+\nu_l}^{b-\mu_l} \left(\Delta_*^{\mu_l} f(s)\right)^2 \right)$$

$$\leq \delta^* \left(\sum_{s=a+\nu_l}^{b-\mu_l} \left(\Delta_*^{\mu_l} f(s)\right)^2 \right) = \delta^* \left(\sum_{s=a+\nu_l}^{b-\mu_l} (C_l(s))^{-1} (C_l(s)) \left(\Delta_*^{\mu_l} f(s)\right)^2 \right)$$

$$\leq \delta^* \rho^* \left(\sum_{s=a+\nu_l}^{b-\mu_l} C_l(s) \left(\Delta_*^{\mu_l} f(s)\right)^2 \right). \tag{35.39}$$

That is

$$\left(\sum_{j=a+m_k}^{b}|f(j)|^r\right)^{\frac{2}{r}} \leq \left(\sum_{j=a+m_l}^{b}|f(j)|^r\right)^{\frac{2}{r}} \leq$$

$$\delta^*\rho^*\left(\sum_{s=a+\nu_l}^{b-\mu_l} C_l(s)\left(\Delta_*^{\mu_l} f(s)\right)^2\right) = \delta^*\rho^* B_l, \quad \text{for } l = 1, ..., k. \tag{35.40}$$

Hence

$$\|f\|_{r,[a+m_k,b]}^2 \leq \delta^*\rho^*\left(\frac{\sum_{l=1}^k B_l}{k}\right), \tag{35.41}$$

proving the claim. ■

36

Discrete Nabla Fractional Calculus with Inequalities

Here we define a Caputo like discrete nabla fractional difference and we give discrete nabla fractional Taylor formulae. We estimate their remainders. Then we derive related discrete nabla fractional Opial, Ostrowski, Poincaré and Sobolev type inequalities. This chapter relies on [51].

36.1 Background

Here we use [105].

We define the rising factorial

$$t^{\overline{n}} = t\,(t+1)\ldots(t+n-1)\,, \qquad n \in \mathbb{N},$$

and $t^{\overline{0}} = 1$. In general, let $\alpha \in \mathbb{R}$, then define $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$, $t \in \mathbb{R} - \{\ldots, -2, -1, 0\}$, and $0^{\overline{\alpha}} = 0$. Note that $\nabla\left(t^{\overline{\alpha}}\right) = \alpha t^{\overline{\alpha-1}}$, where $\nabla y\,(t) = y\,(t) - y\,(t-1)$.

For $k = 2, 3, \ldots$, define ∇^k inductively by $\nabla^k = \nabla\nabla^{k-1}$. Thus $\nabla^k f\,(t) = \sum_{m=0}^{k} (-1)^m \begin{pmatrix} k \\ m \end{pmatrix} f\,(t-m)$.

Call $\rho\,(s) = s-1$, we define the n-th order sum of $f\,(t)$ by

$$\nabla_a^{-n} f\,(t) = \sum_{s=a}^{t} \frac{(t-\rho\,(s))^{\overline{n-1}}}{(n-1)!} f\,(s)\,, \tag{36.1}$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 587–600.
springerlink.com

where $t \geq a$, $n \in \mathbb{N}$.

In general we define the ν-th order fractional sum of f by

$$\nabla_a^{-\nu} f(t) = \sum_{s=a}^{t} \frac{(t - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s), \tag{36.2}$$

where $\nu > 0$ non-integer, $t \geq a$.

We define the fractional Caputo like nabla difference for $\mu > 0$, $m-1 < \mu < m$, $m = \lceil \mu \rceil$, $\lceil . \rceil$ the ceiling of number, $m \in \mathbb{N}$, $\nu = m - \mu$, as follows

$$\nabla_{a*}^{\mu} f(t) = \nabla_a^{-\nu} (\nabla^m f(t)), \quad t \geq a. \tag{36.3}$$

We mention

Theorem 36.1. ([105]) Here Δ^m is the m-th order forward difference operator, $m \in \mathbb{Z}_+$,

$$(\Delta^m f)(t) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(t+k), \quad t \in \mathbb{Z}.$$

Define $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, $t \in \mathbb{R} - \{..., -2, -1\}$, $\alpha > 0$, so that $t^{(n)} = t(t-1)...(t-n+1)$, for $n \in \mathbb{N}$.

Note that $t^{\overline{\alpha}} = (t + \alpha - 1)^{(\alpha)}$.

Define for $\nu > 0$ the operator

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - s - 1)^{(\nu-1)} f(s). \tag{36.4}$$

We also see that

$$\Delta^m f(t-m) = \nabla^m f(t), \quad \forall\, m \in \mathbb{N}.$$

We need the law of exponents.

Theorem 36.2. ([105]) Let f be a real valued function, and let $\mu, \nu > 0$. Then

$$\nabla_a^{-\nu} \left(\nabla_a^{-\mu} f(t) \right) = \nabla_a^{-(\mu+\nu)} f(t) = \nabla_a^{-\mu} \left(\nabla_a^{-\nu} f(t) \right), \tag{36.5}$$

for all $t \geq a$.

We also mention the discrete Taylor formula

Theorem 36.3. ([93]) Let $f : \mathbb{Z} \to \mathbb{R}$ be a function, and let $a \in \mathbb{Z}$. Then, for all $t \in \mathbb{Z}$ with $t \geq a + m$, the representation holds,

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a) + \frac{1}{(m-1)!} \sum_{\tau=a+1}^{t} (t - \tau + 1)^{\overline{m-1}} \nabla^m f(\tau). \tag{36.6}$$

36.2 Main Results

We give the following discrete backward fractional Taylor formula

Theorem 36.4. Let $f : \mathbb{Z} \to \mathbb{R}$ be a function, and let $a \in \mathbb{Z}$. Here $m-1 < \mu < m$, $m = \lceil \mu \rceil$, $\mu > 0$. Then, for all $t \in \mathbb{Z}$ with $t \geq a+m$, the representation holds,

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a) + \frac{1}{\Gamma(\mu)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-1}} \nabla^{\mu}_{(a+1)*} f(\tau). \quad (36.7)$$

Proof. We notice that

$$\nabla_{a+1}^{-\mu} \nabla^{\mu}_{(a+1)*} f(t) = \nabla_{a+1}^{-\mu} \nabla_{a+1}^{-(m-\mu)} \nabla^m f(t)$$

$$\overset{\text{(by (36.5))}}{=} \nabla_{a+1}^{-(\mu+m-\mu)} \nabla^m f(t) = \nabla_{a+1}^{-m} \nabla^m f(t), \quad (36.8)$$

true for $t \geq a+1$.

But

$$\nabla_{a+1}^{-m} \nabla^m f(t) = \frac{1}{(m-1)!} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{m-1}} \nabla^m f(\tau), \quad (36.9)$$

and

$$\nabla_{a+1}^{-\mu} \nabla^{\mu}_{(a+1)*} f(t) = \frac{1}{\Gamma(\mu)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-1}} \nabla^{\mu}_{(a+1)*} f(\tau), \quad (36.10)$$

where $t \geq a+1$.

Then we apply Theorem 36.3.

The claim is proved. ■

Corollary 36.5. (to Theorem 36.4). Additionally suppose that $\nabla^k f(a) = 0$, for $k = 0, 1, ..., m-1$. Then

$$f(t) = \frac{1}{\Gamma(\mu)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-1}} \nabla^{\mu}_{(a+1)*} f(\tau), \quad \forall\, t \geq a+m. \quad (36.11)$$

We need

Lemma 36.6. ([105]) Let $0 \leq m-1 < \nu \leq m$, $m = \lceil \nu \rceil$, $a \in \mathbb{N}$, f defined on $\mathbb{N}_a = \{a, a+1, ...\}$. Then

$$\Delta_a^{-\nu} f(t+\nu) = \nabla_a^{-\nu} f(t), \quad \forall\, t \in \mathbb{N}_a. \quad (36.12)$$

Theorem 36.7. ([106]) Let $p \in \mathbb{N} : \nu > p$. Then

$$\Delta^p \left(\Delta_a^{-\nu} f(t) \right) = \Delta_a^{-(\nu-p)} f(t). \tag{36.13}$$

We give

Theorem 36.8. Let $p \in \mathbb{N} : \nu > p$, $a \in \mathbb{N}$. Then

$$\nabla^p \left(\nabla_\alpha^{-\nu} f(t) \right) = \nabla_a^{-(\nu-p)} f(t), \tag{36.14}$$

for $t \in \mathbb{N}_a$.

Proof. We notice that

$$\begin{aligned} \nabla^p \left(\nabla_a^{-\nu} f(t) \right) &= \Delta^p \left(\nabla_a^{-\nu} f \right)(t-p) \\ \overset{\text{(by (36.12))}}{=} \Delta^p \left(\Delta_a^{-\nu} f(t-p+\nu) \right) &= \left(\Delta^p \Delta_a^{-\nu} f \right)(t-p+\nu) =: A. \end{aligned}$$

Also we see that

$$\nabla_a^{-(\nu-p)} f(t) \overset{\text{(by (36.12))}}{=} \Delta_a^{-(\nu-p)} f(t+\nu-p) =: B.$$

But $A = B$ by (36.13), proving the claim. ∎

We make

Remark 36.9. We have

$$\begin{aligned} \nabla^p \left(\frac{(t-a)^{\overline{k}}}{k!} \right) &= \nabla^p \left(\frac{(t+k-1-a)^{(k)}}{k!} \right) = \\ \Delta^p \left(\frac{(t+k-1-a-p)^{(k)}}{k!} \right) &= \frac{(t+k-1-a-p)^{(k-p)}}{(k-p)!} = \frac{(t-a)^{\overline{k-p}}}{(k-p)!} \end{aligned}$$

for $k \geq p$.

That is

$$\nabla^p \left(\frac{(t-a)^{\overline{k}}}{k!} \right) = \frac{(t-a)^{\overline{k-p}}}{(k-p)!}, \quad \text{for } k \geq p. \tag{36.15}$$

We have proved the following discrete backward fractional extended Taylor's formula.

Theorem 36.10. Let $f : \mathbb{Z} \to \mathbb{R}$ be a function, and let $a \in \mathbb{Z}_+$. Here $m-1 < \mu < m$, $m = \lceil \mu \rceil$, $\mu > 0$. Consider $p \in \mathbb{N} : \mu > p$. Then, for all $t \geq a+m$, $t \in \mathbb{N}$, the representation holds,

$$\nabla^p f(t) = \sum_{k=p}^{m-1} \frac{(t-a)^{\overline{k-p}}}{(k-p)!} \nabla^k f(a) +$$

$$\frac{1}{\Gamma(\mu-p)}\sum_{\tau=a+1}^{t}(t-\tau+1)^{\overline{\mu-p-1}}\nabla_{(a+1)*}^{\mu}f(\tau). \tag{36.16}$$

Proof. By Theorem 36.8 and (36.15). ■

Note. When $a\in\mathbb{Z}_{+}$, and for $p=0$ put on (36.16) we get (36.7).

Corollary 36.11. (to Theorem 36.10). Additionally suppose that $\nabla^{k}f(a)=0$, for $k=p,...,m-1$. Then

$$\nabla^{p}f(t)=\frac{1}{\Gamma(\mu-p)}\sum_{\tau=a+1}^{t}(t-\tau+1)^{\overline{\mu-p-1}}\nabla_{(a+1)*}^{\mu}f(\tau), \quad \forall\, t\geq a+m,\ t\in\mathbb{N}. \tag{36.17}$$

Remark 36.12. (to Theorems 36.4, 36.10). Let f be defined on $[a-m+1,a-m+2,...,b]$, a discrete closed interval, where b is an integer. Then (36.7) and (36.16) are valid only for $t\in[a+m,b]$. Here we must assume that $a+m<b$.

Remark 36.13. We would like to calculate

$$\sum_{\tau=a+1}^{t}(t-\tau+1)^{\overline{\mu-1}}=\sum_{\tau=a+1}^{t-1}(t-\tau+1)^{\overline{\mu-1}}+(1)^{\overline{\mu-1}}=$$

$$\sum_{\tau=a+1}^{t-1}(t-\tau+1)^{\overline{\mu-1}}+\Gamma(\mu)=\sum_{\tau=a+1}^{t-1}\frac{\Gamma(t-\tau+\mu)}{\Gamma(t-\tau+1)}+\Gamma(\mu). \tag{36.18}$$

So still to find

$$A:=\sum_{\tau=a+1}^{t-1}\frac{\Gamma(t-\tau+\mu)}{\Gamma(t-\tau+1)}. \tag{36.19}$$

We will use the following formula

$$\frac{\Gamma(x+1)}{\Gamma(x-k+1)}=\frac{1}{(k+1)}\left(\frac{\Gamma(x+2)}{\Gamma(x-k+1)}-\frac{\Gamma(x+1)}{\Gamma(x-k)}\right), \tag{36.20}$$

where $x>k$, $x,k\in\mathbb{R}:k>-1$, $x>-1$.

So for calculating A we set $x:=t-\tau+\mu-1$, $k:=\mu-1$. We observe here that $x>-1$, $k>-1$ and $x>k$. Also we see that $x+1=t-\tau+\mu$ and $x-k+1=t-\tau+1$. So we get

$$\frac{\Gamma(t-\tau+\mu)}{\Gamma(t-\tau+1)}=\frac{\Gamma(x+1)}{\Gamma(x-k+1)}=\frac{1}{\mu}\left(\frac{\Gamma(t-\tau+\mu+1)}{\Gamma(t-\tau+1)}-\frac{\Gamma(t-\tau+\mu)}{\Gamma(t-\tau)}\right), \tag{36.21}$$

for all $\tau\in\{a+1,...,t-1\}$.

Consequently we obtain

$$\begin{aligned} A &= \frac{1}{\mu}\left\{\left(\frac{\Gamma(t-a+\mu)}{\Gamma(t-a)} - \frac{\Gamma(t-a-1+\mu)}{\Gamma(t-a-1)}\right) + \right. \\ &\left(\frac{\Gamma(t-a-1+\mu)}{\Gamma(t-a-1)} - \frac{\Gamma(t-a-2+\mu)}{\Gamma(t-a-2)}\right) + \\ &\left(\frac{\Gamma(t-a-2+\mu)}{\Gamma(t-a-2)} - \frac{\Gamma(t-a-3+\mu)}{\Gamma(t-a-3)}\right) + \\ &\ldots \\ &\left. + \left(\frac{\Gamma(\mu+2)}{\Gamma(2)} - \frac{\Gamma(\mu+1)}{\Gamma(1)}\right)\right\} \end{aligned}$$

(telescoping sum)

$$= \frac{1}{\mu}\left\{\frac{\Gamma(t-a+\mu)}{\Gamma(t-a)} - \Gamma(\mu+1)\right\} = \frac{\Gamma(t-a+\mu)}{\mu\Gamma(t-a)} - \Gamma(\mu). \tag{36.22}$$

That is

$$A = \frac{\Gamma(t-a+\mu)}{\mu\Gamma(t-a)} - \Gamma(\mu). \tag{36.23}$$

Hence we have found that

$$\sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-1}} = \frac{\Gamma(t-a+\mu)}{\mu\Gamma(t-a)} = \frac{(t-a)^{\overline{\mu}}}{\mu}. \tag{36.24}$$

We give

Corollary 36.14. (to Theorem 36.4). We obtain

$$\left| f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a) \right| \leq \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} \cdot \max_{\tau \in \{a+1,\ldots,t\}} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|. \tag{36.25}$$

Proof. Use of (36.7) and (36.24).

Corollary 36.15. (to Theorem 36.10). It holds

$$\left| \nabla^p f(t) - \sum_{k=p}^{m-1} \frac{(t-a)^{\overline{k-p}}}{(k-p)!} \nabla^k f(a) \right| \leq \frac{(t-a)^{\overline{\mu-p}}}{\Gamma(\mu-p+1)} \cdot \max_{\tau \in \{a+1,\ldots,t\}} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|. \tag{36.26}$$

Proof. Use of (36.16) and (36.24). ■

We present a discrete fractional Opial inequality

Theorem 36.16. Let $\mu > 2$, $m = \lceil \mu \rceil \geq 3$; $p \in \mathbb{Z}_+ : \mu > p$; $a \in \mathbb{Z}_+$. Here f is a real valued function defined on $\{a-m+1, a-m+2, \ldots\}$. Here $t \geq a+m$, $t \in \mathbb{N}$. Suppose that $\nabla^k f(a) = 0$, for $k = p, \ldots, m-1$.

Let $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$; $C(\tau) > 0$ for $\tau = a+1, ..., t$; and $D(t') \geq 0$ for $t' = a+m, ..., t$. Set

$$\theta(t, a, \mu, p, C, \gamma) := \left(\sum_{\tau=a+1}^{t} \left[(t-\tau+1)^{\overline{\mu-p-1}} (C(\tau))^{-1} \right]^{\gamma} \right)^{\frac{1}{\gamma}}, \quad t \geq a+m, \tag{36.27}$$

$$g(t) := \sum_{\tau=a+1}^{t} (C(\tau))^{\delta} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|^{\delta}, \quad t \geq a+1; \tag{36.28}$$

$$G(t, a, m, g) := 2\left(g^2(t) - g^2(a+m-1)\right) + \frac{\left(g^2(t-1) - g^2(a+m-2)\right)}{2} + 2\left[g(t)g(t-1) - g(a+m-1)g(a+m-2)\right], \quad t \geq a+m. \tag{36.29}$$

Call also

$$K(t) := \frac{1}{\Gamma(\mu-p)} \left(\sum_{t'=a+m}^{t} \left[D(t')\left(C(t')\right)^{-1} \theta\left(t', a, \mu, p, C, \gamma\right) \right]^{\gamma} \right)^{\frac{1}{\gamma}}, \ t \geq a+m. \tag{36.30}$$

Then

$$\sum_{t'=a+m}^{t} D(t') \left| \nabla^{p} f(t') \right| \left| \nabla^{\mu}_{(a+1)*} f(t') \right| \leq K(t) \left(G(t, a, m, g)\right)^{\frac{1}{\delta}}, \tag{36.31}$$

for $t \geq a+m$.

Proof. By (36.17) we have

$$|\nabla^{p} f(t)| \leq \frac{1}{\Gamma(\mu-p)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-p-1}} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|$$

$$= \frac{1}{\Gamma(\mu-p)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-p-1}} (C(\tau))^{-1} C(\tau) \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|$$

(by discrete Hölder's inequality)

$$\leq \frac{1}{\Gamma(\mu-p)} \left(\sum_{\tau=a+1}^{t} \left[(t-\tau+1)^{\overline{\mu-p-1}} (C(\tau))^{-1} \right]^{\gamma} \right)^{\frac{1}{\gamma}} \cdot$$

$$\left(\sum_{\tau=a+1}^{t} (C(\tau))^{\delta} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|^{\delta} \right)^{\frac{1}{\delta}}$$

$$= \frac{\theta(t, a, \mu, p, C, \gamma)}{\Gamma(\mu-p)} \left(\sum_{\tau=a+1}^{t} (C(\tau))^{\delta} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|^{\delta} \right)^{\frac{1}{\delta}}, \ \forall \ t \geq a+m. \tag{36.32}$$

We have set

$$g(t)=\sum_{\tau=a+1}^{t}(C(\tau))^{\delta}\left|\nabla_{(a+1)*}^{\mu}f(\tau)\right|^{\delta}, \tag{36.33}$$

which is nondecreasing in $t\geq a+1>a-m+1$.

It holds

$$\nabla g(t)=(C(t))^{\delta}\left|\nabla_{(a+1)*}^{\mu}f(t)\right|^{\delta}, \quad t\in\{a+1,...\}. \tag{36.34}$$

Thus

$$\left|\nabla_{(a+1)*}^{\mu}f(t)\right|=(\nabla g(t))^{\frac{1}{\delta}}(C(t))^{-1}. \tag{36.35}$$

We observe for $a+m\leq t'\leq t$ that

$$\sum_{t'=a+m}^{t}D(t')\left|\nabla^{p}f(t')\right|\left|\nabla_{(a+1)*}^{\mu}f(t')\right|\leq$$

$$\sum_{t'=a+m}^{t}D(t')\frac{\theta(t',a,\mu,p,C,\gamma)}{\Gamma(\mu-p)}(g(t'))^{\frac{1}{\delta}}(\nabla g(t'))^{\frac{1}{\delta}}(C(t'))^{-1}\leq$$

(by discrete Hölder's inequality)

$$\frac{1}{\Gamma(\mu-p)}\left(\sum_{t'=a+m}^{t}\left[D(t')(C(t'))^{-1}\theta(t',a,\mu,p,C,\gamma)\right]^{\gamma}\right)^{\frac{1}{\gamma}}$$

$$\cdot\left(\sum_{t'=a+m}^{t}g(t')\cdot\nabla g(t')\right)^{\frac{1}{\delta}}. \tag{36.36}$$

By $m\geq 3$ notice that $a+m-2\geq a+1$.

We define the discontinuous function

$$\psi(x)=g(t')+\nabla g(t')(x-t'+1), \quad \text{for } x\in[t'-1,t']$$

a closed interval of $\mathbb{R}$, and for $t'=a+m-1,a+m,...$.

So $\psi(x)=g(t'+1)+\nabla g(t'+1)(x-t')$, for $x\in[t',t'+1]$, and notice that $\psi(t'-)=2g(t')-g(t'-1)$, while $\psi(t'+)=g(t'+1)$; thus ψ in general is discontinuous. Also see that $\psi'(x)=\nabla g(t')$, for $x\in[t'-1,t']$, for $t'=a+m-1,...$.

Here $g(t)$, $\nabla g(t)\geq 0$.

We further notice that

$$g(t')\leq\frac{g(t')+(2g(t')-g(t'-1))}{2}=\frac{3g(t')-g(t'-1)}{2}, \tag{36.37}$$

for $t'=a+m-1,...$.

The last means that

$$g\left(t'\right) \leq \int_{t'-1}^{t'} \psi\left(x\right) dx, \quad \text{for } t' = a+m-1, \dots . \tag{36.38}$$

Consequently, we derive

$$g\left(t'\right) \nabla g\left(t'\right) \leq \int_{t'-1}^{t'} \psi\left(x\right) \psi'\left(x\right) dx = \int_{t'-1}^{t'} \psi\left(x\right) d\psi\left(x\right) =$$

$$\left.\frac{\left(\psi\left(x\right)\right)^2}{2}\right|_{t'-1}^{t'} = \frac{1}{2}\left[\left(\psi\left(t'\right)\right)^2 - \left(\psi\left(t'-1\right)\right)^2\right]. \tag{36.39}$$

That is

$$g\left(t'\right) \nabla g\left(t'\right) \leq \frac{1}{2}\left[\left(\psi\left(t'\right)\right)^2 - \left(\psi\left(t'-1\right)\right)^2\right], \quad \text{for } t' = a+m-1, \dots . \tag{36.40}$$

Hence

$$\sum_{t'=a+m}^{t} g\left(t'\right) \nabla g\left(t'\right) \leq \frac{1}{2} \sum_{t'=a+m}^{t} \left[\left(\psi\left(t'\right)\right)^2 - \left(\psi\left(t'-1\right)\right)^2\right]$$

$$= \frac{1}{2}\left[\left(\left(\psi\left(a+m\right)\right)^2 - \left(\psi\left(a+m-1\right)\right)^2\right) + \left(\left(\psi\left(a+m+1\right)\right)^2 - \left(\psi\left(a+m\right)\right)^2\right)\right.$$

$$\left. + \left(\left(\psi\left(a+m+2\right)\right)^2 - \left(\psi\left(a+m+1\right)\right)^2\right) + \dots + \left(\left(\psi\left(t\right)\right)^2 - \left(\psi\left(t-1\right)\right)^2\right)\right]$$

$$= \frac{1}{2}\left[\left(\psi\left(t\right)\right)^2 - \left(\psi\left(a+m-1\right)\right)^2\right]$$

$$= \frac{1}{2}\left[\left(2g\left(t\right) - g\left(t-1\right)\right)^2 - \left(2g\left(a+m-1\right) - g\left(a+m-2\right)\right)^2\right]$$

$$= 2\left(g^2\left(t\right) - g^2\left(a+m-1\right)\right) + \frac{1}{2}\left(g^2\left(t-1\right) - g^2\left(a+m-2\right)\right)$$

$$-2\left[g\left(t\right) g\left(t-1\right) - g\left(a+m-1\right) g\left(a+m-2\right)\right]. \tag{36.41}$$

That is

$$\sum_{t'=a+m}^{t} g\left(t'\right) \nabla g\left(t'\right) \leq 2\left(g^2\left(t\right) - g^2\left(a+m-1\right)\right)$$

$$+\frac{1}{2}\left(g^2\left(t-1\right) - g^2\left(a+m-2\right)\right) -$$

$$2\left[g\left(t\right) g\left(t-1\right) - g\left(a+m-1\right) g\left(a+m-2\right)\right], \quad \forall\, t \geq a+m. \tag{36.42}$$

The last proves the claim. ■

We give

Corollary 36.17. (to Theorem 36.16). Here f is a real valued function defined on $\{-2,-1,0,...\}$, $t \geq 3$, $t \in \mathbb{N}$. Suppose $f(0) = f(-1) = f(-2) = 0$. Set

$$\overline{\theta}(t, 2.5) := \left(\sum_{\tau=1}^{t} \left[(t-\tau+1)^{\overline{1.5}} \right]^2 \right)^{\frac{1}{2}}, \quad t \geq 3, \tag{36.43}$$

$$\overline{g}(t) := \sum_{\tau=1}^{t} \left(\nabla_{1*}^{2.5} f(\tau) \right)^2, \quad t \geq 1; \tag{36.44}$$

$$\overline{G}(t, 3, \overline{g}) := 2\left(\overline{g}^2(t) - \overline{g}^2(2) \right) + \frac{\left(\overline{g}^2(t-1) - \overline{g}^2(1) \right)}{2}$$
$$+2\left[\overline{g}(t)\,\overline{g}(t-1) - \overline{g}(2)\,\overline{g}(1) \right], \quad t \geq 3. \tag{36.45}$$

Call also

$$\overline{K}(t) = \frac{4}{3\sqrt{\pi}} \left(\sum_{t'=3}^{t} \left(\overline{\theta}(t', 2.5) \right)^2 \right)^{\frac{1}{2}}, \quad t \geq 3. \tag{36.46}$$

Then

$$\sum_{t'=3}^{t} \left| f(t') \right| \left| \nabla_{1*}^{2.5} f(t') \right| \leq \overline{K}(t) \left(\overline{G}(t, 3, \overline{g}) \right)^{\frac{1}{2}}, \quad \text{for } t \geq 3. \tag{36.47}$$

Note. Above in (36.45) we have $\overline{g}(1) = \pi (f(1))^2$.

Next we give a discrete fractional nabla Ostrowski type inequality.

Theorem 36.18. Let $m-1 < \mu < m$, $m = \lceil \mu \rceil$, non integer $\mu > 0$; $p, a \in \mathbb{Z}_+$ with $\mu > p$. Consider $b \in \mathbb{N}$ such that $a + m < b$. Let f be a real valued function defined on $[a-m+1, a-m+2, ..., b]$. Here $j \in [a+m, ..., b]$. Suppose that $\nabla^k f(a) = 0$, for $k = p+1, ..., m-1$.

Then

$$\left| \frac{1}{(b-a-m)} \sum_{j=a+m+1}^{b} \nabla^p f(j) - \nabla^p f(a) \right| \leq$$

$$\frac{\left((b-a)^{\overline{\mu-p+1}} - m^{\overline{\mu-p+1}} \right)}{\Gamma(\mu-p+2)(b-a-m)} \cdot \left(\max_{\tau \in \{a+1,...,b\}} \left| \nabla_{(a+1)*}^{\mu} f(\tau) \right| \right). \tag{36.48}$$

Proof. By (36.16) we have

$$\nabla^p f(j) - \nabla^p f(a) = \frac{1}{\Gamma(\mu-p)} \sum_{\tau=a+1}^{j} (j-\tau+1)^{\overline{\mu-p-1}} \nabla_{(a+1)*}^{\mu} f(\tau), \tag{36.49}$$

for all $j \in [a+m+1, a+m+2, ..., b]$.

We obtain that

$$\frac{1}{b-(a+m)}\sum_{j=a+m+1}^{b}\nabla^{p}f(j)-\nabla^{p}f(a)=$$

$$\frac{1}{(b-a-m)}\sum_{j=a+m+1}^{b}\left(\nabla^{p}f(j)-\nabla^{p}f(a)\right)=$$

$$\frac{1}{\Gamma(\mu-p)(b-a-m)}\sum_{j=a+m+1}^{b}\left(\sum_{\tau=a+1}^{j}(j-\tau+1)^{\overline{\mu-p-1}}\nabla_{(a+1)*}^{\mu}f(\tau)\right). \tag{36.50}$$

Therefore we derive

$$\left|\frac{1}{(b-a-m)}\sum_{j=a+m+1}^{b}\nabla^{p}f(j)-\nabla^{p}f(a)\right|\leq$$

$$\frac{1}{\Gamma(\mu-p)(b-a-m)}\sum_{j=a+m+1}^{b}\left(\sum_{\tau=a+1}^{j}(j-\tau+1)^{\overline{\mu-p-1}}\left|\nabla_{(a+1)*}^{\mu}f(\tau)\right|\right)\leq$$

$$\frac{1}{\Gamma(\mu-p)(b-a-m)}\left(\sum_{j=a+m+1}^{b}\left(\sum_{\tau=a+1}^{j}(j-\tau+1)^{\overline{\mu-p-1}}\right)\right)\cdot$$

$$\left(\max_{\tau\in\{a+1,\dots,b\}}\left|\nabla_{(a+1)*}^{\mu}f(\tau)\right|\right)$$

$$\overset{\text{(by (36.24))}}{=}\frac{1}{\Gamma(\mu-p+1)(b-a-m)}\left(\sum_{j=a+m+1}^{b}(j-a)^{\overline{\mu-p}}\right)\cdot$$

$$\left(\max_{\tau\in\{a+1,\dots,b\}}\left|\nabla_{(a+1)*}^{\mu}f(\tau)\right|\right)$$

(by Lemma 19 of [48])

$$=\frac{1}{\Gamma(\mu-p+2)(b-a-m)}\left((b-a)^{\overline{\mu-p+1}}-m^{\overline{\mu-p+1}}\right)\cdot$$

$$\left(\max_{\tau\in\{a+1,\dots,b\}}\left|\nabla_{(a+1)*}^{\mu}f(\tau)\right|\right), \tag{36.51}$$

proving the claim. ■

Next we give a discrete nabla fractional Poincaré inequality.

Theorem 36.19. Let $\mu>p$, $p\in\mathbb{Z}_{+}$, μ non-integer, $m=\lceil\mu\rceil$; $a\in\mathbb{Z}_{+}$. Here $f:[a-m+1,a-m+2,...,b]\rightarrow\mathbb{R}$; $a+m<b$, $b\in\mathbb{N}$, and $\nabla^{k}f(a)=0$, $k=p,...,m-1$.

Let γ, $\delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$. Then

$$\sum_{j=a+m}^{b} |\nabla^p f(j)|^\delta \leq \frac{1}{(\Gamma(\mu-p))^\delta} \left\{ \sum_{j=a+m}^{b} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu-p-1}} \right)^\gamma \right)^{\frac{\delta}{\gamma}} \right\} \cdot \left(\sum_{\tau=a+1}^{b} \left| \nabla^\mu_{(a+1)*} f(\tau) \right|^\delta \right). \tag{36.52}$$

Proof. We have by (36.17) that

$$\nabla^p f(j) = \frac{1}{\Gamma(\mu-p)} \sum_{\tau=a+1}^{j} (j-\tau+1)^{\overline{\mu-p-1}} \nabla^\mu_{(a+1)*} f(\tau), \tag{36.53}$$

$\forall\ j \in [a+m, a+m+1, ..., b]$.

Let γ, $\delta > 1$ such that $\frac{1}{\gamma} + \frac{1}{\delta} = 1$.

We notice that

$$|\nabla^p f(j)| \leq \frac{1}{\Gamma(\mu-p)} \sum_{\tau=a+1}^{j} (j-\tau+1)^{\overline{\mu-p-1}} \left| \nabla^\mu_{(a+1)*} f(\tau) \right|$$

(by discrete Hölder's inequality)

$$\leq \frac{1}{\Gamma(\mu-p)} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu-p-1}} \right)^\gamma \right)^{\frac{1}{\gamma}} \cdot \left(\sum_{\tau=a+1}^{j} \left| \nabla^\mu_{(a+1)*} f(\tau) \right|^\delta \right)^{\frac{1}{\delta}}. \tag{36.54}$$

That is, it holds

$$|\nabla^p f(j)|^\delta \leq \frac{1}{(\Gamma(\mu-p))^\delta} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu-p-1}} \right)^\gamma \right)^{\frac{\delta}{\gamma}} \cdot \left(\sum_{\tau=a+1}^{j} \left| \nabla^\mu_{(a+1)*} f(\tau) \right|^\delta \right) \leq \frac{1}{(\Gamma(\mu-p))^\delta} \cdot \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu-p-1}} \right)^\gamma \right)^{\frac{\delta}{\gamma}} \cdot \left(\sum_{\tau=a+1}^{b} \left| \nabla^\mu_{(a+1)*} f(\tau) \right|^\delta \right), \tag{36.55}$$

$\forall\ j \in [a+m, b]$, a discrete interval.

Applying $\sum_{j=a+m}^{b}$ on both ends of (36.55) we establish (36.52). ■

It follows a discrete nabla Sobolev type fractional inequality.

Theorem 36.20. Let $\mu > p$, $p \in \mathbb{Z}_+$, μ non-integer, $m = \lceil \mu \rceil$; $a \in \mathbb{Z}_+$. Here $f : [a-m+1, ..., b] \to \mathbb{R}$; $a+m < b$, $b \in \mathbb{N}$, and $\nabla^k f(a) = 0$, $k = p, ..., m-1$. Let γ, $\delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, and $r \geq 1$. Then

$$\left(\sum_{j=a+m}^{b} |\nabla^p f(j)|^r \right)^{\frac{1}{r}} \leq \frac{1}{\Gamma(\mu - p)} \left[\sum_{j=a+m}^{b} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu-p-1}} \right)^{\gamma} \right)^{\frac{r}{\gamma}} \right]^{\frac{1}{r}} \cdot$$

$$\left(\sum_{\tau=a+1}^{b} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|^{\delta} \right)^{\frac{1}{\delta}}. \tag{36.56}$$

Proof. By (36.54) and $r \geq 1$ we have

$$|\nabla^p f(j)|^r \leq \frac{1}{(\Gamma(\mu-p))^r} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu-p-1}} \right)^{\gamma} \right)^{\frac{r}{\gamma}} \cdot$$

$$\left(\sum_{\tau=a+1}^{b} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|^{\delta} \right)^{\frac{r}{\delta}}, \quad \forall\ j \in [a+m, ..., b]. \tag{36.57}$$

Consequently we obtain

$$\sum_{j=a+m}^{b} |\nabla^p f(j)|^r \leq \frac{1}{(\Gamma(\mu-p))^r} \left[\sum_{j=a+m}^{b} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu-p-1}} \right)^{\gamma} \right)^{\frac{r}{\gamma}} \right] \cdot$$

$$\left(\sum_{\tau=a+1}^{b} \left| \nabla^{\mu}_{(a+1)*} f(\tau) \right|^{\delta} \right)^{\frac{r}{\delta}}, \tag{36.58}$$

proving the claim. ∎

We finish with the following discrete nabla fractional average Sobolev type inequality.

Theorem 36.21. Let $0 < \mu_1 < \mu_2 < ... < \mu_k$ non-integers; $m_l = \lceil \mu_l \rceil$, $l = 1, ..., k$, $k \in \mathbb{N}$. Assume $\nabla^{\tau} f(a) = 0$, for $\tau = 0, 1, ..., m_k - 1$, where $f : [a - m_k + 1, ..., b] \to \mathbb{R}$; $b \in \mathbb{N}$, $a \in \mathbb{Z}_+$. Let $r \geq 1$; $C_l(s) > 0$ defined on $[a+1, ..., b]$, $l = 1, ..., k$; $a + m_k < b$.

Put

$$B_l := \sum_{\tau=a+1}^{b} C_l(\tau) \left(\nabla^{\mu_l}_{(a+1)*} f(\tau) \right)^2,$$

$$\delta^* := \max_{1 \leq l \leq k} \left\{ \frac{1}{(\Gamma(\mu_l))^2} \left[\sum_{j=a+m_l}^{b} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu_l - 1}} \right)^2 \right)^{\frac{r}{2}} \right]^{\frac{2}{r}} \right\},$$

and

$$\rho^* := \max_{1\le l\le k} \left\| \frac{1}{C_l(\tau)} \right\|_{\infty,[a+1,b]}.$$

Then

$$\|f\|_{r,[a+m_k,b]} \le \sqrt{\delta^*\rho^*} \left(\frac{\sum_{l=1}^k B_l}{k} \right)^{\frac{1}{2}}. \tag{36.59}$$

Proof. It holds also $\nabla^\tau f(a) = 0$, $\tau = 0, 1, ..., m_l - 1$, $l = 1, ..., k-1$. So the assumptions of Theorem 36.20 are fulfilled for f and fractional orders μ_l, $l = 1, ..., k$. Thus by choosing $p = 0$ and $\gamma = \delta = 2$ we apply (36.56), for $l = 1, ..., k$, to get

$$\left(\sum_{j=a+m_l}^{b} |f(j)|^r \right)^{\frac{1}{r}} \le \frac{1}{\Gamma(\mu_l)} \left[\sum_{j=a+m_l}^{b} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu_l - 1}} \right)^2 \right)^{\frac{r}{2}} \right]^{\frac{1}{r}} \cdot$$

$$\left(\sum_{\tau=a+1}^{b} \left(\nabla_{(a+1)*}^{\mu_l} f(\tau) \right)^2 \right)^{\frac{1}{2}}. \tag{36.60}$$

Hence it holds

$$\left(\sum_{j=a+m_l}^{b} |f(j)|^r \right)^{\frac{2}{r}} \le \frac{1}{(\Gamma(\mu_l))^2} \left[\sum_{j=a+m_l}^{b} \left(\sum_{\tau=a+1}^{j} \left((j-\tau+1)^{\overline{\mu_l - 1}} \right)^2 \right)^{\frac{r}{2}} \right]^{\frac{2}{r}} \cdot$$

$$\left(\sum_{\tau=a+1}^{b} \left(\nabla_{(a+1)*}^{\mu_l} f(\tau) \right)^2 \right) \le \delta^* \left(\sum_{\tau=a+1}^{b} \left(\nabla_{(a+1)*}^{\mu_l} f(\tau) \right)^2 \right) =$$

$$\delta^* \left(\sum_{\tau=a+1}^{b} (C_l(\tau))^{-1} (C_l(\tau)) \left(\nabla_{(a+1)*}^{\mu_l} f(\tau) \right)^2 \right) \le$$

$$\delta^*\rho^* \left(\sum_{\tau=a+1}^{b} C_l(\tau) \left(\nabla_{(a+1)*}^{\mu_l} f(\tau) \right)^2 \right).$$

That is

$$\left(\sum_{j=a+m_k}^{b} |f(j)|^r \right)^{\frac{2}{r}} \le \left(\sum_{j=a+m_l}^{b} |f(j)|^r \right)^{\frac{2}{r}} \le$$

$$\delta^*\rho^* \left(\sum_{\tau=a+1}^{b} C_l(\tau) \left(\nabla_{(a+1)*}^{\mu_l} f(\tau) \right)^2 \right) = \delta^*\rho^* B_l, \text{ for } l = 1, ..., k. \tag{36.61}$$

So that

$$\|f\|_{r,[a+m_k,b]}^2 \le \delta^*\rho^* \left(\frac{\sum_{l=1}^k B_l}{k} \right), \tag{36.62}$$

proving the claim. ■

37

About $q-$ Inequalities

We give here forward and reverse $q-$Hölder inequalities, $q-$Poincaré inequality, $q-$Sobolev inequality, $q-$reverse Poincaré inequality, $q-$reverse Sobolev inequality, $q-$Ostrowski inequality, $q-$Opial inequality and $q-$Hilbert-Pachpatte inequality. Some interesting background is mentioned and built in the introduction. This chapter relies on [47].

37.1 Introduction

Here we follow [139], [252].

Let $q \in (0,1)$, $n \in \mathbb{N}$. A $q-$natural number $[n]_q$ is defined by

$$[n]_q := 1 + q + ... + q^{n-1}. \tag{37.1}$$

In general, a $q-$real number $[\alpha]_q$ is

$$[\alpha]_q := \frac{1-q^{\alpha}}{1-q}, \quad \alpha \in \mathbb{R}. \tag{37.2}$$

We define

$$[0]_q! := 1, \quad [n]_q! = [n]_q [n-1]_q ... [1]_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}. \tag{37.3}$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 601–613.
springerlink.com

Also, the $q-$Pochhammer symbol is defined by

$$(z-a)^{(0)}=1, \quad (z-a)^{(k)}=\prod_{i=0}^{k-1}\left(z-aq^{i}\right), \ k\in\mathbb{N}, \ z,a\in\mathbb{R}. \tag{37.4}$$

The $q-$derivative of a function $f(x)$ is

$$(D_q f)(x) := \frac{f(x)-f(qx)}{x-qx} \quad (x\neq 0), \tag{37.5}$$

$$(D_q f)(0) := \lim_{x\to 0}(D_q f)(x),$$

and the high $q-$derivatives

$$D_q^0 f := f, \ \ D_q^k f := D_q\left(D_q^{k-1}f\right), \quad k=1,2,3,... \tag{37.6}$$

From the above definition it is clear that a continuous function on an interval, which does not include 0 is continuously $q-$differentiable.

Here we suppose that the $q-$derivatives we use always exist up to n^{th} order.

Notice that if f is differentiable then $\lim_{q\to 1} D_q f(x) = f'(x)$.

The $q-$integral is defined by

$$(I_{q,0}f)(x)=\int_0^x f(t)\,d_qt = x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right)q^k, \quad (0<q<1). \tag{37.7}$$

We call f $q-$integrable on $[0,a]$, iff $\int_0^x |f(t)|\,d_qt$ exists for all $x\in[0,a]$, $a>0$.

If f is such that, for some $C>0$, $\alpha>-1$, $|f(x)|<Cx^{\alpha}$ in a right neighborhood of $x=0$, then f is $q-$integrable, see [139].

All functions considered in this chapter are assumed to be $q-$integrable.

By [7] it holds

$$(If)(x)=\int_0^x f(t)\,dt = \lim_{q\uparrow 1}(I_{q,0}f)(x), \tag{37.8}$$

given that f is Riemann integrable on $[0,x]$.

Also it holds

$$(D_q I_{q,0}f)(x)=f(x), \tag{37.9}$$

and

$$(I_{q,0}(D_q f))(x)=f(x)-f(0).$$

One can define

$$I_{q,0}^n f = I_{q,0}\left(I_{q,0}^{n-1}f\right), \quad n=1,2,... \tag{37.10}$$

Let $x > 0$, then one has ([7], [156], [197]) the $q-$Taylor formula

$$f(x) = \sum_{k=0}^{n-1} \frac{\left(D_q^k f\right)(0)}{[k]_q!} x^k + \frac{1}{[n-1]_q!} \int_0^x (x - qt)^{(n-1)} D_q^n f(t)\, d_q t. \tag{37.11}$$

Assuming $\left(D_q^k f\right)(0) = 0$, $k = 0, 1, ..., n-1$ we obtain

$$f(x) = \frac{1}{[n-1]_q!} \int_0^x (x - qt)^{(n-1)} D_q^n f(t)\, d_q t. \tag{37.12}$$

Let $u(x) = \alpha x^\beta$, then we get the change of variable formula ([139]),

$$\int_{u(0)}^{u(a)} f(u)\, d_q u = \int_0^a f(u(x))\, D_{q^{\frac{1}{\beta}}} u(x)\, d_{q^{\frac{1}{\beta}}} x. \tag{37.13}$$

In this chapter double $q-$integrals are meant in an iterative way.

Lemma 37.1. ([139]) Let $n \in \mathbb{Z}_+$; $x, t, s, a, b, A, B \in \mathbb{R}$. Then

$$(1) \qquad D_q x^t = [t]_q\, x^{t-1}, \tag{37.14}$$

$$(2) \qquad D_q (Ax + b)^{(n)} = [n]_q\, A\, (Ax + b)^{(n-1)}, \tag{37.15}$$

$$(3) \qquad D_q (a + Bx)^{(n)} = [n]_q\, B\, (a + Bqx)^{(n-1)}. \tag{37.16}$$

We get the $q-$power rule

$$\int_0^x (At + b)^{(n)}\, d_q t = \frac{(Ax + b)^{(n+1)} - b^{(n+1)}}{[n+1]_q\, A}, \tag{37.17}$$

where $b^{(n+1)} = b^{n+1} q^{\frac{n(n+1)}{2}}$.

Furthermore, it holds another $q-$power rule,

$$\int_0^x (a + Bqt)^{(n-1)}\, d_q t = \frac{(a + Bx)^{(n)} - a^n}{[n]_q\, B}. \tag{37.18}$$

Let $f(x) \geq 0$ and f increasing, then

$$\int_0^x f(t)\, d_q t \leq f(x) \cdot x. \tag{37.19}$$

We easily see that $(a > 0,\ 0 < q < 1)$

$$\left| \int_0^a f(x)\, d_q x \right| \leq \int_0^a |f(x)|\, d_q x \tag{37.20}$$

(by $\left|\sum_{i=1}^{\infty} x_i\right| \leq \sum_{i=1}^{\infty} |x_i|$), and

$$\int_0^a (c_1 f_1(x) + c_2 f_2(x))\, d_q x = c_1 \int_0^a f_1(x)\, d_q x + c_2 \int_0^a f_2(x)\, d_q x, \quad c_1, c_2 \in \mathbb{R}. \tag{37.21}$$

Let $0 < x \leq y$ and f increasing. Then

$$x(1-q) \sum_{k=0}^{\infty} f\left(xq^k\right) q^k \leq y(1-q) \sum_{k=0}^{\infty} f\left(yq^k\right) q^k,$$

so that

$$\int_0^x f(t)\, d_q t \leq \int_0^y f(t)\, d_q t. \tag{37.22}$$

Let $f \leq g$, then

$$f\left(xq^k\right) q^k \leq g\left(xq^k\right) q^k$$

and

$$x(1-q) \sum_{k=0}^{\infty} f\left(xq^k\right) q^k \leq x(1-q) \sum_{k=0}^{\infty} g\left(xq^k\right) q^k,$$

that is

$$\int_0^x f(t)\, d_q t \leq \int_0^x g(t)\, d_q t \tag{37.23}$$

($x > 0$, $0 < q < 1$).

Next comes the $q-$Hölder's inequality.

Proposition 37.2. Let $x > 0$, $0 < q < 1$, $p_1, q_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then

$$\int_0^x |f(t)|\, |g(t)|\, d_q t \leq \left(\int_0^x |f(t)|^{p_1} d_q t\right)^{\frac{1}{p_1}} \left(\int_0^x |g(t)|^{q_1} d_q t\right)^{\frac{1}{q_1}}. \tag{37.24}$$

Proof. By the discrete Hölder's inequality we have

$$\int_0^x |f(t)|\, |g(t)|\, d_q t = x(1-q) \sum_{k=0}^{\infty} \left|f\left(xq^k\right)\right| \left|g\left(xq^k\right)\right| q^k$$

$$= x(1-q) \sum_{k=0}^{\infty} \left(\left|f\left(xq^k\right)\right| \left(q^k\right)^{\frac{1}{p_1}}\right) \left(\left|g\left(xq^k\right)\right| \left(q^k\right)^{\frac{1}{q_1}}\right) \leq$$

$$\left(x(1-q) \sum_{k=0}^{\infty} \left|f\left(xq^k\right)\right|^{p_1} q^k\right)^{\frac{1}{p_1}} \left(x(1-q) \sum_{k=0}^{\infty} \left|g\left(xq^k\right)\right|^{q_1} q^k\right)^{\frac{1}{q_1}} =$$

$$\left(\int_0^x |f(t)|^{p_1} d_q t\right)^{\frac{1}{p_1}} \left(\int_0^x |g(t)|^{q_1} d_q t\right)^{\frac{1}{q_1}}.$$

■

Clearly it holds that

$$\int_0^x 1 d_q t = x. \tag{37.25}$$

It follows the reverse $q-$Hölder's inequality.

Proposition 37.3. Let $x > 0$, $0 < q < 1$; $0 < p_1 < 1$, $q_1 < 0 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. Let $f, g \geq 0$ with $\int_0^x (g(t))^{q_1} d_q t > 0$. Then

$$\int_0^x f(t) g(t) d_q t \geq \left(\int_0^x (f(t))^{p_1} d_q t \right)^{\frac{1}{p_1}} \left(\int_0^x (g(t))^{q_1} d_q t \right)^{\frac{1}{q_1}}. \tag{37.26}$$

Proof. Notice that $\int_0^x (g(t))^{q_1} d_q t > 0$, iff $x(1-q) \sum_{k=0}^{\infty} \left(g\left(xq^k \right) \right)^{q_1} q^k > 0$, iff $\sum_{k=0}^{\infty} \left(g\left(xq^k \right) \right)^{q_1} q^k > 0$.

By the discrete reverse Hölder's inequality we have

$$x(1-q) \sum_{k=0}^{\infty} f\left(xq^k \right) g\left(xq^k \right) q^k =$$

$$x(1-q) \sum_{k=0}^{\infty} \left(f\left(xq^k \right) \left(q^k \right)^{\frac{1}{p_1}} \right) \left(g\left(xq^k \right) \left(q^k \right)^{\frac{1}{q_1}} \right) \geq$$

$$\left(x(1-q) \sum_{k=0}^{\infty} \left(f\left(xq^k \right) \right)^{p_1} q^k \right)^{\frac{1}{p_1}} \left(x(1-q) \sum_{k=0}^{\infty} \left(g\left(xq^k \right) \right)^{q_1} q^k \right)^{\frac{1}{q_1}},$$

proving the claim. ■

37.2 Main Results

We give the $q-$Poincaré inequality.

Theorem 37.4. Let $\alpha, \beta > 1 : \frac{1}{\alpha} + \frac{1}{\beta} = 1$, $x > 0$. Suppose $\left(D_q^k f \right)(0) = 0$, $k = 0, 1, ..., n-1$ and $\left| D_q^n f \right|$ be increasing. Then

$$\int_0^x |f(w)|^{\beta} d_q w \leq$$

$$\frac{1}{\left([n-1]_q! \right)^{\beta}} \left(\int_0^x \left(\int_0^w \left((w - qt)^{(n-1)} \right)^{\alpha} d_q t \right)^{\frac{\beta}{\alpha}} d_q w \right) \left(\int_0^x \left| D_q^n f(t) \right|^{\beta} d_q t \right). \tag{37.27}$$

Proof. For $0 \leq w \leq x$, we have

$$f(w) = \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)} D_q^n f(t)\, d_q t.$$

Thus

$$|f(w)| \leq \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)} \left|D_q^n f(t)\right| d_q t$$

(by $q-$Hölder's inequality)

$$\leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{1}{\alpha}} \left(\int_0^w \left|D_q^n f(t)\right|^{\beta} d_q t\right)^{\frac{1}{\beta}}$$

$$\leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{1}{\alpha}} \left(\int_0^x \left|D_q^n f(t)\right|^{\beta} d_q t\right)^{\frac{1}{\beta}}.$$

Hence

$$|f(w)|^{\beta} \leq \frac{1}{\left([n-1]_q!\right)^{\beta}} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{\beta}{\alpha}} \left(\int_0^x \left|D_q^n f(t)\right|^{\beta} d_q t\right). \tag{37.28}$$

Then applying $q-$integration on (37.28) over $[0, x]$, we prove (37.27). ■

We present the $q-$Sobolev inequality.

Theorem 37.5. Let $\alpha, \beta > 1 : \frac{1}{\alpha} + \frac{1}{\beta} = 1$, $x > 0$, $r \geq 1$. Suppose $\left(D_q^k f\right)(0) = 0$, $k = 0, 1, ..., n-1$ and $\left|D_q^n f\right|$ be increasing.

Denote $\|f\|_{q,r,[0,x]} = \left(\int_0^x |f(w)|^r d_q w\right)^{\frac{1}{r}}$. Then

$$\|f\|_{q,r,[0,x]} \leq \frac{1}{[n-1]_q!} \left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{r}{\alpha}} d_q w\right)^{\frac{1}{r}} \left\|D_q^n f\right\|_{q,\beta,[0,x]}. \tag{37.29}$$

Proof. As in the proof of Theorem 37.4 we obtain

$$|f(w)|^r \leq \frac{1}{\left([n-1]_q!\right)^r} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{r}{\alpha}} \left(\int_0^x \left|D_q^n f(t)\right|^{\beta} d_q t\right)^{\frac{r}{\beta}}.$$

Thus

$$\int_0^x |f(w)|^r d_q w \overset{(37.23)}{\leq}$$

$$\frac{1}{\left([n-1]_q!\right)^r} \left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{\alpha} d_q t\right)^{\frac{r}{\alpha}} d_q w\right) \left(\int_0^x \left|D_q^n f(t)\right|^{\beta} d_q t\right)^{\frac{r}{\beta}}. \tag{37.30}$$

Next raise both sides of (37.30) to power $\frac{1}{r}$. Thus proving the claim. ∎

Next we give the reverse $q-$Poincaré inequality.

Theorem 37.6. Let $0 < p_1 < 1$, $q_1 < 0$: $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $x > 0$. Suppose $\left(D_q^k f\right)(0) = 0$, $k = 0, 1, ..., n-1$; $\left|D_q^n f\right|$ be decreasing, and $D_q^n f(t)$ of fixed strict sign on $[0, x]$. Then

$$\int_0^x |f(w)|^{-q_1} d_q w \geq \frac{1}{\left([n-1]_q!\right)^{-q_1}} \cdot$$

$$\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{-q_1}{p_1}} d_q w\right) \left(\int_0^x \left|D_q^n f(t)^{q_1} d_q t\right|\right)^{-1}. \quad (37.31)$$

Proof. Clearly here we have

$$\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t > 0 \quad \text{for all } 0 < w \leq x.$$

Also we have

$$f(w) = \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)} D_q^n f(t) \, d_q t, \quad \text{all } 0 \leq w \leq x.$$

Hence

$$|f(w)| = \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)} \left|D_q^n f(t)\right| d_q t, \quad \text{all } 0 \leq w \leq x.$$

By $q-$reverse Hölder inequality we derive

$$|f(w)| \geq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{1}{p_1}} \left(\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}}.$$

Because $\left|D_q^n f\right|$ is decreasing, we have that $\left|D_q^n f\right|^{q_1}$ is increasing on $[0, x]$. Thus

$$\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t \leq \int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t,$$

and

$$\left(\int_0^w \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}} \geq \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}}, \quad \text{for all } 0 \leq w \leq x.$$

Therefore we derive

$$|f(w)| \geq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{1}{p_1}} \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}},$$

all $0 \leq w \leq x$.

Hence

$$|f(w)|^{-q_1} \geq$$

$$\frac{1}{\left([n-1]_q!\right)^{-q_1}} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{-q_1}{p_1}} \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{-1}, \tag{37.32}$$

all $0 \leq w \leq x$.

At last $q-$integrating (37.32) on $[0, x]$ we obtain (37.31). ■

It follows the reverse $q-$Sobolev inequality.

Theorem 37.7. All assumptions were as in Theorem 37.6 and $r \geq 1$. Then

$$\|f\|_{q,r,[0,x]} \geq$$

$$\frac{1}{[n-1]_q!} \left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{r}{p_1}} d_q w\right)^{\frac{1}{r}} \left\|D_q^n f\right\|_{q,q_1,[0,x]}. \tag{37.33}$$

Proof. As in the proof of Theorem 37.6 we obtain:

$$|f(w)|^r \geq \frac{1}{\left([n-1]_q!\right)^r} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{r}{p_1}} \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{r}{q_1}},$$

all $0 \leq w \leq x$.

Thus

$$\int_0^x |f(w)|^r d_q w \geq \frac{1}{\left([n-1]_q!\right)^r} \cdot$$

$$\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^{p_1} d_q t\right)^{\frac{r}{p_1}} d_q w\right) \left(\int_0^x \left|D_q^n f(t)\right|^{q_1} d_q t\right)^{\frac{r}{q_1}},$$

proving the claim. ■

We continue with a $q-$Ostrowski inequality.

Theorem 37.8. Assume $\left(D_q^k f\right)(0) = 0$, $k = 1, ..., n-1$, $x > 0$, $0 < q < 1$. Then

$$\left|\frac{1}{x} \int_0^x f(w) \, d_q w - f(0)\right| \leq \left\|D_q^n f\right\|_{\infty,[0,x]} \cdot \frac{x^n}{[n+1]_q!}. \tag{37.34}$$

Proof. By assumptions we have

$$f(w) - f(0) = \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)} D_q^n f(t) \, d_q t, \quad \text{all } 0 \leq w \leq x.$$

Hence

$$\Delta(x) := \frac{1}{x}\int_0^x f(w)\, d_q w - f(0) =$$

$$\frac{1}{x}\int_0^x f(w)\, d_q w - \frac{1}{x}\int_0^x f(0)\, d_q w =$$

$$\frac{1}{x}\left(\int_0^x (f(w) - f(0))\, d_q w\right).$$

Thus

$$|\Delta(x)| \le \frac{1}{x}\int_0^x |f(w) - f(0)|\, d_q w. \tag{37.35}$$

However we see that

$$|f(w) - f(0)| \le \frac{1}{[n-1]_q!}\int_0^w (w - qt)^{(n-1)} \left|D_q^n f(t)\right| d_q t$$

$$\le \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n-1]_q!}\int_0^w (w - qt)^{(n-1)}\, d_q t. \tag{37.36}$$

Next we apply (37.13) for $u(t) := -t$.

We notice that $D_q u(t) = -1$.

Therefore it holds

$$\int_0^w (w - qt)^{(n-1)}\, d_q t = -\int_0^w (w + qu(t))^{(n-1)}\, D_q u(t)\, d_q t$$

$$= -\int_0^{-w} (w + qu(t))^{(n-1)}\, d_q u(t) = -\int_0^{-w} (w + qy)^{(n-1)}\, d_q y$$

$$\overset{(37.18)}{=} -\left[\frac{(w + (-w))^{(n)} - w^n}{[n]_q}\right] = \frac{w^n}{[n]_q}.$$

By (37.36) then we have

$$|f(w) - f(0)| \le \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n]_q!} w^n, \quad \text{all } 0 \le w \le x. \tag{37.37}$$

Consequently by (37.35) we derive

$$|\Delta(x)| \le \frac{1}{x}\left(\int_0^x w^n d_q w\right) \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n]_q!}$$

$$\overset{(37.14)}{=} \frac{1}{x}\frac{x^{n+1}}{[n+1]_q}\frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n]_q!} = \frac{\left\|D_q^n f\right\|_{\infty,[0,x]}}{[n+1]_q!} x^n,$$

proving the claim. ∎

Next we present a $q-$Opial type inequality.

Theorem 37.9. Suppose $\left(D_q^k f\right)(0) = 0$, $n \in \mathbb{N}$, $k = 0, 1, ..., n-1$, $x > 0$, $0 < q < 1$; $\alpha, \beta > 1 : \frac{1}{\alpha} + \frac{1}{\beta} = 1$. Also assume $\left|D_q^n f\right|$ is increasing on $[0, x]$. Then

$$\int_0^x |f(w)| \left|D_q^n f(w)\right| d_q w \leq$$

$$\frac{x^{\frac{1}{\beta}}}{[n-1]_q!} \left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right) d_q w\right)^{\frac{1}{\alpha}} \left(\int_0^x \left(D_q^n f(w)\right)^{2\beta} d_q w\right)^{\frac{1}{\beta}}. \tag{37.38}$$

Proof. It holds

$$f(w) = \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)} \left(D_q^n f\right)(t)\, d_q t, \quad \text{all } 0 \leq w \leq x.$$

Thus

$$|f(w)| \leq \frac{1}{[n-1]_q!} \int_0^w (w-qt)^{(n-1)} \left|D_q^n f(t)\right| d_q t$$

(by $q-$Hölder's inequality)

$$\leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right)^{\frac{1}{\alpha}} \left(\int_0^w \left|D_q^n f(t)\right|^\beta d_q t\right)^{\frac{1}{\beta}}.$$

Put

$$z(w) := \int_0^w \left|D_q^n f(t)\right|^\beta d_q t, \quad (z(0) = 0), \quad \text{all } 0 \leq w \leq x.$$

That is

$$|f(w)| \leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right)^{\frac{1}{\alpha}} (z(w))^{\frac{1}{\beta}},$$

with

$$z(w) \leq \left|D_q^n f(w)\right|^\beta w,$$

and

$$(z(w))^{\frac{1}{\beta}} \leq \left|D_q^n f(w)\right| w^{\frac{1}{\beta}}, \quad \text{for all } 0 \leq w \leq x.$$

Consequently we have

$$|f(w)| \leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right)^{\frac{1}{\alpha}} \left|D_q^n f(w)\right| w^{\frac{1}{\beta}},$$

and

$$|f(w)| \left|D_q^n f(w)\right| \leq \frac{1}{[n-1]_q!} \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right)^{\frac{1}{\alpha}} \left(D_q^n f(w)\right)^2 w^{\frac{1}{\beta}},$$

all $0 \leq w \leq x$.

Finally we find

$$\int_0^x |f(w)| \left|D_q^n f(w)\right| d_q w \leq$$

$$\frac{1}{[n-1]_q!}\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right)^{\frac{1}{\alpha}} \left(D_q^n f(w)\right)^2 w^{\frac{1}{\beta}}\right) d_q w$$

(by $q-$Hölder's inequality)

$$\leq \frac{1}{[n-1]_q!}\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right) d_q w\right)^{\frac{1}{\alpha}} \left(\int_0^x \left(D_q^n f(w)\right)^{2\beta} w d_q w\right)^{\frac{1}{\beta}}$$

$$\leq \frac{x^{\frac{1}{\beta}}}{[n-1]_q!}\left(\int_0^x \left(\int_0^w \left((w-qt)^{(n-1)}\right)^\alpha d_q t\right) d_q w\right)^{\frac{1}{\alpha}} \left(\int_0^x \left(D_q^n f(w)\right)^{2\beta} d_q w\right)^{\frac{1}{\beta}},$$

proving the claim. ■

We finish with a $q-$Hilbert-Pachpatte type inequality.

Theorem 37.10. Suppose $\left(D_q^k f\right)(0) = \left(D_q^k g\right)(0) = 0$, $k = 0, 1, ..., n-1$, $n \in \mathbb{N}$; $x, y > 0$, $0 < q < 1$; $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. Also assume $\left|D_q^n f\right|$, $\left|D_q^n g\right|$ are increasing on $[0, x]$, $[0, y]$, respectively. Define

$$\begin{aligned} F(s) &= \int_0^s \left((s-q\sigma)^{(n-1)}\right)^{p_1} d_q\sigma, \quad 0 \leq s \leq x, \\ G(t) &= \int_0^t \left((t-q\tau)^{(n-1)}\right)^{q_1} d_q\tau, \quad 0 \leq t \leq y. \end{aligned}$$

Then

$$\int_0^x \int_0^y \frac{|f(s)| |g(t)|}{\left(\frac{F(s)}{p_1} + \frac{G(t)}{q_1}\right)} d_q s d_q t \leq$$

$$\frac{xy}{\left([n-1]_q!\right)^2}\left(\int_0^x \left|D_q^n f(\sigma)\right|^{q_1} d_q\sigma\right)^{\frac{1}{q_1}} \left(\int_0^y \left|D_q^n g(\tau)\right|^{p_1} d_q\tau\right)^{\frac{1}{p_1}}. \qquad (37.39)$$

Proof. We have

$$\begin{aligned} f(s) &= \frac{1}{[n-1]_q!}\int_0^s (s-q\sigma)^{(n-1)} D_q^n f(\sigma) d_q\sigma, \quad \text{all } 0 \leq s \leq x; \\ g(t) &= \frac{1}{[n-1]_q!}\int_0^t (t-q\tau)^{(n-1)} D_q^n g(\tau) d_q\tau, \quad \text{all } 0 \leq t \leq y. \end{aligned}$$

Thus

$$|f(s)| \leq \frac{1}{[n-1]_q!}\int_0^s (s-q\sigma)^{(n-1)} \left|D_q^n f(\sigma)\right| d_q\sigma \leq$$

$$\frac{1}{[n-1]_q!}\left(\int_0^s\left((s-q\sigma)^{(n-1)}\right)^{p_1}d_q\sigma\right)^{\frac{1}{p_1}}\left(\int_0^s\left|D_q^nf(\sigma)\right|^{q_1}d_q\sigma\right)^{\frac{1}{q_1}}.$$

Also it holds

$$|g(t)|\leq\frac{1}{[n-1]_q!}\int_0^t(t-q\tau)^{(n-1)}\left|D_q^ng(\tau)\right|d_q\tau\leq$$

$$\frac{1}{[n-1]_q!}\left(\int_0^t\left((t-q\tau)^{(n-1)}\right)^{q_1}d_q\tau\right)^{\frac{1}{q_1}}\left(\int_0^t\left|D_q^ng(\tau)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}.$$

Young's inequality for $a,b\geq 0$ says that

$$a^{\frac{1}{p_1}}b^{\frac{1}{q_1}}\leq\frac{a}{p_1}+\frac{b}{q_1}.$$

Therefore we get

$$|f(s)||g(t)|\leq\frac{1}{\left([n-1]_q!\right)^2}(F(s))^{\frac{1}{p_1}}(G(t))^{\frac{1}{q_1}}\cdot$$

$$\left(\int_0^s\left|D_q^nf(\sigma)\right|^{q_1}d_q\sigma\right)^{\frac{1}{q_1}}\left(\int_0^t\left|D_q^ng(\tau)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}$$

$$\leq\frac{1}{\left([n-1]_q!\right)^2}\left(\frac{F(s)}{p_1}+\frac{G(t)}{q_1}\right)\cdot$$

$$\left(\int_0^s\left|D_q^nf(\sigma)\right|^{q_1}d_q\sigma\right)^{\frac{1}{q_1}}\left(\int_0^t\left|D_q^ng(\tau)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}.$$

Hence it holds ($0<s\leq x$, $0<t\leq y$)

$$\frac{|f(s)||g(t)|}{\left(\frac{F(s)}{p_1}+\frac{G(t)}{q_1}\right)}\leq\frac{1}{\left([n-1]_q!\right)^2}\left(\int_0^s\left|D_q^nf(\sigma)\right|^{q_1}d_q\sigma\right)^{\frac{1}{q_1}}\left(\int_0^t\left|D_q^ng(\tau)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}.$$

Therefore

$$\int_0^x\int_0^y\frac{|f(s)||g(t)|}{\left(\frac{F(s)}{p_1}+\frac{G(t)}{q_1}\right)}d_qsd_qt\leq\frac{1}{\left([n-1]_q!\right)^2}\cdot$$

$$\left(\int_0^x\left(\int_0^s\left|D_q^nf(\sigma)\right|^{q_1}d_q\sigma\right)^{\frac{1}{q_1}}d_qs\right)\left(\int_0^y\left(\int_0^t\left|D_q^ng(\tau)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}d_qt\right)\leq$$

$$\frac{1}{\left([n-1]_q!\right)^2}x^{\frac{1}{p_1}}\left(\int_0^x\left(\int_0^s\left|D_q^nf(\sigma)\right|^{q_1}d_q\sigma\right)d_qs\right)^{\frac{1}{q_1}}\cdot$$

$$y^{\frac{1}{q_1}}\left(\int_0^y\left(\int_0^t\left|D_q^n g\left(\tau\right)\right|^{p_1}d_q\tau\right)d_qt\right)^{\frac{1}{p_1}}$$

$$\leq\frac{1}{\left(\left[n-1\right]_q!\right)^2}\left(x^{\frac{1}{p_1}}y^{\frac{1}{q_1}}\right)\left(\int_0^x\left(\int_0^x\left|D_q^n f\left(\sigma\right)\right|^{q_1}d_q\sigma\right)d_qs\right)^{\frac{1}{q_1}}\cdot$$

$$\left(\int_0^y\left(\int_0^y\left|D_q^n g\left(\tau\right)\right|^{p_1}d_q\tau\right)d_qt\right)^{\frac{1}{p_1}}$$

$$=\frac{xy}{\left(\left[n-1\right]_q!\right)^2}\left(\int_0^x\left|D_q^n f\left(\sigma\right)\right|^{q_1}d_q\sigma\right)^{\frac{1}{q_1}}\left(\int_0^y\left|D_q^n g\left(\tau\right)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}},$$

establishing the claim. ■

38

About $q-$ Fractional Inequalities

Here we present $q-$fractional Poincaré type, Sobolev type and Hilbert-Pachpatte type integral inequalities, involving $q-$fractional derivatives of functions. We give also their generalized versions. This chapter relies on [50].

38.1 Background

Here we follow [273] in all of this section, see also [252].

Let $q \in (0,1)$, we define

$$[\alpha]_q := \frac{1-q^\alpha}{1-q}, \qquad (\alpha \in \mathbb{R}). \tag{38.1}$$

The $q-$analog of the Pochhammer symbol ($q-$shifted factorial) is defined by:

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1} \left(1 - aq^i\right) \qquad (k \in \mathbb{N} \cup \{\infty\}).$$

The expansion to reals is

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha \in \mathbb{R}); \tag{38.2}$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 615–625.
springerlink.com

also define the $q-$analog

$$(a-b)^{(\alpha)} = a^{\alpha}\frac{\left(\frac{b}{a};q\right)_{\infty}}{\left(q^{\alpha}\frac{b}{a};q\right)_{\infty}}, \quad a,b\in\mathbb{R},\, a\neq 0.$$

Observe that

$$(a-b)^{(\alpha)} = a^{\alpha}\left(\frac{b}{a};q\right)_{\alpha}.$$

The $q-$gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_{\infty}}{(q^x;q)_{\infty}}(1-q)^{1-x}, \quad (x\in\mathbb{R}-\{0,-1,-2,...\}). \tag{38.3}$$

Clearly

$$\Gamma_q(x+1) = [x]_q\,\Gamma_q(x). \tag{38.4}$$

The $q-$derivative of a function $f(x)$ is defined by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{x-qx}, \quad (x\neq 0), \tag{38.5}$$

$$(D_q f)(0) = \lim_{x\to 0}(D_q f)(x), \tag{38.6}$$

and the $q-$derivatives of higher order:

$$D_q^0 f = f, \quad D_q^n f = D_q\left(D_q^{n-1}f\right), \quad n=1,2,3,... \tag{38.7}$$

The $q-$integral is defined by

$$(I_{q,0}f)(x) = \int_0^x f(t)\,d_qt = x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right)q^k, \quad (0<q<1), \tag{38.8}$$

and

$$(I_{q,a}f)(x) = \int_a^x f(t)\,d_qt = \int_0^x f(t)\,d_qt - \int_0^a f(t)\,d_qt. \tag{38.9}$$

By [171], we see that: if $f(x)\geq 0$, then it is not necessarily true that

$$\int_a^b f(x)\,d_qx \geq 0.$$

In the case of $a = xq^n$, then (38.9) becomes

$$\int_{xq^n}^x f(t)\,d_qt = x(1-q)\sum_{k=0}^{n-1} f\left(xq^k\right)q^k, \tag{38.10}$$

see also [171].

Double $q-$integration is defined the usual iterative way.
Also we define

$$I_{q,a}^{0}f=f,\quad I_{q,a}^{n}f=I_{q,a}\left(I_{q,a}^{n-1}f\right),\qquad n=1,2,3,... \tag{38.11}$$

The following are valid:

$$\left(D_qI_{q,a}f\right)(x)=f(x), \tag{38.12}$$

$$\left(I_{q,a}D_qf\right)(x)=f(x)-f(a). \tag{38.13}$$

Denote

$$\begin{aligned}[n]_q! &= [1]_q[2]_q...[n]_q, \quad n\in\mathbb{N};\\ [0]_q! &= 1, \quad \left[\begin{array}{c} n\\ k\end{array}\right]_q=\frac{[n]_q!}{[k]_q![n-k]_q!}.\end{aligned}$$

In the next we work on $(0,b)$, $b>0$, and let $a\in(0,b)$. Also the required $q-$derivatives and $q-$integrals do exist.

Definition 38.1. The fractional $q-$integral is

$$\left(I_{q,a}^{\alpha}f\right)(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x\left(q\frac{t}{x};q\right)_{\alpha-1}f(t)\,d_qt \tag{38.14}$$

$$=\frac{1}{\Gamma_q(\alpha)}\int_a^x(x-qt)^{(\alpha-1)}f(t)\,d_qt,\qquad \left(a<x,\ \alpha\in\mathbb{R}^+\right).$$

The usual fractional integral (see also [42]) is the limit case of (38.14) as $q\uparrow 1$, since

$$\lim_{q\uparrow 1}x^{\alpha-1}\left(q\frac{t}{x};q\right)_{\alpha-1}=(x-t)^{\alpha-1}. \tag{38.15}$$

Clearly

$$\left(I_{q,a}^{\alpha}f\right)(a)=0. \tag{38.16}$$

We mention

Theorem 38.2. Let $\alpha,\beta\in\mathbb{R}^+$. The $q-$fractional integration has the semigroup property

$$\left(I_{q,a}^{\beta}I_{q,a}^{\alpha}f\right)(x)=\left(I_{q,a}^{\alpha+\beta}f\right)(x),\qquad (a<x). \tag{38.17}$$

Corollary 38.3. For $\alpha\geq n$ $(n\in\mathbb{N})$ it holds

$$\left(D_q^nI_{q,a}^{\alpha}f\right)(x)=\left(I_{q,a}^{\alpha-n}f\right)(x),\qquad (a<x). \tag{38.18}$$

We mention the fractional $q-$derivative of Caputo type:

Definition 38.4. The fractional $q-$derivative of Caputo type is

$$\left({}_*D_{q,a}^{\alpha}f\right)(x)=\begin{cases}\left(I_{q,a}^{-\alpha}f\right)(x), & \alpha\leq 0;\\ \left(I_{q,a}^{\lceil\alpha\rceil-\alpha}D_q^{\lceil\alpha\rceil}f(x)\right), & \alpha>0,\end{cases} \tag{38.19}$$

where $\lceil.\rceil$ denotes the ceiling of the number.

Next we mention the highlight of this introductory section. Again all here come from [273]. So the following is the fractional $q-$Taylor formula of Caputo type.

Theorem 38.5. Let $\alpha\in\mathbb{R}^+-\mathbb{N}$, $a<x$. Then

$$\left(I_{q,a}^{\alpha}\ {}_*D_{q,a}^{\alpha}f\right)(x)=f(x)-\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{\left(D_q^kf\right)(a)}{[k]_q!}x^k\left(\frac{a}{x};q\right)_k. \tag{38.20}$$

Also we present

Theorem 38.6. Let $\alpha\in\mathbb{R}^+-\mathbb{N}$, $\beta\in\mathbb{R}^+$, $\alpha>\beta>0$, $a<x$. Then

$$\left(I_{q,a}^{\beta}\ {}_*D_{q,a}^{\alpha}f\right)(x)=\left({}_*D_{q,a}^{\alpha-\beta}f\right)(x)- \tag{38.21}$$

$$\sum_{k=\lceil\alpha-\beta\rceil}^{\lceil\alpha\rceil-1}\frac{\left(D_q^kf\right)(a)}{\Gamma_q(k-\alpha+\beta+1)}x^{k-\alpha+\beta}\left(\frac{a}{x};q\right)_{k-\alpha+\beta}.$$

38.2 Main Results

We need the following $q-$Hölder's inequality.

Proposition 38.7. Let $x>0$, $0<q<1$; $p_1,q_1>1$ such that $\frac{1}{p_1}+\frac{1}{q_1}=1$; $n\in\mathbb{N}$. Then

$$\int_{xq^n}^x|f(t)||g(t)|d_qt\leq\left(\int_{xq^n}^x|f(t)|^{p_1}d_qt\right)^{\frac{1}{p_1}}\left(\int_{xq^n}^x|g(t)|^{q_1}d_qt\right)^{\frac{1}{q_1}}. \tag{38.22}$$

Proof. By the discrete Hölder's inequality we have

$$\int_{xq^n}^x|f(t)||g(t)|d_qt=x(1-q)\sum_{k=0}^{n-1}\left|f\left(xq^k\right)\right|\left|g\left(xq^k\right)\right|q^k=$$

$$x(1-q)\sum_{k=0}^{n-1}\left(\left|f\left(xq^k\right)\right|\left(q^k\right)^{\frac{1}{p_1}}\right)\left(\left|g\left(xq^k\right)\right|\left(q^k\right)^{\frac{1}{q_1}}\right)\leq$$

$$\left(x\left(1-q\right)\sum_{k=0}^{n-1}\left|f\left(xq^{k}\right)\right|^{p_1}q^{k}\right)^{\frac{1}{p_1}}\left(x\left(1-q\right)\sum_{k=0}^{n-1}\left|g\left(xq^{k}\right)\right|^{q_1}q^{k}\right)^{\frac{1}{q_1}}=$$

$$\left(\int_{xq^n}^{x}\left|f\left(t\right)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\left(\int_{xq^n}^{x}\left|g\left(t\right)\right|^{q_1}d_qt\right)^{\frac{1}{q_1}}.$$

■

We give a $q-$fractional Poincaré type inequality.

Theorem 38.8. Let $x>0$, $0<w\leq x$, $0<q<1$; $\alpha>0$, $p_1,q_1>1$ such that $\frac{1}{p_1}+\frac{1}{q_1}=1$; $n\in\mathbb{N}$. Put

$$\Delta\left(w\right):=f\left(w\right)-\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{\left(D_q^kf\right)\left(wq^n\right)}{\left[k\right]_q!}w^k\left(q^n;q\right)_k.$$

Then

$$\int_0^x\frac{\left|\Delta\left(w\right)\right|^{q_1}}{w^{q_1\left(\alpha-1\right)}}d_qw\leq\frac{1}{\left(\Gamma_q\left(\alpha\right)\right)^{q_1}}\cdot\left(\int_0^x\left(\int_{wq^n}^{w}\left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1}d_qt\right)^{q_1}d_qw\right)^{\frac{1}{p_1}}$$

$$\cdot\left(\int_0^x\left(\int_{wq^n}^{w}\left|{}_*D_{q,wq^n}^{\alpha}f\left(t\right)\right|^{q_1}d_qt\right)^{q_1}d_qw\right)^{\frac{1}{q_1}}. \tag{38.23}$$

Proof. By $q-$fractional Taylor's formula (38.20) we get

$$\Delta\left(w\right)=\left(I_{q,wq^n}^{\alpha}\ {}_*D_{q,wq^n}^{\alpha}f\right)\left(w\right)=\frac{w^{\alpha-1}}{\Gamma_q\left(\alpha\right)}\int_{wq^n}^{w}\left(q\frac{t}{w};q\right)_{\alpha-1}\left({}_*D_{q,wq^n}^{\alpha}f\right)\left(t\right)d_qt. \tag{38.24}$$

Here by (38.14) and (38.19), we see that

$$\left({}_*D_{q,wq^n}^{\alpha}f\right)\left(t\right)=\frac{t^{\lceil\alpha\rceil-\alpha-1}}{\Gamma_q\left(\lceil\alpha\rceil-\alpha\right)}\int_{wq^n}^{w}\left(q\frac{s}{t};q\right)_{\lceil\alpha\rceil-\alpha-1}D_q^{\lceil\alpha\rceil}f\left(s\right)d_q\left(s\right), \tag{38.25}$$

all $wq^n\leq t\leq w$.

Here we observe trivially that

$$\left|\int_{xq^n}^{x}f\left(t\right)d_qt\right|\leq\int_{xq^n}^{x}\left|f\left(t\right)\right|d_qt. \tag{38.26}$$

Furthermore we see that

$$\left(q\frac{t}{w};q\right)_{\alpha-1}=\frac{\left(q\frac{t}{w};q\right)_{\infty}}{\left(q^{\alpha}\frac{t}{w};q\right)_{\infty}}=\frac{\prod_{i=0}^{\infty}\left(1-q\frac{t}{w}q^i\right)}{\prod_{i=0}^{\infty}\left(1-q^{\alpha}\frac{t}{w}q^i\right)}=\frac{\prod_{i=0}^{\infty}\left(1-\frac{t}{w}q^{i+1}\right)}{\prod_{i=0}^{\infty}\left(1-\frac{t}{w}q^{i+\alpha}\right)}>0. \tag{38.27}$$

Hence by (38.22) we obtain

$$|\Delta(w)| \leq \frac{w^{\alpha-1}}{\Gamma_q(\alpha)} \int_{wq^n}^{w} \left(q\frac{t}{w};q\right)_{\alpha-1} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right| d_q t \leq$$

$$\frac{w^{\alpha-1}}{\Gamma_q(\alpha)} \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_q t\right)^{\frac{1}{p_1}} \cdot \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}}. \tag{38.28}$$

Consequently we derive

$$\frac{|\Delta(w)|}{w^{\alpha-1}} \leq \frac{1}{\Gamma_q(\alpha)} \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_q t\right)^{\frac{1}{p_1}} \cdot \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_q t\right)^{\frac{1}{q_1}}, \tag{38.29}$$

and

$$\frac{|\Delta(w)|^{q_1}}{w^{q_1(\alpha-1)}} \leq \frac{1}{(\Gamma_q(\alpha))^{q_1}} \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_q t\right)^{\frac{q_1}{p_1}} \cdot \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_q t\right). \tag{38.30}$$

Applying $q-$Hölder's inequality (which is also valid on $[0,x]$) on (38.30), we observe that

$$\int_0^x \frac{|\Delta(w)|^{q_1}}{w^{q_1(\alpha-1)}} d_q w \leq \frac{1}{(\Gamma_q(\alpha))^{q_1}} \cdot$$

$$\int_0^x \left[\left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_q t\right)^{\frac{q_1}{p_1}} \cdot \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_q t\right)\right] d_q w$$

$$\leq \frac{1}{(\Gamma_q(\alpha))^{q_1}} \cdot \left(\int_0^x \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_q t\right)^{q_1} d_q w\right)^{\frac{1}{p_1}} \cdot \left(\int_0^x \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_q t\right)^{q_1} d_q w\right)^{\frac{1}{q_1}}, \tag{38.31}$$

proving the claim. ■

Next we give a $q-$fractional Sobolev type inequality.

Theorem 38.9. Here all terms and assumptions as in Theorem 38.8. Additionally let $r_1, r_2 > 1 : \frac{1}{r_1} + \frac{1}{r_2} = 1$. Then

$$\left(\int_0^x \left(\frac{|\Delta(w)|}{w^{\alpha-1}}\right)^{r_1} d_q w\right)^{\frac{1}{r_1}} \leq \frac{1}{\Gamma_q(\alpha)} \cdot$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1^2}{p_1}} d_qw\right)^{\frac{1}{r_1^2}} \cdot$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left|{}_*D_{q,wq^n}^{\alpha} f(t)\right|^{q_1} d_qt\right)^{\frac{r_1 r_2}{q_1}} d_qw\right)^{\frac{1}{r_1 r_2}}. \tag{38.32}$$

Proof. As in the proof of Theorem 38.8 we get (38.29), so that

$$\left(\frac{|\Delta(w)|}{w^{\alpha-1}}\right)^{r_1} \leq \frac{1}{(\Gamma_q(\alpha))^{r_1}} \left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1}{p_1}} \cdot \tag{38.33}$$

$$\left(\int_{wq^n}^w \left|\left({}_*D_{q,wq^n}^{\alpha} f\right)(t)\right|^{q_1} d_qt\right)^{\frac{r_1}{q_1}}.$$

Therefore

$$\int_0^x \left(\frac{|\Delta(w)|}{w^{\alpha-1}}\right)^{r_1} d_qw \leq \frac{1}{(\Gamma_q(\alpha))^{r_1}} \cdot \tag{38.34}$$

$$\int_0^x \left[\left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1}{p_1}} \cdot \left(\int_{wq^n}^w \left|\left({}_*D_{q,wq^n}^{\alpha} f\right)(t)\right|^{q_1} d_qt\right)^{\frac{r_1}{q_1}}\right] d_qw$$

(by $q-$Hölder's inequality on $[0,x]$)

$$\leq \frac{1}{(\Gamma_q(\alpha))^{r_1}} \left(\int_0^x \left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1^2}{p_1}} d_qw\right)^{\frac{1}{r_1}} \cdot \tag{38.35}$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left|\left({}_*D_{q,wq^n}^{\alpha} f\right)(t)\right|^{q_1} d_qt\right)^{\frac{r_1 r_2}{q_1}} d_qw\right)^{\frac{1}{r_2}},$$

proving the claim. ■

It follows a $q-$fractional Hilbert-Pachpatte type inequality.

Theorem 38.10. Let for $i = 1, 2$ that $x_i > 0$, $0 < w_i \leq x_i$, $0 < q < 1$; $\alpha > 0$, $p_1, q_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$; $n \in \mathbb{N}$. Set

$$\Delta_i(w_i) = f_i(w_i) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f_i\right)(w_i q^n)}{[k]_q!} w_i^k (q^n;q)_k,$$

$$F(w_1) = \int_{w_1 q^n}^{w_1} \left(q\frac{t_1}{w_1};q\right)_{\alpha-1}^{p_1} d_qt_1, \tag{38.36}$$

and

$$G(w_2) = \int_{w_2 q^n}^{w_2} \left(q \frac{t_2}{w_2}; q \right)_{\alpha-1}^{q_1} d_q t_2.$$

Then

$$\int_0^{x_1} \int_0^{x_2} \frac{|\Delta_1(w_1)| |\Delta_2(w_2)|}{(w_1 w_2)^{\alpha-1} \left(\frac{F(w_1)}{p_1} + \frac{G(w_2)}{q_1} \right)} d_q w_1 d_q w_2 \leq \tag{38.37}$$

$$\frac{x_1^{\frac{1}{p_1}} x_2^{\frac{1}{q_1}}}{(\Gamma_q(\alpha))^2} \left(\int_0^{x_1} \left(\int_{w_1 q^n}^{w_1} \left| {}_* D_{q, w_1 q^n}^{\alpha} f_1 \right|^{q_1} (t_1) \, d_q t_1 \right) d_q w_1 \right)^{\frac{1}{q_1}} \cdot$$

$$\left(\int_0^{x_2} \left(\int_{w_2 q^n}^{w_2} \left| {}_* D_{q, w_2 q^n}^{\alpha} f_2 \right|^{p_1} (t_2) \, d_q t_2 \right) d_q w_2 \right)^{\frac{1}{p_1}}.$$

Proof. We notice by (38.20) that

$$\Delta_i(w_i) = \frac{w_i^{\alpha-1}}{\Gamma_q(\alpha)} \int_{w_i q^n}^{w_i} \left(q \frac{t_i}{w_i}; q \right)_{\alpha-1} \left({}_* D_{q, w_i q^n}^{\alpha} f_i \right)(t_i) \, d_q t_i, \tag{38.38}$$

for $i = 1, 2$.

Therefore we derive

$$|\Delta_1(w_1)| \leq \frac{w_1^{\alpha-1}}{\Gamma_q(\alpha)} \int_{w_1 q^n}^{w_1} \left(q \frac{t_1}{w_1}; q \right)_{\alpha-1} \left| \left({}_* D_{q, w_1 q^n}^{\alpha} f_1 \right)(t_1) \right| d_q t_1 \leq$$

$$\frac{w_1^{\alpha-1}}{\Gamma_q(\alpha)} \left(\int_{w_1 q^n}^{w_1} \left(q \frac{t_1}{w_1}; q \right)_{\alpha-1}^{p_1} d_q t_1 \right)^{\frac{1}{p_1}} \cdot \left(\int_{w_1 q^n}^{w_1} \left| {}_* D_{q, w_1 q^n}^{\alpha} f_1 \right|^{q_1} (t_1) \, d_q t_1 \right)^{\frac{1}{q_1}}. \tag{38.39}$$

Similarly we get

$$|\Delta_2(w_2)| \leq \frac{w_2^{\alpha-1}}{\Gamma_q(\alpha)} \int_{w_2 q^n}^{w_2} \left(q \frac{t_2}{w_2}; q \right)_{\alpha-1} \left| \left({}_* D_{q, w_2 q^n}^{\alpha} f_2 \right)(t_2) \right| d_q t_2 \leq$$

$$\frac{w_2^{\alpha-1}}{\Gamma_q(\alpha)} \left(\int_{w_2 q^n}^{w_2} \left(\left(q \frac{t_2}{w_2}; q \right)_{\alpha-1} \right)^{q_1} d_q t_2 \right)^{\frac{1}{q_1}} \cdot \left(\int_{w_2 q^n}^{w_2} \left| {}_* D_{q, w_2 q^n}^{\alpha} f_2 \right|^{p_1} (t_2) \, d_q t_2 \right)^{\frac{1}{p_1}}. \tag{38.40}$$

Consequently we obtain

$$|\Delta_1(w_1)| |\Delta_2(w_2)| \leq \frac{(w_1 w_2)^{\alpha-1}}{(\Gamma_q(\alpha))^2} (F(w_1))^{\frac{1}{p_1}} (G(w_2))^{\frac{1}{q_1}} \cdot$$

$$\left(\int_{w_1 q^n}^{w_1} \left| {}_* D_{q, w_1 q^n}^{\alpha} f_1 \right|^{q_1} (t_1) \, d_q t_1 \right)^{\frac{1}{q_1}} \cdot \left(\int_{w_2 q^n}^{w_2} \left| {}_* D_{q, w_2 q^n}^{\alpha} f_2 \right|^{p_1} (t_2) \, d_q t_2 \right)^{\frac{1}{p_1}} \tag{38.41}$$

(by Young's inequality)

$$\leq \frac{(w_1 w_2)^{\alpha-1}}{(\Gamma_q(\alpha))^2} \left(\frac{F(w_1)}{p_1} + \frac{G(w_2)}{q_1} \right) \cdot$$

$$\left(\int_{w_1 q^n}^{w_1} \left| {}_* D^{\alpha}_{q, w_1 q^n} f_1 \right|^{q_1} (t_1)\, d_q t_1 \right)^{\frac{1}{q_1}} \cdot \left(\int_{w_2 q^n}^{w_2} \left| {}_* D^{\alpha}_{q, w_2 q^n} f_2 \right|^{p_1} (t_2)\, d_q t_2 \right)^{\frac{1}{p_1}}. \tag{38.42}$$

Therefore

$$\int_0^{x_1} \int_0^{x_2} \frac{|\Delta_1(w_1)|\, |\Delta_2(w_2)|}{(w_1 w_2)^{\alpha-1} \left(\frac{F(w_1)}{p_1} + \frac{G(w_2)}{q_1} \right)} d_q w_1 d_q w_2 \leq$$

$$\frac{1}{(\Gamma_q(\alpha))^2} \left(\int_0^{x_1} \left(\int_{w_1 q^n}^{w_1} \left| {}_* D^{\alpha}_{q, w_1 q^n} f_1 \right|^{q_1} (t_1)\, d_q t_1 \right)^{\frac{1}{q_1}} d_q w_1 \right) \cdot$$

$$\left(\int_0^{x_2} \left(\int_{w_2 q^n}^{w_2} \left| {}_* D^{\alpha}_{q, w_2 q^n} f_2 \right|^{p_1} (t_2)\, d_q t_2 \right)^{\frac{1}{p_1}} d_q w_2 \right) \leq \tag{38.43}$$

$$\frac{x_1^{\frac{1}{p_1}} x_2^{\frac{1}{q_1}}}{(\Gamma_q(\alpha))^2} \left(\int_0^{x_1} \left(\int_{w_1 q^n}^{w_1} \left| {}_* D^{\alpha}_{q, w_1 q^n} f_1 \right|^{q_1} (t_1)\, d_q t_1 \right) d_q w_1 \right)^{\frac{1}{q_1}} \cdot$$

$$\left(\int_0^{x_2} \left(\int_{w_2 q^n}^{w_2} \left| {}_* D^{\alpha}_{q, w_2 q^n} f_2 \right|^{p_1} (t_2)\, d_q t_2 \right) d_q w_2 \right)^{\frac{1}{p_1}}, \tag{38.44}$$

proving the claim. ■

We continue with a generalized $q-$fractional Poincaré type inequality.

Theorem 38.11. Let $x > 0$, $0 < w \leq x$, $0 < q < 1$; $\alpha > \beta > 0$, $p_1, q_1 > 1$: $\frac{1}{p_1} + \frac{1}{q_1} = 1$; $n \in \mathbb{N}$. Put

$$K(w) = \left({}_* D^{\alpha-\beta}_{q, wq^n} f \right)(w) - \sum_{k=\lceil \alpha-\beta \rceil}^{\lceil \alpha \rceil - 1} \frac{\left(D_q^k f \right)(wq^n)}{\Gamma_q(k-\alpha+\beta+1)} w^{k-\alpha+\beta} (q^n; q)_{k-\alpha+\beta}.$$

Then

$$\int_0^x \left(\frac{|K(w)|}{w^{\beta-1}} \right)^{q_1} d_q w \leq \frac{1}{(\Gamma_q(\beta))^{q_1}} \cdot \tag{38.45}$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left(\left(q \frac{t}{w}; q \right)_{\beta-1} \right)^{p_1} d_q t \right)^{q_1} d_q w \right)^{\frac{1}{p_1}} \cdot$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left| {}_* D^{\alpha}_{q, wq^n} f(t) \right|^{q_1} d_q t \right)^{q_1} d_q w \right)^{\frac{1}{q_1}}.$$

Proof. By (38.21) we find

$$K(w) = I^{\beta}_{q,wq^n}\left({}_*D^{\alpha}_{q,wq^n} f\right)(w) = \frac{w^{\beta-1}}{\Gamma_q(\beta)} \int_{wq^n}^{w} \left(q\frac{t}{w};q\right)_{\beta-1} \left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\, d_q t. \tag{38.46}$$

Rest of proof goes as in the proof of Theorem 38.8. ∎

Next comes a generalized $q-$fractional Sobolev's type inequality.

Theorem 38.12. Here all terms and assumptions as in Theorem 38.11. Additionally let $r_1, r_2 > 1 : \frac{1}{r_1} + \frac{1}{r_2} = 1$. Then

$$\left(\int_0^x \left(\frac{|K(w)|}{w^{\beta-1}}\right)^{r_1} d_q w\right)^{\frac{1}{r_1}} \leq \frac{1}{\Gamma_q(\beta)}. \tag{38.47}$$

$$\left(\int_0^x \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\beta-1}\right)^{p_1} d_q t\right)^{\frac{r_1^2}{p_1}} d_q w\right)^{\frac{1}{r_1^2}}.$$

$$\left(\int_0^x \left(\int_{wq^n}^{w} \left|{}_*D^{\alpha}_{q,wq^n} f(t)\right|^{q_1} d_q t\right)^{\frac{r_1 r_2}{q_1}} d_q w\right)^{\frac{1}{r_1 r_2}}.$$

Proof. As in the Theorem 38.9, using (38.46). ∎

We finish with a generalized $q-$fractional Hilbert-Pachpatte type inequality.

Theorem 38.13. Let for $i = 1, 2$ that $x_i > 0$, $0 < w_i \leq x_i$, $0 < q < 1$; $\alpha > \beta > 0$, $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$; $n \in \mathbb{N}$. Put

$$K_i(w_i) = \left({}_*D^{\alpha-\beta}_{q,w_i q^n} f_i\right)(w_i) - \sum_{k=\lceil\alpha-\beta\rceil}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f_i\right)(w_i q^n)}{\Gamma_q(k-\alpha+\beta+1)} w_i^{k-\alpha+\beta} (q^n;q)_{k-\alpha+\beta},$$

$$F^*(w_1) = \int_{w_1 q^n}^{w_1} \left(q\frac{t_1}{w_1};q\right)_{\beta-1}^{p_1} d_q t_1, \tag{38.48}$$

$$G^*(w_2) = \int_{w_2 q^n}^{w_2} \left(q\frac{t_2}{w_2};q\right)_{\beta-1}^{q_1} d_q t_2.$$

Then

$$\int_0^{x_1}\int_0^{x_2} \frac{|K_1(w_1)|\,|K_2(w_2)|}{(w_1 w_2)^{\beta-1}\left(\frac{F^*(w_1)}{p_1} + \frac{G^*(w_2)}{q_1}\right)} d_q w_1 d_q w_2 \leq \frac{x_1^{\frac{1}{p_1}} x_2^{\frac{1}{q_1}}}{(\Gamma_q(\beta))^2}. \tag{38.49}$$

$$\left(\int_0^{x_1} \left(\int_{w_1 q^n}^{w_1} \left|{}_*D^{\alpha}_{q,w_1 q^n} f_1\right|^{q_1}(t_1)\, d_q t_1\right) d_q w_1\right)^{\frac{1}{q_1}}.$$

$$\left(\int_0^{x_2} \left(\int_{w_2 q^n}^{w_2} \left| {}_* D_{q, w_2 q^n}^{\alpha} f_2 \right|^{p_1} (t_2)\, d_q t_2 \right) d_q w_2 \right)^{\frac{1}{p_1}} .$$

Proof. Similar to the proof of Theorem 38.10, using (38.21). ■

39

Inequalities on Time Scales

Here first we collect and develop necessary background on time scales required for this chapter. Then we give time scales integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. We present also the generalized analogs of all these inequalities involving high order delta derivatives of functions on time scales. We finish with many applications: all these inequalities on the specific time scales $\mathbb{R}$, $\mathbb{Z}$ and $q^{\overline{\mathbb{Z}}}$, $q > 1$. This chapter relies on [57].

39.1 Background

Here mainly we use [119]. We are also motivated by [117], [118].

Definition 39.1. A time scale is an arbitrary nonempty closed subset of the real numbers, e.g. $\mathbb{R}$, $\mathbb{Z}$, $q^{\mathbb{N}_0} = \{q^k | k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ q > 1\}$.

Definition 39.2. If $\mathbb{T}$ is a time scale, then we define the forward jump operator $\sigma : \mathbb{T} \longmapsto \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}$, $\forall t \in \mathbb{T}$; the backward jump operator $\rho : \mathbb{T} \longmapsto \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} | s < t\}$, $\forall t \in \mathbb{T}$; and the graininess function $\mu : \mathbb{T} \to \mathbb{R}_+ = [0, \infty)$, by $\mu(t) = \sigma(t) - t$, $\forall t \in \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \to \mathbb{R}$, we define $f^\sigma(t) = f(\sigma(t))$, $\forall t \in \mathbb{T}$; and $f^\rho(t) = f(\rho(t))$, $\forall t \in \mathbb{T}$.

In this definition we use $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if t is the maximum of $\mathbb{T}$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if t is the minimum of $\mathbb{T}$).

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 627–648.
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We call $t \in \mathbb{T}$ right-scattered if $t < \sigma(t)$, $t \in \mathbb{T}$ right-dense if $t = \sigma(t)$, $t \in \mathbb{T}$ left-scattered if $\rho(t) < t$, $t \in \mathbb{T}$ left-dense if $\rho(t) = t$, $t \in \mathbb{T}$ isolated if $\rho(t) < t < \sigma(t)$, $t \in \mathbb{T}$ dense if $\rho(t) = t = \sigma(t)$.

We notice that ρ is an increasing function, so is $\rho^2(t) = \rho(\rho(t)), ...,$ so that $\rho^n(t) = \rho\left(\rho^{n-1}(t)\right)$ is increasing in t for $n \in \mathbb{N}$. Since $\mathbb{T}$ is closed subset of $\mathbb{R}$ we have that $\sigma(t), \rho(t) \in \mathbb{T}$, for $t \in \mathbb{T}$.

Definition 39.3. ([119]) A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous (denoted by C_{rd}) if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits are finite at left-dense points of $\mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is rd-continuous iff f is continuous. Also, if $\mathbb{T} = \mathbb{Z}$, then any function defined on $\mathbb{Z}$ is rd-continuous ([186]).

Definition 39.4. ([119]) If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered, we let $\mathbb{T}^k := \mathbb{T} - \{\sup \mathbb{T}\}$, otherwise we let $\mathbb{T}^k := \mathbb{T}$ the time scale.

In summary, $\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$

Definition 39.5. ([119]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|, \quad \forall\, s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger [187]) derivative of f at t. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta} = f'$, whereas if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, the usual forward difference operator.

Theorem 39.6. ([119]) (Existence of Antiderivatives) Let f be rd-continuous. Then f has an antiderivative F satisfying $F^{\Delta} = f$.

Definition 39.7. ([119]) If f is rd-continuous and $t_0 \in \mathbb{T}$, then we define the integral

$$F(t) = \int_{t_0}^{t} f(\tau)\, \Delta\tau \quad \text{for } t \in \mathbb{T}.$$

Therefore for $f \in C_{rd}(\mathbb{T})$ we have by definition

$$\int_a^b f(\tau)\, \Delta\tau = F(b) - F(a),$$

where $F^{\Delta} = f$.

If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\, \Delta t = \int_a^b f(t)\, dt,$$

where the integral on the right hand side is the Riemann integral ([186]).

If every point in $\mathbb{T}$ is isolated and $a < b$ are in $\mathbb{T}$, then ([186])

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{\rho(b)} f(t)\,\mu(t).$$

Theorem 39.8. ([119]) Let f, g be rd-continuous on $\mathbb{T}$, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then
(1) $\int_a^b (\alpha f(t) + \beta g(t))\,\Delta t = \alpha \int_a^b f(t)\,\Delta t + \beta \int_a^b g(t)\,\Delta t$,
(2) $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$,
(3) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$,
(4) $\int_a^b f(t)\,g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\,g(\sigma(t))\,\Delta t$,
(5) $\int_a^a f(t)\,\Delta t = 0$,
(6) $\int_a^b 1\Delta t = b - a$.

Theorem 39.9. ([4], Hölder's inequality) Let $a, b \in \mathbb{T}$, $a \leq b$, and $f, g : \mathbb{T} \to \mathbb{R}$ be rd-continuous. Then

$$\int_a^b |f(t)|\,|g(t)|\,\Delta t \leq \left(\int_a^b |f(t)|^p\,\Delta t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,\Delta t\right)^{\frac{1}{q}},$$

where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Theorem 39.10. ([119]) Let $f, g \in C_{rd}(\mathbb{T})$, $a, b \in \mathbb{T}$, $a \leq b$. Then
1) if $|f(t)| \leq g(t)$ on $[a, b) \cap \mathbb{T}$, then $\left|\int_a^b f(t)\,\Delta t\right| \leq \int_a^b g(t)\,\Delta t$,
2) if $f(t) \geq 0$, for all $a \leq t < b$ and $t \in \mathbb{T}$, then $\int_a^b f(t)\,\Delta t \geq 0$.

Corollary 39.11. Let $f \in C_{rd}(\mathbb{T})$; $a, b, c \in \mathbb{T}$, with $c \in [a, b]$; $f(t) \geq 0$, $\forall$ $t \in [a, b]$. Then

$$\int_a^c f(t)\,\Delta t \leq \int_a^b f(t)\,\Delta t.$$

Definition 39.12. ([119]) For a function $f : \mathbb{T} \to \mathbb{R}$ we consider the second derivative $f^{\Delta\Delta}$ provided f^{Δ} is differentiable on $\mathbb{T}^{k^2} = \left(\mathbb{T}^k\right)^k$ with derivative $f^{\Delta\Delta} = \left(f^{\Delta}\right)^{\Delta} : \mathbb{T}^{k^2} \to \mathbb{R}$. Similarly we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{k^n} \to \mathbb{R}$.

Similarly we define $\sigma^2(t) = \sigma(\sigma(t)), ..., \sigma^n(t) = \sigma\left(\sigma^{n-1}(t)\right)$, $n \in \mathbb{N}$. For convenience we put $\rho^0(t) = \sigma^0(t) = t$, $f^{\Delta^0} = f$, $\mathbb{T}^{k^0} = \mathbb{T}$.

Notice $\mathbb{T}^{k^n} \subset \mathbb{T}^{k^l}$, $l \in \{0, 1, ..., n\}$.

Theorem 39.13. ([2], Taylor's formula) Let f be n-times differentiable on $\mathbb{T}^{k^n}$, $t \in \mathbb{T}$, and $\alpha \in \mathbb{T}^{k^{n-1}}$; $h_0(r, s) = 1$, $h_{k+1}(r, s) = \int_s^r h_k(\tau, s)\,\Delta\tau$, $k \in \mathbb{N}_0$. Then

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

Corollary 39.14. ([2]) Let f be n-times differentiable on $\mathbb{T}^{k^n}$ and $m \in \mathbb{N}$ with $m < n$. Then, $\forall\ \alpha \in \mathbb{T}^{k^{n-1-m}}$ and $t \in \mathbb{T}^{k^m}$, we have

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} h_k(t,\alpha) f^{\Delta^{k+m}}(\alpha) + \int_{\alpha}^{\rho^{n-m-1}(t)} h_{n-m-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

Denote by $C_{rd}^n(\mathbb{T})$ the space of all functions $f \in C_{rd}(\mathbb{T})$ such that $f^{\Delta^i} \in C_{rd}(\mathbb{T})$ for $i = 1, ..., n \in \mathbb{N}$. In this last case $\mathbb{T}^k = \mathbb{T}$.

We need

Theorem 39.15. ([186], [115], Taylor's formula) Assume $\mathbb{T}^k = \mathbb{T}$ and $f \in C_{rd}^n(\mathbb{T})$, $n \in \mathbb{N}$ and $s,t \in \mathbb{T}$. Here $h_0(t,s) = 1$, $\forall\ s,t \in \mathbb{T}$; $k \in \mathbb{N}_0$, and

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau, \qquad \forall\ s,t \in \mathbb{T}.$$

(then $h_k^{\Delta}(t,s) = h_{k-1}(t,s)$, for $k \in \mathbb{N}$, $\forall\ t \in \mathbb{T}$, for each $s \in \mathbb{T}$ fixed). Then

$$f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(s) h_k(t,s) + \int_s^t h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

Remark 39.16. (to Theorem 39.15) By [186], we have $h_1(t,s) = t - s$, $\forall\ s,t \in \mathbb{T}$.

So if $t \geq s$ then $h_1(t,s) \geq 0$, $h_2(t,s) \geq 0, ..., h_{n-1}(t,s) \geq 0$. However for n odd number $h_{n-1}(t,\sigma(\tau)) \geq 0$ for all $s \leq \tau \leq t$ (see proof of Theorem 39.24).

Also it holds ([2])

$$h_k(t,s) \leq \frac{(t-s)^k}{k!}, \quad \forall\ t \geq s,\ k \in \mathbb{N}_0.$$

Corollary 39.17. (to Theorem 39.15) Suppose $f \in C_{rd}^n(\mathbb{T})$ and $s,t \in \mathbb{T}$. Let $m \in \mathbb{N}$ with $m < n$ Then

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} f^{\Delta^{k+m}}(s) h_k(t,s) + \int_s^t h_{n-m-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

Proof. Use Theorem 39.15 with n and f replaced by $n - m$ and f^{Δ^m}, respectively. ∎

Corollary 39.18. Let $f \in C_{rd}(\mathbb{T})$; $a, b \in \mathbb{T}$, such that $f(t) > 0, \forall\, t \in [a,b] \cap \mathbb{T}$, then $\int_a^b f(t)\,\Delta t > 0$.

Proof. Since $f(t) > 0, \forall\, t \in [a,b] \cap \mathbb{T}$ by Theorem 39.10 (39.2) we get $\int_a^b f(t)\,\Delta t \geq 0$. Assume that $\int_a^b f(t)\,\Delta t = 0$. Then $F(t) = \int_a^t f(t)\,\Delta t = 0$, $\forall\ t \in [a,b] \cap \mathbb{T}$. Thus by ([119]) we get $F^{\Delta}(t) = f(t) = 0$, $\forall\ t \in [a,b] \cap \mathbb{T}$, a contradiction. ∎

We need

Lemma 39.19. Let the time scale $\mathbb{T}$ be such that $\mathbb{T}^k = \mathbb{T}$. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, such that $h_0(t,s) \equiv 1$, $\forall\ s,t \in \mathbb{T}$, and $h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau$, $\forall$ $s,t \in \mathbb{T}$, for all $k \in \mathbb{N}_0$.

Then $h_k(t,s)$ is continuous in $s \in \mathbb{T}$, $k \in \mathbb{N}_0$, for each fixed $t \in \mathbb{T}$; and continuous in $t \in \mathbb{T}$ for each fixed $s \in \mathbb{T}$. Also it holds that $h_k(t,\sigma(s))$ is rd-continuous in $s \in \mathbb{T}$ for each fixed $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$.

Proof. Consider also $g_k : \mathbb{T}^2 \to \mathbb{R}$, $k = 0,1,...,n$, such that $g_0(t,s) \equiv 1$, $\forall$ $s,t \in \mathbb{T}$; and $g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\,\Delta\tau$, $\forall\ s,t \in \mathbb{T}$, for $k \in \mathbb{N}_0$.

By [119], we have that

$$h_k^{\Delta}(t,s) = h_{k-1}(t,s)\,, \quad k \in \mathbb{N}, \forall\ t \in \mathbb{T},$$

for each fixed $s \in \mathbb{T}$.

Also we have

$$g_k^{\Delta}(t,s) = g_{k-1}(\sigma(t),s)\,, \quad k \in \mathbb{N},\ \forall\ t \in \mathbb{T},$$

for each fixed $s \in \mathbb{T}$.

Clearly $g_1(t,s) = h_1(t,s) = t - s$, $\forall\ s,t \in \mathbb{T}$.

By Theorem 1.112 ([119]) we get that

$$h_k(t,s) = (-1)^k g_k(s,t)\,, \quad \forall\ t,s \in \mathbb{T}, \text{for all} \quad k \in \mathbb{N}_0.$$

By Theorem 1.16(i) of [119], we have that since g_k is differentiable for any $t \in \mathbb{T}$ (the first variable), then it is continuous for any $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$. Thus, by the last equation just above, we obtain that $h_k(t,s)$ is continuous in $s \in \mathbb{T}$; and of course h_k is also continuous in $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$.

By Theorem 1.60(iii) of [119], we have that the jump operator σ is rd-continuous, and by the same Theorem 1.60(v) ([119]), we get that $h_k(t,\sigma(s))$ is rd-continuous, for all $k \in \mathbb{N}_0$.

The lemma now is established. ∎

39.2 Main Results

In this chapter we assume $\mathbb{T}^k = \mathbb{T}$.We give first a time scales Poincaré type inequality.

Theorem 39.20. Let $f \in C_{rd}^{n}(\mathbb{T})$, n is an odd number, $a, b \in \mathbb{T}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{\Delta^k}(a) = 0$, $k = 0, 1, ..., n-1$. Here σ is continuous and $h_{n-1}(t, s)$ jointly continuous. Then

$$\int_a^b |f(t)|^q \Delta t \leq \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{q}{p}} \Delta t \right) \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right). \tag{39.1}$$

Proof. Since $f^{\Delta^k}(a) = 0$, $k = 0, 1, ..., n-1$, by Theorem 39.15 we get

$$f(t) = \int_a^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau,$$

$\forall\, t \in [a, b] \cap \mathbb{T}$, where $a, b \in \mathbb{T}$.

Hence

$$|f(t)| \leq \int_a^t h_{n-1}(t, \sigma(\tau)) \left| f^{\Delta^n}(\tau) \right| \Delta\tau$$

$$\leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^t \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}}.$$

Therefore

$$|f(t)|^q \leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{q}{p}} \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right), \tag{39.2}$$

for all $a \leq t \leq b$. Next by integrating (39.2) we are proving the claim. ■

Next we present a time scales Sobolev type inequality.

Theorem 39.21. Here all terms and assumptions are as in Theorem 39.20. Let $r \geq 1$. Denote $\|f\|_r = \left(\int_a^b |f(t)|^r \Delta t \right)^{\frac{1}{r}}$. Then

$$\|f\|_r \leq \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{r}{p}} \Delta t \right)^{\frac{1}{r}} \left\| f^{\Delta^n} \right\|_q. \tag{39.3}$$

Proof. As in the proof of Theorem 39.20 we have $(a \leq t \leq b)$

$$|f(t)| \leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}}.$$

Thus

$$|f(t)|^r \leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{r}{p}} \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{r}{q}},$$

and

$$\int_a^b |f(t)|^r \Delta t \leq \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{r}{p}} \Delta t \right) \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{r}{q}}. \tag{39.4}$$

Next raise both sides of (39.4) to power $\frac{1}{r}$. Thus proving the claim. ■

We give a time scales Opial type inequality.

Theorem 39.22. Let $f \in C_{rd}^n(\mathbb{T})$, n is an odd number, $a, b \in \mathbb{T}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose $f^{\Delta^k}(a) = 0$, $k = 0, 1, ..., n-1$, and that $\left| f^{\Delta^n} \right|$ is increasing on $[a, b] \cap \mathbb{T}$. Here σ is continuous and $h_{n-1}(t, s)$ jointly continuous. Then

$$\int_a^b |f(t)| \left| f^{\Delta^n}(t) \right| \Delta t \leq (b-a)^{\frac{1}{q}} \cdot$$

$$\left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t) \right)^{2q} \Delta t \right)^{\frac{1}{q}}. \tag{39.5}$$

Proof. It holds

$$f(t) = \int_a^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau,$$

$\forall\, t \in [a, b] \cap \mathbb{T}$, where $a, b \in \mathbb{T}$.

Hence

$$|f(t)| \leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^t \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left| f^{\Delta^n}(t) \right| (t-a)^{\frac{1}{q}}.$$

Therefore

$$|f(t)| \left| f^{\Delta^n}(t) \right| \leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(f^{\Delta^n}(t) \right)^2 (t-a)^{\frac{1}{q}},$$

for all $a \leq t \leq b$.

Consequently we obtain

$$\int_a^b |f(t)| \left| f^{\Delta^n}(t) \right| \Delta t \leq \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(f^{\Delta^n}(t) \right)^2 (t-a)^{\frac{1}{q}} \right) \Delta t$$

$$\leq \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t) \right)^{2q} (t-a) \Delta t \right)^{\frac{1}{q}}$$

$$\leq (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t) \right)^{2q} \Delta t \right)^{\frac{1}{q}},$$

proving the claim. ■

We make

Remark 39.23. (to Theorem 39.20-39.22 and their proofs) As we know ([119]), we have that

$$h_{n-1}^{\Delta}(t,\sigma(\tau)) = h_{n-2}(t,\sigma(\tau)), \quad \forall\, t \in [a,b] \cap \mathbb{T}.$$

Also $(h_{n-1}(t,\sigma(t)))^p$ is continuous at (t,t), $t > a$; $p > 1$.

By Chain Rule, Theorem 1.90 [119], we get that $(h_{n-1}(t,\sigma(\tau))^p)^{\Delta}$ exists in $t \in \mathbb{T}$, where τ is fixed in $\mathbb{T}$; $p > 1$, and

$$\begin{aligned}((h_{n-1}(t,\sigma(\tau)))^p)^{\Delta} &= p\left\{ \int_0^1 (h_{n-1}(t,\sigma(\tau)) + \right. \\ &\left. h\mu(t)h_{n-2}(t,\sigma(\tau)))^{p-1}\, dh \right\} h_{n-2}(t,\sigma(\tau)).\end{aligned}$$

Here by assumption σ is continuous and $h_{n-1}(t,s)$ is jointly continuous. So that $(h_{n-1}(t,\sigma(\tau)))^p$ is jointly continuous in (t,τ), that is rd-continuous in t and τ; $p \geq 1$. Here $\mathbb{T}^k = \mathbb{T}$, and by Lemma 39.19 we get that $h_{n-2}(t,\sigma(\tau))$ is continuous in t and τ. By bounded convergence theorem, using the last formula above, we get that $((h_{n-1}(t,\sigma(\tau)))^p)^{\Delta}$ is continuous in t and τ; $p > 1$, and thus rd-continuous in t and τ.

Consider now the function

$$u(t) = \int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau, \quad \forall\, t \in [a,b] \cap \mathbb{T}.$$

Clearly $u(a) = 0$. Furthermore, by Theorem 1.117 of [119], we derive

$$u^{\Delta}(t) = \int_a^t (h_{n-1}(t,\sigma(\tau))^p)^{\Delta} \Delta\tau + (h_{n-1}(\sigma(t),\sigma(t)))^p$$

$$= \int_a^t (h_{n-1}(t,\sigma(\tau))^p)^{\Delta} \Delta\tau.$$

That is $u(t)$ is differentiable, hence continuous and therefore rd-continuous on $[a,b] \cap \mathbb{T}$.

We proceed with a time scales Ostrowski type inequality.

Theorem 39.24. Let $f \in C_{rd}^n(\mathbb{T})$, n is odd, $a,b,c \in \mathbb{T} : a \leq c \leq b$. Suppose that $f^{\Delta^k}(c) = 0$, $k = 1,...,n-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\Delta t - f(c) \right| \leq \frac{[h_{n+1}(a,c) + h_{n+1}(b,c)]}{b-a} \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}}. \quad (39.6)$$

Proof. By assumptions and Theorem 39.15, we get

$$f(t) - f(c) = \int_c^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau, \quad \forall\, t \in [a,b] \cap \mathbb{T}.$$

Hence

$$E(x) := \frac{1}{b-a}\int_a^b f(t)\Delta t - f(c) =$$

$$\frac{1}{b-a}\int_a^b f(t)\Delta t - \frac{1}{b-a}\int_a^b f(c)\Delta t = \frac{1}{b-a}\int_a^b (f(t) - f(c))\Delta t.$$

Thus

$$|E(x)| \le \frac{1}{b-a}\int_a^b |f(t) - f(c)|\Delta t.$$

However we observe that $(c \le t \le b)$

$$|f(t) - f(c)| \le \int_c^t h_{n-1}(t, \sigma(\tau)) \left|f^{\Delta^n}(\tau)\right| \Delta\tau$$

$$\le \left(\int_c^t h_{n-1}(t, \sigma(\tau))\Delta\tau\right) \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Also when $a \le t \le c$, we obtain

$$|f(t) - f(c)| = \left|\int_t^c h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau\right| \le$$

$$\int_t^c |h_{n-1}(t, \sigma(\tau))| \left|f^{\Delta^n}(\tau)\right| \Delta\tau \le \left(\int_t^c |h_{n-1}(t, \sigma(\tau))|\Delta\tau\right) \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Since $h_1(t,s) = t - s$, if $t \le s$ then $h_1(t,s) \le 0$. Then $h_2(t,s) = \int_s^t h_1(\tau, s)\Delta\tau = -\int_t^s h_1(\tau,s)\Delta\tau = \int_t^s (-h_1(\tau,s))\Delta\tau \ge 0$. That is $h_2(t,s) \ge 0$, for any $t, s \in \mathbb{T}$. We continue with $(t \le s)$ $h_3(t,s) = \int_s^t h_2(\tau,s)\Delta\tau = -\int_t^s h_2(\tau,s)\Delta\tau \le 0$. Consequently by induction, we obtain $(t \le s)$

$$|h_k(t,s)| = (-1)^k h_k(t,s), \quad k \in \mathbb{N}_0.$$

Thus $h_k(t,s) \ge 0$, for any $t, s \in \mathbb{T}$, when k is even.

Therefore when $a \le t \le c$, we derive

$$|f(t) - f(c)| \le \left(\int_t^c h_{n-1}(t, \sigma(\tau))\Delta\tau\right) \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

By (1.7), (1.8), (1.9) of [119] and Theorem 1.112 of [119], we notice that $(c \le t \le b)$

$$\int_c^t h_{n-1}(t, \sigma(\tau))\Delta\tau = \int_c^t g_{n-1}(\sigma(\tau), t)\Delta\tau$$

$$= (-1)^n \int_t^c g_{n-1}(\sigma(\tau), t)\, \Delta\tau = (-1)^n g_n(c,t) = h_n(t,c).$$

Also it holds $(a \le t \le c)$

$$(-1)^{n-1} \int_t^c h_{n-1}(t, \sigma(\tau))\, \Delta\tau = \int_t^c g_{n-1}(\sigma(\tau), t)\, \Delta\tau$$

$$= g_n(c,t) = (-1)^n h_n(t,c).$$

So we found that $(c \le t \le b)$

$$|f(t) - f(c)| \le h_n(t,c) \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}},$$

and $(a \le t \le c)$

$$|f(t) - f(c)| \le (-1)^n h_n(t,c) \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Thus we have

$$|E(x)| \le \frac{1}{b-a} \left[\int_a^c |f(t) - f(c)|\, \Delta t + \int_c^b |f(t) - f(c)|\, \Delta t \right] \le$$

$$\frac{1}{b-a} \left[(-1)^n \int_a^c h_n(t,c)\, \Delta t + \int_c^b h_n(t,c)\, \Delta t \right] \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} \le$$

$$\frac{\left[\int_c^a h_n(t,c)\, \Delta t + h_{n+1}(b,c) \right]}{b-a} \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} =$$

$$\frac{\left[h_{n+1}(a,c) + h_{n+1}(b,c) \right]}{b-a} \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}},$$

proving the claim. ∎

It follows a time scales Hilbert-Pachpatte type inequality.

Theorem 39.25. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C_{rd}^n(\mathbb{T}_i)$, n is odd, with $f_i^{\Delta^k}(a_i) = 0$, $k = 0, 1, ..., n-1$; $a_i \le b_i$; $a_i, b_i \in \mathbb{T}_i$, time scale. Let also $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Call

$$F(t_1) = \int_{a_1}^{t_1} h_{n-1}^{(1)}(t_1, \sigma_1(\tau_1))^p\, \Delta\tau_1,$$

for all $t_1 \in [a_1, b_1] \cap \mathbb{T}_1$, and

$$G(t_2) = \int_{a_2}^{t_2} h_{n-1}^{(2)}(t_2, \sigma_2(\tau_2))^q\, \Delta\tau_2,$$

for all $t_2 \in [a_2, b_2] \cap \mathbb{T}_2$ (where $h_{n-1}^{(i)}$, $\sigma^{(i)}$ the corresponding h_{n-1}, σ to $\mathbb{T}_i$,, $i = 1, 2$). Here σ_i is continuous and $h_{n-1}^{(i)}(t_i, s_i)$ jointly continuous in $t_i, s_i \in \mathbb{T}_i$.

We further suppose that

$$\lambda(t_1) = \int_{a_2}^{b_2} \frac{|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \Delta\tau_2$$

is an rd-continuous function on $\mathbb{T}_1$.

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)| \, |f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \Delta t_1 \Delta t_2 \leq (b_1 - a_1)(b_2 - a_2) \cdot$$

$$\left(\int_{a_1}^{b_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}} \tag{39.7}$$

(above double time scales integration is considered in the natural iterative way).

Proof. Since $f_i^{\Delta^k}(a_i) = 0$, $k = 0, 1, ..., n-1$; $i = 1, 2$, by Theorem 39.15 we get

$$f_i(t_i) = \int_{a_i}^{t_i} h_{n-1}^{(i)}(t_i, \sigma_i(\tau_i)) f_i^{\Delta^n}(\tau_i) \Delta\tau_i,$$

$\forall\ t_i \in [a_i, b_i] \cap \mathbb{T}_i$, where $a_i, b_i \in \mathbb{T}_i$.

Hence

$$|f_1(t_1)| \leq \left(\int_{a_1}^{t_1} h_{n-1}^{(1)}(t_1, \sigma_1(\tau_1))^p \Delta\tau_1\right)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}}$$

$$= F(t_1)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}},$$

and

$$|f_2(t_2)| \leq \left(\int_{a_2}^{t_2} h_{n-1}^{(2)}(t_2, \sigma_2(\tau_2))^q \Delta\tau_2\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}}$$

$$= G(t_2)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}}.$$

Young's inequality for $a, b \geq 0$ says that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Consequently we have

$$|f_1(t_1)||f_2(t_2)| \leq$$

$$F(t_1)^{\frac{1}{p}} G(t_2)^{\frac{1}{q}} \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}} \leq$$

$$\left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right) \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}}.$$

The last gives ($\varepsilon > 0$)

$$\frac{|f_1(t_1)||f_2(t_2)|}{\varepsilon + \left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \leq \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}},$$

for all $t_i \in [a_i, b_i] \cap \mathbb{T}_i$, $i = 1, 2$.

Next we see that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)||f_2(t_2)|}{\varepsilon + \left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \Delta t_1 \Delta t_2 \leq$$

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \Delta t_1\right) \left(\int_{a_2}^{b_2} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}} \Delta t_2\right) \leq$$

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right) \Delta t_1\right)^{\frac{1}{q}} (b_1 - a_1)^{\frac{1}{p}} \cdot$$

$$\left(\int_{a_2}^{b_2} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right) \Delta t_2\right)^{\frac{1}{p}} (b_2 - a_2)^{\frac{1}{q}} \leq$$

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{b_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right) \Delta t_1\right)^{\frac{1}{q}} (b_1 - a_1)^{\frac{1}{p}} \cdot$$

$$\left(\int_{a_2}^{b_2} \left(\int_{a_2}^{b_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right) \Delta t_2\right)^{\frac{1}{p}} (b_2 - a_2)^{\frac{1}{q}} =$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}},$$

proving the claim. ■

Based on Corollary 39.17 we get the following results:

First a generalized time scales Poincaré type inequality.

Proposition 39.26. Let $f \in C_{rd}^n(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b \in \mathbb{T}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{\Delta^{k+m}}(a) = 0$,

$k = 0, 1, ..., n - m - 1$. Here σ is continuous and $h_{n-m-1}(t, s)$ jointly continuous. Then

$$\int_a^b \left| f^{\Delta^m}(t) \right|^q \Delta t \leq$$

$$\left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{q}{p}} \Delta t \right) \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right). \tag{39.8}$$

Proof. As in Theorem 39.20. ■

It follows a generalized time scales Sobolev type inequality.

Proposition 39.27. Here all terms and assumptions are as in Proposition 39.26. Let $r \geq 1$. Then

$$\left\| f^{\Delta^m} \right\|_r \leq \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{r}{p}} \Delta t \right)^{\frac{1}{r}} \left\| f^{\Delta^n} \right\|_q. \tag{39.9}$$

Proof. As in Theorem 39.21. ■

Next comes a generalized time scales Opial type inequality.

Proposition 39.28. Let $f \in C_{rd}^n(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b \in \mathbb{T}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{\Delta^{k+m}}(a) = 0$, $k = 0, 1, ..., n - 1$, and that $\left| f^{\Delta^n} \right|$ is increasing on $[a, b] \cap \mathbb{T}$. Here σ is continuous and $h_{n-m-1}(t, s)$ jointly continuous. Then

$$\int_a^b \left| f^{\Delta^m}(t) \right| \left| f^{\Delta^n}(t) \right| \Delta t \leq (b - a)^{\frac{1}{q}} \cdot$$

$$\left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t) \right)^{2q} \Delta t \right)^{\frac{1}{q}}. \tag{39.10}$$

Proof. As in Theorem 39.22. ■

We continue with a generalized Ostrowski type inequality over time scales.

Proposition 39.29. Let $f \in C_{rd}^n(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b, c \in \mathbb{T} : a \leq c \leq b$. Assume that $f^{\Delta^{k+m}}(c) = 0$, $k = 1, ..., n - m - 1$. Then

$$\left| \frac{1}{b - a} \int_a^b f^{\Delta^m} \Delta t - f^{\Delta^m}(c) \right| \leq \frac{\left[h_{n-m+1}(a, c) + h_{n-m+1}(b, c) \right]}{b - a} \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap \mathbb{T}}. \tag{39.11}$$

Proof. As in Theorem 39.24. ■

We finish with the generalized Hilbert-Pachpatte type inequality on time scales.

Proposition 39.30. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C_{rd}^n(\mathbb{T}_i)$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, with $f_i^{\Delta^{k+m}}(a_i) = 0$, $k = 0, 1, ..., n - m - 1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{T}_i$, time scale. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Call

$$F^*(t_1) = \int_{a_1}^{t_1} h_{n-m-1}^{(1)}(t_1, \sigma_1(\tau_1))^p \Delta\tau_1,$$

for all $t_1 \in [a_1, b_1] \cap \mathbb{T}_1$, and

$$G^*(t_2) = \int_{a_2}^{t_2} h_{n-m-1}^{(2)}(t_2, \sigma_2(\tau_2))^q \Delta\tau_2,$$

for all $t_2 \in [a_2, b_2] \cap \mathbb{T}_2$ (where $h_{n-m-1}^{(i)}$, $\sigma^{(i)}$ the corresponding h_{n-m-1}, σ to $\mathbb{T}_i$, $i = 1, 2$). Here σ_i is continuous and $h_{n-m-1}^{(i)}(t_i, s_i)$ jointly continuous in $t_i, s_i \in \mathbb{T}_i$.

We further suppose that

$$\lambda^*(t_1) = \int_{a_2}^{b_2} \frac{\left|f_2^{\Delta^m}(t_2)\right|}{\left(\varepsilon + \frac{F^*(t_1)}{p} + \frac{G^*(t_2)}{q}\right)} \Delta\tau_2$$

is an rd-continuous function on $\mathbb{T}_1$.

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left|f_1^{\Delta^m}(t_1)\right| \left|f_2^{\Delta^m}(t_2)\right|}{\left(\varepsilon + \frac{F^*(t_1)}{p} + \frac{G^*(t_2)}{q}\right)} \Delta t_1 \Delta t_2 \leq (b_1 - a_1)(b_2 - a_2) \cdot$$

$$\left(\int_{a_1}^{b_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}}. \qquad (39.12)$$

Proof. As in Theorem 39.25. ■

39.3 Applications

We need

Remark 39.31. ([119])

i) When $\mathbb{T} = \mathbb{R}$, then $h_k(t, s) = \frac{(t-s)^k}{k!}$, $\forall\ k \in \mathbb{N}_0$, $\forall\ t, s \in \mathbb{R}$, $\sigma(t) = t$, $\int_a^b f(t) \Delta t = \int_a^b f(t) dt$, $f^\Delta(t) = f'(t)$, $f^{\Delta^k} = f^{(k)}$; rd-continuous corresponds to f continuous.

ii) When $\mathbb{T} = \mathbb{Z}$, $h_k(t,s) = \frac{(t-s)^{(k)}}{k!}$, $\forall\ k \in \mathbb{N}_0$, $\forall\ t,s \in \mathbb{Z}$, where $t^{(0)} = 1$, $t^{(k)} = \prod_{i=0}^{k-1}(t-i)$ for $k \in \mathbb{N}$, $\sigma(t) = t+1$,

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t), \quad a < b,$$

$$f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t),$$

$$f^{\Delta^k}(t) = \Delta^k f(t) = \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} (-1)^{k-l} f(t+l),$$

rd-continuous f corresponds to any f.

A Poincaré inequality comes:

Corollary 39.32. Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a,b \in \mathbb{R}$; $a \le b$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{(k)}(a) = 0$, $k = 0,1,...,n-1$. Then

$$\int_a^b |f(t)|^q\,dt \le \frac{(b-a)^{nq}}{((n-1)!)^q\,(p(n-1)+1)^{(q-1)}\,nq} \left(\int_a^b \left|f^{(n)}(t)\right|^q dt\right). \quad (39.13)$$

Proof. Based on Theorem 39.20 and Remark 39.31 (i). ∎

A discrete Poincaré follows:

Corollary 39.33. Let $f : \mathbb{Z} \to \mathbb{R}$, n is odd, $a,b \in \mathbb{Z}$; $a \le b$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\Delta^k f(a) = 0$, $k = 0,1,...,n-1$. Then

$$\sum_{t=a}^{b-1} |f(t)|^q \le \frac{1}{((n-1)!)^q} \left(\sum_{t=a}^{b-1}\left(\sum_{\tau=a}^{t-1}\left((t-\tau-1)^{(n-1)}\right)^p\right)^{\frac{q}{p}}\right)\left(\sum_{\tau=a}^{b-1}|\Delta^n f(\tau)|^q\right). \quad (39.14)$$

Proof. Based on Theorem 39.20 and Remark 39.31 (ii). ∎

A Sobolev inequality comes:

Corollary 39.34. All as in Corollary 39.32. Let $r \ge 1$. Then

$$\left(\int_a^b |f(t)|^r\,dt\right)^{\frac{1}{r}} \le$$

$$\frac{(b-a)^{\left(n-1+\frac{1}{p}+\frac{1}{r}\right)}}{(n-1)!\,((n-1)p+1)^{\frac{1}{p}}\left(\left(n-1+\frac{1}{p}\right)r+1\right)^{\frac{1}{r}}}\left(\int_a^b \left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}. \quad (39.15)$$

Proof. Based on Theorem 39.21 and Remark 39.31 (i). ∎

A discrete Sobolev inequality follows:

Corollary 39.35. All as in Corollary 39.33 and let $r \geq 1$. Then

$$\left(\sum_{t=a}^{b-1} |f(t)|^r\right)^{\frac{1}{r}} \leq$$

$$\frac{1}{(n-1)!}\left(\sum_{t=a}^{b-1}\left(\sum_{\tau=a}^{t-1}\left((t-\tau-1)^{(n-1)}\right)^p\right)^{\frac{r}{p}}\right)^{\frac{1}{r}}\left(\sum_{t=a}^{b-1}|\Delta^n f(t)|^q\right)^{\frac{1}{q}}. \quad (39.16)$$

Proof. Base on Theorem 39.21 and Remark 39.31 (ii). ∎

An Opial inequality comes next:

Corollary 39.36. Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$, and $\left|f^{(n)}\right|$ is increasing on $[a, b]$. Then

$$\int_a^b |f(t)|\left|f^{(n)}(t)\right| dt \leq \frac{(b-a)^{n+\frac{1}{p}}}{(n-1)!\left[((n-1)p+1)((n-1)p+2)\right]^{\frac{1}{p}}}\cdot$$

$$\left(\int_a^b \left(f^{(n)}(t)\right)^{2q} dt\right)^{\frac{1}{q}}. \quad (39.17)$$

Proof. Based on Theorem 39.22 and Remark 39.31 (i). ∎

A discrete Opial inequality follows:

Corollary 39.37. Let $f : \mathbb{Z} \to \mathbb{R}$, n is odd, $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\Delta^k f(a) = 0$, $k = 0, 1, ..., n-1$, and that $|\Delta^n f|$ is increasing on $[a, b]$. Then

$$\sum_{t=a}^{b-1} |f(t)|\,|\Delta^n f(t)| \leq \frac{(b-a)^{\frac{1}{q}}}{(n-1)!}\cdot$$

$$\left(\sum_{t=a}^{b-1}\left(\sum_{\tau=a}^{t-1}\left((t-\tau-1)^{(n-1)}\right)^p\right)^{\frac{1}{p}}\right)\left(\sum_{t=a}^{b-1}(\Delta^n f(t))^{2q}\right)^{\frac{1}{q}}. \quad (39.18)$$

Proof. By Theorem 39.22 and Remark 39.31 (ii). ∎

An Ostrowski inequality comes next:

Corollary 39.38. Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a, b, c \in \mathbb{R} : a \leq c \leq b$. Suppose that $f^{(k)}(c) = 0$, $k = 1, ..., n-1$. Then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(c)\right| \leq \left[\frac{\left((c-a)^{n+1} + (b-c)^{n+1}\right)}{(n+1)!\,(b-a)}\right]\left\|f^{(n)}\right\|_{\infty,[a,b]}. \quad (39.19)$$

Proof. Based on Theorem 39.24 and Remark 39.31 (i). ■

A discrete Ostrowski inequality follows:

Corollary 39.39. Let $f : \mathbb{Z} \to \mathbb{R}$, n is odd, $a, b, c \in \mathbb{Z} : a \leq c \leq b$. Assume that $\Delta^k f(c) = 0$, $k = 1, ..., n-1$. Then

$$\left| \frac{1}{b-a} \sum_{t=a}^{b-1} f(t) - f(c) \right| \leq \left[\frac{(a-c)^{(n+1)} + (b-c)^{(n+1)}}{(n+1)!\,(b-a)} \right] \|\Delta^n f\|_{\infty,[a,b]} . \quad (39.20)$$

Proof. By Theorem 39.24 and Remark 39.31 (ii). ■

A Hilbert-Pachpatte inequality follows:

Corollary 39.40. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, with $f_i^{(k)}(a_i) = 0$, $k = 0, 1, ..., n-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{R}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Call

$$F(t_1) = \frac{1}{((n-1)!)^p} \frac{(t_1 - a_1)^{p(n-1)+1}}{(p(n-1)+1)}, \quad \forall\, t_1 \in [a_1, b_1],$$

$$G(t_2) = \frac{1}{((n-1)!)^q} \frac{(t_2 - a_2)^{q(n-1)+1}}{(q(n-1)+1)}, \quad \forall\, t_2 \in [a_2, b_2].$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} dt_1 dt_2 \leq (b_1 - a_1)(b_2 - a_2) \cdot$$

$$\left(\int_{a_1}^{b_1} \left| f_1^{(n)}(\tau_1) \right|^q d\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left| f_2^{(n)}(\tau_2) \right|^p d\tau_2 \right)^{\frac{1}{p}} . \quad (39.21)$$

Proof. Based on Theorem 39.25 and Remark 39.31 (i). Notice here that $\lambda(t_1)$ is a continuous function on $[a_1, b_1]$ by bounded convergence theorem. ■

It follows a discrete Hilbert-Pachpatte inequality.

Corollary 39.41. Let $\varepsilon > 0$, $i = 1, 2$; $f_i : \mathbb{Z} \to \mathbb{R}$, n is odd, with $\Delta^k f_i(a_i) = 0$, $k = 0, 1, ..., n-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{Z}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Put

$$\overline{F}(t_1) = \frac{\sum_{\tau_1 = a_1}^{t_1 - 1} \left((t_1 - \tau_1 - 1)^{(n-1)} \right)^p}{((n-1)!)^p}, \quad \forall\, t_1 \in [a_1, b_1] \cap \mathbb{Z},$$

and

$$\overline{G}(t_2) = \frac{\sum_{\tau_2 = a_2}^{t_2 - 1} \left((t_2 - \tau_2 - 1)^{(n-1)} \right)^q}{((n-1)!)^q}, \quad \forall\, t_2 \in [a_2, b_2] \cap \mathbb{Z}.$$

Then

$$\sum_{t_1=a_1}^{b_1-1}\sum_{t_2=a_2}^{b_2-1}\frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon+\frac{\overline{F}(t_1)}{p}+\frac{\overline{G}(t_2)}{q}\right)}\le(b_1-a_1)(b_2-a_2)\cdot$$

$$\left(\sum_{\tau_1=a_1}^{b_1-1}|\Delta^n f_1(\tau_1)|^q\right)^{\frac{1}{q}}\left(\sum_{\tau_2=a_2}^{b_2-1}|\Delta^n f_2(\tau_2)|^p\right)^{\frac{1}{p}}. \tag{39.22}$$

Proof. By Theorem 39.25 and Remark 39.31 (ii). ■

Another generalized Poincaré inequality comes:

Corollary 39.42. Let $f\in C^n(\mathbb{R})$, $m,n\in\mathbb{N}$, $m<n$, $a,b\in\mathbb{R}$; $a\le b$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Assume $f^{(k+m)}(a)=0$, $k=0,1,...,n-m-1$. Then

$$\int_a^b\left|f^{(m)}(t)\right|^q dt\le$$

$$\frac{(b-a)^{(n-m)q}}{((n-m-1)!)^q\,(p(n-m-1)+1)^{(q-1)}\,(n-m)\,q}\left(\int_a^b\left|f^{(n)}(t)\right|^q dt\right). \tag{39.23}$$

Proof. By Corollary 39.32, $n\mapsto n-m$, $f\mapsto f^{(m)}$, $f^{(k)}\mapsto f^{(k+m)}$ into (39.13). ■

A generalized discrete Poincaré inequality follows:

Corollary 39.43. Let $f:\mathbb{Z}\to\mathbb{R}$, $m,n\in\mathbb{N}$, $m<n$, $n-m$ is odd, $a,b\in\mathbb{Z}$; $a\le b$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Assume $\Delta^{k+m}f(a)=0$, $k=0,1,...,n-m-1$. Then

$$\sum_{t=a}^{b-1}|\Delta^m f(t)|^q\le$$

$$\frac{1}{((n-m-1)!)^q}\left(\sum_{t=a}^{b-1}\left(\sum_{\tau=a}^{t-1}\left((t-\tau-1)^{(n-m-1)}\right)^p\right)^{\frac{q}{p}}\right)\left(\sum_{\tau=a}^{b-1}|\Delta^n f(\tau)|^q\right). \tag{39.24}$$

Proof. By Corollary 39.33. ■

A generalized Sobolev inequality comes.

Corollary 39.44. All as in Corollary 39.42, $r\ge1$. Then

$$\left(\int_a^b\left|f^{(m)}(t)\right|^r dt\right)^{\frac{1}{r}}\le$$

$$\frac{(b-a)^{\left(n-m-1+\frac{1}{p}+\frac{1}{r}\right)}}{(n-m-1)!\,((n-m-1)p+1)^{\frac{1}{p}}\left(\left(n-m-1+\frac{1}{p}\right)r+1\right)^{\frac{1}{r}}}\left(\int_a^b\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}. \tag{39.25}$$

Proof. By Corollary 39.34. ∎

A generalized discrete Sobolev inequality comes next:

Corollary 39.45. All as in Corollary 39.43, $r \geq 1$. Then

$$\left(\sum_{t=a}^{b-1} |\Delta^m f(t)|^r \right)^{\frac{1}{r}} \leq$$

$$\frac{1}{(n-m-1)!} \left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-m-1)} \right)^p \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \left(\sum_{t=a}^{b-1} |\Delta^n f(t)|^q \right)^{\frac{1}{q}}. \tag{39.26}$$

Proof. By Corollary 39.35. ∎

A generalized Opial inequality follows:

Corollary 39.46. Let $f \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose $f^{(k+m)}(a) = 0$, $k = 0, 1, ..., n-m-1$, and $\left| f^{(n)} \right|$ is increasing on $[a, b]$. Then

$$\int_a^b \left| f^{(m)}(t) \right| \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{(b-a)^{n-m+\frac{1}{p}}}{(n-m-1)! \left[((n-m-1)p+1)((n-m-1)p+2) \right]^{\frac{1}{p}}} \left(\int_a^b \left(f^{(n)}(t) \right)^{2q} dt \right)^{\frac{1}{q}}. \tag{39.27}$$

Proof. By Corollary 39.36. ∎

A generalized discrete Opial inequality follows:

Corollary 39.47. Let $f : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\Delta^{k+m} f(a) = 0$, $k = 0, 1, ..., n-m-1$, and that $|\Delta^n f|$ is increasing on $[a, b]$. Then

$$\sum_{t=a}^{b-1} |\Delta^m f(t)| \, |\Delta^n f(t)| \leq \frac{(b-a)^{\frac{1}{q}}}{(n-m-1)!} \cdot$$

$$\left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-m-1)} \right)^p \right)^{\frac{1}{p}} \right) \left(\sum_{t=a}^{b-1} (\Delta^n f(t))^{2q} \right)^{\frac{1}{q}}. \tag{39.28}$$

Proof. By Corollary 39.37. ∎

A generalized Ostrowski inequality follows:

Corollary 39.48. Let $f \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, $a, b, c \in \mathbb{R} : a \leq c \leq b$. Assume that $f^{(k+m)}(c) = 0$, $k = 1, ..., n-m-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f^{(m)}(t)\, dt - f^{(m)}(c) \right| \leq$$

$$\left[\frac{\left((c-a)^{n-m+1} + (b-c)^{n-m+1} \right)}{(n-m+1)!\,(b-a)} \right] \left\| f^{(n)} \right\|_{\infty,[a,b]}. \tag{39.29}$$

Proof. By Corollary 39.38. ■

A generalized discrete Ostrowski inequality comes next:

Corollary 39.49. Let $f : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b, c \in \mathbb{Z} :$ $a \leq c \leq b$. Assume that $\Delta^{k+m} f(c) = 0$, $k = 1, ..., n-m-1$. Then

$$\left| \frac{1}{b-a} \sum_{t=a}^{b-1} \Delta^m f(t) - \Delta^m f(c) \right| \leq$$

$$\left[\frac{(a-c)^{(n-m+1)} + (b-c)^{(n-m+1)}}{(n-m+1)!\,(b-a)} \right] \|\Delta^n f\|_{\infty,[a,b]}. \tag{39.30}$$

Proof. By Corollary 39.39. ■

A generalized Hilbert-Pachpatte comes:

Corollary 39.50. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, with $f_i^{(k+m)}(a_i) = 0$, $k = 0, 1, ..., n-m-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{R}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Call

$$F^*(t_1) = \frac{1}{((n-m-1)!)^p} \frac{(t_1 - a_1)^{p(n-m-1)+1}}{(p(n-m-1)+1)}, \quad \forall\, t_1 \in [a_1, b_1],$$

$$G^*(t_2) = \frac{1}{((n-m-1)!)^q} \frac{(t_2 - a_2)^{q(n-m-1)+1}}{(q(n-m-1)+1)}, \quad \forall\, t_2 \in [a_2, b_2].$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left| f_1^{(m)}(t_1) \right| \left| f_2^{(m)}(t_2) \right|}{\left(\varepsilon + \frac{F^*(t_1)}{p} + \frac{G^*(t_2)}{q} \right)} dt_1 dt_2 \leq (b_1 - a_1)(b_2 - a_2) \cdot$$

$$\left(\int_{a_1}^{b_1} \left| f_1^{(n)}(\tau_1) \right|^q d\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left| f_2^{(n)}(\tau_2) \right|^p d\tau_2 \right)^{\frac{1}{p}}. \tag{39.31}$$

Proof. By Corollary 39.40. ■

It follows a generalized discrete Hilbert-Pachpatte inequality.

Corollary 39.51. Let $\varepsilon > 0$, $i = 1, 2$; $f_i : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, with $\Delta^{k+m} f_i (a_i) = 0$, $k = 0, 1, ..., n - m - 1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{Z}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Set

$$\overline{F}^* (t_1) = \frac{\sum_{\tau_1 = a_1}^{t_1 - 1} \left((t_1 - \tau_1 - 1)^{(n-m-1)} \right)^p}{((n - m - 1)!)^p}, \quad \forall\ t_1 \in [a_1, b_1] \cap \mathbb{Z},$$

and

$$\overline{G}^* (t_2) = \frac{\sum_{\tau_2 = a_2}^{t_2 - 1} \left((t_2 - \tau_2 - 1)^{(n-m-1)} \right)^q}{((n - m - 1)!)^q}, \quad \forall\ t_2 \in [a_2, b_2] \cap \mathbb{Z}.$$

Then

$$\sum_{t_1 = a_1}^{b_1 - 1} \sum_{t_2 = a_2}^{b_2 - 1} \frac{|\Delta^m f_1 (t_1)| \, |\Delta^m f_2 (t_2)|}{\left(\varepsilon + \frac{\overline{F}^* (t_1)}{p} + \frac{\overline{G}^* (t_2)}{q} \right)} \leq (b_1 - a_1)(b_2 - a_2) \cdot$$

$$\left(\sum_{\tau_1 = a_1}^{b_1 - 1} |\Delta^n f_1 (\tau_1)|^q \right)^{\frac{1}{q}} \left(\sum_{\tau_2 = a_2}^{b_2 - 1} |\Delta^n f_2 (\tau_2)|^p \right)^{\frac{1}{p}}. \tag{39.32}$$

Proof. By Corollary 39.41. ∎

Remark 39.52. ([2], [119]) Consider $q > 1$, $q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\}$, and the time scale $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$, which very important in q-difference equations.

It holds [2], [119] that

$$h_k (t, s) = \prod_{\nu = 0}^{k-1} \frac{t - q^\nu s}{\sum_{\mu = 0}^{\nu} q^\mu}, \quad \forall\ s, t \in \mathbb{T};$$

$$\sigma (t) = qt, \quad \rho (t) = \frac{t}{q}, \quad \forall\ t \in \mathbb{T},$$

$$f^{\Delta} (t) = \frac{f (qt) - f (t)}{(q - 1) t}, \ \ \forall\ t \in \mathbb{T} - \{0\},$$

$$f^{\Delta} (0) = \lim_{s \to 0} \frac{f (s) - f (0)}{s}.$$

We present a related q-Ostrowski type inequality.

Corollary 39.53. Let $f \in C_{rd}^n \left(q^{\overline{\mathbb{Z}}} \right)$, n is odd, $a, b, c \in q^{\overline{\mathbb{Z}}} : a \leq c \leq b$. Suppose that $f^{\Delta^k} (c) = 0$, $k = 1, ..., n - 1$. Then

$$\left| \frac{1}{b - a} \int_a^b f (t) \Delta t - f (c) \right| \leq$$

$$\left[\frac{\prod_{\nu=0}^{n}\frac{a-q^{\nu}c}{\sum_{\mu=0}^{\nu}q^{\mu}}+\prod_{\nu=0}^{n}\frac{b-q^{\nu}c}{\sum_{\mu=0}^{\nu}q^{\mu}}}{b-a}\right]\left\|f^{\Delta^{n}}\right\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}. \quad (39.33)$$

Proof. By Theorem 39.24. ■

We finish with a generalized q-Ostrowski type inequality.

Corollary 39.54. Let $f\in C_{rd}^{n}\left(q^{\overline{\mathbb{Z}}}\right)$, $m,n\in\mathbb{N}$, $m<n$, $n-m$ is odd, $a,b,c\in q^{\overline{\mathbb{Z}}}: a\leq c\leq b$. Assume that $f^{\Delta^{k+m}}(c)=0$, $k=1,...,n-m-1$. Then

$$\left|\frac{1}{b-a}\int_{a}^{b}f^{\Delta^{m}}(t)\,\Delta t-f^{\Delta^{m}}(c)\right|\leq$$

$$\left[\frac{\prod_{\nu=0}^{n-m}\frac{a-q^{\nu}c}{\sum_{\mu=0}^{\nu}q^{\mu}}+\prod_{\nu=0}^{n-m}\frac{b-q^{\nu}c}{\sum_{\mu=0}^{\nu}q^{\mu}}}{b-a}\right]\left\|f^{\Delta^{n}}\right\|_{\infty,[a,b]\cap q^{\overline{\mathbb{Z}}}}. \quad (39.34)$$

Proof. By Corollary 39.53. ■

One can give many similar applications for other time scales.

40

Nabla Inequalities on Time Scales

Here first we collect and develop necessary background on nabla time scales required for this chapter. Then we give nabla time scales integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. We present also the generalized analogs of all these nabla inequalities involving high order nabla derivatives of functions on time scales. We finish with many applications: all these nabla inequalities on the specific time scales $\mathbb{R}$, $\mathbb{Z}$ and $q^{\overline{\mathbb{Z}}}$, $q > 1$. In most of these nabla inequalities the nabla differentiability order is any $n \in \mathbb{N}$, as opposed to delta time scales approach where n is always odd. This chapter relies on [59].

40.1 Preliminaries

Here we use [94], [103], [119], [223]. Let $\mathbb{T}$ be a time scale (a closed subset of $\mathbb{R}$) ([187]), $[a, b]$ be the closed and bounded interval in $\mathbb{T}$, i.e. $[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}$ and $a, b \in \mathbb{T}$.

Clearly, a time scale $\mathbb{T}$ may or may not be connected. Therefore we have the concept of *forward* and *backward jump operators* as follows. Define $\sigma, \rho : \mathbb{T} \longmapsto \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

$(\inf \emptyset := \sup \mathbb{T}, \sup \emptyset := \inf \mathbb{T})$.

If $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, then $t \in \mathbb{T}$ is called *right-dense, right-scattered, left-dense, left-scattered*, respectively. The set $\mathbb{T}_k$ which is

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 649–672.
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derived from $\mathbb{T}$ is as follows: if $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. We also define the *backwards graininess function* $\nu : \mathbb{T} \longmapsto [0, \infty)$ as $\nu(t) = t - \rho(t)$. If $f : \mathbb{T} \longmapsto \mathbb{R}$ is a function, we define the function $f^\rho : \mathbb{T}_k \longmapsto \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}_k$ and $\sigma^0(t) = \rho^0(t) = t$; $\mathbb{T}_{k^{n+1}} := (\mathbb{T}_{k^n})_k$.

Definition 40.1. If $f : \mathbb{T} \longmapsto \mathbb{R}$ is a function and $t \in \mathbb{T}_k$, then we define the nabla derivative of f at a point t to be the number $f^\nabla(t)$ (provided it exists) with the property that, for each $\varepsilon > 0$, there is a neighborhood of U of t such that

$$\left|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]\right| \leq \varepsilon |\rho(t) - s|,$$

for all $s \in U$.

Note that in the case $\mathbb{T} = \mathbb{R}$, then $f^\nabla(t) = f'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\nabla(t) = \nabla f(t) = f(t) - f(t-1)$.

Definition 40.2. A function $F : \mathbb{T} \to \mathbb{R}$ we call a nabla-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_k$. We then define the Cauchy ∇-integral from a to t of f by

$$\int_a^t f(s)\nabla s = F(t) - F(a), \quad \text{for all } t \in \mathbb{T}.$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\int_a^b f(t)\nabla t = \int_a^b f(t)\,dt,$$

and in the case $\mathbb{T} = \mathbb{Z}$ we have

$$\int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k),$$

where $a, b \in \mathbb{T}$ with $a \leq b$.

Definition 40.3. A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous (or ld-continuous) provided that it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points of $\mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous iff f is continuous. If $\mathbb{T} = \mathbb{Z}$, then any function is ld-continuous.

Theorem 40.4. Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$, and $t \in \mathbb{T}_k$. The following holds:

1. If f is nabla differentiable at t, then f is continuous at t.

2. If f is continuous at t and t is left-scaterred, then f is nabla differentiable at t and

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{t - \rho(t)}.$$

3. If t is left-dense, then f is nabla differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4. If f is nabla differentiable at t, then $f(\rho(t)) = f(t) - \nu(t) f^{\nabla}(t)$.

For any time scale $\mathbb{T}$, when f is a constant, then $f^{\nabla} = 0$; if $f(t) = kt$ for some constant k, then $f^{\nabla} = k$.

Theorem 40.5. Suppose $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_k$. Then,

1. the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and $(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t)$;

2. for any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and $(\alpha f)^{\nabla}(t) = \alpha f^{\nabla}(t)$;

3. the product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and

$$(fg)^{\nabla}(t) = f^{\nabla}(t) g(t) + f^{\rho}(t) g^{\nabla}(t) = f^{\nabla}(t) g^{\rho}(t) + f(t) g^{\nabla}(t).$$

Some results concerning ld-continuity are useful:

Theorem 40.6. Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$.
1. If f is continuous, then f is ld-continuous.
2. The backward jump operator ρ is ld-continuous.
3. If f is ld-continuous, then f^{ρ} is also ld-continuous.
4. If $\mathbb{T} = \mathbb{R}$, then f is continuous if and only if f is ld-continuous.
5. If $\mathbb{T} = \mathbb{Z}$, then f is ld-continuous.

Theorem 40.7. Every ld-continuous function has a nabla antiderivative. In particular, if $a \in \mathbb{T}$, then the function F defined by

$$F(t) = \int_a^t f(\tau) \nabla\tau, \quad t \in \mathbb{T},$$

is a nabla antiderivative of f.

The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld}(\mathbb{T},\mathbb{R})$, and the set of all nabla differentiable functions with ld-continuous derivative by $C^1_{ld}(\mathbb{T},\mathbb{R})$.

Theorem 40.8. If $f \in C_{ld}(\mathbb{T},\mathbb{R})$ and $t \in \mathbb{T}_k$, then

$$\int_{\rho(t)}^{t} f(\tau)\,\nabla\tau = \nu(t)\, f(t).$$

Theorem 40.9. If $a, b, c \in \mathbb{T}$, $a \le c \le b$, $\alpha \in \mathbb{R}$, and $f, g \in C_{ld}(\mathbb{T},\mathbb{R})$, then:
1. $\int_a^b (f(t) + g(t))\,\nabla t = \int_a^b f(t)\,\nabla t + \int_a^b g(t)\,\nabla t$;
2. $\int_a^b \alpha f(t)\,\nabla t = \alpha \int_a^b f(t)\,\nabla t$;
3. $\int_a^b f(t)\,\nabla t = -\int_b^a f(t)\,\nabla t$;
4. $\int_a^a f(t)\,\nabla t = 0$;
5. $\int_a^b f(t)\,\nabla t = \int_a^c f(t)\,\nabla t + \int_c^b f(t)\,\nabla t$;
6. If $f(t) > 0$ for all $a < t \le b$, then $\int_a^b f(t)\,\nabla t > 0$;
7. $\int_a^b f^\rho(t)\, g^\nabla(t)\,\nabla t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\nabla(t)\, g(t)\,\nabla t$;
8. $\int_a^b f(t)\, g^\nabla(t)\,\nabla t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\nabla(t)\, g^\rho(t)\,\nabla t$;
9. If $f(t) \ge 0$, $a \le t \le b$, then $\int_a^b f(t)\,\nabla t \ge 0$;
10. If $f(t) \ge 0$, $a \le c \le b$, then $\int_a^b f(t)\,\nabla t \ge \int_a^c f(t)\,\nabla t$;
11. If f and f^∇ are jointly continuous in (t, s), then

$$\left(\int_a^t f(t,s)\,\nabla s\right)^\nabla = f(\rho(t), t) + \int_a^t f^\nabla(t,s)\,\nabla s,$$
$$\left(\int_t^b f(t,s)\,\nabla s\right)^\nabla = -f(\rho(t), t) + \int_t^b f^\nabla(t,s)\,\nabla s;$$

12. If $f(t) \ge g(t)$, then $\int_a^b f(t)\,\nabla t \ge \int_a^b g(t)\,\nabla t$;
13. $\left|\int_a^b f(t)\,\nabla t\right| \le \int_a^b |f(t)|\,\nabla t$;
14. $\int_a^b 1\nabla t = b - a$.

Similarly we define higher order nabla derivatives on $\mathbb{T}_{k^{n+1}}$ by

$$f^{\nabla^{n+1}} := \left(f^{\nabla^n}\right)^\nabla, \quad n \in \mathbb{N}.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\nabla^{n+1}} = f^{(n+1)}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^{\nabla^{n+1}}(t) = \nabla^{n+1} f(t) = \sum_{m=0}^{n+1} (-1)^m \begin{pmatrix} n+1 \\ m \end{pmatrix} f(t-m)$.

Let $\widehat{h}_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, defined recursively as follows:

$$\widehat{h}_0(t,s) = 1, \quad \text{all } s,t \in \mathbb{T},$$

and, given $\widehat{h}_k$ for $k \in \mathbb{N}_0$, the function $\widehat{h}_{k+1}$ is

$$\widehat{h}_{k+1}(t,s) = \int_s^t \widehat{h}_k(\tau,s)\,\nabla\tau, \quad \text{for all } s,t \in \mathbb{T}.$$

Note that $\widehat{h}_k$ are all well defined, since each is ld-continuous in t.

If we let $\widehat{h}_k^{\nabla}(t,s)$ denote for each fixed s the nabla derivative of $\widehat{h}_k(t,s)$ with respect to t, then

$$\widehat{h}_k^{\nabla}(t,s) = \widehat{h}_{k-1}(t,s), \quad \text{for } k \in \mathbb{N},\ t \in \mathbb{T}_k.$$

Observe that $\widehat{h}_1(t,s) = t - s$, for all $s,t \in \mathbb{T}$.

Example 40.10. 1. If $\mathbb{T} = \mathbb{R}$, then $\rho(t) = t$, $t \in \mathbb{R}$, so that $\widehat{h}_k(t,s) = \frac{(t-s)^k}{k!}$ for all $s,t \in \mathbb{R}$, $k \in \mathbb{N}_0$.

2. If $\mathbb{T} = \mathbb{Z}$, then $\rho(t) = t-1$, $t \in \mathbb{Z}$, and $\widehat{h}_k(t,s) = \frac{(t-s)^{\overline{k}}}{k!}$, for all $s,t \in \mathbb{Z}$, $k \in \mathbb{N}_0$, where $t^{\overline{k}} := t(t+1)\ldots(t+k-1)$, $k \in \mathbb{N}$; $t^{\overline{0}} := 1$.

Definition 40.11. The set $C_{ld}^n(\mathbb{T},\mathbb{R})$, $n \in \mathbb{N}$, denotes the set of all n times continuously nabla differentiable functions from $\mathbb{T}$ into $\mathbb{R}$, i.e. all $f, f^{\nabla}, f^{\nabla^2}, \ldots$, $f^{\nabla^n} \in C_{ld}(\mathbb{T},\mathbb{R})$.

This definition requires $\mathbb{T}_k = \mathbb{T}$.

We need

Theorem 40.12. ([93], Nabla Taylor's formula) Suppose f is n times nabla differentiable on $\mathbb{T}_{k^n}$, $n \in \mathbb{N}$. Let $a \in \mathbb{T}_{k^{n-1}}$, $t \in \mathbb{T}$. Then

$$f(t) = \sum_{k=0}^{n-1} \widehat{h}_k(t,a)\, f^{\nabla^k}(a) + \int_a^t \widehat{h}_{n-1}(t,\rho(\tau))\, f^{\nabla^n}(\tau)\,\nabla\tau.$$

If $f \in C_{ld}^n(\mathbb{T},\mathbb{R})$, then nabla Taylor formula is true for all $t,a \in \mathbb{T}$.

Corollary 40.13. (to Theorem 40.12) Suppose $f \in C_{ld}^n(\mathbb{T})$, $n \in \mathbb{N}$, and $s,t \in \mathbb{T}$. Let $m \in \mathbb{N}$ with $m < n$. Then

$$f^{\nabla^m}(t) = \sum_{k=0}^{n-m-1} f^{\nabla^{k+m}}(s)\,\widehat{h}_k(t,s) + \int_s^t \widehat{h}_{n-m-1}(t,\rho(\tau))\, f^{\nabla^n}(\tau)\,\nabla\tau.$$

Proof. Use Theorem 40.12 with n and f replaced by $n-m$ and f^{∇^m}, respectively. ■

Define $[a,b]_k = [a,b]$ if a is right-dense, and $[a,b]_k = [\sigma(a), b]$ if a is right-scattered.

Proposition 40.14. ([223]) Assume $a, b \in \mathbb{T}$, $a < b$, and $f \in C_{ld}([a,b], \mathbb{R})$ is such that $f \geq 0$ on $[a,b]$. If $\int_a^b f(t) \nabla t = 0$, then $f = 0$ on $[a,b]_k$.

Theorem 40.15. (Nabla Hölder's inequality) Let $a, b \in \mathbb{T}$, $a \leq b$. For $f, g \in C_{ld}([a,b])$ we have

$$\int_a^b |f(t)| |g(t)| \nabla t \leq \left(\int_a^b |f(t)|^p \nabla t \right)^{\frac{1}{p}} \cdot \left(\int_a^b |g(t)|^q \nabla t \right)^{\frac{1}{q}},$$

where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.
Proof. For $\alpha, \beta \geq 0$ we have Yang's inequality

$$\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \leq \frac{\alpha}{p} + \frac{\beta}{q}.$$

Assume, without loss of generality, that $\{\int_a^b |f(t)|^p \nabla t\}\{\int_a^b |g(t)|^q \nabla t\} \neq 0$.

Apply Yang's inequality for

$$\begin{aligned} \alpha &= \alpha(t) = \frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \nabla \tau}, \\ \beta &= \beta(t) = \frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \nabla \tau}, \end{aligned}$$

that is for

$$\alpha^{\frac{1}{p}} = (\alpha(t))^{\frac{1}{p}} = \frac{|f(t)|}{\left(\int_a^b |f(\tau)|^p \nabla \tau\right)^{\frac{1}{p}}},$$

and

$$\beta^{\frac{1}{q}} = (\beta(t))^{\frac{1}{q}} = \frac{|g(t)|}{\left(\int_a^b |g(\tau)|^q \nabla \tau\right)^{\frac{1}{q}}},$$

and integrate the resulted inequality from a to b (this is valid since all involved functions are ld-continuous) to obtain

$$\int_a^b \frac{|f(t)|}{\left(\int_a^b |f(\tau)|^p \nabla \tau\right)^{\frac{1}{p}}} \frac{|g(t)|}{\left(\int_a^b |g(\tau)|^q \nabla \tau\right)^{\frac{1}{q}}} \nabla t \leq$$

$$\int_a^b \left[\frac{1}{p} \frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \nabla \tau} + \frac{1}{q} \frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \nabla \tau} \right] \nabla t$$

$$= \frac{1}{p}\int_a^b \left(\frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \nabla\tau} \right) \nabla t + \frac{1}{q}\int_a^b \left(\frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \nabla\tau} \right) \nabla t = \frac{1}{p} + \frac{1}{q} = 1,$$

proving the claim. ■

Next define $\widehat{g}_0(t,s) \equiv 1$,

$$\widehat{g}_{n+1}(t,s) = \int_s^t \widehat{g}_n(\rho(\tau), s) \nabla\tau, \quad n \in \mathbb{N},\ s,t \in \mathbb{T}.$$

Notice that $\widehat{g}_{n+1}^{\nabla}(t,s) = \widehat{g}_n(\rho(t), s)$, $t \in \mathbb{T}_k$; $\widehat{g}_1(t,s) = t - s$, for all $s,t \in \mathbb{T}$.

If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. Similarly define $\mathbb{T}^{k^{n+1}} := \left(\mathbb{T}^{k^n}\right)^k$. Notice $\mathbb{T}_{k^{n+1}} \subset \mathbb{T}_k$ and $\mathbb{T}^{k^{n+1}} \subset \mathbb{T}^k$.

Theorem 40.16. ([93]) Let $t \in \mathbb{T}_k \cap \mathbb{T}^k$, $s \in \mathbb{T}^{k^n}$, and $n \geq 0$. Then

$$\widehat{h}_n(t,s) = (-1)^n \widehat{g}_n(s,t).$$

Remark 40.17. Let the time scale $\mathbb{T}$ be such that $\mathbb{T}^k = \mathbb{T}_k = \mathbb{T}$. Clearly both $\widehat{h}_n$, $\widehat{g}_n$ are nabla differentiable in their first variables, therefore both are continuous in their first variables.

Using now Theorem 40.16 we obtain that also both $\widehat{h}_n$, $\widehat{g}_n$ are continuous in their second variables.

Consequently $\widehat{h}_n(t,s)$ is ld-continuous in each variable and thus $\widehat{h}_n(t,\rho(s))$ is ld-continuous in s.

Notice also in general that if $t \geq s$ then $\widehat{h}_1(t,s) \geq 0$, $\widehat{h}_2(t,s) \geq 0, \ldots,$ $\widehat{h}_{n-1}(t,s) \geq 0$. So that $\widehat{h}_{n-1}(t,\rho(\tau)) \geq 0$ for all $s \leq \tau \leq t$.

Also in general it holds

$$\widehat{h}_k(t,s) \leq (t-s)^k, \ \ \forall\, t \geq s,\ k \in \mathbb{N}_0.$$

We need

Theorem 40.18. ([103]) (Nabla chain rule) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}$. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}$ and the formula

$$(f \circ g)^{\nabla}(t) = \left\{ \int_0^1 f'\left(g(t) + h\nu(t) g^{\nabla}(t)\right) dh \right\} g^{\nabla}(t)$$

holds.

We formulate

Assumption 40.19. Let the time scale $\mathbb{T}$ be such that $\mathbb{T}^k = \mathbb{T}_k = \mathbb{T}$.

Remark 40.20. Suppose that ρ is a continuous function, $\mathbb{T}_k = \mathbb{T}$, $\widehat{h}_{n-1}(t,s)$ and $\widehat{h}_{n-2}(t,s)$ are jointly continuous in $(t,s) \in \mathbb{T}^2$; $p > 1$. Clearly $\widehat{h}_{n-1}^{\nabla}(t,s) = \widehat{h}_{n-2}(t,s)$ in $t \in \mathbb{T}$. Also $\widehat{h}_{n-1}(t,\rho(s))$, $\widehat{h}_{n-2}(t,\rho(s))$ are jointly continuous in $(t,s) \in \mathbb{T}^2$.

By Theorem 40.18 we have that $\left(\left(\widehat{h}_{n-1}(t,\rho(\tau))\right)^p\right)^{\nabla}$ exists in $t \in \mathbb{T}$, where τ is fixed in $\mathbb{T}$, and

$$\left(\left(\widehat{h}_{n-1}(t,\rho(\tau))\right)^p\right)^{\nabla} =$$

$$p\left\{\int_0^1 \left(\widehat{h}_{n-1}(t,\rho(\tau)) + h\nu(t)\,\widehat{h}_{n-2}(t,\rho(\tau))\right)^{p-1} dh\right\} \widehat{h}_{n-2}(t,\rho(\tau)).$$

By bounded convergence theorem we obtain that $\left(\left(\widehat{h}_{n-1}(t,\rho(\tau))\right)^p\right)^{\nabla}$ is jointly continuous in (t,τ), and of course $\left(\widehat{h}_{n-1}(t,\rho(\tau))\right)^p$ is jointly continuous in (t,τ).

Therefore by Theorem 40.9 (40.11), we derive for

$$u(t) = \int_a^t \widehat{h}_{n-1}(t,\rho(\tau))^p \,\nabla\tau$$

$(t \in [a,b] \subset \mathbb{T})$, that

$$u^{\nabla}(t) = \int_a^b \left(\widehat{h}_{n-1}(t,\rho(\tau))^p\right)^{\nabla} \nabla\tau + \left(\widehat{h}_{n-1}(\rho(t),\rho(t))\right)^p.$$

I.e.

$$u^{\nabla}(t) = \int_a^t \left(\widehat{h}_{n-1}(t,\rho(\tau))^p\right)^{\nabla} \nabla\tau.$$

That is $u(t)$ is nabla differentiable, hence continuous and therefore ld-continuous on $[a,b] \subset \mathbb{T}$.

We formulate

Assumption 40.21. We assume that ρ is a continuous function and $\widehat{h}_{n-1}(t,s)$, $\widehat{h}_{n-2}(t,s)$ are jointly continuous in $(t,s) \in \mathbb{T}^2$.

Assumption 40.22. We assume that ρ is a continuous function and $\widehat{h}_{n-m-1}(t,s)$, $\widehat{h}_{n-m-2}(t,s)$ are jointly continuous in $(t,s) \in \mathbb{T}^2$.

40.2 Main Results

In all of the main results we assume Assumption 40.19. We present a Nabla time scales Poincaré type inequality.

Theorem 40.23. Suppose Assumption 40.21. Let $f \in C_{ld}^{n}(\mathbb{T})$, $n \in \mathbb{N}$, $a, b \in \mathbb{T}$; $a \le b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{\nabla^k}(a) = 0$, $k = 0, 1, ..., n-1$. Then

$$\int_a^b |f(t)|^q \nabla t \le \left(\int_a^b \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right)^{\frac{q}{p}} \nabla t \right) \left(\int_a^b \left| f^{\nabla^n}(\tau) \right|^q \nabla \tau \right). \tag{40.1}$$

Proof. Since $f^{\nabla^k}(a) = 0$, $k = 0, 1, ..., n-1$, by Theorem 40.12 we obtain

$$f(t) = \int_a^t \widehat{h}_{n-1}(t, \rho(\tau)) f^{\nabla^n}(\tau) \nabla \tau,$$

$\forall\ t \in [a, b]$, where $a, b \in \mathbb{T}$.

Thus

$$|f(t)| \le \int_a^t \widehat{h}_{n-1}(t, \rho(\tau)) \left| f^{\nabla^n}(\tau) \right| \nabla \tau$$

$$\le \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right)^{\frac{1}{p}} \left(\int_a^t \left| f^{\nabla^n}(\tau) \right|^q \nabla \tau \right)^{\frac{1}{q}}$$

$$\le \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right)^{\frac{1}{p}} \left(\int_a^b \left| f^{\nabla^n}(\tau) \right|^q \nabla \tau \right)^{\frac{1}{q}}.$$

Therefore

$$|f(t)|^q \le \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right)^{\frac{q}{p}} \left(\int_a^b \left| f^{\nabla^n}(\tau) \right|^q \nabla \tau \right). \tag{40.2}$$

for all $a \le t \le b$. Next by integrating (40.2) we are proving the claim. ∎

Next we give a Nabla time scales Sobolev type inequality.

Theorem 40.24. Here all terms and assumptions are in Theorem 40.23. Let $r \ge 1$. Denote

$$\|f\|_r = \left(\int_a^b |f(t)|^r \nabla t \right)^{\frac{1}{r}}.$$

Then

$$\|f\|_r \le \left(\int_a^b \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right)^{\frac{r}{p}} \nabla t \right)^{\frac{1}{r}} \left\| f^{\nabla^n} \right\|_q. \tag{40.3}$$

Proof. As in the proof of Theorem 40.23 we have $(a \le t \le b)$

$$|f(t)| \le \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right)^{\frac{1}{p}} \left(\int_a^b \left| f^{\nabla^n}(\tau) \right|^q \nabla \tau \right)^{\frac{1}{q}}.$$

Hence

$$|f(t)|^r \leq \left(\int_a^t \widehat{h}_{n-1}(t,\rho(\tau))^p \nabla\tau\right)^{\frac{r}{p}} \left(\int_a^b \left|f^{\nabla^n}(\tau)\right|^q \nabla\tau\right)^{\frac{r}{q}},$$

and

$$\int_a^b |f(t)|^r \nabla t \leq \left(\int_a^b \left(\int_a^t \widehat{h}_{n-1}(t,\rho(\tau))^p \nabla\tau\right)^{\frac{r}{p}} \nabla t\right) \left(\int_a^b \left|f^{\nabla^n}(\tau)\right|^q \nabla\tau\right)^{\frac{r}{q}}. \tag{40.4}$$

Next raise both sides of (40.4) to power $\frac{1}{r}$. Thus establishing the claim. ■

We present a Nabla time scales Opial type inequality.

Theorem 40.25. Suppose Assumption 40.21. Let $f \in C_{ld}^n(\mathbb{T})$, $n \in \mathbb{N}$, $a, b \in \mathbb{T}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{\nabla^k}(a) = 0$, $k = 0, 1, ..., n-1$, and that $\left|f^{\nabla^n}\right|$ is increasing on $[a, b]$.

Then

$$\int_a^b |f(t)| \left|f^{\nabla^n}(t)\right| \nabla t \leq$$

$$(b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t \widehat{h}_{n-1}(t,\rho(\tau))^p \nabla\tau\right) \nabla t\right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\nabla^n}(t)\right)^{2q} \nabla t\right)^{\frac{1}{q}}. \tag{40.5}$$

Proof. It holds

$$f(t) = \int_a^t \widehat{h}_{n-1}(t,\rho(\tau)) f^{\nabla^n}(\tau) \nabla\tau,$$

$\forall\, t \in [a, b]$, where $a, b \in \mathbb{T}$.

Hence

$$|f(t)| \leq \left(\int_a^t \widehat{h}_{n-1}(t,\rho(\tau))^p \nabla\tau\right)^{\frac{1}{p}} \left(\int_a^t \left|f^{\nabla^n}(\tau)\right|^q \nabla\tau\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t \widehat{h}_{n-1}(t,\rho(\tau))^p \nabla\tau\right)^{\frac{1}{p}} \left|f^{\nabla^n}(t)\right| (t-a)^{\frac{1}{q}}.$$

Therefore

$$|f(t)| \left|f^{\nabla^n}(t)\right| \leq \left(\int_a^t \widehat{h}_{n-1}(t,\rho(\tau))^p \nabla\tau\right)^{\frac{1}{p}} \left(f^{\nabla^n}(t)\right)^2 (t-a)^{\frac{1}{q}},$$

for all $a \leq t \leq b$.

Consequently we find

$$\int_a^b |f(t)| \left| f^{\nabla^n}(t) \right| \nabla t \leq \int_a^b \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right)^{\frac{1}{p}} \left(f^{\nabla^n}(t) \right)^2 (t-a)^{\frac{1}{q}} \nabla t$$

$$\leq \left(\int_a^b \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right) \nabla t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\nabla^n}(t) \right)^{2q} (t-a) \nabla t \right)^{\frac{1}{q}}$$

$$\leq (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t \widehat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right) \nabla t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\nabla^n}(t) \right)^{2q} \nabla t \right)^{\frac{1}{q}},$$

proving the claim. ■

We proceed with a Nabla time scales Ostrowski type inequality.

Theorem 40.26. Let $f \in C_{ld}^n(\mathbb{T})$, n is an odd number, $a, b, c \in \mathbb{T} : a \leq c \leq b$. Assume that $f^{\nabla^k}(c) = 0$, $k = 1, ..., n-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t) \nabla t - f(c) \right| \leq \frac{\left[\widehat{h}_{n+1}(a, c) + \widehat{h}_{n+1}(b, c) \right]}{b-a} \left\| f^{\nabla^n} \right\|_{\infty, [a,b]}. \quad (40.6)$$

Proof. By assumptions and Theorem 40.12, we get

$$f(t) - f(c) = \int_c^t \widehat{h}_{n-1}(t, \rho(\tau)) f^{\nabla^n}(\tau) \nabla \tau, \quad \forall\ t \in [a, b].$$

Hence

$$E(x) := \frac{1}{b-a} \int_a^b f(t) \nabla t - f(c) =$$

$$\frac{1}{b-a} \int_a^b f(t) \nabla t - \frac{1}{b-a} \int_a^b f(c) \nabla t = \frac{1}{b-a} \int_a^b (f(t) - f(c)) \nabla t.$$

Thus

$$|E(x)| \leq \frac{1}{b-a} \int_a^b |f(t) - f(c)| \nabla t.$$

However we see that $(c \leq t \leq b)$

$$|f(t) - f(c)| \leq \int_c^t \widehat{h}_{n-1}(t, \rho(\tau)) \left| f^{\nabla^n}(\tau) \right| \nabla \tau$$

$$\leq \left(\int_c^t \widehat{h}_{n-1}(t, \rho(\tau)) \nabla \tau \right) \left\| f^{\nabla^n} \right\|_{\infty, [a,b]}.$$

Also when $a \leq t \leq c$, we have

$$|f(t) - f(c)| = \left| \int_t^c \widehat{h}_{n-1}(t, \rho(\tau)) f^{\nabla^n}(\tau) \nabla \tau \right| \leq$$

$$\int_t^c \left|\widehat{h}_{n-1}(t,\rho(\tau))\right| \left|f^{\nabla^n}(\tau)\right| \nabla\tau \le \left(\int_t^c \left|\widehat{h}_{n-1}(t,\rho(\tau))\right| \nabla\tau\right) \left\|f^{\nabla^n}\right\|_{\infty,[a,b]}.$$

Since $\widehat{h}_1(t,s) = t - s$, if $t \le s$ then $\widehat{h}_1(t,s) \le 0$. Then $\widehat{h}_2(t,s) = \int_s^t \widehat{h}_1(\tau,s)\nabla\tau$ $= -\int_t^s \widehat{h}_1(\tau,s)\nabla\tau = \int_t^s \left(-\widehat{h}_1(\tau,s)\right)\nabla\tau \ge 0.$

That is $\widehat{h}_2(t,s) \ge 0$, for any $t,s \in \mathbb{T}$.

We continue with $(t \le s)$

$$\widehat{h}_3(t,s) = \int_s^t \widehat{h}_2(\tau,s)\nabla\tau = -\int_t^s \widehat{h}_2(\tau,s)\nabla\tau \le 0.$$

Consequently by induction, we obtain $(t \le s)$

$$\left|\widehat{h}_k(t,s)\right| = (-1)^k \widehat{h}_k(t,s), \quad k \in \mathbb{N}_0.$$

Thus $\widehat{h}_k(t,s) \ge 0$, for any $t,s \in \mathbb{T}$, when k is even.

Therefore when $a \le t \le c$, we derive

$$|f(t) - f(c)| \le \left(\int_t^c \widehat{h}_{n-1}(t,\rho(\tau))\nabla\tau\right) \left\|f^{\nabla^n}\right\|_{\infty,[a,b]}.$$

By Theorem 40.16 we notice that $(c \le t \le b)$

$$\int_c^t \widehat{h}_{n-1}(t,\rho(\tau))\nabla\tau = \int_c^t \widehat{g}_{n-1}(\rho(\tau),t)\nabla\tau =$$

$$-\int_t^c \widehat{g}_{n-1}(\rho(\tau),t)\nabla\tau = -\widehat{g}_n(c,t) = (-1)^n \widehat{g}_n(c,t) = \widehat{h}_n(t,c).$$

Also it holds $(a \le t \le c)$

$$\int_t^c \widehat{h}_{n-1}(t,\rho(\tau))\nabla\tau = \int_t^c \widehat{g}_{n-1}(\rho(\tau),t)\nabla\tau$$

$$= \widehat{g}_n(c,t) = (-1)^n \widehat{h}_n(t,c).$$

So we found that $(c \le t \le b)$

$$|f(t) - f(c)| \le \widehat{h}_n(t,c) \left\|f^{\nabla^n}\right\|_{\infty,[a,b]},$$

and $(a \le t \le c)$

$$|f(t) - f(c)| \le (-1)^n \widehat{h}_n(t,c) \left\|f^{\nabla^n}\right\|_{\infty,[a,b]}.$$

Thus we have

$$|E(x)| \le \frac{1}{b-a}\left[\int_a^c |f(t) - f(c)|\nabla t + \int_c^b |f(t) - f(c)|\nabla t\right] \le$$

$$\frac{1}{b-a}\left[(-1)^n \int_a^c \widehat{h}_n(t,c)\nabla t + \int_c^b \widehat{h}_n(t,c)\nabla t\right] \left\| f^{\nabla^n} \right\|_{\infty,[a,b]} =$$

$$\frac{\left[\int_c^a \widehat{h}_n(t,c)\nabla t + \widehat{h}_{n+1}(b,c)\right]}{b-a} \left\| f^{\nabla^n} \right\|_{\infty,[a,b]} =$$

$$\frac{\left[\widehat{h}_{n+1}(a,c) + \widehat{h}_{n+1}(b,c)\right]}{b-a} \left\| f^{\nabla^n} \right\|_{\infty,[a,b]},$$

proving the claim. ■

It follows a time scales Nabla Hilbert-Pachpatte type inequality.

Theorem 40.27. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C_{ld}^n(\mathbb{T}_i)$, $n \in \mathbb{N}$, with $f_i^{\nabla^k}(a_i) = 0$, $k = 0, 1, ..., n-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{T}_i$, time scale. Let also $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Put

$$F(t_1) = \int_{a_1}^{t_1} \widehat{h}_{n-1}^{(1)}(t_1, \rho_1(\tau_1))^p \nabla\tau_1, \quad \text{for all } t_1 \in [a_1, b_1],$$

and

$$G(t_2) = \int_{a_2}^{t_2} \widehat{h}_{n-1}^{(2)}(t_2, \rho_2(\tau_2))^q \nabla\tau_2, \quad \text{for all } t_2 \in [a_2, b_2]$$

(where $\widehat{h}_{n-1}^{(i)}$, $\rho^{(i)}$ the corresponding $\widehat{h}_{n-1}, \rho$ to $\mathbb{T}^i$, $i = 1, 2$).

Here $\mathbb{T}_i$, $i = 1, 2$ and their terms fulfill Assumptions 40.19, 40.21.

We further suppose that

$$\lambda(t_1) = \int_{a_2}^{b_2} \frac{|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \nabla\tau_2$$

is an ld-continuous function on $\mathbb{T}_1$.

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)| |f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \nabla t_1 \nabla t_2 \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left| f_1^{\nabla^n}(\tau_1) \right|^q \nabla\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left| f_2^{\nabla^n}(\tau_2) \right|^p \nabla\tau_2\right)^{\frac{1}{p}} \quad (40.7)$$

(above double time scales nabla integration is considered in the natural iterative way).

Proof. Since $f_i^{\nabla^k}(a_i) = 0$, $k = 0, 1, ..., n-1$; $i = 1, 2$, by Theorem 40.12 we get

$$f_i(t_i) = \int_{a_i}^{t_i} \widehat{h}_{n-1}^{(i)}(t_i, \rho_i(\tau_i)) f_i^{\nabla^n}(\tau_i) \nabla\tau_i,$$

$\forall\ t_i \in [a_i, b_i]$, where $a_i, b_i \in \mathbb{T}_i$.

Hence

$$|f_1(t_1)| \leq \left(\int_{a_1}^{t_1} \widehat{h}_{n-1}^{(1)}(t_1, \rho_1(\tau_1))^p \nabla\tau_1\right)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}}$$

$$= F(t_1)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}},$$

and

$$|f_2(t_2)| \leq \left(\int_{a_2}^{t_2} \widehat{h}_{n-1}^{(2)}(t_2, \rho_2(\tau_2))^q \nabla\tau_2\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}}$$

$$= G(t_2)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}}.$$

Young's inequality for $a, b \geq 0$ says that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Therefore we have

$$|f_1(t_1)|\,|f_2(t_2)| \leq$$

$$F(t_1)^{\frac{1}{p}} G(t_2)^{\frac{1}{q}} \left(\int_{a_1}^{t_1} \left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}} \leq$$

$$\left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right) \left(\int_{a_1}^{t_1} \left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}}.$$

The last gives ($\varepsilon > 0$)

$$\frac{|f_1(t_1)|\,|f_2(t_2)|}{\varepsilon + \left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \leq \left(\int_{a_1}^{t_1} \left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}},$$

for all $t_i \in [a_i, b_i]$, $i = 1, 2$.

Next we observe that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \nabla t_1 \nabla t_2 \leq$$

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}} \nabla t_1\right) \left(\int_{a_2}^{b_2} \left(\int_{a_2}^{t_2} \left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}} \nabla t_2\right) \leq$$

$$\left(\int_{a_1}^{b_1}\left(\int_{a_1}^{t_1}\left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)\nabla t_1\right)^{\frac{1}{q}}(b_1-a_1)^{\frac{1}{p}}\cdot$$

$$\left(\int_{a_2}^{b_2}\left(\int_{a_2}^{t_2}\left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)\nabla t_2\right)^{\frac{1}{p}}(b_2-a_2)^{\frac{1}{q}}\leq$$

$$\left(\int_{a_1}^{b_1}\left(\int_{a_1}^{b_1}\left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)\nabla t_1\right)^{\frac{1}{q}}(b_1-a_1)^{\frac{1}{p}}\cdot$$

$$\left(\int_{a_2}^{b_2}\left(\int_{a_2}^{b_2}\left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)\nabla t_2\right)^{\frac{1}{p}}(b_2-a_2)^{\frac{1}{q}}=$$

$$(b_1-a_1)(b_2-a_2)\left(\int_{a_1}^{b_1}\left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}},$$

establishing the claim. ■

Based on Corollary 40.13 we get the following results:

First a generalized time scales nabla Poincaré type inequality.

Proposition 40.28. Suppose Assumption 40.22. Let $f \in C_{ld}^n(\mathbb{T})$, $m,n \in \mathbb{N}$, $m < n$, $a,b \in \mathbb{T}$; $a \leq b$; $p,q > 1 : \frac{1}{p}+\frac{1}{q}=1$. Suppose $f^{\nabla^{k+m}}(a)=0$, $k = 0,1,...,n-m+1$.

Then

$$\int_a^b\left|f^{\nabla^m}(t)\right|^q \nabla t \leq$$

$$\left(\int_a^b\left(\int_a^t \widehat{h}_{n-m-1}(t,\rho(\tau))^p \nabla\tau\right)^{\frac{q}{p}}\nabla t\right)\left(\int_a^b\left|f^{\nabla^n}(\tau)\right|^q \nabla\tau\right). \tag{40.8}$$

Proof. As in Theorem 40.23. ■

It follows a generalized time scales nabla Sobolev type inequality.

Proposition 40.29. Here all terms and assumptions are as in Proposition 40.28. Let $r \geq 1$. Then

$$\left\|f^{\nabla^m}\right\|_r \leq \left(\int_a^b\left(\int_a^t \widehat{h}_{n-m-1}(t,\rho(\tau))^p \nabla\tau\right)^{\frac{r}{p}}\nabla t\right)^{\frac{1}{r}}\left\|f^{\nabla^n}\right\|_q. \tag{40.9}$$

Proof. As in Theorem 40.24. ■

Next comes a generalized time scales nabla Opial type inequality.

Proposition 40.30. Suppose Assumption 40.22. Let $f \in C_{ld}^n(\mathbb{T})$, $m,n \in \mathbb{N}$, $m < n$, $a,b \in \mathbb{T}$; $a \leq b$; $p,q > 1 : \frac{1}{p}+\frac{1}{q}=1$. Assume $f^{\nabla^{k+m}}(a)=0$, $k = 0,1,...,n-1$, and that $\left|f^{\nabla^n}\right|$ is increasing on $[a,b]$.

Then

$$\int_a^b \left|f^{\nabla^m}(t)\right| \left|f^{\nabla^n}(t)\right| \nabla t \leq \tag{40.10}$$

$$(b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t \widehat{h}_{n-m-1}(t,\rho(\tau))^p \nabla\tau\right) \nabla t\right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\nabla^n}(t)\right)^{2q} \nabla t\right)^{\frac{1}{q}}.$$

Proof. As in Theorem 40.25. ■

We continue with a generalized nabla Ostrowski type inequality over time scales.

Proposition 40.31. Let $f \in C_{ld}^n(\mathbb{T})$, $m,n \in \mathbb{N}$, $m < n$, $n-m$ is odd; $a,b,c \in \mathbb{T} : a \leq c \leq b$. Assume that $f^{\nabla^{k+m}}(c) = 0$, $k = 1,...,n-m-1$. Then

$$\left|\frac{1}{b-a}\int_a^b f^{\nabla^m}(t)\nabla t - f^{\nabla^m}(c)\right| \leq \frac{\left[\widehat{h}_{n-m+1}(a,c) + \widehat{h}_{n-m+1}(b,c)\right]}{b-a} \left\|f^{\nabla^n}\right\|_{\infty,[a,b]}. \tag{40.11}$$

Proof. As in Theorem 40.26. ■

We finish with the generalized nabla Hilbert-Pachpatte type inequality on time scales.

Proposition 40.32. Let $\varepsilon > 0$, $i = 1,2$; $f_i \in C_{ld}^n(\mathbb{T}_i)$, $m,n \in \mathbb{N}$, $m < n$, with $f_i^{\nabla^{k+m}}(a_i) = 0$, $k = 0,1,...,n-m-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{T}_i$, time scale. Let also $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Set

$$F^*(t_1) = \int_{a_1}^{t_1} \widehat{h}_{n-m-1}^{(1)}(t_1,\rho_1(\tau_1))^p \nabla\tau_1, \quad \text{for all } t_1 \in [a_1,b_1],$$

and

$$G^*(t_2) = \int_{a_2}^{t_2} \widehat{h}_{n-m-1}^{(2)}(t_2,\rho_2(\tau_2))^q \nabla\tau_2, \quad \text{for all } t_2 \in [a_2,b_2],$$

(where $\widehat{h}_{n-m-1}^{(i)}, \rho^{(i)}$ the corresponding $\widehat{h}_{n-m-1}, \rho$ to $\mathbb{T}^i$, $i = 1,2$).

Here $\mathbb{T}_i$, $i = 1,2$ and terms fulfill Assumptions 40.19, 40.22.

We further suppose that

$$\lambda^*(t_1) = \int_{a_2}^{b_2} \frac{\left|f_2^{\nabla^m}(t_2)\right|}{\left(\varepsilon + \frac{F^*(t_1)}{p} + \frac{G^*(t_2)}{q}\right)} \nabla\tau_2$$

is an ld-continuous function on $\mathbb{T}_i$.

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{\left|f_1^{\nabla^m}(t_1)\right| \left|f_2^{\nabla^m}(t_2)\right|}{\left(\varepsilon + \frac{F^*(t_1)}{p} + \frac{G^*(t_2)}{q}\right)} \nabla t_1 \nabla t_2 \leq \tag{40.12}$$

$$(b_1 - a_1)(b_2 - a_2)\left(\int_{a_1}^{b_1} \left|f_1^{\nabla^n}(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|f_2^{\nabla^n}(\tau_2)\right|^p \nabla\tau_2\right)^{\frac{1}{p}}.$$

Proof. As in Theorem 40.27. ∎

40.3 Applications

A Poincaré inequality comes:

Corollary 40.33. Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$. Then

$$\int_a^b |f(t)|^q \, dt \leq$$

$$\frac{(b-a)^{nq}}{((n-1)!)^q (p(n-1)+1)^{(q-1)} nq} \left(\int_a^b \left|f^{(n)}(t)\right|^q dt\right). \tag{40.13}$$

Proof. Based on Theorem 40.23. ∎

A discrete nabla Poincaré follows:

Corollary 40.34. Let $f : \mathbb{Z} \to \mathbb{R}$, $n \in \mathbb{N}$, $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\nabla^k f(a) = 0$, $k = 0, 1, ..., n-1$. Then

$$\sum_{t=a+1}^{b} |f(t)|^q \leq$$

$$\frac{1}{((n-1)!)^q} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} \left((t-\tau+1)^{\left(\overline{n-1}\right)}\right)^p\right)^{\frac{q}{p}}\right) \left(\sum_{\tau=a+1}^{b} |\nabla^n f(\tau)|^q\right). \tag{40.14}$$

Proof. Based on Theorem 40.23. ∎

A Sobolev inequality comes:

Corollary 40.35. All as in Corollary 40.33. Let $r \geq 1$. Then

$$\left(\int_a^b |f(t)|^r \, dt\right)^{\frac{1}{r}} \leq$$

$$\frac{(b-a)^{\left(n-1+\frac{1}{p}+\frac{1}{r}\right)}}{(n-1)!\,((n-1)p+1)^{\frac{1}{p}} \left(\left(n-1+\frac{1}{p}\right)r+1\right)^{\frac{1}{r}}} \left(\int_a^b \left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}. \tag{40.15}$$

Proof. Based on Theorem 40.24. ∎

A discrete nabla Sobolev inequality follows:

Corollary 40.36. All as in Corollary 40.34 and let $r \geq 1$. Then

$$\left(\sum_{t=a+1}^{b} |f(t)|^r \right)^{\frac{1}{r}} \leq$$

$$\frac{1}{(n-1)!} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} \left((t-\tau+1)^{\left(\overline{n-1}\right)} \right)^p \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \left(\sum_{t=a+1}^{b} |\nabla^n f(t)|^q \right)^{\frac{1}{q}}. \tag{40.16}$$

Proof. Based on Theorem 40.24. ∎

An Opial inequality follows:

Corollary 40.37. Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$, and $\left|f^{(n)}\right|$ is increasing on $[a, b]$.

Then

$$\int_a^b |f(t)| \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{(b-a)^{n+\frac{1}{p}}}{(n-1)! \left[((n-1)p+1)((n-1)p+2) \right]^{\frac{1}{p}}} \left(\int_a^b \left(f^{(n)}(t) \right)^{2q} dt \right)^{\frac{1}{q}}. \tag{40.17}$$

Proof. Based on Theorem 40.25. ∎

A discrete nabla Opial inequality follows:

Corollary 40.38. Let $f : \mathbb{Z} \to \mathbb{R}$, $n \in \mathbb{N}$, $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\nabla^k f(a) = 0$, $k = 0, 1, ..., n-1$, and that $|\nabla^n f|$ is increasing on $[a, b]$.

Then

$$\sum_{t=a+1}^{b} |f(t)| |\nabla^n f(t)| \leq$$

$$\frac{(b-a)^{\frac{1}{q}}}{(n-1)!} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} \left((t-\tau+1)^{\left(\overline{n-1}\right)} \right)^p \right) \right)^{\frac{1}{p}} \left(\sum_{t=a+1}^{b} (\nabla^n f(t))^{2q} \right)^{\frac{1}{q}}. \tag{40.18}$$

Proof. By Theorem 40.25. ∎

An Ostrowski inequality follows:

Corollary 40.39. Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a, b, c \in \mathbb{R} : a \le c \le b$. Suppose that $f^{(k)}(c) = 0$, $k = 1, ..., n-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(c) \right| \le \frac{\left[(c-a)^{n+1} + (b-c)^{n+1} \right]}{(n+1)!\,(b-a)} \left\| f^{(n)} \right\|_{\infty,[a,b]}. \quad (40.19)$$

Proof. Based on Theorem 40.26. ■

A discrete nabla Ostrowski inequality follows:

Corollary 40.40. Let $f : \mathbb{Z} \to \mathbb{R}$, n is an odd number, $a, b, c \in \mathbb{Z} : a \le c \le b$. Assume that $\nabla^k f(c) = 0$, $k = 1, ..., n-1$. Then

$$\left| \frac{1}{b-a} \sum_{t=a+1}^{b} f(t) - f(c) \right| \le \frac{\left[(a-c)^{\left(\overline{n+1}\right)} + (b-c)^{\left(\overline{n+1}\right)} \right]}{(n+1)!\,(b-a)} \left\| \nabla^n f \right\|_{\infty,[a,b]}. \quad (40.20)$$

Proof. By Theorem 40.26. ■

A Hilbert-Pachpatte inequality follows:

Corollary 40.41. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, with $f_i^{(k)}(a_i) = 0$, $k = 0, 1, ..., n-1$; $a_i \le b_i$; $a_i, b_i \in \mathbb{R}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Put

$$F(t_1) = \frac{1}{((n-1)!)^p} \frac{(t_1 - a_1)^{p(n-1)+1}}{(p(n-1)+1)}, \quad \forall\, t_1 \in [a_1, b_1],$$

$$G(t_2) = \frac{1}{((n-1)!)^q} \frac{(t_2 - a_2)^{q(n-1)+1}}{(q(n-1)+1)}, \quad \forall\, t_2 \in [a_2, b_2].$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right)} dt_1 dt_2 \le$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left| f_1^{(n)}(\tau_1) \right|^q d\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left| f_2^{(n)}(\tau_2) \right|^p d\tau_2 \right)^{\frac{1}{p}}. \quad (40.21)$$

Proof. Based on Theorem 40.27.

Notice here that $\lambda(t_1)$ is a continuous function on $[a_1, b_1]$ by bounded convergence theorem. ■

It follows a discrete nabla Hilbert-Pachpatte inequality.

Corollary 40.42. Let $\varepsilon > 0$, $i = 1, 2$; $f_i : \mathbb{Z} \to \mathbb{R}$, $n \in \mathbb{N}$, with $\nabla^k f_i(a_i) = 0$, $k = 0, 1, ..., n-1$; $a_i \le b_i$; $a_i, b_i \in \mathbb{Z}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Set

$$\overline{F}(t_1) = \frac{\sum_{\tau_1 = a_1 + 1}^{t_1} \left((t_1 - \tau_1 + 1)^{\left(\overline{n-1}\right)} \right)^p}{((n-1)!)^p}, \quad \forall\, t_1 \in [a_1, b_1],$$

and

$$\overline{G}(t_2) = \frac{\sum_{\tau_2=a_2+1}^{t_2} \left((t_2 - \tau_2 + 1)^{\left(\overline{n-1}\right)} \right)^q}{((n-1)!)^q}, \quad \forall\, t_2 \in [a_2, b_2].$$

Then

$$\sum_{t_1=a_1+1}^{b_1} \sum_{t_2=a_2+1}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon + \frac{\overline{F}(t_1)}{p} + \frac{\overline{G}(t_2)}{q}\right)} \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\sum_{\tau_1=a_1+1}^{b_1} |\nabla^n f_1(\tau_1)|^q \right)^{\frac{1}{q}} \left(\sum_{\tau_2=a_2+1}^{b_2} |\nabla^n f_2(\tau_2)|^p \right)^{\frac{1}{p}}. \quad (40.22)$$

Proof. By Theorem 40.27. ■

Another generalized Poincaré inequality comes:

Corollary 40.43. Let $f \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$; $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{(k+m)}(a) = 0$, $k = 0, 1, ..., n-m-1$. Then

$$\int_a^b \left| f^{(m)}(t) \right|^q dt \leq$$

$$\frac{(b-a)^{(n-m)q}}{((n-m-1)!)^q \,(p(n-m-1)+1)^{(q-1)}\,(n-m)\,q} \cdot \quad (40.23)$$

$$\left(\int_a^b \left| f^{(n)}(t) \right|^q dt \right).$$

Proof. By Corollary 40.33, $n \mapsto n-m$, $f \mapsto f^{(m)}$, $f^{(k)} \mapsto f^{(k+m)}$ into (40.13). ■

A generalized discrete nabla Poincaré inequality follows:

Corollary 40.44. Let $f : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\nabla^{k+m} f(a) = 0$, $k = 0, 1, ..., n-m-1$. Then

$$\sum_{t=a+1}^{b} |\nabla^m f(t)|^q \leq$$

$$\frac{1}{((n-m-1)!)^q} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} \left((t-\tau+1)^{\left(\overline{n-m-1}\right)} \right)^p \right)^{\frac{q}{p}} \right) \cdot \quad (40.24)$$

$$\left(\sum_{\tau=a+1}^{b} |\nabla^n f(\tau)|^q \right).$$

Proof. By Corollary 40.34. ∎

A generalized Sobolev inequality comes.

Corollary 40.45. All as in Corollary 40.43, $r \geq 1$. Then

$$\left(\int_a^b \left| f^{(m)}(t) \right|^r dt \right)^{\frac{1}{r}} \leq$$

$$\frac{(b-a)^{\left(n-m-1+\frac{1}{p}+\frac{1}{r}\right)}}{(n-m-1)!\left((n-m-1)p+1\right)^{\frac{1}{p}} \left(\left(n-m-1+\frac{1}{p}\right)r+1\right)^{\frac{1}{r}}} \cdot \tag{40.25}$$

$$\left(\int_a^b \left| f^{(n)}(t) \right|^q dt \right)^{\frac{1}{q}}.$$

Proof. By Corollary 40.35. ∎

A generalized discrete nabla Sobolev inequality follows:

Corollary 40.46. All as in Corollary 40.44, $r \geq 1$. Then

$$\left(\sum_{t=a+1}^{b} \left| \nabla^m f(t) \right|^r \right)^{\frac{1}{r}} \leq \frac{1}{(n-m-1)!} \cdot$$

$$\left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} \left((t-\tau+1)^{\left(\overline{n-m-1}\right)} \right)^p \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \left(\sum_{t=a+1}^{b} \left| \nabla^n f(t) \right|^q \right)^{\frac{1}{q}}. \tag{40.26}$$

Proof. By Corollary 40.36. ∎

A generalized Opial inequality follows:

Corollary 40.47. Let $f \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{(k+m)}(a) = 0$, $k = 0, 1, ..., n-m-1$, and $\left| f^{(n)} \right|$ is increasing on $[a, b]$.

Then

$$\int_a^b \left| f^{(m)}(t) \right| \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{(b-a)^{n-m+\frac{1}{p}}}{(n-m-1)!\left[\left((n-m-1)p+1\right)\left((n-m-1)p+2\right)\right]^{\frac{1}{p}}} \left(\int_a^b \left(f^{(n)}(t) \right)^{2q} dt \right)^{\frac{1}{q}}. \tag{40.27}$$

Proof. By Corollary 40.37. ∎

A generalized discrete nabla Opial inequality follows:

Corollary 40.48. Let $f : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\nabla^{k+m} f(a) = 0$, $k = 0, 1, ..., n-m-1$, and that $|\nabla^n f|$ is increasing on $[a, b]$.

Then

$$\sum_{t=a+1}^{b} |\nabla^m f(t)| \, |\nabla^n f(t)| \leq$$

$$\frac{(b-a)^{\frac{1}{q}}}{(n-m-1)!} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} \left((t-\tau+1)^{(\overline{n-m-1})} \right)^p \right)^{\frac{1}{p}} \right) \left(\sum_{t=a+1}^{b} (\nabla^n f(t))^{2q} \right)^{\frac{1}{q}}. \tag{40.28}$$

Proof. By Corollary 40.38. ■

A generalized Ostrowski inequality comes next:

Corollary 40.49. Let $f \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, $a, b, c \in \mathbb{R} : a \leq c \leq b$. Assume that $f^{(k+m)}(c) = 0$, $k = 1, ..., n-m-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f^{(m)}(t)\, dt - f^{(m)}(c) \right| \leq \frac{\left[(c-a)^{n-m+1} + (b-c)^{n-m+1} \right]}{(n-m+1)!\,(b-a)} \left\| f^{(n)} \right\|_{\infty, [a,b]}. \tag{40.29}$$

Proof. By Corollary 40.39. ■

A generalized discrete nabla Ostrowski inequality follows:

Corollary 40.50. Let $f : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b, c \in \mathbb{Z}$: $a \leq c \leq b$. Assume that $\nabla^{k+m} f(c) = 0$, $k = 1, ..., n-m-1$. Then

$$\left| \frac{1}{b-a} \sum_{t=a+1}^{b} \nabla^m f(t) - \nabla^m f(c) \right| \leq$$

$$\frac{\left[(a-c)^{(\overline{n-m+1})} + (b-c)^{(\overline{n-m+1})} \right]}{(n-m+1)!\,(b-a)} \|\nabla^n f\|_{\infty, [a,b]}. \tag{40.30}$$

Proof. By Corollary 40.40. ■

A generalized Hilbert-Pachpatte comes:

Corollary 40.51. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, with $f_i^{(k+m)}(a_i) = 0$, $k = 0, 1, ..., n-m-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{R}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Put

$$F^*(t_1) = \frac{1}{((n-m-1)!)^p} \frac{(t_1 - a_1)^{p(n-m-1)+1}}{(p(n-m-1)+1)}, \quad \forall\, t_1 \in [a_1, b_1],$$

$$G^*(t_2) = \frac{1}{((n-m-1)!)^q} \frac{(t_2 - a_2)^{q(n-m-1)+1}}{(q(n-m-1)+1)}, \quad \forall\, t_2 \in [a_2, b_2].$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{\left|f_1^{(m)}(t_1)\right|\left|f_2^{(m)}(t_2)\right|}{\left(\varepsilon+\frac{F^*(t_1)}{p}+\frac{G^*(t_2)}{q}\right)} dt_1 dt_2 \leq$$

$$(b_1-a_1)(b_2-a_2)\left(\int_{a_1}^{b_1}\left|f_1^{(n)}(\tau_1)\right|^q d\tau_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|f_2^{(n)}(\tau_2)\right|^p d\tau_2\right)^{\frac{1}{p}}. \quad (40.31)$$

Proof. By Corollary 40.41. ■

It follows a generalized discrete nabla Hilbert-Pachpatte inequality.

Corollary 40.52. Let $\varepsilon > 0$, $i = 1, 2$; $f_i : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, with $\nabla^{k+m} f_i(a_i) = 0$, $k = 0, 1, ..., n-m-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{Z}$. Let $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. Set

$$\overline{F}^*(t_1) = \frac{\sum_{\tau_1=a_1+1}^{t_1}\left((t_1-\tau_1+1)^{(\overline{n-m-1})}\right)^p}{((n-m-1)!)^p}, \quad \forall\, t_1 \in [a_1, b_1],$$

and

$$\overline{G}^*(t_2) = \frac{\sum_{\tau_2=a_2+1}^{t_2}\left((t_2-\tau_2+1)^{(\overline{n-m-1})}\right)^q}{((n-m-1)!)^q}, \quad \forall\, t_2 \in [a_2, b_2].$$

Then

$$\sum_{t_1=a_1+1}^{b_1}\sum_{t_2=a_2+1}^{b_2} \frac{|\nabla^m f_1(t_1)|\,|\nabla^m f_2(t_2)|}{\left(\varepsilon+\frac{\overline{F}^*(t_1)}{p}+\frac{\overline{G}^*(t_2)}{q}\right)} \leq$$

$$(b_1-a_1)(b_2-a_2)\left(\sum_{\tau_1=a_1+1}^{b_1}|\nabla^n f_1(\tau_1)|^q\right)^{\frac{1}{q}}\left(\sum_{\tau_2=a_2+1}^{b_2}|\nabla^n f_2(\tau_2)|^p\right)^{\frac{1}{p}}. \quad (40.32)$$

Proof. By Corollary 40.42. ■

We make

Remark 40.53. ([93]) We consider the time scale $\mathbb{T} = q^{\overline{\mathbb{Z}}} = \{0, 1, q, q^{-1}, q^2, q^{-2}, ...\}$, for some $q > 1$. Here $\rho(t) = \frac{t}{q}$, $\forall\, t \in \mathbb{T}$. We have that

$$\widehat{h}_k(t,s) = \prod_{r=0}^{k-1} \frac{q^r t - s}{\sum_{j=0}^{r} q^j}, \text{ for all } s, t \in \mathbb{T},$$

for all $k \in \mathbb{N}_0$.

We give a related nabla q-Ostrowski type inequality.

Corollary 40.54. Let $f \in C_{ld}^{n}\left(q^{\overline{\mathbb{Z}}}\right)$, n is odd, $a, b, c \in q^{\overline{\mathbb{Z}}} : a \leq c \leq b$. Assume that $f^{\nabla^{k}}(c) = 0$, $k = 1, ..., n-1$. Then

$$\left|\frac{1}{b-a}\int_{a}^{b} f(t)\,\nabla t - f(c)\right| \leq \frac{\left[\prod_{\nu=0}^{n} \frac{q^{\nu}a-c}{\sum_{\mu=0}^{\nu} q^{\mu}} + \prod_{\nu=0}^{n} \frac{q^{\nu}b-c}{\sum_{\mu=0}^{\nu} q^{\mu}}\right]}{b-a} \left\|f^{\nabla^{n}}\right\|_{\infty,[a,b]}. \tag{40.33}$$

Proof. By Theorem 40.26. ■

We finish with a generalized nabla q-Ostrowski type inequality.

Corollary 40.55. Let $f \in C_{ld}^{n}\left(q^{\overline{\mathbb{Z}}}\right)$, $m, n \in \mathbb{N}$, $m < n$, $n-m$ is odd, $a, b, c \in q^{\overline{\mathbb{Z}}} : a \leq c \leq b$. Assume that $f^{\nabla^{k+m}}(c) = 0$, $k = 1, ..., n-m-1$. Then

$$\left|\frac{1}{b-a}\int_{a}^{b} f^{\nabla^{m}}(t)\,\nabla t - f^{\nabla^{m}}(c)\right| \leq$$

$$\frac{\left[\prod_{\nu=0}^{n-m} \frac{q^{\nu}a-c}{\sum_{\mu=0}^{\nu} q^{\mu}} + \prod_{\nu=0}^{n-m} \frac{q^{\nu}b-c}{\sum_{\mu=0}^{\nu} q^{\mu}}\right]}{b-a} \left\|f^{\nabla^{n}}\right\|_{\infty,[a,b]}. \tag{40.34}$$

By Corollary 40.54.

One can give many similar applications for other time scales.

41

The Principle of Duality in Time Scales with Inequalities

Here we present and extend the principle of duality in time scales. Using this principle and based on a variety of important delta inequalities we produce the corresponding nabla ones. We give several applications. This chapter relies on [52].

41.1 Preliminaries

Here we use the seminal book by Bohner and Peterson [119].

A time scale is any closed nonempty subset $\mathbb{T}$ of $\mathbb{R}$. The jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

and $\inf \emptyset := \sup \mathbb{T}$, $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$.

The *forward graininess* $\mu : \mathbb{T} \to \mathbb{R}$ is defined by $\mu(t) = \sigma(t) - t$, and the *backward graininess* $\nu : \mathbb{T} \to \mathbb{R}$ is defined by $\nu(t) = t - \rho(t)$.

Given a time scale $\mathbb{T}$, we denote $\mathbb{T}^k := \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$, if $\sup \mathbb{T} < \infty$ and $\mathbb{T}^k := \mathbb{T}$ if $\sup \mathbb{T} = \infty$. Also $\mathbb{T}_k := \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$ if $\inf \mathbb{T} > -\infty$ and $\mathbb{T}_k =: \mathbb{T}$ if $\inf \mathbb{T} = -\infty$. In particular, if $a, b \in \mathbb{T}$ with $a < b$, we denote by $[a, b]$ the interval $[a, b] \cap \mathbb{T}$.

Notice that $\mathbb{R}$ itself is one obvious example of time scale, but one could also take $\mathbb{T}$ to be the Cantor set or the integers $\mathbb{Z}$.

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 673–694.
springerlink.com

Let f be a function defined on $\mathbb{T}$, we say that:

Definition 41.1. f is rd-continuous (or right-dense continuous) we write $f \in C_{rd}$ if it is continuous at the right-dense points and its left-sided limits exist (finite) at all left-dense points; f is ld-continuous (or left-dense continuous) if it is continuous at the left-dense points and its right-sided limits exist (finite) at all right-dense points.

Definition 41.2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be delta differentiable at $t \in \mathbb{T}^k$ if for all $\varepsilon > 0$ there exists U a neighborhood of t such that for some α, the inequality

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| < \varepsilon|\sigma(t) - s| \tag{41.1}$$

is true for all s $\in U$. We write $f^{\Delta}(t) = \alpha$.

Definition 41.3. $f : \mathbb{T} \to \mathbb{R}$ is said to be delta differentiable on $\mathbb{T}$ if $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable for all $t \in \mathrm{T}^k$.

Definition 41.4. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be nabla differentiable at $t \in \mathbb{T}_k$ if for all $\varepsilon > 0$ there exists U a neighborhood of t such that for some β, the inequality

$$|f(\rho(t)) - f(s) - \beta(\rho(t) - s)| < \varepsilon|\rho(t) - s| \tag{41.2}$$

is true for all $s \in U$. We write $f^{\nabla}(t) = \beta$.

Definition 41.5. $f : \mathbb{T} \to \mathbb{R}$ is said to be nabla differentiable on $\mathbb{T}$ if $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable for all $t \in \mathbb{T}_k$.

Definition 41.6. f is rd-continuously delta differentiable (we write $f \in C^1_{rd}$) if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$ and $f^{\Delta} \in C_{rd}$, and f is ld-continuously nabla differentiable (we write $f \in C^1_{ld}$) if $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_k$ and $f^{\nabla} \in C_{ld}$. Similarly one can define higher order such spaces.

Remark 41.7. If $\mathbb{T} = \mathbb{R}$ then the notion of delta derivative and nabla derivative coincide and they denote the standard derivative, however, when $\mathbb{T} = \mathbb{Z}$, then they do not coincide (see [119]) and they are the forward and backward differences (respectively).

41.2 The Dual Time Scale

In this section we mention the definition of a *dual* time scale. (see [127]).

A *dual* time scale is just the "reverse" time scale of a given time scale.

Definition 41.8. Given a time scale $\mathbb{T}$ we define the dual time scale $\mathbb{T}^* := \{s \in \mathbb{R} | -s \in \mathbb{T}\}$.

Let $\mathbb{T}$ be a time scale. If ρ and σ denote its associate jump functions, then we denote by $\hat{\rho}$ and $\hat{\sigma}$ the jump functions associated to $\mathbb{T}^*$. If μ and ν denote respectively the *forward graininess* and *backward graininess* associated to $\mathbb{T}$, then denote by $\hat{\mu}$ and $\hat{\nu}$ respectively the *forward graininess* and the *backward graininess* associated to $\mathbb{T}^*$.

We need

Definition 41.9. ([127]) Given a function $f : \mathbb{T} \to \mathbb{R}$ defined on time scale $\mathbb{T}$ we define the dual function $f^* : \mathbb{T}^* \to \mathbb{R}$ on the time scale $\mathbb{T}^* := \{s \in \mathbb{R} | -s \in \mathbb{T}\}$ by $f^*(s) := f(-s)$ for all $s \in \mathbb{T}^*$.

That is $f^*(-s) = f(s), s \in T$.

Definition 41.10. Given a time scale $\mathbb{T}$ we refer to the delta calculus (resp. nabla calculus) any calculation that involves delta derivatives (resp. nabla derivatives).

41.3 Dual Correspondences

In this section we mention some basic lemmas ([127]) which follow easily from the definitions. These lemmas concern the relationship between *dual* objects. We will use the following notation: given a time scale $\mathbb{T}$ with jumps functions, σ, ρ, and its associated *forward graininess* μ and *backward graininess* ν, hence given the quintuple $(\mathbb{T}, \sigma, \rho, \mu, \nu)$, its dual will be $(\mathbb{T}^*, \hat{\sigma}, \hat{\rho}, \hat{\mu}, \hat{\nu})$, where $\hat{\sigma}, \hat{\rho}, \hat{\mu}$, and $\hat{\nu}$ are given as in Lemmas 41.12, 41.13. Also, Δ and ∇ will denote the derivatives for the time scale $\mathbb{T}$ and $\hat{\Delta}$ and $\hat{\nabla}$ will denote the derivatives for the time scale $\mathbb{T}^*$.

Lemma 41.11. ([127]) Given a time scale $\mathbb{T}$, then

$$(\mathbb{T}^k)^* = (\mathbb{T}^*)_k, \; \textit{and} \; (\mathbb{T}_k)^* = (\mathbb{T}^*)^k. \tag{41.3}$$

Lemma 41.12. ([127]) Given σ, $\rho : \mathbb{T} \to \mathbb{T}$, the jump operators for $\mathbb{T}$, then the jump operators for $\mathbb{T}^*$, $\hat{\sigma}$ and $\hat{\rho} : \mathbb{T}^* \to \mathbb{T}^*$, are given by the following two identities:

$$\begin{aligned} \hat{\sigma}(s) &= -\rho(-s) = -\rho^*(s) \\ \hat{\rho}(s) &= -\sigma(-s) = -\sigma^*(s) \end{aligned} \tag{41.4}$$

for all $s \in \mathbb{T}^*$.

Lemma 41.13. ([127]) Given $\mu : \mathbb{T} \to \mathbb{R}$, the forward graininess of $\mathbb{T}$, then the backwards graininess of $\mathbb{T}^*, \hat{\nu} : \mathbb{T}^* \to \mathbb{R}$ is given by the identity

$$\hat{\nu}(s) = \mu^*(s) \text{ } for \text{ } all \text{ } s \in \mathbb{T}^*. \tag{41.5}$$

Similarly, given $\nu : \mathbb{T} \to \mathbb{R}$, the backward graininess of $\mathbb{T}$, then the forward graininess of $\mathbb{T}^*$, $\hat{\mu} : \mathbb{T}^* \to \mathbb{R}$ is given by the identity

$$\hat{\mu}(s) = \nu^*(s) \text{ } for \text{ } all \text{ } s \in \mathbb{T}^*. \tag{41.6}$$

Lemma 41.14. ([127]) Given $f : \mathbb{T} \to \mathbb{R}, f$ is rd-continuous (resp. ld-continuous) if and only if its dual $f^* : \mathbb{T}^* \to \mathbb{R}$ is ld-continuous (resp. rd-continuous).

The next lemma connects delta derivatives to *nabla* derivatives, showing that the two fundamental concepts of the two types of calculus are, in a certain sense, the dual of each other.

Lemma 41.15. ([127]) Let $f : \mathbb{T} \to \mathbb{R}$ be delta (resp. nabla) differentiable at $t_0 \in \mathbb{T}^k$ (resp. at $t_0 \in \mathbb{T}_k$), then $f^* : \mathbb{T}^* \to \mathbb{R}$ is nabla (resp. delta) differentiable at $-t_0 \in (\mathbb{T}^*)_k$ (resp. at $-t_0 \in (\mathbb{T}^*)^k$), and the following identities hold true

$$f^{\Delta}(t_0) = -(f^*)^{\hat{\nabla}}(-t_0) \text{ (resp. } f^{\nabla}(t_0) = -(f^*)^{\hat{\Delta}}(-t_0)),$$

or,

$$f^{\Delta}(t_0) = -((f^*)^{\hat{\nabla}})^*(t_0) \text{ (resp. } f^{\nabla}(t_0) = -((f^*)^{\hat{\Delta}})^*(t_0)), \tag{41.7}$$

or,

$$(f^{\Delta})^*(-t_0) = -((f^*)^{\hat{\nabla}})(-t_0) \text{ (resp. } (f^{\nabla})^*(-t_0) = -(f^*)^{\hat{\Delta}}(-t_0)),$$

where Δ, ∇ denote the derivatives for the time scale $\mathbb{T}$ and $\hat{\Delta}, \hat{\nabla}$ denote the derivatives for the time scale $\mathbb{T}^*$.

That is

$$(f^{\Delta})^* = -(f^*)^{\hat{\nabla}}, \; (f^{\nabla})^* = -(f^*)^{\hat{\Delta}}. \tag{41.8}$$

More generally we obtain

$$(f^*)^{\hat{\nabla}^n} = (-1)^n (f^{\Delta^n})^*, \text{ and } (f^*)^{\hat{\Delta}^n} = (-1)^n (f^{\nabla^n})^*. \tag{41.9}$$

We need

Lemma 41.16. ([127]) Given a function $f : \mathbb{T} \to \mathbb{R}$, f belongs to C^1_{rd} (resp. C^1_{ld}) if and only if its dual $f^* : \mathbb{T}^* \to \mathbb{R}$ belongs to C^1_{ld} (resp. C^1_{rd}).

Using Lemmas 41.14, 41.15 we get

Proposition 41.17. ([127]) (i) Let $f : [a,b] \to \mathbb{R}$ be a rd-continuous, then the following two integrals are equal

$$\int_a^b f(t)\Delta t = \int_{-b}^{-a} f^*(s)\hat{\nabla} s; \tag{41.10}$$

(ii) Let $f : [a,b] \to \mathbb{R}$ be a ld-continuous, then the following two integrals are equal

$$\int_a^b f(t)\nabla t = \int_{-b}^{-a} f^*(s)\hat{\Delta} s. \tag{41.11}$$

That is,

$$\int_a^b f^*(-t)\nabla t = \int_{-b}^{-a} f^*(s)\hat{\Delta} s, \tag{41.12}$$

and

$$\int_a^b f(t)\nabla t = \int_{-b}^{-a} f(-s)\hat{\Delta} s. \tag{41.13}$$

Notice also that $(f^*)^* = f$.

In this chapter we will be acting under the following

Duality Principle ([127]) *For any statement true in the nabla (resp. delta) calculus in the time scale $\mathbb{T}$ there is an equivalent dual statement in the delta (resp. nabla) calculus for the dual time scale $\mathbb{T}^*$.*

We make

Remark 41.18. We observe that

$$f^{\Delta} = (-1)\left((f^*)^{\hat{\nabla}}\right)^*, \tag{41.14}$$

$$f^{\Delta^2} = (f^{\Delta})^{\Delta} = \left((f^*)^{\hat{\nabla}^2}\right)^*, \tag{41.15}$$

and in general

$$f^{\Delta^k} = (-1)^k\left((f^*)^{\hat{\nabla}^k}\right)^*, \tag{41.16}$$

any $k \in \mathbb{N}$.

Similarly we have

$$f^{\nabla} = -\left((f^*)^{\hat{\Delta}}\right)^*, \tag{41.17}$$

and

$$f^{\nabla^2} = (f^{\nabla})^{\nabla} = \left((f^*)^{\hat{\Delta}^2}\right)^*, \tag{41.18}$$

and in general

$$f^{\nabla^k} = (-1)^k\left((f^*)^{\hat{\Delta}^k}\right)^*, \tag{41.19}$$

any $k \in \mathbb{N}$.

41.4 Dual Generalized Monomials

We make

Remark 41.19. Assume $\hat{h}_o(t,s) \equiv 1$, $\forall\ s,t \in T$, and

$$\hat{h}_{k+1}(t,s) \equiv \int_s^t \hat{h}_k(\tau,s)\nabla\tau, \quad \forall\ s,t \in T. \tag{41.20}$$

Thus

$$\hat{h}_k^{\nabla}(t,s) = \hat{h}_{k-1}(t,s), \qquad \forall\ k \in \mathbb{N},\ t \in T_k. \tag{41.21}$$

Here $\hat{h}_k$ are all well defined, since each is ld-continuous in t, $\forall\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Notice $\hat{h}_1(t,s) = t - s, \quad \forall\ s,t \in T$.

Next assume $h_o^*(t,s) \equiv 1$, $\forall\ s,t \in T^* = -T$, and

$$h_{k+1}^*(t,s) \equiv \int_s^t h_k^*(\theta,s)\hat{\Delta}\theta, \ \ \forall\ s,t \in T^*. \tag{41.22}$$

Furthermore

$$h_k^{*^{\Delta}}(t,s) = h_{k-1}^*(t,s), \qquad \forall\ k \in \mathbb{N}, \quad t \in (T^*)^k. \tag{41.23}$$

Here h_k^* are all well defined, since each is rd-continuous in t, $\forall\ k \in \mathbb{N}_0$.

Notice $h_1^*(t,s) = t - s$, $\forall\ t,s \in T^*$.

Here $s,t \in T$ iff $-s,-t \in T^*$.

We see that

$$\hat{h}_1(t,s) = t - s = -s - (-t) \tag{41.24}$$

$$= (-1)(-t-(-s)) = (-1)h_1^*(-t,-s), \ \ \text{true for } k = 1.$$

Suppose for fixed $k \in \mathbb{N}$ that

$$\hat{h}_k(t,s) = (-1)^k h_k^*(-t,-s), \quad \forall\ t,s \in T. \tag{41.25}$$

That is

$$\hat{h}_k(\tau,s) = (-1)^k h_k^*(-\tau,-s), \quad \forall\ \tau,s \in T. \tag{41.26}$$

Therefore we obtain

$$\hat{h}_{k+1}(t,s) = \int_s^t \hat{h}_k(\tau,s)\nabla\tau =$$

$$(-1)^k \int_s^t h_k^*(-\tau,-s)\nabla\tau \overset{\text{(by (41.12))}}{=}$$

$$(-1)^k \int_{-t}^{-s} h_k^*(\theta,-s)\hat{\Delta}\theta =$$

$$(-1)^{k+1} \int_{-s}^{-t} h_k^*(\theta,-s)\hat{\Delta}\theta =$$

$$(-1)^{k+1} h_{k+1}^*(-t,-s).$$

That is proving

$$\hat{h}_{k+1}(t,s) = (-1)^{k+1} h^*_{k+1}(-t,-s). \tag{41.27}$$

So by mathematical induction we have proved that

$$\hat{h}_k(t,s) = (-1)^k h^*_k(-t,-s), \quad \forall\, k \in \mathbb{N}_0, \tag{41.28}$$

$\forall\, t, s \in T$. That is $\forall\, t, s \in T$, $\forall\, k \in \mathbb{N}_0$ holds

$$h^*_k(-t,-s) = (-1)^k \hat{h}_k(t,s). \tag{41.29}$$

We make

Remark 41.20. Suppose $h_o(t,s) \equiv 1$, $\forall\ s, t \in T$, and

$$h_{k+1}(t,s) \equiv \int_s^t h_k(\tau,s)\Delta\tau, \quad \forall\ s,t \in T. \tag{41.30}$$

That is

$$h_k^{\Delta}(t,s) = h_{k-1}(t,s), \quad \forall\ k \in \mathbb{N}, \quad t \in T^k. \tag{41.31}$$

Also suppose $\hat{h}^*_o(t,s) \equiv 1$, $\forall\ s, t \in T^* = -T$,

$$\hat{h}^*_{k+1}(t,s) \equiv \int_s^t \hat{h}^*_k(\theta,s)\hat{\nabla}\theta, \tag{41.32}$$

$\forall\ s, t \in T^*$. Then

$$\hat{h}_k^{*\nabla}(t,s) = \hat{h}^*_{k-1}(t,s), \quad \forall\, k \in \mathbb{N},\ t \in (T^*)_k. \tag{41.33}$$

One can prove similarly to (41.28) that

$$h_k(t,s) = (-1)^k \hat{h}^*_k(-t-s), \quad \forall\ k \in \mathbb{N}_0,\ \forall\ t,s \in T. \tag{41.34}$$

Indeed we have for $k = 1$ that

$$h_1(t,s) = t - s = -s - (-t) = (-1)(-t-(-s)) = (-1)\hat{h}^*_1(-t,-s). \tag{41.35}$$

Suppose that

$$h_k(t,s) = (-1)^k \hat{h}^*_k(-t,-s), \tag{41.36}$$

true for a fixed $k \in \mathbb{N}_0$.

That is

$$h_k(\tau,s) = (-1)^k \hat{h}^*_k(-\tau,-s), \quad \forall\ \tau, s \in T. \tag{41.37}$$

Therefore

$$\begin{aligned} h_{k+1}(t,s) &= \int_s^t h_k(\tau,s)\Delta\tau = (-1)^k \int_s^t \hat{h}^*_k(-\tau,-s)\Delta\tau \\ &\overset{\text{(by (41.10))}}{=} (-1)^k \int_{-t}^{-s} \hat{h}^*_k(\theta,-s)\hat{\nabla}\theta = \end{aligned}$$

$$(-1)^{k+1} \int_{-s}^{-t} \hat{h}^*_k(\theta,-s)\hat{\nabla}\theta = (-1)^{k+1}\hat{h}^*_{k+1}(-t,-s), \tag{41.38}$$

proving (41.34).

41.5 Time Scales Integral Inequalities

We need the following delta Ostrowski inequality.

Theorem 41.21. ([57]) We assume $T^k = T$. Let f $\in C_{ld}^n(T)$, n is odd, $a, b, c \in T : a \leq c \leq b$.

Suppose $f^{\Delta^k}(c) = 0$, $k = 1, \ldots, n-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t)\Delta t - f(c) \right| \leq \frac{[h_{n+1}(a,c) + h_{n+1}(b,c)]}{b-a} \|f^{\Delta^n}\|_{\infty,[a,b]\cap T} \tag{41.39}$$

We reprove differently the following nabla Ostrowski inequality using (41.39).

Theorem 41.22. ([59]) We assume $T_k = T$. Let $f \in C_{ld}^n(T)$, n is odd, $a, b, c, \in T : a \leq c \leq b$. Suppose $f^{\nabla^k}(c) = 0$, $k = 1, \ldots, n-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t)\nabla t - f(c) \right| \leq \frac{[\hat{h}_{n+1}(a,c) + \hat{h}_{n+1}(b,c)]}{b-a} \|f^{\nabla^n}\|_{\infty,[a,b]\cap T}. \tag{41.40}$$

Proof. See that $a \leq c \leq b$ is equivalent to $-b \leq -c \leq -a$ and $b-a = (-a)-(-b)$.

By assumption $T = T_k$ we get $T^* = (T_k)^* \overset{(by(41.3))}{=} (T^*)^k$, i.e. $T^* = (T^*)^k$.

And by (41.9) we find

$$\begin{aligned} & (f^*)^{\hat{\Delta}^k}(-c) = (-1)^k (f^{\nabla^k})^*(-c) \\ = \quad & (-1)^k (f^{\nabla^k})(c) = 0, \quad k = 1, \ldots, n-1, \end{aligned}$$

by assumption. That is $(f^*)^{\hat{\Delta}^k}(-c) = 0$, $k = 1, \ldots, n-1$.

Also $(f)^* \in C_{rd}^n(T^*)$ iff $f \in C_{ld}^n(T)$, by Lemma 41.16.

Then we see that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)\nabla t - f(c) \right| \overset{(by\ (41.11))}{=} \\ & \left| \frac{1}{(-a)-(-b)} \int_{-b}^{-a} f^*(s)\hat{\Delta}s - f^*(-c) \right| \\ & \overset{(by\ (41.39))}{\leq} \frac{[h^*_{n+1}(-b,-c) + h^*_{n+1}(-a,-c)]}{(-a)-(-b)} \\ & \|(f^*)^{\hat{\Delta}^n}\|_{\infty,[-b,-a]\cap T^*} \\ \overset{(by(41.28))}{=} \quad & \frac{\left[\hat{h}_{n+1}(b,c) + \hat{h}_{n+1}(a,c)\right]}{b-a} \|f^{\nabla^n}\|_{\infty,[a,b]\cap T}, \end{aligned}$$

because by (41.9) we have

$$\|(f^*)^{\hat{\Delta}^n}\|_{\infty,[-b,-a]\cap T^*} =$$
$$\|(f^{\nabla^n})^*\|_{\infty,[-b,-a]\cap T^*} = \|(f^{\nabla^n})\|_{\infty,[a,b]\cap T},$$

proving the claim. ■

Similarly, one can prove (41.39) by the use of (41.40).

We mention a delta Poincare type inequality.

Theorem 41.23. ([57]) Here $T^k = T$.

Let $f \in C^n_{rd}(T)$, n is odd, $a,\ b \in T : a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose $f^{\Delta^k}(a) = 0$, $k = 0, 1, \ldots, n-1$. Here σ is continuous and $h_{n-1}(t,s)$ jointly continuous. Then

$$\int_a^b |f(t)|^q \Delta t \leq \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{q/p} \Delta t \right) \left(\int_a^b |f^{\Delta^n}(t)|^q \Delta t \right). \tag{41.41}$$

We present a nabla Poincare' type inequality.

Theorem 41.24. Here $T_k = T$. Let $f \in C^n_{ld}(T)$, n is odd, $a, b \in T :\ a \leq b;\ p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Suppose $f^{\nabla^k}(b) = 0,\ k = 0, 1, \ldots, n-1$. Here ρ is continuous and $\hat{h}_{n-1}(t,s)$ is jointly continuous. Then

$$\int_a^b |f(t)|^q \nabla t \leq \left(\int_a^b \left(\int_t^b \hat{h}_{n-1}(t, \rho(\tau))^p \nabla\tau \right)^{\frac{q}{p}} \nabla t \right) \left(\int_a^b \left| f^{\nabla^n}(t) \right|^q \nabla t \right). \tag{41.42}$$

Proof. Notice that $\left(f^{\nabla^k}\right)(b) = 0, \quad k = 0, 1, \ldots, n-1$, which is same as

$$\left(f^{\nabla^k}\right)^*(-b) = 0, \quad k = 0, 1, \ldots n-1,$$

and

$$(f^*)^{\hat{\Delta}^k}(-b) \overset{(by (41.9))}{=} (-1)^k \left(f^{\nabla^k}\right)^*(-b) = 0,$$
$$k = 0, 1, \ldots, n-1.$$

That is

$$(f^*)^{\hat{\Delta}^k}(-b) = 0, \quad k = 0, 1, \ldots, n-1. \tag{41.43}$$

Also $T^* = (T^*)^k$.

Observe $f^* \in C^n_{rd}(T^*)$ iff $f \in C^n_{ld}(T)$.

By (41.28) and assumption we get h^*_{n-1} is jointly continuous on $(T^*)^2$.

Also $\hat{\sigma}(s) \overset{(by(41.4))}{=} -\rho(-s),\ s \in T^*$, is continuous. Notice also that $(|f|^q)^*(s) = |f|^q(-s) = |f(-s)|^q,\ s \in T^*$.

We observe that ($|f|^q \in C_{ld}(T)$)

$$\int_a^b |f(t)|^q \nabla t \overset{(by\ (41.13))}{=} \int_{-b}^{-a} |f(-s)|^q \hat{\Delta} s$$
$$\overset{(by\ (41.41))}{\leq} \left(\int_{-b}^{-a} \left(\int_{-b}^{t} h_{n-1}^* \ (t, \hat{\sigma}(\tau))^p \hat{\Delta}\tau \right)^{\frac{q}{p}} \hat{\Delta} t \right) \left(\int_{-b}^{-a} \left| (f^*)^{\hat{\Delta}^n} (t) \right|^q \hat{\Delta} t \right). \tag{41.44}$$

We notice the following

$$\int_{-b}^{-a} \left(\int_{-b}^{-t} h_{n-1}^* \left(t, \hat{\sigma}(\tau)\right)^p \hat{\Delta}\tau \right)^{\frac{q}{p}} \hat{\Delta} t$$
$$= \int_{-b}^{-a} \left(\int_{-b}^{t} h_{n-1}^* \left(t, -\rho(-\tau)\right)^p \hat{\Delta}\tau \right)^{\frac{q}{p}} \hat{\Delta} t$$
$$\overset{(by\ (41.28))}{=} \int_{-b}^{-a} \left(\int_{-b}^{-(-t)} \hat{h}_{n-1}(-t, \rho(-\tau))^p \hat{\Delta}\tau \right)^{\frac{q}{p}} \hat{\Delta} t$$
$$\overset{(by\ (41.13))}{=} \int_{-b}^{-a} \left(\int_{-t}^{b} \hat{h}_{n-1} \left(-t, \rho(\tau)\right)^p \nabla\tau \right)^{\frac{q}{p}} \hat{\Delta} t$$
$$\overset{(by\ (41.13))}{=} \int_{a}^{b} \left(\int_{t}^{b} \hat{h}_{n-1}(t, \rho(\tau))^p \nabla\tau \right)^{q/p} \nabla t,$$

notice the integrand of last integral is continuous in t by dominated convergence theorem.

So we have established that

$$\int_{-b}^{-a} \left(\int_{-b}^{t} h_{n-1}^*(t, \hat{\sigma}(\tau))^p \hat{\Delta}\tau \right)^{q/p} \hat{\Delta} t = \int_{a}^{b} \left(\int_{t}^{b} \hat{h}_{n-1}(t, \rho(\tau))^p \nabla\tau \right)^{q/p} \nabla t. \tag{41.45}$$

Next we observe that

$$\int_{-b}^{-a} \left| (f^*)^{\hat{\Delta}^n} (t) \right|^q \hat{\Delta} t \overset{(by\ (41.9))}{=}$$
$$\int_{-b}^{-a} \left| (f^{\nabla^n})^* (t) \right|^q \hat{\Delta} t =$$
$$\int_{-b}^{-a} \left| f^{\nabla^n} (-t) \right|^q \hat{\Delta} t$$
$$\overset{(by\ (41.13),(41.11))}{=} \int_{a}^{b} \left| f^{\nabla^n} (t) \right|^q \nabla t.$$

Notice here $(t \in T^*)$

$$\left(\left|f^{\nabla^n}\right|^q\right)^*(t) = \left|f^{\nabla^n}\right|^q(-t) = \left|f^{\nabla^n}(-t)\right|^q.$$

So we proved

$$\int_{-b}^{-a} \left|(f^*)^{\hat{\Delta}^n}(t)\right|^q \hat{\Delta}t = \int_a^b \left|f^{\nabla^n}(t)\right|^q \nabla t. \tag{41.46}$$

Finally using (41.45), (41.46) into (41.44) we establish (41.42). ■

We mention the Delta Sobolev type inequality.

Theorem 41.25. ([57]) Here all terms and assumptions as in Theorem 41.23. Let $r \geq 1$. Then

$$\left(\int_a^b |f(t)|^r \Delta t\right)^{\frac{1}{r}} \leq \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau\right)^{\frac{r}{p}} \Delta t\right)^{\frac{1}{r}} \cdot$$
$$\left(\int_a^b \left|f^{\Delta^n}(t)\right|^q \Delta t\right)^{\frac{1}{q}}. \tag{41.47}$$

We give the following Nabla Sobolev type inequality.

Theorem 41.26. Here all as in Theorem 41.24.
Let $r \geq 1$. Then

$$\left(\int_a^b |f(t)|^r \nabla t\right)^{\frac{1}{r}} \leq \left(\int_a^b \left(\int_t^b \hat{h}_{n-1}(t, \rho(\tau))^p \nabla\tau\right)^{\frac{r}{p}} \nabla t\right)^{\frac{1}{r}} \cdot$$
$$\left(\int_a^b \left|f^{\nabla^n}(t)\right|^q \nabla t\right)^{\frac{1}{q}}. \tag{41.48}$$

Proof. Similar to the proof of Theorem 41.24, using (41.47). ■

We mention the following delta Opial type inequality.

Theorem 41.27. ([57]) Here $T^k = T$.
Let $f \in C_{rd}^n(T)$, n is an odd number, a, $b \in T$; $a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.
Suppose $f^{\Delta^*}(a) = 0$, $k = 0, 1, \ldots, n-1$, and that $\left|f^{\Delta^n}\right|$ is increasing on $[a, b] \cap T$.
Here σ is continuous and $h_{n-1}(t, s)$ jointly continuous. Then

$$\int_a^b |f(t)|\, |f^{\Delta^n}(t)| \Delta t \leq (b-a)^{\frac{1}{q}} \cdot \tag{41.49}$$
$$\left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau\right) \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t)\right)^{2q} \Delta t\right)^{\frac{1}{q}}.$$

Comment 41.28. Let f, g: $T \to \mathbb{R}$ and $f^*, g^* : T^* \to \mathbb{R}$, where $T^* = -T$, with $g^*(t) = f(-t)$, $g^*(t) = g(-t)$, $t \in T^*$.

Consider the product $f.g : T \to \mathbb{R}$, then

$$(f.g)^*(t) = (f.g)(-t) = f(-t).g(-t) = f^*(t).g^*(t).$$

I.e.

$$(f.g)^* = f^* \cdot g^*. \tag{41.50}$$

Let $t \leq s$, t, $s \in T^*$, and $f^* \cdot T^* \to \mathbb{R}$ being increasing, i.e. $f^*(t) \leq f^*(s)$, equivalently, $f(-t) \leq f(-s)$, here $-t \geq -s$.

So f^* is increasing on T^* (decreasing) iff f is decreasing on T (increasing).

We give the following nabla Opial type inequality.

Theorem 41.29. Here $T_k = T$. Let $f \in C^n_{ld}(T)$, n is odd, $a, b \in T : a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose $f^{\nabla^k}(b) = 0$, $k = 0, 1, \ldots, n-1$. Here ρ is continuous and $\hat{h}_{n-1}(t, s)$ is jointly continuous. Assume also that $\left| f^{\nabla^n} \right|$ is decreasing on $[a, b] \cap T$.

Then

$$\int_a^b |f(t)| \left| f^{\nabla^n}(t) \right| \nabla t \leq (b-a)^{\frac{1}{q}} \cdot$$
$$\left(\int_a^b \left(\int_t^b \hat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau \right) \nabla t \right)^{\frac{1}{p}} \cdot$$
$$\left(\int_a^b \left(f^{\nabla^n}(t) \right)^{2q} \nabla t \right)^{\frac{1}{q}}. \tag{41.51}$$

Proof. Here again we have $T^* = (T^*)^k$, and $(f^*)^{\widehat{\Delta}^k}(-b) = 0$, $k = 0, 1 \ldots, n-1$. By (41.9) and $|f|^* = |f^*|$ we find that

$$\left| (f^*)^{\hat{\Delta}^n} \right| = \left| (f^{\nabla^n})^* \right| = \left| f^{\nabla^n} \right|^* \tag{41.52}$$

is increasing on T^*.

Notice also that

$$\left| f^{\nabla^n} \right|^*(t) = \left| f^{\nabla^n}(-t) \right|, \tag{41.53}$$

and

$$\left(\left(f^{\nabla^n} \right)^{2q} \right)^*(t) = \left(f^{\nabla^n}(-t) \right)^{2q}, \quad t \in T^*. \tag{41.54}$$

Furthermore by (41.11), (41.50), (41.19) we get

$$\int_a^b |f(t)||f^{\nabla^n}(t)|\nabla t =$$
$$\int_{-b}^{-a} |f(t)|^* \, |f^{\nabla^n}(t)|^* \hat{\triangle} t =$$
$$\int_{-b}^{-a} |f(t)|^* \, |(f^*)^{\hat{\triangle}^n}(t)| \hat{\triangle} t$$
$$\overset{(by(41.53))}{=} \int_{-b}^{-a} |f(-t)| \, |f^{\nabla^n}(-t)| \hat{\triangle}\, t =: I_1. \tag{41.55}$$

We further notice

$$\int_{-b}^{-a} \left((f^*)^{\hat{\triangle}^n}(t)\right)^{2q} \hat{\triangle}\, t \overset{(by(41.9))}{=}$$
$$\int_{-b}^{-a} \left(\left(f^{\nabla^n}\right)^*(t)\right)^{2q} \hat{\triangle}\, t =$$
$$\int_{-b}^{-a} \left(f^{\nabla^n}(-t)\right)^{2q} \hat{\triangle}\, t \overset{(by(41.13))}{=}$$
$$\int_a^b \left(f^{\nabla^n}(t)\right)^{2q} \nabla\, t.$$

That is

$$\int_{-b}^{-a} \left((f^*)^{\hat{\triangle}^n}(t)\right)^{2q} \hat{\triangle} t =$$
$$\int_a^b \left(f^{\nabla^n}(t)\right)^{2q} \nabla t. \tag{41.56}$$

As in the proof of Theorem 41.24 we derive that

$$\int_{-b}^{-a} \left(\int_{-b}^{t} h_{n-1}^*(t, \hat{\sigma}(\tau))^p \hat{\triangle} \tau\right) \hat{\triangle} t =$$
$$\int_a^b \left(\int_t^b \hat{h}_{n-1}(t, \rho(\tau))^p \nabla \tau\right) \nabla t. \tag{41.57}$$

So we apply (41.49) for f^* on $\mathbb{T}^*$ to obtain

$$I_1 \le (b-a)^{1/q} \cdot \left(\int_{-b}^{-a}\left(\int_{-b}^{t} h_{n-1}^*(t,\hat{\sigma}(\tau))^p \hat{\Delta}\tau\right)\hat{\Delta}t\right)^{1/p} \cdot \left(\int_{-b}^{-a}\left((f^*)^{\hat{\Delta}^n}(t)\right)^{2q}\hat{\Delta}t\right)^{1/q} \overset{(by\ (41.57)\ and\ (41.56))}{=} (b-a)^{1/q} \cdot \left(\int_a^b\left(\int_t^b \hat{h}_{n-1}(t,\rho(\tau))^p \nabla\tau\right)\nabla t\right)^{1/p} \cdot \left(\int_a^b \left(f^{\nabla^n}(t)\right)^{2q}\nabla t\right)^{1/q}, \tag{41.58}$$

proving (41.51). ∎

We need the delta Hilbert-Pachpatte type inequality which follows:

Theorem 41.30. ([57]) Let $\varepsilon > 0$, $i = 1,2$; $f_i \in C_{rd}^n(T_i)$, n is odd, with $f_i^{\Delta^k}(a_i) = 0$, $k = 0, 1, \ldots, n-1$; $a_i \le b_i$; $a_i, b_i \in T_i$, time scale. Here $T_i^k = T_i$, $i = 1,2$. Let also $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Put

$$F(t_1) = \int_{a_1}^{t_1} h_{n-1}^{(1)}(t_1,\sigma(\tau_1))^p\,\Delta\tau_1, \tag{41.59}$$

for all $t_1 \in [a_1,b_1] \cap T_1$, and

$$G(t_2) = \int_{a_2}^{t_2} h_{n-1}^{(2)}(t_2,\sigma_2(\tau_2))^q\Delta\tau_2, \tag{41.60}$$

for all $t_2 \in [a_2,b_2] \cap T_2$ (where $h_{n-1}^{(i)}$, $\sigma^{(i)}$ the corresponding h_{n-1}, σ to T_i, $i = 1,2$). Here σ_i is continuous and $h_{n-1}^{(i)}(t_i, s_i)$ jointly continuous in t_i, $s_i \in T_i$.

We further suppose that

$$\lambda(t_1) = \int_{a_2}^{b_2}\left(\frac{|f_2(t_2)|}{\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{p}}\right)\Delta t_2 \tag{41.61}$$

is an rd-continuous function on T_1.

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left(\frac{|f_1(t_1)||f_2(t_2)|}{\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}}\right)\Delta t_1\Delta t_2 \le (b_1-a_1)(b_2-a_2) \cdot \left(\int_{a_1}^{b_1}\left|f_1^{\Delta^n}(\tau_1)\right|^q\Delta\tau_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|f_2^{\Delta^n}(\tau_2)\right|^p\Delta\tau_2\right)^{\frac{1}{p}} \tag{41.62}$$

(above double time scales integration is considered in the natural iterative way).

We give the following nabla Hilbert-Pachpatte type inequality.

Theorem 41.31. Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C^n_{ld}(\mathbb{T}_i)$, n is odd, with $f_i^{\nabla^k}(b_i) = 0$, $k = 0, 1, \ldots, n-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{T}_i$, time scale. Let also $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Put

$$\bar{F}(t_1) = \int_{t_1}^{b_1} \hat{h}^{(1)}_{n-1}(t_1, \rho_1(\tau_1))^p \nabla \tau_1, \quad \text{for all } \ t_1 \in [a_1, b_1] \cap T_1 \tag{41.63}$$

and

$$\bar{G}(t_2) = \int_{t_2}^{b_2} \hat{h}^{(2)}_{n-1}(t_2, \rho_2(\tau_2))^q \nabla \tau_2, \quad \text{for all } \ t_2 \in [a_2, b_2] \cap T_2. \tag{41.64}$$

Here $\hat{h}^{(i)}_{n-1}$, $\rho^{(i)}$ are the corresponding $\hat{h}_{n-1}, \rho$ to $\mathbb{T}_i$, $i = 1, 2$, and are all assumed continuous. Also $\mathbb{T}_i$, $i = 1, 2$, are such that $T_{i_k} = T_i$.

We further suppose that

$$\Theta(t_1) = \int_{a_2}^{b_2} \frac{|f_2(t_2)|}{\left(\varepsilon + \frac{\overline{F(t_1)}}{p} + \frac{\overline{G(t_2)}}{q}\right)} \nabla t_2 \tag{41.65}$$

is an ld-continuous function on $\mathbb{T}_1$.

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)||f_2(t_2)|}{\left(\varepsilon + \frac{\overline{F(t_1)}}{p} + \frac{\overline{G(t_2)}}{q}\right)} \nabla t_1 \nabla t_2 \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} |f_1^{\nabla^n}(t_1)|^q \nabla t_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} |f_2^{\nabla^n}(t_2)|^p \nabla t_2 \right)^{\frac{1}{p}} \tag{41.66}$$

(above double time scales nabla integration is considered in the natural iterative way).

Proof. We have that $(f_i^*)^{\hat{\Delta}^k}(-b_i) = 0$, $k = 0, 1, \ldots, n-1$, and $T_i^* = (T_i^*)^k$, $i = 1, 2$.

Also $f_i^* \in C^n_{rd}(T_i^*)$ iff

$$f_i \in C^n_{ld}(T_i), \quad i = 1, 2.$$

By (41.28) and assumption we get that $h^{*(i)}_{n-1}$ are jointly continuous on $(T_i^*)^2$, $i = 1, 2$.

Also $\hat{\sigma_i}(s_i) \overset{(by (41.4))}{=} -\rho_i(-s_i)$, $s_i \in T_i^*$, is continuous.

We notice that

$$\begin{aligned}
&\overline{F}^*(t_1) = \overline{F}(-t_1) \overset{(41.63)}{=} \int_{-t_1}^{b_1} \hat{h}_{n-1}^{(1)}(-t_1, \rho_1(\tau_1))^p \nabla\tau_1 \\
&\overset{(41.4)}{=} \int_{-t_1}^{b_1} \hat{h}_{n-1}^{(1)}(-t_1, -\hat{\sigma}_1(-\tau_1))^p \nabla\tau_1 \\
&\overset{(41.28)}{=} \int_{-t_1}^{b_1} h_{n-1}^{*(1)}(t_1, \hat{\sigma}_1(-\tau_1))^p \nabla\tau_1 \\
&\overset{(41.13)}{=} \int_{-b_1}^{t_1} h_{n-1}^{*(1)}(t_1, \hat{\sigma}_1(\tau_1))^p \hat{\Delta}\tau_1 \\
= \quad & F(t_1) \quad , \forall\, t_1 \in [-b_1, -a_1] \cap T_1^*,
\end{aligned}$$

where F as in (41.59).

Similarly we get

$$\begin{aligned}
&\overline{G}^*(t_2) = \overline{G}(-t_2) \overset{(41.64)}{=} \int_{-t_2}^{b_2} \hat{h}_{n-1}^{(2)}(-t_2, \rho_2(\tau_2))^q \nabla\tau_2 \\
&\overset{(41.4)}{=} \int_{-t_2}^{b_2} \hat{h}_{n-1}^{(2)}(-t_2, -\hat{\sigma}_2(-\tau_2))^q \nabla\tau_2 \\
&\overset{(41.28)}{=} \int_{-t_2}^{b_2} h_{n-1}^{*(2)}(t_2, \hat{\sigma}_2(-\tau_2))^q \nabla\tau_2 \\
&\overset{(41.13)}{=} \int_{-b_2}^{t_2} h_{n-1}^{*(2)}(t_2, \hat{\sigma}_2(\tau_2))^q \hat{\Delta}\tau_2 \\
= \quad & G(t_2),\ \forall\, t_2 \in [-b_2, -a_2] \cap T_2^*,
\end{aligned}$$

where G as in (41.60).

So we proved that

$$\overline{F}^*(t_1) = F(t_1), \forall\, t_1 \in [-b_1, -a_1] \cap T_1^*, \tag{41.67}$$

and

$$\overline{G}^*(t_2) = G(t_2), \forall\, t_2 \in [-b_2, -a_2] \cap T_2^*. \tag{41.68}$$

Here we have that

$$\begin{aligned}
I_1 := \quad & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)||f_2(t_2)|}{(\varepsilon + \frac{\overline{F}(t_1)}{P} + \frac{\overline{G}(t_2)}{q})} \nabla t_1 \nabla t_2 = \\
& \int_{a_1}^{b_1} |f_1(t_1)| \theta(t_1) \nabla t_1 \overset{(41.11)}{=} \\
& \int_{-b_1}^{-a_1} |f_1^*(t_1)| \theta^*(t_1) \hat{\Delta} t_1.
\end{aligned} \tag{41.69}$$

Next we observe ($\forall\ t_1 \in [-b_1, -a_1] \cap T_1^*$)

$$\begin{aligned}\theta^*(t_1) &= \theta(-t_1)\\ &\overset{(41.65)}{=} \int_{a_2}^{b_2} \frac{|f_2(t_2)|}{\left(\varepsilon + \frac{\overline{F}(-t_1)}{p} + \frac{\overline{G}(t_2)}{q}\right)} \nabla t_2\\ &\overset{(41.67)}{=} \int_{a_2}^{b_2} \frac{|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{\overline{G}(t_2)}{q}\right)} \nabla t_2\end{aligned}$$

(notice $\overline{G}(t_2)$ is continuous in $t_2 \in [a_2, b_2] \cap T_2$,

$$\overset{(by\ (41.11),(41.13))}{=} \int_{-b_2}^{-a_2} \frac{|f_2^*(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{\overline{G}(-t_2)}{q}\right)} \widehat{\Delta} t_2$$

$$\overset{(41.68)}{=} \int_{-b_2}^{-a_2} \frac{|f_2^*(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \hat{\Delta} t_2$$

$$\overset{(41.61)}{=} \lambda(t_1),\ \forall\ t_1 \in [-b_1, -a_1] \cap T_1^*.$$

Clearly here $\lambda(t_1)$ is an rd-continuous function on $[-b_1, -a_1] \cap T_1^*$.

So here

$$\theta^*(t_1) = \lambda(t_1),\ \forall\ t_1 \in [-b_1, -a_1] \cap T_1^*. \tag{41.70}$$

Therefore we obtain

$$\begin{aligned}I_1 &\overset{(41.70)}{=} \int_{-b_1}^{-a_1} |f_1^*(t_1)|\lambda(t_1)\hat{\Delta} t_1\\ &= \int_{-b_1}^{-a_1} |f_1^*(t_1)| \left(\int_{-b_2}^{-a_2} \frac{|f_2^*(t_2)|\ \hat{\Delta} t_2}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \right) \hat{\Delta} t_1\\ &= \int_{-b_1}^{-a_1} \int_{-b_2}^{-a_2} \frac{|f_1^*(t_1)||f_2^*(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \hat{\Delta} t_1 \hat{\Delta} t_2\\ &\overset{(by\ (41.62))}{\leq} (b_1 - a_1)(b_2 - a_2) \left(\int_{-b_1}^{-a_1} |(f_1^*)^{\Delta^n}(t_1)|^q \hat{\Delta} t_1 \right)^{1/q} \cdot\\ &\left(\int_{-b_2}^{-a_2} |(f_2^*)^{\Delta^n}(t_2)|^p \hat{\Delta} t_2 \right)^{1/p}\\ &\overset{(41.46)}{=} (b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} |f_1^{\nabla^n}(t_1)|^q \nabla t_1 \right)^{1/q} \cdot\\ &\left(\int_{a_2}^{b_2} |f_2^{\nabla^n}(t_2)|^p \nabla t_2 \right)^{1/p},\end{aligned} \tag{41.71}$$

proving (41.66). ∎

One can go reverse, and using the nabla inequalities to prove the delta ones, etc.

Also one can prove similarly other inequalities by applying this principle of time scales duality.

41.6 Applications

For applications to delta Ostrowski inequalities, see [57] and to nabla Ostrowski inequalities, see [59].

For applications to the rest of delta inequalities mentioned in this chapter, see [57].

Here we give applications to the rest of derived nabla inequalities.

I) Here $T = \mathbb{R}$, the real numbers, then $\rho(t) = t$, $t \in \mathbb{R}$, $\hat{h}_k(t,s) = \frac{(t-s)^k}{k!}$ for all $s, t \in \mathbb{R}$, $k \in \mathbb{N}_0$.

Also $f^{\nabla^k}(t) = f^{(k)}(t),\ k \in \mathbb{N}_0$, and $\int_a^b f(t)\nabla t = \int_a^b f(t)dt$.

Furthermore $f \in C_{ld}^n(\mathbb{R})$ iff $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}_0$.

A Poincare type inequality follows:

Theorem 41.32. Let $f \in C^n(\mathbb{R})$, n is odd, $a, b \in \mathbb{R} : a \leq b; p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $f^{(k)}(b) = 0$, $k = 0, 1, \ldots, n-1$. Then

$$\int_a^b |f(t)|^q \, dt \leq \frac{(b-a)^{nq}}{nq((n-1)!)^q((n-1)p+1)^{q-1}} \cdot \left(\int_a^b |f^n(t)|^q \, dt \right). \tag{41.72}$$

Proof. By (41.42). ∎

We give next a Sobolev type inequality.

Theorem 41.33. Here all as in Theorem 41.32. Let $r \geq 1$. Then

$$\|f\|_{r,[a,b]} \leq \frac{(b-a)^{(n-1+\frac{1}{p}+\frac{1}{r})}}{(n-1)!((n-1)p+1)^{\frac{1}{p}}((n-1+\frac{1}{p})r+1)^{\frac{1}{r}}} \cdot \left\| f^{(n)} \right\|_{q,[a,b]}. \tag{41.73}$$

Proof. By (41.48). ∎

We give next an Opial type inequality.

Theorem 41.34. Let $f \in C^n(\mathbb{R})$, n odd, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $f^k(b) = 0$, $k = 0, 1, \ldots, n-1$; $\left| f^{(n)} \right|$ is decreasing on

$[a, b]$. Then

$$\int_a^b |f(t)| \left| f^{(n)}(t) \right| dt \leq \frac{(b-a)^{n+\frac{1}{p}}}{(n-1)! \left((p(n-1)+1)(p(n-1)+2) \right)^{\frac{1}{p}}} \cdot$$

$$\left(\int_a^b \left(f^{(n)}(t) \right)^{2q} dt \right)^{\frac{1}{q}}. \quad (41.74)$$

Proof. By (41.51). ■

We continue with a Hilbert-Pachpatte inequality.

Theorem 41.35. Let $\varepsilon > 0$, $i = 1, 2$, $f_i \in C^n(\mathbb{R})$, n is odd, $f_i^{(k)}(b_i) = 0$, $k = 0, 1, \ldots, n-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{R}$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Put

$$\overline{F}(t_1) = \frac{(b_1 - t_1)^{(n-1)p+1}}{((n-1)!)^p (p(n-1)+1)}, \forall\, t_1 \in [a_1, b_1], \quad (41.75)$$

and

$$\overline{G}(t_2) = \frac{(b_2 - t_2)^{(n-1)q+1}}{((n-1)!)^q (q(n-1)+1)}, \forall\, t_2 \in [a_2, b_2]. \quad (41.76)$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)||f_2(t_2)|}{\left(\varepsilon + \frac{\overline{F(t_1)}}{p} + \frac{\overline{G(t_2)}}{q} \right)} dt_1 dt_2 \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left| f_1^{(n)}(t_1) \right|^q dt_1 \right)^{1/q} \left(\int_{a_2}^{b_2} \left| f_2^{(n)}(t_2) \right|^p dt_2 \right)^{1/p}. \quad (41.77)$$

Proof. By (41.66). ■

II) Here $T = \mathbb{Z}$, the integers.

Then

$$\int_a^b f(t) \nabla t = \sum_{k=a+1}^{b} f(k), \quad \text{where}$$

my $f : \mathbb{Z} \to \mathbb{R}$ is ld-continuous.

Also $f^{\nabla^k}(t) = \nabla^k f(t)$

$$= \sum_{m=0}^{k} (-1)^m \binom{k}{m} f(t-m), \quad k \in \mathbb{N}_0.$$

Furthermore here

$$\hat{h}_k(t,s) = \frac{(t-s)^{\overline{k}}}{k!}, \quad \forall \; s,t \in \mathbb{Z}, \; k \in \mathbb{N}_0,$$
$$t^{\overline{k}} = t(t+1)\dots(t+k-1), \quad k \in \mathbb{N};$$
$$t^{\overline{0}} = 1. \text{ Also } \rho(t) = t-1, \quad t \in \mathbb{Z}.$$

We present a nabla discrete Poincare inequality.

Theorem 41.36. Let $f : \mathbb{Z} \to \mathbb{R}$, n is odd, $a,\, b\; \in \mathbb{Z} : a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose $\nabla^k f(b) = 0$, $k = 0,\, 1, \dots,\, n-1$. Then

$$\sum_{t=a+1}^{b} |f(t)|^q \leq \frac{1}{((n-1)!)^q} \cdot \left(\sum_{t=a+1}^{b} \left(\left(\sum_{\tau=t+1}^{b} ((t-t+1)^{(\overline{n-1})} \right)^p \right)^{q/p} \right) \cdot \left(\sum_{t=a+1}^{b} |\nabla^n f(t)|^q \right). \tag{41.78}$$

Proof. By (41.42). ∎

We give a nabla discrete Sobolev inequality.

Theorem 41.37. Same assumptions as in Theorem 41.36. Let $r \geq 1$. Then

$$\left(\sum_{t=a+1}^{b} |f(t)|^r \right)^{1/r} \leq \frac{1}{(n-1)!} \cdot \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=t+1}^{b} \left((t-\tau+1)^{\overline{n-1}} \right)^p \right)^{r/p} \right)^{1/r} \cdot \left(\sum_{t=a+1}^{b} |\nabla^n f(t)|^q \right)^{1/q}. \tag{41.79}$$

Proof. By (41.48). ∎

We give a nabla discrete Opial inequality.

Theorem 41.38. Let $f : \mathbb{Z} \to \mathbb{R}$, n is odd, $a, b \in \mathbb{Z} : a \leq b$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose $\nabla^k f(b) = 0$, $k = 0, 1, \dots, n-1$. Suppose $|\nabla^n f|$ is decreasing on $[a,b]\cap\mathbb{Z}$. Then

$$\sum_{t=a+1}^{b}|f(t)|\,|\nabla^n f(t)| \leq \frac{(b-a)^{1/q}}{n-1!}\cdot$$
$$\left(\sum_{t=a+1}^{b}\left(\sum_{\tau=t+1}^{b}\left((t-\tau+1)^{\overline{n-1}}\right)^p\right)\right)^{1/p}\cdot$$
$$\left(\sum_{t=a+1}^{b}(\nabla^n f(t))^{2q}\right)^{1/q}. \tag{41.80}$$

Proof. By (41.51). ■

We present a nabla discrete Hilbert-Pachpatte inequality.

Theorem 41.39. Let $\varepsilon>0$, $i=1,2$; $f_i:\mathbb{Z}\to\mathbb{R}$, n is odd, $\nabla^k f_i(b_i)=0$, $k=0,1,\ldots,n-1$; $a_i\leq b_i$; $a_i,b_i\in\mathbb{Z}$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Set

$$\overline{F}(t_1)=\sum_{\tau=t_1+1}^{b_1}\frac{\left((t_1-\tau_1+1)^{\overline{n-1}}\right)^p}{((n-1)!)^p},\quad \forall\, t_1\in[a_1,b_1]\cap\mathbb{Z},$$

and

$$\overline{G}(t_2)=\sum_{\tau_2=t_2+1}^{b_2}\frac{\left((t_2-\tau_2+1)^{\overline{n-1}}\right)^q}{((n-1)!)^q},\quad \forall\, t_2\in[a_2,b_2]\cap\mathbb{Z}.$$

Then

$$\sum_{t_1=a_1+1}^{b_1}\sum_{t_2=a_2+1}^{b_2}\frac{|f_1(t_1)||f_2(t_2)|}{\left(\varepsilon+\frac{\overline{F}(t_1)}{p}+\frac{\overline{G(t_2)}}{q}\right)}$$
$$\leq (b_1-a_1)(b_2-a_2)\left(\sum_{t_1=a_1+1}^{b_1}|\nabla^n f_1(t_1)|^q\right)^{1/q}\left(\sum_{t_2=a_2+1}^{b_2}|\nabla^n f_2(t_2)|^p\right)^{1/p}. \tag{41.81}$$

Proof. By (41.66). ■

III) Here $T=q_*^{\overline{\mathbb{Z}}}=\{0,1,q_*,q_*^{-1},q_*^2,q_*^{-2},\ldots\}$, for some $q_*>1$, see [93]. We have $\rho(t)=t/q_*$, $\forall t\in q_*^{\overline{\mathbb{Z}}}$ and

$$\hat{h}_k(t,s)=\prod_{r=0}^{k-1}\frac{q_*^r t-s}{\sum\limits_{j=0}^{r}q_*^j}, \tag{41.82}$$

$\forall\, s,t\in q_*^{\overline{\mathbb{Z}}}$, for all $k\in\mathbb{N}_0$.

We finish with a q-Opial type nabla inequality.

Theorem 41.40. Let $f \in C_{ld}^{n}(\overline{q_*^{\mathbb{Z}}})$, n is odd, $a,\ b \in \overline{q_*^{\mathbb{Z}}} : a \leq b$; $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{\nabla^k}(b) = 0,\ \ k = 0, 1, \ldots, n-1$. Suppose $|f^{\nabla^n}|$ is decreasing on $[a, b] \cap \overline{q_*^{\mathbb{Z}}}$. Then

$$\int_a^b |f(t)||f^{\nabla^n}(t)|\nabla t \leq (b-a)^{1/q} \cdot$$
$$\left(\int_a^b \left(\int_t^b \left(\prod_{r=0}^{n-2} \frac{q_*^r t - \frac{\tau}{q_*}}{\sum\limits_{j=0}^{r} q_*^j} \right)^p \nabla\tau \right) \nabla t \right)^{1/p} \cdot$$
$$\left(\int_a^b \left(f^{\nabla^n}(t) \right)^{2q} \nabla t \right)^{1/q}. \tag{41.83}$$

Proof. By (41.51). ■

One can give many similar applications for other time scales.

42

Foundations of Delta Fractional Calculus on Time Scales with Inequalities

Here we present the Delta Fractional Calculus on Time Scales. Then we prove related integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. At the end we give inequalities applications on the time scale $\mathbb{R}$. This chapter is based on [56].

42.1 Background and Foundation Results

For the basics on time scales we use [119], [113] and [2], [4], [57], [114], [116], [181], [186], [187], [215].

By [282], p. 256, for $\mu, \nu > 0$ we have that

$$\int_t^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \frac{(s-t)^{\nu-1}}{\Gamma(\nu)} ds = \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \tag{42.1}$$

where Γ is the gamma function.

Here we consider time scales T such that $T^k = T$.

Consider the coordinate wise rd-continuous functions $h_\alpha : T \times T \to \mathbb{R}$, $\alpha \geq 0$, such that $h_0(t,s) = 1$,

$$h_{\alpha+1}(t,s) = \int_s^t h_\alpha(\tau, s) \Delta\tau, \tag{42.2}$$

$\forall\ s, t \in T$.

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Here σ is the forward jump operator and $\mu(t) = \sigma(t) - t$.

Furthermore for $\alpha, \beta > 1$ we suppose that

$$\int_{\sigma(u)}^{t} h_{\alpha-1}(t, \sigma(\tau))\, h_{\beta-1}(\tau, \sigma(u))\, \Delta\tau = h_{\alpha+\beta-1}(t, \sigma(u)), \tag{42.3}$$

for all $u < t$; $u, t \in T$.

In the case of $T = \mathbb{R}$; then $\sigma(t) = t$, and $h_k(t, s) = \frac{(t-s)^k}{k!}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and define

$$h_\alpha(t, s) = \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

Notice that

$$\int_s^t \frac{(\tau - s)^\alpha}{\Gamma(\alpha+1)} d\tau = \frac{(t-s)^{\alpha+1}}{\Gamma(\alpha+2)} = h_{\alpha+1}(t, s),$$

fulfilling (42.2).

Furthermore we see that $(\alpha, \beta > 1)$

$$\int_u^t h_{\alpha-1}(t, \tau)\, h_{\beta-1}(\tau, u)\, d\tau = \int_u^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\tau-u)^{\beta-1}}{\Gamma(\beta)} d\tau$$

$$\overset{(\text{by } (42.1))}{=} \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = h_{\alpha+\beta-1}(t, u),$$

fulfilling (42.3).

By Theorem 4.1 of [115], we have for $k, m \in \mathbb{N}_0$ that

$$\int_{t_0}^{t} h_k(t, \sigma(\tau))\, h_m(\tau, t_0)\, \Delta\tau = h_{k+m+1}(t, t_0). \tag{42.4}$$

Let now $T = \mathbb{Z}$, $t \in \mathbb{Z}$, then $\sigma(t) = t + 1$, and $h_k(t, s) = \frac{(t-s)^{(k)}}{k!}$, $\forall\ k \in \mathbb{N}_0$, $\forall\ t, s \in \mathbb{Z}$, where $t^{(0)} = 1$, $t^{(k)} = \prod_{i=0}^{k-1} (t - i)$ for $k \in \mathbb{N}$.

Also $\int_a^b f(t)\, \Delta\tau = \sum_{t=a}^{b-1} f(t)$, $a < b$.

By (42.4) we obtain that

$$\sum_{\tau=t_0}^{t-1} \frac{(t-\tau-1)^{(k)}}{k!} \frac{(\tau-t_0)^{(m)}}{m!} = \frac{(t-t_0)^{(k+m+1)}}{(k+m+1)!},$$

which leads to

$$\sum_{\tau=t_{0+1}}^{t-1} \frac{(t-\tau-1)^{(k-1)}}{(k-1)!} \frac{(\tau-t_0-1)^{(m-1)}}{(m-1)!} = \frac{(t-t_0-1)^{(k+m-1)}}{(k+m-1)!}, \tag{42.5}$$

confirming (42.3).

In general let $\mu, \nu > 0$, and $t \in N_{\mu+\nu} := \{\mu+\nu, \mu+\nu+1, \mu+\nu+2, ...\}$, here $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$ and $\sigma(s) = s+1$. Let $r \in \{0, 1, ..., t-(\mu+\nu)\}$, by proof of Theorem 2.2 of [104] we obtain

$$\frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=r+\nu}^{t-\mu} (t-\sigma(s))^{(\mu-1)} (s-\sigma(r))^{(\nu-1)} = \frac{1}{\Gamma(\nu+\mu)} (t-\sigma(r))^{(\nu+\mu-1)},$$

which is

$$\sum_{s=r+\nu}^{t-\mu} \frac{(t-s-1)^{(\mu-1)}}{\Gamma(\mu)} \frac{(s-r-1)^{(\nu-1)}}{\Gamma(\nu)} = \frac{(t-r-1)^{(\nu+\mu-1)}}{\Gamma(\nu+\mu)}; \quad \mu, \nu > 0, \quad (42.6)$$

that is almost confirming (42.3). By Lemma 19 of [48] for only $-1 < \alpha < 0$, $t, s \in \mathbb{Z}$, $t > s$, we get

$$\sum_{\tau=s}^{t-1} \frac{(\tau-s)^{(\alpha)}}{\Gamma(\alpha+1)} = \frac{(t-s)^{(\alpha+1)}}{\Gamma(\alpha+2)} - \frac{1}{\Gamma(\alpha+2)\Gamma(-\alpha)}, \quad (42.7)$$

missing (42.2).

So in case of $T = \mathbb{Z}$, because of the deficiencies of (42.6) and (42.7) we gave a special treatment to the subject of discrete fractional calculus and inequalities, presented in [48], see also related [51].

We need

Theorem 42.1. (Theorem 1.75 of [119]) If $f \in C_{rd}$ and $t \in T^k$, then

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t) f(t). \quad (42.8)$$

For $\alpha \geq 1$ we define the time scale Δ-Riemann-Liouville type fractional integral $(a, b \in T)$

$$K_a^\alpha f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau)\Delta\tau, \quad (42.9)$$

(by [116] is an integral on $[a, t) \cap T$)

$$K_a^0 f = f,$$

where $f \in L_1([a, b] \cap T)$ (Lebesgue Δ-integrable functions on $[a, b] \cap T$, see [181], [113], [114]), $t \in [a, b] \cap T$.

Notice $K_a^1 f(t) = \int_a^t f(\tau)\Delta\tau$ is absolutely continuous in $t \in [a, b] \cap T$, see [116].

Lemma 42.2. Let $\alpha > 1$, $f \in L_1([a,b] \cap T)$. Suppose $h_{\alpha-1}(s,\sigma(t))$ is additionally Lebesgue Δ-measurable on $([a,b] \cap T)^2$; $a,b \in T$. Then $K_a^\alpha f \in L_1([a,b] \cap T)$.

Proof. Define $\Lambda : \Omega = ([a,b] \cap T)^2 \to \mathbb{R}$, by

$$\Lambda(s,t) = \begin{cases} h_{\alpha-1}(s,\sigma(t)), & \text{if } a \le t \le s \le b, \\ 0, & \text{if } a \le s < t \le b. \end{cases}$$

Clearly $\Lambda(s,t)$ is Lebesgue Δ-measurable on $([a,b] \cap T)^2$.

Then

$$\int_a^b \Lambda(s,t)\,\Delta s = \int_{[a,t)} \Lambda(s,t)\,\Delta s + \int_t^b \Lambda(s,t)\,\Delta s$$

$$= \int_t^b \Lambda(s,t)\,\Delta s = \int_t^b h_{\alpha-1}(s,\sigma(t))\,\Delta s$$

$$= \int_t^{\sigma(t)} h_{\alpha-1}(s,\sigma(t))\,\Delta s + \int_{\sigma(t)}^b h_{\alpha-1}(s,\sigma(t))\,\Delta s$$

$$\overset{\text{(by (42.8) and (42.2))}}{=} \mu(t)\,h_{\alpha-1}(t,\sigma(t)) + h_\alpha(b,\sigma(t)) \in \mathbb{R}.$$

Next we consider the repeated double Lebesgue Δ-integral

$$\int_a^b \left(\int_a^b \Lambda(s,t)\,|f(t)|\,\Delta s\right)\Delta t = \int_a^b |f(t)| \left(\int_a^b \Lambda(s,t)\,\Delta s\right)\Delta t =$$

$$\int_a^b |f(t)| \left\{\mu(t)\,h_{\alpha-1}(t,\sigma(t)) + h_\alpha(b,\sigma(t))\right\}\Delta t =$$

$$\int_a^b |f(t)|\,\mu(t)\,h_{\alpha-1}(t,\sigma(t))\,\Delta t + \int_a^b |f(t)|\,h_\alpha(b,\sigma(t))\,\Delta t,$$

which exists and is finite. Thus the function $(s,t) \to \Lambda(s,t)\,f(t)$ is Lebesgue Δ-integrable over Ω by Tonelli's theorem.

Let now the characteristic function

$$\chi_{[a,s)\cap T}(t) = \begin{cases} 1, & \text{if } t \in [a,s) \cap T \\ 0, & \text{else,} \end{cases}$$

where $s \in [a,b] \cap T$.

Then the function $(s,t) \to \chi_{[a,s)\cap T}(t)\,\Lambda(s,t)\,f(t)$ is Lebesgue Δ-integrable on Ω. Hence by Fubini's theorem we obtain that

$$\int_a^b \chi_{[a,s)\cap T}(t)\,\Lambda(s,t)\,f(t)\,\Delta t = \int_a^s h_{\alpha-1}(s,\sigma(t))\,f(t)\,\Delta t = K_a^\alpha f(s),$$

is Lebesgue Δ-integrable in s on $[a,b] \cap T$, proving the claim. ∎

For $u < t$; $u, t \in T$, we define $(\alpha, \beta > 1)$

$$\theta(t,u) = \int_u^{\sigma(u)} h_{\alpha-1}(t,\sigma(\tau))\, h_{\beta-1}(\tau,\sigma(u))\, \Delta\tau \tag{42.10}$$

$$\overset{\text{(by (42.8))}}{=} \mu(u)\, h_{\alpha-1}(t,\sigma(u))\, h_{\beta-1}(u,\sigma(u)).$$

Next we notice for $\alpha, \beta > 1$; $a, b \in T$, $f \in L_1([a,b] \cap T)$ and $h_{\alpha-1}(s,\sigma(t))$ continuous on $([a,b] \cap T)^2$ for any $\alpha > 1$, that

$$K_a^{\alpha} K_a^{\beta} f(t) = \int_a^t h_{\alpha-1}(t,\sigma(\tau))\, \Delta\tau \int_a^{\tau} h_{\beta-1}(\tau,\sigma(u))\, f(u)\, \Delta u$$

(by Fubini's theorem)

$$= \int_a^t f(u)\, \Delta u \int_u^t h_{\alpha-1}(t,\sigma(\tau))\, h_{\beta-1}(\tau,\sigma(u))\, \Delta\tau =$$

$$\int_a^t f(u)\, \Delta u \cdot \left[\int_u^{\sigma(u)} h_{\alpha-1}(t,\sigma(\tau))\, h_{\beta-1}(\tau,\sigma(u))\, \Delta\tau \right.$$

$$\left. + \int_{\sigma(u)}^t h_{\alpha-1}(t,\sigma(\tau))\, h_{\beta-1}(\tau,\sigma(u))\, \Delta\tau \right]$$

$$= \int_a^t f(u)\, \Delta u \left(h_{\alpha+\beta-1}(t,\sigma(u)) + \theta(t,u) \right)$$

$$= \int_a^t h_{\alpha+\beta-1}(t,\sigma(u))\, f(u)\, \Delta u + \int_a^t f(u)\, \theta(t,u)\, \Delta u$$

$$= K_a^{\alpha+\beta} f(t) + \int_a^t f(u)\, \theta(t,u)\, \Delta u.$$

Thus

$$K_a^{\alpha} K_a^{\beta} f(t) - \int_a^t f(u)\, \theta(t,u)\, \Delta u = K_a^{\alpha+\beta} f(t), \quad \forall\ t \in [a,b] \cap T. \tag{42.11}$$

So we have proved the semigroup property

$$K_a^{\alpha} K_a^{\beta} f(t) - \int_a^t f(u)\, \mu(u)\, h_{\alpha-1}(t,\sigma(u))\, h_{\beta-1}(u,\sigma(u))\, \Delta u = K_a^{\alpha+\beta} f(t), \tag{42.12}$$

$\forall\ t \in [a,b] \cap T$, with $a, b \in T$.

We call the Lebesgue Δ-integral

$$E(f,\alpha,\beta,T,t) = \int_a^t f(u)\, \mu(u)\, h_{\alpha-1}(t,\sigma(u))\, h_{\beta-1}(u,\sigma(u))\, \Delta u, \tag{42.13}$$

$t \in [a,b] \cap T$; $a,b \in T$, the forward graininess deviation functional of $f \in L_1([a,b] \cap T)$.

If $T = \mathbb{R}$, then $E(f,\alpha,\beta,T,t) = 0$.

Putting things together we have

Theorem 42.3. Let $T = T^k$, $a,b \in T$, $f \in L_1([a,b] \cap T)$; $\alpha,\beta > 1$; $h_{\alpha-1}(s,\sigma(t))$ is continuous on $([a,b] \cap T)^2$ for any $\alpha > 1$. Then

$$K_a^{\alpha} K_a^{\beta} f(t) - E(f,\alpha,\beta,T,t) = K_a^{\alpha+\beta} f(t), \tag{42.14}$$

$\forall\, t \in [a,b] \cap T$.

We make

Remark 42.4. Let $\mu > 2 : m-1 < \mu < m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$ (ceiling of the number), $\widetilde{\nu} = m - \mu$ $(0 < \widetilde{\nu} < 1)$.

Here we take $f \in C_{rd}^m([a,b] \cap T)$. Clearly here ([181]) f^{Δ^m} is a Lebesgue Δ-integrable function.

We define the delta fractional derivative on time scale T of order $\mu - 1$ as follows:

$$\Delta_{a*}^{\mu-1} f(t) = \left(K_a^{\widetilde{\nu}+1} f^{\Delta^m}\right)(t) = \int_a^t h_{\widetilde{\nu}}(t,\sigma(\tau)) f^{\Delta^m}(\tau)\Delta\tau, \tag{42.15}$$

$\forall\, t \in [a,b] \cap T$.

Notice here that $\Delta_{a*}^{\mu-1} f \in C([a,b] \cap T)$ by a simple argument using dominated convergence theorem in Lebesgue Δ-sense.

If $\mu = m$, then $\widetilde{\nu} = 0$ and by (42.15) we obtain

$$\Delta_{a*}^{m-1} f(t) = K_a^1 f^{\Delta^m}(t) = f^{\Delta^{m-1}}(t). \tag{42.16}$$

More generally, by [116], given that $f^{\Delta^{m-1}}$ is everywhere finite and absolutely continuous on $[a,b] \cap T$, then f^{Δ^m} exists Δ-a.e. and is Lebesgue Δ-integrable on $[a,t) \cap T$, $\forall\, t \in [a,b] \cap T$ and one can plug it into (42.15).

We see that

$$K_a^{\mu-1} \Delta_{a*}^{\mu-1} f(t) = \left(K_a^{\mu-1} K_a^{\widetilde{\nu}+1} f^{\Delta^m}\right)(t)$$

$$\overset{\text{(by (42.14))}}{=} \left(K_a^{\mu+\widetilde{\nu}} f^{\Delta^m}\right)(t) + \int_a^t f^{\Delta^m}(u)\mu(u) h_{\mu-2}(t,\sigma(u)) h_{\widetilde{\nu}}(u,\sigma(u))\Delta u =$$

$$\left(K_a^m f^{\Delta^m}\right)(t) + \int_a^t f^{\Delta^m}(u)\mu(u) h_{\mu-2}(t,\sigma(u)) h_{\widetilde{\nu}}(u,\sigma(u))\Delta u.$$

Therefore

$$K_a^{\mu-1} \Delta_{a*}^{\mu-1} f(t) - \int_a^t f^{\Delta^m}(u)\mu(u) h_{\mu-2}(t,\sigma(u)) h_{\widetilde{\nu}}(u,\sigma(u))\Delta u =$$

$$\left(K_a^m f^{\Delta^m}\right)(t) = \int_a^t h_{m-1}(t, \sigma(\tau)) f^{\Delta^m}(\tau) \Delta\tau. \quad (42.17)$$

We have established

Theorem 42.5. Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{rd}^m([a, b] \cap T)$, $a, b \in T$, $T^k = T$. Assume $h_{\mu-2}(s, \sigma(t))$, $h_{\widetilde{\nu}}(s, \sigma(t))$ to be continuous on $([a, b] \cap T)^2$.

Then

$$\int_a^t h_{m-1}(t, \sigma(\tau)) f^{\Delta^m}(\tau) \Delta\tau = \quad (42.18)$$

$$-\int_a^t f^{\Delta^m}(u) \mu(u) h_{\mu-2}(t, \sigma(u)) h_{\widetilde{\nu}}(u, \sigma(u)) \Delta u$$

$$+\int_a^t h_{\mu-2}(t, \sigma(\tau)) \Delta_{a*}^{\mu-1} f(\tau) \Delta\tau,$$

$\forall\, t \in [a, b] \cap T$.

We need the delta time scales Taylor formula

Theorem 42.6. ([115], [186]) Let $f \in C_{rd}^m(T)$, $m \in \mathbb{N}$, $T^k = T$; $a, b \in T$. Then

$$f(t) = \sum_{k=0}^{m-1} h_k(t, a) f^{\Delta^k}(a) + \int_a^t h_{m-1}(t, \sigma(\tau)) f^{\Delta^m}(\tau) \Delta\tau, \quad (42.19)$$

$\forall\, t \in [a, b] \cap T$.

Next we present the fractional time scales delta Taylor formula

Theorem 42.7. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m-\mu$; $f \in C_{rd}^m(T)$, $a, b \in T$, $T^k = T$. Assume $h_{\mu-2}(s, \sigma(t))$, $h_{\widetilde{\nu}}(s, \sigma(t))$ to be continuous on $([a, b] \cap T)^2$. Then

$$f(t) = \sum_{k=0}^{m-1} h_k(t, a) f^{\Delta^k}(a) - \quad (42.20)$$

$$\int_a^t f^{\Delta^m}(u) \mu(u) h_{\mu-2}(t, \sigma(u)) h_{\widetilde{\nu}}(u, \sigma(u)) \Delta u +$$

$$\int_a^t h_{\mu-2}(t, \sigma(\tau)) \Delta_{a*}^{\mu-1} f(\tau) \Delta\tau,$$

$\forall\, t \in [a, b] \cap T$.

Corollary 42.8. All as in Theorem 42.7. Additionally suppose $f^{\Delta^k}(a) = 0$, $k = 0, 1, ..., m - 1$. Then

$$B(t) := f(t) + E\left(f^{\Delta^m}, \mu - 1, \widetilde{\nu} + 1, T, t\right) \quad (42.21)$$

$$= f(t) + \int_a^t f^{\Delta^m}(u)\,\mu(u)\,h_{\mu-2}(t,\sigma(u))\,h_{\widetilde{\nu}}(u,\sigma(u))\,\Delta u$$

$$= \int_a^t h_{\mu-2}(t,\sigma(\tau))\,\Delta_{a*}^{\mu-1} f(\tau)\,\Delta\tau,$$

$\forall\, t \in [a,b] \cap T$.

Notice that $E\left(f^{\Delta^m}, \mu-1, \widetilde{\nu}+1, T, t\right) \in C_{rd}([a,b]\cap T)$. Also the R.H.S (42.21) is a continuous function in $t \in [a,b] \cap T$.

42.2 Fractional Delta Inequalities on Time Scales

We give a Poincaré type related inequality.

Theorem 42.9. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{rd}^m(T)$, $a, b \in T$, $a \leq b$, $T^k = T$. Suppose $h_{\mu-2}(s,\sigma(t))$, $h_{\widetilde{\nu}}(s,\sigma(t))$ to be continuous on $([a,b]\cap T)^2$, and $f^{\Delta^k}(a) = 0$, $k = 0, 1, ..., m-1$. Here $B(t) = f(t) + E\left(f^{\Delta^m}, \mu-1, \widetilde{\nu}+1, T, t\right)$, $t \in [a,b] \cap T$; and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\int_a^b |B(t)|^q\,\Delta t \leq \left(\int_a^b \left(\int_a^t |h_{\mu-2}(t,\sigma(\tau))|^p\,\Delta\tau\right)^{\frac{q}{p}} \Delta t\right)\left(\int_a^b \left|\Delta_{a*}^{\mu-1} f(t)\right|^q \Delta t\right). \tag{42.22}$$

Proof. By Corollary 42.8 we obtain that

$$B(t) = \int_a^t h_{\mu-2}(t,\sigma(\tau))\,\Delta_{a*}^{\mu-1} f(\tau)\,\Delta\tau.$$

Hence

$$|B(t)| \leq \int_a^t |h_{\mu-2}(t,\sigma(\tau))|\left|\Delta_{a*}^{\mu-1} f(\tau)\right|\Delta\tau$$

(by Hölder's inequality)

$$\leq \left(\int_a^t |h_{\mu-2}(t,\sigma(\tau))|^p\,\Delta\tau\right)^{\frac{1}{p}}\left(\int_a^t \left|\Delta_{a*}^{\mu-1} f(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t |h_{\mu-2}(t,\sigma(\tau))|^p\,\Delta\tau\right)^{\frac{1}{p}}\left(\int_a^b \left|\Delta_{a*}^{\mu-1} f(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}}.$$

Therefore

$$|B(t)|^q \leq \left(\int_a^t |h_{\mu-2}(t,\sigma(\tau))|^p\,\Delta\tau\right)^{\frac{q}{p}}\left(\int_a^b \left|\Delta_{a*}^{\mu-1} f(\tau)\right|^q \Delta\tau\right), \tag{42.23}$$

$\forall\, t \in [a,b] \cap T$.

Next by integrating (42.23) we are proving the claim. ∎

It follows a related Sobolev inequality.

Theorem 42.10. Here all as in Theorem 42.9. Let $r \geq 1$ and denote

$$\|f\|_r = \left(\int_a^b |f(t)|^r \Delta t \right)^{\frac{1}{r}}.$$

Then

$$\|B\|_r \leq \left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{r}{p}} \Delta t \right)^{\frac{1}{r}} \left\| \Delta_{a*}^{\mu-1} f \right\|_q. \tag{42.24}$$

Proof. As in the proof of Theorem 42.9 we have

$$|B(t)| \leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^b \left| \Delta_{a*}^{\mu-1} f(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}}.$$

Thus

$$|B(t)|^r \leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{r}{p}} \left(\int_a^b \left| \Delta_{a*}^{\mu-1} f(t) \right|^q \Delta t \right)^{\frac{r}{q}},$$

and

$$\int_a^b |B(t)|^r \Delta t \leq \left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{r}{p}} \Delta t \right) \left(\int_a^b \left| \Delta_{a*}^{\mu-1} f(t) \right|^q \Delta t \right)^{\frac{r}{q}}. \tag{42.25}$$

Next raise (42.25) to power $\frac{1}{r}$. Hence proving the claim. ∎

Next we give an Opial type related inequality.

Theorem 42.11. Here all as in Theorem 42.9. Additionally suppose that $\left|\Delta_{a*}^{\mu-1} f\right|$ is increasing on $[a, b] \cap T$. Then

$$\int_a^b |B(t)| \left| \Delta_{a*}^{\mu-1} f(t) \right| \Delta t \leq$$

$$(b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(\Delta_{a*}^{\mu-1} f(t) \right)^{2q} \Delta t \right)^{\frac{1}{q}}. \tag{42.26}$$

Proof. As in the proof of Theorem 42.9 we obtain

$$|B(t)| \leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^t \left| \Delta_{a*}^{\mu-1} f(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{1}{p}} \left| \Delta_{a*}^{\mu-1} f(t) \right| (t-a)^{\frac{1}{q}}.$$

Therefore

$$|B(t)| \left| \Delta_{a*}^{\mu-1} f(t) \right| \leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{1}{p}} \left(\Delta_{a*}^{\mu-1} f(t) \right)^2 (t-a)^{\frac{1}{q}},$$

for all $t \in [a,b] \cap T$.

Consequently we obtain

$$\int_a^b |B(t)| \left| \Delta_{a*}^{\mu-1} f(t) \right| \Delta t \leq$$

$$\int_a^b \left[\left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right)^{\frac{1}{p}} \left(\Delta_{a*}^{\mu-1} f(t) \right)^2 (t-a)^{\frac{1}{q}} \right] \Delta t$$

$$\leq \left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(\Delta_{a*}^{\mu-1} f(t) \right)^{2q} (t-a) \Delta t \right)^{\frac{1}{q}}$$

$$\leq (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(\Delta_{a*}^{\mu-1} f(t) \right)^{2q} \Delta t \right)^{\frac{1}{q}},$$

proving the claim. ■

It follows related Ostrowski type inequalities.

Theorem 42.12. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{rd}^m(T)$, $a, b \in T$, $a \leq b$, $T^k = T$. Assume $h_{\mu-2}(s, \sigma(t))$, $h_{\widetilde{\nu}}(s, \sigma(t))$ to be continuous on $([a,b] \cap T)^2$, and $f^{\Delta^k}(a) = 0$, $k = 1, ..., m-1$. Denote $B(t) = f(t) + E\left(f^{\Delta^m}, \mu - 1, \widetilde{\nu} + 1, T, t \right)$, $t \in [a,b] \cap T$.

Then

$$\left| \frac{1}{b-a} \int_a^b B(t) \Delta t - f(a) \right| \leq$$

$$\frac{1}{b-a} \left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))| \Delta\tau \right) \Delta t \right) \left\| \Delta_{a*}^{\mu-1} f \right\|_{\infty, [a,b] \cap T}. \tag{42.27}$$

Proof. By (42.20) we obtain

$$B(t) - f(a) = \int_a^t h_{\mu-2}(t, \sigma(\tau)) \Delta_{a*}^{\mu-1} f(\tau) \Delta\tau, \quad \forall\, t \in [a,b] \cap T.$$

Then

$$|B(t) - f(a)| \leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))| \Delta\tau \right) \left\| \Delta_{a*}^{\mu-1} f \right\|_{\infty, [a,b] \cap T},$$

$\forall\ t \in [a, b] \cap T$.

Therefore we obtain

$$\left|\frac{1}{b-a}\int_a^b B(t)\,\Delta t - f(a)\right| = \frac{1}{b-a}\left|\int_a^b (B(t) - f(a))\,\Delta t\right|$$

$$\leq \frac{1}{b-a}\int_a^b |B(t) - f(a)|\,\Delta t$$

$$\leq \frac{1}{b-a}\left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|\,\Delta\tau\right)\Delta t\right)\left\|\Delta_{a*}^{\mu-1} f\right\|_{\infty,[a,b]\cap T},$$

proving the claim. ∎

Theorem 42.13. All as in Theorem 42.12. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\frac{1}{b-a}\int_a^b B(t)\,\Delta t - f(a)\right| \leq$$

$$\frac{1}{b-a}\left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p\,\Delta\tau\right)^{\frac{1}{p}}\Delta t\right)\left\|\Delta_{a*}^{\mu-1} f\right\|_{q,[a,b]\cap T}. \qquad (42.28)$$

Proof. By (42.20) we find

$$|B(t) - f(a)| \leq \int_a^t |h_{\mu-2}(t, \sigma(\tau))|\left|\Delta_{a*}^{\mu-1} f(\tau)\right|\Delta\tau$$

$$\leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p\,\Delta\tau\right)^{\frac{1}{p}}\left(\int_a^t \left|\Delta_{a*}^{\mu-1} f(\tau)\right|^q\Delta\tau\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p\,\Delta\tau\right)^{\frac{1}{p}}\left\|\Delta_{a*}^{\mu-1} f\right\|_{q,[a,b]\cap T}.$$

That is we have

$$|B(t) - f(a)| \leq \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p\,\Delta\tau\right)^{\frac{1}{p}}\left\|\Delta_{a*}^{\mu-1} f\right\|_{q,[a,b]\cap T}, \quad \forall\ t \in [a, b] \cap T.$$

Therefore we derive

$$\left|\frac{1}{b-a}\int_a^b B(t)\,\Delta t - f(a)\right| \leq \frac{1}{b-a}\int_a^b |B(t) - f(a)|\,\Delta t$$

$$\leq \frac{1}{b-a}\left(\int_a^b \left(\int_a^t |h_{\mu-2}(t, \sigma(\tau))|^p\,\Delta\tau\right)^{\frac{1}{p}}\Delta t\right)\left\|\Delta_{a*}^{\mu-1} f\right\|_{q,[a,b]\cap T},$$

proving the claim. ■

We finish general fractional delta time scales inequalities with a related Hilbert-Pachpatte type inequality.

Theorem 42.14. Let $\varepsilon > 0$, $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f_i \in C_{rd}^m(T_i)$, $a_i, b_i \in T_i$, $a_i \leq b_i$, $T_i^k = T_i$ time scale, $i = 1, 2$. Assume $h_{\mu-2}^{(i)}(s_i, \sigma_i(t_i)), h_{\widetilde{\nu}}^{(i)}(s_i, \sigma_i(t_i))$ to be continuous on $([a_i, b_i] \cap T_i)^2$, and $f_i^{\Delta^k}(a_i) = 0$, $k = 0, 1, ..., m-1$; $i = 1, 2$. Here $B_i(t_i) = f_i(t_i) + E_i(f_i^{\Delta^m}, \mu - 1, \widetilde{\nu} + 1, T_i, t_i)$, $t_i \in [a_i, b_i] \cap T_i$; $i = 1, 2$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Set

$$F(t_1) = \int_{a_1}^{t_1} \left(\left|h_{\mu-2}^{(1)}(t_1, \sigma_1(\tau_1))\right|\right)^p \Delta\tau_1,$$

for all $t_1 \in [a_1, b_1]$, and

$$G(t_2) = \int_{a_2}^{t_2} \left(\left|h_{\mu-2}^{(2)}(t_2, \sigma_2(\tau_2))\right|\right)^q \Delta\tau_2,$$

for all $t_2 \in [a_2, b_2]$ (where $h_{\mu-2}^{(i)}$, σ_i are the corresponding $h_{\mu-2}$, σ to T_i, $i = 1, 2$).

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|B_1(t_1)| |B_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \Delta t_1 \Delta t_2 \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left|\Delta_{a_1 *}^{\mu-1} f_1(t_1)\right|^q \Delta t_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|\Delta_{a_2 *}^{\mu-1} f_2(t_2)\right|^p \Delta t_2\right)^{\frac{1}{p}}. \tag{42.29}$$

(above double time scales Riemann delta integration is considered in the natural iterative way).

Proof. We notice that

$$\lambda(t_1) = \int_{a_2}^{b_2} \frac{|B_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \Delta t_2$$

is a Riemann Δ-integrable function on $[a_1, b_1] \cap T_1$.

Because $f_i^{\Delta^k}(a_i) = 0$, $k = 0, 1, ..., m - 1$; $i = 1, 2$, by Corollary 42.8 we get that

$$B_i(t_i) = \int_{a_i}^{t_i} h_{\mu-2}^{(i)}(t_i, \sigma_i(\tau_i)) \Delta_{a_i *}^{\mu-1} f_i(\tau_i) \Delta\tau_i,$$

$\forall\ t_i \in [a_i, b_i] \cap T_i$, where $a_i, b_i \in T_i$.

Consequently

$$|B_1(t_1)| \leq \left(\int_{a_1}^{t_1} \left(\left|h_{\mu-2}^{(1)}(t_1, \sigma_1(\tau_1))\right|\right)^p \Delta\tau_1\right)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|\Delta_{a_1 *}^{\mu-1} f_1(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}}$$

$$= F\left(t_1\right)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left| \Delta_{a_1*}^{\mu-1} f_1\left(\tau_1\right) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}},$$

and

$$\left|B_2\left(t_2\right)\right| \leq \left(\int_{a_2}^{t_2} \left(\left| h_{\mu-2}^{(2)}\left(t_2, \sigma_2\left(\tau_2\right)\right) \right| \right)^q \Delta\tau_2 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \Delta_{a_2*}^{\mu-1} f_2\left(\tau_2\right) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}}$$

$$= G\left(t_2\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \Delta_{a_2*}^{\mu-1} f_2\left(\tau_2\right) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}}.$$

Young's inequality for $a, b \geq 0$ says that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Therefore we have

$$\left|B_1\left(t_1\right)\right| \left|B_2\left(t_2\right)\right| \leq$$

$$\left(F\left(t_1\right)\right)^{\frac{1}{p}} \left(G\left(t_2\right)\right)^{\frac{1}{q}} \left(\int_{a_1}^{t_1} \left| \Delta_{a_1*}^{\mu-1} f_1\left(\tau_1\right) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \Delta_{a_2*}^{\mu-1} f_2\left(\tau_2\right) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}}$$

$$\leq \left(\frac{F\left(t_1\right)}{p} + \frac{G\left(t_2\right)}{q} \right) \left(\int_{a_1}^{t_1} \left| \Delta_{a_1*}^{\mu-1} f_1\left(\tau_1\right) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \Delta_{a_2*}^{\mu-1} f_2\left(\tau_2\right) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}}.$$

The last gives ($\varepsilon > 0$)

$$\frac{\left|B_1\left(t_1\right)\right| \left|B_2\left(t_2\right)\right|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right)} \leq \left(\int_{a_1}^{t_1} \left| \Delta_{a_1*}^{\mu-1} f_1\left(\tau_1\right) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \Delta_{a_2*}^{\mu-1} f_2\left(\tau_2\right) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}},$$

for all $t_i \in [a_i, b_i] \cap T_i$, $i = 1, 2$.

Next we see that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left|B_1\left(t_1\right)\right| \left|B_2\left(t_2\right)\right|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right)} \Delta t_1 \Delta t_2 \leq$$

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left| \Delta_{a_1*}^{\mu-1} f_1\left(\tau_1\right) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \Delta t_1 \right) \cdot$$

$$\left(\int_{a_2}^{b_2} \left(\int_{a_2}^{t_2} \left| \Delta_{a_2*}^{\mu-1} f_2\left(\tau_2\right) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}} \Delta t_2 \right) \leq$$

(by Hölder's inequality)

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left| \Delta_{a_1*}^{\mu-1} f_1\left(\tau_1\right) \right|^q \Delta\tau_1 \right) \Delta t_1 \right)^{\frac{1}{q}} \left(b_1 - a_1\right)^{\frac{1}{p}} \cdot$$

$$\left(\int_{a_2}^{b_2}\left(\int_{a_2}^{t_2}\left|\Delta_{a_2*}^{\mu-1}f_2(\tau_2)\right|^p\Delta\tau_2\right)\Delta t_2\right)^{\frac{1}{p}}(b_2-a_2)^{\frac{1}{q}}$$

$$\leq\left(\int_{a_1}^{b_1}\left(\int_{a_1}^{b_1}\left|\Delta_{a_1*}^{\mu-1}f_1(\tau_1)\right|^q\Delta\tau_1\right)\Delta t_1\right)^{\frac{1}{q}}(b_1-a_1)^{\frac{1}{p}}\cdot$$

$$\left(\int_{a_2}^{b_2}\left(\int_{a_2}^{b_2}\left|\Delta_{a_2*}^{\mu-1}f_2(\tau_2)\right|^p\Delta\tau_2\right)\Delta t_2\right)^{\frac{1}{p}}(b_2-a_2)^{\frac{1}{q}}$$

$$=(b_1-a_1)(b_2-a_2)\left(\int_{a_1}^{b_1}\left|\Delta_{a_1*}^{\mu-1}f_1(\tau_1)\right|^q\Delta\tau_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|\Delta_{a_2*}^{\mu-1}f_2(\tau_2)\right|^p\Delta\tau_2\right)^{\frac{1}{p}},$$

proving the claim. ■

42.3 Applications

Here is $T=\mathbb{R}$ case.

Let $\mu>2$ such that $m-1<\mu<m\in\mathbb{N}$, $\widetilde{\nu}=m-\mu$, $f\in C^m([a,b])$, $a,b\in\mathbb{R}$.

The delta fractional derivative on $\mathbb{R}$ of order $\mu-1$ is defined as follows:

$$\Delta_{a*}^{\mu-1}f(t)=\left(K_a^{\widetilde{\nu}+1}f^{(m)}\right)(t)=\frac{1}{\Gamma(\widetilde{\nu}+1)}\int_a^t(t-\tau)^{\widetilde{\nu}}f^{(m)}(\tau)\,d\tau, \tag{42.30}$$

$\forall\ t\in[a,b]$.

Notice that $\Delta_{a*}^{\mu-1}f\in C([a,b])$, and $B(t)=f(t)$, $\forall\ t\in[a,b]$.

We give a Poincaré type inequality.

Theorem 42.15. Let $\mu>2$, $m-1<\mu<m\in\mathbb{N}$, $f\in C^m(\mathbb{R})$, $a,b\in\mathbb{R}$, $a\leq b$. Suppose $f^{(k)}(a)=0$, $k=0,1,...,m-1$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then

$$\int_a^b|f(t)|^q\,dt\leq\frac{(b-a)^{(\mu-1)q}}{(\Gamma(\mu-1))^q(\mu-1)q((\mu-2)p+1)^{q-1}}\left(\int_a^b\left|\Delta_{a*}^{\mu-1}f(t)\right|^q dt\right). \tag{42.31}$$

Proof. By Theorem 42.9. ■

We present a Sobolev type inequality.

Theorem 42.16. All as in Theorem 42.15. Let $r\geq1$. Then

$$\|f\|_r\leq\frac{(b-a)^{\mu-2+\frac{1}{p}+\frac{1}{r}}}{\Gamma(\mu-1)((\mu-2)p+1)^{\frac{1}{p}}\left((\mu-2)r+\frac{r}{p}+1\right)^{\frac{1}{r}}}\left\|\Delta_{a*}^{\mu-1}f\right\|_q. \tag{42.32}$$

Proof. By Theorem 42.10. ■

We continue with an Opial type inequality.

Theorem 42.17. All as in Theorem 42.15. Suppose $\left|\Delta_{a*}^{\mu-1} f\right|$ is increasing on $[a,b]$. Then

$$\int_a^b |f(t)| \left|\Delta_{a*}^{\mu-1} f(t)\right| dt \leq$$

$$\frac{(b-a)^{\mu-\frac{1}{q}}}{\Gamma(\mu-1)\left[((\mu-2)p+1)((\mu-2)p+2)\right]^{\frac{1}{p}}} \left(\int_a^b \left(\Delta_{a*}^{\mu-1} f(t)\right)^{2q} dt\right)^{\frac{1}{q}}. \quad (42.33)$$

Proof. By Theorem 42.11. ■

Some Ostrowski type inequalities follow.

Theorem 42.18. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $f \in C^m(\mathbb{R})$, $a,b \in \mathbb{R}$, $a \leq b$. Suppose $f^{(k)}(a) = 0$, $k = 1, ..., m-1$. Then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right| \leq \frac{(b-a)^{\mu-1}}{\Gamma(\mu+1)} \left\|\Delta_{a*}^{\mu-1} f\right\|_{\infty,[a,b]}. \quad (42.34)$$

Proof. By Theorem 42.12. ■

Theorem 42.19. Here all as in Theorem 42.18. Let $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. Then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right| \leq \frac{(b-a)^{\mu-\frac{1}{q}-1}}{\Gamma(\mu-1)\left(\mu-\frac{1}{q}\right)((\mu-2)p+1)^{\frac{1}{p}}} \left\|\Delta_{a*}^{\mu-1} f\right\|_{q,[a,b]}. \quad (42.35)$$

Proof. By Theorem 42.13. ■

We finish this section and chapter with a Hilbert-Pachpatte inequality on $\mathbb{R}$.

Theorem 42.20. Let $\varepsilon > 0$, $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $i = 1,2$; $f_i \in C^m(\mathbb{R})$, $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$, $f_i^{(k)}(a_i) = 0$, $k = 0,1,...,m-1$; $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$.

Set

$$F(t_1) = \frac{(t_1-a_1)^{(\mu-2)p+1}}{(\Gamma(\mu-1))^p((\mu-2)p+1)},$$

$t_1 \in [a_1,b_1]$,

and

$$G(t_2) = \frac{(t_2-a_2)^{(\mu-2)q+1}}{(\Gamma(\mu-1))^q((\mu-2)q+1)},$$

$t_2 \in [a_2,b_2]$.

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon+\frac{F(t_1)}{p}+\frac{G(t_2)}{q}\right)}dt_1dt_2 \leq$$

$$(b_1-a_1)(b_2-a_2)\left(\int_{a_1}^{b_1}\left|\Delta_{a_1*}^{\mu-1}f_1(t_1)\right|^q dt_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|\Delta_{a_2*}^{\mu-1}f_2(t_2)\right|^p dt_2\right)^{\frac{1}{p}}. \tag{42.36}$$

Proof. By Theorem 42.14. ■

43

Principles of Nabla Fractional Calculus on Time Scales with Inequalities

Here we present the Nabla Fractional Calculus on Time Scales. Then we prove related integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. At the end we give inequalities applications on the time scales $\mathbb{R}$, $\mathbb{Z}$. This chapter relies on [53].

43.1 Background and Foundation Results

For the basics on time scales we use [59], [93], [94], [103], [119], [187], [223], [113], [114], [181].

By [282], p. 256, for $\mu, \nu > 0$ we have that

$$\int_t^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \frac{(s-t)^{\nu-1}}{\Gamma(\nu)} ds = \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \tag{43.1}$$

where Γ is the gamma function.

Here we consider time scales T such that $T_k = T$.

Consider the coordinatewise ld-continuous functions $\widehat{h}_\alpha : T \times T \to \mathbb{R}$, $\alpha \geq 0$, such that $\widehat{h}_0(t,s) = 1$,

$$\widehat{h}_{\alpha+1}(t,s) = \int_s^t \widehat{h}_\alpha(\tau, s) \nabla\tau, \tag{43.2}$$

$\forall\ s, t \in T$.

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Here ρ is the backward jump operator and $\nu(t) = t - \rho(t)$.

Furthermore for $\alpha, \beta > 1$ we suppose that

$$\int_{\rho(u)}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) \widehat{h}_{\beta-1}(\tau, \rho(u)) \nabla\tau = \widehat{h}_{\alpha+\beta-1}(t, \rho(u)), \tag{43.3}$$

valid for all $u, t \in T : u \leq t$.

In the case of $T = \mathbb{R}$; then $\rho(t) = t$, and $\widehat{h}_k(t, s) = \frac{(t-s)^k}{k!}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and define

$$\widehat{h}_\alpha(t, s) = \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

Notice that

$$\int_s^t \frac{(\tau - s)^\alpha}{\Gamma(\alpha+1)} d\tau = \frac{(t-s)^{\alpha+1}}{\Gamma(\alpha+2)} = \widehat{h}_{\alpha+1}(t, s),$$

fulfilling (43.2).

Furthermore we see that $(\alpha, \beta > 1)$

$$\int_u^t \widehat{h}_{\alpha-1}(t, \tau) \widehat{h}_{\beta-1}(\tau, u) d\tau = \int_u^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\tau-u)^{\beta-1}}{\Gamma(\beta)} d\tau$$

$$\overset{(\text{by } (43.1))}{=} \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = \widehat{h}_{\alpha+\beta-1}(t, u),$$

fulfilling (43.3).

By Theorem 2.2 of [251], we have for $k, m \in \mathbb{N}_0$ that

$$\int_{t_0}^{t} \widehat{h}_k(t, \rho(\tau)) \widehat{h}_m(\tau, t_0) \nabla\tau = \widehat{h}_{k+m+1}(t, t_0). \tag{43.4}$$

Let $T = \mathbb{Z}$, then $\rho(t) = t - 1$, $t \in \mathbb{Z}$. Define $t^{\overline{0}} := 1$, $t^{\overline{k}} := t(t+1)\dots(t+k-1)$, $k \in \mathbb{N}$, and by (43.2) we have $\widehat{h}_k(t, s) = \frac{(t-s)^{\overline{k}}}{k!}$, $s, t \in \mathbb{Z}$, $k \in \mathbb{N}_0$.

Here $\int_{t_0}^t \nabla\tau = \sum_{t_0+1}^t$.

Therefore by (43.4) we obtain

$$\sum_{\tau=t_0+1}^{t} \frac{(t-\tau+1)^{\overline{k}}}{k!} \frac{(\tau-t_0)^{\overline{m}}}{m!} = \frac{(t-t_0)^{\overline{k+m+1}}}{(k+m+1)!},$$

which results into

$$\sum_{\tau=t_0}^{t} \frac{(t-\tau+1)^{\overline{k-1}}}{(k-1)!} \frac{(\tau-t_0+1)^{\overline{m-1}}}{(m-1)!} = \frac{(t-t_0+1)^{\overline{k+m-1}}}{(k+m-1)!}, \tag{43.5}$$

confirming (43.3).

Next we follow [105].

Let $a, \alpha \in \mathbb{R}$, define $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$, $t \in \mathbb{R} - \{..., -2, -1, 0\}$, $N_a = \{a, a \pm 1, a \pm 2, ...\}$, notice $N_0 = \mathbb{Z}$, $0^{\overline{\alpha}} = 0$, $t^{\overline{0}} = 1$, and $f : N_a \to \mathbb{R}$. Here $\rho(s) = s - 1$, $\sigma(s) = s + 1$, $\nu(t) = 1$. Also define

$$\nabla_a^{-n} f(t) = \sum_{s=a}^{t} \frac{(t - \rho(s))^{\overline{n-1}}}{(n-1)!} f(s), \quad n \in \mathbb{N},$$

and in general

$$\nabla_a^{-\nu} f(t) = \sum_{s=a}^{t} \frac{(t - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s),$$

where $\nu \in \mathbb{R} - \{..., -2, -1, 0\}$.

Here we put

$$\widehat{h}_\alpha(t, s) = \frac{(t-s)^{\overline{\alpha}}}{\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

We need

Lemma 43.1. Let $\alpha > -1$, $x > \alpha + 1$. Then

$$\frac{\Gamma(x)}{\Gamma(x-\alpha)} = \frac{1}{(\alpha+1)} \left(\frac{\Gamma(x+1)}{\Gamma(x-\alpha)} - \frac{\Gamma(x)}{\Gamma(x-\alpha-1)} \right).$$

Proof. Obvious. ∎

Proposition 43.2. Let $\alpha > -1$. It holds

$$\int_s^t \frac{(\tau - s)^{\overline{\alpha}}}{\Gamma(\alpha+1)} \nabla\tau = \frac{(t-s)^{\overline{\alpha+1}}}{\Gamma(\alpha+2)}, \quad t \geq s.$$

That is $\widehat{h}_\alpha$, $\alpha \geq 0$, on N_a confirm (43.2).

Proof. Let $t > s$. We have that

$$\int_s^t \frac{(\tau - s)^{\overline{\alpha}}}{\Gamma(\alpha+1)} \nabla\tau = \frac{1}{\Gamma(\alpha+1)} \sum_{\tau=s+1}^{t} (\tau - s)^{\overline{\alpha}} = \frac{1}{\Gamma(\alpha+1)} \sum_{\tau=s+1}^{t} \frac{\Gamma(\tau - s + \alpha)}{\Gamma(\tau - s)}$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{\tau=s+1}^{t} \frac{\Gamma(\tau - s + \alpha)}{\Gamma(\tau - s + \alpha - \alpha)} = \frac{1}{\Gamma(\alpha+1)} \sum_{x=\alpha+1}^{t-s+\alpha} \frac{\Gamma(x)}{\Gamma(x-\alpha)}$$

(notice here $\tau - s \geq 1$ and $x \geq \alpha + 1 > 0$)

$$= \frac{1}{\Gamma(\alpha+1)} \left\{ \Gamma(\alpha+1) + \sum_{x=\alpha+2}^{t-s+\alpha} \frac{\Gamma(x)}{\Gamma(x-\alpha)} \right\} = 1 + \frac{1}{\Gamma(\alpha+1)} \sum_{x=\alpha+2}^{t-s+\alpha} \frac{\Gamma(x)}{\Gamma(x-\alpha)}$$

$$\overset{\text{(by Lemma 43.1)}}{=} 1+\frac{1}{\Gamma(\alpha+2)}\sum_{x=\alpha+2}^{t-s+\alpha}\left(\frac{\Gamma(x+1)}{\Gamma(x-\alpha)}-\frac{\Gamma(x)}{\Gamma(x-\alpha-1)}\right)$$

$$=1+\frac{1}{\Gamma(\alpha+2)}\left\{(\Gamma(\alpha+3)-\Gamma(\alpha+2))+\left(\frac{\Gamma(\alpha+4)}{\Gamma(3)}-\Gamma(\alpha+3)\right)+\right.$$

$$\left(\frac{\Gamma(\alpha+5)}{\Gamma(4)}-\frac{\Gamma(\alpha+4)}{\Gamma(3)}\right)+\left(\frac{\Gamma(\alpha+6)}{\Gamma(5)}-\frac{\Gamma(\alpha+5)}{\Gamma(4)}\right)+\ldots+$$

$$\left.\left(\frac{\Gamma(t-s+\alpha)}{\Gamma(t-s-1)}-\frac{\Gamma(t-s+\alpha-1)}{\Gamma(t-s+\alpha-2)}\right)+\left(\frac{\Gamma(t-s+\alpha+1)}{\Gamma(t-s)}-\frac{\Gamma(t-s+\alpha)}{\Gamma(t-s-1)}\right)\right\}$$

(telescoping sum)

$$=1+\frac{1}{\Gamma(\alpha+2)}\left\{\frac{\Gamma(t-s+\alpha+1)}{\Gamma(t-s)}-\Gamma(\alpha+2)\right\}$$

$$=\frac{\Gamma(t-s+\alpha+1)}{\Gamma(\alpha+2)\Gamma(t-s)}=\frac{(t-s)^{\overline{\alpha+1}}}{\Gamma(\alpha+2)}.$$

That is proving the claim. ■

Next for $\mu,\nu>1$, $\tau<t$, from the proof of Theorem 2.1 ([105]) we obtain that

$$\sum_{s=\tau}^{t}\frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}\frac{(s-\rho(\tau))^{\overline{\mu-1}}}{\Gamma(\mu)}=\frac{(t-\rho(\tau))^{\overline{\nu+\mu-1}}}{\Gamma(\mu+\nu)},$$

where $\tau\in\{a,...,t\}$.

So for $t,t_0\in N_a$ with $t_0<t$ we get

$$\sum_{\tau=t_0}^{t}\frac{(t-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)}\frac{(\tau-t_0+1)^{\overline{\mu-1}}}{\Gamma(\mu)}=\frac{(t-t_0+1)^{\overline{\nu+\mu-1}}}{\Gamma(\mu+\nu)}, \tag{43.6}$$

that is confirming (43.3) fractionally on the time scale $T=N_a$.

Notice also here that

$$\int_a^b f(t)\,\nabla t=\sum_{t=a+1}^{b}f(t).$$

So fractional conditions (43.2) and (43.3) are very natural and common on time scales.

For $\alpha\geq 1$ we define the time scale ∇-Riemann-Liouville type fractional integral $(a,b\in T)$

$$J_a^{\alpha}f(t)=\int_a^t \widehat{h}_{\alpha-1}(t,\rho(\tau))\,f(\tau)\,\nabla\tau, \tag{43.7}$$

(by [116] the last integral is on $(a,t]\cap T$)

$$J_a^0 f(t)=f(t),$$

where $f \in L_1([a,b] \cap T)$ (Lebesgue ∇-integrable functions on $[a,b] \cap T$, see [113], [114], [181]), $t \in [a,b] \cap T$.

Notice $J_a^1 f(t) = \int_a^t f(\tau)\nabla\tau$ is absolutely continuous in $t \in [a,b] \cap T$, see [116].

Lemma 43.3. Let $\alpha > 1$, $f \in L_1([a,b] \cap T)$. Suppose that $\widehat{h}_{\alpha-1}(s,\rho(t))$ is Lebesgue ∇-measurable on $([a,b] \cap T)^2$; $a,b \in T$. Then $J_a^\alpha f \in L_1([a,b] \cap T)$.

Proof. Define $K : \Omega := ([a,b] \cap T)^2 \to \mathbb{R}$, by

$$K(s,t) = \begin{cases} \widehat{h}_{\alpha-1}(s,\rho(t)), & \text{if } a \le t \le s \le b, \\ 0, & \text{if } a \le s < t \le b. \end{cases}$$

Clearly $K(s,t)$ is Lebesgue ∇-measurable on $([a,b] \cap T)^2$.

Then

$$\int_a^b K(s,t)\nabla s = \int_{[a,t)} K(s,t)\nabla s + \int_t^b K(s,t)\nabla s$$

$$= \int_t^b K(s,t)\nabla s = \int_t^b \widehat{h}_{\alpha-1}(s,\rho(t))\nabla s$$

$$= \int_{\rho(t)}^b \widehat{h}_{\alpha-1}(s,\rho(t))\nabla s - \int_{\rho(t)}^t \widehat{h}_{\alpha-1}(s,\rho(t))\nabla s$$

$$= \widehat{h}_\alpha(b,\rho(t)) - \nu(t)\widehat{h}_{\alpha-1}(t,\rho(t)) \in \mathbb{R}.$$

Next we consider the repeated double Lebesgue ∇-integral

$$\int_a^b \left(\int_a^b K(s,t)|f(t)|\nabla s\right)\nabla t = \int_a^b |f(t)| \left(\int_a^b K(s,t)\nabla s\right)\nabla t =$$

$$\int_a^b |f(t)| \left\{\widehat{h}_\alpha(b,\rho(t)) - \nu(t)\widehat{h}_{\alpha-1}(t,\rho(t))\right\}\nabla t$$

$$= \int_a^b |f(t)|\widehat{h}_\alpha(b,\rho(t))\nabla t - \int_a^b |f(t)|\nu(t)\widehat{h}_{\alpha-1}(t,\rho(t))\nabla t,$$

which exists and is finite. Thus the function $(s,t) \to K(s,t)f(t)$ is Lebesgue ∇-integrable over Ω by Tonelli's theorem.

Let now the characteristic function

$$\chi_{(a,s]\cap T}(t) = \begin{cases} 1, \text{ if } t \in (a,s] \cap T \\ 0, \text{ else,} \end{cases}$$

where $s \in [a,b] \cap T$.

Then the function $(s,t) \to \chi_{(a,s]\cap T}(t)K(s,t)f(t)$ is Lebesgue ∇-integrable on Ω. Hence by Fubini's theorem we obtain that

$$\int_a^b \chi_{(a,s]\cap T}(t)K(s,t)f(t)\nabla t = \int_a^s \widehat{h}_{\alpha-1}(s,\rho(t))f(t)\nabla t = J_a^\alpha f(s)$$

is Lebesgue ∇-integrable in s on $[a,b]\cap T$, proving the claim. ■

For $u \le t$; $u,t \in T$, we define

$$\varepsilon(t,u) = \int_{\rho(u)}^{u} \widehat{h}_{\alpha-1}(t,\rho(\tau))\,\widehat{h}_{\beta-1}(\tau,\rho(u))\,\nabla\tau$$

$$= \nu(u)\,\widehat{h}_{\alpha-1}(t,\rho(u))\,\widehat{h}_{\beta-1}(u,\rho(u)), \qquad (43.8)$$

where $\alpha,\beta > 1$.

Next we notice for $\alpha,\beta > 1$; $a,b \in T$, $f \in L_1([a,b]\cap T)$, and $\widehat{h}_{\alpha-1}(s,\rho(t))$ is continuous on $([a,b]\cap T)^2$ for any $\alpha > 1$, that

$$J_a^{\alpha} J_a^{\beta} f(t) = \int_a^t \widehat{h}_{\alpha-1}(t,\rho(\tau))\,\nabla\tau \int_a^{\tau} \widehat{h}_{\beta-1}(\tau,\rho(u))\,f(u)\,\nabla u$$

(by Fubini's theorem)

$$= \int_a^t f(u)\,\nabla u \int_u^t \widehat{h}_{\alpha-1}(t,\rho(\tau))\,\widehat{h}_{\beta-1}(\tau,\rho(u))\,\nabla\tau = \int_a^t f(u)\,\nabla u \cdot$$

$$\left[\int_{\rho(u)}^{t} \widehat{h}_{\alpha-1}(t,\rho(\tau))\,\widehat{h}_{\beta-1}(\tau,\rho(u))\,\nabla\tau - \int_{\rho(u)}^{u} \widehat{h}_{\alpha-1}(t,\rho(\tau))\,\widehat{h}_{\beta-1}(\tau,\rho(u))\,\nabla\tau\right]$$

$$\overset{\text{(by (43.3))}}{=} \int_a^t f(u)\,\nabla u\left(\widehat{h}_{\alpha+\beta-1}(t,\rho(u)) - \varepsilon(t,u)\right)$$

$$= \int_a^t \widehat{h}_{\alpha+\beta-1}(t,\rho(u))\,f(u)\,\nabla u - \int_a^t f(u)\,\varepsilon(t,u)\,\nabla u$$

$$= J_a^{\alpha+\beta} f(t) - \int_a^t f(u)\,\varepsilon(t,u)\,\nabla u.$$

Hence

$$J_a^{\alpha} J_a^{\beta} f(t) + \int_a^t f(u)\,\varepsilon(t,u)\,\nabla u = J_a^{\alpha+\beta} f(t), \quad \forall\, t \in [a,b]\cap T.$$

So we have established the semigroup property

$$J_a^{\alpha} J_a^{\beta} f(t) + \int_a^t f(u)\,\nu(u)\,\widehat{h}_{\alpha-1}(t,\rho(u))\,\widehat{h}_{\beta-1}(u,\rho(u))\,\nabla u = J_a^{\alpha+\beta} f(t), \qquad (43.9)$$

$\forall\, t \in [a,b]\cap T$, with $a,b \in T$.

We call the Lebesgue ∇-integral

$$D(f,\alpha,\beta,T,t) = \int_a^t f(u)\,\nu(u)\,\widehat{h}_{\alpha-1}(t,\rho(u))\,\widehat{h}_{\beta-1}(u,\rho(u))\,\nabla u, \qquad (43.10)$$

$t \in [a,b] \cap T$; $a,b \in T$, the backward graininess deviation functional of $f \in L_1([a,b] \cap T)$.

If $T = \mathbb{R}$, then $D(f, \alpha, \beta, \mathbb{R}, t) = 0$.

Putting things together we have

Theorem 43.4. Let $T_k = T$, $a,b \in T$, $f \in L_1([a,b] \cap T)$; $\alpha, \beta > 1$; $\widehat{h}_{\alpha-1}(s, \rho(t))$ is continuous on $([a,b] \cap T)^2$ for any $\alpha > 1$. Then

$$J_a^\alpha J_a^\beta f(t) + D(f, \alpha, \beta, T, t) = J_a^{\alpha+\beta} f(t), \tag{43.11}$$

$\forall\, t \in [a,b] \cap T$.

We make

Remark 43.5. Let $\mu > 2$ such that $m - 1 < \mu < m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$ (ceiling of the number), $\widetilde{\nu} = m - \mu$ $(0 < \widetilde{\nu} < 1)$.

Let $f \in C_{ld}^m([a,b] \cap T)$. Clearly here ([181]) f^{∇^m} is a Lebesgue ∇-integrable function.

We define the nabla fractional derivative on time scale T of order $\mu - 1$ as follows:

$$\nabla_{a*}^{\mu-1} f(t) = \left(J_a^{\widetilde{\nu}+1} f^{\nabla^m}\right)(t) = \int_a^t \widehat{h}_{\widetilde{\nu}}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau, \tag{43.12}$$

$\forall\, t \in [a,b] \cap T$.

Notice here that $\nabla_{a*}^{\mu-1} f \in C([a,b] \cap T)$ by a simple argument using dominated convergence theorem in Lebesgue ∇-sense.

If $\mu = m$, then $\widetilde{\nu} = 0$ and by (43.12) we find

$$\nabla_{a*}^{m-1} f(t) = J_a^1 f^{\nabla^m}(t) = f^{\nabla^{m-1}}(t). \tag{43.13}$$

More generally, by [116], given that $f^{\nabla^{m-1}}$ is everywhere finite and absolutely continuous on $[a,b] \cap T$, then f^{∇^m} exists ∇-a.e. and is Lebesgue ∇-integrable on $(a,t] \cap T$, $\forall\, t \in [a,b] \cap T$, and one can plug it into (43.12).

We observe that

$$J_a^{\mu-1} \nabla_{a*}^{\mu-1} f(t) = \left(J_a^{\mu-1} J_a^{\widetilde{\nu}+1} f^{\nabla^m}\right)(t)$$

$$\overset{\text{(by (43.11))}}{=} \left(J_a^{\mu+\widetilde{\nu}} f^{\nabla^m}\right)(t) - \int_a^t f^{\nabla^m}(u)\, \nu(u)\, \widehat{h}_{\mu-2}(t, \rho(u))\, \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u =$$

$$\left(J_a^m f^{\nabla^m}\right)(t) - \int_a^t f^{\nabla^m}(u)\, \nu(u)\, \widehat{h}_{\mu-2}(t, \rho(u))\, \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u.$$

Hence

$$J_a^{\mu-1} \nabla_{a*}^{\mu-1} f(t) + \int_a^t f^{\nabla^m}(u)\, \nu(u)\, \widehat{h}_{\mu-2}(t, \rho(u))\, \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u =$$

$$\left(J_a^m f^{\nabla^m}\right)(t) = \int_a^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau.$$

We have proved

Theorem 43.6. Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{ld}^m([a,b] \cap T)$, $a, b \in T$, $T_k = T$. Assume $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ to be continuous on $([a,b] \cap T)^2$.

Then

$$\int_a^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau = \tag{43.14}$$

$$\int_a^t f^{\nabla^m}(u) \nu(u) \widehat{h}_{\mu-2}(t, \rho(u)) \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u + \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla\tau,$$

$\forall\, t \in [a,b] \cap T$.

We need the nabla time scales Taylor formula

Theorem 43.7. ([93]) Let $f \in C_{ld}^m(T)$, $m \in \mathbb{N}$, $T_k = T$; $a, b \in T$. Then

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t, a) f^{\nabla^k}(a) + \int_a^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau, \tag{43.15}$$

$\forall\, t \in [a,b] \cap T$.

Next we present the fractional time scales nabla Taylor formula

Theorem 43.8. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m-\mu$; $f \in C_{ld}^m(T)$, $a, b \in T$, $T_k = T$. Assume $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ to be continuous on $([a,b] \cap T)^2$. Then

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t, a) f^{\nabla^k}(a) + \tag{43.16}$$

$$\int_a^t f^{\nabla^m}(u) \nu(u) \widehat{h}_{\mu-2}(t, \rho(u)) \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u + \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla\tau,$$

$\forall\, t \in [a,b] \cap T$.

Corollary 43.9. All as in Theorem 43.8. Additionally suppose $f^{\nabla^k}(a) = 0$, $k = 0, 1, ..., m-1$. Then

$$A(t) := f(t) - D\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T, t\right) \tag{43.17}$$

$$= f(t) - \int_a^t f^{\nabla^m}(u) \nu(u) \widehat{h}_{\mu-2}(t, \rho(u)) \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u$$

$$= \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla\tau,$$

$\forall\, t \in [a, b] \cap T$.

Notice here that $D\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T, t\right) \in C_{ld}([a, b] \cap T)$. Also the R.H.S (43.17) is a continuous function in $t \in [a, b] \cap T$.

43.2 Fractional Nabla Inequalities on Time Scales

We present a Poincaré type related inequality.

Theorem 43.10. Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{ld}^m(T)$, $a, b \in T$, $a \leq b$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ to be continuous on $([a, b] \cap T)^2$, and $f^{\nabla^k}(a) = 0$, $k = 0, 1, ..., m - 1$. Here $A(t) = f(t) - D\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T, t\right)$, $t \in [a, b] \cap T$; and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\int_a^b |A(t)|^q \nabla t \leq \left(\int_a^b \left(\int_a^t \left|\widehat{h}_{\mu-2}(t, \rho(\tau))\right|^p \nabla\tau\right)^{\frac{q}{p}} \nabla t\right) \left(\int_a^b \left|\nabla_{a*}^{\mu-1} f(t)\right|^q \nabla t\right). \tag{43.18}$$

Proof. By Corollary 43.9 we obtain that

$$A(t) = \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla\tau.$$

Hence

$$|A(t)| \leq \int_a^t \left|\widehat{h}_{\mu-2}(t, \rho(\tau))\right| \left|\nabla_{a*}^{\mu-1} f(\tau)\right| \nabla\tau$$

(by Hölder's inequality)

$$\leq \left(\int_a^t \left|\widehat{h}_{\mu-2}(t, \rho(\tau))\right|^p \nabla\tau\right)^{\frac{1}{p}} \left(\int_a^t \left|\nabla_{a*}^{\mu-1} f(\tau)\right|^q \nabla\tau\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t \left|\widehat{h}_{\mu-2}(t, \rho(\tau))\right|^p \nabla\tau\right)^{\frac{1}{p}} \left(\int_a^b \left|\nabla_{a*}^{\mu-1} f(\tau)\right|^q \nabla\tau\right)^{\frac{1}{q}}.$$

Therefore

$$|A(t)|^q \leq \left(\int_a^t \left|\widehat{h}_{\mu-2}(t, \rho(\tau))\right|^p \nabla\tau\right)^{\frac{q}{p}} \left(\int_a^b \left|\nabla_{a*}^{\mu-1} f(\tau)\right|^q \nabla\tau\right), \tag{43.19}$$

$\forall\, t \in [a, b] \cap T$.

Next by integrating (43.19) we are proving the claim. ∎

Next we give a related Sobolev inequality.

Theorem 43.11. Here all as in Theorem 43.10. Let $r \geq 1$ and denote

$$\|f\|_r = \left(\int_a^b |f(t)|^r \nabla t \right)^{\frac{1}{r}}.$$

Then

$$\|A\|_r \leq \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{r}{p}} \nabla t \right)^{\frac{1}{r}} \left\| \nabla_{a*}^{\mu-1} f \right\|_q. \tag{43.20}$$

Proof. As in the proof of Theorem 43.10 we have

$$|A(t)| \leq \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{1}{p}} \left(\int_a^b \left| \nabla_{a*}^{\mu-1} f(\tau) \right|^q \nabla\tau \right)^{\frac{1}{q}}.$$

Therefore

$$|A(t)|^r \leq \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{r}{p}} \left(\int_a^b \left| \nabla_{a*}^{\mu-1} f(t) \right|^q \nabla t \right)^{\frac{r}{q}},$$

and

$$\int_a^b |A(t)|^r \nabla t \leq \int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{r}{p}} \nabla t \left(\int_a^b \left| \nabla_{a*}^{\mu-1} f(t) \right|^q \nabla t \right)^{\frac{r}{q}}. \tag{43.21}$$

Next raise (43.21) to power $\frac{1}{r}$. Hence proving the claim. ■

Next we give an Opial type related inequality.

Theorem 43.12. Here all as in Theorem 43.10. Additionally suppose that $\left| \nabla_{a*}^{\mu-1} f \right|$ is increasing on $[a, b] \cap T$. Then

$$\int_a^b |A(t)| \left| \nabla_{a*}^{\mu-1} f(t) \right| \nabla t \leq$$

$$(b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right) \nabla t \right)^{\frac{1}{p}} \left(\int_a^b \left(\nabla_{a*}^{\mu-1} f(t) \right)^{2q} \nabla t \right)^{\frac{1}{q}}. \tag{43.22}$$

Proof. As in the proof of Theorem 43.10 we obtain

$$|A(t)| \leq \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{1}{p}} \left(\int_a^b \left| \nabla_{a*}^{\mu-1} f(\tau) \right|^q \nabla\tau \right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t \left|\widehat{h}_{\mu-2}(t,\rho(\tau))\right|^p \nabla\tau\right)^{\frac{1}{p}} \left|\nabla_{a*}^{\mu-1} f(t)\right| (t-a)^{\frac{1}{q}}.$$

Therefore

$$|A(t)| \left|\nabla_{a*}^{\mu-1} f(t)\right| \leq \left(\int_a^t \left|\widehat{h}_{\mu-2}(t,\rho(\tau))\right|^p \nabla\tau\right)^{\frac{1}{p}} \left(\nabla_{a*}^{\mu-1} f(t)\right)^2 (t-a)^{\frac{1}{q}},$$

for all $t \in [a,b] \cap T$.

Consequently we derive

$$\int_a^b |A(t)| \left|\nabla_{a*}^{\mu-1} f(t)\right| \nabla t \leq$$

$$\int_a^b \left[\left(\int_a^t \left|\widehat{h}_{\mu-2}(t,\rho(\tau))\right|^p \nabla\tau\right)^{\frac{1}{p}} \left(\nabla_{a*}^{\mu-1} f(t)\right)^2 (t-a)^{\frac{1}{q}}\right] \nabla t$$

$$\leq \left(\int_a^b \left(\int_a^t \left|\widehat{h}_{\mu-2}(t,\rho(\tau))\right|^p \nabla\tau\right) \nabla t\right)^{\frac{1}{p}} \left(\int_a^b \left(\nabla_{a*}^{\mu-1} f(t)\right)^{2q} (t-a) \nabla t\right)^{\frac{1}{q}}$$

$$\leq (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t \left|\widehat{h}_{\mu-2}(t,\rho(\tau))\right|^p \nabla\tau\right) \nabla t\right)^{\frac{1}{p}} \left(\int_a^b \left(\nabla_{a*}^{\mu-1} f(t)\right)^{2q} \nabla t\right)^{\frac{1}{q}},$$

proving the claim. ∎

It follows related Ostrowski type inequalities.

Theorem 43.13. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{ld}^m(T)$, $a,b \in T$, $a \leq b$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s,\rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s,\rho(t))$ to be continuous on $([a,b] \cap T)^2$, and $f^{\nabla^k}(a) = 0$, $k = 1,...,m-1$. Denote $A(t) = f(t) - D\left(f^{\nabla^m}, \mu-1, \widetilde{\nu}+1, T, t\right)$, $t \in [a,b] \cap T$.

Then

$$\left|\frac{1}{b-a}\int_a^b A(t)\nabla t - f(a)\right| \leq$$

$$\frac{1}{b-a}\left(\int_a^b \left(\int_a^t \left|\widehat{h}_{\mu-2}(t,\rho(\tau))\right| \nabla\tau\right) \nabla t\right) \left\|\nabla_{a*}^{\mu-1} f\right\|_{\infty,[a,b]\cap T}. \tag{43.23}$$

Proof. By (43.16) we obtain

$$A(t) - f(a) = \int_a^t \widehat{h}_{\mu-2}(t,\rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla\tau, \quad \forall\, t \in [a,b] \cap T.$$

Then

$$|A(t) - f(a)| \leq \left(\int_a^t |\widehat{h}_{\mu-2}(t,\rho(\tau))| \nabla\tau\right) \left\|\nabla_{a*}^{\mu-1} f\right\|_{\infty,[a,b]\cap T},$$

$\forall\ t \in [a,b] \cap T$.

Therefore we get that

$$\left| \frac{1}{b-a} \int_a^b A(t) \nabla t - f(a) \right| = \frac{1}{b-a} \left| \int_a^b (A(t) - f(a)) \nabla t \right|$$

$$\leq \frac{1}{b-a} \int_a^b |A(t) - f(a)| \nabla t$$

$$\leq \frac{1}{b-a} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right| \nabla\tau \right) \nabla t \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{\infty,[a,b]\cap T},$$

proving the claim. ∎

Theorem 43.14. All as in Theorem 43.13. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \frac{1}{b-a} \int_a^b A(t) \nabla t - f(a) \right| \leq$$

$$\frac{1}{b-a} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{1}{p}} \nabla t \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{q,[a,b]\cap T}. \tag{43.24}$$

Proof. By (43.16) we derive

$$|A(t) - f(a)| \leq \int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right| \left| \nabla_{a*}^{\mu-1} f(\tau) \right| \nabla\tau$$

$$\leq \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{1}{p}} \left(\int_a^t \left| \nabla_{a*}^{\mu-1} f(\tau) \right|^q \nabla\tau \right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{1}{p}} \left\| \nabla_{a*}^{\mu-1} f \right\|_{q,[a,b]\cap T}.$$

That is we have

$$|A(t) - f(a)| \leq \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{1}{p}} \left\| \nabla_{a*}^{\mu-1} f \right\|_{q,[a,b]\cap T}, \quad \forall\ t \in [a,b] \cap T.$$

Therefore we obtain

$$\left| \frac{1}{b-a} \int_a^b A(t) \nabla t - f(a) \right| \leq \frac{1}{b-a} \int_a^b |A(t) - f(a)| \nabla t$$

$$\leq \frac{1}{b-a} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla\tau \right)^{\frac{1}{p}} \nabla t \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{q,[a,b]\cap T},$$

proving the claim. ■

We finish general fractional nabla time scales inequalities with a related Hilbert-Pachpatte type inequality.

Theorem 43.15. Let $\varepsilon > 0$, $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f_i \in C_{ld}^m(T_i)$, $a_i, b_i \in T_i$, $a_i \leq b_i$, $T_{ik} = T_i$ time scale, $i = 1, 2$. Suppose $\widehat{h}_{\mu-2}^{(i)}(s_i, \rho_i(t_i))$, $\widehat{h}_{\widetilde{\nu}}^{(i)}(s_i, \rho_i(t_i))$ to be continuous on $([a_i, b_i] \cap T_i)^2$, and $f_i^{\nabla^k}(a_i) = 0$, $k = 0, 1, ..., m-1$; $i = 1, 2$. Here $A_i(t_i) = f_i(t_i) - D_i(f_i^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T_i, t_i)$, $t_i \in [a_i, b_i] \cap T_i$; $i = 1, 2$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Set

$$F(t_1) = \int_{a_1}^{t_1} \left(\left|\widehat{h}_{\mu-2}^{(1)}(t_1, \rho_1(\tau_1))\right|\right)^p \nabla\tau_1,$$

for all $t_1 \in [a_1, b_1]$, and

$$G(t_2) = \int_{a_2}^{t_2} \left(\left|\widehat{h}_{\mu-2}^{(2)}(t_2, \rho_2(\tau_2))\right|\right)^q \nabla\tau_2,$$

for all $t_2 \in [a_2, b_2]$ (where $\widehat{h}_{\mu-2}^{(i)}$, ρ_i are the corresponding $\widehat{h}_{\mu-2}$, ρ to T_i, $i = 1, 2$).

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|A_1(t_1)| |A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \nabla t_1 \nabla t_2 \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left|\nabla_{a_1 *}^{\mu-1} f_1(t_1)\right|^q \nabla t_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|\nabla_{a_2 *}^{\mu-1} f_2(t_2)\right|^p \nabla t_2\right)^{\frac{1}{p}}. \tag{43.25}$$

(above double time scales Riemann nabla integration is considered in the natural interative way).

Proof. We notice that

$$\lambda(t_1) = \int_{a_2}^{b_2} \frac{|A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \nabla t_2$$

is a Riemann ∇-integrable function on $[a_1, b_1] \cap T_1$.

Since $f_i^{\nabla^k}(a_i) = 0$, $k = 0, 1, ..., m-1$; $i = 1, 2$, by Corollary 43.9 we get that

$$A_i(t_i) = \int_{a_i}^{t_i} \widehat{h}_{\mu-2}^{(i)}(t_i, \rho_i(\tau_i)) \nabla_{a_i *}^{\mu-1} f_i(\tau_i) \nabla\tau_i,$$

$\forall\, t_i \in [a_i, b_i] \cap T_i$, where $a_i, b_i \in T_i$.

Therefore

$$|A_1(t_1)| \leq \left(\int_{a_1}^{t_1} \left(\left|\widehat{h}_{\mu-2}^{(1)}(t_1, \rho_1(\tau_1))\right|\right)^p \nabla\tau_1\right)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|\nabla_{a_1 *}^{\mu-1} f_1(\tau_1)\right|^q \nabla\tau_1\right)^{\frac{1}{q}}$$

$$= F(t_1)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left| \nabla_{a_1*}^{\mu-1} f_1(\tau_1) \right|^q \nabla \tau_1 \right)^{\frac{1}{q}},$$

and

$$|A_2(t_2)| \leq \left(\int_{a_2}^{t_2} \left(\left| \widehat{h}_{\mu-2}^{(2)}(t_2, \rho_2(\tau_2)) \right| \right)^q \nabla \tau_2 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \nabla_{a_2*}^{\mu-1} f_2(\tau_2) \right|^p \nabla \tau_2 \right)^{\frac{1}{p}}$$

$$= G(t_2)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \nabla_{a_2*}^{\mu-1} f_2(\tau_2) \right|^p \nabla \tau_2 \right)^{\frac{1}{p}}.$$

Young's inequality for $a, b \geq 0$ says that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Hence we have

$$|A_1(t_1)| |A_2(t_2)| \leq$$

$$(F(t_1))^{\frac{1}{p}} (G(t_2))^{\frac{1}{q}} \left(\int_{a_1}^{t_1} \left| \nabla_{a_1*}^{\mu-1} f_1(\tau_1) \right|^q \nabla \tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \nabla_{a_2*}^{\mu-1} f_2(\tau_2) \right|^p \nabla \tau_2 \right)^{\frac{1}{p}}$$

$$\leq \left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right) \left(\int_{a_1}^{t_1} \left| \nabla_{a_1*}^{\mu-1} f_1(\tau_1) \right|^q \nabla \tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \nabla_{a_2*}^{\mu-1} f_2(\tau_2) \right|^p \nabla \tau_2 \right)^{\frac{1}{p}}.$$

The last gives ($\varepsilon > 0$)

$$\frac{|A_1(t_1)| |A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right)} \leq \left(\int_{a_1}^{t_1} \left| \nabla_{a_1*}^{\mu-1} f_1(\tau_1) \right|^q \nabla \tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| \nabla_{a_2*}^{\mu-1} f_2(\tau_2) \right|^p \nabla \tau_2 \right)^{\frac{1}{p}},$$

for all $t_i \in [a_i, b_i] \cap T_i$, $i = 1, 2$.

Next we observe that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|A_1(t_1)| |A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right)} \nabla t_1 \nabla t_2 \leq$$

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left| \nabla_{a_1*}^{\mu-1} f_1(\tau_1) \right|^q \nabla \tau_1 \right)^{\frac{1}{q}} \nabla t_1 \right) \cdot$$

$$\left(\int_{a_2}^{b_2} \left(\int_{a_2}^{t_2} \left| \nabla_{a_2*}^{\mu-1} f_2(\tau_2) \right|^p \nabla \tau_2 \right)^{\frac{1}{p}} \nabla t_2 \right) \leq$$

(by Hölder's inequality)

$$\left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left| \nabla_{a_1*}^{\mu-1} f_1(\tau_1) \right|^q \nabla \tau_1 \right) \nabla t_1 \right)^{\frac{1}{q}} (b_1 - a_1)^{\frac{1}{p}} \cdot$$

$$\left(\int_{a_2}^{b_2}\left(\int_{a_2}^{t_2}\left|\nabla_{a_2*}^{\mu-1}f_2(\tau_2)\right|^p\nabla\tau_2\right)\nabla t_2\right)^{\frac{1}{p}}(b_2-a_2)^{\frac{1}{q}}$$

$$\leq\left(\int_{a_1}^{b_1}\left(\int_{a_1}^{b_1}\left|\nabla_{a_1*}^{\mu-1}f_1(\tau_1)\right|^q\nabla\tau_1\right)\nabla t_1\right)^{\frac{1}{q}}(b_1-a_1)^{\frac{1}{p}}\cdot$$

$$\left(\int_{a_2}^{b_2}\left(\int_{a_2}^{b_2}\left|\nabla_{a_2*}^{\mu-1}f_2(\tau_2)\right|^p\nabla\tau_2\right)\nabla t_2\right)^{\frac{1}{p}}(b_2-a_2)^{\frac{1}{q}}$$

$$=(b_1-a_1)(b_2-a_2)\left(\int_{a_1}^{b_1}\left|\nabla_{a_1*}^{\mu-1}f_1(\tau_1)\right|^q\nabla\tau_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|\nabla_{a_2*}^{\mu-1}f_2(\tau_2)\right|^p\nabla\tau_2\right)^{\frac{1}{p}},$$

proving the claim. ■

43.3 Applications

I) Here $T=\mathbb{R}$ case.

Let $\mu>2$ such that $m-1<\mu<m\in\mathbb{N}$, $\widetilde{\nu}=m-\mu$, $f\in C^m([a,b])$, $a,b\in\mathbb{R}$.

The nabla fractional derivative on $\mathbb{R}$ of order $\mu-1$ is defined as follows:

$$\nabla_{a*}^{\mu-1}f(t)=\left(J_a^{\widetilde{\nu}+1}f^{(m)}\right)(t)=\frac{1}{\Gamma(\widetilde{\nu}+1)}\int_a^t(t-\tau)^{\widetilde{\nu}}f^{(m)}(\tau)\,d\tau, \tag{43.26}$$

$\forall\ t\in[a,b]$.

Notice that $\nabla_{a*}^{\mu-1}f\in C([a,b])$, and $A(t)=f(t)$, $\forall\ t\in[a,b]$.

We give a Poincaré type inequality.

Theorem 43.16. Let $\mu>2$, $m-1<\mu<m\in\mathbb{N}$, $f\in C^m(\mathbb{R})$, $a,b\in\mathbb{R}$, $a\leq b$. Suppose $f^{(k)}(a)=0$, $k=0,1,...,m-1$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then

$$\int_a^b|f(t)|^q\,dt\leq\frac{(b-a)^{(\mu-1)q}}{(\Gamma(\mu-1))^q(\mu-1)q((\mu-2)p+1)^{q-1}}\left(\int_a^b\left|\nabla_{a*}^{\mu-1}f(t)\right|^q dt\right). \tag{43.27}$$

Proof. By Theorem 43.10. ■

We present a Sobolev type inequality.

Theorem 43.17. All as in Theorem 43.16. Let $r\geq1$. Then

$$\|f\|_r\leq\frac{(b-a)^{\mu-2+\frac{1}{p}+\frac{1}{r}}}{\Gamma(\mu-1)((\mu-2)p+1)^{\frac{1}{p}}\left((\mu-2)r+\frac{r}{p}+1\right)^{\frac{1}{r}}}\left\|\nabla_{a*}^{\mu-1}f\right\|_q. \tag{43.28}$$

Proof. By Theorem 43.11. ■

We continue with an Opial type inequality.

Theorem 43.18. All as in Theorem 43.16. Assume $\left|\nabla_{a*}^{\mu-1} f\right|$ is increasing on $[a,b]$.

$$\int_a^b |f(t)| \left|\nabla_{a*}^{\mu-1} f(t)\right| dt \leq$$

$$\frac{(b-a)^{\mu-\frac{1}{q}}}{\Gamma(\mu-1)\left[((\mu-2)p+1)((\mu-2)p+2)\right]^{\frac{1}{p}}} \left(\int_a^b \left(\nabla_{a*}^{\mu-1} f(t)\right)^{2q} dt\right)^{\frac{1}{q}}. \quad (43.29)$$

Proof. By Theorem 43.12. ■

Some Ostrowski type inequalities follow.

Theorem 43.19. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $f \in C^m(\mathbb{R})$, $a, b \in \mathbb{R}$, $a \leq b$. Suppose $f^{(k)}(a) = 0$, $k = 1, ..., m-1$. Then

$$\left|\frac{1}{b-a}\int_a^b f(t)\, dt - f(a)\right| \leq \frac{(b-a)^{\mu-1}}{\Gamma(\mu+1)} \left\|\nabla_{a*}^{\mu-1} f\right\|_{\infty,[a,b]}. \quad (43.30)$$

Proof. By Theorem 43.13. ■

Theorem 43.20. Here all as in Theorem 43.19. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\frac{1}{b-a}\int_a^b f(t)\, dt - f(a)\right| \leq \frac{(b-a)^{\mu-\frac{1}{q}-1}}{\Gamma(\mu-1)\left(\mu-\frac{1}{q}\right)((\mu-2)p+1)^{\frac{1}{p}}} \left\|\nabla_{a*}^{\mu-1} f\right\|_{q,[a,b]}. \quad (43.31)$$

Proof. By Theorem 43.14. ■

We finish this subsection with a Hilbert-Pachpatte inequality on $\mathbb{R}$.

Theorem 43.21. Let $\varepsilon > 0$, $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $i = 1, 2$; $f_i \in C^m(\mathbb{R})$, $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$, $f_i^{(k)}(a_i) = 0$, $k = 0, 1, ..., m-1$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Put

$$F(t_1) = \frac{(t_1 - a_1)^{(\mu-2)p+1}}{(\Gamma(\mu-1))^p ((\mu-2)p+1)},$$

$t_1 \in [a_1, b_1]$, and

$$G(t_2) = \frac{(t_2 - a_2)^{(\mu-2)q+1}}{(\Gamma(\mu-1))^q ((\mu-2)q+1)},$$

$t_2 \in [a_2, b_2]$.

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(t_1)||f_2(t_2)|}{\left(\varepsilon+\frac{F(t_1)}{p}+\frac{G(t_2)}{q}\right)}dt_1dt_2 \le$$

$$(b_1-a_1)(b_2-a_2)\left(\int_{a_1}^{b_1}\left|\nabla_{a_1*}^{\mu-1}f_1(t_1)\right|^q dt_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|\nabla_{a_2*}^{\mu-1}f_2(t_2)\right|^p dt_2\right)^{\frac{1}{p}}. \tag{43.32}$$

Proof. By Theorem 43.15. ■

II) Here $T=\mathbb{Z}$ case.

Let $\mu>2$ such that $m-1<\mu<m\in\mathbb{N}$, $\widetilde{\nu}=m-\mu$, $a,b\in\mathbb{Z}$, $a\le b$. Here $f:\mathbb{Z}\to\mathbb{R}$, and $f^{\nabla^m}(t)=\nabla^m f(t)=\sum_{k=0}^m(-1)^k\begin{pmatrix} m \\ k \end{pmatrix}f(t-k)$.

The nabla fractional derivative on $\mathbb{Z}$ of order $\mu-1$ is defined as follows:

$$\nabla_{a*}^{\mu-1}f(t)=\left(J_a^{\widetilde{\nu}+1}(\nabla^m f)\right)(t)=\frac{1}{\Gamma(\widetilde{\nu}+1)}\sum_{\tau=a+1}^{t}(t-\tau+1)^{\overline{\widetilde{\nu}}}(\nabla^m f)(\tau), \tag{43.33}$$

$\forall\ t\in[a,\infty)\cap\mathbb{Z}$.

Notice here that $\nu(t)=1$, $\forall\ t\in\mathbb{Z}$, and

$$A(t)=f(t)-D(\nabla^m f,\mu-1,\widetilde{\nu}+1,\mathbb{Z},t)$$

$$=f(t)-\sum_{u=a+1}^{t}(\nabla^m f(u))\frac{(t-u+1)^{\overline{\mu-2}}}{\Gamma(\mu-1)}, \tag{43.34}$$

$\forall\ t\in[a,\infty)\cap\mathbb{Z}$.

We give a discrete fractional Poincaré type inequality.

Theorem 43.22. Let $\mu>2$, $m-1<\mu<m\in\mathbb{N}$, $a,b\in\mathbb{Z}$, $a\le b$, $f:\mathbb{Z}\to\mathbb{R}$. Assume $\nabla^k f(a)=0$, $k=0,1,...,m-1$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then

$$\sum_{t=a+1}^{b}|A(t)|^q\le$$

$$\frac{1}{(\Gamma(\mu-1))^q}\left(\sum_{t=a+1}^{b}\left(\sum_{\tau=a+1}^{t}(t-\tau+1)^{(\overline{\mu-2})p}\right)\right)\left(\sum_{t=a+1}^{b}\left|\nabla_{a*}^{\mu-1}f(t)\right|^q\right). \tag{43.35}$$

Proof. By Theorem 43.10. ■

We continue with a discrete fractional Sobolev type inequality.

Theorem 43.23. Here all as in Theorem 43.22. Let $r\ge1$ and denote

$$\|f\|_r=\left(\sum_{t=a+1}^{b}|f(t)|^r\right)^{\frac{1}{r}}.$$

Then

$$\|A\|_r \leq \frac{1}{\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{(\overline{\mu-2})p} \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \left\| \nabla_{a*}^{\mu-1} f \right\|_q . \quad (43.36)$$

Proof. By Theorem 43.11. ■

Next we give a discrete fractional Opial type inequality.

Theorem 43.24. Here all as in Theorem 43.22. Suppose that $\left|\nabla_{a*}^{\mu-1} f\right|$ is increasing on $[a, b] \cap \mathbb{Z}$. Then

$$\sum_{t=a+1}^{b} |A(t)| \left|\nabla_{a*}^{\mu-1} f(t)\right| \leq$$

$$\frac{(b-a)^{\frac{1}{q}}}{\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{(\overline{\mu-2})p} \right) \right)^{\frac{1}{p}} \left(\sum_{t=a+1}^{b} \left(\nabla_{a*}^{\mu-1} f(t)\right)^{2q} \right)^{\frac{1}{q}} . \quad (43.37)$$

Proof. By Theorem 43.12. ■

It follows related discrete fractional Ostrowski type inequalities.

Theorem 43.25. Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $a \leq b$, $f : \mathbb{Z} \to \mathbb{R}$. Assume $\nabla^k f(a) = 0$, $k = 1, ..., m-1$.

Then

$$\left| \frac{1}{b-a} \sum_{t=a+1}^{b} A(t) - f(a) \right| \leq$$

$$\frac{1}{(b-a)\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-2}} \right) \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{\infty,[a,b]\cap\mathbb{Z}} . \quad (43.38)$$

Proof. By Theorem 43.13. ■

Theorem 43.26. All as in Theorem 43.25. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \frac{1}{b-a} \sum_{t=a+1}^{b} A(t) - f(a) \right| \leq$$

$$\frac{1}{(b-a)\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{(\overline{\mu-2})p} \right)^{\frac{1}{p}} \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{q,[a,b]\cap\mathbb{Z}} . \quad (43.39)$$

Proof. By Theorem 43.14. ■

We finish chapter with a discrete fractional Hilbert-Pachpatte type inequality.

Theorem 43.27. Let $\varepsilon > 0$, $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$; $i = 1, 2$; $f_i : \mathbb{Z} \rightarrow \mathbb{R}$, $a_i, b_i \in \mathbb{Z}$, $a_i \leq b_i$. Suppose $\nabla^k f_i(a_i) = 0$, $k = 0, 1, ..., m-1$. Here $A_i(t_i) = f_i(t_i) - \sum_{u_i=a_i+1}^{t_i} (\nabla^m f(u_i)) \frac{(t_i - u_i + 1)^{\overline{\mu-2}}}{\Gamma(\mu-1)}$, $\forall\, t_i \in [a_i, \infty) \cap \mathbb{Z}$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Set

$$F(t_1) = \sum_{\tau_1 = a_1+1}^{t_1} \frac{(t_1 - \tau_1 + 1)^{(\overline{\mu-2})p}}{(\Gamma(\mu-1))^p},$$

$\forall\, t_1 \in [a_1, \infty) \cap \mathbb{Z}$, and

$$G(t_2) = \sum_{\tau_2 = a_2+1}^{t_2} \frac{(t_2 - \tau_2 + 1)^{(\overline{\mu-2})q}}{(\Gamma(\mu-1))^q},$$

$\forall\, t_2 \in [a_2, \infty) \cap \mathbb{Z}$.

Then

$$\sum_{t_1=a_1+1}^{b_1} \sum_{t_2=a_2+1}^{b_2} \frac{|A_1(t_1)|\,|A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\sum_{t_1=a_1+1}^{b_1} \left| \nabla_{a_1 *}^{\mu-1} f_1(t_1) \right|^q \right)^{\frac{1}{q}} \left(\sum_{t_2=a_2+1}^{b_2} \left| \nabla_{a_2 *}^{\mu-1} f_2(t_2) \right|^p \right)^{\frac{1}{p}}. \tag{43.40}$$

Proof. By Theorem 43.15. ■

44

Optimal Error Estimate for the Numerical Solution of Multidimensional Dirichlet Problem

For the multivariate Dirichlet problem of the Poisson equation on an arbitrary compact domain, this chapter examines convergence properties with rates of approximate solutions, obtained by a standard difference scheme over inscribed uniform grids. Sharp quantitative estimates are proved by the use of second moduli of continuity of the second single partial derivatives of the exact solution. This is achieved by engaging the probabilistic method of simple random walk. This chapter is based on [63].

44.1 Introduction

Consider $\Omega \subset \mathrm{R}^l,\ l \geq 1$, an open subset with compact closure $\overline{\Omega}$ and a regular boundary $\partial\Omega$, and the Laplacian

$$\Delta := \sum_{i=1}^{l} \partial^2 x_i.$$

The Dirichlet problem in Ω has a solution u on $\overline{\Omega}$ so that

$$\Delta u(x) = -f(x), \quad (\forall)\ x \in \Omega,$$

$$\lim_{x \to y} u(x) = \varphi(y), \quad (\forall)\ y \in \partial\Omega,$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 731–747.
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where f, φ are appropriate real valued functions defined on Ω, $\partial\Omega$, respectively.

Let Ω_h be the inscribed in Ω uniform grid of mesh $h = \frac{1}{n}$, $n \in \mathrm{N}$, with boundary $\partial\Omega_h$. Also, we consider the discrete Dirichlet problem of finding u_h such that

$$\Delta_h u_h(x) = -f(x), \quad (\forall)\ x \in \Omega_h,$$

$$u_h(x) = \varphi(x), \quad (\forall)\ x \in \partial\Omega_h,$$

where

$$\Delta_h u_h(x) := h^{-2} \cdot \left[\sum_{k=1}^{l} u_h(x \pm he_k) - 2l u_h(x) \right], \quad (\forall)\ x \in \Omega_h,$$

is the discrete Laplacian. Here e_k is the natural basis in R^l. Using the probabilistic method of simple random walk we are able to establish that

$$(i) \quad \|u_h - u\|_{\overline{\Omega}_h} \le \frac{1}{4} \sum_{i=1}^{l} w_{2,i}(h, \partial^2_{x_i} u) + D_h,$$

where $\| \cdot \|_{\overline{\Omega}_h}$ is the supremum norm in $\overline{\Omega}_h$.

Here $w_{2,i}$ is the second modulus of continuity of the second single partial of u with respect to x_i, $i = 1, \ldots, l$; and $D_h = \mathrm{Distance}(\Omega, \Omega_h) \to 0$ as $h \to 0$. See Theorems 44.3 (case of $D_h = 0$) and 44.7.

When $\Omega = \{x : \ 0 < x_i < 1\}$, case of $D_h = 0$, inequality (i) is proved to be sharp using a similar method as in [124]. See Theorem 44.6, along with Remark 44.4.

This chapter has been greatly motivated by the pioneering very important work of Büttgenbach, Esser, Lüttgens and Nessel (1992), see [124]. There the above authors worked on a square and produced basic results for the two-dimensional Dirichlet problem, whose generalizations in R^l are found in this chapter. Their method was purely analytical and totally different than the probabilistic approach here.

44.2 Background

44.2.1 Dirichlet Problem: Continuous Case

Let $\Omega \subset \mathrm{R}^l$ be an open subset with compact closure $\overline{\Omega}$ and $\Delta = \partial^2_{x_1} + \ldots + \partial^2_{x_l}$ be the Laplacian. The Dirichlet problem in Ω consists in finding a function on $\overline{\Omega}$ such that for given functions f, defined in Ω, and φ, defined on $\partial\Omega$, we have

$$\begin{cases} \Delta u(x) = -f(x), & (\forall)\ x \in \Omega, \\ \lim_{x\to y} u(x) = \varphi(x), & (\forall)\ y \in \partial\Omega. \end{cases} \tag{44.1}$$

It is well-known fact [147, pp. 8, 49, 85] that if Ω has regular boundary (for example $\partial\Omega$ is a smooth surface) and f is a bounded locally Hölder function and φ is a continuous function then the problem (44.1) has a unique solution $u(x)$, which can be represented in the form

$$u(x) = G_\Omega f(x) + H_\Omega \varphi(x), \tag{44.2}$$

where

$$G_\Omega f(x) = \int_\Omega g_\Omega(x,y) f(y) dy \tag{44.3}$$

$$H_\Omega \varphi(x) = \int_{\partial\Omega} \varphi(y) \Pi_\Omega(x, dy) \tag{44.4}$$

are the Green potential of the function f and the harmonic in Ω function with boundary values φ, respectively. In (44.3) $g_\Omega(x,y)$ is the so-called Green function of Ω which is determined uniquely by the following properties

(i) $\Delta g_\Omega(x,y) = -\delta(x-y),\ x, y \in \Omega$, where δ is the Dirac delta function,

(ii) $g_\Omega(x,y) = 0,\ x \in \partial\Omega \ or \ y \in \partial\Omega.$

The Kernel $\Pi_\Omega(x, dy)$ is the so-called measure of domain Ω or Poisson kernel of Ω and if $\partial\Omega$ is a smooth surface then $\Pi_\Omega(x; dy) = \Pi_\Omega(x,y)d\sigma$, where $d\sigma$ is a surface measure $\partial\Omega$. The function $\Pi_\Omega(x,y),\ x \in \Omega,\ y \in \partial\Omega$ can be defined by the following relation

$$\Pi_\Omega(x,y) = \frac{\partial}{\partial n} g_\Omega(x,y),\ x \in \Omega,\ y \in \partial\Omega, \tag{44.5}$$

where $\frac{\partial}{\partial n}$ is the normal derivative at the boundary $\partial\Omega$.

Following the main idea of this chapter we briefly give here the important probabilistic counterpart of the analytical facts mentioned above. We refer to reader to [151], [152], and [147].

Let $(x_t, P^x),\ x \in \mathrm{R}^l$ be the Wiener process in R^l starting at the point x.

Denote by τ_Ω the first exit time of Ω

$$\tau_\Omega := \inf\{t > 0 : x_t \in \mathrm{R}^l \backslash \Omega\}.$$

As usual we denote by $E_x F(\omega)$ the mathematical expectation corresponding to the measure P^x. We have

$$G_\Omega f(x) = E_x \left(\int_0^{\tau_\Omega} f(x_s)ds \right), \tag{44.6}$$

$$H_\Omega \varphi(x) = E_x \left[\varphi(x_{\tau_\Omega}), \tau_\Omega < \infty \right]. \tag{44.7}$$

In particular, if $\Omega = \{x : |x| < R\}$ is a ball of radius R then it is easy to see that the function $u(x) = \frac{1}{2l}(R^2 - x^2)$ is a solution of the following Dirichlet problem

$$\begin{cases} \Delta u(x) = -1, \; x \in \Omega, \\ u(x) = 0, \; x \in \partial\Omega. \end{cases}$$

Thus

$$G_\Omega 1(x) = \frac{1}{2l}(R^2 - x^2)$$

and we finally get that

$$E_x \left[\tau_\Omega \right] = \frac{1}{2l}(R^2 - x^2) \leq \frac{1}{2l} R^2. \tag{44.8}$$

As an easy but important consequence of this fact, we find that for each bounded domain Ω the corresponding first exit time τ_Ω is finite almost surely and

$$E_x \tau_\Omega \leq \frac{1}{8l} d_\Omega^2,$$

where d_Ω is the diameter of Ω. Indeed, let Ω' be a ball of the radius $\frac{1}{2} d_\Omega$ such that $\Omega \subset \Omega'$. Then $\tau_\Omega \leq \tau_{\Omega'}$ and we obtain

$$E_x \tau_\Omega \leq E_x \tau_{\Omega'} \leq \frac{1}{8l} d_\Omega^2.$$

Another useful fact we would like to mention here is the following property. Let $\Omega_n \subset \Omega$ be an increasing sequence of subdomains of Ω such that $\cup_{n \geq 1} \Omega_n = \Omega$. Let for each $h \geq 1$, τ_n be the first exit time of Ω_n. It is clear that $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\Omega$. Using the continuity of the sample path $t \to x_t$ we easily find that

$$\lim_{n \to \infty} \tau_n = \tau_\Omega.$$

Now assume that the function $\varphi(x)$ is well defined in some neighborhood of $\partial\Omega$ and is Lipschitz continuous there. Denote by u_n the solution of the Dirichlet problem in Ω_n with functions $f_n = f\Big|_{\Omega_n}$ and $\varphi_n := \varphi\Big|_{\partial\Omega_n}$. We estimate the difference $u - u_n$ on the fixed compact $K \subset \Omega$. By (44.6)- (44.7) we derive

$$|u(x) - u_n(x)| \leq E_x\left(\int_{\tau_n}^{\tau_\Omega} f(x_t)dt\right) + E_x|\varphi(x_{\tau_\Omega}) - \varphi(x_{\tau_n})|$$

$$\leq \|f\|_{C(\Omega)}E_x(\tau_\Omega - \tau_n) + L_\Omega \cdot E_x|x_{\tau_\Omega} - x_{\tau_n}|$$

$$\leq \|f\|_{C(\Omega}E_x(\tau_\Omega - \tau_n) + L_\Omega \cdot \sqrt{E_x(x_{\tau_\Omega} - x_{\tau_n})^2}$$

$$= \|f\|_{C(\Omega}E_x(\tau_\Omega - \tau_n) + L_\Omega \cdot \sqrt{E_x(\tau_\Omega - \tau_n)},$$

where

$$L_\Omega := \sup \frac{|\varphi(x) - \varphi(y)|}{|x - y|}$$

is Lipschitz constant of φ. Thus, if we denote by

$$\delta_n(k) := \sup_{x \in K} \sqrt{E_x(\tau_\Omega - \tau_n)},$$

then we derive the following estimate

$$\sup_{x \in K} |u(x) - u_n(x)| \leq f\|_{C(\Omega}\delta_n^2(k) + L_\Omega \cdot \delta_n(k). \tag{44.9}$$

Thus, using the fact that $\delta_n(k) \downarrow 0$ as $n \to \infty$ we find

$$\sup_{x \in K} \|u(x) - u_n(x)| \sim L_\Omega \cdot \delta_n(K). \tag{44.10}$$

We denote that (44.10) gives us the estimation of the speed of the convergence $u_n \to u$ in the geometrical terms $\{\delta_n(K)\}$.

44.2.2 Dirichlet Problem: Discrete Case

Let Z^l be an l- dimensional integer-valued lattice. This lattice consists of points (vectors) of the type $x = x_1e_1 + \ldots + x_le_l$, where $e_1, \ldots, e_l$ comprises the orthonormal basis of R^l, and the coordinates $x_1, \ldots, x_l$ are arbitrary integers. Increasing or decreasing one of the coordinates by one unit and leaving the other coordinates unchanged, we obtain the $2l$ neighboring lattice points to x. Let B a subset of points of a lattice Z^l. We call a point $x \notin B$ a boundary point for the set B if at least one point of the type $x \pm e_k$ belongs to B. The collection of boundary points of the set B is called the boundary of B, denoting it ∂B.

Let f be a function defined at the points of a lattice Z^l. We put

$$Pf(x) := \frac{1}{2l} \sum_{k=1}^{l} f(x \pm e_k).$$

It is logical to call P the averaging operator. It is well known that the linear operator $P - E$, where E is the unit operator, is the discrete analog of the Laplacian Δ. Indeed, for a sufficiently smooth function $f(x)$ specified over the space R^l,

$$\Delta f(x) = \lim_{h \to 0} \frac{\sum_{k=1}^{l} f(x \pm e_k) - 2lf(x)}{h^2}$$

so that the Laplacian is obtained by passing to the limit from the operator $P - E$ as the lattice is infinitely partitioned.

Let $\Omega \subset \mathrm{Z}^l$ be a finite subset (i.e., cardinality $|\Omega| < \infty$). The Dirichlet problem in Ω consists in finding a function $u(x),\ x \in \Omega \cup \partial\Omega$ such that for given functions f defined in Ω and φ defined in $\partial\Omega$ we have

$$\begin{cases} (P - E)u(x) = -f(x),\ x \in \Omega, \\ u(x) = \varphi(x),\ x \in \partial\Omega. \end{cases} \tag{44.11}$$

First of all we note that if u_1 and u_2 are two solutions of the problem (44.11) then $u_1 \equiv u_2$. This fact follows immediately from the well-known minimum principle [152, Ch. 1, Problems 18, 19]: if Ω is connected, i.e., each two points $x, y \in \Omega$ can be connected by a chain of points $x_1 = x, x_2, \ldots, x_n = y$ from Ω, such that each of the differences $x_i - x_{i-1} = \pm e_k$ for some $k \le l$, u is a function on Ω such that $Pu \le u$, and u reaches its minimum value on $\Omega \cup \partial\Omega$ at a point $x \in \Omega$, then u is constant on $\Omega \cup \partial\Omega$.

Next our remark concerns the decomposition $u = u_1 + u_2$ of the solution u of the problem (44.11), where

$$\begin{cases} (P - E)u_1(x) = -f(x),\ x \in \Omega, \\ u_1(x) = 0,\ x \in \partial\Omega, \end{cases} \tag{44.12}$$

$$\begin{cases} (P - E)u_2(x) = 0,\ x \in \Omega, \\ u_2(x) = \varphi(x),\ x \in \partial\Omega. \end{cases} \tag{44.13}$$

This decomposition is a discrete analog of the decomposition (44.2). Following the same reasoning as in Part 44.2.1 of this chapter we give the probabilistic representation of the "discrete" Green potential $u_1 := G_\Omega f$ and the "discrete" harmonic function $u_2 := H_\Omega \varphi$. In this part of our exposition we follow the monographs [152] and [270].

A simple random walk on the lattice Z^l is a random process $(x(n), P)$ with values $x(u) \in \mathrm{Z}^l$ such that the increments $x(n+1) - x(n),\ n = 0, 1, \ldots$, are independent identically distributed random variables and

$$P(x(1) - x(0) = x) = \frac{1}{2l} \ when \ x = \pm e_k,$$

$$= 0 \ when \ x \neq \pm e_k.$$

It is easy to find that for each bounded function $f(x)$

$$E(f(x(1)), x(0) = x) = Pf(x),$$

and more generally

$$E(f(x(n)), x(0) = x) = P^n f(x).$$

Fix the finite subset $\Omega \subset Z^l$ and let τ_Ω be the time of first visit of the "particle" $x(\cdot)$ to the set $Z^l \backslash \Omega$ (first exit time of Ω). Following [270, p.107] we introduce the next functions,

$$\begin{aligned} Q_\Omega(n; x, y) - P(x(n) = y, n < \tau_\Omega; x(0) = x), \ \ x, y \in \Omega, \\ = 0, otherwise, \end{aligned} \tag{44.14}$$

$$g_\Omega(x,y) \quad = \quad \sum_{n=0}^{\infty} Q_\Omega(n,x,y) \tag{44.15}$$

$$\begin{aligned} H_\Omega(x,y) = P(x(\tau_\Omega) = y, \tau_\Omega < \infty, x(0) = x); \ x \in \Omega, \ y \in \partial\Omega, \\ = \delta(x,y), otherwise. \end{aligned} \tag{44.16}$$

Define also the following operators

$$G_\Omega f(x) := \sum_{y \in \Omega} f(y) g_\Omega(x,y) \tag{44.17}$$

$$H_\Omega \varphi(x) := \sum_{y \in \partial\Omega} \varphi(y) H_\Omega(x,y). \tag{44.18}$$

The proofs of the following basic facts can be found in [151, Ch. 1, Problem 21] and [270, p.108].

(A) The function $G_\Omega f(x)$ gives the unique solution of the problem (44.12) and the following representation of $G_\Omega f$ holds true

$$G_\Omega f(x) = E\left\{ \sum_{k=0}^{\tau_\Omega - 1} f(x(k));\ x(0) = x \right\}. \tag{44.19}$$

In particular

$$G_\Omega 1(x) = E\{\tau_\Omega; x(0) = x\}. \tag{44.20}$$

(B) The function $H_\Omega \varphi(x)$ gives the unique solution of the problem (44.13). An auxiliary result that it needed for later follows.

Theorem 44.1. For every function u on $\Omega \cup \partial\Omega$, the following inequality holds true:

$$\|u\|_\Omega \leq c_\Omega \|(P - E)u\|_\Omega + \|u\|_{\partial\Omega}, \tag{44.21}$$

where

$$\|u\|_\Omega := \max\{|u(x)|, x \in \Omega \cup \partial\Omega\}$$

and

$$c_\Omega := \max_{x \in \Omega} G_\Omega 1(x).$$

Proof. Define the following functions

$$\begin{cases} f(x) = -(P - E)u(x),\ x \in \Omega, \\ \varphi(x) = u(x),\ x \in \partial\Omega. \end{cases}$$

Then we have
(i) $(P - E)u(x) = -f(x),\ x \in \Omega$

(ii) $u(x) = \varphi(x),\ x \in \partial\Omega.$
By (A) and (B) we get that

$$u(x) = G_\Omega f(x) + H_\Omega \varphi(x).$$

From this identity we immediately find that

$$\begin{aligned} \|u\|_\Omega &\leq \|G_\Omega 1\|_\Omega\ \|f\|_\Omega + \|H_\Omega \varphi\|_\Omega \\ &\leq c_\Omega \|(P - E)u\|_\Omega + \|\varphi\|_{\partial\Omega} \\ &= c_\Omega \|(P - E)u\|_\Omega + \|u\|_{\partial\Omega}. \end{aligned}$$

■

The following result turns out to be useful in the following considerations.

Theorem 44.2. Let be a finite subset of Z^l. Denote

$$N_\Omega := \sup\{|x_i|,\ i = 1, \ldots, l,\ (x_1, \ldots, x_l) \in \Omega\}.$$

Then

$$c_\Omega := \sup\{G_\Omega 1(x),\ x \in \Omega\} \leq l(N_\Omega + 1)^2.$$

If $\Omega = \{(x_1, \ldots, x_l) \in Z^l : 1 \leq x_i \leq N_\Omega,\ i = 1, \ldots, l\}$ then also

$$c_\Omega \geq \frac{1}{\pi^2}(N_\Omega + 1)^2.$$

Proof. Let $\tilde{\Omega} = \{x \in Z^l : |x_1| \leq N_\Omega\}$ be a minimal slab that contains Ω. Then clearly $\tau_\Omega \leq \tau_{\tilde{\Omega}}$ and consequently

$$G_\Omega 1(x) = E\{\tau_\Omega, x(0) = x\}$$

$$\leq E\{\tau_{\tilde{\Omega}}, x(0) = x\} = G_{\tilde{\Omega}} 1(x).$$

Now note that for each $2 \leq k \leq l$ and $x \in \tilde{\Omega}$ we have

$$G_{\tilde{\Omega}} 1(x \pm e_k) = G_{\tilde{\Omega}} 1(x),$$

thus $G_{\tilde{\Omega}} 1(x) = G_{\tilde{\Omega}} 1(x_1, 0, \ldots, 0)$. Let P_i be the average operator for the one-dimensional random walk on $Z_i^1 := \{ne_i,\ n = 0, \pm 1, \ldots\}$. Then it is clear that

$$P - E = \frac{1}{l}\sum_{i=1}^{l}(P_i - E).$$

Using these remarks we find that the function $m(x_1) := G_{\tilde{\Omega}} 1(x)$ is a solution of the following one-dimensional Dirichlet problem

$$\begin{cases} (P_1 - E)m(x) = -l,\ -N_\Omega \leq x \leq N_\Omega, \\ m(x) = 0,\ x = \pm(N_\Omega + 1). \end{cases} \qquad (*)$$

Now consider the function $n(x) := l \cdot [(N_\Omega + 1)^2 - x^2],\ |x| \leq N_\Omega$ and n(x)=0, and $|x| = \pm(N_\Omega + 1)$. It is easy to see that the function $n(x)$ is a solution to the Dirichlet problem (*). Thus by uniqueness $m(x) \equiv n(x)$ and we finally find

$$c_\Omega = \|G_\Omega 1\|_\Omega \leq \|G_{\tilde{\Omega}} 1\|_{\tilde{\Omega}} \leq l(N_\Omega + 1)^2.$$

To prove the lower bound $c_\Omega \geq \frac{1}{2}l(N_\Omega+1)^2$ for the grid $\Omega = \{x : |x_i| \leq N_\Omega,\ i = 1, 2, \ldots, l\}$ we use the same method. Namely, we consider the function

$$u(s) := \sin \frac{\pi_s}{(N_\Omega + 1)}$$

and denote by

$$\mathcal{U}(x) := \prod_{i=1}^{l} u(x_i),\ x \in \mathrm{Z}^l.$$

Then we get

$$(P - E)\mathcal{U}(x) = \frac{1}{l} \sum_{i=1}^{l} (P_i - E)u(x_i) \cdot \left(\prod_{k \neq i} u(x_k) \right).$$

Now we compute $(P_i - E)u(x_i)$ for $|x_i| \leq N_\Omega$,

$$\begin{aligned}(P_i - E)u(x_i) &= \frac{1}{2}\left[\sin \frac{\pi(x_i+1)}{(N_\Omega+1)} + \sin \frac{\pi(x_i-1)}{(N_\Omega+1)}\right] - \sin \frac{\pi x_i}{(N_\Omega+1)} \\ &= \sin \frac{\pi x_i}{(N_\Omega+1)} \cos \frac{\pi}{(N_\Omega+1)} - \sin \frac{\pi x_i}{(N_\Omega+1)} \\ &= -2\sin^2 \frac{\pi}{2(N_\Omega+1)} \sin \frac{\pi x_i}{(N_\Omega+1)}.\end{aligned}$$

Thus we find that

$$(P - E)\mathcal{U}(x) = -2 \left(\sin^2 \frac{\pi}{2(N_\Omega+1)} \right) \mathcal{U}(x)$$

and moreover

$$\mathcal{U}(x) = 0,\ x \in \partial\Omega.$$

Now we apply the inequality (44.21)

$$0 < \|\mathcal{U}\|_\Omega \leq c_\Omega \cdot 2 \cdot \sin^2 \frac{\pi}{2(N_\Omega+1)} \|\mathcal{U}\|_\Omega \leq c_\Omega \cdot 2 \cdot \frac{\pi^2}{4} \cdot \frac{1}{N_\Omega+1)^2} \|\mathcal{U}\|_\Omega,$$

and finally we derive

$$c_\Omega \geq (N_\Omega+1)^2 \cdot \frac{2}{\pi^2}.$$

The proof is completed. ■

44.3 Main Results

44.3.1 Approximation on the Uniform Grid

We consider the Dirichlet problem in the open unit square $\Omega := \{x \in \mathrm{R}^l;\ 0 < x_i < 1,\ i = 1, 2, \ldots, l\}$

$$\begin{cases} \Delta u(x) = -f(x), & x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{44.22}$$

with φ continuous and f a bounded locally Hölder function. In what follows we restrict our treatment to problem (44.22) for which $u \in C^{(2)}(\tilde{\Omega})$.

Let $h = \frac{1}{n}$ with $n \in \mathrm{N}$, the set of natural numbers. An approximate solution u_h, defined on the uniform grid

$$\tilde{\Omega} := \{x : x_i = \frac{k}{n},\ 0 \le k \le n,\ 1 \le i \le l\},$$

and

$$\Omega_h := \tilde{\Omega}_h \cap \Omega,\ \partial\Omega_h := \tilde{\Omega}_h \cap \partial\Omega,$$

is obtained as the solution of the discrete counterpart to (44.22):

$$\begin{cases} \Delta_h u_h(x) = -f(x), & x \in \Omega_h, \\ u_h(x) = \varphi(x), & x \in \partial\Omega_h, \end{cases} \tag{44.23}$$

where the "discrete" Laplacian Δ_h is given by

$$\Delta_h u_h(x) := h^{-2} \left[\sum_{k=1}^{l} u_h(x \pm h e_k) - 2l u_h(x) \right].$$

The case of the dimension $l = 1, 2$ was investigated in [125] and [124]. Here the main goal is the investigation of this problem for arbitrary $l \ge 1$. The main results resemble those of the above pioneering papers. However, the proving method is the one of random walk which seems very natural, and it is a totally different approach than the one used in the above references.

We denote $h\mathrm{Z}^l = \{x = hz,\ z \in \mathrm{Z}^l\}$ and consider $\{x_h(n), P\}$ the simple random walk on the h-lattice $h\mathrm{Z}^l$

$$x_h(n) := hx(n),$$

where $x(n)$ is the simple random walk on the lattice Z^l. Corresponding to $\{x_h(n), P\}$ values and operators we attach the index h. Thus, for example, the average operator P_h has the following form

$$P_h u(x) = E\{u(x_h(1)), x(0) = x\} = \frac{1}{2l}\sum_{k=1}^{l} u(x \pm he_k).$$

It is easy to see that with these notations the discrete Laplacian Δ_h has the following form

$$\Delta_h = 2lh^{-2}(P_h - E).$$

Now applying the results (A) and (B) of Section 44.2.2 we see that the problem (44.23) has the unique solution u_h which can be represented by the form

$$u_h(x) = \frac{1}{2l} \cdot h^2 G_{\Omega_h,h} f(x) + H_{\Omega_h,h}\varphi(x). \tag{44.24}$$

In what follows, we will use the notation G_h, H_h and etc., instead of $G_{\Omega_h,h}, H_{\Omega_h,h}$ and etc. According to this argument the important inequality (44.21) takes the following form

$$\|u\|_{\Omega_h} \le \frac{1}{2}\|\Delta_h u\|_{\Omega_h} + \|u\|_{\partial\Omega_h}. \tag{44.25}$$

Indeed, by application of (44.21) and Theorem 44.2 we obtain

$$\|u\|_{\Omega_h} \le c_\Omega \|(P_h - E)u\|_{\Omega_h} + \|u\|_{\partial\Omega_h}$$

$$\le ln^2 \cdot \frac{1}{2l} h^2 \|\Delta_h u\|_{\Omega_h} + \|u\|_{\partial\Omega_h}$$

$$= \frac{1}{2}\|\Delta_h u\|_{\Omega_h} + \|u\|_{\partial\Omega_h}.$$

Next we apply inequality (44.25) to $u_h - u$, where u_h and u are the solutions of (44.23) and (44.22), respectively, and we obtain the following error estimate

$$\|u_h - u\|_{\Omega_h} \le \frac{1}{2}\|\Delta_h(u_h - u)\|_{\Omega_h} + \|u_h - u\|_{\partial\Omega_h}$$

$$= \frac{1}{2}\| - f - \Delta_h u\|_{\Omega_h} = \frac{1}{2}\|\Delta u - \Delta_h u\|_{\Omega_h},$$

that is, we have

$$\|u_h - u\|_{\tilde{\Omega}_h} = \|u_h - u\|_{\Omega_h} \le \frac{1}{2}\|\Delta u - \Delta_h u\|_{\Omega_h}. \tag{44.26}$$

Now the rate of convergence of $u_h \to u$ as $h \to 0$ can be measured via the second partial moduli of continuity $w_{2,i},\ i = 1, \ldots, l$, where $w_{2,i}(\delta, v)$ is defined for $v \in C(\tilde{\Omega})$ by

$$w_{2,i}(\delta, v) := \sup\{|v(x+\lambda e_i) - 2v(x) + v(x-\lambda e_i)| : \ x, x+2\lambda e_i \in \tilde{\Omega}, \ |\lambda| \le \delta\}.$$

Theorem 44.3. Assume that the solution u of the problem (44.22) satisfies $u \in C^2(\tilde{\Omega})$.Then for the solution u_h of the problem (44.23) the following inequality holds true

$$\|u_h - u\|_{\tilde{\Omega}_h} \le \frac{1}{4}\sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u). \tag{44.27}$$

Proof. We follow [124, Th.2]. For $u \in C^2(\tilde{\Omega})$ and $1 \le i \le l$ one has

$$u(x \pm he_i) = u(x) \pm h\partial_{x_i} u(x) + \int_0^h (h-s)\partial_{x_i}^2 u(x \pm se_i)ds,$$

which implies

$$(P_{h,i} - E)u(x) = \frac{1}{2}\int_0^h (h-s)[\partial_{x_i}^2 u(x+se_i) + \partial_{x_i}^2 u(x-se_i)]ds.$$

Hence

$$\Delta_h u(x) = 2lh^{-2}(P_h - E)u(x) = 2lh^{-2}\frac{1}{l}\sum_{i=1}^{l}(P_{h,i} - E)u(x)$$

$$= h^{-2}\int_0^h (h-s)\sum_{i=1}^{l}[\partial_{x_i}^2 u(x+se_i) + \partial_{x_i}^2 u(x-se_i)]ds,$$

and finally we find

$$|\Delta u(x) - \Delta_h u(x)| \le h^{-2}\int_0^h (h-s)\sum_{i=1}^{l}|\partial_{x_i}^2 u(x+se_i) - 2\partial_{x_i}^2 u(x) + \partial_{x_i}^2 u(x-se_i)|ds$$

$$\le h^{-2}\int_0^h (h-s)ds\sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u) = \frac{1}{2}\sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u),$$

which with (44.26) imply (44.27). ■

Remark 44.4. The estimate (44.27) is sharp, i.e., there exists a function u such that

$$\liminf_{h\to 0} \|u_h - u\|_{\tilde{\Omega}_h} \Big/ \sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u) > 0. \tag{44.28}$$

Indeed, choose $u(x) := x_1^4$ and compute the left and right hand sides of the inequality (44.27). We have

$$\Delta u(x) = 12x_1^2 \tag{44.29}$$

$$\begin{aligned}\Delta_h u(x) = h^{-2}[(x_1+h)^4 + (x_1-h)^4 - 2x_1^4] &= h^{-2}(12x_1^2h^2 + 2h^4) \\ &= 12x_1^2 + 2h^2,\end{aligned} \tag{44.30}$$

and

$$\Delta u(x) - \Delta_h u(x) = -2h^2. \tag{44.31}$$

Now we apply (44.24) and (44.31) to the function $u - u_h$ which equals zero on $\partial\Omega$ and we find

$$\begin{aligned} u(x) - u_h(x) &= -\frac{1}{2l}h^2 G_h(\Delta_h u - \Delta_h u_h)(x) \\ &= -\frac{1}{2l}h^2 G_h(\Delta_h u - \Delta u)(x) = \\ &= -\frac{1}{2l}h^2 \cdot 2h^2 G_h 1(x) = -\frac{h^4}{l} G_h 1(x). \end{aligned} \tag{44.32}$$

Thus by (44.32) and Theorem 44.2, we have

$$\|u - u_h\|_{\tilde{\Omega}_h} = \frac{h^4}{l}\|G_h 1\|_{\Omega_h} \geq \frac{1}{l} n^{-2} \cdot h^2 \cdot \frac{2}{\pi^2} n^2 = \frac{2}{l\pi^2} h^2. \tag{44.33}$$

On the other hand

$$\sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u) = w_{2,i}(h, \partial_{x_1}^2 u) = 24h^2, \tag{44.34}$$

thus, (44.33) and (44.34) imply (44.28).

44.3.2 Sharpness of the Error Estimates for a Dirichlet Problem

As it was mentioned in Remark 44.4 the error estimate (44.27) is sharp, i.e., there exist a function u such that

$$\|u_h - u\|_{\tilde{\Omega}_h} \approx \sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u) \text{ as } h \downarrow 0. \tag{44.35}$$

The fact that (44.27) is sharp with regard to the rate of convergence is now established in connection to general Lipschitz classes, determined by an abstract modulus of continuity, i.e., by a function w, continuous on $[0,\infty)$ such that

$$0 = w(0) < w(s) \leq w(s+t) \leq w(s) + w(t),\ \ s,t > 0.$$

Here we follow the same technique applied in the papers [125] and [124], which were devoted to the cases of the dimension $l = 1, 2$ and we establish the corresponding fact for arbitrary dimension $l \geq 1$. Our reasoning is based on the following variant of uniform boundedness principle [144]. For a Banach space $(X, \|\cdot\|)$ let X^* be the set of sublinear bounded functionals in X.

Theorem 44.5. Assume that for given $\{T_n\}_{n\in\mathrm{N}} \subset X^*$ and $\{S_\delta,\ \delta > 0\} \subset X^*$ there are given $\{g_n\}_{n\in\mathrm{N}} \subset X$ such that

$$\|g_n\| \leq C_1, \quad n = 1, 2, \ldots, \tag{44.36}$$

$$\lim_{n\to\infty} \inf \|T_n g_n\| > 0, \tag{44.37}$$

and

$$|S_\delta g_n| \leq C_2 \min\left\{1, \frac{\sigma(\delta)}{\varphi_n}\right\},\ \ n = 1, 2, \ldots, \tag{44.38}$$

where $\sigma(\delta)$ is a strictly positive function on $(0,\infty)$, and $\{\varphi_n\}_{n\in\mathrm{N}}$ is a strictly decreasing real sequence with

$$\lim_{n\to\infty} \varphi_n = 0.$$

Then for each modulus of continuity w as above, satisfying

$$\lim_{t\to 0} = \frac{w(t)}{t} = \infty, \tag{44.39}$$

there exists an element $f_w \in X$ such that

$$|S_\delta f_w| \leq C_w \cdot w(\sigma(\delta)),\ \ 0 < \delta < 1, \tag{44.40}$$

$$\lim_{n\to\infty} \inf |T_n f_w| / w(\varphi_n) > 0. \tag{44.41}$$

Next comes the optimal result.

Theorem 44.6. For every modulus of continuity w there exists a function $u_w \in C^2(\tilde{\Omega})$ such that

$$\sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u_w) \leq c \cdot w(\delta^2),\ \ 0 < \delta < 1, \tag{44.42}$$

$$\liminf_{h\to 0} \|u_w - u_{w,h}\|_{\Omega_h}/w(h^2) > 0. \tag{44.43}$$

Proof. To apply Theorem 44.5 we denote by

$$X := C^2(\Omega),$$

$$T_n u := \|u - u_h\|_{\Omega_{1/n}}, \ \left(h = \frac{1}{n}\right),$$

$$S_\delta u := \sum_{i=1}^{l} w_{2,i}(h, \partial_{x_i}^2 u), \ 0 < \delta < 1,$$

and

$$g_n(x) := n^{-2} \sum_{i=1}^{l} \sin^2 \pi n x_i, \quad x = (x_1, \ldots, x_l) \in \tilde{\Omega}, \ n \in \mathrm{N}.$$

Then (44.36) is fulfilled with $c_1 = l$. Since $g_n(x) = g_{n,h}(x)$ for $x \in \partial\Omega_h$, $h = \frac{1}{n}$, and

$$\Delta g_n(x) = 2\pi^2 l, \ \Delta_h g_n(x) = 0, \ for \ x \in \Omega_h$$

one has (cf. (44.24))

$$\begin{aligned} T_n g_n &= \frac{1}{2ln^2} \left\| G_{\Omega_h} \Delta_h (g_n - g_{n,h}) \right\|_{\Omega_h} \\ &= \frac{1}{2ln^2} \left\| G_{\Omega_h} (\Delta_h g_n - \Delta_h g_{n,h}) \right\|_{\Omega_h} \\ &= \frac{1}{2ln^2} \left\| G_{\Omega_h} (\Delta_h g_n - \Delta g_n) \right\|_{\Omega_h} \\ &= \frac{2\pi^2 l}{2ln^2} \|G_{\Omega_h} 1\|_{\Omega_h} = \pi^2 h^2 \|G_{\Omega_h} 1\|_{\Omega_h} \geq \frac{2}{\pi} > 0. \end{aligned}$$

The last inequality comes from Theorem 44.2 and hence condition (44.37) is satisfied. To verify the condition (44.38) we see that

$$S_\delta g_n \leq 8\pi^2 l, \tag{44.44}$$

furthermore,

$$S_\delta g_n \leq 2\delta^2 \sum_{i=1}^{l} \|\partial_{x_i}^4 g_n\|_\Omega \leq \delta^2 n^2 16\pi^4 l, \tag{44.45}$$

which yield (44.38) with

$$\sigma(\delta) := 2\pi^2\delta^2 \text{ and } \varphi_n := n^{-2}.$$

Thus we are able to apply Theorem 4.5 and (44.42), (44.43) are established. ■

44.3.3 Remarks Concerning the Case of a General Domain $\Omega \subset \mathrm{R}^l$.

Let Ω be a domain in R^l with a compact closure $\tilde{\Omega}$ and with a smooth boundary $\partial\Omega$. Without loss of generality we can suppose that $\Omega \subset \{x : \ 0 < \sup\ x_i < 1,\ i = 1, \ldots, l\}$. For $h = \frac{1}{n}$ let $\Omega_h := \Omega \cap h\mathrm{Z}^l$.

For given functions f and φ which are assumed to be Hölder ones in some neighborhood of $\tilde{\Omega}$ we consider the Dirichlet problem

$$\begin{cases} \Delta u(x) = -f(x), \quad x \in \Omega, \\ \lim_{x\to y} u(x) = \varphi(y), \quad y \in \partial\Omega, \end{cases} \tag{44.46}$$

and its discrete counterpart

$$\begin{cases} \Delta_h u_h(x) = -f(x), \quad x \in \Omega_h, \\ u_h(x) = \varphi(x), \quad x \in \partial\Omega_h. \end{cases} \tag{44.47}$$

A related result follows:

Theorem 44.7. Assume that the solution u of the problem (44.46) satisfies $u \in C^2(\tilde{\Omega})$. Then for the solution u_h of the problem (44.47) the following inequality holds true

$$\|u_h - u\|_{\tilde{\Omega}_h} \le \frac{1}{4}\sum_{i=1}^{l} w_{2,i}(h, \partial^2_{x_i} u) + D_h, \tag{44.48}$$

where

$$D_h \sim \sup_{x\in\Omega_h} \sqrt{E_x(\tau_\Omega - \tau_{\Omega_h})},\ \textit{as } h \to 0.$$

Proof. We just apply the estimate (44.10) and the result of Sections 44.2.2 and 44.3.1, which are valid to the case of Ω_h of general configuration. ■

45

Optimal Estimate for the Numerical Solution of Multidimensional Dirichlet Problem for the Heat Equation

For the multidimensional Dirichlet problem of the heat equation on a cylinder, this chapter examines convergence properties with rates of approximate solutions, obtained by a naturally arising difference scheme over inscribed uniform grids. Sharp quantitative estimates are presented by the use of first and second moduli of continuity of some first and second order partial derivatives of the exact solution. This is achieved by using the probabilistic method of an appropriate random walk. This chapter is based on [64].

45.1 Description

Let Ω be the open unit cube in $\mathbb{R}^\ell$, $\ell \geq 1$ and $\dot{\Omega} := \Omega \times I$ be an "interval" in space-time $\dot{\mathbb{R}}^\ell := \mathbb{R}^\ell \times \mathbb{R}$, where $I := (0, T)$, $T > 0$. Let us denote by $\dot{\Delta} = \frac{1}{2}\Delta - \theta_t$ the heat operator, where Δ stands for the Laplacian operator in $\mathbb{R}^\ell$. The Dirichlet problem in $\dot{\Omega}$ has a unique solution u on $\overline{\dot{\Omega}}$ so that

$$\begin{aligned} \dot{\Delta} u(\dot{x}) &= -f(\dot{x}), \quad \forall \dot{x} \in \dot{\Omega}, \\ \lim_{\dot{x} \to \dot{y}} u(\dot{x}) &= \varphi(\dot{y}), \quad \forall \dot{y} \in \partial\dot{\Omega} - \{\dot{x} = (x,t) \colon t = T\}, \end{aligned}$$

where f, φ are appropriate real valued functions defined on $\dot{\Omega}$ and $\partial\dot{\Omega} - \{\dot{x} = (x,t)\colon t = T\}$, respectively. Let $\dot{\Omega}_{h,T}$ be the inscribed in $\dot{\Omega}$ grid, which is uniform in space variables with mesh $h := \frac{1}{n}$, $n \in \mathbb{N}$ and in time variable with mesh $k(h) := \frac{h^2}{\ell}$, and has a boundary $\partial\dot{\Omega}_{h,T}$.

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 749–764.
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We also consider the discrete Dirichlet problem of obtaining u_h such that

$$\begin{aligned}\dot{\Delta}_h u_h(\dot{x}) &= -f(\dot{x}), \quad \forall \dot{x} \in \dot{\Omega}_{h,T}, \\ u_h(\dot{x}) &= \varphi(\dot{x}), \quad \forall \dot{x} \in \dot{\Omega}_{h,T} - \{\dot{x} = (x,t)\colon t = T\},\end{aligned}$$

where

$$\dot{\Delta}_h u_h(\dot{x}) := \frac{1}{2}h^{-2}\left(\sum_{k=1}^{\ell} u_h(x \pm h e_k, t - k(h)) - 2\ell u_h(x,t)\right), \quad \forall \dot{x} := (x,t) \in \dot{\Omega}_{h,T},$$

is the discrete heat operator. Here e_k is the natural basis in $\mathbb{R}^\ell$. Using the probabilistic method of a suitable random walk we are able to show that

(i)

$$\begin{aligned}\|u - u_h\|_{\overline{\dot{\Omega}}_{h,T}} \le\ & \min\left\{T, \frac{1}{4}\right\} \cdot \left\{\frac{1}{4}\sum_{i=1}^{\ell} \omega_{2,i}(h, \partial^2_{x_i} u; \overline{\dot{\Omega}}_{h,T}) \right. \\ & \left. + \frac{1}{2}\sum_{i=1}^{\ell} \omega_1(k, \partial^2_{x_i} u; \overline{\dot{\Omega}}_{h,T}) + \omega_1(k, \partial_t u; \overline{\dot{\Omega}}_{h,T})\right\},\end{aligned}$$

where $\|\cdot\|_{\overline{\dot{\Omega}}_{h,T}}$ is the supremum norm in $\overline{\dot{\Omega}}_{h,T}$. Here $\omega_{2,i}$ is the second modulus of continuity of the second partial of u with respect to x_i, $i = 1, \ldots, \ell$; while ω_1 stands for the first modulus of continuity of the indicated function with respect to the variable t. See Theorem 45.4. Inequality (i) is proved to be sharp, see Remark 45.5 and Theorem 45.7. Sharpness is proved in a similar way as it is established in the related papers [125], [124] and [157]. This chapter has been greatly motivated by the very important and interesting article of Esser, Goebbels, Lüttgens and Nessel (1995), see [157]. They consider the same problem however in the univariate case of space and time, and they investigate different types of discretization than us. Their method is purely analytical however this one is probabilistic.

45.2 Basics

Dirichlet problem for the "heat operator." Let R^ℓ be the Euclidean space, and $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_\ell}$ be the Laplacian. We denote $\dot{\mathbb{R}}^\ell = \{\dot{x} = (x,t)\colon x \in \mathbb{R}^\ell,\ t \in \mathbb{R}^1\}$ and let $\dot{\Delta} = \frac{1}{2}\Delta - \partial_t$ be the "heat operator" (i.e., the parabolic Laplacian). Let $\dot{\Omega} \subset \dot{R}^\ell$ be an open subset with non-empty boundary $\partial\dot{\Omega}$. The Dirichlet problem in $\dot{\Omega}$ consists in finding a function u on $\dot{\Omega}$ such that for given functions f, defined in $\dot{\Omega}$, and φ, defined on $\partial\dot{\Omega}$, we have

$$\begin{aligned}\dot{\Delta} u(\dot{x}) &= -f(\dot{x}), \quad \dot{x} \in \dot{\Omega}, &(45.1)\\ \lim_{\dot{x}\to\dot{y}} u(\dot{x}) &= \varphi(\dot{y}), \quad \forall \dot{y} \in \partial\dot{\Omega}. &(45.2)\end{aligned}$$

It is a well-known fact [147, 1, XVII] that if $\dot{\Omega}$ has compact closure and regular boundary (for example, if every point of $\dot{\Omega}$ in some neighborhood of each boundary point $\dot{y}$ is either above the horizontal hyperplane through $\dot{y}$ or on one side of some other hyperplane through $\dot{y}$) and f is a Hölder function and φ is a continuous function then the problem (45.1)–(45.2) has unique solution $u(\dot{x})$. This solution can be represented in the form

$$u(\dot{x}) = \dot{G}_{\dot{\Omega}} f(\dot{x}) + \dot{H}_{\dot{\Omega}} \varphi(\dot{x}), \tag{45.3}$$

where

$$\begin{aligned}\dot{G}_{\dot{\Omega}} f(\dot{x}) &= \int_{\dot{\Omega}} \dot{g}_{\dot{\Omega}}(\dot{x}, \dot{y}) f(\dot{y}) d\dot{y}, &(45.4)\\ \dot{H}_{\dot{\Omega}} \varphi(\dot{x}) &= \int_{\partial\dot{\Omega}} \varphi(y) \dot{\Pi}_{\dot{\Omega}}(\dot{x}, d\dot{y}), &(45.5)\end{aligned}$$

are the "parabolic" Green potential of the function f and the "parabolic" harmonic function (i.e., a parabolic function) in $\dot{\Omega}$ with boundary values φ, respectively. In (45.4) $\dot{g}_{\dot{\Omega}}(\dot{x}, \dot{y})$ is the so-called Green function of $\dot{\Omega}$ which is determined uniquely by the following properties,

(i) $\dot{\Delta}_{\dot{x}} \dot{g}_{\dot{\Omega}}(\dot{x}, \dot{y}) = -\delta(\dot{x} - \dot{y}), \quad \dot{x}, \dot{y} \in \dot{\Omega},$

where δ is the Dirac delta-function,

(ii) $\dot{g}_{\dot{\Omega}}(\dot{x}, \dot{y}) = 0, \quad \dot{x} \in \partial\dot{\Omega}$ or $\dot{y} \in \partial\dot{\Omega}$.

The kernel $\dot{\Pi}_{\dot{\Omega}}(\dot{x}, d\dot{y})$ is the so-called "parabolic" harmonic measure (i.e., the parabolic measure) of domain $\dot{\Omega}$.

Define the function $\mathcal{E}(\dot{x})$, $\dot{x} = (x, t) \in \dot{R}^\ell$ by

$$\mathcal{E}(\dot{x}) = \begin{cases} (2\pi t)^{-\ell/2} \exp \frac{-x^2}{2t}, & \text{if } t > 0 \\ 0, & \text{if } t \le 0 \end{cases}$$

and set

$$\dot{g}(\dot{x}, \dot{y}) := \mathcal{E}(\dot{x} - \dot{y}), \quad \dot{x}, \dot{y} \in \dot{R}^\ell.$$

The function $\dot{g}(\dot{x}, \dot{y})$ satisfies the equation (i) and condition (ii) at the infinity. This function is called the "parabolic" Green function of the whole space $\Omega = \dot{\mathbb{R}}^\ell$. A connection between $\dot{g}$ and $\dot{g}_{\dot{\Omega}}$ can be expressed by the following relation

$$\dot{g}_{\dot{\Omega}}(\dot{x}, \dot{y}) = \dot{g}(\dot{x}, \dot{y}) - (\dot{H}_{\dot{\Omega}} \dot{g}(\cdot, \dot{y}))(\dot{x}). \tag{45.6}$$

We note here the remarkable property of the parabolic objects $\dot{g}_{\dot{\Omega}}$ and $\dot{\Pi}_{\dot{\Omega}}$ which makes "parabolic" theory different from the "elliptic" one. For $\dot{x} = (x,t)$ and $\dot{y} = (y,t)$ we write $\dot{x} \le \dot{y}$ iff $t \le s$. The property mentioned above can be formulated now as follows: for a connected $\dot{\Omega}$ and for any $\dot{x}, \dot{y} \in \dot{\Omega}$ we have

(i) $\dot{g}_{\dot{\Omega}}(\dot{x}, \dot{y}) \ge 0$, and $= 0$ iff $\dot{x} \le \dot{y}$,

(ii) supp $\dot{\Pi}_{\dot{\Omega}}(\dot{x}, d\dot{y}) = \{\dot{y} \in \partial\dot{\Omega}\colon \dot{y} \le \dot{x}\}$.

Any ball and any convex polyhedron which is situated above of its horizontal face are regular sets [147, 125, XVIII.6]. To the contrary of these examples, such a simple set as a unit square $\dot{\Omega} = \{\dot{x} = (x,t)\colon 0 < x_i < 1,\ 0 < t < 1,\ i = 1, 2, \ldots, \ell\}$ is not regular. To see this it is enough to note that there is no parabolic function with boundary values 1 on the upper side of $\partial\dot{\Omega}$ and 0 otherwise.

According to [147, 125, XVIII] the Dirichlet problem can be well defined for an arbitrary open set $\dot{\Omega}$, but in contrary to the "regular case" in this more general setting the very restrictive condition (45.2) should be replaced by the following condition

$$\lim_{\dot{x}\to\dot{y}} u(\dot{x}) = \varphi(\dot{y}), \quad \forall \dot{y} \in \partial\dot{\Omega}_r, \tag{45.2'}$$

where $\partial\dot{\Omega}_r \subset \partial\dot{\Omega}$ is a set of regular points. Conditions for a given point $\dot{y} \in \partial\dot{\Omega}$ to be regular can be found in [147, 125, XVIII.3, 194]. We note here that if $\dot{\Omega}$ is the unit square then the set of all irregular points of $\partial\dot{\Omega}$ coincides with its upper side $\{\dot{x} = (x,t)\colon t = 1,\ 0 < x_i < 1,\ i = 1, \ldots, \ell\}$. Thus the Dirichlet problem for this special set has the following form

$$\dot{\Delta} u(\dot{x}) = -f(\dot{x}), \quad \forall \dot{x} \in \dot{\Omega}, \tag{45.7}$$

$$\lim_{\dot{x}\to\dot{y}} u(\dot{x}) = \varphi(\dot{y}), \quad \forall \dot{y} \in \partial\dot{\Omega} \setminus \{\dot{y} = (y,s)\colon s = 1\}. \tag{45.8}$$

Coming back to the general case we note that, as in the "regular case", the formula (45.3)–(45.5) giving the representation of the solution of the Dirichlet problem as well as the relation (45.6) for the Green function hold true; property (ii) of the Green function has now a more general form

(ii)′ $\lim_{\dot{x}\to\dot{z}} \dot{g}_{\dot{\Omega}}(\dot{x}, \dot{y}) = 0$, $\forall \dot{z} \in \partial\dot{\Omega}_r$.

We present here briefly the important probabilistic counterpart of the analytical facts mentioned above. We refer the reader to [147, 124, VII] and [194]. Let $\dot{x}_0 = (x_0, t_0)$ be a point of $\dot{R}^\ell$, and let (x_t, P^{x_0}) be a Brownian motion in R^ℓ starting from x_0. The process

$$\{\dot{x}_t, t \in R^+\} := \{(x_t, t_0 - t), t \in R^+\}$$

with state space $\dot{R}^\ell$ is called a space-time Brownian motion starting from $\dot{x}_0$. In this definition space-time Brownian motion moves downward in $\dot{R}^\ell$, that is, in the direction of decreasing ordinate values. Denote by $\tau_{\dot{\Omega}}$ the first exit time of $\dot{\Omega} \subset \dot{R}^\ell$

$$\tau_{\dot{\Omega}} := \inf\{t > 0\colon \dot{x}_t \in \dot{R}^\ell \setminus \dot{\Omega}\}.$$

We notice the special cases:

I. If $\dot{\Omega} = \Omega \times R^1$ is a cylinder with the base $\Omega \subset R^\ell$ then $\tau_{\dot{\Omega}} = \tau_\Omega$, where τ_Ω is the first exit time of Brownian motion x_t of $\Omega \subset R^\ell$.

II. If $\dot{\Omega} = \Omega \times I$, $I = (a, b)$, is an "interval" with the base $\Omega \subset R^\ell$, then $\tau_{\dot{\Omega}} = \tau_\Omega \wedge \tau_I$, where $\tau_I \le b - a$ is the first exit time of the uniform motion $t \to t_0 - t$ of I. Thus, in particular, $\tau_{\dot{\Omega}} \le b - a$.

Property II implies that for any bounded domain $\dot{\Omega} \subset \dot{R}^\ell$ we have $\tau_{\dot{\Omega}} \le d_{\dot{\Omega}}$, where $d_{\dot{\Omega}}$ is a diameter of $\dot{\Omega}$.

As usual we denote by $E_{\dot{x}}F(w)$ (resp., $E_x F(w)$) the math expectation corresponding to this process $\{\dot{x}_t,\, \dot{x}_0 = \dot{x}\}$ (resp., $\{x_t,\, x_0 = x\}$). We have

$$\dot{G}_{\dot{\Omega}} f(\dot{x}) = E_{\dot{x}} \left(\int_0^{\tau_{\dot{\Omega}}} f(\dot{x}_s) ds \right), \tag{45.9}$$

$$\dot{H}_{\dot{\Omega}} \varphi(\dot{x}) = E_{\dot{x}}[\varphi(\dot{x}_{\tau_{\dot{\Omega}}}), \tau_{\dot{\Omega}} < \infty]. \tag{45.10}$$

In particular, if $\dot{\Omega} = \Omega \times (0, T)$ is an "interval", then according to II we will have

$$\dot{G}_{\dot{\Omega}} f(\dot{x}) = E_x \left(\int_0^{\tau_\Omega \wedge t} f(x_s, t - s) ds \right), \tag{45.11}$$

$$\dot{H}_{\dot{\Omega}} \varphi(\dot{x}) = E_x[\varphi(x_t, 0), t < \tau_\Omega] + E_x[\varphi(x_{\tau_\Omega}, t - \tau_\Omega), \tau_\Omega \le t] \tag{45.12}$$

where $\dot{x} = (x, t) \in \dot{\Omega}$.

From the probabilistic point of view a point $\dot{y} \in \partial\dot{\Omega}$ is regular if and only if $P_{\dot{y}}(\tau_{\dot{\Omega}} = 0) = 1$. Kolmogorov's law of iterated logarithm

$$P \left(\limsup_{t \downarrow 0} \frac{|x_t|}{\sqrt{2t \log_{(2)} \frac{1}{t}}} = 1 \right) = 1,$$

gives us a criterion for the regularity of a boundary point of $\dot{\Omega}$ [147, 125, XVIII.6] [194, 7.14]. Namely, let $\dot{\Omega}$ be situated below the abscissa hyperplane defined by the inequalities

$$|x|^2 < 2|t| \left| \log\log \frac{1}{|t|} \right|, \quad -1 < t < 0,$$

then the origin is a regular boundary point of $\dot{\Omega}$.

45.3 Dirichlet Problem: Discrete Case

Let $\dot{\mathbb{Z}}^\ell$ be an $(\ell+1)$-dimensional integer-valued lattice. This lattice consists of points (vectors) of the type $\dot{x} = x_1e_1+\cdots+x_\ell e_\ell+te$, where $e_1,\dots,e_\ell$, e comprises the orthonormal basis of $R^{\ell+1}$, and the coordinates $x_1,\dots,x_\ell$, t are arbitrary integers. Decreasing t-coordinate by one unit and increasing or decreasing each one of the x-coordinates by one unit and leaving the other x-coordinates unchanged, we obtain the 2ℓ neighboring lattice points to $\dot{x}$. Let B be a subset of points of a lattice $\dot{\mathbb{Z}}^\ell$. We call a point $\dot{x} \notin B$ a boundary point for the set B if $\dot{x}$ is a neighboring point for at least one point of B. The collection of boundary points of the set B is called the boundary of B, denoting it ∂B.

Let f be a function defined at the points of a lattice $\dot{\mathbb{Z}}^\ell$. We put

$$\dot{P}f(\dot{x}) := \frac{1}{2\ell}\sum_{k=1}^{\ell} f(\dot{x} \pm e_k - e).$$

It is logical to call $\dot{P}$ the averaging operator. The linear operator $\dot{P}-E$, where E is the unit operator, is the discrete analog of the "parabolic" Laplacian $\dot{\Delta}$. Indeed, for sufficiently smooth function $f(\dot{x})$ specified over all space $\dot{R}^\ell$,

$$\dot{\Delta}f(\dot{x}) = \lim_{h\to 0}\frac{1}{2}h^{-2}\left(\sum_{k=1}^{\ell} f\left(\dot{x} \pm he_k - \frac{h^2}{\ell}e\right) - 2\ell f(\dot{x})\right),$$

so that the "parabolic" Laplacian is obtained by passing to the limit from the operator $\dot{P}-E$ as the lattice is infinitely partitioned.

Let $\dot{\Omega} \subset \dot{\mathbb{Z}}^\ell$ be a finite subset. The Dirichlet problem in $\dot{\Omega}$ consists in finding a function $u(\dot{x})$, $\dot{x} \in \dot{\Omega}\cup\partial\dot{\Omega}$ such that for given functions f defined in $\dot{\Omega}$ and φ defined in $\partial\dot{\Omega}$ we have

$$(\dot{P}-E)u(\dot{x}) = -f(\dot{x}), \quad \dot{x} \in \dot{\Omega}, \tag{45.13}$$
$$u(\dot{x}) = \varphi(\dot{x}), \quad \dot{x} \in \partial\dot{\Omega}. \tag{45.14}$$

First of all we note that if u_1 and u_2 are two solutions of the problem (45.13), (45.14) then $u_1 \equiv u_2$. This fact follows immediately from the following minimum principle.

Theorem 45.1. *Let u be a function on $\dot{\Omega}\cup\partial\dot{\Omega}$ such that $Pu(\dot{x}) \le u(\dot{x})$ for any $\dot{x} \in \dot{\Omega}$. Then u reaches its minimum value on $\dot{\Omega}\cup\partial\dot{\Omega}$ at some point $\dot{y} \in \partial\dot{\Omega}$.*

Proof. Let $u(\dot{x}_0) := \min\{u(\dot{x})\colon \dot{x} \in \dot{\Omega}\cup\partial\dot{\Omega}\}$. If $\dot{x}_0 \in \partial\dot{\Omega}$ then there is nothing to prove. If not, we write the inequality

$$u(\dot{x}_0) \ge \frac{1}{2\ell}\sum_{k=1}^{\ell} u(\dot{x}_0 \pm e_k - e) \ge u(\dot{x}_0),$$

from which we obtain that $u(\dot{x}_0) = u(\dot{x}_0 \pm e_k - e)$ for all $k = 1, \dots, \ell$. If one of the points $\dot{x}_0 \pm e_k - e$ belongs to the boundary $\partial\dot{\Omega}$ then the proof is finished, if not we will repeat the previous reasoning at the point $\dot{x}_1 := \dot{x}_0 + e_1 - e$. It is clear that after a finite number of steps we will meet the boundary $\partial\dot{\Omega}$. The proof is completed. ■

Next our remark concerns the decomposition $u = u_1 + u_2$ of the solution u of the problem (45.13), (45.14), where

$$(\dot{P} - E)u_1(\dot{x}) = -f(\dot{x}), \quad \dot{x} \in \dot{\Omega}, \tag{45.15}$$

$$u_1(\dot{x}) = 0, \quad \dot{x} \in \partial\dot{\Omega}, \tag{45.16}$$

$$(\dot{P} - E)u_2(\dot{x}) = 0, \quad \dot{x} \in \dot{\Omega}, \tag{45.17}$$

$$u_2(\dot{x}) = \varphi(x), \quad \dot{x} \in \partial\dot{\Omega}. \tag{45.18}$$

This decomposition is the discrete analog of the decomposition (45.3), (45.4), (45.5). Following the same reasoning as in Part 45.2 of this chapter, we give the probabilistic representation of the "discrete" Green potential $u_1 := \dot{G}_{\dot{\Omega}} f$ and of the "discrete" parabolic function $u_2 := \dot{H}_{\dot{\Omega}}\varphi$ in (45.15)–(45.18). In this part of our exposition, we follow the monograph [270].

Let $\{x(n), P\}$ be a simple random walk on the lattice $\mathbb{Z}^\ell$, i.e., a random process with independent identically distributed increments $x(n+1) - x(n)$, $n = 0, 1, \dots$, and such that

$$P(x(1) - x(0) = x) \begin{array}{ll} = \frac{1}{2\ell}, & \text{if } x = \pm e_k, \\ = 0, & \text{otherwise.} \end{array}$$

The process

$$\{\dot{x}(n), n = 0, 1, \dots\} := \{(x(n), n_0 - n), n = 0, 1, \dots\}$$

with state space $\dot{\mathbb{Z}}^\ell$ is called a space-time random walk starting from $\dot{x}(0) = (x(0), n_0)$. In this definition space-time random walk moves downward in $\dot{\mathbb{Z}}^\ell$, that is, in the direction of decreasing ordinate values. It is easy to see that for each bounded function f

$$E(f(\dot{x}(1)); \dot{x}(0) = \dot{x}) = \dot{P}f(\dot{x}),$$

and more generally

$$E(f(\dot{x}(n)); \dot{x}(0) = \dot{x}) = \dot{P}^n f(\dot{x}).$$

For a finite set $\dot{\Omega} \subset \dot{\mathbb{Z}}^\ell$ we denote by $\tau_{\dot{\Omega}}$ the first exit time of $\dot{\Omega}$

$$\tau_{\dot{\Omega}} := \inf\{n \geq 1 \colon \dot{x}(n) \in \dot{\mathbb{Z}}^\ell \setminus \dot{\Omega}\}.$$

Following [270, p. 107] we introduce the next functions

$$Q_{\dot{\Omega}}(n; \dot{x}, \dot{y}) \begin{array}{ll} = P(\dot{x}(n) = \dot{y}, n < \tau_{\dot{\Omega}}; \dot{x}(0) = \dot{x}), & \dot{x}, \dot{y} \in \dot{\Omega}, \\ = 0, & \text{otherwise,} \end{array} \tag{45.19}$$

$$\dot g_{\dot\Omega}(\dot x,\dot y)=\sum_{n=0}^{\infty}Q_{\dot\Omega}(n;\dot x,\dot y), \tag{45.20}$$

$$\dot\Pi_{\dot\Omega}(\dot x,\dot y)\begin{array}{ll}=P(\dot x(\tau_{\dot\Omega})=\dot y,\dot x(0)=\dot x), & \dot x\in\dot\Omega,\\ =\delta(\dot x,\dot y), & \text{otherwise.}\end{array} \tag{45.21}$$

Define also the following operators

$$\dot G_{\dot\Omega}f(\dot x):=\sum_{\dot y\in\dot\Omega}f(\dot y)\dot g_{\dot\Omega}(\dot x,\dot y), \tag{45.22}$$

$$\dot H_{\dot\Omega}\varphi(\dot x):=\sum_{\dot y\in\partial\dot\Omega}\varphi(\dot y)\dot\Pi_{\dot\Omega}(\dot x,\dot y). \tag{45.23}$$

The proofs of the following basic facts can be found in [270, p. 108].

(A) The function $\dot G_{\dot\Omega}f(\dot x)$ gives the unique solution of the problem (45.15), (45.16) and the following representation of $\dot G_{\dot\Omega}f(\dot x)$ holds true

$$\dot G_{\dot\Omega}f(\dot x)=E\left\{\sum_{k=0}^{\tau_{\dot\Omega}-1}f(\dot x(k));\dot x(0)=\dot x\right\}. \tag{45.24}$$

In particular,

$$\dot G_{\dot\Omega}1(\dot x)=E\{\tau_{\dot\Omega};\dot x(0)=\dot x\}. \tag{45.25}$$

(B) The function $\dot H_{\dot\Omega}\varphi(\dot x)$ gives the unique solution of the problem (45.17), (45.18). For a function u defined on $\dot\Omega$ we set

$$\|u\|_{\dot\Omega}:=\max\{|u(\dot x)|:\dot x\in\dot\Omega\}.$$

An auxiliary result that is needed for later follows.

Theorem 45.2. *For every function u on $\dot\Omega\cup\partial\dot\Omega$ the following inequality holds true*

$$\|u\|_{\dot\Omega}\le c_{\dot\Omega}\|(\dot P-E)u\|_{\dot\Omega}+\|u\|_{\partial\dot\Omega}, \tag{45.26}$$

where $c_{\dot\Omega}:=\|\dot G_{\dot\Omega}1\|_{\dot\Omega}$.

Proof. Define the following functions

$$\begin{aligned} f(\dot x) &:= -(\dot P-E)u(\dot x), \quad \dot x\in\dot\Omega,\\ \varphi(\dot x) &:= u(\dot x), \quad \dot x\in\partial\dot\Omega.\end{aligned}$$

Then we have

(i) $(\dot P-E)u(\dot x)=-f(\dot x)$, $\dot x\in\dot\Omega$,

(ii) $u(\dot x)=\varphi(\dot x)$, $\dot x\in\partial\dot\Omega$.

By (A) and (B) we obtain that

$$u(\dot{x}) = \dot{G}_{\dot{\Omega}} f(\dot{x}) + \dot{H}_{\dot{\Omega}}\varphi(\dot{x}).$$

From this identity we immediately find that

$$\begin{aligned} \|u\|_{\dot{\Omega}} &\leq \|\dot{G}_{\dot{\Omega}} 1\|_{\dot{\Omega}} \|f\|_{\dot{\Omega}} + \|\dot{H}_{\dot{\Omega}}\varphi\|_{\dot{\Omega}} \\ &\leq c_{\dot{\Omega}} \|(\dot{P} - E)u\|_{\dot{\Omega}} + \|\varphi\|_{\partial\dot{\Omega}} \\ &= c_{\dot{\Omega}} \|(\dot{P} - E)u\|_{\dot{\Omega}} + \|u\|_{\partial\dot{\Omega}}. \end{aligned}$$ ■

The following result turns out to be useful in our further considerations.

Theorem 45.3. *Let* $\dot{\Omega} = \Omega \times I$ *be an interval, where*

$$\begin{aligned} \Omega &= \{(x_1, \ldots, x_\ell) \in \mathbb{Z}^\ell \colon 1 \leq x_i \leq N, i = 1, \ldots, \ell\}, \\ I &= \{1, 2, \ldots, T\}, \quad T \in \mathbb{N}. \end{aligned}$$

Then

$$\frac{1}{2} \min\left\{T, \frac{2}{\pi^2}(N+1)^2\right\} \leq c_{\dot{\Omega}} \leq \min\left\{T, \frac{\ell}{4}(N+1)^2\right\}.$$

Proof. We put $c_\Omega := \sup_{x\in\Omega} E\{\tau_\Omega \colon x(0) = x\}$ and prove the following inequality

$$\frac{1}{2} \min\{T, c_\Omega\} \leq c_{\dot{\Omega}} \leq \min\{T, c_\Omega\} \tag{45.27}$$

holds true. Indeed we notice for a process $\dot{x}(s)$ with $\dot{x}(0) = \dot{x}$, where $\dot{x} = (x, t)$ that we have $\tau_{\dot{\Omega}} = \min\{\tau_\Omega, t\}$. So (45.25) implies the inequality

$$\dot{G}_{\dot{\Omega}} 1(\dot{x}) = E\{\tau_{\dot{\Omega}}, \dot{x}(0) = \dot{x}\} \leq \min\{t, E(\tau_\Omega, x(0) = x)\},$$

from which the right-hand side of (45.27) follows. To prove the left-hand side of inequality (45.27) we consider the functions $u(x) := E\{\tau_\Omega,\ x(0) = x\}$ and $\tilde{u}(\dot{x}) := tu(x)$. It is clear that $\tilde{u}(\dot{x}) = 0$ for any $\dot{x} \in \partial\dot{\Omega}$. Applying to this function (45.26) we obtain

$$\|\tilde{u}\|_{\dot{\Omega}} \leq c_{\dot{\Omega}} \|(\dot{P} - E)\tilde{u}\|_{\dot{\Omega}}. \tag{45.28}$$

Now we note that $\|\tilde{u}\|_{\dot{\Omega}} = T \cdot \|u\|_\Omega = T \cdot c_\Omega$. To calculate $(\dot{P} - E)\tilde{u}$ we observe that τ_Ω coincides with the first exit time of the cylinder $\dot{\Omega}_c := \Omega \times (-\infty, +\infty)$. Thus the function

$$u_c(\dot{x}) := u(x) = E\{\tau_\Omega, x(0) = x\} = E\{\tau_{\dot{\Omega}_c}, \dot{x}(0) = \dot{x}\}$$

satisfies the equation

$$(\dot{P} - E)u_c(\dot{x}) = -1, \quad \dot{x} \in \dot{\Omega}_c.$$

Using the facts mentioned above we derive

$$\begin{aligned}(\dot{P}-E)\tilde{u}(\dot{x}) &= \dot{P}\tilde{u}(\dot{x})-\tilde{u}(\dot{x})=(t-1)\dot{P}u_c(\dot{x})-tu_c(\dot{x})\\ &= (t-1)(u_c(\dot{x}-1)-tu_c(\dot{x})=-(u_c(\dot{x})+(t-1)).\end{aligned}$$

Thus we finally find

$$\|(\dot{P}-E)\tilde{u}\|_{\dot{\Omega}} \le (\|u\|_\Omega + T) = (c_\Omega + T). \tag{45.29}$$

Estimates (45.28) and (45.29) imply that

$$Tc_\Omega \le c_{\dot{\Omega}}(T+c_\Omega),$$

and consequently

$$c_{\dot{\Omega}} \ge \frac{T\cdot c_\Omega}{T+c_\Omega} \ge \frac{1}{2}\min\{T, c_\Omega\}.$$

Thus the left-hand side of (45.27) follows. Now it remains to find upper and lower bounds of the constant $c_\Omega = \max_{x\in\Omega} E\{\tau_\Omega,\, x(0)=x\}$. Let $\Omega_1 = \{x \in \mathbb{Z}^\ell\colon 1 \le x_1 \le N\}$ be a slab. Since $\Omega \subset \Omega_1$ we have $\tau_\Omega \le \tau_{\Omega_1}$ and consequently

$$E\{\tau_\Omega, x(0)=x\} \le E\{\tau_{\Omega_1}, x(0)=x\}.$$

Now we note that the function

$$\tilde{u}_1(\dot{x}) := E\{\tau_{\Omega_1}, x(0)=x\}$$

satisfies the equation $(\dot{P}-E)\tilde{u}_1 = -1$ in a space-time slab $\dot{\Omega}_1 = \Omega_1 \times (-\infty,\infty)$ and has zero boundary values. Consider the function $n(\dot{x}) := \ell x_1((N+1)-x_1)$. It is easy to see that this function satisfies the equation $(\dot{P}-E)n = -1$ in $\dot{\Omega}_1$ and has zero boundary values. Thus by unicity $\tilde{u}_1(\dot{x}) = n(\dot{x})$ for all $\dot{x} \in \dot{\Omega}_1$ and consequently

$$c_\Omega \le \max\{\tilde{u}_1(\dot{x}), \dot{x}\in\dot{\Omega}_1\} = \max\{n(\dot{x}), \dot{x}\in\dot{\Omega}_1\} \le \ell\left(\frac{N+1}{2}\right)^2. \tag{45.30}$$

To find the lower bound of c_Ω we use the same method. Namely, we consider the function $u(s) := \sin\frac{\pi s}{(N+1)}$ and denote

$$U(\dot{x}) := \prod_{i=1}^{\ell} u(x_i), \quad x \in \Omega.$$

It is clear that $U(\dot{x}) = 0$ for every $\dot{x} \in \partial\dot{\Omega}_c$.

Let P_i, $i = 1,\dots,\ell$ be the one-dimensional average operator

$$P_i f(x) := \frac{1}{2}\{f(x+e_i)+f(x-e_i)\}.$$

It is clear that

$$\begin{aligned}(\dot{P}-E)U(\dot{x}) &= \frac{1}{\ell}\sum_{i=1}^{\ell}(P_i-E)U(\dot{x}) \\ &= \frac{1}{\ell}\sum_{i=1}^{\ell}(P_i-E)u(x_i)\prod_{k\neq i}u(x_k).\end{aligned}$$

Now we compute $(P_i-E)u(x_i)$ for $1\le x_i\le N$, $(P_i-E)u(x_i)=-2\sin^2\frac{\pi}{2(N+1)}\cdot u(x_i)$. Thus we find that for $\dot{x}\in\dot{\Omega}_c$

$$(\dot{P}-E)U(\dot{x})=-2\sin^2\frac{\pi}{2(N+1)}\cdot U(\dot{x}).$$

To get the lower bound of $c_\Omega=c_{\dot{\Omega}_c}$ it remains now to apply the inequality (45.26)

$$\begin{aligned}0<\|U\|_{\dot{\Omega}_c} &\le c_\Omega\cdot 2\sin^2\frac{\pi}{2(N+1)}\|U\|_{\dot{\Omega}_c} \\ &\le c_\Omega\cdot\frac{\pi^2}{2(N+1)^2}\cdot\|U\|_{\dot{\Omega}_c}.\end{aligned}$$

Thus we finally find

$$c_\Omega\ge\frac{2}{\pi^2}(N+1)^2. \tag{45.31}$$

Inequalities (45.27), (45.30) and (45.31) together give us the desired result. ■

45.4 Approximation over the Grid

We consider the following Dirichlet problem on the interval $\dot{\Omega}:=\{\dot{x}\in\dot{R}^\ell\colon 0<x_i<1,\ 0<t<\infty,\ i=1,\ldots,\ell\}$

$$\begin{cases}\dot{\Delta}u(\dot{x})=-f(\dot{x}), & \dot{x}\in\dot{\Omega}\\ u(\dot{x})=\varphi(\dot{x}), & \dot{x}\in\partial\dot{\Omega},\end{cases} \tag{45.32}$$

with φ a continuous function and f a bounded locally Hölder function. In what follows we restrict our treatment to problem (45.32) for which $u\in C^{(2)}(\overline{\dot{\Omega}})$.

Let $h:=1/n$ with $n\in\mathbb{N}$, the set of natural numbers, and $k=h^2/\ell$. An approximate solution u_h, defined on the grid $\overline{\dot{\Omega}}_h=\dot{\Omega}_h\cup\partial\dot{\Omega}_h$, where

$$\begin{aligned}\overline{\dot{\Omega}}_h:=\ \{\dot{x}= &\ (x_1,\ldots,x_\ell,t)\in\dot{\mathbb{R}}^\ell\colon x_i=k_ih,\\ t\ = &\ jk, 0\le k_i\le n, i=1,\ldots,\ell, j=0,1,\ldots\},\end{aligned}$$

and

$$\dot{\Omega}_h := \overline{\dot{\Omega}_h} \cap \dot{\Omega},$$

is obtained as the solution of the discrete counterpart to (45.32)

$$\begin{cases} \dot{\Delta}_h u_h(\dot{x}) = -f(\dot{x}), & \dot{x} \in \dot{\Omega}_h \\ u_h(\dot{x}) = \varphi(\dot{x}), & \dot{x} \in \partial\dot{\Omega}_h. \end{cases} \tag{45.33}$$

Here the "discrete" parabolic Laplacian $\dot{\Delta}_h$ is given by

$$\dot{\Delta}_h u_h(x) := \frac{1}{2}h^{-2}\left(\sum_{k=1}^{\ell} f(\dot{x} \pm he_k - k(h)e) - 2\ell f(\dot{x})\right),$$

where $k(h) := h^2/\ell$.

We denote by $\dot{\mathbb{Z}}_h^\ell = \{\dot{x} \in \dot{\mathbb{R}}^\ell\colon x = hz,\ t = k(h)n,\ z \in \mathbb{Z}^\ell,\ n \in \mathbb{Z}^1\}$ and we consider the space-time random walk on $\dot{\mathbb{Z}}_h^\ell$

$$\{\dot{x}(n), n = 0, 1, 2, \ldots\} = \{(x(n), t_0 - nk), n = 0, 1, 2, \ldots\},$$

where $\{x(n), n = 0, 1, \ldots\}$ is a simple random walk on the h-lattice $h\mathbb{Z}^\ell$. Associated to $\{\dot{x}(n),\ n = 0, 1, \ldots\}$ values, functions and operators are attached to the index h. Thus, for example, the average operator $\dot{P}_h$ takes the following form

$$\begin{aligned} \dot{P}_h u(\dot{x}) &= E\{u(\dot{x}(1)), \dot{x}(0) = \dot{x}\} \\ &= \frac{1}{2\ell}\sum_{k=1}^{\ell} u(\dot{x} \pm he_k - ke). \end{aligned}$$

It is easy to see that with these notations the discrete parabolic Laplacian $\dot{\Delta}_h$ takes the following form, $\dot{\Delta}_h = \ell h^{-2}(\dot{P}_h - E)$. Now applying the results (A) and (B) of Section 45.3 we see that the problem (45.33) has a unique solution u_h which can be represented by the form

$$u_h(\dot{x}) = \frac{h^2}{\ell} \cdot \dot{G}_{\dot{\Omega}_h} f(\dot{x}) + \dot{H}_{\dot{\Omega}_h}\varphi(\dot{x}). \tag{45.34}$$

The important inequality (45.26) now takes the following form

$$\|u\|_{\dot{\Omega}_{h,T}} \leq \min\left\{T, \frac{1}{4}\right\} \|\dot{\Delta}_h u\|_{\dot{\Omega}_{h,T}} + \|u\|_{\partial\dot{\Omega}_{h,T}}, \tag{45.35}$$

where $\dot{\Omega}_{h,T} := \{\dot{x} \in \dot{\Omega}_h : t \leq T\}$. Indeed, by application of (45.26) and Theorem 45.3 we obtain

$$\begin{aligned} \|u\|_{\dot{\Omega}_{h,T}} &\leq c_{\dot{\Omega}_{h,T}} \|(\dot{P}_h - E)u\|_{\dot{\Omega}_{h,T}} + \|u\|_{\partial\dot{\Omega}_{h,T}} \\ &\leq \min\left\{\frac{T\ell}{h^2}, \frac{\ell}{4}n^2\right\} \frac{h^2}{\ell} \|\dot{\Delta}_h u\|_{\dot{\Omega}_{h,T}} + \|u\|_{\partial\dot{\Omega}_{h,T}} \\ &= \min\left\{T, \frac{1}{4}\right\} \|\dot{\Delta}_h u\|_{\dot{\Omega}_{h,T}} + \|u\|_{\partial\dot{\Omega}_{h,T}}. \end{aligned}$$

Next we apply inequality (45.35) to $u_h - u$, where u_h and u are the solutions of the problems (45.33) and (45.32), respectively, and we get the following error estimate

$$\begin{aligned} \|(u_h - u)\|_{\dot{\Omega}_{h,T}} &\leq \min\left\{T, \frac{1}{4}\right\} \|\dot{\Delta}_h(u_h - u)\|_{\dot{\Omega}_{h,T}} + \|(u_h - u)\|_{\partial\dot{\Omega}_{h,T}} \\ &= \min\left\{T, \frac{1}{4}\right\} \| - f - \dot{\Delta}_h u\|_{\dot{\Omega}_{h,T}} \\ &= \min\left\{T, \frac{1}{4}\right\} \|\dot{\Delta} u - \dot{\Delta}_h u\|_{\dot{\Omega}_{h,T}}, \end{aligned}$$

i.e., we have derived

$$\|u_h - u\|_{\overline{\dot{\Omega}}_{h,T}} = \|u_h - u\|_{\dot{\Omega}_{h,T}} \leq \min\left\{T, \frac{1}{4}\right\} \|\dot{\Delta} u - \dot{\Delta}_h u\|_{\dot{\Omega}_{h,T}}. \tag{45.36}$$

Now the rate of convergence of $u_h \to u$ as $h \downarrow 0$ can be measured via the partial moduli of continuity ω_1, $\omega_{2,i}$, $i = 1, \ldots, \ell$, where

$$\begin{aligned} \omega_1(\delta, f; \overline{\dot{\Omega}}) &:= \sup\{|f(\dot{x} + \lambda e) - f(\dot{x})| \colon \dot{x}, \dot{x} + \lambda e \in \overline{\dot{\Omega}}, |\lambda| < \delta\}, \\ \omega_{2,i}(\delta, f; \overline{\dot{\Omega}}) &:= \sup\{|f(\dot{x} + \lambda e_i) - 2f(\dot{x}) + f(\dot{x} - \lambda e_i)| \colon \dot{x}, \dot{x} \pm \lambda e_i \in \overline{\dot{\Omega}}, |\lambda| < \delta\}. \end{aligned}$$

Theorem 45.4. *Assume that the solution u of the problem (45.32) satisfies $u \in C^{(2)}(\overline{\dot{\Omega}})$. Then for the solution u_h of the problem (45.33) the following inequality holds true*:

$$\begin{aligned} \|u_h - u\|_{\overline{\dot{\Omega}}_{h,T}} \leq \min\left\{T, \frac{1}{4}\right\} \Bigg[\frac{1}{4}\sum_{i=1}^{\ell} \omega_{2,i}(h, \partial_{x_i}^2 u; \overline{\dot{\Omega}}_{h,T}) \\ + \frac{1}{2}\sum_{i=1}^{\ell} \omega_1(k, \partial_{x_i}^2 u; \overline{\dot{\Omega}}_{h,T}) + \omega_1(k, \partial_t u; \overline{\dot{\Omega}}_{h,T})\Bigg]. \end{aligned} \tag{45.37}$$

Proof. For $u \in C^{(2)}(\overline{\dot{\Omega}})$ and $k = h^2/\ell$ we have

$$\begin{aligned} \dot{\Delta}_h u(x,t) &= \frac{1}{2}h^{-2}\sum_{i=1}^{\ell}[u(x + he_i, t-k) + u(x - he_i, t-k) - 2u(x,t)] \\ &= \frac{1}{2}h^{-2}\left\{\sum_{i=1}^{\ell}[u(x + he_i, t-k) + u(x - he_i, t-k) - 2u(x,t-k)]\right\} \\ &\quad - \frac{1}{k}[u(x,t) - u(x,t-k)] := \sum_{i=1}^{\ell} \Delta_{h,i} u(x,t-k) - \partial_{k,t} u(x,t). \end{aligned}$$

By appropriate Taylor expansions, we get

$$|\Delta_{h,i} u(x,t-k) - \frac{1}{2}\partial_{x_i}^2 u(x,t-k)| \leq \frac{1}{4}\omega_{2,i}(h, \partial_{x_i}^2 u; \overline{\dot{\Omega}}_{h,T}),$$

and

$$|\partial_{k,t}u(x,t) - \partial_t u(x,t)| \le \omega_1(k, \partial_t u; \overline{\dot{\Omega}}_{h,T}).$$

Thus we obtain the following estimate

$$|\dot{\Delta}_h u(x,t) - \dot{\Delta} u(x,t)| \le \frac{1}{4}\sum_{i=1}^{\ell} \omega_{2,i}(h, \partial^2_{x_i} u; \overline{\dot{\Omega}}_{h,T})$$
$$+ \frac{1}{2}\sum_{i=1}^{\ell} \omega_1(k, \partial^2_{x_i} u; \overline{\dot{\Omega}}_T) + \omega_1(k, \partial_t u; \overline{\dot{\Omega}}_{h,T}).$$

Finally we apply this estimate to (45.36) to derive the desired result. ■

Remark 45.5. The estimate (45.37) is sharp, i.e., there exists a function u such that

$$\liminf_{h\to 0} \|u - u_h\|_{\dot{\Omega}_{h,T}} / R(h,u) > 0, \tag{45.38}$$

where $R(h,u)$ is a right-hand side of the inequality (45.37).

Indeed, choose $u(\dot{x}) := x_1^4$ and compute the both sides of the inequality (45.37). We will have

$$\begin{aligned} \dot{\Delta} u(\dot{x}) &= 6x_1^2, \\ \dot{\Delta}_h u(\dot{x}) &= 6x_1^2 + h^2, \end{aligned}$$

that is,

$$\dot{\Delta} u(\dot{x}) - \dot{\Delta}_h u(\dot{x}) = -h^2.$$

Now we apply (45.34) to the function $u - u_h$, which equals zero on the boundary $\partial\dot{\Omega}_h$, and obtain

$$\begin{aligned} u(\dot{x}) - u_h(\dot{x}) &= -\frac{h^2}{\ell}\dot{G}_{\dot{\Omega}_h}(\dot{\Delta}_h u - \dot{\Delta}_h u_h)(\dot{x}) = -\frac{h^2}{\ell}\dot{G}_{\dot{\Omega}_h}(\dot{\Delta}_h u - \dot{\Delta} u)(\dot{x}) \\ &= -\frac{h^4}{\ell}\dot{G}_{\dot{\Omega}_h} 1(\dot{x}). \end{aligned} \tag{45.39}$$

Thus, by (45.39) and Theorem 45.3 we have

$$\begin{aligned} \|u - u_h\|_{\overline{\dot{\Omega}}_{h,T}} &= \frac{h^4}{\ell}\|\dot{G}_{\dot{\Omega}_h} 1\|_{\overline{\dot{\Omega}}_{h,T}} \ge \frac{1}{\ell} n^{-2} h^2 \cdot \frac{1}{2}\min\left\{\frac{T\ell}{h^2}, \frac{2}{\pi^2} n^2\right\} \\ &= \frac{1}{2}\min\left\{T, \frac{2}{\pi^2 \ell}\right\} h^2. \end{aligned} \tag{45.40}$$

On the other hand,

$$\begin{aligned} R(h,u) &= \min\left\{T, \frac{1}{4}\right\} \cdot \frac{1}{4}\sum_{i=1}^{\ell} \omega_{2,i}(h, \partial^2_{x_i} u; \overline{\dot{\Omega}}_{h,T}) \\ &= \min\left\{T, \frac{1}{4}\right\} \cdot \frac{1}{4} \cdot \omega_{2,i}(h, \partial^2_{x_i} u; \overline{\dot{\Omega}}_{h,T}) \\ &= \min\left\{T, \frac{1}{4}\right\} 6h^2. \end{aligned} \tag{45.41}$$

Therefore (45.40) and (45.41) imply (45.38). ■

45.5 Sharpness for the Error Estimates of the Dirichlet Problem for the Heat Equation

As it was mentioned in Remark 45.5 the error estimate (45.37) is sharp, that is, there exists a function u such that

$$\|u - u_h\|_{\overline{\dot{\Omega}}_{h,T}} \asymp R(h,u) \quad \text{as } h \downarrow 0,$$

where $R(h,u)$ is the right-hand side of the inequality (45.37).

The fact that (45.37) is sharp with regard to the rate of convergence is now established in connection to general Lipschitz classes, determined by an abstract modulus of continuity, i.e., by a function ω, continuous on $[0,+\infty)$ such that

$$0 = \omega(0) < \omega(s) \le \omega(s+t) \le \omega(s) + \omega(t), \quad s,t > 0.$$

Here we follow the same technique that was applied in [125], [124] and [157]. These were articles devoted to the elliptic and parabolic Dirichlet problem in dimensions $\ell = 1,2$ and $\ell = 1$, respectively. Our reasoning is based on the following variant of the uniform boundedness principle [144]. For a Banach space $(X, \|\cdot\|)$ let X^* be the set of sublinear bounded functionals on X. We have

Theorem 45.6. *Assume that for given $\{T_n\}_{n\in\mathbb{N}} \subset X^*$ and $\{S_\delta\}_{\delta>0} \subset X^*$ there are $\{g_n\}_{n\in\mathbb{N}} \subset X$ such that*

$$\|g_n\| \le c_1, \quad n = 1,2,\ldots, \tag{45.42}$$

$$\liminf_{n\to\infty} |T_n g_n| > 0, \tag{45.43}$$

$$|S_\delta g_n| \le c_2 \min\left\{1, \frac{\sigma(\delta)}{\varphi_n}\right\}, \quad n = 1,2,\ldots, \tag{45.44}$$

where $\sigma(\delta)$ is a strictly positive function on $(0,\infty)$, and $\{\varphi_n\}_{n\in\mathbb{N}}$ is a strictly decreasing real sequence with $\lim_{n\to\infty}\varphi_n = 0$. Then for each modulus of continuity ω as above, satisfying

$$\lim_{t\to 0}\frac{\omega(t)}{t} = \infty, \tag{45.45}$$

there exists an element $u_\omega \in X$ such that

$$|S_\delta u_\omega| \le c_\omega \omega(\sigma(\delta)), \quad 0 < \delta < 1, \tag{45.46}$$

$$\liminf_{n\to\infty} |T_n u_\omega| / \omega(\varphi_n) > 0. \tag{45.47}$$

Next comes our optimal result.

Theorem 45.7. *For every modulus of continuity* ω *there exists a function* $u_\omega \in C^2(\overline{\dot\Omega})$ *such that*

$$R(\delta, u_\omega) \le c_\omega \omega(\delta^2), \quad 0 < \delta < 1, \tag{45.48}$$

$$\liminf_{h\to 0} \|u_\omega - u_{\omega,h}\|_{\overline{\dot\Omega}_{h,T}} / \omega(h^2) > 0. \tag{45.49}$$

Proof. To apply Theorem 45.6 we denote by

$$X := C^2(\overline{\dot\Omega}),$$
$$T_n u := \|u - u_h\|_{\overline{\dot\Omega}_{h,T}}, \quad h = \frac{1}{n},$$
$$S_\delta u := R(\delta, u), \quad 0 < \delta < 1,$$

and

$$g_n(\dot x) := n^{-2} \sum_{i=1}^{\ell} \sin^2 \pi n x_i, \ \dot x = (x,t), \ x = (x_1, \dots, x_\ell) \in \Omega.$$

Then (45.42) is satisfied with $c_1 = \pi^2 \ell$. We see that $g_n(\dot x) = g_{n,h}(\dot x)$ for $x \in \partial\dot\Omega_{h,T}$, $h = \frac{1}{n}$, and $\dot\Delta g_n(\dot x) = \pi^2\ell$ and $\dot\Delta_h g_n(\dot x) = 0$ for all $\dot x \in \dot\Omega_{h,T}$. Then we have (cf. (45.34))

$$\begin{aligned}
T_n g_n &= \frac{h^2}{\ell} \|\dot G_{\dot\Omega_h} \dot\Delta_h (g_n - g_{n,h})\|_{\overline{\dot\Omega}_{h,T}} \\
&= \frac{h^2}{\ell} \|\dot G_{\dot\Omega_h} (\dot\Delta_h g_n - \dot\Delta_h g_{n,h})\|_{\overline{\dot\Omega}_{h,T}} \\
&= \frac{h^2}{\ell} \|\dot G_{\dot\Omega_h} (\dot\Delta_h g_n - \dot\Delta g_n)\|_{\overline{\dot\Omega}_{h,T}} \\
&= \frac{h^2}{\ell} \cdot \pi^2 \ell \|\dot G_{\dot\Omega_h} 1\|_{\overline{\dot\Omega}_{h,T}} \ge \min\left\{1, \frac{\pi^2 \ell T}{2}\right\}.
\end{aligned}$$

The last inequality comes from Theorem 45.3, and hence condition (45.43) is satisfied. To verify the condition (45.44) we observe that

$$S_\delta g_n \le \frac{1}{2}\pi^2 \ell,$$

and

$$S_\delta g_n \le \frac{\delta^2}{16} \sum_{i=1}^{\ell} \|\partial_{x_i}^{(4)} g_n\|_{\dot\Omega_{h,T}} \le \frac{\delta^2 n^2 \pi^4 \ell}{2}.$$

These upper bounds to $S_\delta g_n$ yield (45.44) with $\sigma(\delta) := \pi^2\delta^2$ and $\varphi_n := n^{-2}$. Thus we are able to apply Theorem 45.6 and (45.48), (45.49) are established. ■

46

Uniqueness of Solution in Evolution in Multivariate Time

46.1 Introduction

The classical time dependent partial differential equations of mathematical physics involve evolution in one dimensional time. Space can be multidimensional, but time stays one dimensional. There are various mathematical situations (such as multiparameter Brownian motion) which suggest that there should be a mathematical theory of evolution in multidimensional time. We formulate a rather general class of equations that involve two "time dimensions" and we prove a related uniqueness theorem.

In Section 46.2 we discuss the formulation of a two dimensional time model. The uniqueness theorem is formulated in Section 46.3 and proved in Section 46.4. Some examples are given in Section 46.6. This chapter is based on [76].

46.2 Bivariate Time

Let s, t be real variables. Then the Partial Differential Equation (PDE)

$$\frac{\partial^2 u}{\partial s \partial t} = f \tag{46.1}$$

becomes

$$\frac{\partial^2 v}{\partial \tau^2} - \frac{\partial^2 v}{\partial x^2} = g \tag{46.2}$$

G.A. Anastassiou: Intelligent Mathematics: Computational Analysis, ISRL 5, pp. 765–772.
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under the change of variables $\tau = s + t$, $x = s - t$; here $v(\tau, x) = u(s, t)$, that is, v is u but thought of as a function of τ and x. Also, $g(\tau, x, v, \nabla v, \ldots) = f(s, t, u, \nabla u, \ldots)$, i.e., g is f but viewed as a function of τ, x, v and derivatives of v. Thus (46.1) is often viewed as a hyperbolic PDE, although the nature of (46.1) [or (46.2)] depends upon the form of f, which itself can involve partial derivatives of u (or v). In studying (46.2) one can view either τ or x as a time variable.

The equation

$$\frac{\partial^2 v}{\partial \tau^2} - \frac{\partial^2 v}{\partial x^2} = f(\tau, x, v, \nabla v)$$

is formally symmetric in τ and x, and in the absence of other considerations it is not clear that τ (respectively x) should be called "the" time. Both τ and x have equivalent status. For

$$\frac{\partial^2 v}{\partial \tau^2} - \frac{\partial^2 v}{\partial x^2} = a\Delta_y v_y$$

where $y \in \Omega \subset \mathbb{R}^n$ is a spatial variable, this equation is, for $a = 1$, hyperbolic if we view τ as the time but not hyperbolic if we view x as the time. It is not of any standard type if $a = i$.

The perspective here is to treat *both* τ and x (or s and t) as time variables. Thus we think of (46.1) as a PDE involving two dimensional time, whenever f depends on u and its derivatives with respect to other variables.

The main goal of this chapter is to formulate and prove a uniqueness theorem for a large class of problems of the form (46.1). The context will be quite general and will include both well-posed and ill-posed initial value problems. We confine this study to two dimensional time; extension to the higher dimensional case can also be done. The main result is stated at the end of Section 46.3.

A main point is that in the uniqueness theorem, the hypotheses (including the initial conditions) are symmetric in both s and t (or τ and x). Thus s and t should have equivalent status; it would be inappropriate to view one as a space variable and the other as a time variable. We specify initial conditions in each variable. This lends strong support to their interpretation as two time variables.

46.3 The Uniqueness Theorem

Let u be a function of $(s, t) \in D \subset \mathbb{R}^2$; u is supposed to take values in some Banach space X and to be sufficiently smooth. We define the (k, j)-*jet* of u to be

$$D^{k,j}u = \left\{ \frac{\partial^{\rho+\sigma}}{\partial s^\rho \partial t^\sigma} : 0 \le \rho \le k,\ 0 \le \sigma \le j \right\};$$

here the dummy indices ρ, σ are integers.

Recall that if $v\colon Y \to X$, where Y, X are Banach spaces, then the derivative $v'(x)$ of v at $x \in Y$ is defined by

$$v(x+h) = v(x) + v'(x)(h) + o(\|h\|)$$

as $h \to 0$; here $v'(x) \in \mathcal{L}(Y, X)$, i.e., $v'(x)$ is a bounded linear operator from Y to X. Similarly,

$$v'(x+h) = v'(x) + v''(x)(h) + o(\|h\|)$$

as $h \to 0$; thus $v''(x) \in \mathcal{L}(Y, \mathcal{L}(Y, X))$. It is now clear how the partial derivatives $\partial^{\rho+\sigma}u/\partial s^\rho \partial t^\sigma$ are defined. It follows that $D^{k,j}u(s,t)$ is a point in a Banach space $Z = Z(X, k, j)$ which can be explicitly constructed from X, k and j. Note that when $X = \mathbb{R}$, Z becomes $\mathbb{R}^L$ where $L = M(k,j)$ is some computable function of k and j.

The Partial Differential Equation (PDE) that we consider has the form

$$\frac{\partial^{n+m}u}{\partial s^n \partial t^m} = F\big(s, t, D^{n-1,m-1}u\big) \tag{46.3}$$

and involves a finite family $\{A_{kj}\}$ of commuting selfadjoint or normal operators on a complex Hilbert space $\mathbb{H}$. Take $X = \mathbb{H}$ and consider the following version of (46.3):

$$\frac{\partial^{n+m}u}{\partial s^n \partial t^m} = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} F_{kj}\big(s, t, D^{k,j}u\big) A_{kj} u, \tag{46.4}$$

where each F_{kj} is a complex valued function, and n, m are positive integers. Now we take $D = [0, a) \times [0, b)$ where $0 < a, b \le \infty$.

Hypothesis 46.1. *Let* $\{A_{kj}\colon 0 \le k \le n-1,\ 0 \le j \le m-1\}$ *be a commuting family of normal operators on* $\mathbb{H}$, *and let, for* $0 \le k \le n-1,\ 0 \le j \le m-1$,

$$F_{kj}\colon D \times Z(\mathbb{H}, k, j) \to \mathbb{C}$$

fulfill the following Caratheodory-Lipschitz condition: F_{kj} *is jointly measurable* (*relative to the Borel sets*) *and*

$$|F_{kj}(x, y) - F_{kj}(x, z)| \le K(x)\|y - z\|_{Z(\mathbb{H},k,j)}$$

where K *is locally Lebesgue integrable on* D.

Note that the definition of the Caratheodory-Lipschitz condition has the obvious extension to $f\colon D_1 \times D_2 \to Y$, where $D_1 \subset \mathbb{R}^\ell$, $D_2 \subset X$, and X, Y are Banach spaces.

By a *solution* of (46.4) we mean a function $u\colon D \to \mathbb{H}$ which is $n-1$ times (resp. $m-1$ times) weakly continuously differentiable in $s \in (0, a]$ [resp. in $t \in [0, b)$], $\frac{\partial^{n+m-2}u}{\partial s^{n-1}\partial t^{m-1}}(s,t)$ is absolutely continuous in each of s, t

and so is its gradient, and the resulting distributional derivative, $\frac{\partial^{n+m}u}{\partial s^n \partial t^m}$, which exists, is equal to the right hand side of (46.4) (and so the equality holds pointwise a.e.).

The main result follows

Theorem 46.2. *Let* Hypothesis 46.1 *hold. Let*

$$f_k\colon [0,a) \to \mathbb{C}, \quad g_j\colon [0,b) \to \mathbb{C}$$

be given continuous functions for $0 \le k \le n-1$, $0 \le j \le m-1$. *Then there is at most one solution of* (46.4) *satisfying the initial conditions*

$$\begin{aligned} \frac{\partial^k u}{\partial s^k}(0,t) &= f_k(t), \quad t \in [0,a),\ 0 \le k \le n-1, \\ \frac{\partial^j u}{\partial t^j}(s,0) &= g_j(s), \quad s \in [0,b),\ 0 \le j \le m-1. \end{aligned}$$

Note that (s,t) lies in the first quadrant of the (s,t) plane, and the initial conditions are specified on (a portion of) the boundary of this quarter plane.

46.4 Proof of Theorem 46.2

By a version of the spectral theorem (see e.g. [173], [203], [254]), there is a unitary operator U from $\mathbb{H}$ to a concrete L^2 space, $L^2(\Lambda, \Sigma, \mu)$, such that

$$A_{kj} = U^{-1} M_{a_{kj}} U \tag{46.5}$$

for all (k,j), where a_{kj} is a Σ-measurable complex valued function on Λ, and $M_{a_{kj}}$ is the corresponding maximal multiplication operator:

$$\big(M_{a_{kj}} f\big)(x) = a_{kj}(x) f(x), \quad x \in \Lambda,$$

$f \in \operatorname{Dom}\big(M_{a_{kj}}\big)$ if and only if $f, a_{kj} f \in L^2(\Lambda, \Sigma, \mu)$. This holds for all k, j.

Let

$$\tilde{u}(s,t,x) = \big(U u(s,t,\cdot)\big)(x), \quad x \in \Lambda. \tag{46.6}$$

Then, using (46.5) and (46.6), we see that (46.4) is equivalent to

$$\frac{\partial^{n+m}\tilde{u}}{\partial s^n \partial t^m} = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \tilde{F}_{kj}(s,t,D^{k,j}\tilde{u}) a_{kj} \tilde{u}, \tag{46.7}$$

where

$$\tilde{F}_{kj}(s,t,\tilde{w}) = F_{kj}(s,t,U^{-1}\tilde{w}).$$

Now, (46.7) is a scalar PDE which can be rewritten as

$$\frac{\partial^{n+m}\tilde{u}}{\partial s^n \partial t^m}(s,t) = G\big(s,t,D^{n-1,m-1}\tilde{u}(s,t)\big).$$

If $n = m = 1$, this reduces to

$$\frac{\partial^2 \tilde{u}}{\partial s \partial t}(s,t) = G\big(s, t, \tilde{u}(s,t)\big).$$

Integration produces

$$\frac{\partial \tilde{u}}{\partial s}(s,t) - \frac{\partial \tilde{u}}{\partial s}(s,0) = \int_0^t G\big(s, t_1, \tilde{u}(s,t_1)\big)\, dt_1,$$

$$\tilde{u}(s,t) - \tilde{u}(0,t) - \big(\tilde{u}(s,0) - \tilde{u}(0,0)\big) = \int_0^s \int_0^t G\big(s_1, t_1, \tilde{u}(s_1,t_1)\big)\, dt_1 ds_1.$$

Because $G(s,t,\tilde{u}(s,t))$ is locally integrable in both s,t (for continuous $\tilde{u}$), this shows that $\{\tilde{u}(0,t)\colon 0 \le t < b\}$, $\{\tilde{u}(s,0)\colon 0 \le s < a\}$ uniquely determine $\tilde{u}$.

If $n = 2$, $m = 1$, the equation is

$$\frac{\partial^3 \tilde{u}}{\partial s^2 \partial t}(s,t) = G\left(s, t, \tilde{u}(s,t), \frac{\partial \tilde{u}}{\partial s}(s,t)\right).$$

Integration implies

$$\frac{\partial^2 \tilde{u}}{\partial s \partial t}(s,t) = \frac{\partial^2 \tilde{u}}{\partial s \partial t}(0,t) + \int_0^s G\left(s, t, \tilde{u}(s_1,t), \frac{\partial \tilde{u}}{\partial s}(s_1,t)\right) ds_1.$$

Since $\frac{\partial u}{\partial s}(0,t) = f_1(t)$, it follows that

$$\frac{\partial^2 u}{\partial s \partial t}(0,t) = f_1'(t), \quad 0 \le t < b.$$

Now we integrate to obtain

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial s}(s,t) &= \frac{\partial \tilde{u}}{\partial s}(s,0) - \left[\frac{\partial \tilde{u}}{\partial s}(0,t) - \frac{\partial \tilde{u}}{\partial s}(0,0)\right] \\
&\quad + \int_0^t \int_0^s G\left(s_1, t_1, \tilde{u}(s_1,t_1), \frac{\partial \tilde{u}}{\partial s}(s_1,t_1)\right) ds_1 dt_1 \\
&= Q(s,t) + \int_0^t \int_0^s G\left(s_1, t_1, \tilde{u}(s_1,t_1), \frac{\partial \tilde{u}}{\partial s}(s_1,t_1)\right) ds_1 dt_1,
\end{aligned}$$

where Q is a known function (since $\frac{\partial \tilde{u}}{\partial s}(0,t)$, $\frac{\partial \tilde{u}}{\partial t}(s,0)$ are given). We rewrite this last equation as an equation in $w = \frac{\partial \tilde{u}}{\partial s}$:

$$w(s,t) = Q(s,t) + \int_0^t \int_0^s G\left(s_1, t_1, \int_{s_1}^{t_1} w(s_2,t_1) ds_2 + w(s_1,t_1), w(s_1,t_1)\right) ds_1 dt_1. \tag{46.8}$$

Letting $(\mathcal{S}w)(s,t)$ be the right hand side of (46.8), the (global) Lipschitz condition assumption on G implies that $\mathcal{S}$ has a unique fixed point in

$C([0,a_1]\times[0,b_1])$ for all $0<a_1<a$, $0<b_1<b$. This fixed point is $\partial\tilde{u}/\partial s$, and this can be integrated to obtain a unique solution $\tilde{u}$ on D (since the initial conditions on $\tilde{u}$ have been properly specified). This proof is basically the standard Ordinary Differential Equation (ODE) proof of existence and uniqueness, suitably modified (see, e.g. [183]).

Now fix $m=1$ and do mathematical induction on n. We checked the result for $n=1,2$. If it holds for $n\le N-1$, then, from

$$\frac{\partial^{N+1}\tilde{u}}{\partial s^N\partial t}=\sum_{k=0}^{N-1}F_{k0}(s,t,D^{k,0}\tilde{u})a_{k0}\tilde{u},$$

integration implies

$$\frac{\partial^{N}\tilde{u}}{\partial s^{N-1}\partial t}(s,t)=\frac{\partial^{N}\tilde{u}}{\partial s^{N-1}\partial t}(0,t)+\sum_{k=0}^{N-1}\int_0^k F_{k0}\big(s_1,t,D^{k,0}\tilde{u}(s_1,t)\big)a_{k0}\tilde{u}(s_1,t)ds_1.$$

Now,

$$\frac{\partial^{N}\tilde{u}}{\partial s^{N-1}\partial t}(0,t)=\tilde{f}'_N(t)$$

is known for $t\in[0,b)$, since

$$\tilde{f}_N(t)=\frac{\partial^{N-1}}{\partial s^{N-1}}\tilde{u}(0,t)=(Uf_N)(t)$$

is determined by the initial data. It follows from the induction hypothesis that $\{\tilde{u}(s,t)\colon (s,t)\in D\}$ is uniquely determined by its initial values, according to the induction hypothesis.

Now we have the result for $(n,1)$ for all $n\ge 1$. The result for all (n,m) now follows by a similar induction argument on m. We omit the details.

The spectral theorem enabled us to replace (46.4) by a family of scalar valued problems (indexed by Λ). The definition of solution shows that uniqueness for the transformed problem (46.7) is equivalent to uniqueness for (46.4). This completes the proof. ∎

46.5 History, Motivation and Related Results

We use the notation of the previous section except for the modifications noted below. Let $D=[0,a]\times[0,b]$ be a compact rectangle in $\mathbb{R}^2$. The pioneering result in this area is due to Agarwal and Pang [5].

Theorem 46.3. [5, p. 360]. *Consider the scalar PDE*

$$\frac{\partial^{n+m}u}{\partial s^n\partial t^m}(s,t)=F\big(s,t,D^{n-1,m-1}u(s,t)\big) \tag{46.9}$$

with initial conditions

$$\frac{\partial^k u}{\partial s^k}(0,t) = f_k(t), \quad t \in [0,b],\ 0 \le k \le n-1, \tag{46.10}$$

$$\frac{\partial^j u}{\partial t^j}(s,0) = g_j(s), \quad s \in [0,a),\ 0 \le j \le m-1. \tag{46.11}$$

Assume $F\colon D \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and satisfies*

$$\begin{aligned}&\left|F(s,t,D^{n-1,m-1}w_1(s,t)) - F(s,t,D^{n-1,m-1}w_2(s,t))\right| \\ &\qquad \le \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} q_{kj}(s,t)\left|\frac{\partial^{k+j}}{\partial s^k \partial t^j}(w_1 - w_2)(s,t)\right|\end{aligned} \tag{46.12}$$

where $0 \le q_{kj} \in C(D,\mathbb{R})$ *for all* $0 \le k \le n-1$, $0 \le j \le m-1$. *Then the problem* (46.9)–(46.11) *has at most one classical solution.*

The authors of Theorem 46.3 used different notation, but we chose to use notation which illustrates its close relationship with the Theorem 46.2. Their proof was based on a two dimensional version of Opial's inequality that they obtained [5, p. 212]. We describe this next.

Proposition 46.4. [5, p. 212]. *Let* $p, q\colon D \to [0,\infty)$ *be Lebesgue measurable. Let* $u \in C^{n-1,m-1}(D)$ *be such that* $\frac{\partial^k}{\partial s^k}u(0,s) = 0$ *for* $0 \le k \le n-1$, $s \in [0,a)$, *and* $\frac{\partial^j}{\partial t^j}u(t,0) = 0$ *for* $0 \le j \le m-1$, $t \in [0,b)$; *and* $\frac{\partial^{n+m}}{\partial s^n \partial t^{m-1}}u$ *and* $\frac{\partial^{n+m} u}{\partial s^{n-1}\partial t^m}$ *are absolutely continuous on* D. *For* $0 \le k \le n$, $0 \le m \le j$, *let* $r_{kj} \in [0,\infty)$ *be such that* $\sigma_1 = \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} r_{kj} > 0$, $r_{nm} > 0$, $r > \max\{1, r_{nm}\}$. *Then*

$$\int_0^a \int_0^b G(q,w)ds\,dt \le C(\sigma_1)\left[\int_0^a \int_0^b H(p,w,r)ds\,dt\right]^{\frac{\sigma_1 + r_{nm}}{r}}, \tag{46.13}$$

where

$$G(q,w) = q\left(\prod_{k=0}^{n-1}\prod_{j=0}^{m-1}\left|\frac{\partial^{k+j} w}{\partial s^n \partial t^j}\right|^{r_{kj}}\right)\left|\frac{\partial^{n+m}}{\partial s^n \partial t^m}w\right|^{r_{nm}}$$

and

$$H(p,w,r) = p\left|\frac{\partial^{n+m}}{\partial s^n \partial t^m}w\right|^r,$$

and $C(\sigma_1)$ *is a constant depending on the parameters but not on* w.

It is of course assumed that the integral on the right hand side of (46.13) is finite; otherwise the result is trivial. If u and v are solutions of (46.9)–(46.11), then Theorem 46.3 is proved by applying Proposition 46.4 to $w = u - v$.

Proposition 46.5. [5, p. 361]. *In Theorem* 46.3, *condition* (46.12) *can be replaced by*

$$\left|F(s,t,D^{n-1,m-1}w_1(s,t)) - F(s,t,D^{n-1,m-1}w_2(s,t))\right| \leq q(s,t)\prod_{k=0}^{n-1}\prod_{j=0}^{m-1}\left|\frac{\partial^{k+j}}{\partial s^k \partial k^j}(w_1 - w_2)(s,t)\right|^{r_{k,j}}$$

where $0 \leq q \in C(D, \mathbb{R})$ *and the* $r_{k,j}$ *are nonnegative constants such that*

$$\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} r_{k,j} \geq 1.$$

Then uniqueness holds for the problem (46.9)–(46.11).

46.6 Examples

Endow the Laplacian with Robin boundary conditions on a domain Ω in Euclidean space: $\Omega \subseteq R^n$, $\alpha u + \beta \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, where $\alpha, \beta \in C(\partial\Omega)$, $\alpha(x)^2 + \beta(x)^2 > 0$ for all x on the boundary. Consider

$$\frac{\partial^2 u}{\partial s \partial t} = a_0(s,t,u) + a_1(s,t,u)\Delta u + a_2(s,t,u,\Delta u)\Delta^2 u, \tag{46.14}$$

or more generally

$$\frac{\partial^2 u}{\partial s \partial t} = a_0(s,t,u) + \sum_{j=1}^{n} a_j(s,t,u,\ldots,\Delta^{j-1}u)\Delta^j u.$$

Equation (46.14) is related to a beam equation when $a_2 \neq 0$. The special case of $a_1 \neq 0$, $a_2 = 0$ is related to the backward heat equation as well as the (forward) heat equation.

Next let $x \in R^n$, $y \in R^m$, and consider

$$\frac{\partial^2 u}{\partial s \partial t} = a(u)\Delta_x u - b(u)\Delta_y u$$

with a and b positive. This can be solved by Fourier transforms when a and b are constants. This is related to heat, backward heat, and wave equations.

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